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Contribution à l'étude du spectre des systèmes dynamiques substitutifs

Contribution to the study of substitutive dynamical systems

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Affidavit

Je soussigné, Juan Guillermo Marshall Maldonado, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique de Alexander I. Bufetov et Pascal Hubert, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte d'Aix-Marseille Université relative à la lutte contre le plagiat.

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I, undersigned, Juan Guillermo Marshall Maldonado, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific direction of Alexander I. Bufetov and Pascal Hubert, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and the Aix-Marseille University charter on the fight against plagiarism.

This work has not been submitted previously either in this country or in aother country in the same or in a similar version to any other examination body.

Marseille, 10/03/2022.

«Ah, people only know what you tell them, Carl.»

«Well, then tell me this, Barry Allen, Secret Service.

How did you know I wouldn't look in your wallet?»

«The same reason the Yankees always win.

Nobody can keep their eyes off the pinstripes.»

«The Yankees win because they have Mickey Mantle.

No one ever bets on the uniform.»

(Frank chuckles)

«You sure about that, Carl?»

«I'll tell you what I am sure of.

You're going to get caught.

One way or another, it's a mathematical fact.

It's-It's like Vegas.

The House always wins.»

«Well, Carl, I'm sorry, but I have to go.»

«Ah. You didn't call just to apologize, did you?»

(laughing)

«What do you mean?»

«You... you... you have no one else to call.»

(laughing)

«Oh, ho, ho.»

(phone bell dings)

(guffaws)

(humming)

«...Morn and night...»

(melancholy melody playing)

(melody fades)

Abstract

This thesis studies the spectrum of systems associated to substitutions, in particular the continuous spectrum. We have based the analysis on the study of the spectral cocycle and twisted Birkhoff sums (and integrals). These tools have been widely used in many recent works to ensure quantitative rates of weak mixing and spectrum singularity in settings such as substitution subshifts, S-adic systems, translations surfaces, deterministic and random substitutive tilings and interval exchange transformations.

The first results are obtained in the case of suspension flows over Salem type substitutions. We prove Hölder decays for correlation measures in the spectral parameters belonging to the algebraic field arising from the Salem number. The proof is based in a fine analysis of the distribution modulo 1 of the sequence $(\eta\alpha^n)_{n\geq 0}$, where $\eta \in \mathbb{Q}(\alpha)$ and α is the corresponding Salem number.

The second set of results are related to the Thue-Morse substitution. We study the behavior of the top Lyapunov exponents of the spectral cocycle associated to the Thue-Morse substitution and its topological factors. We prove that for all topological factors the top Lyapunov exponent is zero, and we also give the sub-exponential behavior of the twisted Birkhoff sums.

Résumé

Cette thèse étudie le spectre des systèmes associés aux substitutions, en particulier le spectre continu. Nous avons basé l'analyse sur l'étude du cocycle spectral et des sommes (et intégrales) de Birkhoff tordues. Ces outils ont été utilisés récemment dans de nombreux travaux pour assurer des taux quantitatifs de mélange faible et singularité du spectre dans des contextes tels que les sous-décalages substitutifs, les systèmes S-adiques, les surfaces de translations, les pavages substitutifs déterministes et aléatoires et les transformations d'échange d'intervalles.

Les premiers résultats sont obtenus dans le cas des flots de suspension sur les substitutions de type Salem. Nous prouvons des décroissances de type Hölder pour les mesures de corrélation sur les paramètres spectraux appartenant au corp algébrique engendré par le nombre de Salem. La preuve est basée sur une analyse fine de la distribution modulo 1 de la suite $(\eta\alpha^n)_{n\geq 0}$, où $\eta \in \mathbb{Q}(\alpha)$ et α est le nombre de Salem correspondant.

La deuxième série de résultats est liée à la substitution de Thue-Morse. Nous étudions le comportement des exposants de Lyapunov maximaux du cocycle spectral associé à la substitution de Thue-Morse et à ses facteurs topologiques. Nous prouvons que pour tous les facteurs topologiques, l'exposant de Lyapunov maximal est nul, et nous donnons également le comportement sous-exponentiel des sommes de Birkhoff tordues.

Contents

Chapter 1

Introduction

The aim of this thesis is to study some aspects of the spectrum of substitutions. Substitutions are simple, but rich combinatorial objects appearing naturally in many domains across mathematics such as symbolic dynamics, number theory, combinatorics of words, Diophantine approximation, and so on. The spectrum of substitutions has been extensively studied (see for example the book of M. Queffélec [61] for an extensive introduction). Nevertheless, the continuous part of the spectrum of general substitutions is still not well understood, and is the focus of this document.

Chapter 1 contains general background necessary for the rest of the chapters: we focus our attention on results concerning the spectrum of substitution subshifts, self-similar and generic suspension flows over them. We finish the chapter by describing our results and open questions associated with them.

In Chapter 2, we study the spectral measures associated to a dynamical system arising from a primitive substitution having as its Perron-Frobenius eigenvalue a Salem number. We obtain a Hölder exponent for those measures at points belonging to the algebraic field generated by the Salem number.

In Chapter 3 we study the invariance by topological conjugacy of the top Lyapunov exponent of the spectral cocycle, in the simple case of the Thue-Morse substitution. We prove that in fact, the exponent vanishes almost everywhere and the same holds for any subshift topological factor.

1.1 Preliminaries

We start by recalling basic definitions and results that give context to the rest of the text. Many results in this section are classic and some of them are just mentioned vaguely. The reader may look for formal statements in the references given. We emphasize in definitions since they will serve as reference for the other chapters.

1.1.1 Measure preserving transformations and flows

Let $(X, \mathcal{B}, \mu)$ be a probability space, i.e., $\mathcal{B}$ is a σ-algebra and $\mu(X) = 1$.

Definition 1.1 *Let $T : X \longrightarrow X$ a measurable transformation of X, in the sense that for each $A \in \mathcal{B}$, $T^{-1}A \in \mathcal{B}$. We will say μ is T-**invariant** (or invariant by T, or preserved by T) if for each $A \in \mathcal{B}$ we have $\mu(T^{-1}A) = \mu(A)$. If T is invertible we will also refer to T as a $\mathbb{Z}$-**action** on X.*

Definition 1.2 *Let $H : X \times \mathbb{R} \longrightarrow X$ be a measurable transformation (for the completion of the product σ-algebra $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra). We will say $(H(\cdot, t))_{t\in\mathbb{R}} = (h_t)_{t\in\mathbb{R}}$ is a **flow** on X if*

- $h_0 = id_X$,
- $h_s \circ h_t = h_{s+t}$ *for all* $s, t \in \mathbb{R}$.

*We will say μ is **invariant** by $(h_t)_{t\in\mathbb{R}}$ if for every $t \in \mathbb{R}$, μ is h_t-invariant. We will also refer to $(h_t)_{t\in\mathbb{R}}$ as an $\mathbb{R}$-**action** on X.*

Definition 1.3 *A (measure-theoretic) discrete time (resp. continuous time) **dynamical system** is a tuple $(X, \mathcal{B}, \mu, T)$ (resp. $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$) such that μ is T-invariant (resp. $(h_t)_{t\in\mathbb{R}}$ invariant).*

Examples

1. **Irrational rotation**. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and consider $X = \mathbb{R}/\mathbb{Z}$ and $T(x) = R_\alpha(x) = x + \alpha \pmod 1$. An invariant Borel probability measure in this case is the Lebesgue measure Leb.
2. **Gauss map**. Consider again $X = \mathbb{R}/\mathbb{Z}$, $T(x) = \{1/x\}$ for $x \neq 0$ and $T(0) = 0$, where $\{x\}$ corresponds to the fractional part of x. An invariant Borel probability is given by
$$\mu(A) = \frac{1}{\log(2)} \int_A \frac{1}{1+s} ds, \quad \text{for each } A \subseteq X \text{ Borel.}$$
3. Let X be a compact differentiable manifold, Vol the volume measure (arising from a volume form) and $h_t : x \mapsto \gamma_x(t)$, where γ_x is the only solution of
$$\begin{cases} \gamma' = F(\gamma(t)) \\ \gamma(0) = x \end{cases}$$
where F is a C^1 vector field such that $\mathrm{Div}(F)$ is identically zero.

Starting from a positive function $f : X \longrightarrow \mathbb{R}_+$, there is a canonical way to construct a flow $(h_t)_{t\in\mathbb{R}}$ from a transformation T of X. We will show this construction in the next subsection.

In Chapter 3 we will need the next notion of "equality" of dynamical systems. We will only use this definition for discrete time dynamical systems.

Definition 1.4 *Let $(X, \mathcal{B}, \mu, T)$, $(Y, \mathcal{V}, \nu, S)$ be two dynamical systems. We will say that they are (measurably) **conjugate** if there exist two measurable sets $\widetilde{X} \subset X$, $\widetilde{Y} \subset Y$ of total mass* *and a bimeasurable map* $\pi : \widetilde{X} \longrightarrow \widetilde{Y}$ *such that*

$$\pi \circ T = S \circ \pi \qquad \textit{and} \qquad \pi_* \mu = \nu,$$

where $\pi_*\mu$ *is the push-forward of* μ*, defined by* $\pi_*\mu(E) = \mu(\pi^{-1}(E))$*, for every* $E \in \mathcal{V} \cap \widetilde{Y}$*. If* π *is measurable and onto, we will say the two systems are* ***semi-conjugate*** *or* $(Y, \mathcal{V}, \nu, S)$ *is a* ***factor*** *of* $(X, \mathcal{B}, \mu, T)$*.*

All previous definitions have their topological counterpart, basically changing the words measurable by continuous. We summarize all these notions in the next definition (for discrete time).

Definition 1.5 *Let* X *be a compact metric space and* $T : X \longrightarrow X$ *a continuous transformation. We call* ***topological dynamical system*** *the pair* (X, T)*. We will say two systems* $(X, T), (Y, S)$ *are* ***topologically conjugate*** *if there exists an homeomorphism* $\pi : X \longrightarrow Y$ *such that*

$$\pi \circ T = S \circ \pi.$$

If π *continuous and onto, we will say the two systems are* ***topologically semi-conjugate*** *or* (Y, S) *is a* ***topological factor*** *of* (X, T)*.*

A basic notion from topological dynamics is the next one.

Definition 1.6 *Let* (X, T) *be a topological dynamical system. We say* (X, T) *is* ***minimal*** *if the forward orbit of every point* $x \in X$ *is dense, i.e.,* $O(x) = \{T^n(x) | n \geq 0\}$ *is a dense subset of* X*.*

From a topological dynamical system we can always obtain a measure-theoretic dynamical system by considering the Borel σ-algebra, since the space of invariant measures is always nonempty, as the Krylov-Bogolyubov theorem shows (see [57]). Now we recall some classical notions in ergodic theory.

Definition 1.7 *We say a dynamical system* $(X, \mathcal{B}, \mu, T)$ *or* T *(resp.* $(X, \mathcal{B}, \mu, (h_t)_{t \in \mathbb{R}})$ *or* $(h_t)_{t \in \mathbb{R}}$*) or* μ *is*

- ***ergodic*** *if*

$$\lim_{n \to \infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \int f \circ T^j(x) g(x) \, d\mu(x) - \int f(x) d\mu(x) \int g(x) \, d\mu(x) \right| = 0,$$

for all $f, g \in L^2(X, \mu)$*.*

$$\left(\textit{resp. if } \lim_{t \to \infty} \left| \frac{1}{t} \int_0^t \left(\int f \circ h_s(x) g(x) \, d\mu(x) \right) ds - \int f(x) d\mu(x) \int g(x) \, d\mu(x) \right| = 0 \right)$$

- ***weakly mixing*** *if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left| \int f \circ T^j(x) g(x) \, d\mu(x) - \int f(x) d\mu(x) \int g(x) \, d\mu(x) \right| = 0,$$

for all $f, g \in L^2(X, \mu)$.

$\left(\textit{resp. if } \lim_{t\to\infty} \dfrac{1}{t} \int_0^t \left| \left(\int f \circ h_s(x) g(x)\, d\mu(x) \right) - \int f(x) d\mu(x) \int g(x)\, d\mu \right| ds = 0 \right)$

- ***mixing*** *if*

$$\lim_{n\to\infty} \int f \circ T^n(x) g(x)\, d\mu(x) - \int f(x) d\mu(x) \int g(x)\, d\mu(x) = 0,$$

for all $f, g \in L^2(X, \mu)$.

$\big(\textit{resp. if } \lim_{t\to\infty} \big(\int f \circ h_t(x) g(x)\, d\mu(x) \big) - \int f(x) d\mu(x) \int g(x)\, d\mu(x) = 0 \big)$.

A related, but of different nature, notion is the next one.

Definition 1.8 *We say a dynamical system* $(X, \mathcal{B}, \mu, T)$ *(resp.* $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$*) is* ***uniquely ergodic*** *if* μ *is the only measure defined on* $\mathcal{B}$ *which is* T*-invariant (resp. which is* $(h_t)_{t\in\mathbb{R}}$*-invariant measure).*

The nomenclature is justified since a uniquely ergodic system is ergodic (see [57]).

1.1.2 Dynamical systems arising from substitutions

The basic notions of substitutions may be found in the books of M. Queffélec [61] or P. Fogg [40] with more detail.

Let $\mathcal{A}$ be a finite set (we will call it **alphabet** and its elements **letters**).

Definition 1.9 *A* ***substitution*** *on* $\mathcal{A}$ *is a map* $\zeta : \mathcal{A} \longrightarrow \mathcal{A}^+$*, where* $\mathcal{A}^+$ *denote the set of finite (nonempty) words on* $\mathcal{A}$*, such that the image of at least one letter has length at least two. By concatenation, it is natural to extend a substitution to* $\mathcal{A}^+$*,* $\mathcal{A}^{\mathbb{N}}$ *(one-sided sequences) or* $\mathcal{A}^{\mathbb{Z}}$ *(two-sided sequences).*

In particular, the iterates $\zeta^n(a) = \zeta(\zeta^{n-1}(a))$ for $a \in \mathcal{A}$, $n \geq 1$, are well-defined. In general, we will take as alphabet the set $\{1, \ldots, d\}$, for some $d \geq 2$.

Examples

1. **Fibonacci substitution**. Let $\mathcal{A} = \{1, 2\}$,

$$\zeta(1) = 12, \quad \zeta(2) = 1.$$

2. **Tribonacci substitution**. Let $\mathcal{A} = \{1, 2, 3\}$,

$$\zeta(1) = 12, \quad \zeta(2) = 13, \quad \zeta(3) = 1.$$

3. **Holton-Zamboni substitution**. Let $\mathcal{A} = \{1, 2, 3, 4\}$,

$$\zeta(1) = 12, \quad \zeta(2) = 14,$$
$$\zeta(3) = 2, \quad \zeta(4) = 3.$$

4. **Bufetov-Solomyak substitution**. Let $\mathcal{A} = \{1, 2, 3\}$,

$$\zeta(1) = 1112, \quad \zeta(2) = 123, \quad \zeta(3) = 2.$$

5. **Thue-Morse substitution**. Let $\mathcal{A} = \{0, 1\}$

$$\zeta(0) = 01, \quad \zeta(1) = 10.$$

For a word $w \in \mathcal{A}^+$, denote its length by $|w|$ and by $|w|_a$ the number of symbols a found in w. A particular class of substitutions is the one that images of letters have the same length.

Definition 1.10 *We will say a substitution ζ over $\mathcal{A}$ is **of constant length** if there exists $q \geq 2$ such that $|\zeta(a)| = q$, for every $a \in \mathcal{A}$.*

An important object associated to substitutions is its "abelianization" defined in the next definition.

Definition 1.11 *Let $\mathcal{A} = \{1, \dots, d\}$ and ζ a substitution over $\mathcal{A}$. We will call **substitution matrix** associated to a substitution ζ the $d \times d$ matrix with integer entries defined by $M_\zeta(a, b) = |\zeta(b)|_a$. A substitution is called **primitive** if its substitution matrix is primitive.*

Example Let ζ be the Holton-Zamboni substitution defined in the examples above. Then its substitution matrix is

$$M_\zeta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and this substitution is primitive.

