UNIVERSITÉ SAVOIE MONT BLANC

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ SAVOIE MONT BLANC

Spécialité : **Mathématiques**

Arrêté ministériel : 25 Mai 2016

Présentée par

Mickaël Nahon

Thèse dirigée par **Dorin Bucur** et
codirigée par **Alessandro Giacomini**

préparée au sein du **Laboratoire LAMA**
dans **l'École Doctorale MSTII**

Problèmes à discontinuité libre de type Robin

Thèse soutenue publiquement le **7 juillet 2022**,
devant le jury composé de :

Emmanuel Russ
Président
Jimmy Lamboley
Rapporteur
Felix Otto
Rapporteur
Antonin Chambolle
Examinateur
Radu Ignat
Examinateur
Yannick Privat
Examinateur

Contents

1 Introduction (version française) 6

1.1 Notion de stabilité d'une inégalité géométrique . . . 10
1.1.1 Stabilité de l'inégalité de Saint-Venant . . . 12
1.1.2 Instabilité de l'inégalité de Weinstock . . . 14
1.1.3 Cas de l'inégalité de Hersch . . . 15
1.2 Optimisation des valeurs propres d'ordres supérieurs . . . 16
1.3 Régularité d'un problème à frontières libres avec obstacle . . . 17
1.4 Un problème d'optimisation de forme pour l'isolation thermique . . . 19
1.5 Positivité des fonctions surharmoniques vérifiant une condition de Robin . . . 21
1.6 Optimisation de la forme d'un obstacle plongé dans un fluide de Stokes . . . 23

2 Introduction (english version) 29

2.1 Notion of stability of a geometric inequality . . . 32
2.1.1 Stability of the Saint-Venant inequality . . . 35
2.1.2 Instability of the Weinstock inequality . . . 37
2.1.3 Case of the Hersch inequality . . . 38
2.2 Optimization of the higher eigenvalues of the Laplacian with Robin boundary condition . . . 38
2.3 Regularity of a free boundary transmission problem with obstacle . . . 40
2.4 A shape optimization problem linked to thermal insulation . . . 41
2.5 Positivity of superharmonic functions verifying a Robin boundary condition . . . 44
2.6 Optimization of an obstacle immerged in a Stokes fluid . . . 45

3 A survey of rectifiable sets and special bounded variation functions 51

3.1 Generalities on geometric measure theory . . . 51
3.2 BV functions - definition, embedding and compactness properties . . . 55
3.3 Sets of finite perimeter . . . 56
3.4 Unidimensional BV functions, structure theorem . . . 58
3.5 Fine properties of BV functions . . . 59
3.6 Compactness of SBV functions . . . 63
3.7 A quick overview of SBD functions . . . 64

4 Degenerate free discontinuity problems and spectral inequalities in quantitative form 67

4.1 Introduction . . . 67
4.2 Introduction of the new functional, main results and strategy of the proofs . . . 70
4.3 Analysis of the analytic-geometric functional . . . 72
4.3.1 Preparatory results . . . 73
4.3.2 Nondegeneracy of the minimizers . . . 75
4.3.3 Closedness of the jump set . . . 77
4.3.4 The optimal function is radially symmetric . . . 79
4.3.5 Existence of minimizers . . . 82
4.4 The ball minimizes the geometric functional . . . 87
4.5 Proof of Theorems 4.1 and 4.2 . . . 87

5 Stability and instability issues of the Weinstock inequality 93
5.1 Introduction . . . 93
5.2 The Steklov problem and some stability properties . . . 95
5.3 A stability result under a priori bounds . . . 102
5.4 Further remarks . . . 106
5.5 Stability of the Hersch inequality . . . 112

6 Existence and regularity of optimal shapes for spectral functionals with Robin boundary conditions 123
6.1 Introduction . . . 123
6.1.1 State of the art . . . 124
6.1.2 Statements of the main results . . . 125
6.2 Relaxed framework . . . 127
6.3 Strictly monotonous functionals . . . 129
6.3.1 A priori estimates . . . 129
6.3.2 Existence of a relaxed minimizer with prescribed measure . . . 135
6.3.3 Regularity of minimizers . . . 137
6.4 The functional $\Omega \mapsto \lambda_k(\Omega;\beta)$ . . . 140
6.4.1 Regularization and perturbation lemma . . . 141
6.4.2 Non-degeneracy lemma and the main result . . . 143
6.4.3 Discussion about the properties of open minimizers . . . 146

7 Regularity of a free boundary problem with transmission condition 150
7.1 Introduction . . . 150
7.2 Uniform regularity estimates . . . 153
7.3 Convergence of minimizers . . . 156
7.4 Monotonicity formula and blow-ups . . . 160
7.5 Epiperimetric inequality at the contact set . . . 164
7.5.1 Estimate near small blow-ups . . . 166
7.5.2 Estimate away from large blow-ups . . . 168
7.5.3 Preliminary computations . . . 168
7.5.4 Erasure of small sectors and high frequencies . . . 172
7.5.5 Gradient descent on low-frequency solutions near large blow-up . . . 175
7.6 Nondegeneracy and regularity of the contact set . . . 178
7.6.1 Nondegeneracy lemma . . . 179
7.6.2 Regularity of the contact set . . . 180
7.7 Study of an example . . . 183

8 Shape optimization of a thermal insulation problem 186
8.1 Introduction . . . 186
8.2 The convection case: proof of Theorem 8.1 . . . 189
8.3 Preparatory results: existence and regularity of relaxed solutions . . . 192
8.4 The penalized problem: proof of Theorem 8.2 . . . 202
8.5 Proof of Theorem 8.3 . . . 206

9 Boundary behaviour of solutions of elliptic equation with Robin boundary conditions on rough domains 213
9.1 Introduction . . . 213
9.2 Global boundary behavior . . . 214
9.2.1 Examples . . . 217
9.3 Local boundary behaviour . . . 219
9.4 Analysis of cusps . . . 221
9.5 Further remarks . . . 229

10 Shape optimization of an obstacle immerged in a creeping flow **231**
10.1 Introduction . . . 231
10.2 Notations and Preliminaries . . . 235
10.2.1 Basic notation . . . 235
10.2.2 Functions of bounded variation and sets of finite perimeter . . . 236
10.2.3 Functions of bounded deformation . . . 237
10.2.4 Hausdorff convergence . . . 238
10.3 Obstacles in Stokes fluids and drag minimization . . . 241
10.3.1 The flow around the obstacle . . . 241
10.3.2 The drag force . . . 242
10.3.3 The optimization problem . . . 243
10.4 A relaxed formulation of the shape optimization problem and existence of optimal obstacles . 244
10.5 Some technical results in SBD . . . 245
10.5.1 An immersion result . . . 245
10.5.2 Closure of the non-penetration constraint . . . 247
10.5.3 A lower semicontinuity result for surface energies in SBD . . . 249
10.6 Proof of Theorem 10.16 . . . 254
10.7 The two dimensional case . . . 256
10.7.1 A two dimensional setting . . . 257
10.7.2 The optimization problem and existence of solutions . . . 260
10.8 Regularity of two-dimensional minimizers . . . 263
10.8.1 Smoothing lemma . . . 264
10.8.2 Decay estimate . . . 269
10.9 Remarks on the Navier-Stokes equation . . . 274

Remerciements

Je tiens à exprimer toute ma gratitude envers Dorin Bucur qui m'a proposé ce beau sujet et m'a beaucoup appris pendant ces quelques années ! Tous ces travaux n'auraient été possibles sans sa disponibilité et sa persévérance à toute épreuve. *Grazie mille* de même à Alessandro Giacomini qui m'a accueilli à Brescia pendant un automne afin de percer tous les secrets des problèmes à discontinuité libre.

Merci également aux rapporteurs, Jimmy Lamboley et Felix Otto qui ont lu attentivement ce manuscrit, pour la précision de leurs commentaires. Je suis honoré qu'ils se soient intéressés à mes travaux et je prends toutes leurs remarques très à coeur. Je suis aussi très reconnaissant envers Antonin Chambolle, Radu Ignat, Yannick Privat et Emmanuel Russ qui se sont rendus disponibles pour faire partie du jury.

Un grand merci à Bozhidar Velichkov qui m'a invité plusieurs fois à Pise et a partagé avec moi ses recherches, pour son hospitalité et pour avoir élargi mes horizons mathématiques.

Tous les travaux suivants sont aussi le fruit d'échanges et de collaborations bravant les pandémies et les distances, avec Iosif Polterovich, Mikhail Karpukhin, Daniel Stern, Carlo Nitsch et Cristina Trombetti, que j'espère avoir l'occasion de tous revoir en personne à l'avenir.

Chapter 1

Introduction (version française)

Sommaire

1.1 Notion de stabilité d'une inégalité géométrique 10
1.1.1 Stabilité de l'inégalité de Saint-Venant 12
1.1.2 Instabilité de l'inégalité de Weinstock 14
1.1.3 Cas de l'inégalité de Hersch 15
1.2 Optimisation des valeurs propres d'ordres supérieurs 16
1.3 Régularité d'un problème à frontières libres avec obstacle 17
1.4 Un problème d'optimisation de forme pour l'isolation thermique 19
1.5 Positivité des fonctions surharmoniques vérifiant une condition de Robin . . . 21
1.6 Optimisation de la forme d'un obstacle plongé dans un fluide de Stokes 23

L'objet de cette thèse est l'étude qualitative de problèmes d'optimisation de forme et de géométrie spectrale issus de situations physiques variées faisant intervenir des discontinuités. Il s'agira principalement de problèmes liés à des solutions d'équations aux dérivées partielles elliptiques avec des conditions de bord dites "de Robin" qui apparaissent entre autres dans les vibrations de membranes fixées élastiquement, dans les phénomènes de transfert thermique par convection, mais aussi sous des formes plus générales (conditions de Navier) dans des phénomènes d'adhérence partielle entre un fluide et une paroi.

Tous ces modèles présentent des discontinuités ; par exemple dans le cas d'un transfert thermique par convection entre un solide et un fluide, il y a une couche limite de transition rapide entre deux zones de températures distinctes qui sera modélisée dans notre cas par une discontinuité de la température selon une surface. On peut aussi citer l'étude des fractures en milieu élastique qui n'apparaîtront pas directement dans ces travaux mais qui sont à l'origine de certaines techniques utilisées. Notamment nous étudierons l'écoulement d'un fluide visqueux autour d'un obstacle, qui présente le même type de discontinuité.

Ces questions seront étudiées sous l'angle des problèmes à discontinuité libre. Plus précisément, il s'agira d'étudier des solutions d'équations aux dérivées partielles et des problèmes d'optimisation variés présentant des discontinuités selon des ensembles de codimension 1. Un exemple emblématique de ce type de problème est la fonctionnelle de Mumford Shah; étant donnés $g : \Omega \to [0, 1]$ une fonction définie sur un domaine $\Omega \Subset \mathbb{R}^n$, et trois constantes $a, b, c > 0$, le problème de Mumford Shah consiste en la minimisation de la fonctionnelle

$$(K, u) \mapsto a \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + b\mathscr{H}^{n-1}(K) + c \int_{\Omega \setminus K} (u - g)^2.$$

parmi tous les fermés $K \subset \Omega$ de dimension $n - 1$, et toutes les fonctions $u \in H^1(\Omega \setminus K)$.

Cette fonctionnelle a été créée à l'origine dans l'étude de la segmentation d'image; Ω est un rectangle de $\mathbb{R}^2$, g représente les niveaux de gris de l'image que l'on veut approximer par des niveaux u plus

lisses dans certaines zones, tout en permettant des transitions rapides (c'est-à-dire des discontinuités) sur un ensemble K de codimension 1. a, b, c sont des constantes choisies selon l'importance accordée à chaque phénomène que sont respectivement la régularité de u, la pénalisation de la longueur des discontinuités et la proximité avec l'image de départ g.

Il existe de nombreux travaux et conjectures autour des minimiseurs de cette fonctionnelle, sur le plan théorique on peut mentionner les livres [1] et [18] qui font un état de l'art d'il y a quelques années autour des questions de régularité des minimiseurs. En particulier, l'une des conjectures centrales est la caractérisation de l'ensemble de ces discontinuités K dans le cas bidimensionnel; est-ce effectivement une union finie de courbes lisses, ne pouvant se rencontrer qu'en des jonctions triples avec des angles de 120 degré ?

Cette fonctionnelle a plus tard trouvé d'autres applications notamment dans l'analyse des déformations et fractures des matériaux élastiques fragiles avec sa généralisation vectorielle

$$(K, u) \mapsto a \int_{\Omega \setminus K} |e(u)|^2 \mathrm{d}\mathscr{L}^n + b\mathscr{H}^{n-1}(K) + c \int_{\Omega \setminus K} (u - g)^2 \mathrm{d}\mathscr{L}^n,$$

où $u \in H^1(\Omega \setminus K, \mathbb{R}^n)$ et $e(u) = \frac{\nabla u + \nabla u^*}{2}$ est la partie symétrique de la différentielle de u. Ce type de problème apparaîtra dans notre cas dans l'étude de la traînée d'un obstacle plongé dans un fluide visqueux, un contexte apparemment éloigné mais régi par des équations similaires.

Une idée centrale dans l'étude de ce type de problème est de changer la donnée de (K, u) en la donnée d'une seule fonction u dans un espace de fonction plus général qui comporte à la fois des parties lisses, et des discontinuités; il s'agit dans l'idée de fonctions "H^1 par morceaux". Les raisons pour cela sont multiples: sauf hypothèse topologique ou géométrique particulière une suite de fermés (K_i) obéissant à certaines contraintes de mesure n'a a priori pas de raison de converger vers un minimiseur. Il n'y a pas de bonne notion de convergence qui fournit à la fois la compacité (comme peut le faire la convergence au sens de Hausdorff) et un contrôle de la mesure, voire des fonctions u_i qui sont dans des espaces dépendant de K_i.

En effaçant la donnée de K et en considérant uniquement la fonction u, on se place dans un espace de fonctions unique (du type $BV(\mathbb{R}^n)$) avec des bonnes propriétés de compacité. En utilisant ensuite l'équation au dérivées partielles ou le problème d'optimisation vérifié en un sens faible par la fonction u, on arrive généralement à montrer que celle-ci est assez régulière pour retomber sur notre problème de départ. Cela est formalisé plus précisément dans la théorie des espaces de fonctions SBV (Special Bounded Variation), un sous-espace particulier des fonctions à variation bornée dont la différentielle faible présente une certaine structure. Il s'agit des fonctions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ telles que Du (la différentielle de u au sens des distributions) soit une mesure vectorielle localement finie qui se décompose sous la forme

$$Du = \nabla u \mathscr{L}^n + (u^+ - u^-)\nu_u \mathscr{H}^{n-1} \lfloor J_u,$$

où $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (et sera en fait $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ dans beaucoup de problèmes que l'on étudiera) et J_u est un ensemble rectifiable - inclus dans une union dénombrable d'hypersurfaces $\mathcal{C}^1$ - appelé l'ensemble des sauts de u, qui est l'ensemble des $x \in \mathbb{R}^n$ tel que $u(x + r\cdot)$ converge dans $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ lorsque $r \to 0$ vers la fonction constante par morceaux $u^+ 1_{\langle \cdot, \nu \rangle > 0} + u^- 1_{\langle \cdot, \nu \rangle < 0}$, où $\nu \in \mathbb{S}^{n-1}$, $u^\pm \in \mathbb{R}$. On y retrouve les fonctions $H^1(\mathbb{R}^n)$, pour lesquelles l'ensemble des sauts est $\mathscr{H}^{n-1}$-negligeable, ou au contraire les indicatrices d'ensembles de périmètres finis E pour lesquelles $D1_E = -\mathscr{H}^{n-1}_{\lfloor \partial^* E} \nu_E$ (où $\partial^* E$ est la frontière réduite de E et ν_E est le vecteur normal sortant), mais aussi la situation "mixte" d'une fonction $u \in H^1(\Omega)$ où $\Omega \Subset \mathbb{R}^n$ est un ouvert lisse (par exemple Lipschitz), que l'on prolonge en $\tilde{u}(x) = \begin{cases} u(x) & \text{si } x \in \Omega \\ 0 & \text{si } x \notin \Omega \end{cases}$, et

$$D\tilde{u} = 1_\Omega \nabla u \mathscr{L}^n - u_{|\partial\Omega} \nu_\Omega \mathscr{H}^{n-1} \lfloor \partial\Omega,$$

où $u_{|\partial\Omega}$ est la trace de u sur $\partial\Omega$ et ν_Ω le vecteur normal sortant de Ω. Mais on y trouve aussi des fonctions n'étant pas issues de ce genre d'exemples, caractérisées par le fait que l'ensemble des sauts n'est pas forcément fermé (à un ensemble négligeable près). Le gain de compacité obtenu en relaxant un problème à discontinuité libre dans un espace SBV n'est donc pas gratuit, et c'est là qu'intervient la théorie de la régularité de ce type de solutions relaxées, qui permet de revenir dans le problème original. Le livre [1] est une référence très complète sur le sujet, et les bases et les résultats sur les fonctions SBV utilisés par la suite sont expliqués dans le chapitre 3.

Dans la suite de l'introduction, je présente dans l'ordre chronologique les différents problèmes que j'ai abordés pendant ma thèse. Le point de départ a été l'étude d'une équation dite "de Poisson" avec une condition au bord de Robin, qui se présente de la façon suivante: étant donnés un domaine borné et lisse Ω dans $\mathbb{R}^n$, une fonction $f \in L^2(\Omega)$, et $\beta > 0$, on cherche une fonction $u \in H^1(\Omega)$ vérifiant

$$\begin{cases} -\Delta u = f & (\Omega) \\ -\partial_\nu u = \beta u & (\partial\Omega) \end{cases}$$

Par des méthodes variationnelles standards (théorème de Lax-Milgram), on a l'existence d'une unique solution u. Voici deux interprétations possibles de cette quantité:

- On peut voir Ω comme un solide de conductivité thermique constante σ, plongé dans un milieu à température constante T^a. On suppose qu'il y a dans Ω une production de chaleur volumique F, et que les échanges de température entre Ω et l'exterieur sont régis par un transfert par convection à travers une couche limite, c'est-à-dire que le flux de température $\partial_\nu T_{|\partial\Omega}$ est proportionnel à $T^a - T_{|\partial\Omega}$. La constante de proportionnalité (notée $\alpha > 0$) dépend des grandeurs caractéristiques de la convection avec l'extérieur (épaisseur de la couche limite, conductivité de l'extérieur...). Dans ce cas, la température à l'équilibre atteinte dans le solide Ω est la fonction $T : \Omega \to \mathbb{R}$ vérifiant

  $$\begin{cases} -\nabla \cdot (\sigma \nabla T) = F & (\Omega) \\ \partial_\nu T = \alpha(T^a - T) & (\partial\Omega) \end{cases}$$

  En renormalisant la température en $u := \frac{T - T^a}{T^a}$, la production de chaleur en $f := \frac{F}{\sigma T^a}$, et la constante de convection en $\beta := \frac{\alpha}{T^a}$, on retrouve bien l'équation précédente. La solution u est en particulier donnée par la minimisation de l'énergie

  $$E_\Omega : v \in H^1(\Omega) \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \frac{\beta}{2}\int_{\partial\Omega} v^2 \mathrm{d}\mathscr{H}^{n-1} - \int_\Omega fv$$

  qui est strictement convexe, coercive, et admet donc un minimum unique par des argument standards. La solution u vérifie

  $$E_\Omega(u) = -\frac{1}{2}\int_\Omega u \mathrm{d}\mathscr{L}^n$$

  et l'énergie de la température à l'équilibre correspond donc, à une constante près, à la température moyenne dans Ω à l'équilibre.

- Si on voit Ω comme la forme au repos d'une membrane de densité surfacique constante, qui est fixée au bord $\partial\Omega$ (qui est ici un squelette rigide) de manière élastique - par exemple par des ressorts de rigidité constante proportionnelle à β répartis uniformément le long de $\partial\Omega$ - et qu'une force orthogonale f est appliquée à cette membrane, alors dans le régime linéaire la solution u représente le déplacement vertical de la membrane engendré par la force f.

Le premier projet a été d'étudier cette équation pour $f = 1$ avec de plus une condition d'obstacle $u \geq c$ changeant légèrement les conditions de bord, dans le but d'obtenir des estimations quantitatives sur la distance entre le domaine Ω d'une boule de même mesure, par la différence d'énergie entre ces deux domaines. Ce projet a été fait en collaboration avec D. Bucur et A. Giacomini et a donné lieu à une publication dans *Archive for Rational Mechanics and Analysis*, qui est résumée dans la section

1.1.1. Celui-ci a ouvert, dans différentes directions, plusieurs des autres problèmes étudiés pendant ma thèse:

- Un problème de stabilité de l'inégalité de Weinstock, faisant intervenir des fonctions harmoniques vérifiant des conditions de Robin (aussi appelées conditions de Steklov dans cette situation), qui s'est révélé être un problème de nature différente présentant notamment une instabilité intrinsèque. Cela a donné lieu à une publication dans *Transactions of the AMS* en collaboration avec D. Bucur, résumé dans la section 1.1.2. La reformulation de ce problème présente une stabilité en un sens plus faible qui a plus tard conduit à l'étude de la stabilité d'inégalités spectrales dans les surfaces compactes, telle que l'inégalité de Hersch sur la sphère, mais aussi sur le plan projectif, et certains tores. Ces travaux sont contenus dans une prépublication écrite en collaboration avec I. Polterovich, M. Karpukhin et D. Stern, décrit en section 1.1.3.

- Un problème d'optimisation de forme à discontinuité libre de nature vectorielle avec l'étude de domaines minimisant une combinaison des valeurs propre du Laplacien avec condition de Robin. Il y a deux composantes dans ce travail, l'étude de fonctionnelles dites "non-dégénérées" dans lesquelles on montre l'existence de domaine optimal en un sens fort, et les fonctionnelles dites "dégénérées" dans lesquelles l'existence est seulement connue dans un sens relaxé (dans l'espace SBV), mais où on montre la dégénérescence des valeurs propres à l'optimum, un problème ouvert dans le cas des conditions de Dirichlet. Ces travaux sont décrits dans la section 1.2.

- Dans l'étude du problème de Poisson avec obstacle, il y a une question de régularité qui a été évitée, sur le lieu de contact avec l'obstacle qui est à mi-chemin entre la fonctionnelle de Mumford-Shah et le problème de Alt-Caffarelli. Bien que cette question reste ouverte, elle a inspiré l'étude d'une fonctionnelle plus abordable présentant un phénomène d'obstacle similaire que l'on a pu étudier. Il s'agit ici d'un problème à frontière libre mais sans discontinuité, qui fait cependant appel à des méthodes similaires. Nous avons pu dans ce cas répondre partiellement à la question de la régularité en dimension deux, ce qui ouvre la porte à l'étude du problème mixte Mumford Shah/Alt-Caffarelli. Ces travaux ont été effectués en commun avec B. Velichkov, et sont décrits en 1.3.

- Comme mentionné précédemment, un passage crucial dans l'étude du problème de Poisson avec obstacle était de contourner la question de la régularité d'un minimiseur en remplaçant notre fonctionnelle par une fonctionnelle approchée pour laquelle on sait montrer la régularité. Cette méthode a permis de résoudre un problème d'optimisation de forme intervenant en isolation thermique, qui consiste à minimiser le flux de chaleur sortant d'un objet à température constante (supérieure à l'extérieur) séparé de l'extérieur par une couche d'isolant ; on établit que dans de nombreux cas la configuration optimale est d'isoler une boule avec une couche d'épaisseur constante. Ces travaux sont trouvables en preprint et sont une collaboration avec D. Bucur, C. Nitsch et C. Trombetti, résumés dans la section 1.4.

- Une condition centrale pour la régularité des solutions relaxées à l'équation de Poisson avec condition de Robin est la stricte positivité de la température u au bord, qui peut aussi se voir comme la positivité du flux de chaleur en chaque point du bord. Nous avons étendu certains résultats notamment de [2], basés sur des méthodes entièrement probabilistes, à des opérateurs elliptiques plus généraux en faisant un lien explicite entre la géométrie du domaine et la positivité des solutions. Nous avons notamment établi un lien entre le profil de l'inégalité isopérimétrique relative dans le domaine et la positivité des solutions d'équations elliptiques générales avec conditions de Robin. Il s'agit d'une collaboration avec D. Bucur et A. Giacomini, expliquée en section 1.5.

Enfin nous étudions, avec D. Bucur, A. Chambolle et A. Giacomini, un problème à discontinuité libre vectoriel issu d'une question d'optimisation de forme en mécanique des fluides : comment minimiser la traînée d'un obstacle plongé dans un fluide de Stokes, avec des conditions de frottement le long de l'obstacle ? Bien qu'ayant une littérature abondante sur le plan numérique, ce type de question a été peu abordé sur le plan théorique ; en se basant un formalisme développé pour les

apparitions de fractures en milieu élastique, on établit l'existence et la régularité des minimiseurs sous certaines hypothèses. Un résumé général est donné en section 1.6.

1.1 Notion de stabilité d'une inégalité géométrique

Un problème d'optimisation de forme est la donnée de deux choses ; d'une part un ensemble

$$\mathscr{A} \subset \{\text{sous-ensembles de } \mathbb{R}^n\}$$

que l'on appelle l'ensemble des formes admissibles. Celui-ci est par exemple donné par des conditions de régularité (ouvert, Lipschitz, $\mathcal{C}^2$), de géométrie (contenant ou contenu dans une certaine forme, étoilé, convexe, quasi-convexe), de topologie (simplement connexe, ayant un groupe fondamental monogène), ou de contrainte sur différentes quantités géométriques (mesure, périmètre).

D'autre part une fonctionnelle $J : \mathscr{A} \to \mathbb{R}$ que l'on cherchera à minimiser (ou à maximiser selon la situation), qui dans notre cas feront souvent intervenir la solution d'une EDP de type elliptique.

Commençons par mentionner quelques exemples classiques qui apparaîtront à nouveau plus loin:

- **Inégalité isopérimétrique**: $\mathrm{Per}(\Omega) \geq \mathrm{Per}(\Omega^*)$ dans $\{\Omega \subset \mathbb{R}^n \text{ mesurable t.q. } |\Omega| = m\}$, où $m > 0$, Ω^* est la boule de mesure m, et Per désigne le périmètre défini par

$$\mathrm{Per}(\Omega) = \sup\left\{\int_\Omega \nabla \cdot \varphi,\ \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n, B_1)\right\}$$

  qui coincide avec $\mathscr{H}^{n-1}(\partial\Omega)$ lorsque Ω est un ensemble suffisament lisse (par exemple à bord Lipschitz). Plus de détails sur la définition de périmètre seront donnés dans le chapitre suivant ; l'inégalité isopérimétrique est un résultat connu de longue date dont la première preuve rigoureuse est attribuée à Weierstrass.

- **Inégalité de Faber-Krahn** (avec conditions de Robin): $\lambda_1(\Omega;\beta) \geq \lambda_1(\Omega^*;\beta)$ dans

$$\{\Omega \subset \mathbb{R}^n \text{ ouvert t.q. } |\Omega| = m\},$$

  où $m > 0$, $\beta \in (0, +\infty]$, Ω^* est la boule de mesure m, et $\lambda_1(\Omega;\beta)$ est la première valeur propre du Laplacien avec condition de Robin, définie par

$$\lambda_1(\Omega;\beta) = \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^{n-1}}{\int_\Omega u^2 \mathrm{d}\mathscr{L}^n}$$

  $\beta = +\infty$ correspond naturellement aux conditions de Dirichlet, dans ce cas la preuve remonte aux travaux de Faber [20] et Krahn [28]; cette preuve peut être vue comme une conséquence de l'inégalité isopérimétrique appliquée aux ensembles de niveaux des fonctions test, à travers l'inégalité de Polya-Szego. Pour le cas des conditions de Robin avec $\beta \in (0, +\infty)$, il faut attendre les travaux de Bossel [4] et [5] par une méthode de réarrangement différente sur laquelle nous reviendrons.

- **Inégalité de Saint-Venant**: $T(\Omega) \leq T(\Omega^*)$ dans $\{\Omega \subset \mathbb{R}^n \text{ ouvert t.q. } |\Omega| = m\}$, où $m > 0$, Ω^* est la boule de mesure m, et $T(\Omega)$ est la moyenne de la fonction u_Ω solution de $-\Delta u_\Omega = 1$ dans Ω avec conditions de Robin $-\partial_\nu u_\Omega = \beta u_\omega$. Le cas des conditions de Dirichlet peut se montrer par les même méthodes que pour l'inégalité de Faber-Krahn, mais pour le cas des conditions de Robin les méthodes de réarrangement de Bossel mentionnées précédemment ne semblent pas s'adapter à ce cas. L'inégalité de Saint-Venant a donc été montrée par une méthode variationnelle dans [10], en montrant qu'un domaine optimal existe, et que tout domaine optimal est nécessairement celui attendu, c'est-à-dire une boule.

- **Inégalité de Weinstock**: $\sigma_1(\Omega) \leq \sigma_1(\mathbb{D})$ dans $\{\Omega \subset \mathbb{R}^2$ simplement connexe t.q. $\mathrm{Per}(\Omega) = 2\pi\}$, où $\mathbb{D}$ est le disque unité et σ_1 désigne la première valeur propre non-triviale du spectre de Steklov définie par

$$\sigma_1(\Omega;\beta) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^2}{\int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^1},\ u \in H^1(\Omega),\ \int_{\partial\Omega} u \mathrm{d}\mathscr{H}^1 = 0\right\}.$$

  La preuve de cette inégalité remonte à Weinstock [34]. L'idée est d'utiliser le théorème de représentation conforme pour construire de bonnes fonctions test. Il existe aussi une version en toute dimension pour les convexes dans [9], qui passe par la comparaison avec une fonctionnelle géométrique (le moment).

- **Inégalité de Hersch**: $\lambda_1(\mathbb{S}^2, g) \leq \lambda_1(\mathbb{S}^2, g_{\mathrm{rond}})$, où $(\mathbb{S}^2, g)$ et $(\mathbb{S}^2, g_{\mathrm{rond}})$ sont des surfaces Riemanniennes homéomorphes à la sphère euclidienne $\mathbb{S}^2 \subset \mathbb{R}^3$, de même aire, telles que g_{rond} soit de courbure constante, et $\lambda_1(\mathbb{S}^2, g)$ désigne la première valeur propre non-triviale du Laplacien $-\Delta_g$, définie par

$$\lambda_1(\mathbb{S}^2, g) = \inf\left\{\frac{\int_{\mathbb{S}^2} |\nabla_g u|^2 \mathrm{d}v_g}{\int_{\mathbb{S}^2} u^2 \mathrm{d}v_g},\ u \in H^1(\mathbb{S}^2, g) : \int_{\mathbb{S}^2} u \mathrm{d}v_g = 0\right\}$$

  où v_g est la mesure d'aire induite par la métrique g. Des inégalités similaires sont connues pour le plan projectif, le tore, la bouteille de Klein, mais pour une surface compacte générale il est seulement connu qu'il existe une métrique optimale ayant un nombre fini de singularités coniques, comme montré par exemple dans [31].

Lorsque l'on résout un problème d'optimisation de forme $\inf_{\Omega\in A} J(\Omega)$ qui admet une solution Ω^{opt}, la question de la stabilité de ce problème peut se formuler de la manière suivante:

$$\textit{Si } J(\Omega) \approx J(\Omega^{opt}), \textit{ est-ce que } \Omega \approx \Omega^{opt} \textit{ ?}$$

Ici $\Omega \approx \Omega^{\mathrm{opt}}$ est à prendre au sens d'une certaine topologie dont est muni l'ensemble des formes admissibles ; un problème peut très bien être stable pour une certaine topologie et instable pour une autre, comme nous le verrons avec l'inégalité de Hersch, et il peut très bien s'agir d'une topologie pour laquelle J n'est pas continue mais qui est satisfaisante sur le plan géométrique, telle que l'asymétrie de Fraenkel définie ci-dessous.

Pour les problèmes d'optimisation dont l'optimum est atteint pour la boule, on utilisera notamment la notion d'asymétrie de Fraenkel

$$\mathscr{A}_F(\Omega) = \inf\left\{|\Omega\Delta(x+\Omega^*)|,\ x \in \mathbb{R}^n\right\},$$

où Δ est la différence symétrique. Un autre exemple est l'asymétrie de Hausdorff

$$\mathscr{A}_H(\Omega) = \inf\left\{\mathrm{d}_H\left(\Omega, x+\Omega^*\right),\ x \in \mathbb{R}^n\right\},$$

où $\mathrm{d}_H(A,B) = \|\mathrm{d}(\cdot,A) - \mathrm{d}(\cdot,B)\|_{L^\infty(\mathbb{R}^n)}$ est la distance de Hausdorff.

Un résultat de stabilité peut se présenter sous forme quantitative; on se demande si pour une certaine notion de distance d entre formes admissibles et pour une certaine fonction croissante $F : \mathbb{R}_+^* \to \mathbb{R}_+^*$, on a

$$J(\Omega) - J(\Omega^{\mathrm{opt}}) \geq F(\mathrm{d}(\Omega, \Omega^{\mathrm{opt}})).$$

On dira qu'un tel résultat est optimal si on peut construire une suite $\Omega_k \to \Omega^{\mathrm{opt}}$ tel que $J(\Omega_k) - J(\Omega^{\mathrm{opt}})$ soit de l'ordre de $F(\mathrm{d}(\Omega_k, \Omega^{\mathrm{opt}}))$.

Commençons par énoncer quelques résultats connus de stabilité; l'inégalité isopérimétrique quantitative optimale a été obtenue dans [22].

Théorème 1.1. *Soit $\Omega \subset \mathbb{R}^n$ un ensemble de mesure finie et Ω^* la boule de même mesure, alors il existe une constante c ne dépendant que de n et $|\Omega|$ telle que*

$$Per(\Omega) - Per(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

Cette inégalité est optimale au sens où pour une suite d'ellipsoïdes qui converge vers une boule, chaque côté de l'inégalité est du même ordre de grandeur.

Pour l'inégalité de Faber-Krahn (et de Saint-Venant) avec conditions de Dirichlet, un résultat du même type a été montré dans [6].

Théorème 1.2 ([6])**.** *Soit $\Omega \subset \mathbb{R}^n$ un ouvert de mesure finie et Ω^* la boule de même mesure, alors il existe une constante c ne dépendant que de n et $|\Omega|$ telle que*

$$\lambda_1(\Omega) - \lambda_1(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

Ce résultat est également optimal en regardant le cas des ellipsoïdes. Pour l'inégalité de Faber-Krahn avec conditions de Robin, le résultat est similaire mais a été établi par des méthodes très différentes dans [5].

Théorème 1.3. *Soit $\Omega \subset \mathbb{R}^n$ un ouvert de mesure finie, $\beta > 0$ et Ω^* la boule de même mesure, alors il existe une constante c ne dépendant que de n, β et $|\Omega|$ telle que*

$$\lambda_1(\Omega;\beta) - \lambda_1(\Omega^*;\beta) \geq c\mathscr{A}_F(\Omega)^2$$

Cette inégalité repose sur un raffinement de la preuve originale de l'inégalité de Faber-Krahn avec conditions de Robin $\lambda_1(\Omega;\beta) \geq \lambda_1(\Omega^*;\beta)$. Plus précisément, il existe une caractérisation de $\lambda_1(\Omega;\beta)$ (aussi valable à la limite $\beta \to \infty$ - pour les conditions de Dirichlet) inspirée par les méthodes de longueurs extrémales de géométrie conforme, établie par Bossel dans [4].
Pour toute fonction $\rho \in L^\infty(\Omega, \mathbb{R}_+)$ que l'on voit comme une métrique multipliant localement les longueurs par ρ, l'aire d'un ensemble $\omega \subset \Omega$ est $A_\rho(\omega) := \int_\omega \rho^2$, et son périmètre est $P_\rho(\Omega) = \int_{\partial\omega\cap\Omega} \rho + \int_{\partial\omega\cap\partial\Omega} \beta$ (notons qu'en particulier pour $\beta = +\infty$, on peut se restreindre aux ensembles à distance positive de $\partial\Omega$). La première valeur propre est alors égale à

$$\lambda_1(\Omega;\beta) = \sup_{\rho\in L^\infty(\Omega,\mathbb{R}_+)} \inf_{\omega\subset\Omega} \frac{P_\rho(\omega) - A_\rho(\omega)}{|\omega|}$$

L'infimum peut en fait être restreint aux ensembles de la forme $\{u > t\}$, et il est montré dans [8] que celui-ci est atteint pour un $\{u > t\}$ et que cela permet d'établir une estimation de la forme

$$\lambda_1(\Omega;\beta) \geq \frac{\beta}{2}\left(\inf_{x\in\Omega} u(x)\right)^2 \left[\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)\right]$$

où u est la première fonction propre normalisée. La différence de périmètre est minorée par $c\mathscr{A}_F(\Omega)^2$, cependant le minimum de u n'est pas nécessairement contrôlé par en dessous. C'est ici qu'intervient un principe de sélection, utilisé notamment dans [16] pour l'inégalité isopérimétrique quantitative et par [6] pour l'inégalité de Faber Krahn (avec conditions de Dirichlet) quantitative ; pour chaque domaine Ω, on peut construire un domaine ω proche de Ω au sens de la distance L^1, de valeur propre proche, mais tel que $\inf_{x\in\omega} u_\omega(x) > 0$.

1.1.1 Stabilité de l'inégalité de Saint-Venant

Dans ce premier article, on s'intéresse aux solutions de l'équation suivante

$$\begin{cases} -\Delta u_\Omega = 1 & (\Omega) \\ -\partial_\nu u_\Omega = \beta u_\Omega & (\partial\Omega) \end{cases}$$

pour un domaine ouvert Ω de mesure fixée à une constante $m > 0$. La solution u_Ω est obtenue comme unique point de minimum de la fonctionnelle

$$E_\Omega(u) := \frac{1}{2}\left(\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \beta u^2 \mathrm{d}\mathscr{H}^{n-1}\right) - \int_\Omega u \mathrm{d}\mathscr{L}^n$$

Et on notera $E(\Omega) := E(u_\Omega; \Omega)$ l'énergie du domaine Ω. Il a été montré dans [7] que parmi les ouverts de mesure fixée, $E(\Omega)$ est minimal sur la boule de même mesure Ω^*. Dans le chapitre 4, on montre une version quantitative de ce résultat sous la forme suivante.

Théorème 1.4. *Soit $\beta \in (0, +\infty)$, $m > 0$, il existe une constante $C_{n,m,\beta} > 0$ telle que pour tout ouvert Ω de mesure m dans $\mathbb{R}^n$,*

$$E(\Omega) - E(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

Ce résultat est obtenu en alliant un principe de sélection et l'inégalité suivante, que l'on obtient en étudiant un problème mélangeant discontinuité libre et frontière libre (d'où le nom; problème à discontinuité libre **dégénéré**).

Théorème 1.5. *Soit Ω un ouvert de mesure finie, alors*

$$E(\Omega) - E(\Omega^*) \geq \frac{\beta}{2}\left(\inf_{x\in\Omega} u_\Omega(x)\right)^2 \Big[Per(\Omega) - Per(\Omega^*)\Big]$$

Cette inégalité est très similaire à celle obtenue pour $\lambda_1(\Omega; \beta)$, notons cependant qu'il n'existe pas de caractérisation par longueur extrémale connue pour l'énergie E qui pourrait permettre de l'obtenir directement, et la méthode pour l'obtenir sera donc très différente.

Cette inégalité peut présenter un intérêt en soi; en effet, la seule constante qui n'y est pas totalement explicite est $\inf_{x\in\Omega} u_\Omega(x)$, qui dépend de la géométrie du domaine d'une manière qui n'est pas encore totalement comprise et sera abordée dans un chapitre ultérieur. Mais si l'on se restreint par exemple aux convexes qui ne sont pas trop éloignés de la boule, cela donne une inégalité quantitative de la forme

$$E(\Omega) - E(\Omega^*) \geq C_{n,|\Omega|,\beta}\Big[\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)\Big]$$

qui est donc plus forte que celle annoncée dans le résultat 1.4, car $\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)$ est minoré par $c_{n,|\Omega|}\mathscr{A}_F(\Omega)^2$.

Pour un domaine Ω quelconque on ne peut cependant pas en tirer une inégalité quantitative directement ; en effet la quantité $\inf_{x\in\Omega} u_\Omega(x)$ peut être nulle dans plusieurs situations. Par exemple, parmi les domaines d'énergie bornée la quantité $\int_{\partial\Omega} u_\Omega^2$ sera bornée, et donc $\inf_{x\in\Omega} u_\Omega(x)$ est petit dans les ensembles de grand périmètre. Un exemple plus fin est le suivant, considérons $\Omega \Subset \mathbb{R}^n$ un domaine lisse en dehors de l'origine tel que

$$\Omega \cap B_1 = \left\{(x_1, x') \in B_1 : x_1 > 0,\ |x'| < x_1^2\right\}$$

dans ce cas le périmètre de Ω reste borné, et pourtant $u_\Omega(0) = 0$. Ce type de phénomène et des critères de positivités pour la solution u_Ω sont étudiés plus en détails dans le chapitre 9.

Dans notre cas, on utilise un principe de sélection qui prend la forme suivante: pour tout domaine Ω, possiblement peu lisse, on résout un problème d'optimisation de forme intermédiaire qui nous donne au final un domaine $\omega \subset \Omega$ tel que $E(\Omega) \gtrsim E(\omega)$ et $\mathscr{A}_F(\Omega) \lesssim \mathscr{A}_F(\omega)$. Celui-ci est obtenu par la minimisation

$$\inf\left\{E(\omega) + k|\omega|,\ \omega \subset \Omega\right\}$$

pour une constante k choisie suffisament petite en fonction des paramètres du problème. Sur la solution ω à ce problème, on est capable de montrer une estimation uniforme

$$\inf_{x\in\omega} u_\omega(x) \geq \delta_{n,\beta,|\Omega|}(> 0)$$

ce qui, à partir du théorème 1.5, permet de conclure. Pour obtenir le théorème 1.5, l'idée est de voir ce résultat de manière variationelle: on définit une énergie modifiée

$$E^c(\Omega) = \inf_{u\in H^1(\Omega), u\geq c} \left(\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \frac{\beta}{2}\int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^{n-1} - \int_\Omega u \mathrm{d}\mathscr{L}^n \right) - \frac{\beta}{2}c^2 \mathrm{Per}(\Omega) + c|\Omega|,$$

où $c > 0$ sera fixée plus tard. On montre que parmi tous les domaines de mesure fixée, la quantité $E^c(\Omega)$ est minimale pour la boule, et une fois ceci obtenu il suffit de poser $c = \inf_{x\in\Omega} u_\Omega(x)$ pour obtenir le théorème 1.5.

Le fait que E^c est minimale sur la boule est obtenu en montrant qu'il existe un minimiseur en un sens relaxé (SBV), que ce minimiseur relaxé est un minimiseur classique par un résultat de régularité, et qu'un minimiseur classique est nécessairement une boule.

La seconde étape de régularité consiste à dire qu'un minimiseur relaxé (au sens SBV) u vérifie $\mathscr{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$, ce qui permet de définir un ouvert Ω tel que u soit une solution au sens classique dans Ω et $J_u \subset \partial\Omega$; une estimation centrale pour obtenir cela est que u vérifie

$$\inf_{\{u>c\}} u > c,$$

c'est-à-dire que u ne vient jamais toucher l'obstacle: cela est faux en général, et la régularité est en fait seulement obtenue pour une fonctionnelle approchée $E^{c,\epsilon}$ qui admet aussi la boule comme minimiseur (car elle a les mêmes propriétés de symétrie). La régularité des minimiseurs de E^c reste en fait un problème ouvert qui se ramène - en linéarisant la fonctionnelle autour de $u \approx c$ - à l'étude de

$$u \in SBV_{\mathrm{loc}}(\Omega, \mathbb{R}_+) \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \alpha \int_{J_u\cap\Omega} (u^+ + u^-)\mathrm{d}\mathscr{H}^{n-1} + k|\{u > 0\}|$$

sur laquelle peu de choses sont actuellement connues.

1.1.2 Instabilité de l'inégalité de Weinstock

Pour tout ouvert connexe, borné et lisse $\Omega \Subset \mathbb{R}^n$, on définit l'opérateur de Steklov comme l'opérateur pseudodifférentiel suivant:

$$S_\Omega : \begin{cases} \mathcal{C}^\infty(\partial\Omega) \to \mathcal{C}^\infty(\partial\Omega) \\ u \mapsto \partial_\nu(\mathcal{H}u) \end{cases},$$

où $\mathcal{H}u$ est l'extension harmonique de u sur Ω et $\partial_\nu f = \nu\cdot\nabla f$, où ν est le vecteur normal sortant de Ω. Bien que ce ne soit pas un opérateur différentiel classique (on constate aisément qu'il est non-local), celui-ci présente des propriétés similaires à $\sqrt{-\Delta_{\partial\Omega}}$ et est dans certain cas égal à celui-ci, notamment lorsque Ω est un demi-espace.

C'est un opérateur auto-adjoint positif dont l'inverse - qui est bien défini dans l'espace des fonctions de moyenne nulles - est compact, et il existe donc une suite de valeurs propres

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \ldots \to \infty$$

associées à des fonctions propres $u_k \in \mathcal{C}^\infty(\partial\Omega)$ (que l'on assimile, par abus de notation, à leurs extensions harmoniques) qui forment une base de $L^2(\partial\Omega)$ telles que

$$\begin{cases} \Delta u_k = 0 & (\Omega) \\ \partial_\nu u_k = \sigma_k(\Omega) u_k & (\partial\Omega) \end{cases}$$

L'inégalité de Weinstock nous dit que parmi les ouverts simplement connexes du plan de perimètre fixé à 2π,

$$\sigma_1(\Omega) \leq \sigma_1(\mathbb{D})(= 1),$$

où $\mathbb{D}$ est le disque unité. Des recherches récentes autour de cette inégalité ont montré une forme de stabilité dans certain cas, citons par exemple le résultat de [23] où il est montré que parmi les convexes en toute dimension, où la boule est toujours maximale, il y a une inégalité quantitative de la forme $\sigma(B) - \sigma(\Omega) \geq F(\mathscr{A}_H(\Omega))$ pour une fonction F strictement croissante explicite dépendant de la dimension.

Dans ce second article, on établit que sans hypothèse autre que la simple connexité, l'inégalité de Weinstock est instable en générale, c'est-à-dire que l'on peut construire des suites d'ouverts simplement connexes $(\Omega_\epsilon)_{\epsilon>0}$ de périmètre 2π tels que $\Omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Omega$ en un sens fort (Ω_ϵ est obtenu comme une perturbation Lipschitz de la frontière de Ω), $\sigma_1(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 1$, mais Ω n'est pas le disque. On montre plus généralement un résultat d'instabilité de tout le spectre.

Théorème 1.6. *Soient Ω, ω deux ouverts bornés et Lipschitz de $\mathbb{R}^2$, tels qu'il existe une application conforme entre les deux. Alors il existe une suite de domaines $\Omega_\epsilon \to \Omega$, tous homéomorphes à Ω et de périmètres uniformément bornés, tels que pour tout $k \geq 0$,*

$$Per(\Omega_\epsilon)\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} Per(\omega)\sigma_k(\omega)$$

L'instabilité de l'inégalité de Weinstock s'obtient en prenant Ω simplement connexe quelconque, et $\omega = \mathbb{D}$. On établit cependant une caractérisation partielle des suites de domaines $(\Omega_\epsilon)_\epsilon$ de périmètre 2π vérifiant $\sigma_1(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 1$; on sait que la dichotomie suivante est toujours vérifiée:

- Soit $\mathscr{A}_H(\Omega_\epsilon) \to 0$.
- Soit $\| \log(|g'_\epsilon|)\|_{\mathcal{C}^\alpha(\mathbb{D})} \underset{\epsilon\to 0}{\longrightarrow} +\infty$ en un sens quantifié, où g_ϵ est n'importe quelle suite d'applications conformes $g_\epsilon : \mathbb{D} \to \Omega_\epsilon$ bijective.

Bien que cette deuxième condition ne soit pas géométriquement explicite, l'idée est que dans une suite de domaines $(\Omega_\epsilon)_\epsilon$ qui sont un contre-exemple à la stabilité de l'inégalité de Weinstock, le bord des Ω_ϵ a forcément des oscillations ou des singularités.

1.1.3 Cas de l'inégalité de Hersch

L'inégalité spectrale de Hersch peut se résumer ainsi: si $(\mathbb{S}^2, g)$ est une variété Riemannienne homeomorphe à la sphère de dimension 2, et que l'on note

$$(0 =)\lambda_0(\mathbb{S}^2, g) < \lambda_1(\mathbb{S}^2, g) \leq \ldots \to \infty$$

les valeurs propres du Laplacien de $(\mathbb{S}^2, g)$, alors nécessairement

$$v_g(\mathbb{S}^2)\lambda_1(\mathbb{S}^2, g) \leq 8\pi,$$

où v_g est la mesure induite par g, le maximum étant exactement atteint par les sphère dotées d'une métrique à courbure constante. On étudie la stabilité de cette inégalité spectrale dans le cadre généralisé des valeurs propres associées à une mesure. Plus précisément, par théorème d'uniformisation on sait qu'il n'existe à difféomorphisme près qu'une classe conforme de métrique sur la sphère, disons engendrée par une métrique de courbure constante notée g, et l'inégalité de Hersch peut se reformuler de la sorte : pour toute fonction $u \in \mathcal{C}^\infty(\mathbb{S}^2)$,

$$\left(\int_{\mathbb{S}^2} e^{2u} \mathrm{d}v_g\right) \lambda_1\left(\mathbb{S}^2, e^{2u}g\right) \leq 8\pi.$$

Cela peut encore être généralisé aux mesures quelconques, grâce au formalisme développé par Kokarev dans [27]: pour toute mesure de Borel positive μ définie sur la sphère, on pose

$$\lambda_k(\mu) := \inf_{V:\dim(V)=k+1} \sup_{v\in V} \frac{\int_{\mathbb{S}^2} |\nabla v|^2 \mathrm{d}v_g}{\int_{\mathbb{S}^2} v^2 \mathrm{d}\mu}.$$

où V est pris parmi les sous-espaces de $H^1(\mathbb{S}^2, g)$. Dans ce cadre, l'inégalité de Hersch devient $\mu(\mathbb{S}^2)\lambda_1(\mu) \leq 8\pi$, et on montre le résultat de stabilité suivant.

Théorème 1.7. *Soit μ une mesure de probabilité sur la sphère $\mathbb{S}^2$, alors il existe deux constantes universelles $c_1, c_2 > 0$ ne dépendant pas de μ, et une métrique h conforme à g de courbure constante telle que*

$$\lambda_1(\mu) \leq 8\pi - c_1 \min\left(c_2, \|\mu - v_h\|^2_{H^{-1}(\mathbb{S}^2,h)}\right).$$

On établit aussi l'existence de mesures de probabilité $(\mu_\epsilon)_\epsilon$ telles que $c\|\mu_\epsilon - v_g\|^2_{H^{-1}(\mathbb{S}^2,g)} \sim 8\pi - \lambda_1(\mu_\epsilon) \to 0$, ce qui montre l'optimalité de l'exposant 2, ainsi que l'existence de mesures de probabilités $(\mu_\epsilon)_\epsilon$ telles que $\lambda_1(\mu_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 8\pi$ mais μ_ϵ ne converge vers aucune métrique ronde dans l'espace $\left(W^{1,2,-\frac{1}{2}}\right)^*$, ce qui montre une forme d'optimalité de l'espace H^{-1} (l'espace $\left(W^{1,2,-\frac{1}{2}}\right)^*$ étant entre autres intercalé entre H^{-1} et tous les H^{-s} pour $s < 1$).

1.2 Optimisation des valeurs propres d'ordres supérieurs

Dans ce chapitre, on s'intéresse aux problème d'optimisation du type

$$\inf\{F(\lambda_1(\Omega;\beta), \ldots, \lambda_k(\Omega;\beta)),\ \Omega \subset \mathbb{R}^n \text{ ouvert t.q. } |\Omega| = m\} \tag{1.1}$$

où $m, \beta > 0$ sont fixés et les $(\lambda_i(\Omega;\beta))_{i\geq 1}$ sont les valeurs propres du Laplacien avec condition de Robin définies par l'équation

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega;\beta) u_k & (\Omega) \\ \partial_\nu u_k + \beta u_k = 0 & (\partial\Omega) \end{cases}$$

On suppose que $F : (0, +\infty)^k \to \mathbb{R}$ est une fonction qui a une dérivée partielle positive ou nulle selon chaque coordonnée, et qui tend vers l'infini lorsque sa dernière coordonnée tend vers l'infini (l'hypothèse de régularité de F peut en fait être allégée; voir le chapitre 6 pour plus de détails). Les deux prototypes de fonctionnelles que l'on peut avoir en tête sont

$$F(\lambda_1, \ldots, \lambda_k) = \lambda_1 + \ldots + \lambda_k$$

et

$$F(\lambda_1, \ldots, \lambda_k) = \lambda_k.$$

On commence par établir un résultat d'existence et de régularité partielle dans le premier cas.

Théorème 1.8. *Soit F une telle fonction, avec des dérivées partielles strictement positives, et $m, \beta > 0$, alors la borne inférieure dans (1.1) est atteinte. De plus, n'importe quel ouvert Ω qui atteint cette borne vérifie $\mathscr{H}^{n-1}(\partial\Omega) \leq C(n, m, \beta)$, et $\partial\Omega$ est Ahlfors-régulier.*

Les étapes principales sont les suivantes:

- On considère une suite minimisante Ω^i, et $u^i = (u^i_1, \ldots, u^i_k)$ un choix des k premières fonctions propres du Laplacien avec conditions de Robin. En prolongeant ces fonctions par 0 à l'exterieur de Ω^i, on obtient une suite de fonctions bornées dans SBV et on peut montrer que celles-ci peuvent s'accumuler autour d'au plus k suites divergentes de points, ce qui permet d'obtenir l'existence d'un minimiseur au sens SBV.

- On montre la régularité des minimiseurs SBV par une méthode similaire à celle de De Giorgi, Carriero et Leaci pour la fonctionnelle de Mumford-Shah, mais dans une version vectorielle ; l'idée principale est qu'en tout point du support, au moins l'une des fonctions propres est supérieure (en valeur absolue) à une valeur seuil, ce qui permet de déduire des bornes sur la longueur des discontinuités partagées.

On s'intéresse ensuite au second cas, la minimisation de la k-ème valeur propre seule. Dans ce cas là on ne sait pas montrer de régularité sur le domaine Ω, et l'existence est seulement connue dans

un cadre relaxé et a été traitée dans [11]. Dans ce cadre relaxé, on ne considère plus des domaines mais des paquets de fonctions $\mathbf{u} = (u_1, \dots, u_k)$ linéairement indépendantes, sur lesquelles ont définit

$$\lambda_i(\mathbf{u}; \beta) = \inf_{V \subset \mathrm{Vect}(\mathbf{u}), \dim(V)=i} \sup_{v \in V} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{u}}} (\overline{v}^2 + \underline{v}^2) \mathrm{d}\mathscr{H}^{n-1}}{\int_{\mathbb{R}^n} v^2 \mathrm{d}\mathscr{L}^n}.$$

Et un paquet de fonctions $\mathbf{u}$ est dit admissible lorsqu'il vérifie certaines conditions de régularité (qui seront précisées plus en détail) ainsi qu'une contrainte de volume $|\{\mathbf{u} \neq 0\}| = m$. Malgré l'absence de régularité, on est cependant capable de montrer la dégénérescence des valeurs propres au minimum, au moins en dimension trois et plus:

Théorème 1.9. *Soit $n \geq 3$, $k \geq 2$, $\beta \in]0, +\infty[$, $m > 0$, et soit $\boldsymbol{u}$ un minimiseur relaxé (SBV) de la fonctionnelle*

$$\boldsymbol{v} \mapsto \lambda_k(\boldsymbol{v}; \beta)$$

parmi les fonctions de $\mathbb{R}^n$ admissibles dont la mesure du support est m. Alors

$$\lambda_{k-1}(\boldsymbol{u}; \beta) = \lambda_k(\boldsymbol{u}; \beta).$$

Dans le cas où $\beta = \infty$, qui s'identifie au problème de Laplacien avec conditions de Dirichlet, il s'agit d'une conjecture citée notamment dans [30], [26] qui a été observée numériquement. Ici, ce résultat est une conséquence du fait que l'on est capable de montrer, si la k-ième valeur propre est simple, que la k-ieme fonction propre ne prend pas de valeurs dans $(-\epsilon, +\epsilon) \setminus \{0\}$ pour ϵ suffisamment petit, ce qui signifie que la solution est nécessairement associée à des fonctions dont le support est disconnexe.

La régularité d'un tel minimiseur est ouverte dû à cette dégénérescence de la valeur propre; il serait intéressant de commencer par comprendre le cas de la minimisation de $\lambda_3(\cdot; \beta)$, où l'on s'attend à ce qu'en n'importe quel minimiseur connexe (c'est-à-dire qui ne soit pas une union disjointe de trois boules, ce qui est une possibilité lorsque β est petit), cette valeur propre ait une multiplicité d'exactement deux.

1.3 Régularité d'un problème à frontières libres avec obstacle

Cette étude est entre autres motivée par la compréhension des problèmes à frontière libre et à discontinuité libre "dégénérés" tels que mentionnés plus tôt, pour les (quasi-)minimiseurs de fonctionnelles de la forme

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{J_u} \left(|u^+| + |u^-| \right) \mathrm{d}\mathscr{H}^{n-1}.$$

On commence par l'étudier dans un cadre possiblement plus simple, d'un problème sans discontinuité libre; étant donnés un domaine ouvert borné et lisse noté D, un ensemble mesurable E_0, une fonction $\varphi \in H^{1/2}(\partial D, \mathbb{R}_+)$, on cherche l'ensemble E et la fonction $u \in H^1(D, \mathbb{R}_+)$ minimisant la quantité

$$J_D(u, E) \mapsto \int_D |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{D \cap \partial^* E \cap \{u > 0\}} u \mathrm{d}\mathscr{H}^{n-1}.$$

parmi les couples (u, E) coincidant avec (φ, E_0) à l'extérieur de D.

On sait qu'il existe des couples minimaux (u, E), et u est automatiquement harmonique sur E et son complémentaire. On peut de plus montrer que $u \in \mathcal{C}^{0, \frac{2}{n+2}}_{\mathrm{loc}}(D)$, et la question de la régularité se pose surtout le long de ∂E:

- Si $x \in \partial^* E \cap \{u > 0\}$, alors E est un quasiminimiseur du périmètre dans un voisinage de x et est donc décomposable en $R \sqcup S$ où R est $\mathcal{C}^{1,\gamma}$ et $\dim(S) \leq n - 8$. Plus précisément, il est montré dans [25] que R est lisse et $u 1_E$, $u 1_{E^c}$ sont lisses jusqu'au bord.

- Si $u_{|B_{x,r}\cap\partial E} \equiv 0$, alors $(1_{E^c} - 1_E)u$ est harmonique dans $B_{x,r}$ et ∂E est donc la ligne de niveau d'une fonction harmonique ; notamment en deux dimensions il s'agit de variétés lisses en dehors d'un ensemble localement fini de points de jonctions.

Il y a donc deux théories de régularités distinctes pour chaque partie de la frontière $\partial^* E \cap \{u > 0\}$ et $\{u = 0\}$, et l'enjeu est de comprendre comment se lient ces deux frontières.

Théorème 1.10. *Soit (u, E) un minimiseur local de J_D dans un ouvert $D \subset \mathbb{R}^2$, et soit $B \Subset D$, alors $Per(E|B) < \infty$, $\partial^* E \cap \{u > 0\}$ est localement $\mathcal{C}^\infty$ et $\{u = 0\}$ est inclus dans l'union d'un nombre fini de courbes $\mathcal{C}^1$.*

Voici le plan de preuve, où les deux premiers points sont valables en dimension $n \geq 2$.

- On établit des estimations de régularité; soit (u, E) un minimiseur de J sur B_4, on a une estimation intérieure

$$\|u\|_{\mathcal{C}^{\frac{2}{n+2}}(B_1)} \leq C_n J_{B_2}(u, E)$$
$$J_{B_1}(u, E) \leq C_n \int_{\partial B_2} u \mathrm{d}\mathscr{H}^{n-1}$$

  et une estimation extérieure

$$\int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} \leq C_n(1 + u(0))$$

- On en déduit en particulier la compacité des suites de minimiseurs ponctuellements bornés; cela permet de montrer l'existence de minimiseurs pour des conditions de bord fixées.

- On établit la monotonie de la fonctionnelle

$$r \to W\left(\frac{u(r\cdot)}{r}, \frac{E}{r}\right),$$

  où $W(u, E) = J_{B_1}(u, E) - \int_{\partial B_1} u^2$. En deux dimensions cela nous permet d'identifier les minimiseurs homogènes obtenus comme blow-up autour de points de $\{u = 0\}$. Ces minimiseurs sont de la forme

$$u(x) = |x \cdot e|, \ E = \{x : x \cdot e > 0\},$$

  où $e \in \mathbb{R}^2 \setminus \{0\}$ est tel que $|e| \leq \frac{1}{4}$, ou de la forme $u \equiv 0$.

- On prouve une version améliorée de la formule de monotonie; pour tout minimiseur (u, E) de J dans B_2 tel que $u(0) = 0$, il existe des constantes universelles $\gamma, \vartheta \in]0, 1[$ telles que

$$\left.\frac{d}{dr}\right|_{r=1} W(u_r, E_r) \geq 2\left(\gamma \wedge W(u, E)^\vartheta\right) W(u, E).$$

  Cette estimation est obtenue en construisant un compétiteur explicite (v, F) dans B_1 tel que $(v, F)_{|\partial B_1} = (u, E)_{|\partial B_1}$ et

$$W(v, F) \leq \left(1 - \gamma \wedge W(u^h, E^h)^\vartheta\right) W(u^h, E^h),$$

  où (u^h, E^h) est l'extension 1-homogène de $(u, E)_{|\partial B_1}$ sur B_1.

- On en déduit des estimations quantitatives de la vitesse de convergence vers les blow-up, et on obtient en particulier l'unicité du blow-up en chaque point de $\{u = 0\}$ avec une dépendance $\mathcal{C}^{1,\log}$, ainsi qu'une propriété de "non-dégénérescence" qui caractérise entièrement les minimiseurs près des petits blow-up.

Le point crucial est l'avant dernier, connu dans des situations similaires comme une **inégalité épipérimètrique** ; des méthodes similaires ont été utilisées avec succès pour les surfaces minimales dans [32], pour le problème de Alt-Caffarelli et le problème d'obstacle fin dans [33].
La question de la régularité est loin d'être résolue, on laisse en particulier les pistes suivantes:

- Peut-on dire que tout ∂E est lisse ? Le résultat obtenu en est très proche, surtout avec la convergence des blow-up sur chaque point de $\{u = 0\}$, mais la continuité de la normale de $\partial^* E \cap \{u > 0\}$ n'est pas claire.
- Que dire en dimension supérieure ? La formule de monotonie reste vraie mais la classification des minimiseurs homogènes n'est pas connue.

1.4 Un problème d'optimisation de forme pour l'isolation thermique

On s'intéresse ici à un problème d'optimisation de forme où l'on essaie d'isoler un solide à température constante (supérieure à la température extérieure, supposée constante) par une couche d'isolant de mesure fixée, avec une loi générale de transfert thermique entre l'isolant et l'extérieur (convection, radiation...). Plus précisément, étant donné un compact à bord lisse $K \subset \mathbb{R}^n$, un ouvert lisse $\Omega \subset \mathbb{R}^n$ contenant K, et une fonction $\Theta : [0,1] \to \mathbb{R}_+$ supposée croissante, semi-continue inférieurement et telle que $\Theta(0) = 0$, on pose

$$E_\Theta(K,\Omega) = \inf\left\{\int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \Theta(v)\mathrm{d}\mathscr{H}^{n-1},\ v \in H^1(\Omega) : 1_K \le v \le 1\right\}.$$

Lorsque Θ est convexe le minimum est atteint pour une unique fonction u qui représente la température à l'équilibre, égale à 1 dans K, 0 à l'extérieur, qui est harmonique dans l'isolant $\Omega \setminus K$, et lorsque Θ est suffisamment lisse on a une condition de bord

$$\begin{cases} -\partial_\nu u = \frac{1}{2}\Theta'(u) & (\partial\Omega \cap \{u > 0\}) \\ -\partial_\nu u \le \frac{1}{2}\Theta'(u) & (\partial\Omega \cap \{u = 0\}) \end{cases}$$

Voici quelques exemples de choix de Θ pertinents:

- $\Theta(u) = \beta u^2$; dans ce cas on obtient des conditions de Robin

  $$-\partial_\nu u = \beta u$$

  au bord de Ω, en particulier lorsque Ω est suffisamment lisse on a $\inf_{x\in\Omega} u(x) > 0$ et le flux de chaleur est proportionnel à la différence de température. Cela correspond par exemple au transfert thermique obtenu lorsque la température extérieure est uniforme par convection (dans un fluide) avec un phénomène de couche limite le long de Ω.
- $\Theta(u) = c1_{u>0}$; il s'agit ici du cas où l'on possède un isolant parfait ayant un coût surfacique c que l'on peut appliquer sur le bord de Ω. La condition de bord obtenue est

  $$u = 0 \text{ ou } \partial_\nu u = 0$$
- $\Theta(u) = 2cu_+$; correspond à un flux de chaleur constant, avec une condition de bord

  $$-\partial_\nu u \begin{cases} = c & \text{dans } \partial\Omega \cap \{u > 0\} \\ \in (0,c] & \text{dans } \partial\Omega \cap \{u = 0\} \end{cases}$$
- $\Theta(u) = \frac{2}{5}u_+^5 + 2Tu_+^4 + 4T^2u_+^3 + 4T^3u_+^2$ où $T \ge 0$ correspond au transfert de chaleur issu d'un phénomène de radiation thermique dans un milieu à température T, associé à la condition de bord

  $$-\partial_\nu u = (T+u)^4 - T^4$$

- Toute combinaison linéaire des fonctions précédentes.

Le problème de l'optimisation de $E_\Theta(K,\Omega)$ à K fixé, parmi tous les domaines Ω vérifiant certaines contraintes de mesure, a été étudié dans [12], [13]. Nous nous intéressons ici au problème d'optimisation de forme où Ω et K peuvent varier ; plus précisément, soit ω_n la mesure de la boule unité de $\mathbb{R}^n$, $M > \omega_n$ et $\Lambda > 0$, on s'interesse aux deux problèmes

$$\inf_{K\subset\Omega,\ |K|=\omega_n,\ |\Omega|\leq M} E_\Theta(K,\Omega), \tag{1.2}$$

et la version pénalisée

$$\inf_{K\subset\Omega,\ |K|=\omega_n} E_\Theta(K,\Omega) + \Lambda|\Omega\setminus K|. \tag{1.3}$$

Nous verrons que selon la loi de transfert Θ, on peut dans certain cas donner une description complète des solutions du premier problème, et pour un Θ général on obtient une solution complète du second problème, qui donne une solution partielle (c'est-à-dire pour seulement certaines valeurs de M) du premier. Un phénomène notable est que dans certains cas il est préférable de ne pas utiliser tout l'isolant, c'est-à-dire que la contrainte $|\Omega| \leq M$ n'est pas nécessairement saturée, particulièrement lorsque M est proche de ω_n. Les résultats principaux sont les suivants

Théorème 1.11 (Cas convectif)**.** *Soit $\Theta(u) = \beta u^2$ et $M = R^n\omega_n$ (où $R \geq 1$), la solution de* (1.2) *consiste en deux boules concentriques, où la boule extérieure est de rayon* 1 *ou R selon le minimum entre $E_\Theta(B_1,B_R)$ et $E_\Theta(B_1,B_1)\}$.*

La preuve de ce premier résultat est pratiquement auto-contenue, basée sur une formulation par longueur extrémale équivalente inspirée par les travaux de Bossel [4], il y a cependant en dimension $n \geq 3$ un cas qui échappe à cette analyse mais est obtenu grâce au résultat suivant. Plus précisément, il y a trois cas possibles:

- Si $\beta \geq n-1$, alors $E_\Theta(K,\Omega) \geq E_\Theta(B_1,B_R)$.
- Si $\beta \in (n-2,n-1)$, alors il existe $R_{n,\beta} > \frac{n-1}{\beta}$ tel que $E_\Theta(K,\Omega) \geq \begin{cases} E_\Theta(B_1,B_R) & \text{si } R \geq R_{n,\beta} \\ E_\Theta(B_1,B_1) & \text{si } R \leq R_{n,\beta} \end{cases}$.
- Si $\beta \leq n-2$, ce qui ne peut se produire qu'en dimension 3 et plus, alors $E_\Theta(K,\Omega) \geq E_\Theta(B_1,B_1)$.

Les deux premiers cas sont obtenus par une méthode directe qui établi un lien entre $E_\Theta(K,\Omega)$ et la longueur extrémale de la famille des courbes (ou hypersurface, en dimension 3 et plus) séparant $\partial\Omega$ de K; on montre que

$$E_\Theta(K,\Omega) = \sup_{\rho\in L^\infty(\Omega,\mathbb{R}_+)} \inf_{K\subset\omega\subset\Omega} \left(\beta \mathrm{Per}(\omega;\partial\Omega) + \int_{\partial^*\omega\cap\Omega} \rho \mathrm{d}\mathscr{H}^{n-1} - \int_\omega \rho^2 \mathrm{d}\mathscr{L}^n\right)$$

où la borne inférieure est prise sur les ouvert ω de périmètre fini. Par des méthodes de réarrangement de la métrique optimale $\rho = |\nabla \log u|$, on obtient des deux premiers points mais pas le dernier. Celui-ci est obtenu comme conséquence du résultat suivant.

Théorème 1.12 (Cas général pénalisé)**.** *Pour tout $\Lambda > 0$ et tout $\Theta : [0,1] \to \mathbb{R}_+$ croissant, semi-continue inférieurement avec $\Theta(0) = 0$, la solution de* (1.3) *consiste en deux boules concentriques avec une température à l'équilibre u radiale.*

Ce résultat est essentiellement obtenu en deux étapes: on montre l'existence d'un minimiseur, et on montre qu'un minimiseur est nécessairement la boule. L'existence est en fait obtenue pour une version approchée de la fonction Θ - ce qui est suffisant pour nos besoins - par le résultat suivant.

Théorème 1.13 (Cas général)**.** *Il existe une constante $c_n > 0$ tel que pour tout $\Theta : [0,1] \to \mathbb{R}_+$ croissant, semi-continue inférieurement avec $\Theta(0) = 0$, $\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} > 0$, et*

$$M < \omega_n + c_n \left(\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_0^1 \frac{t^{2n-1} \mathrm{d}t}{\Theta(t)^n}, \tag{1.4}$$

alors le problème (1.2) *a une solution (K, Ω). Si $|\Omega| < M$ alors (K, Ω) sont des boules concentriques. Sinon Ω est un ouvert de bord rectifiable tel que $\mathscr{H}^{n-1}(\partial\Omega) < \infty$ et K est relativement fermé dans Ω, et de périmètre localement fini. La température à l'équilibre $u \in H^1(\Omega)$ est dans $\mathcal{C}^{0,\frac{2}{n+2}}_{loc}(\Omega)$ et en dimension 2, $\partial K \cap \Omega$ est analytique.*

La preuve de ce résultat est inspirée de la preuve d'existence d'agrégats de périmètre optimaux détaillée dans le livre de Maggi [29]; il s'agit de considérer une suite minimisante (K_i, Ω_i), d'obtenir par des méthodes de concentration-compacité et des estimations de densité que, quitte à translater des parties de (K_i, Ω_i), ceux-ci convergent vers un minimiseur vérifiant les mêmes contraintes de mesure. Une difficulté est ajoutée par le fait qu'on prend en fait une limite sur la température à l'équilibre (u_i) et non directement la suite (K_i, Ω_i); un tel minimiseur est dans un espace SBV plus général et il faut donc aussi montrer la régularité de ces minimiseurs généralisés.

On a enfin un dernier résultat sur le cas où la quantité d'isolant autorisée $M - \omega_n$ est proche de 0.

Théorème 1.14. *Si $\Theta : [0,1] \to \mathbb{R}_+$ est croissante, $\Theta(0) = 0$, et si de plus Θ est $\mathcal{C}^1$ dans un voisinage de 1 et*

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n-1),$$

alors il existe $M > \omega_n$ dépendant de n et Θ tel que la solution de (1.2) *est $K = \Omega = B_1$.*

Il s'agit essentiellement d'un raffinement des estimations précédentes; on montre que pour tout $u \in SBV(\mathbb{R}^n, [0,1])$ tel que $|\{u = 1\}| = \omega_n$, $|\{0 < u < 1\}| \ll 1$, on peut supposer quitte à réduire le support de u sans trop changer son énergie, que $\inf_{\{u>0\}} u \approx 1$, puis on fait un développement limité de Θ. Ce critère est optimal et on retrouve en particulier le phénomène constaté pour les conditions de convection, où il peut être mieux de ne mettre aucun isolant.

Le problème (1.3) est donc entièrement résolu, mais nous laissons ouvert (1.2) dans certains cas: en particulier, si $M = \omega R^n$ est tel que

$$R \notin \bigcup_{\Lambda>0} \operatorname{argmin} \left\{ r \mapsto E_\Theta(B_1, B_R) + \Lambda |B_R \setminus B_1| \right\},$$

peut-on dire dans ce cas que $E_\Theta(K, \Omega) \geq \inf_{1 \leq r \leq R} E_\Theta(B_1, B_r)$?

1.5 Positivité des fonctions surharmoniques vérifiant une condition de Robin

On considère, comme cela a pu être fait précédemment, un domaine ouvert $\Omega \subset \mathbb{R}^n$ dont la frontière topologique est supposée rectifiable et une fonction $f \in L^2(\Omega)$, alors sous de faibles hypothèses de régularité on peut définir au sens variationnel une unique solution u à

$$\begin{cases} -\Delta u = f & (\Omega) \\ -\partial_\nu u = \beta u & (\partial\Omega) \end{cases}$$

Si $f \geq 0$ est non-trivial, alors la solution u est strictement positive à l'intérieur de Ω par principe du maximum fort sur les fonctions harmoniques. De plus, lorsque Ω vérifie une condition de sphère tangente intérieur en chaque point de $\partial\Omega$ - par exemple si $\partial\Omega$ est $\mathcal{C}^2$ - le principe du maximum de

Hopf nous dit que u et $\partial_\nu u$ ne peuvent pas être tous les deux nuls, et comme ces deux quantités sont proportionnelles dans notre cas on déduit donc que $u_{|\partial\Omega} > 0$.

En particulier, cela nous dit que $\partial_\nu u(x) < 0$ en chaque $x \in \partial\Omega$. Si l'on retourne à l'interprétation thermique de ce problème, en voyant u comme la température à l'équilibre d'un solide Ω soumis à une génération de chaleur f et une condition de transfert par convection avec l'extérieur, la condition $\partial_\nu u(x) < 0$ signifie qu'il y a un flux de chaleur non-nul en x. En reprenant la terminologie utilisée par [2], on dit qu'un tel point $x \in \partial\Omega$ est **actif** lorsque $\partial_\nu u(x) < 0$, et le lemme de Hopf entraîne donc que dans un domaine $\mathcal{C}^2$, chaque point du bord est actif.

Lorsque $\Omega \cap B_1 = \{(x_1, x') \in B_1 : x_1 > 0,\ |x'| < x_1^\alpha\}$ pour un $\alpha > 0$, on sait au contraire que l'origine est un point actif si et seulement si $\alpha < 2$.

Dans [2], il est établi par des méthodes probabilistes que dans un large cadre de domaines peu lisses (domaines obtenus par union croissante de domaines Lipschitz avec des constantes uniformes, voir plus de détail dans [2]), il existe un critère nécessaire et suffisant à ce qu'un point soit actif. En particulier, étant donné (D_n) une décomposition en blocs dits hyperboliques d'un voisinage d'un point $x \in \partial\Omega$, le point x est actif si et seulement si $\sum_{n\geq 0} n\mathscr{H}^1(\partial D_n \cap \partial\Omega) < \infty$ en dimension 2 (voir notamment le début de la section 3 et de la section 4 de [2] pour les détails des définitions et la caractérisation plus complexe en dimension quelconque).

Un de nos objectif a été de trouver par des méthodes variationnelles des conditions de positivité faisant plus explicitement appel à la géométrie du domaine, afin d'ensuite les généraliser à des opérateurs elliptiques autres que le Laplacien ; on s'intéresse de manière assez générale à une solution d'un opérateur monotone

$$\begin{cases} -\nabla \cdot \mathcal{A}(x, \nabla u) \gneq 0 & (\Omega) \\ \mathcal{A}(x, \nabla u).\nu_\Omega + \mathcal{B}(x, u) = 0 & (\partial\Omega) \end{cases} \tag{1.5}$$

où $\mathcal{A} \in \mathcal{C}^0(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$, $\mathcal{B} \in \mathcal{C}^0(\partial\Omega \times \mathbb{R}, \mathbb{R})$ vérifient les deux hypothèses suivantes

- Il existe $p \in (1, +\infty)$, $0 < \alpha_1 \leq \alpha_2$ et $\psi \in \mathcal{C}^0(\partial\Omega, \mathbb{R})$ tels que pour tout $x \in \Omega$, $y \in \partial\Omega$, $z \in \mathbb{R}^n$,
$$\alpha_1|z|^p \leq z\mathcal{A}(x, z),\ |\mathcal{A}(x, z)| \leq \alpha_2|z|^{p-1},\ |\mathcal{B}(y, z)| \leq \psi(y)|z|^{p-1}.$$
- Pour tout $x \in \Omega$, $y \in \partial\Omega$, $z_1, z_2 \in \mathbb{R}^n$,
$$(z_1 - z_2)(\mathcal{A}(x, z_1) - \mathcal{A}(x, z_2)) \geq 0,$$
$$(z_1 - z_2)(\mathcal{B}(y, z_1) - \mathcal{B}(y, z_2)) \geq 0.$$

On définit le profil isopérimétrique du domaine Ω comme la fonction qui à tout $m > 0$ suffisamment petit associe

$$I(m) := \inf\left\{\mathscr{H}^{n-1}(\partial^* A \cap \omega),\ A \subset \Omega \text{ ouvert t.q. } |A| \leq \frac{1}{2}|\Omega|,\ |A|^{1-\frac{1}{p}}\mathscr{H}^{n-1}(\partial A \cap \partial\Omega)^{\frac{1}{p}} \geq m\right\}.$$

Puis on établit le critère que pour tout u solution d'un opérateur monotone d'ordre p de la forme (1.5),

$$1/I \text{ est intégrable au voisinage de } 0 \Longrightarrow \inf_{x\in\Omega} u(x) > 0.$$

Cela est par exemple le cas lorsque Ω vérifie une inégalité isopérimétrique relative de type

$$\mathscr{H}^{n-1}(\partial^* A \cap \partial\Omega) \leq C\frac{\mathscr{H}^{n-1}(\partial^* A \cap \Omega)}{|A|^\alpha},$$

pour tous les ensembles $A \subset \Omega$ de périmètre fini, de mesure suffisament petite et pour un $\alpha < \frac{1}{N}\left(1 - \frac{1}{p}\right)$. On étudie aussi une version plus locale de ce résultat, ainsi que le cas particulier des domaines en pointe

$$\Omega = \left\{x = (x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \text{ t.q. } x_1 > 0,\ |x'| \leq h(x_1)\right\}$$

pour une fonction croissante $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ telle que $h(0) = 0$ et h^{n-1} est convexe. On montre dans ce cas que

$$\int_0^1 \left(\frac{\int_0^t h^{n-2}}{\int_0^t h^{n-1}} \right)^{\frac{1}{p}} dt < \infty \text{ implique que tout } \partial\Omega \text{ est actif pour l'équation (1.5)}$$

1.6 Optimisation de la forme d'un obstacle plongé dans un fluide de Stokes

Dans le dernier chapitre, on aborde un autre problème à discontinuité libre vectoriel, qui est issu de la mécanique des fluides. Le modèle est le suivant: on considère un écoulement de fluide incompressible visqueux dans une boite $\Omega \Subset \mathbb{R}^n$ au bord de laquelle le fluide a une vitesse constante u_∞ (sous-entendu "la vitesse à l'infini" ; on ne peux pas prendre $\Omega = \mathbb{R}^n$ pour des raisons techniques). Le fluide s'écoule autour d'un obstacle $E \Subset \Omega$, que l'on suppose Lipschitz au début, et vérifie sur le bord de E une condition dite de Navier que l'on peut interpréter comme provenant d'une friction entre le fluide et le bord de E.

Plus précisément, la vitesse de fluide $u \in H^1(\Omega \setminus E, \mathbb{R}^n)$ et la pression $p \in L^2(\Omega \setminus E, \mathbb{R})$ vérifient l'équation de Stokes incompressible avec condition de Navier de paramètre β, c'est-à-dire que

(a) *Incompressibilité*: div $u = 0$ dans $\Omega \setminus E$.

(b) *Condition à l'infini*: $u = u_\infty$ dans $\partial\Omega$.

(c) *Non-pénétration*: $u \cdot \nu_E = 0$ dans ∂E, où ν_E est le vecteur normal sortant de E.

(d) *Équilibre*: on définit le tenseur de stress $\sigma := -pI_d + 2\mu e(u)$, où $\mu > 0$ est un paramètre de viscosité, $e(u) = \frac{\nabla u + (\nabla u)^*}{2}$ est le gradient symétrisé de u, et p est une pression. On a alors div $\sigma = 0$ dans $\Omega \setminus E$.

(e) *Condition de Navier*: $(\sigma\nu_E)_\tau = \beta u$ dans ∂E, où V_τ désigne la composante tangentielle à ∂E d'un vecteur V et $\beta > 0$ est une constante de friction.

u peut être aussi défini de manière variationnelle comme le minimiseur de l'énergie

$$\mathcal{E}_E(u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{\partial E} |u|^2 \mathrm{d}\mathscr{H}^{n-1}$$

dans la classe des champs de vecteurs suffisament lisses vérifiant les conditions (a,b,c) ci-dessus, notée $\mathcal{V}^{reg}_{E,u_\infty}(\Omega)$; les conditions (c,d) apparaissent comme équation variationnelle de cette énergie.

La traînée de E est la composante dans la direction u_∞ de la force s'exerçant sur K, donc donnée par

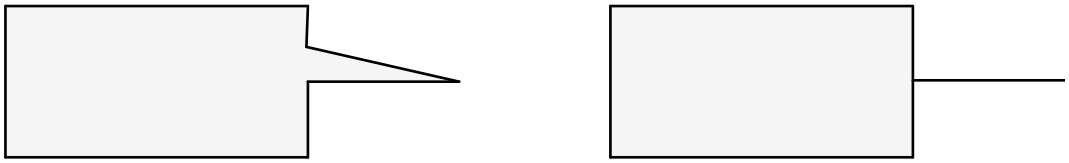
$$Drag(E) := \int_{\partial E} \sigma\nu \cdot u_\infty \mathrm{d}\mathscr{H}^{n-1}$$

et celle-ci peut être en fait définie de façon variationnelle par $Drag(E) = \inf_{u \in \mathcal{V}^{reg}_{E,u_\infty}(\Omega)} \mathcal{E}_E(u)$.

Soient $\Omega, u_\infty, \mu, \beta$ des données du problème, ainsi que $c > 0$, $f : (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ une application semi-continue inférieurement, on s'intéresse à la minimisation de fonctionnelles du type

$$\mathcal{J}(E, u) = \mathscr{E}_E(u) + c\mathscr{H}^{n-1}(\partial E) + f(|E|)$$

pour E Lipschitz, $u \in \mathcal{V}^{reg}_{E,u_\infty}(\Omega)$. On ne s'attend généralement pas à ce qu'il existe un minimum dans cette classe, non seulement à cause de la régularité de E mais aussi à cause de l'apparition possible de structures de dimension inférieure lorsque l'obstacle devient "fin" comme dans la figure ci-dessous, (ce qui était impossible sous des conditions de dirichlet $u_{|\partial E} = 0$, mais l'est avec les conditions de Navier).



E_i E

Cela appelle a une relaxation du problème; pour tout (E,u) admissible, on peut prolonger u par 0 à l'intérieur de E et cela place naturellement u dans l'espace des fonctions SBD (fonctions spéciales à déformation bornées), des fonctions $v \in L^1_{\rm loc}(\mathbb{R}^n)$ telles que $\frac{Dv+(Dv)^*}{2}$ (où Dv désigne la dérivée au sens des distributions) est une mesure de Radon présentant la structure suivante: en notant J_v l'ensemble des $x \in \mathbb{R}^n$ tels qu'il existe $\nu_v(x) \in \mathbb{S}^{n-1}$, $v^\pm(x) \in \mathbb{R}^n$ vérifiant

$$v(x+r\cdot) \underset{L^1_{\rm loc}(\mathbb{R}^n)}{\longrightarrow} v^+(x)1_{\{y:y\cdot\nu_v(x)>0\}} + v^-(x)1_{\{y:y\cdot\nu_v(x)<0\}} \text{ quand } r \to 0$$

alors $v \in SBD_{\rm loc}(\Omega)$ lorsque

$$\frac{Dv+(Dv)^*}{2} = e(v)\mathscr{L}^n + (v^+ - v^-) \odot \nu_v \mathscr{H}^{n-1} \lfloor J_v$$

pour $e(v) \in L^1_{\rm loc}(\mathbb{R}^n)$, et où $\odot$ désigne le produit tensoriel symétrisé $a \odot b = \frac{a\otimes b + b\otimes a}{2}$. On note

$$\mathcal{V}(\Omega) = \left\{ u \in SBD(\Omega) : \mathrm{Tr}[e(u)] = 0,\ \nu_u \cdot u^\pm = 0 \text{ dans } J_u \right\}$$

l'ensemble des flots incompressibles tangents à leurs discontinuités, et pour tout ensemble de périmètre fini E,

$$\mathcal{V}_{E,u_\infty}(\Omega) = \left\{ u \in \mathcal{V}(\Omega) : u_{|\partial\Omega} = u_\infty,\ u_{|E} = 0 \right\}.$$

Notons en particulier que u est autorisé à avoir des discontinuités à l'extérieur de E, et pas seulement sur $\partial^* E$. On pose alors

$$\begin{aligned}\mathcal{J}(E,u) = &\int_\Omega 2\mu|e(u)|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E} (c+\beta|u^+|^2)\mathrm{d}\mathscr{H}^{n-1} \\ &+ \int_{J_u \setminus \partial^* E} (2c + \beta|u^+|^2 + \beta|u^-|^2)\mathrm{d}\mathscr{H}^{n-1} + f(|E|)\end{aligned}$$

et on s'intéresse à la minimisation

$$\inf\left\{ \mathcal{J}(E,u),\ E \text{ de périmètre fini, } u \in \mathcal{V}_{E,u_\infty}(\Omega) \right\}. \tag{1.6}$$

C'est un problème à discontinuité libre similaire au problème d'apparition de fracture de Griffith [24]; la fonctionnelle est essentiellement la même exceptée le terme donnant la condition de Navier, mais la différence est que u est prise dans un espace bien plus restreint correspondant aux flots incompressibles. On établit trois résultats.

Théorème 1.15. (1.6) *admet un minimum.*

Ce résultat a plusieurs composantes; l'idée de base est de considérer une suite admissible (E_i, u_i) dont l'énergie converge vers l'infimum. Il faut alors vérifier les choses suivantes:

- On peut extraire de (u_i) une sous-suite convergente ; c'est possible par les résultats de compacité de [3], et on peut donc supposer que (u_i) converge vers une limite u au sens $L^1_{\rm loc}$.

- La limite u vérifie les conditions d'incompressibilité et de non-pénétration; la première contrainte est directe à la limite, mais la seconde demande de faire appel au résultat récent de semicontinuité des fonctionnelles symétriques conjointement convexes de [21], dont fait partie la fonctionnelle
$$\int_{J_u} \left(|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|\right) \mathrm{d}\mathscr{H}^{n-1}$$
- L'énergie de u est bien minimale ; cela demande de montrer un nouveau résultat de semicontinuité de rentrant pas dans le cadre des précédents, en particulier de la fonctionnelle
$$\int_{J_u} \left(|u^+|^2 + |u^-|^2\right) \mathrm{d}\mathscr{H}^{n-1}.$$

Un minimum de (1.6) est dit fort si $\mathscr{H}^{n-1}(\overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0$; cela signifie qu'en posant F le complémentaire de l'union des composantes connexes de $\Omega \setminus \overline{\partial^* E \cup J_u}$ où $u = 0$, alors F est un fermé à bord topologique rectifiable, $u \in \mathcal{V}_{F,u_\infty}(\Omega)$ avec $u \in H^1_{\mathrm{loc}}(\Omega \setminus F)$, et $J(F,u)$ prend la valeur minimale dans le problème (1.6). En particulier on retrouve que u est une solution de l'équation de Stokes au sens classique dans $\Omega \setminus F$, et u qui vérifie la condition de Navier sur ∂F au sens variationnel.

Théorème 1.16. *En dimension 2, si f est décroissante et Ω est simplement connexe, alors* (1.6) *admet un minimum fort dans la classe des compacts K tels que K et $\mathbb{R}^2 \setminus K$ sont connexes.*

Ce résultat peut être obtenu pratiquement indépendamment du précédent, avec en particulier des simplifications sur la fermeture des contraintes à la limite et la semicontinuité ; l'idée est de formuler la fonctionnelle grâce à la fonction de flux de u, c'est-à-dire la fonction scalaire ψ telle que $\nabla^\perp \psi = u$, ce qui revient à dire que les lignes de niveau de ψ sont les trajectoires du fluide dans le champ u. On dispose sur ψ d'estimation uniforme dans des espaces Hölder, et le théorème de Gołąb sur les compacts connexes de $\mathbb{R}^2$ permet d'obtenir la compacité à moindre frais.

Théorème 1.17. *On suppose que $n = 2$ et que f est Lipschitz. Soit (E,u) un minimiseur de* (1.6)*, alors*
$$\mathscr{H}^1(\Omega \cap \overline{J_u \cup \partial^* E} \setminus (J_u \cup \partial^* E)) = 0.$$

Cela signifie que (E,u) est **localement** un minimiseur fort. Ce résultat n'est ni plus fort, ni plus faible que le précédent; en particulier son raisonnement fait appel à des fonctions test qui ne conservent pas la connexité de l'obstacle. La preuve de ce résultat est similaire à la preuve de la fermeture des sauts des minimiseurs SBV de la fonctionnelle de Mumford-Shah de De Giorgi, Carriero, Leaci [19]. Plus précisément, cette stratégie a été utilisée avec succès pour montrer que les minimiseurs SBD du problème de Griffith statique sont forts, en dimension 2 (dans [17]) et en dimension quelconque (dans [15]). L'idée générale est de montrer la fermeture en établissant une estimation de densité uniforme autour des discontinuités. Plus précisément, notons $Q_{x,r} = x + [-r,r]^n$, on établit l'existence de $\overline{r}, \epsilon, C > 0$ dépendant seulement des constantes $\mu, c, \lambda > 0$ (et étonnament pas de β) telles que si $u \in \mathcal{V}(Q_{x,r})$ est un minimiseur de (1.6) dans $Q_{x,r}$ pour $r \in (0, \overline{r})$, et si
$$\int_{Q_{x,r}} |e(u)|^2 \mathrm{d}\mathscr{L}^2 + \mathscr{H}^1(J_u \cap Q_{x,r}) \leq \epsilon r$$
alors pour tout $\rho \in (0,r)$,
$$\int_{Q_{x,\rho}} |e(u)|^2 \mathrm{d}\mathscr{L}^2 + \mathscr{H}^1(J_u \cap Q_{x,\rho}) \leq C r^{-\frac{1}{2}} \rho^{\frac{3}{2}},$$
ce qui exclut en particulier l'existence de sauts dans un voisinage de x.

Cette borne est obtenue à travers un raisonnement par compacité similaire à [19], dont l'une des composantes principales est le lemme d'approximation suivant, qui peut présenter un intérêt indépendant du reste.

On fixe $\rho \in \mathcal{C}^\infty_c(B_{1/8}, \mathbb{R}_+)$ une fonction radiale d'intégrale 1, et on note $\rho_\delta := \delta^{-2}\rho(\delta^{-1}\cdot)$.

Théorème 1.18. *Il existe des constantes $C, \eta > 0$ telles que pour tout $u \in \mathcal{V}(Q_1)$ vérifiant $\mathscr{H}^1(J_u) \leq \eta$, il existe $r \in [1 - \mathscr{H}^1(J_u)^{\frac{1}{4}}, 1]$, $\varphi \in \mathcal{C}_c^\infty(Q_r, [0,1])$ telle que $Q_{r-\sqrt{\mathscr{H}^1(J_u)}} \subset \{\varphi = 1\}$, et une fonction $v \in \mathcal{V}(Q_1) \cap H^1(Q_r)$ telle que*

- $\{u \neq v\} \subset Q_r$, $\mathscr{H}^1(J_v \setminus J_u) = 0$,
- $\int_{Q_1} \left| e(v) - \left(\rho_{\sqrt{\mathscr{H}^1(J_u)}} * e(u) \right) \varphi \right|^2 \mathrm{d}\mathscr{L}^2 \leq C \mathscr{H}^1(J_u)^{\frac{1}{6}} \int_{Q_1} |e(u)|^2 \mathrm{d}\mathscr{L}^2$.

Autrement dit, un flot incompressible avec peu de discontinuités peut être lissé. L'idée générale de la preuve est de poser $\delta = \sqrt{\mathscr{H}^1(J_u)}$, et choisir un rayon r proche de 1 tel que tous les carrés de tailles $\frac{\delta}{2^k}$ à distance $\frac{\delta}{2^{k-1}}$ de ∂Q_r contiennent peu de sauts (par rapport à leur côté), ce qui est possible en deux dimensions par un argument de recouvrement de Vitali. De là, on commence de manière similaire à [15] en découpant le carré Q_r en carrés de côté δ à l'intérieur, et de côté $\frac{\delta}{2^k}$ près du bord ∂Q_r (par une décomposition dyadique). Sur chacun de ces carrés, on trouve une application affine suffisament proche du flot u par une inégalité de Korn pour les fonctions SBD, issue de [14]. On joint ces approximations et on vérifie enfin que le flot obtenu peut être ajusté pour respecter la condition de divergence nulle.

Cette méthode est ici suffisament robuste pour fonctionner sur des presque quasi-minimiseurs de fonctionnelles plus générales, on pourra voir les détails au chapitre 10.

Ces résultats ouvrent plusieurs pistes. La question de la régularité des minimiseurs relaxés en dimension trois reste ouverte, elle nécessiterait par exemple de montrer une version affaiblie du lemme d'approximation (qui en l'état actuelle n'a aucune chance de se généraliser en dimension supérieure). L'approximation numérique (même en deux dimensions) de la traînée par une méthode de champ de phase, par exemple par une fonctionnelle similaire celle d'Ambrosio-Tortorelli pour le problème de Mumford-Shah, semble abordable grâce au lemme d'approximation ci-dessus, mais elle pose des questions toujours ouvertes sur la formulation des contraintes de non-pénétration. Enfin, il est naturel d'essayer de généraliser ces résultats à l'équation de Navier-Stokes, et il semble qu'on ne puisse s'attendre à obtenir les mêmes conclusions à cause de l'apparition potentielle de microstructures au bord de l'obstacle qui change le comportement du fluide sur les parois.

Bibliography

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[2] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen. On the Robin problem in fractal domains. *Proc. Lond. Math. Soc. (3)*, 96(2):273–311, 2008.

[3] G. Bellettini, A. Coscia, and G. Dal Maso. Compactness and lower semicontinuity properties in SBD(Ω). *Math. Z.*, 228(2):337–351, 1998.

[4] Marie-Hélène Bossel. Longueurs extrémales et fonctionnelles de domaine. *Complex Variables Theory Appl.*, 6(2-4):203–234, 1986.

[5] Marie-Hélène Bossel. Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimétrique de Rayleigh-Faber-Krahn. *Z. Angew. Math. Phys.*, 39(5):733–742, 1988.

[6] Lorenzo Brasco, Guido De Philippis, and Bozhidar Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.*, 164(9):1777–1831, 2015.

[7] D. Bucur and A. Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[8] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. The quantitative Faber-Krahn inequality for the Robin Laplacian. *J. Differential Equations*, 264(7):4488–4503, 2018.

[9] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. Weinstock inequality in higher dimensions. *Journal of Differential Geometry (to appear)*, 2020.

[10] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[11] Dorin Bucur and Alessandro Giacomini. Minimization of the k-th eigenvalue of the Robin-Laplacian. *J. Funct. Anal.*, 277(3):643–687, 2019.

[12] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[13] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[14] Antonin Chambolle, Sergio Conti, and Gilles Francfort. Korn-Poincaré inequalities for functions with a small jump set. *Indiana Univ. Math. J.*, 65(4):1373–1399, 2016.

[15] Antonin Chambolle, Sergio Conti, and Flaviana Iurlano. Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. *J. Math. Pures Appl. (9)*, 128:119–139, 2019.

[16] Marco Cicalese and Gian Paolo Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.*, 206(2):617–643, 2012.

[17] Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Existence of strong minimizers for the Griffith static fracture model in dimension two. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(2):455–474, 2019.

[18] Guy David. *Singular sets of minimizers for the Mumford-Shah functional*, volume 233 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2005.

[19] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

[20] Georg Faber. Beweis, dass unter allen homogenen membranen von gleicher fläche und gleicher spannung die kreisförmige den tiefsten grundton gibt. 1923.

[21] Manuel Friedrich, Matteo Perugini, and Francesco Solombrino. Lower semicontinuity for functionals defined on piecewise rigid functions and on $GSBD$. *J. Funct. Anal.*, 280(7):Paper No. 108929, 45, 2021.

[22] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[23] Nunzia Gavitone, Domenico Angelo La Manna, Gloria Paoli, and Leonardo Trani. A quantitative Weinstock inequality. *Arxiv 1903.04964*, 2019.

[24] Alan Arnold Griffith. Vi. the phenomena of rupture and flow in solids. *Philosophical transactions of the royal society of london. Series A, containing papers of a mathematical or physical character*, 221(582-593):163–198, 1921.

[25] Serena Guarino Lo Bianco, Domenico Angelo La Manna, and Bozhidar Velichkov. A two-phase problem with Robin conditions on the free boundary. *J. Éc. polytech. Math.*, 8:1–25, 2021.

[26] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[27] Gerasim Kokarev. Variational aspects of Laplace eigenvalues on Riemannian surfaces. *Adv. Math.*, 258:191–239, 2014.

[28] E. Krahn. Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.*, 94(1):97–100, 1925.

[29] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[30] Édouard Oudet. Numerical minimization of eigenmodes of a membrane with respect to the domain. *ESAIM Control Optim. Calc. Var.*, 10(3):315–330, 2004.

[31] Romain Petrides. Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces. *Geom. Funct. Anal.*, 24(4):1336–1376, 2014.

[32] E. R. Reifenberg. An epiperimetric inequality related to the analyticity of minimal surfaces. *Ann. of Math. (2)*, 80:1–14, 1964.

[33] Luca Spolaor and Bozhidar Velichkov. On the logarithmic epiperimetric inequality for the obstacle problem. *Math. Eng.*, 3(1):Paper No. 4, 42, 2021.

[34] Robert Weinstock. Inequalities for a classical eigenvalue problem. *J. Rational Mech. Anal.*, 3:745–753, 1954.

Chapter 2

Introduction (english version)

Contents

2.1 Notion of stability of a geometric inequality 32
2.1.1 Stability of the Saint-Venant inequality 35
2.1.2 Instability of the Weinstock inequality 37
2.1.3 Case of the Hersch inequality 38
2.2 Optimization of the higher eigenvalues of the Laplacian with Robin boundary condition 38
2.3 Regularity of a free boundary transmission problem with obstacle 40
2.4 A shape optimization problem linked to thermal insulation 41
2.5 Positivity of superharmonic functions verifying a Robin boundary condition . 44
2.6 Optimization of an obstacle immerged in a Stokes fluid 45

The purpose of this PhD is the qualitative study of shape optimization and spectral geometry problems arising from various physical situations involving discontinuities. These problems are mainly related to solutions of elliptic partial differential equations with so-called "Robin" boundary conditions that appear - among other things - in the vibrations of elastically fixed membranes, in heat transfer phenomena by convection, but also in more general forms (called Navier conditions) in the presence of partial adherence between a fluid and a wall.

All these models have discontinuities; for instance, in the case of heat transfer by convection between a solid and a fluid there is a thin transition layer between two distinct temperature zones which will be modeled in our case by a discontinuity of the temperature along a surface. We can also mention the study of fractures in an elastic medium that will not appear directly in this work but which are at the origin of certain techniques that we use. In particular we will study the flow of a viscous fluid around an obstacle, which presents the same type of discontinuity.

These questions will be studied from the point of view of problems with free discontinuities. More precisely, we will study solutions of partial differential equations and various optimization problems with discontinuities along sets of codimension 1. An emblematic example of this type of problem is the Mumford Shah functional; given $g : \Omega \to [0, 1]$ a function defined on a domain $\Omega \Subset \mathbb{R}^n$, and three constants $a, b, c > 0$, the Mumford Shah problem consists in the minimization of the functional

$$(K, u) \mapsto a \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + b\mathscr{H}^{n-1}(K) + c \int_{\Omega \setminus K} (u - g)^2.$$

among all closed sets $K \subset \Omega$ of dimension 1, and all functions $u \in H^1(\Omega \setminus K)$.

This functional was originally created for the purpose of image segmentation; Ω is a rectangle of $\mathbb{R}^2$, g represents the grayscale of the image that we want to approximate by a smoother grayscale u in some areas, while allowing fast transitions (i.e., discontinuities) on a set K of codimension 1. a, b, c are constants chosen according to the importance given to each phenomenon, which are respectively

the regularity of u, the penalization of the length of the discontinuities and the proximity with the original image g.

There are many works and conjectures around the minimizers of this functional, on the theoretical side we can mention the books [35] and [52] which give a state of the art of a few years ago about the questions of regularity of minimizers. In particular, one of the central conjectures is the characterization of the set of discontinuities K in the two-dimensional case; is it indeed a finite union of smooth segments, which can only meet in triple junctions with angles of 120 degree ?

This functional later turned out to have other applications, notably in the analysis of deformations and fractures of brittle elastic materials with its vectorial generalization

$$(K,u) \mapsto a\int_{\Omega\setminus K} |e(u)|^2 \mathrm{d}\mathscr{L}^n + b\mathscr{H}^{n-1}(K) + c\int_{\Omega\setminus K}(u-g)^2\mathrm{d}\mathscr{L}^n,$$

where $u \in H^1(\Omega\setminus K,\mathbb{R}^n)$ and $e(u) = \frac{\nabla u+\nabla u^*}{2}$ is the symmetric part of the differential of u. This type of problem will appear in our case when studying the drag of an obstacle immersed in a viscous fluid, a context apparently distant from fracture mechanic but governed by similar equations.

A central idea in this type of problem is to change the data of (K,u) into a single function u in a more general functional space that includes both smooth functions and discontinuities; one may imagine these functions as "piecewise H^1". There are several reasons for this: without any particular topological or geometrical hypothesis, a sequence of closed sets (K_i) verifying some measure constraint has a priori no reason to converge to a minimizer in a meaningful sense. There is no good notion of convergence that has both compactness (like Hausdorff convergence) and control on the geometric data of the domain, including the functions u_i that are in functional spaces dependent on K_i.

By deleting the dependence on the data K and considering only the function u, we are in a unique function space (of the type $BV(\mathbb{R}^n)$) with good compactness properties. Then, using the partial differential equation or the optimization problem verified in a weak sense by the function u, we can generally show that it is regular enough to fall back to our starting problem. This is formalized more precisely in the theory of the SBV (Special Bounded Variation) function space, a particular subspace of functions with bounded variation whose weak differential has a certain structure. These are functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that Du (the differential of u in the sense of distributions) is a locally finite vector-valued Radon measure that decomposes into the form

$$Du = \nabla u\mathscr{L}^n + (u^+-u^-)\nu_u\mathscr{H}^{n-1}\lfloor J_u,$$

where $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (and will actually be in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for many of the problems we study) and J_u is a rectifiable set - meaning included in a countable union of $\mathcal{C}^1$ hypersurfaces up to a negligible set - called the jump set of u, defined as the set of $x \in \mathbb{R}^n$ such that $u(x+r\cdot)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $r \to 0$ to the piecewise constant function $u^+1_{\langle\cdot,\nu\rangle>0} + u^-1_{\langle\cdot,\nu\rangle<0}$, where $\nu \in \mathbb{S}^{n-1}$, $u^\pm \in \mathbb{R}$. It contains the subspace $H^1(\mathbb{R}^n)$ for which all functions have a $\mathscr{H}^{n-1}$-negligible jump set, as well as the indicator of sets of finite perimeter E for which $D1_E = -\mathscr{H}^{n-1}_{\lfloor\partial^*E}\nu_E$ (where ∂^*E is the reduced boundary of E and ν_E the outward normal vector). We have also the "mixed" case of a function $u \in H^1(\Omega)$ for some Lipschitz open set $\Omega \Subset \mathbb{R}^n$ that we extend into $\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$, and

$$D\tilde{u} = 1_\Omega\nabla u\mathscr{L}^n - u_{|\partial\Omega}\nu_\Omega\mathscr{H}^{n-1}\lfloor\partial\Omega,$$

where $u_{|\partial\Omega}$ is the trace of u on $\partial\Omega$ and ν_Ω is the outward normal vector of Ω. We also find in it functions that are not of this kind, characterized by the fact that the jump set is not necessarily closed (up to a negligible set). The gain in compactness obtained by relaxing a free discontinuity problem in an SBV space is therefore not gratuitous, and this is where the theory of regularity of this kind of relaxed solutions comes in, which allows us to go back to the original problem. The book [35] is a very complete reference on the subject, and the basics and results on SBV functions used in

the following works are explained in chapter 3.

In the following introduction, I present in chronological order the different problems I have worked on during my PhD. The starting point was the study of a so-called Poisson equation with a Robin boundary condition, which is presented as follows: given a bounded, smooth domain Ω in $\mathbb{R}^n$, a function $f \in L^2(\Omega)$, and $\beta > 0$, we look for a function $u \in H^1(\Omega)$ that satisfies

$$\begin{cases} -\Delta u = f & (\Omega) \\ -\partial_\nu u = \beta u & (\partial\Omega) \end{cases}$$

The existence of a unique solution u may be obtained through standard variational methods (Lax-Milgram theorem). Here are two possible interpretations of this quantity:

- We may see Ω as a solid of constant thermal conductivity σ, immerged in a medium of constant temperature T^a. It is assumed that there is a volumetric heat production F in Ω, and that the temperature exchanges between Ω and the exterior are governed by a convective transfer through a boundary layer, i.e. the temperature flux $\partial_\nu T_{|\partial\Omega}$ is proportional to $T^a - T_{|\partial\Omega}$. The proportionality constant (noted $\alpha > 0$) depends on the quantitative physical characteristic of the convection with the exterior (thickness of the boundary layer, conductivity of the outside...). In this case, the equilibrium temperature reached in the solid Ω is the function $T : \Omega \to \mathbb{R}$ verifying
$$\begin{cases} -\nabla \cdot (\sigma \nabla T) = F & (\Omega) \\ \partial_\nu T = \alpha(T^a - T) & (\partial\Omega) \end{cases}$$
When renormalizing the temperature into $u := \frac{T - T^a}{T^a}$, the heat production into $f := \frac{F}{\sigma T^a}$, and the convection constant $\beta := \frac{\alpha}{T^a}$, we find the previous equation. The solution u is particularly obtained as the minimizer of the energy
$$E_\Omega : v \in H^1(\Omega) \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \frac{\beta}{2}\int_{\partial\Omega} v^2 \mathrm{d}\mathscr{H}^{n-1} - \int_\Omega fv$$
that is strictly convex and coercive. The solution u verifies
$$E_\Omega(u) = -\frac{1}{2}\int_\Omega u \mathrm{d}\mathscr{L}^n$$
so the corresponding energy is, up to a constant, the mean temperature in Ω at the equilibrium.
- If we see Ω as the shape at rest of a membrane of constant area density, that is attached to the boundary $\partial\Omega$ (which is here seen as a rigid skeleton) elastically - for example by springs of constant stiffness proportional to β distributed uniformly along $\partial\Omega$ - and an orthogonal force f is applied to this membrane, then in the linear regime the solution u represents the vertical displacement of the membrane generated by the force f.

The first project was to study this equation when $f = 1$ with an additional obstacle condition $u \geq c$ that changes slightly the boundary conditions, in order to obtain quantitative estimates on the distance between the domain Ω and a ball of same measure, by the energy difference between these two domains. This project has been done in collaboration with D. Bucur and A. Giacomini and has resulted in a publication in *Archive for Rational Mechanics and Analysis*, which is summarized in the section 2.1.1. This project opened, in different directions, several of the other problems studied during my thesis:

- The problem of stability of the Weinstock inequality, involving harmonic functions verifying Robin conditions (also called Steklov conditions in this situation), which turned out to be a problem of a different nature with an intrinsic instability. This led to a publication in *Transactions of the AMS* in collaboration with D. Bucur, summarized in the section 2.1.2. The reformulation of this problem presents a stability in a weaker sense which later led to the study of the stability of spectral inequalities in compact surfaces, such as the Hersch inequality on

the sphere, but also on the projective plane, and some tori. This work is contained in a pre-publication written in collaboration with I. Polterovich, M. Karpukhin and D. Stern, described in section 2.1.3.

- A vectorial optimization problem with free discontinuity, with the study of domains minimizing a combination of the eigenvalues of the Laplacian with Robin boundary condition. There are two components to this work, the study of so-called "nondegenerate" functionals in which we show the existence of optimal domains in a strong sense, and so-called "degenerate" functionals in which the existence is only known in a relaxed sense (in the SBV space), but where we show the degeneracy of the eigenvalues at the optimum, an open problem in the case of Dirichlet conditions. This work is described in section 2.2.

- In the study of the Poisson problem with obstacle, we avoided a question of regularity about the contact set of the function u with the obstacle, a problem which is halfway between the Mumford-Shah functional and the Alt-Caffarelli problem. Although this question remains open, it has inspired the study of a more approachable functional with a similar obstacle phenomenon. This is a free boundary problem without discontinuity, which however uses similar methods. In this case we were able to partially answer the question of regularity in dimension two, which opens the door to the study of the mixed Mumford Shah/Alt-Caffarelli problem. This work has been done jointly with B. Velichkov, and is described in 2.3.

- As mentioned previously, a crucial step in the study of the Poisson problem with obstacle was to avoid the question of regularity of the minimizer. This was done by replacing the energy functional by an approximation for which we know how to prove the regularity. This method allowed us to solve a similar shape optimisation problem from thermal insulation, of the minimization of the heat flux coming out of an object at constant temperature (higher than the exterior) separated to the exterior by a layer of insulating material; it is established that in many cases the optimal configuration is to insulate a ball with a layer of constant thickness. This work can be found in preprint and is a collaboration with D. Bucur, C. Nitsch and C. Trombetti, summarized in the section 2.4

- A central condition for the regularity of the relaxed solutions of the Poisson equation with Robin boundary condition is the strict positivity of the temperature u at the boundary, which can also be seen as the positivity of the heat flux at each point of the boundary. We have extended some results of [36], based on fully probabilistic methods, to more general elliptic operators by making an explicit link between geometry of the domain and the positivity of the solutions. In particular, we establish a link between the profile of the relative isoperimetric inequality in the domain and the positivity of solutions of general elliptic equations with Robin conditions. This is a collaboration with D. Bucur and A. Giacomini, explained in section 2.5.

Finally we study, with D.Bucur, A.Chambolle and A.Giacomini, a vectorial free discontinuity problem arising from a shape optimization question in fluid mechanics: how to minimize the drag of an obstacle immerged in a Stokes fluid, with partial adherence conditions along the wall of the obstacle ? Although there is some literature on the numerical side, this type of question has received few attention from the theoretical side; based on a formalism developed for fracture growth in brittle elastic materials, we establish the existence and regularity of minimizers under certain assumptions. A general summary is given in section 2.6.

2.1 Notion of stability of a geometric inequality

A shape optimization problem is given by two things. On one one hand, a set

$$\mathscr{A} \subset \{ \text{ subset of } \mathbb{R}^n \}$$

usually called the set of admissible shape, given for example by conditions of regularity (open, Lipschitz, $\mathcal{C}^2$), of geometry (containing or contained in a certain shape, star-like, convex, quasi-convex),

of topology (simply connected, or having a monogenous fundamental group), or of constraint on different geometric quantities (measure, perimeter).

On the other hand, we have a functional $J : \mathscr{A} \to \mathbb{R}$ to minimizer (or maximize, depending on the situation), that in our case will often include the solution of an elliptic PDE.

Let us first mention a few classical exemples that will appear again later:

- **Isoperimetric inequality**: $\mathrm{Per}(\Omega) \geq \mathrm{Per}(\Omega^*)$ in $\{\Omega \subset \mathbb{R}^n \text{ measurable s.t. } |\Omega| = m\}$, where $m > 0$, Ω^* is the ball of measure m, and Per designates the De Giorgi perimeter defined by

$$\mathrm{Per}(\Omega) = \sup\left\{\int_\Omega \nabla \cdot \varphi,\ \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n, B_1)\right\}$$

  that coincides with $\mathscr{H}^{n-1}(\partial\Omega)$ when Ω is a sufficiently smooth set (for instance with Lipschitz boundaries). More details on the definition of perimeter will be given in the next chapter; the isoperimetric inequality is a long-known result whose first rigorous proof is attributed to Weierstrass.

- **Faber-Krahn inequality** (with Robin condition): $\lambda_1(\Omega;\beta) \geq \lambda_1(\Omega^*;\beta)$ in

$$\{\Omega \subset \mathbb{R}^n \text{ open s.t. } |\Omega| = m\},$$

  where $m > 0$, $\beta \in (0, +\infty]$, Ω^* is the ball of measure m and $\lambda_1(\Omega;\beta)$ is the first eigenvalue of the Laplacian with Robin boundary condition, defined by

$$\lambda_1(\Omega;\beta) = \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^{n-1}}{\int_\Omega u^2 \mathrm{d}\mathscr{L}^n}$$

  $\beta = +\infty$ corresponds naturally to Dirichlet boundary conditions, for which the proof goes back to the works of Faber [54] and Krahn [62]; this proof may be seen as a consequence of the isoperimetric inequality applied to the superlevel set of test functions through the Polya-Szego inequality. For the case of Robin boundary conditions, this was proved by Bossel in [38] and [39] through a completely different rearrangement method on which we will come back.

- **Saint-Venant inequality**: $T(\Omega) \leq T(\Omega^*)$ in $\{\Omega \subset \mathbb{R}^n \text{ open s.t. } |\Omega| = m\}$, where $m > 0$, Ω^* is the ball of measure m, and $T(\Omega)$ is the average of the function u_Ω solution of $-\Delta u_\Omega = 1$ in Ω with Robin boundary condition $-\partial_\nu u_\Omega = \beta u_\omega$. The case of Dirichlet boundary conditions may be shown with the same methods as the Faber-Krahn inequality, but with Robin boundary conditions, the Bossel rearrangement methods mentioned above do not seem to fit this case. The Saint-Venant inequality has therefore been shown by a variational method in [44], by showing that an optimal domain exists, and that any optimal domain is necessarily the expected one, i.e. a ball.

- **Weinstock inequality**: $\sigma_1(\Omega) \leq \sigma_1(\mathbb{D})$ in $\{\Omega \subset \mathbb{R}^2 \text{ simplement connexe s.t. } \mathrm{Per}(\Omega) = 2\pi\}$, where $\mathbb{D}$ is the unit disk and σ_1 is the first non-zero eigenvalue of the Steklov operator defined by

$$\sigma_1(\Omega;\beta) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^2}{\int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^1},\ u \in H^1(\Omega),\ \int_{\partial\Omega} u \mathrm{d}\mathscr{H}^1 = 0\right\}.$$

  The proof of this inequality goes back to Weinstock [68]. The idea is to use the conformal representation theorem to build good test functions. There is also a version for convex sets in any dimension in [43], that uses a comparison with a geometric functional (called the "moment").

- **Hersch inequality**: $\lambda_1(\mathbb{S}^2, g) \leq \lambda_1(\mathbb{S}^2, g_{\mathrm{round}})$, where $(\mathbb{S}^2, g)$ and $(\mathbb{S}^2, g_{\mathrm{round}})$ are Riemannian surfaces homeomorphic to the Euclidian sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, with same surface area, such that

g_{rond} has constant curvature and $\lambda_1(\mathbb{S}^2, g)$ is the first non-trivial eigenvalue of the Laplacian $-\Delta_g$ defined by

$$\lambda_1(\mathbb{S}^2, g) = \inf\left\{\frac{\int_{\mathbb{S}^2} |\nabla_g u|^2 \mathrm{d}v_g}{\int_S u^2 \mathrm{d}v_g},\ u \in H^1(\mathbb{S}^2, g) : \int_{\mathbb{S}^2} u \mathrm{d}v_g = 0\right\}$$

where v_g is the area measure induced by the metric g. Similar inequalities are known for the projective plane, the torus, the Klein bottle, however for a general compact surface it is only known that there exists an optimal metric with a finite number of conical singularities, as shown for instance in [65].

When we solve a shape optimization problem of the form $\inf_{\Omega \in A} J(\Omega)$ that admits a solution Ω^{opt}, the stability of this problem may be formulated in the following way:

$$\textit{If } J(\Omega) \approx J(\Omega^{opt}), \textit{ is it true that } \Omega \approx \Omega^{opt} \textit{ ?}$$

Here $\Omega \approx \Omega^{\text{opt}}$ is to be taken in the sense of a certain topology on the set of admissible shapes; a shape optimization problem may very well be stable for a topology and unstable for another as we will see with the Hersch inequality, and the "right" topology may very well be one for which J is not continuous but that is satisfying from a geometric view point, such as the Fraenkel asymmetry defined below.

For optimization problems where the optimum is reached for the ball, we will use in particular the notion of Fraenkel asymmetry

$$\mathscr{A}_F(\Omega) = \inf\left\{|\Omega \Delta (x + \Omega^*)|,\ x \in \mathbb{R}^n\right\},$$

where Δ is the symmetric difference. Another example is the Hausdorff asymmetry

$$\mathscr{A}_H(\Omega) = \inf\left\{\mathrm{d}_H\left(\Omega, x + \Omega^*\right),\ x \in \mathbb{R}^n\right\},$$

where $\mathrm{d}_H(A, B) = \|\mathrm{d}(\cdot, A) - \mathrm{d}(\cdot, B)\|_{L^\infty(\mathbb{R}^n)}$ is the Hausdorff distance.

A stability result may be only qualitative, or it may be quantitative; for a certain notion of distance d between admissible shapes and for a certain increasing function $F : \mathbb{R}_+^* \to \mathbb{R}_+^*$,

$$J(\Omega) - J(\Omega^{\text{opt}}) \geq F(\mathrm{d}(\Omega, \Omega^{\text{opt}})).$$

We will say such a result is sharp if we can build a sequence $\Omega_k \to \Omega^{\text{opt}}$ such that $J(\Omega_k) - J(\Omega^{\text{opt}})$ has the same order of magnitude as $F(\mathrm{d}(\Omega_k, \Omega^{\text{opt}}))$.

Let us begin by stating a few known stability results; the sharp quantitative isoperimetric inequality has been obtained in [56].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure and Ω^* be the ball of same measure, then there is a constant $c > 0$ that only depends on n and $|\Omega|$ such that*

$$Per(\Omega) - Per(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

This inequality is sharp in the sense that for a sequence of ellipsoids that converges to a ball, each side of the inequality has the same order of magnitude.

For the Faber-Krahn (and Saint-Venant) inequality with Dirichlet boundary conditions, a similar result has been shown in [40].

Theorem 2.2 ([40])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure and Ω^* be the ball of same measure, then there is a constant $c > 0$ that depends only on n and $|\Omega|$ such that*

$$\lambda_1(\Omega) - \lambda_1(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

This is also optimal due to the ellispoids. For the Faber-Krahn inequality with Robin conditions, the result is similar but has been established by very different methods in [39].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open subset of finite measure, $\beta > 0$ and Ω^* the ball of same measure, then there is a constant $c > 0$ that depends only on n, β and $|\Omega|$ such that*

$$\lambda_1(\Omega;\beta) - \lambda_1(\Omega^*;\beta) \geq c\mathscr{A}_F(\Omega)^2$$

This inequality relies on a refinement of the original proof of the Faber-Krahn inequality with Robin conditions $\lambda_1(\Omega;\beta) \geq \lambda_1(\Omega^*;\beta)$. More precisely, there is a characterization of $\lambda_1(\Omega;\beta)$ (that is also valid in the limit $\beta \to \infty$ - for Dirichlet conditions) inspired by extremal length methods of conformal geometry, established by Bossel in [38].

For every function $\rho \in L^\infty(\Omega, \mathbb{R}_+)$ that we see as a metric that locally multiplies the length by ρ, the area of a subset $\omega \subset \Omega$ is $A_\rho(\omega) := \int_\omega \rho^2$, and its perimeter is $P_\rho(\Omega) = \int_{\partial\omega\cap\Omega} \rho + \int_{\partial\omega\cap\partial\Omega} \beta$ (note in particular that for $\beta = +\infty$ we may restrict ourselves to sets that are bounded away from $\partial\Omega$). The first eigenvalue is then equal to

$$\lambda_1(\Omega;\beta) = \sup_{\rho\in L^\infty(\Omega,\mathbb{R}_+)} \inf_{\omega\subset\Omega} \frac{P_\rho(\omega) - A_\rho(\omega)}{|\omega|}$$

The infimum can in fact be restricted to sets of the form $\{u > t\}$, and it is shown in [42] that this is reached for a $\{u > t\}$, and this allows us to establish an estimate of the form

$$\lambda_1(\Omega;\beta) \geq \frac{\beta}{2}\left(\inf_\Omega u\right)^2 \Big[\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)\Big]$$

where u is the first normalized eigenfunction. The perimeter difference is bounded from below by $c\mathscr{A}_F(\Omega)^2$, however the minimum of u is not necessarily controlled from below. This is where a selection principle comes into play, used in particular in [50] for an alternate proof of the quantitative isoperimetric inequality and by [40] for the quantitative Faber Krahn inequality (with Dirichlet conditions); for each domain Ω, we can construct a domain ω close to Ω in the sense of distance L^1, of similar first eigenvalue, but such that $\inf_{x\in\omega} u_\omega(x) > 0$.

2.1.1 Stability of the Saint-Venant inequality

In this first article, we are interested in the solutions of the following equation:

$$\begin{cases} -\Delta u_\Omega = 1 & (\Omega) \\ -\partial_\nu u_\Omega = \beta u_\Omega & (\partial\Omega) \end{cases}$$

for an open domain Ω of measure $m > 0$. The solution u_Ω is obtained as the unique minimizer of the functional

$$E_\Omega(u) := \frac{1}{2}\left(\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \beta u^2 \mathrm{d}\mathscr{H}^{n-1}\right) - \int_\Omega u \mathrm{d}\mathscr{L}^n$$

and we will denote $E(\Omega) := E(u_\Omega;\Omega)$ the energy of the domain Ω.
It has been shown in [41] that among the open sets of fixed measure, $E(\Omega)$ is minimal on the ball of same measure Ω^*. In chapter 4 we prove a quantitative version of this result in the following form.

Theorem 2.4. *Let $\beta > 0$, $m > 0$, there exists a constant $C_{n,m,\beta} > 0$ such that for any open set Ω of measure m in $\mathbb{R}^n$,*

$$E(\Omega) - E(\Omega^*) \geq c\mathscr{A}_F(\Omega)^2$$

This result is obtained by combining a selection principle and the following inequality, that we get by studying a problem mixing free discontinuity and free boundary (hence the name; **degenerate** free discontinuity problem).

Theorem 2.5. *Let Ω be an open set of finite measure, then*

$$E(\Omega) - E(\Omega^*) \geq \frac{\beta}{2}\left(\inf_{x\in\Omega} u_\Omega(x)\right)^2 \Big[Per(\Omega) - Per(\Omega^*)\Big]$$

This inequality is similar to the one obtained in the proof of stability of $\lambda_1(\Omega;\beta)$, note however that there is no known characterization by extremal length for the energy E, that would allow us to get it directly. The method to find it will thus be different.

This inequality may be interesting in itself; indeed, the only constant that is not completely explicit in it is $\inf_{x\in\Omega} u_\Omega(x)$, which depends on the geometry of the domain in a way that is not yet fully understood and will be discussed in a later chapter. If we restrict ourselves for example to convex sets which are not too far from the ball, this gives a quantitative inequality of the form

$$E(\Omega) - E(\Omega^*) \geq C_{n,|\Omega|,\beta}\Big[\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)\Big]$$

which is stronger than the one announced in 2.4 since $\mathrm{Per}(\Omega) - \mathrm{Per}(\Omega^*)$ is controlled from below by $c_{n,|\Omega|}\mathscr{A}_F(\Omega)^2$.

However for a general domain Ω one cannot derive a quantitative inequality directly from 2.4; indeed the quantity $\inf_\Omega u_\Omega$ can be zero in several situations. For example, among domains of bounded energy the quantity $\int_{\partial\Omega} u_\Omega^2$ will be bounded, so $\inf_{x\in\Omega} u_\Omega(x)$ is small in sets of large perimeter. A more problematic example is the following; consider $\Omega \Subset \mathbb{R}^n$ a domain smooth outside the origin such that

$$\Omega \cap B_1 = \Big\{(x_1, x') \in B_1 : x_1 > 0,\ |x'| < x_1^2\Big\}$$

in this case the perimeter of Ω remains bounded, and yet $u_\Omega(0) = 0$. This kind of phenomenon and positivity criteria for the solution u_Ω are studied in more details in chapter 9.

In our case, we use a selection principle which takes the following form: for any domain Ω, that is possibly not very smooth, we solve an intermediate shape optimization problem which gives us a domain $\omega \subset \Omega$ such that $E(\Omega) \gtrsim E(\omega)$ and $\mathscr{A}_F(\Omega) \lesssim \mathscr{A}_F(\omega)$. This domain is obtained by the minimization

$$\inf\{E(\omega) + k|\omega|,\ \omega \subset \Omega\}$$

for a sufficiently small $k > 0$. We are able to prove the following uniform estimate on the solution:

$$\inf_{x\in\omega} u_\omega(x) \geq \delta_{n,\beta,|\Omega|}(> 0)$$

which, from theorem 2.5, is enough to conclude. To obtain 2.5, the idea is to see this as a minimization problem itself: we define a modified energy

$$E^c(\Omega) = \inf_{u\in H^1(\Omega), u\geq c}\left(\frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \frac{\beta}{2}\int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^{n-1} - \int_\Omega u \mathrm{d}\mathscr{L}^n\right) - \frac{\beta}{2}c^2\mathrm{Per}(\Omega) + c|\Omega|,$$

where $c > 0$ will be fixed later. We show that among every domains of fixed measure, the quantity $E^c(\Omega)$ is minimal for the ball, and once this is proved it is enough to take $c = \inf_{x\in\Omega} u_\Omega(x)$ to obtain the theorem 2.5.

The fact that E^c is minimal on the ball is obtained by showing that there exists a relaxed minimizer (in the SBV sense), that this relaxed minimizer is a classical minimizer by a regularity result, and that a classical minimizer is necessarily a ball.

The regularity step consists in saying that a relaxed minimizer (in the SBV sense) u verifies $\mathscr{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$, which is enough to define an open set Ω on which u is a classical solution and $J_u \subset \partial\Omega$; the central estimate to obtain this is that u verifies

$$\inf_{\{u>c\}} u > c,$$

meaning u never touches the obstacle: this is false in general, and the regularity is in fact only obtained for an approximate functional $E^{c,\epsilon}$ which also admits the ball as a minimizer (because it has the same symmetry properties). The regularity of the minimizers of E^c remains in fact an open problem which reduces - by linearizing the functional around $u \approx c$ - to the study of

$$u \in SBV_{\text{loc}}(\Omega, \mathbb{R}_+) \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \alpha \int_{J_u \cap \Omega} (u^+ + u^-) \mathrm{d}\mathscr{H}^{n-1} + k|\{u > 0\}|$$

on which few things are known.

2.1.2 Instability of the Weinstock inequality

For every open, smooth, connected bounded set $\Omega \Subset \mathbb{R}^n$, we define the Steklov operator as the following pseudodifferential operator:

$$S_\Omega : \begin{cases} \mathcal{C}^\infty(\partial\Omega) \to \mathcal{C}^\infty(\partial\Omega) \\ u \mapsto \partial_\nu(\mathcal{H}u) \end{cases},$$

where $\mathcal{H}u$ is the harmonic extension of u in Ω and $\partial_\nu f = \nu \cdot \nabla f$, where ν is the outward normal vector of Ω. Although it is not a classical differential operator (it is easy to see that it is non-local), it has similar properties as $\sqrt{-\Delta_{\partial\Omega}}$ and is in fact equal to it when Ω is a half-space.

It is a self-adjoint positive operator and its inverse - which is well-defined on the space of function with zero boundary average - is compact, so there is a sequence of eigenvalues

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \ldots \to \infty$$

associated to eigenfunctions $u_k \in \mathcal{C}^\infty(\partial\Omega)$ (that we assimilate - by abuse of notation - to their harmonic extensions) that form a basis of $L^2(\partial\Omega)$ such that

$$\begin{cases} \Delta u_k = 0 & (\Omega) \\ \partial_\nu u_k = \sigma_k(\Omega) u_k & (\partial\Omega) \end{cases}$$

The Weinstock inequality tells us that among every simply connected sets Ω of area 2π in $\mathbb{R}^2$, we have

$$\sigma_1(\Omega) \leq \sigma_1(\mathbb{D})(= 1),$$

where $\mathbb{D}$ is the unit disk. Recent researches around this inequality have shown a form of stability in some cases, let us cite for example the result of [57] where it is shown that among convex sets in any dimension, where the ball is always maximal, there is a quantitative inequality of the form $\sigma(B) - \sigma(\Omega) \geq F(\mathscr{A}_H(\Omega))$ for an explicit strictly increasing function F depending on the dimension.

In this second paper, we establish that without any other hypothesis that simple connexity, the Weinstock inequality is unstable in general, meaning that we may build a sequence of simply connected open sets $(\Omega_\epsilon)_{\epsilon>0}$ of perimeter 2π such that $\Omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Omega$ in a strong sens (Ω_ϵ is obtained as a Lipschitz perturbation of the boundary of Ω), $\sigma_1(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 1$, but Ω is not the disk. We show more generally a result of instability of the whole spectrum.

Theorem 2.6. *Let Ω, ω two open, bounded, Lipschitz sets of $\mathbb{R}^2$ such that there is a conformal application between them. Then there exists a sequence of domains $\Omega_\epsilon \to \Omega$, all homeomorphic to Ω with uniformly bounded perimeter such that for every $k \geq 0$,*

$$Per(\Omega_\epsilon)\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} Per(\omega)\sigma_k(\omega)$$

The instability of the Weinstock inequality is obtained by taking Ω any simply connected smooth set and $\omega = \mathbb{D}$. We establish a partial characterization of sequences of domains $(\Omega_\epsilon)_\epsilon$ of perimeter 2π that verify $\sigma_1(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 1$; we know the following dichotomie is always verified:

- Either $\mathscr{A}_H(\Omega_\epsilon) \to 0$.
- Or $\| \log(|g_\epsilon'|)\|_{\mathcal{C}^\alpha(\mathbb{D})} \underset{\epsilon\to 0}{\longrightarrow} +\infty$ in a quantified sense, where g_ϵ is any sequence of conformal maps $g_\epsilon : \mathbb{D} \to \Omega_\epsilon$.

Although this second condition is not geometrically explicit, the idea is that in a sequence $(\Omega_\epsilon)_\epsilon$ of domains which are a counterexample to the stability of the Weinstock inequality, the boundaries of the Ω_ϵ necessarily has oscillations or singularities.

2.1.3 Case of the Hersch inequality

Hersch's spectral inequality can be summarized as such: let $(\mathbb{S}^2, g)$ be a Riemannian surface that is homeomorphic to the sphere of dimension 2, and denote

$$(0 =)\lambda_0(\mathbb{S}^2, g) < \lambda_1(\mathbb{S}^2, g) \leq \ldots \to \infty$$

the eigenvalues of the Laplacian on $(\mathbb{S}^2, g)$, then necessarily

$$v_g(\mathbb{S}^2)\lambda_1(\mathbb{S}^2, g) \leq 8\pi,$$

where v_g is the measure induced by g, and the maximum 8π is reached exactly for spheres of constant curvature. We study the stability of this inequality in the setting of generalized eigenvalues associated to a measure. More precisely, by uniformization theorem we know there exists (up to diffeomorphism) only one conformal class of metrics on the sphere, that is generated by a metric with constant curvature denoted g, and Hersch's inequality may be rewritten as: for any function $u \in \mathcal{C}^\infty(\mathbb{S}^2)$,

$$\left(\int_{\mathbb{S}^2} e^{2u} \mathrm{d}v_g\right) \lambda_1\left(\mathbb{S}^2, e^{2u} g\right) \leq 8\pi.$$

This may be further generalized to any measure with the formalism developped by Kokarev in [61]: for any positive Borel measure μ défined on the sphere $(\mathbb{S}^2, g)$, we let

$$\lambda_k(\mu) := \inf_{V:\dim(V)=k+1} \sup_{v\in V} \frac{\int_{\mathbb{S}^2} |\nabla v|^2 \mathrm{d}v_g}{\int_{\mathbb{S}^2} v^2 \mathrm{d}\mu}.$$

where V is taken among the subspaces of $H^1(\mathbb{S}^2, g)$. In this setting, Hersch's inequality becomes $\mu(\mathbb{S}^2)\lambda_1(\mu) \leq 8\pi$, and we prove the following stability result.

Theorem 2.7. *Let μ be a probability measure on the sphere $\mathbb{S}^2$, then there are two universal constants $c_1, c_2 > 0$ that do not depend on μ, and a metric h conformal to g and with constant curvature such that*

$$\lambda_1(\mu) \leq 8\pi - c_1 \min\left(c_2, \|\mu - v_h\|^2_{H^{-1}(\mathbb{S}^2,h)}\right).$$

We also establish the existence of probability measures $(\mu_\epsilon)_\epsilon$ such that $c\|\mu_\epsilon - v_g\|^2_{H^{-1}(\mathbb{S}^2,g)} \sim 8\pi - \lambda_1(\mu_\epsilon) \to 0$, which proves the optimality of the exponent 2, as well as the existence of probability measure $(\mu_\epsilon)_\epsilon$ such that $\lambda_1(\mu_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 8\pi$ but μ_ϵ do not converge to a round metric in the space $\left(W^{1,2,-\frac{1}{2}}\right)^*$, which shows a form of optimality of the space H^{-1} (the space $\left(W^{1,2,-\frac{1}{2}}\right)^*$ being inserted between H^{-1} and the H^{-s} for $s < 1$).