We recall that a primitive matrix with nonnegative integer entries always has a simple dominant real eigenvalue of modulus greater than 1 admitting a coordinatewise positive eigenvector by the Perron-Frobenius theorem (see e.g. [61]). This is called the **Perron-Frobenius eigenvalue** (resp. **Perron-Frobenius eigenvector**), and in the context of primitive substitutions, we will call it the Perron-Frobenius eigenvalue (resp. Perron-Frobenius eigenvector) of the substitution.

Now we define a first class of dynamical systems arising from substitutions.

Definition 1.12 *The **substitution subshift space** associated to ζ is the set $\mathfrak{X}_\zeta$ of sequences $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ exist $a \in \mathcal{A}$ and $n \in \mathbb{N}$ such that $x_i \dots x_{i+k}$ is a subword of some $\zeta^n(a)$.*

Note that for any power of a primitive substitution, the substitution subshift space does not change, i.e., $\mathfrak{X}_{\zeta^k} = \mathfrak{X}_\zeta$ for any $k \geq 1$. In particular, considering a suitable power of the substitution, we may always suppose there is a **fixed point** of the substitution: there exists $\mathbf{u} \in \mathfrak{X}_\zeta$ such that $\zeta(\mathbf{u}) = \mathbf{u}$. If for some letter $a \in \mathcal{A}$, $\zeta(a)$ starts with the letter a, then, for every $n \geq 1$, $\zeta^{n-1}(a)$ is a prefix of $\zeta^n(a)$. Iterating the substitution indefinitely, this produce a unique one-sided sequence $\mathbf{u}$ such that for every $N \geq 1$ there exist $n \geq 1$ such that $\mathbf{u}_{[0,N]}$ is prefix of $\zeta^n(a)$. We denote this (one-sided) fixed point by $\zeta^\infty(a)$.

We can decompose $\mathfrak{X}_\zeta$ in (one letter) **cylinders**:

$$\mathfrak{X}_\zeta = \bigcup_{a \in \mathcal{A}} \mathfrak{X}_a, \quad \mathfrak{X}_a = \{\mathbf{x} \in \mathfrak{X}_\zeta \mid x_0 = a\} = [a].$$

Analogously, for $w = w_0 \dots w_k \in \mathcal{A}^+$, define the cylinder $[w] = \{\mathbf{x} \in \mathfrak{X}_\zeta \mid x_0 = w_0, \dots, x_k = w_k\}$. We can endow $\mathcal{A}^\mathbb{Z}$ with the product topology (of the discrete topology in each copy of $\mathcal{A}$). It is metrizable and makes $\mathcal{A}^\mathbb{Z}$ a compact metric space. The same holds for $\mathfrak{X}_\zeta$, since $\mathfrak{X}_\zeta$ is a closed subspace of $\mathcal{A}^\mathbb{Z}$.

A classical result is that the (continuous) $\mathbb{Z}$-action on $\mathfrak{X}_\zeta$ given by the **left-shift** $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ is minimal and uniquely ergodic when ζ is primitive (see [61]). In fact, the only invariant Borel probability measure μ is characterized, for each cylinder $[w] = [w_0 \dots w_k]$, by the frequency it appears in any $\mathbf{x} \in \mathfrak{X}_\zeta$, i.e.,

$$\mu([w]) = \lim_{n\to\infty} \frac{N(w, x_0 \dots x_{n-1})}{n},$$

where $N(w, x_0 \dots x_{n-1})$ is the number of occurrences of w in $x_0 \dots x_{n-1}$. Also, the vector of positive entries $(\mu([a]))_{a\in\mathcal{A}}$ is a Perron-Frobenius eigenvector of the matrix M_ζ.

A sequence $\mathbf{x} \in \mathcal{A}^\mathbb{Z}$ is **shift-periodic** if there exists $n \geq 1$ such that $\mathbf{x} = T^n\mathbf{x}$. For a primitive substitution, every point of $\mathfrak{X}_\zeta$ is shift-periodic (and $\mathfrak{X}_\zeta$ is finite) or every point of $\mathfrak{X}_\zeta$ is not shift-periodic (and $\mathfrak{X}_\zeta$ is infinite). We will be interested in the latter case, therefore we will always assume $\mathfrak{X}_\zeta$ does not contain shift-periodic points. In this case, we say the substitution ζ is **aperiodic**.

Now we define one of the main dynamical systems we will be dealing with.

Definition 1.13 *Let ζ be a primitive aperiodic substitution. The **substitution subshift** associated to ζ is the tuple $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$.*

Now we turn to the continuous time counterpart of the substitution subshift. Let start by recalling how to construct a continuous time dynamical system from a discrete time one and from a positive function. Let (X, T) be topological dynamical system, with T invertible and $f : X \longrightarrow \mathbb{R}_+$ a positive function.

Definition 1.14 *Consider $F : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ defined by $F(x,t) = (T(x), t - f(x))$. The **suspension flow** of (X,T) with roof function f, is the topological dynamical system*

$(X^f, (h_t)_{t\in\mathbb{R}})$ defined by

$$X^f = (X \times \mathbb{R})/\sim,$$
$$h_t(x, t') = (x, t' + t) \pmod{\sim},$$

where $\sim$ is the equivalence relation defined by $(x, t) \sim (x', t')$ if and only if $F^n(x, t) = (x', t')$, for some $n \in \mathbb{Z}$.

We may endow $(X^f, (h_t)_{t\in\mathbb{R}})$ with its Borel σ-algebra $\mathcal{B}(X^f)$ and an invariant measure μ. In fact, one can construct an invariant measure on X^f for the flow from an invariant measure on X for the transformation T (see [57], Chapter 3). Therefore, we also refer as suspension flow to the tuple $(X^f, \mathcal{B}(X^f), \mu, (h_t)_{t\in\mathbb{R}})$.

We will identify X^f with a *fundamental domain* (see for example [58], Chapter 3). We will take as fundamental domain

$$\mathcal{D} = \{(x, s) \in X \times \mathbb{R} \mid 0 \le s < f(x)\}.$$

With this identification, the flow h_t (for $t \ge 0$) acts by increasing the second coordinate of a point (x, s) until it arrives to the "top" of its suspension: the point $(x, f(x))$, where we identify this point by the relation $\sim$ to $(T(x), 0) \in \mathcal{D}$ (see Figure 1.1). We will make no further reference to the fundamental domain, and we will simply denote it by X^f.

For the particular case of substitution dynamics, we will focus our attention on the piecewise constant functions on $\mathfrak{X}_\zeta$, where ζ is a primitive aperiodic substitution on $\mathcal{A}$. Let $\vec{p} = (p_a)_{a\in\mathcal{A}}$ be an entrywise positive vector, with components indexed by $\mathcal{A}$.

Definition 1.15 *Consider $f : \mathfrak{X}_\zeta \longrightarrow \mathbb{R}_+$ defined by $f(\mathbf{x}) = p_a$, for any $\mathbf{x} \in \mathfrak{X}_a$. The **suspension flow** of $\mathfrak{X}_\zeta$ with roof vector $\vec{p}$ is the suspension flow $(\mathfrak{X}_\zeta^f, (h_t)_{t\in\mathbb{R}})$ and it will be denoted by $(\mathfrak{X}_\zeta^{\vec{p}}, (h_t)_{t\in\mathbb{R}})$.*

A special choice for $\vec{p}$ is the normalized (positive) Perron-Frobenius eigenvector of the substitution matrix M_ζ^T.

Definition 1.16 *Let $\vec{p} = (p_a)_{a\in\mathcal{A}}$ be the positive normalized Perron-Frobenius eigenvector of M_ζ^T. We call $(\mathfrak{X}_\zeta^{\vec{p}}, (h_t)_{t\in\mathbb{R}})$ the **self-similar suspension flow**.*

Once again, we may again decompose a suspension flow in cylinders:

$$\mathfrak{X}_\zeta^{\vec{p}} = \bigcup_{a\in\mathcal{A}} \mathfrak{X}_a^{\vec{p}}, \quad \mathfrak{X}_a^{\vec{p}} = \left\{(\mathbf{x}, s) \in \mathfrak{X}_\zeta^{\vec{p}} \,\middle|\, x_0 = a\right\}.$$

1.1.3 Conjugacy and semi-conjugacy of substitution subshifts

In this subsection we recall some results and questions related to the (semi-)conjugacy of substitutions subshifts. A good reference is the recent work of F. Durand and J. Leroy [35]. Let us recall the general definition of a subshift.

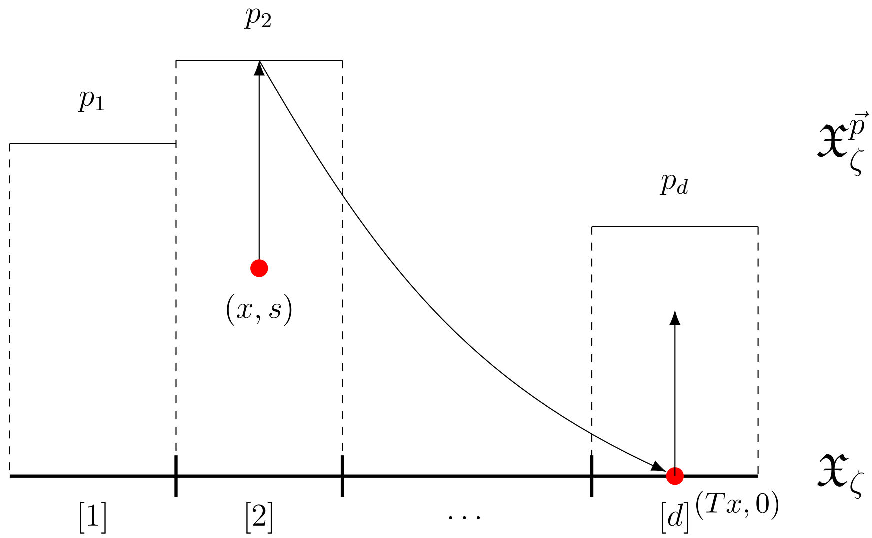

Figure 1.1: Suspension flow over a substitution subshift defined by ζ with a piecewise constant roof function defined by the positive vector $\vec{p} = (p_a)_{a\in\mathcal{A}}$, representing the heights of the rectangles. The arrows indicate the action by the flow $(h_t)_{t\in\mathbb{R}}$.

Definition 1.17 *Let $\mathcal{A}$ be a finite set. A closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ (or the pair (X, T)) is called a* ***subshift*** *if $T(X) \subseteq X$, where T as always denote the left-shift. If $X = \mathfrak{X}_\zeta$, for some substitution ζ on $\mathcal{A}$, we will say X is a* ***substitution subshift****.*

The first natural general question we could ask is the next decision problem.

Question 1.18 *Let ζ, σ be substitutions on $\mathcal{A}, \mathcal{B}$ respectively. Decide whether the associated subshifts are conjugate. What about semi-conjugacy?*

Passing to topological dynamics makes this problem much more accesible since topological factors maps to a subshift have a simple form by the Curtis-Hedlund-Lyndon theorem. Before state the results in this direction, let us recall two results related to the decision problem of conjugacy and semi-conjugacy in substitution dynamics. The first due to F. Durand, B. Host and C. Skau and the second one to F. Durand.

Theorem 1.19 ([34]) *All subshift (topological) factors of substitution systems are substitution subshifts.*

Theorem 1.20 ([33]) *There exist finitely many subshift (topological) factors of a substitution subfhift up to topological conjugacy.*

In the topological setting, we have next two problems.

Question 1.21 *Let ζ, σ be substitutions on $\mathcal{A}, \mathcal{B}$ respectively. Decide whether the associated subshifts are toplogically conjugate. What about topological semi-conjugacy?*

Question 1.22 *Let ζ be a substitution on $\mathcal{A}$. Give the complete list (modulo conjugacy)* *of substitution subshifts topologically conjugate to $(\mathfrak{X}_\zeta, T)$. Give the complete list (modulo conjugacy) of substitution subshifts topologically semi-conjugate to $(\mathfrak{X}_\zeta, T)$.*

The first question has been recently answered by F. Durand and J. Leroy, in the more general setting of uniformly recurrent morphic subshifts, which includes aperiodic primitive substitutions as a particular case.

Theorem 1.23 ([35]) *For two uniformly recurrent substitution subshifts (X, T) and (Y, T'), it is decidable whether they are semi-conjugate. Moreover, if (Y, T') is aperiodic, then there exists a computable constant r such that for any topological factor $\pi : X \longrightarrow Y$ there exist $k \in \mathbb{Z}$ and a factor $\pi' : X \longrightarrow Y$ of radius less than r, such that $\pi = (T')^k \circ \pi'$.*

It is tempting to think we may list all factors from a given substitution system by considering all possible radii, but the radius r in the last result depends strongly on the factor system.

On the other hand, in the constant length case, this is partially solved: it is possible to give the list of all factors of constant length (see [27, 35]). Nevertheless, a conjugate system is not necessary of constant length, as it is shown by the next result of M. Dekking.

Theorem 1.24 ([30]) *There exist infinitely many non constant length, primitive, injective substitutions with Perron-Frobenius eigenvalue equal to 2, topologically conjugate to the Thue-Morse substitution.*

In Chapter 3 we will study some aspects of factors of the substitution subshift arising from the Thue-Morse substitution defined above. Examples of conjugate substitution systems to the Thue-Morse subshift which are not constant length are given in [30].

Example Consider the substitutions ζ_1, ζ_2 and ζ_3 defined below. Both ζ_1, ζ_2 are topologically conjugate to the Thue-Morse substitution, and ζ_3 is topologically semi-conjugate.

$$\begin{aligned}
\zeta_1 &: 0 \mapsto 01,\ 1 \mapsto 20,\ 2 \mapsto 10; \\
\zeta_2 &: 0 \mapsto 012,\ 1 \mapsto 02,\ 2 \mapsto 1; \\
\zeta_3 &: 0 \mapsto 01,\ 1 \mapsto 00.
\end{aligned}$$

The last substitution shows the subtlety in considering measurable and topological factors: the subshift associated to ζ_3 is a topological factor of the Thue-Morse subshift. The former is also measurably conjugate to the odometer in two symbols (which is also a topological factor of Thue-Morse), but those systems are not topologically conjugate. In fact, the action on the odometer is an isometry and the one in the subshift is an expansive transformation.

1.1.4 Number theoretic preliminaries

Here we recall some number-theoretic notions which will play an important role in the study of the spectrum of substitutions. As a reference, see for example [15, 23].

Diophantine approximation

We start defining two important classes of real numbers.

Definition 1.25 *We call $\alpha \in \mathbb{R}$ a* ***Pisot-Vijayaraghavan number*** *or* ***PV number*** *if α is a real algebraic integer greater than 1 all of whose Galois conjugates are less than 1 in absolute value.*

Definition 1.26 *We call $\alpha \in \mathbb{R}$ a* ***Salem number*** *if α is a real algebraic integer greater than 1 whose Galois conjugates have absolute value no greater than 1 and at least one of which has absolute value exactly 1.*

This definition actually forces that the inverse α^{-1} is a Galois conjugate of α, and the rest of them are on the unit circle. These numbers come up in many areas in mathematics, in general because of some remarkable arithmetic properties they enjoy. A nice survey on Salem numbers is [64]. We state two main features which will be essential for us. Denote as usual by $\{x\}$ the fractional part of x, and $||x||_{\mathbb{R}/\mathbb{Z}} = \min(\{x\}, 1 - \{x\})$ the distance to the integers.

Let us recall two Diophantine properties of these sets of numbers: The first result is due independently to C. Pisot and T. Vijayaraghavan.

Theorem 1.27 ([59, 68]) *Let $\alpha > 1$ be an algebraic number. Are equivalent:*

- *α is a PV number.*
- *There exists $\eta \in \mathbb{Q}(\alpha) \setminus \{0\}$ such that $\lim_{n\to\infty} ||\eta\alpha^n||_{\mathbb{R}/\mathbb{Z}} = 0$.*

Theorem 1.28 (cf. [23], Theorem 3.9) *Let α be a Salem number and $\varepsilon > 0$, then there exists $\eta = \eta(\varepsilon) \in \mathbb{Q}(\alpha)$ different from zero such that*

$$||\eta\alpha^n||_{\mathbb{R}/\mathbb{Z}} < \varepsilon,$$

for all $n \geq 0$.

These results are surprising since a theorem due to H. Weyl says that almost surely, the sequences $(\eta\alpha^n)_{n\geq 0}$ are uniformly distributed modulo 1 (in fact, there exist three variants depending on α, η and (α, η), see [23], Chapter 1).

Let us focus on Salem numbers, which is the case of the Perron-Frobenius eigenvalues of substitution matrices studied in Chapter 2. We will see that the behavior of the sequence $(\eta\alpha^n)_{n\geq 0}$ modulo 1 is intimately related with the spectrum of the dynamical system associated to a substitution, which arises naturally the number α as its Perron-Frobenius eigenvalue.

The next result is due to C. Pisot and R. Salem.

Theorem 1.29 ([60]) *Let α be a Salem number. Then the sequence $(\alpha^n)_{n\geq 1} \pmod 1$ is dense in $[0, 1]$ but is not uniformly distributed modulo 1.*

In spite of the above result, S. Akiyama and Y. Tanigawa showed in [2] that the sequence

$(\alpha^n)_{n\geq 1}$ is not far from being uniformly distributed modulo 1.

Theorem 1.30 ([2]) *Let α be a Salem number of degree $d = 2m+2 \geq 8$ and $J = [a, b] \subseteq [0, 1]$, then*

$$\left| \lim_{N\to\infty} \frac{\#\{n \leq N \mid \{\alpha^n\} \in J\}}{N} - (b-a) \right| \leq 2\zeta(m/2)(2\pi)^{-m}(b-a),$$

where ζ denotes the Riemann zeta function.

For degree $d = 4$ and $d = 6$ there are similar estimates we omit here. As an example of the distribution, see Figure 1.2. The proof of the main theorems of Chapter 2 is based on the

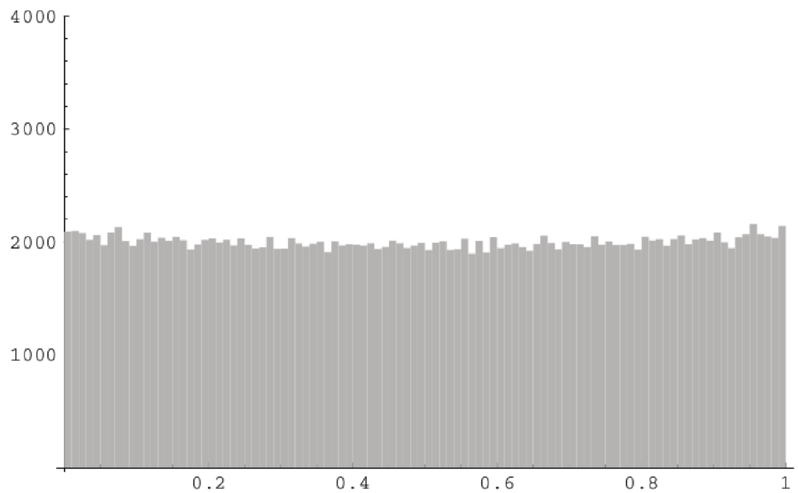

Figure 1.2: **Akiyama-Tanigawa [2]**: distribution modulo 1 of $(\alpha^n)_{n\geq 0}$, α the largest solution of $X^8 - 2X^7 + X^6 - X^4 + X^2 - 2X + 1 = 0$.

proof of this last theorem: we are able to prove a similar result for the sequences $(\eta\alpha^n)_{n\geq 0}$, for $\eta \in \mathbb{Q}(\alpha)$. In Chapter 2, the next result is easily deduced from Corollary 2.21 and from Lemmas 2.22, 2.23, 2.24 and 2.25.

Theorem 1.31 (M-M.) *Denote $J(\delta) = [\delta, 1-\delta]$, for $\delta < 1/2$. Let α be a Salem number of degree d and $\eta = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha)$, with $L \geq 1$ and $l_0, \ldots, l_{d-1} \in \mathbb{Z}$. Consider $\sigma_0 : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ the embedding corresponding to $\alpha \mapsto \alpha^{-1}$. Assume $\gcd(L, l_0, \ldots, l_{d-1}) = 1$. Then there exists an explicit $\delta = \delta(L, |\eta|, |\sigma_0(\eta)|) > 0$ such that*

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid \{\eta\alpha^n\} \in J(\delta)\}}{N} \geq 1/2.$$

We can not finish this section without state two of the most famous open problems concerning PV and Salem numbers.

Question 1.32 (Hardy problem) *Is there any transcendental number α such that $||\alpha^n||_{\mathbb{R}/\mathbb{Z}} \to 0$ as $n \to \infty$?*

For the second one we need a definition first.

Definition 1.33 *Let $P(X) \in \mathbb{C}[X]$. The **Mahler measure** of the polynomial $P(X) = a(X - \alpha_1) \dots (X - \alpha_d)$ is defined by*

$$\mathfrak{m}(P) = |a| \prod_{j=1}^{d} \max(1, |\alpha_j|).$$

For an algebraic number α with minimal polynomial P, we usually write $\mathfrak{m}(\alpha) = \mathfrak{m}(P)$.

Question 1.34 (Lehmer problem) *Is there any $c > 1$ such that $c \leq \mathfrak{m}(P)$ for all $P \in \mathbb{Z}[X]$ which are not a product of cyclotomic polynomials?*

D. H. Lehmer found the smallest known Mahler measure, given by the polynomial

$$L(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

Due to a result of C. J. Smyth, for non-palindromic polynomials, a lower bound for the Mahler measure is given by $\mathfrak{m}(\rho)$, ρ being the *plastic number*, the only real solution of $P(X) = X^3 - X - 1$. For a Salem number α, α^{-1} is always a Galois conjugate, and therefore the minimal polynomial is palindromic. Since for a Salem number the Mahler measure of its minimal polynomial coincides with it, Lehmer problem is in particular asking if the set of Salem numbers is bounded away from 1. The strong Lehmer conjecture asserts it is the case, and in fact one can take $c = \mathfrak{m}(L)$, implying the largest real root of L is in fact the smallest Salem number.

Digits sum and Thue-Morse polynomials

Chapter 3 is devoted to the *top Lyapunov exponent* of the *spectral cocycle* (see subsection 1.1.6) of the Thue-Morse substitution and its topological factors. As it is well-known, there is an arithmetic characterization of the fixed points of the Thue-Morse substitution involving the function s_2 (defined below) over the nonnegative integers that sums the digits of the base 2 expansion. We develop some of these connections in the present subsection.

Definition 1.35 *Let k be a nonnegative integer. Define the **sum of digits** function $s_2 : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ by $s_2(k) = k_0 + \cdots + k_{n-1}$, where $k = k_0 + k_1 2 + \cdots + k_{n-1} 2^{n-1}$ ($k_i \in \{0, 1\}$) is the base 2 expansion of k.*

Proposition 1.36 *Let $\mathbf{u}$ be the (one sided) **Thue-Morse word**, i.e., $\mathbf{u} = \zeta^{\infty}(0)$, where ζ is the Thue-Morse substitution over $\{0, 1\}$. Then, for every $k \geq 0$,*

$$\mathbf{u}_k = s_2(k) \pmod 2.$$

Let us consider the next sums on the Thue-Morse word associated with the indicator functions

$f = \mathbb{1}_{[0]}, g = \mathbb{1}_{[1]}$ and a parameter $\omega \in [0, 1)$. We use the notation $e(x) := e^{2\pi i x}$.

$$S^f_{2^n}(\omega, \mathbf{u}) = \sum_{k=0}^{2^n-1} \mathbb{1}_{[0]}(T^k\mathbf{u})e(k\omega), \tag{1.1}$$

$$S^g_{2^n}(\omega, \mathbf{u}) = \sum_{k=0}^{2^n-1} \mathbb{1}_{[1]}(T^k\mathbf{u})e(k\omega). \tag{1.2}$$

This is a particular case of a *twisted Birkhoff sum*, a notion we will define in subsection 1.1.5. For reasons we will clarify later, we are interested in the asymptotic growth of the twisted Birkhoff sums. Note that adding both sums yields simply a (bounded) geometric sum for $\omega \neq 0$. Fortunately, the difference has been well studied: by the arithmetic characterization of the Thue-Morse word we have

$$S^f_{2^n}(\omega, \mathbf{u}) - S^g_{2^n}(\omega, \mathbf{u}) = \sum_{k=0}^{2^n-1} (-1)^{s_2(k)} e(k\omega) =: p_n(\omega).$$

The trigonometric polynomials $p_n(\omega)$ are called **Thue-Morse polynomials**. Lot of attention has been given to the asymptotics of L^p norms of p_n: an ingredient in the proof of a prime number theorem for the sum of digits function due to C. Maduit and J. Rivat in [55] is the asymptotic

$$||p_n||_1 \sim 2^{n\delta}, \text{ with } \delta = 0.40325...$$

See for example [62] for related questions. We will be interested in the asymptotics of p_n for fixed ω as n goes to infinity. Apparently contradictory with the latter result, the growth is sub-exponential almost everywhere for the Lebesgue measure. The next result gives an explicit sub-exponential bound, and is a consequence of a bounded iterated logarithm law for the map $x \mapsto 2x$. The classical setting ensure such a law works with bounded observables, which is not our case. But the result holds due to a deep result of J. Dedecker, S. Gouëzel and F. Merlèvede [28]. More details on this are found in Chapter 3.

Proposition 1.37 (M-M.) *There exists a positive constant B such that for almost all ω, there is a positive integer $n_0(\omega)$ such that for all $n \geq n_0(\omega)$,*

$$\max(|p_n(\omega)|, |p_n(\omega)|^{-1}) \leq e^{B\sqrt{n \log\log(n)}}.$$

In consequence, the twisted Birkhoff sums of the Thue-Morse substitution will have, almost surely, a sub-exponential growth. We will also study in Chapter 3 a generalization of the twisted Birkhoff sum (which we will call *twisted correlation*) for which we can find similar estimates. We summarize those results in subsection 1.3.

In order to study twisted correlations sums, we have used some results about the solution set of the equation $s_2(k + a) - s_2(k) = d$, $a \in \mathbb{N}$ and $d \in \mathbb{Z}$. Denote by $\mathcal{S}_{a,d}$ its solution set. To change between words and numbers represented by digits consider the next notation: $\underline{k}_2$ denotes the word associated to the digits of k in base 2, i.e., if $k = k_0 + \cdots + k_{n-1}2^{n-1}$ then $\underline{k}_2 = k_0 \ldots k_{n-1} \in \{0, 1\}^*$. Similarly, for a word $w = w_0 \ldots w_{n-1} \in \{0, 1\}^*$, denote by $\overline{w}^2$ the number $w_0 + \cdots + w_{n-1}2^{n-1}$.

Lemma 1.38 (cf. [38]) *There exists a finite set of words $\mathcal{P}_{a,d} = \{\mathfrak{p}_a^d(1), \ldots, \mathfrak{p}_a^d(s)\} \subset \{0,1\}^*$, such that*

$$k \in \mathcal{S}_{a,d} \iff \underline{k}_2 \in \bigcup_{i=1}^{s} [\mathfrak{p}_a^d(i)].$$

1.1.5 Some notions on spectral theory

Here we develop some of the background in spectral theory of unitary operators associated to dynamical systems. The focus on the definition of the *spectral measures*, specially relevant in Chapter 2. References for spectral theory of dynamical systems are [61, 48, 36, 47]. We use again the notation $e(x) = e^{2\pi i x}$.

General spectral theory, Koopman representations and eigenvalues

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space and $V : \mathcal{H} \longrightarrow \mathcal{H}$ a bounded linear operator.

Definition 1.39 *We call V a **normal** operator if $VV^* = V^*V$, where V^* denotes the adjoint operator. V is called an **isometry** if $VV^* = id_{\mathcal{H}}$. Finally, V is called **unitary** if it is an isometry and normal.*

The existence and definition of spectral measures in the next theorem is a consequence of the Bochner-Herglotz theorem. Since in Chapter 2 we will study the spectral measures of the self-similar suspension flow over a substitution subshift, we will restrict ourselves to recall mostly the spectral theory of $\mathbb{R}$-actions. Similar statements are valid for $\mathbb{Z}$-actions (see [61]), and in more generality, for second countable locally compact Abelian groups (see [47]).

Let $(V_t)_{t\in\mathbb{R}}$ be a one-parameter group of unitary operators in a Hilbert space $\mathcal{H}$ continuous in the strong operator topology.

Theorem 1.40 *Let $u, v \in \mathcal{H}$. There exists a finite complex measure $\nu_{u,v}$ with support in $\mathbb{R}$, called **spectral measure**, such that*

$$\widehat{\nu_{u,v}}(-t) := \int_{\mathbb{R}} e^{2\pi i t\omega} d\nu_{u,v}(\omega) = \langle V_t u, v\rangle.$$

*We denote the diagonal measures $\nu_{u,u}$ by ν_u. Moreover, there exist a **unitary equivalence** W between the action of V_t on the **cyclic space** $\mathcal{H}_u = \mathrm{cl}\left(\{V_t u \,|\, t \in \mathbb{R}\}\right)$ and the multiplication operator $M_t : f \mapsto e(\cdot t)f$ on $L^2(\mathbb{R}, \nu_u)$ and such that $Wu = 1$; that is, W is a unitary operator such that the next diagram commutes.*

$$\begin{array}{ccc} \mathcal{H}_u & \xrightarrow{V_t} & \mathcal{H}_u \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle W} \\ L^2(\mathbb{R}, \nu_u) & \xrightarrow{M_t} & L^2(\mathbb{R}, \nu_u) \end{array}$$

Note that a diagonal measure is a positive measure. In the special case we are considering the indicator function $f = \mathbb{1}_{\mathfrak{X}_a^{\vec{p}}}$ (for $a \in \mathcal{A}$), we will denote ν_f by ν_a and we will call it **correlation measure**.

Spectral measures allow us to make a unitary equivalence between the action of V_t on the cyclic space generated by some function $f \in \mathcal{H}$ and the multiplication operator on $L^2(\mathbb{R}, \nu_f)$ by the character $e(t\cdot)$. By an induction argument, it is possible to decompose the whole space:

Theorem 1.41 (cf. [48]) *There exists a family of diagonal spectral measures on $\mathbb{R}$, ordered (by absolute continuity relation) as*

$$\nu_1 \gg \nu_2 \gg \cdots \gg \nu_n \gg \dots,$$

such that the action of V_t on $\mathcal{H}$ is unitarily equivalent to the action of the multiplication operator M_t defined below:

$$\begin{aligned} M_t : \bigoplus_{i\geq 1} L^2(\mathbb{R}, \nu_i) &\longrightarrow \bigoplus_{i\geq 1} L^2(\mathbb{R}, \nu_i) \\ (f_i)_{i\geq 1} &\longmapsto (e(t\cdot)f_i)_{i\geq 1} \end{aligned}$$

*The decomposition is unique modulo the measures type (same null-sets). The type of ν_1 is called the **maximal spectral type**, denoted in general by ν_{max}.*

Consider the system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ arising from the continuous $\mathbb{R}$-action of a flow $(h_t)_{t\in\mathbb{R}}$. We associate the Hilbert space $L^2(X, \mu)$.

Definition 1.42 *We call the **Koopman representation** of $(h_t)_{t\in\mathbb{R}}$ the representation $\mathcal{U} : \mathbb{R} \longrightarrow \mathfrak{B}(L^2(X, \mu))$ (the set of bounded linear operators) defined by $\mathcal{U}(t)(f) = U_t(f) = f \circ h_t$.*

For any $t \in \mathbb{R}$, the operator U_t is unitary since the flow is invertible and measure preserving.