2.2 Optimization of the higher eigenvalues of the Laplacian with Robin boundary condition

In this chapter, we are interested in the following type of optimization problem

$$\inf\left\{F(\lambda_1(\Omega;\beta), \ldots, \lambda_k(\Omega;\beta)),\ \Omega \subset \mathbb{R}^n \text{ ouvert s.t. } |\Omega| = m\right\} \tag{2.1}$$

where $m, \beta > 0$ are fixed and the $(\lambda_i(\Omega;\beta))_{i\geq 1}$ are the eigenvalues of the Laplacian with Robin boundary condition defined by the equation

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega;\beta) u_k & (\Omega) \\ \partial_\nu u_k + \beta u_k = 0 & (\partial\Omega) \end{cases}$$

We suppose $F : (0,+\infty)^k \to \mathbb{R}$ is a smooth function that has nonnegative partial derivative in each coordinate, and that tends to infinity when its last coordinate goes to infinity (the regularity hypothesis of F may be lightened; see chapter 6 for more details). The two functionals that we have in mind are

$$F(\lambda_1, \ldots, \lambda_k) = \lambda_1 + \ldots + \lambda_k$$

and

$$F(\lambda_1, \ldots, \lambda_k) = \lambda_k.$$

We begin by establishing an existence and partial regularity result for the first case.

Theorem 2.8. *Let F be such a function, with strictly positive partial derivative and $m, \beta > 0$, then the lower bound in (2.1) is attained. Moreover, any open set Ω that reaches that bound verifies $\mathscr{H}^{n-1}(\partial\Omega) \leq C(n, m, \beta)$, and $\partial\Omega$ is Ahlfors-regular.*

The main steps are the following:

- Consider a minimizing sequence Ω^i, and $u^i = (u_1^i, \ldots, u_k^i)$ a choice of the first k eigenfunctions of the Laplacian with Robin boundary condition. By extending these functions by 0 at the exterior of Ω^i, we obtain a sequence of function that are bounded in SBV and we prove an accumulation of this sequence of functions around at most k diverging sequences of points, which implies the existence of a minimizer in the SBV sense.
- We prove the regularity of SBV minimizers by a method similar to the one of De Giorgi, Carriero and Leaci ([53]) for the Mumford-Shah functional, but in a vectorial version; the main idea is that at every point of the support at least one of the eigenfunctions is larger (in absolute value) than a positive threshold, which allows us to deduce bounds on the length of the shared discontinuities.

We are then interested in the second case, the minimization of the k-th eigenvalue alone. In this case we do not prove the general regularity of the domain Ω, and the existence is only known in a relaxed setting and has been treated in [45]. In this relaxed setting, we do not consider domains but instead bundles of linearly independant functions $\mathbf{u} = (u_1, \ldots, u_k)$ on which we define

$$\lambda_i(\mathbf{u};\beta) = \inf_{V \subset \mathrm{span}(\mathbf{u}), \dim(V)=i} \sup_{v \in V} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{J_\mathbf{u}} (\overline{v}^2 + \underline{v}^2) \mathrm{d}\mathscr{H}^{n-1}}{\int_{\mathbb{R}^n} v^2 \mathrm{d}\mathscr{L}^n}.$$

And a bundle of functions $\mathbf{u}$ is said to be admissible when it verifies some regularity conditions (that will be made more precise later) as well as a measure constraint $|\{\mathbf{u} \neq 0\}| = m$. Despite the absence of regularity, we are able to prove the degeneracy of the eigenvalue at the minimum in dimension three or more:

Theorem 2.9. *Let $n \geq 3$, $k \geq 2$, $\beta \in]0, +\infty[$, $m > 0$, and let $\boldsymbol{u}$ be a relaxed (SBV) minimizer of the functional*

$$\boldsymbol{v} \mapsto \lambda_k(\boldsymbol{v};\beta)$$

among the admissible functions of $\mathbb{R}^n$ for which the measure of the support is m. Then

$$\lambda_{k-1}(\boldsymbol{u};\beta) = \lambda_k(\boldsymbol{u};\beta).$$

In the case $\beta = \infty$, that identifies to the eigenvalues of the Laplacian with Dirichlet boundary conditions, this is a conjecture that was cited in [64], [60] with numerical evidences. Here this result is a consequence of the following fact: we prove that when the k-th eigenvalue is simples then the

k-th eigenfunction do not take values in $(-\epsilon, +\epsilon) \setminus \{0\}$ for some small enough ϵ, meaning that the solution is necessarily associated to functions with a disconnected support.

The regularity of such a minimizer is still open due to the degeneracy of the eigenvalue ; it would be interesting to start by understanding the minimization of $\lambda_3(\cdot;\beta)$, where we expect that any connected minimizer (meaning any minimizer that is not a disjoint union of three balls, which is a possibility when β is small), this eigenvalue has a multiplicity of exactly two.

2.3 Regularity of a free boundary transmission problem with obstacle

This work is motivated, between other things, by the understanding of free boundary and free discontinuity problems that are "degenerate" as mentioned ealier, for (quasi-)minimizers of functionals of the form

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{J_u} (|\overline{u}| + |\underline{u}|)\, \mathrm{d}\mathscr{H}^{n-1}.$$

We begin by studying a possibly simpler case, of a problem with no free discontinuity; given a smooth bounded open set $D \subset \mathbb{R}^n$, a measurable set E_0, a function $\varphi \in H^{1/2}(\partial D, \mathbb{R}_+)$, we look for the set E and the function $u \in H^1(D, \mathbb{R}_+)$ that minimize the quantity

$$J_D(u, E) \mapsto \int_D |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{D \cap \partial^* E \cap \{u>0\}} u \mathrm{d}\mathscr{H}^{n-1}$$

among couples (u, E) that coincide with (φ, E_0) at the exterior of D.

We know there are optimal couples (u, E), and u is automatically harmonic on the interior of E and E^c. We may moreover prove that $u \in \mathcal{C}^{0,\frac{2}{n+2}}_{\mathrm{loc}}(D)$, and the regularity of (u, E) is unclear near ∂E only:

- If $x \in \partial^* E \cap \{u > 0\}$, then E is a quasiminimizer of the perimeter in a neighbourhood of x and so it may be decomposed into $R \sqcup S$ where R is $\mathcal{C}^{1,\gamma}$ and $\dim(S) \leq n - 8$. More precisely, it was proven in [59] that R is smooth and $u1_E$, $u1_{E^c}$ are smooth to the boundary.
- If $u_{|B_{x,r} \cap \partial E} \equiv 0$, then $(1_{E^c} - 1_E)u$ is harmonic in $B_{x,r}$ and ∂E is thus the level set of a harmonic function ; in two dimensions it is a smooth curve outside a locally finite set of equiangular junction points.

There are thus two distinct regularity theory for each part of the boundary $\partial^* E \cap \{u > 0\}$ and $\{u = 0\}$, and our goal is to understand how these two boundaries join.

Theorem 2.10. *Let (u, E) be a local minimizer of J_D in an open set $D \subset \mathbb{R}^2$, and let $B \Subset D$, then $Per(E|B) < \infty$, $\partial^* E \cap \{u > 0\}$ is locally $\mathcal{C}^\infty$ and $\{u = 0\}$ is included in a finite number of $\mathcal{C}^{1,\log}$ segments.*

Here is the proof plan, where the first two points are valid in dimension $n \geq 2$.

- We establish regularity estimates; let u be a minimizer of J in B_4, we have an interior estimate

$$\|u\|_{\mathcal{C}^{\frac{2}{n+2}}(B_1)} \leq C_n J_{B_2}(u, E)$$
$$J_{B_1}(u, E) \leq C_n \int_{\partial B_2} u \mathrm{d}\mathscr{H}^{n-1}$$

  and an exterior estimate

$$\int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} \leq C_n(1 + u(0))$$

- We deduce the compactness of sequences of punctually bounded minimizers; this allows us to prove the existence of minimizers for fixed boundary data.

- We establish the monotonicity of the functional

$$r \to W\left(\frac{u(r\cdot)}{r}, \frac{E}{r}\right),$$

where $W(u,E) = J_{B_1}(u,E) - \int_{\partial B_1} u^2$. In two dimension this is sufficient to identify the homogeneous minimizers obtained as blow-up around the points of $\{u=0\}$. These minimizers are of the form

$$u(x) = |x \cdot e|, \ E = \{x : x \cdot e > 0\},$$

where $e \in \mathbb{R}^2 \setminus \{0\}$ is such that $|e| \leq \frac{1}{4}$, or of the form $u \equiv 0$.

- We prove an improved version of the monotonicity formula; for any minimizer (u,E) of J in B_2 such that $u(0)=0$, there exists universal constants $\gamma, \vartheta \in]0,1[$ such that

$$\left.\frac{d}{dr}\right|_{r=1} W(u_r, E_r) \geq 2\left(\gamma \wedge W(u,E)^{\vartheta}\right) W(u,E).$$

This estimate is obtained by building an explicit competitor (v,F) in B_1 such that $(v,F)_{|\partial B_1} = (u,E)_{|\partial B_1}$ and

$$W(v,F) \leq \left(1 - \gamma \wedge W(u^h, E^h)^{\vartheta}\right) W(u^h, E^h),$$

where (u^h, E^h) is the 1-homogeneous extension of $(u,E)_{|\partial B_1}$ on B_1.

- We deduce quantitative estimates of the speed of convergence to the blow-ups, thereby obtaining the uniqueness of the blow-up at each point of $\{u=0\}$ with a $\mathcal{C}^{1,\log}$ dependency on the point, as well as a "non-degeneracy" property that characterizes entirely the minimizers near small blow-ups.

The crucial point is the second-to-last point, known in similar situations as a **epiperimetric inequality** ; similar methods have been used with success for minimal surfaces in [66], for the Alt-Caffarelli problem and the thin obstacle problem in [67].
The regularity question is far from being fully answered, we leave open the following problems:

- Can we say that all of ∂E is smooth ? The result we obtained is very close to this, especially with the uniform convergence of the blow-ups on each point of $\{u=0\}$, but the continuity of the tangent space on $\partial^* E \cap \{u>0\}$ is unclear.

- What can we say in higher dimension ? The monotonicity formula is still true but the classification of homogeneous solutions is not known.

2.4 A shape optimization problem linked to thermal insulation

We are now interested in a shape optimization problem where we attempt to insulate a solid at constant temperature (higher than the outside temperature, supposed constant) by a layer of insulator of fixed total measure, with a general heat transfer condition between the insulator and the outside (by convection, radiation...). More precisely, given a smooth compact set $K \subset \mathbb{R}^n$, a smooth open set $\Omega \subset \mathbb{R}^n$ that contains K, and a function $\Theta : [0,1] \to \mathbb{R}_+$ that we suppose to be increasing, lower semi-continuous and such that $\Theta(0) = 0$, we let

$$E_\Theta(K,\Omega) = \inf\left\{\int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \Theta(v)\mathrm{d}\mathscr{H}^{n-1}, \ v \in H^1(\Omega) : 1_K \leq v \leq 1\right\}.$$

When Θ is convex, the minimum is reached for a unique function u that represent the temperature at rest, equal to 1 in K, 0 in the exterior, that is harmonic in the insulation $\Omega \setminus K$, and when Θ is smooth enough we have a boundary condition

$$\begin{cases} -\partial_\nu u = \frac{1}{2}\Theta'(u) & (\partial\Omega \cap \{u > 0\}) \\ -\partial_\nu u \leq \frac{1}{2}\Theta'(u) & (\partial\Omega \cap \{u = 0\}) \end{cases}$$

Here are a few relevant choices of Θ:

- $\Theta(u) = \beta u^2$; in this case we get the Robin boundary condition

  $$-\partial_\nu u = \beta u$$

  at the boundary of Ω, in particular when Ω is smooth enough we have $\inf_\Omega(u) > 0$ and the heat flow is proportional to the temperature difference. This corresponds for instance to the heat transfer obtained when the outside temperature is uniform by convection (in a fluid) with a boundary layer phenomenon along Ω.

- $\Theta(u) = c1_{u>0}$; This is the case when we have a perfect insulator with a surface cost c that we can apply on the boundary of Ω. The resulting boundary condition is

  $$u = 0 \text{ or } \partial_\nu u = 0$$

- $\Theta(u) = 2cu_+$; this corresponds to a constant heat flux, with a boundary condition

  $$-\partial_\nu u \begin{cases} = c & \text{in } \partial\Omega \cap \{u > 0\} \\ \in (0, c] & \text{in } \partial\Omega \cap \{u = 0\} \end{cases}$$

- $\Theta(u) = \frac{2}{5}u_+^5 + 2Tu_+^4 + 4T^2u_+^3 + 4T^3u_+^2$ where $T \geq 0$ corresponds to the heat transfer resulting from a thermal radiation in a medium at temperature T, associated with the edge condition

  $$-\partial_\nu u = (T + u)^4 - T^4$$

- Any linear combination of the previous choices.

The optimisation problem of $E_\Theta(K, \Omega)$ at fixed K, among all domains Ω verifying some measure constraint, has been studied in [46], [47]. We are interested here in the shape optimization problem where Ω and K may both vary ; more precisely, let ω_n be the measure of the unit ball of $\mathbb{R}^n$, $M > \omega_n$ and $\Lambda > 0$, we are interested in the two problems

$$\inf_{K \subset \Omega,\ |K| = \omega_n,\ |\Omega| \leq M} E_\Theta(K, \Omega), \tag{2.2}$$

and the penalized version

$$\inf_{K \subset \Omega,\ |K| = \omega_n} E_\Theta(K, \Omega) + \Lambda|\Omega \setminus K|. \tag{2.3}$$

As we will see, according to the heat transfer law Θ, we can give a complete description of the solutions of the first problem in some cases, and for a general Θ we give a complete solution of the second problem, which gives a partial solution (i.e. for only some values of M) of the first. A notable phenomenon is that it might be sometimes better to not use all the insulator, i.e. the $|\Omega| \leq M$ constraint is not necessarily saturated, especially when M is close to ω_n. The main results are the following.

Theorem 2.11 (Convective case)**.** *Let $\Theta(u) = \beta u^2$ and $M = R^n\omega_n$ (where $R \geq 1$), the solution of (2.2) consists in two concentric balls, where the exterior ball has radius 1 or R depending on the minimum between $E_\Theta(B_1, B_R)$ et $E_\Theta(B_1, B_1)$}.*

The proof of this first result is practically self-contained, based on an equivalent extremal length formulation inspired by the work of Bossel [38], there is however in dimension $n \geq 3$ a case which escapes this analysis but is obtained thanks to a later result. More precisely, there are three possible cases:

- If $\beta \geq n-1$, then $E_\Theta(K,\Omega) \geq E_\Theta(B_1, B_R)$.
- If $\beta \in (n-2, n-1)$, then there exists $R_{n,\beta} > \frac{n-1}{\beta}$ such that $E_\Theta(K,\Omega) \geq \begin{cases} E_\Theta(B_1,B_R) & \text{if } R \geq R_{n,\beta} \\ E_\Theta(B_1,B_1) & \text{if } R \leq R_{n,\beta} \end{cases}$.
- If $\beta \leq n-2$, something that can only happen in dimension 3 and more, then $E_\Theta(K,\Omega) \geq E_\Theta(B_1,B_1)$.

The first two cases are obtained by a direct method where we establish a link between $E_\Theta(K,\Omega)$ and the extremal length of a family of curves (or hypersurface, in dimension 3 and more) that separate $\partial\Omega$ from K; we prove that

$$E_\Theta(K,\Omega) = \sup_{\rho \in L^\infty(\Omega,\mathbb{R}_+)} \inf_{K \subset \omega \subset \Omega} \left(\beta \mathrm{Per}(\omega;\partial\Omega) + \int_{\partial^*\omega \cap \Omega} \rho \mathrm{d}\mathscr{H}^{n-1} - \int_\omega \rho^2 \mathrm{d}\mathscr{L}^n \right)$$

where the infimum is taken among all sets ω with finite perimeter. By some a method of rearrangement of the optimal metric $\rho = |\nabla \log u|$, we get the first two cases but not the last one. This one is obtained as a corollary of the following result.

Theorem 2.12 (General penalized case)**.** *For any $\Lambda > 0$ and any increasing, lower semi-continuous $\Theta : [0,1] \to \mathbb{R}_+$ with $\Theta(0) = 0$, the solution of (2.3) consists of two concentric balls.*

This result is proved in two steps: we first prove the existence of a minimizer, and then prove that such a minimizer is necessarily the ball. The existence is in fact merely obtained for an approximated version of the function Θ - which is enough for our needs - by the following result.

Theorem 2.13 (General case)**.** *There exists a constant $c_n > 0$ such that for any increasing, lower semi-continuous $\Theta : [0,1] \to \mathbb{R}_+$ with $\Theta(0) = 0$, $\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} > 0$, and*

$$M < \omega_n + c_n \left(\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_0^1 \frac{t^{2n-1}\mathrm{d}t}{\Theta(t)^n}, \tag{2.4}$$

then the problem (2.2) admits a solution (K,Ω). If $|\Omega| < M$ then (K,Ω) are concentric balls. Else Ω in an open set with rectifiable topological boundary such that $\mathscr{H}^{n-1}(\partial\Omega) < \infty$ and K is relatively closed in Ω, with locally finite perimeter. The equilibrium temperature $u \in H^1(\Omega)$ is in $\mathcal{C}^{0,\frac{2}{n+2}}_{loc}(\Omega)$ and in dimension 2, $\partial K \cap \Omega$ is analytic.

The proof of this result is inspired by the proof of existence of isoperimetric clusters as detailed in the book of Maggi [63]; the main idea is to consider a minimizing sequence (K_i,Ω_i), to obtain by concentration-compactness and density estimate that after cutting and translating some parts of (K_i,Ω_i), they converge to a minimizer that verify the same measure constraint. One difficulty is added by the fact that we take the limite of the equilibrium temperature (u_i) instead of taking directly a limit of domains (K_i,Ω_i), thereby relaxing the problem; a minimizer obtained through this process is in a more general SBV space and we need to prove the regularity of such a minimizer.

Finally, we have a result on the case where the total quantity of insulating material $M - \omega_n$ is low.

Theorem 2.14. *If $\Theta : [0,1] \to \mathbb{R}_+$ is increasing, $\Theta(0) = 0$, and moreover if Θ is $\mathcal{C}^1$ in a neighbourhood of 1 with*

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n-1),$$

then there exists $M > \omega_n$ dependent on n and Θ such that the solution of (2.2) is $K = \Omega = B_1$.

This is essentially a refinement of previous estimates; we prove that for any $u \in SBV(\mathbb{R}^n, [0,1])$ such that $|\{u = 1\}| = \omega_n$, $|\{0 < u < 1\}| \ll 1$, we may suppose up to reducing the support of u without changing its energy too much that $\inf_{\{u>0\}} u \approx 1$, then we make a Taylor development of Θ. This criterion is optimal and we find in particular the phenomenon observed for convection conditions, where it may be better to put no insulation.

The problem (2.3) is thus entirely solved, but we leave open (2.2) in some cases: in particular, what if $M = \omega R^n$ is such that

$$R \notin \bigcup_{\Lambda>0} \operatorname{argmin} \{r \mapsto E_\Theta(B_1, B_R) + \Lambda |B_R \setminus B_1|\},$$

can we say in this case that $E_\Theta(K, \Omega) \geq \inf_{1 \leq r \leq R} E_\Theta(B_1, B_r)$?

2.5 Positivity of superharmonic functions verifying a Robin boundary condition

We consider, as may have been done previously, an open set $\Omega \subset \mathbb{R}^n$ with a rectifiable topological boundary and a function $f \in L^2(\Omega)$, then under some weak regularity hypotheses we may define in the variational sense a unique solution u to

$$\begin{cases} -\Delta u = f & (\Omega) \\ -\partial_\nu u = \beta u & (\partial\Omega) \end{cases}$$

If $f \geq 0$ is non-trivial, then the solution u is strictly positive in the interior of Ω by strong maximum principle on harmonic functions. Moreover, when Ω verifies an interior tangent sphere condition at each point of $\partial\Omega$ - for example when $\partial\Omega$ is $\mathcal{C}^2$ - then Hopf's maximum principle tells us that u and $\partial_\nu u$ cannot be both zero, and since these quantities are proportional in our case we deduce that $u_{|\partial\Omega} > 0$.

In particular, this tells us that $\partial_\nu u(x) < 0$ at each $x \in \partial\Omega$. In the "thermal" interpretation of this problem, when we see u as the temperature at rest of a solid Ω with a volumetric heat production f and a heat transfer by convection with the exterior, the condition $\partial_\nu u(x) < 0$ means that there is a non-zero heat flow through x. Borrowing the terminology used in [36], we say that a point $x \in \partial\Omega$ is **active** when $\partial_\nu u(x) < 0$, and Hopf's lemma implies that a $\mathcal{C}^2$ domain is active at every point of the boundary.

When $\Omega \cap B_1 = \{(x_1, x') \in B_1 : x_1 > 0,\ |x'| < x_1^\alpha\}$ for some $\alpha > 0$, we know instead that the origin is active if and only if $\alpha < 2$.

In [36], it is established by probabilistic methods that in a wide setting of non-smooth domains (more precisely domain obtained as an increasing union of Lipschitz sets with uniform Lipschitz constant, more details may be found in [36]), there exists a necessary a sufficient criterion for a point to be active. In particular, given (D_n) a so-called "decomposition in hyperbolic blocs" in the neighbourhood of a point $x \in \partial\Omega$, the point x is active if and only if $\sum_{n \geq 0} n\mathscr{H}^1(\partial D_n \cap \partial\Omega) < \infty$ in dimension 2 (one may consult section 3 and 4 of [36] for more details and for the result in higher dimension).

One of our objective was to find positivity conditions through purely variational means under conditions that rely more explicitly the geometry of the domain, so that we can then generalize it to elliptic operators other than the Laplacian ; we are more generally interested in the solution of a monotone operator

$$\begin{cases} -\nabla \cdot \mathcal{A}(x, \nabla u) \gneq 0 & (\Omega) \\ \mathcal{A}(x, \nabla u).\nu_\Omega + \mathcal{B}(x, u) = 0 & (\partial\Omega) \end{cases} \tag{2.5}$$

where $\mathcal{A} \in \mathcal{C}^0(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$, $\mathcal{B} \in \mathcal{C}^0(\partial\Omega \times \mathbb{R}, \mathbb{R})$ verify the two following hypotheses

- There exists $p \in (1, +\infty)$, $0 < \alpha_1 \leq \alpha_2$ and $\psi \in \mathcal{C}^0(\partial\Omega, \mathbb{R})$ such that for any $x \in \Omega$, $y \in \partial\Omega$, $z \in \mathbb{R}^n$,
$$\alpha_1 |z|^p \leq z\mathcal{A}(x,z),\ |\mathcal{A}(x,z)| \leq \alpha_2 |z|^{p-1},\ |\mathcal{B}(y,z)| \leq \psi(y)|z|^{p-1}.$$
- For any $x \in \Omega$, $y \in \partial\Omega$, $z_1, z_2 \in \mathbb{R}^n$,
$$(z_1 - z_2)(\mathcal{A}(x,z_1) - \mathcal{A}(x,z_2)) \geq 0,$$
$$(z_1 - z_2)(\mathcal{B}(y,z_1) - \mathcal{B}(y,z_2)) \geq 0.$$

We define the isoperimetric profile of the domain Ω as the function which for any sufficiently small $m > 0$ associate

$$I(m) := \inf \left\{ \mathscr{H}^{n-1}(\partial^* A \cap \omega),\ A \subset \Omega \text{ open s.t. } |A| \leq \frac{1}{2}|\Omega|,\ |A|^{1-\frac{1}{p}} \mathscr{H}^{n-1}(\partial A \cap \partial\Omega)^{\frac{1}{p}} \geq m \right\}.$$

Then we establish the criterion that for any u solution of a monotonous operator of order p of the form (2.5),

$$1/I \text{ is integrable in a neighbourhood of } 0 \Longrightarrow \inf_{x \in \Omega} u(x) > 0.$$

This is for instance the case when Ω verifies an isoperimetric inequality of the type

$$\mathscr{H}^{n-1}(\partial^* A \cap \partial\Omega) \leq C \frac{\mathscr{H}^{n-1}(\partial^* A \cap \Omega)}{|A|^\alpha},$$

for every subset $A \subset \Omega$ of finite perimeter and sufficiently small measure, and for some $\alpha < \frac{1}{N}\left(1 - \frac{1}{p}\right)$. We also establish a more local version of this result, and study the particular case of cusp domains:

$$\Omega = \left\{ x = (x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \text{ s.t. } x_1 > 0,\ |x'| \leq h(x_1) \right\}$$

for an increasing function $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $h(0) = 0$ and h^{n-1} is convex. We prove in this case that

$$\int_0^1 \left(\frac{\int_0^t h^{n-2}}{\int_0^t h^{n-1}} \right)^{\frac{1}{p}} dt < \infty \text{ implies that } \partial\Omega \text{ is active for equation (2.5)}$$

2.6 Optimization of an obstacle immerged in a Stokes fluid

In this last chapter, we address another vectorial free discontinuity problem, which is derived from fluid mechanics. The model is the following: consider a viscous incompressible fluid flow in a box $\Omega \Subset \mathbb{R}^n$ at the boundary of which the fluid has a constant velocity u_∞ (taken in the sense of "the velocity at infinity"; we can't take $\Omega = \mathbb{R}^n$ for technical reasons). The fluid flows around an obstacle $E \subset \Omega$ which is assumed to be Lipschitz at the beginning, and verifies on the edge of E a so-called Navier condition which can be interpreted as a partial adherence between the fluid and the boundary of E.

More precisely, the velocity of the fluid $u \in H^1(\Omega \setminus E, \mathbb{R}^n)$ and the pressure $p \in L^2(\Omega \setminus E, \mathbb{R})$ verify the incompressible Stokes equation with Navier boundary condition of parameter β, meaning

(a) *Incompressibility*: $\operatorname{div} u = 0$ in $\Omega \setminus E$.

(b) *Condition at infinity*: $u = u_\infty$ in $\partial\Omega$.

(c) *Non-penetration*: $u \cdot \nu_E = 0$ in ∂E, where ν_E is the outward normal vector of E.

(d) *Equilibrium*: we define the stress tensor $\sigma := -pI_d + 2\mu e(u)$, where $\mu > 0$ is a viscosity parameter, $e(u) = \frac{\nabla u + (\nabla u)^*}{2}$ is the symmetrised gradient of u and p is a pressure. We then have $\operatorname{div} \sigma = 0$ in $\Omega \setminus E$.

(e) *Navier condition*: $(\sigma\nu_E)_\tau = \beta u$ in ∂E, where V_τ denotes the tangential component to ∂E of a vector V and $\beta > 0$ is an adherence constant.

u may also be defined variationaly as the minimizer of the energy

$$\mathcal{E}_E(u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{\partial E} |u|^2 \mathrm{d}\mathscr{H}^{n-1}$$

in the class of admissible fields verifying conditions (a,b,c) above, denoted $\mathcal{V}^{reg}_{E,u_\infty}(\Omega)$; the conditions (c,d) appear as a variational equation of the energy.

The drag E is the component of the force exerted on K in the direction of u_∞, given by

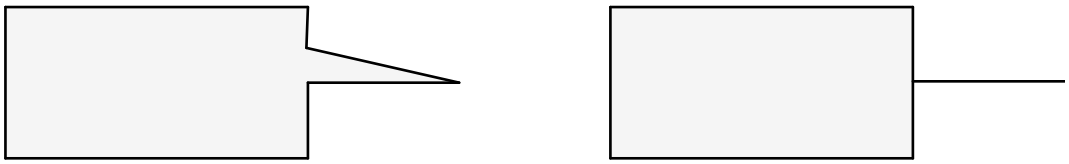
$$Drag(E) := \int_{\partial E} \sigma\nu \cdot u_\infty \mathrm{d}\mathscr{H}^{n-1}$$

and it may also be defined variationaly by $Drag(E) = \inf_{u\in\mathcal{V}^{reg}_{E,u_\infty}(\Omega)} \mathcal{E}_E(u)$.

Let $\Omega, u_\infty, \mu, \beta$ the parameters of the problem, as well as $c > 0$, $f : (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function, we are interested in the minimization of functionals of the type

$$J(E, u) = \mathscr{E}_E(u) + c\mathscr{H}^{n-1}(\partial E) + f(|E|)$$

among Lipschitz sets E, $u \in \mathcal{V}^{reg}_{E,u_\infty}(\Omega)$. We do not expect that there is a minimizer in this class, not only because of the regularity of E but also due to the possible appearance of lower-dimensional structures when the obstacle becomes "thin" in some parts, as in the figure below. This is impossible under Dirichlet (or "no-slip") boundary condition $u_{|\partial E} = 0$, but may be possible with Navier conditions.



E_i $\qquad$ E

This calls for a relaxation of the problem; for any admissible (E, u), we may extend u in a natural way by 0 in E, and this places u in the space of SBD functions (special bounded deformation functions), meaning functions $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\frac{Dv+(Dv)^*}{2}$ (where Dv is the differential in the sense of distribution) is a Radon measure that has the following structure: denote J_v the set of points $x \in \mathbb{R}^n$ such that there exists $\nu_v(x) \in \mathbb{S}^{n-1}$, $v^\pm(x) \in \mathbb{R}^n$ verifying

$$v(x + r\cdot) \underset{L^1_{\mathrm{loc}}(\mathbb{R}^n)}{\longrightarrow} v^+(x)1_{\{y:y\cdot\nu_v(x)>0\}} + v^-(x)1_{\{y:y\cdot\nu_v(x)<0\}} \text{ when } r \to 0$$

then $v \in SBD_{\mathrm{loc}}(\Omega)$ when

$$\frac{Dv + (Dv)^*}{2} = e(v)\mathscr{L}^n + (v^+ - v^-) \odot \nu_v \mathscr{H}^{n-1}\lfloor J_v$$

for some $e(v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and where $\odot$ designates the symmetrized tensor product $a \odot b = \frac{a\otimes b+b\otimes a}{2}$. We write

$$\mathcal{V}(\Omega) = \left\{u \in SBD(\Omega) : \mathrm{Tr}[e(u)] = 0,\ \nu_u \cdot u^\pm = 0 \text{ in } J_u\right\}$$

the set of incompressible flows that are tangent to their discontinuities, and for any set of finite perimeter E,

$$\mathcal{V}_{E,u_\infty}(\Omega) = \left\{u \in \mathcal{V}(\Omega) : u_{|\partial\Omega} = u_\infty,\ u_{|E} = 0\right\}.$$

Note in particular that u may have discontinuities outside of E, and not only on $\partial^* E$. We then let

$$J(E,u) = \int_\Omega 2\mu|e(u)|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E} (c + \beta|u^+|^2)\mathrm{d}\mathscr{H}^{n-1} + \int_{J_u \setminus \partial^* E} (2c + \beta|u^+|^2 + \beta|u^-|^2)\mathrm{d}\mathscr{H}^{n-1} + f(|E|)$$

and we are interested in the minimization

$$\inf \{J(E,u),\ E \text{ of finite perimeter, } u \in \mathcal{V}_{E,u_\infty}(\Omega)\}. \tag{2.6}$$

This is a free discontinuity problem similar to the Griffitch fracture model ([58]); the functional is essentially the same excepted for the polynomial boundary term, but u is in a much more restricted admissible space (corresponding to incompressible flows) in our case. We establish the following three results.

Theorem 2.15. (2.6) *admits a minimum.*

This result has several components; the basic idea is to consider an admissible sequence (E_i, u_i) that converges to an infimum. Then we need to check the following points:

- We can extract a converging subsequence from (u_i) ; this is possible by the compactness results of [37], and we may suppose that (u_i) converges to a limit u in L^1_{loc}.
- The limit u verifies the incompressibility and nonpenetration constraint; the first one is direct, but for the second we make use of a recent lower-semicontinuity result on jointly symmetric functionals of [55], among which is the functional

$$\int_{J_u} \left(|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|\right) \mathrm{d}\mathscr{H}^{n-1}$$

- The energy of u is minimal ; this means we need a new lower semicontinuity result of the functional

$$\int_{J_u} \left(|u^+|^2 + |u^-|^2\right) \mathrm{d}\mathscr{H}^{n-1}.$$

A minimizer of (2.6) is said to be strong if $\mathscr{H}^{n-1}(\overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0$; this means that setting F to be the complementary of the union of the connected components of $\Omega \setminus \overline{\partial^* E \cup J_u}$ where $u = 0$, then F is a closed set with rectifiable topological boundary, $u \in \mathcal{V}_{F,u_\infty}(\Omega)$ with $u \in H^1_{\mathrm{loc}}(\Omega \setminus F)$, and $J(F,u)$ takes the minimal value in (2.6). In particular we get that u is a solution of the Stokes equation in the classical sense in $\Omega \setminus F$, and u verifies the Navier boundary conditions on ∂F in the variational sense.

Theorem 2.16. *In 2 dimensions, if additionally f is nonincreasing and Ω is simply connected, then* (2.6) *admits a strong minimizer in the class of compact set K such that K and $\mathbb{R}^2 \setminus K$ are connected.*

This result can be obtained almost independently of the previous one, with simplifications on the closure of the boundary constraints and the semicontinuity; the idea is to reformulate the functional and the constraints with the stream function of u, i.e. the scalar function ψ such that $\nabla^\perp \psi = u$, which amounts to saying that the level sets of ψ are the trajectories of the fluid in the field u. Uniform estimates in Hölder spaces are available on ψ, and the Gołąb theorem on connected compacts sets of $\mathbb{R}^2$ allows to obtain the compactness at a lower cost.

Theorem 2.17. *Suppose that $n = 2$ and f is Lipschitz. Let (E,u) be a minimizer of* (2.6)*, then*

$$\mathscr{H}^1(\Omega \cap \overline{J_u \cup \partial^* E} \setminus (J_u \cup \partial^* E)) = 0.$$

This means that (E, u) is **locally** a strong minimizer. This result is neither stronger or weaker than the previous one, in particular its reasoning uses test functions that do not preserve the connexity of the obstacle. The proof of this result is similar to the proof of the closure of the jump set of SBV minimizers of Mumford-Shah by De Giorgi, Carriero, Leaci [53]. More precisely, this strategy was used with success to prove that SBD minimizers of the Griffith functional are strong, in dimension 2 (in [51]) and later in any dimension (by [49]). The general idea is to prove the closure by establishing a uniform density estimate around the discontinuities. More precisely, denote $Q_{x,r} = x + [-r, r]^n$, we establish the existence of $\overline{r}, \epsilon, C > 0$ depending only on the constants $\mu, c, \lambda > 0$ (and suprisingly not of β) such that if $u \in \mathcal{V}(Q_{x,r})$ is a local minimizer of (2.6) in $Q_{x,r}$ for $r \in (0, \overline{r})$, and if

$$\int_{Q_{x,r}} |e(u)|^2 \mathrm{d}\mathscr{L}^2 + \mathscr{H}^1(J_u \cap Q_{x,r}) \leq \epsilon r$$

then for every $\rho \in (0, r)$,

$$\int_{Q_{x,\rho}} |e(u)|^2 \mathrm{d}\mathscr{L}^2 + \mathscr{H}^1(J_u \cap Q_{x,\rho}) \leq C r^{-\frac{1}{2}} \rho^{\frac{3}{2}},$$

which excludes in particular the existence of jump in a neighbourhood of x.

This bound is obtained through a compactness argument similar to [53], for which one of the main component is the following approximation lemma that may have other uses beyond that.

We fix $\rho \in \mathcal{C}_c^\infty(B_{1/8}, \mathbb{R}_+)$ a radial function of integral 1, and we write $\rho_\delta := \delta^{-2}\rho(\delta^{-1}\cdot)$.

Theorem 2.18. *There exists constants $C, \eta > 0$ sucht that for every $u \in \mathcal{V}(Q_1)$ verifying $\mathscr{H}^1(J_u) \leq \eta$, there exists $r \in [1 - \mathscr{H}^1(J_u)^{\frac{1}{4}}, 1]$, $\varphi \in \mathcal{C}_c^\infty(Q_r, [0, 1])$ such that $Q_{r-\sqrt{\mathscr{H}^1(J_u)}} \subset \{\varphi = 1\}$, and a function $v \in \mathcal{V}(Q_1) \cap H^1(Q_r)$ such that*

- $\{u \neq v\} \subset Q_r$, $\mathscr{H}^1(J_v \setminus J_u) = 0$,
- $\int_{Q_1} \left| e(v) - \left(\rho_{\sqrt{\mathscr{H}^1(J_u)}} * e(u)\right)\varphi \right|^2 \mathrm{d}\mathscr{L}^2 \leq C\mathscr{H}^1(J_u)^{\frac{1}{6}} \int_{Q_1} |e(u)|^2 \mathrm{d}\mathscr{L}^2$.

Said differently, an incompressible flow with few discontinuities may be smoothed. THe general idea of the proof is to let $\delta = \sqrt{\mathscr{H}^1(J_u)}$, and choose a radius r close to 1 such that any squares of size $\frac{\delta}{2^k}$ at distance $\frac{\delta}{2^{k-1}}$ of ∂Q_r contains few jump compared to the length of its boundary, whichi is only possible in two dimension by a Vitali covering argument. From this we begin similarly as [49] by dividing the square Q_r in smaller squares of side $r\delta$ inside, and side $\frac{r\delta}{2^k}$ near the boundary ∂Q_r (by a dyadic decomposition). On each square we find an affine function sufficiently close to the flow u through a generalized Korn inequality for SBD functions, proved in [48]. We join these approximations and verify that the flow we obtain can be adjusted to respect the zero-divergence constraint.

This method is robust enough to work on so-called "almost quasi-minimizers" of more general functionals, details can be found at the end of chapter 10.

These results open several new questions. The regularity of the relaxed minimizers in dimension three remains open, and would require for example to prove a weakened version of the approximation lemma (which in the current state cannot generalize to higher dimension). The numerical approximation (even in two dimensions) of the drag by a phase field method with a functional similar to the Ambrosio-Tortorelli one for the Mumford-Shah problem, seems to be affordable thanks to the above approximation lemma, but it still raises open questions about the formulation of the non-penetration constraints. Finally, it is natural to try to generalize these results to the Navier-Stokes equation, and it seems that one cannot expect to obtain the same conclusions because of the potential appearance of microstructures at the boundary of the obstacle that change the behavior of the fluid on the walls.

Bibliography

[35] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[36] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen. On the Robin problem in fractal domains. *Proc. Lond. Math. Soc. (3)*, 96(2):273–311, 2008.

[37] G. Bellettini, A. Coscia, and G. Dal Maso. Compactness and lower semicontinuity properties in SBD(Ω). *Math. Z.*, 228(2):337–351, 1998.

[38] Marie-Hélène Bossel. Longueurs extrémales et fonctionnelles de domaine. *Complex Variables Theory Appl.*, 6(2-4):203–234, 1986.

[39] Marie-Hélène Bossel. Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimétrique de Rayleigh-Faber-Krahn. *Z. Angew. Math. Phys.*, 39(5):733–742, 1988.

[40] Lorenzo Brasco, Guido De Philippis, and Bozhidar Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.*, 164(9):1777–1831, 2015.

[41] D. Bucur and A. Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[42] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. The quantitative Faber-Krahn inequality for the Robin Laplacian. *J. Differential Equations*, 264(7):4488–4503, 2018.

[43] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. Weinstock inequality in higher dimensions. *Journal of Differential Geometry (to appear)*, 2020.

[44] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[45] Dorin Bucur and Alessandro Giacomini. Minimization of the k-th eigenvalue of the Robin-Laplacian. *J. Funct. Anal.*, 277(3):643–687, 2019.

[46] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[47] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[48] Antonin Chambolle, Sergio Conti, and Gilles Francfort. Korn-Poincaré inequalities for functions with a small jump set. *Indiana Univ. Math. J.*, 65(4):1373–1399, 2016.

[49] Antonin Chambolle, Sergio Conti, and Flaviana Iurlano. Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. *J. Math. Pures Appl. (9)*, 128:119–139, 2019.

[50] Marco Cicalese and Gian Paolo Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.*, 206(2):617–643, 2012.

[51] Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Existence of strong minimizers for the Griffith static fracture model in dimension two. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(2):455–474, 2019.

[52] Guy David. *Singular sets of minimizers for the Mumford-Shah functional*, volume 233 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2005.

[53] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

[54] Georg Faber. Beweis, dass unter allen homogenen membranen von gleicher fläche und gleicher spannung die kreisförmige den tiefsten grundton gibt. 1923.

[55] Manuel Friedrich, Matteo Perugini, and Francesco Solombrino. Lower semicontinuity for functionals defined on piecewise rigid functions and on $GSBD$. *J. Funct. Anal.*, 280(7):Paper No. 108929, 45, 2021.

[56] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[57] Nunzia Gavitone, Domenico Angelo La Manna, Gloria Paoli, and Leonardo Trani. A quantitative Weinstock inequality. *Arxiv 1903.04964*, 2019.

[58] Alan Arnold Griffith. Vi. the phenomena of rupture and flow in solids. *Philosophical transactions of the royal society of london. Series A, containing papers of a mathematical or physical character*, 221(582-593):163–198, 1921.

[59] Serena Guarino Lo Bianco, Domenico Angelo La Manna, and Bozhidar Velichkov. A two-phase problem with Robin conditions on the free boundary. *J. Éc. polytech. Math.*, 8:1–25, 2021.

[60] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[61] Gerasim Kokarev. Variational aspects of Laplace eigenvalues on Riemannian surfaces. *Adv. Math.*, 258:191–239, 2014.

[62] E. Krahn. Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.*, 94(1):97–100, 1925.

[63] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[64] Édouard Oudet. Numerical minimization of eigenmodes of a membrane with respect to the domain. *ESAIM Control Optim. Calc. Var.*, 10(3):315–330, 2004.

[65] Romain Petrides. Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces. *Geom. Funct. Anal.*, 24(4):1336–1376, 2014.

[66] E. R. Reifenberg. An epiperimetric inequality related to the analyticity of minimal surfaces. *Ann. of Math. (2)*, 80:1–14, 1964.

[67] Luca Spolaor and Bozhidar Velichkov. On the logarithmic epiperimetric inequality for the obstacle problem. *Math. Eng.*, 3(1):Paper No. 4, 42, 2021.

[68] Robert Weinstock. Inequalities for a classical eigenvalue problem. *J. Rational Mech. Anal.*, 3:745–753, 1954.

Chapter 3

A survey of rectifiable sets and special bounded variation functions

Contents

3.1 Generalities on geometric measure theory 51
3.2 BV functions - definition, embedding and compactness properties 55
3.3 Sets of finite perimeter 56
3.4 Unidimensional BV functions, structure theorem 58
3.5 Fine properties of BV functions 59
3.6 Compactness of SBV functions 63
3.7 A quick overview of SBD functions 64

In this section we introduce some notions of geometric measure theory and more specifically we present rectifiability, sets of finite perimeter and special bounded variation functions, that will be used extensively in most of the following chapters. The source of this exposition is mainly extracted from [72] for the basics of geometric measure theory and [70] for the fine properties of BV and SBV functions. The proofs are sketched but not given in full details.

3.1 Generalities on geometric measure theory

In all the following, an outer measure on $\mathbb{R}^n$ is a function $\mu : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and $E \subset \cup_{i\in\mathbb{N}} E_i$ implies $\mu(E) \leq \sum_{i\in\mathbb{N}} \mu(E_i)$. An outer measure μ is said to be a Borel measure when open sets (and by extension Borel sets) are measurable for μ, and it is said to be a Radon measure when it is Borel and finite on every compact set.

For any measure μ and any measurable set E, $\mu\lfloor E$ will be the measure $\mu\lfloor E(F) := \mu(E \cap F)$, in particular if μ is Borel and $\mu(E) < \infty$ then $\mu\lfloor E$ is a Radon measure.

We remind a fundamental property of Radon measure, the Besicovitch differentiation theorem that we will use extensively.

Theorem 3.1 (Besicovitch's differentiation theorem)**.** *Let μ, ν two Radon measure in $\mathbb{R}^n$, then the quantity*

$$\frac{d\nu}{d\mu}(x) = \lim_{r\to 0} \frac{\nu(B_{x,r})}{\mu(B_{x,r})}$$

is defined with value in $\mathbb{R}_+ \cup \{+\infty\}$ for μ-almost every $x \in \mathbb{R}^n$, with $\frac{d\nu}{d\mu} \in L^1(\mu)$. Moreover,

$$\nu = \frac{d\nu}{d\mu}\mu + \nu^s_\mu,$$

where $\nu^s_\mu \perp \mu$, meaning that there is a Borel set E such that $\mu(E) = \nu^s_\mu(\mathbb{R}^n \setminus E) = 0$.

The statement of the Besicovitch differentiation theorem of [72] has more details on the support of ν_μ^s. One consequence that will prove useful later is that for $\nu = 1_E\mu$ (where E is some Borel set), we have $\frac{d\nu}{d\mu} = 1_E$ μ-a.e, so E has density (with regard to μ) 1 for μ-a.e. point of E, and 0 for μ-a.e. point of $\mathbb{R}^n \setminus E$.

Likewise, for $\mu = \mathscr{L}^n$ and $\nu = f\mathscr{L}^n$ for some $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we get that $⨍_{B_{x,r}} |f - f(x)| d\mathscr{L}^n \underset{r\to 0}{\longrightarrow} 0$ for $\mathscr{L}^n$-almost every $x \in \mathbb{R}^n$. We shall call S_f the set of $x \in \mathbb{R}^n$ such that there is no $c \in \mathbb{R}$ verifying $⨍_{B_{x,r}} |f - c| d\mathscr{L}^n \underset{r\to 0}{\longrightarrow} 0$ - or equivalently that $y \mapsto f(x + ry)$ does not converge in $L^1_{\text{loc}}(\mathbb{R}^n)$ to a finite constant as $r \to 0$ - and the Besicovitch differentiation theorem implies that $\mathscr{L}^n(S_f) = 0$.

We will also extensively use Riesz's theorem applied to vector-valued Radon measures. Let U an open subset of $\mathbb{R}^n$, we say a linear application $L : \mathcal{C}^0_c(U, \mathbb{R}^m) \to \mathbb{R}$ is continuous if for every compact set $K \Subset U$, there is some constant $C_K > 0$ such that for every $f \in \mathcal{C}^0(U, \mathbb{R}^m)$,

$$\{f \neq 0\} \subset K \text{ implies } |L(f)| \leq C_K \|f\|_{L^\infty(U,\mathbb{R}^m)}.$$

Theorem 3.2 (Riesz's theorem)**.** *Let $L : \mathcal{C}^0_c(U, \mathbb{R}^m) \to \mathbb{R}$ be continuous, then there is a Radon measure denoted $|L|$ and a $|L|$-measurable function $g : U \to \mathbb{R}^m$ such that $|g| = 1$ $|L|$-a.e. and*

$$\forall f \in \mathcal{C}^0_c(U, \mathbb{R}^m),\ L(f) = \int_U (f \cdot g) \mathrm{d}|L|.$$

A proof may be found at [29, Th 4.7].

Definition 3.3 (Hausdorff's measures)**.** Let $0 \leq k \leq n$, $E \Subset \mathbb{R}^n$, we define

$$\mathscr{H}^k(E) = \sup_{\delta > 0} \mathscr{H}^k_\delta(E), \text{ where}$$
$$\mathscr{H}^k_\delta(E) = \inf \left\{ \omega_k \sum_{i\in\mathbb{N}} \left(\frac{\operatorname{diam}(U_i)}{2} \right)^k,\ U_i \text{ open s.t. } \operatorname{diam}(U_i) \leq \delta,\ E \subset \cup_{i\in\mathbb{N}} U_i \right\}.$$

Here ω_k is a normalizing constant given by $\omega_k = \frac{\pi^{k/2}}{\Gamma(1+s/2)}$, chosen such that $\mathscr{H}^k([0,1]^k \times \{0\}^{n-k}) = 1$.

It is known that $\mathscr{H}^k$ is a Borel measure that coincides with the Lebesgue measure when $k = n$, with the counting measure when $k = 0$, and when $k \in \{1, \dots, n-1\}$ it coincides with the surface measure of dimension k; more precisely, let $f \in \operatorname{Lip}(U, \mathbb{R}^n)$ where U is an open subset of $\mathbb{R}^k$, then

$$\mathscr{H}^k(f(U)) = \int_U Jf(x) \mathrm{d}\mathscr{L}^k,$$

where $Jf(x) = \sqrt{\det(Df(x)^* Df(x))}$ is defined almost everywhere by the following theorem of Rademacher.

Theorem 3.4 (Rademacher's theorem)**.** *Let $f \in Lip(\mathbb{R}^k, \mathbb{R}^n)$, then f is differentiable almost everywhere.*

More precisely we prove that f is differentiable at every Lebesgue point of its weak gradient.

Proof. Let $\nabla f \in L^\infty(\mathbb{R}^k)$ be the distributional gradient of f, and let $S_{\nabla f}$ be the points of $\mathbb{R}^k$ that are not Lebesgue points of ∇f; the Lebesgue-Besicovitch differentiation theorem states that $\mathscr{L}^n(S_f) = 0$. Let now $x \in \mathbb{R}^k \setminus S_f$ and $f_{x,r}(y) := \frac{f(x+ry)-f(x)}{r}$. Then $(f_{x,r})_{r\to 0}$ is a family of uniformly Lipschitz functions, and by choice of x

$$\nabla f_{x,r} = (\nabla f)(x + r\cdot) \underset{r\to 0}{\longrightarrow} \nabla f(x) \cdot \text{ in } L^1_{\text{loc}}$$

By Ascoli theorem, let g be a limit of a subsequence of $(f_{x,r})_{r\to 0}$, then $\nabla g = \nabla f(x)$ almost everywhere, so $g(y) = \nabla f(x) y$ and $(f_{x,r})_{r\to 0}$ converges uniformly to a linear function, which means exactly that f is differentiable at x. □

Note that for $k < n$, $\mathscr{H}^k$ is a Borel measure, but not a Radon measure as $\mathscr{H}^k([0,1]^n) = +\infty$. We will restrict it to sets E such that $\mathscr{H}^k(E) < \infty$, with in general the additional (weak) regularity hypothesis that E is rectifiable.

Definition 3.5 (Rectifiability)**.** Let $0 \leq k \leq n$ two integers, we say $E \subset \mathbb{R}^n$ is a countably (k-)rectifiable set if there exists a countable set of functions $(f_i)_{i\in\mathbb{N}}$ in $\mathrm{Lip}(\mathbb{R}^k, \mathbb{R}^n)$ such that

$$\mathscr{H}^k\left(E \setminus \cup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)\right) = 0$$

Moreover, the f_i maybe be supposed to be $\mathcal{C}^1$, although we will not use it in our case. We may also suppose that for every $x \neq y$ in $\mathbb{R}^k$, and every i, $\frac{|f(x)-f(y)|}{|x-y|} \in [1-\epsilon, 1+\epsilon]$ for any arbitrarily small ϵ. We say E is locally (k-)rectifiable if it is countably rectifiable with finite $\mathscr{H}^k$ measure on every compact set, meaning that $\mathscr{H}^k \lfloor E$ is a Radon measure.

In our case we will mostly use $(n-1)$ countably rectifiable set, although for the sake of the presentation we keep k-rectifiable sets for any integer k in this subsection. Later on, when we say a set is countably rectifiable, we implicitely assume it is countably k-rectifiable for $k = n-1$.

In general it is hard to verify directly the definition of rectifiability, so we start by proving a few criteria for rectifiability. We begin by a geometric criteria.

Theorem 3.6 (A geometric rectifiability criteria)**.** *For any k-plane $\pi \subset \mathbb{R}^n$ and $M > 0$, we let p_π the orthogonal projection on π, $p_\pi^\perp = p_{\pi^\perp}$, and*

$$K_M(\pi) = \left\{x \in \mathbb{R}^n : |p_\pi^\perp(x)| \leq M|p_\pi(x)|\right\}$$

Let $R \subset \mathbb{R}^n$ be such that for $\mathscr{H}^k$-almost every $x \in R$, there exists $r > 0$, $M > 0$, and π a k-plane such that

$$R \cap B_{x,r} \subset x + K_M(\pi)$$

Then R is (k-) countably rectifiable.

Proof. For any x we let $\pi(x)$, $M(x)$, $r(x)$ the associated constants. It is enough to prove that R is rectifiable when r, M are bounded from below, since a countable union of rectifiable sets is rectifiable. Moreover, up to considering a dense subsequence of the k-planes, it is enough to suppose that $\pi(x)$ is constant (which up to rotation we suppose to be $\mathbb{R}^k \times \{0\}^{n-k}$). For any $x \in \mathbb{R}^n$, we decompose $x = (x', x'')$ in $\mathbb{R}^k \times \mathbb{R}^{n-k}$, and so our (strengthened) hypothesis is that for any $x \in R$,

$$B_r \cap (R - x) \subset \{y : |y''| \leq M|y'|\}$$

Without loss of generality we suppose $0 \in R$ and $R \Subset B_{r/2}$, then for any $x' \in \mathbb{R}^{n-1}$, let

$$u(x') = \inf_{y\in R} (y'' + M|x' - y'|)$$

Automatically, u is M-Lipschitz and $R \subset \{(x', u(x'))\}$, hence the conclusion. □

We now prove a blow-up criteria, meaning a critera for rectifiability from the local information obtained by zooming in on every point. A notation that will be frequently useful for us is the k-dimensional density: for any Borel measure μ in $\mathbb{R}^n$, and $x \in \mathbb{R}^n$, we write

$$\Theta_k^*(\mu; x) = \limsup_{r\to 0} \frac{\mu(B_{x,r})}{\omega_k r^k},\ \Theta_{k*}(\mu; x) = \liminf_{r\to 0} \frac{\mu(B_{x,r})}{\omega_k r^k}$$

and if both coincide we write $\Theta_k(\mu; x)$ their common limit. For a set $E \subset \mathbb{R}^n$, its k-dimensional density is by convention the density of the measure $\mathscr{H}^k \lfloor E$. A useful fact about densities, which can be obtain by Besicovitch covering lemma, is the following; let μ be a Radon measure and M a Borel set.

- If for every $x \in M$, $\Theta_k^*(\mu; x) \geq 1$, then $\mu(M) \geq \mathscr{H}^k(M)$.
- If for every $x \in M$, $\Theta_k^*(\mu; x) \leq 1$, then $\mu(M) \leq 2^k \mathscr{H}^k(M)$.

Definition 3.7 (Tangent plane)**.** Let $M \subset \mathbb{R}^n$ and π a subspace of dimension k of $\mathbb{R}^n$. We say M admits the k-tangent plane π at x if

$$\mathscr{H}^k \lfloor \left(\frac{M - x}{r} \right) \underset{r \to 0}{\rightharpoonup} \mathscr{H}^k \lfloor \pi,$$

in the sense that for any $\varphi \in \mathcal{C}_c^0(\mathbb{R}^n)$,

$$\frac{1}{r^k} \int_M \varphi \left(\frac{y - x}{r} \right) \mathrm{d}\mathscr{H}^k(y) \underset{r \to 0}{\longrightarrow} \int_\pi \varphi(y) \mathrm{d}\mathscr{H}^k(y)$$

Similarly, let μ be a Radon measure in $\mathbb{R}^n$, we say π_x is a k-tangent space (or blow-up) of μ at x if the sequence of measure

$$E \mapsto \frac{\mu(x + rE)}{r^k}$$

converges weakly to $\mathscr{H}^k \lfloor \pi_x$.

In particular, a measure μ that admits a k-tangent plane at x verifies $\Theta_k(\mu; x) = 1$.

Theorem 3.8 (Rectifiability by blow-up)**.**
- *Let $R \subset \mathbb{R}^n$ be a locally k-rectifiable set, then for $\mathscr{H}^k$-almost every $x \in \mathbb{R}^n$ there is a unique k-plane π_x that is tangent to R at x.*
- *Reciprocally, let μ be a Radon measure in $\mathbb{R}^n$ and R a Borel set of $\mathbb{R}^n$ on which μ is concentrated such that for every $x \in R$ we have a unique blow-up $\mathscr{H}^k \lfloor \pi_x$ for some k-plane π_x, then R is locally k-rectifiable and $\mu = \mathscr{H}^k \lfloor R$.*

Proof.
- First see that if $R = f(E)$ for some $f \in Lip(\mathbb{R}^k, \mathbb{R}^n)$ with $\frac{|f(x) - f(y)|}{|x - y|} \in (1/2, 2)$ for any x, y, and $E \subset \mathbb{R}^k$, then for $\mathscr{H}^k$ almost every $x \in R$, there is a $y \in E$ that is a Lebesgue point of ∇f such that $f(y) = x$. Then from the fact that it is a Lebesgue point, we obtain the result. Let us go back to the general case. Since R is rectifiable we may write that $R = R_0 \cup_{i \in \mathbb{N}} R_i$ where $R_i = f_i(E_i)$ as previously and $\mathscr{H}^k(R_0) = 0$. Then its is a consequence of Besicovitch differentiation theorem that for $\mathscr{H}^k$-almost every $x \in R \setminus R_i$, $\Theta_k(R_i; x) = 0$. In particular, for almost every $x \in R_i$ we have such a blow-up for R_i and the same blow-up for R.
- We use the previous geometric criteria to prove the rectifiability. Notice that for any two k-space π, σ that are close enough and for some small enough $\lambda \in (0, 1)$, we have that for any $w \in \mathbb{R}^n \setminus K_2(\sigma)$, then $B(w, \lambda|w|) \cap K_1(\pi) = \emptyset$. Consider $\sigma_1, \ldots, \sigma_N$ a set of k-spaces such that any k-space π is close enough to one of the σ_i for this property to hold.

  Suppose first that for every $x \in M$, the limits

  $$\frac{\mu(B_{x,r})}{\omega_k r^k} \to 1, \ \frac{\mu(B_{x,r} \setminus (x + K_1(\pi_x))}{\omega_k r^k} \to 0$$

  are uniform in x. Let $\epsilon > 0$, $\delta > 0$ such that for any $r \in (0, \delta)$ and any $x \in M$ we have

  $$\frac{\mu(B_{x,r})}{\omega_k r^k} \geq 1 - \epsilon, \ \frac{\mu(B_{x,r} \setminus (x + K_1(\pi_x))}{\omega_k r^k} \leq \epsilon$$

  We denote M_i the set of points $x \in M$ such that π_x is closest to σ_i, and we apply the geometric criteria of rectifiability to M_i. Indeed we claim that for any $x \in M_i$,

  $$M_i \cap B_{x,\delta} \subset (x + K_2(\sigma_i)).$$

If it were not the case we could find some $y \in M_i \cap B_{x,\delta} \setminus (x + K_2(\sigma_i))$, so $B(y, \lambda|x-y|) \subset \mathbb{R}^n \setminus (x + K_1(\pi_x))$, and since $\lambda|x-y| < \delta$ we have

$$(1-\epsilon)\omega_k \lambda^k |x-y|^k \leq \mu(B_{y,\lambda|x-y|}) \leq \mu(B_{x,|x-y|}) \leq \epsilon\omega_k |x-y|^k$$

so for a small enough ϵ (depending only on k, λ) we get a contradiction, meaning that each M_i (so M) is rectifiable.

Let us go back to the general case. The sequences of functions

$$f_r : x \in M \mapsto \frac{\mu(B_{x,r})}{\omega_k r^k}, \; g_r : x \in M \mapsto \frac{\mu(B_{x,r} \setminus (x + K_1(\pi_x))}{\omega_k r^k}$$

converges punctually to $1, 0$ respectively as $r \to 0$, so by Egoroff theorem we may find for any $\epsilon > 0$ some subset $M' \subset M$ with $\mu(M \setminus M') < \epsilon$ where the convergence is uniform, taking this for some sequence $\epsilon_k \to 0$ gives that M is countably rectifiable.

Finally, the fact that the k-density of μ at every point of M is 1 gives that $\mu(M) \geq \mathscr{H}^k(M)$, thus M is rectifiable and $\mathscr{H}^k \lfloor M$ is a Radon measure. $\mu \lfloor M = \mathscr{H}^k \lfloor M$ is then obtained by Besicovitch differentiation theorem.

□

3.2 BV functions - definition, embedding and compactness properties

Definition 3.9 (BV space)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, we let $BV(\Omega)$ be the set of functions $u \in L^1(\Omega)$ such that the distributional derivative Du is a finite Borel measure, meaning

$$\sup \left\{ \int_\Omega u \mathrm{div}(\varphi) \mathrm{d}\mathscr{L}^n, \; \varphi \in \mathcal{C}^1_c(\Omega, B_1) \right\} =: V(u;\Omega) < \infty$$

And in term of measure, we may write $V(u;\Omega) = |Du|(\Omega)$. In general we will not see $V(\cdot,\Omega)$ as a (semi-)norm, because its topology is too strong for our purposes. We will prefer to look at the L^1_{loc} topology on BV, it is for instance easy to prove that

- If $u_i \underset{L^1(\Omega)}{\longrightarrow} u$, then $V(u;\Omega) \leq \liminf_{i\to\infty} V(u_i;\Omega)$. Moreover, if the right-hand side is finite then $Du_i \rightharpoonup Du$
- Let $u \in BV(\Omega)$, $\rho \in \mathcal{C}^\infty_c(B_1, \mathbb{R}_+)$ that is radial with integral 1, and $\rho_\epsilon = \frac{1}{\epsilon^n}\rho\left(\frac{\cdot}{\epsilon}\right)$, then the sequence $u_\epsilon := \rho_\epsilon * u$ is smooth, converges to u in $L^1_{\mathrm{loc}}(\Omega)$, with $Du_\epsilon \rightharpoonup Du$.

In the following, we denote $K^i_r = \Pi_{k=1}^n [i_k r, i_k r + r]$ the cube of side r based in ir for some $i \in \mathbb{Z}^n$. For any compact K and $u \in L^1(K)$, we denote $u_K := \frac{1}{|K|}\int_K u$.

Proposition 3.10 (Poincaré inequality in BV)**.** *Let $u \in BV(\mathbb{R}^n)$, then*

$$\int_{K_r} |u - u_{K_r}| \leq r \sum_{l=1}^n |\partial_l u|(K_r)$$

Proof. By scaling we may suppose $r = 1$. By convolution we may suppose that $u \in \mathcal{C}^1(\mathbb{R}^n)$ and proceed by induction on the dimension; in dimension 1 it is direct that $|u - u_{[0,1]}| \leq \int_0^1 |u'|$. Suppose now that $n \geq 2$, and denote $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$.
Let $v(x) = \int_{[0,1]} u(x', y_n) dy_n$, then we apply the induction on v to see that

$$\int_{[0,1]^n} |v - v_{[0,1]^n}| \leq \sum_{l=1}^{n-1} \int_{[0,1]^{n-1}} |\partial_l v|(y') dy' \leq \sum_{l=1}^{n-1} \int_{[0,1]^n} |\partial_l u|(y) dy$$

We apply the result for $n = 1$ to $u(x', \cdot)$ for each x', then by integrating along x', we get

$$\int_{[0,1]^n} |u - v| \leq \int_{[0,1]^n} |\partial_n u|$$

Notice that $u_{[0,1]^n} = v_{[0,1]^n}$, so we get the result. □

This has an interesting consequence when applied to a function $u = 1_E$; it means that

$$\frac{|E \cap K_r|}{|K_r|} \times \frac{|E^c \cap K_r|}{|K_r|} \leq \frac{\sqrt{n}|D1_E|(K_r)}{r^{n-1}} \tag{3.1}$$

In particular when the right-hand side is small enough, then the density of E in K_r is either close to 0 or to 1.

Corollary 3.11. *Let $u \in BV(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and let u_r the function equal to $u_{K_r^i}$ in K_r^i and 0 elsewhere. Then*

$$\|u - u_r\|_{L^1(\mathbb{R}^n)} \leq r \sum_{l=1}^{n} |\partial_l u|(\mathbb{R}^n)$$

This is direct by applying the previous proposition to each square.

Corollary 3.12 (Compactness of BV)**.** *Let $(u_i)_i$ be a sequence of $BV(\mathbb{R}^n)$ with support in some open bounded set Ω, such that $\limsup_{i\to\infty} V(u_i; \Omega) < \infty$. Then there exists a subsequence $(i_k)_{k\in\mathbb{N}}$ and a function $u \in L^1(\mathbb{R}^n)$ such that*

$$u_{i_k} \underset{L^1(\mathbb{R}^n)}{\longrightarrow} u$$

This is obtained by diagonal extraction: for any fixed $r \in \mathbb{Q}_+^*$, the sequence $(u_i^r)_{i\in\mathbb{N}}$ is bounded in $L^1(\Omega)$ in some finite dimensional subspace (of functions that are constant on every square that meet Ω and zero elsewhere). Thus we find a convergent subsequence $(i_k)_k$ such that $(u_{i_k}^r)_{k\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^n)$ for every $r \in \mathbb{Q}_+^*$, and with the uniform estimate of the previous corollary we get the result.

Proposition 3.13 (Embedding of BV)**.** *Let $u \in BV(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then*

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq V(u; \mathbb{R}^n)$$

Proof. When $u \in \mathcal{C}_c^1(\mathbb{R}^n, \mathbb{R})$, this is a consequence of the classical Gagliardo-Nirenberg inequality,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \mathrm{d}\mathscr{L}^n \leq \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i u| \mathrm{d}\mathscr{L}^n \right)^{\frac{1}{n-1}}$$

and the general case is obtained by convolution and cut-off. □

The constant here is not actually optimal, the optimal one being given by $u = 1_B$ where B is a ball.

3.3 Sets of finite perimeter

Definition 3.14. Let $E \subset \mathbb{R}^n$ measurable and Ω an open set, we define

$$\mathrm{Per}(E; \Omega) = \sup \left\{ \int_\Omega 1_E \mathrm{div}(\varphi) \mathrm{d}\mathscr{L}^n,\ \varphi \in \mathcal{C}_c^1(\Omega, B_1) \right\}$$

and we say that E has finite perimeter (in Ω) when this quantity is finite.

Notice that this is just a special case of BV functions of the form 1_E, in particular $\mathrm{Per}(E;\Omega) = V(1_E;\Omega) = |D1_E|(\Omega)$.

When E has finite perimeter, we will denote it μ_E the Radon measure $-D1_E$. Notice that for any Borel set A we may extend the definition of $\mathrm{Per}(E;\cdot)$ by

$$\mathrm{Per}(E;A) := |\mu_E|(A)$$

If E is a smooth open set then $\mu_E = \nu_E|\mu_E|$ for ν_E is its the outward normal vector of E (this is the reason for the minus sign in the definition of μ_E) and $|\mu_E| = \mathscr{H}^{n-1}\lfloor\partial E$. We will see however that for a general set, the topological boundary is too large and we need to define a notion of boundary more dependent on the measure μ_E.

Definition 3.15 (Boundaries)**.** Let E be a set of finite perimeter in Ω and $\mu_E = -D1_E$, we define the following three notions of boundary.

- According to the Besicovitch differentiation theorem, the limit $\lim_{\rho\to 0}\frac{\mu_E(B_{x,\rho})}{|\mu_E|(B_{x,\rho})}$ exists and has unit norm $|\mu_E|$-almost everywhere. We call $\partial^* E$ the set where it exists and has unit norm, and we write $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ its value. $\partial^* E$ is called the reduced boundary.
- We let $E^{(1/2)} = \{x \in \mathbb{R}^n : \Theta_n^*(E;x) = \Theta_{n*}(E;x) = \frac{1}{2}\}$ the points of density exactly $\frac{1}{2}$.
- We define the essential boundary $\partial^e E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$, where

$$E^{(0)} = \{x \in \mathbb{R}^n : \Theta_n^*(E;x) = 0\},\ E^{(1)} = \{x \in \mathbb{R}^n : \Theta_{n*}(E;x) = 1\}$$

An advantage of each of these notions is that they only depend on E almost everywhere. As we will see, $\partial^* E$ is a strong enough notion of boundary such that the blow-up of E around any point of the reduced boundary is a half-plane, we have an inclusion

$$\partial^* E \subset E^{(1/2)} \subset \partial^e E,$$

and $\mathscr{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0$ so all of these notions coincide up to a negligible set.

Proposition 3.16. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, then for any $x \in \partial^* E$, we have*

$$\frac{E-x}{r} \underset{r\to 0}{\longrightarrow} \{y : y \cdot \nu_E(x) < 0\} \text{ in } L^1_{loc}(\mathbb{R}^n)$$

as well as $\Theta^*_{n-1}(|\mu_E|;x) < \infty$.

Proof. Suppose $0 \in \partial^* E$, we assume without loss of generality that $\nu(0) = e_n$, and we write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for any $x \in \mathbb{R}^n$. We claim that $\mu_{E\cap B_r} = \mu_E\lfloor B_r + \mu_{B_r}\lfloor E$, the proof of this may be found in [72].

We first prove a control on the perimeter of E in B_r for any small enough r. Indeed, according to the previous claim one has $\mu_E(B_r) = -\mu_{B_r}(E)$. For any small enough r, $\frac{|\mu_E(B_r)|}{|\mu_E|(B_r)} \geq \frac{1}{2}$, so

$$|\mu_E|(B_r) \leq 2|\mu_E(B_r)| \leq 2|\mu_{B_R}(E)| \leq 2n\omega_n r^{n-1}$$

This implies the second conclusion, that $\Theta^*_{n-1}(|\mu_E|;x) < \infty$. Let now $m(r) = |E \cap B_r|$, then $m(r)$ is absolutely continuous with $m'(r) = \mathscr{H}^{n-1}(\partial B_r \cap E)$, and

$$m(r)^{\frac{n-1}{n}} \leq \mathrm{Per}(E \cap B_r) \leq 3\mathscr{H}^{n-1}(\partial B_r \cap E) = 3m'(r),$$

so integrating from 0 to r (small enough), which is possible since $m > 0$ on $(0,r)$, we get $|E\cap B_r|^{1/n} \geq \frac{r}{3n}$. Notice that the same reasonning works seamlessly on the complementary of E.

Let now $E_r = \frac{E}{r}$, then we have prove that $(E_r)_{r\to 0}$ is a sequence of sets with locally uniformly bounded perimeter, such that $|E_r \cap B_1|$ and $|E_r^c \cap B_1|$ are strictly bounded from below for any small

enough r. As a consequence, from any subsequence of radius converging to 0 we may re-extract a subsequence $r_i \to 0$ such that $E_{r_i} \underset{i\to\infty}{\longrightarrow} F$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ for some set F with locally finite perimeter, and $|F \cap B_1| \wedge |F^c \cap B_1| > 0$. It is then enough to prove that F is the half-plane $\{x_n < 0\}$.

We know $\mu_{E_{r_i}} \rightharpoonup \mu_F$ (by uniform bound on the perimeter). By using the fact that x is in the reduced boundary with normal vector e_n,

$$\text{Per}(F; B_R) \leq \liminf_{i\to\infty} \text{Per}(E_{r_i}; B_R) = \liminf_{i\to\infty} e_n \cdot \mu_{E_{r_i}}(B_R) = e_n \cdot \mu_F(B_R) \leq |\mu_F|(B_R),$$

so this is a chain of equality and

$$0 = |\mu_F|(B_R) - e_n \cdot \mu_F(B_R) = \int_{B_R} (1 - e_n \cdot \nu_F)\, d|\mu_F|.$$

as a consequence $\nu_F(x) = e_n$ for $|\mu_F|$-almost every $x \in \mathbb{R}^n$, or equivalently that $|\partial_i 1_F|(\mathbb{R}^n) = 0$ for $i = 1, \dots, n-1$, and that $\partial_n 1_F$ is a nonpositive measure. Consider $(\rho_\epsilon)_{\epsilon\to 0}$ a radial smooth mollifyer and $u_\epsilon := 1_F * \rho_\epsilon$, then u_ϵ only depend on x_n and is decreasing, and so does its limit. This implies that F is a half-plane $\{x : x_n < \alpha\}$ and clearly α must be 0, which concludes this proof. □

Corollary 3.17 (Structure of the boundary)**.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, then $\partial^* E$ is rectifiable and $\mu_E = \nu_E \mathscr{H}^{n-1} \lfloor \partial^* E$.*

Proof. The previous proposition implies that for any $x \in \partial^* E$, the sequence of measures $\frac{\mu_E(x+r\cdot)}{r^{n-1}}$ converges weakly to $\mathscr{H}^{n-1} \lfloor \nu_E(x)^\perp$. Using the rectifiability criteria by blow-up this implies the result. □

Proposition 3.18 (Identification of boundaries)**.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, then $\partial^e E$ is rectifiable and $\mathscr{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0$.*

Proof. using the formula (3.1) (note that it was for cubes instead of balls, but the proof does not change up to a constant) we know that for any ball $B_{x,r}$,

$$\frac{|E \cap B_{x,r}|}{|B_{x,r}|} \cdot \frac{|E^c \cap B_{x,r}|}{|B_{x,r}|} \leq \frac{c_n |\mu_E|(B_{x,r})}{r^{n-1}}$$

Since $|\mu_E| = \mathscr{H}^{n-1} \lfloor \partial^* E$, then for $\mathscr{H}^{n-1}$-almost every $x \notin \partial^* E$ we have $\Theta^*_{n-1}(\mu_E; x) = 0$, meaning (with the formula above) that $x \in E^{(0)} \cup E^{(1)}$. □

3.4 Unidimensional BV functions, structure theorem

In this whole section, I is a bounded interval on $\mathbb{R}$, and we study in particular the decomposition of BV functions on an interval.

When $u \in BV(I)$, then for any $t \in I$ the approximate right-limit and left-limit are well defined (since having two possible values would implies infinite variation), and we shall denote them $u^+(t)$ and $u^-(t)$. We will write $[u](t) = u^+(t) - u^-(t)$; note that $[u](t)$ is zero except for a possibly countable subset of I, since $V(u; I) \geq \sum_{t\in I} |[u](t)|$.

Proposition 3.19 (Decomposition of BV)**.** *Let $u \in BV(I)$, then there exists, up to a constant, a unique decomposition of u in three parts*

$$u = u^a + u^j + u^c$$

where $u^a \in W^{1,1}(I)$, $Du^j = \sum_{t\in I} [u](t)$, and u^c is continuous and Du^c is singular with respect to the Lebesgue measure.

This is simply an application of the Besicovitch differentiation theorem on the derivative Du with respect to the Lebesgue measure.

Definition 3.20. Let $u \in BV(I)$, we will write $Du^a = \nabla u \mathscr{L}^1$, and $J_u = \{t \in I : [u](t) \neq 0\}$. We say that u is in $SBV(I)$ when $Du^c = 0$, meaning that

$$Du = \nabla u \mathscr{L}^1 + \left(u^+ - u^-\right) \mathscr{H}^0 \lfloor J_u.$$

The space $SBV(I)$ is not closed for the L^1-induced topology (under bounded total variation) ; indeed the most well-known example of function with non-constant u^c is the Cantor staircase, which is obtained as a limit of (non-uniformly) increasing Lipschitz functions. Similarly, a piecewise constant function may approximate any Lipschitz function. As a consequence we need a stronger constraint on both the absolutely continuous part and the jump part to ensure that a sequence of such functions stay in SBV at the limit. We do not state the Ambrosio compactness theorem in its full generality, we refer to the classical reference [70] for this.

Theorem 3.21 (One-dimensional compactness)**.** *Let I a bounded open interval of $\mathbb{R}$, $(u_i)_{i\in\mathbb{N}}$ be a sequence in $SBV(I)$, let $p > 1$ and suppose that*

$$\limsup_{i\to\infty} \left(V(u_i, I) + \int_I |\nabla u_i|^p + \mathscr{H}^0(J_{u_i}) \right) < \infty,$$

and suppose that $u_i \underset{L^1(I)}{\longrightarrow} u \in BV(I)$. Then $u \in SBV(I)$ and

$$\nabla u_i \underset{i\to\infty}{\longrightarrow} \nabla u \text{ weakly in } L^p(I)$$
$$\mathscr{H}^0(J_u) \leq \liminf_{i\to\infty} \mathscr{H}^0(J_{u_i})$$

Proof. We decompose $u_i = c_i + u_i^a + u_i^j$ where $c_i \in \mathbb{R}$ converges to some limit $c \in \mathbb{R}$, $u_i^a \in W^{1,p}(I)$ and $u_i^j = \sum_{t\in J_{u_i}} [u_i](t)$, such that both u_i^a and u_i^j have zero average. Now we may suppose that u_i^a converges weakly in $W^{1,p}(I)$ to some limit u^a, and likewise with the uniform bound on the number of jump we get that u_i^j converges in $L^1(I)$ to some jump function u^j (such that the jumps of u^j are included in the Hausdorff limit of $(J_{u_i})_i$). Then $u = c + u^a + u^j$ is in $SBV(I)$ with the two required inequalities. □

3.5 Fine properties of BV functions

We let $\Omega \subset \mathbb{R}^n$ be an open set in all this section, for $n \geq 2$. Let us first mention this useful result that makes stronger link between sets of finite perimeter and BV functions.

Proposition 3.22 (Coarea formula for BV functions)**.** *Let $u \in BV(\Omega)$, then*

$$V(u;\Omega) = \int_{-\infty}^{+\infty} Per(\{u > t\};\Omega)\mathrm{d}t$$

Proof. One side ($\leq$) is obtained by direct application of Fatou's lemma on the definition of $V(\cdot;\Omega)$ and using the formula $u = \int_0^{+\infty} 1_{u>t} dt$. For the other side, we consider a convolution $u_\epsilon := \rho_\epsilon * u$ for a standard mollifyer $(\rho_\epsilon)_{\epsilon\to 0}$, and since the classic coarea formula is valid on (u_ϵ) we get

$$V(u;\Omega) = \lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} \mathrm{Per}(\{u_\epsilon > t\};\Omega)dt \geq \int_{-\infty}^{+\infty} \liminf_{\epsilon\to 0} \mathrm{Per}(\{u_\epsilon > t\};\Omega)dt \geq \int_{-\infty}^{+\infty} \mathrm{Per}(\{u > t\};\Omega)dt$$

where the last inequality is due to the fact that $\{u_\epsilon > t\} \underset{L^1(\Omega)}{\longrightarrow} \{u > t\}$ for every t such that $\{u = t\}$ has zero Lebesgue measure, which is $\mathscr{L}^1$-a.e. t. □

A consequence of this formula is that BV function may be uniformly approximated from below by combination of indicator of sets with finite perimeter. Indeed, let $\delta > 0$, for every $k \in \mathbb{Z}$ there is some $t_k \in [k\delta, (k+1)\delta)$ such that

$$\int_{k\delta}^{(k+1)\delta} \mathrm{Per}(\{u > t\}; \Omega) \mathrm{d}t \geq \delta \mathrm{Per}(\{u > t_k\}; \Omega)$$

We then let $u_\delta = \sum_{k\in\mathbb{N}}(t_k - t_{k-1})1_{u>t_k}$, meaning that $u_\delta = t_k$ on $\{t_k < u \leq t_{k+1}\}$. Then $V(u_\delta, \Omega) \leq 2V(u;\Omega)$ and $|u - u_\delta| \leq 2\delta$. This means that some properties of sets of finite perimeter directly transfer to BV functions with this.

We define the following singular and jump sets.

Definition 3.23 (Jump set)**.** Let $u \in BV(\Omega)$, we say $x \in \Omega$ is singular (for u), and we write $x \in S_u$, if and only if there is no $z \in \mathbb{R}$ such that

$$⨍_{B(x,r)} |u - z| \to 0$$

We define the jump point J_u to be the particular subset of points $x \in S_u$ where moreover there exists $\nu_u \in \mathbb{S}^{n-1}$, $u^+, u^- \in \mathbb{R}^n$ distincts, such that

$$u(x + r\cdot) \underset{r\to 0}{\longrightarrow} u^+ 1_{\{z:z\cdot\nu>0\}} + u^- 1_{\{z:z\cdot\nu<0\}} \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

The idea here is that J_u represents the points at which u changes from one value u^- to another u^+ along a flat hypersurface. What we see with the following two results is that in general $\mathscr{H}^{n-1}$-almost every point of S_u are of the form of J_u.

Theorem 3.24 (Structure of the singular set)**.** *Let $u \in BV(\Omega)$, then S_u is countably rectifiable.*

Proof. We let

$$Z = \left\{ x \in \Omega \text{ s.t. } ⨍_{B(x,r)} |u|^{\frac{n}{n-1}} \underset{r\to 0}{\longrightarrow} +\infty \right\}$$

By the BV coarea formula there is a countable dense subset D of $\mathbb{R}_+$ such that $\mathrm{Per}(\{u > t\}; \Omega) < \infty$ for all $t \in D$.

Claim 1: $S_u \setminus Z \subset \cup_{t\in D} \partial^e\{u > t\}$, where we remind that $\partial^e E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$.

Claim 2: $\mathscr{H}^{n-1}(Z) = 0$.
This implies the rectifiability of S_u, since the $\partial^e\{u > t\}$ are rectifiable.

- Proof of Claim 1: Let x that is neither in Z or $\cup_{t\in D}\partial^e\{u > t\}$, we prove that x is a Lebesgue point of u. Let $z = \sup\{t \in D : x \in \{u > t\}^{(1)}\}$ then for any $s < z < t$, $s, t \in D$, we have $z \in \{u > s\}^{(1)} \cap \{u > t\}^{(0)}$ by definition of x and z. Now let $\rho > 0$ be small enough such that $B_{x,\rho} \Subset \Omega$, then for any $\epsilon > 0$,

$$\begin{aligned}\rho^{-n} \int_{B_{x,\rho}} |u - z| &= \rho^{-n} \int_{B_{x,\rho}\cap\{|u-z|\geq\epsilon\}} |u - z| + \rho^{-n} \int_{B_{x,\rho}\cap\{|u-z|<\epsilon\}} |u - z| \\ &\leq \rho^{-n} |B_{x,\rho} \cap \{|u - z| \geq \epsilon\}|^{\frac{1}{n}} \left(\int_{B_{x,\rho}} |u - z|^{\frac{n}{n-1}} \right)^{1-\frac{1}{n}} + |B_1|\epsilon\end{aligned}$$

  which is less than $|B_1|\epsilon$ as $\rho \to 0$, and since ϵ is taken arbitrarily small this means that x is a Lebesgue point of u with value z.

- Proof of Claim 2: we will need the following lemma.

Lemma 3.25. *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of measurable subset of $\mathbb{R}^n$ such that $|E_i|, Per(E_i) \to 0$, then*

$$\mathscr{H}^{n-1}(\mathbb{R}^n \setminus \cup_i E_i^{(0)}) = 0.$$

Proof. Let $E \subset \mathbb{R}^n$ such that $|E|$ and $\mathrm{Per}(E)$ are finite, $\alpha \in (0, 1/2)$ and $\delta = \left(\frac{|E|}{\omega_n \alpha}\right)^{1/n}$, we prove that

$$\mathscr{H}^{n-1}_{2\delta}(\{x : \Theta^*_n(E, x) > \alpha\}) \leq \frac{C_n}{\alpha}\mathrm{Per}(E)$$

which is sufficient. For any x such that $\Theta^*_n(E, x) > \alpha$ we know there are arbitrarily small ρ such that $|E \cap B_{x,\rho}| > \alpha|B_\rho|$. By our choice of δ we also know that $|E \cap B_{x,\delta}| \leq |E| = \alpha|B_\delta|$, so in particular we find by intermediate value theorem a $\rho_x \in (0, \delta)$ such that

$$\frac{|E \cap B_{x,\rho_x}|}{\omega_n \rho_x^n} = \alpha$$

The Poincaré inequality applied to 1_E on $B_{x,\rho}$ gives

$$\left(1 - \frac{|E \cap B_{x,\rho}|}{|B_\rho|}\right)\frac{|E \cap B_{x,\rho}|}{|B_\rho|} \leq \frac{C_n \rho \mathrm{Per}(E; B_{x,\rho})}{|B_\rho|},$$

so in particular $\mathrm{Per}(E; B_{x,\rho_x}) \geq c_n \alpha \rho_x^{n-1}$. The balls $(B_{x,\rho_x})_{x:\Theta^*_n(E,x)>\alpha}$ cover the set $\{x : \Theta^*_n(E, x) > \alpha\}$ so we may extract a Besicovitch covering $(B_{x_i,\rho_{x_i}})$ (meaning a covering with at most ξ_n intersections for ξ_n depending only on the dimension), obtaining

$$\mathscr{H}^{n-1}_{2\delta}(\{x : \Theta^*_n(E, x) > \alpha\}) \leq \sum_i \omega_{n-1}\rho_{x_i}^{n-1} \leq \frac{\omega_{n-1}}{c_n \alpha}\sum_i \mathrm{Per}(E; B_{x_i,\rho_{x_i}}) \leq \frac{\xi_n \omega_{n-1}}{c_n \alpha}\mathrm{Per}(E)$$

□

Now that this lemma is proved, suppose without loss of generality that u is nonnegative and for each $k \in \mathbb{N}$, there is some $t_k \in [k, k+1)$ such that

$$\mathrm{Per}(\{u > t_k\}; \Omega) \leq \int_k^{k+1} \mathrm{Per}(\{u > t\}; \Omega)\mathrm{d}t$$

and so by integrability of u, the sets $(\{u > t_k\})_{k\in\mathbb{N}}$ verify the hypothesis of the lemma.

Let $D = \{x : \Theta^*_{n-1}(|Du|, x) = \infty\}$, then by Besicovitch covering argument $|Du| \geq +\infty\mathscr{H}^{n-1}\lfloor D$ and since $|Du|$ is a Radon measure, D must be $\mathscr{H}^{n-1}$-negligible. We prove that if $x \in \mathbb{R}^n$ is not of upper density 1 in all of the $\{u > t_k\}$ and not in D, then x in not is Z, which prove the claim.

The Poincaré inequality gives that

$$⨍_{B_{x,\rho}} |u - (u)_{x,\rho}|^{\frac{n}{n-1}} \leq C_n \left(\frac{|Du|(B_{x,\rho})}{\rho^{n-1}}\right)^{\frac{n}{n-1}}$$

And so for $x \notin D$ the right-hand side is bounded as $\rho \to 0$; to prove that $x \notin Z$ it is then enough to prove that $((u)_{x,\rho})_{\rho\to 0}$ is bounded. Suppose it is unbounded, meaning there is some subsequence $\rho_i \to 0$ such that $c_i := (u)_{x,\rho_i}$ diverges to infinity. Consider then $v_i(y) = u(x + \rho_i y) - c_i$, we know

$$\int_{B_1} v_i = 0,\ V(v_i, B_1) = \frac{V(u; B_{x,\rho_i})}{\rho_i^{n-1}} = O_{i\to\infty}(1) \text{ because } x \notin D,$$

so v_i is bounded in $BV(B_1)$ and by compactness converges in $L^1(B_1)$ and almost everywhere to some limit v that is finite almost everywhere. In particular, for any $k \in \mathbb{N}$,

$$\Theta^*_n(\{u > t_k\}, x) \geq \limsup_i \frac{|\{u > t_k\} \cap B_{x,\rho_i}|}{\omega_n \rho_i^n} \geq \limsup_i \omega_n^{-1}|\{v_i > t_k - c_i\} \cap B_1| = 1$$

where the last equality is because $v = \lim(v_i)$ is finite almost everywhere. As a consequence, x must be in $\cap_k(\mathbb{R}^n \setminus \{u > t_k\}^{(0)})$. This concludes the result.

□

Theorem 3.26 (Trace on rectifiable sets)**.** *Let $u \in BV(\Omega)$ and R a countable rectifiable set oriented by ν, then for $\mathscr{H}^{n-1}$-almost every $x \in R$ there exists $u^+_{|R}, u^-_{|R} \in \mathbb{R}$, such that*

$$u(x + r\cdot) \underset{r\to 0}{\longrightarrow} u^+_{|R} 1_{\{z : z\cdot\nu(x) > 0\}} + u^-_{|R} 1_{\{z : z\cdot\nu(x) < 0\}} \text{ in } L^1_{loc}(\mathbb{R}^n).$$

When $R = J_u$, we just write $u^\pm := u^\pm_{|J_u}$ and this is coherent with the notations of the definition of J_u.

Proof. Suppose first that $u = 1_E$ for some set E with finite perimeter. Then we know from the structure theorem that $\mathscr{H}^{n-1}$-a.e. point x is either in $E^{(0)}$, $E^{(1)}$, or $\partial^* E$. Moreover, $\nu_E(x) = \pm\nu(x)$ for $\mathscr{H}^{n-1}$ almost every $x \in R \cap \partial^* E$. Such for $\mathscr{H}^{n-1}$-almost every x in R we are in one of the three following cases:

- If $x \in E^{(0)}$, $u^\pm_{|R}(x) = 0$ works.
- If $x \in E^{(1)}$, $u^\pm_{|R}(x) = 1$ works.
- If $x \in \partial^* E$ with $\nu_E = \nu$ (resp $-\nu$), then we let $u^\pm_{|R} = u^\pm$ (resp $u^\mp$)

Now, for a general $u \in BV(\Omega)$, that we suppose to be bounded without loss of generality, with the BV coarea formula we may approximate it uniformly by a function of the forme $\sum_{i=1}^N t_i 1_{E_i}$ for sets of finite perimeter E_i, and the definition of $u^\pm_{|R}$ follows.

□

As a corollary of the two previous result we directly obtain the following.

Theorem 3.27 (Rectifiability of the singular set)**.** *Let $u \in BV(\Omega)$, then $\mathscr{H}^{n-1}(S_u \setminus J_u) = 0$.*

Proof. S_u is countably rectifiable so for $\mathscr{H}^{n-1}$-almost every $x \in S_u$ there exists $u^+_{|S_u}(x), u^-_{|S_u}(x) \in R$ as previously. However $u^+_{|S_u}(x) = u^-_{|S_u}(x)$ implies that $x \notin S_u$, so necessarily $u^+_{|S_u}(x) \neq u^-_{|S_u}(x)$ and $x \in J_u$. □

Definition 3.28 (Decomposition of BV functions)**.** Let $u \in BV(\Omega)$, we write

$$Du = D^a u + D^j u + D^c u$$

where

$$\begin{aligned} &D^a u = \nabla u \mathscr{L}^n \text{ where } \nabla u \in L^1(\Omega) \text{ is the absolutely continuous part of } Du, \\ &D^j u = (u^+ - u^-)\nu_u \mathscr{H}^{n-1} \lfloor J_u \text{ is the jump part}, \\ &D^c u = Du - D^a u - D^j u \text{ is the Cantor part.} \end{aligned}$$

$D^c u$ verifies that $D^c u \perp \mathscr{L}^n$, and $|Du^c|(R) = 0$ for any rectifiable R. The terminology "Cantor part" originates from the fact that the most classical example of a function with non-zero Cantor part is the Cantor staircase. A corollary of the trace theorem is that $D^c u$ is purely unrectifiable, meaning that for any rectifiable R we have $|D^c u|(R) = 0$. In this sense it is foreign to both $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$; this is for instance what happens for the Cantor staircase, for which the derivative is a measure that has support on the Cantor set (which is $\mathscr{L}^1$-negligible) with no atom (so singular to $\mathscr{H}^0$).

Definition 3.29. Let $u \in BV(\Omega)$, we say u is in $SBV(\Omega)$ when $D^c u = 0$.

3.6 Compactness of SBV functions

This space SBV is the one we will use in most of the following chapters. It is a non-trivial subspace of BV (because of, says, the Cantor staircase) but notice that it is not closed for the L^1 topology: the uniform approximation of any BV function with combinations of indicators of finite perimeter sets - which are actually in SBV with $D^c u = D^a u = 0$ - proves this. Thus for a compactness theorem we will ask a slightly stronger control of both the gradient and the jump ; it is actually the same result as in $SBV(\mathbb{R})$ due to the following result that identify the decomposition of Du with the decomposition of its one-dimensional sections.

Definition 3.30. Let $u : \Omega \to \mathbb{R}$, $y \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{n-1}$, we call Ω_y^ξ the open set of $\mathbb{R}$ such that

$$\Omega \cap (y + \mathbb{R}\xi) = y + \Omega_y^\xi \xi$$

And we define the slice of u as

$$u_y^\xi : t \in \Omega_y^\xi \mapsto u(y + t\xi)$$

Theorem 3.31 (Slicing)**.** *Let $u \in BV(\Omega)$, $\xi \in \mathbb{S}^{n-1}$, then for $\mathscr{H}^{n-1}$-almost every $y \in \xi^\perp$ we have $u_y^\xi \in BV(\Omega_y^\xi)$ and for every $\varphi \in \mathcal{C}^0(\Omega, \mathbb{R})$,*

$$\int_\Omega \varphi(x) d(\xi \cdot D^a u)(x) = \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi(y + t\xi) dD^a u_y^\xi(t) \right) d\mathscr{H}^{n-1}(y)$$
$$\int_\Omega \varphi(x) d(\xi \cdot D^j u)(x) = \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi(y + t\xi) dD^j u_y^\xi(t) \right) d\mathscr{H}^{n-1}(y)$$
$$\int_\Omega \varphi(x) d(\xi \cdot D^c u)(x) = \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi(y + t\xi) dD^c u_y^\xi(t) \right) d\mathscr{H}^{n-1}(y)$$

For this we refer to [70, Th. 3.107, 3.108].

An important corollary is that if for a basis $(\xi_i)_{i=1,\dots,n}$ of $\mathbb{R}^n$, and for $\mathscr{H}^{n-1}$-almost every $y \in \xi_i^\perp$, $u_y^{\xi_i} \in SBV(\Omega_y^{\xi_i})$, then $u \in SBV(\Omega)$. With this we can prove the very useful compactness theorem.

Theorem 3.32 (Compactness of SBV functions)**.** *Let Ω be a bounded open set of $\mathbb{R}^n$, $(u_i)_{i\in\mathbb{N}}$ be a sequence in $SBV(\Omega))$, let $p > 1$ and suppose that*

$$\limsup_{i\to\infty} \left(V(u_i, \Omega) + \int_\Omega |\nabla u_i|^p \mathrm{d}\mathscr{L}^n + \mathscr{H}^{n-1}(J_{u_i}) \right) < \infty,$$

and suppose that $u_i \underset{L^1(\Omega)}{\longrightarrow} u \in BV(\Omega)$. Then $u \in SBV(\Omega)$ and

$$\nabla u_i \underset{i\to\infty}{\longrightarrow} \nabla u \text{ weakly in } L^p(\Omega)$$
$$\mathscr{H}^{n-1}(J_u) \le \liminf_{i\to\infty} \mathscr{H}^{n-1}(J_{u_i})$$

Proof. We know $u \in BV(\Omega)$ and then we use the slicing lemma along a basis (ξ_i) and the one-dimensional compactness theorem for $\mathscr{H}^{n-1}$-almost every section; this tells us that almost every section is in SBV, thus the full function is in SBV. The lower semicontinuity is obtained by a section argument as well: for any $\xi \in \mathbb{S}^{n-1}$, the semi-continuity of each term in the one-dimensional case gives the lower semi-continuity of

$$\int_\Omega |\xi \cdot \nabla u|^p \mathrm{d}\mathscr{L}^n \text{ and } \int_{J_u} |\xi \cdot \nu_u| \mathrm{d}\mathscr{H}^{n-1}.$$

In particular the first one implies the lower semi-continuity of the gradient term. More precisely the weak convergence $\nabla u_i \hookrightarrow \nabla u$ is a consequence of the weak convergence of each $e_k \cdot \nabla u_i$ to $e_k \cdot \nabla u$.

Now for the lower semi-continuity of the second term ; let $\epsilon > 0$, for every Lebesgue point $x \in J_u$ for ν_u and for any arbitrarily small $r > 0$,

$$\int_{B_{x,r} \cap J_u} \nu_u(x) \cdot \nu_u(y) \mathrm{d}\mathscr{H}^{n-1}(y) \geq (1-\epsilon)\mathscr{H}^{n-1}(B_{x,r} \cap J_u)$$

By Vitali covering lemma (see for instance [72, cor. 5.5]) we deduce a covering of Ω by balls $(B_k)_{k\in\mathbb{N}}$ of arbitrarily small radius, of vectors $\xi_k \in \mathbb{S}^{n-1}$, such that $\mathscr{H}^{n-1}(J_u \setminus \cup_k B_k) = 0$ and

$$(1-\epsilon)\mathscr{H}^{n-1}(J_u) \leq \sum_k \int_{B_k \cap J_u} \xi_k \cdot \nu_u \mathrm{d}\mathscr{H}^{n-1} \leq \liminf_{i\to\infty} \sum_k \int_{B_k \cap J_{u_i}} \xi_k \cdot \nu_{u_i} \mathrm{d}\mathscr{H}^{n-1} \leq \liminf_{i\to\infty} \mathscr{H}^{n-1}(J_{u_i}),$$

and we get the result as $\epsilon \to 0$. □

3.7 A quick overview of SBD functions

Bounded deformation functions (abbreviated as BD) are a vectorial generalization of BV functions for which we only have control on the symmetric gradient. This is something that arose from elastic material theory in which one only has control on the deformation tensor, meaning the symmetrised gradient.

Definition 3.33. Let Ω be an open set of $\mathbb{R}^n$ and $u \in L^1(\Omega, \mathbb{R}^n)$, we say $u \in BD(\Omega)$ when

$$Eu := \frac{Du + (Du)^*}{2}$$

is a S_n-valued Radon measure, where S_n is the set of real $n \times n$ symmetric matrices..

We can define the singular set S_u and the jump set J_u the same way we do for BV functions, and we let $SBD(\Omega)$ be the space of function $u \in BD(\Omega)$ such that

$$Eu = e(u)\mathscr{L}^n + (u^+ - u^-) \odot \nu_u \mathscr{H}^{n-1} \lfloor J_u,$$

where $e(u) \in L^2(\Omega)$ and $(a \odot b)_{ij} = \frac{a_i b_j + a_j b_i}{2}$. While there are analogies between the properties of SBD and SBV functions, there are also some knowledge gap ; for instance it is unknown whether for $u \in BD(\Omega)$ we have $\mathscr{H}^{n-1}(S_u \setminus J_u) = 0$. Let us state without proof basic properties of SBD functions, mainly taken from [69] and [71].

The slices of BD functions are done differently ; for any $y \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{n-1}$, we still define u_y^ξ as previously, however there is no reason its components should have bounded variation. Instead we let

$$\hat{u}_y^\xi(t) = \xi \cdot u(y + t\xi),$$

defined for $t \in \Omega_y^\xi$. Then a similar theorem as its equivalent for BV functions holds.

Theorem 3.34. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in SBD(\Omega)$, and let $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Then for $\mathscr{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have*

$$\hat{u}_y^\xi \in SBV(\Omega_y^\xi)$$

with $\nabla \hat{u}_y^\xi(t) = (e(u)\xi \cdot \xi)(y + t\xi)$ and $J_{\hat{u}_y^\xi} = (J_u^\xi)_y^\xi$, where $J_u^\xi = \{x \in J_u : \xi \cdot (u^+(x) - u^-(x)) \neq 0\}$.

Similarly, we have the following useful compactness theorem.

Theorem 3.35. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with a Lipschitz boundary, and let $(u_i)_{i\in\mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that*

$$\sup_i \left[|Eu_i|(\Omega) + \|u_i\|_{L^1(\Omega;\mathbb{R}^n)} + \|e(u_i)\|_{L^p(\Omega;\mathrm{M}^n_{S_n})} + \mathscr{H}^{n-1}(J_{u_i}) \right] < +\infty$$

for some $p > 1$. Then there exists $u \in SBD(\Omega)$ and a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ such that

$$u_{i_k} \to u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^n),$$

$$e(u_{i_k}) \rightharpoonup e(u) \qquad \text{weakly in } L^p(\Omega;\mathrm{M}^n_{\mathrm{sym}}),$$

and

$$\mathscr{H}^{n-1}(J_u) \leq \liminf_{k\to+\infty} \mathscr{H}^{n-1}(J_{u_{i_k}}).$$

Bibliography

[69] Luigi Ambrosio, Alessandra Coscia, and Gianni Dal Maso. Fine properties of functions with bounded deformation. *Arch. Rational Mech. Anal.*, 139(3):201–238, 1997.

[70] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[71] G. Bellettini, A. Coscia, and G. Dal Maso. Compactness and lower semicontinuity properties in SBD(Ω). *Math. Z.*, 228(2):337–351, 1998.

[72] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

Chapter 4

Degenerate free discontinuity problems and spectral inequalities in quantitative form

This is a joint work with Dorin Bucur and Alessandro Giacomini.

Contents

4.1 Introduction 67
4.2 Introduction of the new functional, main results and strategy of the proofs . . 70
4.3 Analysis of the analytic-geometric functional 72
4.3.1 Preparatory results 73
4.3.2 Nondegeneracy of the minimizers 75
4.3.3 Closedness of the jump set 77
4.3.4 The optimal function is radially symmetric 79
4.3.5 Existence of minimizers 82
4.4 The ball minimizes the geometric functional 87
4.5 Proof of Theorems 4.1 and 4.2 87

4.1 Introduction

Free discontinuity problems emerged in the context of the analysis of the Mumford-Shah functional, later on around different crack propagation models of Francfort and Marigo type and more recently around shape optimization problems of Robin type. The common feature of all those problems is, roughly speaking, the minimization of a sum between an energy term corresponding to a certain state equation issued from the model and of some more geometric terms involving the volume of the domain of the PDE, the length of the jump set or some more complex jump energy. A formal example could be written as

$$\min\{E(u) + \mathcal{H}^{n-1}(J_u) : u \in SBV_{loc}(\mathbb{R}^n)\}.$$

The balance between the energy $E(\cdot)$ of the PDE and the geometric term (above the length of the jump set) is the key phenomenon leading to a solution of the free discontinuity problem.

The main focus of this paper is to introduce and analyse a new analytic-geometric functional involving both an energy of a PDE and the length of the jump set, in which the geometric term appears with negative sign. The exact description is given in the next section but, formally, this could be written as

$$\min\{E(u) - \mathcal{H}^{n-1}(J_u) : u \in SBV_{loc}(\mathbb{R}^n)\}.$$

Of course, at a first sight this may appear surprising! Presumably, the negative sign would lead to non-existence of a solution and ill posedness. However, this is not always the case, as the presence

of the jump energy with a negative sign can sometimes be balanced by the energy of the PDE. As we will show in the next section, this is the case if the jump set acts as an obstacle and the energy contains some mass of the state function on the jump set. Robin boundary conditions can be suitably adapted to play this role. Ultimately, this leads to a new (degenerate) problem which takes the form of a free discontinuity problem above the obstacle and of a free boundary problem at the level of the obstacle.

This kind of problems pops up naturally in the context of searching quantitative forms of spectral isoperimetric inequalities for eigenvalues of nonlinear Robin Laplacian problems, in which the ball is expected to be a solution. Proving that the minimizer of the associated analytic-geometric functional is the ball, gives straight away a spectral isoperimetric inequality in a quantitative form.

In order to introduce the functional, we recall our objectives.

The context of quantitative isoperimetric inequalities. The sharp quantitative isoperimetric inequality proved by Fusco, Maggi and Pratelli in 2008 (see [87]) reads

$$|\Omega|^{\frac{1-n}{n}}\mathrm{Per}(\Omega) - |B|^{\frac{1-n}{n}}\mathrm{Per}(B) \geq C(n)\,\mathcal{A}(\Omega)^2, \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a measurable set, B is a ball of the same volume as Ω, $\mathrm{Per}(\Omega)$ is the generalized perimeter of Ω and

$$\mathcal{A}(\Omega) = \inf\left\{\frac{|\Omega \triangle B|}{|\Omega|} : B \subset \mathbb{R}^n, |B| = |\Omega|\right\},$$

is the Fraenkel asymmetry.

In the vein of this inequality, in the last decade intensive research was carried to obtain quantitative versions of some classical spectral inequalities, like Faber-Krahn, Szegö-Weinberger, Saint-Venant, Weinstock and many others. We refer the reader to the recent survey by Brasco and De Philippis [75] for an overview of the topic.

In [78], Brasco and Pratelli prove a sharp quantitative form for the Szegö-Weinberger inequality

$$|B|^{2/N}\mu_1(B) - |\Omega|^{2/n}\mu_1(\Omega) \geq C(n)\,\mathcal{A}(\Omega)^2,$$

and in [76] Brasco, De Philippis and Ruffini found a similar quantitative form of the Brock-Weinstock inequality. The common feature of both results is that the ball corresponds to a *maximal* value. Loosely speaking, the strategy to prove such an inequality relies on studying some weighted form of (4.1), via a suitable choice of test functions.

Spectral inequalities where the ball is *minimal*, like the Faber-Krahn inequality for the Dirichlet Laplacian, requires a completely different approach, since the use of fixed test functions is not anymore useful. The first results on the quantitative form of the Faber-Krahn inequalities were obtained by Melas [90] and Hansen and Nadirshvili [89] for simply connected sets in dimension 2 and convex sets in $\mathbb{R}^n$, but the complete proof of the sharp form of the quantitative inequality was given only in 2015 by Brasco, De Philippis and Velichkov [77]. A fundamental idea in their proof is to use a selection principle, in the spirit of Cicalese and Leonardi [85], which, roughly speaking, reduces the class of sets Ω for which the inequality has to be proved to a much smaller one, consisting on smooth, small graph perturbations of the ball which can be handled by local perturbation arguments. The selection of those sets is done by solving a suitable *auxiliary free boundary problem*; this part concentrates the most of the technicalities. Following the same strategy, nonlinear eigenvalues were discussed by Fusco and Zhang in [88].

The purpose of this paper is to get quantitative isoperimetric inequalities for the best constants of Sobolev-Poincaré inequalities with trace terms. Those constants are fundamental semilinear eigenvalues of the Laplace operator with Robin boundary conditions and can be expressed by minimization of suitable Rayleigh quotients. Our objective could be compared to the quantitative inequalities of Faber-Krahn type obtained for Dirichlet boundary conditions in [77] and [88], but from a technical point of view the solution is completely different.

Quantitative spectral inequalities for the Robin Laplacian. Let $\beta > 0$. For every bounded,

open Lipschitz set $\Omega \subseteq \mathbb{R}^n$ and for every $q \in [1, \frac{2n}{n-1})$ one defines

$$\lambda_q(\Omega) = \inf_{u \in H^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 d\mathscr{L}^n + \int_{\partial\Omega} \beta u^2 d\mathscr{H}^{n-1}}{\left(\int_\Omega |u|^q d\mathscr{L}^n\right)^{2/q}}. \tag{4.2}$$

Our objective is to prove that

$$\lambda_q(\Omega) - \lambda_q(B) \geq C\, \mathcal{A}(\Omega)^2 \tag{4.3}$$

where the constant $C > 0$ depends on n, β, q and $|\Omega|$, but not on Ω.

The non-quantitative version of (4.3)*: the case* $C = 0$. Before proving (4.3) in its quantitative form, with $C > 0$, it is convenient to recall that the inequality is true with $C = 0$ for every $q \in [1, 2]$. The minimality of the ball among all Lipschitz sets of the same volume for the first Robin eigenvalue of the Laplacian (i.e. $q = 2$)

$$\lambda_2(\Omega) - \lambda_2(B) \geq 0 \tag{4.4}$$

was proved in two steps, by Bossel in $\mathbb{R}^2$ in 1986 (see [74]) and by Daners in $\mathbb{R}^n$ in 2006 (see [86]). The proofs are quite involved and definitely require new ideas with respect to the Faber-Krahn inequality, namely the analysis of the so called *H-function*. As we do not use this function here and because it is quite technical, we shall not detail it here (the reader is referred to [86]). Nevertheless, it is important to say that intensive efforts were done to build similar H-functions for other values of $q \neq 2$, in particular for the special case $q = 1$ corresponding to the torsional rigidity, with the objective to extend the Saint-Venant inequality. Up to now, they were not successful and it is likely that such an H-function may not exist, so that a proof similar to Bossel-Daners in the case $q \neq 2$ cannot be produced. However, the inequality $\lambda_q(\Omega) - \lambda_q(B) \geq 0$ has been proved, using a different strategy, for any $q \in [1, 2]$ in [81] (see also [80]), while for $q \in (2, \frac{2n}{n-1})$ it has been proved in a slightly weaker form. The proof is based on a free discontinuity approach in which the inequality is seen as a minimization problem in the class of special functions of bounded variation in $\mathbb{R}^n$ (see Section 4.3 below).

The quantitative version of (4.3)*: the case* $C > 0$. Coming back to the quantitative form (4.3), in [79] the result was proved for $q = 2$, *only*. The reason was of technical nature. Precisely, the proof makes crucial use of the H-function, available only for $q = 2$. Indeed, there are two steps in the proof ([79]): the first step is based on a deeper analysis of the H-function of Bossel and Daners which led to the *intermediate* inequality

$$\lambda_2(\Omega) - \lambda_2(B) \geq \frac{\beta}{2} \inf_{x \in \Omega} u^2(x)(\mathrm{Per}(\Omega) - \mathrm{Per}(B)). \tag{4.5}$$

By itself, this inequality is interesting and already quantitative, but not uniform, as the difference of the perimeters on the right hand side is multiplied by the infimum of an L^2-normalized eigenfunction u, which depends on Ω. In a second step, one uses the selection principle to replace Ω by a new set which, roughly speaking, has lower eigenvalue, comparable Fraenkel asymmetry and a controlled, uniform, lower bound of the eigenfunction. The new set is build as a minimizer of a suitable *auxiliary free discontinuity problem*. This last step recalls both the strategy of Cicalese and Leonardi for the quantitative isoperimetric inequality and the one of Brasco, De Philippis and Velichkov for the first Dirichlet eigenvalue. The difference is however fundamental, as one has to solve a free discontinuity problem with a completely different objective. Indeed, one aims to compare a general set with another set, with a comparable Fraenkel asymmetry and lower eigenvalue, for which the lower bound of the state function is controlled from below (in order to use the intermediate inequality of Step 1). Meanwhile, in [85] and [77] the solutions of the associated free boundary problems had the objective to compare a general set with a set which is graph over the ball (in order to use second order differential inequalities).

A new functional to handle quantitative inequalities. The main purpose of the paper is to obtain the quantitative inequality (4.3). While the selection principle in association with the auxiliary free discontinuity problem can be extended to the case $1 \leq q < 2$, it turns out that the main difficulty is to prove an *intermediate* inequality similar to (4.5). Indeed, the absence of H-functions requires a completely new strategy.

The key idea is to introduce a new analytic-geometric functional involving both the energy of an obstacle problem and geometric terms. Precisely, we add a perimeter term with *negative sign*, which may appear surprising for a minimization problem. However, this term is balanced by the obstacle energy. Indeed, the PDE and the geometric terms interact in the minimization process, which can be carried out in the framework of free discontinuity problems. We prove that the minimizer corresponds to a ball and, quite directly, this fact provides the intermediate quantitative inequality.

4.2 Introduction of the new functional, main results and strategy of the proofs

Let $q \in [1,2)$ and $\beta > 0$ be given. Let Ω be a bounded, open, Lipschitz set and B a ball such that $|B| = |\Omega|$. Instead of working with the functionnal $\lambda_q(\Omega)$, we work with the following: for all $u \in H^1(\Omega), u \geq 0$, we define

$$E(u;\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\mathscr{L}^n + \frac{\beta}{2}\int_{\partial\Omega} u^2 d\mathscr{H}^{n-1} - \frac{1}{q}\int_\Omega u^q d\mathscr{L}^n,$$

and

$$E(\Omega) = \min\{E(u;\Omega) : u \in H^1(\Omega), u \geq 0\}.$$

$E(\Omega)$ and $\lambda_q(\Omega)$ are linked by the relation:

$$E(\Omega) = \frac{q-2}{2q}\lambda_q(\Omega)^{\frac{q}{q-2}}, \tag{4.6}$$

Here are the main results of the paper.

Theorem 4.1. *For every, Ω, β and q as above, and let u_Ω be a minimizer of $E(\cdot;\Omega)$, then the intermediate inequality*

$$E(\Omega) - E(B) \geq \frac{\beta}{2}\left(\inf_{x\in\Omega} u_\Omega(x)\right)^2 (Per(\Omega) - Per(B)) \tag{4.7}$$

holds true.

This intermediate inequality may have its own interest although it is not uniform. Using the relation between E and λ_q, we obtain the following inequality on $\lambda_q(\Omega)$

$$\frac{2-q}{2q}\left(\left(\frac{\lambda_q(\Omega)}{\lambda_q(B)}\right)^{\frac{q}{2-q}} - 1\right)\lambda_q(\Omega) \geq \frac{\beta}{2}\left(\inf_{x\in\Omega} u(x)\right)^2 (\mathrm{Per}(\Omega) - \mathrm{Per}(B)),$$

where u is a minimizer of the Rayleigh quotient (4.2) that is normalized in L^q. Notice that when $q \to 2$, the left-hand side diverges while the right-hand side converges; we do not recover (4.5) with this. However, we still obtain the following.

Theorem 4.2. *For every, Ω, β and q as above:*

$$\lambda_q(\Omega) - \lambda_q(B) \geq C\mathcal{A}(\Omega)^2,$$

where $C > 0$ depends on n, β, q and $|\Omega|$, only.

In fact, both results are proved in a more general framework than stated above. They take the form of a Sobolev-Poincaré inequality with trace terms in SBV (see Theorem 4.15 in Section 4.5) with improved constant. However, as most readers are interested only by the classical setting, we prefer to present our result for Lipschitz sets, and push technicalities in the second part of the paper.

To obtain the (intermediate) inequality in Theorem 4.1, we will study a different problem that depends on a parameter $c \geq 0$. For every $c \geq 0$, for any nonnegative $u \in H^1(\Omega)$, we set

$$\begin{aligned} E^c(u;\Omega) &= E(c+u;\Omega) - \frac{\beta}{2}c^2\mathrm{Per}(\Omega) + \frac{c^q}{q}|\Omega| \\ E^c(\Omega) &= \min\{E^c(u;\Omega) : u \in H^1(\Omega), u \geq 0\} \\ &= \min\{E(u;\Omega) : u \in H^1(\Omega), u \geq c\} - \frac{\beta}{2}c^2\mathrm{Per}(\Omega) + \frac{c^q}{q}|\Omega|. \end{aligned}$$

This functional involves both an obstacle problem and geometric terms. When we minimize $E^c(\Omega)$ among sets of constant measure, the perimeter term, coming with negative sign, will interact with the solution of the obstacle problem, while the measure part does not play any role. At fixed Ω, the geometric terms do not play any role in the obstacle problem.

Clearly, for every $c \geq 0$

$$E^c(\Omega) \geq E(\Omega) - \frac{\beta}{2}c^2\mathrm{Per}(\Omega) + \frac{c^q}{q}|\Omega|. \tag{4.8}$$

If $0 \leq c \leq \inf_\Omega u_\Omega$ then the equality sign occurs in (4.8) since $u_\Omega - c$ is also solution of the obstacle problem. If $c > \inf_\Omega u_\Omega$ then the solution of obstacle problem is different from $u_\Omega - c$ and the inequality is strict. As an example, with $q = 1$ and $\Omega = B_R$ (the ball of radius R), the minimizer u of $E^c(\cdot; B_R)$ takes the form:

$$u(x) = \left(\frac{R}{n\beta} - c\right)_+ + \frac{R^2 - |x|^2}{2n},$$

and

$$E^c(B_R) = -\frac{|B_R|}{2}\left(\left(\frac{R}{n\beta} - c\right)_+ + \frac{R^2}{n(n+2)}\right).$$

The strategy to prove Theorems 4.1 and 4.2 is based on the following steps.

Step 1. Minimization of $\Omega \mapsto E^c(\Omega)$. We prove that the ball minimizes E^c, i.e.

$$E^c(\Omega) \geq E^c(B) \qquad \text{where } |\Omega| = |B|. \tag{4.9}$$

Before describing how we do it, we point out that (4.9) leads quite directly to the intermediate quantitative inequality (4.7) in Theorem 4.1. Indeed, it is enough take $c = \inf_\Omega u_\Omega$ in (4.9) and use (4.8) to get

$$E(\Omega) - E(B) \geq \frac{\beta}{2}\left(\inf u_\Omega\right)^2\left(\mathrm{Per}(\Omega) - \mathrm{Per}(B)\right). \tag{4.10}$$

The proof of (4.9) requires the most of our work, and is based the following arguments.

- We naturally relax the original shape optimization problem

  $$\min\{E^c(\Omega) : \Omega \subseteq \mathbb{R}^n, |\Omega| = m\},$$

  as a new free discontinuity problem in the space of special functions of bounded variation (see [80, 81] and Section 4.3 below). Precisely, we consider

  $$\min\{\mathcal{E}^c(v) : v \in SBV^{1/2}(\mathbb{R}^n), |\{v > 0\}| = m\},$$

  where

  $$\mathcal{E}^c(v) = \frac{1}{2}\int_{\mathbb{R}^n}|\nabla v|^2 d\mathscr{L}^n + \frac{\beta}{2}\int_{J_v}\left[(\underline{v}^2 + 2c\underline{v}) + (\overline{v}^2 + 2c\overline{v})\right]d\mathscr{H}^{n-1} - \int_{\mathbb{R}^n}\left(\frac{(c+v)^q - c^q}{q}\right)d\mathscr{L}^n$$

  is chosen such that, if v is in $H^1(\Omega)$ with $v \geq 0$ on Ω then, when extended by 0 outside Ω, we have

  $$\mathcal{E}^c(v) = E^c(v;\Omega) = E(c+v;\Omega) - \frac{\beta}{2}c^2\mathrm{Per}(\Omega) + \frac{c^q}{q}|\Omega|.$$

  In particular, for $c = 0$ we write $\mathcal{E} = \mathcal{E}^0$ and for any such function u we have $\mathcal{E}^0(u) = E(u,\Omega)$.

- The presence of the perimeter term with negative sign leads to a critical behavior of the boundary energy of the solution near the contact with the obstacle c. This is managed by approximation of the boundary energy: the terms of the form

$$\int_{J_v} (v^2 + 2cv) d\mathscr{H}^{n-1} \qquad \text{are replaced by} \qquad \int_{J_v} (v^2 + 2cv^{1+\varepsilon}) d\mathscr{H}^{n-1}$$

  for $\varepsilon > 0$, small (see Definition 4.3).

- We prove that the minimizer of the approximating functional is a radial function with support on a ball. First, we study qualitative properties of minimizers (non-degeneracy, closedness of the jump set, radial symmetry) and, second, we show the existence of a solution. The approximation of the jump terms involving the parameter ϵ is in particular fundamental for the nondegeneracy result (see Lemma 4.8).

- Pass to the limit $\varepsilon \to 0$ and get that the minimizer of $\mathcal{E}^c$ is a radial function with support on a ball.

Step 2. Use of the selection principle to control uniformly $\inf_{x\in\Omega} u_\Omega$**.** The intermediate inequality (4.10) together with the quantitative isoperimetric inequality leads to

$$E(\Omega) - E(B) \geq \frac{\beta}{2} C_n \left(\inf_{\Omega} u_\Omega \right)^2 \mathcal{A}(\Omega)^2.$$

This quantitative inequality is not uniform in Ω since the right hand side is multiplied by $\inf_\Omega u_\Omega$. We regularize Ω by replacing it with Ω^{opt}, a minimizer of

$$\omega \mapsto E(\omega) + k|\omega|$$

among all $\omega \subset \Omega$ for some small enough $k > 0$. Following the main lines of [79], we prove that $\inf_{x\in\Omega^{\text{opt}}} u_{\Omega^{\text{opt}}}(x) \geq \alpha > 0$ where α depends on n, β, q and $|\Omega|$, while $\mathcal{A}(\Omega^{\text{opt}})$ is comparable to $\mathcal{A}(\Omega)$. This will conclude the proof.

The paper is organized as follows. In Sections 4.3 we study the minimization of the geometric functional $\Omega \mapsto E^c(\Omega)$, by relaxation in SBV and approximation. In Section 4.4 we prove that the minimizer corresponds to a ball. These two sections concentrate most of the technicalities of the paper. In the last section we prove Theorems 4.1 and 4.2.

It should be noted that for the quantitative inequality with Dirichlet boundary condition, in [77] the authors reduced their study to the sole study of the torsion functional $\Omega \mapsto \inf_{u\in H^1_0(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - u \right) d\mathscr{L}^n$ (corresponding to the case $q = 1$). This is due to a hierarchy of the eigenvalues relying on the Kohler-Jobin inequality (see [75, Chapter 7, Section 7.8.1]). To our knowledge, there is no such inequality with Robin boundary conditions, the reason for which we have to directly work on the general case.

4.3 Analysis of the analytic-geometric functional

In this section we study the minimization of $\Omega \mapsto E^c(\Omega)$ in the class of open, bounded, Lipschitz sets of measure m. For that purpose, we introduce the relaxed form of the functional in the space of special functions of bounded variation. We refer the reader to [73] for an introduction to the SBV space, as subspace in $BV(\mathbb{R}^n)$. Below, we denote by Du the distributional gradient of u and recall that

$$SBV(\mathbb{R}^n) = \{u \in BV(\mathbb{R}^n) : Du \text{ is absolutely continuous with respect to } dx + \mathscr{H}^{n-1}\lfloor_{J_u}\}.$$

The following space was introduced in [80],

$$SBV^{1/2}(\mathbb{R}^n) = \{u \in BV_{loc}(\mathbb{R}^n) : u \geq 0, u^2 \in SBV(\mathbb{R}^n)\}.$$

We refer to [80, 81] for the main properties of $SBV^{1/2}(\mathbb{R}^n)$. In particular, we recall the following Poincaré inequality with trace term proved in [81] for $q \in [1,2]$; for all $u \in SBV^{1/2}(\mathbb{R}^n)$, $|\{u>0\}| \leq m$:

$$\lambda_q(B^m)\left(\int_{\mathbb{R}^n} u^q dx\right)^{\frac{2}{q}} \leq \int_{\mathbb{R}^n} |\nabla u|^2 d\mathscr{L}^n + \beta \int_{J_u} (\underline{u}^2 + \overline{u}^2) d\mathscr{H}^{n-1}. \tag{4.11}$$

Where B^m is the ball of volume m, and for every $x \in J_u$, $\underline{u}(x)$ and $\overline{u}(x)$ refer to the lower and upper approximate limits of u at x. The constant $\lambda_q(B^m)$ is optimal.

Definition 4.3. Let $c \geq 0$. For any function $v \in SBV^{1/2}(\mathbb{R}^n)$ we set:

$$\Theta(v) = \left(\frac{(c+v)^q - c^q}{q}\right).$$

We introduce the following regularization of $\mathcal{E}^c$

$$\mathcal{E}^{c,\epsilon}(v) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 d\mathscr{L}^n + \frac{\beta}{2}\int_{J_v} \left[(\underline{v}^2 + 2c\underline{v}^{1+\epsilon}) + (\overline{v}^2 + 2c\overline{v}^{1+\epsilon})\right] d\mathscr{H}^{n-1} - \int_{\mathbb{R}^n} \Theta(v) d\mathscr{L}^n,$$

which is well posed thanks to (4.11). We shall also work with the penalised version

$$\mathcal{E}_k^{c,\epsilon}(v) = \mathcal{E}^{c,\epsilon}(v) + k|\{v>0\}|,$$

where $k>0$ is a positive constant.

Below is the key result of this section. Let us denote by

$$\mathcal{U}_m := \{u \in SBV^{1/2}(\mathbb{R}^n) : |\{u>0\}| = m\}$$

the class of admissible functions.

Theorem 4.4. *For every $\varepsilon > 0$ and $c \geq 0$ the minimizer of*

$$\inf\{\mathcal{E}^{c,\epsilon}(u) : u \in \mathcal{U}_m\}$$

is a radial function and its support is a ball of measure m.

The rest of the section is devoted to the proof of this theorem. The strategy is as follows.

- We first assume the existence of a minimizer u and study its properties to arrive at the conclusion that it is a radial function with the support being a ball of measure m.
- We then prove the existence of a minimizer by analyzing a minimizing sequence using the a priori properties proved before. The key point is to show that, up to a subsequence and up to translations, a minimizing sequence necessarily has to concentrate the mass around the origin and to converge.

4.3.1 Preparatory results

We write $\mathcal{E}^{c,\epsilon}$ as

$$\mathcal{E}^{c,\epsilon}(u) = Q(u) + N^{c,\epsilon}(u) - \int_{\mathbb{R}^n} \Theta(u) d\mathscr{L}^n,$$

where

$$Q(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 d\mathscr{L}^n + \frac{\beta}{2}\int_{J_v} (\underline{v}^2 + \overline{v}^2) d\mathscr{H}^{n-1}$$

and

$$N^{c,\epsilon}(v) = \beta c \int_{J_v} (\underline{v}^{1+\epsilon} + \overline{v}^{1+\epsilon}) d\mathscr{H}^{n-1}.$$

For any non trivial $u \in SBV^{1/2}$ such that $\mathcal{E}^{c,\epsilon}(u) < \infty$, we have for small $t > 0$

$$\mathcal{E}^{c,\epsilon}(tu) < 0.$$

While $\mathcal{E}^{c,\epsilon}(u)$ is not necessarily positive, its terms coming with positive and negative sign, control each other in certain cases. We summarize this observation as follows.

Lemma 4.5. *For every function $u \in \mathcal{U}_m$ there is constant $C > 0$, depending on $n, m, \beta, q, c, \mathcal{E}^{c,\epsilon}(u)$ such that*

$$Q(u) + N^{c,\epsilon}(u) + \|\Theta(u)\|_{L^1} < C.$$

Proof. For any such u, we have, using (4.11)

$$\int_{\mathbb{R}^n} u d\mathscr{L}^n \leq \lambda_1(B^m)^{-\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2 d\mathscr{L}^n + \beta \int_{J_u} (\underline{u}^2 + \overline{u}^2) d\mathscr{H}^{n-1} \right)^{\frac{1}{2}}$$

and

$$\left(\int_{\mathbb{R}^n} u^q d\mathscr{L}^n \right)^{1/q} \leq \lambda_q(B^m)^{-\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2 d\mathscr{L}^n + \beta \int_{J_u} (\underline{u}^2 + \overline{u}^2) d\mathscr{H}^{n-1} \right)^{\frac{1}{2}},$$

where B^m is the ball of volume m. We then know that for a certain constant C,

$$\int_{\mathbb{R}^n} \Theta(u) d\mathscr{L}^n \leq C(Q(u)^{1/2} + Q(u)^{q/2}),$$

so that

$$Q(u) + N^{c,\epsilon}(u) - C(Q(u)^{1/2} + Q(u)^{q/2}) \leq \mathcal{E}^{c,\epsilon}(u).$$

This means that $Q(u) + N^{c,\epsilon}(u)$ is bounded by a constant that depends only on $\mathcal{E}^{c,\epsilon}(u)$ and C (which depends on n, m, β, q, c); this proves the result. □

Let us now study the monotonicity of $\mathcal{E}^{c,\epsilon}$ on scale change. This will imply that minimizing $\mathcal{E}^{c,\epsilon}(u)$ under the constraint $|\{u > 0\}| = m$ or $|\{u > 0\}| \leq m$ is equivalent.

Lemma 4.6. *Let $u \in SBV^{1/2}(\mathbb{R}^n)$ be a non-trivial function and $t > 1$. Then:*

$$\mathcal{E}^{c,\epsilon}\left(u\left(\cdot/t\right)\right) < t^n \mathcal{E}^{c,\epsilon}(u)$$

Proof. It is immediate by a change of variable and using $t^{n-1}, t^{n-2} < t^n$. □

We have to compare a minimizer u with a function whose support does not necessarily have the same measure. The following lemma allows us to use $\mathcal{E}_k^{c,\epsilon}$, the volume-penalised version of $\mathcal{E}^{c,\epsilon}$ with a suitable k.

Lemma 4.7. *Let $u \in \mathcal{U}_m$ be a minimizer of the functional $\mathcal{E}^{c,\epsilon}$. Then*

- *For $k = \frac{-\mathcal{E}^{c,\epsilon}(u)}{m}$, u is a minimizer of $\mathcal{E}_k^{c,\epsilon}$ in the class*

$$\{v \in SBV^{1/2}(\mathbb{R}^n) : |\{v > 0\}| \leq m\}.$$

- *For $k = \frac{2\|\Theta(u)\|_{L^1}}{m}$, u is a minimizer of $\mathcal{E}_k^{c,\epsilon}$ in the class*

$$\{v \in SBV^{1/2}(\mathbb{R}^n) : |\{v > 0\}| \geq m,\ \|\Theta(v)\|_{L^1} \leq 2\|\Theta(u)\|_{L^1}\}.$$

Proof. **First case**: Let v be such a function, we write $\frac{|\{v>0\}|}{m} = 1 - \eta$ with $\eta \in]0, 1[$. Let

$$w(x) := v((1-\eta)^{1/n} x).$$

Notice that w is in $\mathcal{U}_m$ with

$$\mathcal{E}^{c,\epsilon}(w) = (1-\eta)^{-1+\frac{2}{n}}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla v|^2 d\mathscr{L}^n + (1-\eta)^{-1+\frac{1}{n}}\frac{\beta}{2}\int_{J_v}\left[(\underline{v}^2+2c\underline{v}^{1+\epsilon})+(\overline{v}^2+2c\overline{v}^{1+\epsilon})\right]d\mathscr{H}^{n-1} - (1-\eta)^{-1}\int_{\mathbb{R}^n}\Theta(v)d\mathscr{L}^n,$$

so that $\mathcal{E}^{c,\epsilon}(w) \leq (1-\eta)^{-1}\mathcal{E}^{c,\epsilon}(v)$. The minimality of u yields

$$(1-\eta)\mathcal{E}^{c,\epsilon}(u) \leq \mathcal{E}^{c,\epsilon}(v),$$

which, with our definition of k and η, is exactly

$$\mathcal{E}_k^{c,\epsilon}(u) \leq \mathcal{E}_k^{c,\epsilon}(v).$$

Second case: We proceed in the same way. Let us consider such a function v, write $\frac{|\{v>0\}|}{m} = 1+\eta$ where $\eta > 0$. Let

$$w(x) := v((1+\eta)^{1/n}x).$$

Again w is in $\mathcal{U}_m$ with

$$\mathcal{E}^{c,\epsilon}(w) = (1+\eta)^{-1+\frac{2}{n}}\frac{1}{2}\int_{\mathbb{R}^n}|\nabla v|^2 d\mathscr{L}^n + (1+\eta)^{-1+\frac{1}{n}}\frac{\beta}{2}\int_{J_v}\left[(\underline{v}^2+2c\underline{v}^{1+\epsilon})+(\overline{v}^2+2c\overline{v}^{1+\epsilon})\right]d\mathscr{H}^{n-1} - (1+\eta)^{-1}\int_{\mathbb{R}^n}\Theta(v)d\mathscr{L}^n.$$

For any $\eta > 0$, we have the inequalities

$$(1+\eta)^{-1+\frac{2}{n}}, (1+\eta)^{-1+\frac{1}{n}} \leq 1, \quad (1+\eta)^{-1} \geq 1-\eta.$$

From the minimality of u and the choice of v we can write

$$\begin{aligned}\mathcal{E}^{c,\epsilon}(u) &\leq \mathcal{E}^{c,\epsilon}(w)\\ &\leq \mathcal{E}^{c,\epsilon}(v) + \eta\int_{\mathbb{R}^n}\Theta(v)d\mathscr{L}^n\\ &\leq \mathcal{E}^{c,\epsilon}(v) + 2\eta\int_{\mathbb{R}^n}\Theta(u)d\mathscr{L}^n,\end{aligned}$$

which, following the definition of k and η yields

$$\mathcal{E}_k^{c,\epsilon}(u) \leq \mathcal{E}_k^{c,\epsilon}(v).$$

□

4.3.2 Nondegeneracy of the minimizers

In this part, we prove that a minimizer lies above a strictly positive threshold, on the set where it is non vanishing. We refer to [83], [84], [81], [79] for similar arguments.

Lemma 4.8. *Let $u \in \mathcal{U}_m$ be a minimizer of the functional $\mathcal{E}^{c,\epsilon}$. There exists $\delta > 0$, depending on $n, c, \beta, m, \epsilon, \mathcal{E}^{c,\epsilon}(u)$, such that $u \geq \delta 1_{\{u>0\}}$.*

Proof. In this proof, which follows the main lines of [84, Theorem 3.2], we denote by C a positive constant that may change from line to line which depends on the parameters only. Let k be defined as in the first part of the previous result. We introduce the following function

$$f_{t_{\min}}(t) = \int_{\{t_{\min}\leq u\leq t\}} u^\epsilon|\nabla u|d\mathscr{L}^n,$$

where $0 < t_{\min} < t$. Since $(u - t_{\min})_+$ is in SBV, we can apply the coarea area for SBV functions (see [84]) to get

$$f_{t_{\min}}(t) = \int_{t_{\min}}^{t} s^{\epsilon} \mathrm{Per}(\{u \le s\}; \mathbb{R}^n \setminus J_u) ds.$$

Define $f(t) = f_{t_{\min}=0}(t)$; by taking the limit $t_{\min} \to 0^+$ in the above formula, we see that by monotone convergence

$$f(t) = \int_{\{0 \le u \le t\}} u^{\epsilon} |\nabla u| d\mathscr{L}^n = \int_0^t s^{\epsilon} \mathrm{Per}(\{u \le s\}; \mathbb{R}^n \setminus J_u) ds.$$

The first form yields the fact that f is bounded (for bounded t at least). Indeed, it is controlled by the positive part of $\mathcal{E}^{c,\epsilon}(u)$, and more precisely by $Q(u) + k|\{u > 0\}|$

$$f(t) \le t^{\epsilon} \int_{\{u \le t\}} |\nabla u| d\mathscr{L}^n \le t^{\epsilon} |\{0 < u \le t\}|^{1/2} \left(\int_{\{0 < u \le t\}} |\nabla u|^2 d\mathscr{L}^n \right)^{1/2}.$$

We can then apply Lemma 4.5 to infer that the positive part of $\mathcal{E}^{c,\epsilon}(u)$ is controlled by a constant that depends on $\mathcal{E}^{c,\epsilon}(u)$ and the parameters of the problem. As u is a minimizer, f is bounded by a constant that only depends on the parameters.