Definition 1.43 *We call **spectrum** of the system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ the Gelfand spectrum of its Koopman representation, i.e., the set of **approximate eigenvalues***

$$\sigma(\mathcal{U}) = \{\omega \in \mathbb{R} \,|\, \exists (f_n)_{n\geq 1} \subset L^2(X, \mu), ||f_n|| = 1 : ||U_t(f_n) - e(t\omega)f_n|| \to 0,\ \forall t \in \mathbb{R}\}.$$

In particular, if the sequence $(f_n)_{n\geq 1}$ is constant, we recover the classic notion of eigenvalue:

Definition 1.44 *We say $\omega \in \mathbb{R}$ is an **eigenvalue** of $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ if there exists $f \in L^2(X, \mu)$ (called **eigenvector** or **eigenfunction**) such that for all $t \in \mathbb{R}$, $U_t(f) = e(t\omega)f$ (in $L^2(X, \mu)$). On a discrete time system $(X, \mathcal{B}, \mu, T)$, one takes $\omega \in [0, 1)$ and the analogous condition is $f \circ T = e(\omega)f$.*

The set of all eigenvalues is called the **discrete spectrum** of the system, and is usually denoted by σ_{disc}. Note that $\omega = 0$ is always an eigenvalue with a constant function as eigenvector. In general, the discrete spectrum is exactly the set of atoms of the maximal spectral type.

To finish this subsection, let us recall the connection of eigenvalues with some of the definitions of ergodic theory introduced in subsection 1.1.1.

Proposition 1.45 (cf. [47]) *The system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ is ergodic if and only if* 1 *is a simple eigenvalue of the system. The system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ is weakly mixing if and only if* 1 *is the only eigenvalue of the system. If the system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ is ergodic, then all eigenvalues are simple.*

Continuous spectrum and spectral measures

We can always decompose the maximal spectral type in its discrete part and continuous part:

$$\nu_{\max} = \nu_{\text{disc}} + \nu_{\text{cont}}.$$

The set of eigenvalues provides a first invariant to distinguish two measures, since ν_{disc} is essentially a sum of Dirac deltas corresponding to eigenvalues of the system. The complement of the set of eigenvalues is called the **continuous spectrum** of the system. Instead of studying the set $\sigma(\mathcal{U}) \setminus \sigma_{\text{disc}}$, we will study the measure ν_{cont}.

A second invariant to distinguish two measures comes from the decomposition of ν_{cont} into an absolutely continuous part and a singular continuous part respect to the natural Lebesgue measure on $\mathbb{R}$. Then

$$\nu_{\max} = \nu_{\text{disc}} + \nu_{\text{abs}} + \nu_{\text{sing}}.$$

We will make use of a third invariant defined below.

Definition 1.46 *Let ν be a a positive measure on the real line. The* ***local lower dimension*** *of ν at a point $\omega \in \mathbb{R}$ is defined by*

$$\underline{d}(\nu, \omega) = \liminf_{r\to 0^+} \frac{\log(\nu([\omega - r, \omega + r]))}{\log(r)}.$$

Let us make a connection with ergodic theory. Given a function $\delta : \mathbb{R}_+ \longrightarrow [0, 1)$ satisfying $\lim_{r\to 0^+} \delta(r) = 0$. Call $(h_t)_{t\in\mathbb{R}}$ δ-**partially weakly mixing**, if there exists a constant C and unit $f \in L^2(X, \mu)$ such that for all unit $g \in L^2(X, \mu)$ and all $R > 0$,

$$\frac{1}{R}\int_0^R \left| \int f\, g \circ h_t \, d\mu - \int f \, d\mu \int g \, d\mu \right| dt \leq C\delta(1/R).$$

We say the measure ν is δ-**continuous** at ω if there exist $C, r_0 > 0$ such that

$$\nu([\omega - r, \omega + r]) \leq C\delta(r),\ \forall\, 0 < r \leq r_0.$$

If the measure is δ-continuous at every ω, we will say it is δ-**continuous**. The next theorem due to Y. Last shows a relation between the two notions above (see also the second reference from the work of O. Knill for the discrete time case).

Theorem 1.47 ([52, 49]) *If there exists a spectral measure ν_f which is δ-continuous, then $(h_t)_{t\in\mathbb{R}}$ is δ-partially weakly mixing. If $(h_t)_{t\in\mathbb{R}}$ is δ-partially weakly mixing, then there exists a spectral measure ν_f which is $\sqrt{\delta}$-continuous.*

Obviously, if the maximal spectral type is δ**-continuous** at every $\omega \neq 0$, then $(h_t)_{t\in\mathbb{R}}$ is weakly mixing. If δ is an explicit function, this phenomenon will be called **quantitative** or **effective weak mixing**.

Twisted Birkhoff sums and twisted correlations

The study of the spectral measures for substitution subshifts and its suspension flows will be done by studying the growth rate of the next kind of sums (resp. integrals).

Definition 1.48 *Let $(X, \mathcal{B}, \mu, T)$ (resp. $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$) be a discrete time (resp. continuous time) dynamical system. Let f a measurable function, $x, \in X$, $\omega \in [0, 1)$ and $N \geq 1$ (resp. $\omega \in \mathbb{R}$ and $R > 0$). The* ***twisted Birkhoff sum*** *(resp.* ***twisted Birkhoff integral****) associated to f is defined by*

$$S_N^f(x,\omega) = \sum_{k=0}^{N-1} f(T^k x) e(k\omega) \ \left(\textit{resp. } S_R^f(x,\omega) = \int_0^R f(h_t(x)) e(\omega t) dt \right).$$

The next generalization of this definition will be used in Chapter 3.

Definition 1.49 *Let $(X, \mathcal{B}, \mu, T)$ be a discrete time dynamical system, f a measurable function. Let $a_1 < \cdots < a_t$ positive integers and $\omega \in [0, 1)$. The (accumulated) twisted correlation $C_N^f(a_1, \ldots, a_t, \omega, x)$ (at time N, of parameters $a_1, \ldots, a_t$ and ω) of f at a point $x \in X$ is defined by*

$$C_N^f(a_1, \ldots, a_t, \omega, x) = \sum_{k=0}^{N-1} f(T^k x) f(T^{k+a_1} x) \ldots f(T^{k+a_t} x) e(k\omega).$$

We will see in Chapter 3 that the twisted Birkhoff sums of topological factors of the Thue-Morse substitution may be expressed as a sum of twisted correlations on the Thue-Morse subshift.

Let us state one result due to A. Hof that enlight the connection between spectral measures of the system $(X, \mathcal{B}, \mu, (h_t)_{t\in\mathbb{R}})$ and its twisted Birkhoff integrals.

Proposition 1.50 ([43, 18]) *Let Ω be a continuous increasing function on $[0, 1)$ with $\Omega(0) = 0$, $\omega \in \mathbb{R}$ and $f \in L^2(X, \mu)$. Suppose there exist $C, R_0 > 0$ such that*

$$\sup_{x\in X} \left| S_R^f(x,\omega) \right| \leq C R\, \Omega(1/R) \ \textit{for all } R \geq R_0.$$

Then

$$\nu_f([\omega - r, \omega + r]) \leq \frac{\pi^2 C}{4} \Omega(2r) \ \textit{for all } r \leq (2R_0)^{-1}.$$

1.1.6 Cocycles

Definition 1.51 *Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system and $d \in \mathbb{N}$. We call* ***cocycle*** *(of dimension d) over $(X, \mathcal{B}, \mu, T)$ or (X, T), a sequence of maps $\mathscr{A} = (\mathscr{A}(\cdot, n))_{n\geq 1}$, where $\mathscr{A}(\cdot, n) : X \longrightarrow \mathcal{M}_d(\mathbb{C})$, satisfying the relations*

$$\mathscr{A}(x, n+m) = \mathscr{A}(T^n(x), m)\mathscr{A}(x, n), \quad \forall m, n \in \mathbb{N},\, x \in X.$$

Example Let M^d be a differentiable manifold of dimension d and f and automorphism of it. The application $p \in M \mapsto Df_p \in \mathcal{L}(T_pM, T_{f(p)}M)$ defines a cocycle over M. Indeed, the chain rule give us for $v \in T_pM$,

$$D^{m+n}f_p(v) = D^m f_{f^n(p)}\, D^n f_p(v).$$

Let ζ be a substitution over $\mathcal{A} = \{1, \dots, d\}$ and let $\zeta(a) = w_1 \dots w_{k_a}$. Note that the transpose of the substitution matrix M_ζ^T defines an endomorphism of $\mathbb{R}^d/\mathbb{Z}^d$, denoted by this same matrix. In [21], A. I. Bufetov and B. Solomyak introduced the next cocycle, which will play an important role for studying the spectral measures of subshifts arising from substitution and their suspension flows.

Definition 1.52 *The* ***spectral cocycle*** *is the cocycle over $(\mathbb{R}^d/\mathbb{Z}^d, M_\zeta^T)$ given by*

$$\begin{aligned}\mathscr{C}_\zeta(\xi, 1)(a, b) &= \sum_{j=1}^{k_a} \delta_{w_j b}\, e(\xi_{w_1} + \dots + \xi_{w_{j-1}}), \quad \xi = (\xi_1, \dots \xi_d)^T \in \mathbb{R}^d/\mathbb{Z}^d \\ \mathscr{C}_\zeta(\xi, n) &= \mathscr{C}_\zeta((M_\zeta^T)^{n-1}\xi, 1) \dots \mathscr{C}_\zeta(\xi, 1).\end{aligned}$$

The spectral cocycle is suitable for studying the spectrum of suspension flows over substitution subshifts: if $\vec{p}$ is the positive vector defining the suspension and $\omega \in \mathbb{R}$, we will set $\xi = \omega\vec{p} \,(\mathrm{mod}\ \mathbb{Z}^d)$. We refer the reader to [21] for some results that justify this decomposition.

Example Following [21], let ζ be the substitution defined on $\mathcal{A} = \{1, 2, 3\}$ by

$$\zeta(1) = 121321, \quad \zeta(2) = 2231, \quad \zeta(3) = 31123.$$

Then, denoting $z_j = e(\xi_j)$,

$$\mathscr{C}_\zeta(\xi, 1) = \begin{pmatrix} 1 + z_1z_2 + z_1^2z_2^2z_3 & z_1 + z_1^2z_2z_3 & z_1^2z_2 \\ z_2^2z_3 & 1 + z_2 & z_2^2 \\ z_3 + z_1z_3 & z_1^2z_3 & 1 + z_1^2z_2z_3 \end{pmatrix}.$$

If we want to study the $\mathbb{Z}$-action, we can consider $\vec{p} = \vec{1} = (1, \dots, 1)^T$ and the decomposition $\xi = \omega\vec{p}$. If the substitution is of constant length q, the endomorphism given by the substitution matrix acts on $\omega\vec{p}$ in $(\mathbb{R}/\mathbb{Z})^d$ in the same way the q-times map acts on ω in $\mathbb{R}/\mathbb{Z}$. Similar is true in the case of the self-similar flow: the action of the endomorphism on $\omega\vec{p}$ is just the multiplication by the Perron-Frobenius eigenvalue of the substitution (since $\vec{p}$ is the left-eigenvector of the substitution matrix), which leaves a cocycle on $\mathbb{R}$. In both latter cases we will refer to the cocycle over $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ as the **restricted** spectral cocycle.

Remark The spectral cocycle may be defined to study *S-adic systems* (see for example [14] for background), which include substitution subshifts as a particular case; and the corresponding suspension flows over them. Other systems for which it may be used are *interval exchange transformations* and *translation flows on flat surfaces* (for background, see for example [67]), since it is an extension of the well studied *Rauzy-Veech cocycle*. This is explained in detail in [21], and we refer to this paper for the interested reader.

Let us recall the Furstenberg-Kesten theorem in complete generality. Denote $\log^+(x) = \max(\log(x), 0)$.

Theorem 1.53 *Let $\mathscr{A} = (\mathscr{A}(\cdot, n))_{n\geq 1}$ be a cocycle over $(X, \mathcal{B}, \mu, T)$ such that $\log^+(||\mathscr{A}^{(1)}||) \in L^1(X, \mu)$. Then the limit*

$$\chi^+(x) = \lim_{n\to\infty} \frac{1}{n} \log(||\mathscr{A}(\cdot, n)||),$$

exists for μ-almost $x \in X$, $\chi^+ \in L^1(X, \mu)$, χ^+ is T-invariant and

$$\int_X \chi^+(x) d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \int_X \log(||\mathscr{A}(x, n)||) d\mu(x) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \log(||\mathscr{A}(x, n)||) d\mu(x).$$

In particular, if T is ergodic, χ^+ is almost everywhere equal to a constant $\chi^+(\mathscr{A})$ and

$$\chi^+(\mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} \int_X \log(||\mathscr{A}(x, n)||) d\mu(x) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \log(||\mathscr{A}(x, n)||) d\mu(x).$$

The function $\chi^+(x)$ is called the **top Lyapunov exponent** of the cocycle $\mathscr{A}$. For the null set where the top Lyapunov exponent is possibly not defined, set

$$\chi^+(x) = \limsup_{n\to\infty} \frac{1}{n} \log(||\mathscr{A}(x, n)||).$$

Coming back to substitutions, when considering the function $\mathbb{1}_{[a]}$ ($a \in \mathcal{A}$) for $\mathfrak{X}_\zeta$, the associated diagonal spectral measure is called **correlation measure**, and is denoted by ν_a. There is a relation between the top Lyapunov exponent of the spectral cocycle and the dimension of the correlation measures proved by A. I. Bufetov and B. Solomyak (see also [21]), which we state in the next theorem.

If we are studying the $\mathbb{Z}$-action (i.e., $\vec{p} = \vec{1}$), we will use the next notation for the top Lyapunov exponent of the spectral cocycle:

$$\chi_\zeta^+(\omega) := \limsup_{n\to\infty} \frac{1}{n} \log(||\mathscr{C}_\zeta(\omega\vec{1}, n)||), \quad \omega \in [0, 1).$$

As a consequence of the Birkhoff-Khinchin ergodic theorem, if ζ is a constant length substitution, the above is in fact a limit (for almost every point) and it is almost everywhere constant.

Theorem 1.54 ([18]) *Let ζ be an aperiodic primitive substitution on $\mathcal{A}$, with Perron-Frobenius eigenvalue equal to α. Let $\chi_\zeta^+(\omega)$ be the top Lyapunov exponent of the spectral cocycle at $\omega \in [0, 1)$. Then for all $\omega \in [0, 1)$ and $a \in \mathcal{A}$,*

$$\underline{d}(\nu_a, \omega) \geq 2 - 2\max(0, \chi_\zeta^+(\omega)/\log(\alpha))$$

In fact, for a full measure subset, taking max is not necessary since $\chi_\zeta^+(\omega)$ is in general (Lebesgue) almost surely nonnegative:

Proposition 1.55 (M-M.) *Let ζ be an arbitrary substitution. Then $\chi_\zeta^+(\omega) \geq 0$ for Lebesgue almost every $\omega \in [0,1)$.*

This is proved in Chapter 3, see Proposition 3.11. This is also the case for other suspensions (i.e., $\vec{p} \neq \vec{1}$) in a natural situation, as it was mentioned to us by B. Solomyak by personal communication.

Theorem 1.56 *Let ζ be a primitive substitution such that of M_ζ has no eigenvalues which are roots of unity. Then the top Lyapunov exponent of the spectral cocycle is almost everywhere constant and $\chi^+(\mathscr{C}_\zeta) \geq 0$.*

This is also proved in Chapter 3, see Theorem 3.13.

We finish this section by citing a relation between the top Lyapunov exponent of the spectral cocycle and Question 1.34, found in [5]. A **Borwein polynomial** is a polynomial with coefficients in $\{-1, 0, 1\}$ with non zero constant coefficient. The next result is due to M. Baake, M. Coons and N. Mañibo.

Theorem 1.57 ([5]) *Let ζ be a primitive constant length substitution on $\{0,1\}$. Then*

$$\chi_\zeta^+ = \log(\mathfrak{m}(P)),$$

where $P = P(\zeta)$ is an explicit Borwein polynomial.

In particular, it is possible to deduce the top Lyapunov exponent for the Thue-Morse subshift is zero.

Coming back to the conclusion of this theorem, starting from a Borwein polynomial it is possible to construct a primitive constant length substitution on two letters, as it is also shown in [5]. Consequently, the Mahler measure of any Borwein polynomial may be seen as the top Lyapunov exponent of the spectral cocycle of a substitution subshift. This fact allows us to write a dynamical analog of Lehmer problem for the class of Borwein polynomials:

Question 1.58 ([5]) *Is there any $c > 0$ such that $c \leq \chi_\zeta^+$ for all primitive constant length substitution on $\{0,1\}$ with $\chi_\zeta^+ > 0$?*

1.2 Recent results

In this section we intend to give a brief (very incomplete) summary of the results concerning the spectrum of the system $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$ and its suspension flows by a positive vector $\vec{p}$.

1.2.1 Discrete spectrum

Let us start with the case of $\mathbb{Z}$-actions. In the constant length case the discrete part of the spectrum is never trivial: using the notion of *height* (see [61]) of a substitution, an explicit characterization is given by the next theorem proved by M. Dekking and J. C. Martin by different methods.

Theorem 1.59 ([29, 54]) *Let ζ be a primitive aperiodic substitution of constant length equal to $q \geq 2$ and height h. Then the set of eigenvalues of $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$ is*

$$\sigma_{disc} = (\mathbb{Z}[1/q] + \frac{1}{h}\mathbb{Z})/\mathbb{Z}.$$

As an example we have the Thue-Morse substitution: the set of eigenvalues is exactly the set of dyadic rationals $\mathbb{Z}[1/2]/\mathbb{Z}$ (a direct calculation is made in [40]).

Another remarkable class of substitutions are the ones which have as its Perron-Frobenius eigenvalue a PV number. We will call those substitutions of **Pisot type** or a **Pisot substitution**. For a Pisot substitution, it is also the case that the discrete spectrum of the subshift is not trivial. Those associated with the trivial *coboundary* (see [40]) are given by a result of B. Host [45].

In the general case, S. Ferenczi, C. Mauduit and A. Nogueira proved the next theorem which characterize eigenvalues of a primitive aperiodic substitution subshift. Call a **generalized return word** of the subshift $\mathfrak{X}_\zeta$ any word $w = w_1 \dots w_{l-1}$ appearing in a fixed point $\mathbf{u}$ of ζ such that if ww_l is a factor of $\mathbf{u}$, then for N large enough and for every $n \geq N$,

$$\zeta^n(w_1) = \zeta^n(w_l), \qquad \zeta^n(w_1) \neq \zeta^n(w_j) \quad \forall\, 1 < j < l$$

For a generalized return word $w = w_1 \dots w_{l-1}$, call the **return time sequence** the next sequence of integers $(r_n(w))_{n\geq 1}$:

$$r_n(w) = |\zeta^n(w_1)| + \cdots + |\zeta^n(w_{l-1})|.$$

Example Following [39], let ζ be the substitution defined on $\mathcal{A} = \{1, 2, 3, 4\}$ by

$$\begin{aligned} \zeta(1) &= 1244, & \zeta(2) &= 23, \\ \zeta(3) &= 4, & \zeta(4) &= 1. \end{aligned}$$

Then $w = 412$ is a generalized return word. Note that

$$\zeta^\infty(1) = 1244231123412441244\dots$$

Then $w4$ appears in a fixed point and the condition needed is simple to verify.

Theorem 1.60 ([39]) *Let ζ be a primitive aperiodic substitution. $\omega \in [0, 1)$ is an eigenvalue of the system $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$ if and only if*

$$\lim_{n\to\infty} e^{2\pi i \omega r_n(w)} = 1,$$

for every generalized return word w.

A similar result was proved before by B. Host [45].

For self-similar suspension flows, let us cite a result of M. Barge and J. Kwapisz for the Pisot case:

Theorem 1.61 ([12]) *Let ζ be a primitive aperiodic substitution of Pisot type on $\mathcal{A} = \{1, \ldots, d\}$ such that $\det(M_\zeta) = \pm 1$. Then the set of eigenvalues of the self-similar suspension flow $(\mathfrak{X}_\zeta^{\vec{p}}, \mathcal{B}(\mathfrak{X}_\zeta^{\vec{p}}), \mu, (h_t)_{t\in\mathbb{R}})$ is*

$$\sigma_{disc} = \left\{ \sum_{i=1}^{d} k_i q_i \;\middle|\; k_i \in \mathbb{Z} \right\},$$

where $(q_1, \ldots, q_d)^T$ is the normalized right Perron-Frobenius eigenvector of M_ζ.

For general suspension flows see [25, 26] and references therein.

Let us finish commenting the problem of deciding whether $\sigma(\mathcal{U}) = \sigma_{\text{disc}}$ (we say in that case the spectrum is **pure point** or **pure discrete**). An example is given by the Fibonacci substitution: it is possible to prove that $(\mathfrak{X}_{\zeta_F}, \mathcal{B}(\mathfrak{X}_{\zeta_F}), \mu, T)$ is measurably conjugate to the irrational rotation $([0,1], \mathcal{B}([0,1]), \text{Leb}, R_\phi)$ (ϕ being the golden ratio), which has pure discrete spectrum.

In [63], G. Rauzy found an explicit measurable conjugacy between the substitution subshift associated to the Tribonacci substitution and a toral translation, by introducing a domain exchange on what is now called the *Rauzy fractal*. This construction can be generalized to **unimodular** (such that $\det(M_\zeta) = \pm 1$) Pisot type substitutions satisfying a (some) *coincidence condition* (see [1]). None of these conditions is known to hold in general. Therefore, the main conjecture is the one we state below. We say a substitution ζ is **irreducible** if the characteristic polynomial of M_ζ is irreducible over $\mathbb{Q}$.

Conjecture 1.62 (**Pisot substitution conjecture**) *The spectrum of the substitution subshift associated a irreducible Pisot substitution is pure point.*

An extensive survey on the Pisot substitution conjecture is [1].

1.2.2 Continuous spectrum

In this subsection we recall some results on the continuous part of the spectrum, that is, we will study the measure ν_{cont} or, more precisely, the continuous part of ν_f. A first result to contextualize the spectrum is the next one, proved by F. M. Dekking and M. Keane for the $\mathbb{Z}$-action, and A. Clark and L. Sadun for the $\mathbb{R}$-action.

Theorem 1.63 ([31],[25]) *The dynamical systems $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$, $(\mathfrak{X}_\zeta^{\vec{p}}, \mathcal{B}(\mathfrak{X}_\zeta^{\vec{p}}), \mu, (h_t)_{t\in\mathbb{R}})$ are never mixing.*

Note that in terms of the spectral measures that means there exists f such that $\widehat{\nu_f}(t) \nrightarrow 0$

when $|t| \to \infty$.

In the case of constant length substitutions, a characterizatiom of the maximal spectral type is done by M. Queffélec (see [61]). In that case the set of eigenvalues is always not trivial.

Now we focus on the purely continuous spectrum, i.e., weakly mixing systems. For $\mathbb{Z}$-actions it is not enough to ensure weak mixing to have a non PV number as the Perron-Frobenius eigenvalue of the substitution matrix (see [39]). But it is the case for the self-similar suspension flows, as shown by B. Solomyak in the next theorem.

Theorem 1.64 ([65]) *The self-similar flow $(\mathfrak{X}_\zeta^{\vec{p}}, \mathcal{B}(\mathfrak{X}_\zeta^{\vec{p}}), \mu, (h_t)_{t\in\mathbb{R}})$ is weakly mixing if and only if the Perron-Frobenius eigenvalue of the substitution matrix is not a Pisot number.*

For $\mathbb{Z}$-actions the problem of characterize weakly mixing substitution subshifts is completely solved by the work of S. Ferenczi, C. Mauduit and A. Nogueira [39]. The condition is much more complicated to state than for the self-similar suspension flow and we omit it here, although we need to mention it is completely explicit in terms of the Perron-Frobenius eigenvalue and its Galois conjugates.

Finally, for generic suspensions there is also a simple criterion for weak mixing, although is not written explicitly. The next result follows from the works of A. Clark and L. Sadun [25, 26], and the one of M. Barge and B. Diamond [11].

Theorem 1.65 ([25, 26, 11]) *The suspension flow $(\mathfrak{X}_\zeta^{\vec{p}}, \mathcal{B}(\mathfrak{X}_\zeta^{\vec{p}}), \mu, (h_t)_{t\in\mathbb{R}})$ is weakly mixing for generic $\vec{p} \in \mathbb{R}_+^m$ if the Perron-Frobenius eigenvalue of the substitution matrix is a Salem number or there exist at least two eigenvalues outside the closed unit disk.*

1.2.3 Quantitative weak mixing and finer properties

Having reviewed results concerning weak mixing, we finish this section recalling some results about quantitative weak mixing and finer properties of the spectral measures, like their decomposition with respect to Lebesgue measure. Recall that if $f = \mathbb{1}_{\mathfrak{X}_a^{\vec{p}}}$ (for $a \in \mathcal{A}$) we will denote ν_f by ν_a and we will call it correlation measure. The next asymptotic was proved by A. I. Bufetov and B. Solomyak.

Theorem 1.66 ([18]) *Let ζ be a primitive aperiodic substitution on $\mathcal{A}$, with at least two eigenvalues of its substitution matrix outside the closed unit disk. Suppose also the characteristic polynomial is irreducible over the rationals. Then there exists $\gamma \in (0,1)$ only depending on ζ such that for all $B > 1$, $a \in \mathcal{A}$ and almost every normalized $\vec{p} \in \mathbb{R}_+^m$, there exist $C = C(B, \vec{p}) > 0$ and $r_0 = r_0(B, \vec{p}) > 0$ such that*

$$\nu_a([\omega - r, \omega + r]) \leq C r^\gamma,$$

for all $|\omega| \in [B^{-1}, B]$ and $0 < r < r_0$.

The exponent γ is called a **Hölder** exponent.

In the same article, the authors proved for the self-similar suspension flow the next decay for the correlation measures.

Theorem 1.67 ([18]) *Let ζ be a primitive aperiodic substitution on $\mathcal{A}$, with at least two eigenvalues of its substitution matrix outside the closed unit disk. Then there exists $\gamma \in (0,1)$ only depending on ζ such that for all $B > 1$, $a \in \mathcal{A}$, there exist $C = C(B) > 0$ and $r_0 = r_0(B) > 0$ such that*

$$\nu_a([\omega - r, \omega + r]) \leq C \log(1/r)^{\gamma},$$

for all $|\omega| \in [B^{-1}, B]$ and $0 < r < r_0$.

In this last case, the exponent γ is called a **log-Hölder** exponent.

As for decomposition of measures, let us cite some results on this aspect. We say a system (continuous or discrete) has **purely singular spectrum** (or the action is **purely singular**) if the absolutely continuous part of the decomposition of the maximal spectral type is null, i.e., $\nu_{\text{abs}} = 0$. The next theorem was also proved by A. I. Bufetov and B. Solomyak.

Theorem 1.68 ([21]) *Let ζ be a primitive aperiodic substitution and α its Perron-Frobenius eigenvalue. Assume the toral endomorphism defined by M_{ζ}^{T} is ergodic. Then the spectral cocycle has a constant top Lyapunov exponent $\chi^{+}(\mathscr{C}_{\zeta})$ and*

- $\chi^{+}(\mathscr{C}_{\zeta}) \leq \log(\alpha)/2$.
- *If $\chi^{+}(\mathscr{C}_{\zeta}) < \log(\alpha)/2$ and $\det(\mathscr{C}_{\zeta}(\cdot, 1)) \neq 0$, then for almost every $\vec{p} \in \mathbb{R}_{+}^{m}$ the flow $(\mathfrak{X}_{\zeta}^{\vec{p}}, \mathcal{B}(\mathfrak{X}_{\zeta}), \mu, (h_t)_{t \in \mathbb{R}})$ is purely singular.*

Similar results were obtained by M. Baake and collaborators for the self-similar suspension flow, see [6, 9, 7]. They involve the concept of *diffraction spectrum*, but this is closely related to the spectrum we have defined in this text. More details on these connections in [10]. The next result is due to M. Baake, F. Gähler and N. Mañibo.

Theorem 1.69 ([7]) *Let ζ be a primitive aperiodic substitution, α its Perron-Frobenius eigenvalue and $\vec{p}$ its normalized Perron-Frobenius eigenvector. If for some $\varepsilon > 0$, $\chi_{\zeta}^{+}(\omega) + \varepsilon < \log(\alpha)/2$ for Lebesgue almost all $\omega \in \mathbb{R}$, then the diffraction spectrum of the self-similar suspension flow $(\mathfrak{X}_{\zeta}^{\vec{p}}, \mathcal{B}(\mathfrak{X}_{\zeta}), \mu, (h_t)_{t \in \mathbb{R}})$ is purely singular.*

Concerning the $\mathbb{Z}$-action, we have the following condition ensuring pure singularity of the spectrum, due again to A. I. Bufetov and B. Solomyak.

Theorem 1.70 ([20]) *Let ζ be a primitive aperiodic substitution such that the substitution matrix has characteristic polynomial irreducible over $\mathbb{Q}$. Let α be the Perron-Frobenius eigenvalue of ζ. If*

$$\chi^{+}(\mathscr{C}_{\zeta}) < \log(\alpha)/2,$$

then the system $(\mathfrak{X}_{\zeta}, \mathcal{B}(\mathfrak{X}_{\zeta}), \mu, T)$ has purely singular spectrum.

The next result due to A. Berlinkov and B. Solomyak gives also a sufficient condition for pure singularity of the substitution subshift, but restricted to the class of constant length substitutions.

Theorem 1.71 ([13]) *Let ζ be a primitive aperiodic substitution of constant length q. If the substitution matrix M_ζ has no eigenvalue whose absolute value equals $\sqrt{q}$, then the maximal spectral type of the substitution system $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$ is singular.*

The results above are based on dimension estimates of the spectral measures in terms of the top Lyapunov of the spectral cocycle, like Theorem 1.54. An example of purely singular spectrum for substitution subshift $(\mathfrak{X}_\zeta, \mathcal{B}(\mathfrak{X}_\zeta), \mu, T)$ is the Bufetov-Solomyak substitution (this is proved in [20]).

We finish by remarking a particular case of absolutely continuous component: the classical example is given by the Rudin-Shapiro substitution:

Example (**Rudin-Shapiro substitution**). Let $\mathcal{A} = \{1, 2, 3, 4\}$,

$$\begin{aligned} \zeta(1) &= 12, \quad \zeta(3) = 42, \\ \zeta(2) &= 13, \quad \zeta(4) = 43. \end{aligned}$$

When considering the $\mathbb{Z}$-action, there exists $f \in L^2(\mathfrak{X}_\zeta, \mu)$ such that ν_f is equivalent to Leb. For a proof, see Proposition 5.3.3 in [40].

1.3 Results

In this subsection we summarize the results of the thesis. We start with the results obtained in Chapter 2 on Salem type substitutions. In [18] the authors ask if a similar result to Theorem 1.67 holds if the Perron-Frobenius eigenvalue of the substitution is a Salem number: we will call such a substitution of **Salem type** or **Salem substitution**. An example of a Salem substitution is the Holton-Zamboni substitution.

We are not able to prove an analog estimate which holds for every non zero spectral parameter $\omega \in \mathbb{R}$. Instead, we had to reduce ourselves to study the local dimension of the correlation measures on points belonging to $\mathbb{Q}(\alpha)$, where α is the Perron-Frobenius eigenvalue of the Salem substitution. For integers $l_0, \dots, l_n$, we denote by $(l_0, \dots, l_n)$ the greatest common divisor of $l_0, \dots, l_n$. Also, for a number $\eta = (l_0 + \cdots + l_{d-1}\alpha^{d-1})/L$ (with $L \in \mathbb{N}$ and $l_0, \dots, l_{d-1} \in \mathbb{Z}$) belonging to a number field $\mathbb{Q}(\alpha)$, we say it is in **reduced form** if $(l_0, \dots, l_{d-1}, L) = 1$.

Theorem 1.72 (M-M.) *Let ζ be a Salem type, aperiodic and primitive substitution on $\mathcal{A}$, α its Perron-Frobenius eigenvalue and $\vec{p}$ the positive (left-)eigenvector of the substitution matrix. Let $\mathfrak{X}_\zeta^{\vec{p}}$ be the corresponding self-similar suspension flow and for any $a \in \mathcal{A}$, denote by ν_a the correlation measure associated to a. Consider $\sigma_0 : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ the embedding corresponding to $\alpha \mapsto \alpha^{-1}$. Fix $A, B, C > 1$ and suppose $|\omega| \in \mathbb{Q}(\alpha) \cap [B^{-1}, B]$ satisfies $|\sigma_0(\omega)| \leq C$ and $L \leq A$, where $L \in \mathbb{N}$ is defined by the expression in reduced form $\omega = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1})$.*

Then there exist $\gamma = \gamma(A, B, C)$, $c = c(\zeta)$, $r_0 = r_0(\omega) > 0$ *such that*

$$\nu_a([\omega - r, \omega + r]) \leq cr^\gamma,$$

for all $0 < r < r_0$ *and* $a \in \mathcal{A}$.