Next, we use the optimality of u against $u1_{\{u>t\}}$ in view of Lemma 4.7, which gives

$$\mathcal{E}_k^{c,\epsilon}(u) \le \mathcal{E}_k^{c,\epsilon}\left(u1_{\{u>t\}}\right)$$

or, after computations,

$$Q\left(u1_{\{u\le t\}}\right) + N^{c,\epsilon}\left(u1_{\{u\le t\}}\right) + k|\{0 < u \le t\}| - \int_{\{0<u\le t\}} \Theta(v) d\mathscr{L}^n \le \frac{\beta}{2} \int_{\partial^*\{u>t\}\setminus J_u} (u^2 + 2cu^{1+\epsilon}) d\mathscr{H}^{n-1}.$$

In the rest of the proof, we will only consider t small enough

$$t < \Theta^{-1}(k/2) \wedge c^{\frac{1}{1-\epsilon}},$$

which allows us to write

$$k|\{0 < u \le t\}| - \int_{\{0<u\le t\}} \Theta(v) d\mathscr{L}^n \ge \frac{k}{2} |\{0 < u \le t\}|,$$

and

$$\begin{aligned} \frac{\beta}{2} \int_{\partial^*\{u>t\}\setminus J_u} (u^2 + 2cu^{1+\epsilon}) d\mathscr{H}^{n-1} &= \frac{\beta}{2} (t^2 + 2ct^{1+\epsilon}) \mathscr{H}^{n-1}(\partial^*\{u > t\} \setminus J_u) \\ &= \frac{\beta}{2} (t^{2-\epsilon} + 2ct) f'(t) \le \frac{3}{2} \beta ct f'(t). \end{aligned}$$

With these, the optimality condition becomes

$$Q\left(u1_{\{u\le t\}}\right) + N^{c,\epsilon}\left(u1_{\{u\le t\}}\right) + \frac{k}{2}|\{0 < u \le t\}| \le \frac{3}{2}\beta ct f'(t). \tag{4.12}$$

By Hölder,

$$f(t) \le |\{0 < u \le t\}|^{\frac{n-(n-1)\epsilon}{2n}} \|1_{\{0<u<t\}} \nabla u\|_{L^2} \, \|1_{\{0<u\le t\}} u^2\|^{\frac{\epsilon}{2}}_{L^{\frac{n}{n-1}}} \le C\Big[tf'(t)\Big]^{1-\frac{n-1}{2n}\epsilon} \, \|u^2 1_{\{0<u\le t\}}\|^{\frac{\epsilon}{2}}_{L^{\frac{n}{n-1}}}.$$

To estimate the last factor, we use the continuity of the embedding $BV(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and (4.12) to get

$$\begin{aligned} c_n \|u^2 1_{\{0<u\le t\}}\|_{L^{\frac{n}{n-1}}} &\le [u^2 1_{\{0<u\le t\}}]_{BV} = |D(u^2 1_{\{0<u\le t\}})|(\mathbb{R}^n) \\ = \int_{\{0<u\le t\}} 2u|\nabla u|\, d\mathscr{L}^n &+ \int_{J_u\cap\{u<t\}^0} (\underline{u}^2+\overline{u}^2) d\mathscr{H}^{n-1} + \int_{J_u\cap\partial^*\{u>t\}} \underline{u}^2 d\mathscr{H}^{n-1} + \int_{\partial^*\{u>t\}\setminus J_u} u^2 d\mathscr{H}^{n-1} \\ &\le C\Big[Q\left(u1_{\{u\le t\}}\right) + |\{0<u\le t\}| + t^2 \mathrm{Per}(\{u>t\};\mathbb{R}^n\setminus J_u)\Big] \le Ctf'(t). \end{aligned}$$

Coming back to the estimate of $f(t)$, we obtain

$$f(t) \le C\Big[tf'(t)\Big]^{1+\frac{\epsilon}{2n}}.$$

This implies that, for all t such that $f(t)>0$, we have:

$$\frac{d}{dt}\Big[f(t)^{\frac{\epsilon}{2n+\epsilon}}\Big] \ge \frac{1}{Ct}.$$

Let t_0 be such that $f(t_0)>0$ and $t_1 := \min(c^{\frac{1}{1-\epsilon}}, \Theta^{-1}(k/2))$. We integrate on $[t_0,t_1]$

$$(f(t_1)^{\frac{\epsilon}{2n+\epsilon}} \ge) f(t_1)^{\frac{\epsilon}{2n+\epsilon}} - f(t_0)^{\frac{\epsilon}{2n+\epsilon}} \ge \frac{1}{C}\log(t_1/t_0).$$

Using our uniform bound on $f(t_1)$, we obtain a lower bound δ on t_0 that only depends on the parameters of the problem, meaning that $f(\delta)=0$ for an explicit $\delta>0$. We apply the optimality condition in $t=\delta$:

$$Q\left(u1_{\{u\le\delta\}}\right) + N^{c,\epsilon}\left(u1_{\{u\le\delta\}}\right) + \frac{k}{2}|\{0<u\le\delta\}| \le 0,$$

and so $|\{0<u\le\delta\}|=0$.

□

4.3.3 Closedness of the jump set

We prove below that the support of the minimizer u is an open set Ω with finite perimeter. For this we prove that the jump set of u is closed as J_u is identified with $\partial\Omega$.

Lemma 4.9. *Let $u\in\mathcal{U}_m$ be a minimizer of the functional $\mathcal{E}^{c,\epsilon}$. Then $u\in SBV(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $\mathscr{H}^{n-1}(J_u)<\infty$. Moreover, $\mathscr{H}^{n-1}(\overline{J_u}\setminus J_u)=0$.*

Proof. We divide the proof in several steps.

Step 1. We prove that $u\in L^\infty(\mathbb{R}^n)$. Let us set

$$u_M = (u-M)_+, \qquad f(M) = \int_{\mathbb{R}^n} u_M d\mathscr{L}^n \qquad \text{and} \qquad \alpha(M) = |\{u_M>0\}|,$$

and let $\lambda_{2,\beta}(B^m)$ be the best constant in (4.11) with $q=2$ and β as boundary parameter, among sets with mass m. We suppose that u is not bounded, and so that $f(M)>0$ and $\alpha(M)>0$ for all $M>0$. Let

$$g(M) = \frac{\lambda_{2,\alpha(M)^{1/n}\beta}(B^1)}{\alpha(M)^{1/n}}.$$

The results we will use here is that

$$\lambda_{2,\beta}(B^{\alpha(M)}) = \alpha(M)^{-\frac{1}{n}} g(M) \text{ with } \liminf_{M\to\infty} g(M) > 0.$$

We test the optimality of the function u against the function $u \wedge M$. We get

$$\frac{1}{2}\int_{\{u>M\}} |\nabla u|^2 d\mathscr{L}^n + \frac{\beta}{2}\int_{\{M\leq \underline{u}<\overline{u}\}} (\overline{u}^2 + 2c\overline{u}^{1+\epsilon} + \underline{u}^2 + 2c\underline{u}^{1+\epsilon}) d\mathscr{H}^{n-1}$$
$$+ \frac{\beta}{2}\int_{\{\underline{u}<M\leq \overline{u}\}} (\overline{u}^2 + 2c\overline{u}^{1+\epsilon} - M^2 - 2cM^{1+\epsilon}) d\mathscr{H}^{n-1} \leq \int_{\{u>M\}} (\Theta(u) - \Theta(M)) d\mathscr{L}^n.$$

This implies:

$$Q(u_M) \leq \int_{\{u>M\}} (\Theta(u) - \Theta(M)) d\mathscr{L}^n \tag{4.13}$$

and (4.11) gives

$$g(M)\alpha(M)^{-\frac{1}{n}} \int_{\mathbb{R}^n} u_M^2 d\mathscr{L}^n \leq Q(u_M). \tag{4.14}$$

Using $1 \leq q < 2$, we know that for all $1 \leq a \leq b$, we have $b^q - a^q \leq b^2 - a^2$, which implies, for $M \geq 1$, that

$$\int_{\{u>M\}} (\Theta(u) - \Theta(M)) d\mathscr{L}^n \leq \frac{1}{q}\int_{\{u>M\}} ((c + M + u_M)^2 - (c+M)^2) d\mathscr{L}^n$$
$$= \frac{1}{q}\int_{\{u>M\}} (u_M^2 + 2(c+M)u_M) d\mathscr{L}^n.$$

Combining this with the estimates (4.13),(4.14), we get:

$$\left(g(M)\alpha(M)^{-\frac{1}{n}} - \frac{1}{q}\right)\int_{\mathbb{R}^n} u_M^2 d\mathscr{L}^n \leq \frac{2}{q}(c+M)f(M).$$

Since $\liminf_{M\to\infty} g(M) > 0$ and $\alpha(M) \underset{M\to\infty}{\longrightarrow} 0$, we know that for all big enough M,

$$g(M)\alpha(M)^{-\frac{1}{n}} - \frac{1}{q} \geq \frac{1}{2}g(M)\alpha(M)^{-\frac{1}{n}}.$$

Holder's inequality gives:

$$\alpha(M)^{-1} f(M)^2 \leq \int_{\mathbb{R}^n} u_M^2 d\mathscr{L}^n.$$

Thus we get:

$$\frac{1}{2}g(M)\alpha(M)^{-\frac{n+1}{n}} f(M)^2 \leq \frac{2}{q}(c+M)f(M),$$

which can be rewritten as

$$\left(\frac{q}{4}\frac{g(M)}{c+M}\right)^{\frac{n}{n+1}} \leq \alpha(M) f(M)^{-\frac{n}{n+1}}.$$

The left side is not integrable because $\liminf g > 0$. Since $f(M) \to 0$ and $f'(M) = -\alpha(M)$, the right side is integrable (its integral on $[M_0, +\infty[$ is $\frac{1}{n+1} f(M_0)^{\frac{1}{n+1}} < \infty$): this is a contradiction. We deduce that u is bounded by a certain constant $M > 0$.

Step 2. We get $\mathscr{H}^{n-1}(J_u) < \infty$. Indeed, by Lemma 4.8 we have $u > \delta 1_{\{u>0\}}$ which implies

$$\delta^2 \mathscr{H}^{n-1}(J_u) \leq \int_{J_u} (\overline{u}^2 + \underline{u}^2) d\mathscr{H}^{n-1} < \infty.$$

Step 3. The function u belongs to $SBV(\mathbb{R}^n)$. The proof follows the same arguments as in [81], by considering the function $u^\eta = \sqrt{u^2 + \eta^2}$ for $\eta \to 0$.

Step 4. We show the closedness of the jump set $\mathscr{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$. Following [83], we only need to show that u is a local almost quasi-minimizer of the Mumford-Shah functional. We use $\delta 1_{\{u>0\}} \leq u \leq M$ to prove this.

Indeed, consider $v \in SBV(\mathbb{R}^n)$ such that $\{u \neq v\} \Subset B_r$ for a ball B_r of radius $r > 0$ small enough. Let $v' = v \wedge M$. Since $u \leq M$, then $\{u \neq v'\} \Subset B_r$ and v' still belongs to SBV. Applying Lemma 4.7, either $|\{v' > 0\}| \leq m$ and we are in the first case or $|\{v' > 0\}| \geq m$ and we are in the second one. In the second case we need to verify that $\|\Theta(v')\|_{L^1} \leq 2\|\Theta(u)\|_{L^1}$; this is true for small enough r since $\|\Theta(v')\|_{L^1} \leq \|\Theta(u)\|_{L^1} + |B_r|\Theta(M)$. Thus for a small enough r there exists $k > 0$ depending only on the parameters such that

$$\mathcal{E}_k^{c,\epsilon}(u) \leq \mathcal{E}_k^{c,\epsilon}(v').$$

This can be rewritten

$$\begin{aligned}
&\frac{1}{2}\int_{B_r} |\nabla u|^2 d\mathscr{L}^n - \int_{B_r} \Theta(u) d\mathscr{L}^n \\
&\quad + \frac{\beta}{2}\int_{J_u \cap B_r} \left[(\underline{u}^2 + 2c\underline{u}^{1+\epsilon}) + (\overline{u}^2 + 2c\overline{u}^{1+\epsilon})\right] d\mathscr{H}^{n-1} + k|\{u > 0\} \cap B_r| \\
\leq &\frac{1}{2}\int_{B_r} |\nabla v'|^2 d\mathscr{L}^n - \int_{B_r} \Theta(v') d\mathscr{L}^n \\
&\quad + \frac{\beta}{2}\int_{J_{v'} \cap B_r} \left[(\underline{v'}^2 + 2c\underline{v'}^{1+\epsilon}) + (\overline{v'}^2 + 2c\overline{v'}^{1+\epsilon})\right] d\mathscr{H}^{n-1} + k|\{v' > 0\} \cap B_r|.
\end{aligned}$$

Using $u > \delta 1_{\{u>0\}}$ and $v' \leq M$, as well as $|\nabla v'| \leq |\nabla v|$, $J_{v'} \subset J_v$, we get

$$\begin{aligned}
&\int_{B_r} |\nabla u|^2 d\mathscr{L}^n + \beta(\delta^2 + 2c\delta^{1+\epsilon})\mathscr{H}^{n-1}(J_u \cap B_r) \\
&\qquad\qquad \leq \int_{B_r} |\nabla v|^2 d\mathscr{L}^n + 2\beta(M^2 + 2cM^{1+\epsilon})\mathscr{H}^{n-1}(J_v \cap B_r) + 2\alpha_n\left(k + \Theta(M)\right) r^n,
\end{aligned}$$

where $\alpha_n = |B_1|$. Up to a renormalization of u, this is exactly the definition of a local quasi-almost minimizer. Following [83, Theorem 3.1], this implies

$$\mathscr{H}^{n-1}(\overline{J_u} \setminus J_u) = 0.$$

□

Lemma 4.10. *Let $u \in \mathcal{U}_m$ be a minimizer of the functional $\mathcal{E}^{c,\epsilon}$, then there exists an open domain Ω such that the following items hold true.*

(a) $\partial\Omega = \overline{J_u}$, $\mathscr{H}^{n-1}(\partial\Omega \setminus J_u) = 0$ and $u = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$

(b) $u_{|\Omega} \in H^1(\Omega)$ and verifies $\delta < u_{|\Omega} < M$ for certain constants $\delta, M > 0$ and

$$-\Delta u = (c + u)^{q-1} \text{ in } \mathcal{D}'(\Omega).$$

In particular, u is analytic on its support.

Proof. The proof is the same as Theorem 6.15 of [81]. □

4.3.4 The optimal function is radially symmetric

Now we prove that the optimal function is radial and is supported on a ball of measure m.

Lemma 4.11. *Let $u \in \mathcal{U}_m$ be a minimizer of the functional $\mathcal{E}^{c,\epsilon}$, then u is radial and its support is a ball.*

Proof. Using Lemma 4.10, we know that u is an analytic function on an open domain Ω with finite perimeter. We divide the proof in several steps.

Step 1. The set Ω is connected. We first show that Ω is connected: suppose that $\Omega = V \sqcup W$ for two open sets V, W, then we write $v = u1_V$ and $w = u1_W$: these functions are in $\mathcal{U}_{|V|}$ and $\mathcal{U}_{|W|}$. We also set

$$\widetilde{v}(x) := v\left(\left[\frac{|V|}{m}\right]^{1/n} x\right) \qquad \text{and} \qquad \widetilde{w}(x) := w\left(\left[\frac{|W|}{m}\right]^{1/n} x\right).$$

The functions $\widetilde{v}$ and $\widetilde{w}$ are in $\mathcal{U}_m$ so, by comparison to u and Lemma 4.6,

$$\mathcal{E}^{c,\epsilon}(u) \leq \frac{|V|}{m}\mathcal{E}^{c,\epsilon}(\widetilde{v}) + \frac{|W|}{m}\mathcal{E}^{c,\epsilon}(\widetilde{w}) < \mathcal{E}^{c,\epsilon}(v) + \mathcal{E}^{c,\epsilon}(w) = \mathcal{E}^{c,\epsilon}(u).$$

This is a contradiction which implies that Ω is connected.

Step 2. The function u is locally radial. Here we call *locally radial* any function u that is locally the restriction of a radial function on a support that is not necessarily radial; when u is smooth on its support, which is the case here, it is equivalent, up to a translation, to $\nabla u(x)$ being proportional to x for every $x \in \Omega$.

We first build a symmetrized version of Ω called Ω^s such that any hyperplane going through the origin cuts Ω^s in two parts of same volume. The procedure is the following: take $\lambda_1 \in \mathbb{R}$ such that the hyperplane $\{x_1 = \lambda_1\}$ cuts the support of u in two parts of equal volume. Let S_1 be the symmetry across this hyperplane, and

$$u_1^+ = \begin{cases} u & \text{in } \{x_1 > \lambda_1\} \\ u \circ S_1 & \text{in } \{x_1 < \lambda_1\} \end{cases}$$

$$u_1^- = \begin{cases} u & \text{in } \{x_1 < \lambda_1\} \\ u \circ S_1 & \text{in } \{x_1 > \lambda_1\} \end{cases}$$

Then $u_1^\pm$ both have a support of volume m and $\mathcal{E}^{c,\epsilon}(u_1^+) + \mathcal{E}^{c,\epsilon}(u_1^-) \leq 2\mathcal{E}^{c,\epsilon}(u)$ (the inequality is only there because some part of J_u lying on $\{x_1 = \lambda_1\}$ might be deleted in $u_1^\pm$: when this does not happen, there is equality). In particular u_1^+ and u_1^- are also minimizers of $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_m$. Now apply the same procedure to u_1^+ across an hyperplane $\{x_2 = \lambda_2\}$, and so on: we get in the end a minimizer $u^s \in \mathcal{U}_m$ that coincides with u on the quadrant $\{x : x_i \geq \lambda_i,\ \forall i = 1, \dots, n\}$, and that is symmetric relative to every hyperplane $\{x_i = \lambda_i\}$.

Without loss of generality, we can suppose that every λ_i is 0. Let Ω^s be the support of u^s, we know that it is symmetric relative to every $\{x_i = 0\}$. The composition of all those symmetries is the central symmetry relative to the origin, and so Ω^s has a central symmetry. In particular, any hyperplane that goes through the origin cuts the volume of Ω^s in half.

We now show that u^s is locally radial: this implies that u is locally radial because $u = u^s$ on $\Omega \cap \Omega^s (\neq \emptyset)$, u and u^s are analytic, and Ω is connected, so we can use analytic continuation. To show that u^s is locally radial, take Π any hyperplane going through 0, and S the symmetry across this hyperplane. Let Π^+ and Π^- be the two half-space defined from Π, we define

$$u^{s,\pm} = \begin{cases} u^s & \text{in } \Pi^\pm, \\ u \circ S & \text{in } \Pi^\mp. \end{cases}$$

Moreover, $\mathcal{E}^{c,\epsilon}(u^{s,+}) + \mathcal{E}^{c,\epsilon}(u^{s,-}) \leq 2\mathcal{E}^{c,\epsilon}(u^s)$, so $u^{s,+}$ (and $u^{s,-}$) is a minimizer. Since it is a minimizer, it is $\mathcal{C}^1$ on its support which is an open set. u^s and $u^{s,+}$ are both $\mathcal{C}^1$ and coincide on $\Pi^- \cap \Omega$: this implies that ∇u^s is purely normal to Π on $\Pi \cap \Omega$. Since this is true for any hyperplane, we have shown that u^s is locally radial, which implies that u is locally radial by analyticity.

We know that u is locally radial on a connected open set, and verifies $-\Delta u = (c+u)^{q-1}$, which means that it can be written $u(x) = \psi(|x|)1_\Omega(x)$ where ψ verifies the following ordinary differential equation:

$$\psi'' + \frac{n-1}{r}\psi' + (\psi + c)^{q-1} = 0. \tag{4.15}$$

Step 3. The set Ω is radial. To prove this we use the argument developed in [82]. We show that the topological boundary of Ω coincides with the measure theoretical one, up to a $\mathscr{H}^{n-1}$-negligible set. Since $u = \psi(|\cdot|)1_\Omega$ where ψ is analytic and bounded below, the singular part (relative to the Lebesgue measure) of the derivative of u is:

$$D^s u = \psi(|\cdot|)D^s 1_\Omega.$$

And so its support (which is $\overline{J_u}$ up to a $\mathscr{H}^{n-1}$-negligible set) is the support of $D^s 1_\Omega$, which is $\partial^*\Omega$ up to a $\mathscr{H}^{n-1}$-negligible set. Since $\partial\Omega = \overline{J_u}$, it means that $\mathscr{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$, and so Ω has no inner boundaries. This will allow us to use symmetrization technique on Ω in the framework of sets with finite perimeter, to show that Ω itself is radial.

Let $\widetilde{\Omega}$ be the symmetrization of Ω by spherical caps in the direction e_1, and $\widetilde{u}(x) = \psi(|x|)1_{\widetilde{\Omega}}(x)$. We aim to show that Ω is necessarily radial, which is implied by the property $|\widetilde{\Omega} \cap \{x_1 < 0\}| = |\widetilde{\Omega} \cap \{x_1 > 0\}|$. We suppose it is not the case, then there is a point $p = \lambda e_1$ for $\lambda > 0$ such that all hyperplanes going through p cuts $\widetilde{\Omega}$ in two parts with same volume. This is because it is the case for n orthogonal hyperplanes, namely the $\{x_i = 0\}$ for $i \geq 2$ (by property of the spherical cap rearrangement), and a certain $\{x_1 = \lambda\}$ for $\lambda > 0$ (because $|\widetilde{\Omega} \cap \{x_1 < 0\}| < |\widetilde{\Omega} \cap \{x_1 > 0\}|$).

Moreover, $\widetilde{u}$ is a minimizer. This is a consequence of the properties of the spherical rearrangement and the fact that u is radial with no inner jump, which implies

$$\begin{aligned}
\mathcal{E}^{c,\epsilon}(\widetilde{u}) &= \int_{\widetilde{\Omega}} \left(\frac{1}{2}|\psi'(|x|)| - \Theta(\psi(|x|))\right) d\mathscr{L}^n + \int_{\partial\widetilde{\Omega}} \left(\psi(|x|)^2 + 2c\psi(|x|)^{1+\epsilon}\right) d\mathscr{H}^{n-1} \\
&\leq \int_{\Omega} \left(\frac{1}{2}|\psi'(|x|)| - \Theta(\psi(|x|))\right) d\mathscr{L}^n + \int_{\partial^*\Omega} \left(\psi(|x|)^2 + 2c\psi(|x|)^{1+\epsilon}\right) d\mathscr{H}^{n-1} \\
&= \mathcal{E}^{c,\epsilon}(u).
\end{aligned}$$

Where in the last line we used that $J_u = \partial^*\Omega$ up to a $\mathscr{H}^{n-1}$-negligible set, and either $\overline{u}$ or $\underline{u}$ is 0 for $\mathscr{H}^{n-1}$-every point of $\partial\Omega$ (since there are no inner boundaries).

We know from the above procedure that $\widetilde{u}$ is locally radial in 0 and in $p \neq 0$. Thus, for any point $x \in \widetilde{\Omega} \setminus (0, p)$, $\nabla u(x)$ is proportional to x and $x - p$, which means that it is 0. This means that $\widetilde{u}$ is locally constant, which is not the case because of the equation $-\Delta u = (c+u)^{q-1}$ verified by u.

The consequence is that $|\widetilde{\Omega} \cap \{x_1 < 0\}| = |\widetilde{\Omega} \cap \{x_1 > 0\}|$ which proves that Ω is a radial set.

Step 4. The set Ω is a ball. We already know that Ω is a connected radial set. To show that Ω is a ball, it suffices to exclude that it is an annulus. Suppose by contradiction $\Omega = B_{r_2} \setminus \overline{B}_{r_1}$, where we can assume that $r_1 > 0$, since a point has zero capacity.

We know that u is of the form $\psi(|x|)1_\Omega(x)$ where ψ verifies (4.15), with the modified Robin boundary condition

$$(-1)^i \psi'(r_i) + \beta\left(\psi(r_i) + c(1+\epsilon)\psi(r_i)^\epsilon\right) = 0, \quad i = 1, 2.$$

We also know that $\psi > \delta > 0$ on $\overline{\Omega}$. This means that ψ can be extended slightly to a neighbourhood of $[r_1, r_2]$ as a solution of (4.15), as long as $\psi > 0$. We can thus extend u as $\widetilde{u}(x) = \psi(|x|)$. Let:

$$g(\varrho_1, \varrho_2) = \mathcal{E}^{c,\epsilon}(\widetilde{u}1_{B_{\varrho_2} \setminus B_{\varrho_1}}),$$

where ρ_1, ρ_2 are taken close to r_1, r_2. We know that g is minimal in (r_1, r_2) among the set $\{(\varrho_1, \varrho_2) : \varrho_2^n - \varrho_1^n = r_2^n - r_1^n\}$. This implies:

$$\frac{\partial_{\varrho_1} g(r_1, r_2)}{r_1^{n-1}} + \frac{\partial_{\varrho_2} g(r_1, r_2)}{r_2^{n-1}} = 0.$$

This gives the following inequality, in which we shortened $\psi_i = \psi(r_i)$, $\psi_i' = \psi'(r_i)$

$$\frac{1}{2}(|\psi_2'|^2 - |\psi_1'|^2) + (\Theta(\psi_1) - \Theta(\psi_2)) + \frac{\beta}{2}(n-1)\left(\frac{\psi_1^2 + 2c\psi_1^{1+\epsilon}}{r_1} + \frac{\psi_2^2 + 2c\psi_2^{1+\epsilon}}{r_2}\right) + \beta\left(\psi_1 + c(1+\epsilon)\psi_1^\epsilon\right)\psi_1' + \beta\left(\psi_2 + c(1+\epsilon)\psi_2^\epsilon\right)\psi_2' = 0.$$

We know that the (modified) Robin boundary condition is verified in r_1 and r_2, which means that $\beta\left(\psi_i + c(1+\epsilon)\psi_i^\epsilon\right) = \pm\psi_i'$ ($+$ for $i = 1$, $-$ for $i = 2$). We rewrite the previous inequality as:

$$\left(\frac{1}{2}|\psi_1'|^2 + \frac{1}{q}(c+\psi_1)^q\right) - \left(\frac{1}{2}|\psi_2'|^2 + \frac{1}{q}(c+\psi_2)^q\right) + \frac{\beta}{2}(n-1)\left(\frac{\psi_1^2 + 2c\psi_1^{1+\epsilon}}{r_1} + \frac{\psi_2^2 + 2c\psi_2^{1+\epsilon}}{r_2}\right) = 0.$$

In particular, this means

$$\frac{1}{2}|\psi_1'|^2 + \frac{1}{q}(c+\psi_1)^q < \frac{1}{2}|\psi_2'|^2 + \frac{1}{q}(c+\psi_2)^q.$$

This is a contradiction because ψ verifies (4.15), which implies:

$$\frac{d}{d\varrho}\left(\frac{1}{2}|\psi'|^2 + \frac{1}{q}(c+\psi)^q\right) = -\frac{n-1}{r}|\psi'|^2 \leq 0.$$

This means that a nontrivial annulus cannot be stationary for $\mathcal{E}^{c,\epsilon}$, which implies in particular that it cannot be a minimizer. □

4.3.5 Existence of minimizers

Before proving the existence of minimizer, we need some technical results to localize the mass of a function in $SBV^{1/2}$. Below, we denote by K_p the unit cube centered in $p \in \mathbb{Z}^n$.

Lemma 4.12. *Let u be a non-trivial positive function on $SBV^{1/2}(\mathbb{R}^n)$. There exists $p \in \mathbb{Z}^n$ such that*

$$|\{u > 0\} \cap K_p| \geq \left(\frac{C\|u\|_{L^2}^2}{\|u\|_{L^2}^2 + Q(u)}\right)^n,$$

where $C = C(n) > 0$.

Proof. The family $(K_p)_{p\in\mathbb{Z}}$ is a covering of $\mathbb{R}^n$. For all p, we have

$$\|u\|_{L^2(K_p)}^2 \leq |\{u > 0\} \cap K_p|^{1/n} \|u^2\|_{L^{\frac{n}{n-1}}(K_p)} \leq C\left(\sup_q |\{u > 0\} \cap K_q|\right)^{1/n}\left(\|u\|_{L^2(K_p)}^2 + Q_{|K_p}(u)\right)$$

since $u_{|K_p}^2 \in BV(K_p)$ for each p. Summing over p, we get, for possibly a different constant,

$$\|u\|_{L^2}^2 \leq C\left(\sup_q |\{u > 0\} \cap K_q|\right)^{1/n}\left(\|u\|_{L^2}^2 + Q(u)\right).$$

□

Below, we give an obvious variation on the compactness result of [80].

Lemma 4.13. *Let (u_i) be a sequence of $SBV^{1/2}(\mathbb{R}^n)$ such that*

$$\limsup_i \left(Q(u_i) + N^{c,\epsilon}(u_i) + \|u_i\|_{L^2}^2\right) < \infty.$$

Then there exists a subsequence, still denoted with the same index, and a function $u \in SBV^{1/2}(\mathbb{R}^n)$ such that $u_i \to u$ in $L^2_{loc}(\mathbb{R}^n)$ and for all open set $A \Subset \mathbb{R}^n$,

$$Q_{|A}(u) \leq \liminf_{i\to\infty} Q_{|A}(u_i) \qquad \text{and} \qquad N^{\epsilon}_{|A}(u) \leq \liminf_{i\to\infty} N^{\epsilon}_{|A}(u_i).$$

We are now in a position to prove Theorem 4.4

Proof of Theorem 4.4. In view of the preceding results, it suffices to prove the existence of a minimizer. The proof goes as follows. Let (u_i) be a minimizing sequence.

- We show that up to a translation and up to subsequences, $(u_i)_i$ converges in some sense to a non-trivial function u such that $|\{u > 0\}| =: m' \in]0, m]$.
- We show that u minimizes $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_{m'}$.
- We suppose by contradiction that $m' < m$: this allows us to find a sequence (p_i) such that $|p_i| \to \infty$ and $(u_i(\cdot - p_i))_i$ converges to another non-trivial minimizer v in $\mathcal{U}_{m''}$ for $m'' \in]0, m-m']$.
- Then, knowing the structure of minimizers (smooth function defined on balls), we modify the sequence $(u_i)_i$ to build a new minimizing sequence that converges to a function whose support is given by the union of two disjoint balls, which is absurd. Thus $m' = m$ and $u \in \mathcal{U}_m$ is the minimizer we were looking for.

We proceed in several steps.

Step 1: Convergence of $(\mathbf{u_i})$. We remind that, according to Lemma 4.5, a minimizing sequence $(u_i)_i$ verifies

$$\sup_{i\geq 1} \left(Q(u_i) + N^{c,\epsilon}(u_i) + \|\Theta(u_i)\|_{L^1}\right) < \infty.$$

Moreover, since for all $w \in \mathcal{U}_m$, $\mathcal{E}^{c,\epsilon}(tw) < 0$ for small enough t, we can suppose without loss of generality that $-\mathcal{E}^{c,\epsilon}(u_i)$ is bounded away from 0. In particular this implies that $\|\Theta(u_i)\|_{L^1}$ is bounded away from 0. This with the fact that $u \in \mathcal{U}_m$ implies that $\|u_i\|_{L^2}$ is bounded away from 0. We let

$$S_i := \sup_{p\in\mathbb{Z}^n} \int_{K_p} u_i^2 d\mathscr{L}^n.$$

Then (S_i) is a bounded sequence and we can suppose that it converges up to subsequences to a limit $S \geq 0$.

We show that $S > 0$. Indeed, assume by contradiction that $S = 0$. Let K_{p_i} be a cube chosen by applying Lemma 4.12 to u_i, with $p_i \in \mathbb{Z}^n$. Since $Q(u_i)$ is bounded and $\|u_i\|_{L^2}$ is bounded below, then for a constant $\delta > 0$ that does not depend on i

$$|K_{p_i} \cap \{u_i > 0\}| \geq 2\delta.$$

Let

$$v_i := u_i 1_{\mathbb{R}^n \setminus \widetilde{K}_{p_i}},$$

where $\widetilde{K}_{p_i}$ is a slightly smaller version of K_{p_i} chosen so that the integral of u_i^2 on $\partial\widetilde{K}_{p_i}$ goes to 0 as $i \to \infty$ (using the assumption that $S = 0$), and $|\widetilde{K}_{p_i} \cap \{u_i > 0\}| \geq \delta$. Then, since $S = 0$

$$\mathcal{E}^{c,\epsilon}(v_i) \leq \mathcal{E}^{c,\epsilon}(u_i) + o_{i\to\infty}(1)$$

so that (v_i) is also a minimizing sequence. Since $|\{v_i > 0\}| \leq m - \delta$, letting $t := \left(\frac{m}{m-\delta}\right)^{1/n}$, $v_i(\cdot/t)$ is in $\mathcal{U}_m$ and we get

$$\inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} \leq \liminf_{i\to\infty} \mathcal{E}^{c,\epsilon}\left(v_i(\cdot/t)\right) \leq t^n \liminf_{i\to\infty} \mathcal{E}^{c,\epsilon}(v_i) = t^n \inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon},$$

which is absurd (recall that $\inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} < 0$). Hence, for all i large enough there exists $p_i \in \mathbb{Z}^n$ such that

$$\int_{K_{p_i}} u_i^2 d\mathscr{L}^n \geq \frac{S}{2} > 0. \tag{4.16}$$

We translate each u_i in $\mathbb{R}^n$ and assume that $p_i = 0$ for all i. Using Lemma 4.13, there exists $u \in SBV^{1/2}(\mathbb{R}^n)$ such that

$$u_i \underset{L^2_{\text{loc}}}{\longrightarrow} u \neq 0$$

with local lower semicontinuity on Q and $N^{c,\epsilon}$.

Step 2: The limit of $(\mathbf{u_i})_\mathbf{i}$ is a minimizer of $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_{m'}$ where $m' := |\{u > 0\}| \in]0, m]$. By Step 1 we know that u is nontrivial. Let $v \in \mathcal{U}_{m'}$, and let us consider the functions in $SBV^{1/2}(\mathbb{R}^n)$ given by

$$v^r = 1_{B_r} v + 1_{\mathbb{R}^n \setminus B_r} u \qquad \text{and} \qquad v_i^r = 1_{B_r} v + 1_{\mathbb{R}^n \setminus B_r} u_i.$$

We claim that we can find a set $D \subseteq \mathbb{R}$ with $|D| = 0$ such that for $r \notin D$

$$\mathcal{E}^{c,\epsilon}(u) - \mathcal{E}^{c,\epsilon}(v^r) \leq C \liminf_{i\to\infty} \left(1 - \frac{m}{m_i^r}\right)_+, \tag{4.17}$$

and

$$\liminf_{i\to\infty} (m_i^r - m) \leq e_r \underset{r\to\infty}{\longrightarrow} 0, \tag{4.18}$$

where $m_i^r := |\{v_i^r > 0\}|$, and C does not depend on i and k.

We can then find $r_k \to +\infty$ with $r_k \notin D$ such that

$$\mathcal{E}^{c,\epsilon}(v^{r_k}) \underset{k\to\infty}{\longrightarrow} \mathcal{E}^{c,\epsilon}(v). \tag{4.19}$$

Indeed we have

$$J_{v^r} = (J_u \cap (\mathbb{R}^n \setminus \overline{B_r})) \cup (J_v \cap B_r) \cup J^r,$$

where J^r is the subset of points in ∂B_r where $\overline{u1_{\mathbb{R}^n \setminus B_r}} \neq \overline{v1_{B_r}}$. Choose r such that $\mathscr{H}^{n-1}(\partial B_r \cap J_u) = \mathscr{H}^{n-1}(\partial B_r \cap J_v) = 0$, which amounts to choosing almost any $r > 0$. Then

$$\mathcal{E}^{c,\epsilon}(v^r) - \mathcal{E}^{c,\epsilon}(v) = \mathcal{E}^{c,\epsilon}(u1_{|\mathbb{R}^n \setminus B_r}) - \mathcal{E}^{c,\epsilon}(v1_{|\mathbb{R}^n \setminus B_r}) + S^r$$

where

$$S^r = \frac{\beta}{2} \int_{J^r} \left[(u^2 + 2cu^{1+\epsilon}) + (v^2 + 2cv^{1+\epsilon})\right] d\mathscr{H}^{n-1}. \tag{4.20}$$

$\mathcal{E}^{c,\epsilon}(u1_{|\mathbb{R}^n \setminus B_r}) - \mathcal{E}^{c,\epsilon}(v1_{|\mathbb{R}^n \setminus B_r})$ goes to zero as r goes to infinity by dominated convergence. Since u and v belong to L^2 with supports of volume m', they belong also to $L^{1+\epsilon}$, which means that $\int_0^\infty S^r dr < \infty$; in particular this implies that we can find $r_k \to +\infty$ with $r_k \notin D$ such that $S^{r_k} \to 0$, so that (4.19) follows.

Taking (4.19), (4.17) and (4.18) into account, we infer easily that

$$\mathcal{E}^{c,\epsilon}(u) \leq \mathcal{E}^{c,\epsilon}(v),$$

i.e. u is a minimizer of $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_{m'}$. In particular, u is supported on a ball.

In order to conclude Step 2, we prove claims (4.17) and (4.18).

(a) Let us consider claim (4.17). Comparing v^r and u we get

$$\mathcal{E}^{c,\epsilon}(u) - \mathcal{E}^{c,\epsilon}(v^r) = \mathcal{E}^{c,\epsilon}(u1_{|B_r}) - \mathcal{E}^{c,\epsilon}(v1_{|B_r}) - S^r,$$

where S^r is given by (4.20). Setting

$$S_i^r = \frac{\beta}{2}\int_{\partial B_r \cap \{u \neq v\}^1} \left[(u_i^2 + 2cu_i^{1+\epsilon}) + (v^2 + 2cv^{1+\epsilon})\right] d\mathscr{H}^{n-1}$$

we have for a.e. $r > 0$

$$S_i^r \underset{i\to\infty}{\longrightarrow} S^r$$

as $u_i \underset{L^2(\partial B_r)}{\to} u$ for a.e. $r > 0$. We infer

$$\begin{aligned}
\mathcal{E}^{c,\epsilon}(u) - \mathcal{E}^{c,\epsilon}(v^r) &\leq \liminf_{i\to\infty} \left(\mathcal{E}^{c,\epsilon}_{|B_r}(u_i) - \mathcal{E}^{c,\epsilon}_{|B_r}(v)\right) - S^r \\
&= \liminf_{i\to\infty} \left(\mathcal{E}^{c,\epsilon}(u_i) - \mathcal{E}^{c,\epsilon}(v_i^r) + S_i^r\right) - S^r \\
&= \liminf_{i\to\infty} \left(\mathcal{E}^{c,\epsilon}(u_i) - \mathcal{E}^{c,\epsilon}(v_i^r)\right) \\
&= \inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} - \limsup_{i\to\infty} \mathcal{E}^{c,\epsilon}(v_i^r).
\end{aligned} \tag{4.21}$$

If $m_i^r \leq m$, we know this last quantity is nonpositive. Suppose now that $m_i^r > m$, and let

$$w_i^r(x) := v_i^r\left(\left[\frac{m_i^r}{m}\right]^{1/n} x\right).$$

Then w_i^r has a support of volume m, and so with a change of variable

$$\begin{aligned}
\inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} \leq \mathcal{E}^{c,\epsilon}(w_i^r) &\leq \mathcal{E}^{c,\epsilon}(v_i^r) + \left(1 - \frac{m}{m_i^r}\right) \|\Theta(v_i^r)\|_{L^1} \\
&\leq \mathcal{E}^{c,\epsilon}(v_i^r) + \left(1 - \frac{m}{m_i^r}\right) \|\Theta(v) + \Theta(u_i)\|_{L^1} \\
&\leq \mathcal{E}^{c,\epsilon}(v_i^r) + C\left(1 - \frac{m}{m_i^r}\right),
\end{aligned}$$

where C does not depend on i, r. Then

$$\inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} \leq \limsup_{i\to\infty} \mathcal{E}^{c,\epsilon}(v_i^r) + C \liminf_{i\to\infty} \left(1 - \frac{m}{m_i^r}\right),$$

so that coming back to (4.21) we deduce

$$\mathcal{E}^{c,\epsilon}(u) - \mathcal{E}^{c,\epsilon}(v^r) \leq C \liminf_{i\to\infty} \left(1 - \frac{m}{m_i^r}\right).$$

Collecting both cases $m_i^r \leq m$ and $m_i^r > m$, we get precisely claim (4.17).

(b) Let us come to claim (4.18). We have for every $r > 0$

$$\begin{aligned}
\liminf_{i\to\infty}(m_i^r - m) &= \liminf_{i\to\infty} \left(|\{v > 0\} \cap B_r| - |\{u_i > 0\} \cap B_r|\right) \\
&= |\{v > 0\} \cap B_r| - \limsup_{i\to\infty} \left(|\{u_i > 0\} \cap B_r|\right) \\
&\leq |\{v > 0\} \cap B_r| - |\{u > 0\} \cap B_r| =: e_r \underset{r\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Step 3: Existence of another sequence $(\mathbf{u_i})_\mathbf{i}$ such that $(\mathbf{u_i}(\cdot - \mathbf{p_i}))_\mathbf{i}$ converges if $m' < m$. Let $R > 0$ be such that the support of u is contained in B_R, so that

$$u_i \underset{L^2_{loc}(\mathbb{R}^n \setminus B_R)}{\longrightarrow} 0. \tag{4.22}$$

We can suppose that

$$\|u_i\|_{L^2(\partial B_R)} \to 0.$$

Let

$$v_i := u_i 1_{\mathbb{R}^n \setminus B_{2R}}.$$

Notice that

$$Q(v_i) + N^{c,\epsilon}(v_i) \leq C. \tag{4.23}$$

We claim

$$\liminf_{i\to\infty} \|v_i\|^2_{L^2} > 0 \tag{4.24}$$

Indeed by contradiction, we would have (up to a subsequence) $\|v_i\|_{L^2(\mathbb{R}^n)} \to 0$, which implies strong L^2 convergence of (u_i) to u and so $\|\Theta(u_i)\|_{L^1} \to \|\Theta(u)\|_{L^1}$. This implies by lower semicontinuity of Q and $N^{c,\epsilon}$

$$\mathcal{E}^{c,\epsilon}(u) \leq \liminf_{i\to\infty} \mathcal{E}^{c,\epsilon}(u_i) = \inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon}.$$

But since $m' < m$, then for $t := \left[\frac{m}{m'}\right]^{1/n} > 1$, we get

$$\mathcal{E}^{c,\epsilon}(u(\cdot/t)) < t^n \inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon}$$

which is absurd since $u(\cdot/t)$ is in $\mathcal{U}_m$ (recall $\inf_{\mathcal{U}_m} \mathcal{E}^{c,\epsilon} < 0$). Then (4.24) follows.

This gives, in particular, that

$$\liminf_{i\to\infty} \left(\frac{c\|v_i\|^2_{L^2}}{\|v_i\|^2_{L^2} + Q(v_i)} \right)^n > 0.$$

Taking into account (4.23), we can then proceed as in Step 2 (by defining the "concentration" S) to show that for each i there exists $p_i \in \mathbb{R}^n$ such that

$$\liminf_{i\to\infty} \left(\|v_i\|_{L^2(K_{p_i})} \right) > 0.$$

Notice that in view of (4.22) we know that necessarily

$$|p_i| \to +\infty. \tag{4.25}$$

As previously, we can show that $u_i(\cdot - p_i)$ converges in the L^2_{loc} sense to a function v in $\mathcal{U}_{m''}$, where $m'' \in]0, m - m']$ that is a minimizer of $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_{m''}$. As a consequence, its support is also a ball.

Step 4: Construction of an incompatible minimizing sequence in the case $m' < m$. The idea is now to bring together u and v constructed in Step 2 and 3. While each ball is optimal, the union of two disjoint balls is not (because it is not a ball, which is the only possible minimizer). Let $R > 0$ be big enough such that the supports of u and v are contained in B_R. Let

$$w_i(x) := (1_{B_R(p_i)} u_i)(p_i + x).$$

We can choose R slightly bigger such that $\|w_i\|_{L^2(\partial B_R(p_i))} \to 0$. Let

$$\widetilde{u_i}(x) := 1_{\mathbb{R}^n \setminus (B_R(2Re_n) \cup B_R(p_i))}(x) u_i(x) + w_i(x - 2Re_n).$$

Then, up to choosing a slightly bigger R (so that the boundary terms all go to 0), we get

$$\mathcal{E}^{c,\epsilon}(\widetilde{u_i}) \leq \mathcal{E}^{c,\epsilon}(u_i) + o_{i\to\infty}(1).$$

By construction $|\{\widetilde{u_i} > 0\}| \leq m$, so up to a dilation that can only decrease $\mathcal{E}^{c,\epsilon}$, $(\widetilde{u_i})$ is also a minimizing sequence.

By similar arguments, we know $\widetilde{u_i}$ converges to a minimizer $\widetilde{u}$ for a certain volume less than m. But by construction, the support of this limit is exactly the union of two balls, so it cannot be a minimizer. This means that our assumption $m' < m$ is false.

The consequence is that $m' = m$, and so that u is a minimizer of $\mathcal{E}^{c,\epsilon}$ in $\mathcal{U}_m$. □

4.4 The ball minimizes the geometric functional

Proposition 4.14. *Let $u \in \mathcal{U}_m$, then*

$$\mathcal{E}^c(u) \geq E^c(B^m).$$

Proof. Let $u_m^{c,\epsilon}$ be a minimizer obtained in Theorem 4.4 whose support is B^m. Then

$$E^c(B^m) \leq \mathcal{E}^c(u_m^{c,\epsilon}).$$

Moreover:

$$\mathcal{E}^c(u_m^{c,\epsilon}) - \mathcal{E}^{c,\epsilon}(u_m^{c,\epsilon}) = c\beta \int_{\partial B^m} \left[u_m^{c,\epsilon} - (u_m^{c,\epsilon})^{1+\epsilon}\right] d\mathscr{H}^{n-1}.$$

A sufficient condition for this term to go to 0 when ϵ goes to 0 is

$$\limsup_{\epsilon \to 0} \|u_m^{c,\epsilon}\|_{L^\infty(\partial B^m)} < \infty.$$

Since $u_m^{c,\epsilon}$ is radial,

$$C(n,\beta,c,m) = \mathcal{E}^{c,\epsilon}(1_{B^m}) \geq \mathcal{E}^{c,\epsilon}(u_m^{c,\epsilon}) = \left[Q(u_m^{c,\epsilon}) - \int_{B^m} \Theta(u_m^{c,\epsilon}) d\mathscr{H}^{n-1}\right] + N^{c,\epsilon}(u_m^{c,\epsilon}).$$

The first term is bounded below uniformly in ϵ due to inequality (4.11) and

$$N^{c,\epsilon}(u_m^{c,\epsilon}) = c\beta \mathscr{H}^{n-1}(\partial B^m) \|u_m^{c,\epsilon}\|^{1+\epsilon}_{L^\infty(\partial B^m)},$$

so $\|u_m^{c,\epsilon}\|_{L^\infty(\partial B^m)}$ is bounded uniformly as $\epsilon \to 0$.

Take u in $\mathcal{U}_m$. We know from Theorem 4.4 that

$$\mathcal{E}^{c,\epsilon}(u) \geq \mathcal{E}^{c,\epsilon}(u_m^{c,\epsilon})$$

The previous discussion gives

$$\liminf_{\epsilon \to 0} \mathcal{E}^{c,\epsilon}(u) \geq E^c(B^m).$$

In order to conclude, we may assume $\mathcal{E}^c(u) < +\infty$ and it suffices to check that

$$\mathcal{E}^{c,\epsilon}(u) \underset{\epsilon \to 0}{\longrightarrow} \mathcal{E}^c(u),$$

which amounts to show

$$\lim_{\epsilon \to 0} \int_{J_u} (\overline{u}^{1+\epsilon} + \underline{u}^{1+\epsilon}) d\mathscr{H}^{n-1} = \int_{J_u} (\overline{u} + \underline{u}) d\mathscr{H}^{n-1}.$$

Since $u^{1+\epsilon} \leq u + u^2$ for all u and all $\epsilon \in]0,1[$, the dominated convergence leads to the result. □

4.5 Proof of Theorems 4.1 and 4.2

Proof of Theorem 4.1. Recall that

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 d\mathscr{L}^n + \frac{\beta}{2} \int_{J_v} \left[\underline{u}^2 + \overline{u}^2\right] d\mathscr{H}^{n-1} - \frac{1}{q} \int_{\mathbb{R}^n} u^q \, dx.$$

We shall prove the following stronger result.

Theorem 4.15. *Let $u \in \mathcal{U}_m$. Then*

$$\mathcal{E}(u) - E(B^m) \geq \frac{\beta}{2} \left(\operatorname*{essinf}_{\{u>0\}} u\right)^2 \left[\int_{J_u} \left(1_{\{\overline{u}>0\}} + 1_{\{\underline{u}>0\}}\right) d\mathscr{H}^{n-1} - \mathscr{H}^{n-1}(\partial B^m)\right].$$

Remark 4.16. Note that $\int_{J_u}\left(1_{\{\overline{u}>0\}}+1_{\{\underline{u}>0\}}\right)d\mathscr{H}^{n-1}$ is larger than $\mathscr{H}^{n-1}(J_u)$, since it counts twice the points in J_u on which both $\overline{u}$ and $\underline{u}$ are positive.

Proof. We may assume $\operatorname*{essinf}_{\{u>0\}} u > 0$. Let $c \in]0, \operatorname*{essinf}_{\{u>0\}} u[$. Then $(u-c)_+$ belongs to $\mathcal{U}_m$ and Proposition 4.14 gives

$$\mathcal{E}^c((u-c)_+) \geq E^c(B^m).$$

Moreover

$$\mathcal{E}^c((u-c)_+) = \mathcal{E}(u) - \frac{\beta}{2}c^2\int_{J_u}\left(1_{\{\overline{u}>0\}}+1_{\{\underline{u}>0\}}\right)d\mathscr{H}^{n-1} + \frac{c^q}{q}m$$

and

$$E^c(B^m) \geq E(B^m) - \frac{\beta}{2}c^2\mathscr{H}^{n-1}(B^m) + \frac{c^q}{q}m,$$

so that the result follows. This proves as well Theorem 4.1 taking u to be the minimizer of $E(\cdot;\Omega)$, extended by 0 on $\mathbb{R}^n \setminus \Omega$. □

Proof of Theorem 4.2. It is enough to show:

$$E(\Omega) - E(B) \geq C\mathcal{A}(\Omega)^2, \tag{4.26}$$

where $C > 0$ depends on n, β, q and $|\Omega|$, only. Indeed we know that $E(\Omega) = \frac{q-2}{2q}\lambda_q(\Omega)^{\frac{q}{q-2}}$, so

$$C\mathcal{A}(\Omega)^2 \leq \frac{q-2}{2q}\lambda_q(\Omega)^{\frac{q}{q-2}} - \frac{q-2}{2q}\lambda_q(B)^{\frac{q}{q-2}}.$$

Since $\mathcal{A}(\Omega) \leq 2$, up to replacing C with a smaller constant, we loose no generality by supposing that $\lambda_q(\Omega)$ is close to $\lambda_q(B)$. For $\lambda_q(\Omega)$ close enough to $\lambda_q(B)$, we have:

$$\frac{q-2}{2q}\lambda_q(\Omega)^{\frac{q}{q-2}} - \frac{q-2}{2q}\lambda_q(B)^{\frac{q}{q-2}} \leq \lambda_q(B)^{\frac{2}{q-2}}\left(\lambda_q(\Omega) - \lambda_q(B)\right),$$

which proves the result. Let us now show (4.26).

From now on, we let $m := |\Omega|$. The idea is to use an intermediate set A such that $\mathcal{A}(A)$ and $\mathcal{A}(\Omega)$ are of the same order, but $\inf\ u_A \geq C(n, \beta, |\Omega|)$ for a certain positive constant. This may be done by considering the following auxiliary minimisation problem

$$\inf\{E(A) + k|A|,\ A \subset \Omega\}, \tag{4.27}$$

for a suitable constant $k > 0$ (that may be chosen such that if we restrict the admissible sets to balls of volume less than m, the optimal ball has the measure equal to m). We need to check such a minimizer exists. Showing the existence and regularity of such a minimizer is similar to the work that has been done in Section 4.3, we outline the main steps.

Step 1. We introduce the following relaxed version in $SBV^{\frac{1}{2}}$

$$\inf\{\mathcal{E}_k(u),\ |\{u>0\} \setminus \Omega| = 0\},$$

where $\mathcal{E}_k(u) = \mathcal{E}(u) + k|\{u>0\}|$.

Step 2. We show the existence of a minimizer of the relaxed problem. Take a minimizing sequence (u_i), up to subsequences we can suppose it converges in $L^2_{\text{loc}}(\mathbb{R}^n)$ to a function u with support in Ω. Since Q is lower semi-continuous, we only need to show that (u_i) is tight in the sense that

$$\limsup_{R\to\infty}\sup_{i\geq 1}\left(\|u_i\|_{L^2(\Omega\setminus B_R)}\right) = 0.$$

With the L^2_{loc} convergence, this will prove that $\|u_i\|_{L^q(\mathbb{R}^n)} \underset{i\to\infty}{\longrightarrow} \|u\|_{L^q(\mathbb{R}^n)}$. Let χ_R be a smooth function such that

$$\begin{cases} 0 \le \chi_R \le 1 \\ \chi_R \equiv 1 \text{ in } \mathbb{R}^n \setminus B_R \\ \chi_R \equiv 0 \text{ in } B_{R-1} \\ \|\nabla \chi_R\|_{L^\infty(\mathbb{R}^n)} \le 2, \end{cases}$$

and let $v_{i,R} = \chi_R u_i$. Then:

$$\begin{aligned} \|u_i\|^2_{L^2(\mathbb{R}^n\setminus B_R)} &\le \|v_i\|^2_{L^2(\mathbb{R}^n\setminus B_R)} \\ &\le \lambda_2\left(B^{|\{v_i>0\}|}\right)^{-1} Q(v_i) && \text{using inequality (4.11) for } q=2 \\ &\le \lambda_2\left(B^{|\Omega\setminus B_{R-1}|}\right)^{-1} (2Q(u_i) + 8\|u_i\|^2_{L^2(\mathbb{R}^n)}) && \text{since } |\nabla v_i|^2 \le 2|\nabla u_i|^2 + 8u_i^2 \\ &\le C\lambda_2\left(B^{|\Omega\setminus B_{R-1}|}\right)^{-1} Q(u_i) && \text{where } C \text{ depends on } n, m. \end{aligned}$$

$Q(u_i)$ is bounded uniformly in i and $\lambda_2\left(B^{|\Omega\setminus B_{R-1}|}\right)^{-1}$ goes to 0 when $R \to \infty$: this proves what we wanted to show.

Thus we know that

$$\mathcal{E}_k(u) \le \liminf_{i\to\infty} \mathcal{E}_k(u_i)$$

which means u is a minimizer.

Step 3. We show that $u \ge c1_{\{u>0\}}$ for a certain $c = c(n, \beta, k, E(\Omega)) > 0$; this is essentially the same proof as Lemma 4.8 in the case $\epsilon = 1$. We then show that u is bounded and defines an open domain A the same way we did in Lemma 4.9 and Lemma 4.10.

Step 4. The map $\rho \mapsto E(B_\rho)$ is a strictly decreasing function since for any ball $B' \Subset B$, one has:

$$E(B) \le \frac{|B|}{|B'|} E(B'),$$

which implies

$$\frac{d}{d\rho} E(B_\rho) \le \frac{nE(B_\rho)}{\rho} (< 0).$$

We deduce that for any $k > 0$ small enough depending only on the parameters (n, q, β, m), the function

$$\begin{cases} [0, r_m] \to \mathbb{R} \\ \rho \mapsto E(B_\rho) + 2k|B_\rho| \end{cases} \tag{4.28}$$

admits a minimum in r_m (where r_m is such that $|B_{r_m}| = m$).

Step 5 As remarked at the beginning of the proof, we can suppose without loss of generality that $E(\Omega) \in [E(B), \frac{1}{2}E(B)]$. Letting k be given by Step 4, the solution u of the relaxed problem (4.27) is thus such that $u \ge c$ on its support A, where c depends only on the parameters (n, q, β, m).

We can then write (with c being a constant only depending on (n, q, β, m) that may not be the same from line to line):

$$\begin{aligned} E(\Omega) &\ge E(A) - k|\Omega \setminus A| && \text{because } A \text{ solves (4.27)}, \\ &\ge E(B^{|A|}) - k|\Omega \setminus A| + c\Big[\mathscr{H}^{n-1}(\partial A) - \mathscr{H}^{n-1}(\partial B^{|A|})\Big] && \text{in view of Theorem 4.1}, \\ &\ge E(B^{|A|}) - k|\Omega \setminus A| + c|A \triangle B^{|A|}|^2 && \text{using the inequality (4.1)}, \\ &\ge E(B^{|\Omega|}) + k|\Omega \setminus A| + c|A \triangle B^{|A|}|^2 && \text{using that (4.28) has a minimum in } r_m, \\ &\ge E(B^{|\Omega|}) + c\Big[2|\Omega \setminus A| + |A \triangle B^{|A|}|\Big]^2 && \text{because } |\Omega \setminus A| \ge |\Omega|^{-1}|\Omega \setminus A|^2, \\ &\ge E(B^{|\Omega|}) + c\mathcal{A}(\Omega)^2 && \text{we detail that point below}, \end{aligned}$$

so that Theorem 4.2 is proved.

We detail the last inequality. Let $B^{|\Omega|}$ be any ball containing $B^{|A|}$. Then:

$$2|\Omega \setminus A| + |A \triangle B^{|A|}| = \left(|\Omega \setminus A| + |A \setminus B^{|A|}|\right) + \left(|\Omega \setminus A| + |B^{|A|} \setminus A|\right).$$

On one hand

$$|\Omega \setminus A| + |A \setminus B^{|A|}| \geq |\Omega \setminus B^{|A|}| \geq |\Omega \setminus B^{|\Omega|}|.$$

On the other hand

$$\begin{aligned} |\Omega \setminus A| + |B^{|A|} \setminus A| &= |\Omega| - |A| + |B^{|A|}| - |B^{|A|} \cap A| \\ = |B^{|\Omega|}| - |B^{|A|} \cap A| &= |B^{|\Omega|} \setminus (A \cap B^{|A|})| \geq |B^{|\Omega|} \setminus \Omega|. \end{aligned}$$

Bibliography

[73] N. Ambrosio, L. Fusco and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[74] Marie-Hélène Bossel. Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):47–50, 1986.

[75] Lorenzo Brasco and Guido De Philippis. Spectral inequalities in quantitative form. In *Shape optimization and spectral theory*, pages 201–281. De Gruyter Open, Warsaw, 2017.

[76] Lorenzo Brasco, Guido De Philippis, and Berardo Ruffini. Spectral optimization for the Stekloff-Laplacian: the stability issue. *J. Funct. Anal.*, 262(11):4675–4710, 2012.

[77] Lorenzo Brasco, Guido De Philippis, and Bozhidar Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.*, 164(9):1777–1831, 2015.

[78] Lorenzo Brasco and Aldo Pratelli. Sharp stability of some spectral inequalities. *Geom. Funct. Anal.*, 22(1):107–135, 2012.

[79] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. The quantitative Faber-Krahn inequality for the Robin Laplacian. *J. Differential Equations*, 264(7):4488–4503, 2018.

[80] Dorin Bucur and Alessandro Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[81] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[82] Dorin Bucur, Alessandro Giacomini, and Paola Trebeschi. Best constant in Poincaré inequalities with traces: a free discontinuity approach. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(7):1959–1986, 2019.

[83] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[84] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[85] Marco Cicalese and Gian Paolo Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.*, 206(2):617–643, 2012.

[86] Daniel Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.

[87] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[88] Nicola Fusco and Yi Ru-Ya Zhang. A quantitative form of Faber-Krahn inequality. *Calc. Var. Partial Differential Equations*, 56(5):Paper No. 138, 44, 2017.

[89] W. Hansen and N. Nadirashvili. Isoperimetric inequalities in potential theory. In *Proceedings from the International Conference on Potential Theory (Amersfoort, 1991)*, volume 3, pages 1–14, 1994.

[90] Antonios D. Melas. The stability of some eigenvalue estimates. *J. Differential Geom.*, 36(1):19–33, 1992.

Chapter 5

Stability and instability issues of the Weinstock inequality

This is a joint work with Dorin Bucur.

Contents

5.1 Introduction **93**
5.2 The Steklov problem and some stability properties **95**
5.3 A stability result under a priori bounds **102**
5.4 Further remarks **106**
5.5 Stability of the Hersch inequality **112**

5.1 Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, open, connected set with a Lipschitz boundary (generically called throughout the paper *smooth set*). We consider the Steklov eigenvalue problem on Ω

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

For $k = 0, 1, 2, \dots$ we denote by $\sigma_k(\Omega)$ the k-th eigenvalue defined by

$$\sigma_k(\Omega) = \inf_{U \in \mathcal{U}_k(\Omega)} \sup_{u \in U \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_{\partial\Omega} u^2 \mathrm{d}s},$$

where $\mathcal{U}_k(\Omega)$ is the family of subspaces of dimension $k+1$ of $H^1(\Omega)$. Then

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \cdots \to +\infty.$$

Below, we denote by d_H the Hausdorff distance between two sets. The first result of the paper reads as follows.

Theorem 5.1. *Let Ω, ω be two smooth conformal sets. Then there exists a sequence (Ω_ϵ) of smooth sets homeomorphic to Ω such that $d_H(\partial\Omega_\epsilon, \partial\Omega) \underset{\epsilon\to 0}{\longrightarrow} 0$ and*

$$|\partial\Omega_\epsilon| \sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} |\partial\omega| \sigma_k(\omega)$$

for any k.

In other words, for every $k \in \mathbb{N}$, in any tubular neighbourhood of Ω, one can find a smooth set Ω_ϵ which has almost the same first (normalized) k eigenvalues as ω.

Theorem 5.1 answers in the negative a question raised by Girouard and Polterovich in [103] (see also [102, Open problem 5.21]) concerning the stability of the Weinstock inequality. Weinstock proved in [110] that the disk maximizes the product between the perimeter and the first Steklov eigenvalue in the class of simply connected planar sets. For the disk, the value of this product equals 2π. The open problem of Girouard and Polterovich reads

"Let Ω be a planar simply connected domain such that the difference $2\pi - |\partial\Omega|\sigma_1(\Omega)$ is small. Show that Ω must be close to a disk (in the sense of Fraenkel asymmetry or some other measure of proximity)."

Theorem 5.1 gives a negative answer to the question and, even more, it states that the maximal value 2π can be asymptotically achieved in the geometric neigbourhood of any smooth simply connected set.

However, in the recent years, several spectral isoperimetric inequalities have been proved to be stable, in the vein of the quantitative isoperimetric inequality proved by Fusco, Maggi and Pratelli in 2008 [97]. Stability, involving the Fraenkel asymmetry, holds for the Faber-Krahn, the Saint-Venant or the Sezgö-Weinberger inequalites, but also many others. For the Steklov problem, it was proved by Brasco, De Phillipis and Ruffini in 2012 [93] that the Brock version of the Weinstock inequality is stable. This inequality involves the volume of the set instead of the volume of the boundary as a constraint. Precisely, it is proved in [93] that in $\mathbb{R}^N$, for every open, bounded and smooth set, it holds

$$|B|^{\frac{1}{N}}\sigma_1(B) - |\Omega|^{\frac{1}{N}}\sigma_1(\Omega) \geq C_N \mathcal{A}^2(\Omega),$$

where B is a ball and $\mathcal{A}$ is the Fraenkel asymmetry.

Coming back in $\mathbb{R}^2$ to the original Weinstock inequality, if one restricts to the class of convex sets, Weinstock himself implicitly found a stable version of the inequality. Precisely, he proved that if $\Omega \subseteq \mathbb{R}^2$ is a bounded, convex set containing the origin, then

$$\pi \int_{\partial\Omega} |x|^2 \mathrm{d}s - |\Omega||\partial\Omega| \geq \frac{|\partial\Omega|}{2} \int_{S^1} (h - \overline{h})^2 \mathrm{d}s,$$

where h is the support function of the convex set, and $\overline{h}$ is its average. This inequality, readily gives a quantitative form of the inequality, in the class of convex sets

$$2\pi - |\partial\Omega|\sigma_1(\Omega) \geq \frac{|\partial\Omega|}{\int_{\partial\Omega} |x|^2 \mathrm{d}x} \int_{S^1} (h - \overline{h})^2 \mathrm{d}s. \tag{5.1}$$

In a recent paper [98] (still in the class of convex sets) the right hand side is replaced by $C\mathcal{A}^{\frac{5}{2}}(\Omega)$. We point out that, very surprisingly, inequality (5.1) is published only in the preprint version [111] and does not figure in the final version the paper [110].

In $\mathbb{R}^N$, for $N \geq 3$, under a boundary volume constraint, a similar version to the Weinstock inequality is proved to hold in the class of convex sets (see [94]) and it is proved, by Fraser and Schoen, not to hold in the class of contractible domains (see [96]). Under convexity hypotheses (see [94]) one has

$$\frac{\int_{\partial\Omega} |x|^2 \mathrm{d}s}{|\partial\Omega||\Omega|^{\frac{2}{N}}} \geq \omega_N^{-\frac{2}{N}}, \tag{5.2}$$

which by standard use of test functions leads to the N-dimensional version of the Weinstock inequality (above ω_N is the volume of the unit ball in $\mathbb{R}^N$, and the right hand side corresponds to the ball). A quantitative version in terms of the Fraenkel asymmetry has been established in [98].

Theorem 5.1 in this paper asserts that there is no general stability of the Weinstock inequality in the class of simply connected sets in $\mathbb{R}^2$. We analyse this issue and give a stability result provided some rigidity is a priori known on the boundary. The convexity assumption in [98] is such a rigidity. Our result involves some control of the norm of the conformal mapping but allows oscillations of

the boundary. Precisely, for all $K > 0$, $\alpha \in]0,1]$, we denote $\mathcal{A}(K,\alpha)$ the family of smooth simply connected sets Ω with perimeter 2π such that there exists a conformal mapping $g : \mathbb{D} \to \Omega$ with

$$\| \log(|g'|) \|_{\mathcal{C}^{0,\alpha}(\partial\mathbb{D})} \leq K.$$

Theorem 5.2. *There exists a constant $C = C(K,\alpha) > 0$ such that for any $\Omega \in \mathcal{A}(K,\alpha)$*

$$\sigma_1(\mathbb{D}) - \sigma_1(\Omega) \geq C\Big(\overline{d_H}(\Omega,\mathbb{D})\Big)^{2(1+\alpha^{-1})}.$$

Above, $\overline{d_H}(\Omega,\omega) = \inf_{z\in\mathbb{R}^2} d_H(\Omega,\omega+z)$. We shall prove that this result is sharp in the sense that no weaker a priori estimate is enough to obtain stability; in particular, a bound on $\| \log(|g'|) \|_{\mathcal{C}^0(\partial\mathbb{D})}$ is not sufficient. However, there is no indication that the exponent $2(1+\alpha^{-1})$ is sharp.

The paper is organized as follows: in the second section, we prove Theorem 5.1. The key of the proof is an approximation result of a weighted Steklov problem by a sequence of Steklov problems without weights but with oscillating boundaries. In the third section, we study the stability of the Steklov eigenvalue on the disk in terms of perturbation of the constant weight on the boundary. The geometric interpretation of this result gives us Theorem 5.2. The last section is devoted to the study of the sharpness of this result with some explicit computations.

5.2 The Steklov problem and some stability properties

Throughout the paper we identify $\mathbb{R}^2$ with $\mathbb{C}$ by $(x_1,x_2) \to z = x_1 + ix_2$. For any smooth set Ω of $\mathbb{R}^2$ and any non-negative function $\Theta \in L^\infty(\partial\Omega), \Theta \neq 0$ (generically called below *weight*) we define the (generalized) Steklov eigenvalues with the weight Θ on the boundary by

$$\sigma_k(\Omega,\Theta) = \inf_{U\in\mathcal{U}_k(\Omega)} \sup_{u\in U\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_{\partial\Omega} \Theta u^2 \mathrm{d}s},$$

where $\mathcal{U}_k(\Omega)$ stands for the family of subspaces of dimension $k+1$ in $H^1(\Omega)$. The spectrum is well defined, as a direct consequence of the continuous embedding $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\partial\Omega)$ and of the compact embedding $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

In particular, we have

$$\sigma_1(\Omega,\Theta) = \inf_{\substack{u\in H^1(\Omega) \\ \int_{\partial\Omega}\Theta u=0}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_{\partial\Omega} \Theta u^2 \mathrm{d}s}.$$

The instability result lies on two observations:

- Assume Ω and ω are two smooth domains with weights $\Theta_\Omega \in L^\infty(\partial\Omega,(0,+\infty))$, $\Theta_\omega \in L^\infty(\partial\omega,(0,+\infty))$ on the boundary, such that there exists a conformal mapping $g : \Omega \to \omega$ with $g \in \mathcal{C}^1(\overline{\Omega},\overline{\omega})$ and $g' \neq 0$ everywhere on $\overline{\Omega}$, and $|g'(z)|\Theta_\omega(g(z)) = \Theta_\Omega(z)$. Under this condition, one has $\sigma_k(\Omega,\Theta_\Omega) = \sigma_k(\omega,\Theta_\omega)$.
- By a boundary homogeneization process, a domain Ω with a bounded weight $\Theta \geq 1$ can be geometrically approximated by domains (Ω_ϵ) such that $\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \sigma_k(\Omega,\Theta)$. Roughly speaking, the sets (Ω_ϵ) are perturbations of Ω with boundaries $\partial\Omega_\epsilon$ which are locally build as graphs of functions f_ϵ on $\partial\Omega$, with small L^∞ norm, and such that $\sqrt{1+|f_\epsilon'|^2} \approx \Theta$.

As a notation, throughout the paper, by (Ω_ϵ) we denote a sequence of open sets consisting of geometric perturbations of a set Ω, where ϵ is a sequential, vanishing, positive parameter. We start by introducing a notion of convergence of domains.

Definition 5.3. We say that a sequence of smooth sets (Ω_ϵ) converges uniformy to a smooth set Ω if

- $\partial\Omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \partial\Omega$ in the Hausdorff sense
- (Ω_ϵ) verify a uniform cone condition (see for instance [104]), meaning that there exists $\delta \in]0, \pi/2[$ such that for any $\epsilon > 0$ and any $x \in \partial\Omega_\epsilon$, there exists $\zeta_{x,\epsilon}$ a unitary vector such that for all $y \in \overline{\Omega_\epsilon} \cap B(x,\delta)$
$$\{z \in B(y,\delta) \text{ with } \langle z-y, \zeta_{x,\epsilon}\rangle \geq \cos(\delta)|z-y|\} \subset \overline{\Omega_\epsilon}.$$

In particular (see [104]), this convergence implies that

$$\limsup_{\varepsilon\to 0} |\partial\Omega_\epsilon| < +\infty.$$

Under the hypotheses of Definition 5.3, the set Ω also satisfies the uniform cone condition and there exists a constant $M > 0$ such that all u in $H^1(\Omega)$ or $H^1(\Omega_\epsilon)$ can be extended to a function of $H^1(\mathbb{R}^2)$ with

$$\|\tilde{u}\|_{H^1(\mathbb{R}^2)} \leq M\|u\|_{H^1(\Omega_\epsilon) \text{ or } H^1(\Omega)}.$$

In addition, there exist a finite number of squares $(C_i)_{i=1,\dots,K}$ centered in $p_i \in \mathbb{R}^2$ with radius r_i, that cover $\partial\Omega$ and all $\partial\Omega_\epsilon$, and such that for any i there exist an orthonormal basis (e_i, f_i) such that

$$\begin{aligned} C_i &= \{p_i + te_i + sf_i,\ t,s \in [-r_i, r_i]\},\\ \partial\Omega \cap C_i &= \{p_i + te_i + g_i(t)f_i,\ t \in [-r_i, r_i]\},\\ \partial\Omega_\epsilon \cap C_i &= \{p_i + te_i + g_{i,\epsilon}(t)f_i,\ t \in [-r_i, r_i]\}, \end{aligned}$$

where the (g_i) and $(g_{i,\epsilon})$ are uniformly Lipschitz functions. Moreover, the Hausdorff convergence of $(\partial\Omega_\epsilon)$ gives

$$\sup_{1\leq i\leq K} \|g_i - g_{i,\epsilon}\|_{L^\infty} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

The following proposition extends a result from [92].

Proposition 5.4. *Let (Ω_ϵ) converge uniformly to Ω and the weights $\Theta_\epsilon \in L^\infty(\partial\Omega_\epsilon)$, $\Theta \in L^\infty(\partial\Omega)$ be such that*

$$\limsup_{\epsilon\to 0} \|\Theta_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} < \infty \text{ and } \Theta_\epsilon \mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \Theta\mathscr{H}^1_{\lfloor\partial\Omega}$$

weakly-$*$ in the sense of measures. Let $(u_\epsilon)_\epsilon$ be a sequence of functions in $H^1(\mathbb{R}^2)$ converging weakly to u. Then

$$\int_{\partial\Omega_\epsilon} \Theta_\epsilon u_\epsilon^2 \mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta u^2 \mathrm{d}s.$$

Proof. We fix a $i \in \{1,\dots,K\}$, and we let $v(t,s) = u(p_i + te_i + sf_i)$ and $v_\epsilon(t,s) = u_\epsilon(p_i + te_i + sf_i)$. We also take $(\psi_i)_i$ a partition of unity associated to the covering (C_i), and

$$\begin{aligned} J(t) &= \psi_i(t, g_i(t))\sqrt{1 + |g_i'(t)|^2}\Theta(t, g_i(t)),\\ J_\epsilon(t) &= \psi_i(t, g_i(t))\sqrt{1 + |g_{i,\epsilon}'(t)|^2}\Theta_\epsilon(t, g_{i,\epsilon}(t)). \end{aligned}$$

Denoting $I = [-r_i, r_i]$, we only need to show that

$$\int_I v_\epsilon(t, g_{i,\epsilon}(t))^2 J_\epsilon(t)dt \underset{\epsilon\to 0}{\longrightarrow} \int_I v(t, g_i(t))^2 J(t)dt.$$

We split the difference in three and use the bound $|J|, |J_\epsilon| \leq M$ for a certain $M > 0$ that only depends on the Lipschitz constant of the $(g_{i,\epsilon})$, on $\|\Theta\|_{L^\infty(\partial\Omega)}$ and on $\sup_\epsilon \|\Theta_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)}$

$$\begin{aligned}\left|\int_I (v_\epsilon(t, g_{i,\epsilon}(t))^2 J_\epsilon(t) - v(t, g_i(t))^2 J(t))dt\right| \leq M \int_I |v_\epsilon(t, g_{i,\epsilon}(t))^2 - v_\epsilon(t, g_i(t))^2|dt \\ + M \int_I |v_\epsilon(t, g_i(t))^2 - v(t, g_i(t))^2|dt \\ + |\int_I v(t, g_i(t))^2 (J_\epsilon(t) - J(t))dt|.\end{aligned}$$

- **First term**. We first show that $v_\epsilon(t, g_{i,\epsilon}(t)) - v_\epsilon(t, g_i(t)) \underset{\epsilon\to 0}{\longrightarrow} 0$ in $L^2(I)$. Since $v_\epsilon \in H^1(\mathbb{R}^2)$, Fubini theorem gives that that $s \mapsto v_\epsilon(t, s)$ belongs to $H^1(\mathbb{R})$ for almost every $t \in I$, with derivative $\partial_s v$. Thus for almost every $t \in I$, we may write

$$\begin{aligned}|v_\epsilon(t, g_{i,\epsilon}(t)) - v_\epsilon(t, g_i(t))| &\leq \int\limits_{[g_{i,\epsilon}(t), g_i(t)]} |\partial_s v_\epsilon(t, s)|ds \\ &\leq |g_{i,\epsilon}(t) - g_i(t)|^{1/2} \left(\int\limits_{[g_{i,\epsilon}(t), g_i(t)]} |\nabla v_\epsilon(t, s)|^2 ds\right)^{1/2}.\end{aligned}$$

  Summing for $t \in I$ and using $\|\nabla v_\epsilon\|^2_{L^2(\mathbb{R}^2)} = \|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^2)}$,

$$\int_I |v_\epsilon(t, g_{i,\epsilon}(t)) - v_\epsilon(t, g_i(t))|^2 dt \leq \|g_i - g_{i,\epsilon}\|_{L^\infty(I)} \|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^2)} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

  Now, we write

$$\begin{aligned}v_\epsilon(t, g_{i,\epsilon}(t))^2 - v_\epsilon(t, g_i(t))^2 =& \Big[v_\epsilon(t, g_{i,\epsilon}(t)) - v_\epsilon(t, g_i(t))\Big]^2 \\ &+ 2v_\epsilon(t, g_i(t))\Big[v_\epsilon(t, g_{i,\epsilon}(t)) - v_\epsilon(t, g_i(t))\Big],\end{aligned}$$

  and we use the fact that (u_ϵ) is bounded in $L^2(\partial\Omega)$, since it is bounded in $H^1(\mathbb{R}^2)$. Consequently, $(v_\epsilon(t, g_i(t))$ is bounded in $L^2(I)$ by a constant $C > 0$, so that

$$\begin{aligned}\int_I |v_\epsilon(t, g_{i,\epsilon}(t))^2 - v_\epsilon(t, g_i(t))^2|dt &\leq \|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^2)} \|g_{i,\epsilon} - g_i\|_{L^\infty(I)} + 2C\|\nabla u_\epsilon\|_{L^2(\mathbb{R}^2)} \|g_{i,\epsilon} - g_i\|^{\frac{1}{2}}_{L^\infty(I)} \\ &\underset{\epsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

- **Second term**. u_ϵ converges weakly to u in $H^1(\mathbb{R}^2)$, so their traces converge weakly in $H^{\frac{1}{2}}(\partial\Omega)$ and strongly in $L^2(\partial\Omega)$. This implies that $v_\epsilon(t, g_i(t))$ converges to $v(t, g_i(t))$ in $L^2(I)$, which gives that the second term goes to 0.

- **Third term**. We use here the hypothesis that $\Theta_\epsilon \mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \Theta\mathscr{H}^1_{\lfloor\partial\Omega}$ weakly-$*$ in the sense of measures. This gives that for any continuous function $\varphi \in \mathcal{C}^0(I)$, we have

$$\int_I \varphi(t)(J_\epsilon(t) - J(t))dt \underset{\epsilon\to 0}{\longrightarrow} 0.$$

  Since $v(t, g_i(t))^2 \in L^1(I)$, we can approximate it with a continuous function φ in the L^1 sense. Then

$$\left|\int_I v^2(J_\epsilon - J)dt - \int_I \varphi(J_\epsilon - J)dt\right| \leq 2M \int_I |v^2 - \varphi|dt,$$

  where M is an upper bound for the $L^\infty(I)$ norms of J_ϵ and J. This gives the result.

□

We are now in position to prove the following homogeneization result.

Proposition 5.5. *Let (Ω_ϵ) be a sequence of domains that converges to Ω uniformly and the weights $\Theta_\epsilon \in L^\infty(\partial\Omega_\epsilon)$, $\Theta \in L^\infty(\partial\Omega)$ be such that*

$$\limsup_{\epsilon\to 0} \|\Theta_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} < \infty \text{ and } \Theta_\epsilon \mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \Theta\mathscr{H}^1_{\lfloor\partial\Omega}$$

weak-$*$ in the measure sense. Then for all $k \geq 1$

$$\sigma_k(\Omega_\epsilon, \Theta_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \sigma_k(\Omega, \Theta).$$

Proof. We prove the result in two steps.
Lower semicontinuity. For all $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}_k(\Omega_\epsilon)$ be a subspace that attains $\sigma_k(\Omega_\epsilon)$, and let $(u_{p,\epsilon})_{p=0,\dots,k}$ be an adapted basis of it (that is, a basis that is orthonormal relative to the quadratic form $\int_{\partial\Omega_\epsilon} \Theta_\epsilon u^2$ and orthogonal relative to $\int_{\Omega_\epsilon} |\nabla u|^2$).

Since $|\partial\Omega_\epsilon|$ converges, and $|\partial\Omega_\epsilon|\sigma_k(\Omega_\epsilon)$ is bounded by $2\pi lk$ according to [101, Theorem 1.2] (where l is the number of connected components of the boundary, which does not depend on ϵ due to the uniform cone condition), we know that the $(u_{p,\epsilon})$ are bounded in $H^1(\mathbb{R}^2)$. Up to an extraction we suppose they converge weakly to some $(u_p)_{p=0,\dots,k} \in H^1(\mathbb{R}^2)$, in particular we have the following semi-continuity inequality: for all $(a_0, \dots, a_k)$ with $\sum_p a_p^2 = 1$

$$\int_\Omega |\nabla \sum_p a_p u_p|^2 \mathrm{d}x \leq \liminf_{\epsilon\to 0} \int_\Omega |\nabla \sum_p a_p u_{p,\epsilon}|^2 \mathrm{d}x \leq \liminf_{\epsilon\to 0} \sigma_k(\Omega_\epsilon, \Theta_\epsilon).$$

Moreover, according to the previous theorem, for all p, p', we have

$$\delta_{p,p'} = \int_{\partial\Omega_\epsilon} \Theta_\epsilon u_{p,\epsilon} u_{p',\epsilon} \mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta u_p u_{p'} \mathrm{d}s.$$

This shows in particular that (u_p) is orthonormal for this scalar product. Thus $\mathrm{Span}(u_0, \dots, u_p)$ lies in $\mathcal{U}_k(\Omega)$ and

$$\sigma_k(\Omega, \Theta) \leq \sup_{\sum_p a_p^2 = 1} \int_\Omega |\nabla \sum_p a_p u_p|^2 \mathrm{d}x \leq \liminf_{\epsilon\to 0} \sigma_k(\Omega_\epsilon, \Theta_\epsilon).$$

Upper semicontinuity. Let $V \in \mathcal{U}_k(\Omega)$ be a subspace that attains $\sigma_k(\Omega, \Theta)$, and let $(v_0, \dots, v_p)$ be an adapted basis of it. When we extend it, $(v_0, \dots, v_k)$ is a linearly independant family of $H^1(\Omega_\epsilon)$ for small enough ϵ. For all ϵ, let $a_\epsilon = (a_{p,\epsilon})_{p=0,\dots,k} \in \mathbb{S}^k$ be chosen such that

$$\sup_{w \in \mathrm{Span}(v_0,\dots,v_k)} \frac{\int_{\Omega_\epsilon} |\nabla w|^2 \mathrm{d}x}{\int_{\partial\Omega_\epsilon} \Theta_\epsilon w^2 \mathrm{d}s} = \frac{\int_{\Omega_\epsilon} |\nabla \sum_p a_{p,\epsilon} v_p|^2 \mathrm{d}x}{\int_{\partial\Omega_\epsilon} \Theta_\epsilon |\sum_p a_{p,\epsilon} v_p|^2 \mathrm{d}s}.$$

We can suppose each $a_{p,\epsilon}$ converge to a certain a_p. Using the previous theorem, we know that

$$\int_{\partial\Omega_\epsilon} \Theta_\epsilon \left|\sum_p a_{p,\epsilon} v_p\right|^2 \mathrm{d}s \underset{\epsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta \left|\sum_p a_p v_p\right|^2 \mathrm{d}s.$$

And the fact that $\Omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Omega$ gives

$$\int_{\Omega_\epsilon} |\nabla \sum_p a_{p,\epsilon} v_p|^2 \mathrm{d}x \underset{\epsilon\to 0}{\longrightarrow} \int_\Omega |\nabla \sum_p a_p v_p|^2 \mathrm{d}x.$$

Using the two previous limits

$$\limsup_{\epsilon\to 0}\sigma_k(\Omega_\epsilon,\Theta_\epsilon)\leq\frac{\int\limits_{\Omega}\left|\nabla\sum_p a_p v_p\right|^2\mathrm{d}x}{\int\limits_{\partial\Omega}\Theta\left|\sum_p a_p v_p\right|^2\mathrm{d}s}\leq\sigma_k(\Omega,\Theta).$$

□

We refer the reader to the recent paper [95], where continuity of the Steklov eigenvalues under geometric domain perturbations is studied in the absence of homogeneization of the boundaries, but under weaker geometrical assumptions not requiring uniform cone condition.

Lemma 5.6. *Let Ω,ω be two smooth sets of $\mathbb{C}$ and $g:\Omega\to\omega$ a conformal map such that $g\in\mathcal{C}^1(\overline{\Omega},\overline{\omega})$. Let $\Theta\in L^\infty(\partial\omega)$ be a positive function. Then*

$$\sigma_k(\omega,\Theta)=\sigma_k(\Omega,g^*\Theta),$$

where $g^*\Theta=|g'|\Theta\circ g$.

Proof. For a function $u\in H^1(\Omega)$, we introduce $v=u\circ g^{-1}$. By change of variable we get

$$\int\limits_{\omega}|\nabla v|^2\mathrm{d}x=\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\text{ and }\int\limits_{\partial\omega}\Theta v^2\mathrm{d}s=\int\limits_{\partial\Omega}(g^*\Theta)u^2\mathrm{d}s.$$

Moreover, if U is in $\mathcal{U}_k(\Omega)$, then the subspace $V=\{u\circ g^{-1},u\in U\}$ is in $\mathcal{U}_k(\omega)$ and the Rayleigh quotient are the same on both space: we have a bijection between $\mathcal{U}_k(\Omega)$ and $\mathcal{U}_k(\omega)$ that preserves the Rayleigh quotient, which proves the result. □

The question on how to approximate a Lipschitz set by a sequence of smooth sets, is a standard question in analysis. The most technical part of the conclusion of the next lemma is contained in [109, Theorem 1.12].

Lemma 5.7. *Let Ω be a smooth domain and $\Theta\in L^\infty(\partial\Omega,[1,+\infty))$. There exists a sequence of domains Ω_ϵ that converges uniformly to Ω and such that*

$$\mathscr{H}^1_{\lfloor\partial\Omega_\epsilon}\rightharpoonup\Theta\mathscr{H}^1_{\lfloor\partial\Omega}$$

weakly-$*$ in the sense of measures.

Proof. Roughly speaking, the proof of the lemma has two components:

- prove first the assertion for a C^2-domain Ω and for a smooth weight $\Theta\in C^1(\partial\Omega)$.
- second, use an approximation procedure to approach both a smooth (Lipschitz) domain Ω uniformly by C^2 domains (Ω_ϵ) and the L^∞-weight Θ by $\Theta_\epsilon\in C^1(\partial\Omega_\epsilon,(1,+\infty))$ such that

$$\Theta_\epsilon\mathscr{H}^1_{\lfloor\partial\Omega_\epsilon}\rightharpoonup\Theta\mathscr{H}^1_{\lfloor\partial\Omega}.$$

  Finally, use a diagonal procedure.

Assume that Ω is of class $\mathcal{C}^2$ and $\Theta\in\mathcal{C}^1(\partial\Omega)$. We construct the sequence (Ω_ϵ) such that $\partial\Omega_\epsilon$ is locally a graph on $\partial\Omega$. Roughly speaking, we shall add oscillations on the boundary of Ω whose (local) amplitudes are chosen such that the perimeter of Ω is locally multiplied by Θ. This is the reason why we need Θ to be larger than 1.

Let us define the basic oscillation $d : \frac{\mathbb{R}}{\mathbb{Z}} \to \mathbb{R}$ by

$$d : \begin{cases} x \in [0, 1/2] \mapsto x \\ x \in [1/2, 1] \mapsto 1 - x. \end{cases}$$

Each connected component of $\partial\Omega$, denoted $B_1, \dots, B_l$, has a unit-length parametrization $c_i : \frac{\mathbb{R}}{|B_i|\mathbb{Z}} \to B_i$. For $\epsilon = 1/n$ (where $n \in \mathbb{N}^*$) we define $d_\epsilon : \partial\Omega \to \mathbb{R}$ by:

$$d_\epsilon(x) = \epsilon |B_i| d\left(\frac{c_i^{-1}(x)}{\epsilon |B_i|} \right), \text{ for } x \in B_i.$$

Finally, let $\lambda : \partial\Omega \to \mathbb{R}_+^*$ be a smooth $\mathcal{C}^1$-function that will be made precise later. We define the oscillations (f_ϵ) by:

$$f_\epsilon : \begin{cases} \partial\Omega \to \mathbb{R}_+ \\ x \mapsto \lambda(x) d_\epsilon(x), \end{cases}$$

and observe that f_ϵ verifies $\|f_\epsilon\|_{L^\infty(\partial\Omega)} = \mathcal{O}(\epsilon)$ and $f_\epsilon' = \pm\lambda + d_\epsilon \lambda'(x) = \pm\lambda + \mathcal{O}(\epsilon)$. Here we use the fact that λ belongs to $\mathcal{C}^1$.

We consider the domain Ω_ϵ defined as a perturbation of Ω, with boundary given by

$$\partial\Omega_\epsilon = \{x + f_\epsilon(x)\nu(x),\ x \in \partial\Omega\}.$$

Roughly speaking, this means that Ω_ϵ is defined for any small enough ϵ as the union of Ω and of the segments $[x, x + \nu(x) f_\epsilon(x)]$ for $x \in \partial\Omega$. It has a piecewise smooth boundary, and since it is defined as the graph of a Lipschitz function (where the Lipschitz constant does not depend on ϵ) on the smooth compact set $\partial\Omega$, then by [104, Theorem 2.4.7], the sequence (Ω_ϵ) verifies a uniform cone condition.

We now show that $\mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \Theta\mathscr{H}^1_{\lfloor\partial\Omega}$ weakly-$*$ in the sense of measures for a suitable choice of λ. Let φ be any continuous function, then

$$\begin{aligned} \left(\mathcal{H}^1_{\lfloor\partial\Omega_\epsilon}\right)(\varphi) &= \int_{\partial\Omega_\epsilon} \varphi(x) \mathrm{d}s \\ &= \int_{\partial\Omega} \varphi(x + \nu(x) f_\epsilon(x)) \left(\sqrt{1 + \lambda^2} + \mathcal{O}(\epsilon) \right) \mathrm{d}s \\ &\underset{\epsilon \to 0}{\longrightarrow} (\sqrt{1 + \lambda^2} \mathscr{H}^1_{\lfloor\partial\Omega})(\varphi). \end{aligned}$$

By explicit computation, one can notice that the $\mathcal{O}(\epsilon)$-term contains not only the derivative of λ, but also the curvature of Ω; we implicitly use the fact that $\partial\Omega$ is $\mathcal{C}^2$. Choosing $\lambda = \sqrt{\Theta^2 - 1}$ (which is possible since Θ is $\mathcal{C}^1$), we get the result.

Assume now that the set Ω is (only) Lipschitz and the weight $\Theta \in L^\infty(\partial\Omega)$. We proceed by regularization and use the previous result. Following [109, Theorem 1.12], there exists a sequence of $\mathcal{C}^2$ domains (Ω_ϵ) that converges uniformly to Ω. In particular $\partial\Omega$, $\partial\Omega_\epsilon$ may be parametrized by a finite number of uniformly Lipschitz functions $\gamma_i, \gamma_{i,\epsilon} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ such that $\gamma_{i,\epsilon}$ converges uniformly to γ_i and $|\nabla\gamma_{i,\epsilon}|$ converges pointwise a.e. to $|\nabla\gamma_i|$. In particular, this implies that $\mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \mathscr{H}^1_{\lfloor\partial\Omega}$ weakly-$*$ in the sense of measures. We build now a sequence $\Theta_\epsilon \in \mathcal{C}^1(\partial\Omega_\epsilon, (1, +\infty))$ which is uniformly bounded in L^∞ and such that

$$\Theta_\epsilon \mathscr{H}^1_{\lfloor\partial\Omega_\epsilon} \rightharpoonup \Theta\mathscr{H}^1_{\lfloor\partial\Omega},$$

weakly-$*$ in the sense of measures. We let (ρ_ϵ) be a standard mollifying sequence on $\mathbb{R}/\mathbb{Z}$, and we define the regularization $\Theta_\epsilon \circ \gamma_\epsilon := \rho_\epsilon * (\Theta \circ \gamma_i)$. Then the sequence $(\Omega_\epsilon, \Theta_\epsilon)$ suits our purposes.

For each ϵ, we apply the previous result to get a sequence $\Omega_{\epsilon,\eta} \underset{\eta \to 0}{\longrightarrow} \Omega_\epsilon$ with

$$\mathscr{H}^1_{\lfloor\partial\Omega_{\epsilon,\eta}} \underset{\eta \to 0}{\rightharpoonup} \Theta_\epsilon \mathscr{H}^1_{\lfloor\partial\Omega_\epsilon}.$$

Using the fact that the space of Radon measures endowed with the weak-$*$ convergence is metrizable, by a diagonal extraction we can find a $\eta(\epsilon) > 0$ for each ϵ such that

$$\mathscr{H}^1_{\lfloor \partial\Omega_{\epsilon,\eta(\epsilon)}} \rightharpoonup \Theta \mathscr{H}^1_{\lfloor \partial\Omega}.$$

□

Theorem 5.8. *Let Ω, ω be two smooth sets and $g : \Omega \to \omega$ be a conformal map between the two. Then there exists a sequence (Ω_ϵ) of smooth sets homeomorphic to Ω such that $d_H(\partial\Omega_\epsilon, \partial\Omega) \underset{\epsilon\to 0}{\longrightarrow} 0$ and*

$$|\partial\Omega_\epsilon|\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} |\partial\omega|\sigma_k(\omega)$$

for any k.

Remark 5.9. Before the proof, let us point out that when Ω and ω are such that there exists a conformal mapping $g : \Omega \to \omega$ with $g \in \mathcal{C}^1(\overline{\Omega}, \overline{\omega})$ and $g' \neq 0$ everywhere on $\overline{\Omega}$ (this is for instance the case when Ω and ω are Dini-smooth or $\mathcal{C}^{1,\alpha}$, see [107, Theorem 3.5]), then a stronger conclusion can be obtained, namely that $\Omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Omega$ uniformly and $(|\partial\Omega_\epsilon|)_{\epsilon>0}$ is uniformly bounded. This is implicitly proved in the first step below.

Proof. (of Theorem 5.8) We split the proof in two steps.
Step. 1. Let us assume first that there exists a conformal mapping $g : \Omega \to \omega$ such that $g \in \mathcal{C}^1(\overline{\Omega}, \overline{\omega})$ and g' does not vanish on $\overline{\Omega}$. We let $\Theta(z) = \Lambda|g'(z)|$ for $z \in \partial\Omega$, where $\Lambda > 0$ is a constant chosen large enough to have $\Theta > 1$ everywhere on $\overline{\Omega}$. From Lemma 5.7 and Proposition 5.5 there exists a sequence (Ω_ϵ) that converges uniformly to Ω such that for all k

$$\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \sigma_k(\Omega, \Theta).$$

A consequence of Lemma 5.6 is that for all k

$$\sigma_k(\Omega, \Theta) = \sigma_k(\omega, \Lambda) = \Lambda^{-1}\sigma_k(\omega),$$

and

$$|\partial\Omega_\epsilon| \underset{\epsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta \mathrm{d}s = \Lambda|\partial\omega|,$$

which proves the result.

Step 2. Let us now assume that $g : \Omega \to \omega$ is a conformal mapping, without any information about its boundary behavior. We use a smoothing argument, as follows. Let (ω_ϵ) be a sequence of smooth sets such that

- $\forall \epsilon > \epsilon' > 0,\ \omega_\epsilon \subset \omega_{\epsilon'} \Subset \omega$ and ω_ϵ is homeomorphic to ω.
- $\omega_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \omega$ uniformly.
- $\mathscr{H}^1_{\lfloor \partial\omega_\epsilon} \rightharpoonup \mathscr{H}^1_{\lfloor \partial\omega}$.

In particular, the second and third conditions imply $|\partial\omega_\epsilon| \underset{\epsilon\to 0}{\longrightarrow} |\partial\omega|$ and $\sigma_k(\omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \sigma_k(\omega)$ for all k. Let $\Omega_\epsilon := g^{-1}(\omega_\epsilon)$, the sequence (Ω_ϵ) verifies

$$d_H(\partial\Omega_\epsilon, \partial\Omega) \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Moreover, $g_{|\overline{\Omega_\epsilon}} \in \mathcal{C}^1(\overline{\Omega}_\epsilon, \overline{\omega}_\epsilon)$ and $g'_{\overline{\Omega_\epsilon}}$ does not vanish; we may proceed as in Step 1, and build for each $\epsilon > 0$ a sequence $(\Omega_{\epsilon,\delta})_{\delta>0}$ such that

$$\Omega_{\epsilon,\delta} \underset{\delta\to 0}{\longrightarrow} \Omega_\epsilon \text{ uniformly},$$
$$\forall k \geq 0,\ |\partial\Omega_{\epsilon,\delta}|\sigma_k(\Omega_{\epsilon,\delta}) \underset{\delta\to 0}{\longrightarrow} |\partial\omega_\epsilon|\sigma_k(\omega_\epsilon).$$

By a diagonal extraction we find for each $\epsilon > 0$ a small enough $\delta(\epsilon) > 0$ such that

$$d_H(\partial\Omega_{\epsilon,\delta(\epsilon)}, \partial\Omega) \underset{\epsilon\to 0}{\longrightarrow} 0,$$
$$\forall k \geq 0,\ |\partial\Omega_{\epsilon,\delta(\epsilon)}|\sigma_k(\Omega_{\epsilon,\delta(\epsilon)}) \underset{\epsilon\to 0}{\longrightarrow} |\partial\omega|\sigma_k(\omega),$$

so that $(\Omega_{\epsilon,\delta(\epsilon)})$ verifies the conclusion. □

A corollary of Theorem 5.8, for $k = 1$, leads to a negative answer to the stability question raised by Girouard and Polterovich, for the Weinstock inequality. In order to fit the hypotheses of the inequality, below we assume that Ω is simply connected.

Theorem 5.10. *Let Ω be a simply connected smooth set of $\mathbb{R}^2$. Then there exists a sequence of simply connected smooth sets (Ω_ϵ) such that $d_H(\partial\Omega_\epsilon, \partial\Omega) \underset{\epsilon\to 0}{\longrightarrow} 0$ and*

$$\forall k \in \mathbb{N},\ |\partial\Omega_\epsilon|\sigma_k(\Omega_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 2\pi\sigma_k(\mathbb{D}).$$

5.3 A stability result under a priori bounds

In this section, we show that the stability of Weinstock's inequality could be obtained provided some a priori information on the conformal map $g : \mathbb{D} \to \Omega$ is known. The family of convex sets, where stability occurs e.g. [98], can be described in terms of conformal mappings by the constraint

$$\mathrm{Re}\left(1 + z\frac{g''(z)}{g'(z)}\right) \geq 0.$$

Our a priori condition will only include smooth enough sets but will allow for non-convex and non-starlike domains.

The strategy, based on the relationship between the Steklov problem on Ω and the weighted Steklov problem on $\mathbb{D}$, with weight $\Theta = |g'|$ on $\partial\mathbb{D}$, is as follows.

- We begin by the study of the stability of $\Theta \mapsto \sigma_1(\mathbb{D}, \Theta)$. From the information that $\left(\int_{\partial\mathbb{D}} \Theta \mathrm{d}s\right)\sigma_1(\mathbb{D}, \Theta)$ is close to $|\partial\mathbb{D}|\sigma_1(\mathbb{D})$ we obtain that Θ is close to a constant in the $H^{-\frac{1}{2}}(\partial\mathbb{D})$ norm. This assertion holds provided that Θ is normalized to have its center of mass in 0, which amounts to replacing g with $g \circ \phi$ where ϕ is a certain conformal automorphism of the disk.
- Using the a priori information on g, we improve the norm and get that Θ is close to a constant in $L^\infty(\partial\mathbb{D})$.
- We give a geometric interpretation and transfer this result to sets Ω by showing that the Hausdorff asymmetry of Ω is small when $|g'|$ is close to a constant in $L^\infty(\partial\mathbb{D})$.

In view of Theorem 5.10, the a priori information on the conformal mapping is crucial.

As previously mentioned, we first begin by the study of the Steklov problem of the disk with weight Θ. For all $\Theta \in L^\infty(\partial\mathbb{D}, \mathbb{R}_+^*)$ it will be practical to introduce the deficit

$$\mathrm{def}(\Theta) = \frac{1}{\sigma_1(\partial\mathbb{D}, \Theta)} - 1.$$

If $f \in L^2(\partial\mathbb{D})$, we denote by $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt$ its Fourier coefficients, and we define the H^s semi-norm by

$$\|f\|^2_{H^s(\partial\mathbb{D})} = \sum_{n\in\mathbb{Z}^*} |n|^{2s}|\widehat{f}(n)|^2.$$

Provided we restrict ourselves to functions that verify $\widehat{f}(0) = 0$, this becomes a norm. Using this definition, it can be checked with Fourier series decomposition that for all $f \in H^{\frac{1}{2}}(\partial\mathbb{D})$, the harmonic extension $\mathcal{H}f$ of f is well-defined and

$$\int_{\mathbb{D}} |\nabla\mathcal{H}f|^2 \mathrm{d}x = 2\pi\|f\|^2_{H^{\frac{1}{2}}(\partial\mathbb{D})}.$$

Proposition 5.11. *Let $\Theta \in L^\infty(\partial\mathbb{D}, \mathbb{R}_+)$ be such that $\widehat{\Theta}(0) = 1$ and $\widehat{\Theta}(\pm 1) = 0$. Assume morever that def$(\Theta) \leq \delta_0$ for a small enough constant $\delta_0 \in]0, 1[$. Then, for some constant $C > 0$ independent of Θ, we have*

$$\|\Theta - 1\|_{H^{-\frac{1}{2}}(\partial\mathbb{D})} \leq C\sqrt{\mathrm{def}(\Theta)}.$$

Proof. Let $u(z) = z$ be the first (complex) eigenfunction associated to $\Theta \equiv 1$. Let ϕ be a normalized, real-valued $H^{\frac{1}{2}}(\partial\mathbb{D})$ function and assume, without restricting generality, that $\widehat{\phi}(0) = 0$. Let also $\zeta \in \mathbb{C}$ be a number that will be fixed later. Then, from the definition of def(Θ)

$$(1 + \mathrm{def}(\Theta)) \int_{\mathbb{D}} |\nabla(u + \zeta\mathcal{H}\phi)|^2 \mathrm{d}x \geq \int_{\partial\mathbb{D}} \Theta|u + \zeta\phi|^2 \mathrm{d}s - \frac{1}{2\pi}\left|\int_{\partial\mathbb{D}} \Theta(u + \zeta\phi)\mathrm{d}s\right|^2.$$

This can be rewritten as

$$(1 + \mathrm{def}(\Theta))\left(2\pi + 2\pi|\zeta|^2 + 2\mathrm{Re}\left(\overline{\zeta}\int_{\partial\mathbb{D}} u\phi \mathrm{d}s\right)\right)$$

$$\geq 2\pi + \int_{\partial\mathbb{D}} \Theta\left(2\mathrm{Re}(\overline{\zeta}u\phi) + |\zeta|^2\phi^2\right)\mathrm{d}s - \frac{1}{2\pi}|\zeta|^2\left(\int_{\partial\mathbb{D}} \Theta\phi\mathrm{d}s\right)^2.$$

Thus

$$2\mathrm{Re}\left(\overline{\zeta}\int_{\partial\mathbb{D}} (\Theta - 1)u\phi\mathrm{d}s\right)$$

$$\leq 2\mathrm{def}(\Theta)\mathrm{Re}\left(\overline{\zeta}\int_{\partial\mathbb{D}} u\phi\mathrm{d}s\right) + 2\pi\mathrm{def}(\Theta) + |\zeta|^2\left(2\pi(1 + \mathrm{def}(\Theta)) + \frac{1}{2\pi}\left(\int_{\partial\mathbb{D}} \Theta\phi\mathrm{d}s\right)^2\right).$$

Using def$(\Theta) \leq 1$, this can be simplified to

$$\mathrm{Re}\left(\frac{\zeta}{|\zeta|}\int_{\partial\mathbb{D}} (\Theta - 1)u\phi\mathrm{d}s\right) \leq \mathrm{def}(\Theta)\mathrm{Re}\left(\frac{\overline{\zeta}}{|\zeta|}\int_{\partial\mathbb{D}} u\phi\mathrm{d}s\right) + \frac{\pi\mathrm{def}(\Theta)}{|\zeta|} + \frac{|\zeta|}{2}\left(4\pi + \frac{1}{2\pi}\left(\int_{\partial\mathbb{D}} \Theta\phi\mathrm{d}s\right)^2\right).$$

Moreover

$$\int_{\partial\mathbb{D}} \Theta\phi\mathrm{d}s = \int_{\partial\mathbb{D}} (\Theta - 1)\phi\mathrm{d}s \leq 2\pi\|\Theta - 1\|_{H^{-\frac{1}{2}}(\mathbb{D})},$$

and

$$\left|\int_{\partial\mathbb{D}} u\phi\mathrm{d}s\right| = |2\pi\widehat{\phi}(1)| \leq 2\pi.$$

Thus, after normalization by 2π we get

$$\mathrm{Re}\left(\frac{\zeta}{|\zeta|}\fint_{\partial\mathbb{D}} (\Theta - 1)u\phi\mathrm{d}s\right) \leq \mathrm{def}(\Theta) + \frac{\mathrm{def}(\Theta)}{2|\zeta|} + \frac{|\zeta|}{2}\left(2 + \|\Theta - 1\|^2_{H^{-\frac{1}{2}}(\mathbb{D})}\right).$$

We optimize with respect to ζ, and get

$$\left|⨍_{\partial\mathbb{D}}(\Theta-1)u\phi\mathrm{d}s\right| \leq \mathrm{def}(\Theta)+\sqrt{\mathrm{def}(\Theta)\left(2+\|\Theta-1\|^2_{H^{-\frac{1}{2}}(\mathbb{D})}\right)}.$$

From the hypothesis we have $\mathrm{def}(\Theta)\leq\sqrt{\mathrm{def}(\Theta)}$, so that

$$\left|⨍_{\partial\mathbb{D}}(\Theta-1)u\phi\mathrm{d}s\right| \leq \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{\mathrm{def}(\Theta)\left(2+\|\Theta-1\|^2_{H^{-\frac{1}{2}}(\mathbb{D})}\right)}.$$

Since ϕ is free (but normalized in $H^{1/2}(\partial\mathbb{D})$), we get

$$\|(\Theta-1)u\|_{H^{-1/2}(\partial\mathbb{D})} \leq \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{\mathrm{def}(\Theta)\left(2+\|\Theta-1\|^2_{H^{-\frac{1}{2}}(\mathbb{D})}\right)}.$$

By direct computation, using $\widehat{\Theta}(1)=\widehat{\Theta}(-1)=0$,

$$\|(\Theta-1)\|_{H^{-\frac{1}{2}}(\partial\mathbb{D})} \leq \sqrt{\frac{3}{2}}\|u(\Theta-1)\|_{H^{-\frac{1}{2}}(\partial\mathbb{D})},$$

so that

$$\frac{\|\Theta-1\|_{H^{-1/2}(\partial\mathbb{D})}}{\sqrt{2+\|\Theta-1\|^2_{H^{-\frac{1}{2}}(\mathbb{D})}}} \leq \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{3\mathrm{def}(\Theta)},$$

which gives the conclusion with

$$\delta_0=\frac{1}{6(3+\sqrt{8})}$$

□

In order to get the improved estimate on $\|\Theta-1\|_{L^\infty(\partial\mathbb{D})}$, we assume some knowledge on the smoothness of Θ.

Proposition 5.12. *Let $\Theta\in\mathcal{C}^{0,\alpha}(\partial\mathbb{D},\mathbb{R})$ for $\alpha\in]0,1]$ be such that $\widehat{\Theta}(0)=1$, $\widehat{\Theta}(\pm1)=0$, and $\mathrm{def}(\Theta)\leq\delta_0$. Then*

$$\|\Theta-1\|_{L^\infty(\partial\mathbb{D})} \leq C_\alpha(\|\Theta\|_{\mathcal{C}^{0,\alpha}(\partial\mathbb{D})})\|\Theta-1\|^{\frac{1}{1+\alpha^{-1}}}_{H^{-\frac{1}{2}}(\partial\mathbb{D})}.$$

Proof. Let ϕ be a positive function, smooth with support in $[-1,+1]$. Let $\phi_\epsilon(e^{it})=\phi(t/\epsilon)$. One can check that

$$\|\phi_\epsilon\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}=\mathcal{O}_{\epsilon\to0}(1).$$

Indeed, denoting $\mathcal{F}$ the Fourier transform in $\mathbb{R}$, we have

$$\|\phi_\epsilon\|^2_{H^{\frac{1}{2}}(\partial\mathbb{D})}=\sum_{n\in\mathbb{Z}}|n||\widehat{\phi_\epsilon}(n)|^2=\sum_{n\in\mathbb{Z}}\epsilon|\epsilon n||\mathcal{F}\phi(\epsilon n)|^2\underset{\epsilon\to0}{\longrightarrow}\int_{\mathbb{R}}|\xi||\mathcal{F}\phi(\xi)|^2d\xi<\infty.$$

Suppose now that Θ reaches its maximum $1+m$ in e^{it_0}. With $\epsilon=\left(\frac{m}{2[\Theta]_{C^\alpha}}\right)^{\alpha^{-1}}$, we have $(\Theta-1)\phi_\epsilon(e^{i(t_0+\cdot)})\geq\frac{1}{2}m\phi_\epsilon(e^{i(t_0+\cdot)})$, and thus

$$\int_{\partial\mathbb{D}}(\Theta-1)\phi_\epsilon(e^{i(t_0+\cdot)})\mathrm{d}s\geq\frac{1}{2}m\int_{\partial\mathbb{D}}\phi_\epsilon(e^{i(t_0+\cdot)})\mathrm{d}s=cm^{1+\alpha^{-1}}.$$

For a certain constant c that only depends on ϕ, $[\Theta]_{C^\alpha}$ and α.

This gives $m\leq C\|\Theta-1\|^{\frac{1}{1+\alpha^{-1}}}_{H^{-\frac{1}{2}}(\partial\mathbb{D})}$. We can do the same for the minimum of $\Theta-1$ by taking $-\phi_\epsilon$ instead, getting the result.

□

Definition 5.13. For all $K > 0$, $\alpha \in]0,1]$, we denote $\mathcal{A}(K,\alpha)$ the family of simply connected sets Ω with perimeter 2π such that there exists a conformal map $g : \mathbb{D} \to \Omega$ with

$$\| \log(|g'|) \|_{\mathcal{C}^{0,\alpha}(\partial\mathbb{D})} \leq K.$$

Remark 5.14. In particular, these domains are $\mathcal{C}^{1,\alpha}$, so this class excludes domains with angles (e.g. polygonal sets).

Our main stability result reads as follows.

Theorem 5.15. *Let $K > 0$, $\alpha \in]0,1]$. Then there exists a constant $M = M(K,\alpha) > 0$ such that for any $\Omega \in \mathcal{A}(K,\alpha)$*

$$\sigma_1(\mathbb{D}) - \sigma_1(\Omega) \geq M\left(\overline{d_H}(\Omega,\mathbb{D})\right)^{2(1+\alpha^{-1})}.$$

We previously showed that $\|\Theta - 1\|_{L^\infty(\partial\mathbb{D})} = \||g'| - 1\|_{L^\infty(\partial\mathbb{D})}$ is small when $\sigma_1(\Omega)$ is close to $\sigma_1(\mathbb{D})$. The link between this estimate and its geometric interpretation in terms of the Hausdorff asymmetry of Ω is given by the following lemma.

Lemma 5.16. *Let g be a conformal map that sends $\mathbb{D}$ to Ω, then*

$$\overline{d_H}(\Omega,\mathbb{D}) \leq 3\||g'| - 1\|_{L^\infty(\partial\mathbb{D})}.$$

Remark 5.17. The main argument of the proof relies on the Bloch theorem that we recall here for the convenience of the reader (see [91]): *let f be any holomorphic function defined on a disk $\mathbb{D}_{z,r}$. Then the image of f contains a disk of radius $L|f'(z)|r$ where $L > 0$ is a universal constant called the Landau constant.* A version of this result is proven in [91], and asserts that $L \geq \frac{1}{2}$.

Proof. We write $\epsilon = \||g'| - 1\|_{L^\infty(\partial\mathbb{D})}$. Using the maximum principle, we know that for all $z \in \mathbb{D}$:

$$1 - \varepsilon \leq |g'| \leq 1 + \varepsilon.$$

For all $z \in \mathbb{D}$, the disk $\mathbb{D}_{z,1-|z|}$ is in $\mathbb{D}$, and so, according to Bloch's theorem, the image of g' constains a disk of radius $L|g''(z)|(1-|z|)$ where L is the Landau constant. Since the image of g' contains no disk with radius larger than ϵ, we deduce that for all $z \in D$

$$|g''(z)| \leq \frac{\varepsilon}{L(1-|z|)}.$$

Now, this implies in particular that for all $z \in \mathbb{D}$

$$\begin{aligned} |g(z) - g'(0)z| &\leq \int_0^1 |(1-t)|z|^2 g''(tz)|dt \\ &\leq \int_0^1 (1-t)|z|^2 \frac{\epsilon}{L(1-t|z|)}dt \\ &\leq \frac{\epsilon}{L}. \end{aligned}$$

In particular,

$$\mathbb{D}_{1-(1+L^{-1})\epsilon} \subset g(\mathbb{D}) \subset \mathbb{D}_{1+(1+L^{-1})\epsilon}.$$

Since $L \geq \frac{1}{2}$, we conclude the proof of the lemma. □

We can now return to the proof of Theorem 5.15.

Proof. (of Theorem 5.15) The set Ω belongs to $\mathcal{A}(K,\alpha)$, so there exists a conformal representation g of Ω for which $\| \log(|g'|) \|_{C^{0,\alpha}} \leq K$. We let $\Theta = |g'|_{|\partial\mathbb{D}}$. The perimeter constraint gives $\widehat{\Theta}(0) = 1$ and we assume for now that $\widehat{\Theta}(\pm 1) = 0$.

The hypothesis $\Omega \in \mathcal{A}(K,\alpha)$ implies that $\|\Theta\|_{\mathcal{C}^{0,\alpha}}$ is controlled by a constant depending only on K. Since $\overline{d_H}(\Omega,\mathbb{D})$ is bounded above, we only need to show the inequality when $\sigma_1(\mathbb{D}) - \sigma_1(\Omega)$ is small enough, so we may suppose that $\mathrm{def}(\Theta) < \delta_0$. From Proposition 5.12, there exists a constant $M = M(K) > 0$ such that

$$\||g'| - 1\|_{L^\infty(\partial\mathbb{D})} \leq M|\sigma_1(\mathbb{D}) - \sigma_1(\Omega)|^{\frac{1}{2(1+\alpha^{-1})}},$$

so according to the previous lemma, we get

$$\overline{d_H}(\Omega,\mathbb{D}) \leq 3M|\sigma_1(\mathbb{D}) - \sigma_1(\Omega)|^{\frac{1}{2(1+\alpha^{-1})}}.$$

Let us show that we can always normalize g without changing $\|\log(|g'|)\|_{\mathcal{C}^\alpha(\partial\mathbb{D})}$ too much: the idea is that there exists a unique $\zeta_g \in \mathbb{D}$ such that $g \circ \phi_{\zeta_g}^{-1}$ is well-normalized, where $\phi_\zeta(z) := \frac{z+\zeta}{1+\overline{\zeta}z}$ is a conformal automorphism of the disk. It could happen that the $\mathcal{C}^{0,\alpha}$ norm of $g \circ \phi_{\zeta_g}^{-1}$ explodes when $|\zeta_g|$ gets too close to 1. The following lemma shows that it is not the case.

Lemma 5.18. *Let g be a conformal map, then there exists $r = r\left(\|\log(|g'|)\|_{L^\infty(\mathbb{D})}\right)$ in $[0,1[$ such that $|\zeta_g| \leq r$.*

Proof. We let $\Theta = |g'_{|\partial\mathbb{D}}|$, and $K = \|\log(|g'|)\|_{L^\infty(\mathbb{D})}$. This means that $e^{-K} \leq \Theta \leq e^K$. We are interested in the quantity

$$F(\zeta) = \int_{\partial\mathbb{D}} |(g \circ \phi_{-\zeta})'(z)| z \mathrm{d}s = \int_{\partial\mathbb{D}} \Theta(\phi_{-\zeta}(z)) \frac{1-|\zeta|^2}{|1-\overline{\zeta}z|^2} z \mathrm{d}s.$$

To estimate $|\zeta_g|$, we use the following topological criteria: if $\langle F(\zeta), \zeta\rangle > 0$ for all $\zeta \in \partial\mathbb{D}_r$, then $|\zeta_g| \leq r$. Let $r > 0$, we have

$$\begin{aligned}
\frac{1}{1-r^2}\langle F(re^{it_0}), e^{it_0}\rangle &= \int_{\partial\mathbb{D}} \Theta(\phi_{-re^{it_0}}(z)) \frac{\langle z, e^{it_0}\rangle}{|1-rz|^2} \mathrm{d}s \\
&= \int_0^{2\pi} \Theta(\phi_{-re^{it_0}}(e^{i(t_0+t)})) \frac{\cos(t)}{|1-re^{it}|^2} dt \\
&\geq \int_{|t|\leq\pi/2} e^{-K} \frac{\cos(t)}{|1-re^{it}|^2} dt + \int_{|t|\geq\pi/2} e^{K} \frac{\cos(t)}{|1-re^{it}|^2} dt \\
&\geq e^{-K} \int_{|t|\leq\pi/2} \frac{\cos(t)}{|1-re^{it}|^2} dt - \frac{\pi}{2} e^K \\
&\underset{r\to 1}{\longrightarrow} +\infty \text{ (uniformly in } t_0).
\end{aligned}$$

This proves that for a certain $r \in]0,1[$ that only depends on K, we have $|\zeta_g| \leq r$. □

□

5.4 Further remarks

The study of the stability of $\Theta \mapsto \sigma_1(\mathbb{D},\Theta)$ is independent of the fact that $\Theta = |g'|$ for a certain conformal map g. A natural question is to understand whether there is an equivalence between the study of the Steklov problem on domains Ω and the study of the weighted Steklov operator on the disk. We prove below that this is the case provided we slightly relax the setting and formally allow domains that might overlap, meaning they are immerged in the plane and are seen as images of holomorphic functions with non-vanishing derivative which are not necessarily globally injective. We also show that in our stability result, the domains cannot overlap when the first Steklov eigenvalue is close to that of the disk.

Proposition 5.19. *Let $\Theta \in \mathcal{C}^0(\partial\mathbb{D}, \mathbb{R}_+^*)$, then there exists an holomorphic function with non-vanishing derivative $g : \mathbb{D} \to \mathbb{C}$ such that $|g'_{|\partial\mathbb{D}}| = \Theta$ which, moreover, is unique up to an affine isometry.*

Proof. Let u be the harmonic extension of $\log(\Theta)$ on $\mathbb{D}$. Let v be a harmonic conjugate of u (it can be unique if we fix $v(0) = 0$). Then take g as an integral of e^{u+iv} (it exists because $\mathbb{D}$ is simply connected). Then $|g'| = e^u$, which is equal to Θ on $\partial\mathbb{D}$.

For the uniqueness, consider f and g two such functions, then $\log(|f'|)$ and $\log(|g'|)$ are two harmonic functions that coincide on $\partial\mathbb{D}$, so they are the same. Then $\mathrm{Im}(\log(f'))$ and $\mathrm{Im}(\log(g'))$ (which are defined up to a constant in $2\pi\mathbb{Z}$) are conjugated to the same harmonic functions, so they differ by a constant. □

Here is a criterium to get a domain that does not overlap, meaning that the function g defined above is injective.

Proposition 5.20. *Let $\Theta \in \mathcal{C}^0(\partial\mathbb{D}, \mathbb{R}_+^*)$. If $\|\Theta - 1\|_{L^\infty(\partial\mathbb{D})} \leq \frac{1}{5}$, then Θ defines a domain that does not overlap.*

Remark 5.21. In particular, for all $\alpha \in]0, 1[$, $K > 0$, there exists $\delta > 0$ such that if $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic function with non-vanishing derivative that verifies $\int_{\partial\mathbb{D}} |g'| \mathrm{d}s = 2\pi$, $\| \log(|g'|) \|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq K$, $\sigma_1(\mathbb{D}, |g'_{|\partial\mathbb{D}}|) \geq 1 - \delta$ then g is injective and defines a domains that does not overlap. Indeed, Proposition 5.12 shows that $\| |g'| - 1 \|_{L^\infty(\partial\mathbb{D})} \leq C(K, \alpha)\delta^{2(1+\alpha^{-1})}$, which is less than $\frac{1}{5}$ when δ is small enough.

Proof. (of Proposition 5.20) Write $\|\Theta - 1\|_{L^\infty(\partial\mathbb{D})} = \epsilon$, g a conformal map such that $|g'_{|\partial\mathbb{D}}| = \Theta$. We want to show that g is univalent. Using [107, Theorem 1.11] it is enough to show that for all $z \in \mathbb{D}$

$$(1 - |z|^2) \left| z \frac{g''(z)}{g'(z)} \right| \leq 1.$$

Since $||g'| - 1| \leq \epsilon$, it means that $g'(\mathbb{D})$ contains no disk with larger radius than ϵ, and so, using Bloch's theorem

$$L(1 - |z|)|g''(z)| \leq \epsilon.$$

Thus we can verify the univalence criteria

$$(1 - |z|^2) \left| z \frac{g''(z)}{g'(z)} \right| \leq (1 - |z|^2) \frac{\epsilon}{L(1 - |z|)(1 - \epsilon)} \leq \frac{2\epsilon}{L(1 - \epsilon)},$$

the right hand side being less than 1 when $\epsilon \leq \frac{L}{2+L}$. Since $L \geq \frac{1}{2}$, it is enough to chose $\epsilon \leq \frac{1}{5}$. □

Sharpness of the stability result. One could wonder if an a priori bound on $\| \log(|g'|) \|_{L^\infty(\partial\mathbb{D})}$ is enough to obtain stability. We prove below that this is not the case.

Proposition 5.22. *Let Ω be the image of a smooth conformal map $g \in \mathcal{C}^1(\overline{\mathbb{D}})$ such that*

$$\max_{\partial\mathbb{D}} |g'| < \frac{4}{\pi} \min_{\partial\mathbb{D}} |g'|.$$

Then there exists a sequence of domains $\Omega_n = g_n(\mathbb{D})$ with $g_n \in \mathcal{C}^1(\overline{\mathbb{D}})$ such that

$$|\partial\Omega_n| \sigma_k(\Omega_n) \underset{n\to\infty}{\longrightarrow} |\partial\mathbb{D}| \sigma_k(\mathbb{D}), \text{ for all } k \geq 0,$$
$$\Omega_n \underset{n\to\infty}{\longrightarrow} \Omega \text{ uniformly},$$
$$\sup_{n\in\mathbb{N}} \| \log |g'_n| \|_{L^\infty(\mathbb{D})} < \infty.$$

Proof. Consider f an holomorphic function defined on $\mathbb{D}$ that never vanishes and such that $|f| < |g'|$ on $\partial\mathbb{D}$. We introduce

$$g_n(z) = g(z) + \frac{z^{n+1}}{n+1} f(z)$$

and show that, starting with some n large enough, this defines a sequence of domains that converges uniformly to Ω. We only need to check the uniform cone condition. Since Ω is $\mathcal{C}^1$, there exists a finite number of squares that cover $\partial\Omega$, called (C_i) and oriented by the orthonormal basis (e_i, f_i) such that

$$\begin{aligned} C_i &= \{p_i + te_i + sf_i,\ t, s \in [-r_i, r_i]\}, \\ \partial\Omega \cap C_i &= \{p_i + te_i + h_i(t)f_i,\ t \in [-r_i, r_i]\} \end{aligned}$$

for some $\|h_i\|_{\mathcal{C}^1(\partial\mathbb{D})} \leq \delta$ (with δ chosen arbitrarily small) . We also write $c(t) = g(e^{it})$ and $c_n(t) = g_n(e^{it})$, as well as $I_i = c^{-1}(C_i)$. Then, for all $t \in I_i$ we have

$$\left| \frac{\langle c', f_i \rangle}{\langle c', e_i \rangle} \right| \leq \delta.$$

For a large enough n, there exists $\eta > 0$ such that $|(c_n - c)'| < (1-\eta)|c'|$, because of the condition that $|f| < |g'|$ (any η smaller than $\inf \frac{|g'|-|f|}{|g'|}$ works for n large enough). In particular, $|c_n'| \leq 2|c'|$ and

$$\begin{aligned} |\langle c_n', e_i \rangle| &\geq |\langle c', e_i \rangle| - |c_n' - c'| \\ &\geq |\langle c', e_i \rangle| - (1-\eta)|c'| \\ &\geq |\langle c', e_i \rangle| - (1-\eta)|\langle c', e_i \rangle| - (1-\eta)|\langle c', f_i \rangle| \\ &= (\eta - (1-\eta)\delta)\, |\langle c', e_i \rangle| \,. \end{aligned}$$

We choose $\delta \leq \frac{\eta}{2(1-\eta)}$ so that

$$\left| \frac{\langle c_n', f_i \rangle}{\langle c_n', e_i \rangle} \right| \leq \frac{4}{\eta} \frac{|c'|}{|\langle c', e_i \rangle|}.$$

Consequently, letting $M = \max_i \left(\frac{4}{\eta} \max_{I_i} \frac{|c'|}{|\langle c', e_i \rangle|} \right)$, there exists a sequence of M-Lipschitz functions $(h_{i,n})$ such that for all i

$$\partial\Omega_n \cap C_i = \{p_i + te_i + h_{i,n}(t)f_i,\ t \in [-r_i, r_i]\}.$$

Which proves that the sequence (Ω_n) verifies a uniform cone condition.

Lemma 5.23. *For the sequence (Ω_n) build above, we have that*

$$\mathscr{H}^1_{\lfloor \partial\Omega_n} \underset{n\to\infty}{\rightharpoonup} \Theta \mathscr{H}^1_{\lfloor \partial\Omega} \text{ weakly-}* \text{ in the sense of measures,}$$

where

$$\Theta = P\left(\left| \frac{f \circ g^{-1}}{g' \circ g^{-1}} \right| \right) \text{ and } P(a) := \fint_0^{2\pi} \sqrt{1 + a^2 + 2a\cos(t)}dt.$$

Moreover, P is a strictly increasing function with $P(0) = 1$ and $P(1) = \frac{4}{\pi}$.

Proof. We already know that g_n converges uniformly to g, and so it is easier to prove the lemma on the pullback of these measures through g and g_n on $\partial\mathbb{D}$. Let $\varphi \in \mathcal{C}^0(\partial\mathbb{D}, \mathbb{R})$, we search for the limit

of $\int_{\partial\mathbb{D}} \varphi|g_n'|\mathrm{d}s$. We compute

$$\begin{aligned}\int_{\partial\mathbb{D}} \varphi|g_n'|\mathrm{d}s &= \int_{\partial\mathbb{D}} \varphi|g' + z^n f + \mathcal{O}(1/n)|\mathrm{d}s \\ &= \int_{\partial\mathbb{D}} \varphi|g' + z^n f|\mathrm{d}s + \mathcal{O}(1/n) \\ &= \int_{\partial\mathbb{D}} \varphi|g'|\left|1 + z^n\frac{f}{g'}\right|\mathrm{d}s + \mathcal{O}(1/n) \\ &\underset{n\to\infty}{\longrightarrow} \int_{\partial\mathbb{D}} \varphi|g'|P\left(\left|\frac{f}{g'}\right|\right)\mathrm{d}s.\end{aligned}$$

Above, we used the fact that for any continuous and 2π-periodic function ψ, the sequence $x \xrightarrow{\psi_n} \psi(nx)$ converges weakly-* in L^∞ to $⨍\,\psi$. □

Consequently, in order to have $|\partial\Omega_n|\sigma_1(\Omega_n) \to 2\pi$, we chose f such that $\left(\int_{\partial\Omega}\Theta\mathrm{d}s\right)\sigma_1(\Omega,\Theta) = \left(\int_{\partial\mathbb{D}} g^*\Theta\mathrm{d}s\right)\sigma_1(\mathbb{D}, g^*\Theta)$ is equal to 2π, which is the case of equality in Weinstock's inequality. In other words, we introduce the constant $\Lambda > 0$ such that $g^*\Theta = \Lambda$ or, equivalently,

$$P\left(\left|\frac{f}{g'}\right|\right) = \frac{\Lambda}{|g'|}.$$

This is only possible when $\frac{\Lambda}{|g'|}$ takes value in $\left(1, \frac{4}{\pi}\right)$, and so there exists such a constant Λ if and only if $\max_{\partial\mathbb{D}}|g'| < \frac{4}{\pi}\min_{\partial\mathbb{D}}|g'|$. For such a value Λ, the previous relation can be written

$$|f| = |g'|P^{-1}\left(\frac{\Lambda}{|g'|}\right).$$

This drives us to define f as $f = e^{u+iv}$, where $u = \mathcal{H}\log\left(|g'|P^{-1}\left(\frac{\Lambda}{|g'|}\right)\right)$ and v is an harmonic conjugate of u. With this definition, since $0 < P^{-1} < 1$, we get that $0 < |f| < |g'|$ on $\partial\mathbb{D}$, and with the maximum principle we get $0 < |f| < |g'|$ on $\mathbb{D}$. With f defined in such a way, we obtain that for all $k \geq 0$

$$|\partial\Omega_n|\sigma_k(\Omega_n) \underset{n\to\infty}{\longrightarrow} \left(\int_{\partial\Omega}\Theta\mathrm{d}s\right)\sigma_k(\Omega,\Theta) = \left(\int_{\partial\mathbb{D}}\Lambda\mathrm{d}s\right)\sigma_k(\mathbb{D},\Lambda) = |\partial\mathbb{D}|\sigma_k(\mathbb{D}).$$

□

Remark 5.24. A consequence of this instability result is that no inequality of the form

$$\overline{d_H}(g(\mathbb{D}),\mathbb{D}) \leq \epsilon\left(\||g'|-1\|^{\lambda}_{H^{-\frac{1}{2}}(\partial\mathbb{D})}\||g'|-1\|^{1-\lambda}_{L^\infty(\partial\mathbb{D})}\right),$$

can hold true, where $\lambda \in]0,1]$, $\epsilon(t)$ is a function that goes to 0 as $t \to 0$, and g is any conformal map. Indeed, such an inequality would lead to a stability result with a priori bound in L^∞, which was shown to fail.

Lower bound on the exponent. We now compute the spectrum on the special case where $\Theta - 1$ is a sine function. This allows us to prove that the optimal exponent of stability is not less than 2, whereas the exponent we are able to obtain is $2(1+\alpha^{-1})$.

Lemma 5.25. *Let $\alpha \in]0,1[$, $N \geq 4$, and $\Theta(e^{it}) = 1 + \alpha\cos(Nt)$. Then*

$$\mathit{def}(\Theta) \leq \frac{\alpha^2}{N-3}.$$

Remark 5.26. *This result may be compared to proposition 5.11; when* $\frac{\alpha^2}{N-3} < \delta_0$*, it applies to* Θ *and it gives:*

$$def(\Theta) \geq \frac{c\alpha^2}{N}$$

For a constant $c > 0$ *that does not depend on* α *nor* N*. This means that in this case, proposition 5.11 and lemma 5.25 are sharp up to a constant.*

Proof. Let $u \in \mathcal{C}^\infty(\partial\mathbb{D})$ be a finite sum of trigonometric functions, we denote c_n its Fourier coefficients (only a finite number are not vanishing). Let (Θ_n) be the Fourier coefficients of Θ. The harmonic extension of u is given by

$$\mathcal{H}u(re^{it}) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{int},$$

and we can computed explicitly

$$\begin{aligned}
\int_D |\nabla \mathcal{H}u|^2 \mathrm{d}x &= 2\pi \sum_{n\in\mathbb{Z}^*} |nc_n^2|, \\
\int_{\partial\mathbb{D}} \Theta u^2 \mathrm{d}s &= 2\pi \sum_{k,l\in\mathbb{Z}} c_k \overline{c_l} \Theta_{l-k}, \\
\int_{\partial\mathbb{D}} \Theta u \mathrm{d}s &= 2\pi \sum_{n\in\mathbb{Z}} c_n \overline{\Theta_k}.
\end{aligned}$$

Since we are only interested in test functions such that $\int_{\partial\mathbb{D}} \Theta u = 0$, we can assume from now on that

$$c_0 = -\sum_{n\in\mathbb{Z}^*} c_n \overline{\Theta_k}.$$

Under the previous hypotheses, $\mathrm{def}(\Theta)$ is the smallest value that verifies, for all finitely supported $(c_n)_{n\in\mathbb{Z}^*}$ and $c_0 := -\sum_{n\in\mathbb{Z}^*} c_n \overline{\Theta_k}$:

$$(1 + \mathrm{def}(\Theta)) \sum_{n\in\mathbb{Z}^*} |nc_n^2| \geq \sum_{k,l\in\mathbb{Z}} c_k \overline{c_l} \Theta_{l-k}.$$

This inequality may be rewritten as:

$$(1 + \mathrm{def}(\Theta)) \sum_{n\in\mathbb{Z}^*} |nc_n^2| \geq \sum_{k,l\in\mathbb{Z}^*} c_k \overline{c_l} \Theta_{l-k} - \left| \sum_{n\in\mathbb{Z}^*} c_n \overline{\Theta_n} \right|^2.$$

If we find a value δ such that for all finitely supported sequence $(c_n)_{n\in\mathbb{Z}^*}$, we have

$$\sum_{k,l\in\mathbb{Z}^*} c_k \overline{c_l} \Theta_{l-k} \leq (1+\delta) \sum_{n\in\mathbb{Z}^*} |nc_n^2|,$$

then $\mathrm{def}(\Theta) \leq \delta$.

$$\text{Here } \Theta_n = \begin{cases} 1 \text{ for } n = 0 \\ \alpha/2 \text{ for } n = \pm N. \\ 0 \text{ elsewhere} \end{cases}$$

We write, for a parameter $\epsilon > 0$ that will be chosen below,

$$\begin{aligned}
\sum_{k,l\in\mathbb{Z}^*} c_k\overline{c_l}\Theta_{l-k} &= \sum_{n\in\mathbb{Z}^*} |c_n|^2 + \frac{\alpha}{2}\sum_{k,l\in\mathbb{Z}^*,\ |k-l|=N} c_k\overline{c_l} \\
&\leq \sum_{n\in\mathbb{Z}^*} |c_n|^2 + \alpha \sum_{k,l\in\mathbb{Z}^*,\ |k-l|=N,\ |k|\leq N/2} |c_k c_l| + \frac{\alpha}{2}\sum_{k,l\in\mathbb{Z}^*,\ |k-l|=N,\ |k|,|l|>N/2} |c_k c_l| \\
&\leq \sum_{n\in\mathbb{Z}^*} |c_n|^2 + \alpha \sum_{k,l\in\mathbb{Z}^*,\ |k-l|=N,\ |k|\leq N/2} \frac{\epsilon}{2}|c_k|^2 + \frac{1}{2\epsilon}|c_l|^2 \\
&+ \frac{\alpha}{2}\sum_{k,l\in\mathbb{Z}^*,\ |k-l|=N,\ |k|,|l|>N/2} \frac{1}{2}|c_k|^2 + \frac{1}{2}|c_l|^2 \\
&\leq \sum_{n\in\mathbb{Z}^*,\ |n|\leq N/2} (1+\alpha\epsilon)\,|c_n|^2 + \sum_{n\in\mathbb{Z}^*,\ |n|>N/2} \left(1+\alpha+\frac{\alpha}{2\epsilon}\right)|c_n|^2 \\
&\leq \sum_{n\in\mathbb{Z}^*,\ |n|\leq N/2} (1+\alpha\epsilon)\,|c_n|^2 + \sum_{n\in\mathbb{Z}^*,\ |n|>N/2} \left(2+\frac{\alpha}{2\epsilon}\right)|c_n|^2.
\end{aligned}$$

We choose ϵ such that $\left(2+\frac{\alpha}{2\epsilon}\right) = \frac{N+1}{2}$, so we take $\epsilon = \frac{\alpha}{N-3}$. Hence

$$\begin{aligned}
\sum_{k,l\in\mathbb{Z}^*} c_k\overline{c_l}\Theta_{l-k} &\leq \sum_{n\in\mathbb{Z}^*,\ |n|\leq N/2} \left(1+\frac{\alpha^2}{N-3}\right)|c_n|^2 + \sum_{n\in\mathbb{Z}^*,\ |n|>N/2} |nc_n^2| \\
&\leq \left(1+\frac{\alpha^2}{N-3}\right)\sum_{n\in\mathbb{Z}^*} |nc_n^2|,
\end{aligned}$$

□

Proposition 5.27. *Let $\epsilon > 0$, $\alpha > 0$. There exists a sequence of domains Ω_n converging to $\mathbb{D}$ such that*

$$\begin{aligned}
&|\partial\Omega_n| = 2\pi \text{ for all } n, \\
&\sup_{n\in\mathbb{N}} \|\log|g_n'|\|_{\mathcal{C}^{0,\alpha}} < \infty, \\
&\sup_{n\geq 0} \frac{\sigma_1(\mathbb{D}) - \sigma_1(\Omega_n)}{\left(\overline{d_H}(\Omega_n,\mathbb{D})\right)^{2-\epsilon}} < \infty.
\end{aligned}$$

Remark 5.28. Above, α can be arbitrarily large, which means we can ask for a much stronger a priori bound and that the optimal exponent is still larger than 2.