This is a pointwise result, but from the estimates from Lemmas 2.23, 2.24 and 2.25 in Chapter 2, it is possible to obtain uniformity of γ of the last theorem on the variables B, C for a family of points $\omega \in \mathbb{Q}(\alpha)$ satisfying an arithmetic property, which leaves the exponent only depending on a "naive height" of the spectral parameter. We state this the next result.

Theorem 1.73 (M-M.) *Let* ζ *be a Salem type, aperiodic and primitive substitution and* ν_a *the correlation measure associated to the letter* $a \in \mathcal{A}$ *on the self-similar suspension flow. There exists* $\kappa \in \mathbb{Q}(\alpha)$ *an explicit positive constant such that the next statement holds: for fixed* $A > 1$, *suppose*

- *There exists* $n \in \{0, \ldots, d-1\}$ *such that* $Tr(L\kappa\omega\alpha^n) \not\equiv 0 \,(mod\ L)$, *and*
- $L \leq A$,

where $L \in \mathbb{N}$ *is defined by the expression in reduced form* $\omega\kappa = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1})$. *Then there exist* $\gamma = \gamma(A)$, $c = c(\zeta)$, $r_0 = r_0(\omega) > 0$ *such that*

$$\nu_a([\omega - r, \omega + r]) \leq cr^\gamma,$$

for all $0 < r < r_0$, $a \in \mathcal{A}$.

The next step is to study how restrictive is the first condition. In fact, we can prove that except for a finite set of values of L (depending only on α), the condition holds.

Proposition 1.74 (M-M.) *Suppose* $\eta = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha)$ *is in reduced form and* $Tr(L\eta\alpha^n) \equiv 0 \,(mod\ L)$ *for all* $n = 0, \ldots, d-1$. *Then* L *divides* $E(\alpha)$, *where* $E(\alpha)$ *is the least common multiple of the denominators of the dual basis of* $\{1, \alpha, \ldots, \alpha^{d-1}\}$ *(expressed in reduced form).*

A major difference with Theorem 1.67 is also the dependence of r_0. In Theorem 1.67 r_0 only depends on the parameter B, that is, on the absolute value of the spectral parameter. In our case the dependence is much more subtle, since r_0 is related with the convergence of some Birkhoff sums over a toral translation. In fact, we are not able to give a lower bound for r_0 in full generality, but only when $\deg(\alpha) = 4$.

Proposition 1.75 (M-M.) *Let* $A, B, C > 1$ *and* $\omega \in \mathbb{Q}(\alpha) \setminus \{0\}$ *satisfying the conditions of Theorem 1.72. Suppose* α *is a Salem number of degree equal 4 and let* $\alpha_1 = e^{2\pi i\theta_1}$ *be a Galois conjugate on the unit circle. Then* θ_1 *is Diophantine and if* τ *is an upper bound for its type, there exists a constant* $c_\alpha > 0$ *only depending on* α *such that* $r_0(\omega)$ *appearing in Theorem 1.72 satisfies*

$$r_0(\omega) > c_\alpha / \alpha^{A^4 \max(\lceil \log_\alpha(C/\delta) \rceil, H^\tau)},$$

where $H > 0$ *is explicit and depends only on the substitution, and* $\delta = \delta(A, B, C) > 0$ *is as in Theorem 1.72.*

We continue with the results obtained in Chapter 3. The aim is to study the behavior of the top Lyapunov exponent of the spectral cocycle when passing to a factor (in particular, the conjugacy class) of a substitutive subshift. We focus our attention on topological dynamics. As there are no previous results of this kind in the literature to our knowledge, we begin by analysing the particular example of the Thue-Morse substitution. We have already recalled that it is known that the top Lyapunov exponent for the spectral cocycle is null almost everywhere. We show this is the case in fact for any subshift topological factor.

Theorem 1.76 (M-M.) *For every topological factor of the Thue-Morse subshift coming from an aperiodic primitive substitution ζ, we have $\chi_{\zeta}^{+}(\omega) = 0$ (Lebesgue) almost surely.*

To obtain this theorem, we study the twisted correlation of the indicator functions associated to the cylinders [0] and [1] on the Thue-Morse word since, as it is shown in Chapter 3, the twisted Birkhoff sums of topological factors may be expressed as a linear combination of different twisted correlations on the Thue-Morse word.

The growth of this latter kind of sum is summarized in the next result.

Theorem 1.77 (M-M.) *Let $1 \leq a_1 < \dots a_t$ and $C_{2^n}^{f}(a_1, \dots, a_t, \omega, \mathbf{u})$ be the twisted correlation of the function $f = \mathbb{1}_{[0]} - \mathbb{1}_{[1]}$ at the fixed point of the Thue-Morse substitution $\zeta_{TM}^{\infty}(0) = \mathbf{u}$, defined by*

$$C_{2^n}^{f}(a_1, \dots, a_t, \omega, \mathbf{u}) = \sum_{k=0}^{2^n-1} (-1)^{s_2(k)+s_2(k+a_1)+\cdots+s_2(k+a_t)} e(k\omega).$$

Then,

- *if t is even, there exists $B > 0$ depending only on f such that for almost every ω,*

$$C_{2^n}^{f}(a_1, \dots, a_t, \omega, \mathbf{u}) = O_{a_t,\omega}\left(n^t e^{B\sqrt{n \log\log(n)}}\right).$$

- *if t is odd, for every $\varepsilon > 0$ and almost every ω,*

$$C_{2^n}^{f}(a_1, \dots, a_t, \omega, \mathbf{u}) = O_{a_t,\omega}\left(n^{t+1+\varepsilon}\right).$$

1.4 Future directions

For the self-similar suspension flow over a Salem type substitution, Theorem 1.72 and Theorem 1.73 give a lower bound for the local dimension of the correlation measures only for spectral parameters ω belonging to $\mathbb{Q}(\alpha)$, different from Theorem 1.67 which holds for every $\omega \in \mathbb{R}$.

Question 1.78 *Can we find a lower bound for the local dimension of correlation measures of the self-similar suspension flow over a Salem type substitution for spectral parameters outside $\mathbb{Q}(\alpha)$?*

Leaving the self-similar realm, it will be interesting to study the correlation measures for generic suspensions flows over the subsfhift of a Salem type substitution, in the same spirit of Theorem 1.66.

Question 1.79 *Can we get stronger results than Theorem 1.72 and Theorem 1.73 for generic suspensions over Salem substitutions?*

Again, it will be interesting to get a similar result to Theorem 1.72 and Theorem 1.73 for the subshift associated to a Salem type substitution.

Question 1.80 *Do similar results to Theorem 1.72 and Theorem 1.73 hold for Salem substitutions in the case of $\mathbb{Z}$-actions: does weak mixing imply a Hölder exponent for the spectral parameters in $\mathbb{Q}(\alpha)$?*

In [19, 22, 21], the authors generalize many results from [18] to the S-adic setting. As a fundamental hypothesis, the cocycles they consider have always a positive *second Lyapunov exponent*. In the context of substitutions, this is just the existence of a Galois conjugate of α outside of the closed unit disk.

Question 1.81 *Do similar results to Theorem 1.72 and Theorem 1.73 hold for cocycles with a vanishing second Lyapunov exponent in the S-adic setting, in the spirit of [19, 22, 21]?*

Other direction is to pass to higher dimension, that is *substitutive tilings* (see [46] for an introduction). In the work of J. Emme [37], it is proved asymptotic of the spectral measure at zero for substitutive tilings of $\mathbb{R}^n$, under the hypothesis that the expansion has at least two eigenvalues outside the closed unit disk. In the paper of R. Treviño [66], under the same hypothesis and other technical ones, it is proved a result similar to Theorem 1.66 in the context of random substitutive tiling.

Question 1.82 *Extend results to self-similar or generic substitutive tilings of $\mathbb{R}^n$ having as the Perron-Frobenius eigenvalue of the expansion a Salem number.*

Concerning Chapter 3, the methods used should apply for some class of substitution related to automata.

Question 1.83 *Is it possible to generalize the methods used to prove Theorem 1.76 to a whole class of substitutions (such as bijective abelian)?*

The proof of Theorem 1.76 and Theorem 1.77 suggest the next conjecture on the top Lyapunov exponent of the spectral cocycle when considering topological factors, twisted Birkhoff sums and twisted correlations.

Question 1.84 *May the top Lyapunov exponent of the spectral cocycle increase when passing to (measurable, topological) factors? Is it invariant by (measurable, topological) conjugacy? Does the growth of the twisted Birkhoff sums dominates the growth of the twisted correlations?*

Finally, another natural aspect to look at is the rest of the *Lyapunov spectrum* of the spectral cocycle: a strong improvement of the Furstenberg-Kesten theorem is Oseledets theorem (see for example [57], Chapter 3).

Question 1.85 *For general substitutions, what can we say about the rest of the Lyapunov spectrum? What dynamical information can we deduce from the rest of the Lyapunov spectrum of the spectral cocycle?*

Chapter 2

Modulus of continuity for spectral measures of suspension flows over Salem type substitutions

The content of this chapter has been submitted for publication.

Substitutions appear as natural objects in many different research areas such as symbolic dynamics, number theory, combinatorics of words, Diophantine approximation, and so on. For instance, the study of the spectrum of dynamical systems arising from substitutions has left longstanding open problems. One of the most important is the *Pisot substitution conjecture*, which asserts that if a substitution is irreducible and of *Pisot type*, then the corresponding subshift has pure discrete spectrum (see [1]).

A continuous counterpart is found studying tilings of the Euclidean space, and similar questions for the spectrum emerge again in this case. A suspension flow of a substitution subshift can be seen as a special kind of tiling of the real line. Previous work on this type of systems are the papers of Clark and Sadun [25], [26] and Barge and Diamond [11]. By relating the eigenvalues of the system with the eigenvalues of the matrix representing the cohomological action of the substitution map, it is possible to conclude that generic tile suspensions over non Pisot irreducible substitutions are weakly mixing. In this case, the spectral measures do not have any atoms (except for the trivial one at the origin), and moduli of continuity of these measures are linked with rates of weak mixing (see [49]). Decay rates of spectral measures also give information on its absolutely continuous and singular components.

In the weak mixing case, the work of Bufetov and Solomyak [18] analyses the case in which the Perron-Frobenius eigenvalue of the substitution matrix has at least one conjugate outside the closed unit disk. They prove a Hölder decay of the spectral measures for a typical suspension flow (if the characteristic polynomial of the substitution matrix is irreducible), and a log-Hölder one for a self-similar suspension flow with this hypothesis. In both cases, for spectral parameters away from the origin. They have broadened many of the tools used in that article from this simple setting to more complex systems such as the ones associated to Bratelli-Vershik diagrams or translations flows on flat surfaces (see [19],[22]). A different approach in this last setting to this problem is found in [41], and the study for higher rank actions is in [66].

The main objective of this paper is to study the spectrum of the self-similar suspension flow when the substitution is of Salem type, i.e., when the dominant eigenvalue of the substitution matrix is a Salem number. This is a question raised in [18]. This could be thought as a limit case, since we do not have homoclinic points (which give rise to eigenvalues in the Pisot case) nor an unstable subspace ensuring the absence of atoms (e.g., if there is a different conjugate of the Perron-Frobenius eigenvalue of the substitution matrix outside the unit disk). Instead, it acts as an isometry on the invariant subspace complementary to the subspace generated by the eigenvectors associated to the dominating eigenvalue and its inverse.

Salem substitutions arise naturally in the study of Veech groups: on each non arithmetic primitive Veech surface of genus two, there exists a pseudo-Anosov diffeomorphism whose dilatation is a Salem number of degree 4. The associated interval exchange transformation (by zippered rectangles) is self-similar and is (more precisely, the loop in its Rauzy diagram) defined by a substitution of Salem type (see [17]). Explicit examples appear in [17] and [3].

The difference between the analogous result in [18] and ours is reflected in the complicated dependence we have found on the parameters controlling the decay of the spectral measure. In fact, we are only able to find this decay when the spectral parameter belongs to the number field generated by the principal eigenvalue, as we show in the next result. For integers $l_0, \ldots, l_n$, we denote by $\gcd(l_0, \ldots, l_n)$ the greatest common divisor of $l_0, \ldots, l_n$. Also, for a number $\eta = (l_0 + \cdots + l_{d-1}\alpha^{d-1})/L$ (with $L \in \mathbb{N}$ and $l_0, \ldots, l_{d-1} \in \mathbb{Z}$) belonging to a number field $\mathbb{Q}(\alpha)$, we say it is in *reduced form* if $\gcd(l_0, \ldots, l_{d-1}, L) = 1$.

Theorem 2.1 *Let ζ be a Salem type, aperiodic and primitive substitution on $\mathcal{A}$, α its Perron-Frobenius eigenvalue and $\vec{p}$ the positive (left-)eigenvector of the substitution matrix. Let $\mathfrak{X}_\zeta^{\vec{p}}$ be the corresponding self-similar suspension flow and for any $a \in \mathcal{A}$, denote by ν_a the correlation measure associated to a. Consider $\sigma_0 : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ the embedding corresponding to $\alpha \mapsto \alpha^{-1}$. Fix $A, B, C > 1$ and suppose $|\omega| \in \mathbb{Q}(\alpha) \cap [B^{-1}, B]$ satisfies $|\sigma_0(\omega)| \leq C$ and $L \leq A$, where $L \in \mathbb{N}$ is defined by the expression in reduced form $\omega = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1})$. Then there exist $\gamma = \gamma(A, B, C)$, $c = c(\zeta)$, $r_0 = r_0(\omega) > 0$ such that*

$$\nu_a([\omega - r, \omega + r]) \leq cr^\gamma,$$

for all $0 < r < r_0$ and $a \in \mathcal{A}$.

Under an arithmetic condition there is a uniform dependence of γ on the variables B, C, as we state in our second result.

Theorem 2.2 *Let ζ be a Salem type, aperiodic and primitive substitution and ν_a the correlation measure associated to the letter $a \in \mathcal{A}$ on the self-similar suspension flow. Let $\kappa \in \mathbb{Z}[\alpha]$*

be the explicit positive constant of Lemma 2.9. Fix $A > 1$ and suppose

- *There exists $n \in \{0, \dots, d-1\}$ such that $Tr(L\kappa\omega\alpha^n) \not\equiv 0 \,(mod\ L)$, and*
- *$L \leq A$,*

where $L \in \mathbb{N}$ is defined by the expression in reduced form $\omega\kappa = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1})$. Then there exist $\gamma = \gamma(A)$, $c = c(\zeta)$, $r_0 = r_0(\omega) > 0$ such that

$$\nu_a([\omega - r, \omega + r]) \leq c r^{\gamma},$$

for all $0 < r < r_0$, $a \in \mathcal{A}$.

The first condition above is generic, in the sense that the complement of the set of ω's satisfying it, is contained in a finite union of lattices of $\mathbb{Q}(\alpha)$, according to

Proposition 2.3 *Suppose $\eta = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha)$ is in reduced form and $Tr(L\eta\alpha^n) \equiv 0 \,(mod\ L)$ for all $n = 0, \dots, d-1$. Then L divides $E(\alpha)$, where $E(\alpha)$ is the least common multiple of the denominators of the dual basis of $\{1, \alpha, \dots, \alpha^{d-1}\}$ (expressed in reduced form).*

An important difference with the results in [18] is the complex dependence of $r_0 > 0$ on ω, in both Theorem 2.1 and 2.2. A lower bound is derived explicitly when $\deg(\alpha) = 4$, but it does not have a simple expression in terms of the spectral parameter. We will summarize this in Proposition 2.29.

The proof of Theorems 2.1 and 2.2 is based in the distribution modulo 1 of the sequence $(\omega\alpha^n)_{n\geq 0}$. The link between a modulus of continuity for the spectral measures and the distribution modulo one of such a sequence may be seen from Lemma 2.9 and Proposition 2.8. S. Akiyama and Y. Tanigawa showed in [2] that the sequence $(\alpha^n)_{n\geq 1}$ is not far from being uniformly distributed modulo 1. We recall this result in Theorem 2.11 . We are able to prove a similar result for the sequences $(\omega\alpha^n)_{n\geq 0}$, for $\omega \in \mathbb{Q}(\alpha)$.

Theorem 2.4 *Denote $J(\delta) = [\delta, 1-\delta]$, for $\delta < 1/2$. Let α be a Salem number of degree d and $\eta = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha)$, with $L \geq 1$ and $l_0, \dots, l_{d-1} \in \mathbb{Z}$. Consider $\sigma_0 : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ the embedding corresponding to $\alpha \mapsto \alpha^{-1}$. Assume $\gcd(l_0, \dots, l_{d-1}, L) = 1$. Then there exists an explicit $\delta = \delta(L, |\eta|, |\sigma_0(\eta)|) > 0$ such that*

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid \{\eta\alpha^n\} \in J(\delta)\}}{N} \geq 1/2.$$

This result is easily deduced from Corollary 2.21 and from Lemmas 2.22, 2.23, 2.24 and 2.25.

The rest of the paper is organized as follows. In Section 2 we provide a background material around substitutions, spectral theory of dynamical systems, algebraic number theory and harmonic analysis. In particular, we recall the definition of special trigonometric polynomials used in the proof of Theorems 2.1 and 2.2. Section 3 sketches the proof of the two main theorems since it is rather technical in full generality. Section 4 is devoted to the proof of Theorems 2.1, 2.2 and Proposition 2.3. Finally, in Section 5 we study the nature of r_0 appearing in both main results, ending with the proof of Proposition 2.29.

2.1 Background

2.1.1 Dynamical systems arising from substitutions

The basic notions of substitutions may be found in [61, 40] with more detail. Let us start by fixing a positive even integer $d \geq 4$ and a finite alphabet $\mathcal{A} = \{1, \dots, d\}$. A *substitution* on the alphabet $\mathcal{A}$ is a map $\zeta : \mathcal{A} \longrightarrow \mathcal{A}^+$, where $\mathcal{A}^+$ denote the set of finite (nonempty) words on $\mathcal{A}$. By concatenation, it is natural to extend a substitution to $\mathcal{A}^+$, to $\mathcal{A}^{\mathbb{N}}$ (one-sided sequences) or $\mathcal{A}^{\mathbb{Z}}$ (two-sided sequences). In particular, the iterates $\zeta^n(a) = \zeta(\zeta^{n-1}(a))$ for $a \in \mathcal{A}$, are well defined.

Example (see [44]) Let $\mathcal{A} = \{1, 2, 3, 4\}$ and define ζ by

$$\begin{aligned} \zeta(1) &= 12, \quad \zeta(3) = 2, \\ \zeta(2) &= 14, \quad \zeta(4) = 3. \end{aligned}$$

For a word $w \in \mathcal{A}^+$ denote its length by $|w|$ and by $|w|_a$ the number of symbols a found in w. The *substitution matrix* associated to a substitution ζ is the $d \times d$ matrix with integer entries defined by $M_\zeta(a, b) = |\zeta(b)|_a$. A substitution is called *primitive* if its substitution matrix is primitive.

Example Let ζ be the substitution defined in Example 2.1. Then its substitution matrix is

$$M_\zeta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and this substitution is primitive.

The *substitution subshift* associated to ζ is the set $\mathfrak{X}_\zeta$ of sequences $(x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ exist $a \in \mathcal{A}$ and $n \in \mathbb{N}$ such that $x_i \dots x_{i+k}$ is a subword of some $\zeta^n(a)$. A classical result is that the $\mathbb{Z}$-action by the *left-shift* $T((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ on this subshift is minimal and uniquely ergodic when ζ is primitive. From now on we only consider primitive and *aperiodic* substitutions, which means in the primitive case that the subshift is not finite.

Now we turn to the continuous counterpart of the substitution subshift. The study of the discrete part of the spectrum of the suspensions over substitutions (not only the self-similar case) is done in the article [25]. For a primitive substitution ζ, denote by $\vec{p} = (p_a)_{a\in\mathcal{A}}$ a positive Perron-Frobenius left-eigenvector of M_ζ. For convenience, we will normalize the vector $\vec{p}$ in such a way that its components belong to $\mathbb{Z}[\alpha]$ (it is enough to multiply for some positive integer). This allows us to assume the constant κ appearing in Lemma 2.9 belongs to $\mathbb{Z}[\alpha]$ (see the remark below Lemma 2.9). Set $F : \mathfrak{X}_\zeta \times \mathbb{R} \longrightarrow \mathfrak{X}_\zeta \times \mathbb{R}$ defined by $F(x,t) = (T(x), t - p_{x_0})$.

Definition 2.5 *The self-similar suspension flow is the pair $(\mathfrak{X}_\zeta^{\vec{p}}, (h_t)_{t\in\mathbb{R}})$ given by*

$$\mathfrak{X}_\zeta^{\vec{p}} = (\mathfrak{X}_\zeta \times \mathbb{R})/\sim,$$
$$h_t(x,t') = (x, t'+t) \quad (mod\ \sim),$$

where $\sim$ is the equivalence relation defined by $(x,t) \sim (x',t')$ if and only if $F^n(x,t) = (x',t')$, for some $n \in \mathbb{Z}$.

We will identify $\mathfrak{X}_\zeta^{\vec{p}}$ with a *fundamental domain* (see for example [58], Chapter 3). We will take as fundamental domain

$$\mathcal{D} = \left\{(x,t) \in \mathfrak{X}_\zeta^{\vec{p}} \times \mathbb{R} \mid 0 \le t < f(x)\right\}.$$

We will make no further reference to the fundamental domain, and we will simply denote it by $\mathfrak{X}_\zeta^{\vec{p}}$. We may decompose

$$\mathfrak{X}_\zeta^{\vec{p}} = \bigcup_{a\in\mathcal{A}} \mathfrak{X}_a^{\vec{p}}, \quad \mathfrak{X}_a^{\vec{p}} = \left\{(x,t) \in \mathfrak{X}_\zeta^{\vec{p}} \,\middle|\, x_0 = a\right\}.$$

Once again, this flow is uniquely ergodic and the only (Borelian) measure invariant for the flow $(h_t)_{t\in\mathbb{R}}$ will be denoted als by μ. Our results will concern the spectral measures (we recall its definition in the next subsection) associated to the indicator functions of this partition (in measure), i.e., $f = \mathbb{1}_{\mathfrak{X}_a^{\vec{p}}}$, for each $a \in \mathcal{A}$.

2.1.2 Spectral theory

We define the main objects of study of this work, namely, the spectral measures associated to the self-similar suspension flow. We restrict ourselves to stating a dynamical version of the spectral theorem taken from [48]. A more extensive introduction may also be found in [61, 40].

Theorem 2.6 *Let $f, g \in L^2(\mathfrak{X}_\zeta^{\vec{p}}, \mu)$. There exists a complex measure $\nu_{f,g}$ with support in $\mathbb{R}$, called* ***spectral measure****, such that for all $t \in \mathbb{R}$*

$$\widehat{\nu_{f,g}}(-t) := \int_{\mathbb{R}} e^{2\pi i t\omega} d\nu_{f,g}(\omega) = \langle U^t f, g\rangle_{L^2(\mathfrak{X}_\zeta^{\vec{p}},\mu)},$$

where $U^t f(y) = f(h_t(y))$ for $y \in \mathfrak{X}_\zeta^{\vec{p}}$.

We will denote $\nu_{f,f}$ by ν_f, and for $f = \mathbb{1}_{\mathfrak{X}_a^{\vec{p}}}$ we will write $\nu_f = \nu_a$. The measure ν_a will be called the *correlation measure* associated to a. To study the asymptotics of spectral measures, we look into a special kind of Birkhoff integral.

Definition 2.7 *Let $f \in L^2(\mathfrak{X}_\zeta^{\vec{p}}, \mu)$, $(x,s) \in \mathfrak{X}_\zeta^{\vec{p}}$, $\omega \in \mathbb{R}$, $R > 0$. The twisted Birkhoff integral associated to f is defined by*

$$S_R^f((x,s),\omega) = \int_0^R e^{-2\pi i\omega t} f(h_t(x,s))dt.$$

The relation between these two concepts is clarified by the next proposition.

Proposition 2.8 ([18]) *Denote $G_R(f,\omega) = \dfrac{1}{R}||S_R^f(\cdot,\omega)||_{L^2}^2$. Suppose there exists $0 < \gamma < 1$ such that $G_R(f,\omega) \leq CR^{2\gamma-1}$, for some constant $C > 0$ and $R \geq R_0$. Then there exists $r_0 > 0$ (depending only on R_0) such that for every $0 < r \leq r_0$ holds*

$$\nu_f([\omega - r, \omega + r]) \leq \pi^2 C r^{2(1-\gamma)}.$$

Finally, the problem of finding bounds for the twisted Birkhoff sums may be addressed solving a problem on Diophantine approximation, according to the next lemma. We denote the distance of $x \in \mathbb{R}$ to the nearest integer by $||x||_{\mathbb{R}/\mathbb{Z}} = \min(\{x\}, 1 - \{x\})$, where as usual $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Lemma 2.9 ([18]) *Let ζ be a primitive substitution (with α being its Perron-Frobenius eigenvalue and $\vec{p}$ the eigenvector normalized as before) and $\mathfrak{X}_\zeta^{\vec{p}}$ the corresponding self-similar suspension flow. Let $a \in \mathcal{A}$ and f be the indicator function of $\mathfrak{X}_a^{\vec{p}}$. Then there exist $\lambda \in (0,1)$, $C_1 > 0$ and $\kappa \in \mathbb{Z}[\alpha]$ an explicit positive constant, all depending only on the substitution ζ, such that*

$$|S_R^f((x,s),\omega)| \leq C_1 R \prod_{n=0}^{\lfloor \log_\alpha(R) \rfloor} (1 - \lambda||\omega\kappa\, \alpha^n||_{\mathbb{R}/\mathbb{Z}}^2),$$

for all $R > 0$, $(x,s) \in \mathfrak{X}_\zeta^{\vec{p}}$ and $\omega \in \mathbb{R}$.

Remark As we remarked before, if $\vec{p} \in (\mathbb{Z}[\alpha])^d$, then we may assume $\kappa \in \mathbb{Z}[\alpha]$, since by definition $\kappa = \langle \mathrm{Ab}(w), \vec{p} \rangle$, where w is an appropriate *return word* and $\mathrm{Ab}(w)$ its *population vector* indexed by $\mathcal{A}$ with components $(\mathrm{Ab}(w))_a = |w|_a$. More details in [18].

2.1.3 Salem numbers

Recall the definition of a *Salem number*: a real algebraic integer greater than 1 having all its Galois conjugates inside the closed unit disk, with at least one conjugate on the unit circle. This definition actually forces that the inverse is a conjugate, and the rest of them are on the unit circle. For a survey on Salem numbers see [64]. We will say a primitive substitution is of *Salem type* if the dominant eigenvalue is a Salem number. An example of substitution of Salem type is the one of Example 2.1.1.

We now state some results regarding Salem numbers.

Proposition 2.10 (see [23]) *Let α be a Salem number and $\varepsilon > 0$, then there exists $\eta = \eta(\varepsilon) \in \mathbb{Q}(\alpha)$ different from zero such that*

$$||\eta\alpha^n||_{\mathbb{R}/\mathbb{Z}} < \varepsilon,$$

for all $n \geq 0$.

In fact, in [70] there is a characterization of numbers $\eta \in \mathbb{R}$ such that $\limsup_n ||\eta\alpha^n||_{\mathbb{R}/\mathbb{Z}} < \varepsilon$, with $\varepsilon \leq \delta_1(\alpha) = 1/\mathscr{L}(\alpha)$, where $\mathscr{L}(\alpha)$ denotes the *length* of α, i.e., the sum of the absolute values of the coefficients of its minimal polynomial. Proposition 2.10 shows the difficulty to find a universal exponent for the spectral measure as in [18] in the case of Salem type substitutions, at least by the methods we are using.

A classical result states that the sequence $(\alpha^n)_{n\geq 1}$ is dense but not uniformly distributed. In spite of this result, S. Akiyama and Y. Tanigawa showed in [2] that the sequence $(\alpha^n)_{n\geq 1}$ is not far from being uniformly distributed modulo 1.

Theorem 2.11 ([2]) *Let α be a Salem number of degree $d = 2m + 2 \geq 8$ and $J = [\mathfrak{a}, \mathfrak{b}] \subseteq [0, 1]$, then*

$$\left| \lim_{N\to\infty} \frac{\#\{n \leq N \mid \{\alpha^n\} \in J\}}{N} - (\mathfrak{b} - \mathfrak{a}) \right| \leq 2\zeta\,(m/2)\,(2\pi)^{-m}(\mathfrak{b} - \mathfrak{a}),$$

where ζ denotes the Riemann zeta function.

For degree $d = 4$ and $d = 6$ there are similar estimates we omit here. An extension of this work is the heart of the proof of Lemmas 2.22, 2.23, 2.24 and 2.25, which is the essential step to prove Theorems 2.1 and 2.2.

Proposition 2.12 (see [23]) *Let α be a Salem number of degree $d = 2m+2$ and $e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_m}$ the conjugates on the upper-half of the unit circle. Then $1, \theta_1, \ldots, \theta_m$ are rationally independent.*

Let us recall the notion of *trace* of an algebraic number: for an algebraic number $\eta \in \mathbb{Q}(\alpha)$ define

$$\mathrm{Tr}(\eta) := \sum_{\sigma:\mathbb{Q}(\alpha)\hookrightarrow\mathbb{C}} \sigma(\eta),$$

where the sum runs over all embeddings of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$. In particular, for an algebraic integer η we have $\mathrm{Tr}(\eta) \in \mathbb{Z}$.

Finally, we state a classic inequality about evaluation of integer polynomials on algebraic numbers. For $P(X) = a_0 + \cdots + a_d X^d \in \mathbb{Z}[X]$ the (naive) *height* of P is defined as $\mathrm{Height}(P) = \max(|a_0|, \ldots, |a_d|)$.

Lemma 2.13 ([42]) *Let ξ be an algebraic integer and $Q \in \mathbb{Z}[X]$ of degree at most $n \geq 1$ such that $Q(\xi) \neq 0$. Let $\xi_1, \ldots, \xi_d$ be the other conjugates of ξ and m the number of i's such that $|\xi_i| = 1$. Then*

$$|Q(\xi)| \geq \frac{\prod_{|\xi_i| \neq 1} ||\xi_i| - 1|}{(n+1)^m \left(\prod_{|\xi_i|>1} |\xi_i|\right)^{n+1} \operatorname{Height}(Q)^d}.$$

2.1.4 Polynomial approximation of functions

One of the main technical tools we use is a family of trigonometric polynomials (Selberg polynomials) which approximate the indicator function $\mathbb{1}_J$ of an interval $J \subset [0,1]$. A detailed reference is found in [56]. In order to introduce it, we define also several other families, starting with the well-known Fejer kernel.

Definition 2.14 *The Fejer kernel of degree $N-1$ is defined by*

$$\Delta_N(z) = \sum_{0 \leq |k| \leq N-1} \left(1 - \frac{|k|}{N}\right) e^{2\pi i k z}$$

Definition 2.15 *The Vaaler polynomial of degree N is defined by*

$$\mathcal{V}_N(z) = \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) \sin(2\pi k z),$$

where the function f is given by $f(x) = -(1-x)\cot(\pi x) - 1/\pi$ and satisfies for every $\xi < 1/2$ the inequalities

$$|f(x)| \leq \begin{cases} \dfrac{\pi\xi}{\sin(\pi\xi)} \dfrac{1}{\pi x} + \dfrac{1}{\pi} & \text{if } 0 < x \leq \xi, \\[2ex] \dfrac{1-\xi}{\sin(\pi(1-\xi))} + \dfrac{1}{\pi} & \text{if } \xi < x < 1. \end{cases} \tag{2.1}$$

Definition 2.16 *The Beurling polynomial of degree N is defined by*

$$\mathcal{B}_N(z) = \mathcal{V}_N(z) + \frac{1}{2N+1} \Delta_N(z).$$

The Beurling polynomials provide approximation to the sawtooth function $s(x) = \{x\} - 1/2$ if $x \notin \mathbb{Z}$, and $s(x) = 0$ if $x \in \mathbb{Z}$. The Vaaler lemma (see [56]) ensures the choice of these polynomials is optimal in certain sense. If we denote by $\mathbb{1}_J$ the periodic extension to $\mathbb{R}$ of the indicator function of some interval $J = [\mathfrak{a}, \mathfrak{b}] \subseteq [0,1]$, we have the equality $\mathbb{1}_J(z) = \mathfrak{b} - \mathfrak{a} + s(z - \mathfrak{b}) + s(\mathfrak{a} - z)$. This fact justifies the next definition.