Proof. Consider the sequence $\Omega_n = g_n(\mathbb{D})$ defined by the weight $\Theta_n(t) = 1 + a_n\cos(nt)$, for a sequence a_n that will go to 0. Since $\|\Theta_n - 1\|_{L^\infty(\partial\mathbb{D})}$ is less than $\frac{1}{5}$, for n large enough this defines a domain that does not overlap. The estimate above gives us that

$$\operatorname{def}(\Theta_n) \leq \frac{a_n^2}{n-3} \leq C\frac{a_n^2}{n},$$

for some constant $C > 0$. Let h_n be defined by $h_n = \log(|g_n'|)$. This is the unique harmonic function verifying $h_n = \log(\Theta_n)$ on the boundary. We develop it as

$$\begin{aligned}
h_n(e^{it}) &= \log\left(1 + a_n\cos(nt)\right) \\
&= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} a_n^k \cos(nt)^k \\
&= \sum_{k\geq 1} \left(\frac{(-1)^{k-1}}{k} a_n^k \sum_{p=0}^{\lfloor k/2\rfloor} \frac{1}{2^k}\binom{k}{p}(2\cos(n(2k-p)t)1_{p\neq k/2} + 1_{p=k/2})\right).
\end{aligned}$$

In particular, since h_n is harmonic, its expression in the disk is

$$h_n(re^{it}) = \sum_{k\geq 1}\left(\frac{(-1)^{k-1}}{k}a_n^k\sum_{p=0}^{\lfloor k/2\rfloor}\frac{1}{2^k}\binom{k}{p}(2r^{n(2k-p)}\cos(n(2k-p)t)1_{p\neq k/2}+1_{p=k/2})\right).$$

We can verify the a priori hypothesis on $h_n = \log(|g_n'|)$, precisely, there is a constant $C_\alpha > 0$ such that

$$[\log(|g_n'|)]_{\mathcal{C}^{0,\alpha}(\partial\mathbb{D})} \leq C_\alpha \sum_{k\geq 1}(nk)^\alpha a_n^k.$$

The right hand side is bounded as soon as $n^\alpha a_n$ is bounded: this will be verified later. Then h_n is the real part of the holomorphic function

$$\log(g_n'(z)) = ib_n + \sum_{k\geq 1}\left(\frac{(-1)^{k-1}}{k}a_n^k\sum_{p=0}^{\lfloor k/2\rfloor}\frac{1}{2^k}\binom{k}{p}(2z^{2k-p}1_{p\neq k/2}+1_{p=k/2})\right),$$

for a certain branch of the logarithm and $b_n \in \mathbb{R}$. In particular, for a certain constant $C > 0$

$$|g_n(z) - ib_n - a_n z^n| \leq \sum_{k\geq 2}\frac{1}{k}a_n^k \leq Ca_n^2.$$

This means that $g_n'(z) = e^{\log(g_n'(z))} = e^{ib_n}(1 + a_n z^n + k_n(z))$ where k_n is an holomorphic function that verifies $|k_n(z)| \leq Ca_n^2$ for a certain constant C.

We lose no generality if we assume that $b_n = 0$ for all n. Suppose now that n is odd, $d_H(\Omega_n, \mathbb{D})$ can be estimated from below by:

$$g_n(1) = \int_0^1 g_n'(r)dr = 1 + \frac{a_n}{n} + \mathcal{O}\left(\frac{a_n^2}{n}\right),$$

$$g_n(-1) = \int_0^1 g_n'(r)dr = -1 - \frac{a_n}{n} + \mathcal{O}\left(\frac{a_n^2}{n}\right).$$

Thus, for a certain constant $c > 0$

$$\overline{d_H}(\Omega_n, \mathbb{D}) \geq c\frac{a_n}{n}.$$

Using this and the upper bound $\mathrm{def}(\Theta_n) \leq C\frac{a_n^2}{n}$, we obtain

$$\frac{\sigma_1(\mathbb{D}) - \sigma_1(\Omega_n)}{\left(\overline{d_H}(\Omega_n, \mathbb{D})\right)^{2-\epsilon}} \leq Ca_n^\epsilon n^{1-\epsilon},$$

which is bounded for $a_n = n^{-\frac{1-\epsilon}{\epsilon}}$. For a small enough ϵ, $(n^\alpha a_n)$ is bounded, so the a priori condition holds, which proves the result. □

5.5 Stability of the Hersch inequality

This section is extracted from a work in common with Iosif Polterovich, Mikhail Karpukhin and Daniel Stern, that was at the origin motivated by estimates like the one of proposition 5.11 in a more general setting.

In this section, we prove a quantitative version of the Hersch inequality with elementary tools. We also discuss the sharpness of this quantitative inequality.

We call $\mathbb{S}^2$ the unit sphere of $\mathbb{R}^3$, $g = \frac{1}{4\pi} g^e$ (where g^e is the euclidian induced metric), v_g its surface measure (note that g was normalized such that $v_g(\mathbb{S}^2) = 1$). We say that a measure μ on $\mathbb{S}^2$ is admissible if it is positive, and if the identity of $\mathcal{C}^1(\mathbb{S}^2, \mathbb{R})$ continuously extends to a compact embedding $H^1(\mathbb{S}^2, g) \hookrightarrow L^2(\mathbb{S}^2, \mu)$, where $H^1(\mathbb{S}^2, g)$ is the usual Sobolev space associated to the metric g with the norm

$$\|u\|_{H^1(\mathbb{S}^2,g)} := \left(\int_{\mathbb{S}^2} \left(u^2 + |du|_g^2 \right) \mathrm{d}v_g \right)^{\frac{1}{2}}.$$

For any (nonnegative) measure μ on $\mathbb{S}^2$, we define the eigenvalues of the Laplacian with respect to μ as

$$\lambda_k(\mu) = \inf \left\{ \sup_{u \in U} \frac{\int_{\mathbb{S}^2} |du|_g^2 \mathrm{d}v_g}{\int_{\mathbb{S}^2} |u|^2 \mathrm{d}\mu},\ U \text{ subspace of } H^1(\mathbb{S}^2, g) : \dim(U) = k+1 \right\}.$$

When μ is admissible these are reached for eigenfunctions (u_k) such that $-\Delta u_k = \lambda_k(\mu) u_k \mu$ in the weak sense. Hersch's inequality in the framework of eigenvalues of measures (as developped in [105]) is stated as follows:

For any admissible probability measure μ on the sphere, $\lambda_1(\mu) \leq 8\pi$.

The proof is in two steps: first, up to replacing μ with $\Phi_* \mu$ for a certain conformal automorphism of $(\mathbb{S}^2, [g])$ (that is uniquely defined by μ as long as it has no atom of measure equal or greater than $\frac{\mu(S)}{2}$, which is the case since it is admissible), we may suppose that μ is balanced, meaning

$$\int_{\mathbb{S}^2} z \mathrm{d}\mu(z) = 0.$$

The inequality is then obtained using the coordinate functions as test functions. We prove the following quantitative version:

Proposition 5.29. *Let μ be an admissible and balanced probability measure on the sphere. Then there are constants $c_1, c_2 > 0$ such that*

$$\lambda_1(\mu) \leq 8\pi - c_1 \min \left(c_2, \|\mu - v_g\|_{H^{-1}(\mathbb{S}^2,g)}^2 \right).$$

Since any admissible measure has no atom and may be sent conformally to a balance measure, it directly implies the corollary that for any admissible probability measure μ on $\mathbb{S}^2$:

$$\lambda_1(\mu) \leq 8\pi - c_1 \min \left(c_2, \inf_{\varphi \in \mathrm{Conf}(\mathbb{S}^2)} \|\varphi_\sharp \mu - v_g\|_{H^{-1}(\mathbb{S}^2,g)}^2 \right).$$

Proof. Let $\delta > 0$ be defined by $\lambda_1(\mu) =: \frac{8\pi}{1+\delta}$; we prove that if δ is small enough (not depending on μ) then

$$\|\mu - v_g\|_{H^{-1}(\mathbb{S}^2,g)} \leq C\sqrt{\delta},$$

for a certain constant $C > 0$ that does not depend on μ. $\lambda_1(\mu) = \frac{8\pi}{1+\delta}$ implies that, for all $u \in H^1(\mathbb{S}^2, g)$,

$$\int_{\mathbb{S}^2} u^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^2} u \mathrm{d}\mu \right)^2 \leq \frac{1+\delta}{8\pi} \int_{\mathbb{S}^2} |du|_g^2 \mathrm{d}v_g.$$

We let X, Y, Z be the coordinate functions and φ be any $H^1(\mathbb{S}^2, g)$ function that verifies

$$\int_{\mathbb{S}^2} \varphi \mathrm{d}v_g = 0,\ \int_{\mathbb{S}^2} \left(|d\varphi|_g^2 + \varphi^2 \right) \mathrm{d}v_g = 1.$$

Let $t \in \mathbb{R}$ that will be fixed later, taking $u = X + t\varphi$ above we obtain

$$\int_{\mathbb{S}^2} (X + t\varphi)^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^2} (X + t\varphi) \mathrm{d}\mu \right)^2 \leq \frac{1+\delta}{8\pi} \int_{\mathbb{S}^2} |d(X + t\varphi)|_g^2 \mathrm{d}v_g.$$

Using the hypothesis on μ, φ and the fact that X is an eigenfunction for v_g with eigenvalue 8π,

$$\begin{aligned}\int_{\mathbb{S}^2}\left(X^2+2tX\varphi+t^2\varphi^2\right)\mathrm{d}\mu \leq &(1+\delta)\left(\int_{\mathbb{S}^2}X^2\mathrm{d}v_g+2t\int_{\mathbb{S}^2}X\varphi\mathrm{d}v_g+\frac{t^2}{8\pi}\right)\\ &+t^2\left(\int_{\mathbb{S}^2}\varphi\mathrm{d}(\mu-v_g)\right)^2.\end{aligned}$$

This implies, after a few simplifications,

$$\begin{aligned}\int_{\mathbb{S}^2}\left(2tX\varphi+X^2\right)\mathrm{d}(\mu-v_g) &\leq \delta\int_{\mathbb{S}^2}X^2\mathrm{d}v_g+2\delta t\int_{\mathbb{S}^2}X\varphi\mathrm{d}v_g+(1+\delta)\frac{t^2}{8\pi}\\ &\quad+t^2\left(\int_{\mathbb{S}^2}\varphi\mathrm{d}(\mu-v_g)\right)^2-t^2\int_{\mathbb{S}^2}\varphi^2\mathrm{d}\mu\\ &\leq \frac{\delta}{3}+\frac{2}{\sqrt{3}}\delta t+\frac{t^2}{4\pi}\left(1+4\pi\|\mu-v_g\|^2_{H^{-1}(\mathbb{S}^2,g)}\right).\end{aligned}$$

We may do the same for Y and Z that share the same properties, and by summing these three inequalities (and denoting $W=X+Y+Z$) we obtain

$$2t\int_{\mathbb{S}^2}W\varphi\mathrm{d}(\mu-v_g)\leq\delta+2\sqrt{3\pi}\delta t+\frac{3t^2}{4\pi}\left(1+4\pi\|\mu-v_g\|^2_{H^{-1}(\mathbb{S}^2,g)}\right).$$

After optimization in t:

$$\left|\int_{\mathbb{S}^2}W\varphi\mathrm{d}(\mu-v_g)\right|\leq C_1\sqrt{\delta\left(1+4\pi\|\mu-v_g\|^2_{H^{-1}(\mathbb{S}^2,g)}\right)}. \tag{5.3}$$

This may be done for any $L^2(v_g)$-unitary function W in $\mathrm{Span}(X,Y,Z)$ and for any φ that verifies

$$\int_{\mathbb{S}^2}\varphi\mathrm{d}v_g=0,\ \|\varphi\|_{H^1(\mathbb{S}^2,g)}=1.$$

The first condition may be dropped (up to changing C_1 in a larger constant C_2), since the left-hand side of the estimate (5.3) is invariant when adding a constant to φ by orthogonality of $\mu-v_g$ with W. Now write, for any $\varphi\in H^1(\mathbb{S}^2,g)$ with unit norm:

$$\begin{aligned}\int_{\mathbb{S}^2}\varphi\mathrm{d}(\mu-v_g)&=\int_{\mathbb{S}^2}\left(X(X\varphi)+Y(Y\varphi)+Z(Z\varphi)\right)\mathrm{d}(\mu-v_g)\\ &\leq C_3\sqrt{\delta\left(2+\|\mu-v_g\|^2_{H^{-1}(\mathbb{S}^2,g)}\right)}.\end{aligned}$$

Where $C_3=C_2\sup_{\|\varphi\|_{H^1}=1}(\|X\varphi\|_{H^1}+\|Y\varphi\|_{H^1}+\|Z\varphi\|_{H^1})$. And so

$$\|\mu-v_g\|_{H^{-1}(\mathbb{S}^2,g)}\leq C_3\sqrt{\delta\left(2+\|\mu-v_g\|^2_{H^{-1}(\mathbb{S}^2,g)}\right)},$$

which ends the proof. □

We now prove that the proposition 5.29 is sharp in two ways: the exponent is optimal, and so is the choice of distance in a certain way.
As will be detailed later, the eigenvalue functional $\mu\mapsto\lambda_1(\mathbb{S}^2,\mu)$ is not continuous in $W^{-1,2}:=W^{-1,2}(\mathbb{S}^2,g)$ norm. In general, we may see in the definition

$$\lambda_1(\mu)^{-1}=\sup_{\|du\|_{L^2(\mathbb{S}^2,g)}=1}\left(\int_{\mathbb{S}^2}u^2\,d\mu-\left(\int_{\mathbb{S}^2}u\,d\mu\right)^2\right)$$

that $\lambda_1(\mu)$ is naturally continuous with respect to the dual of the space of the squares of $W^{1,2}$ functions. Following [100, Proposition 4.11] we introduce the Orlicz-Sobolev space below.

Definition 5.30. For a function $u \in L^1(\mathbb{S}^2, g)$ let

$$\|u\|_{L^2(\mathrm{Log}L)^{-\frac{1}{2}}} := \inf\left\{\eta > 0 : \int_{\mathbb{S}^2} \frac{|u/\eta|^2}{\log(2+|u/\eta|)}\, dv_g \leqslant 1\right\}.$$

The *Orlicz-Sobolev space* $W^{1,2,-\frac{1}{2}} := W^{1,2,-\frac{1}{2}}(\mathbb{S}^2, g)$ is defined to be the space of functions such that

$$\|u\|_{W^{1,2,-\frac{1}{2}}} := \|u\|_{L^2(\mathrm{Log}L)^{-\frac{1}{2}}} + \|du\|_{L^2(\mathrm{Log}L)^{-\frac{1}{2}}} < \infty.$$

In what follows, the only property property of $W^{1,2,-\frac{1}{2}}$ we are using is that there exists a constant C_{Or} such that for any $u \in W^{1,2}$ one has

$$\|u^2\|_{W^{1,2,-\frac{1}{2}}} \leqslant C_{\mathrm{Or}} \|u\|^2_{W^{1,2}},$$

which implies the continuity of $\mu \mapsto \lambda_1(\mu) := \lambda_1(\mathbb{S}^2, \mu)$ in the dual of $W^{1,2,-\frac{1}{2}}$ (see [100] for more details). In the next proposition, we prove that if a balanced measure μ is close enough to v_g in the dual of $W^{1,2,-\frac{1}{2}}$ (and hence in $W^{-1,2}$ as well), then the exponent two in the stability estimate 5.29 is sharp on a finite codimension subspace of admissible measures.

Proposition 5.31. *There exist $\epsilon_0, c > 0$ such that, for any balanced admissible measure μ, if*

$$\int_{\mathbb{S}^2} \varphi^2\, d(\mu - v_g) \leqslant \epsilon_0 \|\varphi\|^2_{W^{1,2}} \tag{5.4}$$

for any $\varphi \in W^{1,2}$, then

$$\lambda_1(\mu) \geqslant \frac{8\pi}{1 + c\left(\|\mu - v_g\|_{W^{-1,2}} + \|\mu - v_g\|^2_{W^{-1,2}}\right)}. \tag{5.5}$$

Moreover, if in addition one has $\int_{\mathbb{S}^2} w^2\, d(\mu - v_g) = 0$ for any $w \in E_1$, then

$$\lambda_1(\mu) \geqslant \frac{8\pi}{1 + c\|\mu - v_g\|^2_{W^{-1,2}}}. \tag{5.6}$$

In particular, the exponent in the stability estimate 5.29 is sharp.

Remark 5.32. *Assumption* (5.4) *is verified whenever $\|\mu - v_g\|_{\left(W^{1,2,-\frac{1}{2}}\right)^*}$ is smaller than $C_{Or}^{-1}\epsilon_0$.*

Remark 5.33. *Inequality* (5.5) *can be interpreted as continuity of $\lambda_1(\mu)$ at $\mu = dv_g$ with respect to the $W^{-1,2}$ distance in the class of measures satisfying* (5.4)*. We will see in Corollary 5.39 below that an additional assumption* (5.4) *is necessary.*

Proof. In this proof, we denote by E_1 the eigenspace corresponding to the eigenvalue $\lambda_1(\mathbb{S}^2, g)$ spanned by the coordinate functions. Since

$$\lambda_1(\mu) = \inf_{\varphi \in W^{1,2}} \frac{\int_{\mathbb{S}^2} |d\varphi|^2_g\, dv_g}{\int_{\mathbb{S}^2} \left(\varphi - \int_{\mathbb{S}^2} \varphi\, d\mu\right)^2 d\mu},$$

and both numerator and denominator are invariant up to the addition of a constant to φ, $\lambda_1(\mu)$ may be written as

$$\lambda_1(\mu) = \inf_{\varphi \in 1^\perp} \frac{\int_{\mathbb{S}^2} |d\varphi|^2_g\, dv_g}{\int_{\mathbb{S}^2} \varphi^2\, d\mu - \left(\int_{\mathbb{S}^2} \varphi\, d\mu\right)^2},$$

where $1^\perp = \{\varphi : \int_{\mathbb{S}^2} \varphi\, dv_g = 0\}$. For any $\delta > 0$, in order to prove that $\lambda_1(\mu) \geqslant \frac{8\pi}{1+\delta}$ it suffices to show that for any $\varphi \in 1^\perp$ one has

$$\int_{\mathbb{S}^2} \varphi^2\, d\mu \leqslant \frac{1+\delta}{8\pi} \int_{\mathbb{S}^2} |d\varphi|^2_g\, dv_g. \tag{5.7}$$

We write $\varphi = w + \psi$, where $w \in E_1$, $\psi \in \{1, E_1\}^\perp$. Notice that, since $\lambda_1(dv_g) = \lambda_3(dv_g) = 8\pi$ and $\lambda_4(dv_g) = 24\pi$, it follows that $24\pi \int_{\mathbb{S}^2} \psi^2 \, dv_g \leqslant \int_{\mathbb{S}^2} |d\psi|_g^2 \, dv_g$ and $8\pi \int_{\mathbb{S}^2} w^2 \, dv_g = \int_{\mathbb{S}^2} |dw|_g^2 \, dv_g$. Thus, one obtains

$$\begin{aligned}
&\frac{1+\delta}{8\pi} \int_{\mathbb{S}^2} |d\varphi|_g^2 \, dv_g = \frac{1+\delta}{8\pi} \int_{\mathbb{S}^2} \left(|dw|_g^2 + |d\psi|_g^2 \right) \, dv_g \\
&\geqslant (1+\delta) \int_{\mathbb{S}^2} \left(w^2 + \psi^2 \right) \, dv_g + \frac{1+\delta}{12\pi} \int_{\mathbb{S}^2} |d\psi|_g^2 \, dv_g \\
&\geqslant \int_{\mathbb{S}^2} \left(\varphi^2 + \delta w^2 \right) \, dv_g + \frac{1}{12\pi} \int_{\mathbb{S}^2} |d\psi|_g^2 \, dv_g
\end{aligned}$$

Thus, to obtain (5.7) it is sufficient to show that

$$\int_{\mathbb{S}^2} \varphi^2 \, d(\mu - v_g) \leqslant \delta \int_{\mathbb{S}^2} w^2 \, dv_g + \frac{1}{12\pi} \int_{\mathbb{S}^2} |d\psi|_g^2 \, dv_g. \tag{5.8}$$

Before proving this inequality, let us first remark that since E_1 is finite dimensional, there exists a constant C such that $\|w\|_{W^{1,\infty}} \leqslant C\|w\|_{L^2}$. Similarly, for any $\psi \in \{1, E_1\}^\perp$, since $\lambda_4(dv_g) = 24\pi$ we have $\|\psi\|_{W^{1,2}}^2 \leqslant (1 + (24\pi)^{-1}) \, \|d\psi\|_{L^2}^2$, and for a certain constant $C > 0$

$$\|w\psi\|_{W^{1,2}} \leqslant C\|w\|_{L^2}\|d\psi\|_{L^2}.$$

Let us first verify (5.6). With the hypothesis on μ, $\int_{\mathbb{S}^2} w^2 \, d(\mu - v_g) = 0$, thus, the l.h.s. of (5.8) may be estimated as follows,

$$\begin{aligned}
&\int_{\mathbb{S}^2} (w+\psi)^2 \, d(\mu - v_g) = \int_{\mathbb{S}^2} \left(2w\psi + \psi^2 \right) \, d(\mu - v_g) \\
&\leqslant 2\|\mu - v_g\|_{W^{-1,2}} \|w\psi\|_{W^{1,2}} + \epsilon_0 \|\psi\|_{W^{1,2}}^2 \\
&\leqslant 2C\|\mu - v_g\|_{W^{-1,2}} \|w\|_{L^2} \|d\psi\|_{L^2} + \left(1 + \frac{1}{24\pi} \right) \epsilon_0 \|d\psi\|_{L^2}^2 \\
&\leqslant 24\pi C^2 \|\mu - v_g\|_{W^{-1,2}}^2 \|w\|_{L^2}^2 + \left(\frac{1}{24\pi} + \left(1 + \frac{1}{24\pi} \right) \epsilon_0 \right) \|d\psi\|_{L^2}^2,
\end{aligned}$$

where we used the arithmetic-geometric mean inequality in the last step. As a result, we obtain (5.8) with $\delta = 24\pi C^2 \|\mu - v_g\|_{W^{-1,2}}^2$ as long as we choose $\epsilon_0 = \dfrac{1}{1 + 24\pi}$. Substituting this δ in (5.7) completes the proof.

To show (5.5) it is sufficient to note that

$$\int_{\mathbb{S}^2} w^2 \, d(\mu - v_g) \leqslant \|\mu - v_g\|_{W^{-1,2}} \|w^2\|_{W^{1,2}} \leqslant C\|\mu - v_g\|_{W^{-1,2}} \|w\|_{L^2}^2.$$

Adding this term to the computation above, we obtain (5.8) with

$$\delta = C\|\mu - v_g\|_{W^{-1,2}} (1 + 24\pi C \|\mu - v_g\|_{W^{-1,2}}).$$

□

We now prove that the Hersch inequality is not stable in $\left(W^{1,2,-\frac{1}{2}} \right)^*$, and, consequently, is also not stable in $(W^{1,2-\epsilon})^*$ for any $\varepsilon > 0$. Note that by Sobolev embedding theorem, it is thus not stable in $(W^{1-\varepsilon,2})^*$ either. We claim that in order to show this it is sufficient to prove the following theorem.

Theorem 5.34. *There exists a sequence $(\mu_\epsilon)_\epsilon$ of admissible, balanced probability measures on $\mathbb{S}^2$, such that $\lambda_1(\mu_\epsilon) \underset{\epsilon \to 0}{\longrightarrow} 8\pi$ and*

$$\liminf_{\epsilon \to 0} \|\mu_\epsilon - v_g\|_{\left(W^{1,2,-\frac{1}{2}} \right)^*} > 0.$$

Indeed, assume that there exist conformal automorphisms $\Phi_\epsilon \in \mathrm{Conf}(\mathbb{S}^2)$ such that $(\Phi_\epsilon)_*\mu_\epsilon \to v_g$ in $\left(W^{1,2,-\frac{1}{2}}\right)^*$. Note that since $\bar{\lambda}_1(\mu_\epsilon) \to 8\pi$ and μ_ϵ are balanced, an application of proposition 5.29 with $u = \mathrm{id}\colon \mathbb{S}^2 \to \mathbb{S}^2$ yields that $\mu_\epsilon \to v_g$ in $W^{-1,2}$. Suppose that Φ_ϵ converges (smoothly up to a choice of a subsequence) to a conformal automorphism Φ_0. Then $\mu_\epsilon \to (\Phi_0^{-1})_* v_g$ in $\left(W^{1,2,-\frac{1}{2}}\right)^*$ and, at the same time, $\mu_\epsilon \to v_g$ in $W^{-1,2}$. Since the space $\left(W^{1,2,-\frac{1}{2}}\right)^*$ embeds in $W^{-1,2}$, one has $(\Phi_0^{-1})_* v_g = v_g$ in contradiction with the conclusion of Theorem 5.34. If Φ_ϵ does not converge, then (up to a choice of a subsequence) Φ_ϵ sends most of $\mathbb{S}^2$ into a shrinking neighbourhood of a single point. In particular, $\mu_\epsilon \rightharpoonup \delta_p$ for some $p \in \mathbb{S}^2$, which contradicts $\mu_\epsilon \to v_g$ in $W^{-1,2}$.

In order to prove Theorem 5.34 we need the following lemma, where we use the notation $a_\epsilon \underset{\epsilon\to 0}{\sim} b_\epsilon$ to mean that $\lim_{\epsilon\to 0} \frac{a_\epsilon}{b_\epsilon} = 1$.

Lemma 5.35. *Let B_ϵ be a disk of radius ϵ in $\mathbb{R}^2$, then*

$$\sup_{u\in W_0^{1,2}(B_1),\ \|du\|_{L^2(B_1)}=1} \int_{B_\epsilon} u^2 \,\mathrm{d}x \underset{\epsilon\to 0}{\sim} \frac{1}{2}\epsilon^2 \log(1/\epsilon), \tag{5.9}$$

$$\sup_{u\in W_0^{1,2}(B_1),\ \|du\|_{L^2(B_1)}=1} \int_{B_\epsilon} u \,\mathrm{d}x \underset{\epsilon\to 0}{\sim} \epsilon^2 \sqrt{\frac{\pi}{2}\log(1/\epsilon)}, \tag{5.10}$$

where $W_0^{1,2}(B_1)$ is the completion of $C_0^\infty(B_1,\mathbb{R})$ with its respective norm.

Proof. We start with the proof of (5.9). The inverse of this quantity may be rewritten as the eigenvalue problem

$$\lambda_\epsilon := \inf_{u\in W_0^{1,2}(B_1)} \frac{\int_{B_1} |du|^2 \,\mathrm{d}x}{\int_{B_\epsilon} u^2 \,\mathrm{d}x}.$$

By compactness of $W_0^{1,2}(B_1) \hookrightarrow L^2(B_1)$ the infimum is attained for some function $u_\epsilon \in W_0^{1,2}(B_1)$. Its Schwarz rearrangement has lower energy, so we may suppose without loss of generality that $u_\epsilon(x) = \psi_\epsilon(|x|)$ for a certain function $\psi_\epsilon(r)$ defined for $r \in [0,1]$. The Euler-Lagrange equation associated to this problem is

$$\psi_\epsilon'' + \frac{1}{r}\psi_\epsilon' = -\lambda_\epsilon 1_{r<\epsilon}\psi_\epsilon,$$

which implies that $\psi_\epsilon \in C^1((0,1])$ by elliptic regularity. On $B_1 \setminus B_\epsilon$ we know that for a certain constant $k > 0$:

$$u_\epsilon(x) = k\frac{\log(1/|x|)}{\log(1/\epsilon)}.$$

Notice also that at $|x| = \epsilon$ we have $\frac{\partial_r u_\epsilon}{u_\epsilon} = -\frac{1}{\epsilon\log(1/\epsilon)}$, thus $u_\epsilon|_{B_\epsilon}$ is an eigenfunction associated to the first eigenvalue of the Laplacian with Robin condition of parameter $\frac{1}{\epsilon\log(1/\epsilon)}$, denoted $\lambda_1\left(B_\epsilon; \frac{1}{\epsilon\log(1/\epsilon)}\right)$. Now by applying Stokes' formula to u_ϵ on $B_1 \setminus B_\epsilon$ we see that

$$\lambda_\epsilon = \frac{\int_{B_1} |\nabla u_\epsilon|^2}{\int_{B_\epsilon} u_\epsilon^2} = \frac{\int_{B_\epsilon} |\nabla u_\epsilon|^2 + \frac{1}{\epsilon\log(1/\epsilon)}\int_{\partial B_\epsilon} u_\epsilon^2}{\int_{B_\epsilon} u_\epsilon^2} = \lambda_1\left(B_\epsilon; \frac{1}{\epsilon\log(1/\epsilon)}\right).$$

At the same time, $\lambda_1\left(B_\epsilon; \frac{1}{\epsilon\log(1/\epsilon)}\right) = \epsilon^{-2}\lambda_1\left(B_1; \frac{1}{\log(1/\epsilon)}\right) \sim \frac{2}{\epsilon^2\log(1/\epsilon)}$, where we use in the first step the general scaling property

$$\lambda_1(r\Omega;\alpha) = r^{-2}\lambda_1(\Omega; r\alpha),$$

and in the second step the well-known asymptotic formula $\lambda_1(\Omega;\alpha) \underset{\alpha\to 0^+}{\sim} \alpha\frac{|\partial\Omega|}{|\Omega|}$ ([108], see also [106, 99]).

Similarly, let us now prove (5.10). By compactness of $W_0^{1,2}(B_1) \hookrightarrow L^1(B_1)$, we find that the supremum is attained for a positive function u_ϵ. Using the Schwarz rearrangement argument as earlier, we may suppose that u_ϵ is radial and satisfies the Euler-Lagrange equation $-\Delta u_\epsilon = c_\epsilon 1_{B_\epsilon}$ for

a certain constant c_ϵ. In particular, $u \in C^1(B_1)$ by elliptic regularity. Thus, up to multiplication by a scalar, u_ϵ is given by:

$$u_\epsilon(x) = \begin{cases} \log(1/|x|) & \text{if } \epsilon \leqslant |x| \leqslant 1 \\ \log(1/\epsilon) + \frac{\epsilon^2 - |x|^2}{2\epsilon^2} & \text{if } |x| \leqslant \epsilon. \end{cases}$$

An explicit computation yields the result. □

Consider two antipodal points x_1, x_2 on $\mathbb{S}^2$, e.g. the north pole $N = x_1$ and south pole $S = x_2$. Let $r > 0$ be such that $B_g(x_i, 2r)$ is the corresponding hemisphere. Using the stereographic projection from x_i onto the equatorial plane one can construct conformal flat metrics $g_i := e^{2w_i} g$ on $B_g(x_i, 2r)$, such that

- $w_i(x)$ is bounded and only depends on $\mathrm{dist}_g(x, x_i)$;
- $B_g(x_i, r) = B_{g_i}(x_i, 1)$;
- if $\rho_i(\epsilon)$ is such that $B_g(x_i, \epsilon) = B_{g_i}(x_i, \rho_i(\epsilon))$, then $\rho_i(\epsilon) \underset{\epsilon \to 0}{\sim} e^{w_i(x_i)}\epsilon$.

Corollary 5.36. *There exists a constant $C_0 > 0$ such that for any $\epsilon > 0$ small enough one has*

$$C_0^{-1}\epsilon^2 \log(1/\epsilon) \leqslant \sup_{\|u\|_{W^{1,2}}=1} \int_{\cup_{i=1}^2 B_g(x_i,\epsilon)} u^2 \, dv_g \leqslant C_0 \epsilon^2 \log(1/\epsilon), \tag{5.11}$$

$$C_0^{-1}\epsilon^2 \sqrt{\log(1/\epsilon)} \leqslant \sup_{\|u\|_{W^{1,2}}=1} \int_{\cup_{i=1}^2 B_g(x_i,\epsilon)} u \, dv_g \leqslant C_0 \epsilon^2 \sqrt{\log(1/\epsilon)}. \tag{5.12}$$

Proof. We prove (5.11), the proof of (5.12) is identical. Let $\chi_i \in C^\infty(\mathbb{S}^2, [0,1])$ be such that $\{\chi_i \neq 0\} \Subset B_g(x_i, r)$ and $B_g(x_i, r/2) \Subset \{\chi_i = 1\}$. The estimate (5.9) implies that there exists a constant $C_1 > 0$ such that for any small enough $\epsilon > 0$,

$$C_1^{-1}\epsilon^2 \log(1/\epsilon) \leqslant \sup_{u \in W_0^{1,2}(B_1),\ \|du\|_{L^2(B_1)}=1} \int_{B_\epsilon} u^2 \, \mathrm{d}x \leqslant C_1 \epsilon^2 \log(1/\epsilon).$$

Since the functions w_i are bounded, for any $\varphi \in W^{1,2}$, one has

$$\begin{aligned} &\int_{\cup_{i=1}^2 B_g(x_i,\epsilon)} \varphi^2 \, dv_g \leqslant C \sum_{i=1}^2 \int_{B_{g_i}(x_i,\rho_i(\epsilon))} (\chi_i \varphi)^2 \, dv_{g_i} \\ &\leqslant C C_1 \sum_{i=1}^2 \rho_i(\epsilon)^2 \log(1/\rho_i(\epsilon)) \|d(\chi_i\varphi)\|^2_{L^2(\mathbb{S}^2, g_i)} \\ &\leqslant C C_1 \left(1 + \|d\chi_i\|^2_{L^\infty}\right) \epsilon^2 \log(1/\epsilon)(1 + o(1)) \|\varphi\|^2_{W^{1,2}}, \end{aligned}$$

where C is a constant depending only on $\|w_i\|_{L^\infty}$, possibly changing from line to line.

Conversely, for each $\epsilon > 0$ there is a function $\varphi_\epsilon \in C_0^\infty(B_g(x_1, r))$ such that $\int_{B_{g_1}(x_1,\rho_1(\epsilon))} \varphi_\epsilon^2 \, dv_{g_1} \geqslant C_1^{-1} \rho_1(\epsilon)^2 \log(1/\rho_1(\epsilon)) \|d\varphi_\epsilon\|^2_{L^2}$. As a result,

$$\begin{aligned} &\int_{\cup_{i=1}^2 B_g(x_i,\epsilon)} \varphi_\epsilon^2 \, dv_g \geqslant C C_1^{-1} \rho_1(\epsilon)^2 \log(1/\rho_1(\epsilon)) \|d\varphi_\epsilon\|^2_{L^2} \\ &\geqslant \frac{C C^{-1}(1 + o(1))}{(1 + \lambda_*(B_g(x_1, r), g))} \epsilon^2 \log(1/\epsilon) \|\varphi_\epsilon\|^2_{W^{1,2}}, \end{aligned}$$

where $\lambda_*(\Omega, g)$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions on $\partial\Omega$.

The proof of (5.12) is the same, except φ^2 is replaced with φ.

□

We are now ready to define our sequence of measures. For any $M, \epsilon > 0$ set

$$\nu_\epsilon = \frac{1_{\cup_{i=1}^2 B_g(x_i,\epsilon)}}{\epsilon^2 \log(1/\epsilon)} dv_g, \ \mu_\epsilon^M = \frac{v_g + M\nu_\epsilon}{1 + M\nu_\epsilon(\mathbb{S}^2)}.$$

The following lemma motivates the definition of ν_ϵ.

Lemma 5.37. *For any $M, \epsilon > 0$ the measures ν_ϵ, μ_ϵ^M possess the following properties,*

1. *$\|\nu_\epsilon\|_{W^{-1,2}} \to 0$ as $\epsilon \to 0$. In particular, one has*
$$\|\mu_\epsilon^M - v_g\|_{W^{-1,2}} \to 0;$$
2. *There exists $c > 0$ such that*
$$\liminf_{\epsilon\to 0} \|\mu_\epsilon^M - v_g\|_{\left(W^{1,2,-\frac{1}{2}}\right)^*} > cM > 0.$$

Proof. The upper bound (5.12) implies that

$$\|\nu_\epsilon\|_{W^{-1,2}} \leqslant C_0 \left(\log(1/\epsilon)\right)^{-\frac{1}{2}} \to 0.$$

As a result,

$$\|\mu_\epsilon^M - v_g\|_{W^{-1,2}} = \left\| \frac{M\nu_\epsilon(\mathbb{S}^2) dv_g + M\nu_\epsilon}{1 + M\nu_\epsilon(\mathbb{S}^2)} \right\|_{W^{-1,2}} \to 0,$$

since $\nu_\epsilon(\mathbb{S}^2) \leqslant C \left(\log(1/\epsilon)\right)^{-1} \to 0$. This completes the proof of (1).

To show (2) we write

$$\begin{aligned}
\|\mu_\epsilon^M - v_g\|_{\left(W^{1,2,-\frac{1}{2}}\right)^*} &\geqslant \sup_{\varphi \in W^{1,2}} \frac{\int_{\mathbb{S}^2} \varphi^2 \, d(\mu_\epsilon^M - v_g)}{\|\varphi^2\|_{W^{1,2,-\frac{1}{2}}}} \\
&\geqslant \sup_{\varphi \in W^{1,2}} \frac{M}{1 + M\nu_\epsilon(\mathbb{S}^2)} \left(\frac{\int_{\mathbb{S}^2} \varphi^2 \, d\nu_\epsilon}{C_{\mathrm{Or}} \|\varphi\|_{W^{1,2}}^2} - \frac{\nu_\epsilon(\mathbb{S}^2) \int_{\mathbb{S}^2} \varphi^2 dv_g}{\|\varphi^2\|_{\left(W^{1,2,-\frac{1}{2}}\right)^*}} \right) \\
&\geqslant \frac{cM}{1 + M\nu_\epsilon(\mathbb{S}^2)} (1 - C\nu_\epsilon(\mathbb{S}^2)),
\end{aligned}$$

where in the last step we used the lower bound (5.11). Since $\nu_\epsilon(\mathbb{S}^2) \to 0$, the proof is complete. □

Proposition 5.38. *There are constants $M_0, C > 0$ such that*

1. *$\limsup_{\epsilon\to 0} \lambda_1(\mu_\epsilon^M) \leqslant \frac{C}{M}$;*
2. *If $M < M_0$, then $\lambda_1(\mu_\epsilon^M) \to 8\pi$ as $\epsilon \to 0$.*

Proof. To prove (1) we write

$$\lambda_1(\mu_\epsilon^M)^{-1} = \sup_{\int_{\mathbb{S}^2} |d\varphi|^2 dv_g = 1} \left(\int_{\mathbb{S}^2} \varphi^2 d\mu_\epsilon^M - \left(\int_{\mathbb{S}^2} \varphi d\mu_\epsilon^M \right)^2 \right)$$

This formula is invariant up to the addition of a constant to φ, so we may take $\varphi \in 1^\perp := \left\{\psi : \int_{\mathbb{S}^2} \psi dv_g = 0\right\}$ without loss of generality. Let $H = \{\varphi \in 1^\perp, \int_{\mathbb{S}^2} |d\varphi|^2 dv_g = 1\}$, this may be rewritten

$$\begin{aligned}
&\lambda_1(\mu_\epsilon^M)^{-1} = \\
&= \frac{1}{1 + M\nu_\epsilon(\mathbb{S}^2)} \sup_{\varphi \in H} \left(\int_{\mathbb{S}^2} \varphi^2 d(v_g + M\nu_\epsilon) - \frac{M^2}{1 + M\nu_\epsilon(\mathbb{S}^2)} \left(\int_{\mathbb{S}^2} \varphi d\nu_\epsilon \right)^2 \right) \\
&\geqslant \frac{M}{1 + M\nu_\epsilon(\mathbb{S}^2)} \sup_{\varphi \in H} \int_{\mathbb{S}^2} \varphi^2 d\nu_\epsilon - \frac{(1 + (8\pi)^{-1}) M^2 \|\nu_\epsilon\|_{W^{-1,2}}^2}{(1 + M\nu_\epsilon(\mathbb{S}^2))^2}.
\end{aligned}$$

The second term goes to 0 by Lemma 5.37. Moreover, according to the estimate (5.9), we may find a function ψ_ϵ with support on $B_g(x_1, r)$ such that $\|d\psi_\epsilon\|_{L^2(\mathbb{S}^2, g_1)} = 1$ and

$$\int_{B_{g_1}(x_1, \rho_1(\epsilon))} \psi_\epsilon^2 dv_{g_1} \geqslant \frac{1}{2} C_1^{-1} \epsilon^2 \log(1/\epsilon).$$

Let ϕ_ϵ be the same function on $B_g(x_2, r)$ (which is isometric to $B_g(x_1, r)$), then $\frac{\phi_\epsilon - \psi_\epsilon}{\sqrt{2}} \in H$ and there is a constant $c > 0$ such that

$$\sup_{\varphi \in H} \int_{\mathbb{S}^2} \varphi^2 d\nu_\epsilon \geqslant \int_{\mathbb{S}^2} \left(\frac{\phi_\epsilon - \psi_\epsilon}{\sqrt{2}} \right)^2 d\nu_\epsilon \geqslant c.$$

Thus, as $\epsilon \to 0$, we obtain $\lambda_1(\mu_\epsilon^M)^{-1} \geqslant cM$.

To show (2) we note that (5.11) implies that for any $\varphi \in W^{1,2}$,

$$\int_{\mathbb{S}^2} \varphi^2 \, d\nu_\epsilon \leqslant C_0 M \|\varphi\|_{W^{1,2}}.$$

Therefore, the assumption (5.4) of Proposition 5.31 is satisfied for $M < M_0 := C_0^{-1} \epsilon_0$. An application of (5.5) concludes the proof. □

Let M be large enough so that $\limsup_{\epsilon \to 0} \lambda_1(\mu_\epsilon^M) \leqslant \frac{C}{M} < 8\pi$. Then by Lemma 5.37 part (1) the sequence $(\mu_\epsilon^M)_{\epsilon \to 0}$ gives the proof of the following corollary.

Corollary 5.39. *$\mu \mapsto \lambda_1(\mu)$ is not continuous in $W^{-1,2}$ and the Proposition 5.31 fails without the assumption* (5.4).

At the same time, when $M < M_0$ we obtain Theorem 5.34.

Proof of Theorem 5.34. Consider $M < M_0$ and $(\mu_\epsilon^M)_\epsilon$ as defined in proposition 5.38, then $\lambda_1(\mu_\epsilon^M) \underset{\epsilon \to 0}{\longrightarrow} 8\pi$. An application of Lemma 5.37, part (2) completes the proof the corollary.

□

Bibliography

[91] Lars V. Ahlfors. An extension of Schwarz's lemma. *Trans. Amer. Math. Soc.*, 43(3):359–364, 1938.

[92] Beniamin Bogosel. The Steklov spectrum on moving domains. *Appl. Math. Optim.*, 75(1):1–25, 2017.

[93] Lorenzo Brasco, Guido De Philippis, and Berardo Ruffini. Spectral optimization for the Stekloff-Laplacian: the stability issue. *J. Funct. Anal.*, 262(11):4675–4710, 2012.

[94] Dorin Bucur, Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti. Weinstock inequality in higher dimensions. *Journal of Differential Geometry (to appear)*, 2020.

[95] Dorin Bucur, Alessandro Giacomini, and Paola Trebeschi. L^∞-bounds of Steklov eigenfunctions and spectrum stability under domain variation. *J. Diff. Equations (to appear)*, 2020.

[96] Ailana Fraser and Richard Schoen. Shape optimization for the Steklov problem in higher dimensions. *Adv. Math.*, 348:146–162, 2019.

[97] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[98] Nunzia Gavitone, Domenico Angelo La Manna, Gloria Paoli, and Leonardo Trani. A quantitative Weinstock inequality. *Arxiv 1903.04964*, 2019.

[99] Tiziana Giorgi and Robert Smits. Bounds and monotonicity for the generalized Robin problem. *Z. Angew. Math. Phys.*, 59(4):600–618, 2008.

[100] Alexandre Girouard, Mikhail Karpukhin, and Jean Lagacé. Continuity of eigenvalues and shape optimisation for Laplace and Steklov problems. *Geom. Funct. Anal.*, 31(3):513–561, 2021.

[101] Alexandre Girouard and Iosif Polterovich. Upper bounds for Steklov eigenvalues on surfaces. *Electron. Res. Announc. Math. Sci.*, 19:77–85, 2012.

[102] Alexandre Girouard and Iosif Polterovich. Spectral geometry of the Steklov problem [reprint of MR3662010]. In *Shape optimization and spectral theory*, pages 120–148. De Gruyter Open, Warsaw, 2017.

[103] Alexandre Girouard and Iosif Polterovich. Spectral geometry of the Steklov problem (survey article). *J. Spectr. Theory*, 7(2):321–359, 2017.

[104] Antoine Henrot and Michel Pierre. *Shape variation and optimization*, volume 28 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2018. A geometrical analysis, English version of the French publication [MR2512810] with additions and updates.

[105] Gerasim Kokarev. Variational aspects of Laplace eigenvalues on Riemannian surfaces. *Adv. Math.*, 258:191–239, 2014.

[106] A. A. Lacey, J. R. Ockendon, and J. Sabina. Multidimensional reaction diffusion equations with nonlinear boundary conditions. *SIAM J. Appl. Math.*, 58(5):1622–1647, 1998.

[107] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[108] René P. Sperb. Untere und obere Schranken für den tiefsten Eigenwert der elastisch gestützten Membran. *Z. Angew. Math. Phys.*, 23:231–244, 1972.

[109] Gregory Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.*, 59(3):572–611, 1984.

[110] Robert Weinstock. Inequalities for a classical eigenvalue problem. *J. Rational Mech. Anal.*, 3:745–753, 1954.

[111] Robert Weinstock. Inequalities for a classical eigenvalue problem. *Department of Math., Stanford Univ., Tech. Rep.*, 37, 1954.

Chapter 6

Existence and regularity of optimal shapes for spectral functionals with Robin boundary conditions

Contents

6.1 Introduction 123
6.1.1 State of the art 124
6.1.2 Statements of the main results 125
6.2 Relaxed framework 127
6.3 Strictly monotonous functionals 129
6.3.1 A priori estimates 129
6.3.2 Existence of a relaxed minimizer with prescribed measure 135
6.3.3 Regularity of minimizers 137
6.4 The functional $\Omega \mapsto \lambda_k(\Omega;\beta)$ 140
6.4.1 Regularization and perturbation lemma 141
6.4.2 Non-degeneracy lemma and the main result 143
6.4.3 Discussion about the properties of open minimizers 146

6.1 Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $\beta > 0$ a parameter that is constant throughout the paper, and $f \in L^2(\Omega)$. The Poisson equation with Robin boundary conditions is

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_\nu u + \beta u = 0 & \text{in } \partial\Omega, \end{cases}$$

where ∂_ν is the outward normal derivative that may only have a meaning in the sense that for all $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \beta u v \mathrm{d}\mathscr{H}^{n-1} = \int_\Omega f v \mathrm{d}\mathscr{L}^n.$$

This equation (and in particular its boundary conditions) has several interpretations: we may see the solution u as the temperature obtained in an homogeneous solid Ω with the volumetric heat source f, and insulator on the boundary (more precisely, a width $\beta^{-1}\epsilon$ of insulator of conductivity ϵ for $\epsilon \to 0$) that separates the solid Ω from a thermostat.
Another interpretation is to see u as the vertical displacement of a membrane with shape Ω on which we apply a volumetric normal force f, and the membrane is fixed on its boundary by elastic with stiffness proportional to β.

This equation is associated to a sequence of eigenvalues

$$0 < \lambda_1(\Omega;\beta) \leq \lambda_2(\Omega;\beta) \leq \ldots \to +\infty,$$

with eigenfunctions $u_k(\Omega;\beta)$ that verify

$$\begin{cases} \Delta u_k(\Omega;\beta) + \lambda_k(\Omega;\beta)u_k(\Omega;\beta) = 0 & \text{in } \Omega, \\ \partial_\nu u_k(\Omega;\beta) + \beta u_k(\Omega;\beta) = 0 & \text{in } \partial\Omega. \end{cases}$$

The quantities $(\lambda_k(\Omega;\beta))_k$ may be extended to any open set Ω in a natural way, see Section 2 for more details.

In this paper, we study some shape optimization problems involving the eigenvalues $(\lambda_k(\Omega;\beta))_k$ with measure constraint on general open sets. In particular we prove that when $F(\lambda_1,\ldots,\lambda_k)$ is a function with positive partial derivative in each λ_i (such as $F(\lambda_1,\ldots,\lambda_k) = \lambda_1 + \ldots + \lambda_k$), then for any $m, \beta > 0$ the optimisation problem

$$\min\left\{F\left(\lambda_1(\Omega;\beta),\ldots,\lambda_k(\Omega;\beta)\right),\ \Omega \subset \mathbb{R}^n \text{ open such that } |\Omega| = m\right\}$$

has a solution. Moreover the topological boundary of an optimal set is rectifiable, Ahlfors-regular, with finite $\mathscr{H}^{n-1}$-measure. For functionals of the form $F(\lambda_1,\ldots,\lambda_k) = \lambda_k$, while minimizers are only known to exist in a relaxed SBV setting (that will be detailed in the second section), we show that any SBV minimizer verifies

$$\lambda_k(\Omega;\beta) = \lambda_{k-1}(\Omega;\beta)$$

in any dimension $n \geq 3$.

6.1.1 State of the art

The link between the eigenvalues of the Laplace operator (or other differential operators) on a domain and the geometry of this domain is a problem that has been widely studied, in particular in the field of spectral geometry.

The earliest and most well-known result in this direction dates back to the Faber-Krahn inequality, that states that the first eigenvalue of the Laplacian with Dirichlet boundary conditions is, among sets of given measure, minimal on the disk. The same result was shown for Robin boundary conditions with positive parameter in [115] in the two-dimensional case, then in [125] in any dimension for a certain class of domains on which the trace may be defined, using dearrangement methods. It was extended in [117], [118] in the SBV framework that we will describe in the next section, such that the first eigenvalue with Robin boundary condition is minimal on the ball among all open sets of given measure. In order to handle the lack of uniform smoothness of the admissible domains, the method here is to consider a relaxed version of the problem, so as to optimize an eigenfunction instead of a shape. Once it is known a minimizer exists in the relaxed framework, it is shown by regularity and symmetry arguments that this minimizer corresponds to the disk.

Similar problems of spectral optimization with Neumann boundary conditions or Robin conditions with negative parameter have been shown to be different in nature, in the former case the first eigenvalue is maximal on the disk, and this is shown with radically different method, mainly building appropriate test functions since the eigenvalues are defined as an infimum through the Courant-Fischer min-max formula. Let us also mention several maximization result for Robin boundary condition with parameter that scales with the perimeter, obtained in [134], [127] with similar methods.

The existence and partial regularity for minimizers of functions $F(\lambda_1^D(\Omega),\ldots,\lambda_k^D(\Omega))$ (where $\lambda_i^D(\Omega)$ is the i-th eigenvalue of the Laplacien with Dirichlet boundary conditions) with measure constraint or penalization has been achieved in [116], [135], [132], [133]: it is known that if F is increasing

and bi-Lipschitz in each λ_i then there is an optimal open set that is $\mathcal{C}^{1,\alpha}$ outside of a singular set of codimension at least three, and if F is merely nondecreasing in each coordinate then there is an optimal quasiopen set that has analytic boundary outside of a singular set of codimension three and points with Lebesgue density one. It has been shown in [122], [133] that a shape optimizer for the k-th eigenvalue with Dirichlet boundary conditions and measure constraint admits Lipschitz eigenfunctions. In these papers the monotonicity and scaling properties of the eigenvalues with Dirichlet boundary condition ($\omega \mapsto \lambda_k^D(\omega)$ is decreasing in ω) plays a crucial role, however eigenvalues with Robin boundary conditions have no such properties so the same methods cannot be extended in a straightforward way.

The minimization of $\lambda_2(\Omega;\beta)$ under measure constraint on Ω was treated in [130]; as in the Dirichlet case, the minimizer is the disjoint union of two balls of same measure. For the minimization of $\lambda_k(\Omega;\beta)$ or other functionals of $\lambda_1(\Omega;\beta),\dots,\lambda_k(\Omega;\beta)$, nothing is known except for the existence of a minimizer in the relaxed setting with bounded support for $\lambda_k(\Omega;\beta)$, see [119]. The regularity theory for minimizers of functionals involving Robin boundary conditions was developped in [123], [131] and we will relay on some of its results in our vectorial setting.

Numerical simulations in [113] for two-dimensional minimizers of $\lambda_k(\cdot;\beta)$ (for $3 \leq k \leq 7$) with prescribed area suggest a bifurcation phenomena in which the optimal shape is a union of k balls for every small enough β, and it is connected for any large enough β. In [113], the connected minimizers were searched by parametric optimization among perturbations of the disk, however a consequence of our analysis in the last section is that minimizers of $\lambda_3(\cdot;\beta)$ are never homeomorphic to the disk.

6.1.2 Statements of the main results

In the first part of the paper, we are concerned in what we call the non-degenerate case; consider

$$F : \left\{\lambda \in \mathbb{R}^k : 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k\right\} \to \mathbb{R}_+$$

a Lipschitz function with directional derivatives - in the sense that for any $\lambda \in \mathbb{R}^k$ there is some positively homogeneous function F_0 such that $F(\lambda+\nu) = F(\lambda) + F_0(\nu) + o_{\nu\to 0}(|\nu|)$ - such that for all $i \in \{1,\dots,n\}$, and all $0 < \lambda_1 \leq \ldots \leq \lambda_k$

$$\frac{\partial F}{\partial^{\pm}\lambda_i}(\lambda_1,\dots,\lambda_k) > 0,\ F(\lambda_1,\dots,\lambda_{k-1},\mu_k) \underset{\mu_k\to\infty}{\longrightarrow} +\infty, \tag{6.1}$$

where $\frac{\partial}{\partial^{\pm}\lambda_i}$ designates the directional partial derivatives in λ_i. This applies in particular to any of these:

$$F_p(\lambda_1,\dots,\lambda_k) = \left(\sum_{i=1}^k \lambda_i^p\right)^{\frac{1}{p}}.$$

Our first main result is the following.

Theorem 6.1. *Let F be such a function, $m > 0$, then there exists an open set that minimizes the functional*

$$\Omega \mapsto F(\lambda_1(\Omega;\beta),\dots,\lambda_k(\Omega;\beta))$$

among open sets of measure m in $\mathbb{R}^n$. Moreover any minimizing set is bounded, verifies $\mathscr{H}^{n-1}(\partial\Omega) \leq C$ for some constant $C > 0$ depending only on (n, m, β, F), and $\partial\Omega$ is Ahlfors-regular.

Here are the main steps of the proof:

- **Relaxation**. We relax the problem in the SBV framework; this is introduced in the next subsection, following [117], [118], [119]. The idea is that the eigenfunctions on a domain Ω are expected to be zero almost nowhere on Ω; we extend these eigenfunctions by zero outside of Ω (thereby creating a discontinuity along $\partial\Omega$) and reformulate the optimization problem on general functions defined in $\mathbb{R}^n$ that may have discontinuities, with measure constraint on

their support. The advantage is that we have some compactness and lower semi-continuity results to obtain the existence of minimizers in the relaxed framework, however a sequence of eigenfunctions extended by zero may converge to a function that does not correspond to the eigenfunction of an open domain, so we will need to show some regularity on relaxed minimizers.

- **A priori estimates and nondegeneracy**. We obtain a priori estimates for relaxed interior minimizers (meaning minimizers compared to any set that it contains). More precisely for any interior minimizer that corresponds to the eigenfunctions $(u_1, \dots, u_k)$, we show that for almost any point in the support of these eigenfunctions, at least one of them is above a certain positive threshold. We also obtain L^∞ bounds of these eigenfunctions and deduce a lower estimate for the Lebesgue density of the support, from which we obtain the boundedness of the support.

- **Existence of minimizers**. We consider a minimizing sequence and show that, up to a translation, it either converges to a minimizer or it splits into two minimizing sequences of similar functionals depending on p and $k-p$ (where $1 \leq p < k$) eigenvalues respectively, and we know minimizers of these exists by induction on k.

- **Regularity**. Finally, we show the regularity of relaxed minimizer, meaning that a relaxed minimizer corresponds to the eigenfunctions of a certain open domain that were extended by zero, by showing that the singular set of relaxed minimizers is closed up to a $\mathscr{H}^{n-1}$-negligible set.

Notice that in the second step we do not show that the first eigenfunction (or one of the l first in the case where the minimizer has l connected components) is positive, which is what we expect in general for sufficiently smooth sets; if u_1 is the first (positive) eigenfunction on a connected $\mathcal{C}^2$ set Ω, and suppose $u_1(x) = 0$ for some $x \in \partial\Omega$ then by Hopf's lemma $\partial_\nu u_1(x) < 0$, which breaks the Robin condition at x, so $\inf_\Omega u_1 > 0$. In our case, we get instead a "joint non-degeneracy" of the eigenfunctions in the sense that at every point of their joint support, at least one is positive.

Notice also that the second hypothesis in (6.1) is not superfluous: without it, a minimizing sequence (Ω^i) could have some of its first k eigenvalues diverge. This is because, unlike the Dirichlet case, there is no upper bound for $\frac{\lambda_k(\cdot;\beta)}{\lambda_1(\cdot;\beta)}$ in general even among sets with fixed measure. While $\lambda_k(\cdot;\beta)$ is not homogeneous by dilation, we still have the scaling property

$$\lambda_k(r\Omega;\beta) = r^{-2}\lambda_k(\Omega;r\beta).$$

Consider a connected smooth open set Ω. Since each $\lambda_k(\Omega;r\beta)$ converges to $\lambda_k(\Omega,0)$ (the eigenvalues with Neumann boundary conditions) as $r \to 0$, and $0 = \lambda_1(\Omega,0) < \lambda_2(\Omega,0)$, then for any $k \geq 2$, $\frac{\lambda_k(r\Omega;\beta)}{\lambda_1(r\Omega;\beta)} \underset{r\to 0}{\longrightarrow} +\infty$. A counterexample among sets of fixed measure may be obtained with the disjoint union of $r\Omega$ for small r and a set ω with prescribed measure such that $\lambda_1(\omega;\beta) > \lambda_k(r\Omega;\beta)$, such as a disjoint union of enough balls of radius $\rho > 0$, chosen small enough to have $\lambda_1(\omega,\beta) = \lambda_1(\mathbb{B}_\rho;\beta) > \lambda_k(r\Omega;\beta)$.

In the second part of the paper, we study the minimizers of the functional

$$\Omega \mapsto \lambda_k(\Omega;\beta).$$

A minimizer in the SBV framework (see the introduction below) was shown to exist in [119], and aside from the fact that its support is bounded nothing more is known. We show that, in this SBV framework, a minimizer necessarily verify that $\lambda_{k-1}(\Omega;\beta) = \lambda_k(\Omega;\beta)$, in the context of definition 6.6.

This is a long lasting open problem for minimizers of λ_k with Dirichlet boundary condition (see [129, open problem 1] and [136]).

However, although we prove it for Robin conditions, we do not expect this result to directly extend to the Dirichlet case ; simply put, even if some smooth sequence of minimizers Ω^β of $\lambda_k(\cdot;\beta)$ approached a minimizer Ω of λ_k^D that is a counterexample of the conjecture, then there is no reason why the upper semi-continuity $\lambda_{k-1}^D(\Omega) \geq \limsup_{\beta\to\infty} \lambda_{k-1}(\Omega^\beta;\beta)$ should hold.

Theorem 6.2. *Suppose $n \geq 3$, $k \geq 2$. Let $m > 0$, and let $\boldsymbol{u}$ be a relaxed minimizer of*

$$\boldsymbol{v} \mapsto \lambda_k(\boldsymbol{v};\beta)$$

among admissible functions with support of measure m. Then

$$\lambda_{k-1}(\boldsymbol{u};\beta) = \lambda_k(\boldsymbol{u};\beta).$$

Here are the main steps and ideas of the proof:

- First, we replace the minimizer $\mathbf{u} = (u_1, \dots, u_k)$ with another minimizer $\mathbf{v} = (v_1, \dots, v_k)$, with the property that $v_1 \geq 0$, $\lambda_k(\mathbf{v};\beta) > \lambda_{k-1}(\mathbf{v};\beta)$, and $\mathbf{v} \in L^\infty(\mathbb{R}^n)$. One might think this estimate also holds for $\mathbf{u}$, however there is no particular reason why $\mathrm{span}(\mathbf{u})$ should contain eigenfunctions for $\lambda_1(\mathbf{u};\beta), \dots, \lambda_{k-1}(\mathbf{u};\beta)$ in a variational sense.
  This phenomenon may be easily understood in a finite-dimensional setting as follows: consider the matrix $A = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ with $\lambda_1 < \lambda_2 < \lambda_3$. Then $\Lambda_2\Big[A\Big]$ is given by:

$$\lambda_2 = \inf_{V\subset\mathbb{R}^3, \dim(V)=2} \sup_{x\in V} \frac{(x,Ax)}{(x,x)}.$$

  This infimum is reached by the subspace $\mathrm{span}(e_1, e_2)$, but also by any subspace $\mathrm{span}(e_1 + te_3, e_2)$ for $|t| \leq \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_2}$, and these subspaces do not contain the first eigenvector e_1.

- Then we obtain a weak optimality condition on u_k using perturbations on sets with a small enough perimeter. The reason for this is that we have no access to any information on $u_1, \dots, u_{k-1}$ apart from the fact that their Rayleigh quotient is strictly less than $\lambda_k(\Omega;\beta)$, so we must do perturbations of u_k that do not increase dramatically the Rayleigh quotient of $u_1, \dots, u_{k-1}$.

- We apply this to sets of the form $\mathbb{B}_{x,r} \cap \{|u_k| \leq t\}$ where r is chosen small enough for each t. With this we obtain that $|u_k| \geq c1_{\{u_k\neq 0\}}$.

- We deduce the result by showing that the support of $\mathbf{u}$ is disconnected, so the k-th eigenvalue may be decreased without changing the volume by dilations.

While the existence of open minimizers is not yet known, we end with a few observations on the topology of these minimizer, in particular with the fact that a bidimensionnal minimizer of $\lambda_3(\cdot;\beta)$ with prescribed measure is never simply connected.

6.2 Relaxed framework

Throughout the paper, we use the relaxed framework of SBV functions to define Robin eigenvalues on any open set without regularity condition, and more importantly to transform our shape optimization problem into a free discontinuity problem on functions that are not defined on a particular domain any more. The SBV space was originally developed to handle relaxations of free discontinuity problems such as the Mumford-Shah functional that will come into play later, we refer to [112] for a complete introduction. SBV functions may be thought of as "$W^{1,1}$ by part" functions, and this space is defined as a particular subspace of BV as follows:

Definition 6.3. A SBV function is a function $u \in BV(\mathbb{R}^n, \mathbb{R})$ such that the distributional derivative Du (which is a finite vector-valued Radon measure) may be decomposed into

$$Du = \nabla u \mathscr{L}^n + (\overline{u} - \underline{u})\nu_u \mathscr{H}^{n-1}_{\lfloor J_u},$$

where $\nabla u \in L^1(\mathbb{R}^n)$, J_u is the jump set of u defined as the set of point $x \in \mathbb{R}^n$ for which there is some $\overline{u}(x) \neq \underline{u}(x) \in \mathbb{R}$, $\nu_u(x) \in \mathbb{S}^{n-1}$, such that

$$(y \mapsto u(x + ry)) \underset{L^1_{\text{loc}}(\mathbb{R}^n)}{\longrightarrow} \overline{u}(x)1_{\{y : y \cdot \nu_u(x) > 0\}} + \underline{u}(x)1_{\{y : y \cdot \nu_u(x) < 0\}} \text{ as } r \to 0.$$

We will not work directly with the SBV space but with an L^2 analog defined below, that was studied in [117].

Definition 6.4. Let $\mathcal{U}_k$ be the space of functions $\mathbf{u} \in L^2(\mathbb{R}^n, \mathbb{R}^k)$ such that

$$D\mathbf{u} = \nabla \mathbf{u} \mathscr{L}^n + (\overline{\mathbf{u}} - \underline{\mathbf{u}})\nu_{\mathbf{u}} \mathscr{H}^{n-1}_{\lfloor J_{\mathbf{u}}},$$

where $\nabla \mathbf{u} \in L^2(\mathbb{R}^n, \mathbb{R}^{nk})$ and $\int_{J_{\mathbf{u}}} (|\overline{\mathbf{u}}|^2 + |\underline{\mathbf{u}}|^2) \mathrm{d}\mathscr{H}^{n-1} < \infty$. The second term will be written $D^s\mathbf{u}$ (s stands for singular). The function $\mathbf{u}$ is said to be linearly independant if its components span a k-dimensional space of $L^2(\mathbb{R}^n)$.
We will also say that a function $\mathbf{u} \in \mathcal{U}_k$ is disconnected if there is a measurable partition Ω, ω of the support of $\mathbf{u}$ such that $\mathbf{u}1_\Omega$ and $\mathbf{u}1_\omega$ are in $\mathcal{U}_k$, and:

$$D^s(\mathbf{u}1_\Omega) = (\overline{\mathbf{u}1_\Omega} - \underline{\mathbf{u}1_\Omega})\nu_{\mathbf{u}} \mathscr{H}^{n-1}_{\lfloor J_{\mathbf{u}}},$$
$$D^s(\mathbf{u}1_\omega) = (\overline{\mathbf{u}1_\omega} - \underline{\mathbf{u}1_\omega})\nu_{\mathbf{u}} \mathscr{H}^{n-1}_{\lfloor J_{\mathbf{u}}}.$$

In this case we will write $\mathbf{u} = (\mathbf{u}1_\Omega) \oplus (\mathbf{u}1_\omega)$.

The following compactness theorem is a reformulation of Theorem 2 from [117].

Proposition 6.5. *Let $(\boldsymbol{u}^i)$ be a sequence of $\mathcal{U}_k$ such that*

$$\sup_i \int_{\mathbb{R}^n} |\nabla \boldsymbol{u}^i|^2 \mathrm{d}\mathscr{L}^n + \int_{J_{\boldsymbol{u}}} (|\underline{\boldsymbol{u}^i}|^2 + |\overline{\boldsymbol{u}^i}|^2) \mathrm{d}\mathscr{H}^{n-1} + \int_{\mathbb{R}^n} |\boldsymbol{u}^i|^2 \mathrm{d}\mathscr{L}^n < \infty,$$

then there exists a subsequence $(\boldsymbol{u}^{\phi(i)})$ and a function $\boldsymbol{u} \in \mathcal{U}_k$ such that

$$\boldsymbol{u}^{\phi(i)} \underset{L^2_{loc}}{\longrightarrow} \boldsymbol{u},$$
$$\nabla \boldsymbol{u}^{\phi(i)} \underset{L^2_{loc}-weak}{\rightharpoonup} \nabla \boldsymbol{u},$$

Moreover for any bounded open set $A \subset \mathbb{R}^n$

$$\int_A |\nabla \boldsymbol{u}|^2 \mathrm{d}\mathscr{L}^n \leq \liminf_{i \to +\infty} \int_A |\nabla \boldsymbol{u}_\phi(i)|^2 \mathrm{d}\mathscr{L}^n,$$
$$\int_{J_u \cap A} (|\overline{\boldsymbol{u}}|^2 + |\underline{\boldsymbol{u}}|^2) \mathrm{d}\mathscr{H}^{n-1} \leq \liminf_{i \to +\infty} \int_{J_u \cap A} (|\overline{\boldsymbol{u}^{\phi(i)}}|^2 + |\underline{\boldsymbol{u}^{\phi(i)}}|^2) \mathrm{d}\mathscr{H}^{n-1}.$$

Proof. The proof is an adaptation of [117, theorem 2] to a multidimensional case. □

We define a notion of i-th eigenvalue of the Laplace operator with Robin boundary conditions that allows us to speak of the functional $\lambda_k(\cdot; \beta)$ with no pre-defined domain, and to define the k-th eigenvalue on any open set even when the trace of H^1 functions is not well-defined.

Definition 6.6. Let $\mathbf{u} \in \mathcal{U}_k$ be linearly independant. We define the two Gram matrices:

$$A(\mathbf{u}) = \left(\langle u_i, u_j\rangle_{L^2(\mathbb{R}^n,\mathscr{L}^n)}\right)_{1\le i,j\le k},$$
$$B(\mathbf{u}) = \left(\langle \nabla u_i, \nabla u_j\rangle_{L^2(\mathbb{R}^n,\mathscr{L}^n)} + \beta\langle \overline{u_i}, \overline{u_j}\rangle_{L^2(J_{\mathbf{u}},\mathscr{H}^{n-1})} + \beta\langle \underline{u_i}, \underline{u_j}\rangle_{L^2(J_{\mathbf{u}},\mathscr{H}^{n-1})}\right)_{1\le i,j\le k}.$$

We then define the i-th eigenvalue of the vector-valued function $\mathbf{u}$ as

$$\lambda_i(\mathbf{u};\beta) = \inf_{V\subset \mathrm{span}(\mathbf{u}),\dim(V)=i}\ \sup_{v\in V} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \beta\int_{J_{\mathbf{u}}} (\overline{v}^2 + \underline{v}^2)\mathrm{d}\mathscr{H}^{n-1}}{\int_{\mathbb{R}^n} v^2 \mathrm{d}\mathscr{L}^n} = \Lambda_i\left[A(\mathbf{u})^{-\frac{1}{2}} B(\mathbf{u}) A(\mathbf{u})^{-\frac{1}{2}}\right], \tag{6.2}$$

where Λ_i designates the i-th eigenvalue of a symmetric matrix.
We will say that $\mathbf{u}$ is normalized if $A(\mathbf{u}) = I_k$ and $B(\mathbf{u})$ is the diagonal $(\lambda_1(\mathbf{u};\beta),\dots,\lambda_k(\mathbf{u};\beta))$. Following the spectral theorem, for any linearly independant $\mathbf{u} \in \mathcal{U}_k$ there exists $P \in \mathrm{GL}_k(\mathbb{R})$ such that $P\mathbf{u}$ is normalized.
Although we expect the optimal sets to have rectifiable boundary, we may define the eigenvalues with Robin boundary conditions for any open set $\Omega \subset \mathbb{R}^n$ as

$$\lambda_k(\Omega;\beta) := \inf\left[\lambda_k(\mathbf{u};\beta), \mathbf{u} \in \mathcal{U}_k \text{ linearly independant} : \mathscr{H}^{n-1}(J_{\mathbf{u}} \setminus \partial\Omega) = \mathscr{L}^n(\{\mathbf{u} \neq 0\} \setminus \Omega) = 0\right]. \tag{6.3}$$

It may be checked that for any bounded Lipschitz domain, the admissible space corresponds to linearly independant functions $\mathbf{u} \in H^1(\Omega)^k$ so this definition is coherent with the usual.

6.3 Strictly monotonous functionals

Let us first restate the first main result in the SBV framework. We define the admissible set of functions as

$$\mathcal{U}_k(m) = \{\mathbf{v} \in \mathcal{U}_k : \ \mathbf{v} \text{ is linearly independant and } |\{\mathbf{v} \neq 0\}| = m\}.$$

For any linearly independant $\mathbf{u} \in \mathcal{U}_k$, we let:

$$\mathcal{F}(\mathbf{u}) := F(\lambda_1(\mathbf{u};\beta),\dots,\lambda_k(\mathbf{u};\beta)),$$
$$\mathcal{F}_\gamma(\mathbf{u}) := \mathcal{F}(\mathbf{u}) + \gamma|\{\mathbf{u} \neq 0\}|.$$

Our goal is now to show that $\mathcal{F}$ has a minimizer in $\mathcal{U}_k(m)$, and that any minimizer of $\mathcal{F}$ in $\mathcal{U}_k(m)$ is deduced from an open set, meaning there is an open set Ω that essentially contains $\{\mathbf{u} \neq 0\}$ such that $\mathbf{u}_{|\Omega} \in H^1(\Omega)^k$. This is not the case for every SBV functions: some may have a dense and non-closed jump set, while $\partial\Omega$ is closed and not dense.
The lemma 6.10 will make a link between minimizers of $\mathcal{F}$ in $\mathcal{U}_k(m)$ and minimizers of $\mathcal{F}_\gamma$ among linearly independant $\mathbf{u} \in \mathcal{U}_k$ for which the support's measure is less than m.

6.3.1 A priori estimates

An internal relaxed minimizer of $\mathcal{F}_\gamma$ is a linearly independant function $\mathbf{u} \in \mathcal{U}_k$ such that for any linearly independant $\mathbf{v} \in \mathcal{U}_k$ verifying $|\{\mathbf{v} \neq 0\} \setminus \{\mathbf{u} \neq 0\}| = 0$:

$$\mathcal{F}_\gamma(\mathbf{u}) \le \mathcal{F}_\gamma(\mathbf{v}).$$

To shorten some notations, we introduce the function $G : S_k^{++}(\mathbb{R}) \to \mathbb{R}$ such that

$$\mathcal{F}_\gamma(\mathbf{u}) = G\left[A(\mathbf{u})^{-\frac{1}{2}} B(\mathbf{u}) A(\mathbf{u})^{-\frac{1}{2}}\right] + \gamma|\{\mathbf{u} \neq 0\}|,$$

meaning that for any positive definite symmetric matrix S, $G\left[S\right] = F\left(\Lambda_1\left[S\right],\dots,\Lambda_k\left[S\right]\right)$. The smoothness of F does not imply the smoothness of G in general, because of the multiplicities of

eigenvalues. However the monotonicity of F implies the monotonicity of G in the following sense: suppose M, N are positive symmetric matrices, then:

$$G\Big[M+N\Big] \leq G\Big[M\Big] + \left(\max_{i=1,\dots,k} \sup_{\Lambda_j\Big[M\Big] \leq \lambda_j \leq \lambda_j(M+N)} \frac{\partial F}{\partial^{\pm}\lambda_i}(\lambda_1,\dots,\lambda_k) \right) \mathrm{Tr}\Big[N\Big],$$

$$G\Big[M+N\Big] \geq G\Big[M\Big] + \left(\min_{i=1,\dots,k} \inf_{\lambda_j\Big[M\Big] \leq \lambda_j \leq \lambda_j\Big[M+N\Big]} \frac{\partial F}{\partial^{\pm}\lambda_i}(\lambda_1,\dots,\lambda_k) \right) \mathrm{Tr}\Big[N\Big].$$

Above $\frac{\partial F}{\partial^{\pm}\lambda_i}$ designates the directional partial derivatives of F. Moreover, G has directional derivative everywhere; let $M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$ be a diagonal matrix with p distincts eigenvalues and $1 \leq i_1 < i_2 < i_p \leq k$ be such that for any $i \in I_l := [i_l, i_{l+1})$:

$$\lambda_{i_l} = \lambda_i < \lambda_{i_{l+1}}.$$

Then for each $i \in I_l$ the function $N \mapsto \Lambda_i\Big[N\Big]$ admits the following directional derivative at M:

$$\Lambda_i\Big[M+N\Big] = \Lambda_i\Big[M\Big] + \Lambda_{i-i_l+1}\Big[N_{|I_l}\Big] + \underset{N\to 0}{o}(N),$$

where $N_{|I} := (N_{i,j})_{i,j\in I}$. Since F has a directional derivative everywhere, this means that G admits a directional derivative

$$G\Big[M+N\Big] = G\Big[M\Big] + F_0\left(\Lambda_1\Big[N_{|I_1}\Big], \dots, \Lambda_{k-i_k+1}\Big[N_{|I_p}\Big]\right) + \underset{N\to 0}{o}(N), \tag{6.4}$$

where F_0 is a positiverly homogeneous function that is the directional derivative of F at $(\lambda_1,\dots,\lambda_k)$.

Proposition 6.7. *Let $\boldsymbol{u}$ be a relaxed internal minimizer of $\mathcal{F}_\gamma$, suppose it is normalized. Then there exists constants $M, \delta, R > 0$ that only depend on $(n, k, \beta, \mathcal{F}_\gamma(\boldsymbol{u}), F)$ such that*

$$\delta 1_{\{\boldsymbol{u}\neq 0\}} \leq |\boldsymbol{u}| \leq M.$$

Moreover, up to translation of its connected component, $\boldsymbol{u}$ is supported in a set of diameter bounded by R.

Estimates of the form $|\mathbf{u}| \geq \delta 1_{\{\mathbf{u}\neq 0\}}$ for solution of elliptic equations with Robin boundary conditions appear in [121], [118], [123], see also [114] in a context without free discontinuity. It is a crucial steps to show the regularity of the function $\mathbf{u}$; once $\mathbf{u}$ is known to take values between two positive bounds, then it may be seen as a quasi-minimizer of the Mumford-Shah functional $\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 \mathrm{d}\mathscr{L}^n + \mathscr{H}^{n-1}(J_{\mathbf{u}})$ on which the techniques used to show the regularity of Mumford-Shah minimizers (see [126]) may be extended (see [123], [121]).

Proof. We show, in order, that the eigenvalues $\lambda_i(\mathbf{u};\beta)$ are bounded above and below, the L^∞ bound on $\mathbf{u}$, the lower bound on $\mathbf{u}_{|\{\mathbf{u}\neq 0\}}$, a lower bound on the Lebesgue density of $\{\mathbf{u} \neq 0\}$, and then the boundedness of the support.

- Since $|\{\mathbf{u} \neq 0\}| \leq \mathcal{F}_\gamma(\mathbf{u})/\gamma$, then by the Faber-Krahn inequality with Robin Boundary conditions (as proved in [117]) $\lambda_1(\mathbf{u};\beta) \geq \lambda_1(\mathbb{B}^{|\mathcal{F}_\gamma(\mathbf{u})|/\gamma};\beta) =: \lambda$.
  In a similar way, since

$$F(\lambda,\dots,\lambda,\lambda_k(\mathbf{u};\beta)) \leq F(\lambda_1(\mathbf{u};\beta),\dots,\lambda_k(\mathbf{u};\beta)) \leq \mathcal{F}_\gamma(\mathbf{u})$$

and F diverges when its last coordinate does, so $\lambda_k(\mathbf{u};\beta)$ is bounded by a constant $\Lambda > 0$ that only depends on the behaviour of F and $\mathcal{F}_\gamma(\mathbf{u})$. Let us write:

$$a = \inf_{\frac{1}{2}\lambda \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le 2\Lambda} \inf_{i=1,\ldots,k} \frac{\partial F}{\partial^{\pm}\lambda_i}(\lambda_1,\ldots,\lambda_k),$$

$$b = \sup_{\frac{1}{2}\lambda \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le 2\Lambda} \sup_{i=1,\ldots,k} \frac{\partial F}{\partial^{\pm}\lambda_i}(\lambda_1,\ldots,\lambda_k).$$

a and b are positive and only depend on $\mathcal{F}_\gamma(\mathbf{u})$ and the behaviour of F.

- For the L^∞ bound we use a Moser iteration procedure (see for instance [128, Th 4.1] for a similar method). We begin by establishing that u_i is an eigenfunction of $\lambda_i(\mathbf{u};\beta)$ in a variational sense. Let $v_i \in \mathcal{U}_1$ be such that $\{v_i \neq 0\} \subset \{\mathbf{u} \neq 0\}$ and $J_{v_i} \subset J_{\mathbf{u}}$, we show that $V(u_i, v_i) = 0$, where

$$V(u_i, v_i) := \int_{\mathbb{R}^n} \nabla u_i \cdot \nabla v_i \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{u}}} u_i v_i \mathrm{d}\mathscr{H}^{n-1} - \lambda_i(\mathbf{u};\beta) \int_{\mathbb{R}^n} u_i v_i \mathrm{d}\mathscr{L}^n.$$

For this consider $\mathbf{u}_t = \mathbf{u} - t(v_i - \sum_{j\neq i} V(v_i,u_j)u_j)e_i$. Since $A(\mathbf{u}_t)$ converges to I_k, $\mathbf{u}_t$ is linearly independant for a small enough t and

$$\mathcal{F}_\gamma(\mathbf{u}) \le \mathcal{F}_\gamma(\mathbf{u}_t).$$

This implies, since $\{\mathbf{u}_t \neq 0\} \subset \{\mathbf{u} \neq 0\}$, that

$$F(\lambda_1(\mathbf{u};\beta),\ldots,\lambda_k(\mathbf{u};\beta)) \le F(\lambda_1(\mathbf{u}_t;\beta),\ldots,\lambda_k(\mathbf{u}_t;\beta)),$$

which may also be written

$$G\Big[B(\mathbf{u})\Big] \le G\Big[A(\mathbf{u}_t)^{-\frac{1}{2}} B(\mathbf{u}_t) A(\mathbf{u}_t)^{-\frac{1}{2}}\Big].$$

Now, $A(\mathbf{u}_t)^{-\frac{1}{2}} B(\mathbf{u}_t) A(\mathbf{u}_t)^{-\frac{1}{2}} = B(\mathbf{u}) - (e_i e_i^*) V(u_i,v_i)t + \mathcal{O}(t^2)$. Suppose that $V(u_i,v_i) > 0$. Let i' be the lowest index such that $\lambda_{i'}(\mathbf{u};\beta) = \lambda_i(\mathbf{u};\beta)$. Then knowing the directional derivative of G given in (6.4) we obtain (for $t > 0$)

$$G\Big[A(\mathbf{u}_t)^{-\frac{1}{2}} B(\mathbf{u}_t) A(\mathbf{u}_t)^{-\frac{1}{2}}\Big] = G\Big[B(\mathbf{u})\Big] + tV(u_i,v_i) F_0\left(0,0,\ldots,0,-1,0,\ldots,0\right) + \underset{t\to 0}{o}(t),$$

which is less than $G\Big[B(\mathbf{u})\Big]$ for a small enough t: this is a contradiction. When $V(u_i,v_i) \le 0$ we may do the same by replacing v_i with $-v_i$. Thus for all v_i with support and jump set included in the support and jump set of $\mathbf{u}$

$$\int_{\mathbb{R}^n} \nabla u_i \cdot \nabla v_i \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{u}}} u_i v_i \mathrm{d}\mathscr{H}^{n-1} = \lambda_i(\mathbf{u};\beta) \int_{\mathbb{R}^n} u_i v_i \mathrm{d}\mathscr{L}^n.$$

Now we use Moser iteration methods. Let $\alpha \ge 2$ be such that $u_i \in L^\alpha$, then by taking v_i to be a truncation of $|u_i|^{\alpha-2}u_i$ in $[-M,M]$ for $M \to \infty$ in the variational equation above, we obtain

$$\int_{\mathbb{R}^n} (\alpha-1)|u_i|^{\alpha-2}|\nabla u_i|^2 \mathrm{d}\mathscr{L}^n + \int_{J_{\mathbf{u}}} (|\overline{u_i}|^\alpha + |\underline{u_i}|^\alpha)\mathrm{d}\mathscr{H}^{n-1} = \lambda_i(\mathbf{u};\beta) \int_{\mathbb{R}^n} |u_i|^\alpha \mathrm{d}\mathscr{L}^n.$$

Using the embedding $BV(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ we have:

$$\begin{aligned}
\|u_i^\alpha\|_{L^{\frac{n}{n-1}}} &\le C_n \|u_i^\alpha\|_{BV} \\
&\le C_n \left(\int_{\mathbb{R}^n} |\nabla(|u_i|^{\alpha-1}u_i)| \mathrm{d}\mathscr{L}^n + \int_{J_{\mathbf{u}}} (|\overline{u_i}|^\alpha + |\underline{u_i}|^\alpha)\mathrm{d}\mathscr{H}^{n-1} \right) \\
&\le C_n \left(\int_{\mathbb{R}^n} \alpha \left(|u_i|^\alpha + |u_i|^{\alpha-2}|\nabla u_i|^2 \right) \mathrm{d}\mathscr{L}^n + \int_{J_{\mathbf{u}}} (|\overline{u_i}|^\alpha + |\underline{u_i}|^\alpha)\mathrm{d}\mathscr{H}^{n-1} \right) \\
&\le C_{n,\beta} \left(\alpha + \lambda_i(\mathbf{u};\beta) \right) \|u_i\|_{L^\alpha}^\alpha.
\end{aligned}$$

And so $\|u_i\|_{L^{\frac{n}{n-1}\alpha}} \leq \left[C_{n,\beta}\left(\alpha+\lambda_i(\mathbf{u};\beta)\right)\right]^{\frac{1}{\alpha}}\|u_i\|_{L^\alpha}$. We may apply this iteratively with $\alpha_p = 2\left(\frac{n}{n-1}\right)^p$ to obtain an L^∞ bound of u_i that only depends on n,β and $\lambda_i(\mathbf{u};\beta)$. In fact using the Faber-Krahn inequality for Robin conditions $\lambda_i(\mathbf{u};\beta) \geq \lambda_1\left(\mathbb{B}^{|\{\mathbf{u}\neq 0\}|};\beta\right)$ the previous inequality applied to α_p may be simplified into

$$\log\left(\frac{\|u_i\|_{L^{\alpha_{p+1}}}}{\|u_i\|_{L^{\alpha_p}}}\right) \leq \left(C(n,\beta,|\{\mathbf{u}\neq 0\}|)(p+1)+\frac{1}{2}\log\lambda_i(\mathbf{u};\beta)\right)\left(\frac{n-1}{n}\right)^p,$$

and summing in p we obtain an estimate of the form $\|u_i\|_{L^\infty} \leq C(n,\beta,|\{\mathbf{u}\neq 0\}|)\lambda_i(\mathbf{u};\beta)^{\frac{n}{2}}$.

- Lower bound on $\mathbf{u}$: our goal is first to obtain an estimate of the form

$$\mathrm{Tr}\Big[B_t\Big] + |\{0<|\mathbf{u}|\leq t\}| \leq \frac{1}{\epsilon}\mathrm{Tr}\Big[\beta_t\Big], \tag{6.5}$$

where $\epsilon>0$ is a constant that only depends on the parameters and

$$(B_t)_{i,j} = \int_{|\mathbf{u}|\leq t}\nabla u_i\cdot\nabla u_j \mathrm{d}\mathscr{L}^n + \beta\int_{J_{\mathbf{u}}}\left(\overline{u_i 1_{|\mathbf{u}|\leq t}}\cdot\overline{u_j 1_{|\mathbf{u}|\leq t}} + \underline{u_i 1_{|\mathbf{u}|\leq t}}\cdot\underline{u_j 1_{|\mathbf{u}|\leq t}}\right)\mathrm{d}\mathscr{H}^{n-1},$$
$$(\beta_t)_{i,j} = \beta\int_{\partial^*\{|\mathbf{u}|>t\}\setminus J_{\mathbf{u}}} u_i u_j \mathrm{d}\mathscr{H}^{n-1}.$$

This is intuitively what we obtain by comparing $\mathbf{u}$ and $\mathbf{u}1_{\{|\mathbf{u}|>t\}}$. From this we will derive a lower bound of $\inf_{\mathbf{u}\neq 0}|\mathbf{u}|$ with similar arguments as what was done in [123]. Suppose (6.5) does not hold. This means that, since $B_t \leq B$ and $|\{\mathbf{u}\neq 0\}| \leq \mathcal{F}_\gamma(\mathbf{u})$,

$$\beta_t \leq \left(B(\mathbf{u})+\mathcal{F}_\gamma(\mathbf{u})I_k\right)k\epsilon \leq c\epsilon B(\mathbf{u}),$$

for a certain $c>0$ since $B(\mathbf{u}) \geq \lambda I_k$. Let us now compare $\mathbf{u}$ with $\mathbf{u}_t = \mathbf{u}1_{\{|\mathbf{u}|>t\}}$; this function is admissible for a small enough t because $A(\mathbf{u}_t) = I_k - A\left(\mathbf{u}1_{\{0<|\mathbf{u}|\leq t\}}\right)$, so

$$\|A(\mathbf{u}_t)-I_k\| \leq Ct^2|\{0<|\mathbf{u}|\leq t\}|.$$

Notice also that $B(\mathbf{u}_t) = B(\mathbf{u}) - B_t + \beta_t$. Then the optimality condition $\mathcal{F}_\gamma(\mathbf{u}) \leq \mathcal{F}_\gamma(\mathbf{u}_t)$ gives

$$G\Big[B(\mathbf{u})\Big] + \gamma|\{0<|\mathbf{u}|\leq t\}| \leq G\Big[A(\mathbf{u}_t)^{-\frac{1}{2}}(B(\mathbf{u})-B_t+\beta_t)A(\mathbf{u}_t)^{-\frac{1}{2}}\Big]. \tag{6.6}$$

We first show that B_t is small enough for small t. With our hypothesis on β_t and the fact that $A(\mathbf{u}_t)^{-\frac{1}{2}} \leq I_k + Ct^2 I_k \leq (1+c\epsilon)I_k$ for a small enough t

$$G\Big[B(\mathbf{u})\Big] + \gamma|\{0<|\mathbf{u}|\leq t\}| \leq G\Big[\Big[1+2c\epsilon\Big]B(\mathbf{u})-B_t\Big].$$

So $G\Big[B(\mathbf{u})\Big] \leq G\Big[(1+2c\epsilon)B(\mathbf{u})-B_t\Big]$. Now, there exists $i\in\{1,\dots,k\}$ such that

$$\Lambda_i\Big[(1+c\epsilon)B(\mathbf{u})-B_t\Big] \leq (1+c\epsilon)\Lambda_i\Big[B(\mathbf{u})\Big] - \frac{1}{k}\mathrm{Tr}\Big[B_t\Big].$$

And so, using the monotonicity of F and the definition of a,b in the first part of the proof:

$$\begin{aligned}
G\Big[B(\mathbf{u})\Big] &\leq G\Big[(1+c\epsilon)B(\mathbf{u})-B_t\Big] \\
&\leq F\left((1+c\epsilon)\lambda_1(\mathbf{u};\beta),\dots,(1+c\epsilon)\lambda_{i-1}(\mathbf{u};\beta),(1+c\epsilon)\lambda_i(\mathbf{u};\beta)-\frac{1}{k}\mathrm{Tr}\Big[B_t\Big],\dots,(1+c\epsilon)\lambda_k(\mathbf{u};\beta)\right) \\
&\leq G\Big[B(\mathbf{u})\Big] + bc\epsilon\mathrm{Tr}\Big[B(\mathbf{u})\Big] - a\min\left(\frac{1}{k}\mathrm{Tr}\Big[B_t\Big],\frac{\lambda}{2}\right).
\end{aligned}$$

With a small enough ϵ, we obtain $\mathrm{Tr}\Big[B_t\Big] \leq \frac{\lambda}{2}$. Now we may come back to (6.6), and using the fact that $A(\mathbf{u}_t)^{-\frac{1}{2}} \leq (1 + Ct^2|\{0 < |\mathbf{u}| \leq t\}|)I_k$ we obtain

$$G\Big[B(\mathbf{u})\Big] + \gamma|\{0 < |\mathbf{u}| \leq t\}| \leq G\Big[B(\mathbf{u}) + \beta_t - B_t + Ct^2|\{0 < |\mathbf{u}| \leq t\}|I_k\Big],$$

and so with the monotonicity of G

$$G\Big[B(\mathbf{u})\Big] + \gamma|\{0 < |\mathbf{u}| \leq t\}| \leq G\Big[B(\mathbf{u})\Big] + b\mathrm{Tr}\Big[\beta_t\Big] - a\mathrm{Tr}\Big[B_t\Big] + Cbt^2|\{0 < |\mathbf{u}| \leq t\}|.$$

In particular, for a small enough $t > 0$ (depending only on the parameters)

$$a\mathrm{Tr}\Big[B_t\Big] + \frac{\gamma}{2}|\{0 < |\mathbf{u}| \leq t\}| \leq b\mathrm{Tr}\Big[\beta_t\Big],$$

and so we obtained that there is a big enough constant $C > 0$, and a small enough $t_1 > 0$, such that for any $t \in (0, t_1]$

$$\mathrm{Tr}\Big[B_t\Big] + |\{0 < |\mathbf{u}| \leq t\}| \leq C\mathrm{Tr}\Big[\beta_t\Big]. \tag{6.7}$$

Now we let

$$V := |\mathbf{u}| = \sqrt{u_1^2 + \ldots + u_k^2}\left(\geq \delta 1_{\{\mathbf{u}\neq 0\}}\right).$$

Let $f(t) = \int_0^t \tau\mathscr{H}^{n-1}(\partial^*\{V > t\} \setminus J_{\mathbf{u}})\mathrm{d}\tau$. Notice that the right-hand side of (6.7) is $Ctf'(t)$. Then for any $t \leq t_1$:

$$\begin{aligned}
f(t) &\leq \int_0^t \tau\mathscr{H}^{n-1}(\partial^*\{V > \tau\} \setminus J_V)d\tau = \int_{\omega_t} V|\nabla V|\mathrm{d}\mathscr{L}^n \\
&\leq |\{0 < V \leq t\}|^{\frac{1}{2n}} \left(\int_{\{0<V\leq t\}} |\nabla V|^2\mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} \left(\int_{\{0<V\leq t\}} (V^2)^{\frac{n}{n-1}}\mathrm{d}\mathscr{L}^n\right)^{\frac{n-1}{2n}} \\
&\leq C(tf'(t))^{\frac{1}{2n}+\frac{1}{2}} \left(|D(V^2)|(\{0 < V \leq t\})\right)^{\frac{1}{2}} \\
&\leq C(tf'(t))^{\frac{1}{2n}+\frac{1}{2}} \left(\int_{\{0<V\leq t\}} V|\nabla V|\mathrm{d}\mathscr{L}^n + \int_{J_V\cap\{0<V\leq t\}} V^2\mathrm{d}\mathscr{H}^{n-1}\right)^{\frac{1}{2}} \\
&\leq C(tf'(t))^{\frac{1}{2n}+\frac{1}{2}} \left(t|\{0 < V \leq t\}|^{\frac{1}{2}} \left(\int_{\{0<V\leq t\}} |\nabla V|^2\mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} + \int_{J_V\cap\{0<V\leq t\}} V^2\mathrm{d}\mathscr{H}^{n-1}\right)^{\frac{1}{2}} \\
&\leq C(tf'(t))^{1+\frac{1}{2n}}.
\end{aligned}$$

The constant $C > 0$ above depends only on the parameters and may change from line to line. This implies that $f'(t)f(t)^{-\frac{2n}{2n+1}} \geq ct^{-1}$, so for any $t \in]0, t_1[$ such that $f(t) > 0$ this may be integrated from t to t_1 to obtain

$$\frac{1}{2n+1} f(t_1)^{\frac{1}{2n+1}} \geq c\log(t_1/t).$$

Since $f(t_1) \leq |\{\mathbf{u} \neq 0\}|^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla V|^2\right)^{\frac{1}{2}} \leq \sqrt{k\mathcal{F}_\gamma(\mathbf{u})\Lambda/\gamma}$, then t is bounded below in terms of the parameters of the problem. This means that $f(\delta) = 0$ for a certain explicit $\delta > 0$. In particular, $(\nabla\mathbf{u})1_{\{|\mathbf{u}|\leq\delta\}} = 0$, and by comparing $\mathbf{u}$ with $\mathbf{u}_\delta$, (6.5) becomes $|\{0 < |\mathbf{u}| \leq \delta\}| \leq 0$, so we obtained

$$|\mathbf{u}| \geq \delta 1_{\{\mathbf{u}\neq 0\}}.$$

- To show the support $\{\mathbf{u} \neq 0\}$ (or its connected components) is bounded, we begin by showing a lower estimate for the Lebesgue density on this set. This is obtained by comparing $\mathbf{u}$ with $\mathbf{u}_r := u1_{\mathbb{R}^n\setminus\mathbb{B}_r}$ where $\mathbb{B}_r$ is a ball of radius $r > 0$.

As previously, we first need to check that $\mathbf{u}_r$ is admissible for any small enough $r > 0$. With the L^∞ bound on $\mathbf{u}$, we get $|A(\mathbf{u}_r) - I_k| \leq C|\{\mathbf{u} \neq 0\} \cap \mathbb{B}_r|$. In particular, $|A(\mathbf{u}_r) - I_k| \leq Cr^n$, which proves that $A(\mathbf{u}_r)$ is invertible for a small enough r.

Let $f(r) = |\{\mathbf{u} \neq 0\} \cap \mathbb{B}_r|$. By comparing $\mathbf{u}$ with $\mathbf{u}_r$ we obtain

$$G\Big[B(\mathbf{u})\Big] + \gamma|\{\mathbf{u} \neq 0\} \cap \mathbb{B}_r| \leq G\Big[A_r(B - B_r + \beta_r)A_r\Big],$$

where A_r, B_r, β_r are defined as previously: $A_r = A(\mathbf{u}_r)^{-\frac{1}{2}}$ and

$$(B_r)_{i,j} = \int_{\mathbb{B}_r} \nabla u_i \cdot \nabla u_j \mathrm{d}\mathscr{L}^n + \beta \int_{J_\mathbf{u}} \left(\overline{u_i 1_{\mathbb{B}_r}} \cdot \overline{u_j 1_{\mathbb{B}_r}} + \underline{u_i 1_{\mathbb{B}_r}} \cdot \underline{u_j 1_{\mathbb{B}_r}}\right) \mathrm{d}\mathscr{H}^{n-1},$$
$$(\beta_r)_{i,j} = \beta \int_{\partial\mathbb{B}_r \setminus J_\mathbf{u}} u_i u_j \mathrm{d}\mathscr{H}^{n-1}.$$

With the same argument as what we did to obtain the lower bound, this estimate implies that for any $r \in]0, r_0]$ where r_0 is small enough

$$c\mathrm{Tr}\Big[B_r\Big] \leq \mathrm{Tr}\Big[\beta_r\Big] + f(r),$$

for a certain $c > 0$. With the L^∞ bound and the lower bound on $\mathbf{u}$, we deduce that for a certain constant $C > 0$:

$$\mathscr{H}^{n-1}(\mathbb{B}_r \cap J_\mathbf{u}) \leq C\left(f(r) + \mathscr{H}^{n-1}(\partial\mathbb{B}_r \cap \{\mathbf{u} \neq 0\})\right).$$

Notice that $f'(r) = \mathscr{H}^{n-1}(\partial\mathbb{B}_r \cap \{\mathbf{u} \neq 0\})$, so with the isoperimetric inequality

$$c_n f(r)^{1-\frac{1}{n}} \leq \mathscr{H}^{n-1}(\mathbb{B}_r \cap J_\mathbf{u}) + \mathscr{H}^{n-1}(\partial\mathbb{B}_r \cap \{\mathbf{u} \neq 0\}) \leq C(f(r) + f'(r)).$$

Since $f(r) \leq Cr^n \to 0$, we deduce that for a certain constant $C > 0$ and any small enough r ($r < r_0$) we have

$$f(r)^{1-\frac{1}{n}} \leq Cf'(r).$$

Suppose now that $f(r) > 0$ for any $r > 0$. Then by integrating the above estimate from 0 to r_0, we obtain that for a certain constant $c > 0$ and any $r \in [0, r_0]$

$$|\{\mathbf{u} \neq 0\} \cap \mathbb{B}_{x,r}| \geq cr^n.$$

Consider now a system of points $S \subset \mathbb{R}^n$ such that for any $x \in S$ and any $r > 0$, $|\{\mathbf{u} \neq 0\} \cap \mathbb{B}_{x,r}| > 0$, and such that for any distinct $x, y \in S$, $|x - y| \geq 2r_0$. Then

$$\mathcal{F}_\gamma(\mathbf{u}) \geq \gamma|\{\mathbf{u} \neq 0\}| \geq \gamma \sum_{x \in S} |\{\mathbf{u} \neq 0\} \cap \mathbb{B}_{x,r_0}| \geq c\gamma r_0^n \mathrm{Card}(S),$$

so $\mathrm{Card}(S)$ is bounded. Then by taking a maximal set of separated points S as above, the balls $(\mathbb{B}_{x,2r_0})_{x \in S}$ cover $\{\mathbf{u} \neq 0\}$. This means in particular that the support of u is bounded by a constant only depending on the parameters, up to a translation of the its connected components.

□

6.3.2 Existence of a relaxed minimizer with prescribed measure

This section is dedicated to the proof of the following result.

Proposition 6.8. *Let $m, \beta > 0$, then there exists $\boldsymbol{u} \in \mathcal{U}_k$ that minimizes $\mathcal{F}$ in the admissible set $\mathcal{U}_k(m)$.*

We begin with a lemma that will help us to show that any minimizing sequence of $\mathcal{F}$ in $\mathcal{U}_k(m)$ has concentration points, meaning points around which the measure of the support is bounded below by a positive constant.

Lemma 6.9. *Let $\boldsymbol{u} \in \mathcal{U}_k$, we let $K_p := p + [-\frac{1}{2}, \frac{1}{2}]^n$, then there exists $p \in \mathbb{Z}^n$ such that*

$$|\{\boldsymbol{u} \neq 0\} \cap K_p| \geq \left(\frac{c_n \|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)}}{\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} |\nabla \boldsymbol{u}|^2 \mathrm{d}\mathscr{L}^n + \int_{J_{\boldsymbol{u}}} (|\overline{\boldsymbol{u}}|^2 + |\underline{\boldsymbol{u}}|^2)\, \mathrm{d}\mathscr{H}^{n-1}} \right)^n .$$

Proof. It is the consequence of the $BV(K_p) \hookrightarrow L^{\frac{n}{n-1}}(K_p)$ embedding, see [120, lemma 12]. □

The following lemma makes a straightforward link between minimizers of $\mathcal{F}$ with fixed volume and interior minimizers of $\mathcal{F}_\gamma$ for a sufficiently small γ, which means that all the a priori estimates apply.

Lemma 6.10. *Let $\boldsymbol{u} \in \mathcal{U}_k$ be a minimizer of $\mathcal{F}$ in the admissible set*

$$\{\boldsymbol{v} \in \mathcal{U}_k : \ \boldsymbol{v} \text{ is linearly independant and } |\{\boldsymbol{v} \neq 0\}| = m\}.$$

Then there exists $\gamma > 0$ depending only on $(n, m, \beta, \mathcal{F}(u), F)$ such that $\boldsymbol{u}$ is a minimizer of $\mathcal{F}_\gamma$ in the admissible set

$$\{\boldsymbol{v} \in \mathcal{U}_k : \ \boldsymbol{v} \text{ is linearly independant and } |\{\boldsymbol{v} \neq 0\}| \in]0, m]\}.$$

Proof. Consider a linearly independant $\mathbf{v} \in \mathcal{U}_k$ such that $\delta := \frac{|\{\mathbf{v}\neq 0\}|}{|\{\mathbf{u}\neq 0\}|} \in]0, 1]$. Let $\mathbf{w}(x) := \mathbf{v}(x\delta^{1/n})$. Then the support of $\mathbf{w}$ has the same measure as $\mathbf{u}$ and so $\mathcal{F}(\mathbf{u}) \leq \mathcal{F}(\mathbf{w})$. Looking how the matrices A and B scale with the change of variable $x \to x\delta^{-\frac{1}{n}}$ we obtain

$$A(\mathbf{w})^{-\frac{1}{2}} B(\mathbf{w}) A(\mathbf{w})^{-\frac{1}{2}} \leq \delta^{\frac{1}{n}} A(\mathbf{v})^{-\frac{1}{2}} B(\mathbf{v}) A(\mathbf{v})^{-\frac{1}{2}},$$

hence

$$\mathcal{F}(\mathbf{u}) \leq F\left(\delta^{\frac{1}{n}} \lambda_1(\mathbf{v}; \beta), \ldots, \delta^{\frac{1}{n}} \lambda_k(\mathbf{v}; \beta)\right).$$

By the Faber-Krahn inequality for Robin eigenvalues, $\lambda_1(\mathbf{v}; \beta) \geq \lambda_1(\mathbb{B}^{|\{\mathbf{v}\neq 0\}|}; \beta)$. Moreover since F diverges when its last coordinate does, we may suppose without loss of generality that $\lambda_k(\mathbf{v}; \beta)$ is bounded by a certain constant $\Lambda > 0$ that does no depend on $\mathbf{v}$. This in turn means that $|\{\mathbf{v} \neq 0\}|$ is bounded below by a positive constant depending only on n, β, Λ by the Faber-Krahn inequality, so δ is bounded below. Then by denoting a the minimum of the partial derivatives of F on $[\delta^{\frac{1}{n}} \lambda_1(\mathbb{B}^m; \beta), \Lambda]^k$, we obtain

$$\mathcal{F}(\mathbf{u}) \leq \mathcal{F}(\mathbf{v}) - a(\lambda_1(\mathbf{v}; \beta) + \ldots + \lambda_k(\mathbf{v}; \beta))(1 - \delta^{1/n}) \leq \mathcal{F}(\mathbf{v}) - \frac{ka\lambda_1(\mathbb{B}^m; \beta)}{nm}(|\{\mathbf{u} \neq 0\}| - |\{\mathbf{v} \neq 0\}|).$$

This concludes the proof. □

We may now prove the main result of this section.

Proof. We proceed by induction on k. The main idea is that we either obtain the existence of a minimizer by taking the limit of a minimizing sequence, or we don't and in this case the minimizer is disconnected so it is the union of two minimizers of different functionals depending on strictly less than k eigenvalues.
The initialisation for $k = 1$ amounts to showing there is a minimizer for $\lambda_1(\mathbf{u}; \beta)$ in $\mathcal{U}_1(m)$: this has

been done in [118] and it is known to be the first eigenfunction of a ball of measure m.
Suppose now that $k \geq 2$ and the result is true up to $k-1$. Consider $(\mathbf{u}^i)_i$ a minimizing sequence for $\mathcal{F}$ in $\mathcal{U}_k(m)$. Then the concentration lemma 6.9 may be applied to each $\mathbf{u}^i$ to find a sequence $(p^i)_i$ in $\mathbb{Z}^n$ such that

$$\liminf_{i\to\infty} |K_{p_i} \cap \{\mathbf{u}^i \neq 0\}| > 0. \tag{6.8}$$

We lose no generality in supposing, up to a translation of each $\mathbf{u}^i$, that $p^i = 0$. Now with the compactness lemma 6.5, we now up to extraction that $\mathbf{u}^i$ converges in L^2_{loc} to a certain function $\mathbf{u} \in \mathcal{U}_k$ with local lower semicontinuity of its Dirichlet-Robin energy.
We now split $\mathbf{u}^i$ into a "local" part and a "distant" part; we may find an increasing sequence $R^i \to \infty$ such that

$$\mathbf{u}^i 1_{\mathbb{B}_{p^i,R^i}} \underset{L^2(\mathbb{R}^n)}{\longrightarrow} \mathbf{u}.$$

Up to changing each R^i with a certain $\tilde{R}^i \in [\frac{1}{2}R^i, R^i]$, we may suppose that

$$\int_{\partial\mathbb{B}_{R^i}\setminus J_{\mathbf{u}^i}} |\mathbf{u}^i|^2 \mathrm{d}\mathscr{H}^{n-1} = \underset{i\to\infty}{o}(1),$$

so that for each $i \in \{1,\dots,k\}$

$$\lambda_i\left((\mathbf{u}1_{\mathbb{B}_{R^i}}, \mathbf{u}1_{\mathbb{B}^c_{R^i}}); \beta\right) \leq \lambda_i(\mathbf{u};\beta) + \underset{i\to\infty}{o}(1).$$

Since $A\left(\mathbf{u}1_{\mathbb{B}_{R^i}}, \mathbf{u}1_{\mathbb{B}^c_{R^i}}\right)$ and $B\left(\mathbf{u}1_{\mathbb{B}_{R^i}}, \mathbf{u}1_{\mathbb{B}^c_{R^i}}\right)$ are block diagonal (with two blocks of size $k\times k$), then up to extraction on i there is a certain $p \in \{0,1,\dots,k\}$ such that

$$\left[\lambda_1(\mathbf{u}^i 1_{\mathbb{B}_{R^i}};\beta),\dots,\lambda_p(\mathbf{u}^i 1_{\mathbb{B}_{R^i}};\beta), \lambda_1(\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}};\beta),\dots,\lambda_p(\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}};\beta)\right]^{\mathfrak{S}_k} \leq (\lambda_1(\mathbf{u}^i;\beta),\dots,\lambda_k(\mathbf{u}^i;\beta)) + \underset{i\to\infty}{o}(1),$$

where $\left[a_1,\dots,a_k\right]^{\mathfrak{S}_k}$ designate the ordered list of the values $(a_1,\dots,a_k)$. There are now three cases:

- $p = 0$: we claim this can not occur. Indeed this would mean that $\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}$ is such that

  $$\mathcal{F}(\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}) \underset{i\to\infty}{\longrightarrow} \inf_{\mathcal{U}_k(m)} \mathcal{F}.$$

  However, because of (6.8) we know there is a certain $\delta > 0$ such that for all big enough i the measure of the support of $\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}$ is less than $m-\delta$. Letting $\mathbf{v}^i = \mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}\left(\left[\frac{m-\delta}{m}\right]^{\frac{1}{n}}\cdot\right)$, $\mathbf{v}^i$ is a linearly independant sequence of $\mathcal{U}_k$, with support of volume less than m, such that $\mathcal{F}(\mathbf{v}^i) < \inf_{\mathcal{U}_k(m)} \mathcal{F}$ for a big enough i: this is a contradiction.
- $p = k$. In this case $\mathbf{u}(= \lim_i \mathbf{u}^i 1_{\mathbb{B}_{R^i}})$ is a minimizer of $\mathcal{F}$ with measure less than m. This is because, in addition to the fact that $\mathbf{u}^i 1_{\mathbb{B}_{R^i}}$ converges to $\mathbf{u}$ in L^2, the lower semi-continuity result tells us that for each $z \in \mathbb{R}^k$:

  $$z^* B(\mathbf{u}) z \leq \liminf_i z^* B(\mathbf{u}^i 1_{\mathbb{B}_{R^i}}) z,$$

  thus for any $j = 1,\dots,k$, $\lambda_j(\mathbf{u};\beta) \leq \liminf_i \lambda_j(\mathbf{u}^i 1_{\mathbb{B}_{R^i}};\beta)$. And $|\{\mathbf{u}\neq 0\}| \leq \liminf |\{\mathbf{u}^i 1_{B_{R^i}} \neq 0\}| \leq m$.
- $1 \leq p \leq k-1$. This is where we will use the induction hypothesis. We let:

  $$\lambda_j = \lim_{i\to\infty} \lambda_j(\mathbf{u}^i 1_{\mathbb{B}_{R^i}};\beta),\ \forall j = 1,\dots,p \qquad m_{\text{loc}} = \lim_{i\to\infty} |\{\mathbf{u}^i 1_{B_{R^i}} \neq 0\}|,$$
  $$\mu_j = \lim_{i\to\infty} \lambda_j(\mathbf{u}^i 1^c_{\mathbb{B}_{R^i}};\beta),\ \forall j = 1,\dots,k-p \qquad m_{\text{dist}} = \lim_{i\to\infty} |\{\mathbf{u}^i 1_{B^c_{R^i}} \neq 0\}|.$$

Then by continuity of F

$$\inf_{\mathcal{U}_p(m)} \mathcal{F} = F\left(\left[\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_{k-p}\right]^{\mathfrak{S}_k}\right).$$

Let us introduce

$$\mathcal{F}_{\text{loc}} : \mathbf{v} \in \mathcal{U}_p(m_{\text{loc}}) \mapsto F\left(\left[\lambda_1(\mathbf{v};\beta), \dots, \lambda_p(\mathbf{v};\beta), \mu_1, \dots, \mu_{k-p}\right]^{\mathfrak{S}_k}\right).$$

This functional verify the hypothesis (6.1), so following the induction hypothesis we know it has a minimizer $\mathbf{v}$. Moreover, according to the a priori bounds, $\mathbf{v}$ is known to have bounded support. Since $|\{\mathbf{u}^i 1_{\mathbb{B}_{R^i} \neq 0}\}| \underset{i\to\infty}{\to} m_{\text{loc}}$, then by the optimality of $\mathbf{v}$ we get

$$\mathcal{F}_{\text{loc}}(\mathbf{v}) \leq \liminf_{i\to\infty} \mathcal{F}_{\text{loc}}(\mathbf{u}^i 1_{\mathbb{B}_{R^i}}) = F\left(\left[\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_{k-p}\right]^{\mathfrak{S}_k}\right) = \inf_{\mathcal{U}_k(m)} \mathcal{F}.$$

Now consider the functional

$$\mathcal{F}_{\text{dist}} : \mathbf{w} \in \mathcal{U}_{k-p}(m_{\text{dist}}) \mapsto F\left(\left[\lambda_1(\mathbf{v};\beta), \dots, \lambda_p(\mathbf{v};\beta), \lambda_1(\mathbf{w};\beta), \dots, \lambda_{k-p}(\mathbf{w};\beta)\right]^{\mathfrak{S}_k}\right).$$

With the same arguments, there is a minimizer $\mathbf{w}$ with bounded support. By comparing $\mathbf{w}$ with $\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}$ we obtain

$$\mathcal{F}_{\text{dist}}(\mathbf{w}) \leq \liminf_{i\to\infty} \mathcal{F}_{\text{dist}}(\mathbf{u}^i 1_{\mathbb{B}^c_{R^i}}) = \mathcal{F}_{\text{loc}}(\mathbf{v}) \left(\leq \inf_{\mathcal{U}_k(m)} \mathcal{F}\right).$$

Since both $\mathbf{v}$ and $\mathbf{w}$ have bounded support we may suppose up to translation that their support are a positive distance from each other. Consider $\mathbf{u} = \mathbf{v} \oplus \mathbf{w}$, then $\mathcal{F}(\mathbf{u}) = \mathcal{F}_{\text{dist}}(\mathbf{w})$ so $\mathbf{u}$ is a minimizer of $\mathcal{F}$ in $\mathcal{U}_k(m)$.

□

6.3.3 Regularity of minimizers

Here we show that the relaxed global minimizer $\mathbf{u}$ that we found in the previous section corresponds to the eigenfunctions of an open set. What this means is that there is an open set Ω that contains almost all the support of $\mathbf{u}$ such that $\mathbf{u}_{|\Omega} \in H^1(\Omega)^k$ and $\lambda_1(\Omega;\beta), \dots, \lambda_k(\Omega;\beta)$ as defined in (6.3) are reached for $u_{1|\Omega}, \dots, u_{k|\Omega}$ respectively (provided $\mathbf{u}$ is normalized). Moreover we show that this open set $\partial\Omega$ is Ahlfors regular and $\mathscr{H}^{n-1}(\partial\Omega) < \infty$.

The main step is to show that $J_{\mathbf{u}}$ is essentially closed, meaning $\mathscr{H}^{n-1}\left(\overline{J_{\mathbf{u}}} \setminus J_{\mathbf{u}}\right) = 0$. This is obvious for functions $\mathbf{u}$ that are eigenfunctions of a smooth open set Ω, since $J_{\mathbf{u}} = \partial\Omega$, however an SBV function could have a dense jump set.

This is dealt using similar methods as in [126], [123]; we show that for every point $x \in \mathbb{R}^n$ with sufficiently low $(n-1)$ dimensional density in $J_{\mathbf{u}}$, the energy of $\mathbf{u}$ decreases rapidly around that point (this is lemma 6.12). This is obtained by contradiction and blow-up methods, by considering a rescaling of a sequence of function that do not verify this estimate. As a consequence we obtain uniform lower bound on the $(n-1)$ dimensional density of $J_{\mathbf{u}}$, which implies that it is essentially closed. We point out that in similar problems (see [121]), the essential closedness of the jump set is obtained using the monotonicity of $\frac{1}{r^{n-1}}\left(\int_{\mathbb{B}_r} |\nabla u|^2 + \mathscr{H}^{n-1}(J_u \cap \mathbb{B}_r)\right) \wedge c + c' r^\alpha$ for some constants $c, c', \alpha > 0$ (where u is a scalar solution of some similar free discontinuity problem). However our optimality condition (see (6.11) below) does not seem to be enough to establish a similar monotonicity property, namely due to the remainder on the right-hand side and the multiplicities of eigenvalues.

Proposition 6.11. *Let $\boldsymbol{u}$ be a relaxed minimizer $\mathcal{F}_\gamma$. Then $\mathscr{H}^{n-1}\left(\overline{J_{\boldsymbol{u}}} \setminus J_{\boldsymbol{u}}\right) = 0$ and $\Omega := \left\{\overline{|\boldsymbol{u}|} > 0\right\} \setminus \overline{J_{\boldsymbol{u}}}$ is an open set such that $(u_1, \dots, u_k)$ are the first k eigenfunctions of the Laplacian with Robin boundary conditions on Ω.*

Since the proof is very similar to what was done in [123], we only sketch the specific parts of the proof that concern the vectorial character of our problem.

Proof. We first establish an optimality conditions for perturbations of $\mathbf{u}$ on balls with small diameter. We suppose $\mathbf{u}$ is normalized and, using the same notations as in (6.4) for $M = B(\mathbf{u})$ we denote

$$G_0\Big[N\Big] = F_0\left(\Lambda_1\Big[N_{|I_1}\Big], \dots, \Lambda_{k-i_k+1}\Big[N_{|I_p}\Big]\right), \tag{6.9}$$

such that:

$$G\Big[B(\mathbf{u}) + N\Big] = G\Big[B(\mathbf{u})\Big] + G_0\Big[N\Big] + \underset{N\to 0}{o}(N). \tag{6.10}$$

While G_0 is not linear (except in the particular case where $\frac{\partial F}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_j}$ for each i, j such that $\lambda_i(\mathbf{u}; \beta) = \lambda_j(\mathbf{u}; \beta)$), it is positively homogeneous. We let

$$E_0\Big[N\Big] = \max\left(G_0\Big[N\Big], \mathrm{Tr}\Big[N\Big]\right).$$

E_0 is also positively homogeneous and verify that for any non-zero $S \in S_k^+(\mathbb{R})$, $E_0\Big[-S\Big] < 0$. We show that:

For any $\boldsymbol{v} \in \mathcal{U}_k$ that differs from $\boldsymbol{u}$ on a ball $\mathbb{B}_{x,r}$ where r is small enough, we have

$$E_0\Big[B(\mathbf{v}; \mathbb{B}_{x,r}) - B(\mathbf{u}; \mathbb{B}_{x,r})\Big] \geq -\Lambda r^n - \delta(r)|B(\mathbf{u}; \mathbb{B}_{x,r})|. \tag{6.11}$$

Where $\Lambda > 0$, $\delta(r) \underset{r\to 0}{\to} 0$, and

$$B(\mathbf{w}; \mathbb{B}_{x,r})_{i,j} := \int_{\mathbb{B}_{x,r}} \nabla w_i \cdot \nabla w_j \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{w}}} \left(\overline{w_i 1_{\mathbb{B}_{x,r}}} \cdot \overline{w_j 1_{\mathbb{B}_{x,r}}} + \underline{w_i 1_{\mathbb{B}_{x,r}}} \cdot \underline{w_j 1_{\mathbb{B}_{x,r}}}\right) \mathrm{d}\mathscr{H}^{n-1}.$$

To show (6.11), we may suppose that $\mathrm{Tr}\Big[B(\mathbf{v}; \mathbb{B}_{x,r})\Big] \leq \mathrm{Tr}\Big[B(\mathbf{u}; \mathbb{B}_{x,r})\Big]$ (or else it is automatically true) and that $\mathbf{v}$ is bounded in L^∞ by the same bound as $\mathbf{u}$. The optimality condition of $\mathbf{u}$ gives

$$\mathcal{F}_\gamma(\mathbf{u}) \leq \mathcal{F}_\gamma(\mathbf{v}),$$

where the right-hand side is well defined for any small enough $r > 0$ since $|A(\mathbf{v}) - I_k| \leq Cr^n$. This implies

$$G\Big[B(\mathbf{u})\Big] \leq G\Big[(1 + Cr^n)(B(\mathbf{u}) - B(\mathbf{u}; \mathbb{B}_{x,r}) + B(\mathbf{v}; \mathbb{B}_{x,r}))\Big] + \gamma|\mathbb{B}_{x,r}|.$$

Thus, using the monotonicity of G and the developpement (6.10) we obtain the estimate (6.11). Let us now show that this estimate, along with the a priori estimate

$$\delta 1_{\{\mathbf{u}\neq 0\}} \leq |\mathbf{u}| \leq M, \tag{6.12}$$

implies the closedness of $J_{\mathbf{u}}$, following arguments of [123] that were originally developped in [126] for minimizers of the Mumford-Shah functional. The crucial argument is the following decay lemma.

Lemma 6.12. *For any small enough $\tau \in]0, 1[$, there exists $\overline{r} = \overline{r}(\tau), \epsilon = \epsilon(\tau) > 0$, such that for any $x \in \mathbb{R}^n$, $r \in]0, \overline{r}]$, $\boldsymbol{w} \in \mathcal{U}_k$ verifying the a priori estimates* (6.12) *and the optimality condition* (6.11)

$$\left(\mathscr{H}^{n-1}(J_{\boldsymbol{w}} \cap \mathbb{B}_{x,r}) \leq \epsilon r^{n-1},\ \ Tr\Big[B(\boldsymbol{w}; \mathbb{B}_{x,r})\Big] \geq r^{n-\frac{1}{2}}\right) \text{ implies } Tr\Big[B(\boldsymbol{w}; \mathbb{B}_{x,\tau r})\Big] \leq \tau^{n-\frac{1}{2}} Tr\Big[B(\boldsymbol{w}; \mathbb{B}_{x,r})\Big].$$

Proof. The proof is sketched following the same steps as [123]. Consider a sequence of functions $\mathbf{w}^i \in \mathcal{U}_k$ with a sequence $r_i, \epsilon_i \to 0$ and a certain $\tau \in]0,1[$ that will be fixed later, such that:

$$\mathscr{H}^{n-1}(J_{\mathbf{w}^i} \cap \mathbb{B}_{r_i}) = \epsilon_i r_i^{n-1}, \tag{6.13}$$

$$\mathrm{Tr}\Big[B(\mathbf{w}^i; \mathbb{B}_{r_i})\Big] \geq r_i^{n-\frac{1}{2}}, \tag{6.14}$$

$$Tr\Big[B(\mathbf{w}^i; \mathbb{B}_{x,\tau r_i})\Big] \geq \tau^{n-\frac{1}{2}} Tr\Big[B(\mathbf{w}; \mathbb{B}_{r_i})\Big]. \tag{6.15}$$

And let

$$\mathbf{v}^i(x) = \frac{\mathbf{w}^i\left(x/r_i\right)}{\sqrt{r_i^{2-n}\mathrm{Tr}\Big[B(\mathbf{w}^i; \mathbb{B}_{r_i})\Big]}}.$$

Then, since $\int_{\mathbb{B}_1} |\nabla \mathbf{v}^i|^2 \mathrm{d}\mathscr{L}^n \leq 1$ and $\mathscr{H}^{n-1}(J_{\mathbf{v}^i} \cap \mathbb{B}_1) = \epsilon_i \to 0$, we know there exists some sequences $\tau_i^- < m_i < \tau_i^+$ such that the function: $\tilde{\mathbf{v}^i} := \min(\max(\mathbf{v}^i, \tau_i^-), \tau_i^+)$ (where the min and max are taken for each component) verifies:

$$\begin{aligned} \|\tilde{\mathbf{v}^i} - m_i\|_{L^{\frac{2n}{n-2}}(\mathbb{B}_1)} &\leq C_n \|\nabla \mathbf{v}\|_{L^2(\mathbb{B}_1)} && (\leq 1)\,, \\ \mathscr{L}^n(\{\tilde{\mathbf{v}^i} \neq \mathbf{v}^i\}) &\leq C_n \mathscr{H}^{n-1}(J_{\mathbf{v}^i} \cap \mathbb{B}_1)^{\frac{n}{n-1}} && \left(= C_n \epsilon_i^{\frac{n}{n-1}}\right). \end{aligned}$$

One may prove (using a BV and a $L^{\frac{2n}{n-2}}$ bound) that $\tilde{\mathbf{v}}^i - m_i$ converges in L^2 with lower semi-continuity for the Dirichlet energy to some $\mathbf{v} \in H^1(\mathbb{B}_1)$. We claim $\mathbf{v}$ is harmonic as a consequence of (6.11): for this consider a function $\boldsymbol{\varphi} \in H^1(\mathbb{B}_1)^k$ that coincides with $\mathbf{v}$ outside a ball $\mathbb{B}_\rho$ for some $\rho < 1$. Let $\rho' \in]\rho, 1[$, $\eta \in \mathcal{C}^\infty_{\text{compact}}(\mathbb{B}_{\rho'}, [0,1])$ such that $\eta = 1$ on $\mathbb{B}_\rho$ and $|\nabla \eta| \leq 2(\rho' - \rho)^{-1}$. Then we define

$$\begin{aligned} \boldsymbol{\varphi}^i &= (m_i + \boldsymbol{\varphi})\eta + \tilde{\mathbf{v}}^i(1-\eta)1_{\mathbb{B}_{\rho'}} + \mathbf{v}^i 1_{\mathbb{R}^n \setminus \mathbb{B}_{\rho'}}, \\ \boldsymbol{\Phi}^i(x) &= \sqrt{r_i^{2-n}\mathrm{Tr}\Big[B(\mathbf{w}^i; \mathbb{B}_{r_i})\Big]}\,\boldsymbol{\varphi}^i(r_i x). \end{aligned}$$

$\boldsymbol{\Phi}^i$ coincides with $\mathbf{w}^i$ outside of a ball of radius $\rho' r_i$, so it may be compared to $\mathbf{w}^i$ using the optimality condition (6.11). With the same computations as in [123] we obtain, as $\rho \nearrow \rho'$, that

$$E_0\Big[B(\boldsymbol{\varphi}; \mathbb{B}_{\rho'}) - B(\mathbf{v}; \mathbb{B}_{\rho'})\Big] \geq 0.$$

Taking $\boldsymbol{\varphi}$ to be the harmonic extension of $\mathbf{v}_{|\partial \mathbb{B}_\rho}$ in $\mathbb{B}_\rho$, we find that $B(\boldsymbol{\varphi}; \mathbb{B}_{\rho'}) \leq B(\mathbf{v}; \mathbb{B}_{\rho'})$ with equality if and only if $\mathbf{v}$ is equal to its harmonic extension. If it is not, then

$$E_0\Big[B(\boldsymbol{\varphi}; \mathbb{B}_{\rho'}) - B(\mathbf{v}; \mathbb{B}_{\rho'})\Big] < 0,$$

which contradicts the optimality. This means that the components of $\mathbf{v}$ are harmonic. Since $\int_{\mathbb{B}_1} |\nabla \mathbf{v}|^2 \mathrm{d}\mathscr{L}^n \leq 1$, then $|\nabla \mathbf{v}| \leq \sqrt{1/|\mathbb{B}_{1/2}|}$ on $\mathbb{B}_{1/2}$, so for any $\tau < \frac{1}{2^n |\mathbb{B}_1|}$ we find that $\int_{\mathbb{B}_\tau} |\nabla u|^2 \mathrm{d}\mathscr{L}^n < \tau^{n-\frac{1}{2}}$; this contradicts the condition (6.15). □

The decay lemma implies the existence of $r_1, \epsilon_1 > 0$ such that for any $x \in J_{\mathbf{u}}^{\text{reg}}$ and $r \in]0, r_1[$:

$$\mathscr{H}^{n-1}(J_{\mathbf{u}} \cap \mathbb{B}_{x,r}) \geq \epsilon_1 r^{n-1}. \tag{6.16}$$

Suppose indeed that it is not the case for some $x \in J_{\mathbf{u}}$. Let $\tau_0 \in]0,1[$ be small enough to apply lemma 6.12. Then for a small enough τ_1,

$$\mathrm{Tr}\Big[B(\mathbf{u} : \mathbb{B}_{x,\tau_1 r}\Big] \leq \delta^2 \epsilon(\tau_0)(\tau_1 r)^{n-1}.$$

Indeed, either $\mathrm{Tr}\Big[B(\mathbf{u};\mathbb{B}_{x,r})\Big]$ is less than $r^{n-\frac{1}{2}}$ and this is direct provided we take $r_1 < \delta^4\epsilon(\tau_0)^2\tau_1^{2(n-1)}$, or it is not and then by application of the lemma (and using the fact that $\mathrm{Tr}\Big[B(\mathbf{u};\mathbb{B}_{x,r})\Big] \leq C(\mathbf{u})r^{n-1}$, which is obtained by comparing $\mathbf{u}$ with $\mathbf{u}1_{\mathbb{R}^n\setminus\mathbb{B}_{x,r}}$) we get

$$\mathrm{Tr}\Big[B(\mathbf{u};\mathbb{B}_{x,\tau_1 r})\Big] \leq C(\mathbf{u})\tau_1^{n-\frac{1}{2}}r^{n-1} \leq \delta^2\epsilon(\tau_0)(\tau_1 r)^{n-1},$$

provided we choose $\tau_1 \leq C(\mathbf{u})^{-2}\delta^4\epsilon(\tau_0)^2$ (and $\epsilon_1 = \epsilon(\tau_1), r_1 < \overline{r}(\tau_1)$ so that the lemma may be applied). Then we may show by induction that for all $k \in \mathbb{N}$,

$$\mathrm{Tr}\Big[B(\mathbf{u};\mathbb{B}_{x,\tau_0^k\tau_1 r})\Big] \leq \delta^2\epsilon(\tau_0)\tau_0^{k(n-\frac{1}{2})}(\tau_1 r)^{n-1}. \tag{6.17}$$

Indeed (6.17) implies that $\mathscr{H}^{n-1}(J_{\mathbf{u}} \cap \mathbb{B}_{\tau_0^k\tau_1 r}) \leq \epsilon(\tau_0)(\tau_0^k\tau_1 r)^{n-1}$, so with the same dichotomy as above we may apply the lemma 6.12 again to obtain (6.17) by induction.

Overall this means that $\frac{1}{\rho^{n-1}}\left(\int_{\mathbb{B}_{x,\rho}}|\nabla\mathbf{u}|^2\mathrm{d}\mathscr{L}^n + \mathscr{H}^{n-1}(J_{\mathbf{u}}\cap\mathbb{B}_{x,r})\right) \underset{\rho\to 0}{\to} 0$, which is not the case when $x \in J_{\mathbf{u}}$ (see [126], Theorem 3.6), so (6.16) holds. By definition it also holds for $x \in \overline{J_{\mathbf{u}}}$ with a smaller constant, however according to [126], lemma 2.6, $\mathscr{H}^{n-1}$-almost every x such that $\liminf_{r\to 0}\frac{\mathscr{H}^{n-1}(J_{\mathbf{u}}\cap\mathbb{B}_{x,r})}{r^{n-1}} > 0$ is in $J_{\mathbf{u}}$, which ends the proof. □

As a consequence of the existence of a relaxed minimizer and the regularity of relaxed minimizers, we obtain the theorem 6.1.

Proof. We know from the proposition 6.8 that there exists a relaxed minimizer $\mathbf{u}$ of $\mathcal{F}$ in $\mathcal{U}_k(m)$, and from lemma 6.10 that $\mathbf{u}$ is an internal relaxed minimizer of $\mathcal{F}_\gamma$ for some $\gamma > 0$ that only depends on the parameters. From the proposition 6.7 we obtain that for certain constants $\delta, M, R > 0$ only depending on the parameters, $\delta 1_{\{\mathbf{u}\neq 0\}} \leq |\mathbf{u}| \leq M$ and the diameter of the support of $\mathbf{u}$ (up to translation of its components) is less than R. From proposition 6.11 we know that $\mathscr{H}^{n-1}(\overline{J_{\mathbf{u}}}\setminus J_{\mathbf{u}}) = 0$. Since $|\mathbf{u}| \geq \delta 1_{\{\mathbf{u}\neq 0\}}$, we obtain

$$\begin{aligned}\mathscr{H}^{n-1}(\overline{J_{\mathbf{u}}}) = \mathscr{H}^{n-1}(J_{\mathbf{u}}) &\leq \delta^{-2}\int_{J_{\mathbf{u}}}\left(|\overline{\mathbf{u}}|^2 + |\underline{\mathbf{u}}|^2\right)\mathrm{d}\mathscr{H}^{n-1} \\ &\leq \beta^{-1}\delta^{-2}\left(\lambda_1(\mathbf{u};\beta) + \ldots + \lambda_k(\mathbf{u};\beta)\right) \leq C(n,m,\beta,F).\end{aligned}$$

Let Ω be the union of the connected components of $\mathbb{R}^n\setminus\overline{J_{\mathbf{u}}}$ on which $\mathbf{u}$ is not zero almost everywhere. By definition $\partial\Omega = \overline{J_{\mathbf{u}}}$, and $\mathbf{u}$ is continuous on $\mathbb{R}^n\setminus J_{\mathbf{u}}$ and do not take the values $\pm\frac{\delta}{2}$, thus $|\mathbf{u}| \geq \delta$ on Ω. In particular, $\{\mathbf{u}\neq 0\}$ and Ω differ by a $\mathscr{L}^n$-negligible set, and $J_{\mathbf{u}} \subset \partial\Omega$, so $\mathbf{u}_{|\Omega} \in H^1(\Omega)^k$. This means that for every $i = 1,\ldots,k$, $\lambda_i(\Omega;\beta) \leq \lambda_i(\mathbf{u};\beta)$, so Ω is optimal for $\mathcal{F}$.

In the proof of proposition 6.11 we obtained the existence of a certain $\epsilon_1, r_1 > 0$ such that for every $x \in \partial\Omega(=\overline{J_{\mathbf{u}}})$, $r < r_1$, then $\mathscr{H}^{n-1}(\mathbb{B}_{x,r}\cap\partial\Omega) \geq \epsilon_1 r^{n-1}$. By comparing $\mathbf{u}$ with $\mathbf{u}1_{\mathbb{R}^n\setminus B_{x,r}}$ (similarly to what was done in the proof of the proposition 6.7), we obtain the upper bound $\mathscr{H}^{n-1}(\mathbb{B}_{x,r}\cap\partial\Omega) \leq Cr^{n-1}$; this concludes the proof. □

6.4 The functional $\Omega \mapsto \lambda_k(\Omega;\beta)$

We are now interested by the specific functional

$$\Omega \mapsto \lambda_k(\Omega;\beta).$$

While it is not covered by the previous existence result, relaxed minimizers of this functional were shown to exist in [119]. To understand its regularity, it might be tempting to consider a sequence of relaxed minimizers with the function $F(\lambda_1,\ldots,\lambda_k) = \lambda_k + \epsilon(\lambda_1 + \ldots + \lambda_{k-1})$ where $\epsilon \to 0$, however

while the L^∞ bound does not depend on ϵ, the lower bound does and it seems to degenerate to 0 as ϵ goes to 0.
This prevents us to obtain any regularity on relaxed minimizers of this functional. We are, however, able to treat the specific case where the k-th eigenvalue would be simple, and this analysis allows us to prove that this does not happen in general. In particular, we shall prove $\lambda_k(\mathbf{u};\beta) = \lambda_{k-1}(\mathbf{u};\beta)$.

6.4.1 Regularization and perturbation lemma

We begin with a density result that allows us to suppose without loss of generality that $\mathbf{u}$ is bounded in L^∞. This relies on the same procedure as [119, Theorem 4.3].
We remind the notation for admissible functions used previously:

$$\mathcal{U}_k(m) = \{\mathbf{v} \in \mathcal{U}_k : \ \mathbf{v} \text{ is linearly independant and } |\{\mathbf{v} \neq 0\}| = m\},$$

as well as the fact that if $\mathbf{u}$ is a relaxed minimizer of $\lambda_k(\cdot;\beta)$ in $\mathcal{U}_k(m)$ then according to lemma 6.10 there is a constant $\gamma > 0$ such that $\mathbf{u}$ is a minimizer of

$$\mathbf{v} \mapsto \lambda_k(\mathbf{v};\beta) + \gamma|\{\mathbf{v} \neq 0\}|$$

for linearly independant function $\mathbf{v}$ such that $|\{\mathbf{v} \neq 0\}| \in]0, m]$.

Lemma 6.13. *Let $\boldsymbol{u} = (u_1, \dots, u_k)$ be a relaxed minimizer of $\lambda_k(\boldsymbol{u};\beta)$ in $\mathcal{U}_k(m)$. Suppose that $\lambda_k(\boldsymbol{u};\beta) > \lambda_{k-1}(\boldsymbol{u};\beta)$. Then there exists another minimizer $\boldsymbol{v} \in \mathcal{U}_k(m)$ that is linearly independant, normalized, such that $v_1 \geq 0$, $\boldsymbol{v} \in L^\infty(\mathbb{R}^n)$, and*

$$\lambda_{k-1}(\boldsymbol{v};\beta) < \lambda_k(\boldsymbol{v};\beta).$$

This justifies that in all the following propositions we may suppose that $\mathbf{u} \in L^\infty(\mathbb{R}^n)$ without loss of generality.

Proof. Without loss of generality suppose that $\mathbf{u}$ is normalized. Then according to [119], which itself relies on the Cortesani-Toader regularization (see [124]), there exists a sequence of bounded polyhedral domains (Ω^p) along with a sequence $\mathbf{u}^p \in \mathcal{U}_k \cap H^1(\Omega^p)^k$ such that $\mathbf{u}^p \underset{p\to\infty}{\to} \mathbf{u}$ in L^2, and

$$\limsup_{p\to\infty} B(\mathbf{u}^p) \leq B(\mathbf{u}), \ \limsup_{p\to\infty} |\Omega^p| \leq |\{\mathbf{u} \neq 0\}|.$$

Let $\mathbf{v}^p = (v_1^p, \dots, v_k^p)$ be the first k eigenfunctions of Ω^p (with an arbitrary choice in case of multiplicity; notice v_1^p may be chosen positive), then $B(\mathbf{v}^p) \leq B(\mathbf{u}^p)$ and with Moser iteration $\mathbf{v}^p$ is bounded in L^∞ by $C_{n,\beta,m}\lambda_k(\mathbf{u};\beta)^{\frac{n}{2}}$ (which, in particular, does not depend on p). Using the compactness result 6.5, we find that up to an extraction $\mathbf{v}^p$ converges in L^2 and almost everywhere to $\mathbf{v} \in \mathcal{U}_k$ with lower semi-continuity on its Dirichlet-Robin energy, thus $\mathbf{v}$ is a minimizer in L^∞ with $v_1 \geq 0$. Moreover,

$$\lambda_{k-1}(\mathbf{v};\beta) \leq \liminf_{p\to\infty} \lambda_{k-1}(\mathbf{v}^p;\beta) \leq \liminf_{p\to\infty} \lambda_{k-1}(\mathbf{u}^p;\beta) \leq \lambda_{k-1}(\mathbf{u};\beta) < \lambda_k(\mathbf{u};\beta) \leq \lambda_k(\mathbf{v};\beta).$$

□

Lemma 6.14. *Let $\boldsymbol{u} = (u_1, \dots, u_k) \in \mathcal{U}_k(m) \cap L^\infty(\mathbb{R}^n)$ be an internal relaxed minimizer of $\lambda_k(\boldsymbol{u};\beta)$ in $\mathcal{U}_k(m)$, that we suppose to be normalized. Suppose that $\lambda_k(\boldsymbol{u};\beta) = \lambda_{k-l+1}(\boldsymbol{u};\beta) > \lambda_{k-l}(\boldsymbol{u};\beta)$. Then there exists $\delta, \gamma > 0$ such that, for all $\omega \subset \mathbb{R}^n$ that verify*

$$|\omega| + Per(\omega; \mathbb{R}^n \setminus J_{\boldsymbol{u}}) < \delta,$$

there exists $\alpha \in (\{0\}^{k-l} \times \mathbb{R}^l) \cap \mathbb{S}^{k-1}$ such that

$$\int_\omega |\nabla u_\alpha|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\boldsymbol{u}}} \left(\overline{u_\alpha 1_\omega}^2 + \underline{u_\alpha 1_\omega}^2\right) \mathrm{d}\mathscr{H}^{n-1} + \gamma|\omega| \leq 2\beta \int_{\partial^*\omega \setminus J_{\boldsymbol{u}}} u_\alpha^2 \mathrm{d}\mathscr{H}^{n-1} + 2\lambda_k(\boldsymbol{u};\beta) \int_\omega u_\alpha^2 \mathrm{d}\mathscr{L}^n. \tag{6.18}$$

As may be seen in the proof, the factors 2 on the right-hand side may be replaced by $1 + \underset{\delta\to 0}{o}(1)$, however this will not be useful for us.
This result will only be applied in the particular case where $l = 1$: when $l > 1$ it gives a very weak information on the eigenspace of $\lambda_k(\mathbf{u};\beta)$ and it would be interesting to see if the regularity of one of the eigenfunctions might be deduced from it as was done in [122] (in the same problem with Dirichlet boundary conditions). In this case better estimates were obtained by perturbing the functional into $(1-\epsilon)\lambda_k + \epsilon\lambda_{k-1}$, considering a minimizer Ω^ϵ that contains the minimizer Ω of λ_k, and separating the cases where $\lambda_k(\Omega^\epsilon)$ is simple or not. However these arguments use crucially the monotonicity and scaling properties of λ_i, which are not available for Robin boundary conditions.

Proof. Let us denote $\mathbf{v} = \mathbf{u}1_{\mathbb{R}^n\setminus\omega}$, $A, B = A(\mathbf{v}), B(\mathbf{v})$, and for any $\alpha, \beta \in \mathbb{R}^k$,

$$A_{\alpha,\beta} = \sum_{i=1}^k \alpha_i\beta_i A_{i,j},$$
$$B_{\alpha,\beta} = \sum_{i=1}^k \alpha_i\beta_i B_{i,j}.$$

We study the quantity

$$\lambda_k(\mathbf{v};\beta) = \max_{\alpha\in\mathbb{S}^{k-1}} \frac{B_{\alpha,\alpha}}{A_{\alpha,\alpha}}.$$

Due to the L^∞ bound on $\mathbf{u}$ and the fact that $|\omega| + \mathrm{Per}(\omega;\mathbb{R}^n\setminus J_{\mathbf{u}}) \le \delta$:

$$\inf_{\alpha\in\{0\}^{k-l}\times\mathbb{R}^l\cap\mathbb{S}^{k-1}} \frac{B_{\alpha,\alpha}}{A_{\alpha,\alpha}} \underset{\delta\to 0}{\longrightarrow} \lambda_k(\mathbf{u};\beta),$$
$$\sup_{\eta\in\mathbb{R}^{k-l}\times\{0\}^l\cap\mathbb{S}^{k-2}} \frac{B_{\eta,\eta}}{A_{\eta,\eta}} \underset{\delta\to 0}{\longrightarrow} \lambda_{k-l}(\mathbf{u};\beta) (< \lambda_k(\mathbf{u};\beta))$$

Thus for a small enough δ the maximum above is attained for a certain $\frac{\alpha+t\eta}{\sqrt{1+t^2}}$ where $\alpha \in \{0\}^{k-l}\times\mathbb{R}^l\cap\mathbb{S}^{k-1}$, $\eta \in \mathbb{R}^{k-l}\times\{0\}^l\cap\mathbb{S}^{k-1}$ and $t\in\mathbb{R}$. α and η are fixed in what follows and so

$$\lambda_k(\mathbf{v};\beta) = \max_{t\in\mathbb{R}} \frac{B_{\alpha,\alpha} + 2tB_{\alpha,\eta} + t^2B_{\eta,\eta}}{A_{\alpha,\alpha} + 2tA_{\alpha,\eta} + t^2A_{\eta,\eta}}.$$

We let

$$b_{\alpha,\eta} = \frac{B_{\alpha,\eta}}{B_{\alpha,\alpha}}, \qquad b_{\eta,\eta} = \frac{B_{\eta,\eta}}{B_{\alpha,\alpha}},$$
$$a_{\alpha,\eta} = \frac{A_{\alpha,\eta}}{A_{\alpha,\alpha}}, \qquad a_{\eta,\eta} = \frac{A_{\eta,\eta}}{A_{\alpha,\alpha}},$$

$$F(t) = \frac{1 + 2tb_{\alpha,\eta} + t^2b_{\eta,\eta}}{1 + 2ta_{\alpha,\eta} + t^2a_{\eta,\eta}}.$$

Then we may rewrite

$$\lambda_k(\mathbf{v};\beta) = \frac{B_{\alpha,\alpha}}{A_{\alpha,\alpha}} \max_{t\in\mathbb{R}} F(t). \tag{6.19}$$

Moreover,

$$a_{\eta,\eta} \underset{\delta\to 0}{\longrightarrow} 1, \ \limsup_{\delta\to 0} b_{\eta,\eta} \le \frac{\lambda_{k-l}(\mathbf{u};\beta)}{\lambda_k(\mathbf{u};\beta)} < 1.$$

We look for the critical points of F; $F'(t)$ has the same sign as

$$(a_{\alpha,\eta}b_{\eta,\eta} - a_{\eta,\eta}b_{\alpha,\eta})t^2 - (a_{\eta,\eta} - b_{\eta,\eta})t + (b_{\alpha,\eta} - a_{\alpha,\eta}).$$

Since F has the same limit in $\pm\infty$, this polynomial has two real roots given by:

$$t^{\pm} = \frac{a_{\eta,\eta} - b_{\eta,\eta}}{2(a_{\alpha,\eta}b_{\eta,\eta} - a_{\eta,\eta}b_{\alpha,\eta})}\left(1 \pm \sqrt{1 - 4\frac{(b_{\alpha,\eta} - a_{\alpha,\eta})(a_{\alpha,\eta}b_{\eta,\eta} - a_{\eta,\eta}b_{\alpha,\eta})}{(a_{\eta,\eta} - b_{\eta,\eta})^2}}\right).$$

Since F' has the same sign as $(a_{\alpha,\eta}b_{\eta,\eta} - a_{\eta,\eta}b_{\alpha,\eta})$ in $\pm\infty$, we find that the maximum of F is attained in t^-. For any small enough δ we obtain

$$|t^-| \leq C_1|a_{\alpha,\eta} - b_{\alpha,\eta}|,$$

where C_1 only depends on $\lambda_k(\mathbf{u};\beta), \lambda_{k-1}(\mathbf{u};\beta)$. We evaluate F in t^- to obtain, for small enough δ,

$$F(t^-) \leq 1 + C_2(A^2_{\alpha,\eta} + B^2_{\alpha,\eta}),$$

where C_2 is another such constant. With the Cauchy Schwarz inequality we obtain

$$A^2_{\alpha,\eta} = \underset{\delta\to 0}{o}\left(\int_\omega u_\alpha^2 \mathrm{d}\mathscr{L}^n\right),$$
$$B^2_{\alpha,\eta} = \underset{\delta\to 0}{o}\left(\int_\omega |\nabla u_\alpha|^2 + \beta\int_{J_{\mathbf{u}}\cup\partial^*\omega}\left(\underline{u_\alpha 1_\omega}^2 + \overline{u_\alpha 1_\omega}^2\right)\mathrm{d}\mathscr{H}^{n-1}\right).$$

Moreover,

$$B_{\alpha,\alpha} = B(\mathbf{u})_{\alpha,\alpha} - \int_\omega |\nabla u_\alpha|^2\mathrm{d}\mathscr{L}^n - \beta\int_{J_{\mathbf{u}}}\left(\underline{u_\alpha 1_\omega}^2 + \overline{u_\alpha 1_\omega}^2\right)\mathrm{d}\mathscr{H}^{n-1} + \int_{\partial^*\omega\setminus J_{\mathbf{u}}} u_\alpha^2\mathrm{d}\mathscr{H}^{n-1},$$
$$A_{\alpha,\alpha} = 1 - \int_\omega u_\alpha^2\mathrm{d}\mathscr{L}^n.$$

Thus for a small enough δ, we obtained the following estimate in (6.19)

$$\left(1 - \int_\omega u_\alpha^2\mathrm{d}\mathscr{L}^n\right)\lambda_k(\mathbf{v};\beta) \leq B(\mathbf{u})_{\alpha,\alpha} - (1 - \underset{\delta\to 0}{o}(1))\left(\int_\omega |\nabla u_\alpha|^2\mathrm{d}\mathscr{L}^n + \beta\int_{J_{\mathbf{u}}}\left(\underline{u_\alpha 1_\omega}^2 + \overline{u_\alpha 1_\omega}^2\right)\mathrm{d}\mathscr{H}^{n-1}\right)$$
$$+ (1 + \underset{\delta\to 0}{o}(1))\int_{\partial^*\omega\setminus J_{\mathbf{u}}} u_\alpha^2\mathrm{d}\mathscr{H}^{n-1} + \underset{\delta\to 0}{o}\left(\int_\omega u_\alpha^2\mathrm{d}\mathscr{L}^n\right).$$

The optimality condition on $\mathbf{u}$ ($\lambda_k(\mathbf{u};\beta) + \gamma|\omega| \leq \lambda_k(\mathbf{v})$ for a certain $\gamma > 0$ that does not depend on ω, obtained through Lemma 6.10) coupled with the fact that $\lambda_k(\mathbf{u}) = B(\mathbf{u})_{\alpha,\alpha}$ gives us the estimate (6.18) for any small enough δ. □

6.4.2 Non-degeneracy lemma and the main result

Proposition 6.15. *Let $\boldsymbol{u} = (u_1,\dots,u_k) \in \mathcal{U}_k \cap L^\infty(\mathbb{R}^n)$ an internal relaxed minimizer of $\lambda_k(\boldsymbol{u};\beta) + \gamma|\{\boldsymbol{u}\neq 0\}|$. Suppose $n \geq 3$, and that $\lambda_k(\boldsymbol{u};\beta) > \lambda_{k-1}(\boldsymbol{u};\beta)$. Then there exists $c > 0$ such that $|u_k| \geq c1_{\{u_k\neq 0\}}$.*

Proof. We actually prove that there exists $r, t > 0$ such that for any $x \in \mathbb{R}^n$, $|u_k| \geq t1_{\mathbb{B}_{x,r}\cap\{u_k\neq 0\}}$, since this is sufficient to conclude. We suppose $x = 0$ to simplify the notations. We cannot proceed as in the proof of result 6.7 because we do not know whether $\mathrm{Per}(\{|u_k| > t\};\mathbb{R}^n\setminus J_{\mathbf{u}})$ is less than a constant δ or not. The idea is to compare $\mathbf{u}$ with $\mathbf{u}1_{\mathbb{R}^n\setminus\omega_t}$ where

$$\omega_t = \mathbb{B}_{r(t)} \cap \{|u_k| \leq t\},$$

for $t > 0$ and $r(t) > 0$ chosen sufficiently small such that $\mathrm{Per}(\omega_t;\mathbb{R}^n\setminus J_u)$ is sufficiently small.

Lemma 6.16. *Under these circumstances, there exists $t_1 > 0$ such that for all $t < t_1$,*

$$\int_{\omega_t} |\nabla u_k|^2\mathrm{d}\mathscr{L}^n + \beta\int_{J_{\boldsymbol{u}}}\left(\overline{u_k 1_{\omega_t}}^2 + \underline{u_k 1_{\omega_t}}^2\right)\mathrm{d}\mathscr{H}^{n-1} + \frac{1}{2}\gamma|\omega_t| \leq 2\beta\int_{\partial^*\omega_t\setminus J_{\boldsymbol{u}}} u_k^2\mathrm{d}\mathscr{H}^{n-1}, \tag{6.20}$$

where $\omega_t = \{|u_k| \leq t\}\cap\mathbb{B}_{r(t)}$ with $r(t) := \epsilon t^{\frac{2}{n-1}}$ for a small enough $\epsilon > 0$.

Proof. As we said previously, this estimate will be obtained by comparing $\mathbf{u}$ and $\mathbf{u}1_{\mathbb{R}^n\setminus\omega_t}$ where $\omega_t = \mathbb{B}_{r(t)} \cap \{|u_k| \leq t\}$. This is direct if we can apply Lemma 6.14, we only need to show the hypothesis

$$\mathscr{H}^{n-1}(\partial^*\omega_t \setminus J_u) < \delta.$$

Suppose that

$$\beta t^2 \mathscr{H}^{n-1}(\partial^* \{|u_k| \leq t\} \cap \mathbb{B}_{r(t)} \setminus J_{\mathbf{u}}) \leq \int_{\omega_t} |\nabla u_k|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{u}}} \left(\overline{u_k 1_{\omega_t}}^2 + \underline{u_k 1_{\omega_t}}^2\right) \mathrm{d}\mathscr{H}^{n-1} + \gamma|\omega_t|.$$

Indeed if this inequality is false then we obtained the result. Then, comparing $\mathbf{u}$ with $\mathbf{u}1_{\mathbb{R}^n\setminus\mathbb{B}_{r(t)}}$ with the lemma 6.14 (which is allowed for any small enough $r > 0$) we obtain the estimate

$$\begin{aligned}\int_{\mathbb{B}_{r(t)}} |\nabla u_k|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{J_{\mathbf{u}}} \left(\overline{u_k 1_{\mathbb{B}_{r(t)}}}^2 + \underline{u_k 1_{\mathbb{B}_{r(t)}}}^2\right) \mathrm{d}\mathscr{H}^{n-1} + \frac{1}{2}\gamma|\mathbb{B}_{r(t)}| &\leq 2\beta \int_{\partial\mathbb{B}_{r(t)}\setminus J_{\mathbf{u}}} u_k^2 \mathrm{d}\mathscr{H}^{n-1} \\ &\leq C(n, \beta, \|u_k\|_{L^\infty}) r(t)^{n-1}.\end{aligned}$$

Combining the two previous inequalities,

$$\begin{aligned}\mathscr{H}^{n-1}(\partial^* \{|u_k| \leq t\} \cap \mathbb{B}_{r(t)} \setminus J_{\mathbf{u}}) &\leq C(n, \beta, \|u_k\|_{L^\infty}) \frac{r(t)^{n-1}}{t^2} \\ &= C(n, \beta, \|u_k\|_{L^\infty}) \epsilon^{n-1} \\ &\leq \frac{1}{2}\delta \text{ for a small enough } \epsilon.\end{aligned}$$

And so Lemma 6.14 may be applied, concluding the proof. □

We introduce the sets

$$\omega^{\sup} = \left\{x : |u_k(x)| \geq |x/\epsilon|^{\frac{n-1}{2}}\right\}, \ \omega^{\inf} = \left\{x : |u_k(x)| \leq |x/\epsilon|^{\frac{n-1}{2}}\right\}$$

and the function

$$f(t) = \int_{\omega_t} \left(|\nabla u_k| 1_{\omega^{\sup}} + 1_{\omega^{\inf}}\right) |u_k| \mathrm{d}\mathscr{L}^n.$$

From the coarea formula we get

$$f(t) = \int_0^t \left(\int_{\partial^*\{|u_k|\leq\tau\}\cap\mathbb{B}_{r(\tau)}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1}\right) \mathrm{d}\tau + \int_0^{r(t)} \left(\int_{\partial\mathbb{B}_r \cap \left\{|u_k|\leq(r/\epsilon)^{\frac{n-1}{2}}\right\}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1}\right) \mathrm{d}r.$$

So f is absolutely continuous and

$$f'(t) = \int_{\partial^*\{|u_k|\leq t\}\cap\mathbb{B}_{r(t)}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1} + \frac{2\epsilon}{n-1} t^{-\frac{n-3}{n-1}} \int_{\partial\mathbb{B}_{r(t)}\cap\{|u_k|\leq t\}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1}.$$

We use here the fact that $n \geq 3$, so that for all small enough t we get

$$\frac{1}{\epsilon} f'(t) \geq \int_{\partial^*\{|u_k|\leq t\}\cap\mathbb{B}_{r(t)}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1} + \int_{\partial\mathbb{B}_{r(t)}\cap\{|u_k|\leq t\}\setminus J_{\mathbf{u}}} |u_k| \mathrm{d}\mathscr{H}^{n-1}.$$

We will now estimate f in a similar manner as in result 6.7.

$$\begin{aligned}
c_n\left(\int_{\omega_t}|u_k|^{2\frac{n}{n-1}}\mathrm{d}\mathscr{L}^n\right)^{\frac{n-1}{n}} &\leq D(|u_k|^2 1_{\omega_t})(\mathbb{R}^n)\\
&= \int_{\omega_t} 2|u_k\nabla u_k|\mathrm{d}\mathscr{L}^n + \int_{J_{\mathbf{u}}}\left(\overline{u_k 1_{\omega_t}}^2 + \underline{u_k 1_{\omega_{r,t}}}^2\right)\mathrm{d}\mathscr{H}^{n-1}\\
&+ \int_{\partial^*\omega_t\setminus J_u}|u_k|^2\mathrm{d}\mathscr{H}^{n-1}\\
&\leq |\omega_t| + \int_{\omega_t}|\nabla u_k|^2\mathrm{d}\mathscr{L}^n + \int_{J_{\mathbf{u}}}\left(\overline{u_k 1_{\omega_t}}^2 + \underline{u_k 1_{\omega_{r,t}}}^2\right)\mathrm{d}\mathscr{H}^{n-1}\\
&+ \int_{\partial^*\omega_t\setminus J_u}|u_k|^2\mathrm{d}\mathscr{H}^{n-1}\\
&\leq C_{\beta,\gamma}\int_{\partial^*\omega_t\setminus J_u}|u_k|^2\mathrm{d}\mathscr{H}^{n-1}\\
&\leq \frac{C_{\beta,\gamma}}{\epsilon}tf'(t).
\end{aligned}$$

We used the lemma 6.16 in the penultimate line, which is only valid for small enough t. The hypothesis that $n \geq 3$ was used in the last line. Finally,

$$\begin{aligned}
f(t) &= \int_{\omega_t}\left(|\nabla u_k|1_{\omega^{\sup}} + 1_{\omega^{\inf}}\right)|u_k|\mathrm{d}\mathscr{L}^n\\
&\leq |\omega_t|^{\frac{1}{2n}}\left(\int_{\omega_t}|\nabla u_k|^2\mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}}\left(\int_{\omega_t}|u|^{2\frac{n}{n-1}}\mathrm{d}\mathscr{L}^n\right)^{\frac{n-1}{2n}} + \gamma|\omega_t|^{\frac{n+1}{2n}}\left(\int_{\omega_t}|u_k|^{2\frac{n}{n-1}}\mathrm{d}\mathscr{L}^n\right)^{\frac{n-1}{2n}}\\
&\leq C_{n,\beta,\gamma}\left(tf'(t)\right)^{\frac{2n+1}{2n}},
\end{aligned}$$

which implies for a certain $t > 0$ that $f(t) = 0$. Let $r = \epsilon t^{\frac{n-1}{2}}$, we show $|u_k| \geq t1_{\mathbb{B}_{x,r}\cap\{u_k\neq 0\}}$. From $f(t) = 0$ we get that $u_k = 0$ on $\mathbb{B}_r \cap \left\{x : |u_k(x)| \leq |x/\epsilon|^{\frac{n-1}{2}}\right\}$. In particular, up to reducing slightly r and t we may suppose

$$\mathscr{H}^{n-1}(\partial\mathbb{B}_r \cap \{|u_k| \leq t\}) = 0.$$

Moreover, $f(t) = 0$ also gives that $\nabla u_k = 0$ on $\mathbb{B}_r \cap \{0 < u \leq t\}$. Consider $\mathbf{u}' = \mathbf{u}1_{\mathbb{R}^n\setminus\omega}$ where $\omega = \mathbb{B}_r \cap \{|u_k| \leq t\}$. Note that $J_{\mathbf{u}'} \subset J_{\mathbf{u}}$, and for any small enough $t > 0$,

$$\lambda_k(\mathbf{u};\beta') \leq \lambda_k(\mathbf{u};\beta) + 2t^2|\omega| - \frac{1}{2}\beta\int_{J_{\mathbf{u}}}\left(\overline{u_k 1_\omega}^2 + \underline{u_k 1_\omega}^2\right)\mathrm{d}\mathscr{H}^{n-1}.$$

This contradicts the minimality of $\lambda_k(\mathbf{u};\beta) + \gamma|\{\mathbf{u} \neq 0\}|$ as soon as $|\omega| > 0$. This concludes the proof.

□

Note that the proof fails when $n = 2$; we need to choose $r(t) \ll t^{\frac{2}{n-1}}$ to ensure that the competitor $u1_{\mathbb{R}^n\setminus\omega_t}$ yields information, but later we use $\inf_{t<1} r'(t) > 0$ in a crucial way. When $n = 2$ the inequalities are weakened to instead yield $f(t) \geq ct^5$, which is not enough to conclude.

We now deduce the second main result as a consequence.

Proposition 6.17. *Suppose $n \geq 3, k \geq 2$. Let $\boldsymbol{u} = (u_1, \ldots, u_k)$ a relaxed minimizer of $\lambda_k(\boldsymbol{u};\beta)$ in $\mathcal{U}_k(m)$. Then*

$$\lambda_k(\boldsymbol{u};\beta) = \lambda_{k-1}(\boldsymbol{u};\beta).$$

Proof. Suppose that $\lambda_k(\mathbf{u};\beta) > \lambda_{k-1}(\mathbf{u};\beta)$. We may apply lemma 6.13 to assume without loss of generality that $u_1 \geq 0$ and $\mathbf{u} \in L^\infty$, so that all the previous estimates apply.
Let Ω be the support of u_k, with $\Omega^+ = \{u_k > 0\}$ and $\Omega^- = \{u_k < 0\}$.

We first notice that $|\{\mathbf{u} \neq 0\} \setminus \Omega| = 0$. Suppose indeed that it is not the case, and let $\omega = \{\mathbf{u} \neq 0\} \setminus \Omega$. Since $|u_k| \geq \delta 1_\Omega$, $\mathbf{u}$ may be written as a disconnected sum of two $\mathcal{U}_k$ functions

$$\mathbf{u} = (\mathbf{u}1_\Omega) \oplus (\mathbf{u}1_\omega).$$

We may translate Ω and ω so that they have a positive distance from each other. Then consider $t > 1$ and $s = s(t) < 1$ chosen such that

$$|t\Omega| + |s\omega| = |\Omega| + |\omega|,$$

and $\mathbf{u}_t$ the function built by dilation of $\mathbf{u}$ on $t\Omega \cup s\omega$. Then for $t = 1 + \epsilon$ with a small enough ϵ we have $\lambda_k(\mathbf{u}_t; \beta) < \lambda_k(\mathbf{u}; \beta)$ with support of same measure, which is absurd by minimality of $\mathbf{u}$. Thus $|\omega| = 0$.

Since u_1 is nonnegative, has support in Ω, and $\langle u_1, u_k \rangle_{L^2} = 0$, this means that $|\Omega^+|, |\Omega^-| > 0$. We may again decompose $\mathbf{u}$ into

$$\mathbf{u} = (\mathbf{u}1_{\Omega^+}) \oplus (\mathbf{u}1_{\Omega^-}).$$

Consider $\mathbf{v} \in \mathcal{U}_p(m)$ for some $p \in \{k, \dots, 2k\}$ an extraction of $(\mathbf{u}1_{\Omega^+}, \mathbf{u}1_{\Omega^-})$, such that it spans the same space in $L^2(\mathbb{R}^n)$ and $\mathbf{v}$ is linearly independant. Then for each $i \in \{1, \dots, k\}$,

$$\lambda_i(\mathbf{v}; \beta) \leq \lambda_i(\mathbf{u}; \beta),$$

with equality if $i = k$ by optimality of $\mathbf{u}$. Since $A(\mathbf{v})$ and $B(\mathbf{v})$ are block diagonals we may suppose $\mathbf{v}$ is normalized such that its components have support in either Ω^+ or Ω^-: say v_k is supported in Ω^+. This means that $\mathbf{v} = (v_1, \dots, v_k)$ is a minimizer in L^∞ such that $\lambda_k(\mathbf{v}; \beta) > \lambda_{k-1}(\mathbf{v}; \beta)$, and by the previous arguments we know that up to a negligible set $\{\mathbf{v} \neq 0\} \subset \{v_k \neq 0\}$, thus $|\Omega^-| = 0$: this is a contradiction.

□

6.4.3 Discussion about the properties of open minimizers

Here we make a few observations on the properties of minimizing open sets, provided we know such sets exist.

Proposition 6.18. *Let Ω be an open minimizer of $\lambda_k(\Omega; \beta)$ among opens sets of measure m, for $k \geq 2$, with eigenfunction $u_1, \dots, u_k$. Suppose $\lambda_{k-l}(\Omega; \beta) < \lambda_{k-l+1}(\Omega; \beta) = \lambda_k(\Omega; \beta)$. Then we know that*

$$\cap_{i=k-l+1}^{k} \left(u_i^{-1}(\{0\}) \cap \Omega \right) = \emptyset.$$

In particular, for $k = 3$ and $n = 2$, Ω is not simply connected.

Proof. By contradiction, consider $x \in \cap_{i=k-l+1}^{k} u_i^{-1}(\{0\})$, and $\mathbf{u}_r = (u_1, \dots, u_k)1_{\mathbb{R}^n \setminus \mathbb{B}_{x,r}}$. For a small enough r, $\mathbf{u}_r$ is admissible and, with the same estimate as in Lemma 6.14, there is a (L^2-normalized) eigenfunction u_α associated to $\lambda_k(\Omega; \beta)$ such that

$$\int_{\mathbb{B}_{x,r}} |\nabla u_\alpha|^2 \mathrm{d}\mathscr{L}^n + \gamma |\mathbb{B}_{x,r}| \leq 2\beta \int_{\partial \mathbb{B}_{x,r}} u_\alpha^2 \mathrm{d}\mathscr{H}^{n-1}.$$

This implies that for any small enough $r > 0$,

$$⨍_{\partial \mathbb{B}_{x,r}} u_\alpha^2 \mathrm{d}\mathscr{H}^{n-1} \geq \frac{r\gamma}{2n\beta}.$$

However, if x is at the intersection of every nodal line associated to eigenfunctions of $\lambda_k(\Omega; \beta)$, and since these eigenfunctions are $\mathcal{C}^1$, there is a constant $C > 0$ such that for all α, $|u_\alpha(y)| \leq C|x - y|$, thus

$$⨍_{\partial \mathbb{B}_{x,r}} u_\alpha^2 \mathrm{d}\mathscr{H}^{n-1} \leq C^2 r^2,$$

which is a contradiction.

Let us now suppose that $n = 2$, $k = 3$, and that Ω is simply connected. Since any eigenfunction related to $\lambda_3(\Omega;\beta)$ has a non-empty nodal set, we know that

$$\lambda_1(\Omega;\beta) < \lambda_2(\Omega;\beta) = \lambda_3(\Omega;\beta).$$

Let u_1, u, v be the associated eigenfunctions. Every non-trivial linear combination of u and v is an eigenfunction associated to $\lambda_2(\Omega;\beta)$ so it has a non-empty nodal set and no more than two nodal domains, thus, with the simple connectedness of Ω, its nodal set is connected (either a circle or a curve) and the eigenfunction changes sign at the nodal set.
Let us parametrize the eigenspace with

$$w_t(x) = \cos(t)u(x) + \sin(t)v(x).$$

We show that the nodal sets $(\{w_t = 0\})_{t\in\frac{\mathbb{R}}{\pi\mathbb{Z}}}$ are a partition of Ω and that there is a continuous open function $T : \Omega \to \frac{\mathbb{R}}{\pi\mathbb{Z}}$ such that $x \in \left\{w_{T(x)} = 0\right\}$ for all $x \in \Omega$. Indeed, the sets $(\{w_t = 0\})_{t\in\frac{\mathbb{R}}{\pi\mathbb{Z}}}$ are disjoints because u and v have no common zeroes, and for any x we may define

$$T(x) = -\arctan\left(\frac{u(x)}{v(x)}\right),$$

where $\arctan(\infty) = \frac{\pi}{2}[\pi]$ by convention. The function T is continuous, $x \in \left\{w_{T(x)} = 0\right\}$, and since eigenfunctions change sign at their nodal lines then T is open. Since Ω is simply connected T may be lifted into

$$\Omega \underset{T'}{\longrightarrow} \mathbb{R} \underset{p}{\longrightarrow} \mathbb{R}/\pi\mathbb{Z}.$$

Let I be the image of T', since T is open, then T' is too so I is an open interval. If $T(x) = T(y)$, then x and y are in the same nodal line and since these are connected we know $T'(x) = T'(y)$. In particular, if t is in I, then $t \pm \pi \notin I$; this implies that $I =]a, b[$ where $a < b$ and $b - a \leq \pi$. However every w_t has a non-empty nodal set so $\frac{\mathbb{R}}{\pi\mathbb{Z}} = T(\Omega) = p(]a, b[)$: this is a contradiction, thus Ω is not simply connected.

□

Bibliography

[112] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[113] Pedro Ricardo Simão Antunes, Pedro Freitas, and James Bernard Kennedy. Asymptotic behaviour and numerical approximation of optimal eigenvalues of the Robin Laplacian. *ESAIM Control Optim. Calc. Var.*, 19(2):438–459, 2013.

[114] Serena Guarino Lo Bianco, Domenico Angelo La Manna, and Bozhidar Velichkov. A two-phase problem with Robin conditions on the free boundary. *J. Éc. polytech. Math.*, 8:1–25, 2021.

[115] Marie-Hélène Bossel. Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimétrique de Rayleigh-Faber-Krahn. *Z. Angew. Math. Phys.*, 39(5):733–742, 1988.

[116] Dorin Bucur. Minimization of the k-th eigenvalue of the Dirichlet Laplacian. *Arch. Ration. Mech. Anal.*, 206(3):1073–1083, 2012.

[117] Dorin Bucur and Alessandro Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[118] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[119] Dorin Bucur and Alessandro Giacomini. Minimization of the k-th eigenvalue of the Robin-Laplacian. *J. Funct. Anal.*, 277(3):643–687, 2019.

[120] Dorin Bucur, Alessandro Giacomini, and Mickaël Nahon. Degenerate free discontinuity problems and spectral inequalities in quantitative form. *Arch. Ration. Mech. Anal.*, 242(1):453–483, 2021.

[121] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[122] Dorin Bucur, Dario Mazzoleni, Aldo Pratelli, and Bozhidar Velichkov. Lipschitz regularity of the eigenfunctions on optimal domains. *Arch. Ration. Mech. Anal.*, 216(1):117–151, 2015.

[123] L.A. Caffarelli and D. Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[124] Guido Cortesani and Rodica Toader. A density result in SBV with respect to non-isotropic energies. *Nonlinear Anal.*, 38(5, Ser. B: Real World Appl.):585–604, 1999.

[125] Daniel Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.

[126] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

[127] Alexandre Girouard and Richard S. Laugesen. Robin spectrum: two disks maximize the third eigenvalue, 2019.

[128] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[129] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[130] James Kennedy. An isoperimetric inequality for the second eigenvalue of the Laplacian with Robin boundary conditions. *Proc. Amer. Math. Soc.*, 137(2):627–633, 2009.

[131] Dennis Kriventsov. A free boundary problem related to thermal insulation: flat implies smooth. *Calc. Var. Partial Differential Equations*, 58(2):Paper No. 78, 83, 2019.

[132] Dennis Kriventsov and Fanghua Lin. Regularity for shape optimizers: the nondegenerate case. *Comm. Pure Appl. Math.*, 71(8):1535–1596, 2018.

[133] Dennis Kriventsov and Fanghua Lin. Regularity for shape optimizers: the degenerate case. *Comm. Pure Appl. Math.*, 72(8):1678–1721, 2019.

[134] Richard S. Laugesen. The Robin Laplacian—Spectral conjectures, rectangular theorems. *J. Math. Phys.*, 60(12):121507, 31, 2019.

[135] Dario Mazzoleni and Aldo Pratelli. Existence of minimizers for spectral problems. *J. Math. Pures Appl. (9)*, 100(3):433–453, 2013.

[136] Édouard Oudet. Numerical minimization of eigenmodes of a membrane with respect to the domain. *ESAIM Control Optim. Calc. Var.*, 10(3):315–330, 2004.

Chapter 7

Regularity of a free boundary problem with transmission condition

This is a joint work with Bozhidar Velichkov.

Contents

7.1 Introduction . . . 150
7.2 Uniform regularity estimates . . . 153
7.3 Convergence of minimizers . . . 156
7.4 Monotonicity formula and blow-ups . . . 160
7.5 Epiperimetric inequality at the contact set . . . 164
7.5.1 Estimate near small blow-ups . . . 166
7.5.2 Estimate away from large blow-ups . . . 168
7.5.3 Preliminary computations . . . 168
7.5.4 Erasure of small sectors and high frequencies . . . 172
7.5.5 Gradient descent on low-frequency solutions near large blow-up . . . 175
7.6 Nondegeneracy and regularity of the contact set . . . 178
7.6.1 Nondegeneracy lemma . . . 179
7.6.2 Regularity of the contact set . . . 180
7.7 Study of an example . . . 183

7.1 Introduction

Given an open set $D \subset \mathbb{R}^n$, a non-negative function $u \in H^1(D)$ and a measurable set E such that $\mathrm{Per}(E; \{u > t\}) < \infty$ for every $t > 0$, we define the energy

$$J_D(u, E) := \int_D |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E \cap \{u>0\} \cap D} u \mathrm{d}\mathscr{H}^{n-1}$$

where the trace of u on $\partial^* E \cap \{u > 0\}$ is well-defined by the rectifiability of $\partial^* E \cap \{u > 0\}$. Our goal in this paper is to understand the properties of a local minimizer of this functional. This study is partially motivated by the interaction between two different kind of boundary problems similar to what is mentioned in [138] in a free discontinuity setting, and more particularly about the minimizers of a functional of the form $u \in SBV(D, \mathbb{R}_+) \mapsto \int_D |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{J_u \cap D} (u^+ + u^-) \, \mathrm{d}\mathscr{H}^{n-1} + k|\{u > 0\} \cap D|$ for some constant $k > 0$, that comes from a variational elliptic equation with Robin boundary condition and an obstacle from below.

Our case is closer to a free boundary version of the transmission problem studied in [139], with the addition of a natural obstacle phenomena. It is also similar to the free boundary problem [140] wich deals instead an energy of the form $\int_D |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E \cap \{u>0\} \cap D} u^2 \mathrm{d}\mathscr{H}^{n-1}$, and the regularity

theory is perfectly analog as long as $u > 0$. This still applies to our functional on the parts where u is larger than a constant, but in their case the solution u is guaranteed to be positive everywhere.

We establish a $\mathcal{C}^1$ regularity up to the boundary of the contact set $\{u = 0\}$ in two dimensions, through an epiperimetric inequality of a monotonicity formula of Weiss type. This epiperimetric inequality is obtained by careful construction of competitors away from the critical blow-up, and by a formulation in terms of gradient descent near the critical blow-up.

Definition 7.1. For any open set D, we say (u, E) is a minimizer of J in D if and only if

- $u \in H^1_{\text{loc}}(D, \mathbb{R}_+) \cap \mathcal{C}^0(D)$ and for all $t > 0$, and any open subset $\omega \Subset D$, we have
$$\text{Per}(E|\{u > t\} \cap \omega) < \infty.$$
- $J_\omega(u, E) \leq J_\omega(v, F)$ for every (v, F) verifying the first point and such that $\{v \neq u\}, E\Delta F \Subset \omega \Subset D$ for some open ω.

In this case we will write $(u, E) \in \mathscr{M}(D)$. For any measurable set A, we will write $(u, E) \in \mathscr{M}(A)$ as soon as there is a neighbourhood D of A and $(\tilde{u}, \tilde{E}) \in \mathscr{M}(D)$ such that $(\tilde{u}_{|A}, \tilde{E} \cap A) = (u, E)$. Similarly, we write $(u, E) \in \mathscr{M}_m(A)$ when $u \geq m$ and (u, E) is a minimizer when compared with any competitor (v, F) such that $v \geq m$.

We will see later how, for any sufficiently smooth domain D and boundary data, there exists an associated minimizer. Let us summarize that in particular

- For any $x \in \{u > 0\} \cap \partial E$, ∂E may be decomposed in a neighbourhood of x as the union of a $\mathcal{C}^\infty$ hypersurface R and a singular set S of codimension at least 8, with $u \in \mathcal{C}^\infty(R)$ as well as in a neighbourhood of R in E and in E^c. At R, we have the so-called transmission condition
$$\nu_E \cdot \left(\nabla u_{|E^c} - \nabla u_{|E}\right) = \frac{1}{2}$$
on R, where ν_E designates the outward normal vector on $\partial^* E$ relative to E, and $\nabla u_{|E}$ (resp $\nabla u_{|E^c}$) is the trace of the gradient of $u_{|E}$ (resp $u_{|E^c}$) on $\partial^* E$.
- For any $x \in \{u = 0\}$ such that $\partial^* E \cap \{u > 0\}$ is bounded away from x, then $(1_E - 1_{E^c})$ is a harmonic function and as a consequence, ∂E is a zero set of an harmonic function near x with u smooth on both sides of the boundary, with moreover the weak transmission condition
$$\nu_E \cdot \left(\nabla u_{|E^c} - \nabla u_{|E}\right) \leq \frac{1}{2}$$

So there are two "competing" regularity theory for the free boundary; either a free boundary transmission problem, which is very similar to a minimal surface, or a zero of harmonic function that may have multiple branching points even in two dimensions. Our subject of interest in this paper is to understand how these two regularity join up at the junction between $\partial^* E \cap \{u > 0\}$ and $\{u = 0\}$. Here is the main result of the paper.

Theorem 7.2. *Let (u, E) be a minimizer of J in a domain $D \subset \mathbb{R}^2$, and let $\Omega \Subset D$ be a subdomain, then Per$(E|\Omega) < \infty$, $\partial^* E \cap \{u > 0\} \cap \Omega$ is locally $\mathcal{C}^\infty$ and $\{u = 0\} \cap \Omega$ is included in a finite union of $\mathcal{C}^{1,\log}$ curves.*

We actually we prove a quantitative flatness estimate for the **full** boundary $(\partial^* E \cap \{u > 0\}) \cup \{u = 0\}$ (that we may simply write $\partial^* E$ with the knowledge that E has locally finite perimeter) around any point of the contact set $\{u = 0\}$. This means more precisely that there are constants $C > 0$, $\overline{r} \in \left(0, \frac{1}{2}\text{dist}(B, \partial\Omega)\right)$ depending only on B, Ω, and a universal exponent $\alpha > 0$ such that for any $x \in \{u = 0\} \cap B$ and any $r \in (0, \overline{r})$,
$$\partial^* E \cap B_{x,r} \subset \left\{y \in \Omega : |(y - x) \cdot \nu_x| \leq Cr \log\left(\frac{1}{r}\right)^{-\alpha}\right\}$$

where ν_x is the normal vector of ∂E at x.
Here is the plan of the proof. Some sections work in any dimension, that we denote by the letter $n \geq 2$.

- We begin by establishing basic regularity estimates on minimizers (u, E), namely that if (u, E) is a minimizer of J in D and $B \Subset D$, then

$$\|u\|_{\mathcal{C}^{\frac{1}{n+1}}(B)} \leq C(B, D) J_D(u, E)$$
$$J_B(u, E) \leq C(B, D) \int_{\partial D} u \mathrm{d}\mathscr{H}^{n-1}$$

  Moreover, if (u, E) is a minimizer of J in the centered ball B_4, we establish the existence of a constant $C_n > 0$ such that

$$\int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} \leq C_n(1 + u(0))$$

  From this we get in particular the compactness of punctually bounded sequences of minimizers.

- We prove several properties of compactness of uniformly bounded minimizers, that will also give us the existence of minimizers with prescribed boundary data.

- We establish the monotonicity of the Weiss-type functionnal

$$r \to W\left(\frac{u(r\cdot)}{r}, \frac{E}{r}\right),$$

  where $W(u, E) = J_{B_1}(u, E) - \int_{\partial B_1} u^2$. All up until this point works in any dimension, and from now on we work in two dimension. In two dimensions this monotonicity formula allows us to identify the blow-ups of minimizer (u, E) around a point in $\{u = 0\}$ as being either of the form

$$u(x) = |x \cdot e|, \ E = \{x : x \cdot e > 0\},$$

  for some $e \in \mathbb{R}^2 \setminus \{0\}$ such that $|e| \leq \frac{1}{4}$, or of the form $u \equiv 0$, and E being whatever.

- We prove a refinement of the monotonicity formula; for any (u, E) minimizer of J in B_2 such that $u(0) = 0$, there are some universal constants $\gamma, \vartheta \in]0, 1[$ such that

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1} W(u_r, E_r) \geq 2\left(\gamma \wedge W(u, E)^{\vartheta}\right) W(u, E)$$

  This is obtained by building an explicit competitor (v, F) in B_1 such that

$$W(v, F) \leq \left(1 - \gamma \wedge W(u^h, E^h)^{\vartheta}\right) W(u^h, E^h),$$

  where (u^h, E^h) is the 1-homogenenous extension of $(u, E)_{|\partial B_1}$ on B_1.

- We deduce quantitative estimates for the speed of convergence to the blow-up, and thus obtain $\mathcal{C}^{1,\log}$ regularity, as well as a "non-degeneracy" property that fully characterizes the behaviour of minimizers near small blow-ups.

Notations:

- For two quantities a, b, we will say that $a \lesssim b$ when $a \leq Cb$ for some $C > 0$ that only depends on the dimension n.

7.2 Uniform regularity estimates

By uniform estimate, we mean in particular estimates that will be true in particular for all minimizers $(u, E) \in \mathscr{M}(B_1)$ that vanishes at the origin, which is the case we will focus on later.

We begin with a few non-uniform estimate. The presence of the parameter m is merely here for the proof of existence later on.

Lemma 7.3. *Let $m \geq 0$ and $(u, E) \in \mathscr{M}_m(B_1)$ with $u \in H^1(B_1)$, then*

$$\|u\|_{\mathcal{C}^{0,\frac{1}{n+1}}(B_{1/2})} + J_{B_{1/2}}(u, E) \leq C_n \int_{\partial B_1} u \mathrm{d}\mathscr{L}^n$$

Proof. The proof is divided in several steps.

- u is subharmonic. Indeed, let $\varphi \in \mathcal{C}_c^\infty(B_1, \mathbb{R}_+)$, and $t > 0$, we compare (u, E) to $((u - t\varphi) \vee m, E)$. Notice this is the only place in the proof where we use a competitor that is lower than u. The comparison gives

$$\int_{\{u \leq t\varphi + m\}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\{u > t\varphi + m\}} \left(2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2\right) \mathrm{d}\mathscr{L}^n + \int_{\partial^* E} (u \wedge t\varphi) \mathrm{d}\mathscr{H}^{n-1} \leq 0$$

  Dividing by t and taking $t \to 0^+$, we obtain

$$\int_{\{u > m\}} \nabla u \cdot \nabla\varphi \mathrm{d}\mathscr{L}^n \leq 0$$

  And since $1_{\{u=m\}} \nabla u = 0$, this proves the subharmonicity of u.

- We then establish the L^∞ bound. This is direct by subharmonicity of u; for any $z \in B_1$,

$$u(z) \leq \fint_{\partial B_1} \frac{1 - |z|^2}{|x - z|^n} u(x) \mathrm{d}\mathscr{H}^{n-1} \leq \frac{2}{(1 - |z|)^{n-1}} \fint_{\partial B_1} u \mathrm{d}\mathscr{L}^n.$$

- Here we establish the Hölder bound, and the bound on $J_{B_{1/2}}(u, E)$ is obtained as a byproduct.

  Let $x \in B_{1/2}$, $r > 0$ that will be chosen arbitrarily small later, and $R > 0$ such that $0 < 2r < R < \frac{1}{4}$. Let $M := \|u\|_{L^\infty(B_{3/4})} (< \infty)$ and h be the function that coincides with u on $B_1 \setminus B_{x,R}$ and with the harmonic extension of $u_{|\partial B_{x,R}}$ on $B_{x,R}$. Then the minimality of (u, E) compared to $(h, E \setminus B_{x,r})$ gives

$$\int_{B_{x,R}} |\nabla(u - h)|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E \cap B_{x,R}} u \mathrm{d}\mathscr{H}^{n-1} \leq \int_{\partial B_{x,R}} u \mathrm{d}\mathscr{H}^{n-1},$$

  so, using the harmonicity of h (and the Cacciopoli inequality) in $B_{x,R}$,

$$\begin{aligned}
J_{B_{x,r}}(u, E) &\leq 2 \int_{B_{x,r}} |\nabla(u - h)|^2 \mathrm{d}\mathscr{L}^n + 2 \int_{B_{x,r}} |\nabla h|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E \cap B_{x,R}} u \mathrm{d}\mathscr{H}^{n-1} \\
&\leq 2 \int_{B_{x,R}} |\nabla(u - h)|^2 \mathrm{d}\mathscr{L}^n + 2 \left(\frac{r}{R/2}\right)^n \int_{B_{x,R/2}} |\nabla h|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial^* E \cap B_{x,R}} u \mathrm{d}\mathscr{H}^{n-1} \\
&\leq 2 \int_{\partial B_{x,R}} u \mathrm{d}\mathscr{H}^{n-1} + 2^{n+6} \frac{r^n}{R^{n+2}} \int_{B_{x,R}} h^2 \mathrm{d}\mathscr{L}^n \\
&\leq 2n\omega_n M R^{n-1} + 2^{n+6} \omega_n M^2 \frac{r^n}{R^2}
\end{aligned}$$

  Then letting $R := r^{\frac{n}{n+1}}$ which is valid for any small enough r we obtain

$$J_{B_{x,r}}(u, E) \leq C_n M^{\frac{2n}{n+1}} r^{\frac{n-1}{n+1} n}$$

In particular, this implies that for any small enough r and for any $x \in B_{1/2}$, $⨍_{B_{x,r}} |\nabla u|^2 \leq C_n M^{\frac{2n}{n+1}} r^{-\frac{2n}{n+1}}$, which by classical embedding results (see for instance [141, cor 3.2]) implies the Hölder bound.

□

We then establish the following uniform estimate (we call it uniform because in the case that interests us the most in the following sections - when $u(0) = 0$ - then the bound on $\int_{\partial B_1} u$ depends only on the dimension).

Proposition 7.4. *Let $m \geq 0$ and $(u, E) \in \mathscr{M}_m(B_4)$, then*

$$\int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} \leq C_n(1 + u(0))$$

Remark 7.5. *Notice how in this proof (u, E) is only compared with competitor larger than u.*

Proof. In the following, we let $M = 1 + 8u(0)$ and C_n designates constants that depend only on n and that may change from line to line. For every $t \in [0, 4]$, define

$$\omega_t := \{x \in B_1 : u(x)/M + |x| < t\}$$

Notice a few consequences of the definition of ω_t:

$$\omega_t \subset B_t,\ \omega_{1/8} \neq \emptyset,\ \omega_2 \supset B_1 \cap \{u < M\}$$

The proof is in two steps; we give a bound from below of $|\omega_1|$ that depends only on (n, M), and then use a capacity argument to prove that $\int_{\partial B_1} u$ is uniformly bounded as a result.
We let $t_0 \in [0, 1)$ be the smallest real number such that $\omega_{t_0} \neq \emptyset$; we know it is smaller than $1/8$ by our choice of M. Then we let

$$f(t) = \int_{t_0}^{t} \mathrm{Per}(\omega_\tau) d\tau \left(= \int_{\omega_t} \left| \nabla \left(\frac{u}{M} + |x| \right) \right| \mathrm{d}\mathscr{L}^n \right),\ g(t) = \int_{\omega_t} \left| \nabla \left(\frac{u}{M} + |x| \right) \right|^2 \mathrm{d}\mathscr{L}^n$$

The strategy is to bound f from below and g from above with the following estimate. The comparison between (u, E) and $(u \vee \left[M(t - |x|)\right], E \setminus \omega_t)$ gives:

$$\int_{\omega_t} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \leq M^2 |\omega_t| + \int_{\partial^* \omega_t} u \mathrm{d}\mathscr{H}^{n-1}$$

Which may be simplified into

$$\int_{\omega_t} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \leq M^2 |\omega_t| + Mt\mathrm{Per}(\omega_t)$$

So with triangular and Cauchy-Schwarz inequality we get

$$f(t) \leq 2|\omega_t| + (M^{-1} t |\omega_t| \mathrm{Per}(\omega_t))^{\frac{1}{2}} \tag{7.1}$$

Using the isoperimetric inequality $|\omega|^{\frac{n-1}{n}} \leq C_n \mathrm{Per}(\omega)$ and the estimate $|\omega_t| \leq |B_t| \leq C_n t^n$, we obtain the following estimates:

$$f(t) \leq C_n t^{\frac{1}{2}} f'(t)^{\frac{2n-1}{2n-2}}.$$

Integrating from t_0 to $t \in (t_0, 4)$, we get the lower bound

$$f(t)^{\frac{1}{2n-1}} \geq c_n \left(t^{\frac{n}{2n-1}} - t_0^{\frac{n}{2n-1}} \right)$$

Let us now estimate g from above. We let $G(t) = \int_{t_0}^t g$, and we use that

$$g(t) \leq \int_{\omega_t} \left| \nabla \left(\frac{u}{M} + |x| \right) \right|^2 \mathrm{d}\mathscr{L}^n \leq 4|\omega_t| + 2M^{-1} t \mathrm{Per}(\omega_t)$$

Now,

$$G(t) \leq \int_{t_0}^t \left(4|\omega_\tau| + 2M^{-1}\tau \mathrm{Per}(\omega_\tau) \right) d\tau \leq 4t|\omega_t| + 2tM^{-1} f(t)$$

Now, for any $t > 2t_0$, we have $f(t) \geq c_n t^n \geq c_n |\omega_t|$ so $|\omega_t| \leq C_n g(t)$. As a consequence,

$$G(t) \leq C_n t |\omega_t|^{\frac{1}{2}} g(t)^{\frac{1}{2}}$$

Reminding that $G' = g$, we integrate this from t to $2t$ for some $t \in (2t_0, 2)$;

$$\frac{1}{G(t)} \geq \frac{1}{G(t)} - \frac{1}{G(2t)} \geq \frac{C_n}{t|\omega_{2t}|}$$

And so for any $t \in (2t_0, 1)$,

$$g(t) \leq \frac{1}{t} \int_t^{2t} g \leq C_n |\omega_{4t}|,$$

so using the previous estimate on f we get

$$c_n t^n \leq f(t) \leq |\omega_t|^{\frac{1}{2}} g(t)^{\frac{1}{2}} \leq C_n |\omega_{4t}|.$$

From this we obtain two important estimates; with $t = 1/4$ and $t = 1$ we get

$$|B_1 \cap \{u < M\}| \geq c_n, \ \int_{B_1 \cap \{u<M\}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \leq C_n M^2$$

Now we use the inequality of [137, lem. 3.2] that states that for any $v \in H^1(B_1)$, denoting Hv the harmonic extension of $v_{|\partial B_1}$ on B_1 we have

$$|\{v = 0\}| \left(\int_{\partial B_1} |v| \mathrm{d}\mathscr{H}^{n-1} \right)^2 \leq C_n \int_{B_1} |\nabla(v - Hv)|^2 \mathrm{d}\mathscr{L}^n \tag{7.2}$$

We will apply this to $(u - M)_+$. The optimality condition $J(u, E) \leq J(Hu, E \setminus B_1)$ gives

$$\int_{B_1} |\nabla(u - Hu)|^2 \mathrm{d}\mathscr{L}^n \leq \int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1}$$

Now, separate $u = (u - M)_+ + u \wedge M$ and $Hu = H(u - M)_+ + H(u \wedge M)$. The inequality of Alt-Caffarelli inequality (7.2) and the estimate $|B_1 \cap \{u < M\}| > c_n$ give

$$\left(\int_{\partial B_1} (u - M)_+ \mathrm{d}\mathscr{H}^{n-1} \right)^2 \leq C_n \int_{B_1} |\nabla((u - M)_+ - H(u - M)_+|^2 \mathrm{d}\mathscr{L}^n$$

Thus for a certain constant $C_n > 0$ (different than above):

$$\begin{aligned}
\left(\fint_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1}\right)^2 &\leq 2M^2 + 2\left(\int_{\partial B_1} (u-M)_+ \mathrm{d}\mathscr{H}^{n-1}\right)^2 \\
&\leq 2M^2 + C_n \int_{B_1} |\nabla((u-M)_+ - H(u-M)_+|^2 \mathrm{d}\mathscr{L}^n \\
&\leq 2M^2 + 2C_n \int_{B_1} |\nabla(u - Hu)|^2 \mathrm{d}\mathscr{L}^n + 2C_n \int_{B_1} |\nabla(u\wedge M - H(u\wedge M))|^2 \mathrm{d}\mathscr{L}^n \\
&\leq 2M^2 + 2C_n \int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} + 2C_n \int_{B_1} (|\nabla(u\wedge 1)|^2 - |\nabla H(u\wedge 1)|^2) \mathrm{d}\mathscr{L}^n \\
&\leq 2M^2 + 2C_n \int_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1} + 2C_n \int_{B_1\cap\{u<M\}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \\
&\leq A_n M^2 + B_n \fint_{\partial B_1} u \mathrm{d}\mathscr{H}^{n-1}
\end{aligned}$$

The inequality $X^2 \leq A_n M^2 + B_n X$ implies $X \leq C_n M$ for some constant $C_n > 0$; this proves the result.

□

Corollary 7.6. *Let $(u, E) \in \mathscr{M}(B_1)$, $r \in (0,1)$, then there is a constant $C_n(r) > 0$ such that*

$$\|u\|_{\mathcal{C}^{0,\frac{1}{n+1}}(B_r)} + J_{B_r}(u, E) \leq C_n(r)\,(1 + u(0))$$

This is obtained by combining the two previous results.

7.3 Convergence of minimizers

We establish a compactness argument for minimizers. We first prove the following density estimate.

Proposition 7.7. *Let $(u, E) \in \mathscr{M}_m(B_1)$ for some $m \geq 0$ such that $|B_{1/2} \cap E| > 0$. Then there is a constant $c_n > 0$ such that*

$$|B_1 \cap E| \geq c_n \left(\frac{\inf_{B_1}(u)}{\sup_{B_1}(u)}\right)^n$$

Remark 7.8. *The same conclusion holds for E^c by symmetry of the problem.*

Proof. Consider $f(r) = |B_r \cap E|$, then by the isoperimetric inequality applied to $B_r \cap E$ and with the comparison $J(u, E) \leq J(u, E \setminus B_r)$,

$$f(r)^{\frac{n-1}{n}} \leq C_n \mathrm{Per}(B_r \cap E) \leq C_n \frac{\sup_{B_1}(u)}{\inf_{B_1}(u)} f'(r)$$

If $f > 0$ on $]1/2, 1[$ we may integrate $f^{-\frac{n-1}{n}} f'$ from $1/2$ to 1 to get the result. □

Proposition 7.9. *Let (u_i, E_i) be a sequence of $\mathscr{M}_{m_i}(B_1)$, such that*

$$\limsup_i u_i(0) < \infty,\ m_i \to m (\geq 0).$$

Then either (u_i) converges to 0 strongly in $H^1(B_1) \cap \mathcal{C}^0(B_1)$, or there is a subsequence $(u_{\phi(i)}, E_{\phi(i)})$ such that $u_{\phi(i)}$ converges in $H^1_{loc}(B_1) \cap \mathcal{C}^0_{loc}(B_1)$ to a non-zero function u, and $E_{\phi(i)}$ converges in $L^1_{loc}(\{u > 0\})$ and in the local Hausdorff sense to a set E in $\{u > 0\}$, such that $(u, E) \in \mathscr{M}_m(B_1)$ and for almost every $r \in (0,1)$:

$$J_{B_r}(u, E) = \lim_{i\to\infty} J_{B_r}(u_i, E_i)$$

Proof. Due to the bound on $u_i(0)$, u_i is bounded in $\mathcal{C}^{\frac{1}{n+1}}_{\text{loc}}(B_1) \cap H^1_{\text{loc}}(B_1)$, so up to a subsequence we may suppose that (u_i) converges to a function $u \in H^1_{\text{loc}}(B_1)$ in the $\mathcal{C}^0_{\text{loc}}(B_1)$, $L^2_{\text{loc}}(B_1)$ sense, and weakly in $H^1_{\text{loc}}(B_1)$. We will straight-away suppose that u is non-zero.

For any $t > 0$ and for any large enough i that depends on t, $\|u_i - u\|_{L^\infty(B_1)} \leq \frac{t}{2}$, meaning $u_i \geq \frac{t}{2}$ on the open set $\{u > t\}$ (that is non-empty for some any small enough t). Thus

$$\text{Per}(E_i; \{u > t\}) \leq \frac{2}{t} \int_{B_1 \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1} \leq \frac{C_n}{t}$$

Thus we may suppose up to an extraction that E_i converges in $L^1_{\text{loc}}(\{u > t\})$ to a certain $E^t \subset \{u > t\}$. Taking $E = \cup_{t>0} E^t$, we know that E_i converges in $L^1_{\text{loc}}(\{u > 0\})$ to E. As a consequence, using the uniform convergence of (u_i), we have $u_i 1_{E_i} \to u 1_E$ in $L^1_{\text{loc}}(B_1)$, with

$$\int_{B_r \cap \partial^* E} u \mathrm{d}\mathscr{H}^{n-1} \leq \liminf_{i \to \infty} \int_{B_r \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1},$$

for every $r \in]0, 1[$. The density estimate on the sets (E_i) yield that this convergence is in the local Hausdorff sense in $\{u > 0\}$. Let us now show that the convergence $u_i \to u$ is strong in H^1, and $J(u_i, E_i | B_r) \to J(u, E | B_r)$; since we already have a weak convergence and semicontinuity we merely need to show that

$$J(u, E | B_r) \geq \limsup_{i \to \infty} J(u_i, E_i | B_r)$$

for almost every r. For this we use the minimality of each u_i. Let $0 < r < R < 1$, $\eta \in \mathcal{C}^\infty(B_1)$ such that $\eta = 1$ on $B_1 \setminus B_R$, $\eta = 0$ on B_r.

- Let us first check that the conclusion holds when $\inf_{B_1}(u) =: \epsilon > 0$. Notice that, for any large enough i, $u_i \geq \frac{\epsilon}{2}$ on B_R, so we have the uniform perimeter estimates

  $$\text{Per}(E_i | B_{x,r}) \leq \frac{C_n}{\epsilon} r^{n-1}$$

  by comparing (u, E) and $(u, E \setminus B_{x,r})$. Moreover, we have $E_i \to E$ in L^1_{loc}, so for almost every $R > 0$, $\mathscr{H}^{n-1}(E \Delta E_i \cap \partial B_R) \to 0$. We now make the comparison

  $$J(u_i, E_i | B_R) \leq J(\hat{u_i}, \hat{E}_i | B_R),$$

  where

  $$\hat{u_i} = \eta u_i + (1 - \eta) u, \;\; \hat{E}_i = (E_i \setminus B_R) \cup (E \cap B_R),$$

  where $0 < r < R < 1$, r is a point of continuity of $\rho \mapsto \text{Per}(E | B_\rho)$, and R is chosen such that $\mathscr{H}^{n-1}(E \Delta E_i \cap \partial B_R) \to 0$. This comparison gives

  $$\begin{aligned}
  0 &\leq \int_{B_R} \left(|\nabla \hat{u_i}|^2 - |\nabla u_i|^2 \right) \mathrm{d}\mathscr{L}^n + \int_{B_\tau \cap \partial^* \hat{E}_i} \hat{u_i} \mathrm{d}\mathscr{H}^{n-1} - \int_{B_\tau \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1} \\
  &= \int_{B_R} \left((\eta^2 - 1) |\nabla u_i|^2 + (1 - \eta^2) |\nabla u|^2 + 2\eta(1 - \eta) \nabla u_i \cdot \nabla u \right) \mathrm{d}\mathscr{L}^n + o_{i \to \infty}(1) \\
  &+ \int_{B_r \cap \partial^* E} u \mathrm{d}\mathscr{H}^{n-1} + \int_{B_R \setminus B_r \cap \partial^* E} \hat{u_i} \mathrm{d}\mathscr{H}^{n-1} + \int_{E \Delta E_i \cap \partial B_R} u_i \mathrm{d}\mathscr{H}^{n-1} - \int_{B_R \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1} \\
  &= \int_{B_R} (1 - \eta^2) \left(|\nabla u|^2 - |\nabla u_i|^2 \right) \mathrm{d}\mathscr{L}^n + o_{i \to \infty}(1) \\
  &+ \int_{B_r \cap \partial^* E} u \mathrm{d}\mathscr{H}^{n-1} + \int_{B_R \setminus B_r \cap \partial^* E} u \mathrm{d}\mathscr{H}^{n-1} + o_{i \to \infty}(1) - \int_{B_R \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1} \\
  &= \int_{B_R} (1 - \eta^2) \left(|\nabla u|^2 - |\nabla u_i|^2 \right) \mathrm{d}\mathscr{L}^n + \int_{B_R \cap \partial^* E} u \mathrm{d}\mathscr{H}^{n-1} - \int_{B_R \cap \partial^* E_i} u_i \mathrm{d}\mathscr{H}^{n-1} + o_{i \to \infty}(1)
  \end{aligned}$$

where we used dominated convergence (since $\mathrm{Per}(E|B_R)$ is bounded on $\int_{B_R\setminus B_r\cap\partial^* E}\hat{u}_i$, as well as the fact that $\int_{E\Delta E_i\cap\partial B_R}u_i\to 0$). Thus, taking the limit $i\to\infty$, and then $R\to r$ (using the fact that r is a continuity point of $r\mapsto\mathrm{Per}(E|B_r)$), we get that for almost every $r\in]0,1[$,

$$\liminf_{i\to\infty}\left(J(u,E|B_r)-J(u_i,E_i|B_r)\right)\geq 0$$

- Let us now handle the case where $\inf(u)$ may be zero.

  We make the comparison

  $$J(u_i,E_i|B_R)\leq J(\tilde{u_i},\tilde{E}_i|B_R),$$

  where

  $$\tilde{u_i}=\eta u_i+(1-\eta)v_i,\ \ v_i=u\wedge 2u_i,\ \ \tilde{E}_i=(E_i\setminus B_r)\cup(E\cap B_r)$$

  Here r,R are chosen such that $s\mapsto\int_{B_s\cap\partial^* E}u$ is continuous at r and $\int_{\partial B_r\cap E\Delta E_i}u_i\to 0$. Let us make a few preliminary estimates: let ρ be a smooth function that takes the value 1 in B_r, 0 outside B_R, and $t>0$, then the comparison between (u_i,E_i) and $((u_i-t\rho)_+,E_i)$ gives:

  $$\begin{aligned}\int_{B_r\cap\{u_i<t\}}|\nabla u_i|^2\mathrm{d}\mathscr{L}^n+\int_{\partial^* E\cap B_r\cap\{u_i<t\}}u_i\mathrm{d}\mathscr{H}^{n-1}&\leq\int_{\{u_i<t\rho\}}|\nabla u_i|^2\mathrm{d}\mathscr{L}^n+\int_{\partial^* E\cap\{u_i<t\rho\}}u_i\mathrm{d}\mathscr{H}^{n-1}\\&\leq J(u_i,E_i|B_R)-J((u_i-t\rho)_+,E_i|B_R)\\&+\int_{\{u_i>t\rho\}}\left(-2t\nabla\rho\cdot\nabla u_i+t^2|\nabla\rho|^2\right)\mathrm{d}\mathscr{L}^n\\&\leq C_\rho\sqrt{J(u_i,E_i|B_R)}t+\tilde{C}_\rho t^2\\&\leq C_\rho t,\end{aligned}$$

  where the last inequality is only true for $t<1$. Notice that as a consequence, since $\{u>2u_i\}\subset\{u_i<\|u-u_i\|_{L^\infty}\}$, then $v_i\to u$ in L^2_{loc} and H^1_{loc}; indeed,

  $$\begin{aligned}\int_{B_r}|\nabla(u-v_i)|^2\mathrm{d}\mathscr{L}^n&=\int_{B_r}|\nabla(u-2u_i)|^2 1_{\{u>2u_i\}}\mathrm{d}\mathscr{L}^n\\&\leq 2\int_{B_r}|\nabla u|^2 1_{\{u>2u_i\}}\mathrm{d}\mathscr{L}^n+8C_\rho\|u-u_i\|_{L^\infty(B_r)}\underset{i\to\infty}{\longrightarrow}0\end{aligned}$$

  Now, the comparison between (u_i,E_i) and $(\tilde{u_i},\tilde{E}_i)$ gives

  $$\int_{B_R}\left(|\nabla\tilde{u_i}|^2-|\nabla u_i|^2\right)\mathrm{d}\mathscr{L}^n+\int_{B_R\cap\partial^*\tilde{E}_i}\tilde{u_i}\mathrm{d}\mathscr{H}^{n-1}-\int_{B_R\cap\partial^* E_i}u_i\mathrm{d}\mathscr{H}^{n-1}\geq 0$$

  The first term is handled as previously; since $v_i\to u$ in H^1_{loc} we get

  $$\int_{B_R}\left(|\nabla\tilde{u_i}|^2-|\nabla u_i|^2\right)\mathrm{d}\mathscr{L}^n=\int_{B_R}\left(1-\eta^2\right)\left(|\nabla u|^2-|\nabla u_i|^2\right)\mathrm{d}\mathscr{L}^n+o_{i\to 0}(1)$$

  Now, for the perimeter term, it may be decomposed into

  $$\int_{B_R\setminus B_r\cap\partial^* E_i}\tilde{u_i}\mathrm{d}\mathscr{H}^{n-1}+\int_{\partial B_r\cap E\Delta E_i}v_i\mathrm{d}\mathscr{H}^{n-1}+\int_{B_r\cap\partial^* E}v_i\mathrm{d}\mathscr{H}^{n-1}-\int_{B_R\cap\partial^* E_i}u_i\mathrm{d}\mathscr{H}^{n-1}$$

  The most problematic term is the first one that we treat separately. Notice first that

  - By our choice of r, $\int_{\partial B_r\cap E\Delta E_i}v_i\mathrm{d}\mathscr{H}^{n-1}\leq 2\int_{\partial B_r\cap E\Delta E_i}u_i\to 0$.
  - By dominated convergence (since $v_i\leq u$), $\int_{B_r\cap\partial^* E}v_i\mathrm{d}\mathscr{H}^{n-1}\to\int_{B_r\cap\partial^* E}u\mathrm{d}\mathscr{H}^{n-1}$.

- Finally, let $\epsilon > 0$ that will be taken arbitrarily small. We suppose i is large enough such that $\|u - u_i\|_{L^\infty(B_R)} \leq \epsilon$.

$$\begin{aligned}\int_{B_R \setminus B_r \cap \partial^* E_i} \tilde{u}_i \mathrm{d}\mathscr{H}^{n-1} &\leq \int_{B_R \setminus B_r \cap \partial^* E_i \cap \{u_i > \epsilon\}} \tilde{u}_i \mathrm{d}\mathscr{H}^{n-1} + \int_{B_R \setminus B_r \cap \partial^* E_i \cap \{u_i < \epsilon\}} 2u_i \mathrm{d}\mathscr{H}^{n-1} \\ &\leq \int_{B_R \setminus B_r \cap \partial^* E_i \cap \{u > \frac{\epsilon}{2}\}} \tilde{u}_i \mathrm{d}\mathscr{H}^{n-1} + C_\rho \epsilon\end{aligned}$$

Then, applying the previous case to the open set $\{u > \frac{\epsilon}{2}\}$, we know that.

$$\int_{B_R \setminus B_r \cap \partial^* E_i \cap \{u > \frac{\epsilon}{2}\}} \tilde{u}_i \mathrm{d}\mathscr{H}^{n-1} \underset{i \to \infty}{\longrightarrow} \int_{B_R \setminus B_r \cap \partial^* E \cap \{u > \frac{\epsilon}{2}\}} u \mathrm{d}\mathscr{H}^{n-1}$$

To summarize, we get

$$\begin{aligned}0 &\leq J_{B_R}(\tilde{u}_i, \tilde{E}_i) - J_{B_r}(u_i, E_i) \\ &\leq \int_{B_R} \left(1 - \eta^2\right) \left(|\nabla u|^2 - |\nabla u_i|^2\right) \mathrm{d}\mathscr{L}^n + o_{i \to \infty}(1) \\ &+ \int_{B_R \cap \partial^* E\}} u \mathrm{d}\mathscr{H}^{n-1} - \int_{B_R \cap \partial^* E_i\}} u_i \mathrm{d}\mathscr{H}^{n-1}\end{aligned}$$

Then proceeding as previously by choosing r a continuity point of $r \mapsto \int_{B_r \cap \partial^* E} u$ and $R \searrow r$ we get the result.

As a consequence, $u_i \to u$ in H^1_{loc}. It follows by the same method that (u, E) is a minimizer; let (v, F) be any admissible competitor, that coincides with (u, E) in a ball B_r, then with the same computations as earlier we get $J(v, F|B_r) \geq \lim_i J(u_i, E_i|B_r) = J(u, E|B_r)$. □

Corollary 7.10. *Let $\tilde{E} \Subset \mathbb{R}^n$ and $\tilde{u} \in H^{1/2}(\partial B_1, \mathbb{R}_+)$. Then there exists $(u, E) \in \mathscr{M}(B_1)$ such that $J_{B_1}(u, E) < \infty$ and $u_{|\partial B_1} = \tilde{u}$, $E \setminus B_1 = \tilde{E} \setminus B_1$.*

Proof. Let $m > 0$, we claim there exists $(u_m, E_m) \in \mathscr{M}_m(B_1)$ such that $u_{m|\partial B_1} = \tilde{u} \wedge m$, $E_m \setminus B_1 = \tilde{E} \setminus B_1$, and $J_{B_1}(u_m, E_m)$ is bounded independently of m (for $m \leq 1$).
Indeed, notice first that denoting h the harmonic extension of $\tilde{u}$ in B_1, then for any $m \geq 0$, $(h \wedge m, \tilde{E} \setminus B_1)$ is a valid competitor so the minimal energy is less that

$$J(h \wedge m, \tilde{E} \setminus B_1) \leq \|\tilde{u}\|^2_{H^{1/2}(\partial B_1)} + |\partial B_1|$$

Consider (u^i_m, E^i_m) a minimizing sequence (it exists since $\tilde{u} \in H^{1/2}(\partial B_1)$ so it admits some $H^1(B_1)$ extension with finite energy). Due to the previous point we may assume that $J(u^i_m, E^i_m)$ is bounded independently of m by a constant C that depends on the boundary conditions only. Then the perimeter of E^i_m in B_1 is less that $\frac{C}{m}$, so by compactness we may assume E^i_m converges strongly in L^1 to some set E_m.

Similarly, $\int_{B_1} |\nabla u^i_m|^2 \mathrm{d}\mathscr{L}^n \leq C$, so we may assume u^i_m converges strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$ to some limit u_m.

Lemma 7.11. *$(u_m, E_m) \in \mathscr{M}_m(B_1)$.*

For the proof of this we refer to [140] as the functional is essentially the same when u is bounded from below. Now that we have the existence of (u_m, E_m), then using the previous result we know it converges to some $(u, E) \in \mathscr{M}(B_1)$ that verifies the right boundary condition. □

7.4 Monotonicity formula and blow-ups

Now that the existence of minimizers is established, we shall only deal with $(u, E) \in \mathscr{M}(\Omega)$ instead of $\mathscr{M}_m(\Omega)$ for some $m \geq 0$. We begin by establishing a few classical computations about the stationnary condition of minimizers (u, E) along a flow ξ.

Lemma 7.12. *Let $(u, E) \in \mathscr{M}(D)$ for some domain D and $\zeta \in \mathcal{C}^1_c(D, \mathbb{R}^n)$; Let also $\zeta_t(x) = x + t\zeta(x)$, which is a $\mathcal{C}^1$-diffeomorphism for any small enough t. Then the comparison:*

$$J_D(u, E) \leq J_D(u \circ \zeta_t^{-1}, \zeta_t(E)), \ \forall t \in (-\epsilon, \epsilon)$$

yields, as $t \to 0$, the condition:

$$\int_D \left(\mathrm{div}(\zeta)|\nabla u|^2 - 2\nabla u \cdot D\zeta \nabla u\right) \mathrm{d}\mathscr{L}^n + \int_{\partial^* E} u \mathrm{div}_E(\zeta) \mathrm{d}\mathscr{H}^{n-1} = 0$$

This is obtained by a simple change of variable, we refer to [142, Ch. 17] for more details on the second term.

From this we deduce the two following result; the first one will help us establish a precise monotonicity formula while the second one will be useful in the classification of blow-ups.

Corollary 7.13. *Let $(u, E) \in \mathscr{M}(D)$ for some open set D, suppose that $u > 0$ in D and $\partial^* E$ is a $\mathcal{C}^2$ submanifold with mean curvature H (for the outward normal vector with respect to E). Then*

$$|\nabla u_{|E}|^2 - |\nabla u_{|E^c}|^2 = uH \text{ on } \partial^* E$$

Proof. Let ξ be a vector field defined on a neighbourhood of a point of $\partial^* E$, then the comparison between (u, E) and $(u \circ (I + t\xi)^{-1}, (I + t\xi)(E))$ as $t \to 0$ gives

$$\int_D \left(\mathrm{div}(\xi)|\nabla u|^2 - 2\nabla u D\xi \nabla u\right) \mathrm{d}\mathscr{L}^n + \int_{\partial^* E} u \mathrm{div}_E \xi \mathrm{d}\mathscr{H}^{n-1} = 0$$

Since u is harmonic on E and E^c, notice that $\mathrm{div}(\xi)|\nabla u|^2 - 2\nabla u D\xi \nabla u = \mathrm{div}\left(\xi|\nabla u|^2 - 2(\xi \cdot \nabla u)\nabla u\right)$, so

$$\int_\Gamma \left(\nu \cdot \xi \left(|\nabla u_E|^2 - |\nabla u_{E^c}|^2\right) - 2(\xi \cdot \nabla u_E)(\nu \cdot \nabla u_E) + 2(\xi \cdot \nabla u_{E^c})(\nu \cdot \nabla u_{E^c}) + \mathrm{div}_E(\xi) u\right) \mathrm{d}\mathscr{H}^{n-1} = 0$$

We choose $\xi = \psi \nu_E$, for a scalar function ψ and where $\nu_E = \nabla\left((1_{E^c} - 1_E)\mathrm{dist}(\cdot, \partial^* E)\right)$ is an extension of the outward normal vector of E that is well-defined and $\mathcal{C}^1$ in a neighbourhood of $\partial^* E$. This way $\mathrm{div}_E(\xi) = H_E \xi$ on $\partial^* E$. This simplifies into

$$\int_\Gamma \left(|\nabla u_{E^c}|^2 - |\nabla u_E|^2 + u H_E\right) \psi \mathrm{d}\mathscr{H}^{n-1} = 0$$

thus we get the result. □

In all the following, for any $(u, E) \in \mathscr{M}(D)$, and any $x \in D$, $r > 0$ such that $B(x, r) \subset D$ we write

$$(u_{x,r}, E_{x,r}) := \left(\frac{u(x + r\cdot)}{r}, \frac{E - x}{r}\right)$$

And when x is not mentioned, it is implicitely assumed to be the origin. Notice that $(u_{x,r}, E_{x,r}) \in \mathscr{M}(B_1)$. Now, for any $(u, E) \in \mathscr{M}(B_1)$, we define the normalized functional

$$W(u, E) = J_{B_1}(u, E) - \int_{\partial B_1} u^2 \mathrm{d}\mathscr{H}^{n-1}$$

We establish the monotonicity of the normalized functional, inspired by the Weiss monotonicity formula.

Proposition 7.14. *Let D be an open set and $(u,E)\in\mathscr{M}(D)$, then the function*

$$\begin{cases}\{(x,r)\in D\times\mathbb{R}_+ : B_{x,r}\Subset D\} & \to\mathbb{R}\\ (x,r) & \mapsto W(u_{x,r},E_{x,r})\end{cases}$$

is continuous and nondecreasing is its second coordinate. Moreover the absolutely continuous part of its derivative is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}W(u_{x,r},E_{x,r}) &= \frac{2}{r}\int_{\partial B_1}|z\nabla u_{x,r}-u_{x,r}|^2\mathrm{d}\mathscr{H}^{n-1}+\frac{1}{r^n}\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{B_{x,r}\cap\partial^*E}(x\cdot\nu_E)^2u\mathrm{d}\mathscr{H}^{n-1}\right)\\ &\geq \frac{n}{r}\left(W(u^h_{x,r},E^h_{x,r})-W(u_{x,r},E_{x,r})\right)(\geq 0)\end{aligned}$$

Where $(u^h_{x,r},E^h_{x,r})$ is the 1-homogeneous extension of $(u_{x,r},E_{x,r})$ in B_1. Moreover for any $(u,E)\in\mathscr{M}(\mathbb{R}^n)$, $x\in\mathbb{R}^n$, $r\mapsto W(u_{x,r},E_{x,r})$ is constant if and only if (u,E) is 1-homogeneous around x.

Proof. The continuity of $(x,r)\mapsto W(u_{x,r},E_{x,r})$ is a direct consequence of dominated convergence theorem, and in the following we will lose no generality in supposing that $x=0$, $r\in(0,1)$, and that (u,E) is a minimizer of B_1. In this proof we denote

$$p(r)=\int_{B_r\cap\partial^*E\cap\{u>0\}}u\mathrm{d}\mathscr{H}^{n-1},\ q(r)=\int_{B_r\cap\partial^*E\cap\{u>0\}}|x\cdot\nu_E|^2u\mathrm{d}\mathscr{H}^{n-1}$$

We first obtain two relations from the minimality of (u,E). The first is obtained by the equation verified by u with the Stokes formula

$$\int_{B_r}|\nabla u|^2\mathrm{d}\mathscr{L}^n=\int_{\partial B_r}u\partial_r u\mathrm{d}\mathscr{H}^{n-1}+\frac{1}{2}p(r) \tag{7.3}$$

The second is obtain by variation of E, using lemma 7.12 on the vector field $\xi=\xi_r$ where:

$$\xi_r(x)=\varphi_r(x)x,$$

where $\varphi_r(x)=\varphi\left(\frac{|x|}{r}\right)$ for some smooth function $\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^*_+,\mathbb{R}_+)$. Then we have the formulas:

$$\begin{aligned}D\zeta_r &= \varphi\left(\frac{|x|}{r}\right)I_n+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\frac{x}{|x|}\otimes\frac{x}{|x|}\\ \mathrm{div}(\zeta_r) &= n\varphi\left(\frac{|x|}{r}\right)+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\\ \nabla u\cdot D\zeta_r\nabla u &= \varphi\left(\frac{|x|}{r}\right)|\nabla u|^2+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left|\frac{x}{|x|}\cdot\nabla u\right|^2\\ \mathrm{div}_E(\zeta_r) &= (n-1)\varphi\left(\frac{|x|}{r}\right)+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left(1-\frac{(x\cdot\nu_E(x))^2}{|x|^2}\right)\end{aligned}$$

The optimality condition becomes:

$$\begin{aligned}&\int_D(n-2)\varphi\left(\frac{|x|}{r}\right)|\nabla u|^2\mathrm{d}\mathscr{L}^n+\int_{\partial^*E}\left((n-1)\varphi\left(\frac{|x|}{r}\right)+\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\right)u\mathrm{d}\mathscr{H}^{n-1}\\ &=\int_D\frac{|x|}{r}\varphi'\left(\frac{|x|}{r}\right)\left((\partial_r u)^2-(\partial_\tau u)^2\right)\mathrm{d}\mathscr{L}^n+\int_{\partial^*E}\frac{1}{r|x|}\varphi'\left(\frac{|x|}{r}\right)(x\cdot\nu_E(x))^2u\mathrm{d}\mathscr{H}^{n-1}\end{aligned}$$

We now take the function $\varphi=\varphi_\epsilon$, where $\epsilon\in]0,1[$ and $\varphi_\epsilon(t)=\begin{cases}1 & \text{when } t<1-\epsilon\\ \frac{1-t}{\epsilon} & \text{when } 1-\epsilon\leq t\leq 1\\ 0 & \text{when } t>1\end{cases}$. The

optimality condition becomes, as $\epsilon \to 0^+$ (and for $r = 1$ without loss of generality):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1^-}\left(\int_{B_r\cap\partial^* E}\left[1-(x\cdot\nu_E(x))^2\right]u\mathrm{d}\mathscr{H}^{n-1}\right) &= (n-2)\int_{B_1}|\nabla u|^2\mathrm{d}\mathscr{L}^n + (n-1)p(1)\\ &\quad+\int_{\partial B_1}\left((\partial_r u)^2-(\partial_\tau u)^2\right)\mathrm{d}\mathscr{H}^{n-1}\end{aligned} \tag{7.4}$$

Notice the left-hand side is a left derivative that is known to exist because the derivative of the right-hand side exists. Now for the derivative of $W(u_r, E_r)$ there are two cases ; either r is a Lebesgue point of $r \mapsto \int_{B_r\cap\partial^* E\cap\{u>0\}} u$ and then directly the derivative of $W(u_r, E_r)$ in the sense of distributions has a positive atom in r (which may happen for at most a countable number of r), or we are at a Lebesgue point, which we will suppose from now on. Combining (7.3) and (7.4) we get

$$p'(1) - q'(1) = \int_{\partial B_1}\left((n-2)u\partial_r u + 2|\partial_r u|^2 - |\nabla u|^2\right)\mathrm{d}\mathscr{H}^{n-1} + \frac{n}{2}p(1) \tag{7.5}$$

The proof of the monotonicity is now direct using equation (7.3), (7.5). Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1} W(u_r, E_r) = -n\int_{B_1}|\nabla u|^2\mathrm{d}\mathscr{L}^n + \int_{\partial B_1}\left(|\nabla u|^2 + 2u^2 - 2u\partial_r u\right)\mathrm{d}\mathscr{H}^{n-1} + p'(1) - np(1)$$

So replacing $p'(1)$ by what is given in (7.5), and $\int_{B_1}|\nabla u|^2\mathrm{d}\mathscr{L}^n$ with what is given in (7.3) we get exactly

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1} W(u_r, E_r) = q'(1) + \int_{\partial B_1}(u - \partial_r u)^2\mathrm{d}\mathscr{H}^{n-1}$$

which is the first part of the result. From this relation we directly get that if $W(u_r E_r)$ is constant, then $x\cdot\nabla u = u$ so u is homogeneous, and $u(x)(\nu_E(x)\cdot x)^2 = 0$ for $\mathscr{H}^{n-1}$-almost every x so $\{u > 0\}\cap\partial^* E$ is homogeneous, meaning E is homogeneous. Now for the second part of the result, compute

$$\begin{aligned}\int_{B_1}|\nabla u^h|^2\mathrm{d}\mathscr{L}^n &= \frac{1}{n}\int_{\partial B_1}(u^2 + |\nabla_\tau u|^2)\mathrm{d}\mathscr{H}^{n-1}\\ \int_{B_1\cap\partial^* E^h} u^h\mathrm{d}\mathscr{H}^{n-1} &= \frac{1}{n}\int_{\partial B_1\cap\partial^* E} u\mathrm{d}\mathscr{H}^{n-2}\end{aligned}$$

Thus:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1} W(u_r, E_r) - n\left(W(u^h, E^h) - W(u, E)\right) &= \int_{\partial B_1}|\nabla u|^2\mathrm{d}\mathscr{H}^{n-1} + \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=1}\int_{\partial^* E\cap B_r} u\mathrm{d}\mathscr{H}^{n-1}\\ &\quad+2\int_{\partial B_1}(u^2 - u\partial_r u)\mathrm{d}\mathscr{H}^{n-1} - \int_{\partial B_1}(u^2 + |\nabla_\tau u|^2)\mathrm{d}\mathscr{H}^{n-1}\\ &\quad-\int_{\partial B_1\cap\partial^* E} u\mathrm{d}\mathscr{H}^{n-2}\\ &\geq \int_{\partial B_1}|u - \partial_r u|^2\mathrm{d}\mathscr{H}^{n-1} \geq 0\end{aligned}$$

□

Up until now we worked in general dimension $n \geq 1$; from now on everything will be in two dimension. In this cas we can fully caracterise the possible blow-ups.

Proposition 7.15. *Let $(u, E) \in \mathscr{M}(B_1)$ be 1-homogeneous around the origin. Then either $u = 0$ and E is whatever, or there exists some vector $|e| \leq \frac{1}{4}$ such that*

$$u(x) = |x\cdot e|,\ E = \{x : x\cdot e > 0\}$$

Remark 7.16. *Notice the peculiar fact that in the blow-ups, $\partial^* E \cap \{u > 0\}$ is empty, meaning that the "perimeter" part of the problem vanishes at the limit. For any homogeneous minimizers (u, E) in general dimension the transmission condition on u gives*

$$W(u, E) = \frac{1}{2} \int_{\partial^* E \cap \{u>0\} \cap B_1} u \mathrm{d}\mathscr{H}^{n-1}$$

and in particular, for any homogeneous minimizers in two dimension we have $W(u, E) = 0$.

We will see in proposition 7.37 that reciprocally, these are all minimizers.

Proof. Up to rescaling we may suppose (u, E) is a minimizer on an arbitrarily large ball.

Let $x, y \in \partial B_1 \cap \partial^* E$, we suppose $x \neq \pm y$. Then by the comparison

$$J(u, E) \leq \min(J(u, E \cup T_{x,y}), J(u, E \setminus T_{x,y}))$$

where $T_{x,y}$ is the triangle formed by $0, x, y$, we find:

$$\frac{1}{2}(u(x) + u(y)) \leq \frac{1}{2}(u(x) + u(y))|x - y|$$

Thus $|x - y| \geq 1$; we know E and E^c are a finite union of sectors of angle at least $\frac{\pi}{3}$. We denote $(S_l)_l$ the sectors of E and E^c enumerated in the trigonometric order, each directed by two unitary vectors (x_l, x_{l+1}). Since u is harmonic and 1-homogeneous on each sector, then for each l there is a vector $w_l (= \nabla u_{|S_l})$ such that $u(x) = x \cdot w_l$ on S_l. Then we claim that for each l,

$$w_l \text{ is the orthogonal reflexion of } w_{l-1} \text{ through } \mathbb{R}x_l$$

Indeed, the continuity at x_l gives $x_l \cdot (w_l - w_{l-1}) = 0$, and the first variation of the boundary of lemma 7.12 gives $|w_l| = |w_{l-1}|$.
Notice moreover that in the case $w_l \cdot x_l = 0$, then $u_{|E} - u_{|E^c}$ extends as an harmonic function in a neighbourhood of x_l so necessarily $w_l \propto^+ x_l^\perp$, $w_{l-1} = -w_l$, and in the case $w_l \cdot x_l > 0$ the transmission conditions gives $|w_l - w_{l-1}| = \frac{1}{2}$.

We know that for every l, $\arg(x_{l+1}/x_l) \in [\frac{\pi}{3}, \pi]$, we now show that $\arg(x_{l+1}/x_l) > \frac{\pi}{3}$. Suppose indeed that there is equality for some l. By symmetry of the problem we may suppose that the triangle T_l formed by $(0, x_l, x_{l+1})$ is in E and that $u(x_l) > 0$. Then with the same computations as previously,

$$J(u, E) = J(u, E \setminus T_l)$$

So $(u, E \setminus T_l)$ is a minimizer too. In particular this means that u is harmonic on $S_{l-1} \cup T_l$: this implies that $w_{l-1} = w_l$, which contradicts the transmission condition at x_l. Thus every sector is strictly larger that $\frac{\pi}{3}$, so there are either 2 or 4 sectors.

If there are 2 sectors, it is straightforward that $u(x) = |x \cdot w|$ and $E = \{x : x \cdot w > 0\}$ for some vector $|w| \leq \frac{1}{4}$. Let us now focus on the case with 4 sector. Then the conditions on (x_l, w_l) implies that the (w_l) (as well as the (x_l)) form a square and up to rotation

$$E = (\mathbb{R}_+)^2 \cup (\mathbb{R}_-)^2, \ (w_l)_{l=0,1,2,3} = \frac{(\pm 1, \pm 1)}{4}$$

We let C be the square of vertices $e_1, e_2, -e_1, -e_2$; notice that with this choice u is constant on ∂C, with $u_{|\partial C} \equiv \frac{1}{4}$. We let $F = E \setminus C$, $v = u \vee \frac{1}{4} = \begin{cases} u & \text{outside } C \\ \frac{1}{4} & \text{inside } C \end{cases}$. Then by explicit computation we see (v, F) has lower energy than (u, E) in C;

$$J_C(v, F) = \frac{1}{\sqrt{2}} < \frac{3}{4} = J_C(u, E)$$

□

Using the monotonicity, we actually get the stronger, Bernstein-type result

Corollary 7.17. *Let $(u,E) \in \mathscr{M}(\mathbb{R}^2)$ such that $u(0) = 0$, then either $u = 0$ and E is whatever, or there exists some vector $|e| \leq \frac{1}{4}$ such that*

$$u(x) = |x \cdot e|, \ E = \{x : x \cdot e > 0\}$$

Proof. Consider $(v,F),(w,G)$ a limit of $\left(\frac{u(r\cdot)}{r}, \frac{E}{r}\right)$ for $r \to 0, \infty$ respectively. Then (v,F) and (w,G) are homogeneous minimizers and in particular $W(v,F) = W(w,G) = 0$. By monotonicity, $W(u_r, E_r) = 0$ for all $r > 0$, so (u,E) is homogeneous. □

7.5 Epiperimetric inequality at the contact set

In this section we establish the refinement of the monotonicity formula, through an epiperimetric inequality that quantifies the speed of convergence to the blow-up.

Theorem 7.18. *There are constants $\gamma > 0$, $\vartheta \in (0,1)$, such that for any $(u,E) \in \mathscr{M}(B_2)$ with $u(0) = 0$, then*

$$W(u,E) \leq \left(1 - \gamma \wedge W(u^h, E^h)^\vartheta\right) W(u^h, E^h),$$

where (u^h, E^h) is the 1-homogeneous extension of (u,E) on B_1.

The result is established in five successive steps, that is mostly due to the fact that the set of blow-ups is not compact and small blow-ups must be treated separately. In a way the epiperimetric is obtainable through the explicit construction of "rough" competitor near any blow-up that is bounded away from the blow-ups $\left\{\frac{1}{4}|x \cdot e|, \ e \in \mathbb{S}^1\right\}$ which act as a critical case. Denoting $\mathcal{B}$ the set of blow-ups in $\mathbb{R}^2$, we separate this set in

$$\mathcal{B} = \mathcal{B}^{\text{low}} \cup \mathcal{B}^{\text{high}},$$

with $\mathcal{B}^{\text{low}}$ consisting of blow-ups (u,E) such that $\int_{\partial B_1} u \leq \frac{1}{2}$ (including the case $u = 0$ and E being whatever) and $\mathcal{B}^{\text{high}}$ consisting of blow-ups (u,E) such that $\frac{1}{4} \leq \int_{\partial B_1} u \leq 1$. In particular $\mathcal{B}^{\text{high}}$ is compact, and this will be used in the last step.

- We first establish the result along minimizers (u,E) with $\inf_{\partial B_1 \cap \partial^* E} u$ small enough (which is the case near any blow-up) and $\int_{\partial B_1} u \leq \frac{1}{4}$ (which is the case near $\mathcal{B}^{\text{low}}$) by explicit construction of a minimizer that vanishes on $\partial^* E$. This is what is done in lemma 7.21.

- Then by compactness argument we establish the inequality outside any neighbourhood of $\mathcal{B}^{\text{high}}$. Indeed, if the inequality is wrong, then there is some sequence of minimizers that approach the equality case, and since it must converge to some element of $\mathcal{B}^{\text{low}}$ we get a contradiction with the previous point. This is done in lemma 7.22

- Near any point of $\mathcal{B}^{\text{high}}$, we construct a competitor that has only two sectors in a small disk. This is done by first damping the high-frequency modes on small sectors and then removing said sector. Either this operation concludes the inequality or the energy of the two-sector competitor we built is close to the original and we proceed with the next point. The computations for this are done in subsection 7.5.3 and the proof of this is done in lemma 7.27.

- Near any point of $\mathcal{B}^{\text{high}}$ that has only two sectors with openness close to π, we construct a competitor that depend only on a finite number of parameters in a smaller disk. This is done by damping the high-frequency modes on each sector. A subtle point is that on the smaller disk the high-frequency modes are not zero but instead depend on the parameters being the openness of the sectors, the low-frequency mode on each side, and the values of u on the boundary of the sector. As previously, either this concludes the inequality or the change of energy is actually small compared to the previous competitor and we may proceed with the last step. The computations are very similar to the previous step, and this is done in lemma 7.28.

- Now we are left with (u, E) that only depends on five parameters, so it is in a five-dimensional submanifold in a neighbourhood of $\mathcal{B}^{\text{high}}$. Since $\mathcal{B}^{\text{high}}$ is compact, then for a small enough neigbourhood we may see the energy functional as an analytic function of five variables that verify the Lojasiewicz inequality, and then we implicitely build a competitor by controlled gradient descent. A subtle point is that there are two constraints on the parameters, and once gradient descent hit those constraints we need to continue a gradient descent on the constrained submanifold. This is proved in lemma 7.29.

Let us introduce a few shorthand and notations that will be useful to us in this section in our use of polar coordinate. For a function $\phi : \mathbb{S}^1 \to \mathbb{R}$ and $E \subset \partial B_1$, we will write ϕ^h, E^h the 1-homogeneous extension of (ϕ, E) on B_1, meaning $\phi^h(x) := |x|\phi(x/|x|)$ and $E^h := \cup_{0<r\leq 1} rE$. While the trace of $u \in H^1(B_1)$ on $\mathbb{S}^1$ is in $H^{1/2}(\mathbb{S}^1)$, notice that a homogeneous function is in $H^1(B_1)$ if and only if its trace is also in $H^1(\mathbb{S}^1)$.

It will be useful to see a function $\phi \in H^1(B_1)$ as a choice, for every $r \in (0, 1)$, of a function

$$\phi_{|r} := \phi_{r|\mathbb{S}^1}$$

in $H^{1/2}(\mathbb{S}^1, \mathbb{R}+)$ (that is actually in $H^1(\mathbb{S}^1)$ for almost every r), meaning $\phi_{|r}$ is the usual rescaling $\phi_r(x) := \frac{\phi(rx)}{r}$ restricted to $\mathbb{S}^1$. Likewise, we will write $E_{|r} = \frac{E^{(1)}}{r} \cap \mathbb{S}^1$. This decomposition is useful because it allows us to decompose the energy of (ϕ, E^h) itself as an integral of energies of $\phi_{|r}$ (restricted to the circle) with a penalization on the variation $\partial_r \phi_{|r}$ (seen as a derivative of $\frac{\phi(rx)}{r}$ in r, and not as a radial derivative).

Lemma 7.19. *Let $\phi \in H^1(B_1, \mathbb{R}_+)$ and $E \subset \mathbb{S}^1$, then*

$$W(\phi, E^h) = \int_0^1 \left(r\mathcal{F}(\phi_{|r}) + r^3 \int_{\mathbb{S}^1} |\partial_r \phi_{|r}|^2 \mathrm{d}\theta \right) \mathrm{d}r, \text{ where}$$
$$\mathcal{F}(\phi, E) = \int_{\mathbb{S}^1} \left(|\partial_\theta \phi|^2 - |\phi|^2 \right) \mathrm{d}\theta + \sum_{x \in \partial^* E_1} \phi(x)$$

$\mathcal{F}(\cdot, E)$ is defined on $H^1(\mathbb{S}^1)$, and when we apply it to a $H^1(B_1, \mathbb{R}_+)$ function we implicitely assume we apply it to its trace on $\mathbb{S}^1$ (with the convention that it is infinity if the trace is not in H^1).

Proof. This is done by writing in polar coordinates $|\nabla \phi|^2 = (\phi_r + r\partial_r \phi_r)^2 + (\partial_\theta \phi_r)^2$, and with one integration by part along r. □

Remark 7.20. *Notice that in the last three steps of the proof the minimizer becomes simpler and simpler, the main idea behind this "iterative" process is the following: suppose (ϕ, E) is an homogeneous function and set, and suppose that we may build a competitor (ψ, F) such that*

- *There is some radius $\rho > 0$ such that (ψ, F) is homogeneous on B_ρ.*
- *There is a constant $\nu > 0$ such that either $W(\psi, F) \leq (1 - \nu)W(\phi, E)$ or we have both $W(\psi, F) \leq W(\phi, E)$ and $\mathcal{F}(\psi_{|\rho}, F_{|\rho}) \geq \frac{1}{2}\mathcal{F}(\phi_{|1}, E_{|1})$.*
- *We are able to build (χ, G) such that $(\chi_{|1}, G_{|1}) = (\psi_{|\rho}, F_{|\rho})_{|\mathbb{S}^1}$ and*

$$W(\chi, G) \leq \left(1 - \gamma \wedge W(\psi^h_{|\rho}, F^h_{|\rho})^\vartheta\right) W(\psi^h_{|\rho}, F^h_{|\rho}).$$

Then the logarithmic epiperimetric inequality holds for (ϕ, E). To be more precise, in the second point the constant $1/2$ could be replaced by any positive constant and the epiperimetric inequality

could be replaced by the weaker logarithmic epiperimetric inequality, but this is what we will prove. Indeed, if $W(\psi,F) > (1-\nu)W(u,E)$ *then letting* $(\overline{\chi},\overline{G})_{|r} = \begin{cases}(\psi,F)_{|r} & \text{if } r>\rho\\ (\chi,G)_{|\rho r} & \text{if } r\le\rho\end{cases}$*, we have*

$$\begin{aligned}W(\overline{\chi},\overline{G}) - W(\phi,E) &= W(\overline{\chi},\overline{G}) - W(\psi,F) + W(\psi,F) - W(\phi,E)\\ &\le \rho^2\left(W(\chi,G) - W(\psi^h_{|\rho},F^h_{|\rho})\right)\\ &\le -\rho^2(\gamma \wedge W(\psi^h_{|\rho},F^h_{|\rho})^\vartheta)W(\psi^h_{|\rho},F^h_{|\rho}),\end{aligned}$$

and $W(\psi^h_{|\rho},F^h_{|\rho}) = \frac{1}{2}\mathcal{F}(\psi_{|\rho},F_{|\rho}) \ge \frac{1}{2}W(\phi,E)$*, hence*

$$W(\overline{\chi},\overline{G}) - W(\phi,E) \le -\frac{1}{2}\rho^2(\gamma \wedge (W(\phi,E)/2)^\vartheta)W(\phi,E).$$

7.5.1 Estimate near small blow-ups

Lemma 7.21. *There exists two universal constants* ν*,* δ *such that, for any* $u \in H^1(\mathbb{S}^1,\mathbb{R}_+)$*,* $E \subset \mathbb{S}^1$*, with* $W(u^h,E^h) < \infty$*, if*

$$\int_{\mathbb{S}^1} u\mathrm{d}\theta \le \frac{1}{2},\ u_{|\partial^* E} \le \delta,$$

then there exists (v,F) *that coincides with* (u,E) *on the boundary and such that*

$$W(v,F) \le (1-\nu)W(u^h,E^h)$$

Proof. We let $m = \sup_{\partial E} u$; we know that $m \le \delta_0$. Let us begin with the case $m = 0$; then $u(1_E - 1_{E^c})$ is smooth and we consider its fourier serie decomposition:

$$u(1_E - 1_{E^c})(e^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

Let v be the harmonic extension of $u(1_E - 1_{E^c})$ in the ball, and denote $F = \{v > 0\}$, then

$$\begin{aligned}W(v(1_F - 1_{F^c}),F) &= \sum_{n\in\mathbb{Z}}(|n|-1)|c_n|^2\\ W(u^h,E^h) &= \sum_{n\in\mathbb{Z}}\frac{|n|^2-1}{2}|c_n|^2\end{aligned}$$

From $(|n|-1) \le \frac{2}{3}\frac{|n|^2-1}{2}$ we deduce that $W(v(1_F-1_{F^c}),F) \le \frac{2}{3}W(u^h,E^h)$. Note that this is still the case if $W(u^h,E^h) < 0$, though in this case the inequality is trivial.

We now suppose that $m > 0$, the idea is to first build a first competitor v by its section $v_{|r}$ such that $\sup_{\partial^* E} v_{|\rho} = 0$ for some small enough ρ, and then use the previous harmonic extension to conclude. Let $\rho \in]0,1[$ that will be fixed later, as well as a function $f : [0,1] \to [0,1]$ with value 1 on $[0,\rho]$, 0 in 1, and that is decreasing. We let

$$v_{|r} = (u - f(r)m)_+$$

We compare v and u^h. We have

$$\begin{aligned}
W(v,E^h)-W(u^h,E^h) &= \int_0^1\int_{\mathbb{S}^1}\left(r\left(|\partial_\theta v_{|r}|^2-|\partial_\theta u|^2\right)-r\left(|v_{|r}|^2-|u|^2\right)+r^3|\partial_r v_{|r}|^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad - P \text{ where } P:=\sum_{x\in\partial^* E}\int_0^1 r(mf(r)\wedge u(x))\mathrm{d}r\\
&= \int_0^1\int_{\mathbb{S}^1}1_{u>mf(r)}\left(r\left(|\partial_\theta v_{|r}|^2-|\partial_\theta u|^2\right)-r\left(|v_{|r}|^2-|u|^2\right)+r^3|\partial_r v_{|r}|^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad + \int_0^1\int_{\mathbb{S}^1}1_{u\le mf(r)}\left(r\left(|\partial_\theta v_{|r}|^2-|\partial_\theta u|^2\right)-r\left(|v_{|r}|^2-|u|^2\right)+r^3|\partial_r v_{|r}|^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad - P\\
&= \int_0^1\int_{\mathbb{S}^1}1_{u>mf(r)}\left(2urf(r)m-rf(r)^2m^2+r^3f'(r)^2m^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad - \int_0^1\int_{\mathbb{S}^1}1_{u\le mf(r)}\left(r|\partial_\theta u|^2-r|u|^2\right)\mathrm{d}\theta\mathrm{d}r - P\\
&\le 2\left(\int_0^1\int_{\mathbb{S}^1}rf(r)u1_{u>mf(r)}\mathrm{d}\theta\mathrm{d}r\right)m\\
&\quad - \int_0^1\int_{\mathbb{S}^1}1_{u\le mf(r)}\left(r|\partial_\theta u|^2-r|u|^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad - P+C_f m^2
\end{aligned}$$

Where C_f is a constant that depends only on the choice of f. Let now $(w_{|r},F_{|r})$ be defined as à $(v_{|r},E_{|r})$ for $r>\rho$, and with the same harmonic extension as previously for $r<\rho$. Through the same Fourier serie decomposition we have

$$\begin{aligned}
W(w,F)-W(v,E^h) &\le -\frac{\rho^2}{6}\int_{\mathbb{S}^1}\left(|\partial_\theta v_{|\rho}|^2-|v_{|\rho}|^2\right)\mathrm{d}\theta\\
&= -\frac{\rho^2}{6}\int_{\mathbb{S}^1}1_{u>m}\left(|\partial_\theta u|^2-|u-m|^2\right)\mathrm{d}\theta\\
&= -\frac{\rho^2}{6}\int_{\mathbb{S}^1}1_{u>m}\left(|\partial_\theta u|^2-|u|^2\right)\mathrm{d}\theta-\frac{\rho^2}{3}\left(\int_{\mathbb{S}^1}u1_{u>m}\mathrm{d}\theta\right)m+Cm^2
\end{aligned}$$

So, regrouping the two previous computations,

$$\begin{aligned}
W(w,F)-W(u,E^h) &\le 2\left(\int_0^1\int_{\mathbb{S}^1}rf(r)u1_{u>mf(r)}\mathrm{d}\theta\mathrm{d}r\right)m-\int_0^1\int_{\mathbb{S}^1}1_{u\le mf(r)}\left(r|\partial_\theta u|^2-r|u|^2\right)\mathrm{d}\theta\mathrm{d}r\\
&\quad -\frac{\rho^2}{6}\int_{\mathbb{S}^1}1_{u>m}\left(|\partial_\theta u|^2-|u|^2\right)\mathrm{d}\theta\mathrm{d}r-\frac{\rho^2}{3}\left(\int_{\mathbb{S}^1}u1_{u>m}\mathrm{d}\theta\mathrm{d}r\right)m+C_fm^2-P\\
&\le 2\left(\int_0^1 rf(r)\mathrm{d}r\right)\left(\int_{\mathbb{S}^1}u\mathrm{d}\theta\right)m-\frac{1}{2}\int_{\mathbb{S}^1}1_{u\le m}\left(|\partial_\theta u|^2-|u|^2\right)\mathrm{d}\theta\\
&\quad -\frac{\rho^2}{6}\int_{\mathbb{S}^1}1_{u>m}\left(|\partial_\theta u|^2-|u|^2\right)\mathrm{d}\theta\mathrm{d}r-\frac{\rho^2}{3}\left(\int_{\mathbb{S}^1}u\mathrm{d}\theta\right)m+C_fm^2-P\\
&\le -\frac{\rho^2}{6}\int_{\mathbb{S}^1}(|\partial_\theta u|^2-|u|^2)\mathrm{d}\theta+\left(\int_{\mathbb{S}^1}u\right)\left(2\int_0^1 rf(r)\mathrm{d}r-\frac{\rho^2}{3}\right)m+C_fm^2-P\\
&= -\frac{\rho^2}{3}W(u^h,E^h)+\left(\int_{\mathbb{S}^1}u\right)\left(2\int_0^1 rf(r)\mathrm{d}r-\frac{\rho^2}{3}\right)m+C_fm^2-\frac{2}{3}P
\end{aligned}$$

where C_f is a constant depending only on f that may change from line to line. Now we just have to prove that the sum of the three last term is negative. For this, notice that since $m = \sup_{\partial^* E} u$, then there is some $x \in \partial^* E$ such that $u(x) = m$ (it is actually reached, else the energy would be infinite), so

$$P \geq \left(\int_0^1 r f(r) \mathrm{d}r \right) m,$$

so it is enough to prove

$$\frac{2}{3} \int_0^1 r f(r) \mathrm{d}r \geq \left(\int_{\mathbb{S}^1} u \right) \left(2 \int_0^1 r f(r) \mathrm{d}r - \frac{\rho^2}{3} \right) + C_f m^2$$

We fix $\rho \in (0,1)$, and $f(r) = 1 - \left(\frac{r-\rho}{1-\rho} \right)_+^{1/K}$ for a large enough K such that $\int_0^1 r f(r) \mathrm{d}r \approx \frac{\rho^2}{2}$. Then when $\int_{\mathbb{S}^1} u \leq \frac{1}{2}$ and m is small enough this condition is true and we get the result. □

7.5.2 Estimate away from large blow-ups

Lemma 7.22. *For any small $\eta > 0$, there is a constant $\nu_\eta > 0$ such that, for any $(u, E) \in \mathscr{M}(B_2)$ with $u(0) = 0$, $\mathrm{dist}(u_{|B_1}, \mathcal{B}^{high}) \geq \eta$, if we denote (u^h, E^h) the 1-homogeneous extension of u in B_1, then*

$$W(u, E) \leq (1 - \nu_\eta) W(u^h, E^h)$$

An appropriate $\eta > 0$ will be fixed at the very end of the proof.

Proof. Suppose there exists a sequence $\nu_i \to 0$, and a sequence of minimizers (u_i, E_i) on B_2 such that for every i:

$$W(u_i, E_i) > (1 - \nu_i) W(u_i^h, E_i^h)$$

Then up to a subsequence, we may suppose that (u_i, E_i) converges to a minimizer (u, E) (we remind that the convergence for u is strong in $H^1_{\mathrm{loc}}(B_2) \cap \mathcal{C}^0_{\mathrm{loc}}(B_2)$ and that $W(u_i, E_i) \to W(u, E)$). Under these circumstances,

$$W(u^h, E^h) \leq \liminf_{i \to \infty} W(u_i^h, E_i^h)$$

by lower semicontinuity. In particular, $W(u^h, E^h) \leq W(u, E)$, so (u^h, E^h) is a homogeneous minimizer on B_1; in particular we know that $W(u^h, E^h) = 0$, which in turn implies that $W(u_r, E_r) = 0$ for every $r \in]0, 1[$ by monotonicity, so (u, E) is an homogenous minimizer. In particular, for any large enough i the hypothesis of lemma 7.21 are verified, meaning that

$$(1 - \nu) W(u_i^h, E_i^h) \geq W(u_i, E_i) > (1 - \nu_i) W(u_i^h, E_i^h),$$

which is a contradiction for any large enough i for which $\nu_i < \nu$. □

7.5.3 Preliminary computations

Consider $\phi \in H^1(\mathbb{S}^1)$, $E \subset \mathbb{S}^1$, such that $\mathcal{F}(\phi, E) < \infty$. Suppose that E and E^c each form a countable number of sectors $(S^i)_{i \in I}$, on which, up to a rotation that makes $S^i = [0, \pi\alpha_i]$, ϕ is equal to

$$\phi_i = m^i + l_i \cos\left(\frac{\theta}{\alpha_i} \right) + \sum_{n \geq 1} c_{n,i} \sin\left(\frac{n\theta}{\alpha_i} \right)$$

Then the energy $\mathcal{F}(\phi, E)$ decomposed into the contribution of each sector as

$$\mathcal{F}(\phi, E) = \sum_{i \in I} \left(\frac{\phi_i(0) + \phi_i(\alpha_i)}{2} + \int_0^{\alpha_i} (|\partial_\theta \phi_i|^2 - |\phi_i|^2) \mathrm{d}\theta \right) =: \sum_{i \in I} \mathcal{F}^{\alpha_i}(\phi_i)$$

In this subsection we fix $\alpha \in (0, 2)$, $\varphi \in H^1([0, \pi\alpha])$, and compute $F^\alpha(\varphi)$ for several parametrizations.

Lemma 7.23. *Let $\alpha \in (0,2)$, and let $\varphi \in H^1([0,\alpha])$, $\mathcal{F}^\alpha$ can be decomposed as*

$$\varphi = m + l\cos\left(\frac{\theta}{\alpha}\right) + \sum_{n\geq 1} c_n \sin\left(\frac{n\theta}{\alpha}\right)$$

Then

$$\begin{aligned}\mathcal{F}^\alpha(\varphi) =& m - \pi\alpha m^2 + \frac{\pi}{2}\left(\alpha^{-1}-\alpha\right) l^2 \\ &+ \sum_{\text{odd } n}\left(\frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2 - \frac{4}{n}\alpha m c_n\right) \\ &+ \sum_{\text{even } n}\left(\frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2 + \frac{4n}{n^2-1}\left(\alpha^{-1}-\alpha\right) l c_n\right)\end{aligned}$$

Proof. We compute separately:

$$\begin{aligned}\int_0^{\pi\alpha}\varphi^2 &= \int_0^\alpha \left| m + l\cos\left(\frac{\theta}{\alpha}\right) + \sum_{n\geq 1} c_n \sin\left(\frac{n\theta}{\alpha}\right)\right|^2 \mathrm{d}\theta \\ &= \int_0^{\pi\alpha}\Big[m^2 + l^2\cos(\theta/\alpha)^2 + \sum_{n\geq 1} c_n^2 \sin(n\theta/\alpha)^2 \\ &+ 2m\sum_{n\geq 1} c_n \sin(n\theta/\alpha) + 2l\cos(\theta/\alpha)\sum_{n\geq 1} c_n\sin(n\theta/\alpha)\Big]\mathrm{d}\theta\mathrm{d}\theta \\ &= \alpha\int_0^\pi \left(m^2 + l^2\cos(\theta)^2 + \sum_{n\geq 1} c_n^2\sin(n\theta)^2 + 2m\sum_{n\geq 1} c_n\sin(n\theta) + 2l\cos(\theta)\sum_{n\geq 1} c_n\sin(n\theta)\right)\mathrm{d}\theta \\ &= \alpha\left(\pi m^2 + \frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} c_n^2 + 2m\sum_{n\geq 1} c_n\int_0^\pi \sin(n\theta)\mathrm{d}\theta + 2l\sum_{n\geq 1} c_n\int_0^\pi \cos(\theta)\sin(n\theta)\mathrm{d}\theta\right)\end{aligned}$$

Then notice that

$$\int_0^\pi \sin(n\theta)\mathrm{d}\theta = \frac{2}{n}1_{\text{odd } n},\ \int_0^\pi \cos(\theta)\sin(n\theta)\mathrm{d}\theta = -n\int_0^\pi \sin(\theta)\cos(n\theta)\mathrm{d}\theta = \frac{2n}{n^2-1}1_{\text{even } n}$$

Thus

$$\int_0^{\pi\alpha}\varphi^2\mathrm{d}\theta = \alpha\left(\pi m^2 + \frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} c_n^2 + 4m\sum_{\text{odd } n}\frac{c_n}{n} + 4l\sum_{\text{even } n}\frac{nc_n}{n^2-1}\right)$$

Similarly,

$$\begin{aligned}\int_0^{\pi\alpha}|\partial_\theta\varphi|^2\mathrm{d}\theta &= \alpha^{-2}\int_0^\alpha \left|-l\sin\left(\frac{\theta}{\alpha}\right) + \sum_{n\geq 1} nc_n\cos\left(\frac{n\theta}{\alpha}\right)\right|^2\mathrm{d}\theta \\ &= \alpha^{-1}\int_0^\pi \left|-l\sin(\theta) + \sum_{n\geq 1} nc_n\cos(n\theta)\right|^2\mathrm{d}\theta \\ &= \alpha^{-1}\left(\frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} n^2c_n^2 - 2l\sum_{n\geq 1} nc_n\int_0^\pi \sin(\theta)\cos(n\theta)\mathrm{d}\theta\right) \\ &= \alpha^{-1}\left(\frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} n^2c_n^2 + 4l\sum_{\text{even}}\frac{nc_n}{n^2-1}\right)\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{F}^{\alpha}(\varphi) &= m + \alpha^{-1}\left(\frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} n^2 c_n^2 + 4l\sum_{\text{even}} \frac{nc_n}{n^2-1}\right)\\ &\quad - \alpha\left(\pi m^2 + \frac{\pi}{2}l^2 + \frac{\pi}{2}\sum_{n\geq 1} c_n^2 + 4m\sum_{\text{odd } n}\frac{c_n}{n} + 4l\sum_{\text{even } n}\frac{nc_n}{n^2-1}\right)\\ &= m - \pi\alpha m^2 + \frac{\pi}{2}\left(\alpha^{-1}-\alpha\right)l^2 + \sum_{\text{odd } n}\left(\frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2 - \frac{4}{n}\alpha m c_n\right)\\ &\quad + \sum_{\text{even } n}\left(\frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2 + \frac{4n}{n^2-1}\left(\alpha^{-1}-\alpha\right)lc_n\right)\end{aligned}$$

□

We now introduce a change of variable that will be practical to isolate high-frequencies contribution, meaning either every $(c_n)_{n\geq 2}$ for $\alpha \approx 1$ or every $(c_n)_{n\geq 1}$ for $\alpha \approx 0$.

Lemma 7.24 (small sector)**.** *Let $\alpha \in (0, 1/\sqrt{2})$, and let $\varphi \in H^1([0,\alpha])$ be decomposed as*

$$\varphi = m + l\cos\left(\frac{\theta}{\alpha}\right) + \sum_{n\geq 1} c_n \sin\left(\frac{n\theta}{\alpha}\right)$$

for any $n \geq 1$ we let

$$c_n = b_n + \begin{cases} \frac{4\alpha^2 m}{\pi n(n^2-\alpha^2)} & \text{for odd } n \\ \frac{4n(1-\alpha^2)l}{\pi(n^2-1)(n^2-\alpha^2)} & \text{for even } n \end{cases},$$

such that

$$\begin{aligned}\varphi(\theta) &= \left(1 + \sum_{\text{odd } n>1}\frac{4\alpha^2}{\pi n\left(n^2-\alpha^2\right)}\sin\left(\frac{n\theta}{\alpha}\right)\right)m\\ &\quad + \left(\cos\left(\frac{\theta}{\alpha}\right) - \sum_{\text{even } n}\frac{4n\left(1-\alpha^2\right)}{\pi(n^2-1)\left(n^2-\alpha^2\right)}\sin\left(\frac{n\theta}{\alpha}\right)\right)l\\ &\quad + \sum_{n\geq 1} b_n \sin\left(\frac{n\theta}{\alpha}\right)\end{aligned}$$

Then $\mathcal{F}^{\alpha}(\varphi) = \overline{\mathcal{F}}^{\alpha}(\varphi) + \sum_{n\geq 1}\mathcal{F}_n^{\alpha}(\varphi)$, where

$$\begin{aligned}\overline{\mathcal{F}}^{\alpha}(\varphi) =& m - \left(\pi + \frac{8\alpha^2}{\pi}\sum_{\text{odd } n}\frac{1}{n^2(n^2-\alpha^2)}\right)\alpha m^2\\ &+ \left(\frac{\pi}{2} - \frac{8(1-\alpha^2)}{\pi}\sum_{\text{even } n}\frac{n^2}{(n^2-1)^2(n^2-\alpha^2)}\right)\frac{1-\alpha^2}{\alpha}l^2\\ \mathcal{F}_n^{\alpha}(\varphi) =& \frac{\pi}{2\alpha}(n^2-\alpha^2)b_n^2\end{aligned}$$

While $\overline{\mathcal{F}}^{\alpha}$ might seem complicated, notice that it is an analytic function that only depends on a finite number of variables, while the high-frequency term are now much easier.

Proof. Let $n \geq 1$ (if α is small), then we may write

$$\begin{aligned}\frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2-\frac{4}{n}\alpha m c_n &= \frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)\left(c_n-\frac{4\alpha^2 m}{\pi n(n^2-\alpha^2)}\right)^2 \\ &\quad -\frac{8\alpha^3 m^2}{\pi n^2(n^2-\alpha^2)} \quad \text{for odd } n \\ \frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)c_n^2+\frac{4n}{n^2-1}\left(\alpha^{-1}-\alpha\right)l c_n &= \frac{\pi}{2}\left(\alpha^{-1}n^2-\alpha\right)\left(c_n+\frac{4n(1-\alpha^2)l}{\pi(n^2-1)(n^2-\alpha^2)}\right)^2 \\ &\quad -\frac{8n^2(1-\alpha^2)^2 l^2}{\pi\alpha(n^2-1)^2(n^2-\alpha^2)} \quad \text{for even } n\end{aligned}$$

Replacing these in the formula for $\mathcal{F}^\alpha$, we obtain the result. □

For large sector ($\alpha \approx 1$) we have the completely analogous decomposition.

Lemma 7.25 (large sector)**.** *Let $\alpha \in [1/\sqrt{2}, 2)$, and let $\varphi \in H^1([0,\alpha])$ be decomposed as*

$$\varphi = m + l\cos\left(\frac{\theta}{\alpha}\right) + \sum_{n\geq 1} c_n \sin\left(\frac{n\theta}{\alpha}\right)$$

for any $n \geq 2$ we let

$$c_n = b_n + \begin{cases} \frac{4\alpha^2 m}{\pi n(n^2-\alpha^2)} & \text{for odd } n \\ \frac{4n(1-\alpha^2)l}{\pi(n^2-1)(n^2-\alpha^2)} & \text{for even } n \end{cases},$$

such that

$$\begin{aligned}\varphi(\theta) &= \left(1+\sum_{\text{odd } n>1}\frac{4\alpha^2}{\pi n\left(n^2-\alpha^2\right)}\sin\left(\frac{n\theta}{\alpha}\right)\right)m \\ &\quad +\left(\cos\left(\frac{\theta}{\alpha}\right)-\sum_{\text{even } n}\frac{4n\left(1-\alpha^2\right)}{\pi(n^2-1)\left(n^2-\alpha^2\right)}\sin\left(\frac{n\theta}{\alpha}\right)\right)l \\ &\quad +c_1\sin\left(\frac{\theta}{\alpha}\right)+\sum_{n>1}b_n\sin\left(\frac{n\theta}{\alpha}\right)\end{aligned}$$

Then $\mathcal{F}^\alpha(\varphi) = \overline{\mathcal{F}}_1^\alpha(\varphi) + \sum_{n>1}\mathcal{F}_n^\alpha(\varphi)$, where

$$\begin{aligned}\overline{\mathcal{F}}_1^\alpha(\varphi) =& m-\left(\pi+\frac{8\alpha^2}{\pi}\sum_{\text{odd } n>1}\frac{1}{n^2(n^2-\alpha^2)}\right)\alpha m^2 \\ &+\left(\frac{\pi}{2}-\frac{8(1-\alpha^2)}{\pi}\sum_{\text{even } n}\frac{n^2}{(n^2-1)^2(n^2-\alpha^2)}\right)\frac{1-\alpha^2}{\alpha}l^2 \\ &+\frac{\pi}{2}\frac{\alpha^2-1}{\alpha}c_1^2-4\alpha m c_1 \\ \mathcal{F}_n^\alpha(\varphi) =& \frac{\pi}{2\alpha}(n^2-\alpha^2)b_n^2\end{aligned}$$

The proof is the same, only we isolate the first fourier coefficient.

Now, if we consider a function $\varphi \in H^1(B_1)$ and $E \subset \mathbb{S}^1$ that we extend homogeneously, with countable sectors $(S_i)_{i\in I}$ (of both E and E^c) of angle $\pi\alpha_i$, then on each sector and for each $r \in (0,1)$ we can parametrize the restriction of $\varphi_{|r}$ to S_i by parameters $(m_i(r), l_i(r), \alpha_i, (b_{i,n}(r))_{n\geq 1})$ following 7.24 or 7.25. We make the hypothesis that for all i, $m_i(r)$ and $l_i(r)$ do not depend on r. Then we have a decomposition

$$W(\varphi, E) = \sum_{i\in I} W^{\alpha_i}(\varphi|S_i),$$

where each term is decomposed as

$$W^{\alpha_i}(\varphi|S_i) = \overline{W}^{\alpha_i}(\varphi|S_i) + \sum_{n\geq 1} W_n^{\alpha_i}(\varphi|S_i)$$

where (in the small sector case, the large sector case is completely analog using 7.24) each term is defined by

$$\overline{W}^{\alpha_i}(\varphi|S_i) = \overline{W}^{\alpha_i}(\varphi^h_{|1}|S_i) = \frac{1}{2}\overline{F}^{\alpha_i}(\varphi_{|1}|S_i) \text{ as in lemma 7.24}$$

$$W_n^{\alpha_i}(\varphi|S_i) = \frac{\pi}{2}\alpha_i \int_0^1 \left(r^3 b'_{i,n}(r)^2 + r\left(\frac{n^2}{\alpha^2} - 1\right) b_{i,n}(r)^2 \right) \mathrm{d}r$$

Lemma 7.26. *There exists $\rho \in (0,1)$, that may be taken arbitrarily small, such that, denoting*

$$h(r) = \frac{1}{r}\left(\frac{r-\rho}{1-\rho}\right)_+^{1+\rho^{2/3}}$$

Then for any $\kappa \geq 1$, we have

$$\int_0^1 \left(r^3 h'(r)^2 + r\kappa h(r)^2 \right) \mathrm{d}r \leq \left(1 - \frac{1}{2}\rho^{2/3}\right)\frac{\kappa}{2}$$

We see in particular this lemma applies to the quantity $W_n^{\alpha_i}$ when $\frac{n^2}{\alpha_i^2} - 1 \geq 1$, meaning any $n \geq 1$ when $\alpha_i \leq \frac{1}{\sqrt{2}}$, and for any $n \geq 2$ in general (we may suppose that we are only dealing with sectors of angle no more than $\sqrt{2}\pi$, since we already know the epiperimetric inequality away from large blow-ups, which are flat).

7.5.4 Erasure of small sectors and high frequencies

Lemma 7.27. *There exists three universal constants ρ, ν, η such that, for any $\phi \in H^1(\mathbb{S}^1)$, $E \subset \mathbb{S}^1$, with $\mathcal{F}(\phi, E) < \infty$, dist$((\phi, E), \mathcal{B}^{high}) \leq \eta$, then there exists $\psi \in H^1(B_1)$, $F \subset B_1$ that coincides with (ϕ, E) on the boundary, such that $W(\psi, F) \leq W(\phi^h, E^h)$ and more precisely*

- *(ψ, F) is 1-homogeneous on B_ρ and has only two sectors with angles close to π.*
- *Or $W(\psi, F) \leq (1-\nu)W(\phi^h, E^h)$ or $\mathcal{F}(\psi_\rho, F_\rho) \geq \frac{1}{2}\mathcal{F}(\phi_1, E_1)$.*

Proof. When η is small enough, we may assume that the angles of the sectors of E and E^c are arbitrarily close to either 0 or π. As previously, we let $(S_i)_{i\in I}$ the (countable) sectors of E and E^c. We write $I = \{1,2\} \sqcup J$ where S_1, S_2 are the two large sectors (with $S_1 \subset E$, $S_2 \subset E^c$) and the S_js are the small sectors (we suppose that J is non-empty, otherwise there is nothing to prove).

We define as previously the associated parameters $(m_j, l_j, \alpha_j, (b_{j,n})_{n\geq 1})$ associated to $\phi|S_j$ for any $j \in J$, and we build ψ by variation of the parameters $(b_{j,n})$ on each sector. We also let $\rho \in (0,1)$, and h chosen as in the lemma 7.26. Now for any sector S_i we define $\psi_{|r}$ by the following:

$$\psi_{|r}|S_i = \begin{cases} \phi|S_i & \text{if } i = 1,2, \\ \phi|S_i - (1-h(r))\sum_{n\geq 1} b_n^j \sin\left(\frac{n\theta}{\alpha}\right) & \text{if } i \in J. \end{cases}$$

meaning that the Fourier coefficient associated to $\psi_{|r}|S_j$ for a small sector S_j are $b_{j,n}(r) = h(r)b_{j,n}$, which is in particular zero for $r \in (0, \rho]$, and

$$\begin{aligned}\psi_{|r}|S_j(\theta) = &\left(1 + \sum_{\text{odd } n>1} \frac{4\alpha_j^2}{\pi n\left(n^2 - \alpha_j^2\right)} \sin\left(\frac{n\theta}{\alpha_j}\right)\right) m_j \\ &+ \left(\cos\left(\frac{\theta}{\alpha_j}\right) - \sum_{\text{even } n} \frac{4n\left(1-\alpha_j^2\right)}{\pi(n^2-1)\left(n^2-\alpha_j^2\right)} \sin\left(\frac{n\theta}{\alpha_j}\right)\right) l_j\end{aligned}$$

Notice that there is a universal constant $C > 0$ such that for any small sector S_j,

$$\psi_\rho | S_j \leq C m_j$$

Up to switching E and E^c, we suppose

$$\sum_{j \in J, S^j \subset E} m^j \geq \frac{1}{2} \sum_{j \in J} m^j$$

We then define F to be

$$F_{|r} = \begin{cases} E & \text{if } r > \rho \\ S_1 & \text{if } r \leq \rho \end{cases}$$

Such that (ψ, F) is a 2-sector homogeneous function on B_ρ. Notice then that for a small enough η (depending on C only), all the angles $(\alpha_j)_{j \in J}$ are small enough such that,

$$W(\psi, F) \leq W(\psi, E^h) - \rho^2 \sum_{j \in J, S^j \subset E} \left(\frac{1}{2} - C\alpha^j \right) m^j \leq W(\psi, E^h) - \frac{1}{8} \rho^2 \sum_{j \in J} m^j$$

Now, we decompose $W(\psi, E_1^h)$ using the notations of lemma 7.24

$$W(\psi, E^h) = W^{\alpha_1}(\psi | S_1) + W^{\alpha_2}(\psi | S_2) + \sum_{j \in J} \left(\overline{W}^{\alpha_j}(\psi | S_j) + \sum_{n \geq 1} W_n^{\alpha_j}(\psi | S_j) \right)$$

and likewise for $\mathcal{F}$. Then by the choice of h in lemma 7.26,

$$\begin{aligned} W(\psi, E^h) &= W^{\alpha_1}(\phi^h | S_1) + W^{\alpha_2}(\phi^h | S_2) + \sum_{j \in J} \left(\overline{W}^{\alpha_j}(\phi^h | S_j) + \sum_{n \geq 1} W_n^{\alpha_j}(\psi | S_j) \right) \\ &= W^{\alpha_1}(\phi^h | S_1) + W^{\alpha_2}(\phi^h | S_2) \\ &+ \sum_{j \in J} \left(\overline{W}^{\alpha_j}(\phi^h | S_j) + \left(1 - \frac{1}{2} \rho^{2/3} \right) \sum_{n \geq 1} W_n^{\alpha_j}(\phi^h | S_j) \right) \\ &= W(\phi^h, E^h) - \frac{1}{4} \rho^{2/3} \sum_{j \in J} \sum_{n \geq 1} \mathcal{F}_n^{\alpha_j}(\phi | S_j) \end{aligned}$$

And so

$$W(\psi, F) \leq W(\phi^h, E^h) - \frac{1}{4} \rho^{2/3} \sum_{j \in J} \sum_{n \geq 1} \mathcal{F}_n^{\alpha_j}(\phi | S_j) - \frac{1}{8} \rho^2 \sum_{j \in J} m^j$$

let $\nu > 0$ that will be fixed later, suppose that the right-hand side is not lower than $(1 - \nu) W(\phi_1^h, E_1^h)$, meaning that

$$\frac{1}{4} \rho^{2/3} \sum_{j \in J} \sum_{n \geq 1} \mathcal{F}_n^{\alpha_j}(\phi | S_j) + \frac{1}{8} \rho^2 \sum_{j \in J} m^j \leq \frac{\nu}{2} \mathcal{F}(\phi, E)$$

Then

$$\begin{aligned} \mathcal{F}(\psi_{|\rho}, F) &= \mathcal{F}(\psi_{|\rho}, E) + (\mathcal{F}(\psi_{|\rho}, F) - \mathcal{F}(\psi_{|\rho}, E)) \\ &= \mathcal{F}(\phi, E) - \sum_{j \in J} \sum_{n \geq 1} \mathcal{F}_n^{\alpha_j}(\phi | S_j) - \sum_{j \in J : S_j \subset E} m_j \\ &\geq \left(1 - \frac{4\nu}{\rho^2} \right) \mathcal{F}(\phi, E) \end{aligned}$$

Choosing a small enough ν (depending on ρ, which is fixed) we obtain $\mathcal{F}(\psi_\rho, F) \geq \frac{1}{2} \mathcal{F}(\phi, E)$, which is the result. □

Lemma 7.28. *There exists three universal constants ρ, ν, η such that, for any $\phi \in H^1(\mathbb{S}^1)$, $E \subset \mathbb{S}^1$ with only two sectors with angles close to π, and $W(\phi^h, E^h) < \infty$, dist$((\phi^h, E^h), \mathcal{B}^{high}) \leq \eta$, then there exists $\psi \in H^1(B_1)$, $F \subset B_1$ that coincides with (ϕ, E) on the boundary, such that $W(\psi, F) \leq W(\phi^h, E^h)$ and more precisely*

- *(ψ, F) is 1-homogeneous on B_ρ and up to rotation is of the form*

$$\begin{aligned}
F_\rho =& [0, \pi(1+\delta)] \\
\psi_\rho(\theta) =& \begin{cases} \psi_\rho^+(+\theta) & \text{in } F_\rho \\ \psi_\rho^-(-\theta) & \text{in } F_\rho^c \end{cases}, \quad \text{where} \\
\psi^\pm(\theta) =& \left(1 + \sum_{\text{odd } n>1} \frac{4(1\pm\delta)^2}{\pi n\left(n^2-(1\pm\delta)^2\right)} \sin\left(\frac{n\theta}{(1\pm\delta)}\right)\right) m \\
&+ \left(\cos\left(\frac{\theta}{(1\pm\delta)}\right) - \sum_{\text{even } n} \frac{4n\left(1-(1\pm\delta)^2\right)}{\pi(n^2-1)\left(n^2-(1\pm\delta)^2\right)} \sin\left(\frac{n\theta}{(1\pm\delta)}\right)\right) l \\
&+ c_1^\pm \sin\left(\frac{\theta}{(1\pm\delta)}\right)
\end{aligned}$$

  for some $\delta, m, l, c_1^+, c_1^- \in \mathbb{R}^5$.

- *Either $W(\psi, F) \leq (1-\nu) W(\phi^h, E^h)$ or $\mathcal{F}(\psi_\rho, F_\rho) \geq \frac{1}{2}\mathcal{F}(\phi, E)$.*

Proof. We proceed the same way as previously. Write $S^\pm = \pm[0, \pi(1\pm\delta)]$ the two sectors, and decompose ϕ on each sector as in lemma 7.25. Let h be as defined in lemma 7.26, then we let

$$\psi_{|r}|S^\pm(\theta) = \phi|S^\pm(\theta) - (1-h(r)) \sum_{n\geq 1} b_n^\pm \sin\left(\frac{n\theta}{1\pm\delta}\right)$$

We use the notations of lemma 7.25, and we will denote $W^\pm$ (resp $\mathcal{F}^\pm$) for $W^{\pi(1\pm\delta)}$. Then

$$\begin{aligned}
W(\psi, E^h) &= \left(\overline{W}_1^+(\psi|S^+) + \overline{W}_1^-(\psi|S^-)\right) + \sum_{n\geq 2}\left(W_n^+(\psi|S^+) + W_n^-(\psi|S^-)\right) \\
&\leq \left(\overline{W}_1^+(\phi^h|S^+) + \overline{W}_1^-(\phi^h|S^-)\right) \\
&+ \left(1 - \frac{1}{2}\rho^{2/3}\right) \sum_{n\geq 2}\left(W_n^+(\phi^h|S^+) + W_n^-(\phi^h|S^-)\right)
\end{aligned}$$

So if the right-hand side is larger than $(1-\nu)W(\phi^h, E^h)$ then

$$\frac{1}{2}\rho^{2/3} \sum_{n\geq 2}\left(\mathcal{F}_n^+(\phi|S^+) + \mathcal{F}_n^-(\phi|S^-)\right) \leq \nu \mathcal{F}(\phi, E),$$

so

$$\begin{aligned}
\mathcal{F}(\psi_{|\rho}, E) &= \mathcal{F}(\phi, E) - \sum_{n\geq 2}\left(\mathcal{F}_n^+(\phi|S^+) + \mathcal{F}_n^-(\phi|S^-)\right) \\
&\geq \left(1 - \frac{2\nu}{\rho^{2/3}}\right)\mathcal{F}(\phi, E) \\
&\geq \frac{1}{2}\mathcal{F}(\phi_1, E_1) \text{ when } \nu \leq \frac{1}{4}\rho^{2/3}
\end{aligned}$$

□

7.5.5 Gradient descent on low-frequency solutions near large blow-up

Here we treat the most critical case; the case where E and E^c are both one sector of angle $\pi\alpha = \pi(1 \pm \delta) \approx \pi$ for a small δ, and where the "b_n" coefficient defined previously are all 0, meaning that (ϕ, E) takes the following form:

$$\begin{aligned}
E =& [0, \pi(1+\delta)] \\
\phi(\theta) =& \begin{cases} \phi^+(+\theta) & \text{in } E \\ \phi^-(-\theta) & \text{in } E^c \end{cases}, \quad \text{where} \\
\phi^\pm(\theta) =& \left(1 + \sum_{\text{odd } n>1} \frac{4(1\pm\delta)^2}{\pi n\left(n^2 - (1\pm\delta)^2\right)} \sin\left(\frac{n\theta}{(1\pm\delta)}\right)\right) m \\
&+ \left(\cos\left(\frac{\theta}{(1\pm\delta)}\right) - \sum_{\text{even } n} \frac{4n\left(1-(1\pm\delta)^2\right)}{\pi(n^2-1)\left(n^2-(1\pm\delta)^2\right)} \sin\left(\frac{n\theta}{(1\pm\delta)}\right)\right) l \\
&+ c_1^\pm \sin\left(\frac{\theta}{(1\pm\delta)}\right)
\end{aligned} \tag{7.6}$$

meaning that (ϕ, E) is a smooth function of the five parameters $P := (m, l, \delta, c_1^+, c_1^-)$ (that we write $(\phi(P), E(P))$), with the constraint that $0 \le |l| \le m$. In this situation, $\mathcal{F}(\phi, E)$ is actually an analytic function of the parameters P;

$$\begin{aligned}
\mathscr{F}(P) :=& \mathcal{F}(\phi(P), E(P)) \\
=& 2m - \left(2\pi + \frac{8}{\pi} \sum_{\text{odd } n>1} \left(\frac{(1+\delta)^3}{n^2(n^2-(1+\delta)^2)} + \frac{(1-\delta)^3}{n^2(n^2-(1-\delta)^2)}\right)\right) m^2 \\
&+ \left(\frac{\pi\delta^2}{1-\delta^2} - \frac{8}{\pi} \sum_{\text{even } n} \left(\frac{n^2(1-(1+\delta)^2)^2}{(1+\delta)(n^2-1)^2(n^2-(1+\delta)^2)} + \frac{n^2(1-(1-\delta)^2)^2}{(1-\delta)(n^2-1)^2(n^2-(1-\delta)^2)}\right)\right) l^2 \\
&+ \frac{\pi}{2}\frac{(1+\delta)^2-1}{(1+\delta)}|c_1^+|^2 - 4(1+\delta)mc_1^+ + \frac{\pi}{2}\frac{(1-\delta)^2-1}{(1-\delta)}|c_1^-|^2 - 4(1-\delta)mc_1^-
\end{aligned}$$

Notice however two important facts; first there is a constraint on the parameter P, it must evolve in the admissible subspace

$$\left\{(m, l, \delta, c_1^+, c_1^-) : 0 \le |l| \le m\right\}$$

Secondly, lemma 7.19 was only valid for a homogeneous E; here if we consider (ϕ, E) that is smoothly parametrized by a parameter curve $P(r)$, meaning $\phi_{|r} = \phi(P(r))$, $E_{|r} = E(P(r))$ for some curve $(P(r))_{0<r<1}$ in the admissible parameter space, then $W(\phi, E)$ is given by

$$\begin{aligned}
W(\phi, E) = \int_0^1 \left(r\mathcal{F}(\phi_r) + rR(\phi_r, E_r) + r^3 \int_{\mathbb{S}^1} |\partial_r \phi_r|^2 \mathrm{d}\theta\right) \mathrm{d}r, \\
\text{where } R(\phi_r, E_r) = \pi(m(r) - l(r))\left(\sqrt{1 + r^2\delta'(r)^2} - 1\right)\mathrm{d}r
\end{aligned}$$

So it doesn't split as nicely as the case where E is homogeneous. In our case since m and l will be arbitrarily small, then the additional term R verifies $R \ll \delta'(r)$.

We start by a logarithmic reparametrization of the radius; let $\kappa > 0$ that will be fixed later, and $P = P(t)_{t\ge 0}$ a curve in the parameter space

$$P(t) = (m(t), l(t), \delta(t), c_1^+(t), c_1^-(t)),$$

meaning a smooth curve in $\mathbb{R}^5$. We associate to it the function $\phi \in H^1(B_1)$ and the set E given by

$$(\phi_{|r}, E_{|r}) := (\phi(P(e^{-\kappa t})), E(P(e^{-\kappa t}))).$$

In these circumstances, $W(\phi, E)$ takes the expression

$$\begin{aligned} W(\phi, E) &=: \mathscr{G}(P(t)_{t\geq 0}) \\ &= \int_0^{+\infty} \left(\frac{1}{\kappa} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \phi \circ P(t) \right\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{\kappa} \mathscr{R}_\kappa(t, P(t), P'(t)) + \kappa \mathscr{F}(P(t)) \right) e^{-2\kappa t} \mathrm{d}t \end{aligned}$$

Where $\mathscr{R}_\kappa(t, P(t), P'(t)) = \kappa^2 \pi (m(t) - l(t)) \left(\sqrt{1 + \frac{1}{\kappa^2} \delta'(t)^2} - 1 \right)$ is defined as R above, and notice in particular that $|\mathscr{R}_\kappa(t, P(t), P'(t))| \leq \pi m(t) \delta'(t)^2$

Lemma 7.29. *There exists three universal positive constants ϑ, γ, η, such that for any ϕ, E of the form (7.6) with dist$((\phi, E), \mathcal{B}^{high}) \leq \eta$, then there exists (ψ, F) such that $(\psi_{|1}, F_{|1}) = (\phi, E)$ and*

$$W(\psi, F) \leq \left(1 - \gamma \wedge W(\phi^h, E^h)^\vartheta\right) W(\phi^h, E^h).$$

Proof. With our change of variable in the parameter space, it is equivalent to prove the following; there exists three universal positive constants ϑ, γ, η, such that for any admissible $P_0 \in \mathbb{R}^5$ with

$$\mathrm{dist}\,(P_0, Z) \leq \eta, \text{ where } Z := \left\{ (0, 0, 0, t, t), \frac{1}{4} \leq t \leq 1 \right\}$$

then there exists $P \in W^{1,1}(\mathbb{R}_+, \mathbb{R}^5)$ such that $P(0) = P_0$, $P(t)$ is admissible for all t and

$$\mathscr{G}(P) \leq \left(1 - \gamma \wedge \mathscr{G}(P_0)^\vartheta\right) \mathscr{G}(P_0),$$

where by abuse of notation P_0 is seen as the constant function egal to P_0.

This will be done by a gradient descent on $\mathscr{F}$. Notice that $\mathscr{F}$ is analytic in a neighbourhood of $Z \subset \mathscr{F}^{-1}(0)$, so by the Lojasiewicz inequality there exists a $\eta > 0$, $\vartheta \in (0, 1)$, such that in a 2η-neighbourhood of this set (denoted $Z_{2\eta}$) we have

$$|\nabla \mathscr{F}|^2 \geq |\mathscr{F}|^{1+\vartheta}$$

Let $\overline{T} := \frac{\eta}{\|\nabla \mathscr{F}\|_{L^\infty(Z_{2\eta})}}$. Then if $P_0 \in Z_\eta$, let $\overline{P}$ the variation defined on $[0, \overline{T}]$ as

$$\begin{cases} \overline{P}(0) = P_0 \\ \overline{P}'(t) = -\nabla \mathscr{F}(\overline{P}(t)) \quad (0 \leq t \leq \overline{T}) \end{cases}$$

the definition of $\overline{T}$ makes it so $\overline{P}$ is well-defined and in $Z_{2\eta}$ at all time. Then we define a stopping time T; either $\mathscr{F}(\overline{P}(\overline{T})) \geq \frac{1}{2} \mathscr{F}(P_0)$ and we let $T := \overline{T}$, or we let T the smallest time for which $\mathscr{F}(\overline{P}(T)) = \frac{1}{2} \mathscr{F}(P_0)$, and

$$P(t) = \begin{cases} \overline{P}(t) & \text{if } 0 \leq t \leq T \\ \overline{P}(T) & \text{if } t \geq T \end{cases}$$

For the sake of clarity let us suppose for now that $P(t)$ is admissible for all t, meaning that at all time we have $0 \leq |l(t)| \leq m(t)$. Then, just using the fact that P is constant for $t \geq T$ we get

$$\begin{aligned} \mathscr{G}(P) - \mathscr{G}(P_0) &= \int_0^{+\infty} \left(\frac{1}{\kappa} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \phi \circ P(t) \right\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{\kappa} \mathscr{R}_\kappa(t, P(t), P'(t)) + \kappa(\mathscr{F}(P(t)) - \mathscr{F}(P_0)) \right) e^{-2\kappa t} \mathrm{d}t \\ &= \int_0^{T} \left(\frac{1}{\kappa} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \phi \circ P(t) \right\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{\kappa} \mathscr{R}_\kappa(t, P(t), P'(t)) + \frac{1}{2} \langle \nabla \mathscr{F}(P(t)), P'(t) \rangle \right) e^{-2\kappa t} \mathrm{d}t \end{aligned}$$

Then,

$$\frac{1}{\kappa} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \phi \circ P(t) \right\|^2_{L^2(\mathbb{S}^1)} = \frac{1}{\kappa} \left\| D\phi(P(t)) \cdot P'(t) \right\|^2_{L^2(\mathbb{S}^1)} \leq \frac{C}{\kappa} |P'(t)|^2 \leq \frac{1}{8} |P'(t)|^2$$

for some universal constant C that depends only on η, and for a large enough κ (depending on η) that is now fixed (notice that we used the equation of P here). Similarly,

$$\frac{1}{\kappa}\mathscr{R}_\kappa(t,P(t),P'(t)) \leq \frac{1}{\kappa}\pi m(t)\delta'(t)^2 \leq \frac{1}{8}|P'(t)|^2$$

when m (so η) is small enough and if we only consider $t \leq 1$ (which we can do without loss of generality), then we get as a consequence that

$$\mathscr{G}(P) - \mathscr{G}(P_0) \leq \frac{1}{4}\int_0^T \langle \nabla \mathscr{F}(P(t)), P'(t)\rangle e^{-2\kappa t}\mathrm{d}t$$

Let us now separate two cases:

- If $\mathscr{F}(P(T)) = \frac{1}{2}\mathscr{F}(P_0)$, then by a simple integration by part

$$\begin{aligned}\mathscr{G}(P) - \mathscr{G}(P_0) &\leq \frac{1}{4}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathscr{F}(P(t)) - \mathscr{F}(P_0)\Big]e^{-2\kappa t}\mathrm{d}t \\ &= \frac{e^{-2\kappa T}}{4}\Big[\mathscr{F}(P(t)) - \mathscr{F}(P_0)\Big] + \frac{\kappa}{2}\int_0^T \Big[\mathscr{F}(P(t)) - \mathscr{F}(P_0)\Big]e^{-2\kappa t}\mathrm{d}t \\ &\leq -\frac{e^{-2\kappa T}}{4}\mathscr{G}(P_0)\end{aligned}$$

- If $\mathscr{F}(P(T)) > \frac{1}{2}\mathscr{F}(P_0)$, then

$$\begin{aligned}\mathscr{G}(P) - \mathscr{G}(P_0) &\leq \frac{1}{4}\int_0^T \langle \nabla \mathscr{F}(P(t)), P'(t)\rangle e^{-2\kappa t}\mathrm{d}t \\ &= -\frac{1}{4}\int_0^T |\nabla \mathscr{F}(P(t))|^2 e^{-2\kappa t}\mathrm{d}t \\ &\leq -\frac{1}{4}\int_0^T |\mathscr{F}(P(t))|^{1+\vartheta} e^{-2\kappa t}\mathrm{d}t \\ &\leq -\frac{1-e^{-2\kappa T}}{2^{2-\vartheta}\kappa}\mathscr{G}(P_0)^{1+\vartheta}\end{aligned}$$

This ends the proof in the case where $P(t)$ stays in the admissible space

$$\Big\{(m,l,\delta,c_1^+,c_1^-) : 0 \leq |l| \leq m\Big\}$$

Let us first detail what happens should there exists some $\tau \in [0,T)$ (supposed to be minimal) such that $l(\tau) = \pm m(\tau)$ with $m(t) > 0$ on $[0,\tau]$. Then instead of stopping P at time T we stop it at time τ. This way we obtain (ψ, F) a competitor for (ϕ^h, E^h) such that

- (ψ, F) is 1-homogeneous on B_ρ with $\rho = e^{-\kappa\tau} \geq e^{-\kappa\overline{T}}$.
- $W(\psi,F) - W(\phi_1^h, E_1^h) = \mathscr{G}(P) - \mathscr{G}(P_0) \leq -\frac{1}{4}\int_0^\tau |\nabla\mathscr{F}\circ P|^2\mathrm{d}t \leq 0$.
- $\mathcal{F}(\psi_\rho, F_\rho) = \mathscr{F}(P(\tau)) \geq \frac{1}{2}\mathscr{F}(P_0) = \mathcal{F}(\phi_1^h, E_1^h)$.

So by the remark 7.20, it is enough to prove the epiperimetric inequality for $(\psi_{|\rho}, F_{|\rho})$; this is done in the exact same way as previously except the gradient descent is done in the submanifold

$$\{(m,l,\delta,c_1^+,c_1^-) : 0 < |l| = m\}$$

which is the union of two hyperplanes. From this we define new constants η', ϑ' and we do the exact same proof; in the end we need to take the minimum between all these constants. If we hit the constraint $m = 0$, then we stop the descent at the time the constraint is hit and do a new gradient descent on $\{(0,0,\delta,c_1^+,c_1^-)\}$ (notice this one may also be done by "by hand" by simple harmonic extension of $(1_{E^c} - 1_E 0)\phi$ and we get a "strong" epiperimetric inequality, although it is not needed in our case). □

7.6 Nondegeneracy and regularity of the contact set

Proposition 7.30. *Let $(u,E) \in \mathscr{M}(\overline{B_1})$. We let $r(u,E)$ be the smallest $r \in [0,1]$ such that $W(u_r,E_r)^\vartheta \geq \gamma$. Then*

$$r(u,E) \geq 1 \wedge \left(\frac{\gamma^{1/\vartheta}}{W(u,E)}\right)^{\frac{1}{2\gamma}}$$

Moreover, for any $r(u,E) \leq s < r \leq 1$,

$$\frac{W(u_s,E_s)}{W(u_r,E_r)} \leq \left(\frac{s}{r}\right)^{2\gamma},$$

and for any $0 < s < r \leq r(u,E)$,

$$W(u_s,E_s)^{-\vartheta} - W(u_r,E_r)^{-\vartheta} \geq 2\vartheta \log\left(\frac{r}{s}\right).$$

Proof. Suppose that $r(u,E) < 1$. Let $r \in (r(u,E),1)$, then the monotonocity formula and the epiperimetric inequality give

$$\frac{\mathrm{d}}{\mathrm{d}r} W(u_r,E_r) \geq \frac{2}{r}\left(W(u_r^h,E_r^h) - W(u_r,E_r)\right) \geq \frac{2\gamma}{r} W(u_r,E_r)$$

Integrating this from $r(u,E)$ to 1 we get the lower bound on $r(u,E)$, and then integrating this from s to r for $r(u,E) \leq s < r \leq 1$ we get the first inequality. For the second inequality, with the same computation we get

$$\frac{\mathrm{d}}{\mathrm{d}r} W(u_r,E_r) \geq \frac{2}{r}\left(W(u_r^h,E_r^h) - W(u_r,E_r)\right) \geq \frac{2}{r} W(u_r,E_r)^{1+\vartheta},$$

and the rest follows by integration. □

Proposition 7.31. *Let $(u,E) \in \mathscr{M}(\overline{B_2})$ be such that $W(u_2,E_2)^\vartheta \leq \gamma$ and $u(0) = 0$. Then there exists a universal $C > 0$ such that for any $0 < s < r \leq 1$ we have*

$$\|u_r - u_s\|_{L^1(\mathbb{S}^1)} \leq C \log\left(\frac{1}{r}\right)^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)}$$

In all the following we will denote $\alpha := \frac{1}{2}\left(\frac{1}{\vartheta} - 1\right)$ the exponent that appears here.

Proof. We have $W(u_2,E_2)^\vartheta \leq \gamma$, so the logarithmic epiperimetric applies for any radius lower than 2. In particular, using the previous proposition we get for any $r \leq 1$ that

$$W(u_r,E_r)^{-\vartheta} - W(u_1,E_1)^{-\vartheta} \geq 2\vartheta \log\left(\frac{1}{r}\right),$$

so

$$W(u_r,E_r) \leq (2\vartheta)^{-\frac{1}{\vartheta}} \log\left(\frac{1}{r}\right)^{-\frac{1}{\vartheta}}$$

Let $0 < s < r \leq 1$, we write $w(r) = W(u_r,E_r)$ and $D(r) = \int_{\mathbb{S}^1} |x \cdot \nabla u - u|^2$. Then

$$\begin{aligned}\|u_r - u_s\|_{L^1(\mathbb{S}^1)} &= \int_{\mathbb{S}^1} \left|\int_s^r \frac{\mathrm{d}}{\mathrm{d}\rho} u_\rho(x)\mathrm{d}\rho\right| \mathrm{d}\mathscr{H}^{n-1} = \int_{\mathbb{S}^1} \left|\int_s^r \frac{x\dot{\nabla} u_\rho(x) - u_\rho(x)}{\rho}\mathrm{d}\rho\right| \mathrm{d}\mathscr{H}^{n-1} \\ &\leq \sqrt{2\pi}\int_s^r \frac{\sqrt{D(\rho)}}{\rho}\mathrm{d}\rho \leq \sqrt{\pi}\int_s^r \sqrt{\frac{w'(\rho)}{\rho}}\mathrm{d}\rho \leq \sqrt{\pi \log\left(\frac{r}{s}\right) w(r)}\end{aligned}$$

In particular, for $s \geq r^2$, we get

$$\|u_r - u_s\|_{L^1(\mathbb{S}^1)} \leq \pi^{\frac{1}{2}}(2\vartheta)^{-\frac{1}{2\vartheta}} \log\left(\frac{1}{r}\right)^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)}$$

Denoting $r_k := e^{-2^k}$, we thus get that for any $k > l$ in $\mathbb{N}^*$,

$$\begin{aligned}\|u_{r_l} - u_{r_k}\|_{L^1(\mathbb{S}^1)} &\leq \sum_{l \leq j < k} \|u_{r_j} - u_{r_j^2}\|_{L^1(\mathbb{S}^1)} \leq \pi^{\frac{1}{2}}(2\vartheta)^{-\frac{1}{2\vartheta}} \sum_{l \leq j < k} 2^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)j} \\ &\leq C 2^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)l} = C \log\left(\frac{1}{r_l}\right)^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)}\end{aligned}$$

And so for any $0 < s < r < e^{-1}$, by finding l, k such that $r \in [r_l^2, r_l]$ and $s \in [r_l^2, r_l]$, we obtain

$$\|u_r - u_s\|_{L^1(\mathbb{S}^1)} \leq C \log\left(\frac{1}{r}\right)^{-\frac{1}{2}\left(\frac{1}{\vartheta}-1\right)}$$

For $r \in [e^{-1}, 1]$ it is obvious up to taking a large enough C due to the uniform bound on $\|u_r\|_{L^1(\mathbb{S}^1)}$, so we get the result. □

Corollary 7.32. *There exists $\mathfrak{R} > 1, C > 0$ such that, for any $(u, E) \in \mathscr{M}(B_{\mathfrak{R}})$ with $u(0) = 0$, and for any $0 < s < r \leq 1$ we have*

$$\|u_r - u_s\|_{L^1(\mathbb{S}^1)} \leq C \log\left(\frac{1}{r}\right)^{-\alpha}$$

Proof. Due to the uniform bound of proposition 7.4 we know that $W(u_{\mathfrak{R}/2}, E_{\mathfrak{R}/2}) \leq C$ for some uniform C. Then, we know $r(u_{\mathfrak{R}/2}, E_{\mathfrak{R}/2}) \geq c$ for some uniform $c > 0$, and we choose $\mathfrak{R}$ large enough such that $\mathfrak{R}c > 2$, meaning that for any $(u, E) \in \mathscr{M}(B_{\mathfrak{R}})$ with $u(0) = 0$ we have $W(u_1, E_1)^{\vartheta} \leq \gamma$, so the previous result applies. □

$\mathfrak{R}$ is now a constant that is fixed in all the following.

7.6.1 Nondegeneracy lemma

Proposition 7.33. *There exists $\epsilon, r > 0$ such that, if $(u, E) \in \mathscr{M}(B_1)$ verifies $\|u\|_{L^\infty(B_1)} \leq \epsilon$ then*

$$\partial^* E \cap \{u > 0\} \cap B_r = \emptyset$$

Proof. We let $\epsilon, r > 0$ that will be fixed small enough at the end, with ϵ depending on r. Suppose that $\partial^* E \cap \{u > 0\} \cap B_r$ contains some point x. In all the following, we suppose $r \leq \frac{1}{24}$.

- Claim: $\{u = 0\} \cap B_{2r} \neq \emptyset$. Indeed, suppose that it is not the case. Then $\partial^* E \cap \{u > 0\}$ contains an arc from $\partial^* E \cap B_r$ to $\partial^* E \cap B_{2r}$, of length at least r. Let $\varphi \in \mathcal{C}_c^\infty(B_1, \mathbb{R}_+)$ be such that $\varphi = 1$ on B_{2r}, then

  $$\frac{1}{2} r \leq \int_{B_{2r} \setminus B_r} \varphi \Delta u \mathrm{d}\mathscr{L}^2 \leq \int_{B_1} \varphi \Delta u \mathrm{d}\mathscr{L}^2 \leq \|\Delta \varphi\|_{L^\infty(B_1)} \epsilon$$

  which is absurd for a small enough ϵ depending on r. In all the following we let x_0 be a projection of x on $\{u = 0\}$, and so we know that $|x - x_0| < 3r$.

- Since x_0 is a projection of x on $\{u = 0\}$, this means that $\mathscr{H}^1(\partial^* E \cap \{u > 0\} \cap B_{x,r}) \geq 2|x - x_0|$. We will consider the rescaling

  $$(v, F) = (u_{x_0, 3|x-x_0|}, F_{x_0, 3|x-x_0|}).$$

We write $y := \frac{x-x_0}{3|x-x_0|}$, and we know that $\Delta v(B_{y,1/3}) \geq 1/3$. We let $\varphi \in \mathcal{C}_c^\infty(B_1, \mathbb{R}_+)$ be such that $\varphi_{|B_{y,1/3}} \equiv 1$, and integrating φ against Δv we get

$$\frac{1}{3} \leq \int_{B_1} v \Delta\varphi \mathrm{d}\mathscr{L}^2 \leq \|\Delta\varphi\|_{L^\infty(B_1)} \int_{B_1} v \mathrm{d}\mathscr{L}^2$$

so by subharmonicity of v there is a constant $c > 0$ such that $\int_{\mathbb{S}^1} v \geq c$.

- Using the a priori bounds of proposition 7.3, we have $J_{B_{1/2}}(u, E) \leq C\|u\|_{L^\infty(B_1)} \leq C\epsilon$. In particular

$$W(u_{x_0, \frac{1}{2}-|x_0|}, E_{x_0, \frac{1}{2}-|x_0|}) \leq C\epsilon$$

so for a small enough ϵ we get $r(u_{x_0, \frac{1}{2}-|x_0|}, E_{x_0, \frac{1}{2}-|x_0|}) = 1$, meaning the logarithmic epiperimetric inequality applies. We apply it from $3|x - x_0|$ to $\rho \in [3|x - x_0|, \frac{1}{2} - |x_0|]$, and we get

$$\|u_{x_0,\rho} - v\|_{L^1(\mathbb{S}^1)} \leq C \log\left(\frac{1}{\rho}\right)^{-\alpha}$$

And so

$$c \leq \|v\|_{L^1(\mathbb{S}^1)} \leq \|u_{x_0,\rho}\|_{L^1(\mathbb{S}^1)} + \|u_{x_0,\rho} - v\|_{L^1(\mathbb{S}^1)} \leq C\left(\frac{\epsilon}{\rho} + \log\left(\frac{1}{\rho}\right)^{-\alpha}\right)$$

Choosing $\rho := 10r$ (which is possible since $r \leq \frac{1}{24}$ so $3|x - x_0|(\leq 9r) \leq 10r \leq \frac{1}{2} - |x_0|$) we obtain that for some universal $c > 0$,

$$c \leq \frac{\epsilon}{10r} + \log\left(\frac{1}{10r}\right)^{-\alpha}$$

which for a small enough r and a small enough $\epsilon \ll r$ is a contradiction.

□

Corollary 7.34. *Let $r, \epsilon > 0$ be defined as previously, then for any $(u, E) \in \mathscr{M}(B_1)$ with $\|u\|_{L^\infty(B_1)} \leq \epsilon$ and $u(0) = 0$, there exists $n \in \mathbb{N}^*$ and a conformal map $g : B_\rho \to \mathbb{R}^2$ for some $\rho \in (0, r]$ with $g(0) = 0$, such that*

$$u_{|B_\rho} \equiv |\mathrm{Re}\,(g^n)|\,,\ E \cap B_\rho = g^{-1}\left(\{w : \mathrm{Re}\,(w^n) > 0\}\right)$$

Proof. With the previous result, we know that for a small enough ϵ there is a harmonic function such that $u = |h|$ and $E = \{h > 0\}$ in a neighbourhood of the origin. Let f be the unique holomorphic function such that $f(0) = 0$ and $\mathrm{Re}(f) = h$, then there exists a number $n \in \mathbb{N}^*$, a radius $\rho \in (0, r]$ and a holomorphic function g defined on B_ρ such that $g(0) = 0$, $g' \neq 0$, and $f = g^n$, which proves the result (up to taking a slighty smaller ρ so that g is injective). □

7.6.2 Regularity of the contact set

For two sets $E, F \subset \mathbb{R}^2$, we write

$$d_{B_1}(E, F) = \max\left(\sup_{x \in E \cap B_1} \inf_{y \in F} |x - y|, \sup_{y \in F \cap B_1} \inf_{x \in E} |x - y|\right)$$

Proposition 7.35. *Let $(u, E) \in \mathscr{M}(D)$ for some open set D and $\Omega \Subset D$. There exists $\overline{r} = \overline{r}(\Omega, D)$ such that*

- *For any $x \in \{u = 0\} \cap \Omega$, there is a unique blow-up (u_x, E_x) at x and for any $r \in (0, \overline{r})$,*

$$\|u_{x,r} - u_x\|_{L^1(\mathbb{S}^1)} \lesssim \log\left(\frac{1}{r}\right)^{-\alpha}$$
$$d_{B_1}(E_{x,r}, E_x) \lesssim \frac{\log\left(\frac{1}{r}\right)^{-\frac{\alpha}{3}}}{\|u_x\|^{\frac{1}{3}}_{L^1(\mathbb{S}^1)}}$$

  with the second inequality being trivial if $u_x \equiv 0$.

- *For any $x, y \in \{u = 0\} \cap \Omega$ such that $|x - y| \leq \overline{r}$, we have*

$$\|u_x - u_y\|_{L^1(\mathbb{S}^1)} \lesssim \log\left(\frac{1}{|x-y|}\right)^{-\alpha}$$
$$d_{B_1}(E_x, E_y) \lesssim \frac{\log\left(\frac{1}{|x-y|}\right)^{-\frac{\alpha}{3}}}{\|u_x\|^{\frac{1}{3}}_{L^1(\mathbb{S}^1)}}$$

Proof. We let $\mathfrak{R} > 0$ be defined as in corollary 7.32. $\overline{r}$ will be different in the three following points; in the end it is enough to take the minimum of the three.

- We chose $\overline{r} = \min\left(1, \frac{1}{\mathfrak{R}}\text{dist}(\partial D, \partial\Omega)\right)$, such that for any $x \in \Omega \cap \{u = 0\}$, $B_{x,\overline{r}} \subset B_{x,\mathfrak{R}\overline{r}} \subset D$. Thus, corollary 7.32 applies to (u, E) on $B_{x,\overline{r}}$ and as a consequence $(u_{x,r})_{r\to 0}$ is a Cauchy sequence in $L^1(\mathbb{S}^1)$ and has a unique limit as $r \to 0$. Passing the estimate of corollary 7.32 to the limit this gives the first point of the proposition.

  Moreover, either $u_x = 0$ and (u, E) is thus given by corollary 7.34 in a neighbourhood of E, so the limit of $E_{x,r}$ is also unique, or $u_x \neq 0$ and the convergence of $E_{x,r}$ to the blow-up E_x is in the local Hausdorff sense. The quantification of $d_{B_1}(E_x, E_{x,r})$ will be treated separately later.

- Let $r := \sqrt{|x - y|}$, which is supposed small enough such that $B_{x,Rr} \cup B_{y,Rr} \cup \text{Conv}(B_{x,r} \cup B_{y,r}) \Subset D$, which is possible by compactness when $|x - y| \leq \overline{r}(\Omega, D)$. Then we compare directly

$$\|u_x - u_y\|_{L^1(\mathbb{S}^1)} \leq \|u_x - u_{x,r}\|_{L^1(\mathbb{S}^1)} + \|u_y - u_{y,r}\|_{L^1(\mathbb{S}^1)} + \|u_{x,r} - u_{y,r}\|_{L^1(\mathbb{S}^1)}$$

  The first two terms are bounded by $C \log\left(\frac{1}{r}\right)^{-\alpha}$ by corollary 7.32. For the third one, notice $u_{y,r}(z) = u_{x,r}\left(z + \frac{y-x}{r}\right)$, so using the uniform $\mathcal{C}^{0,\frac{1}{n+1}}(B_2)$ bound on $u_{x,r}$ we get

$$\|u_{y,r} - u_{x,r}\|_{L^1(\mathbb{S}^1)} \leq \int_{\mathbb{S}^1} \left|u_{x,r}\left(z + \frac{y-x}{r}\right) - u_{x,r}(z)\right| \mathrm{d}\mathscr{H}^1(z) \lesssim r^{-\frac{1}{n+1}}|x - y|^{\frac{1}{n+1}}$$

  As a consequence,

$$\|u_x - u_y\|_{L^1(\mathbb{S}^1)} \lesssim \left(\log\left(\frac{1}{|x-y|}\right)^{-\alpha} + |x - y|^{\frac{1}{2(n+1)}}\right) \lesssim \log\left(\frac{1}{|x-y|}\right)^{-\alpha}$$

- We prove the bound on $d_{B_1}(E_x, E_{x,r})$. It is enough to prove that the boundaries of these two sets verify the same estimate. Let $z \in B_1 \cap \partial E_{x,r}$ and $\rho := \text{dist}(z, \partial E_x) \in [0, 1]$, that we suppose to be strictly positive (or else there is nothing to prove). Now $\partial E_{x,r} \cap B_{z,\rho/2}$ has length at least ρ. Moreover:

- $\Delta u_{x,r} \geq \frac{1}{2}\mathscr{H}^1_{\lfloor \partial^* E_{x,r} \cap \{u_{x,r}>0\}}$,
- For all $w \in \{u_{x,r} = 0\} \cap B_{z,\rho}$,

$$\|(u_{x,r})_w - u_x\|_{L^1(\mathbb{S}^1)} \lesssim \log\left(\frac{1}{r}\right)^{-\alpha}$$

So for a small enough r,

$$\Delta u_{x,r} \geq \frac{\|u_x\|_{L^1(\mathbb{S}^1)}}{4}\mathscr{H}^1_{\lfloor \{u=0\}\cap B_{z,\rho}}$$

We let $\varphi \in \mathcal{C}^\infty_c(B_1, \mathbb{R}_+)$ such that $\varphi = 1$ on $B_{1/2}$. We also let $\varphi_{z,\rho}(w) := \varphi(z + \rho w)$. Then

$$\begin{aligned}\frac{\|u_x\|_{L^1(\mathbb{S}^1)}}{4}\rho &\leq \int_{B_1} \varphi_{z,\rho}\Delta u_{x,r}\mathrm{d}\mathscr{L}^2 = \int_{B_1} \varphi_{z,\rho}\Delta(u_{x,r} - u_x)\mathrm{d}\mathscr{L}^2 \\ &= \int_{B_1}(u_{x,r} - u_x)\Delta\varphi_{z,\rho}\mathrm{d}\mathscr{L}^2 \lesssim \frac{1}{\rho^2}\|u_{x,r} - u_x\|_{L^1(\mathbb{S}^1)},\end{aligned}$$

so $\|u_x\|_{L^1(\mathbb{S}^1)}\rho^3 \lesssim \|u_{x,r} - u_x\|_{L^1(\mathbb{S}^1)}$.

The distance of any point of $B_1 \cap \partial E_x$ to $\partial E_{x,r}$ is obtained similarly, so we obtain the bound on $d_{B_1}(E_x, E_{x,r})$.

- Finally, taking $r := \sqrt{|x-y|}$,

$$d_{B_1}(E_x, E_y) \leq d_{B_1}(E_x, E_{x,r}) + d_{B_1}(E_y, E_{y,r}) + C\sqrt{|x-y|}$$

and we use the previous estimate.

□

We can summarize our result in the following.

Theorem 7.36. *Let $(u, E) \in \mathscr{M}(D)$ for some open set $D \subset \mathbb{R}^2$, and $\Omega \Subset D$ an open set. Then $Per(E|\Omega) < \infty$, $\partial^* E \cap \{u > 0\}$ is locally $\mathcal{C}^\infty$ and $\{u = 0\} \cap \Omega$ is included in a finite union of $\mathcal{C}^{1,\log}$ curves. Moreover,*

- *the set $Z := \{z \in D : u_z = 0\}$ is locally finite and we know that for every $x \in Z$, (u, E) has the form*

$$u = |Re(z^n)| \circ g,\ E = g^{-1}\left(\{z : Re(z^n) > 0\}\right)$$

where $n \in \mathbb{N}^*$ and g is a conformal map defined ina neighbourhood of x such that $g(x) = 0$.

- *For every $x \in \{u = 0\} \cap \Omega$ and any small enough r (depending only on (Ω, D)) we have*

$$\|u_{x,r} - u_x\|_{L^1(B_1)} \leq C\log\left(\frac{1}{r}\right)^{-\alpha},\ d_{B_1}(E_x, E_{x,r}) \leq \frac{C\log\left(\frac{1}{r}\right)^{-\frac{\alpha}{3}}}{\|u_x\|^{\frac{1}{3}}_{L^1(\mathbb{S}^1)}}$$

- *A connected component of $\{u = 0\} \cap \Omega$ is a $\mathcal{C}^{1,\log}$ segment that has, at its end x, either $x \in \partial\Omega$, $u_x = 0$, or $u_x(z) = \frac{1}{4}|\nu_x \cdot z|$ where ν_x is the unit normal vector of E_x.*

Proof. The fact that $\{u = 0\} \cap \Omega$ is included in a finite number of $\mathcal{C}^{1,\log}$ curves is obtained by applying the previous proposition 7.35 on a coverinf of $\{u = 0\} \cap \Omega$ with small enough balls. In particular this gives a lower bound on $\mathscr{H}^1(\{u = 0\} \cap \Omega)$, and since $\Delta u \geq \frac{1}{2}\mathscr{H}^1 \lfloor \partial E \cap \{u > 0\}$ this gives us a uniform bound on the whole boundary ∂E. We also get the rectifiability (since both $\partial E \cap \{u > 0\}$ and $\{u = 0\}$ are rectifiable).

Now, the first point is just a rewriting of 7.34, the second point of 7.35, and for the third point ; suppose that we are not is the first two cases, then there is a sequence $z_n \in \partial E \cap \{u > 0\}$ that converges to x. Consider $r_n = 2|x - z_n|$, then up to rotation and extraction on $(z_n)_n$ we get that

$$u_{x,r_n} \underset{n\to\infty}{\longrightarrow} \left(w \mapsto \frac{\lambda}{4}|w_2|\right) \text{ for some } \lambda \in [0,1],$$

with $u_{x,r_n} > 0$ on $B_{\frac{1}{2}e_1,\frac{1}{2}}$, and $\frac{1}{2}e_1 \in \partial E_{x,r_n}$. Since $\partial E_{x,r_n} \cap \{u > 0\} \cap B_{\frac{1}{2}e_1,\frac{1}{2}}$ has length at least 1, then by the transmission condition we obtain

$$\Delta u_{x,r_n}(B_{\tilde{z}_n,1/2}) \geq \frac{1}{2}.$$

So u_x verifies the same estimate, meaning $\lambda = 1$. □

7.7 Study of an example

In this section we study a simple explicit example, which has the advantage of spanning all the (non-zero) blow-ups. This implies that the blow-ups that were classified previously are also solutions, something which wasn't known previously.

Proposition 7.37. *Let $t > 0$, then there is a unique $(u^t, E^t) \in \mathscr{M}(B_1)$ such that $u^t_{|\partial B_1} = t$, $E^t \setminus B_1 = \{x_2 > 0\} \setminus B_1$. Moreover, (u_t, E_t) verifies:*

- *$E^t = \{x_2 > 0\}$, u^t is symmetric around $\{x_1 = 0\}$ and $\{x_2 = 0\}$.*
- *$u^t_{|\{x_2>0\}} \in \mathcal{C}^{1,\alpha}_{loc}(B_1 \cap \{x_2 > 0\})$ for some $\alpha \in (0,1)$.*
- *There exists $0 < t_0 < t_1 < \infty$ and $c_1 > 0$ such that*

$$\inf(u^t) \begin{cases} = 0 & \text{if } t < t_0 \\ \geq c_1(t - t_1) & \text{if } t > t_1 \end{cases}$$

- *There exists $c > 0$ such that $u^t(x_1, 0) > 0$ for some $x_1 \leq 1 - e^{-c/t}$.*

Proof. Notice that for any admissible (u, E), u may be written as $t - \varphi$ for some $\varphi \in H^1_0(B_1, \mathbb{R}_+)$, and then we can always lower the energy by taking the Steiner rearrangement of φ and replacing E with $\{x_2 > 0\}$, as in the example of [140]. For a fixed E the problem is strictly convex, so it has a unique solution. Then

- The symmetry in $\{x_2 = 0\}$ follows from the Steiner rearrangement and the symmetry in $\{x_1 = 0\}$ follows from the uniquness.
- This can be obtained for instance through classic viscosity methods that we shall not detail here. In the next corollary we will only use the continuity of the blow-up on the contact set, which we already know through the epiperimetric inequality in this arguably much simpler case.
- The first point can be obtained applying the non-degeneracy lemma for any small enough t. For the second point, suppose the minimum of u^t is attained is some x; by the transmission conditions and minimum principle we know $x_2 = 0$. The uniform estimate of proposition 7.4 gives that $u^t_{x,1-|x|} \leq C(1 + u^t(x)/(1 - |x|))$ on $B_{1/2}$, meaning

$$u^t \leq C(1 + u^t(x)) \text{ in } B_{x,\frac{1-|x|}{2}}.$$

  Then notice that, extending u^t naturally by t outside B_1, and denoting C the square of center $\frac{1+|x|}{2}$ and side $1 - |x|$ we get

$$\int_{B_1} |\nabla u^t|^2 \mathrm{d}\mathscr{L}^2 \geq \int_C |\nabla u^t|^2 \mathrm{d}\mathscr{L}^2 \geq \left(t - C(1 + \inf(u^t))\right)^2.$$

On the other hand,

$$\int_{B_1} |\nabla u^t|^2 \mathrm{d}\mathscr{L}^2 \leq J_{B_1}(u^t, \{x_2 > 0\}) \leq 2t$$

by using the constant competitor, hence

$$\Big(t - C(1 + \inf(u^t))\Big)^2 \leq 2t.$$

- This is obtained by iterating the previous reasonning in $x_k = (1 - 2^{-k}, 0)$ as long as $u^t(x_k) = 0$.

□

Corollary 7.38. *Let $e \in \mathbb{S}^1$, $\lambda \in (0, 1]$, and*

$$u(x) = \frac{\lambda}{4}|x \cdot e|, \ E = \{x : x \cdot e > 0\}$$

Then (u, E) is a global minimizer.

Notice that all along we knew that any homogenous minimizer necessarily had this form, but not the converse that these are minimizers.

Proof. Let $t > 0$, and u^t the minimizer defined previously, for a small enough t. Let $r(t)$ be the projection of $(0, 0)$ on $\{x_2 = 0, x_1 > 0\} \cap \{u^t > 0\}$ which is non-empty according to the previous result, then

$$r \in [0, r(t)] \mapsto u^t_{(r,0),0}$$

is a continuum of homogeneous non-zero minimizer. Observe that

$$u^t_{(r(t),0),0} = \frac{1}{4}|x_2|, \ \|u^t_{(0,0),0}\|_{L^\infty(B_1)} \underset{t \to 0}{\longrightarrow} 0,$$

hence the result. □

Bibliography

[137] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[138] Dorin Bucur, Alessandro Giacomini, and Mickaël Nahon. Degenerate free discontinuity problems and spectral inequalities in quantitative form. *Arch. Ration. Mech. Anal.*, 242(1):453–483, 2021.

[139] Luis A. Caffarelli, María Soria-Carro, and Pablo Raúl Stinga. Regularity for $C^{1,\alpha}$ interface transmission problems. *Arch. Ration. Mech. Anal.*, 240(1):265–294, 2021.

[140] Serena Guarino Lo Bianco, Domenico Angelo La Manna, and Bozhidar Velichkov. A two-phase problem with Robin conditions on the free boundary. *J. Éc. polytech. Math.*, 8:1–25, 2021.

[141] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[142] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

Chapter 8

Shape optimization of a thermal insulation problem

This is a joint work with Dorin Bucur, Carlo Nitsch, Cristina Trombetti.

Contents

8.1 Introduction 186
8.2 The convection case: proof of Theorem 8.1 189
8.3 Preparatory results: existence and regularity of relaxed solutions 192
8.4 The penalized problem: proof of Theorem 8.2 202
8.5 Proof of Theorem 8.3 206

8.1 Introduction

Given a measurable set $K \subseteq \mathbb{R}^n$ along with a Lipschitz open set $\Omega \supset K$, we consider the energy functional

$$E_\Theta(K,\Omega) := \min\left\{\int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \Theta(v)\mathrm{d}\mathscr{H}^{n-1},\ v \in H^1(\Omega,[0,1]) : v = 1 \text{ a.e. on } K\right\} \tag{8.1}$$

where $\Theta : [0,1] \to \mathbb{R}_+$ is a rather general nondecreasing function that vanishes at 0. If all data are smooth, meaning Ω is a smooth open set, K is a smooth compact set and $\Theta \in \mathcal{C}^1$, a minimizer u satisfies

$$\begin{cases} \Delta u = 0 & \Omega \setminus K, \\ -\frac{\partial u}{\partial \nu} = \frac{1}{2}\Theta'(u) & \partial\Omega \cap \{u > 0\} \setminus \partial K, \\ -\frac{\partial u}{\partial \nu} \le \frac{1}{2}\Theta'(u) & \partial\Omega \cap \{u = 0\}, \\ u \equiv 1 & K. \end{cases} \tag{8.2}$$

A classical prototype, $\Theta(u) = \beta u^2$ for some $\beta > 0$, corresponds to the so called Robin boundary conditions $\frac{\partial u}{\partial \nu} + \beta u = 0$ for the harmonic function u minimizing the energy, ν being the outward normal at the boundary. The physical motivation which leads us to study the functional $E_\Theta(K,\Omega)$ and related shape optimization problems can be found in optimal design in thermal insulation.
Consider a body K of given constant fixed temperature T_K, surrounded by an insulator A. The temperature distribution T inside the insulator satisfies the classical heat equation (Laplace equation) with the condition that T is continuous across the boundary which separates A and K. If we now assume that the body $\Omega = K \cup A$ is immersed into an environment of fixed temperature T_e, the heat exchange rate across the surface of Ω (between the body and the environment) has to be modeled according to the physical process governing the mechanism. The most common assumption is to assume that the temperature at the boundary of Ω is kept constant (T_e), leading to the so-called Dirichlet boundary conditions, which corresponds to conduction heat transfer. But if we assume that the environment is a fluid (gas or liquid) then, convection heat transfer, radiation heat transfer or even more general laws, have to be taken into account. If for instance, convection heat transfer is

the leading mechanism, then the rate of heat flux per unit of surface, across the solid fluid interface, is proportional to $T|_{\partial\Omega} - T_e$ (also known as Newton's law of cooling), $T|_{\partial\Omega}$ being the temperature T at the solid surface. While for radiation heat transfer (according to Stefan-Boltzmann law), the heat flux per unit of surface, across the solid fluid interface, is proportional to $T^4_{\partial\Omega} - T^4_e$.

On the other hand, the heat flux across a solid surface is proportional (Fourier's law) to the normal derivative of the temperature $-\frac{\partial T}{\partial \nu}$.

Assume that $T_K > T_e$ and denote by $u = \frac{T-T_e}{T_K-T_e}$. One can easily check that, up to a constant of proportionality, convection corresponds to

$$\Theta(u) = u^2.$$

Radiation, on the other hand, is modeled up to a multiplicative constant by

$$\Theta(u) = \frac{u^5}{5} + \gamma u^4 + 2\gamma^2 u^3 + 2\gamma^3 u^2, \tag{8.3}$$

where $\gamma = \frac{T_e}{T_K-T_e}$.

Both mechanisms can be taken into account simultaneously, upon considering a linear combination of the previous functions. When considering a quadratic Θ (pure convection heat transfer at the boundary), E_Θ is proportional to the heat loss rate. In general, regardless of the choice of Θ, the energy functional E_Θ can be considered a measure of the goodness of the thermal insulation; the less the energy the better the insulation.

In this article we are interested in the shape optimization problem of both K and Ω of prescribed volume which lead to a minimal energy configuration. For a given set K, when only the geometry of Ω is unknown, the problem has already been considered in the literature in [151], [152], [156], with a different purpose, namely to obtain qualitative information on the free boundary of Ω. Its analysis relies on the fine study of optimal configurations in the framework of free discontinuity problems in SBV.

We search to optimize both K and Ω and the purpose is to understand whether or not an optimal configuration is given by two concentric balls (as common sense suggests). From this perspective, our problem is more of isoperimetric type. As we shall prove, depending on the dissipation law Θ, we can give either full description of the optimal sets as two concentric balls (for instance in the convection case) or some partial answer leading to the same geometric configuration for more general dissipation laws. A striking phenomenon, which has been observed in some specific geometric configurations in [154], and that we have to handle is the following: if only a small amount of insulator is available, then in some cases it is better to not use it ($\Omega = K$).

Let us denote by ω_n the volume of the unit ball and by $M := R^n\omega_n$ the volume of the ball of radius $R \geq 1$. Without restricting generality, we shall fix the measure of K to be ω_n and leave the other parameters free (the general case is obtained by rescaling). Given $M \geq \omega_n$ and $\Lambda > 0$, we are interested in the following minimization problem: find an open set Ω and a relatively closed set K, solutions of

$$\inf_{K\subset\Omega,\ |K|=\omega_n,\ |\Omega|\leq M} E_\Theta(K,\Omega), \tag{8.4}$$

or in its penalized version

$$\inf_{K\subset\Omega,\ |K|=\omega_n} E_\Theta(K,\Omega) + \Lambda|\Omega|. \tag{8.5}$$

Throughout the paper we assume that $\Theta : [0,1] \to \mathbb{R}$ is a lower semicontinuous, nondecreasing function such that $\Theta(0) = 0$. Here are our main results.

Theorem 8.1 (The convection case)**.** *For $\Theta(u) = \beta u^2$ and $M = R^n\omega_n$, the solutions of problems* (8.4) *consist of two concentric balls. The radius of the outer ball equals either R or* 1*, according to* $\min\{E_\Theta(B_1, B_R), E_\Theta(B_1, B_1)\}$ *and the associated state function u is radial.*

Theorem 8.2. *For any $\Lambda > 0$ and any admissible Θ, the solution of problem* (8.5) *consists of two concentric balls and the associated state function u is radial.*

We introduce the following hypothesis

$$\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} > 0. \tag{8.6}$$

Theorem 8.3 (The general case)**.** *There exists some dimensional constant c_n, such that if Θ satisfies hypothesis* (8.6) *and*

$$M < \omega_n + c_n \left(\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_0^1 \frac{t^{2n-1}\mathrm{d}t}{\Theta(t)^n}, \tag{8.7}$$

then problem (8.4) *has a solution (K, Ω). If $|\Omega| < M$ then (K, Ω) are two concentric balls. Otherwise, Ω is an open set with rectifiable topological boundary such that $\mathscr{H}^{n-1}(\partial\Omega) < \infty$ and K is relatively closed in Ω with locally finite perimeter in Ω. The temperature $u \in H^1(\Omega)$ is $\mathcal{C}^{0,\frac{2}{n+2}}_{loc}(\Omega)$. In two dimensions, $\partial K \cap \Omega$ is analytic.*

If moreover Θ is $\mathcal{C}^1$ in a neighbourhood of 1 and

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n-1),$$

then there exists $M > \omega_n$ depending on n and Θ such that the solution of problem (8.4) *is $K = \Omega = B_1$.*

Remark 8.4. Notice that any homogenous functions $\Theta(u) = u^\alpha$, $\alpha > 0$ satisfies the assumption (8.6). Moreover, if $\Theta(u) = O(u^2)$ then in (8.7) any value $M \in [\omega_n, +\infty)$ is accepted. In the convection case $\Theta(u) = \beta u^2$ the full picture of the solutions is understood. Other interesting choices of Θ may be given by (8.3), corresponding to thermal radiation, or $\Theta(u) = cu$ corresponding to a constant heat flux. Moreover, discontinuous functions Θ, with $\Theta(0_+) > 0$, are admissible. This means that one can consider functions like $\Theta(u) = c_1 1_{u>0} + c_2 u^\alpha$ on $[0, 1]$ which models a cost of the highly insulating material on $\partial(A \cup K)$, cost which is proportional to its surface measure.

The function Θ may be extended to $\mathbb{R}$ as a nondecreasing and nonnegative function, and the constraint $1_K \le v \le 1$ a.e. in (8.1) may be relaxed into $v_{|K} \ge 1$ a.e. Whatever the extension of Θ is outside $[0, 1]$, the new problem is equivalent to the original one, by truncation below 0 and above 1.

Clearly, we know many more things on the penalized problem (8.5) than on the constrained problem (8.4), in particular that the solutions are always concentric balls. This stills leaves open the following question:

Under reasonable hypotheses on Θ, is it true that the solution of (8.4) *always consists on two concentric balls?*

The organisation of the paper is as follows.

- In Section 8.2 we discuss the convection case and we prove Theorem 8.1. The proof follows the strategy of Bossel and Daners for the Faber-Krahn inequality for the Robin Laplacian and is based on the construction of a so called H-function. Up to knowledge of the authors, this is the only case (aside from Faber-Krahn) where this strategy works. However, we point out that this strategy is fully working only in dimension 2, while for $n \ge 3$ it works only for $\beta > n - 2$. This section is mostly independent from the rest of the paper, except for the case $n \ge 3$, $\beta \in (0, n-2]$ which is a consequence of the analysis by free discontinuity techniques, for which we refer to Corollary 8.21 and Remark 8.22.

- Section 8.3 is devoted to the analysis of the existence of relaxed solutions for the constrained problem (8.4) in the context of a general dissipation functions Θ, and to the regularity of the free boundaries. These results are rather technical and they prepare the proofs of Theorems 8.2 and 8.3. The key idea is that once we know that problem (8.4) has a sufficiently smooth solution, we can extract qualitative information out of its optimality. We work in the framework of free discontinuity problems, based on the analysis of special functions with bounded variation.

- In Section 8.4 we prove Theorem 8.2 and in Section 8.5 we prove Theorem 8.3.

8.2 The convection case: proof of Theorem 8.1

In this section we consider $\Theta(s) = \beta s^2$. In this case, the energy E_Θ is simply denoted by E_β and takes the form

$$E_\beta(K,\Omega) := \inf_{v\in H^1(\Omega), v\geq 1_K} \int_\Omega |\nabla v|^2 \mathrm{d}\mathscr{L}^n + \beta \int_{\partial\Omega} v^2 \mathrm{d}\mathscr{H}^{n-1}. \tag{8.8}$$

For fixed K, minimizers of the functional $\Omega \to E_\beta(K,\Omega) + \Lambda|\Omega|$ have been studied in [151], [152] and [156], in particular the existence of an optimal set Ω and its regularity.

The claim of Theorem 8.1 is that the solution of the constrained problem (8.4) consists of two centered balls for every $M > \omega_n$, and the size of the balls is then given by the study of the function

$$R \mapsto E_\beta(B_1, B_R).$$

Let us first describe this function in detail. We let

$$\Phi_n(\rho) = \begin{cases} \log(\rho) & \text{if } n = 2 \\ -\frac{1}{(n-2)\rho^{n-2}} & \text{if } n \geq 3, \end{cases}$$

where the sign convention is taken such that Φ_n is always increasing. Relying on the expression of radial harmonic functions ($x \mapsto a + b\Phi_n(|x|)$) and the boundary conditions we obtain that the temperature associated to (B_1, B_R) is given by

$$u(x) = 1 - \frac{\beta(\Phi_n(|x|) - \Phi_n(1))_+}{\Phi_n'(R) + \beta(\Phi_n(R) - \Phi_n(1))}$$

and

$$E_\beta(B_1, B_R) = \frac{\beta \mathrm{Per}(B_1)\Phi_n'(1)}{\Phi_n'(R) + \beta(\Phi_n(R) - \Phi_n(1))}.$$

Notice in particular that

$$\frac{d}{dR} E_\beta(B_1, B_R) \leq 0 \text{ iff } \frac{d}{dR}(\Phi_n'(R) + \beta\Phi_n(R)) \geq 0 \text{ iff } R \geq \frac{n-1}{\beta}.$$

Moreover the extremal values are

$$E_\beta(B_1, B_1) = \beta n \omega_n, \ \lim_{R\to\infty} E_\beta(B_1, B_R) = (n-2)n\omega_n.$$

In two dimension there are two cases

- if $\beta \geq 1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ is decreasing.
- if $\beta < 1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ increases on $[1, \beta^{-1}]$ and decreases on $[\beta^{-1}, +\infty[$, with the existence of a unique $R_\beta > \beta^{-1}$ such that $E_\beta(B_1, B_{R_\beta}) = E_\beta(B_1, B_1)$.

In higher dimension there are three cases

- if $\beta \geq n-1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ is decreasing.
- if $n-1 > \beta > n-2$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ increases on $[1, \frac{n-1}{\beta}]$, decreases on $[\frac{n-1}{\beta}, +\infty)$, with the existence of a unique $R_\beta > \frac{n-1}{\beta}$ such that $E_\beta(B_1, B_{R_\beta}) = E_\beta(B_1, B_1)$.
- if $\beta \leq n-2$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ reaches its minimum at $R = 1$.

We postpone the analysis of the last case, and start by proving the result for the first two cases. For this we will need the following.

Lemma 8.5. *Let $R > 1$, $\beta > 0$, and let u^* be the (unique) minimizer in (8.8) on (B_1, B_R). Then $\frac{|\nabla u^*|}{u^*} \leq \beta$ on $B_R \setminus B_1$ if and only if*

$$\forall \rho \in [1, R],\ E_\beta(B_1, B_\rho) \geq E_\beta(B_1, B_R).$$

Proof. We remind the expression

$$u^*(x) = 1 - \frac{\beta(\Phi_n(|x|) - \Phi_n(1))}{\Phi_n'(R) + \beta(\Phi_n(R) - \Phi_n(1))},$$

so by straightforward computations

$$\frac{|\nabla u^*|}{u^*} \leq \beta \text{ iff } \Phi_n'(\rho) + \beta\Phi_n(\rho) \leq \Phi_n'(R) + \beta\Phi_n(R) \text{ iff } E_\beta(B_1, B_\rho) \geq E_\beta(B_1, B_R).$$

□

We may now prove the result. The proof relies on the study of the so called H function introduced by Bossel [145], see also [153] and [148], for the study of the Faber-Krahn inequality involving the first eigenvalue of the Laplace operator with Robin boundary conditions.

Proof of Theorem 8.1. Let (K, Ω) be smooth sets such that $K \subseteq \Omega$, $|K| = \omega_n$, $|\Omega| \leq M$ (and R the radius defined by $M = |B_R|$). Let u be the minimizer of $E_\beta(K, \Omega)$ and denote $\Omega_t = \{u > t\}$. We decompose $\partial\Omega_t$ into two disjoint (up to a $\mathscr{H}^{n-1}$-negligible set) sets; $\partial^i\Omega_t = \{u = t\} \cap \Omega$, and $\partial^e\Omega_t = \partial\Omega_t \cap \partial\Omega$. Then for a.e. $t \in [0, 1]$

$$\begin{aligned}
0 = \int_{\{t<u<1\}} \frac{\Delta u}{u} \mathrm{d}\mathscr{L}^n &= \int_{\{t<u<1\}} \left(\nabla \cdot \left(\frac{\nabla u}{u}\right) - \nabla u \cdot \nabla \frac{1}{u}\right) \mathrm{d}\mathscr{L}^n \\
&= \int_{\partial\{t<u<1\}} \nu_{\Omega_t} \cdot \frac{\nabla u}{u} \mathrm{d}\mathscr{H}^{n-1} + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \mathrm{d}\mathscr{L}^n \\
&= \int_{\partial K \cap \Omega} |\nabla u| \mathrm{d}\mathscr{H}^{n-1} - \int_{\partial^i\Omega_t} \frac{|\nabla u|}{u} \mathrm{d}\mathscr{H}^{n-1} \\
&\quad - \beta\mathscr{H}^{n-1}(\partial\{t < u < 1\} \cap \partial\Omega) + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \mathrm{d}\mathscr{L}^n.
\end{aligned}$$

Since

$$\begin{aligned}
E_\beta(K, \Omega) &= \int_{\Omega\setminus K} \nabla \cdot (u\nabla u) \mathrm{d}\mathscr{L}^n + \beta \int_{\partial\Omega} u^2 \mathrm{d}\mathscr{H}^{n-1} \\
&= \int_{\partial K \cap \Omega} |\nabla u| \mathrm{d}\mathscr{H}^{n-1} + \int_{\partial\Omega\setminus\partial K} \left(\beta u^2 + u\partial_\nu u\right) \mathrm{d}\mathscr{H}^{n-1} + \int_{\partial K \cap \partial\Omega} \beta u^2 \\
&= \int_{\partial K \cap \Omega} |\nabla u| \mathrm{d}\mathscr{H}^{n-1} + \beta\mathscr{H}^{n-1}(\partial K \cap \partial\Omega)
\end{aligned}$$

then injecting this in the previous equation we obtain

$$E_\beta(K, \Omega) = \beta\mathscr{H}^{n-1}(\partial^e\Omega_t) + \int_{\partial^i\Omega_t} \frac{|\nabla u|}{u} \mathrm{d}\mathscr{H}^{n-1} - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \mathrm{d}\mathscr{L}^n.$$

Let us define, for all $t \in [0, 1]$ and $\phi \geq 0$,

$$H(t, \phi) = \beta\mathscr{H}^{n-1}(\partial^e\Omega_t) + \int_{\partial^i\Omega_t} \phi \mathrm{d}\mathscr{H}^{n-1} - \int_{\Omega_t} \phi^2 \mathrm{d}\mathscr{L}^n.$$

Lemma 8.6. *For any nonnegative L^∞ function ϕ, there exists some $t \in]0, 1[$ for which $H(t, \phi) \leq E_\beta(K, \Omega)$.*

Proof. Let $w = \phi - \frac{|\nabla u|}{u}$, then

$$\begin{aligned} H(t,\phi) - E_\beta(K,\Omega) &= H(t,\phi) - H\left(t, \frac{|\nabla u|}{u}\right) \\ &= \int_{\partial^i \Omega_t} w \mathrm{d}\mathscr{H}^{n-1} + \int_{\Omega_t} \left(\left(\frac{|\nabla u|}{u}\right)^2 - \phi^2 \right) \mathrm{d}\mathscr{L}^n \\ &\leq \int_{\partial^i \Omega_t} w \mathrm{d}\mathscr{H}^{n-1} - 2\int_{\Omega_t} \frac{|\nabla u|}{u} w \mathrm{d}\mathscr{L}^n \\ &= -\frac{1}{t}\frac{d}{dt}\left(t^2 \int_{\Omega_t} \frac{|\nabla u|}{u} w \mathrm{d}\mathscr{L}^n \right), \end{aligned}$$

where the last line is obtained by coarea formula on the level sets of u. So in particular

$$\int_0^1 t(H(t,\phi) - E_\beta(K,\Omega))dt \leq -\left[t^2 \int_{\Omega_t} \frac{|\nabla u|}{u} w \mathrm{d}\mathscr{L}^n \right]_{t=0}^{t=1} = 0,$$

which proves the lemma. □

Proof of Theorem 8.1, (continuation). Recall that u^* is the solution on (B_1, B_R), which is radially symmetric and decreasing. Let ϕ be the dearrangement of $\frac{|\nabla u^*|}{u^*}$ on Ω following the level sets of u. To be more precise, for any $x \in \Omega \setminus K$, let $r(x) > 0$ be defined by the formula

$$|B_{r(x)}| = |\{u > u(x)\}|,$$

then we let $\phi(x)$ be the value of $\frac{|\nabla u^*|}{u^*}$ on $\partial B_{r(x)}$.

Let t be chosen as in Lemma 8.6, so that $H(t,\phi) \leq E_\beta(K,\Omega)$. Assuming that $\frac{|\nabla u^*|}{u^*} \leq \beta$ on $\partial B_{r(x)}$, we have

$$\begin{aligned} H(t,\phi) &= \beta \mathscr{H}^{n-1}(\partial^e \Omega_t) + \int_{\partial^i \Omega_t} \phi \mathrm{d}\mathscr{H}^{n-1} - \int_{\Omega_t} \phi^2 \mathrm{d}\mathscr{L}^n \\ &\geq \int_{\partial \Omega_t} \left(\frac{|\nabla u^*|}{u^*} \right)_{|\partial B_{r(x)}} \mathrm{d}\mathscr{H}^{n-1} - \int_{\Omega_t^*} \frac{|\nabla u^*|^2}{(u^*)^2} \mathrm{d}\mathscr{L}^n \\ &\geq \int_{\partial \Omega_t^*} \frac{|\nabla u^*|}{u^*} \mathrm{d}\mathscr{H}^{n-1} - \int_{\Omega_t^*} \frac{|\nabla u^*|^2}{(u^*)^2} \mathrm{d}\mathscr{L}^n \\ &= H^*\left(u^*_{|\partial B_{r(x)}}, |\nabla u^*|/u^* \right) \\ &= E_\beta(B_1, B_R) \end{aligned}$$

The inequalities above rely on the rearrangement properties, the isoperimetric inequality and on the hypothesis $\beta \geq \frac{|\nabla u^*|}{u^*}$.

We conclude by discussing when the assumption $\frac{|\nabla u^*|}{u^*} \leq \beta$ is verified. If $\beta \geq n-1$ the inequality $\frac{|\nabla u^*|}{u^*} \leq \beta$ is verified since $R \mapsto E_\beta(B_1, B_R)$ is decreasing on $[1, +\infty)$, by Lemma 8.5.

If instead $n-2 < \beta < n-1$, two situations occur. If $M \geq |B_{R(n,\beta)}|$, then Lemma 8.5 still works and the previous computation applies, with the same result. If $M \in [\omega_n, |B_{R(n,\beta)}|]$ then we may consider u^* the solution on $(B_1, B_{R(n,\beta)})$ restricted to B_R; it verifies $\frac{|\nabla u^*|}{u^*} \leq \beta$ and the same argument applies. We obtain that

$$E_\beta(K,\Omega) \geq E_\beta(B_1, B_{R(n,\beta)}).$$

Since $E_\beta(B_1, B_{R(n,\beta)}) = E_\beta(B_1, B_1)$, we conclude with the minimality of the couple (B_1, B_1).

The case $n \geq 3$, $\beta \leq n-2$ can not be treated directly with the H function, but is a direct consequence of Corollary 8.21. Indeed, it is enough to take $\Lambda \to 0$ in Corollary 8.21, since $E_\beta(B_1, B_R) \geq E_\beta(B_1, B_1)$ for all $R \geq 1$ in this case.

□

Remark 8.7. Finally we have the following picture for problem (8.4) with $M = R^n\omega_n$; the minimizer is (B_1, B_r), where r is defined through the following case disjunction.

(a) If $n-1 \leq \beta$ then $r = R$.

(b) If $n-2 < \beta < n-1$, then defining $R(n,\beta) > \frac{n-1}{\beta}$ as the unique non-trivial solution of the equation $E_\Theta\left(B_1, B_{R(n,\beta)}\right) = n\beta\omega_n$ we have

- if $R(n,\beta) > R \geq 1$ then $r = 1$
- if $R = R(n,\beta)$ then $r = 1$ or $r = R$
- if $R > R(n,\beta)$ then $r = R$

(c) If $\beta \leq n-2$ then $r = 1$.

8.3 Preparatory results: existence and regularity of relaxed solutions

We analyze in this section the existence of a solution of problem (8.4) for general Θ. Precisely, our purpose is to prove the following.

Proposition 8.8. *Let $M > \omega_n$ and Θ admissible, such that, for some dimensional constant c_n (that will be specified)*

$$M < \omega_n + c_n \left(\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)}\right)^{2n} \int_0^1 \frac{t^{2n-1}\mathrm{d}t}{\Theta(t)^n}, \tag{8.9}$$

Then problem (8.4) has a (sufficiently regular) solution (K, Ω).

The word "sufficiently" above will be described later, and refers to the regularity that we need in the proofs of Theorems 8.2 and 8.3. The proof of this proposition relies on ideas inspired from the the relaxation strategy in the SBV framework (defined below) as introduced in [148] and from [157, Chapter 29] on the existence proof of minimal clusters for the perimeter. The main difficulty in our case comes from both the generality of the function Θ and from the fact that two measure constraints have to be satisfied simultaneously.

As mentioned previously, the functional space used for our relaxation procedure is the SBV space, on which more information may be found in the books [146, 143]. In few words, for an open set $\Omega \subset \mathbb{R}^n$, $SBV(\Omega)$ is defined as the set of functions $u \in L^1(\Omega)$ such that Du (the differential of u in the sense of distribution) is a Radon measure that decomposes as

$$Du = \nabla u \mathscr{L}^n + (u_+ - u_-)\nu_u \mathscr{H}^{n-1} \lfloor J_u,$$

where $\nabla u \in L^1(\Omega)$ and J_u is the set of jump points, meaning points $x \in \mathbb{R}^n$ such that $y \mapsto u(x+ry)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as r goes to 0 to

$$u_+ 1_{y:y\cdot\nu_u>0} + u_- 1_{y:y\cdot\nu_u<0}$$

for some $u_+ \neq u_- \in \mathbb{R}$, $\nu_u \in \mathbb{S}^{n-1}$. The set J_u turns out to be rectifiable. We will make use of the following compactness result, which can be obtained by combination of [146, Th. 2.3] and [146, Th. 2.12] applied to $\phi(z,w) = \Theta(z) + \Theta(w)$.

Proposition 8.9. *Let $(u_i)_{i\in\mathbb{N}}$ be a sequence in $SBV_{loc}(\mathbb{R}^n)$ with values in $[0,1]$ such that for any $R>0$,*

$$\sup_{i\in\mathbb{N}}\left(\int_{B_R}|\nabla u_i|^2+\mathscr{H}^{n-1}(J_{u_i}\cap B_R)\right)<\infty$$

then up to the extraction of a subsequence, there is a function $u\in SBV_{loc}(\mathbb{R}^n)$ such that

$$u_i\underset{a.e.}{\longrightarrow}u$$
$$\nabla u_i\underset{L^2_{loc}}{\rightharpoonup}\nabla u$$
$$\forall R>0,\ \int_{J_u\cap B_R}[\Theta(u_+)+\Theta(u_-)]\,d\mathscr{H}^{n-1}\le\liminf_{i\to\infty}\int_{J_{u_i}\cap B_R}[\Theta((u_i)_+)+\Theta((u_i)_-)]\,d\mathscr{H}^{n-1}$$

Here is the main strategy.

Step 1. (Relaxation) We change the problem into finding a minimizer for the following problem

$$\inf_{u\in SBV(\mathbb{R}^n),|\{u=1\}|\ge\omega_n,|\{u>0\}|\le M}\mathscr{E}_\Theta(u),\tag{8.10}$$

where

$$\mathscr{E}_\Theta(u)=\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}\mathscr{L}^n+\int_{J_u}(\Theta(u_+)+\Theta(u_-))\mathrm{d}\mathscr{H}^{n-1}.\tag{8.11}$$

This means that we will need afterward to prove that such a relaxed minimizer (meaning a minimizer of the relaxed problem) corresponds to a classical minimizer.

Step 2. (A priori regularity) We prove that we may restrict to considering functions that verify an a priori estimate of the form $u\ge\delta 1_{\{u>0\}}$ for some explicit δ; for Robin boundary conditions this kind of estimate is a cornerstone of the regularity theory. This is the only place where we use the hypothesis (8.9). Similar estimates may be found in [151], [149] and [152] (see also [144] for a related problem). Moreover, we prove a concentration lemma that says all the support of u - up to a set of measure ϵ - is contained in a union of $C\epsilon^{-n}$ unit cubes for some explicit constant C, and we prove a cut-off lemma that says that one may replace u by 0 outside a large enough cube and lower its energy by a controlled amount.

Step 3. (Existence of a relaxed solution) We rely on Step 2 to prove the existence of a minimizer. This is done by the direct method, considering a minimizing sequence, applying the concentration lemma a first time to translate the minimizing sequence such that it converges to a non-trivial solution, and then applying again the concentration lemma and cut-off lemma for an appropriately small ϵ.

Step 4. (Regularity of the relaxed solution) Finally we prove the regularity of such a relaxed minimizer by using the theory of almost quasiminimizers of the Mumford-Shah functionnal to handle the support of u, and the theory of the regularity of Alt-Caffarelli problem inside Ω.

As mentioned above, we are first interested in the relaxed problem (8.10). Notice that the generalized energy (8.11) takes into account "cracks", meaning part of the jump set where u may be nonzero on both sides. The general energy (8.11) may also be used to define $E_\Theta(K,\Omega)$ in a more general setting ; for any open set Ω, and any measurable $K\subset\Omega$, we let

$$E_\Theta(K,\Omega)=\inf\left\{\mathscr{E}_\Theta(u),\ u\in SBV(\mathbb{R}^n,[0,1]):|K\setminus\{u=1\}|=|\{u\neq 0\}\setminus\Omega|=\mathscr{H}^{n-1}(J_u\setminus\partial\Omega)=0\right\}$$

When Ω is a smooth open set, this is coherent with the first definition.

Lemma 8.10. *Assume that (8.9) is verified. There is a constant $\delta=\delta_{n,\Theta,M}>0$ such that for any admissible u in (8.10), there is some $t>\delta$ verifying $\mathscr{E}_\Theta(u1_{\{u>t\}})\le\mathscr{E}_\Theta(u)$.*

Proof. We actually show the stronger result; there is some $t > s > e^{-c_{n,\Theta}M}$ such that

$$\mathscr{E}_\Theta(u1_{\{s<u<t\}^c}) \leq \mathscr{E}_\Theta(u).$$

Let $\epsilon > 0$, suppose $\mathscr{E}_\Theta(u1_{\{s<u<t\}^c}) > \mathscr{E}_\Theta(u)$ for every $\epsilon < s < t < 1$. We write

- $\Omega(s,t) = \{s < u \leq t\}$.
- $\gamma(s,t) = \int_{J_u} \left(1_{s<u_+\leq t} + 1_{s<u_-\leq t}\right) \mathrm{d}\mathscr{H}^{n-1}$.
- $h(t) = \mathscr{H}^{n-1}(\{u = t\} \setminus J_u)$.

Our hypothesis becomes that for every $\eta \in]2\epsilon, \frac{2}{3}[$, $t \in]0, \frac{\eta}{2}[$, $\mathscr{E}_\Theta(u1_{\{\eta-t<u<\eta+t\}^c}) > \mathscr{E}_\Theta(u)$ so

$$\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \Theta\left(\frac{1}{2}\eta\right)\gamma(\eta-t,\eta+t) \leq \Theta\left(\frac{3}{2}\eta\right)(h(\eta-t)+h(\eta+t)). \tag{8.12}$$

The proof is based on a lower bound of $\int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n$ and an upper bound of $\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n$ that are in contradiction when ϵ is small enough.

- For all $t \in]0, \frac{1}{2}\eta[$ we let:

$$f_\eta(t) = \int_{\eta-t}^{\eta+t} h(s)\mathrm{d}s = \int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n.$$

f_η is absolutely continuous and

$$f'_\eta(t) = h(\eta-t) + h(\eta+t).$$

Moreover,

$$\begin{aligned}
\text{Then, } f_\eta(t) &\leq |\Omega(\eta-t,\eta+t)|^{\frac{1}{2}} \left(\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} \\
&\leq C_n \mathrm{Per}(\Omega(\eta-t,\eta+t))^{\frac{n}{2(n-1)}} \left(\Theta\left(\frac{3}{2}\eta\right)(h(\eta-t)+h(\eta+t))\right)^{\frac{1}{2}} \\
&\quad \text{by isoperimetric inequality} \\
&\leq C_n (h(\eta-t)+\gamma(\eta-t,\eta+t)+h(\eta+t))^{\frac{n}{2(n-1)}} \left(\Theta\left(\frac{3}{2}\eta\right)(h(\eta-t)+h(\eta+t))\right)^{\frac{1}{2}} \\
&\leq C_n \Theta\left(\frac{3}{2}\eta\right)^{\frac{2n-1}{2n-2}} \Theta\left(\frac{1}{2}\eta\right)^{-\frac{n}{2n-2}} f'_\eta(t)^{\frac{2n-1}{2n-2}}.
\end{aligned}$$

Raising to power $\frac{2n-2}{2n-1}$, after summation on $[0,t]$

$$\left(\int_{\Omega\left(\frac{1}{2}\eta,\frac{3}{2}\eta\right)} |\nabla u| \mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2n-1}} \geq c_n t \Theta\left(\frac{3}{2}\eta\right)^{-1} \Theta\left(\frac{1}{2}\eta\right)^{\frac{n}{2n-1}}.$$

And so

$$\int_{\Omega\left(\frac{1}{2}\eta,\frac{3}{2}\eta\right)} |\nabla u| \mathrm{d}\mathscr{L}^n \geq c_n \eta^{2n-1} \Theta\left(\frac{3}{2}\eta\right)^{-(2n-1)} \Theta\left(\frac{1}{2}\eta\right)^{n}.$$

- We let

$$g_\eta(t) = \int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n, \; G_\eta(t) = \int_0^t g_\eta.$$

We begin by finding a upper bound for G_η. We take t in $[0, \eta/2]$, then:

$$\begin{aligned} G_\eta(t) &\leq \int_0^t \Theta\left(\frac{3}{2}\eta\right)(h(\eta - s) + h(\eta + s))ds \\ &\leq \Theta\left(\frac{3}{2}\eta\right)|\Omega(\eta - t, \eta + t)|^{\frac{1}{2}} G'_\eta(t)^{\frac{1}{2}}. \end{aligned}$$

Thus:

$$G'_\eta(t)G_\eta(t)^{-2} \geq \Theta\left(\frac{3}{2}\eta\right)^{-2}\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|^{-1}.$$

We integrate from t to $2t$ (up to supposing $t < \eta/4$)

$$G_\eta(t)^{-1} \geq t\Theta\left(\frac{3}{2}\eta\right)^{-2}\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|^{-1}.$$

Thus:

$$G_\eta(t) \leq t^{-1}\Theta\left(\frac{3}{2}\eta\right)^{2}\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|.$$

Since g_η is increasing, then up to supposing $t < \eta/8$

$$g_\eta(t) \leq \frac{1}{t}\int_t^{2t} g_\eta \leq \frac{G_\eta(2t)}{t} \leq t^{-2}\Theta\left(\frac{3}{2}\eta\right)^{2}\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|.$$

Combining the previous inequalities, that are valid for $t \in [0, \eta/8]$, we get

$$\begin{aligned} c_n\eta^{2n-1}\Theta\left(\frac{3}{2}\eta\right)^{-(2n-1)}\Theta\left(\frac{1}{2}\eta\right)^{n} &\leq \int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n \\ &\leq |\Omega(\eta - t, \eta + t)|^{\frac{1}{2}}\left(\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} \\ &\leq t^{-1}\Theta\left(\frac{3}{2}\eta\right)\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|. \end{aligned}$$

Taking $t = \eta/8$, we get:

$$\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right| \geq \frac{c_n\eta^{2n}}{\Theta\left(\frac{3}{2}\eta\right)^{2n}\Theta\left(\frac{1}{2}\eta\right)^{-n}},$$

for a constant c_n that may be made explicit. Now, suppose that the quantity ϵ chosen at the start was of the form 3^{-K} for some $K \in \mathbb{N}^*$, and then by taking $\eta = \frac{2}{3^k}$ for $k = 1, \ldots, K$, we get

$$\begin{aligned} M - \omega_n \geq |\{u > 3^{-K}\}| &\geq \sum_{k=1}^{K}\left|\Omega\left(\frac{1}{3^p}, \frac{1}{3^{p-1}}\right)\right| \\ &\geq c_n\sum_{k=1}^{K}\frac{3^{-2nk}}{\Theta\left(3^{-(k-1)}\right)^{2n}\Theta\left(3^{-k}\right)^{-n}} \\ &\geq c_n\left(\inf_{0<s<1}\frac{\Theta(s/3)}{\Theta(s)}\right)^{2n}\int_{3^{-K}}^{1}\frac{t^{2n-1}\mathrm{d}t}{\Theta(t)^n}. \end{aligned}$$

Thus if the hypothesis (8.9) in the result is verified, this gives an upper bound on K. □

With this, we may replace any minimizing sequence (u_i) by a minimising sequence $(u_i 1_{\{u_i > t_i\}})$ such that

$$\inf_{\{u_i>0\}}(u_i) \geq \delta$$

so we will now suppose that all the functions we consider verify this property.

For any $p \in \mathbb{Z}^n$, let us write $K_p = p + [0, 1]^n$.

Lemma 8.11. *Let u be admissible for (8.10) and $\delta > 0$ such that $u \geq \delta 1_{\{u>0\}}$. Let $\epsilon > 0$. Then there exists a set $F = \cup_{i\in I} K_i$ where $I \Subset \mathbb{Z}^n$ is such that*

$$|I| \leq C_n M \left(\frac{\Theta(\delta)^{-1}\mathscr{E}_\Theta(u) + M}{\epsilon} \right)^n, \ |\{u > 0\} \setminus F| \leq \epsilon.$$

Proof. We first prove the following technical result. Let $(K_i)_{i\in I}$ be a finite family of unit cubes, $F = \cup_{i\in I} K_i$, then

$$\max_{p\notin I} |\{u > 0\} \cap K_p| \geq c_n \left(\frac{|\{u > 0\} \setminus \cup_{p\notin I} K_p|}{|\{u > 0\}| + \Theta(\delta)^{-1}\mathscr{E}_\Theta(u)} \right)^n.$$

For any $p \notin I$, we may write

$$\begin{aligned}
|\{u > 0\} \cap K_p| &= |\{u > 0\} \cap K_p|^{\frac{1}{n}} |\{u > 0\} \cap K_p|^{1-\frac{1}{n}} \\
&\leq C_n \left(\max_{q\notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\} \cap K_p| + \mathscr{H}^{n-1}(J_u \cap K_p) \right) \\
&\text{by the embedding } \mathrm{BV}(K_p) \hookrightarrow L^{\frac{n}{n-1}}(K_p) \\
&\leq C_n \left(\max_{q\notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\} \cap K_p| + \Theta(\delta)^{-1}\mathscr{E}_\Theta(u|K_p) \right).
\end{aligned}$$

where the last term is defined as

$$\mathscr{E}_\Theta(u|\Omega) := \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{J_u} (\Theta((u1_\Omega)_+) + \Theta((u1_\Omega)_-)) \, \mathrm{d}\mathscr{H}^{n-1}$$

And so, by summing in $p \in \mathbb{Z}^n \setminus I$:

$$|\{u > 0\} \setminus \cup_{q\notin I} K_q| \leq C_n \left(\max_{q\notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\}| + \Theta(\delta)^{-1}\mathscr{E}_\Theta(u) \right),$$

which is the result.

We now construct F by induction starting with $F_0 = \emptyset$ and as long as $|\{u > 0\} \setminus F_k| \geq \epsilon$, we take $I_{k+1} = I_k \cup \{p\}$ where $p \notin F_k$ is chosen with the previous lemma such that $|\{u > 0\} \cap K_p| \geq c_n \left(\frac{\epsilon}{|\{u>0\}|+\Theta(\delta)^{-1}\mathscr{E}_\Theta(u)} \right)^n$. Suppose that this goes on until a rank N, then

$$M \geq |\{u > 0\}| \geq \sum_{i\in I_N} |\{u > 0\} \cap K_i| \geq c_n N \left(\frac{\epsilon}{|\{u > 0\}| + \Theta(\delta)^{-1}\mathscr{E}_\Theta(u)} \right)^n,$$

so N is bounded uniformly and conclude the proof of the lemma. □

For a closed set F, we shall write $d_F(x) = \inf_{y\in F} |x - y|$.

Lemma 8.12. *Let u be admissible for (8.10) and $\delta > 0$ such that $u \geq \delta 1_{\{u>0\}}$. Then there exist constants $\tau_{n,\Theta,\delta}$, $C_{n,\Theta,\delta}$ such that for any closed set F there exists some $r \in \left[0, C_{n,\Theta,\delta}|\{u > 0\} \setminus F|^{\frac{1}{n}}\right]$ such that*

$$\mathscr{E}_\Theta(u1_{\{d_F<r\}}) \leq \mathscr{E}_\Theta(u) - \tau_{n,\Theta,\delta}|\{u > 0\} \cap \{d_F > r\}|^{1-\frac{1}{n}}.$$

Proof. Let $\Omega_r = \{u > 0\} \cap \{d_F \geq r\}$, and $m(r) = |\Omega_r|$. Suppose that the result we want to prove is not true for $r \in [0, r_1]$ for some $r_1 > 0$, meaning that for a constant $\tau > 0$ that will be chosen later and for any $r \in]0, r_1[$,

$$\mathscr{E}_\Theta(u|\Omega_r) \leq \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \tau|\Omega_r|^{1-\frac{1}{n}},$$

where we remind $\mathscr{E}_\Theta(u|\Omega) := \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{J_u} (\Theta((u1_\Omega)_+) + \Theta((u1_\Omega)_-))\,\mathrm{d}\mathscr{H}^{n-1}$. Now, for any $r \in]0, r_1[$,

$$\begin{aligned} m(r)^{\frac{n-1}{n}} &\leq C_n \mathrm{Per}(\Omega_r) \text{ (by isoperimetry)} \\ &\leq C_n \Theta(\delta)^{-1} \left(\int_{\partial^*\Omega_r \setminus J_u} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \int_{\partial^*\Omega_r \cap J_u} (\Theta((u1_{\Omega_r})_-) + \Theta((u1_{\Omega_r})_+)) \mathrm{d}\mathscr{H}^{n-1} \right) \\ &\leq C_n \Theta(\delta)^{-1} \left(\int_{\partial^*\Omega_r \setminus J_u} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \mathscr{E}_\Theta(u|\Omega_r) \right) \\ &\leq C_n \Theta(\delta)^{-1} \left(2 \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \tau |\Omega_r|^{1-\frac{1}{n}} \right), \end{aligned}$$

so for $\tau := \frac{\Theta(\delta)}{2C_n}$ this implies

$$m(r)^{\frac{n-1}{n}} \leq 4C_n \Theta(\delta)^{-1} \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} \leq 4C_n \frac{\Theta(1)}{\Theta(\delta)} \mathscr{H}^{n-1}(\partial^*\Omega_r \setminus J_u).$$

Moreover $\mathscr{H}^{n-1}(\partial^*\Omega_r \setminus J_u) \leq -m'(r)$. So by integrating this for every $r \in]0, r_1[$, we get

$$m(r_1)^{\frac{1}{n}} \leq m(0)^{\frac{1}{n}} - \frac{\Theta(\delta)}{4C_n\Theta(1)} r_1.$$

Since $m(r_1) \geq 0$ this means that necessarily

$$r_1 \leq 4C_n \frac{\Theta(\delta)}{\Theta(1)} |\{u > 0\} \setminus F|^{\frac{1}{n}}.$$

Thus, there is always some $r \in \left[0, 8C_n \frac{\Theta(\delta)}{\Theta(1)} |\{u > 0\} \setminus F|^{\frac{1}{n}}\right]$ such that

$$\mathscr{E}_\Theta(u1_{\{d_F < r\}}) \leq \mathscr{E}_\Theta(u) - \tau |\{u > 0\} \cap \{d_F > r\}|^{1-\frac{1}{n}},$$

where $\tau = \frac{\Theta(\delta)}{2C_n}$ as defined earlier. □

In all that follows we will need the following lemma for controlled infinitesimal volume exchange between measurable set, that may be found for instance in [157, Lemma 29.13].

Lemma 8.13. *Let U be a connected open set, $E_1, \dots, E_N$ be a measurable partition of U such that, for every i, $|E_i \cap U| > 0$. Then there are vector fields X_{ij} with disjoint support such that for any i, j, k,*

$$\int_{E_k} \mathrm{div}(X_{ij}) = \begin{cases} +1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{else.} \end{cases}$$

Proof. Consider the linear application

$$L : \begin{cases} \mathcal{C}_c^\infty(U) \to \mathbb{R}^N, \\ X \mapsto \left(\int_{E_i} \mathrm{div}(X) \right)_{i=1,\dots,N}. \end{cases}$$

The range of L is included in $\{(m_1, \dots, m_N) : \sum_{i=1}^N m_i = 0\}$. If it were not equal to this subspace, there would be a non trivial vector $(a_1, \dots, a_N)$ independant of $(1, \dots, 1)$ such that, for any X, $\sum_{i=1}^N a_i \int_{E_i} \mathrm{div}(X) = 0$, meaning that $\nabla \left(\sum_{i=1}^N a_i 1_{E_i} \right) = 0$ in $\mathcal{D}'(U)$, which is a contradiction by the connectedness of U. □

Proposition 8.14. *Under the hypotheses of Proposition 8.8, there exists a minimizer of $\mathscr{E}_\Theta$ in Problem* (8.10).

Proof. Consider a minimizing sequence (u_i); up to truncation we may suppose that $\inf_{\{u_i>0\}}(u_i) \geq \delta_{n,\Theta,m,M}$. We may apply Lemma 8.11 with $\epsilon = m/2$ to find sequences $(p_i^k)_{1 \leq k \leq N, i \geq 0}$ such that $|\{u_i > 0\} \setminus \cup_k K_{p_i^k}| \leq \frac{\omega_n}{2}$. In particular, $|\{u_i = 1\} \cap \cup_k K_{p_i^k}| \geq \frac{\omega_n}{2}$ so for each i there is some k_i such that $|K_{p_i^{k_i}} \cap \{u = 1\}| \geq \frac{\omega_n}{2N}$. Up to a translation we will suppose that $p_i^{k_i} = 0$, such that $K_{p_i^{k_i}} = [0,1]^n$.

Now, by compactness arguments in SBV, see theorem 8.9, we know that up to an extraction (u_i) converges almost everywhere (with lower semicontinuity on the energy) to a non-trivial limit $u \in SBV$, in particular such that $|\{u = 1\}| > 0$ and Fatou's lemma tells us that for any cube K, $|\{u = 1\} \cap K| \geq \limsup_{i\to\infty} |\{u_i = 1\} \cap K|$. We also denote $R := (2M)^{1/n}$ to be such that $|[0, R]^n \cap \{u = 0\}| \geq M$.

Lemma 8.15. *Up to extraction, either $|\{u > 0\} \cap [0, R]^n| > \limsup_{i\to\infty} |\{u_i > 0\} \cap [0, R]^n|$, which we call the loose case, or*

$$1_{\{u_i=1\}\cap[0,R]^n} \to 1_{\{u=1\}\cap[0,R]^n}$$

in the weak-$*$ sense, which we call the saturated case.

Proof. Suppose that $|\{u > 0\}\cap[0, R]^n| = \limsup_{i\to\infty} |\{u_i > 0\}\cap[0, R]^n|$. We denote ν a weak limit of the sequence of measures given by the density $1_{\{u_i=1\}}$; by hypothesis $\nu([0, R]^n) = |\{u = 1\} \cap [0, R]^n|$. Moreover for any nonnegative continuous function $\varphi \in \mathcal{C}^0([0, R]^n, \mathbb{R})$, by Fatou's lemma,

$$\nu(\varphi) = \lim_{i\to\infty} \int_{[0,R]^n} \varphi 1_{\{u_i=1\}} \mathrm{d}\mathscr{L}^n \leq \int_{[0,R]^n} \varphi \limsup_{i\to\infty} 1_{\{u_i=1\}} \mathrm{d}\mathscr{L}^n \leq \int_{[0,R]^n} \varphi 1_{\{u=1\}} \mathrm{d}\mathscr{L}^n,$$

which concludes the lemma. □

We now let $\epsilon > 0$ to be a small number (it will be fixed later), and we find a union of $N(= N_{n,\Theta,m,M,\epsilon})$ cube (we also include cubes to cover $[0, R]^n$), denoted F_i, such that $|\{u_i > 0\} \setminus F_i| \leq \epsilon$; by applying Lemma 8.12 to (u_i, F_i), we find a radius $r_i \leq C_{n,\Theta,M}|\{u_i > 0\} \setminus F_i|^{\frac{1}{n}}$ such that

$$\mathscr{E}_\Theta(u_i 1_{\{d_{F_i}<r_i\}}) \leq \mathscr{E}_\Theta(u_i) - \tau_{n,\Theta,M}|\{u_i > 0\} \cap \{d_{F_i} > r_i\}|^{1-\frac{1}{n}}.$$

We let $v_i = u_i 1_{\{d_F<r_i\}}$, and we now differentiate between the loose and the saturated case.

- Loose case: here we may choose $\epsilon < |\{u = 1\} \cap [0, R]^n| - \limsup_{i\to\infty} |\{u_i = 1\} \cap [0, R]^n|$. Since the support of v_i is on a finite (not depending on i) number of cubes, they may be moved around so that v_i is supported in a compact set. Then by SBV compactness theorem 8.9 we obtain that $v_i \to v$, such that $\mathscr{E}_\Theta(v) \leq \liminf_{i\to\infty} \mathscr{E}_\Theta(v_i) = \inf_{|\{u=1\}|\geq\omega_n, |\{u>0\}|\leq M} \mathscr{E}_\Theta(u)$. Moreover, $|\{v > 0\}| = \lim_{i\to\infty} |\{v_i > 0\}| \leq M$ and

$$\begin{aligned}
|\{v = 1\}| &\geq |\{u = 1\} \cap [0, R]^n| - \limsup_{i\to\infty} |\{u_i = 1\} \cap [0, R]^n| + \limsup_{i\to\infty} |\{v_i = 1\}| \\
&\geq |\{u = 1\} \cap [0, R]^n| - \limsup_{i\to\infty} |\{u_i = 1\} \cap [0, R]^n| - \epsilon + \omega_n \\
&\geq \omega_n,
\end{aligned}$$

  so v is admissible and this proves the result.

- Saturated case: based on the partition $\{u = 0\}, \{0 < u < 1\}, \{u = 1\}$ of $[0, R]^n$, where the first and last set have positive measure, there exists a vector field $\xi \in \mathcal{C}^\infty_c((0, R)^n, \mathbb{R}^n)$ such that

$$\int_{\{u=1\}} \mathrm{div}(\xi) \mathrm{d}\mathscr{L}^n = 1, \ \int_{\{0<u<1\}} \mathrm{div}(\xi) \mathrm{d}\mathscr{L}^n = 0, \ \int_{\{u=0\}} \mathrm{div}(\xi) \mathrm{d}\mathscr{L}^n = -1.$$

  Moreover notice that $\phi_t(x) := x + t\xi(x)$ is a diffeomorphism with compact support for any small enough t; ϕ_t will be used to regulate the measure of $\{u_i = 1\}$ after truncation. Using the weak convergence of the measure of the supports of the (u_i), we may suppose that for any large enough i and any small enough t (not depending on i)

$$\begin{aligned}
|\{u_i \circ \phi_t^{-1} = 1\}| &= \int_{\{u_i=1\}} \det(D\phi_t) \mathrm{d}\mathscr{L}^n = \int_{\{u_i=1\}} \left(1 + t\mathrm{div}(\xi) + t^2 P_\xi(t)\right) \mathrm{d}\mathscr{L}^n \\
&\geq |\{u_i = 1\}| + \frac{t}{2}, \\
|\{0 < u_i \circ \phi_t^{-1} < 1\}| &= \int_{\{0<u_i<1\}} \det(D\phi_t) \mathrm{d}\mathscr{L}^n = \int_{\{0<u_i<1\}} \left(1 + t\mathrm{div}(\xi) + t^2 P_\xi(t)\right) \mathrm{d}\mathscr{L}^n \\
&\leq |\{0 < u_i < 1\}| + Ct^2.
\end{aligned}$$

where P_ξ is a polynomial of degree $n-2$ that depends on ξ.

Now, let $t_i = 8(\omega_n - \{v_i = 1\})$. We know that $t_i = 8|\{u_i > 0\} \setminus \{d_{F_i} > r_i\}| \leq 8\epsilon$; when ϵ is small enough, for any $|t| < 8\epsilon$ we know ϕ_t is a diffeomorphism and the estimates above hold for any large enough i. Consider

$$w_i(x) = v_i \circ \phi_{t_i}^{-1}\left(\left[\frac{\omega_n}{\omega_n - \frac{1}{4}t_i}\right]^{1/n} x\right).$$

Then

$$\begin{aligned}
|\{w_i = 1\}| &= \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}|\{v_i \circ \phi_{t_i}^{-1} = 1\}| \geq \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}\left(|\{v_i = 1\}| + \frac{1}{2}t_i\right), \\
&\geq |\{v_i = 1\}| + \frac{t_i}{8} = \omega_n \\
|\{0 < w_i < 1\}| &= \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}|\{0 < v_i \circ \phi_{t_i}^{-1} < 1\}| \leq \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}\left(|\{0 < v_i < 1\}| + Ct_i^2\right) \\
&\leq |\{0 < v_i < 1\}| \leq |\{0 < u_i < 1\}|,
\end{aligned}$$

where in both lines we use that t_i is taken arbitrarily small (less than $8\epsilon > 0$, not depending on i). Thus (w_i) is admissible. Then it may be computed with a similar method that

$$\begin{aligned}
\mathscr{E}_\Theta(w_i) - \mathscr{E}_\Theta(v_i) &= \int_{\{\xi \neq 0\}} \left(\left|\left((I + t_i D\xi)^*\right)^{-1} \nabla u_i\right|^2 \det(I_n + t_i D\xi) - |\nabla u|^2\right) \mathrm{d}\mathscr{L}^n \\
&+ \int_{J_u \cap \{\xi \neq 0\}} \left(\Theta(u_i^+) + \Theta(u_i^-)\right)\left(\nu_{J_{u_i}}^*(I_n + t_i D\xi)\nu_{J_{u_i}} - 1\right) \mathrm{d}\mathscr{H}^{n-1} \\
&\leq C_0 t_i \text{ where } C_0 \text{ does not depend on } i.
\end{aligned}$$

And then, due to Lemma 8.12,

$$\mathscr{E}_\Theta(w_i) \leq \mathscr{E}_\Theta(u_i) + 8C_0|\{u_i > 0\} \setminus \{d_{F_i} > r_i\}| - \tau_{n,\Theta,M}|\{u_i > 0\} \cap \{d_{F_i} > r_i\}|^{1-\frac{1}{n}} \leq \mathscr{E}_\Theta(u_i),$$

where the last inequality is due to $|\{u_i > 0\} \cap \{d_{F_i} > r_i\}| \leq \epsilon$ and ϵ is chosen small enough (depending on the flow, which was defined before ϵ). So (w_i) is an admissible minimizing sequence that is confined in a disjoint union of N unit cubes, and up to moving these cubes we may suppose that w_i has support in a certain ball B not depending on i. So with the compactness result 8.9, it converges to a minimizer.

□

Lemma 8.16. *Let u be a relaxed minimizer of (8.10). Then J_u is $\mathscr{H}^{n-1}$-essentially closed, and there exists a bounded open set Ω with $\partial\Omega = \overline{J_u}$ and a relatively closed set $K \subset \Omega$ such that (K, Ω) is a solution of (8.4), associated to the function u, with $u \in H^1(\Omega) \cap \mathcal{C}^{0,\frac{2}{n+2}}_{loc}(\Omega)$ and $K = \{u = 1\}$.*

Proof. We assume without loss of generality that $|\{0 < u < 1\}| > 0$; otherwise the minimizer is directly identified by the isoperimetric inequality.

- We first prove that u is an almost quasiminimizer of the Mumford-Shah functionnal, meaning that there are constants $c_u, r_u > 0$ such that for any ball $B_{x,r}$ with $r < r_u$ and any function $v \in SBV(\mathbb{R}^n)$ that differs from u on $B_{x,r}$ only,

$$\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \Theta(\delta)\mathscr{H}^{n-1}(B_{x,r} \cap J_u) \leq \int_{B_{x,r}} |\nabla v|^2 \mathrm{d}\mathscr{L}^n + 2\Theta(1)\mathscr{H}^{n-1}(B_{x,r} \cap J_v) + c_u|\{u \neq v\}|. \tag{8.13}$$

Let U be the union of two balls of arbitrarily small radius δ such that

$$|U \cap \{u = 1\}|,\ |U \cap \{0 < u < 1\}|,\ |U \cap \{u = 0\}| > 0 \tag{8.14}$$
$$|U^c \cap \{u = 1\}|,\ |U^c \cap \{0 < u < 1\}|,\ |U^c \cap \{u = 0\}| > 0. \tag{8.15}$$

The second condition (8.15) is automatic as soon as δ is small enough, and the first condition (8.14) may be obtained by continuity of $x \mapsto |B_{x,\delta} \cap A|$ for $A = \{u = 1\}, \{0 < u < 1\}, \{u = 0\}$. We may apply lemma 8.13 for these three sets in U, thus obtaining three smooth vector fields (X, Y, Z) with support in U that transfer measure between $(\{u = 0\}, \{0 < u < 1\}, \{u = 1\})$, in particular

$$\int_{\{u=1\}} \operatorname{div}(X) = -\int_{\{u=0\}} \operatorname{div}(X) = 1,\ \int_{\{0<u<1\}} \operatorname{div}(X) = 0$$
$$\int_{\{u=1\}} \operatorname{div}(Y) = -\int_{\{u=0\}} \operatorname{div}(Y) = 1,\ \int_{\{0<u<1\}} \operatorname{div}(Y) = 0$$

Write $\phi_t(x) = x + tX(x)$, $\psi_t(x) = x + tY(x)$, and $\Phi_{s,t} = \phi_s \circ \psi_t$. Consider the application

$$G : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (s,t) \mapsto \left(|\Phi_{s,t}(\{u = 1\})| - |\{u = 1\}|, |\Phi_{s,t}(\{0 < u < 1\})| - |\{0 < u < 1\}|\right). \end{cases}$$

Then G is smooth and $DG(0,0) = I_2$; there exists $\epsilon_0 > 0$ such that G is invertible on B_{ϵ_0} with $\frac{1}{2}|(s,t)| \leq |G(s,t)| \leq 2|(s,t)|$ and $G(B_{\epsilon_0}) \supset B_{\epsilon_0/2}$. Let $r > 0$ and $x \in \mathbb{R}^n$ be such that $|B_{x,r} \cap U| = 0$ and $|B_{x,r}| < \epsilon_0/4$, and $v \in SBV$ be such that $\{u \neq v\} \Subset B_{x,r}$. Up to truncating v from above (by 1, that can only decrease the energy) and from below (by some $t \geq \delta$, by lemma 8.10) we assume that $\delta 1_{\{v>0\}} \leq v \leq 1$.

Let $(a,b) = (|\{u = 1\}| - |\{v = 1\}|, |\{0 < u < 1\}| - |\{0 < v < 1\}|)(\in B_{\epsilon_0/2})$, and $(s,t) = G^{-1}_{|B_{\epsilon_0}}(a,b)$. Then $v \circ \Phi^{-1}_{s,t}$ satisfies the measure constraints (ω_n, M), so

$$\mathscr{E}_\Theta(u) \leq E_\Theta(v \circ \Phi^{-1}_{s,t}).$$

So

$$\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \Theta(\delta)\mathscr{H}^{n-1}(B_{x,r} \cap J_u) \leq \int_{B_{x,r}} |\nabla v|^2 \mathrm{d}\mathscr{L}^n + 2\Theta(1)\mathscr{H}^{n-1}(B_{x,r} \cap J_v) + R,$$

where

$$\begin{aligned} R &= \int_U \left(\left| (D\Phi^*_{s,t})^{-1} \nabla u \right|^2 \det(D\Phi_{s,t}) - |\nabla u|^2 \right) \mathrm{d}\mathscr{L}^n \\ &\quad + \int_{U \cap J_u} \left(\nu_u^\perp D\Phi_{s,t} \nu_u^\perp - 1 \right) (\Theta(u_-) + \Theta(u_+)) \, \mathrm{d}\mathscr{H}^{n-1} \\ &\leq C(|s| + |t|) \leq 2C(|a| + |b|). \end{aligned}$$

This proves that u is an almost quasiminimizer at any positive distance of U. Now by our choice of U, the condition (8.15) allows us to choose similarly a set $U' \subset \mathbb{R}^n \setminus \overline{U}$ that is a union of two balls of arbitrarily small radii such that

$$|U' \cap \{u = 1\}|,\ |U' \cap \{0 < u < 1\}|,\ |U' \cap \{u = 0\}| > 0$$

and by choosing similarly vector fields X', Y', Z' with support in U' that transit measure between these three sets, u is an almost quasiminimizer at any positive distance of U'. Thus u is an almost quasiminimizer, and this concludes the proof of the estimate (8.13).

- By [151, Theorem 3.1], u being an almost quasiminimizer of Mumford-Shah implies that $\overline{J_u}$ is essentially closed, meaning $\mathscr{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$. We now let Ω be the union of the bounded connected components of $\mathbb{R}^n \setminus \overline{J_u}$, then $u \in H^1(\Omega)$ and $\partial\Omega = \overline{J_u}$.

- Let us prove Ω is bounded by proving an explicit lower density estimate; indeed, we let $r := r_u$ as defined in the first point and considering $v = u1_{\mathbb{R}^n \setminus B_{x,\rho}}$ for any $x \in \mathbb{R}^n$ and $\rho \in]0, r[$ in (8.13) we get

$$\int_{B_{x,\rho}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \Theta(\delta)\mathscr{H}^{n-1}(B_{x,\rho} \cap J_u) \leq 2\Theta(1)\mathscr{H}^{n-1}(\partial B_{x,\rho} \cap \{u > 0\} \setminus J_u) + c_u |B_{x,\rho} \cap \{u > 0\}|.$$

  Let $m(\rho) = |\Omega \cap B_{x,\rho}|$, then by the isoperimetric inequality

$$c_n m(\rho)^{1-\frac{1}{n}} \leq \mathscr{H}^{n-1}(\partial\Omega \cap B_\rho) + \mathscr{H}^{n-1}(\Omega \cap \partial B_\rho) \leq \left(1 + 2\frac{\Theta(1)}{\Theta(\delta)}\right) m'(\rho) + c_u m(\rho).$$

  So if $|B_{x,r/2} \cap \Omega| > 0$ then by integrating this estimate, $|B_{x,r} \cap \Omega| \geq cr^n$ for a constant c that does not depend on x. Since $|\Omega| \leq M$ then there are at most $N \leq \frac{M}{cr^n}$ points $(x_i)_{i=1,\dots,N}$ such that $|x_i - x_j| \geq r$ for all $i \neq j$ and $|\Omega \cap B(x_i, r/2)| > 0$, meaning that Ω is bounded.

- We prove that u is locally Hölder, which implies the relative closedness of K with $K = \{u = 1\}$. Indeed let $K = \{u = 1\}$ and $B \Subset \Omega$ be a small ball inside Ω such that $|B \cap K|$ and $|B \setminus K|$ are positive (such a ball exists as soon as $|\Omega| > |K|$, otherwise K and $\Omega \setminus K$ would be disconnected, and (K, K) would have strictly lower energy than (K, Ω)). Let $\xi \in \mathcal{C}_c^\infty(B, \mathbb{R}^n)$ be such that

$$\int_{B \cap K} \mathrm{div}(\xi)\mathrm{d}\mathscr{L}^n = 1, \ \int_{B \setminus K} \mathrm{div}(\xi)\mathrm{d}\mathscr{L}^n = -1,$$

  and let $\phi_t(x) = x + t\xi(x)$ be the associated diffeomorphism for a small enough t. Consider then any ball $B_{x,r} \Subset \Omega$ such that

$$B_{x,r^{\frac{n}{n+2}}} \Subset \Omega \setminus B,$$

  we prove that provided r is small enough (depending only on the choice of the flow and the parameters of the problem and not on x) there is a constant $C > 0$ not depending on x and r such that

$$\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \leq Cr^{\frac{n^2}{n+2}}.$$

  This directly implies the local Hölder continuity in $\Omega \setminus B$ and the relative closedness of K using classical integral growth argument (see for instance [155, cor. 3.2]), and the same may be done for another small ball B' that has positive distance from B which conclude the result. Let us now focus on this estimate.

  Let $R = r^{\frac{n}{n+2}}$ and h be the harmonic extension of $u_{|\partial B_{x,R}}$ on $B_{x,R}$ (that we extend simply by u outside $B_{x,R}$). We suppose that r is small enough such that $r < \frac{R}{2}$. Notice that h might not be admissible, however $|\{h = 1\}| \geq \omega_n - |B_R|$; we let $t = 2|B_R|$ and for a small enough R, we have

$$|\{h \circ \phi_t^{-1} = 1\}| \geq \omega_n,$$

  so $h \circ \phi_t^{-1}$ is admissible. By comparison with the minimizer u we get

$$\int_{B_{x,R}} |\nabla(u - h)|^2 \mathrm{d}\mathscr{L}^n \leq \int_B \left(\left| \left((I + tD\xi)^*\right)^{-1} \nabla u \right|^2 \det(I + tD\xi) - |\nabla u|^2 \right) \mathrm{d}\mathscr{L}^n \leq CR^n.$$

for some constant C that depends on n, $u_{|B}$ and on the flow ξ. Now, using the subharmonicity of $|\nabla h|^2$ on $B_{x,R}$,

$$\begin{aligned}\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}\mathscr{L}^n &\leq 2\int_{B_{x,r}} |\nabla(u-h)|^2 \mathrm{d}\mathscr{L}^n + 2\int_{B_{x,r}} |\nabla h|^2 \mathrm{d}\mathscr{L}^n \\ &\leq CR^n + 2\left(\frac{r}{R/2}\right)^n \int_{B_{x,R/2}} |\nabla h|^2 \mathrm{d}\mathscr{L}^n \\ &\leq CR^n + C'\frac{r^n}{R^2} \text{ by Cacciopoli inequality on the second term.}\end{aligned}$$

This ends the proof.

□

Remark 8.17. Summarizing our results, we know that Ω is an open set with rectifiable topological boundary such that $\mathscr{H}^{n-1}(\partial\Omega) < \infty$ and that the temperature $u \in H^1(\Omega)$ is $\mathcal{C}^{0,\frac{2}{n+2}}_{\mathrm{loc}}(\Omega)$.

8.4 The penalized problem: proof of Theorem 8.2

In this section we prove Theorem 8.2. For any $\Lambda > 0$, we denote

$$E_{\Theta,\Lambda}(K,\Omega) = E_\Theta(K,\Omega) + \Lambda|\Omega \setminus K|,\ \mathscr{E}_{\Theta,\Lambda}(u) = \mathscr{E}_\Theta(u) + \Lambda|\{0 < u < 1\}|.$$

We claim that it is enough to prove the result in the case for a function Θ which satisfies (8.6) and such that $\Theta(u) = \mathcal{O}_{u\to 0}(u^2)$. Indeed, for any small $\epsilon > 0$ one may replace Θ on $[0,1]$ with a perturbation that verifies (8.6), namely

$$\Theta^\epsilon(u) = \min\left((\Theta(1)+\epsilon)\left(\frac{u}{\epsilon}\right)^2, \Theta(u) + \epsilon 1_{(0,1]}\right)$$

and define E_{Θ^ϵ}, $E_{\Theta^\epsilon,\Lambda}$, accordingly. Then we know $E_{\Theta^\epsilon,\Lambda}$ is minimal on some (B_1, B_{R^ϵ}), and that the associated state function has the form

$$u^\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 1 - (1-u^\epsilon(R^\epsilon))\frac{\Phi_n(|x|)-\Phi_n(1)}{\Phi_n(R^\epsilon)-\Phi_n(1)} & \text{if } 1 \leq |x| \leq R^\epsilon, \\ 0 & \text{if } |x| > R^\epsilon, \end{cases}$$

due to the general form of radial harmonic functions ($x \mapsto a + b\Phi_n(|x|)$). The associated energy takes the form

$$E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon}) = (1-u^\epsilon(R^\epsilon))^2 \frac{\mathrm{Per}(B_r)\Phi_n'(1)}{\Phi_n(R^\epsilon)-\Phi_n(1)} + \mathrm{Per}(B_{R^\epsilon})\Theta^\epsilon\left(u^\epsilon(R^\epsilon)\right).$$

R^ϵ is bounded uniformly in ϵ; indeed, due to the penalization term,

$$|B_{R^\epsilon}| \leq \Lambda^{-1} E_{\Theta^\epsilon}(B_1, B_{R^\epsilon}) \leq \Lambda^{-1} E_{\Theta^\epsilon}(B_1, B_1) = \Lambda^{-1}(\Theta(1)+\epsilon)\mathrm{Per}(B_1)$$

so we may suppose without loss of generality that $R^\epsilon \underset{\epsilon\to 0}{\longrightarrow} R$ and $u^\epsilon(R^\epsilon) \underset{\epsilon\to 0}{\longrightarrow} l \in [0,1]$, and define

$$u(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 1 - (1-l)\frac{\Phi_n(|x|)-\Phi_n(1)}{\Phi_n(R)-\Phi_n(1)} & \text{if } 1 < |x| < R, \\ 0 & \text{if } |x| > R, \end{cases}$$

Then by lower semicontinuity of Θ and the fact that $\Theta(0) = 0$ we get

$$E_{\Theta,\Lambda}(B_1, B_R) \leq \mathscr{E}_{\Theta,\Lambda}(u) \leq \liminf_{\epsilon\to 0} E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon})$$

and then for any admissible v we get by dominated convergence

$$\mathscr{E}_{\Theta,\Lambda}(v) = \lim_{\epsilon\to 0} \mathscr{E}_{\Theta^\epsilon,\Lambda}(v) \geq \liminf_{\epsilon\to 0} E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon}) \geq E_{\Theta,\Lambda}(B_1, B_R)$$

This is a similar method as what was used in [150] to handle a similar function with $\Theta(v) \sim 2\beta c v$ near 0 for some constants $c, \beta > 0$.

Lemma 8.18. *Problem* (8.5) *has a solution.*

Proof. We let $\Theta^\epsilon(u)$ defined above. Notice that

$$\inf_{0<s<1} \frac{\Theta^\epsilon(s/3)}{\Theta^\epsilon(s)} \geq \min\left(\frac{1}{9}, \inf_{0<s<1} \frac{\Theta(s/3)+\epsilon}{\Theta(s)+\epsilon}\right) \geq \min\left(\frac{1}{9}, \frac{\epsilon}{\Theta(1)+\epsilon}\right) > 0$$

and $\Theta^\epsilon(v) = \mathcal{O}_{s\to 0}(s^2)$ so the hypothesis (8.7) is automatically verified. For what comes next we drop the ϵ to lighten the notations.
The results of Section 8.3 apply; for any $M \geq \omega_n$, there exists a minimizer (K^M, Ω^M) of problem (8.4). $M \mapsto E_\Theta(K^M, \Omega^M)$ is nonincreasing, and we prove that it is lower semicontinuous. Indeed, consider $M_i \underset{i\to\infty}{\longrightarrow} M$, and consider the sequence of minimizers (K^{M_i}, Ω^{M_i}), with their state function (u_i). Then proceeding as in the proof of Proposition 8.14, we find a modified sequence (w_i) obtained from (u_i) through truncation with uniformly bounded support, such that its limit w verifies $|\{w = 1\}| \geq \omega_n$, $|\{w > 0\}| \leq M$, and $\mathscr{E}_\Theta(w) \leq \liminf_{i\to\infty} \mathscr{E}_\Theta(u_i)$; this prove the lower semi-continuity. Since

$$E_{\Theta,\Lambda}(K^M, \Omega^M) \geq \Lambda(M - \omega_n) \underset{M\to+\infty}{\longrightarrow} +\infty,$$

then there is a (non-necessarily unique) $M_\Lambda > 0$ such that $[\omega_n, +\infty) \ni M \mapsto E_{\Theta,\Lambda}(K^M, \Omega^M)$ is minimal at $M = M_\Lambda$ and $(K^{M_\Lambda}, \Omega^{M_\Lambda})$ is a minimum of $E_{\Theta,\Lambda}$ with $|\Omega| \leq M$. □

We may now prove the main result of this section. We will say a set A has the center property if any hyperplane that goes through the origin divides A in two parts of same measure. This is in particular the case if A has central symmetry. We will also call a minimizer (K, Ω) "centered" if K has the center property.

Proof of Theorem 8.2. Consider u a minimizer of $\mathscr{E}_{\Theta,\Lambda}$ with the associated sets $\Omega = \{u > 0\}$, $K = \{u = 1\}$. We remind that we know Ω to be open and $u \in H^1(\Omega)$. When e is a unit vector and $\lambda \in \mathbb{R}$, we will write

$$u^+_{e,\lambda}(x) = \begin{cases} u(x) & \text{if } x\cdot e \geq \lambda \\ u(S_{e,\lambda}(x)) & \text{if } x\cdot e < \lambda \end{cases}, \quad u^-_{e,\lambda}(x) = \begin{cases} u(S_{e,\lambda}(x)) & \text{if } x\cdot e \geq \lambda \\ u(x) & \text{if } x\cdot e < \lambda \end{cases},$$

where $S_{e,\lambda}$ is the reflexion relative to the hyperplane $\{x : x\cdot e = \lambda\}$.

- We first show that there exists a minimizer with the center property by building a minimizer with central symmetry. Indeed, consider u a minimizer associated to (K, Ω), $\lambda_1 \in \mathbb{R}$ such that $\{x : x_1 = \lambda_1\}$ cuts K in half. Then $u^+_{e_1,\lambda_1}$ and $u^-_{e_1,\lambda_1}$ are also minimizers; we may in particular replace u with $u^+_{e_1,\lambda_1}$ or $u^-_{e_1,\lambda_1}$ (the choice does not matter here) to suppose u has a symmetry along $\{x : x_1 = \lambda_1\}$. We do the same for $\{x : x_i = \lambda_i\}$ successively for $i = 2, \dots, n$; in the end we arrive to a minimizer

  $$\tilde{u} = \left(\dots\left(u^\pm_{e_1,\lambda_1}\right)^\pm_{e_2,\lambda_2}\right)\dots\Big)^\pm_{e_n,\lambda_n},$$

  that is symmetric relative to every S_{e_i,λ_i}. Up to a translation of λ, $\tilde{u}$ is invariant for every $S_{e_i,0}$, and so it is invariant by their composition which is the central symmetry $x \mapsto -x$.

- Suppose that u is a minimizer (with $K = \{u = 1\}$, $\Omega = \{u > 0\}$) where K has the center property (we know such a minimizer exists from the previous point). We prove that the free boundary $\partial K \cap \Omega$ is a union of spherical arcs centered at the origin.

If $\{0 < u < 1\}$ is empty then the problem is equivalent to the isoperimetric inequality and we are done, so we suppose that it is not. Consider A a connected component of $\{0 < u < 1\}$, which is open by the regularity of minimizers. The set $\partial A \cap \partial K$ is not empty, since it would otherwise mean that $\mathscr{E}_\Theta(u1_{\mathbb{R}^n \setminus A}) < \mathscr{E}_\Theta(u)$.

Let $H_e = \{x : x \cdot e = 0\}$ be a hyperplane going through the origin such that $H_e \cap A \neq \emptyset$, then $u^+_{e,0}$ is also a minimizer; in particular this means by analyticity of u in A that $\nabla u_{|H_e}$ is colinear to H_e. Since this is true for every e, then $u_{|A}$ is a radial function (restricted to a set A that may not be radial), and since it is harmonic it has an expression of the form

$$u_{|A}(x) = a_A - b_A \Phi_n(|x|),$$

where Φ_n is the fundamental solution of the Laplacian (taken with the sign convention that it is increasing). Moreover, since u is 1 on $\partial K \cap \partial A$, we know this set is an arc of circle around 0, and we call its radius r_A. While r_A might depend on the component A, notice that the optimality condition for the Alt-Caffarelli problem near $\partial K \cap \Omega$ is exactly that for a certain constant $\lambda > 0$ that does not depend on A,

$$\forall x \in \partial K \cap \partial\{0 < u < 1\},\ |\nabla u(x)| = \lambda,$$

where $\nabla u(x)$ refers in this case to the limit of $\nabla u(y)$ as $y(\in A) \to x$. With $u(x) = 1$ this exactly reduces to

$$\begin{cases} a_A - b_A \Phi_n(r_A) & = 1, \\ -b_A \Phi_n'(r_A) & = \lambda, \end{cases}$$

so (a_A, b_A) are fully determined by r_A.

- We prove a reflexion lemma that will be useful for successive reflections.

Lemma 8.19. *Let u be a centered minimizer. Then for any vector e, $u^+_{e,0}$ is also a centered minimizer.*

Proof. We let $v = u^+_{e,0}$. If $\{0 < v < 1\}$ has zero measure then as a solution of the isoperimetric problem we know v is the indicator of a ball so we are done; we suppose it is not the case. We may consider A a connected component of $\{0 < v < 1\}$ as previously and $Z \Subset \partial A \cap \partial K$ a small spherical cap that is open with respect to ∂B_{r_A}. Consider another vector f that is completed in an orthogonal basis $f = f_1, f_2, \dots, f_n$ of $\mathbb{R}^n$. As previously we may reflect v successively as

$$\tilde{v} = \left(\dots \left(v^{\pm}_{f_1,\lambda_1} \right)^{\pm}_{f_2,\lambda_2} \right) \dots \Big)^{\pm}_{f_n,\lambda_n},$$

Where the signs are chosen such that v and $\tilde{v}$ coincide on a quadrant that has non-empty intersection with Z. By construction $\tilde{v}$ is centrally symmetric around the point $\lambda = (\lambda_1, \dots, \lambda_n)$, so Z is a spherical cap both around the origin and λ; this implies that $\lambda = 0$ so in particular $\lambda_1 = 0$, which means that $H_{f,0}$ cuts $\{u^+_{e,0} = 1\}$ in half; this is what we wanted to prove. □

- Consider a centered minimizer u (it exists due to the first step, since there exists a centrally symmetric minimizer), such that $|\{0 < u < 1\}| > 0$. We construct another centered minimizer v such that for some connected component A of $\{0 < v < 1\}$, $\partial A \cap \partial\{v = 1\}$ contains a centered sphere. We do this by iteration of the following lemma.

Lemma 8.20. *Let u be a centered minimizer, let A be a connected component of $\{0 < u < 1\}$ and let $\lambda \in]1, 2[$ and $D^{\partial B_r}_{x,\rho} \Subset \partial A \cap \partial K$ be a small spherical disk in the sphere of radius $r := r_A$. Then there exists another centered minimizer v obtained by a finite number of reflexions on u, associated to the sets $(K', \Omega') = (\{v = 1\}, \{v > 0\})$, such that for some connected component A' of $\{0 < v < 1\}$, $D^{\partial B_r}_{x,\lambda\rho} \subset \partial K' \cap \partial A'$.*

Proof. We will denote $D(x, \rho)$ the ball relative the the unit sphere. If $\rho > \frac{\pi}{2} r$ then one reflexion along the hyperplane $x^\perp$ is enough; we suppose that $\rho \leq \frac{\pi}{2} r$.

Let $\epsilon = 1 - \frac{\lambda}{2}$. For every $y \in \partial D(x,(1-\epsilon)\rho)$, the reflexion by the hyperplane going through y and tangent to $\partial D(x,(1-\epsilon)\rho)$ reflects a neighbourhood of $[x,y]$ on a neighbourhood $N(y)$ of $[x',y]$ for some point x' in the continuation of the geodesic $[xy)$ at distance $\lambda\rho$ from x. By compactness there is some finite set $(y_i)_{i\in I}$ in $\partial D(x,(1-\epsilon)\rho)$ such that $\cup_{i\in I} N(y_i) \supset D(x,\lambda\rho)$. By iterating the associated reflexions we obtain the result. □

- There exists a centered minimizer u such that $\{u=1\}$ is a ball. Indeed, consider a minimizer (K,Ω) with a function u as previously, such that there is some $r>0$ for which $\partial B_r \subset \partial K$. For every such r, the function $u_r = \max(u, 1_{B_r})$ is also a minimizer associated to $(K\cup B_r, \Omega\cup B_r)$ that has the same energy as u, thus $u=1$ on B_r. In particular this means that u is radially decreasing on the whole component of Ω that contains a small neighbourhood of B_r; let us denote Ω_1 this component and $\Omega_2 = \Omega\setminus\Omega_1$, as well as $u_i = u1_{\Omega_i}$. Then $u = u_1 + u_2$ where u_1, u_2 have disjoint support and $J_{u_1}\cup J_{u_2} \subset J_u$. If u_2 is not zero this means that u is a disconnected minimizer; let us prove that this is not possible.

  Indeed, let $m_i = |\{u_i = 1\}|$ and $v_i(\cdot) = u_i\left(\left(\frac{m_i}{\omega_n}\right)^{1/n}\cdot\right)$. Then $|\{v_i=1\}| = \omega_n$ and

  $$\mathscr{E}_{\Theta,\Lambda}(u_i) = \left(\frac{m_i}{\omega_n}\right)^{1-\frac{2}{n}}\int_{\mathbb{R}^n}|\nabla v_i|^2 \mathrm{d}\mathscr{L}^n + \left(\frac{m_i}{\omega_n}\right)^{1-\frac{1}{n}}\int_{J_{v_i}}\Theta(v_i)\mathrm{d}\mathscr{H}^{n-1} + \frac{m_i}{\omega_n}\Lambda|\{v_i>0\}| > \frac{m_i}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_i),$$

  where the inequality is strict because $\mathscr{E}_\Theta(v_i) > 0$ and $m_i < \omega_n$ (or else one of the v_i is zero, which we supposed is not the case). Now,

  $$\mathscr{E}_{\Theta,\Lambda}(u) = \mathscr{E}_{\Theta,\Lambda}(u_1) + \mathscr{E}_{\Theta,\Lambda}(u_2) > \frac{m_1}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_1) + \frac{m_2}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_2) \geq \inf \mathscr{E}_{\Theta,\Lambda}.$$

  This contradicts the fact that u is a minimizer. Thus $u_2 = 0$ and $u = u_1$ with $\{u_1=1\} = B_r$, which turns out to be the ball of volume ω_n, this determines r to be 1 and in turns means that u is of the form

  $$u(x) = \varphi(|x|)1_\Omega,$$

  for some radial function φ that takes the value 1 on $[0,r]$ and some set A that contains B_1, and takes the value $1 - c(\Phi_n(|x|) - \Phi_n(1))$ on Ω for some $c>0$.

- Consider such a minimizer u associated with the sets (K,Ω), such that $\{u=1\}$ is the ball B_1, $\{u>0\}$ contains a neighbourhood of B_1 and $u(x)$ is a radial function $\varphi(|x|)$ restricted to a non-necessarily radial set Ω. Then we prove that for some $R>0$, $\varphi 1_{B_R}$ is also a minimizer.

  First replace Ω with its spherical cap rearrangement Ω^s around the axis $\{te_1, t>0\}$, meaning that for every $t>0$, $\Omega^s\cap\partial B_t$ is a spherical cap centered in te_1 with the same $\mathscr{H}^{n-1}$-measure as $\Omega\cap\partial B_t$. Said differently this means that $\mathscr{H}^{n-1}(\Omega\cap\partial B_t) = \mathscr{H}^{n-1}(\Omega^s\cap\partial B_t)$ for all $t>0$ with $\partial\Omega^s = \{f(e\cdot e_1)e, e\in\mathbb{S}^{n-1}\}$ for some nondecreasing $f:[-1,1]\to(r,+\infty)$ that is continuous in 1 by openness of Ω^s (it may also be checked afterward that in our case, f is continuous at -1 by minimality although we will not need it). A property of spherical cap rearrangement (see for instance [158, Prop. 3 and Rem. 4]) is that

  $$\int_{\partial^*\Omega^s}\varphi(|x|)d\mathscr{H}^{n-1} \leq \int_{\partial^*\Omega}\varphi(|x|)d\mathscr{H}^{n-1}$$

  and $|\Omega^s| = |\Omega|$, $\int_{\Omega^s}|\varphi'(|x|)|^2dx = \int_\Omega|\varphi'(|x|)|^2dx$. As a consequence, $\varphi(|\cdot|)1_{\Omega^s}$ is still a minimizer. Write $R := f(1)$, for any small $\epsilon>0$, there is a small enough $\rho>0$ such that

  $$f([1-\rho,1]) \subset [R-\epsilon, R].$$

We may iterate Lemma 8.20 again to obtain a minimizer $(u_\epsilon, K_\epsilon, \Omega_\epsilon)$ such that u_ϵ is still given by the same radial function φ on a non-necessarily radial set, and such that

$$K_\epsilon = B_r,\ B_{R-\epsilon} \subset \Omega_\epsilon \subset B_R.$$

Since this may be done for any arbitrarily small ϵ, by lower semi-continuity as $\epsilon \to 0$ we obtain that $(\varphi(|\cdot|)1_{B_R}, B_r, B_R)$ is a minimizer.

Notice we could have done the same for $(B_1, B_{F(-e_1)})$; if $F(-e_1) < F(e_1)$ it means we are in a case in which $\operatorname{argmin}\{R \geq 1 \mapsto E_{\Theta,\Lambda}(B_1, B_R)\}$ is not uniquely defined.

□

Corollary 8.21. *If, additionally to the hypotheses of Theorem 8.2, Θ is such that $[1, +\infty[\ni R \to E_\Theta(B_1, B_R)$ is minimal at 1, then for any (K, Ω)*

$$E_\Theta(K, \Omega) \geq E_\Theta(B_1, B_1).$$

Proof. (B_1, B_1) is always the minimizer of $E_{\Theta,\Lambda}$ as $\Lambda \to 0$, hence the result. □

Remark 8.22. This covers the case $\beta \leq n-2$ for $\Theta(u) = \beta u^2$.

8.5 Proof of Theorem 8.3

Part of the assertions of the Theorem 8.3, namely the existence of a solution to problem (8.4) and its regularity, have already been proved in the preparatory section 8.3 (see Proposition 8.8 and Remark 8.17).

Proof of Theorem 8.3. In order to complete the proof of Theorem 8.3, we give the following.

Lemma 8.23. *Let (K, Ω) a solution of (8.4). Then K has locally finite perimeter in Ω and $\partial K \cap \Omega$ is analytic when $n = 2$.*

Proof. We refer to [147] where a closely related problem is treated. The same arguments apply in our case. □

Lemma 8.24. *Let Θ be l.s.c. and nondecreasing. Let (K, Ω) be a solution of (8.4) for $M = R^n\omega_n$. Then either $|\Omega| = M$ or $E_\Theta(K, \Omega) = \inf_{r\in[1,R]} E_\Theta(B_1, B_r)$.*

Proof. Suppose there is a minimizer with measure strictly less than M. Roughly speaking, in this case the measure constraint is not saturated so that the problem behaves under many aspects an unconstrained one.

By compactness, there exists (K, Ω) that is the minimizer with the lowest volume, associated to a function u. Indeed, consider (K_i, Ω_i) a sequence of such minimizers, associated with functions (u_i), then reproducing the existence proof in Section 8.3 on the sequence (u_i), we obtain a new minimizer with minimal volume, still denoted (K, Ω).

If $|\Omega| = \omega_n$ then we are done, so we suppose that this is not the case.

Let $\mathcal{H}$ be the set of every hyperplane that cuts K in half. It is not quite identified with $\mathbb{RP}^{n-1}$ since there may be several parallel hyperplanes that cut K in half if K is not connected. It is, however, straightforward that $\mathcal{H}$ is a connected set for the natural topology given by $\{(H, v) \in \mathbb{RP}^{n-1} \times \mathbb{R}^n \text{ s.t. } v \in H^\perp\}$. For every hyperplane H, we write

$$m(H) = \sup\left\{\left|H^+ \cap \Omega\right|, \left|H^- \cap \Omega\right|\right\}$$

and we let

$$\mathcal{H}_< = \left\{ H \in \mathcal{H} : m(H) < \frac{M}{2} \right\},$$
$$\mathcal{H}_= = \left\{ H \in \mathcal{H} : m(H) = \frac{M}{2} \right\},$$
$$\mathcal{H}_> = \left\{ H \in \mathcal{H} : m(H) > \frac{M}{2} \right\}.$$

Then $\mathcal{H}_<$ and $\mathcal{H}_>$ are open relatively to $\mathcal{H}$ by the continuity of m. The minimality of the volume of Ω and the fact that $|\Omega| < M$ implies that $\mathcal{H}_=$ is empty (otherwise we could construct a minimizer with stricly lower volume by reflection around an element of $\mathcal{H}_=$).

Finally, $\mathcal{H}_<$ is not empty, as may be seen by considering, for each $\theta \in \frac{\mathbb{R}}{\pi\mathbb{Z}}$, an hyperplane orthogonal to $\cos(\theta)e_1 + \sin(\theta)e_2$ and using intermediate value theorem after making a full turn. By connectedness, this directly implies that $\mathcal{H}_> = \emptyset$ and so $\mathcal{H}_< = \mathcal{H}$. Now, let $H \in \mathcal{H}$, then Ω may be reflected on both sides of H: since the measure of Ω is minimal among minimizers, then

$$\left|H^- \cap \Omega\right| = \left|H^+ \cap \Omega\right|.$$

Thus, for every hyperplane that cuts K in half, Ω is also cut in half; this means that we can use the same arguments than for the penalized problem in Theorem 8.2: by successive reflections, we find a solution with minimal volume such that the free boundary of u is ∂B_1. By spherical rearrangement around $\{te_1, t > 0\}$ we obtain a new minimizer (K, Ω). Now, if Ω is not a ball, then $\left|H^+_{e_1} \cap \Omega\right| > \left|H^-_{e_1} \cap \Omega\right|$, which is a contradiction. □

We prove now the last assertion of Theorem 8.3. We begin with an a priori estimate that is similar to Lemma 8.10, only that it is done with in mind the idea to find a lower bound arbitrarily close to 1.

Proposition 8.25. *Let Θ be l.s.c and nondecreasing. There exists a constant $C_n > 0$ such that, if $\delta \in]0, 1[$ verifies*

$$\delta + C_n \frac{\Theta(1)}{\sqrt{\Theta(\delta)}} (M - \omega_n)^{\frac{1}{2n}} < 1,$$

then for all u such that $|\{u \geq 1\}| \geq \omega_n$, $|\{u > 0\}| \leq M$, there is some $t > \delta$ such that $\mathscr{E}_\Theta(u1_{\{u>t\}}) \leq \mathscr{E}_\Theta(u)$.

This lemma is applicable in particular when δ is close to 1 and $M - \omega_n$ is small.

Proof. This proof is very similar to Lemma 8.10, so some steps are only briefly described. We actually show the stronger result; there is some $t > s > \delta$ such that

$$\mathscr{E}_\Theta\left(u1_{\{s<u<t\}^c}\right) \leq \mathscr{E}_\Theta(u).$$

Let $\epsilon = 1 - \delta > 0$, suppose $\mathscr{E}_\Theta(u1_{\{s<u<t\}^c}) > \mathscr{E}_\Theta(u)$ for every $1 - \epsilon < s < t < 1$. We write

- $\Omega(s,t) = \{s < u \leq t\}$.
- $\gamma(s,t) = \int_{J_u} \left(1_{s<u_+\leq t} + 1_{s<u_-\leq t}\right) \mathrm{d}\mathscr{H}^{n-1}$.
- $h(t) = \mathscr{H}^{n-1}(\{u = t\} \setminus J_u)$.

Let $\eta := 1 - \frac{\epsilon}{2}$, our hypothesis becomes that for every, $t \in]0, \frac{\epsilon}{2}[$, $\mathscr{E}_\Theta(u1_{\{\eta-t<u<\eta+t\}^c}) > \mathscr{E}_\Theta(u)$ so

$$\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \Theta(1-\epsilon)\gamma(\eta-t,\eta+t) \leq \Theta(1)\left(h\left(\eta-t\right)+h\left(\eta+t\right)\right). \tag{8.16}$$

The proof is, as previously, based on a lower bound of $\int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n$ and an upper bound of $\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n$ that are in contradiction when ϵ is too large.

- For all $t \in]0, \frac{1}{2}\epsilon[$ we let:
$$f(t) = \int_{\eta-t}^{\eta+t} h = \int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n.$$
f is absolutely continuous and
$$f'(t) = h\left(\eta-t\right)+h\left(\eta+t\right).$$
Moreover,
$$\begin{aligned} f(t) &\leq |\Omega\left(\eta-t,\eta+t\right)|^{\frac{1}{2}} \left(\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2}} \\ &\leq C_n \mathrm{Per}(\Omega\left(\eta-t,\eta+t\right))^{\frac{n}{2(n-1)}} \left(\Theta(1)(h\left(\eta-t\right)+h\left(\eta+t\right))\right)^{\frac{1}{2}} \quad \text{by isoperimetric inequality} \\ &\leq C_n \Theta(1)^{\frac{1}{2}} \left(h\left(\eta-t\right)+\gamma\left(\eta-t,\eta+t\right)+h\left(\eta+t\right)\right)^{\frac{n}{2(n-1)}} \left(h\left(\eta-t\right)+h\left(\eta+t\right)\right)^{\frac{1}{2}} \\ &\leq C_n \left(\frac{\Theta(1)}{\Theta(1-\epsilon)}\right)^{\frac{n}{2(n-1)}} \Theta(1)^{\frac{1}{2}} f'(t)^{\frac{2n-1}{2n-2}}, \end{aligned}$$
so:
$$f'(t) f(t)^{-\frac{2n-2}{2n-1}} \geq c_n \Theta(1)^{-\frac{n-1}{2n-1}} \left(\frac{\Theta(1)}{\Theta(1-\epsilon)}\right)^{-\frac{n}{2n-1}}.$$
We integrate on $[0,t]$:
$$\left(\int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n\right)^{\frac{1}{2n-1}} \geq c_n \Theta(1)^{-\frac{n-1}{2n-1}} \left(\frac{\Theta(1)}{\Theta(1-\epsilon)}\right)^{-n} t.$$
And so:
$$\int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n \geq c_n \Theta(1)^{-(n-1)} t^{2n-1}.$$

- We let:
$$g(t) = \int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n, \; G(t) = \int_0^t g$$
We begin by finding a upper bound for G. We take t in $[0, \epsilon/2]$, then:
$$\begin{aligned} G(t) &\leq \int_0^t \Theta(1)(h(\eta-s)+h(\eta+s))\mathrm{d}s \\ &\leq c_n \Theta(1) |\Omega\left(\eta-t,\eta+t\right)|^{\frac{1}{2}} G'(t)^{\frac{1}{2}}. \end{aligned}$$
Thus:
$$G'(t) G(t)^{-2} \geq c_n \Theta(1)^{-2} |\Omega\left(1-\epsilon,1\right)|^{-1}.$$
We integrate from t to $2t$ (up to supposing $t < \epsilon/4$):
$$G(t)^{-1} \geq c_n t \Theta(1)^{-2} |\Omega\left(1-\epsilon,1\right)|^{-1}.$$
Thus:
$$G(t) \leq C_n \Theta(1)^2 t^{-1} |\Omega\left(1-\epsilon,1\right)|.$$
Since g is increasing, then up to supposing $t < \epsilon/8$:
$$g(t) \leq \frac{1}{t}\int_t^{2t} g \leq \frac{G(2t)}{t} \leq c_n \Theta(1)^2 t^{-2} |\Omega\left(1-\epsilon,1\right)|.$$

Combining the previous inequalities, that are valid for $t \in [0, \epsilon/8]$, we get:

$$\begin{aligned} c_n \Theta(1)^{-(n-1)} \left(\frac{\Theta(1)}{\Theta(1-\epsilon)} \right)^{-n} t^{2n-1} &\leq \int_{\Omega(\eta-t,\eta+t)} |\nabla u| \mathrm{d}\mathscr{L}^n \\ &\leq |\Omega(\eta-t, \eta+t)|^{\frac{1}{2}} \left(\int_{\Omega(\eta-t,\eta+t)} |\nabla u|^2 \mathrm{d}\mathscr{L}^n \right)^{\frac{1}{2}} \\ &\leq C_n \Theta(1) t^{-1} |\Omega(1-\epsilon, 1)|. \end{aligned}$$

Taking $t = \epsilon/8$, and with $|\Omega(1-\epsilon, 1)| \leq M - \omega_n$, we get:

$$\epsilon \leq C_n \Theta(1) \Theta(1-\epsilon)^{-\frac{1}{2}} (M - \omega_n)^{\frac{1}{2n}},$$

for a constant $C_n > 0$ that only depends on n; this proves the result. □

Proof of Theorem 8.3 (continuation). Let Θ be l.s.c. and nondecreasing. Suppose moreover that it is of class $\mathcal{C}^1$ near 1 and

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n-1).$$

Then there is some $M_0 > \omega_n$ depending on n, Θ such that, for any $M \in [\omega_n, M_0]$ and for any admissible u with $|\{u \geq 1\}| \geq \omega_n$, $|\{u > 0\}| \leq M$,

$$\mathscr{E}_\Theta(u) \geq E_\Theta(B_1, B_1) \, (= \Theta(1) n \omega_n).$$

Denote for simplicity $\Omega = \{u > 0\}$, $K = \{u = 1\}$, and let δ be defined as in the previous result (it is always possible when $M - \omega_n$ is small enough), and suppose that it is close enough to 1 such that Θ in $\mathcal{C}^1$ on $[\delta, 1]$. Using the previous lemma, we lose no generality in supposing $u_{|\Omega} \geq \delta$, and $|\Omega| = M$. Then

$$\begin{aligned} \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \int_\Omega \Theta'(u) |\nabla u| \mathrm{d}\mathscr{L}^n &= \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}; \Omega) \mathrm{d}t \\ &= \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} - \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}; \partial\Omega) \mathrm{d}t \\ &\quad + \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}) \mathrm{d}t \\ &= \Theta(\delta) \mathrm{Per}(\Omega) + \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}) \mathrm{d}t \\ &\geq \Theta(\delta) \left(\frac{M}{\omega_n} \right)^{1-\frac{1}{n}} \mathrm{Per}(B_1) + \int_\delta^1 \Theta'(t) \mathrm{Per}(B_1) \mathrm{d}t \\ &= \Theta(1) \mathrm{Per}(B_1) + \Theta(\delta) \mathrm{Per}(B_1) \left(\left(\frac{M}{\omega_n} \right)^{\frac{n-1}{n}} - 1 \right). \end{aligned}$$

Now,

$$\int_\Omega \Theta'(u) |\nabla u| \mathrm{d}\mathscr{L}^n \leq \int_\Omega \|\Theta'\|_{\infty,[\delta,1]} |\nabla u| \mathrm{d}\mathscr{L}^n \leq \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \frac{\|\Theta'\|^2_{\infty,[\delta,1]}}{4} (M - \omega_n),$$

Thus,

$$\begin{aligned} \int_\Omega |\nabla u|^2 \mathrm{d}\mathscr{L}^n + \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} - \Theta(1) \mathrm{Per}(B_1) \geq & \Theta(\delta) \mathrm{Per}(B_1) \left(\left(\frac{M}{\omega_n} \right)^{\frac{n-1}{n}} - 1 \right) \\ & - \frac{\|\Theta'\|^2_{\infty,[\delta,1]}}{4} (M - \omega_n). \end{aligned}$$

So we obtain the result as soon as

$$\frac{\|\Theta'\|^2_{\infty,[\delta,1]}}{\Theta(\delta)} \leq 4\mathrm{Per}(B_1)\frac{\left(\frac{M}{\omega_n}\right)^{\frac{n-1}{n}}-1}{M-\omega_n}.$$

As $M \to \omega_n$, we may take $\delta \to 1$, and this gives the result.

□

Remark 8.26. Conversely, suppose that

$$\frac{\Theta'(1)^2}{\Theta(1)} > 4(n-1),$$

then for every $R > 1$ close enough to 1, $E_\Theta(B_1, B_R) < E_\Theta(B_1, B_1)$. Indeed, let $\epsilon > 0$ and

$$u_\epsilon(x) = \begin{cases} 1 & (B_1), \\ 1 - \frac{|x|-r}{2}\Theta'(1) & (B_{1+\epsilon} \setminus B_1). \end{cases}$$

Then

$$\begin{aligned} E_\Theta(B_1, B_{1+\epsilon}) \leq \mathscr{E}_\Theta(u_\epsilon) &= n\omega_n(1+\epsilon)^{n-1}\Theta\left(1 - \frac{1}{2}\Theta'(1)\epsilon\right) + \omega_n\left((1+\epsilon)^n - r^n\right)\frac{\Theta'(1)^2}{4} \\ &= E_\Theta(B_1, B_1) + \left((n-1)\Theta(1) - \frac{1}{4}\Theta'(1)^2\right)\epsilon\mathrm{Per}(B_1) + o_{\epsilon\to 0}(\epsilon). \end{aligned}$$

The first-order term is negative and this proves the converse for a small enough ϵ.

Bibliography

[143] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[144] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen. On the Robin problem in fractal domains. *Proc. Lond. Math. Soc. (3)*, 96(2):273–311, 2008.

[145] Marie-Hélène Bossel. Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimétrique de Rayleigh-Faber-Krahn. *Z. Angew. Math. Phys.*, 39(5):733–742, 1988.

[146] Andrea Braides. *Approximation of free-discontinuity problems*, volume 1694 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[147] Tanguy Briançon and Jimmy Lamboley. Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1149–1163, 2009.

[148] Dorin Bucur and Alessandro Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[149] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[150] Dorin Bucur, Alessandro Giacomini, and Mickaël Nahon. Degenerate Free Discontinuity Problems and Spectral Inequalities in Quantitative Form. *Arch. Ration. Mech. Anal.*, 242(1):453–483, 2021.

[151] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[152] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[153] Daniel Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.

[154] Francesco Della Pietra, Carlo Nitsch, and Cristina Trombetti. An optimal insulation problem. *Math Ann*, Online, 2020.

[155] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[156] Dennis Kriventsov. A free boundary problem related to thermal insulation: flat implies smooth. *Calc. Var. Partial Differential Equations*, 58(2):Paper No. 78, 83, 2019.

[157] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[158] Frank Morgan, Sean Howe, and Nate Harman. Steiner and Schwarz symmetrization in warped products and fiber bundles with density. *Rev. Mat. Iberoam.*, 27(3):909–918, 2011.

Chapter 9

Boundary behaviour of solutions of elliptic equation with Robin boundary conditions on rough domains

This is a joint work with Dorin Bucur and Alessandro Giacomini

Contents

9.1 Introduction **213**
9.2 Global boundary behavior **214**
9.2.1 Examples 217
9.3 Local boundary behaviour **219**
9.4 Analysis of cusps **221**
9.5 Further remarks **229**

9.1 Introduction

The purpose of this paper is to study the boundary behavior of the solution of a Robin problem for the p-Laplace operator associated to some non-negative right hand side in a non-smooth domain. For instance, for the Laplace operator, if both the domain and the solution were smooth, a consequence of the boundary Hopf principle is that the solution is strictly positive on the boundary (and of course in the interior of the domain). In this paper, we are interested in this question in the context of non smooth boundaries where the strong form of the boundary condition may not hold.

To introduce the problem, let $\Omega \subset \mathbb{R}^N$ be an open, connected, bounded set with a boundary which, for the moment and for expository reasons, is assumed to be of class C^2 (this regularity assumption will be removed later). Given a parameter $\beta > 0$ and a smooth function $f : \Omega \to \mathbb{R}$, we consider the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$

ν being the outer normal at the boundary. Under the hypothesis $f \geq 0$ in Ω one gets from the maximum principle that $u \geq 0$ in Ω and, if $f \not\equiv 0$, u is strictly positive inside Ω. The minimum of u has to be searched on the boundary of Ω where, as a consequence of the Hopf lemma, the normal derivative has to be strictly negative. Consequently, from the strong form of the Robin boundary condition $\frac{\partial u}{\partial n} + \beta u = 0$ one gets that u has to be strictly positive at its minimum. Finally, there exists some $\delta > 0$ such that

$$\forall x \in \Omega, \quad u(x) \geq \delta > 0. \tag{9.2}$$

If Ω is not of class C^2 (for instance it is Lipschitz, or less regular but smooth enough to give sense to a weak form of the problem), then the Hopf lemma can not be anymore used to arrive to

the same conclusion. The normal direction at the boundary might not be properly defined at some specific point of the boundary, and the strong pointwise form of the Robin boundary conditions may not apply. Nevertheless, lower bounds as (9.2) may hold provided that, intuitively, there is no "too high" concentration of the boundary around one point.

Looking for strict positive estimates similar to (9.2) in non smooth domains, the first result is due to Bass, Burdzy and Chen [162], by probabilistic methods in the context of harmonic functions. They identify a class of nonsmooth domains, including the Lipschitz ones, for which (9.2) holds for non-negative harmonic functions solving a Robin problem (see also [164, 165] for a similar question in the context of free discontinuity problems). The result of [162] relies on a probabilistic method and applies to the Laplace operator. The key argument strongly uses the linearity of the equation, involving the study of Green functions and the use of a uniform boundary Harnack inequality. In particular, the geometric conditions given in [162] require that the domain should be written as union of sets with Lipschitz boundaries sharing the same bound on the Lipschitz norm, up to a set of $\mathscr{H}^{N-1}-$ Hausdorff measure equal to zero. This covers the case of Lipschitz domains, of some domains with cusps and of some fractal domains.

Two more recent results by Gesztesy, Mitrea and Nichols [166] on the one hand, and by Arendt and ter Elst [161] on the other, show that a first non-negative eigenfunction of the Robin Laplacian in a Lipschitz set Ω satisfies the bound from below (9.2), as soon as it is is not identically equal to 0. Their arguments are based on the analysis of related semigroups acting on $C(\partial\Omega)$ and are a consequence of a regularity property of the eigenfunctions of the Robin Laplacian in Lipschitz domains, in particular their continuity up to the boundary.

The purpose of this paper is to analyze the boundary strict positivity inequality (9.2) in a more general context of non-smooth domains and of nonlinear PDEs of p-Laplacian type (even if they are not of energy type). Our technique is based on variational arguments, it allows to handle global and local results and to give quantitative estimates of the lowest value in terms of some average sum of the solution. The key ingredient is the behaviour of a kind of global (or local) isoperimetric profile of the set, which depends on p.

We deal with bounded, open, connected sets with a rectifiable topological boundary which have finite $(N-1)$-Hausdorff measure. In this case, traces of $W^{1,p}$-Sobolev functions are $\mathscr{H}^{N-1}$-pointwise well defined on $\partial\Omega$ and the Robin problem is well posed in a weak form for operators of p-Laplacian type. The geometric properties which play a crucial role in the validity of (9.2) can, in some situations, be related to the local control of the L^1 norm of the trace of an BV function on $\partial\Omega$ and to local reinforced isoperimetric inequalites via the summability of an isoperimetric profile function. Of course, a class of domains which satisfy naturally these geometric properties includes all Lipschitz sets. Nevertheless, Lipschitz regularity is not, in general, required for the property to hold. Our analysis provides a quantitative estimate for the constant δ in (9.2) in terms of the geometry and of the mass of the solution u on low sublevels.

For simplicity, we focus on the p-Laplacian equation which obeys an energetic variational principle. It turns out that the energetic principle is useful for the comprehension of the existence of a solution in a nonsmooth domain, but it is not a key tool for the boundary behaviour. We discuss in the last section how the results for the p-Laplacian extend (under the assumption that a weak solution exists) to more general monotone elliptic operators in divergence form, which are not necessarily related to energy minimization.

9.2 Global boundary behavior

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, connected set such that $\partial\Omega$ is rectifiable and $\mathscr{H}^{N-1}(\partial\Omega) < +\infty$. Let $\beta > 0$ be a constant, $p \in (1, +\infty)$, $p' = \frac{p}{p-1}$. Let $f \in L^{p'}(\Omega)$, $f \geq 0$ not identically equal to 0.

Let $v \in W^{1,p}(\Omega)$. Then for $\mathscr{H}^{N-1}$-a.e. point $x \in \partial\Omega$ the function v extended by 0 outside Ω has upper and lower approximate limits, denoted v_+ and v_-. Moreover,

$$v \to \int_{\partial\Omega} (|v_+|^p + |v_-|^p) d\mathscr{H}^{N-1}$$

is lower semicontinuous for the weak topology in $W^{1,p}(\Omega)$. There exists a function u which minimises in $W^{1,p}(\Omega)$ the following energy

$$E(v) = \frac{1}{p}\left(\int_\Omega |\nabla v|^p \mathrm{d}\mathscr{L}^N + \beta \int_{\partial\Omega} (|v_+|^p + |v_-|^p) d\mathscr{H}^{N-1}\right) - \int_\Omega f v \mathrm{d}\mathscr{L}^N, \tag{9.3}$$

and the minimizer is unique, from the strict convexity of E. For the existence part, the key results are the Poincaré inequality with trace term which occurs in this setting together with the compactness result in $L^p(\Omega)$ which holds for a sequence of functions with bounded energy. For all these results we refer to [159] and more specifically to [163]. Alternatively, relying on the space of special functions with bounded variation $SBV^{1/p}(\mathbb{R}^n, \mathbb{R}^+)$ (see [159]), this procedure to define weak solutions can be seen by the minimization of E over $SBV(\mathbb{R}^n, \mathbb{R}^+)$-functions with support in Ω and jump in $\partial\Omega$ (see [163]). As the existence part is not relevant for our purpose in this paper, we shall not detail it.

If Ω is Lipschitz, then u is the (weak) variational solution of

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

meaning that $u \in W^{1,p}(\Omega)$ and for every $v \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}\mathscr{L}^N + \beta \int_{\partial\Omega} |u|^{p-2} u v d\mathscr{H}^{N-1} = \int_\Omega f v \mathrm{d}\mathscr{L}^N.$$

If Ω has rectifiable boundary with finite Hausdorff measure, the weak solution $u \in W^{1,p}(\Omega)$ satisfies for every $v \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}\mathscr{L}^N + \beta \int_{\partial\Omega} \left(|u^+|^{p-2}u^+ v^+ + |u^-|^{p-2}u^- v^-\right) d\mathscr{H}^{N-1} = \int_\Omega f v \mathrm{d}\mathscr{L}^N,$$

where $u^+(x), u^-(x)$ (and $v^+(x), v^-(x)$) denote the traces of u at $x \in \partial\Omega$, ordered by the choice of a normal vector normal to $\partial\Omega$ at x which is well-defined $\mathscr{H}^{N-1}$-almost everywhere.

For a measurable set with finite perimeter $\omega \subset \Omega$, we denote $\partial^*\omega$ its reduced boundary and

$$P^e(\omega) := \int_{\partial\Omega} \left(1^+_\omega + 1^-_\omega\right) d\mathscr{H}^{N-1} \qquad \text{and} \qquad P^i(\omega) := P(\omega, \Omega),$$

the exterior and the interior perimeter of ω, respectively. Note that the exterior perimeter counts the both sides of a crack of $\partial\Omega$ which crosses ω. In particular, $P^e(\Omega)$ equals the sum between the Hausdorff measure of the topological boundary points of density $\frac{1}{2}$ in Ω and twice the Hausdorff measure of topological boundary points of density 1. For simplicity, we denote $\partial^i\omega = \partial^*\omega \cap \Omega$ and $\partial^e\Omega = \partial^*\omega \cap \partial\Omega$.

We define the p-isoperimetric profile of Ω as follows

$$I : \left(0, \frac{1}{2}P^e(\Omega)^{\frac{1}{p}}|\Omega|^{\frac{1}{p'}}\right) \to \mathbb{R} \cup \{+\infty\},$$

$$I(m) = \inf\left\{P^i(\omega) :\ \omega \subset \Omega, |\omega| \le \frac{|\Omega|}{2}, m \le P^e(\omega)^{\frac{1}{p}}|\omega|^{\frac{1}{p'}}\right\}.$$

The value of $I(m)$ is well defined for every $0 < m \le \frac{1}{2}P^e(\Omega)^{\frac{1}{p}}|\Omega|^{\frac{1}{p'}}$ since the set above is not empty. Moreover, the function I is non decreasing. Note that the main interest is related to in the behavior of I when m is small.

Theorem 9.1. *Assume $\frac{1}{I}$ is summable. Then*

$$\inf_{x\in\Omega} u(x) \geq \frac{T}{2} > 0,$$

where $T > 0$ denotes the largest number such that

$$|\{u < T\}| \leq \frac{1}{2}|\Omega| \tag{9.4}$$

and

$$\beta^{\frac{1}{p}} \int_0^{P^e(\Omega)^{\frac{1}{p}}|\{u<T\}|^{\frac{1}{p'}}} \frac{dm}{I(m)} \leq \frac{1}{2}. \tag{9.5}$$

Before the proof, let us make some remarks. The summability condition on the isoperimetric profile seems abstract and difficult to handle. However, as we shall point out in some examples, there are situations where efficient estimates of I can be obtained under controllable geometric assumptions, for instance in the case of cusps. Note as well that T is indeed strictly positive, otherwise the solution u would vanish on a set of positive measure, which is impossible for a nonvanishing, nonnegative, p-superharmonic function ([167, Theorem 7.12]).

Let also point out that a similar result could be obtained for a nonconstant $\beta \in L^1(\partial\Omega, \mathbb{R}^+)$. However, in this case, a fine study would require the analysis of an isoperimetric profile involving the function β, which is not easy in practical situations. In the particular case in which the function β is bounded, our result applies to the constant $\|\beta\|_\infty$. In the next section, where we perform a local analysis, considering such functions β which are locally bounded may be of more interest.

Proof. We note first that $u \geq 0$. For every $t > 0$, let $\omega_t = \{u < t\}$ and $g(t) = \int_{\omega_t} |\nabla u|$. From the minimizing property of u compared to $\max(u, t)$, we have the estimate

$$\int_{\omega_t} |\nabla u|^p \leq \beta t^p P^e(\omega_t). \tag{9.6}$$

Consequently, Hölder inequality gives

$$g(t) \leq t(\beta P^e(\omega_t))^{\frac{1}{p}} |\omega_t|^{\frac{1}{p'}}. \tag{9.7}$$

We denote $t_0 = \sup\{s \geq 0 : g(s) = 0\} = \inf(u)$, our goal being to estimate t_0 from below. If $t_0 \leq t \leq T$, thanks to (9.4) and to the monotonicity of I we get

$$I\left(\frac{g(t)}{\beta^{\frac{1}{p}} t}\right) \leq I\left(P^e(\omega_t)^{\frac{1}{p}} |\omega_t|^{\frac{1}{p'}}\right) \leq P^i(\omega_t) = g'(t).$$

Summing from t_0 to t, we obtain (using again the monotonicity of I)

$$\int_{t_0}^t \frac{g'(\tau)d\tau}{I(g(\tau)/\beta^{\frac{1}{p}} t)} \geq \int_{t_0}^t \frac{g'(\tau)d\tau}{I(g(\tau)/\beta^{\frac{1}{p}} \tau)} \geq t - t_0,$$

and the change of variable $m = g(\tau)/\beta^{\frac{1}{p}} t$, the inequality becomes

$$\beta^{\frac{1}{p}} t \int_0^{g(t)/\beta^{\frac{1}{p}} t} \frac{dm}{I(m)} \geq t - t_0.$$

With the initial estimate (9.7) and equation (9.5) for T, we finally obtain

$$1 - \frac{t_0}{T} \leq \int_0^{P^e(\Omega)^{\frac{1}{p}} |\omega_T|^{\frac{1}{p'}}} \frac{dm}{I(m)} \leq \frac{1}{2},$$

so that $t_0 \geq \frac{T}{2}$. □

9.2.1 Examples

To be more precise, we discuss now some global geometric assumptions on Ω which can be verified and which lead to strict positivity. We denote by $\partial\Omega$ the topological boundary and by $\partial^*\Omega$ the measure theoretical boundary of Ω.

Geometric interpretation. Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, connected set such that $\partial\Omega$ is rectifiable and $\mathscr{H}^{N-1}(\partial\Omega) < +\infty$. Below, the constant $G > 0$ may change from line to line. Assume that that there exist constants $G > 0$ and $1 \geq a, b \geq 0$ such that for all $\omega \subset \Omega$ with finite perimeter satisfying $|\omega| \leq \frac{|\Omega|}{2} \wedge 1$ we have

$$P^e(\omega) \leq G\frac{P^i(\omega)}{|\omega|^a}. \tag{9.8}$$

$$|\omega|^{\frac{N-1}{N}} \leq G\left(P^i(\omega) + P^e(\omega)\right)\left(\frac{P^i(\omega)}{P^e(\omega)}\right)^b. \tag{9.9}$$

The first inequality gives a control for the ratio between the boundary and inner perimeters by a constant which can blow up when the measure of the domain is small, while the second inequality can be interpreted as an improved isoperimetric inequality for domains with small measure with a large isoperimetric quotient. Then, Theorem 9.1 applies for certain values of a, b, provided (9.8) and (9.9) are satisfied. We point out the following examples.

- In general, if

$$M := \frac{1}{p} + \frac{\frac{1}{p'} - \frac{a}{p}}{\frac{N-1}{N} + a(1-b)} > 1, \tag{9.10}$$

  then the integrability hypothesis of Theorem 9.1 is satisfied and strict positivity occurs. Indeed, assume that (9.10) occurs. We study the integrability of the p-isoperimetric profile of Ω. Take $\omega \subset \Omega$ such that

$$m \leq P^e(\omega)^{\frac{1}{p}}|\omega|^{\frac{1}{p'}}.$$

  We deduce from (9.8) and (9.9) a lower bound on $P^i(\omega)$, which implies a lower bound on $I(m)$ by taking the infinimum among all admissible ω.

  If $P^e(\omega) \leq P^i(\omega)$, then the usual isoperimetric inequality applies and so there is a constant $c_N > 0$ such that $|\omega|^{\frac{N-1}{N}} \leq c_N \mathrm{Per}(\omega) \leq 2c_N P^i(\omega)$, so

$$P^i(\omega) = P^i(\omega)^{\frac{Np-N}{Np-1}} P^i(\omega)^{\frac{N-1}{Np-1}} \geq A_N|\omega|^{\frac{(N-1)(p-1)}{Np-1}} P^i(\omega)^{\frac{N-1}{Np-1}} \geq A_N m^{\frac{Np-p}{Np-1}}$$

  for some $A_N > 0$. On the contrary if $P^i(\omega) \leq P^e(\omega)$ then we have

$$P^i(\omega)^b \geq c|\omega|^{\frac{N-1}{N}} P^e(\omega)^{b-1}$$
$$P^i(\omega) \geq c|\omega|^a P^e(\omega)$$

  for some $c > 0$, so combining those as previously we obtain

$$P^i(\omega)^M = P^i(\omega)^{b\frac{1-\frac{1+a}{p}}{\frac{N-1}{N}+a(1-b)}} P^i(\omega)^{\frac{1}{p}+\frac{\left(1-\frac{1+a}{p}\right)(1-b)}{\frac{N-1}{N}+a(1-b)}} \geq c|\omega|^{\frac{1}{p'}} P^e(\omega)^{\frac{1}{p}} \geq m$$

  In conclusion

$$I(m) \geq c\left(m^{\frac{1}{M}} \wedge m^{\frac{Np-p}{Np-1}}\right).$$

  Hence $\int_0^{m_0} \frac{1}{I(m)} dm < +\infty$.

- For $N = 2$, the case $b = \frac{1}{2}$, $a = \frac{\alpha-1}{\alpha+1}$, $\alpha \in (1, 2)$ covers the cuspidal domains in $\mathbb{R}^2$ $\{(x, y) : 1 > x > 0, |y| < x^\alpha\}$. The proof it is not direct, we refer to Section 9.4 for a general approach of cusps. For the Laplace operator, the case $a = \frac{1}{3}$ and $b = \frac{1}{2}$ is critical as M defined above equals to 1, and the positivity property does not hold (see [162]).

- For a Lipschitz set Ω, inequalities (9.8)-(9.9) are satisfied with $a = b = 0$. Indeed (9.9) reduces to the isoperimetric inequality. Concerning (9.8), the trace theorem in BV applied to 1_ω together with the relative isoperimetric inequality in a Lispchitz domain (recall $|\omega| \leq \frac{1}{2}|\Omega|$) yield

$$P^e(\omega) \leq C_\Omega \left[P^i(\omega) + |\omega|\right] \leq C_\Omega \left[P^i(\omega) + |\Omega|^{\frac{1}{N}}|\omega|^{\frac{N-1}{N}}\right] \leq \hat{C}_\Omega P^i(\omega),$$

  for some constants $C_\Omega, \hat{C}_\Omega > 0$.

Remark 9.2. More generally, we notice that inequality (9.8) entails an inner density estimate for the set Ω in the case $a = 0$. Indeed, taking a boundary point $x_0 \in \partial\Omega$, and $A = B(x_0, t)$, from the isoperimetric inequality and (9.8) we get by the co-area formula that for a.e. $t \in (0, t_0)$ that

$$m(t)^{\frac{N-1}{N}} \leq C_N(1 + G)m'(t).$$

Above, t_0 is the radius of the ball of the same measure as $\frac{|\Omega|}{2}$ and $m(t) = |\Omega \cap B(x_0, t)|$. Then, by summing from 0 to r, with $r \leq t_0$, one gets

$$|\Omega \cap B(x_0, r)| \geq \frac{r^N}{\Big(NC_N(1 + G)\Big)^N},$$

giving the uniform inner density estimate. More generally $a \in [0, 1/N)$ this implies an estimate

$$|\Omega \cap B(x_0, r)| \geq cr^{\frac{N}{1-N\alpha}}.$$

If $a = 0$, inequality (9.8) is in fact related to the BV-trace theorem of Anzellotti and Giaquinta, for which we refer the reader to [160]. Indeed, if the topological boundary of Ω coincides $\mathscr{H}^{N-1}$-a.e. with its reduced boundary, the existence of a constant C_1 above is related to the following inequality

$$C \int_{\partial\Omega} |u| \mathrm{d}\mathscr{L}^N \leq \int_\Omega |Du| + \int_\Omega |u| \mathrm{d}\mathscr{L}^N,$$

by $|Du|$ denoting the total variation measure associated to the BV function u. If $\partial\Omega$ differs from $\partial^*\Omega$ by a set of positive $\mathscr{H}^{N-1}$-measure (for instance Ω has cracks), then the result of Anzellotti and Giaquinta does not apply directly. It is not our goal here to analyze the existence of continuous traces.

Remark 9.3. Let us sketch briefly a direct argument to prove that $\inf_\Omega u > 0$ in the Lipschitz case, which employes the standard relative isoperimetric inequality and the trace operator in BV (see the considerations above). Using the same notation as in the proof of Theorem 9.1, let

$$\omega_t := \{x \in \Omega \,:\, u(x) < t\}$$

and let us assume by contradiction that $\omega_t \neq \emptyset$ for every $t > 0$. Below C indicates a constant depending only on Ω which can very from line to line. The Lipschitz regularity of the boundary entails for t small

$$P^e(\omega_t) \leq CP^i(\omega_t) \qquad \text{and} \qquad |\omega_t| \leq CP^i(\omega_t)^{\frac{N}{N-1}},$$

so that the comparison between the functions u and $u \wedge t$ leads to

$$\int_{\omega_t} |\nabla u| \,\mathrm{d}\mathscr{L}^N \leq \left(\int_{\omega_t} |\nabla u|^p \,\mathrm{d}\mathscr{L}^N\right)^{\frac{1}{p}} |\omega_t|^{\frac{1}{p'}} \leq [\beta t^p P^e(\omega_t)]^{\frac{1}{p}} |\omega_t|^{\frac{1}{p'}} \leq CtP^i(\omega_t)^{1+\epsilon}, \tag{9.11}$$

where $\epsilon > 0$. Setting $h(t) := P^i(\omega_t)$, the coarea formula yields

$$\int_{\omega_t} |\nabla u| \,\mathrm{d}\mathscr{L}^N = \int_0^t h(\tau)\, d\tau =: H(t),$$

so that

$$\frac{1}{t^{1-\eta}} \leq C \frac{H'(t)}{H(t)^{1-\eta}},$$

where $0 < \eta < 1$. Summing from 0 to t we get

$$t \leq C \int_0^t h(\tau)\, d\tau. \tag{9.12}$$

Coming back to (9.11), we may write

$$\frac{C}{t^p |\omega_t|^{\frac{p}{p'}}} \leq \frac{h(t)}{\left(\int_0^t h(\tau)\, d\tau\right)^p},$$

so that, using the monotony of $|\omega_t|$ and integrating on $[t_1, t_2]$

$$\frac{C(t_2 - t_1)}{t_2^p |\omega_{t_2}|^{\frac{p}{p'}}} \leq \frac{1}{p-1} \frac{1}{\left(\int_0^{t_1} h(\tau)\, d\tau\right)^{p-1}}.$$

Setting $t_2 = 2t_1$ we obtain

$$\int_0^{t_1} h(\tau)\, d\tau \leq C t_1 |\omega_{2t_1}|.$$

In view of (9.12) we get

$$t_1 \leq C \int_0^{t_1} h(\tau)\, d\tau \leq C t_1 |\omega_{t_1}|$$

which is false if t_1 is small enough, so that the result follows.

A similar reasoning can be used starting from the more general inequalities (9.8) and (9.9), but the approach through the isoperimetric profile $I(m)$ encompasses all the situations in a very elegant way.

9.3 Local boundary behaviour

In this section, we give a localized positivity result. In particular, this applies to the case in which β is a function, locally bounded from above. In the computations below we assume without loosing generality that $\beta \leq 1$.

Under the previous hypotheses on Ω, let Γ be a relatively open subset $\Gamma \subset \partial\Omega$. For a measurable set $\omega \subset \Omega$ with finite perimeter, we denote the exterior/interior perimeters relative to Γ

$$P_\Gamma^e(\omega) = \int_\Gamma (1_\omega^+ + 1_\omega^-) d\mathscr{H}^{N-1},\ P_\Gamma^i(\omega) = \mathscr{H}^{N-1}(\Omega \cap J_{1_\omega}) + \int_{\partial\Omega\setminus\Gamma} (1_\omega^+ + 1_\omega^-) d\mathscr{H}^{N-1},$$

and the associated isoperimetric profile $I_\Gamma : (0, \frac{1}{2} P_\Gamma^e(\Omega)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}}] \to \mathbb{R} \cup \{+\infty\}$

$$I_\Gamma(m) = \inf \left\{ P_\Gamma^i(\omega), \omega \subset \Omega \text{ s.t. } |\omega| \leq \frac{1}{2}|\Omega|,\ P_\Gamma^e(\omega)^{\frac{1}{p}} |\omega|^{\frac{1}{p'}} \geq m \right\}.$$

We start with a general result, which applies, in particular, to the minimizers of (9.3), as soon as $f \geq 0$, $f \neq 0$ and $\beta \leq 1$.

Theorem 9.4. *Let $u \in W^{1,p}(\Omega)$, $u \neq 0$, be a non negative function such that for any $v \in W^{1,p}(\Omega)$ with $v \geq u$, $dist(\{v > u\}, \partial\Omega \setminus \Gamma) > 0$,*

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathscr{L}^N \leq \int_\Omega |\nabla v|^p \, \mathrm{d}\mathscr{L}^N + \int_\Gamma [v_+^p + v_-^p] \, d\mathscr{H}^{N-1}. \tag{9.13}$$

Suppose also that $1/I_\Gamma$ is summable in a neigbourhood of the origin. Then for any compact set K in Γ,

$$\operatorname*{ess-liminf}_{z(\in\Omega)\to K} u(z) > 0,$$

meaning that there exist $r, \epsilon > 0$ such that

$$|\{u < \epsilon\} \cap \{z \in \Omega : dist(z, K) < r\}| = 0.$$

We start with a technical result.

Lemma 9.5. *Let I_Γ be as defined above and let J_Γ be defined by*

$$J_\Gamma(m) := \inf \left\{ P_\Gamma^i(\omega), \omega \subset \Omega \text{ s.t. } |\omega| \leq \frac{1}{2}|\Omega|, \ P_\Gamma^e(\omega)^{\frac{1}{p}} |\omega|^{\frac{1}{p'}} + |\omega| \geq m \right\}.$$

Then $1/I_\Gamma$ is summable if and only if $1/J$ is summable.

Proof. Let $m > 0$. Clearly $I_\Gamma(m) \geq J_\Gamma(m)$. We claim that there exist $C, \varepsilon > 0$, that depend only on N and $|\Omega|$, such that

$$J_\Gamma(m) \geq I_\Gamma(\varepsilon m) \wedge C m^{\frac{N-1}{N}}. \tag{9.14}$$

Then since

$$\frac{1}{I_\Gamma(m)} \leq \frac{1}{J_\Gamma(m)} \leq \frac{1}{I_\Gamma(\varepsilon m)} + \frac{1}{C m^{\frac{N-1}{N}}},$$

the conclusion follows.

Let us check claim (9.14). Indeed, let $\omega \subset \Omega$ be a domain such that $|\omega| \leq \frac{1}{2}|\Omega|$, and $|\omega| + |\omega|^{\frac{1}{p'}} P_\Gamma^e(\omega)^{\frac{1}{p}} \geq m$. Let $\eta > 0$ that will be fixed later, and suppose that

$$P_\Gamma^e(\omega) \leq \eta |\omega|.$$

Then, by the classical isoperimetric inequality, there is a constant $a_N > 0$ such that

$$a_N |\omega|^{\frac{N-1}{N}} \leq P_\Gamma^i(\omega) + P_\Gamma^e(\omega) \leq P_\Gamma^i(\omega) + \eta |\omega|^{\frac{N-1}{N}} |\Omega|^{\frac{1}{N}}.$$

We fix $\eta := \frac{a_N}{2|\Omega|^{\frac{1}{N}}}$. Then

$$P_\Gamma^i(\omega) \geq \frac{a_N}{2} |\omega|^{\frac{N-1}{N}} \geq \frac{a_N}{2(1+\eta^{\frac{1}{p}})^{\frac{N-1}{N}}} m^{\frac{N-1}{N}}.$$

On the other hand, suppose that $P_\Gamma^e(\omega) > \eta|\omega|$. Then

$$m \leq |\omega|^{\frac{1}{p'}} P_\Gamma^e(\omega)^{\frac{1}{p}} + |\omega| \leq (\eta^{-\frac{1}{p}} + 1) |\omega|^{\frac{1}{p'}} P_\Gamma^e(\omega)^{\frac{1}{p}},$$

so

$$P_\Gamma^i(\omega) \geq I_\Gamma \left(\frac{m}{\eta^{-\frac{1}{p}} + 1} \right).$$

Claim (9.14) follows by choosing $\varepsilon := \frac{1}{\eta^{-\frac{1}{p}}+1}$ and $C := \frac{a_N}{2(1+\eta^{\frac{1}{p}})^{\frac{N-1}{N}}}$.

□

Proof of Theorem 9.4. Since K is compact, let $x_0 \in K$ and $r > 0$ be small enough such that

$$r < dist(x_0, \partial\Omega \setminus \Gamma) \qquad \text{and} \qquad |B_{x_0,r} \cap \Omega| \leq \frac{1}{2}|\Omega|.$$

It is enough (by compactness) to prove that u is essentially bounded below on a smaller ball $B_{x_0,r'} \cap \Omega$ for some $r' < r$. We proceed by contradiction and suppose without loss of generality that

$$\operatorname*{ess-liminf}_{x(\in\Omega)\to x_0} u(x) = 0.$$

Let $\gamma > 0$ that will be fixed small enough at the end of the proof. For any $t \in (0, r\gamma)$ we set

$$\omega_t := \{x \in \Omega : u(x) < t - \gamma|x - x_0|\}$$

and

$$g(t) := \int_{\omega_t} |\nabla(u(x) + \gamma|x - x_0|)| \, \mathrm{d}\mathscr{L}^N.$$

Notice that $\omega_t \subset B_{x_0,r}$. By our hypothesis, ω_t has positive measure for all $t \in (0, r\gamma)$, and $g(t) = 0$ for some $t > 0$ implies that $x \mapsto u(x) + \gamma|x - x_0|$ would be locally constant in ω_t, which is at most true for one γ, thus we suppose that $g(t) > 0$ for all $t \in (0, r\gamma)$.

Testing u against $\max(u, t - \gamma|\cdot - x_0|)$ gives

$$\int_{\omega_t} |\nabla u|^p \, \mathrm{d}\mathscr{L}^N \le \gamma^p |\omega_t| + t^p P^e_\Gamma(\omega_t),$$

which can be simplified further into

$$\int_{\omega_t} |\nabla u|^p \, \mathrm{d}\mathscr{L}^N \le \gamma^p |\omega_t| + r^p \gamma^p P^e_\Gamma(\omega_t).$$

Now,

$$g(t) \le \gamma|\omega_t| + |\omega_t|^{\frac{1}{p'}} \left(\int_{\omega_t} |\nabla u|^p \, \mathrm{d}\mathscr{L}^N \right)^{\frac{1}{p}} \le 2\gamma|\omega_t| + r\gamma|\omega_t|^{\frac{1}{p'}} P^e_\Gamma(\omega_t)^{\frac{1}{p}},$$

so in particular $\frac{g(t)}{(2+r)\gamma} \le |\omega_t| + |\omega_t|^{\frac{1}{p'}} P^e_\Gamma(\omega_t)^{\frac{1}{p}}$. By monotonicity of J_Γ, and since ω_t is at positive distance from $\partial\Omega \setminus \Gamma$ we get

$$J_\Gamma\left(\frac{g(t)}{(2+r)\gamma}\right) \le P^i_\Gamma(\omega_t) = g'(t).$$

Summing from $t = 0$ to $r\gamma$ we obtain

$$r\gamma \le \int_0^{r\gamma} \frac{g'(t)dt}{J_\Gamma\left(\frac{g(t)}{(2+r)\gamma}\right)} = (2+r)\gamma \int_0^{\frac{g(r\gamma)}{(2+r)\gamma}} \frac{dm}{J_\Gamma(m)}.$$

Using our previous estimate on g, we obtain

$$\int_0^{|\omega_{r\gamma}| + |\omega_{r\gamma}|^{\frac{1}{p'}} P^e_\Gamma(\omega_{r\gamma})^{\frac{1}{p}}} \frac{dm}{J_\Gamma(m)} \ge \frac{r}{r+2}.$$

In particular this means that $|\omega_{r\gamma}| + |\omega_{r\gamma}|^{\frac{1}{p'}} P^e_\Gamma(\omega_{r\gamma})^{\frac{1}{p}}$ is bounded below by a constant that does not depend on γ. However $P^e_\Gamma(\omega_{r\gamma}) \le 2\mathscr{H}^{N-1}(\Gamma) < \infty$ and $|\omega_{r\gamma}| \le |B_{x_0,r} \cap \{u < r\gamma\}| \underset{\gamma\to 0}{\longrightarrow} 0$, which is a contradiction. □

9.4 Analysis of cusps

We focus in this section on the behaviour of the solution of the Robin problem in a cusp in $\mathbb{R}^N$. We consider $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$, an increasing function such that $h(0) = 0$, $h'(0) = 0$, $\sup_{\mathbb{R}_+} |h'| < \infty$ and assume that h^{N-1} is convex. Let

$$\Omega := \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \text{ s.t. } \ x_1 > 0, \ |x'| < h(x_1)\},$$

and for $t \in (0,1)$ let us set

$$\Omega_t := \Omega \cap \{(x_1, x') \in \Omega : x_1 \leq t\}.$$

We introduce the isoperimetric profile of Ω restricted to revolution sets by

$$I_{sym}(m) = \inf \left\{ P^i(\omega) \ : \ \omega \in \mathscr{A} \text{ and } m \leq P^e(\omega)^{\frac{1}{p}} |\omega|^{\frac{1}{p'}} \right\}, \tag{9.15}$$

where

$$\mathcal{A} := \{A \subset \Omega \text{ open, rotationally symmetric with respect to the } x_1\text{-axes, and } |A| < +\infty\}.$$

The following technical result, whose proof will be given at the end of the section, will be essential to recover the summability property related to the isoperimetric profile of Ω_t, which is the key ingredient of our approach. Of course, the summability depends on the behaviour near the origin of the function h which defines the cusp.

Lemma 9.6. *We have*

$$\liminf_{t \to 0} \frac{I_{sym}\left(2P^e(\Omega_t)^{\frac{1}{p}} |\Omega_t|^{\frac{1}{p'}}\right)}{P^i(\Omega_t)} > 0.$$

The main result of the section is the following estimate near the cusp of Ω. We fix $t = 1$.

Theorem 9.7. *Let $f \in L^{p'}(\Omega_1)$, $f \geq 0$, $f \not\equiv 0$ and let $u \in W^{1,p}(\Omega_1)$ be a minimizer in $W^{1,p}(\Omega_1)$ of*

$$E(v) = \frac{1}{p}\left(\int_{\Omega_1} |\nabla v|^p \mathrm{d}\mathscr{L}^N + \beta \int_{\partial^e \Omega_1} v^p d\mathscr{H}^{N-1}\right) - \int_{\Omega_1} f v \mathrm{d}\mathscr{L}^N,$$

under the constraint $v = 1$ on $\partial^i \Omega_1$. Then

$$\left(\int_0^1 \left(\frac{\int_0^t h^{N-2}}{\int_0^t h^{N-1}}\right)^{\frac{1}{p}} dt < \infty\right) \Rightarrow \left(\inf_{\Omega_{1/2}} u > 0\right).$$

Proof. Let I_{sym} be the restricted isoperimetric profile associated to h as defined in (9.15). Note that for $t \in (0,1)$ we have

$$\begin{aligned} |\Omega_t| &= \alpha_{N-1} \int_0^t h^{N-1}, \\ P^e(\Omega_t) &= N\alpha_{N-1} \int_0^t h^{N-2}\sqrt{1+h'^2}, \\ P^i(\Omega_t) &= \alpha_{N-1} h^{N-1}(t). \end{aligned}$$

So according to the estimate on I_{sym} given by Lemma 9.6 , and since $|h'|$ is bounded by a constant, we know that there exist two constants $a, b > 0$ such that for any $t \in (0,1)$,

$$I_{sym}\left(a\left(\int_0^t h^{N-1}\right)^{\frac{1}{p'}}\left(\int_0^t h^{N-2}\right)^{\frac{1}{p}}\right) \geq b h(t)^{N-1}. \tag{9.16}$$

We set

$$m_0 := a\left(\int_0^1 h^{N-1}\right)^{\frac{1}{p'}}\left(\int_0^1 h^{N-2}\right)^{\frac{1}{p}}.$$

By change of variable, an easy computation which relays on (9.16) and the fact that h is increasing, shows that

$$\int_0^{m_0} \frac{dm}{I_{sym}(m)} = \int_0^1 \frac{\frac{d}{dt}\left(a\left(\int_0^t h^{N-1}\right)^{\frac{1}{p'}}\left(\int_0^t h^{N-2}\right)^{\frac{1}{p}}\right)}{I_{sym}\left(a\left(\int_0^t h^{N-1}\right)^{\frac{1}{p'}}\left(\int_0^t h^{N-2}\right)^{\frac{1}{p}}\right)}dt \tag{9.17}$$
$$\leq \frac{a}{b}\int_0^1 \left(\frac{\int_0^t h^{N-2}}{\int_0^t h^{N-1}}\right)^{\frac{1}{p}} dt < \infty.$$

Let $\Gamma := \partial\Omega \cap \{1/4 \leq x_1 \leq 3/4\}$. We apply Theorem 9.4 to $\Omega \cap \{1/4 \leq x_1 \leq 3/4\}$ and Γ and get that $\inf_{\Omega\cap\{1/3\leq x_1\leq 2/3\}}(u) > 0$. In particular,

$$\inf_{\Omega\cap\{x_1=1/2\}}(u) =: c > 0.$$

Let v be the p-harmonic function on $\Omega_{1/2}$ that verifies $v = 1$ on $\Omega\cap\{x_1 = 1/2\}$ and a Robin boundary condition on $\partial\Omega \cap \{x_1 \leq 1/2\}$. By comparison principle,

$$u \geq cv \text{ on } \Omega_{1/2}.$$

Now, as v is the unique minimizer of $w \mapsto \int_{\Omega_{1/2}} |\nabla w|^p \,\mathrm{d}\mathscr{L}^N + \beta \int_{\partial^e\Omega_{1/2}} |w|^p \, d\mathscr{H}^{N-1}$ among functions that verify the constraint $w = 1$ on $\partial^i\Omega_1$, then it is rotationally symmetric; in particular its sublevel sets $\{v < t\}$ belong to $\mathcal{A}$ for $t \in (0, 1)$.

Suppose then $\inf v = 0$, value which is reached approaching the cusp. For any $t \in]0, 1[$ let $\omega_t := \{v < t\}$, and $f(t) := \int_{\omega_t} |\nabla v| \,\mathrm{d}\mathscr{L}^N$. With the same computations as in the proof of Theorem 9.1, we find that

$$f(t) \leq \beta^{1/p} t P^e(\omega_t)^{\frac{1}{p}} |\omega_t|^{\frac{1}{p'}}.$$

so that

$$I_{sym}(f(t)/\beta^{1/p}t) \leq f'(t).$$

The contradiction follows by integration, taking into account the summability property (9.17). □

Remark 9.8. Theorem 9.7 particularly applies to $h(t) = t^\alpha$ provided that $1 \leq \alpha < p$. In fact, $\alpha \in [1, p)$ is also a necessary condition to get the bound from below.

Let indeed u be the p-harmonic function on Ω_1 equal to 1 on $\partial^i\Omega_1$ and with Robin boundary conditions on $\partial^e\Omega_1$. We claim that u extends continuously to $\overline{\Omega_1}$. Indeed the continuity on the boundary $\partial\Omega_1\backslash\{(0, 0)\}$ is a consequence of boundary elliptic regularity (see for example [168, Theorem 4.4]). Let now for $0 < t \leq 1$

$$\delta_t^+ = \sup_{x\in\Omega_1 : x_1 = t}(u) \qquad \text{and} \qquad \delta_t^- = \inf_{x\in\Omega_1 : x_1 \leq t}(u).$$

Notice that δ_t^+ is nondecreasing: indeed, if $\delta_{t_1}^+ > \delta_{t_2}^+$ for some $t_1 < t_2$, we can consider the admissible function

$$v(x) := \begin{cases} u(x) & \text{if } x_1 \geq t_2 \\ u(x) \wedge \delta_{t_2}^+ & \text{if } x_1 < t_2 \end{cases}$$

for which $E(v) < E(u)$, a contradiction. Let us denote with $\delta_0^\pm$ the limits of $\delta_t^\pm$ as $t \to 0$. To prove the continuity up to $(0, 0)$, it suffices to show $\delta_0^+ = \delta_0^-$. Let $(t_i)_i$ be a sequence converging to 0 such that

$$\inf_{x\in\Omega_1 : x_1 = t_i} u =: \delta_{t_i} \to \delta_0^-. \tag{9.18}$$

Let

$$\tilde{\Omega}_i := \frac{\Omega - (t_i, 0)}{h(t_i)} \cap \{y : |y_1| \leq 1\},$$

which is a domain that is $\mathcal{C}^1$-close to the cylinder $[-1,1]\times B_1^{N-1}$ where B_1^{N-1} is the unit ball of $\mathbb{R}^{N-1}$. Let

$$u_i(x) := u(t_i + h(t_i)x_1, h(t_i)x')$$

be defined on $\tilde{\Omega}_i$. Since the sequence u_i is uniformly bounded and $\partial_\nu u_i \in [0, h(t_i)\beta]$, then by boundary elliptic regularity (see for instance the proof of [168, Theorem 4.4], thanks to which u_i can be extended across the boundary satisfying an extension of our PDE) one can use Harnack inequality up to the boundary and infer that there exists a constant $C>0$ such that

$$\sup(u_i - \delta^-_{t_i+h(t_i)}) \le C\inf(u_i - \delta^-_{t_i+h(t_i)}) \qquad \text{on } \tilde{\Omega}_i \cap \{x\in\mathbb{R}^N : |x_1|\le 1/2\}.$$

In view of (9.18), this yields

$$\delta^+_{t_i} \le \delta^-_{t_i+h(t_i)} + C(\delta_{t_i} - \delta^-_{t_i+h(t_i)}) \underset{i\to\infty}{\longrightarrow} \delta^-_0,$$

from which we get $\delta^+_t \to \delta^-_0$ as $t\to 0^+$, leading to the desired equality $\delta^+_0 = \delta^-_0$.

Suppose now that u satisfies a bound from below, that is $(\delta^-_0 \ge)\inf_{\Omega_1} u > 0$. Then for any $t>0$, and $\epsilon\in(0,1)$ to be fixed small enough later, consider the competitor

$$u_t(x) = \begin{cases} u(x) & \text{if } t\le x_1\le 1\\ \min\left((1-\epsilon)\delta^-_0 + \left(\frac{2x_1}{t}-1\right)\left(\delta^+_t - (1-\epsilon)\delta^-_0\right), u\right) & \text{if } \frac{t}{2}\le x_1\le t\\ (1-\epsilon)\delta^-_0 & \text{if } 0\le x_1\le \frac{t}{2}. \end{cases}$$

Notice that for a small enough t (depending on ϵ) we have $u_t\le u$ (as $u\ge\left(1-\frac{\epsilon}{2}\right)\delta^-_0$ for $x_1\le t$). The energy comparison gives

$$\beta\int_{\partial^e\Omega_t}[u^p - u_t^p]\,d\mathscr{H}^{N-1} \le \int_{\Omega_t}[|\nabla u_t|^p - |\nabla u|^p]\,\mathrm{d}\mathscr{L}^N$$

which is simplified into

$$c_p\beta\epsilon(\delta^-_0)^p P^e(\Omega_{t/2}) \le \frac{2^p}{t^p}\left|(\delta^+_t - \delta^-_0) + \epsilon\delta^-_0\right|^p |\Omega_t|.$$

Taking into account the expression of $h(t)=t^\alpha$, this gives

$$c_{p,\alpha}\beta^{\frac{1}{p}} t^{1-\frac{\alpha}{p}} \le \frac{\delta^+_t - \delta^-_0}{\delta^-_0\,\epsilon^{\frac{1}{p}}} + \epsilon^{1-\frac{1}{p}}.$$

Letting $t\to 0^+$, and since ϵ is arbitrary, we conclude that $\alpha<p$.

Remark 9.9. *For the Laplace operator, in [162, Examples 3.4 and 4.13] the authors discuss the cusps corresponding to $h(t)=t^\alpha$. Their analysis is based on estimates of Green function and relies on a uniform boundary Harnack inequality. Although general functions h are not considered [162], it seems that for functions $h\in\mathcal{C}^0([0,1],\mathbb{R}_+)$ with $h(0)=0$, $h(t/2)>ch(t)$ for any $t>0$ and some $c>0$, the decomposition of the cusp Ω given by into blocs $D_n=\{x\in\Omega : t_{n+1}\le x_1\le t_n\}$ where the sequence t_n is chosen such that each D_n is close to a square, links their criteria to the summability of $\frac{\int_0^t h^{N-2}}{h(t)^{N-1}}$. This is not equivalent to our criteria of summability of $\sqrt{\frac{\int_0^t h^{N-2}}{\int_0^t h^{N-1}}}$, as may be seen from the example $h(t)=t^2\log(1/t)^\alpha$ for $\alpha\in(1,2]$. Presumably, the reason our criteria is weaker lies in our method that gives sharp inequalities as long as the sets $\{u<t\}$ are not too far from a minimizer of the relative perimeter, which is not what happens in this kind of cusps.*

We conclude the section with the proof of the technical Lemma 9.6.

Proof of Lemma 9.6. We rely on the classification of constant mean curvature revolution surfaces in $\mathbb{R}^n$, meaning connected surfaces of $\mathbb{R}^n$ with constant mean curvature which are invariant for any

isometry that fixes e_1. These surfaces are given by the revolution of a curve $(x(t), y(t))$ in $\mathbb{R}^2$ which, after parametrization by unit length, is defined by the equation

$$\begin{cases} x' = \cos(\sigma),\ y' = \sin(\sigma) \\ \sigma' = -NH + (N-1)y^{-1}\cos(\sigma). \end{cases}$$

Above, σ is the angle of the tangent vector and H is the mean curvature. Moreover the quantity $T := y^{N-1}\cos(\sigma) - Hy^N$ is constant and the signs of (H, T) fully classify the type of surface (see [170, Proposition 2.4]).

The main idea of the proof is the following : we consider a minimizing sequence (A_k), and we change it into another sequence $(\tilde{A}_k)$ that is quasi-minimizing (it verifies the same constraint and $P^i(\tilde{A}_k)/P^i(A_k)$ is bounded) and that decomposes into the union of a set of the form Ω_t and a set that is far from the axis of revolution, on which the analysis is simpler.

Below we will write $a \lesssim b$ when $a \leq Cb$ for a positive constant C that may depend on N and h but not on the other quantities.

Given $t > 0$, let us set

$$m_t := 2P^e(\Omega_t)^{\frac{1}{p}}|\Omega_t|^{\frac{1}{p'}}.$$

Consider $(A_k)_k$ a minimizing sequence for $I_{sym}(m_t)$. We lose no generality in replacing each A_k with the minimizer of

$$\inf\left\{P^i(A),\ A \in \mathscr{A} \text{ s.t. } A\Delta A_k \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{k}, x_1 \leq k\},\ |A| \geq |A_k|\right\}.$$

This problem is well posed and has a solution, by classical compactness arguments. Due to the classification of constant mean curvature revolution surfaces, we know that in the set $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}, x_1 < k\}$, A_k is smooth and ∂A_k is a union of nonnegative constant mean curvature surfaces (when the normal vector is oriented outward). We denote $\partial^e A_k := \partial^* A_k \cap \partial\Omega$ and $\partial^i A_k = \partial^* A_k \cap \Omega$. The perimeter of $(A_k)_k$ is locally bounded so there is a subsequence of $(A_k)_k$ (that we denote with the same index) and a set $A \in \mathscr{A}$ such that $A_k \to A$ in $L^1_{\text{loc}}(\Omega)$, with also a local Hausdorff convergence in Ω due to the interior minimality of the sets (A_k).

- Note that $|A_k|$ and $P^e(A_k)$ are bounded from above and below. To prove that they are bounded from above we consider the projection $\Phi : (x_1, x') \in \Omega \mapsto (h^{-1}(|x'|), x')$. It may be checked that

  $$\sup_{|x'| \geq h(1)} \|D\Phi\| < \infty \text{ and } \mathscr{H}^{N-1}\left(\partial^e A_k \setminus \Phi(\partial^i A_k)\right) = 0$$

  so

  $$P^e(A_k) \leq P^e(\Omega_1) + \mathscr{H}^{N-1}(\partial^e A_k \cap \{|x_1| \geq 1\}) \lesssim P^e(\Omega_1) + I_{sym}(m_t).$$

  Similarly,

  $$|A_k|^{1-\frac{1}{N}} \lesssim \operatorname{Per}(A_k) \lesssim P^e(\Omega_1) + I_{sym}(m_t),$$

  by the classical isoperimetric inequality. In particular we know that $|A| < \infty$.

- We have $I_{sym}(m_t) > 0$. Suppose indeed that $I_{sym}(m_t) = 0$. Then for any $\tau > 0$ we get with the same reasonning on $\Omega \cap \{x_1 \geq \tau\}$ that

  $$P^e(A_k \cap \{x_1 > \tau\}) \leq \left(\sup_{|x'| \geq h(\tau)} \|D\Phi\|\right) P^i(A_k) \underset{k\to\infty}{\longrightarrow} 0$$

  for every τ, so that $P^e(A_k) \to 0$, which contradicts the previous point.

- ∂A is a revolution surface of constant mean curvature, such that its section

  $$A \cap \{(x_1, x_2, 0, \ldots, 0), x_1, x_2 > 0\}$$

  is a union of convex sets that meet $\partial\Omega$ with an angle less than $\frac{\pi}{2}$ as in Figures 9.1 and 9.2. Indeed, by regularity argument on minimal surfaces we know ∂A is a union of rotationally symmetric surfaces with (nonnegative) constant mean curvature, which are moreover bounded (because $|A| < \infty$). In view of the classification of such surfaces, the components of A may only be the intersection of Ω with

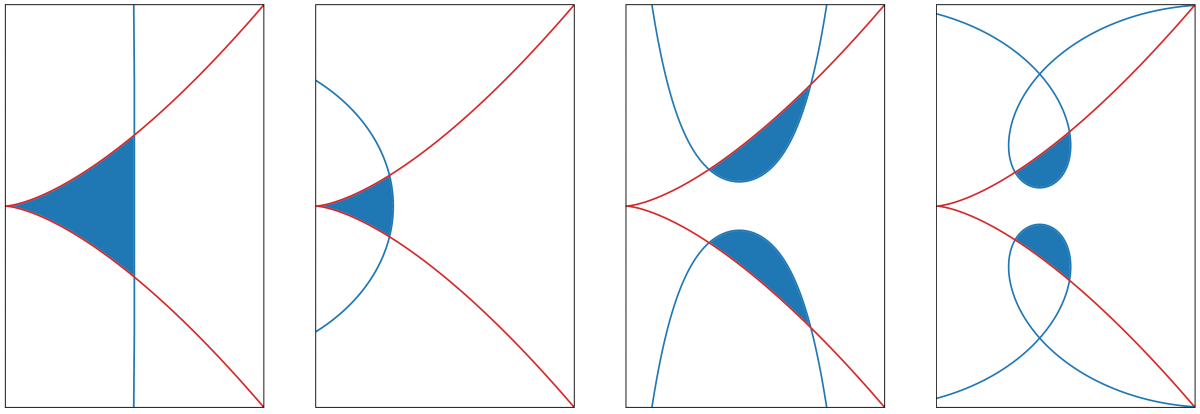


Figure 9.1: Possible connected components of A, seen in the section $\mathbb{R}_+ \times \mathbb{R} \times \{0\}^{N-2}$. From left to right: hyperplane, sphere, catenoid, nodoid.

1) a half-space $\{x_1 \leq \lambda\}$ or a ball centered on $\{x' = 0\}$ (observe that the boundary angle condition and the interior regularity implies that the ball meets $\partial\Omega$).
2) the exterior of a catenoid or the interior of the loop of a nodoid (see the figure below).

In particular each component of the section is convex. Finally, let us check the angle condition; let $\tilde{A}$ be one of the connected components of A, and parametrize $\partial^i \tilde{A} \cap (\mathbb{R}^2_+ \times \{0\}^{N-2})$ by a smooth curve $c : [0,T] \to \mathbb{R}^2_+$ that rotates clockwise where $c(0), c(T) \in \partial\Omega$. There exists a sequence of curves $c_k : [0,T] \to \mathbb{R}^2_+$ that converges in $\mathcal{C}^1$ to c_k such that $c_k((0,T)) \subset \partial^i A_k$. Suppose the angle condition is not verified at $c(0) = (x_1, x_2)$ ($c(T)$ is handled similarly), meaning $c'(0) \cdot (1, h'(x_2)) > 0$. Then there is a small $\delta \in (0,1)$ such that $c'(t) \cdot (1, h'(x_2)) > 2\delta$ for $t \in (0, \delta)$, which implies that for any large enough k,

$$c_k'(t) \cdot (1, h'(x_2)) > \delta, \ \forall t \in [0, \delta].$$

Let p_k, q_k be the orthogonal projections of $c_k(0)$, $c_k(\delta)$ on the graph of h (and similarly p, q the projection of $c(0), c(\delta)$; notice $p = c(0)$) and let T_k be the trapezoid formed by the graph of h between p_k and q_k, $[q_k, c_k(\delta)]$, $c_k([0,\delta])$, $[c_k(0), p_k]$. Let $T_k^{rev} \subset \mathbb{R}^N$ be the revolution of T_k around the axis. We claim that the sequence $(A_k \cup T_k^{rev})_k$ still verifies the constraint (this is direct because the measure and exterior perimeter can only increase) while

$$\liminf_{k\to\infty} P^i(A_k \cup T_k^{rev}) < \liminf_{k\to\infty} P^i(A_k)$$

which is a contradiction because (A_k) is already a minimizing sequence. This last estimate is obtained from

$$\begin{aligned} P^i(A_k \cup T_k^{rev}) - P^i(A_k) &\leq \mathscr{H}^{N-1}([p_k, c_k(0)]^{rev}) + \mathscr{H}^{N-1}([q_k, c_k(\delta)]^{rev}) - \mathscr{H}^{N-1}(c_k([0,\delta])^{rev}) \\ &\underset{k\to\infty}{\longrightarrow} \mathscr{H}^{N-1}([q, c(\delta)]^{rev}) - \mathscr{H}^{N-1}(c([0,\delta])^{rev}) \\ &< 0 \text{ when } \delta \text{ is small enough.} \end{aligned}$$

Let

$$\Omega' = \{(x_1, x') \in \Omega : |x'| \leq \frac{1}{2} h(x_1)\}, \quad \Omega'' = \{(x_1, x') \in \Omega : |x'| \leq \frac{1}{4} h(x_1)\}.$$

We will now make a few modifications on the sequence $(A_k)_k$, to obtain a new sequence $\tilde{A}_k$ verifying the same constraints such that

$$\tilde{A}_k = \Omega_{t_k} \sqcup D_k,$$

where $t_k > 0$, $D_k \subset \Omega \setminus \Omega''$, and $P^i(\tilde{A}_k) \lesssim P^i(A_k)$.

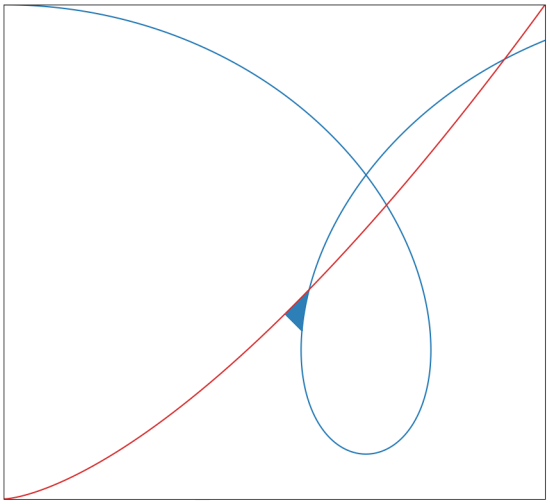


Figure 9.2: The shaded region is added to A_k.

- First modification: let $\delta_t > 0$ be such that $P^i(\Omega_{\delta_t}) = I_{sym}(m_t)$ (note that we do not know a priori how to compare δ_t and t). change A_k into $A'_k := A_k \cup \Omega_{\delta_t}$. Notice that A'_k still verifies the constraint and by choice of δ_t,

$$P^i(A'_k) \leq 2I_{sym}(m_t) + o_{k\to\infty}(1).$$

- Second modification: suppose there exists a point $(\tau, 0) \in \partial A$ for some $\tau > \delta_t$, with τ chosen maximal. This means that for any large enough k, ∂A_k contains

$$S_k \cap \left\{ x \in \Omega \,:\, x_1 \leq k \text{ and } dist(x, \partial\Omega) \geq \frac{1}{k} \right\},$$

where S_k is either a hyperplane or a sphere going through $(\tau_k, 0)$, where $\tau_k \to \tau$ (the hyperplane is orthogonal to the x_1 axes, while the center of the sphere is on the x_1-axes, and has an abscissa less that τ_k). In this case we let $A''_k := A'_k \cup \Omega_{\tau_k}$. Again, we may assume τ_k to be chosen maximal.

Consider the projection

$$(x_1, x') \in S_k \mapsto \left(\tau_k, \frac{h(\tau_k)}{h(x_1)} x' \right) \in \{x_1 = \tau_k\}.$$

Its differential is locally bounded on S_k because for any such $(x_1, x') \in S_k$ one has $h(x_1) \geq \frac{1}{1+\|h'\|^2_{L^\infty}} h(\tau_k)$. Indeed the minimal value on x_1 here is given by the intersection of S_k with $\partial\Omega$; the boundary angle condition gives that $(\tau_k - x_1, -h(x_1)) \cdot (1, h'(x_1)) \leq 0$, so $h(\tau_k) \leq h(x_1 + h(x_1)h'(x_1)) \leq (1 + \|h'\|^2_{L^\infty})\, h(x_1)$. As a consequence,

$$P^i(A''_k) \lesssim I_{sym}(m_t) + o_{k\to\infty}(1).$$

- Third modification: suppose $A \cap \{x_1 > \tau\} \cap \Omega''$ is not empty, where τ is the same as in the previous point. Then ∂A contains a part of catenoid or nodoid C that passes through $\partial\Omega'$. We now make a disjunction of two cases.

Case 1. The rightmost point of C is in $\Omega \setminus \Omega''$. Then, for a large enough k we know ∂A_k contains a part of a catenoid or nodoid C_k that approaches C in $\mathcal{C}^1$ such that its rightmost point is in $\Omega \setminus \Omega''$, and it meets $\partial\Omega''$ at some $(T_k, \frac{1}{4}h(T_k))$. Then we let

$$\tilde{A}_k = A''_k \cup \Omega_{T_k}.$$

Again $\tilde{A}_k$ verifies the constraint and with the same horizontal projection argument as in the previous point

$$\mathscr{H}^{N-1}(\partial^i \Omega_{T_k}) \lesssim \mathscr{H}^{N-1}(\partial^i \Omega_{T_k} \cap (\Omega \setminus \Omega'')) \lesssim P^i(A_k)$$

so $P^i(\tilde{A}_k) \lesssim P^i(A_k)$, where the second inequality is obtained by projection of C_k on the annulus $\left\{(T_k, x') : \frac{1}{4}h(T_k) \le |x'| \le h(T_k)\right\}$ through the application $\psi : (x_1, x') \in C_k \mapsto \left(T_k, \frac{h(T_k)}{h(x_1)}x'\right) \in \{x : x_1 = T_k\}$.

Case 2. Or the rightmost point of C is reached in Ω'', so the rightmost point of C_k is reached in Ω'. We then denote T_k its abscissa and we let

$$\tilde{A}_k = A_k'' \cup \Omega_{T_k}$$

as previously. With the same projection, $P^i(\tilde{A}_k) \lesssim P^i(A_k)$.

After having performed the previous modifications, we went from A_k to $\tilde{A}_k$ such that

$$\tilde{A}_k = \Omega_{t_k} \sqcup D_k,$$

where $t_k \ge \delta_t$,

$$D_k \subset \Omega \setminus \Omega'',$$

and

$$P^i(\tilde{A}_k) \lesssim I_{sym}(m_t) + o_{k\to\infty}(1).$$

Notice that the "classical" relative isoperimetric inequality applies to D_k. If we consider indeed the projection $\pi : \Omega \setminus \Omega'' \mapsto \partial\Omega$ such that

$$\pi(x_1, x') := \left(x_1, \frac{x'}{|x'|}h(x_1)\right)$$

then its differential is bounded on $\Omega \setminus \Omega''$. Moreover $\partial^e D_k \subset \pi(\partial^i D_k)$ up to a $\mathscr{H}^{N-1}$-negligible set, from which we get

$$P^e(D_k) + |D_k|^{1-\frac{1}{N}} \lesssim P^i(D_k).$$

As well, for k large enough

$$P^i(\Omega_{t_k}) \lesssim P^i(\tilde{A}_k) \qquad \text{and} \qquad P^i(\tilde{A}_k) \lesssim I_{sym}(m_t). \tag{9.19}$$

If $t_k \ge t$ then inequality (9.6) follows. Assume that $t_k < t$. Then

$$\begin{aligned} 2P^e(\Omega_t)^{\frac{1}{p}}|\Omega_t|^{\frac{1}{p'}} = m_t &\lesssim P^e(\tilde{A}_k)^{\frac{1}{p}}|\tilde{A}_k|^{\frac{1}{p'}} \\ &= (P^e(\Omega_{t_k}) + P^e(D_k))^{\frac{1}{p}}(|\Omega_{t_k}| + |D_k|)^{\frac{1}{p'}} \\ &\le \left(P^e(\Omega_t) + cP^i(D_k)\right)^{\frac{1}{p}}\left(|\Omega_t| + cP^i(D_k)^{\frac{N}{N-1}}\right)^{\frac{1}{p'}} \end{aligned}$$

for some constant $c > 0$. We thus infer

$$P^i(D_k) \gtrsim |\Omega_t|^{1-\frac{1}{N}} \wedge P^e(\Omega_t).$$

Since in view of (9.19)

$$P^i(D_k) \le P^i(\Omega_{t_k}) + P^i(\tilde{A}_k) \lesssim P^i(\tilde{A}_k) \lesssim I_{sym}(m_t) =: \sigma_t P^i(\Omega_t)$$

for some $\sigma_t > 0$, we deduce

$$\sigma_t \gtrsim \frac{|\Omega_t|^{1-\frac{1}{N}} \wedge P^e(\Omega_t)}{P^i(\Omega_t)}.$$

The conclusion follows if we estimate σ_t from below. Since $P^i(\Omega_t) = o_{t\to 0}\left(P^e(\Omega_t)\right)$ it is enough to bound from below $\frac{|\Omega_t|^{1-\frac{1}{N}}}{P^i(\Omega_t)}$. Using the convexity of h^{N-1} we deduce

$$|\Omega_t| = \int_0^t h^{N-1}(x_1) \ge \frac{1}{2}\frac{h(t)^{2N-2}}{\frac{d}{dt}h^{N-1}(t)} = \frac{h(t)^N}{(2N-2)h'(t)},$$

so we get $\sigma_t \gtrsim 1 \wedge \frac{1}{h'(t)^{1-\frac{1}{N}}}$, which ends the proof. □

9.5 Further remarks

More general operators. The results of the paper extend naturally to more general elliptic problems with Robin boundary conditions, which are not of energy type. So let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, open set, with rectifiable boundary, such that $\mathscr{H}^{N-1}(\partial\Omega) < +\infty$.

Let

$$\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$$

$$\mathcal{B} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$$

$$\psi : \partial\Omega \to \mathbb{R}$$

be three continuous functions such that for some $0 < \alpha_1 \leq \alpha_2$, and every $x \in \Omega, y \in \partial\Omega, z \in \mathbb{R}^N, v \in \mathbb{R}$,

$$\alpha_1|z|^p \leq z\mathcal{A}(x,z), |\mathcal{A}(x,z)| \leq \alpha_2|z|^{p-1},$$

$$|\mathcal{B}(y,v)| \leq \psi(y)|v|^{p-1}.$$

Assume moreover that for every $x \in \Omega, y \in \partial\Omega, z_1, z_2 \in \mathbb{R}^N, v_1, v_2 \in \mathbb{R}$

$$(z_1 - z_2)(\mathcal{A}(x,z_1) - \mathcal{A}(x,z_2)) \geq 0,$$

$$(v_1 - v_2)(\mathcal{B}(y,v_1) - \mathcal{B}(y,v_2)) \geq 0.$$

We consider the (formal) problem

$$\begin{cases} -div(\mathcal{A}(x,\nabla u)) = f & \text{in } \Omega \\ \mathcal{A}(x,\nabla u)\cdot \mathbf{n} + \mathcal{B}(x,u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.20}$$

We do not develop around the question of existence of a (weak) solution in this framework and refer to the paper by R. Nittka [169] for an introduction to general Robin problems in Lipschitz sets, to [167] for the analysis of $\mathcal{A}$-superharmonic functions and to [163] for details on the framework of domains with rectifiable boundary.

Let $f \in L^{p'}(\Omega)$, $f \geq 0$, $f \neq 0$. Assume that a weak solution exists in our nonsmooth context, namely that there exists $u \in W^{1,p}(\Omega)$ such that $\forall v \in W^{1,p}(\Omega)$

$$\int_\Omega \mathcal{A}(x,\nabla u)\nabla v \mathrm{d}\mathscr{L}^N + \int_{\partial\Omega} [\mathcal{B}(x,u^+)v^+ + \mathcal{B}(x,u^-)v^-] d\mathscr{H}^{N-1} = \int_\Omega fv \mathrm{d}\mathscr{L}^N.$$

The fact that $u \geq 0$ is a consequence of the properties of $\mathcal{A}, \mathcal{B}$ and can be noticed by testing the equation with $u \wedge 0$. Taking $u \vee t$ as test function for $t > 0$, and using again the properties of $\mathcal{A}, \mathcal{B}$ and ψ one gets directly an inequality similar to (9.6)

$$\int_{\omega_t} |\nabla u|^p \,\mathrm{d}\mathscr{L}^N \leq \frac{|\psi|_\infty}{\alpha_1} t^p P^e(\omega_t), \tag{9.21}$$

which is the key ingredient of our results. The proofs of Theorems 9.1 and 9.4 can be continued from this point on.

More general open sets. It could be possible to deal with more general open sets, removing the rectifiability hypothesis, but some important drawbacks occur. However, this removes the Robin problem from Sobolev spaces, the natural context being the one of the SBV functions, the very first question being its well posedness. The traces can occur only on the rectifiable part of the boundary; on the purely non-rectifiable part, the functions that we consider do not have jumps ! In other words, they do not behave as boundary points, even though they belong to the topological boundary.

Bibliography

[159] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[160] G. Anzellotti and M. Giaquinta. BV functions and traces. *Rend. Sem. Mat. Univ. Padova*, 60:1–21 (1979), 1978.

[161] Wolfgang Arendt and A. F. M. ter Elst. The Dirichlet-to-Neumann operator on $C(\partial\Omega)$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 20(3):1169–1196, 2020.

[162] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen. On the Robin problem in fractal domains. *Proc. Lond. Math. Soc. (3)*, 96(2):273–311, 2008.

[163] D. Bucur and A. Giacomini. Shape optimization problems with Robin conditions on the free boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(6):1539–1568, 2016.

[164] Dorin Bucur and Stephan Luckhaus. Monotonicity formula and regularity for general free discontinuity problems. *Arch. Ration. Mech. Anal.*, 211(2):489–511, 2014.

[165] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[166] F. Gesztesy, M. Mitrea, and R. Nichols. Heat kernel bounds for elliptic partial differential operators in divergence form with Robin-type boundary conditions. *J. Anal. Math.*, 122:229–287, 2014.

[167] Juha Heinonen, Tero Kilpeläinen, and Olli Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[168] R. Nittka. Quasilinear elliptic and parabolic Robin problems on Lipschitz domains. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):1125–1155, 2013.

[169] R. Nittka. Quasilinear elliptic and parabolic Robin problems on Lipschitz domains. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):1125–1155, 2013.

[170] César Rosales. Isoperimetric regions in rotationally symmetric convex bodies. *Indiana Univ. Math. J.*, 52(5):1201–1214, 2003.

Chapter 10

Shape optimization of an obstacle immerged in a creeping flow

This is a joint work with Dorin Bucur, Antonin Chambolle and Alessandro Giacomini.

Contents

10.1 Introduction . . . 231
10.2 Notations and Preliminaries . . . 235
10.2.1 Basic notation . . . 235
10.2.2 Functions of bounded variation and sets of finite perimeter . . . 236
10.2.3 Functions of bounded deformation . . . 237
10.2.4 Hausdorff convergence . . . 238
10.3 Obstacles in Stokes fluids and drag minimization . . . 241
10.3.1 The flow around the obstacle . . . 241
10.3.2 The drag force . . . 242
10.3.3 The optimization problem . . . 243
10.4 A relaxed formulation of the shape optimization problem and existence of optimal obstacles . . . 244
10.5 Some technical results in SBD . . . 245
10.5.1 An immersion result . . . 245
10.5.2 Closure of the non-penetration constraint . . . 247
10.5.3 A lower semicontinuity result for surface energies in SBD . . . 249
10.6 Proof of Theorem 10.16 . . . 254
10.7 The two dimensional case . . . 256
10.7.1 A two dimensional setting . . . 257
10.7.2 The optimization problem and existence of solutions . . . 260
10.8 Regularity of two-dimensional minimizers . . . 263
10.8.1 Smoothing lemma . . . 264
10.8.2 Decay estimate . . . 269
10.9 Remarks on the Navier-Stokes equation . . . 274

10.1 Introduction

Consider an obstacle $E \subset \mathbb{R}^d$ ($d = 2, 3$ in real applications) contained in a (finite) channel Ω in which a fluid with viscosity coefficient $\mu > 0$ is flowing. Assume that the flow is stationary and incompressible, and that the associated velocity field u is equal to a constant vector V_∞ on the walls of the channel. The obstacle E experiences a *drag force*, whose component in direction of V_∞ will be denoted by $Drag(E)$. If we assume that the velocity of the fluid is tangent and satisfies the Stokes

equation with *Navier boundary conditions* on ∂E with coefficient $\beta > 0$, as detailed in Subsection 10.3.2, up to a multiplicative constant we have

$$Drag(E) = 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, d\mathscr{L}^d + \beta \int_{\partial E} |u|^2 \, d\mathscr{H}^{d-1}, \tag{10.1}$$

where $e(u) := \frac{1}{2}(Du + (Du)^*)$ denotes the symmetrized gradient of u.

We are interested in minimizing the drag force under some penalizations concerning the volume $|E|$ and the perimeter $\mathscr{H}^{d-1}(\partial E)$. The model problem we are interested in is of the form

$$\min_E \left\{ Drag(E) + c\mathscr{H}^{d-1}(\partial E) + f(|E|) \right\}, \tag{10.2}$$

where $c > 0$ and $f : (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous. Roughly speaking, the terms involving perimeter and volume can be thought as a price to pay in order to build the obstacle E, and we can give the two relevant choices of function f:

$$f(m) = +\infty 1_{\{m \neq m_0\}} \text{ for some } m_0 \in (0, |\Omega|), \text{ or } f(m) = -\lambda m \text{ for some } \lambda > 0.$$

Many similar optimisation problems have been considered under the "no-slip" boundary condition, meaning flows for which $u = 0$ at ∂E. Under measure constraint this has been studied in [199] on smooth surfaces, in [195] where it was conjectured that the optimal profile in three dimension is a prolate spheroid with sharp ends of angle 120 degree. Let us also mention the slender body approximation of [198].

The Navier boundary condition gives many new challenges, namely the possible apparition of lower dimensional structures in the obstacle that minimize the drag, something which was absent under the no-slip condition. The Navier boundary condition may be seen as a partial adherence to the boundary of the obstacle, and it may be asymptotically obtained as a limit of flows with no adherence on an obstacle with rough boundary. More precisely, a periodic microstructure with the right scaling on the boundary is modelled at the limit by a Navier boundary condition, as was proved in [183]. In dimension higher than two it is also necessary to take into account more complex geometries for the microstructure, which at the limit produce an anisotropic factor that favors certain directions of the flow.

The minimization of the drag for a fixed obstacle E, with respect to the parameter of the Navier conditions (meaning, with respect to the microstructure on the boundary) has been studied in [176], for both Stokes and Navier-Stokes flows, in particular for the monotony of the drag with respect to the Navier parameter β which is not true for Navier-Stokes flows in general.

Since the stationary velocity field associated to E turns out to be characterized variationally as the minimizer of the right hand side of (10.1) in the class of admissible velocities $\mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega)$ (see the definition 10.21 in subsection 10.3.1), we can conveniently rephrase the minimization problem by letting also the velocity fields intervene explicitely in the form

$$\min_{E, u \in \mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega)} \left\{ 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, d\mathscr{L}^d + \beta \int_{\partial E} |u|^2 \, d\mathscr{H}^{d-1} + c\mathscr{H}^{d-1}(\partial E) + f(|E|) \right\}. \tag{10.3}$$

The main goal of the paper is to find suitable relaxations of problem (10.3) for which we can prove the existence of minimizers.

In order to avoid unnatural geometric restrictions on the obstacle E, it is natural in view of the third term appearing in (10.3) to let it vary within the class of sets of *finite perimeter* (see Subsection 10.2.2), and replace the topological boundary with reduced one $\partial^* E$.

In order to describe properly obstacles with very narrow spikes which in the limit degenerate to $(d-1)$-surfaces and that cannot be taken into account through the reduced boundary, it is convenient

to consider admissible velocity fields which can be *discontinuous* outside E (see Subsection 10.3.3). Since the symmetrized gradient $e(u)$ is involved explicitly in (10.3), a natural family for the admissible velocities is given by the space of *functions of bounded deformation* SBD. The natural relaxation of the energy takes the form

$$\begin{aligned}\mathcal{J}(E,u) :=& 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, d\mathscr{L}^d + \beta \int_{\partial^* E} |u^+|^2 \, d\mathscr{H}^{d-1} + \beta \int_{J_u \setminus \partial^* E} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^{d-1} \\ &+ c\mathscr{H}^{d-1}(\partial^* E) + 2c\mathscr{H}^{d-1}(J_u \setminus \partial^* E) + f(|E|),\end{aligned} \tag{10.4}$$

where u is set equal to zero a.e. in E, while J_u denotes the discontinuity set of u and $u^\pm$ are the traces of u on $\partial^* E$ and J_u (the trace u^- vanishes on $\partial^* E$ by choice of orientation).

Within this framework the *global* obstacle is given by $E \cup J_u$, so that it contains also *lower dimensional parts*, namely $J_u \setminus \partial^* E$: the admissible velocities must be tangent to the obstacles, meaning the $u^\pm$ are orthogonal to the normal ν_u along the jump set. The contribution of the *Navier surface term* takes naturally into account the contribution from both sides given by $u^\pm$. Concerning the perimeter term, we count twice the lower dimensional parts because we see the relaxed obstacle as a limit of regular obstacles, such that points of $J_u \setminus \partial^* E$ correspond to thin parts of the regular obstacle that collapse on a lower-dimensional structure. We could also see the perimeter term as a price to pay in order to construct the obstacle and just keep $\mathscr{H}^{d-1}(\partial^* E \cup J_u)$ instead, in particular the lower semi-continuity results will not be affected.

The optimization problem can be seen as a minimization problem on the pairs (E, u) which has the features of classical geometrical problems for E coupled with a *free discontinuity problem* for u, with a surface term depending on the traces which are subject to suitable tangency constraints and boundary conditions.

The main results of the paper are the following. The first main result is theorem 10.1, where we prove the existence of a minimizer of (10.4) in a relaxed setting, and that is valid in any dimension.

We say that a couple (E, u) of a set E in Ω and a function $u \in SBD(\Omega)$ is admissible when

- E is in the subset of Ω of finite perimeter, denoted $\mathcal{A}(\Omega)$.
- $u \in \mathcal{V}_{E,V_\infty}(\Omega)$ which means that $u 1_E = 0$, $\operatorname{div}(u) = 0$, $u^\pm_{|J_u} \cdot \nu_u = 0$ on the jump set J_u, and $u_{|\partial\Omega} = V_\infty$.

See the definition 10.13 for more details.

Theorem 10.1 (**Existence of optimal obstacles**). *The minimization problem*

$$\min_{E\in\mathcal{A}(\Omega), u\in\mathcal{V}_{E,V_\infty}(\Omega)} \mathcal{J}(E,u) \tag{10.5}$$

admits a solution.

The main difficulties we have to face in order to prove that the problem is well posed are the following:

(a) the closure of the tangency constraint for the velocity on $\partial^* E \cup J_u$ under the natural weak convergence of the problem;

(b) the lower semicontinuity of energies of the form

$$\int_{J_u} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^{d-1} \tag{10.6}$$

associated to the Navier conditions.

Point (a) is a consequence of a lower semicontinuity result for the energy

$$\int_{J_u} \left[|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|\right] \, d\mathscr{H}^{d-1}$$

which is proved in Theorem 10.18, by resorting to recent lower semicontinuity results for functionals on SBD by Friedrich, Perugini and Solombrino [193].

The energy of point (b) naturally appears in a scalar setting when dealing with shape optimization problems involving *Robin boundary conditions* (see e.g. [178, 181, 180, 182]), and it is easily seen to enjoy lower semicontinuity properties by working with sections. The lower semicontinuity result in the vectorial SBD setting is given by Theorem 10.20 and cannot rely on an easy argument by sections, which would yield the lower semicontinuity of an energy of the form

$$\int_{J_u} \left[|u^+ \cdot \xi|^2 + |u^- \cdot \xi|^2 \right] |\xi \cdot \nu_u| \, d\mathscr{H}^{d-1}$$

with $\xi \in \mathbb{R}^d$ with $|\xi| = 1$: the optimization in ξ in order to recover (10.6) does not seem feasible in dimension $d \geq 3$. We thus follow a different strategy based on a blow up argument in which we reconstruct the vector quantities $u^\pm$ by controlling them along a sufficiently high number of directions (see Subsection 10.5.3 for details): in this way we can deal with more general energy densities of the form $\phi(u^+) + \phi(u^-)$, where ϕ is a lower semi-continuous function.

In section 10.7 we will consider a different relaxed form of problem (10.2) in dimension two, in which the admissible obstacles are given by a suitable family of closed connected sets, and prove the existence of minimizers (Theorem 10.2). The issue of the tangency condition and its closure explained in point (a) can be dealt with the use of a stream function (Lemma 10.27). The lower semicontinuity result of point (b) can be obtained in this case through a classical slicing argument which takes advantage of the two dimensional setting and of the tangency condition (Theorem 10.28)

A couple (K, u) of a connected compact set K in Ω and a function $u \in SBD(\Omega)$ is in the regular admissible space when

- $K \in \mathcal{K}(\Omega)$, the set of connected compact sets K such that $\mathbb{R}^2 \setminus K$ is connected and $\mathscr{H}^1(\partial K) < \infty$.
- $u \in \mathcal{V}^{reg}_{K,V_\infty}(\Omega)$ which means that $u 1_K = 0$, $\operatorname{div}(u) = 0$, $u^\pm_{|\partial K} \cdot \nu_K = 0$ on the boundary ∂K, and most importantly that $\mathscr{H}^1(J_u \setminus \partial K) = 0$.

More details are given in the definition 10.22.

Theorem 10.2 (**Existence of optimal obstacles**). *Suppose $d = 2$ and f is non-increasing. The minimization problem*

$$\min_{K \in \mathcal{K}(\Omega), u \in \mathcal{V}^{reg}_{K,V}(\Omega)} \mathcal{J}(K, u) \tag{10.7}$$

admits a solution.

We prove this result almost independantly of the previous one. In particular we heavily use the structure of connected compact sets of finite length in two dimension (that they are automatically rectifiable, that the length is lower semi-continuous with respect to Hausdorff convergence). We also use the stream function of u, meaning the scalar function ψ defined up to a constant by the equation $\nabla^\perp \psi = u$; this and the fact that ψ may be extended continuously as a constant in K encapsulates both the divergence-free and the non-penetration constraint on u.

The second result of the paper concerns the regularity of the relaxed minimizers of (10.4). Provided that f is Lipschitz and that we are in two dimensions, we prove that a minimizer (E, u) (that we know exists from Theorem 10.1) is locally a strong minimizer in the classical sense. Precisely, the optimal obstacle is a closed set and the velocity is a Sobolev function ; we prove that

$$\mathscr{H}^1(\Omega \cap \overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0,$$

so setting $F = \Omega \setminus \cup_i C_i$ where the (C_i)s are the connected components of $\Omega \setminus \overline{J_u \cup \partial^* E}$ where u is nonzero, F is a closed set of Ω with $\Omega \cap \partial F \subset \overline{J_u \cup \partial^* E}$, $u \in H^1_{\mathrm{loc}}(\Omega \setminus F, \mathbb{R}^d)$ and (F, u) is a minimizer

of the functional

$$\mathcal{J}^{\text{strong}}(G,v) = \int_\Omega 2\mu|e(v)|^2 d\mathscr{L}^2 + \int_{\Omega\cap\partial^* G} (c+\beta|v^+|^2) d\mathscr{H}^1 + \int_{\Omega\cap G^{(0)}\cap G} (2c+\beta|v^+|^2+\beta|v^-|^2) d\mathscr{H}^1 + f(|G|), \tag{10.8}$$

defined on the strong admissible space of compact perturbations of (F,u):

$$\Big\{(G,v):\ G \text{ closed with } \Omega\cap\partial G \text{ rectifiable, } \mathscr{H}^1(\Omega\cap\partial G)<\infty, v\in H^1_{\text{loc}}(\Omega\setminus G),\ \operatorname{div}(v)=0, \int_{\Omega\setminus G}|e(v)|^2 d\mathscr{L}^2<+\infty,\ v^{\pm}_{|\partial G}\cdot\nu_G = 0 \text{ on } \partial G,\ (G,v)\equiv(F,u) \text{ on a neighbourhood of } \partial\Omega,\Big\}. \tag{10.9}$$

Above, the normal ν_F is well-defined by rectifiability of ∂F and when F is smooth we may apply Korn's inequality on $\Omega\setminus F$ to prove that the trace of v on ∂F are well-defined. However for a general F, it is not obvious that these are well-defined and will be the subject of the beginning of section 10.8.

Theorem 10.3. *Let (E,u) be a minimizer of the functional (10.4) in dimension $d=2$, and suppose that f is Lipschitz. Then*

$$\mathscr{H}^1(\Omega\cap\overline{J_u\cup\partial^* E}\setminus(J_u\cup\partial^* E))=0.$$

The technical ideas to prove Theorem 10.2 stem in the pioneering result of De Giorgi, Carriero and Leaci on the Mumford-Shah problem in [190], where the key of the proof is a decay estimate obtained by a contradiction/compactness argument. For vectorial problems, a similar strategy, but definitely more involved, was used for the Griffith fracture problem in [188] (for the two-dimensional case) and [185] (for higher dimension). In the fracture problem, the key compactness result relies on the possibility to approximate a field $u\in SBD([-1,1]^d)$ with a small jump set by a Sobolev function which is locally controlled in H^1 (via the classical Korn inequality).

In our case, within the approximation procedure, we have to handle two additional constraints: incompressibility and non-penetration at the jumps. From a technical point of view, this is problematic since the bound in [188] in not strong enough to stay in divergence-free vector fields and the method in [185] creates new jumps on which the non-penetration constraint is not verified.

However, when restricted to two dimensions, the method of [185] leads to a stronger result, where both both constraints can be handled.

The paper is organized as follows. In Section 10.2 we recall fix the notation and recall some basic facts concerning sets of finite perimeter, functions of bounded deformation and Hausdorff convergence of compact sets. Section 10.3 is devoted to the precise exposition of the drag optimization problem. In Section 10.4 we detail the relaxation of the problem in the family of obstacle of finite perimeter and with velocities of bounded deformation, and formulate the main existence result in Theorem 10.16. The proof is given in Section 10.6, and it is based on some technical results for SBD functions collected in Section 10.5. Section 10.7 is devoted to the formulation of the relaxation in dimension two with topologically closed obstacle, and is almost independant from the rest aside from the compactness result of SBD. The last main result is proved in section 10.8, and finally we discuss generalizations to Navier-Stokes in 10.9.

10.2 Notations and Preliminaries

10.2.1 Basic notation

If $E\subseteq\mathbb{R}^d$, we will denote with $|E|$ or $\mathscr{L}^d(E)$ its d-dimensional Lebesgue measure, and by $\mathscr{H}^{d-1}(E)$ its $(d-1)$-dimensional Hausdorff measure: we refer to [191, Chapter 2] for a precise definition,

recalling that for sufficiently regular sets $\mathscr{H}^{d-1}$ coincides with the usual area measure. Moreover, we denote by E^c the complementary set of E, and by 1_E its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise.

If $A \subseteq \mathbb{R}^d$ is open and $1 \leq p \leq +\infty$, we denote by $L^p(A)$ the usual space of p-summable functions on A with norm indicated by $\|\cdot\|_p$. $W^{1,p}(A)$ will stand for the Sobolev space of functions in $L^p(A)$ whose gradient in the sense of distributions belongs to $L^p(A, \mathbb{R}^d)$. Finally, given a finite dimensional unitary space Y, we will denote by $\mathcal{M}_b(A; Y)$ will denote the space of Y-valued Radon measures on A, which can be identified with the dual of Y-valued continuous functions on A vanishing at the boundary.

We will denote by $M^{d\times m}$ the set of $d \times m$ matrices with values in $\mathbb{R}$: when $d = m$ we will simply write M^d, and we will denote by $\mathrm{M}^d_{\mathrm{sym}}$ the subspace of symmetric matrices. For $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^m$ we will denote with $a \otimes b$ the element of $M^{d\times m}$ such that

$$(a \otimes b)_{ij} = a_i b_j,$$

while if $a, b \in \mathbb{R}^d$ we denote with $a \odot b$ the matrix in $\mathrm{M}^d_{\mathrm{sym}}$ such that

$$(a \odot b)_{ij} = \frac{a_i b_j + a_j b_i}{2}.$$

Given $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, we denote with $\xi^\perp$ the hyperplane through the origin orthogonal to ξ. If $E \subseteq \mathbb{R}^d$, we set

$$E^\xi := \pi_{\xi^\perp}(E),$$

where π denotes the orthogonal projection, and for $y \in \xi^\perp$ we set

$$E^\xi_y := \{t \in \mathbb{R} \,:\, y + t\xi \in E\}. \tag{10.10}$$

10.2.2 Functions of bounded variation and sets of finite perimeter

If $\Omega \subseteq \mathbb{R}^d$ is open, we say that $u \in BV(\Omega; \mathbb{R}^m)$ if $u \in L^1(\Omega; \mathbb{R}^m)$ and its derivative in the sense of distributions is a finite Radon measure on Ω, i.e., $Du \in \mathcal{M}_b(\Omega; M^{d\times m})$. $BV(\Omega; \mathbb{R}^m)$ is called the space of *functions of bounded variation* on Ω with values in $\mathbb{R}^m$. $BV(\Omega; \mathbb{R}^m)$ is a Banach space under the norm $\|u\|_{BV(\Omega;\mathbb{R}^m)} := \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \|Du\|_{\mathcal{M}_b(\Omega;M^{d\times m})}$. We call $|Du|(\Omega) := \|Du\|_{\mathcal{M}_b(\Omega;M^{d\times m})}$ the *total variation* of u. We refer the reader to [171] for an exhaustive treatment of the space BV.

We say that $u \in SBV(\Omega; \mathbb{R}^m)$ if $u \in BV(\Omega; \mathbb{R}^m)$ and its distributional derivative can be written in the form

$$Du = \nabla u \, \mathscr{L}^d + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{d-1} \lfloor J_u,$$

where $\nabla u \in L^1(\Omega; M^{d\times m})$ denotes the approximate gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, and $\nu_u(x)$ is the normal to J_u at x.

Note that if $u \in SBV(\Omega; \mathbb{R}^m)$, then the singular part of Du is concentrated on J_u which is a countably $\mathscr{H}^{d-1}$-rectifiable set: there exists a set E with $\mathscr{H}^{d-1}(E) = 0$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^d$ such that $J_u \subseteq E \cup \bigcup_{i\in\mathbb{N}} M_i$.

We will say that $E \subseteq \mathbb{R}^d$ with $|E| < +\infty$ has finite perimeter if $1_E \in BV(\mathbb{R}^d)$. The perimeter of E is defined as

$$Per(E) = |D1_E|(\mathbb{R}^d).$$

It turns out that

$$D1_E = \nu_E \mathscr{H}^{d-1} \lfloor \partial^* E, \qquad Per(E) = \mathscr{H}^{d-1}(\partial^* E),$$

where $\partial^* E$ is called the *reduced boundary* of E, and ν_E is the associated inner approximate normal (see [171, Section 3.5]). We have that $\partial^* E \subseteq \partial E$, but the topological boundary can in in general be much larger than the reduced one. If $A \subseteq \mathbb{R}^d$ is open and bounded with $\mathscr{H}^{d-1}(A) < +\infty$, then A has finite perimeter with $Per(A) \leq \mathscr{H}^{d-1}(\partial A)$.

10.2.3 Functions of bounded deformation

If $\Omega \subseteq \mathbb{R}^d$ is open, we say that $u \in BD(\Omega)$ if $u \in L^1(\Omega;\mathbb{R}^d)$ and its symmetric gradient $Eu := \frac{Du+(Du)^*}{2}$ in the sense of distributions is a finite Radon measure on Ω, i.e., $Eu \in \mathcal{M}_b(\Omega;\mathrm{M}^d_{\mathrm{sym}})$. $BD(\Omega)$ is called the space of *functions of bounded deformation* on Ω. We refer the reader to [197, 196] for the main properties of the space BD.

We will make use of a subspace of $BD(\Omega)$ called the space of *special functions of bounded deformation* introduced in [172]. We say that $u \in SBD(\Omega)$ if $u \in BD(\Omega)$ and its symmetrized distributional derivative can be written in the form

$$Eu = e(u)\,\mathscr{L}^d + (u^+ - u^-) \odot \nu_u \mathscr{H}^{d-1} \lfloor J_u,$$

where $e(u) \in L^1(\Omega;\mathrm{M}^d_{\mathrm{sym}})$ denotes the approximate symmetrized gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, and $\nu_u(x)$ is the normal to J_u at x. As in the case of functions of bounded variation, J_u is a $\mathscr{H}^{d-1}$-countably rectifiable set.

We will use the following compactness and lower semicontinuity result proved in [174].

Theorem 10.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and with a Lipschitz boundary, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that*

$$\sup_n \Big[|Eu_n|(\Omega) + \|u_n\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u_n)\|_{L^p(\Omega;\mathrm{M}^d_{\mathrm{sym}})} + \mathscr{H}^{d-1}(J_{u_n})\Big] < +\infty$$

for some $p > 1$. Then there exists $u \in SBD(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^d),$$

$$e(u_{n_k}) \rightharpoonup e(u) \qquad \text{weakly in } L^p(\Omega;\mathrm{M}^d_{\mathrm{sym}}),$$

and

$$\mathscr{H}^{d-1}(J_u) \le \liminf_{k\to+\infty} \mathscr{H}^{d-1}(J_{u_{n_k}}).$$

We will need also some properties of the *sections* of SBD-functions. If $\Omega \subseteq \mathbb{R}^d$ is open and $u \in SBD(\Omega)$, let us consider the scalar function on Ω^ξ_y given by

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi$$

and the set

$$J^\xi_u := \{x \in J_u \,:\, (u^+(x) - u^-(x)) \cdot \xi \neq 0\}$$

The following result holds true (see [172]).

Theorem 10.5 (One dimensional sections of SBD-functions). *Let $\Omega \subseteq \mathbb{R}^d$ be open, $u \in SBD(\Omega)$, and let $\xi \in \mathbb{R}^d$ with $|\xi| = 1$. Then for $\mathscr{H}^{d-1}$-a.e. $y \in \Omega^\xi$ we have*

$$\hat{u}^\xi_y \in SBV(\Omega^\xi_y)$$

with

$$(\hat{u}^\xi_y)'(t) = (e(u)\xi \cdot \xi)(y + t\xi) \qquad \text{for a.e. } t \in \Omega^\xi_y$$

and

$$J_{\hat{u}^\xi_y} = (J^\xi_u)^\xi_y.$$

10.2.4 Hausdorff convergence

The family of closed sets in $\mathbb{R}^d$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{ \sup_{x\in K_1} \operatorname{dist}(x, K_2), \sup_{y\in K_2} \operatorname{dist}(y, K_1) \right\}$$

with the conventions $\operatorname{dist}(x, \emptyset) = +\infty$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = +\infty$ if $K \neq \emptyset$.

The Hausdorff metric has good compactness properties (see [173, Theorem 4.4.15]).

Proposition 10.6 (Compactness). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets contained in a fixed compact set of $\mathbb{R}^d$. Then there exists a compact set $K \subseteq \mathbb{R}^d$ such that up to a subsequence*

$$K_n \to K \qquad \text{in the Hausdorff metric.}$$

Compact and connected sets with finite length in the plane. In Section 10.7, we will deal with the the family of compact and connected sets in $\mathbb{R}^2$ with finite $\mathscr{H}^1$-measure. For our analysis we will need the following lower semicontinuity result for the length due to Gołąb: for the proof we refer the reader to [192, Theorem 3.18] or [173, Theorem 4.4.17].

Theorem 10.7 (Gołąb). *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of compact and connected sets in $\mathbb{R}^2$ with*

$$\Gamma_n \to \Gamma \qquad \text{in the Hausdorff metric.}$$

Then

$$\mathscr{H}^1(\Gamma) \leq \liminf_{n\to+\infty} \mathscr{H}^1(\Gamma_n).$$

The following result contains some rectifiablity property of compact and connected sets with finite length which will be important for our analysis: recall that a set Γ is said to be $\mathscr{H}^1$-countably rectifiable if it is contained up to a $\mathscr{H}^1$-negligible set into the union of a countable family of C^1 curves.

Theorem 10.8. *Let $\Gamma \subseteq \mathbb{R}^2$ be compact, connected and with $\mathscr{H}^1(\Gamma) < +\infty$. Then the following items hold true.*

(a) Γ is $\mathscr{H}^1$-countably rectifiable, so that in particular it admits an approximate unit normal $\nu_\Gamma(x)$ for $\mathscr{H}^1$-a.e. $x \in \Gamma$.

(b) Let $x \in \Gamma$ be a point for which $\nu_\Gamma(x)$ exists. Then for $\epsilon \to 0^+$

$$\Gamma_{x,\epsilon} \to \ell_{\Gamma,x} \qquad \text{locally in the Hausdorff metric,} \tag{10.11}$$

where

$$\Gamma_{x,\epsilon} := \frac{1}{\epsilon}[\Gamma - x] := \left\{ \frac{y-x}{\epsilon} \ : \ y \in \Gamma \right\},$$

and $\ell_{\Gamma,x}$ is the line orthogonal to $\nu_\Gamma(x)$ through 0.

Proof. Point (a) can be found in [192, Lemma 3.13] (indeed this property holds true even if $\Gamma \subset \mathbb{R}^d$). The existence of the approximate normal yields that for $\mathscr{H}^1$-a.e. $x \in \Gamma$

$$\mathscr{H}^1 \lfloor \Gamma_{x,\epsilon} \overset{*}{\rightharpoonup} \mathscr{H}^1 \lfloor \ell_{\Gamma,x} \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\mathbb{R}^2), \tag{10.12}$$

The reinforced property of point (b) can be found e.g. in Step 2 of the proof of [175, Proposition 2.6] and we report here the argument for completeness.

Up to a translation, we may assume $x = 0$ and write Γ_ϵ and l in place of $\Gamma_{x,\epsilon}$ and l_x. Given any sequence $\epsilon_n \to 0$, by the compactness of Hausdorff convergence and using a diagonal argument, we can find a subsequence $(\epsilon_{n_h})_{h\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $m \geq 1$

$$\Gamma_{\epsilon_{n_h}} \cap \bar{Q}_m(0) \to \Gamma_0^m \qquad \text{in the Hausdorff metric.}$$

It is readily checked that for every $m \geq 1$

$$\Gamma_0^m \subseteq \Gamma_0^{m+1} \qquad \text{and} \qquad \Gamma_0^m \cap Q_m(0) = \Gamma_0^{m+1} \cap Q_m(0). \tag{10.13}$$

Let us set $\Gamma_0 := \bigcup_{m=1}^\infty \Gamma_0^m$. The conclusion follows by showing that

$$\Gamma_0 = l. \tag{10.14}$$

(a) We have $\Gamma_0 \subseteq l$. Indeed, assume by contradiction that $\xi \in \Gamma_0 \setminus l$ with $\overline{B}_\eta(\xi) \cap l = \emptyset$. Using the measure convergence (10.12), we obtain that

$$\mathscr{H}^1(\Gamma_{\epsilon_{n_h}} \cap B_\eta(\xi)) \to 0. \tag{10.15}$$

But $\Gamma_{\epsilon_{n_h}}$ is connected by arcs (see [192, Lemma 3.12]), so that the points $\xi_{n_h} \in \Gamma_{\epsilon_{n_h}}$ such that $\xi_{n_h} \to \xi$ are connected to 0 through an arc contained in $\Gamma_{\epsilon_{n_h}}$, against (10.15).

(b) We have on the contrary $l \subseteq \Gamma_0$. Indeed, assume by contradiction that $\xi \in l \setminus \Gamma_0$. Then there exists $\eta > 0$ such that $\Gamma_{\epsilon_{n_h}} \cap B_\eta(\xi) = \emptyset$ for h large, against (10.12).

In view of (10.13) and (10.14) we deduce that for $\epsilon \to 0$ and for every $m \geq 1$

$$\Gamma_\epsilon \cap \bar{Q}_m(0) \to l \cap \bar{Q}_m(0) \qquad \text{in the Hausdorff metric,}$$

i.e., convergence (10.11) holds true. □

The following result concerning Hausdorff convergence by slicing will be useful.

Lemma 10.9 (Hausdorff convergence by sections). *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of compact and connected sets in $\mathbb{R}^2$ with $\sup_n \mathscr{H}^1(\Gamma_n) < +\infty$ and*

$$\Gamma_n \to \Gamma \qquad \textit{in the Hausdorff metric.}$$

Then given $\xi \in \mathbb{R}^2$ with $|\xi| = 1$, there exists an at most countable set $E \subset \xi^\perp$ such that for every $y \notin E$

$$(\Gamma_n)_y^\xi \to (\Gamma)_y^\xi \qquad \textit{in the Hausdorff metric.}$$

Proof. We may assume up to a rotation that $\xi = e_1$, where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. To simplify the notation, we will write $(\Gamma_n)_y$ for $(\Gamma_n)_{(0,y)}^{e_1}$ to denote the section of Γ_n along the line $\{(x, y) : x \in \mathbb{R}\}$.

Let us consider the maps $f_n : \mathbb{R}^2 \to [0, +\infty]$

$$f_n(x, y) := dist(x, (\Gamma_n)_y),$$

where we adopt the convention $dist(x, \emptyset) = +\infty$. If $f := \liminf_{n\to+\infty} f_n$, we have that

$$f(x, y) = dist(x, H_y),$$

where H_y is the Kuratowski limsup of the sequence $((\Gamma_n)_y)_{n\in\mathbb{N}}$: in view of the Hausdorff convergence of Γ_n to Γ we have $H_y \subseteq \Gamma_y$. If we consider

$$g(x, y) := dist(x, \Gamma_y)$$

we thus deduce $f \geq g$.

For every $x \in \mathbb{R}$, let

$$E_x := \{y \in \mathbb{R} \,:\, f(x,y) > g(x,y)\}. \tag{10.16}$$

We claim that E_x is at most countable. If we set

$$E := \cup_{x\in\mathbb{Q}} E_x,$$

we infer that E is at most countable and that $f(x,y) = g(x,y)$ for $y \notin E$: indeed if $x_n \in \mathbb{Q}$ with $x_n \to x$, then as $y \notin E_{x_n}$ and f, g are continuous (Lipschitz) with respect to the first component,

$$f(x,y) = \lim_n f(x_n, y) = \lim_n g(x_n, y) = g(x,y).$$

We conclude that $H_y = \Gamma_y$ for every $y \notin E$: this entails that on every section except an at most countable family, Hausdorff convergence takes place towards the section of Γ, so that the conclusion follows.

In order to conclude the proof, we need to show that E_x defined in (10.16) is at most countable for every $x \in \mathbb{R}$.

It is not restrictive to assume $x = 0$. Given $k, h \geq 1$ let us consider the set

$$E_0^{k,h} := \left\{ y \in \mathbb{R} \,:\, \frac{h}{4k} \leq g(0,y) < \frac{h+1}{4k} \text{ and } f(0,y) > g(0,y) + \frac{1}{k} \right\}.$$

We claim that the set $E_0^{k,h}$ has at most $2(Ck+1)$ points, where $C := \sup_n \mathscr{H}^1(\Gamma_n)$. Since

$$E_0 := \{(0,y) \,:\, f(0,y) > g(0,y)\} = \bigcup_{h,k} E_0^{h,k},$$

we deduce that E_0 is at most countable, so that the conclusion follows.

In order to prove the claim, assume that $E_0^{k,h}$ contains $\{y_1, \dots, y_{2N}\}$, with $y_i < y_j$ for $i < j$, for some $N \geq 1$. Let us consider for $j = 1, \dots, 2N$ the rectangles

$$R_j := \left\{ (x,y) \,:\, |x| < \max\{g(0,y_j), g(0,y_{j+1}\} + \frac{1}{4k},\ y_j < y < y_{j+1} \right\}$$

and the larger ones

$$\hat{R}_j := \left\{ (x,y) \,:\, |x| < \max\{g(0,y_j), g(0,y_{j+1}\} + \frac{1}{2k},\ y_j < y < y_{j+1} \right\}.$$

Let $\hat{S}_j^1$ and $\hat{S}_j^2$ denote the horizontal segments of $\partial \hat{R}_j$ at height y_j and y_{j+1} respectively. Notice that for n large enough we have that

$$\hat{S}_j^h \cap \Gamma_n = \emptyset.$$

Indeed if for example $\hat{S}_j^1 \cap \Gamma_{n_k} \neq \emptyset$ along a subsequence $(n_k)_{k\in\mathbb{N}}$, there would exist $x_j \in H_{y_j}$ such that (recall that $y_j, y_{j+1} \in E_0^{h,k}$)

$$f(0,y_j) \leq |x_j| \leq \max\{g(0,y_j), g(0,y_{j+1}\} + \frac{1}{2k} < g(0,y_j) + \frac{1}{4k} + \frac{1}{2k} = g(0,y_j) + \frac{3}{4k},$$

which is against the definition of $E_0^{k,h}$.

Notice that if $\Gamma_n \cap R_i \neq \emptyset$ and $\Gamma_n \cap R_j \neq \emptyset$ for some $i \neq j$, then

$$\mathscr{H}^1(\Gamma_n \cap R_i) \geq \frac{1}{4k}. \tag{10.17}$$

Indeed since Γ_n is connected by arcs (because it is connected and with finite $\mathscr{H}^1$-measure), there is a curve contained in Γ_n with the first extreme in R_i and the second in R_j. Since Γ_n cannot intersect the horizontal segments $\hat{S}_i^h$ for every $i = 1, \dots, 2N$ and $h = 1, 2$, then we get that Γ_n intersects one vertical segment of $\partial \hat{R}_i$ and one of $\partial \hat{R}_j$, so that inequality (10.17) follows by projection onto the horizontal axis.

For every $i = 1, \dots, 2N$, let $z_n^i = (x_n^i, y_n^i) \in \Gamma_n$ be such that

$$x_n^i \to g(0, y_i) \qquad \text{and} \qquad y_n^i \to y_i.$$

Notice that for n large enough we have that at least $N-1$ points among the z_n^i belongs to different rectangles of the type R_j. In view of (10.17) we deduce

$$C \geq \mathscr{H}^1(\Gamma_n) \geq \frac{N-1}{k}$$

which yields that the set $E_0^{k,h}$ contains at most $2(Ck+1)$ points, so that the claim is proved. □

Remark 10.10. Notice that exceptional directions can indeed occur in a countable number.

In a first step, if we consider Γ_n of the form

$$\Gamma_n := \partial B \cup L_{\xi_n}$$

where B is a ball and L_{ξ_n} is its diameter in the direction $\xi_n \to \xi$ with $\xi_n \neq \xi$, then $\Gamma_n \to \Gamma$ in the Hausdorff metric with

$$\Gamma := \partial B \cup L_\xi.$$

We see that along every line in the ξ direction except the one containing L_ξ we have Hausdorff convergence of the corresponding sections; along the line r_ξ through L_ξ the sections of Γ_n converge indeed to the two point set $\partial B \cap r_\xi$ which is strictly contained into the section of Γ given by L_ξ.

The example can be constructed by considering a countable family of disjoint smaller and smaller disks accumulating to a point, inside which we consider varying diameters which approach a fixed direction belonging to a countable family of directions: we can connect the various disks in order to create a sequence of compact and connected sets with bounded length.

10.3 Obstacles in Stokes fluids and drag minimization

In this section we explain the drag problem for an obstacle immersed in a stationary flow.

10.3.1 The flow around the obstacle

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary, and let $V \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ be a divergence free vector field. Given $E \Subset \Omega$ open and with a Lipschitz boundary, let us consider the stationary flow for a viscous incompressible fluid around E with boundary conditions on $\partial\Omega$ given by V, and with Navier boundary conditions on ∂E. More precisely, if $u : \Omega \setminus E \to \mathbb{R}^d$ is the velocity field, we require that the following items hold true.

(a) *Incompressibility*: $\operatorname{div} u = 0$ in $\Omega \setminus E$.

(b) *Boundary conditions*: we have

$$u = V \text{ on } \partial\Omega \qquad \text{and non-penetration} \qquad u \cdot \nu = 0 \text{ on } \partial E,$$

where ν denotes the exterior normal to E.

(c) *Equilibrium*: considering the stress

$$\sigma := -pI_d + 2\mu e(u), \tag{10.18}$$

where $\mu > 0$ is a viscosity parameter, $e(u)$ the symmetrized gradient of u (also denoted by $D(u)$) and p is the pressure, we require

$$\operatorname{div} \sigma = 0 \qquad \text{in } \Omega \setminus E. \tag{10.19}$$

(d) *Navier conditions on the obstacle*: we have

$$(\sigma\nu)_\tau = \beta u \qquad \text{on } \partial E,$$

where $\beta > 0$ is a friction parameter, and $(\sigma\nu)_\tau$ denotes the tangential component of force $\sigma\nu$.

The stationary flow has the following variational characterization: u is the minimizer of the energy

$$\mathscr{E}(u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, d\mathscr{L}^d + \beta \int_{\partial E} |u|^2 \, d\mathscr{H}^{d-1} \tag{10.20}$$

among the class of (sufficiently regular) admissible fields

$$\mathcal{V}^{\rm reg}_{E,V}(\Omega) := \{v \in H^1(\Omega \setminus E, \mathbb{R}^d) \, : \, v \text{ satisfies points (a) and (b)}\}, \tag{10.21}$$

where $\mathscr{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measures, which reduces to the area measure on sufficiently regular sets. Indeed if u is a minimizer, and φ is an admissible variation, so that $\varphi = 0$ on $\partial\Omega$, we get

$$\begin{aligned} 0 &= 2\mu \int_{\Omega\setminus E} e(u) : e(\varphi) \, d\mathscr{L}^d + \beta \int_{\partial E} u \cdot \varphi \, d\mathscr{H}^{d-1} \\ &= 2\mu \int_{\Omega\setminus E} e(u) : \nabla\varphi \, d\mathscr{L}^d + \beta \int_{\partial E} u \cdot \varphi \, d\mathscr{H}^{d-1} \\ &= -2\mu \int_{\Omega\setminus E} \text{div } e(u) \cdot \varphi \, d\mathscr{L}^d + \int_{\partial E} [-2\mu e(u)\nu + \beta u] \cdot \varphi \, d\mathscr{H}^{d-1} \end{aligned}$$

In particular, choosing φ with compact support in $\Omega \setminus E$ we have

$$2\mu \text{div} e(u) = \nabla p$$

for some pressure field p: as a consequence $\sigma := -pI_d + 2\mu e(u)$ satisfies (10.19) of condition (c).

Since the admissible functions φ are tangent to ∂E, the optimality condition reduces to

$$0 = \int_{\partial E} [-2\mu e(u)\nu + \beta u] \cdot \varphi \, d\mathscr{H}^{d-1} = \int_{\partial E} [-\sigma\nu + \beta u] \cdot \varphi \, d\mathscr{H}^{d-1}. \tag{10.22}$$

Notice that every tangential vector field η on ∂E can be extended to a divergence free vector field on $\Omega \setminus E$ which vanishes on $\partial\Omega$, hence it is the trace of an admissible variation φ: indeed any extension W which vanishes on $\partial\Omega$ has a divergence with zero mean, so that considering W_1 with $\text{div}W_1 = \text{div}W$ with $W_1 = 0$ on $\partial\Omega$ and on ∂E (whose existence is guaranteed, for example by [177, Theorem IV.3.1])), the required extension is given by $W - W_1$. We conclude that the optimality condition (10.22) yields the Navier condition of point (b).

10.3.2 The drag force

Assume now that the external vector field V is equal to a constant $V_\infty \in \mathbb{R}^d \setminus \{0\}$, i.e. the obstacle E is immersed in a uniform flow. The flow is perturbed near E, assuming the value u, and the obstacle experiences a force which has a component in the direction V_∞ which is given, up to a multiplicative constant (V_∞ has not unit norm in general), by

$$Drag(E) := \int_{\partial E} \sigma\nu \cdot V_\infty \, d\mathscr{H}^{d-1},$$

which is called the *drag force* on E in the direction of the flow.

We claim that

$$Drag(E) = \mathscr{E}(u), \tag{10.23}$$

where $\mathscr{E}(u)$ is the energy defined in (10.20). Using the facts that σ is symmetric and with zero divergence (so that also the vector field σV_∞ is divergence free), and that $u = V_\infty$ on $\partial\Omega$, we may write

$$\begin{aligned}\int_{\partial E}\sigma\nu\cdot V_\infty\,d\mathscr{H}^{d-1}&=\int_{\partial E}\sigma V_\infty\cdot\nu\,d\mathscr{H}^{d-1}=\int_{\partial\Omega}\sigma V_\infty\cdot\nu\,d\mathscr{H}^{d-1}=\int_{\partial\Omega}\sigma u\cdot\nu\,d\mathscr{H}^{d-1}\\&=\int_{\Omega\setminus E}\operatorname{div}(\sigma u)\,d\mathscr{L}^d+\int_{\partial E}\sigma u\cdot\nu\,d\mathscr{H}^{d-1}=\int_{\Omega\setminus E}\sigma:\nabla u\,d\mathscr{L}^d+\int_{\partial E}\sigma\nu\cdot u\,d\mathscr{H}^{d-1}.\end{aligned}\tag{10.24}$$

Using again that σ is symmetric and that u is divergence free, together with the constitutive equation (10.18), we have

$$\begin{aligned}\int_{\Omega\setminus E}\sigma:\nabla u\,d\mathscr{L}^d&=\int_{\Omega\setminus E}\sigma:e(u)\,d\mathscr{L}^d=\int_{\Omega\setminus E}(-p\,I_d+2\mu e(u)):e(u)\,d\mathscr{L}^d\\&=\int_{\Omega\setminus E}(-p\operatorname{div}u+2\mu|e(u|^2)\,d\mathscr{L}^d=2\mu\int_{\Omega\setminus E}|e(u)|^2\,d\mathscr{L}^d,\end{aligned}$$

while in view of the Navier conditions on ∂E and the fact that u is tangent to the obstacle

$$\int_{\partial E}\sigma\nu\cdot u\,d\mathscr{H}^{d-1}=\int_{\partial E}(\sigma\nu)_\tau\cdot u\,d\mathscr{H}^{d-1}=\beta\int_{\partial E}|u|^2\,d\mathscr{H}^{d-1}.$$

Inserting into (10.24), we get that (10.23) follows.

10.3.3 The optimization problem

Let $c>0$ and $f:(0,|\Omega|)\to\mathbb{R}\cup\{+\infty\}$ a lower semi-continuous functions that is not identically equal to $+\infty$. We are concerned with the following optimization problem:

$$\min_E\left\{Drag(E)+c\mathscr{H}^{d-1}(\partial E)+f(|E|)\right\}.$$

We are thus interested in finding the optimal shape of an obstacle which minimizes the drag force, under a penalization involving its perimeter and its volume.

In view of the energetic characterization of the drag force established in Subsection 10.3.2, we can formulate the problem as a minimization problem among the pairs (E,u), where u is a velocity field belonging to the family $\mathcal{V}^{\rm reg}_{E,V_\infty}(\Omega)$ defined in (10.21):

$$\min_{E,u\in\mathcal{V}^{\rm reg}_{E,V_\infty}(\Omega)}\left\{2\mu\int_{\Omega\setminus E}|e(u)|^2\,d\mathscr{L}^d+\beta\int_{\partial E}|u|^2\,d\mathscr{H}^{d-1}+c\mathscr{H}^{d-1}(\partial E)+f(|E|)\right\}.$$

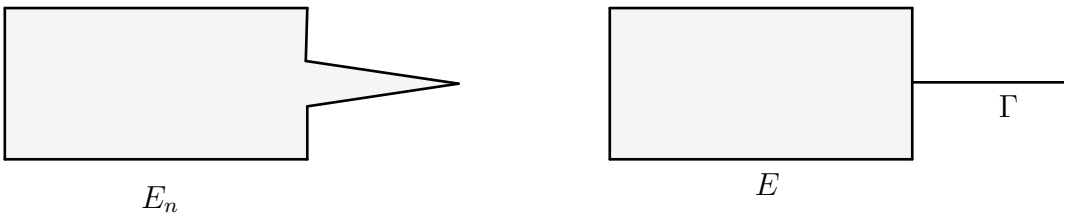
Setting all the constants equal to 1, and replacing V_∞ by a given divergence free velocity vector field V as in Subsection 10.3.1, the drag minimization problem above is a particular case of the following shape optimization problem

$$\min_{E,u\in\mathcal{V}^{\rm reg}_{E,V_\infty}(\Omega)}\left\{\int_{\Omega\setminus E}|e(u)|^2\,d\mathscr{L}^d+\int_{\partial E}|u|^2\,d\mathscr{H}^{d-1}+\mathscr{H}^{d-1}(\partial E)+f(|E|)\right\}.\tag{10.25}$$

If we want to apply the direct method of the calculus of variations to the problem, i.e., if we want to recover a minimizer by looking at minimizing sequences $(E_n,u_n)_{n\in\mathbb{N}}$, the following considerations are quite natural.

(a) Since the problem involves the perimeter of E, the sequence $(E_n)_{n\in\mathbb{N}}$ is relatively compact in the family of *sets of finite perimeter* (see Section 10.2).

(b) Concerning the velocities, it turns out naturally that it is convenient to consider also *discontinuous* vector fields. Indeed if $u_n\to u$ in some sense, and ∂E_n collapses in some parts generating a surface Γ *outside* the limit set E, the limit velocity field u can present, in general, discontinuities across Γ.



We thus expect an extra term in the surface integral related to the Navier conditions, which amounts at least to

$$\int_{\Gamma\setminus\partial E} [|u^+|^2 + |u^-|^2]\, d\mathscr{H}^{d-1},$$

where $u^\pm$ are the two traces from both sides of Γ.

The previous considerations yield to formulate a relaxed version of problem (10.25) in which the obstacle E varies among the family of sets of finite perimeter contained in Ω, while the family of associated admissible velocity fields is naturally contained in the space of *special functions of bounded deformation* $SBD(\Omega)$ (see Section 10.2).

In Section 10.4, we will give a precise formulation of problem in this weak setting, which guarantees existence of optimal solutions, describing in particular how the boundary conditions on $\partial\Omega$ and on the obstacle have to be rephrased in this context.

10.4 A relaxed formulation of the shape optimization problem and existence of optimal obstacles

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and with a Lipschitz boundary, and let $V \in C^1(\Omega';\mathbb{R}^d)$ be a divergence free vector field defined on some bounded open set Ω' such that $\overline{\Omega} \subset \Omega'$.

Definition 10.11 (**Admissible obstacles).** The class of admissible obstacles in Ω is given by

$$\mathcal{A}(\Omega) := \{E \subseteq \Omega \,:\, \mathrm{Per}(E) < +\infty\},$$

i.e., by the subsets of Ω which have finite perimeter.

Definition 10.12 (**Admissible fields).** Given an open set Ω, we denote by $\mathcal{V}(\Omega)$ the family of velocity vector fields

$$u \in SBD(\Omega) \cap L^2(\Omega;\mathbb{R}^d)$$

that verifies the two following constraints:

(a) The flow is divergence free: we have

$$\operatorname{div} u = 0 \text{ in } \mathcal{D}'(\Omega).$$

(b) Non-penetration condition on the discontinuity set: we have

$$u^\pm \perp \nu_u \text{ on } J_u$$

Definition 10.13 (**Admissible set-field pairs).** Given $E \in \mathcal{A}(\Omega)$ and V as above, we denote by $\mathcal{V}_{E,V}(\Omega)$ the family of vector fields $u \in \mathcal{V}(\Omega)$ such that $u = 0$ a.e. on E and by extending u by V on a neighbourhood Ω' of $\overline{\Omega}$, it is in $\mathcal{V}(\Omega')$.

The crucial difference between $\mathcal{V}_{E,V}(\Omega)$ and $\mathcal{V}^{\mathrm{reg}}_{E,V}(\Omega)$ is that $u \in \mathcal{V}_{E,V}(\Omega)$ may have discontinuities outside of E. Let $E \in \mathcal{A}(\Omega)$ and $u \in \mathcal{V}_{E,V}(\Omega)$, since $\partial^* E$ is rectifiable (with outward normal vector

ν_E) the traces of u on both sides are well-defined $\mathscr{H}^{d-1}$-almost everywhere and we will orient them such that u^+ is in the direction of ν_E, so $u^- = 0$ a.e. on $\partial^* E$. The non-penetration constraint is automatically verified; for any $x \in \partial^* E$, either $u^+(x) = 0$ and it is verified, or $x \in J_u$ so outside of a $\mathscr{H}^{d-1}$-negligible set and up to a choice of orientation we have $\nu_E(x) = \nu_u(x)$ and the condition (b) of definition 10.12 applies.

Remark 10.14. Note that the obstacle E may touch $\partial\Omega$ only on those part where V is tangent to Ω: this is due to the fact that on $\partial^* E \cap \partial\Omega$ the sets E and Ω share $\mathscr{H}^{d-1}$-a.e. the same exterior normal, and $u^+ = V$. A similar property holds true for the discontinuity set J_u.

Let $f : [0, |\Omega|] \to [0, +\infty]$ be a lower semicontinuous functions not identically equal to $+\infty$. For every $E \in \mathcal{A}(\Omega)$ and $u \in \mathcal{V}_{E,V}(\Omega)$, let us set

$$\begin{aligned}\mathcal{J}(E,u) := &\int_{\Omega\setminus E} |e(u)|^2 \, d\mathscr{L}^d + \beta \int_{\partial^* E} |u^+|^2 \, d\mathscr{H}^{d-1} + \beta \int_{J_u \setminus \partial^* E} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^{d-1} \\ &+ \mathscr{H}^{d-1}(\partial^* E) + 2\mathscr{H}^{d-1}(J_u \setminus \partial^* E) + f(|E|),\end{aligned} \tag{10.26}$$

Remark 10.15. Let $E \Subset \Omega$ be open and with a Lipschitz boundary. Then we can find $W \in H^1(\Omega \setminus E; \mathbb{R}^d)$ such that $W = V$ on $\partial\Omega$, $W = 0$ on ∂E and $\mathrm{div} W = 0$. Indeed if $\varphi \in C^\infty(\mathbb{R}^d)$ is such that $\varphi = 1$ on a neighborhood of $\mathbb{R}^d \setminus \Omega$ and $\varphi = 0$ on a neighborhood of E, we can consider the vector field $V_1 := \varphi V$, whose divergence has zero mean on $\Omega \setminus E$ (by Gauss theorem). Then we can find $V_2 \in H^1_0(\Omega \setminus E; \mathbb{R}^d)$ such that $\mathrm{div} V = \mathrm{div} V_1$ (see [177, Theorem IV.3.1]), so that the field $W := V_1 - V_2$ is an admissible choice. In particular we get that $W \in \mathcal{V}_{E,V}$, so that the class of admissible velocities is not empty.

The following is the first main result of the paper.

Theorem 10.16 (**Existence of optimal obstacles**). *The minimization problem*

$$\min_{E \in \mathcal{A}(\Omega), u \in \mathcal{V}_{E,V}(\Omega)} \mathcal{J}(E,u) \tag{10.27}$$

admits a solution.

The proof of Theorem 10.16 will be given in Section 10.6, based on some technical results for SBD functions collected in Section 10.5.

10.5 Some technical results in SBD

In this section we collect some technical properties concerning the space SBD that will be fundamental in the proof of etiquette Theorem 10.1. In particular in Theorem 10.17 we will prove that admissible velocity vector fields enjoy higher summability properties (indeed they belong to $L^{\frac{2d}{d-1}}$). In Theorem 10.19 we will prove that velocity fields u with $u^\pm$ tangent to the discontinuity set J_u form a closed set under the natural convergence of minimizing sequences for the main optimization problem. Finally in Theorem 10.20 we will prove a lower semicontinuity result for surface energies depending on the traces, which entails in particular the lower semicontinuity of the term associated to the Navier conditions.

10.5.1 An immersion result

The following embedding result holds true.

Theorem 10.17. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $u \in SBD(\mathbb{R}^d)$ be supported in Ω such that*

$$\mathscr{E}(u) := \int_\Omega |e(u)|^2 \, d\mathscr{L}^d + \int_{J_u} \left[|u^+|^2 + |u^-|^2\right] \, d\mathscr{H}^{d-1} < +\infty.$$

Then $u \in L^{\frac{2d}{d-1}}(\Omega)$ with

$$\|u\|_{\frac{2d}{d-1}} \le C\sqrt{\mathscr{E}(u)},$$

where C depends on d and $diam(\Omega)$ only.

Proof. It suffices to follow the strategy of the proof of the classical embedding of BD into $L^{d/d-1}$ explained in [196], but concentrating on the square of the components.

Let us consider the unit vector

$$\xi := \frac{1}{\sqrt{d}}(1, 1, \dots, 1) \in \mathbb{R}^d.$$

Employing the characterization by sections recalled in Section 10.2, for $\mathscr{H}^{d-1}$-a.e. $y \in \xi^\perp$ we have

$$\hat{u}_y^\xi \in SBV(\Omega_y^\xi)$$

with

$$\int_{\Omega_y^\xi} |(\hat{u}_y^\xi)'|^2 \, dt + \sum_{t \in J_{\hat{u}_y^\xi}} \left[|(\hat{u}_y^\xi)^+(t)|^2 + |(\hat{u}_y^\xi)^-(t)|^2\right] < +\infty.$$

Then we can write for a.e. $t \in \mathbb{R}$

$$\begin{aligned} \|\hat{u}_y^\xi\|^2_{L^\infty(\Omega_y^\xi)} &\le \left|D(\hat{u}_y^\xi)^2\right|(\Omega_y^\xi) = \int_{\Omega_y^\xi} 2|\hat{u}_y^\xi(\hat{u}_y^\xi)'|dt + \sum_{t \in J_{\hat{u}_y^\xi}} \left||(\hat{u}_y^\xi)^+(t)|^2 - |(\hat{u}_y^\xi)^-(t)|^2\right| \\ &\le \frac{1}{2}\|\hat{u}_y^\xi\|^2_{L^\infty(\Omega_y^\xi)} + 2|\Omega_y^\xi| \int_{\Omega_y^\xi} \left|(\hat{u}_y^\xi)'\right|^2 dt + \sum_{t \in J_{\hat{u}_y^\xi}} \left(\left|(\hat{u}_y^\xi)^+(t)\right|^2 + \left|(\hat{u}_y^\xi)^-(t)\right|^2\right), \end{aligned} \tag{10.28}$$

Let us set

$$g_\xi(x) := \int_{\Omega_y^\xi} |(\hat{u}_y^\xi)'|^2 \, dt + \sum_{t \in J_{\hat{u}_y^\xi}} \left[|(\hat{u}_y^\xi)^+(t)|^2 + |(\hat{u}_y^\xi)^-(t)|^2\right],$$

where $y := \pi^{\xi^\perp}(x)$, i.e., the projection of x on the hyperplane $\xi^\perp$. $g_\xi(x)$ only depends on the projection of x on $\xi^\perp$ and

$$\begin{aligned} \int_{\xi^\perp} g_\xi d\mathscr{H}^{d-1} &= \int_\Omega |e(u)\xi \cdot \xi|^2 \, d\mathscr{L}^d + \int_{J_u} \left[|u^+|^2 + |u^-|^2\right] |\xi \cdot \nu| \, d\mathscr{H}^{d-1} \\ &\le C\left[\int_\Omega |e(u)|^2 \, d\mathscr{L}^d + \int_{J_u} \left[|u^+|^2 + |u^-|^2\right] \, d\mathscr{H}^{d-1}\right] \end{aligned}$$

where C depends only on d. Thanks to (10.28) we have

$$|\xi \cdot u|^2 \le C g_\xi \qquad \text{a.e. on } \Omega, \tag{10.29}$$

where C depends on the diameter of Ω, and from now on all the constants C that appear depend on $n, \mathrm{diam}(\Omega)$. For every $k = 1, \dots, d-1$, we can write

$$\xi = \frac{1}{\sqrt{d}} e_k + \sqrt{\frac{d-1}{d}} h_k,$$

where e_k is the k-th vector of the canonical base, and h_k is the unit vector in the direction $\sqrt{d}\xi - e_k$. Reasoning as above on the decomposition

$$\xi \cdot u = \sqrt{\frac{d-1}{d}} h_k \cdot u + \frac{1}{\sqrt{d}} e_k \cdot u$$

we obtain a similar estimate

$$|\xi \cdot u|^2 \leq C \left(g_{h_k} + g_{e_k} \right), \tag{10.30}$$

Multiplying inequality (10.29) with inequalities (10.30) for $k = 1, \dots, d-1$, we obtain reasoning as in [196, Chapter II, Theorem 1.2]

$$\|(\xi \cdot u)^2\|_{\frac{d}{d-1}} \leq C \left[\int_\Omega |e(u)|^2 \, d\mathscr{L}^d + \int_{J_u} \left[|u^+|^2 + |u^-|^2 \right] \, d\mathscr{H}^{d-1} \right].$$

Working similarly with $\tilde{\xi} = \frac{1}{\sqrt{d}}(1, -1, -1, \dots, -1)$ and using that

$$\frac{1}{d} u_1^2 = (\xi \cdot u + \tilde{\xi} \cdot u)^2 \leq 2 \left[|\xi \cdot u|^2 + |\tilde{\xi} \cdot u|^2 \right],$$

we get the embedding result for the first component, and hence for the entire function u reasoning similarly on the other components.

□

10.5.2 Closure of the non-penetration constraint

In the context of equi-Lipschitz boundaries, the preservation of the non-penetration property for a sequence of Sobolev functions converging weakly, comes rather directly via the divergence theorem (we refer the reader, for instance, to [179]). However, in the case of collapsing boundaries, so that the limit function lives on both sides of a surface and in absence of any smoothness of the limit set, this technique does not work. The proof of the non-penetration preservation requires different technical arguments that we handle in the SBD context.

Let us start with the following lower semicontinuity result.

Theorem 10.18. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that*

$$\sup_n \left[\int_\Omega |e(u_n)|^2 \, d\mathscr{L}^d + \mathscr{H}^{d-1}(J_{u_n}) \right] < +\infty$$

with

$$u_n \to u \qquad \text{in measure}$$

for some $u \in SBD(\Omega)$. Then

$$\int_{J_u} \left[|u^+ \cdot \nu_u| + |u^- \cdot \nu_u| \right] \, d\mathscr{H}^{d-1} \leq \liminf_{n \to +\infty} \int_{J_{u_n}} \left[|u_n^+ \cdot \nu_{u_n}| + |u^- \cdot \nu_{u_n}| \right] \, d\mathscr{H}^{d-1}.$$

Proof. Let us consider a countable set of functions $\{\varphi_h : h \in \mathbb{N}\}$ which is dense with respect to $\|\cdot\|_\infty$ inside the set

$$\left\{ f \in C_c^0(]0, +\infty[) : \int_0^{+\infty} f \, \mathrm{d}t = 0 \text{ and } \|f\|_\infty \leq 1 \right\}.$$

Given $\epsilon > 0$, let us consider

$$g_{h,k}(x) := \int_0^{\frac{1}{2}|x - x_k|^2} \varphi_h(t) \, dt,$$

where $\{x_k : k \in \mathbb{N}\}$ is a countable and dense set in $B_\epsilon(0) \subset \mathbb{R}^d$ with $x_0 = 0$. Clearly $g_{h,k} \in C_c^1(\mathbb{R}^d)$ with

$$G_{h,k}(x) := \nabla g_{h,k}(x) = \varphi_h \left(\frac{1}{2} |x - x_k|^2 \right) (x - x_k).$$

We have that $G_{h,k}$ is a continuous conservative vector field with compact support on $\mathbb{R}^d$.

Let us set for $(i,j) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\nu \in \mathbb{R}^d$ with $|\nu| = 1$

$$f_\epsilon(i,j,\nu) := \sup_{h,k} (G_{h,k}(i) - G_{h,k}(j)) \cdot \nu.$$

By construction f_ϵ is a *symmetric jointly convex function* according to [193, Definition 3.1]. We claim that for $i \neq j$

$$|i \cdot \nu| + |j \cdot \nu| \leq f_\epsilon(i,j,\nu) \leq |i \cdot \nu| + |j \cdot \nu| + 2\epsilon. \tag{10.31}$$

In view of the lower semicontinuity result [193, Theorem 5.1] we have

$$\liminf_{n \to +\infty} \int_{J_{u_n}} f_\epsilon(u_n^+, u_n^-, \nu_{u_n})\, d\mathscr{H}^{d-1} \geq \int_{J_u} f_\epsilon(u^+, u^-, \nu_u)\, d\mathscr{H}^{d-1}.$$

We can thus write

$$\begin{aligned}
&\liminf_{n \to +\infty} \left[\int_{J_{u_n}} \left[|u_n^+ \cdot \nu_{u_n}| + |u_n^- \cdot \nu_{u_n}| \right] d\mathscr{H}^{d-1} + 2\epsilon \mathscr{H}^{d-1}(J_{u_n}) \right] \geq \liminf_{n \to +\infty} \int_{J_{u_n}} f_\epsilon(u_n^+, u_n^-, \nu_{u_n})\, d\mathscr{H}^{d-1} \\
&\geq \int_{J_u} f_\epsilon(u^+, u^-, \nu_u)\, d\mathscr{H}^{d-1} \geq \int_{J_u} \left[|u^+ \cdot \nu_u| + |u^- \cdot \nu_u| \right] d\mathscr{H}^{d-1},
\end{aligned}$$

so that the result follows taking into account the bound on $\mathscr{H}^{d-1}(J_{u_n})$ and letting $\epsilon \to 0$.

In order to complete the proof, we need to show claim (10.31). The estimate from above follows from

$$[G_{h,k}(i) - G_{h,k}(j)] \cdot \nu \leq |(i - x_k) \cdot \nu| + |(j - x_k) \cdot \nu| \leq |i \cdot \nu| + |j \cdot \nu| + 2\epsilon$$

since $\|\varphi_h\|_\infty \leq 1$ and $|x_k| < \epsilon$. Let us prove the estimate from below. We select $x_{k_n} \to 0$ such that $|i - x_{k_n}| \neq |j - x_{k_n}|$ (which is always possibile in view of the density of $\{x_k : k \in \mathbb{N}\}$ inside $B_\epsilon(0)$ and since $i \neq j$) and then φ_{h_n} such that for $n \to +\infty$

$$\varphi_{h_n}\left(\frac{1}{2}|i - x_{k_n}|^2\right) \to \frac{i \cdot \nu}{|i \cdot \nu| + \eta} \qquad \text{and} \qquad \varphi_{h_n}\left(\frac{1}{2}|j - x_{k_n}|^2\right) \to -\frac{j \cdot \nu}{|j \cdot \nu| + \eta},$$

where $\eta > 0$. By definition of f_ϵ we infer that

$$f_\epsilon(i,j,\nu) \geq |i \cdot \nu| + |j \cdot \nu| - 2\eta,$$

so that the estimate from below follows by sending $\eta \to 0$. □

We are now in a position to prove the main result of the section.

Theorem 10.19 (Closure of the non-penetration constraint on the jump set). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that*

$$\sup_n \left[\int_\Omega |e(u_n)|^2\, d\mathscr{L}^d + \mathscr{H}^{d-1}(J_{u_n}) \right] < +\infty$$

and

$$u_n \to u \qquad \text{in measure}$$

for some $u \in SBD(\Omega)$. If

$$u_n^\pm \cdot \nu_{u_n} = 0 \qquad \mathscr{H}^{d-1}\text{-a.e. on } J_{u_n},$$

then

$$u^\pm \cdot \nu_u = 0 \qquad \mathscr{H}^{d-1}\text{-a.e. on } J_u.$$

Proof. By Theorem 10.18 we may write

$$\int_{J_u} \left[|u^+ \cdot \nu_u| + |u^- \cdot \nu_u| \right] d\mathscr{H}^{d-1} \leq \liminf_{n \to +\infty} \int_{J_{u_n}} \left[|u_n^+ \cdot \nu_{u_n}| + |u^- \cdot \nu_{u_n}| \right] d\mathscr{H}^{d-1} = 0,$$

so that the result follows. □

10.5.3 A lower semicontinuity result for surface energies in SBD

In this section we deal with the lower semicontinuity of the surface term of the functional J in (10.26) connected with the Navier conditions on the obstacle. The following lower semicontinuity result holds true.

Theorem 10.20. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $u_n, u \in SBD(\Omega)$ such that*

$$u_n \to u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^d)$$

and

$$\sup_n \left[\int_\Omega |e(u_n)|^2\, d\mathscr{L}^d + \mathscr{H}^{d-1}(J_{u_n})\right] < +\infty.$$

Then if $\phi : \mathbb{R}^d \to [0,+\infty]$ is a lower semi- continuous function, we have

$$\int_{J_u} [\phi(u^+) + \phi(u^-)]\, d\mathscr{H}^{d-1} \le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1}.$$

This applies in particular to $\phi(u) = |u|^2$ and $\phi(u) = 1_{\{u\neq 0\}}$, which will be of interest to us.

Proof. Notice first that ϕ may be supposed to be continuous. Indeed for any lower-semicontinuous nonnegative ϕ, there is a sequence of continuous nonnegative functions $\phi_k \nearrow \phi$ (meaning with punctual convergence and $0 \le \phi_k \le \phi$). Then once the property is known for each ϕ_k,

$$\begin{aligned}\int_{J_u} [\phi(u^+) + \phi(u^-)]\, d\mathscr{H}^{d-1} &= \liminf_{k\to\infty} \int_{J_u} [\phi_k(u^+) + \phi_k(u^-)]\, d\mathscr{H}^{d-1}\\ &\le \liminf_{k\to\infty} \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi_k(u_n^+) + \phi_k(u_n^-)]\, d\mathscr{H}^{d-1}\\ &\le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1}\end{aligned}$$

Through a blow-up approach, we can reduce the problem to the following lower semicontinuity result.

Let $Q_1 \subseteq \mathbb{R}^d$ be the unit square centred at 0, and let us set

$$H := Q_1 \cap \{x_d = 0\} \qquad \text{and} \qquad Q_1^\pm := Q_1 \cap \{x_d \gtrless 0\}.$$

Given $u^\pm \in \mathbb{R}^d$ with $u^+ \neq u^-$ and $u_n \in SBD(Q_1)$ with

$$u_n \to u := u^+ 1_{Q_1^+} + u^- 1_{Q_1^-} \qquad \text{strongly in } L^1(Q_1;\mathbb{R}^d), \tag{10.32}$$

$$\sup_n \mathscr{H}^{d-1}(J_{u_n}) < +\infty \tag{10.33}$$

and

$$e(u_n) \to 0 \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^{d\times d}_{sym}), \tag{10.34}$$

then it is enough to prove that

$$\phi(u^+) + \phi(u^-) \le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1}.$$

Indeed, let $u_n, u \in SBD(\Omega)$ be as in the statement of Theorem 10.20. Let $\mu_n = [\phi(u_n^+) + \phi(u_n^-)]\mathscr{H}^{d-1}\lfloor J_{u_n}$, then μ_n is a sequence of Radon measures with uniformly bounded mass, so we may suppose that up to taking a subsequence, $\mu_n \rightharpoonup \mu$ for some Radon measure μ. Our goal is to prove that

$$\int_{J_u} [\phi(u^+) + \phi(u^-)]\, d\mathscr{H}^{d-1} \le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1}. = \mu\left(\mathbb{R}^d\right),$$

and for this it is sufficient to prove that the Besicovitch differential of μ in $\mathscr{H}^{d-1}\lfloor J_u$ is larger that $\phi(u^+)+\phi(u^-)$; we want to prove for $\mathscr{H}^{d-1}$-almost every $x\in J_u$ that,

$$\frac{\mu(x+rQ_1)}{\mathscr{H}^{d-1}(J_u\cap(x+rQ_1))}\geq \phi(u^+(x))+\phi(u^-(x))+o_{r\to 0}(1).$$

and we assume without loss of generality that we are at a point x for which $\nu_u(x)=e_n$ (up to a rotation of the square Q_1, using [171, Th. 5.52]), with $\mathscr{H}^{d-1}(J_u\cap(x+rQ_1))\sim r^{d-1}$, meaning that it is enough to prove

$$\frac{\mu(x+rQ_1)}{r^{d-1}}\geq \phi(u^+(x))+\phi(u^-(x))+o_{r\to 0}(1).$$

We may additionnaly suppose, up to the exclusion of a $\mathscr{H}^{d-1}$-negligible set, that the left-hand side limit exists (so it is enough to prove the result for a subsequence) and

$$\frac{1}{r^{d-1}}\int_{x+rQ_1}|e(u)|d\mathscr{L}^d\underset{r\to 0}{\longrightarrow}0,$$
$$\limsup_{r\to 0}\frac{\alpha(x+rQ_1)}{r^{d-1}}<+\infty,\ \text{where } \alpha \text{ is the weak limit of } \mathscr{H}^{d-1}\lfloor J_{u_n}$$

which is true for $\mathscr{H}^{d-1}$-almost every x since the Besicovitch derivative of $e(u)\mathscr{L}^d$ to $\mathscr{H}^{d-1}\lfloor J_u$ is zero, and the one of α to $\mathscr{H}^{d-1}\lfloor J_u$ is finite $\mathscr{H}^{d-1}$-almost everywhere. Let r_k be some sequence converging to 0. Using the measure convergence of $\mathscr{H}^{d-1}\lfloor J_{u_n}\to\alpha$, $(\phi(u_n^+)+\phi(u_n^-))\mathscr{H}^{d-1}\lfloor J_{u_n}\to\mu$ and of $|e(u_n)|\mathscr{L}^d\to|e(u)|\mathscr{L}^d$ (due to the uniform L^2 bound on $|e(u_n)|$), and the strong L^1 convergence of u_n to u, there exists for each $k\in\mathbb{N}$ some $n_k(\to\infty)$ such that

$$\mathscr{H}^{d-1}(J_{u_{n_k}}\cap(x+r_kQ_1))\leq\alpha(x+r_kQ_1)+r_k^d$$
$$\int_{J_{u_{n_k}}\cap(x+r_kQ_1)}[\phi(u_{n_k}^+)+\phi(u_{n_k}^-)]d\mathscr{H}^{d-1}\leq\mu(x+r_kQ_1)+r_k^d$$
$$\int_{x+r_kQ_1}|e(u_{n_k})|d\mathscr{L}^d\leq\int_{x+r_kQ_1}|e(u)|d\mathscr{L}^d+r_k^d$$
$$\|u-u_{n_k}\|_{L^1(x+r_kQ_1)}\leq r_k^{d+1}$$

Then we let $v_k(y)=u_{n_k}(x+r_ky)\in SBD(Q_1)$, such that the conditions (10.32), (10.33), (10.34) are verified and

$$\frac{\mu(x+r_kQ_1)}{r_k^{d-1}}\geq -r_k+\int_{J_{v_k}\cap Q_1}[\phi(v_k^+)+\phi(v_k^-)]d\mathscr{H}^{d-1}$$

So once we know the inferior limit of the right-hand side is larger than $\phi(u^+(x))+\phi(u^-(x))$, we are done as claimed.

We now divide the proof in several steps, and we will employ the characterization by sections of SBD functions explained in Section 10.2.

Step 1. Let $\epsilon>0$ be given. We fix $\delta>0$ and $N\in\mathbb{N}$ with $N>d$: these numbers will be subject to several constraints that will appear during the proof.

Let us fix N unit vectors $\{\xi_i\}_{1\leq i\leq N}$ such that

$$|e_d\cdot\xi_i-1|<\delta \tag{10.35}$$

and such that any subset of d of them forms a basis of $\mathbb{R}^d$. Moreover, we may assume in addition that

$$(u^+-u^-)\cdot\xi_i\neq 0 \tag{10.36}$$

for every $i=1,\dots,N$.

Thanks to (10.32) and (10.33), we can fix $a>0$ small such that setting $H^\pm=H\times\{\pm a\}$, we have

$$(u_n)_{|H^\pm}\to u^\pm\qquad\text{strongly in } L^1(H^\pm;\mathbb{R}^d)$$

and

$$\forall n \in \mathbb{N} : \mathscr{H}^{d-1}(J_{u_n} \cap H^{\pm}) = 0. \tag{10.37}$$

Step 2. We claim that, up to a subsequence, we can find $H^-_\epsilon \subset H^-$ with

$$\mathscr{H}^{d-1}(H^- \setminus H^-_\epsilon) < \epsilon \tag{10.38}$$

such that for every $i = 1, \dots, N$, for every $y \in H^-_\epsilon$ and for every $n \in \mathbb{N}$

$$H^-_\epsilon \cap J_{u_n} = \emptyset, \tag{10.39}$$

and

$$1 \leq \mathscr{H}^0((J_{u_n})^{\xi_i}_y) < +\infty. \tag{10.40}$$

Moreover setting

$$\widehat{(u_n)}^{\xi_i}_y := u_n(y + t\xi_i) \cdot \xi_i,$$

for every $y \in H^-_\epsilon$ we have

$$\widehat{(u_n)}^{\xi_i}_y \in SBV((Q_1)^{\xi_i}_y), \tag{10.41}$$

$$J_{\widehat{(u_n)}^{\xi_i}_y} = (J_{u_n})^{\xi_i}_y \tag{10.42}$$

$$\|[\widehat{(u_n)}^{\xi_i}_y]'\|_{L^1} \to 0 \qquad \text{uniformly for } y \in H^-_\epsilon, \tag{10.43}$$

and

$$(u_n)_{|H^-} \to u^- \qquad \text{uniformly on } H^-_\epsilon. \tag{10.44}$$

Indeed, if the number δ appearing in (10.35) is small enough, we can find $A^-_\epsilon \subseteq H^-$ with

$$\mathscr{H}^{d-1}(H^- \setminus A^-_\epsilon) < \frac{\epsilon}{2} \tag{10.45}$$

and such that for every $y \in A^-_\epsilon$ the lines $\{y + t\xi_i : t \in \mathbb{R}\}$ intersect H^+ for every $i = 1, \dots, N$. In view of (10.32), (10.33) and (10.34), and since pointwise convergence implies almost uniform convergence, we can find $N_\epsilon \subset A^-_\epsilon$ with

$$\mathscr{H}^{d-1}(N_\epsilon) < \frac{\epsilon}{2} \tag{10.46}$$

and such that, up to a subsequence

$$\|\widehat{(u_n)}^{\xi_i}_y - \widehat{u}^{\xi_i}_y\|_{L^1} \to 0 \qquad \text{uniformly for } y \in A^-_\epsilon \setminus N_\epsilon \tag{10.47}$$

$$\|[\widehat{(u_n)}^{\xi_i}_y]'\|_{L^1} \to 0 \qquad \text{uniformly for } y \in A^-_\epsilon \setminus N_\epsilon \tag{10.48}$$

$$(u_n)_{|H^-} \to u^- \qquad \text{uniformly on } A^-_\epsilon \setminus N_\epsilon, \tag{10.49}$$

and for every $y \in A^-_\epsilon \setminus N_\epsilon$

$$\mathscr{H}^0((J_{u_n})^{\xi_i}_y) < +\infty. \tag{10.50}$$

Notice that for n large enough and for every $y \in A^-_\epsilon \setminus N_\epsilon$ we have

$$(J_{u_n})^{\xi_i}_y \neq \emptyset. \tag{10.51}$$

Indeed otherwise, we would get for $n_k \to +\infty$ the existence of $y_k \in A^-_\epsilon \setminus N_\epsilon$ with $\widehat{(u_{n_k})}^{\xi_i}_{y_k} \in W^{1,1}((Q_1)^{\xi_i}_{y_k})$, and (10.49) together with (10.48) would yield

$$\|\widehat{(u_{n_k})}^{\xi_i}_{y_k} - u^-\|_1 \to 0$$

against (10.47) (recall that by the choice (10.36) of the ξ_i, the functions $\widehat{u}_y^{\xi_i}$ have a jump). The claim follows by setting

$$H_\epsilon^- := A_\epsilon \setminus \left[N_\epsilon \cup \bigcup_n (J_{u_n} \cap H^-) \right].$$

Indeed (10.38) follows from (10.45), (10.46) and (10.37), while (10.39) is clearly satisfied. Relation (10.40) follows by (10.50) and (10.51), while relation (10.43) follows from (10.48). Finally relation (10.44) follows from (10.49).

Step 3. For every $i = 1, \dots, N$, let us consider the set $J_n^{i,-}$ given by the first point of intersection (with $t > 0$) of the line $\{y + t\xi^i : t \in \mathbb{R}\}$ with the jump set J_{u_n} as y varies in the set H_ϵ^- defined in Step 2 (recall (10.40) and (10.42)). In view of (10.43) and (10.44), we can find $\eta_n \to 0$ such that for every $x \in J_n^{i,-}$ with $\nu_{u_n} \cdot \xi_i > 0$

$$|u_n^-(x) \cdot \xi_i - u^- \cdot \xi_i| < \eta_n. \tag{10.52}$$

Step 4. We claim that, for δ small enough and N large enough, up to a subsequence, we can find $\tilde{J}_n^- \subseteq J_{u_n}$ with

$$\mathscr{H}^{d-1}(\tilde{J}_n^-) \geq 1 - c_\epsilon, \tag{10.53}$$

where $c_\epsilon \to 0$ as $\epsilon \to 0$, and such that for every $x \in \tilde{J}_n^-$

$$x \in J_n^{i,-} \text{ for } d \text{ different indices } i \in \{1, \dots, N\}, \tag{10.54}$$

where $J_n^{i,-}$ is defined in Step 3. Moreover, we can orient ν_{u_n} on $\tilde{J}_n^-$ in such a way that

$$e_d \cdot \nu_{u_n} > 0 \qquad \text{and} \qquad \xi_i \cdot \nu_{u_n} > 0 \text{ for every } i = 1, \dots, N. \tag{10.55}$$

Intuitively speaking, the points in $\tilde{J}_n^-$ are seen from H_ϵ^- under d different directions: moreover the associated lines cut the jump transversaly, from the "lower" to the "upper" part.

Indeed, in view of the very definition of ξ_i (which form a very small angle with e_d as $\delta \to 0$) and of the area formula, we can assume that δ is so small that for every $i = 1, \dots, N$

$$\mathscr{H}^{d-1}(J_n^{i,-}) \geq \int_{J_n^{i,-}} |\nu_{u_n} \cdot \xi_i| \, d\mathscr{H}^{d-1} = \mathscr{H}^{d-1}((H_\epsilon^-)^{\xi_i}) = \frac{1}{1 + \hat{c}_\delta} \mathscr{H}^{d-1}(H_\epsilon^-), \tag{10.56}$$

where $\hat{c}_\delta \to 0$, so that, taking into account (10.38), for small δ we have

$$\mathscr{H}^{d-1}(J_n^{i,-}) \geq 1 - 2\epsilon. \tag{10.57}$$

By Lemma 10.21 below (with $X = J_{u_n}$, $\mu = \mathscr{H}^{d-1}$, and $\mathcal{M}$ given by the family of Borel sets) if N is large enough we can find an index $\bar{i}$ such that

$$\mathscr{H}^{d-1}\left(J_n^{\bar{i},-} \setminus \bigcup_{\substack{i_1 < i_2 < \cdots < i_d \\ i_h = 1, \dots, N}} \left(J_n^{i_1,-} \cap J_n^{i_2,-} \cap \cdots \cap J_n^{i_d,-} \right) \right) < \epsilon. \tag{10.58}$$

Intuitively speaking, most of the points in $J_n^{\bar{i},-}$ are seen from H_ϵ^- at least under d different directions: we call this set $\tilde{J}_n^-$, i.e.,

$$\tilde{J}_n^- := J_n^{\bar{i},-} \cap \bigcup_{\substack{i_1 < i_2 < \cdots < i_d \\ i_h = 1, \dots, N}} \left(J_n^{i_1,-} \cap J_n^{i_2,-} \cap \cdots \cap J_n^{i_d,-} \right). \tag{10.59}$$

In view of (10.57) and (10.58) we get

$$\mathscr{H}^{d-1}(\tilde{J}_n^-) \geq 1 - 3\epsilon. \tag{10.60}$$

Finally, if we set

$$G_{n,\epsilon} := \{x \in \tilde{J}_n^- \, : \, |\nu_{u_n} \cdot \xi_{\bar{i}}| > \epsilon\} \qquad \text{and} \qquad B_{n,\epsilon} := \tilde{J}_n^- \setminus G_{n,\epsilon},$$

coming back to (10.56) we have

$$\mathscr{H}^{d-1}(G_{n,\epsilon}) + \epsilon^2 \mathscr{H}^{d-1}(B_{n,\epsilon}) > 1 - 3\epsilon,$$

so that

$$\mathscr{H}^{d-1}(G_{n,\epsilon}) > 1 - 3\epsilon - \epsilon^2 C,$$

where $C := \sup_n \mathscr{H}^{d-1}(J_{u_n}) < +\infty$. Finally we orient the normal ν_{u_n} on $G_{n,\epsilon}$ in such a way that

$$\nu_{u_n} \cdot \xi_{\bar{i}} > \epsilon.$$

The inequalities (10.55) then also hold true on $G_{n,\epsilon}$ if δ is small enough thanks to (10.35). Reducing $\tilde{J}_n^-$ to $G_{n,\epsilon}$ if necessary, the full claim follows taking into account (10.59) and (10.60).

Step 5. Let $\tilde{J}_n^- \subseteq J_{u_n}$ be the set given by Step 4. Since the points of this set are seen from H_ϵ^- under d different directions, in view of (10.52) we infer that there exists $\tilde{\eta}_n \to 0$ such that for every $x \in \tilde{J}_n^-$

$$|u_n^-(x) - u^-| < \tilde{\eta}_n.$$

Reasoning in a similar way starting from the upper part H_ϵ^+, and employing the opposite directions $\{-\xi_i : i = 1, \dots, N\}$, we can construct $\tilde{J}_n^+ \subseteq J_{u_n}$ with ν_{u_n} oriented such that again

$$e_d \cdot \nu_{u_n} > 0 \qquad \text{and} \qquad \xi_i \cdot \nu_{u_n} > 0 \text{ for every } i = 1, \dots, N,$$

such that

$$\mathscr{H}^{d-1}(\tilde{J}_n^+) \geq 1 - c_\epsilon \tag{10.61}$$

with $c_\epsilon \to 0$ as $\epsilon \to 0$, and such that for every $x \in \tilde{J}_n^+$

$$|u_n^+(x) - u^+| < \tilde{\eta}_n.$$

Notice that for $x \in \tilde{J}_n^- \cap \tilde{J}_n^+$, the orientation chosen is compatible with that of (10.55), so that indeed $u_n^-(x)$ and $u_n^+(x)$ are the two traces of u_n at x.

We can thus write, in view of the continuity of ϕ

$$\begin{aligned}
\int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1} &\geq \int_{\tilde{J}_n^+ \cap \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1} + \int_{\tilde{J}_n^+ \Delta \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1} \\
&\geq \int_{\tilde{J}_n^+ \cap \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1} + \int_{\tilde{J}_n^+ \setminus \tilde{J}_n^-} \phi(u_n^+)\, d\mathscr{H}^{d-1} + \int_{\tilde{J}_n^- \setminus \tilde{J}_n^+} \phi(u_n^-)\, d\mathscr{H}^{d-1} \\
&\geq \int_{\tilde{J}_n^+} \phi(u_n^+)\, d\mathscr{H}^{d-1} + \int_{\tilde{J}_n^-} \phi(u_n^-)\, d\mathscr{H}^{d-1} \\
&\geq [\phi(u^+) - \tilde{\eta}_n]\mathscr{H}^{d-1}(\tilde{J}_n^+) + [\phi(u^-) - \tilde{\eta}_n]\mathscr{H}^{d-1}(\tilde{J}_n^-)
\end{aligned}$$

where $\tilde{\eta}_n \to 0$, so that, taking into account (10.53) and (10.61)

$$\liminf_{n \to +\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathscr{H}^{d-1} \geq [\phi(u^+) + \phi(u^-)](1 - 2c_\epsilon).$$

The conclusion follows by letting $\epsilon \to 0$. □

We made use of the following abstract lemma.

Lemma 10.21. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\epsilon > 0$ and $d \geq 2$. Then there exists $N \in \mathbb{N}$ that only depends on $\mu(X), \epsilon, d$ such that if $\{E_i\}_{i=1,\dots,N}$ is a family of sets in $\mathcal{M}$, we can find $\bar{i}$ such that*

$$\mu\left(E_{\bar{i}} \setminus \bigcup_{j_1 < j_2 < \cdots < j_d} (E_{j_1} \cap E_{j_2} \cap \cdots \cap E_{j_d})\right) < \epsilon.$$

Proof. Up to dividing ϵ by $\mu(X)$ we suppose without loss of generality that $\mu(X) = 1$. It is enough to prove that for any $d \geq 2, \epsilon > 0$, there is some $N(d, \epsilon)$ such that any family of $N \geq N(d, \epsilon) \in \mathbb{N}^*$ of sets $(E_i)_{1 \leq i \leq N}$ there is some i that verifies

$$\mu\left(E_i \setminus \bigcup_{J \subset [1,N] \setminus \{i\}, |J| = d-1} \bigcap_{j \in J} E_j\right) < \epsilon.$$

meaning that there is some i such that every point of E_i outside a set of measure ϵ is in (at least) $d-1$ other sets E_j (for $j \neq i$). We prove it by recursion. If $d = 2$, let $N = \left\lceil \frac{1}{\epsilon} \right\rceil$ and sets $(E_i)_{1 \leq i \leq N}$, then consider the sets $\left(E_i \setminus \bigcup_{1 \leq j \leq N, j \neq i} E_j\right)_{1 \leq i \leq N}$. These are disjoint and $\mu(X) = 1$ so there is some i such that

$$\mu\left(E_i \setminus \bigcup_{1 \leq j \leq N, j \neq i} E_j\right) \leq \frac{1}{N} \leq \epsilon,$$

which proves the initialisation. Let now $d \geq 2$ and

$$N := N\left(d, \frac{\epsilon}{2}\right), \ M := \left\lceil \frac{2}{\epsilon} \right\rceil,$$

consider $N \times M$ sets that we classify into N groups of M sets, written $(E_{k,i})_{1 \leq k \leq N, 1 \leq i \leq M}$. For every $k \in [1, N]$, the sets $\left(E_{k,i} \setminus \bigcup_{1 \leq j \leq M, j \neq i} E_{k,j}\right)_{1 \leq i \leq M}$ are disjoints so there is some i_k such that

$$\mu\left(E_{k,i_k} \setminus \bigcup_{1 \leq i \leq M, i \neq i_k} E_{k,i}\right) \leq \frac{1}{M} \leq \frac{\epsilon}{2}.$$

Considering the sets $(E_{k,i_k})_{1 \leq k \leq N}$, since $N = N\left(d, \frac{\epsilon}{2}\right)$ we find some $\overline{k}$ such that

$$\mu\left(E_{\overline{k},i_{\overline{k}}} \setminus \bigcup_{K \subset [1,N] \setminus \{\overline{k}\}, |K| = d-1} \bigcap_{k \in K} E_{k,i_k}\right) \leq \frac{\epsilon}{2},$$

so outside a set of measure at most $\frac{\epsilon}{2}$, every point of $E_{\overline{k},i_{\overline{k}}}$ is in $d-1$ sets of the form E_{k,i_k} for $k \neq \overline{k}$. Similarly every point outside a set of measure at most $\frac{\epsilon}{2}$ is also in one set of the form $E_{\overline{k},i}$ for some $i \neq i_{\overline{k}}$. We conclude that outside of measure at most ϵ, every point of $E_{\overline{k},i_{\overline{k}}}$ is in d other sets, meaning $N(d+1, \epsilon)$ is well-defined and $N(d+1, \epsilon) \leq N\left(d, \frac{\epsilon}{2}\right)\left\lceil \frac{2}{\epsilon} \right\rceil$. □

As a consequence, we obtain the lower semi-continuity of our energy defined on set-functions pairs (10.26).

10.6 Proof of Theorem 10.16

We are now in a position to prove the first main result of the paper.

Proof of Theorem 10.16. Let $(E_n, u_n)_{n \in \mathbb{N}}$ be a minimizing sequence: since the function f is not identically equal to $+\infty$, and in view of Remark 10.15, there exists $C > 0$ such that

$$\mathcal{J}(E_n, u_n) \leq C.$$

Since $u_n = 0$ a.e. on E_n we may write

$$\int_{\partial^* E_n} |u_n^+|^2 \, d\mathscr{H}^{d-1} + \int_{J_{u_n} \setminus \partial^* E_n} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^{d-1} = \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^{d-1}$$

so that we infer

$$\mathscr{H}^{d-1}(\partial^* E_n) \le C \qquad \text{and} \qquad \int_\Omega |e(u_n)|^2 \, d\mathscr{L}^d + \mathscr{H}^{d-1}(J_{u_n}) + \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^{d-1} \le C.$$

Notice that

$$\begin{aligned}
|E(u_n)|(\Omega') &= \int_{\Omega'} |e(u_n)| \, d\mathscr{L}^d + \int_{J_{u_n}} |u_n^+ - u_n^-| \, d\mathscr{H}^{d-1} \\
&\le \int_{\Omega' \setminus \Omega} |e(V)| \, d\mathscr{L}^d + \int_\Omega |e(u_n)| \, d\mathscr{L}^d + \int_{J_{u_n}} [|u_n^+| + |u_n^-|] \, d\mathscr{H}^{d-1} \\
&\le \int_{\Omega' \setminus \Omega} |e(V)| \, d\mathscr{L}^d + \frac{1}{2} \left[|\Omega| + \int_\Omega |e(u_n)|^2 \, d\mathscr{L}^d + \mathscr{H}^{d-1}(J_{u_n}) + \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^{d-1} \right] \\
&\le \tilde{C},
\end{aligned}$$

for some $\tilde{C} > 0$. Moreover, thanks to Theorem 10.17 applied to $u - V$ we may assume also that

$$\|u_n\|_{L^{\frac{2d}{d-1}}(\Omega')} \le \tilde{C}. \tag{10.62}$$

By the compactness result in SBD (see Theorem 10.4), there exist a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and $u \in SBD(\Omega')$ with $u = V$ on $\Omega' \setminus \Omega$ and such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Omega'; \mathbb{R}^d), \tag{10.63}$$

$$e(u_{n_k}) \rightharpoonup e(u) \qquad \text{weakly in } L^2(\Omega'; \mathbb{R}^{d \times d}_{sym}), \tag{10.64}$$

and

$$\mathscr{H}^{d-1}(J_u) \le \liminf_{k \to +\infty} \mathscr{H}^{d-1}(J_{u_{n_k}}). \tag{10.65}$$

Concerning the sets E_{n_k}, we may assume, up to a further subsequence if necessary, that there exists a set of fine perimeter $E \subseteq \Omega$ such that

$$1_{E_{n_k}} \to 1_E \qquad \text{strongly in } L^1(\mathbb{R}^d) \tag{10.66}$$

with

$$\mathscr{H}^{d-1}(\partial^* E) \le \liminf_{k \to +\infty} \mathscr{H}^{d-1}(\partial^* E_{n_k}).$$

In particular we get

$$f(|E|) \le \liminf_{n \to +\infty} f(|E_n|). \tag{10.67}$$

Let us prove that

$$u \in \mathcal{V}_{E,V}(\Omega). \tag{10.68}$$

Indeed, in view of (10.62) we infer that $u \in L^{\frac{2d}{d-1}}(\Omega'; \mathbb{R}^d)$ so that in particular $u \in L^2(\Omega'; \mathbb{R}^d)$. Moreover $u = V$ on $\Omega' \setminus \Omega$, while $u = 0$ a.e. on E thanks to (10.63) and (10.66).

Since the divergence constraint is intended in the sense of distributions on Ω, this passes easily to the limit thanks to (10.63). Moreover, in view of Theorem 10.19 we deduce

$$u^\pm \perp \nu_u \qquad \text{on } J_u.$$

In particular this entails

$$u^+ \perp \nu_E \qquad \text{on } \partial^* E \cap \Omega,$$

since for $x \in \partial^* E$ we have either $x \in J_u$ or $u^+(x) = 0$. We conclude that the tangential constraint for the velocity field holds on $\partial^* E$ and on $J_u \setminus \partial^* E$. We conclude that (10.68) holds true.

Let us prove the pair (E, u) is a minimizer for the problem. Thanks to (10.64) we get

$$\int_\Omega |e(u)|^2 \, d\mathscr{L}^d \le \liminf_{k \to +\infty} \int_\Omega |e(u_{n_k})|^2 \, d\mathscr{L}^d,$$

while in view of Theorem 10.20 we have that

$$\int_{J_u} [|u^+|^2 + |u^-|^2]\, d\mathscr{H}^{d-1} \le \liminf_{k\to+\infty} \int_{J_{u_{n_k}}} [|u^+_{n_k}|^2 + |u^-_{n_k}|^2]\, d\mathscr{H}^{d-1},$$

which entails

$$\begin{aligned} &\int_{\partial^* E} |u^+|^2\, d\mathscr{H}^{d-1} + \int_{J_u\setminus\partial^* E} [|u^+|^2 + |u^-|^2]\, d\mathscr{H}^{d-1} \\ &\le \liminf_{k\to+\infty} \left[\int_{\partial^* E_{n_k}} [u^+_{n_k}|^2\, d\mathscr{H}^{d-1} + \int_{J_{u_{n_k}}\setminus\partial^* E_{n_k}} [|u^+_{n_k}|^2 + |u^-_{n_k}|^2]\, d\mathscr{H}^{d-1} \right] \end{aligned} \tag{10.69}$$

since $u = 0$ a.e. on E and $u_{n_k} = 0$ a.e. on E_{n_k}.

Let us prove that

$$2\mathscr{H}^{d-1}(J_u \setminus \partial^* E) + \mathscr{H}^{d-1}(\partial^* E) \le \liminf_{i\to\infty} \left(2\mathscr{H}^{d-1}(J_{u_i} \setminus \partial^* E_i) + \mathscr{H}^{d-1}(\partial^* E_i) \right). \tag{10.70}$$

We claim that that we can find $h \in \mathbb{R}^d$ such that setting

$$v := u + h1_E \qquad \text{and} \qquad v_{n_k} = u_{n_k} + h1_{E_{n_k}}$$

we have

$$J_v = J_u \cup J_{1_E} = \partial^* E \cup J_u \qquad \text{and} \qquad J_{v_{n_k}} = J_{u_{n_k}} \cup J_{1_{E_{n_k}}} = \partial^* E_{n_k} \cup J_{u_{n_k}} \tag{10.71}$$

up to $\mathscr{H}^{d-1}$-negligible sets. The first relation follows from the fact that the sets

$$\{x \in J_u \cup \partial^* E \,:\, u^+(x) = -h \text{ or } u^-(x) = -h\}$$

are disjoint as h varies, so that, except for a countable family of possible vectors, they all have zero $\mathscr{H}^{d-1}$-measure. This yields $v^+ - v^- \neq 0$ $\mathscr{H}^{d-1}$-a.e. on $J_u \cup \partial^* E$ which gives the first relation in (10.71), since $J_v \subseteq J_u \cup \partial^* E$. The second follows from a similar argument, so that the claim follows since in the choice of the vector h we simply need to avoid a countable family of vectors.

Then the previous lower semicontinuity result applied to (v_{n_k}) with $\phi(v) = 1_{v\neq 0}$ gives

$$\int_{J_{\tilde u}} [\phi(\tilde u^+) + \phi(\tilde u^-)] d\mathscr{H}^{d-1} \le \liminf_{i\to\infty} \int_{J_{\tilde u_i}} [\phi(\tilde u_i^+) + \phi(\tilde u_i^-)] d\mathscr{H}^{d-1}$$

so that (10.70) follows. In view of (10.64), (10.69), (10.67) and (10.70), we deduce

$$\mathcal{J}(E,u) \le \liminf_{k\to+\infty} \mathcal{J}(E_{n_k}, u_{n_k})$$

so that, taking into account (10.68), the pair (E,u) is a minimizer of the main problem (10.27). □

10.7 The two dimensional case

In this section we deal with a different formulation of the optimization problem in a two dimensional context, in which the obstacles are topologically closed: some arguments in the proof of the existence of optimal obstacles turn out to be greatly simplified.

10.7.1 A two dimensional setting

Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and with a Lipschitz boundary, and let us consider $\Omega' \subseteq \mathbb{R}^2$ open such that $\Omega \Subset \Omega'$.

Definition 10.22 (Admissible obstacles). We say that $K \subseteq \overline{\Omega}$ is an admissible obstacle, and we write $K \in \mathcal{K}(\Omega)$, if K is compact and connected, $\mathbb{R}^2 \setminus K$ is connected and $\mathscr{H}^1(\partial K) < +\infty$.

Remark 10.23. The connectedness of $\Omega \setminus K$ implies that K has no cavities, which are indeed not relevant for the drag problem. Notice that in general $\overline{int(K)} \subset K$, so that ∂K may contain thin parts which depart from the "main bulk" of the obstacle.

Remark 10.24. If $K \in \mathcal{K}(\Omega)$, we have that ∂K is compact and connected with finite $\mathscr{H}^1$-measure. In particular thanks to point (a) of Theorem 10.8 we have that ∂K is $\mathscr{H}^1$-countably rectifiable, so that an approximate tangent line and an approximate normal are well defined at $\mathscr{H}^1$-a.e. $x \in \partial K$.

Let $V \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ be a divergence free vector field.

Definition 10.25 (Admissible velocities). Let $K \in \mathcal{K}(\Omega)$. We say that $u \in L^2(\Omega';\mathbb{R}^2)$ is an admissible velocity for K and V, and we write $u \in \mathcal{V}^{reg}_{K,V}(\Omega)$, if the following conditions are satisfied.

(a) Boundary condition: $u = V$ a.e. on $\Omega' \setminus \overline{\Omega}$.

(b) Existence of a stream function: There exists $\psi \in H^1(\Omega')$ with $\psi = 0$ c_2-q.e. on K and such that
$$u = \nabla^\perp \psi,$$
where $\nabla^\perp \psi$ denotes the rotation of 90 degrees counterclockwise of $\nabla\psi$, and c_2 denotes the two-capacity measure.

(c) Regularity: we have $u \in SBD(\Omega')$ with $\mathscr{H}^1(J_u \setminus K) = 0$ and
$$\int_{\Omega'\setminus K} |e(u)|^2 \, d\mathscr{L}^d + \int_{\partial K} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^1 < +\infty. \tag{10.72}$$

Remark 10.26. The function ψ is a stream function describing the associated admissible velocity field. If K has a regular boundary and u is sufficiently smooth on $\Omega' \setminus K$, we get that u is divergence free in Ω' and tangent to ∂K. In particular, ∂K may touch $\partial\Omega$, but on those parts where V is tangent to $\partial\Omega$.

The following result collects the main properties of the family of admissible velocities.

Proposition 10.27 (Properties of admissible velocities). *Let $K \in \mathcal{K}(\Omega)$ and $u \in \mathcal{V}^{reg}_{K,V}(\Omega)$. The following items hold true.*

(a) Divergence condition*: we have* $\mathrm{div} u = 0$ *in* $\mathcal{D}'(\Omega')$.

(b) Slip condition on the obstacle*: for $\mathscr{H}^1$-a.e. $x \in \partial K$ we have*
$$u^\pm(x) \perp \nu_{\partial K}(x),$$
where $\nu_{\partial K}$ denotes the approximate normal to the rectifiable set ∂K, while $u^\pm(x)$ denote the one sided Lebesgue values of u at x with respect to ∂K.

Proof. Point (a) follows immediately from the equality $u = \nabla^\perp\psi$, where ψ is the stream function for u. Let us come to point (b). We proceed through a blow-up argument. Let us fix $x \in \partial K$ such that there exist the approximate normal ν and the one sided Lebesgue limits $u^\pm$: up to a translation and a rotation, we may assume $x = 0$ and $\nu = e_2$.

Since ∂K is connected, from point (b) of Theorem 10.8 we have that

$$\frac{1}{r}\partial K \to l \qquad \text{locally in the Hausdorff metric as } r \to 0^+, \tag{10.73}$$

where l denotes the horizontal axis.

Let us fix $\epsilon > 0$. For r small enough we get

$$\Gamma_r := \frac{1}{r}\partial K \cap \overline{Q} \subseteq \overline{Q} \cap S_\epsilon,$$

where S_ϵ is the strip $\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < \epsilon\}$ and $Q := [-1, 1]^2$. Let us consider the two vertical segments

$$H^\epsilon_{-1} := \{(-1, x_2) : |x_2| < \epsilon\} \qquad \text{and} \qquad H^\epsilon_1 := \{(1, x_2) : |x_2| < \epsilon\}.$$

In view of (10.73) and since ∂K is arcwise connected (since it is connected with finite length), we get that for r small enough every point in Γ_r can be connected by means of a continuous curve contained in Γ_r either to a point of H^ϵ_{-1} or to a point of H^ϵ_{-1}. This implies that if we denote by I_r the set of abscissa x_1 such that the vertical line through $(x_1, 0)$ does not intersect Γ_r, we have that I_r is an interval and

$$|I_r| \to 0. \tag{10.74}$$

In view of the properties of the one sided Lebesgue values, setting

$$u_r(x) := u(rx),$$

we have

$$u_r \to u^+ 1_{Q^+} + u^- 1_{Q^-} \qquad \text{strongly in } L^1(Q, \mathbb{R}^2) \text{ as } r \to 0^+,$$

where $Q^\pm := \{(x_1, x_2) \in Q : x_2 \gtrless 0\}$.

Let $r_n \to 0$, and let us write Γ_n, I_n and u_n for Γ_{r_n}, I_{r_n} and u_{r_n}. If we consider the continuous functions

$$\varphi_n(x) := \frac{1}{r_n}\psi(r_n x),$$

since $\nabla\varphi_n(x) = \nabla\psi(r_n x)$, we get that

$$\nabla\varphi_n \to Ru^+ 1_{Q^+} + Ru^- 1_{Q^-} \qquad \text{strongly in } L^1(Q; \mathbb{R}^2), \tag{10.75}$$

where R denotes a rotation of 90 degrees clockwise.

By integrating along the vertical lines, and recalling that $\varphi_n = 0$ on Γ_n we obtain for a.e. $x = (x_1, x_2) \in Q$ with $x_1 \notin I_n$

$$|\varphi_n(x_1, x_2)| \le \int_{-1}^{1} |\partial_2\varphi_n(x_1, s)|\, ds,$$

so that

$$\int_{I_n^c \times]-1,1[} |\varphi_n(x)|\, d\mathscr{L}^2 \le \int_Q |\nabla\varphi_n|\, d\mathscr{L}^2. \tag{10.76}$$

For a.e. $x = (x_1, x_2) \in Q$ with $x_1 \in I_n$, by integrating horizontally we have that for a.e. $\xi \notin I_n$

$$|\varphi_n(x_1, x_2)| \le |\varphi_n(\xi, x_2)| + \int_{-1}^{1} |\partial_1\varphi_n(s, x_2)|\, ds,$$

so that by integrating in $x_1 \in I_n$, $\xi \in]-1, 1[\setminus I_n$, and since $\mathscr{H}^1(I_n) \to 0$, we get

$$\int_{I_n \times]-1,1[} |\varphi_n|\, d\mathscr{L}^2 \le \int_{I_n^c \times]-1,1[} |\varphi_n| d\mathscr{L}^2 + 2\int_Q |\nabla\varphi_n|\, d\mathscr{L}^2 \le 3\int_Q |\nabla\varphi_n|\, d\mathscr{L}^2. \tag{10.77}$$

In view of (10.76), (10.77) and (10.75), we get that φ_n is bounded in $W^{1,1}(Q)$, and we already know $\nabla\varphi_n = Ru_n \underset{L^1(Q)}{\longrightarrow} Ru^+1_{Q^+} + Ru^-1_{u^-}$. Thus up to a subsequence there is some $\varphi \in L^1(Q)$ such that $\varphi_n \to \varphi$ in $W^{1,1}(Q)$ with

$$\nabla\varphi = Ru^+1_{Q^+} + Ru^-1_{Q^-}. \tag{10.78}$$

Notice that φ is Lipschitz continuous, and it vanishes on the horizontal axis. Indeed, since

$$\Gamma_n \to l \cap \overline{Q} \qquad \text{in the Hausdorff metric,}$$

reasoning as above through integrations on vertical lines we get for a.e. $x_1 \notin I_n$,

$$|\varphi_n(x_1,0)| \le \int_{-\epsilon}^{\epsilon} |\partial_2\varphi_n(x_1,s)|\,ds.$$

and then by horizontal integration to reach $(x_1,0)$ for $x_1 \in I_n$ as previously,

$$\int_{-1}^{1} |\varphi_n(x_1,0)|dx_1 \le C \int_{]-1,1[\times]-\epsilon,\epsilon[} |\nabla\varphi_n|$$

which converges to 0 as $\epsilon \to 0$ uniformly in n by uniform integrability of $(\nabla\varphi_n)_n$. As a consequence, and by continuity of the trace $W^{1,1}(Q) \hookrightarrow L^1(l)$, we get that φ is zero on l. It is also Lipschitz so we get directly that Ru^+, Ru^- must be in the direction of $\pm e_2$, which is the result. □

The following lower semicontinuity result concerns the surface term in the main problem associated to the Navier conditions on the obstacle.

Theorem 10.28 (**Lower semicontinuity of Navier surface energies**). *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of compact and connected sets in Ω with $\mathscr{H}^1(\Gamma_n) \le C$ and*

$$\Gamma_n \to \Gamma \qquad \textit{in the Hausdorff metric}$$

for some $\Gamma \subset \Omega$. Let $u_n \in SBD(\Omega)$ be such that $J_{u_n} \subseteq \Gamma_n$, $u_n^\pm \perp \nu_{\Gamma_n}$ $\mathscr{H}^1$-a.e. on Γ_n,

$$\sup_n \left[\int_\Omega |e(u_n)|^2\,d\mathscr{L}^d + \int_{\Gamma_n} [|u_n^+|^2 + |u_n^-|^2]\,d\mathscr{H}^1\right] < +\infty$$

and

$$u_n \to u \qquad \textit{strongly in } L^1(\Omega;\mathbb{R}^2)$$

for some $u \in SBD(\Omega)$ with $J_u \subseteq \Gamma$ and $u^\pm \perp \nu_\Gamma$ $\mathscr{H}^1$-a.e. on Γ. Then we have

$$\int_\Gamma [|u^+|^2 + |u^-|^2]\,d\mathscr{H}^1 \le \liminf_{n\to+\infty} \int_{\Gamma_n} [|u_n^+|^2 + |u_n^-|^2]\,d\mathscr{H}^1. \tag{10.79}$$

Proof. Fix a unit vector $\xi \in \mathbb{R}^2$. Thanks to Lemma 10.9 we know that

$$(\Gamma_n)_y^\xi \to (\Gamma)_y^\xi \qquad \text{in the Hausdorff metric} \tag{10.80}$$

for every $y \in \xi^\perp$ except a countable set.

Working by sections, for every $\epsilon > 0$ we have that

$$\begin{aligned}
&\int_{\Omega^\xi} \liminf_{n\to+\infty} \left[\epsilon \int_{\Omega_y^\xi} |(\widehat{u_n}_y^\xi)'|^2\,dt + \epsilon\mathscr{H}^0((\Gamma_n)_y^\xi) + \int_{(\Gamma_n)_y^\xi} [|(\widehat{u_n}_y^\xi)^+|^2 + |(\widehat{u_n}_y^\xi)^-|^2]\,d\mathscr{H}^1\right] \\
&\le \liminf_{n\to+\infty} \int_{\Omega^\xi} \left[\epsilon \int_{\Omega_y^\xi} |(\widehat{u_n}_y^\xi)'|^2\,dt + \epsilon\mathscr{H}^0((\Gamma_n)_y^\xi) + \int_{(\Gamma_n)_y^\xi} [|(\widehat{u_n}_y^\xi)^+|^2 + |(\widehat{u_n}_y^\xi)^-|^2]\,d\mathscr{H}^1\right] \\
&= \liminf_{n\to+\infty} \left[\epsilon \int_\Omega |e(u_n)\xi\cdot\xi|^2\,d\mathscr{L}^d + \epsilon \int_{\Gamma_n} |\xi\cdot\nu|\,d\mathscr{H}^1 + \int_{\Gamma_n} [|u_n^+\cdot\xi|^2 + |u_n^-\cdot\xi|^2]|\xi\cdot\nu_n|\,d\mathscr{H}^1\right] \\
&\le \liminf_{n\to+\infty} \left[\epsilon \int_\Omega |e(u_n)|^2\,d\mathscr{L}^d + \epsilon\mathscr{H}^1(\Gamma_n) + \int_{\Gamma_n} [|u_n^+|^2 + |u_n^-|^2]\,d\mathscr{H}^1\right] \le C,
\end{aligned}$$

for some $C > 0$ independent of n. Let $A \subseteq \Omega$ be an open set. Taking into account (10.80), by lower semicontinuity in one dimension we get for a.e. y

$$\liminf_{n\to+\infty} \left[\epsilon \int_{A_y^\xi} |(\widehat{u_n}_y^\xi)'|^2 \, dt + \epsilon \mathscr{H}^0(A_y^\xi \cap (\Gamma_n)_y^\xi) + \int_{A_y^\xi \cap (\Gamma_n)_y^\xi} [|(\widehat{u_n}_y^\xi)^+|^2 + |(\widehat{u_n}_y^\xi)^-|^2] \, d\mathscr{H}^1 \right]$$
$$\geq \epsilon \int_{A_y^\xi} |(\widehat{u}_y^\xi)'|^2 \, dt + \epsilon \mathscr{H}^0(A_y^\xi \cap \Gamma_y^\xi) + \int_{A_y^\xi \cap \Gamma_y^\xi} [|(\widehat{u}_y^\xi)^+|^2 + |(\widehat{u}_y^\xi)^-|^2] \, d\mathscr{H}^1.$$

By integrating in y and letting $\epsilon \to 0$ we deduce

$$\int_{A\cap\Gamma} [|u^+ \cdot \xi|^2 + |u^- \cdot \xi|^2] |\xi \cdot \nu_\Gamma| \, d\mathscr{H}^1 \leq \liminf_{n\to+\infty} \int_{A\cap\Gamma_n} [|u_n^+ \cdot \xi|^2 + |u_n^- \cdot \xi|^2] |\xi \cdot \nu_{\Gamma_n}| \, d\mathscr{H}^1.$$

In view of the tangency condition for u on Γ and u_n on Γ_n, and since we are in dimension two, we may write

$$\int_{A\cap\Gamma} [|u^+|^2 + |u^-|^2](1 - |\xi \cdot \nu_\Gamma|^2) |\xi \cdot \nu_\Gamma| \, d\mathscr{H}^1$$
$$\leq \liminf_{n\to+\infty} \int_{A\cap\Gamma_n} [|u_n^+|^2 |u_n^-|^2](1 - |\xi \cdot \nu_{\Gamma_n}|^2) |\xi \cdot \nu_{\Gamma_n}| \, d\mathscr{H}^1$$
$$\leq C_1 \liminf_{n\to+\infty} \int_{A\cap\Gamma_n} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^1,$$

where

$$C_1 := \max_{|\xi|=1} \xi_1^2 |\xi_2|.$$

We can now employ a covering argument optimizing with respect to the direction ξ to obtain the factor C_1 also on the left hand side, so that (10.79) follows. □

Remark 10.29. The lower semicontinuity result of Theorem 10.20 covers the case of general surface energies ϕ. The simpler arguments of the previous proof take advantage of the choice $\phi(u) = |u|^2$ (indeed a homogeneous polinomial is admissible), of the tangency condition and of the two dimensional context (to cope with the jacobians arising by working with sections). Notice finally that the integration is carried out on a set Γ which is in principle larger than the jump set J_u.

10.7.2 The optimization problem and existence of solutions

We are now in a position to propose a reformulation of the optimization drag problem in this new two dimensional context. Given an obstacle $K \in \mathcal{K}(\Omega)$ and a velocity field $V \in \mathcal{V}_{K,V}^{reg}(\Omega)$, we can rephrase the energy (10.26) as

$$\mathcal{J}(K,u) := \int_{\Omega\setminus K} |e(u)|^2 \, d\mathscr{L}^d + \int_{\partial K} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^1 + \mathscr{H}^1(\partial K) + f(|K|),$$

where $f : [0, |\Omega|] \to [0, +\infty]$ is a lower semicontinuous *non increasing* function not identically equal to $+\infty$.

Remark 10.30. Let $K \in \mathcal{K}(\Omega)$ be the closure of an open set with Lipschitz boundary well contained in Ω. Then we can find $W \in H^1(\Omega \setminus K; \mathbb{R}^2)$ such that $W = V$ on $\partial\Omega$, $W = 0$ on ∂K and $\mathrm{div} W = 0$. Indeed if $\varphi \in C^\infty(\mathbb{R}^2)$ is such that $\varphi = 1$ on a neighborhood of $\mathbb{R}^2 \setminus \Omega$ and $\varphi = 0$ on a neighborhood of K, we can consider the vector field $V_1 := \varphi V$, whose divergence has zero mean on $\Omega \setminus K$ (by Gauss theorem). Then we can find $V_2 \in H_0^1(\Omega \setminus K; \mathbb{R}^d)$ such that $\mathrm{div} V = \mathrm{div} V_1$ (see [177, Theorem IV.3.1]), so that the field $W := V_1 - V_2$ is an admissible choice with $\mathcal{J}(K, W) < +\infty$. In particular the class $\mathcal{V}(K, V)$ is not empty.

We can now state the main result of the section.

Theorem 10.31 (Existence of optimal obstacles). *Suppose f is non-increasing, the minimization problem*

$$\min_{K\in\mathcal{K}(\Omega),u\in\mathcal{V}^{reg}_{K,V}(\Omega)} \mathcal{J}(K,u) \tag{10.81}$$

admits a solution.

Remark 10.32. *When V is a constant, the hypothesis that f is non-decreasing may be replaced by the hypothesis that Ω is star-like. Indeed, in the proof we obtain at some point a compact set H that is a Hausdorff limit of sets $K_n \in \mathcal{K}(\Omega)$, with $|K_n| \to |H|$, however $\mathbb{R}^2 \setminus H$ might not be connected and we instead define K to be the complementary of the unbounded connected component of $\mathbb{R}^2 \setminus H$, meaning $|K| \geq |H|$. Then if Ω is star-like we may define $u_t(x) = u(x/t)$ for an appropriately chosen $t \in (0,1)$ such that $|tK| = |H|$, extending u by the constant V outside Ω, and (tK, u_t) will be the minimizer.*

Proof. Let $(K_n, u_n)_{n\in\mathbb{N}}$ be a minimizing sequence: in view of Remark 10.30 and since the function f is not identically equal to $+\infty$, there exists $C > 0$ such that

$$\mathcal{J}(K_n, u_n) \leq C.$$

Up to a subsequence we may assume that

$$K_n \to H \qquad \text{in the Hausdorff metric} \tag{10.82}$$

and

$$\Gamma_n := \partial K_n \to \Gamma \qquad \text{in the Hausdorff metric}$$

for some compact and connected sets $H, \Gamma \subseteq \overline{\Omega}$. Notice that $\partial H \subseteq \Gamma$, so that thanks to Gołąb Theorem 10.7 we infer

$$\mathscr{H}^1(\partial H) \leq \mathscr{H}^1(\Gamma) \leq \liminf_{n\to+\infty} \mathscr{H}^1(\partial K_n) \leq C. \tag{10.83}$$

We have

$$|K_n| \to |H|. \tag{10.84}$$

Indeed, if $A \Subset \Omega' \setminus H$ is open, then $A \Subset \Omega' \setminus K_n$ for n large, which yields

$$\limsup_n |K_n| \leq |H|.$$

On the other hand, if $A \Subset int(H)$, then A is at a positive distance from Γ, hence also from ∂K_n for n large. We infer that $A \Subset int(K_n)$ for n large so that (as $|\Gamma| = 0$)

$$|H| = |int(H)| \leq \liminf_{n\to+\infty} |K_n|,$$

so that we recover (10.84). In particular we get

$$f(|H|) \leq \liminf_{n\to+\infty} f(|K_n|). \tag{10.85}$$

Concerning the velocities $u_n \in SBD(\Omega')$, since $J_{u_n} \subseteq \partial K_n$, we get

$$\begin{aligned} |E(u_n)|(\Omega') &= \int_{\Omega'} |e(u_n)|\, d\mathscr{L}^d + \int_{J_{u_n}} |u_n^+ - u_n^-|\, d\mathscr{H}^1 \leq \int_{\Omega'} |e(u_n)|\, d\mathscr{L}^d + \int_{\partial K_n} |u_n^+ - u_n^-|\, d\mathscr{H}^1 \\ &\leq |\Omega'| + \int_{\Omega'} |e(u_n)|^2\, d\mathscr{L}^d + \mathscr{H}^{d-1}(\partial K_n) + \int_{\partial K_n} [|u_n^+|^2 + |u_n^-|^2]\, d\mathscr{H}^{d-1} \leq \tilde{C}. \end{aligned}$$

Moreover thanks to Theorem 10.17 we may assume also

$$\|u_n\|_{L^4(\Omega')} \leq \tilde{C}. \tag{10.86}$$

By compactness in SBD (see Theorem 10.4), up to a subsequence, we have

$$u_n \to v \qquad \text{strongly in } L^1(\Omega';\mathbb{R}^2) \tag{10.87}$$

for some $v \in SBD(\Omega') \cap L^4(\Omega';\mathbb{R}^2)$. Clearly $v = V$ a.e. on $\Omega' \setminus \overline{\Omega}$, and in particular $v \in L^2(\Omega';\mathbb{R}^2)$.

Thanks to the Hausdorff convergence (10.82) we get that

$$\int_{\Omega'\setminus H} |e(v)|^2 \, d\mathscr{L}^d \le \liminf_{n\to+\infty} \int_{\Omega'\setminus K_n} |e(v_n)|^2 \, d\mathscr{L}^d. \tag{10.88}$$

Indeed, since $(e(u_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega', M^2_{sym})$, we may assume

$$e(u_n) \rightharpoonup W \qquad \text{weakly in } L^2(\Omega', M^2_{sym}).$$

If $A \Subset \Omega' \setminus H$, we get that $A \Subset \Omega' \setminus K_n$ for n large, and from (10.87) we get

$$W = e(v) \qquad \text{on } A.$$

Letting A invade $\Omega' \setminus H$, inequality (10.88) easily follows.

Let ψ_n be the stream function associated to u_n. Thanks to (10.86), we infer that $\nabla\psi_n$ is uniformly bounded in $L^4(\Omega';\mathbb{R}^2)$. Since Ω' has a Lipschitz boundary, by setting

$$\tilde{\psi}_n := \psi_n - \frac{1}{|\Omega'|}\int_{\Omega'} \psi_n \, d\mathscr{L}^d,$$

we get $(\tilde{\psi}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,4}(\Omega')$. Up to a subsequence we may assume that

$$\tilde{\psi}_n \rightharpoonup \tilde{\psi} \qquad \text{weakly in } W^{1,4}(\Omega')$$

for some $\tilde{\psi} \in W^{1,4}(\Omega')$. By Sobolev embedding, we get that the functions are continuous and that the convergence is also uniform: this yields in particular that $\tilde{\psi}$ is equal to a constant $c_{\tilde{\psi}}$ on H and on Γ. Passing to the limit in $u_n = \nabla^\perp\tilde{\psi}_n$, we deduce that

$$v = \nabla^\perp\tilde{\psi}.$$

Finally from the lower semicontinuity result of Theorem 10.28 we finally deduce

$$\int_\Gamma [|v^+|^2 + |v^-|^2] \, d\mathscr{H}^1 \le \liminf_{n\to+\infty} \int_{\Gamma_n} [|u_n^+|^2 + |u_n^-|^2] \, d\mathscr{H}^1. \tag{10.89}$$

Let us define

$$K := (A_{H^c})^c,$$

where A_{H^c} is the unbounded connected component of H^c. Notice that $K \in \mathcal{K}(\Omega)$ with $\partial K \subseteq \Gamma$. Intuitively, K is obtained form H by adding the possible (open) cavities created by the Hausdorff convergence of K_n to H.

Let us consider the stream function $\psi \in H^1(\Omega')$ such that $\psi = \tilde{\psi}$ on $\Omega' \setminus K$ and $\psi = c_{\tilde{\psi}}$ on K. The function ψ differs from $\tilde{\psi}$ on the open inner cavities of H, where it is set equal to a constant. Let us set

$$u := \nabla^\perp\psi \in \mathcal{V}^{reg}_{K,V}(\Omega).$$

Notice that

$$u = v \qquad \text{on } \Omega \setminus K$$

while $v = 0$ on $K \setminus H$. Since $\partial K \subseteq \Gamma$,

$$\int_{\Omega\setminus K} |e(u)|^2 \, d\mathscr{L}^d \le \int_{\Omega\setminus H} |e(v)|^2 \, d\mathscr{L}^d \quad \text{and} \quad \int_{\partial K} [|u^+|^2 + |u^-|^2] \, d\mathscr{H}^1 \le \int_\Gamma [|v^+|^2 + |v^-|^2] \, d\mathscr{H}^1,$$

taking into account (10.88), (10.89), (10.83), (10.84), and since f is non increasing, we deduce that

$$\mathcal{J}(K,u) \le \liminf_{n\to+\infty} \mathcal{J}(K_n, u_n),$$

so that (K,u) is a minimizer for the problem, and the proof is concluded. □

10.8 Regularity of two-dimensional minimizers

In this section we are interested in the regularity of the minimizers given by Theorem 10.1, precisely whether they are of classical type, in the sense that the obstacle is a closed set and the flow is Sobolev outside the obstacle.

We begin by discussing the strong setting for closed obstacle that are not necessarily smooth;

Definition 10.33. (F,u) is said to be strongly admissible on Ω when

- F is closed in Ω, $\Omega\cap\partial F$ is rectifiable and $\mathscr{H}^1(\Omega\cap\partial F)<+\infty$.
- $u\in H^1_{\mathrm{loc}}(\Omega\setminus F)$, $\mathrm{div}(u)=0$, $\int_{\Omega\setminus K}|e(u)|^2d\mathscr{L}^d<+\infty$.
- $u^\pm_{|\partial F}\cdot\nu_F=0$ on $\Omega\cap\partial F$.

and we define their energy

$$\begin{aligned}\mathcal{J}^{\mathrm{strong}}(F,u):=&2\mu\int_{\Omega\setminus F}|e(u)|^2\,d\mathscr{L}^d+\beta\int_{\Omega\cap\partial^*F}|u^+|^2\,d\mathscr{H}^{d-1}+\beta\int_{\Omega\cap F^{(0)}\cap F}[|u^+|^2+|u^-|^2]\,d\mathscr{H}^{d-1}\\&+c\mathscr{H}^{d-1}(\Omega\cap\partial^*F)+2c\mathscr{H}^{d-1}(\Omega\cap F^{(0)}\cap F)+f(|F|),\end{aligned}\tag{10.90}$$

It is not clear the traces of u are well-defined and it is a consequence of the following lemma. Since before knowing the traces are defined we have a-priori control only over the total elastic energy of u and the length of ∂F, we have to work in an "unbounded" version of SBD, the $GSBD$ setting as defined in [189]. More precisely we will make use of the compactness theorem [186, Th 1.1].

Lemma 10.34. *Let $F\subset\Omega$ be a relatively closed set such that $\mathscr{H}^1(\Omega\cap\partial F)<\infty$ and $u\in H^1_{loc}(\Omega\setminus F)$ such that $\int_{\Omega\setminus F}|e(u)|^2<\infty$, and let $\tilde{u}=\begin{cases}u & \text{in } \Omega\setminus F\\ 0 & \text{in } F\end{cases}$, then $\tilde{u}\in GSBD(\Omega)$ with $\mathscr{H}^1(J_{\tilde{u}}\setminus\partial F)=0$.*

So up to a choice of orientation of $\Omega\cap\partial F$ (by rectifiability) there is no ambiguity to defining the traces $u^\pm_{|\partial F}$ of u on $\mathscr{H}^1$-almost all ∂F.

Proof. Since $\mathscr{H}^1(\Omega\cap\partial F)<\infty$, we may find for any small ϵ some covering of ∂F by a finite union of balls of radius less than ϵ, denoted $(B^\epsilon_i)_{1\le i\le N^\epsilon}$, such that $\sum_{i=1}^{N^\epsilon}\frac{\mathrm{diam}(B^\epsilon_i)}{2}\le C$ for some $C>0$ that does not depend on ϵ. Let B^ϵ be the union of these balls - which is a Lipschitz set up to a small perturbation of the radii - and $u^\epsilon=u1_{\Omega\setminus B^\epsilon}$. Then $u^\epsilon\in SBD(\Omega)$ with

$$Eu^\epsilon=e(u)\mathscr{L}^2\lfloor(\Omega\setminus(F\cup B^\epsilon))+u\mathscr{H}^1\lfloor\partial B^\epsilon$$

and we have

$$u^\epsilon\xrightarrow[a.e.]{}\tilde{u},\ \limsup_{\epsilon\to0}\int_\Omega|e(u^\epsilon)|^2+\mathscr{H}^1(J_{u^\epsilon})<\infty.$$

We apply [186, Th 1.1] to (u^ϵ); since $\tilde{u}$ is finite almost everywhere, we directly identify $\tilde{u}$ with the limit that is obtained, and $\tilde{u}\in GSBD(\Omega)$. Moreover up to a negligible set, $J_{\tilde{u}}\subset\partial F$ by construction. □

As a consequence of this lemma, if (F,u) is as in definition 10.33 and $\mathcal{J}^{\mathrm{strong}}(F,u)<\infty$, then $\tilde{u}\in\mathbf{v}(\Omega)$ and $(F,\tilde{u})$ is admissible in the sense of definitions 10.11,10.12. Indeed it is enough to prove that $\tilde{u}\in SBD(\Omega)$, which is the case since for every direction $\xi\in\mathbb{S}^{d-1}$ we have a bound

$$\mathcal{J}^{\mathrm{strong}}(F,u)\ge\int_{\xi^\perp}\left[2\mu\int_{\Omega^\xi_y}|(\tilde{u}^\xi_y)'(t)|^2dt+\sum_{t\in J_{\tilde{u}^\xi_y}}\left(c+\beta|(\tilde{u}^\xi_y)^+(t)|^2+\beta|(\tilde{u}^\xi_y)^-(t)|^2\right)\right]d\mathscr{H}^1(y)$$

What we prove in this section is that when $d = 2$, relaxed minimizer of (10.26) in the relaxed setting defined in definitions 10.11,10.12 are **locally** strong in the sense that $\mathscr{H}^1(\Omega \cap \overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0$; as a consequence, setting F to be the complementary of the union of the connected components of $\Omega \setminus \overline{\partial^* E \cup J_u}$ where u is nonzero, we find that $(F, u_{|\Omega\setminus F})$ is a strong local minimizer in the sense given above. There is a missing piece to get a full correspondance with the strong problem, which is a boundary regularity $\mathscr{H}^1(\partial\Omega \cap \overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0$.

The general strategy used by De Giorgi, Carriero and Leaci for the Mumford-Shah problem in [190] faces the new difficulties given by the vectorial context and the extra conditions satisfied by the competing fields: incompressibility and non-penetration. We follow the main lines of [188] and [185] developed for the Griffith fracture problem. However, for now, technical difficulties allow us to deal only with dimension 2.

For simplicity, we shall assume f is a Lipschitz function (although $\mathcal{C}^{0,\alpha}$ regularity, for some $\alpha > \frac{1}{2}$, would be enough).

We begin with a technical result which allows us to approximate any function $u \in \mathcal{V}([-1,1]^2)$ with few jumps by a Sobolev function in a slightly smaller domain, while keeping control on the energy. This result is inspired by [185]. In the second part, we prove the decay estimate and conclude.

10.8.1 Smoothing lemma

We fix a standard radial, smooth, nonnegative mollifier ρ with support in a disc of radius $1/8$ and denote $\rho_\delta(x) = \delta^{-2}\rho(\frac{x}{\delta})$.

Instead of working directly with functions $u \in \mathcal{V}(\Omega)$, we have to work with pairs (J, u); the reason for this is that we will apply this result in the end to $\partial^* E \cup J_u$ for an admissible set-function pair (E, u), and some points of $\partial^* E$ may not be jump points of u. We will write that the pair (J, u) of a rectifiable set J and a function $u \in \mathcal{V}(\Omega)$ is admissible, and we write $(J, u) \in \overline{\mathcal{V}}(\Omega)$, when in addition to $u \in \mathcal{V}(\Omega)$ one has

$$\mathscr{H}^1(J_u \setminus J) = 0,\ u^{\pm}_{|J} \cdot \nu_J = 0$$

where ν_J is the normal vector of J and $u^{\pm}_{|J}$ are the traces of u on J, both of which are well-defined by rectifiability of J . However all that follows may be adapted seamlessly to the case where J is simply J_u.

de definit $\mathcal{V}(\Omega)$ si Q_r

Proposition 10.35. *There exists constants $C, \eta > 0$ such that for any $(J, u) \in \overline{\mathcal{V}}(Q_1)$ such that $\mathscr{H}^1(J) < \eta$, then denoting $\delta = \sqrt{\mathscr{H}^1(J)}$, there is a radius $r \in [1 - \sqrt{\delta}, 1]$, a function $v \in \mathcal{V}(Q_1) \cap H^1(Q_r)$, and a cut-off function $\varphi \in \mathcal{C}^\infty(Q_r, [0,1])$ such that*

(a) $\{u \neq v\} \subset Q_r$, $\mathscr{H}^1(J_v \setminus J) = 0$,

(b) $\varphi = 0$ *on* $Q_r \setminus Q_{r-\frac{\delta}{2}}$, $\varphi = 1$ *on* $Q_{r-2\delta}$.

(c) $\int_{Q_r} |e(v) - (\rho_\delta * e(u))\varphi|^2 \, d\mathscr{L}^2 \leq C\delta^{\frac{1}{3}} \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$,

In particular $(J \setminus Q_r, v) \in \overline{\mathcal{V}}(Q_1)$.

Proof. In this proof, we write $a \lesssim b$ when $a \leq Cb$ for some universal constant C. There is no compactness argument in this proof so the constants involved are fully explicitable, although we shall not attempt it. We begin by a lemma due to Nečas that will allow us to project a H^1 field into divergence-free fields.

Lemma 10.36. *Let Ω be a bounded, connected Lipschitz open set, we let $L^2_0(\Omega)$ be the set of zero-average L^2-functions. Then there is a continuous linear map $\Phi : L^2_0(\Omega) \to H^1_0(\Omega, \mathbb{R}^d)$ such that $div \circ \Phi = Id_{L^2_0(\Omega)}$.*

We shall also write, for any $u \in H^1(\Omega, \mathbb{R}^d)$ such that $\int_\Omega \mathrm{div}(u) d\mathscr{L}^2 = 0$ (or equivalently that $\int_{\partial\Omega} u \cdot \nu_\Omega d\mathscr{H}^1 = 0$),

$$\mathbb{P}_\Omega[u] = u - \Phi[\ \mathrm{div}\ (u)]$$

the "projection" of u on divergence-free vector fields. The proof of this lemma may be found for instance in [177, Th IV.3.1]. We let $N := 1 + \lfloor 1/\sqrt{\mathscr{H}^1(J)} \rfloor$, in the following it will be supposed arbitrarily large. We also let $\delta := 1/N$. Then for any $k \geq 0$ we write

$$C_r^k = Q_r \setminus Q_{r - \frac{\delta r}{2^k}}$$

We now make the choice of radius r such that the density of jump near ∂Q_r is small, following a similar method as [187, th 2.1].

Lemma 10.37. *There is a choice of $r \in [1 - \sqrt{\delta}, 1]$ such that*

$$\forall k \geq 0,\ \mathscr{H}^1(J \cap C_r^k) \leq 20 \mathscr{H}^1(J)^{\frac{3}{4}} \frac{\delta r}{2^k}$$
$$\int_{C_r^{-2}} |e(u)|^2 d\mathscr{L}^2 \leq 80 \mathscr{H}^1(J)^{\frac{1}{4}} r \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$$

Proof. We prove that this is verified for a set of $r \in [1 - \sqrt{\delta}, 1]$ of measure larger than $\frac{\sqrt{\delta}}{2}$. Define the measure μ on $[0, 1]$ as

$$\mu(E) = \frac{\mathscr{H}^1(J \cap Q_E)}{\mathscr{H}^1(J)} + \frac{\int_{Q_E} |e(u)|^2 d\mathscr{L}^2}{\int_{Q_1} |e(u)|^2 d\mathscr{L}^2},$$

where $Q_E := \cup_{r \in R} \partial Q_r$ is the cubic shell associated to $E \subset [0, 1]$. Denoting $I_r^k = [r - \frac{\delta r}{2^k}, r]$ (such that $Q_{I_r^k} = C_r^k$), our goal is to prove that for some choice of r, we have

$$\forall k \geq -2,\ \mu(I_r^k) \leq \frac{20}{\mathscr{H}^1(J)^{1/4}} |I_r^k|$$

Indeed, this implies

$$\forall k \geq 0,\ \mathscr{H}^1(J \cap C_r^k) \leq \mathscr{H}^1(J) \frac{20\delta r}{2^k \mathscr{H}^1(J)^{\frac{1}{4}}} \leq 20 \mathscr{H}^1(J)^{\frac{3}{4}} \frac{\delta r}{2^k}$$
$$\int_{C_r^{-2}} |e(u)|^2 d\mathscr{L}^2 \leq 80\delta \mathscr{H}^1(J)^{-\frac{1}{4}} r \int_{Q_1} |e(u)|^2 d\mathscr{L}^2 \leq 80 \mathscr{H}^1(J)^{\frac{1}{4}} r \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$$

Let I be the union of all intervals that do not verify this, notice that any $r \in [1 - \sqrt{\delta}, 1] \setminus I$ yields an answer so we only need to prove $|I| < \sqrt{\delta}$. Let $(I_{r_i}^{k_i})$ be a Vitali covering of I, then

$$2 = \mu([0, 1]) \geq \sum_i \mu(I_{r_i}^{k_i}) \geq \frac{20}{\mathscr{H}^1(J)^{1/4}} \sum_i |I_{r_i}^{k_i}| \geq \frac{4|I|}{\mathscr{H}^1(J)^{1/4}}$$

hence $|I| < \frac{1}{2}\sqrt{\delta}$ and we get the result.

□

In all the following, r is fixed and we denote $\delta_k := \frac{\delta r}{2^k}$, $r_k = \left(N - \frac{1}{2^k}\right) \delta r = r - \delta_k$. Then we consider a partition of Q_r by cubes, by filling Q_{r_0} with cubes of side δ_0 denoted $(\tilde{q}_{0,j})_j$ and then each $Q_{r_k} \setminus Q_{r_{k-1}}$ by cubes of side δ_k denoted $(\tilde{q}_{k,j})_j$ (note that there is only one way to do this).

For any square $q = z + [-t,t]^2$, we write $q' = z + [-\frac{8}{7}t, \frac{8}{7}t]^2$ and $q'' = (q')'$. We will denote $q_{j,k} = (\tilde{q}_{j,k})'$.

We may notice that with our choices, $q''_{j,k} \Subset Q_{r_{k+1}} \setminus Q_{r_{k-1}}$ and $(q''_{j,k})$ is a covering of Q_r with a fixed finite number of overlapping; each $(q''_{j,k})$ meets at most 8 neighbours $(q''_{i,p})$, and they all verify $|k-p| \leq 1$, meaning $\delta_k/\delta_p \in \left\{\frac{1}{2}, 1, 2\right\}$. This is because the factor $\frac{8}{7}$ above is chosen such that $\left(\frac{8}{7}\right)^3 < \frac{3}{2}$.

With our choice of r, we know that for every $k \geq 1$, $\mathscr{H}^1(J_u \cap q''_{j,k}) \lesssim \delta_k \sqrt{\mathscr{H}^1(J)}$, and $\mathscr{H}^1(J_u \cap q''_{j,0}) \lesssim \delta_0 \sqrt{\mathscr{H}^1(J)}$. This means that the jump set of u in every cube is arbitrarily small compared to its sides, when $\mathscr{H}^1(J)$ gets smaller.

We apply [185, Proposition 3.1], for every (j,k) there is a set $\omega_{j,k} \subset q'_{j,k}$ and an affine function $a_{j,k}$ with $e(a_{j,k}) = 0$, such that

- $|\omega_{j,k}| \lesssim \delta_k \mathscr{H}^1(J_u \cap q''_{j,k})$. Notice that as a consequence of the choice of r, $|\omega_{j,k}| \lesssim \delta_k^2 \mathscr{H}^1(J)^{\frac{1}{2}} \ll |q''_{j,k}|$: this is where we most use the fact that we are in two dimensions.
- $\int_{q'_{j,k} \setminus \omega_{j,k}} |u - a_{j,k}|^4 \lesssim \left(\delta_k \int_{q''_{j,k}} |e(u)|^2\right)^2$.
- The function $v_{j,k} = u + (a_{j,k} - u)1_{\omega_{j,k}}$ verifies

$$\begin{aligned}\int_{q_{j,k}} |e(v_{j,k} * \rho_{\delta_k}) - e(u) * \rho_{\delta_k}|^2 d\mathscr{L}^2 &\lesssim \left(\frac{\mathscr{H}^1(J_u \cap q''_{j,k})}{\delta_k}\right)^{\frac{1}{3}} \int_{q''_{j,k}} |e(u)|^2 d\mathscr{L}^2 \\ &\lesssim \delta^{\frac{1}{3}} \int_{q''_{j,k}} |e(u)|^2 d\mathscr{L}^2,\end{aligned}$$

  where ρ is the mollifier defined previously.

We now let $(\varphi_{j,k})$ be an approximation of unity associated to the covering $(q_{j,k})$ and such that $|\nabla \varphi_{j,k}| \lesssim \frac{1}{\delta_k}$. We then let $w_{j,k} = \rho_{\delta_k} * v_{j,k}$ and

$$w = 1_{Q_1 \setminus Q_r} u + 1_{Q_r} \sum_{j,k} \varphi_{j,k} w_{j,k}$$

Lemma 10.38. *The function w defined above verifies*

- $w \in SBD(Q_1) \cap H^1(Q_r)$,
- $\{u \neq w\} \subset Q_r$, $\mathscr{H}^1(J_w \setminus J) = \mathscr{H}^0(J \cap \partial Q_r) = 0$,
- $\int_{Q_r} \left|e(w) - \sum_{j,k} \varphi_{j,k} \rho_{\delta_k} * e(u)\right|^2 \lesssim \delta^{\frac{1}{3}} \int_{Q_1} |e(u)|^2$.

The proof of this lemma follows the strategy introduced in [185]. Some steps of the proof are referred directly to [185]. However, since our conclusion is slightly different, we prefer do develop some computations in detail.

Proof. Notice that as soon as $q_{j,k}$ and $q_{i,p}$ intersects, then $|q_{j,k} \cap q_{i,p}| \gtrsim \max(|q_{j,k}|, |q_{j,p}|)$,

$$\|a_{j,k} - a_{i,p}\|_{L^4(q_{j,k} \cap q_{i,p})} \lesssim \sqrt{\delta_k} \|e(u)\|_{L^2(q''_{j,k} \cup q''_{j,p})}$$

Indeed, since $|q_{j,k} \cap q_{i,p} \cap (\omega_{j,k} \cup \omega_{i,p})| \ll |q_{j,k} \cap q_{i,p}|$ and $a_{j,k}, a_{i,p}$ are affine, then using [185, Lemma 3.4],

$$\begin{aligned}\|a_{j,k} - a_{i,p}\|_{L^4(q_{j,k} \cap q_{i,p})} &\lesssim \|a_{j,k} - a_{i,p}\|_{L^4(q_{j,k} \cap q_{i,p} \setminus (\omega_{j,k} \cup \omega_{i,p}))} \leq \|a_{j,k} - u\|_{L^4(q_{j,k} \setminus \omega_{j,k})} + \|a_{i,p} - u\|_{L^4(q_{i,p} \setminus \omega_{i,p})} \\ &\lesssim \sqrt{\delta_k} \|e(u)\|_{L^2(q''_{j,k})} + \sqrt{\delta_p} \|e(u)\|_{L^2(q''_{i,p})} \lesssim \sqrt{\delta_k} \|e(u)\|_{L^2(q''_{j,k} \cup q''_{i,p})}.\end{aligned}$$

And so in particular $\|a_{j,k}-a_{i,p}\|_{L^2(q_{j,k}\cap q_{i,p})}\lesssim \delta_k\|e(u)\|_{L^2(q''_{j,k}\cup q''_{i,p})}$. Now,

$$\begin{aligned} e(w)&=\sum_{j,k}\left(\varphi_{j,k}e(w_{j,k})+\nabla\varphi_{j,k}\odot w_{j,k}\right)\\ &=\sum_{j,k}\varphi_{j,k}e(w_{j,k})+\sum_{q_{j,k}\cap q_{i,p}\neq\emptyset}\nabla\varphi_{j,k}\odot(w_{j,k}-w_{i,p}) \qquad \text{in } q_{i,p}. \end{aligned}$$

So

$$\begin{aligned} e(w)-\sum_{j,k}\varphi_{j,k}\rho_{\delta_k}*e(u)=&\sum_{j,k}\varphi_{j,k}\Big[e(w_{j,k})-\rho_{\delta_k}*e(u)\Big]\\ &+\sum_{q_{j,k}\cap q_{i,p}\neq\emptyset}\nabla\varphi_{j,k}\odot(w_{j,k}-w_{i,p}) \text{ in } q_{i,p}. \end{aligned} \tag{10.91}$$

We begin by an estimate of the second term of the right-hand side of (10.91).

- If $q''_{i,p}\Subset Q_{r_{-1}}$, then $q_{j,k}\cap q_{i,p}\neq\emptyset$ means that $\delta_k=\delta_p=\delta$, $k=p=0$, and we may rewrite

  $$|\nabla\varphi_{j,0}\odot(w_{i,0}-w_{j,0})|\leq\|\rho_\delta*\nabla\varphi_{j,0}\|_{L^\infty}|v_{i,0}-v_{j,0}|.$$

  On one hand $\|\rho_\delta*\nabla\varphi_{j,0}\|_{L^\infty}\lesssim\frac{1}{\delta}$, on the other hand

  $$|v_{i,0}-v_{j,0}|\leq|u-a_{i,0}|1_{\omega_{j,0}\setminus\omega_{i,0}}+|u-a_{j,0}|1_{\omega_{i,0}\setminus\omega_{j,0}}+|a_{j,0}-a_{i,0}|1_{\omega_{j,0}\cap\omega_{i,0}}.$$

  So integrating on $q_{j,0}\cap q_{i,0}$ we get

  $$\begin{aligned} \|v_{i,0}-v_{j,0}\|_{L^2(q_{j,0}\cap q_{i,0})}&\leq\|(u-a_{i,0})1_{\omega_{j,0}}\|_{L^2(q_{i,0}\setminus\omega_{i,0})}+\|(u-a_{j,0})1_{\omega_{i,0}}\|_{L^2(q_{j,0}\setminus\omega_{j,0})}\\ &+\|(a_{j,0}-a_{i,0})1_{\omega_{i,0}\cup\omega_{j,0}}\|_{L^2(q_{i,0}\cap q_{j,0})}\\ &\leq|\omega_{j,0}|^{\frac14}\|u-a_{i,0}\|_{L^4(q_{i,0}\setminus\omega_{i,0})}+|\omega_{i,0}|^{\frac14}\|u-a_{j,0}\|_{L^4(q_{j,0}\setminus\omega_{j,0})}\\ &+|\omega_{i,0}\cup\omega_{j,0}|^{\frac14}\|a_{j,0}-a_{i,0}\|_{L^4(q_{i,0}\cap q_{j,0})}\\ &\lesssim\delta^{1+\frac14}\|e(u)\|_{L^2(q''_{i,0}\cup q''_{j,0})}. \end{aligned}$$

  Integrating everything we get

  $$\begin{aligned} \int_{Q_{r_{-1}}}\left|\sum_{j,k}\nabla\varphi_{j,k}\odot w_{j,k}\right|^2d\mathscr{L}^2&\lesssim\int_{Q_{r_{-1}}}\sum_{i,p}1_{q_{i,p}}\left|\sum_{q_{j,k}\cap q_{i,p}\neq\emptyset}\nabla\varphi_{j,k}\odot(w_{j,k}-w_{i,p})\right|^2d\mathscr{L}^2\\ &\lesssim\int_{Q_{r_{-1}}}\sum_i1_{q_{i,0}}\sum_{j:q_{j,0}\cap q_{i,0}\neq\emptyset}|\rho_\delta*\nabla\varphi_{j,0}|^2 1_{q_{j,0}}|v_{i,0}-v_{j,0}|d\mathscr{L}^2\\ &\lesssim\delta^{-2}\sum_i\sum_{j:q_{j,0}\cap q_{i,0}\neq\emptyset}\|v_{j,0}-v_{i,0}\|^2_{L^2(q_{i,0}\cap q_{j,0})}d\mathscr{L}^2\\ &\lesssim\delta^{\frac12}\sum_i\sum_{j:q_{j,0}\cap q_{i,0}\neq\emptyset}\|e(u)\|^2_{L^2(q''_{i,0}\cup q''_{j,0})}d\mathscr{L}^2\\ &\lesssim\delta^{\frac12}\int_{Q_{r_0}}|e(u)|^2d\mathscr{L}^2 \end{aligned}$$

  where we used several times the finite overlapping property of the covering.

- If $q_{i,p}\not\Subset Q_{r_{-1}}$, then for any $q_{j,k}$ that meets $q_{i,p}$, we decompose

  $$w_{i,p}-w_{j,k}=\rho_{\delta_p}*(v_{i,p}-a_{i,p})-\rho_{\delta_k}*(w_{j,k}-a_{j,k})+(a_{i,p}-a_{j,k}).$$

  Notice the crucial step that $\rho_{\delta_k}*a_{j,k}=a_{j,k}$ due to the fact that $a_{j,k}$ is harmonic (since it is affine). Then we have:

  $$\begin{aligned} \|\rho_{\delta_k}*(v_{i,p}-a_{i,p})\|_{L^2(q_{j,k}\cap q_{i,p})}&\leq\|v_{i,p}-a_{i,p}\|_{L^2(q'_{i,p})}\lesssim\delta_p\|e(u)\|_{L^2(q''_{i,p})}\\ \|\rho_{\delta_k}*(v_{j,k}-a_{j,k})\|_{L^2(q_{j,k}\cap q_{i,p})}&\leq\|v_{j,k}-a_{j,k}\|_{L^2(q'_{j,k})}\lesssim\delta_p\|e(u)\|_{L^2(q''_{j,k})}\\ \|a_{i,p}-a_{j,k}\|_{L^2(q_{j,k}\cap q_{i,p})}&\lesssim\delta_p\|e(u)\|_{L^2(q''_{j,k}\cup q''_{i,p})} \end{aligned}$$

where we also used the fact that δ_p and δ_k differ from at most a factor 2. And so we obtain with the same computations as the previous point that

$$\int_{Q_r\setminus Q_{r-1}}\left|\sum_{j,k}\nabla\varphi_{j,k}\odot w_{j,k}\right|^2 d\mathscr{L}^2\lesssim\int_{Q_r\setminus Q_{r-2}}|e(u)|^2d\mathscr{L}^2.$$

Thus, using the choice of r in Lemma 10.37,

$$\int_{Q_r\setminus Q_{r-1}}\left|\sum_{j,k}\nabla\varphi_{j,k}\odot w_{j,k}\right|^2 d\mathscr{L}^2\lesssim\sqrt{\delta}\int_{Q_1}|e(u)|^2d\mathscr{L}^2.$$

And so the second term of (10.91) is bounded by $C\sqrt{\delta}\int_{Q_1}|e(u)|^2$. For the first term, we have

$$\begin{aligned}\int_{Q_r}\left|\sum_{j,k}\varphi_{j,k}e(w_{k,j})-\sum_{j,k}\varphi_{j,k}\rho_{\delta_k}*e(u)\right|^2 d\mathscr{L}^2&\leq\int_{Q_r}\left|\sum_{j,k}\varphi_{j,k}\left(e(\rho_{\delta_k}*v_{j,k})-\rho_{\delta_k}*e(u)\right)\right|^2 d\mathscr{L}^2\\&\lesssim\sum_{j,k}\int_{q_{j,k}}\varphi_{j,k}^2\left|e(\rho_{\delta_k}*v_{j,k})-\rho_{\delta_k}*e(u)\right|^2 d\mathscr{L}^2\\&\lesssim\delta^{\frac{1}{3}}\sum_{j,k}\int_{q''_{j,k}}|e(u)|^2d\mathscr{L}^2\\&\lesssim\delta^{\frac{1}{3}}\int_{Q_r}|e(u)|^2d\mathscr{L}^2.\end{aligned}$$

This proves the lemma. □

Now, in all the following we let

$$\varphi:=\sum_j\varphi_{j,0}$$

which is a cut-off function as stated in the proposition. This way,

$$e(w)-\sum_{j,k}\varphi_{j,k}\rho_{\delta_k}*e(u)=\Big[e(w)-\varphi(\rho_\delta*e(u))\Big]-\sum_{j,k>0}\varphi_{j,k}\rho_{\delta_k}*e(u).$$

And this last term is estimated by

$$\begin{aligned}\left\|\sum_{j,k>0}\varphi_{j,k}\rho_{\delta_k}*e(u)\right\|^2_{L^2(Q_r)}=\left\|\sum_{j,k>0}\varphi_{j,k}\rho_{\delta_k}*e(u)\right\|^2_{L^2(Q_r\setminus Q_{r-1})}&\lesssim\int_{Q_r\setminus Q_{r-1}}\sum_{j,k}\varphi_{j,k}^2(\rho_{\delta_k}*e(u))^2d\mathscr{L}^2\\&\lesssim\int_{Q_r\setminus Q_{r-2}}|e(u)|^2d\mathscr{L}^2\lesssim\sqrt{\delta}\int_{Q_1}|e(u)|^2d\mathscr{L}^2,\end{aligned}$$

where we used the property of our initial choice of r in the last line. To summarize, we obtained $w\in\mathcal{C}^\infty(Q_r)$ such that

$$\|e(w)-\varphi(\rho_\delta*e(u))\|_{L^2(Q_r)}\lesssim\delta^{\frac{1}{6}}\|e(u)\|_{L^2(Q_1)}.$$

From this estimate we deduce that the divergence of w is small, indeed

$$\int_{Q_r}|\mathrm{div}(w)|^2d\mathscr{L}^2=\int_{Q_r}\left|\mathrm{Tr}(e(w))\right|^2d\mathscr{L}^2=\int_{Q_r}\left|\mathrm{Tr}\left(e(w)-\varphi\rho_\delta*e(u)\right)\right|^2d\mathscr{L}^2\lesssim\delta^{\frac{1}{3}}\int_{Q_1}|e(u)|^2d\mathscr{L}^2.$$

Lemma 10.39. *As it is defined, $w\in H^1(Q_r)$ and the trace of w on both sides of ∂Q_r is the same.*

Proof. We refer to [185]. □

We may now enforce the divergence-free condition; since u is divergence-free in the sense of distribution and it admits a trace on Q_r, then $\int_{\partial Q_r} u \cdot \nu = 0$, where ν is the outward normal vector of Q_r. Since the trace of u coincide with the trace of w here, then $\int_{\partial Q_r} w \cdot \nu = 0$, meaning that:

$$\int_{Q_r} \mathrm{div}(w) d\mathscr{L}^2 = 0$$

Thus $\mathrm{div}(w)$ is a $L^2_0(Q_r)$ functions; using lemma 10.36, there exists a vector field $q \in H^1_0(Q_r)$ such that

$$\mathrm{div}(q) = \mathrm{div}(w), \ \|\nabla q\|^2_{L^2(Q_r)} \lesssim \|\mathrm{div}(w)\|^2_{L^2(Q_r)} \lesssim \delta^{\frac{1}{3}} \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$$

And thus the function $v := w - q$ verifies the conclusion of the result.

□

10.8.2 Decay estimate

In all the following, we will denote

$$G(J, u|\Omega) = \int_\Omega |e(u)|^2 d\mathscr{L}^2 + \mathscr{H}^1(J \cap \Omega).$$

Definition 10.40. For any open set $\Omega \Subset \mathbb{R}^2$, we say $(J, u) \in \overline{\mathcal{V}}(\Omega)$ is a $(\Lambda, \vartheta, \overline{r})$ almost quasi-minimizer (in Ω) if $G(J, u|\omega) < +\infty$ for any open $\omega \Subset \Omega$, and for any $v \in \mathcal{V}(\Omega)$ such that

- $\{u \neq v\} \subset Q_{x,r}$ for some cube $Q_{x,r} \Subset \Omega$ with $r \in (0, \overline{r})$,
- $\mathscr{H}^1(J_v \setminus J) = 0$ and $v_{|Q_{x,r}} \in H^1(Q_{x,r})$,
- $\limsup_{s \to 0^+} \frac{1}{s} \mathscr{H}^1 \left(J \cap (Q_{x,r} \setminus Q_{x,r-s}) < 1 \right.$,

then we have

$$\int_{Q_{x,r}} |e(u)|^2 d\mathscr{L}^2 + \mathscr{H}^1(J \cap Q_{x,r}) \leq (1 + \vartheta) \int_{Q_{x,r}} |e(v)|^2 d\mathscr{L}^2 + \Lambda r^2.$$

The last inequality may be written $G(J, u|Q_{x,r}) \leq (1 + \vartheta) G(J \setminus Q_{x,r}, v|Q_{x,r}) + \Lambda r^2$. This definition is tailor-made for the application of the smoothing lemma, in particular it is only useful in two dimension for functions with few jumps so we know that we can find squares with no jump on the boundary. More precisely when applied to the case of an admissible pair (E, u) we will prove the third condition implies that $\partial Q_{x,r}$ separates E in two connected components, in the sense that the reduced boundary of either $E \cup Q_{x,r}$ or $E \setminus Q_{x,r}$ is included (up to a $\mathscr{H}^1$-negligible set) in $\partial^* E \setminus Q_{x,r}$. The third condition is automatically verified for the competitor built through the smoothing lemma since the choice of r implies $\limsup_{s \to 0} \frac{\mathscr{H}^1(J \cap Q_r \setminus Q_{r-s})}{s} \leq 40 \mathscr{H}^1(J \cap Q_1)^{\frac{3}{4}} < 1$.

This notion is weak enough to include any local minimizer of a functional of the form

$$\int_\Omega |e(u)|^2 d\mathscr{L}^2 + \int_{\Omega \cap J_u} \Theta(\nu_u, u^+, u^-) d\mathscr{H}^1$$

where Θ is a measurable function such that $\inf(\Theta) \geq 1$ (or, $\inf(\Theta) > 0$ up to scaling).

Proposition 10.41. *There exists a universal constant $\overline{\tau} \in (0, 1)$ such that for any $\tau \in (0, \overline{\tau})$, $\Lambda, \overline{r} > 0$, then there exists $\hat{r} \in (0, \overline{r})$, $\vartheta, \epsilon > 0$ such that for any $(\Lambda, \vartheta, \overline{r})$-almost quasiminimizer (J, u) in Q_r for some $r < \hat{r}$, we have*

$$\Big[G(J, u|Q_r) \geq r^{3/2}, \ \mathscr{H}^1(J \cap Q_r) \leq \epsilon r \Big] \Rightarrow \Big[G(J, u|Q_{\tau r}) \leq \tau^{3/2} G(u|Q_r). \Big]$$

Lemma 10.42. *There exists a constant $C_0 > 0$ such that for any divergence-free vector field $u \in H^1(Q_1, \mathbb{R}^2)$ such that $\Delta u = \nabla p$ for some pressure $p \in L^2(Q_1)$, we have*

$$\forall \tau \in (0, 1/2], \ \int_{Q_\tau} |e(u)|^2 d\mathscr{L}^2 \leq C_0 \tau^2 \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$$

As a consequence, for any $\tau \leq \overline{\tau} := \frac{1}{4C_0^2} \wedge \frac{1}{2}$ we get $\int_{Q_\tau} |e(u)|^2 d\mathscr{L}^2 \leq \frac{1}{2}\tau^{3/2} \int_{Q_1} |e(u)|^2 d\mathscr{L}^2$.

Proof. Notice that $e(u)$ is invariant by the addition of an asymmetric affine function a. Up to a translation by such a function, Korn's inequality tells us that

$$\int_{Q_1} u^2 d\mathscr{L}^2 \leq C \int_{Q_1} |e(u)|^2 d\mathscr{L}^2.$$

The equations verified by u are equivalent to the existence of $\varphi \in H^2(Q_1)$ such that $\varphi(0) = 0$, $u = \nabla^\perp \varphi$, and $\Delta^2 \varphi = 0$. By elliptic regularity there is a constant C' such that

$$\sup_{Q_{1/2}} \left|\nabla^2 \varphi\right|^2 \leq C' \int_{Q_1} |\nabla \varphi|^2 d\mathscr{L}^2$$

and so for any $\tau \leq 1/2$,

$$\int_{Q_\tau} |e(u)|^2 d\mathscr{L}^2 \leq 4|Q_\tau| \sup_{Q_{1/2}} \left|\nabla^2 \varphi\right|^2 \leq 4CC'|Q_1|\tau^2 \int_{Q_1} |e(u)|^2 d\mathscr{L}^2.$$

□

Proof. We reason by contradiction; suppose that there exists $\tau > 0$, and sequences $r_i \to 0$, $\vartheta_i \to 0$ $\epsilon_i \to 0$, of $(\Lambda, \vartheta_i, \overline{r})$-minimizers (K_i, w_i) in Q_{r_i} such that for all i,

$$G(K_i, w_i | Q_{r_i}) \geq r_i^{3/2}, \ \mathscr{H}^1(K_i \cap Q_{r_i}) \leq \epsilon_i r_i, \ G(K_i, w_i | Q_{\tau r_i}) > \tau^{3/2} G(K_i, w_i | Q_{r_i})$$

We let $g_i = G(K_i, w_i | Q_{r_i})$, and

$$J_i = \frac{K_i}{r_i}, \ u_i(x) = \frac{w_i(r_i x)}{\sqrt{g_i}}.$$

Then (J_i, u_i) is a $(\Lambda\sqrt{r_i}, \vartheta_i, 1)$-minimizer of $G_i(\cdot | Q_1)$, where

$$G_i(J, u | \Omega) := \int_\Omega |e(u)|^2 d\mathscr{L}^2 + \frac{r_i}{g_i} \mathscr{H}^1(J \cap \Omega)$$

with

$$G_i(J_i, u_i | Q_1) = 1, \ G_i(J_i, u_i | Q_\tau) > \tau^{3/2}, \ \mathscr{H}^1(J_i \cap Q_1) = \epsilon_i$$

Let $\rho_i \in [1 - \sqrt{\epsilon_i}, 1]$ and v_i be as defined in the smoothing lemma, such that

- $\{v_i \neq u_i\} \subset Q_{\rho_i}$,
- $v_i \in H^1(Q_{\rho_i}) \cap \mathcal{V}(Q_1)$, $\operatorname{div}(v_i) = 0$,
- $\int_{Q_1} |e(v_i)|^2 \leq (1 + c\epsilon_i^{\frac{1}{6}}) \int_{Q_1} |e(u_i)|^2$,
- $\int_{Q_{1/2}} |e(v_i) - \rho_{\sqrt{\epsilon_i}} * e(u_i)|^2 \lesssim \epsilon_i^{\frac{1}{6}} \int_{Q_1} |e(u_i)|^2$.

By the classical Korn inequality on Q_{ρ_i} there is an asymmetric affine function a_i such that $\int_{Q_{\rho_i}} (v_i - a_i) d\mathscr{L}^2 = 0$ and

$$\int_{Q_{\rho_i}} |\nabla(v_i - a_i)|^2 d\mathscr{L}^2 \lesssim \int_{Q_{\rho_i}} |e(v_i)|^2 d\mathscr{L}^2.$$

So $(v_i - a_i)$ is bounded in $H^1(Q_{\rho_i})$; by Rellich theorem we can take a subsequence (that we denote by the same indice to not overload the notations) such that, for some $w \in H^1(Q_1)$, we have

$$v_i - a_i \underset{L^2}{\longrightarrow} w$$

with weak convergence in H^1 (notice that these are taken on square Q_{ρ_i} of slightly different radius, but we can do a minor rescaling). Since every $v_i - a_i$ has zero divergence, then so does w. Then for any $\varphi \in \mathcal{C}_c^\infty(Q_1, \mathbb{R}^2)$ with zero divergence, and for some $\eta \in \mathcal{C}_c^\infty(Q_1, [0,1])$ that will be fixed later (and such that $\{\varphi \neq 0\} \Subset \{\eta = 1\}$), we can compare u_i and $\mathbb{P}_{Q_{\rho_i}}\Big[(1-\eta)v_i + \eta(a_i + w + \varphi)\Big]$;

$$G_i(u_i|Q_1) \leq (1+\vartheta_i) G_i\left(\mathbb{P}_{Q_{\rho_i}}\Big[(1-\eta)v_i + \eta(a_i + w + \varphi)\Big]|Q_1\right) + \Lambda\sqrt{r_i},$$

meaning that

$$\begin{aligned}
G_i(u_i|Q_{\rho_i}) &\leq (1+\vartheta_i)\left\|e\left(\mathbb{P}_{Q_{\rho_i}}\Big[(1-\eta)v_i + \eta(a_i + w + \varphi)\Big]\right)\right\|^2_{L^2(Q_{\rho_i})} + \Lambda\sqrt{r_i} \\
&\leq (1+\vartheta_i)\left(\|e\left((1-\eta)v_i + \eta(a_i + w + \varphi)\right)\|_{L^2(Q_{\rho_i})} + C\|\mathrm{div}((1-\eta)v_i + \eta(a_i + w + \varphi))\|_{L^2(Q_{\rho_i})}\right)^2 \\
&\quad + \Lambda\sqrt{r_i} \\
&\leq (1+\vartheta_i)\left(\|e\left((1-\eta)v_i + \eta(a_i + w + \varphi)\right)\|_{L^2(Q_{\rho_i})} + C\|\nabla\eta\cdot(w + a_i - v_i)\|_{L^2(Q_{\rho_i})}\right)^2 + \Lambda\sqrt{r_i}.
\end{aligned}$$

Then, notice first that

$$\|\nabla\eta\cdot(w + a_i - v_i)\|_{L^2(Q_{\rho_i})} \underset{i\to\infty}{\longrightarrow} 0.$$

Moreover,

$$\begin{aligned}
\|e\left((1-\eta)v_i + \eta(a_i + w + \varphi)\right)\|_{L^2(Q_{\rho_i})} &= \|(1-\eta)e(v_i) + \eta e(w+\varphi) + \nabla\eta\odot(w + a_i - v_i)\|_{L^2(Q_{\rho_i})} \\
&\leq \|(1-\eta)e(v_i) + \eta e(w+\varphi)\|_{L^2(Q_{\rho_i})} + o_{i\to\infty}(1).
\end{aligned}$$

And so we obtain,

$$G_i(u_i|Q_{\rho_i}) \leq \|(1-\eta)e(v_i) + \eta e(w+\varphi)\|^2_{L^2(Q_{\rho_i})} + o_{i\to\infty}(1),$$

Now, we have

$$\int_{Q_{\rho_i}} |e(v_i - a_i)|^2 d\mathscr{L}^2 \leq (1+\epsilon_i^{\frac{1}{6}})\int_{Q_{\rho_i}} |e(u_i)|^2 d\mathscr{L}^2 \leq (1+\epsilon_i^{\frac{1}{6}}) G_i(u_i|Q_{\rho_i}),$$

and so, keeping in mind that $\varphi = 0$ on $\{\eta \neq 1\}$,

$$\int_{Q_{\rho_i}} \left(1 - (1-\eta)^2\right)|e(v_i - a_i)|^2 d\mathscr{L}^2 \leq \int_{Q_{\rho_i}} \left(2\eta(1-\eta)e(v_i - a_i) : e(w) + \eta^2|e(w+\varphi)|^2\right) d\mathscr{L}^2 + o_{i\to\infty}(1).$$

Taking the limit we get

$$\limsup_{i\to\infty} \int_{Q_1} \left(1 - (1-\eta)^2\right)|e(v_i - a_i)|^2 d\mathscr{L}^2 \leq \int_{Q_1} \left(2\eta(1-\eta)e(w) : e(w) + \eta^2|e(w+\varphi)|^2\right) d\mathscr{L}^2.$$

With $\varphi = 0$ this gives the strong convergence of $v_i - a_i$ to w in H^1_{loc}, and with a general φ this can be rearranged into

$$\int_{Q_1} \eta^2 |e(w)|^2 d\mathscr{L}^2 \leq \int_{Q_1} \eta^2 |e(w+\varphi)|^2 d\mathscr{L}^2$$

for every such η, meaning $\int_{Q_1} |e(w)|^2 d\mathscr{L}^2 \leq \int_{Q_1} |e(w+\varphi)|^2 d\mathscr{L}^2$. In particular w is a minimizer of this functional ; $\Delta w = \nabla p$ for some pressure p. Using the previous lemma, we have for any $\tau \in [0, \overline{\tau}]$ (that in particular does not depend on w):

$$\int_{Q_\tau} |e(w)|^2 d\mathscr{L}^2 \leq \frac{1}{2}\tau^{\frac{3}{2}} \int_{Q_1} |e(w)|^2 d\mathscr{L}^2 \leq \frac{1}{2}\tau^{\frac{3}{2}}$$

Since $e(v_i) - \rho_{\sqrt{\epsilon_i}} * e(u_i)$ tends to 0 in $L^2(Q_{1/2})$, $v_i - a_i$ converges in H^1_{loc} to w, and $e(u_i)$ is bounded in $L^2(Q_1)$, then $e(u_i)$ converges weakly to $e(w)$, and so we get

$$\liminf_{i\to\infty} \int_{Q_\tau} |e(u_i)|^2 d\mathscr{L}^2 \leq \frac{1}{2}\tau^{\frac{3}{2}}.$$

And so for any large enough i we get the contradiction:

$$\tau^{\frac{3}{2}} \leq \int_{Q_\tau} |e(u_i)|^2 d\mathscr{L}^2 + \mathscr{H}^1(J_u \cap Q_\tau) \leq \frac{1}{2}\tau^{\frac{3}{2}} + o_{i\to\infty}(1) + \epsilon_i$$

□

Let us remind the following useful fact about rectifiable sets: for any rectifiable set $J \subset \mathbb{R}^2$ (and there is an equivalent for k-rectifiable set in $\mathbb{R}^d$), we may denote

$$J^+ = \left\{ x \in J : \limsup_{r\to 0} \frac{\mathscr{H}^1(J \cap Q_{x,r})}{r} > 0 \right\} \tag{10.92}$$

and it is known that $\mathscr{H}^1(J \setminus J^+) = 0$ (see for instance [194, Th 10.2]).

Proposition 10.43. *Let $\Lambda, \overline{r} > 0$,then there exists $\vartheta, \epsilon, \hat{r} > 0$ such that of any $(\Lambda, \vartheta, \overline{r})$-almost quasiminimizer (J, u) in Ω for some $\Omega \Subset \mathbb{R}^2$, and for any $Q_{x,r} \Subset \Omega$ with $r < \hat{r}$,*

$$G(J, u|Q_{x,2r}) \leq \epsilon r \text{ implies } \mathscr{H}^1(J \cap Q_{x,r}) = 0.$$

Proof. We begin by a lemma on the decay rate of $G(J, u|Q_{x,r})$ as r goes to 0.

Lemma 10.44. *Let τ_0 be small enough such that the decay lemma applies, associated to $(r_0, \vartheta_0, \epsilon_0)$. Let $\tau_1 \in (0, \epsilon_0^2)$ be small enough such that the decay lemma applies with the associated $(r_1, \vartheta_1, \epsilon_1)$. Finally, let*

$$\hat{r} := \min\left(r_0, r_1, \epsilon_0^2\tau_1^2, \epsilon_0^2\tau_0^3\overline{r}^{-1}\right), \vartheta := \min(\vartheta_0, \vartheta_1).$$

Suppose that (J, u) is a (Λ, ϑ, r)-minimizer and $G(J, u|Q_{x,r}) \leq \epsilon_1 r$ for some $r \in (0, \hat{r})$, then for all $k \in \mathbb{N}$,

$$G(J, u|Q_{x,\tau_0^k\tau_1 r}) < \epsilon_0 \tau_0^{\frac{3}{2}k} \tau_1 r.$$

Proof. In the following, we write $g(r) = G(J, u|Q_{x,r})$. We start with the initialization, and we fix in the proof of this lemma some $r \in (0, \hat{r})$.

- Either $g(r) > r^{3/2}$ so $g(\tau_1 r) \leq \epsilon_1 \tau_1^{3/2} r \leq \epsilon_0 \tau_1 r$ by definition of τ_1.
- Or $g(r) \leq r^{3/2}$, so $g(\tau_1 r) \leq r^{3/2} \leq \epsilon_0 \tau_1 r$ by definition of $\hat{r}$.

Then we prove the induction; suppose that it is true for some $k \in \mathbb{N}$, notice that by definition of G we have

$$\mathscr{H}^1(J \cap Q_{\tau_0^k\tau_1 r}) \leq g(\tau_0^k \tau_1 r) \leq \epsilon_0 \tau_0^k \tau_1 r,$$

so the decay lemma may be applied. Then again we separate two cases

- Either $g(\tau_0^k \tau_1 r) > (\tau_0^l \tau_1 r)^{3/2}$ so applying the decay lemma we have

$$g(\tau_0^{k+1}\tau_1 r) \leq \tau_0^{3/2} g(\tau_0^k \tau_1 r) \leq \epsilon_0 \tau_0^{\frac{3}{2}(k+1)} \tau_1 r,$$

- Or $g(\tau_0^k \tau_1 r) \leq (\tau_0^k \tau_1 r)^{3/2}$ and in this case,

$$g(\tau_0^{k+1}\tau_1 r) \leq g(\tau_0^k \tau_1 r) \leq \sqrt{\frac{\tau_1 r}{\tau_0^3 \epsilon_0^2}} \epsilon_0 \tau_0^{\frac{3}{2}(k+1)} \tau_1 r \leq \epsilon_0 \tau_0^{\frac{3}{2}(k+1)} \tau_1 r.$$

□

In particular this may be rewritten as: if $G(J,u|Q_{x,r}) \leq \epsilon_1 r$, then for any $\rho \in (0,r)$,

$$G(J,u|Q_{x,\rho}) \leq C_0 r^{-\frac{1}{2}} \rho^{\frac{3}{2}}, \text{ where } C_0 = \max\left(\epsilon_1 \tau_1^{-\frac{3}{2}}, \epsilon_0 \tau_0^{-\frac{3}{2}} \tau_1^{-\frac{1}{2}}\right).$$

Let now u be a $(\Lambda, \vartheta, \overline{r})$ minimizer in $Q_{x,2r}$ for some $r < \hat{r}$ and suppose that

$$G(J,u|Q_{x,2r}) \leq \epsilon_1 r.$$

By rectifiability of J, let J^+ be defined as in (10.92). Notice that for any $y \in Q_{x,r}$, we have $G(J,u|Q_{y,r}) \leq \epsilon r$ so $G(J,u|Q_{y,\rho}) = o_{\rho\to 0}(\rho)$, meaning $J^+ \cap Q_{x,r} = \emptyset$. As a consequence, we get $\mathscr{H}^1(J \cap Q_{x,r}) = 0$. □

Proposition 10.45. *Let $\Lambda, \overline{r} > 0$, then there exists $\vartheta > 0$ such that for any $(\Lambda, \vartheta, \overline{r})$-almost quasi-minimizer (J,u) of $G(\cdot|\Omega)$ for some $\Omega \Subset \mathbb{R}^2$, we have*

$$\mathscr{H}^1\left(\Omega \cap \overline{J^+} \setminus J^+\right) = 0.$$

Proof. Let ϵ, r be chosen as previously. There is a $\mathscr{H}^1$-negligible set $N \subset \Omega \setminus J$ such that

$$\forall x \in \Omega \setminus (J \cup N), \ \frac{G(J,u|Q_{x,\rho})}{\rho} \underset{\rho\to 0}{\longrightarrow} 0.$$

Let us quickly remind the proof of this: let μ be the Radon measure on Ω defined by

$$\mu(E) = \int_E |e(u)|^2 d\mathscr{L}^2 + \mathscr{H}^1 \lfloor J(E),$$

let $N^t = \left\{x \in \Omega \setminus J : \limsup_{\rho\to 0} \frac{\mu(Q_{x,\rho})}{2\rho} \geq t\right\}$, then by a classical Besicovitch covering argument,

$$\mu \lfloor N^t \geq t \mathscr{H}^1 \lfloor N^t,$$

so $\mathscr{H}^1(N^t) < \infty$, meaning $\mathscr{L}^2(N^t) = 0$, which implies

$$\mathscr{H}^1(N^t) \leq t^{-1} \int_{N^t} |e(u)|^2 d\mathscr{L}^2 = 0,$$

and we let $N = \cup_{t \in \mathbb{Q}_+^*} N_u^t$.

According to the previous corollary, $\Omega \cap \overline{J^+} \subset J \cup N = J^+ \cup (J \setminus J^+) \cup N$, and the last two sets are $\mathscr{H}^1$-negligible so $\mathscr{H}^1(\Omega \cap \overline{J^+} \setminus J^+) = 0$.

□

We now arrive at the conclusion of this section, with the proof of the second main theorem 10.2.

Proof of etiquette Theorem 10.2. In all the following we let $\Lambda := 4\mathrm{Lip}(f)$ and $J = J_u \cup \partial^* E$. We also suppose (up to multiplying u by $c^{-\frac{1}{2}}$) that the constant c of (10.26) is 1. We first prove that (J,u) is an almost quasiminimizer according to Definition 10.40. Indeed, let $v \in \mathcal{V}(\Omega)$ be a perturbation of u on a square $Q_{x,r}$ as in Definition 10.40, notice then that either

$$\mathscr{H}^1(\partial^*(E \setminus Q_{x,r}) \cap (Q_{x,r} \setminus \partial^* E)) = 0 \text{ or } \mathscr{H}^1(\partial^*(E \cup Q_{x,r}) \cap (Q_{x,r} \setminus \partial^* E)) = 0$$

Indeed suppose both conditions are false, then there exists $p \in E^{(1)} \cap \partial Q_{x,r}$ and $q \in E^{(0)} \cap \partial Q_{x,r}$ that are not in one of the corners. Without loss of generality we suppose $p, q \in \{x - re_2 + \mathbb{R}e_1\}$ with $p_1 < q_1$, the case when both are in different sides is analog. We let:

- $\epsilon > 0$ that will be chosen small later,
- $C_p = p + [-s,0] \times [0,s]$, $C_q = q + [0,s]^2$, $C_{p,q}$ the convex envelope of $C_p \cup C_q$,

- $g_s : [p_1 - s, q_1 + s] \to [0, 1]$ that is zero on the boundaries, affine on $[p_1 - s, p_1]$ and $[q_1, q_1 + s]$ and equal to 1 on $[p_1, q_1]$,
- $f_s : [0, s] \to [0, 1]$ such that f is zero on the boundary and $f = 1$ on $[\epsilon s, (1 - \epsilon)s]$,
- $\varphi_s(x) = g_s(x_1)f_s(x_2)$.

Then

$$\begin{aligned}\mathscr{H}^1(J \cap (Q_{x,r} \setminus Q_{x,r-s})) &\geq \int_{\partial^* E} \varphi_s(\nu_E)_1 dx = \int_E \partial_1 \varphi_s = \frac{1}{r}\int_{C_p \cap E} f_s(x_2)dx - \frac{1}{r}\int_{C_q \cap E} f_s(x_2)dx \\ &\geq \frac{|E \cap C_p|}{|C_p|}(1 - 2\epsilon)^2 r - \frac{|E \cap C_q|}{|C_q|} r\end{aligned}$$

Since $\frac{|E\cap C_p|}{|C_p|} \to 1$, $\frac{|E\cap C_q|}{|C_q|} \to 0$, and ϵ may be taken arbitrarily small we get $\limsup_{s\to 0} \frac{\mathscr{H}^1(J\cap(Q_{x,r}\setminus Q_{x,r-s}))}{s} \geq 1$ which is contrary to the third hypothesis of definition 10.40. In the case where $\mathscr{H}^1(\partial^*(E \cup Q_{x,r}) \cap (Q_{x,r} \setminus \partial^* E)) = 0$, then taking a competitor $F = E \cup Q_{x,r}$, $w = u1_{Q^c_{x,r}}$ implies by minimality that $u_{|Q_{x,r}} = 0$ almost everywhere and $\mathscr{H}^1(\partial^* E \cap Q_{x,r}) = 0$ so there is nothing to prove. In the second case then using the competitor $(E \setminus Q_{x,r}, v)$ we get:

$$\mathcal{J}(E, u) \leq \mathcal{J}(E \setminus Q_{x,r}, v),$$

which gives exactly

$$\int_{Q_{x,r}} |e(u)|^2 d\mathscr{L}^2 + \mathscr{H}^1(J \cap Q_{x,r}) \leq \int_{Q_{x,r}} |e(v)|^2 d\mathscr{L}^2 + \Lambda r^2$$

so according to the previous result, $\mathscr{H}^1(\Omega \cap \overline{J^+} \setminus J^+) = 0$. Let B be an open ball in $\Omega \setminus \overline{J^+}$, then $u \in SBD(B)$ with $\mathscr{H}^1(J_u \cap B) \leq \mathscr{H}^1(J^+ \cap B) = 0$, so $u \in H^1(B)$ and it is locally a solution of Stokes equation. In particular it is analytic, so $J_u \cap B = \emptyset$. This last fact implies that $\Omega \cap J_u \subset \overline{J^+}$, and every point of $\partial^* E$ is in J^+ by definition of the reduced boundary (see [194, Th 15.5, Th 15.9]). Notice finally that $J^+ \subset J_u \cup \partial^* E$. As a consequence

$$\Omega \cap J^+ \subset \Omega \cap (J_u \cup \partial^* E) \subset \overline{J^+}$$

so $\Omega \cap \overline{J_u \cup \partial^* E} = \Omega \cap \overline{J^+}$ and we get the conclusion.

□

10.9 Remarks on the Navier-Stokes equation

Some of the previous results extend to the Navier-Stokes equation to some extent. Say indeed that for any obstacle with finite perimeter E, we say $u \in \mathcal{V}_{E,V}(\Omega)$ is a solution of the (stationnary) Navier-Stokes equation when for every $\varphi \in \mathcal{V}_{E,0}(\Omega)$ with $J_\varphi \subset J_u$,

$$\int_{\Omega\setminus K} (2\mu e(u) : e(\varphi) + \nabla_u u \cdot \varphi)\, d\mathscr{L}^d + \beta \int_{\partial^* E} u^+ \cdot \varphi^+ d\mathscr{H}^{d-1} + \beta \int_{J_u \setminus \partial^* E} \left(u^+ \cdot \varphi^+ + u^- \cdot \varphi^-\right) d\mathscr{H}^{d-1} = 0$$

and the drag exerted on E associated to u is still given by

$$\mathscr{E}(u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2\, d\mathscr{L}^d + \beta \int_{\partial^* E} |u^+|^2\, d\mathscr{H}^{d-1} + \beta \int_{J_u\setminus\partial^* E} \left(|u^+|^2 + |u^-|^2\right) d\mathscr{H}^{d-1}$$

Now, consider (E_i, u_i) a sequence of solutions of Navier Stokes that minimizes (10.26) ; then using the compactness and lower semicontinuity result we obtain that for some subsequence of (E_i, u_i), there is a set of finite perimeter E and a function $u \in \mathcal{V}_{E,V}(\Omega)$ such that $E_i \to E$ in $L^1(\Omega)$, $u_i \to u$ in $L^2(\Omega)$, with $e(u_i)$ converging $L^2(\Omega)$-weakly to $e(u)$ and

$$\mathcal{J}(E, u) \leq \liminf_{i\to\infty} \mathcal{J}(E_i, u_i)$$

Suppose then that we are in the two-dimensional setting where E is compact, connected and $J_u \subset E$, with a Hausdorff convergence of E_i to E. Then for any $\varphi \in \mathcal{V}^{reg}_{E,0}(\Omega)$ with support in $\Omega \setminus E$, φ has support in $\Omega \setminus E_i$ for any large i and passing to the limit the variational Navier-Stokes equation on u_i we get

$$\int_{\Omega \setminus K} (2\mu e(u) : e(\varphi) + \nabla_u u \cdot \varphi)\, d\mathscr{L}^d = 0$$

so u is a variational solution of Navier-Stokes in $\Omega \setminus E$.

However we do not necessarily expect that the limit u would verify the Navier condition of parameter β on ∂E. One argument against this is that the drag associated to the Navier-Stokes equation is generally not increasing in term of β as was demonstrated in [176] (while it is clearly for the Stokes equation, due to the variational caracterization of the solution). More precisely, it was proved in [183],[184] that fast, small periodic oscillation of the boundary in the (Navier-)Stokes equation creates at the limit a Navier condition with a new friction $\tilde{\beta} \in [0, +\infty]$ that may be higher than the initial friction β, and even possibly a anisotropic factor that favors certain directions for the flow (in dimension three and more). In other words, the minimum of the drag may be reached through the addition of friction - corresponding to Navier condition with different characteristic than ours - that is asymptotically obtained by the apparition of a microstructure on the boundary.

Bibliography

[171] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[172] Luigi Ambrosio, Alessandra Coscia, and Gianni Dal Maso. Fine properties of functions with bounded deformation. *Arch. Rational Mech. Anal.*, 139(3):201–238, 1997.

[173] Luigi Ambrosio and Paolo Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[174] G. Bellettini, A. Coscia, and G. Dal Maso. Compactness and lower semicontinuity properties in SBD(Ω). *Math. Z.*, 228(2):337–351, 1998.

[175] B. Bogosel, D. Bucur, and A. Giacomini. Optimal shapes maximizing the Steklov eigenvalues. *SIAM J. Math. Anal.*, 49(2):1645–1680, 2017.

[176] Matthieu Bonnivard and Dorin Bucur. Microshape control, riblets, and drag minimization. *SIAM J. Appl. Math.*, 73(2):723–740, 2013.

[177] Franck Boyer and Pierre Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[178] D. Bucur and A. Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[179] Dorin Bucur, Eduard Feireisl, and Šárka Nečasová. Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions. *Arch. Ration. Mech. Anal.*, 197(1):117–138, 2010.

[180] Dorin Bucur and Alessandro Giacomini. Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach. *Arch. Ration. Mech. Anal.*, 218(2):757–824, 2015.

[181] Dorin Bucur and Alessandro Giacomini. Shape optimization problems with Robin conditions on the free boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(6):1539–1568, 2016.

[182] Luis A. Caffarelli and Dennis Kriventsov. A free boundary problem related to thermal insulation. *Comm. Partial Differential Equations*, 41(7):1149–1182, 2016.

[183] J. Casado-Díaz, M. Luna-Laynez, and F. J. Suárez-Grau. Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall. *Math. Models Methods Appl. Sci.*, 20(1):121–156, 2010.

[184] J. Casado-Díaz, M. Luna-Laynez, and F. J. Suárez-Grau. Asymptotic behavior of the Navier-Stokes system in a thin domain with Navier condition on a slightly rough boundary. *SIAM J. Math. Anal.*, 45(3):1641–1674, 2013.

[185] Antonin Chambolle, Sergio Conti, and Flaviana Iurlano. Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. *J. Math. Pures Appl. (9)*, 128:119–139, 2019.

[186] Antonin Chambolle and Vito Crismale. Compactness and lower semicontinuity in *gsbd*. *Journal of the European Mathematical Society*, 23(3):701–719, 2020.

[187] Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Integral representation for functionals defined on SBD^p in dimension two. *Arch. Ration. Mech. Anal.*, 223(3):1337–1374, 2017.

[188] Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Existence of strong minimizers for the Griffith static fracture model in dimension two. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(2):455–474, 2019.

[189] Gianni Dal Maso. Generalised functions of bounded deformation. *J. Eur. Math. Soc. (JEMS)*, 15(5):1943–1997, 2013.

[190] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

[191] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[192] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[193] Manuel Friedrich, Matteo Perugini, and Francesco Solombrino. Lower semicontinuity for functionals defined on piecewise rigid functions and on $GSBD$. *J. Funct. Anal.*, 280(7):108929, 45, 2021.

[194] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[195] O. Pironneau. On optimum profiles in Stokes flow. *J. Fluid Mech.*, 59:117–128, 1973.

[196] R. Temam. Variational problems of plasticity. In *Trends in applications of pure mathematics to mechanics, Vol. III (Edinburgh, 1979)*, volume 11 of *Monographs Stud. Math.*, pages 209–217. Pitman, Boston, Mass.-London, 1981.

[197] Roger Temam and Gilbert Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75(1):7–21, 1980/81.

[198] E. O. Tuck. Toward the calculation and minimization of stokes drag on bodies of arbitrary shape. In *3rd Australasian Conference on Hydraulics and Fluid Mechanics (Institute of Engineers, Australia, 1970)*, pages 29–32, 1968.

[199] S. R. Watson. Towards the minimum drag on a body of given volume in slow viscous flow. *J. Inst. Math. Appl.*, 7:367–376, 1971.