Definition 2.17 *The Selberg polynomials of degree N of an interval $J = [\mathfrak{a}, \mathfrak{b}] \subseteq [0,1]$ are defined by*

$$\mathcal{S}_N^+(z) = \mathfrak{b} - \mathfrak{a} + \mathcal{B}_N(z - \mathfrak{b}) + \mathcal{B}_N(-z + \mathfrak{a}),$$
$$\mathcal{S}_N^-(z) = \mathfrak{b} - \mathfrak{a} - \mathcal{B}_N(-z + \mathfrak{b}) - \mathcal{B}_N(z - \mathfrak{a}).$$

These polynomials satisfy for all $N \geq 1$,

$$\mathcal{S}_N^- \leq \mathbb{1}_J \leq \mathcal{S}_N^+ \tag{2.2}$$

2.1.5 Bessel functions

We will denote by $J_0(x)$ the *Bessel function of order zero*, that is, the unique solution to

$$\begin{cases} xy'' + y' + xy = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

We summarize two classic properties used later in the next

Proposition 2.18 (see [16]) *For $x \geq 0$,*

$$J_0(x) = \int_0^1 e^{ix\cos(2\pi t)}dt, \quad |J_0(x)| \leq \min\left(1, \sqrt{\frac{2}{\pi x}}\right).$$

To finish this background section, we prove an equality involving the Bessel function which will be used repeatedly in the technical calculations from Lemmas 2.22 to 2.25, to calculate the integral of the Beurling polynomial.

Proposition 2.19 *Let $H_1, \ldots, H_m$ be positive real numbers and*

$$z(x_1, \ldots, x_m) = 2\sum_{j=1}^m H_j \cos(2\pi x_j),$$

with every $x_j \in [0,1]$. Then, for every integer $k \geq 1$, we have

$$\int_{(\mathbb{R}/\mathbb{Z})^m} e^{2\pi i k z(x_1,\ldots,x_m)} dx_1 \ldots dx_m = \prod_{j=1}^m J_0(4\pi k H_j).$$

Proof.

$$\begin{aligned}\int_{(\mathbb{R}/\mathbb{Z})^m} e^{2\pi i k z(x_1,\ldots,x_m)} dx_1 \ldots dx_m &= \prod_{j=1}^m \int_{\mathbb{R}/\mathbb{Z}} e^{4\pi i k H_j \cos(2\pi x_j)} dx_j \\ &= \prod_{j=1}^m J_0(4\pi k H_j).\end{aligned}$$

□

2.2 Outline of the proof

In this section we sketch the basic steps forward the proof of Theorem 2.1 and Theorem 2.2 in a simpler case to fix ideas, and leave the formal proof to the next section.

By Lemma 2.9, we are interested in the distribution of the sequence $(\omega\kappa\alpha^n)_{n\geq 0}$ modulo 1. Let us outline the strategy when $\eta = \omega\kappa$ belongs to $\mathbb{Z}[\alpha]$. Let α be a Salem number of degree $d = 2m+2$ and $\alpha_1 = \sigma_1(\alpha) = e^{2\pi i\theta_1}, \ldots, \alpha_m = \sigma_m(\alpha) = e^{2\pi i\theta_m}$ the Galois conjugates on the upper half of S^1. Denote the embeddings of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$ by σ_j, where $\sigma_0(\alpha) = \alpha^{-1}$ and $\sigma_j(\alpha) = \alpha_j = e^{2\pi i\theta_j}$ for $j = 1, \ldots, m$. In this case, for all $n \geq 0$

$$\mathrm{Tr}(\eta\alpha^n) = \eta\alpha^n + \sigma_0(\eta)\alpha^{-n} + 2\mathcal{R}_n,$$

where $\mathcal{R}_n = \sum_{j=1}^m |\sigma_j(\eta)| \cos(2\pi n\theta_j + \phi_j)$ for some $\phi_j \in \mathbb{R}$. Since $\eta\alpha^n \in \mathbb{Z}[\alpha]$, we have $\mathrm{Tr}(\eta\alpha^n) \in \mathbb{Z}$. This implies

$$\{\eta\alpha^n\} + 2\mathcal{R}_n \ (\mathrm{mod}\ 1) \longrightarrow 0, \text{ as } n \to \infty.$$

This convergence implies the next fact (this is proved formally in Corollary 2.21): let $J \subseteq [0,1]$ be an interval. Then

$$\lim_{N\to\infty} \frac{\#\{n \leq N | \{\eta\alpha^n\} \in J\}}{N} = \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_J \left(-2\sum_{j=1}^m |\sigma_j(\eta)| \cos(x_j)\right) d\vec{x}.$$

The last equality leave us the problem of understanding the integral of the indicator function of some interval. This is the same strategy used in [2] to prove Theorem 2.11. The extra difficulty in our case is that we have to manage the parameters $|\sigma_j(\eta)|$, whereas in [2] they only work the case $\eta = 1$, which implies $|\sigma_j(\eta)| = 1$ for all $j = 1, \ldots, m$.

The solution is proving a much weaker inequality (in fact it is not possible to obtain the same one according to Proposition 2.10): we will see in the next section that for some a suitable $\delta > 0$, if we consider the interval $J(\delta) = [\delta, 1-\delta]$, then

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(-2\sum_{j=1}^m |\sigma_j(\eta)| \cos(x_j)\right) d\vec{x} \geq 1/2. \tag{2.3}$$

This is the technical part of the proof, but the strategy is the same as in [2]: we use the family of Selberg polynomials defined in subsection 2.1.4 to apporximate the indicator function of an interval. To manage the coefficients $|\sigma_j(\eta)|$ we follow the strategy used in [18], using an inequality due to Garsia (Proposition 2.13).

Finally, once we have proved 2.3, the proof will be straithforward: an application of Lemma 2.9 and Proposition 2.8 will yield the desired modulus of continuty for the spectral measure.

2.3 Proof of Theorem 2.1 and Theorem 2.2

In this section we give the proof of Theorem 2.1. We begin with fixing the notation and hypothesis for the rest of this section. Let α be a Salem number of degree $d = 2m+2$

and $\alpha_1 = \sigma_1(\alpha) = e^{2\pi i\theta_1}, \ldots, \alpha_m = \sigma_m(\alpha) = e^{2\pi i\theta_m}$ the Galois conjugates on the upper half of S^1, and $\sigma_1, \ldots, \sigma_m$ the respective embeddings. Let $\sigma_0 : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ be the embedding corresponding to $\sigma_0(\alpha) = \alpha^{-1}$. Let $\eta = \dfrac{1}{L}(l_0 + \cdots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha) \setminus \{0\}$, with $l_j \in \mathbb{Z}$ and $L \in \mathbb{N}$. We will denote $\vec{x} = (x_1, \ldots, x_m) \in (\mathbb{R}/\mathbb{Z})^m$.

2.3.1 Preliminary results

We begin with a proposition from [32] (see also [69]).

Proposition 2.20 *Consider*

$$\begin{aligned} U(z) &= l_0 + \cdots + l_{d-1}\cos(2\pi(d-1)z), \\ V(z) &= l_1 \sin(2\pi z) + \cdots + l_{d-1}\sin(2\pi(d-1)z), \\ \phi(z) &= \arctan(U(z)/V(z)). \end{aligned}$$

Define $\mathcal{R}_n = \sum_{j=1}^{m} \sqrt{U^2(\theta_j) + V^2(\theta_j)}\cos(2\pi n\theta_j - \phi(\theta_j))$. *There exists a positive integer* $P \leq L^d$ *such that for every* $j \in \{0, \ldots, P-1\}$ *there exists* $a_j \in \{0, \ldots, L-1\}$ *satisfying*

$$\left\{\eta\alpha^{Pn+j}\right\} + \frac{2\mathcal{R}_{Pn+j}}{L} \pmod 1 \longrightarrow \frac{a_j}{L} \quad \textit{as } n \to \infty.$$

Proof. We only sketch the proof, more details in [32]. Note that $U(\theta_j) = \Re(\sigma_j(L\eta))$ and $V(\theta_j) = \Im(\sigma_j(L\eta))$ and so $\sqrt{U^2(\theta_j) + V^2(\theta_j)} = |\sigma_j(L\eta)|$. By definition of the trace, for every $n \geq 0$

$$\mathrm{Tr}(L\eta\alpha^n) = L\eta\alpha^n + \sigma_0(L\eta)\alpha^{-n} + 2\mathcal{R}_n. \tag{2.4}$$

The sequence $(\mathrm{Tr}(L\eta\alpha^n) \pmod L)_{n\geq 0}$ is purely periodic of period $P \leq L^d$, since it satisfies an integer linear recurrence given by the minimal polynomial of α. Rearranging the first calculation we conclude

$$\{\eta\alpha^n\} + \frac{2\mathcal{R}_n}{L} \pmod 1 \longrightarrow \frac{a_j}{L}, \text{ as } n \to \infty$$

where $a_j \in \{0, \ldots, L-1\}$ and $n \equiv j \pmod P$. □

Corollary 2.21 *Let* $J \subseteq [0,1]$ *be an interval and set*

$$\mathcal{R}(x_1, \ldots, x_m) = \sum_{j=1}^{m} \sqrt{U^2(\theta_j) + V^2(\theta_j)}\cos(2\pi x_j).$$

Then

$$\lim_{N\to\infty} \frac{\#\{n \leq N | \{\eta\alpha^n\} \in J\}}{N} = \frac{1}{P}\sum_{j=0}^{P-1}\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_J\left(\frac{-2\mathcal{R}(\vec{x}) + a_j}{L}\right) d\vec{x}.$$

Proof. By Proposition 2.12, the numbers $1, \theta_1, \ldots, \theta_m$ are rationally independent, then the uniform distribution mod $\mathbb{Z}^m$ of $(n\theta_1, \ldots, n\theta_m)_{n\geq 1}$ justifies the last of the next equalities.

We can ignore the term $\sigma_0(\eta)\alpha^{-n}$ in (2.7) since this term goes to zero and the sequences $z_n^{(j)} = \dfrac{-2\mathcal{R}_{Pn+j}}{L} + \dfrac{a_j}{L}$ have continuous asymptotic distribution functions.

$$\lim_{N\to\infty} \frac{\#\{n \le N | \{\eta\alpha^n\} \in J\}}{N} \tag{2.5}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{P-1} \#\{n \le N,\, n \equiv j \pmod P | \{\eta\alpha^n\} \in J\} \tag{2.6}$$

$$= \sum_{j=0}^{P-1} \lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N,\, n \equiv j \pmod P \middle| \frac{-2\mathcal{R}_n}{L} + \frac{a_j}{L} - \sigma_0(\eta)\alpha^{-n} \pmod 1 \in J \right\} \tag{2.7}$$

$$= \frac{1}{P} \sum_{j=0}^{P-1} \lim_{N\to\infty} \frac{1}{N/P} \# \left\{ n \le N,\, n \equiv j \pmod P \middle| \frac{-2\mathcal{R}_n}{L} + \frac{a_j}{L} \pmod 1 \in J \right\} \tag{2.8}$$

$$= \frac{1}{P} \sum_{j=0}^{P-1} \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_J \left(\frac{-2\mathcal{R}(\vec{x}) + a_j}{L} \right) d\vec{x}. \tag{2.9}$$

□

For a fixed $1/2 > \delta > 0$, denote by $J(\delta)$ the interval $[\delta, 1-\delta] \subseteq [0,1]$. Denote $H_j = \sqrt{U^2(\theta_j) + V^2(\theta_j)}$ and $H = \sum_{j=1}^m H_j = \max_{\vec{x}\in(\mathbb{R}/\mathbb{Z})^m} |\mathcal{R}(\vec{x})|$. The next lemmas show that all integrals in the last corollary are greater than a fixed positive constant for a suitable choice of δ. We will divide the analysis depending on the size of $2H/L$ with respect to $\delta_1(\alpha) := 1/\mathscr{L}(\alpha)$, where $\mathscr{L}(\alpha)$ denotes the sum of the absolute values of the coefficients of the minimal polynomial of α.

Lemma 2.22 *Let $\beta = \alpha^P$ and $\tilde{\eta} = \eta\alpha^j$ for some $j \in \{0, \dots, P-1\}$, in order to have $\eta\alpha^{Pn+j} = \tilde{\eta}\beta^n$ for all $n \ge 0$. Suppose $Tr(L\tilde{\eta}\beta^n) = 0 \pmod L$ for all $n \ge 0$ and $2H/L < \delta_1(\beta)/2$. Then there exists $\delta = \delta(L, |\eta|, |\sigma_0(\eta)|) > 0$ such that*

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(\frac{-2\mathcal{R}(\vec{x})}{L} \right) d\vec{x} \ge 1/2.$$

Remark From the proof of Lemma 2.22 it is easy to deduce that in the dependence of $\delta = \delta(L, |\eta|, |\sigma_0(\eta)|)$ what is important is an upper bound of L and $|\sigma_0(\eta)|)$ and both bounds for $|\eta|$.

Proof. Note that by definition β is also a Salem number of degree d, fact that is used implicitly. Let us follow [18] and write

$$\tilde{\eta}\beta^n = K_n(\tilde{\eta}, \beta) + \varepsilon_n(\tilde{\eta}, \beta), \quad K_n \in \mathbb{Z},\ -1/2 < \varepsilon_n \le 1/2.$$

Let n_0 be the smallest nonnegative integer such that $|\tilde{\eta}|\beta^{n_0} \ge 1$. Proposition 2.20 and $2H/L < \delta_1(\beta)/2$ implies $|\varepsilon_n| < \delta_1(\beta)$ for all $n \ge n_1$, where $n_1 = \lceil \log_\beta(2|\sigma_0(\tilde{\eta})|/\delta_1(\beta) \rceil$. Indeed, $|\varepsilon_n| = \min(\{\tilde{\eta}\beta^n\}, 1 - \{\tilde{\eta}\beta^n\})$. Then, for all $n \ge n_1$, we have

$$|\varepsilon_n| = \left\| -\sigma_0(\tilde{\eta})\beta^{-n} - \frac{2\mathcal{R}_{Pn+j}}{L} \right\|_{\mathbb{R}/\mathbb{Z}} \le |\sigma_0(\tilde{\eta})|\beta^{-n} + \left| \frac{2\mathcal{R}_{Pn+j}}{L} \right| < \delta_1(\beta).$$

Denote $n_2 = n_2(|\tilde{\eta}|, |\sigma_0(\tilde{\eta})|) = \max(n_0, n_1)$ and define also the vectors

$$\vec{\varepsilon}_n = \begin{pmatrix} \varepsilon_n \\ \varepsilon_{n+1} \\ \vdots \\ \varepsilon_{n+d-2} \\ \varepsilon_{n+d-1} \end{pmatrix}, \quad \vec{K}_n = \begin{pmatrix} K_n \\ K_{n+1} \\ \vdots \\ K_{n+d-2} \\ K_{n+d-1} \end{pmatrix}.$$

As shown in [18], we may prove that if $|\varepsilon_{n_2+n}| < \delta_1(\beta)$ for all $n \geq 0$, then $\vec{\varepsilon}_{n_2+n} = \mathfrak{C}(\beta)^n \vec{\varepsilon}_{n_2}$ for all $n \geq 0$, with $\mathfrak{C}(\beta)$ is the companion matrix of the minimal polynomial of β: let $X^d - c_{d-1}X^{d-1} - \cdots - c_0$ be the minimal polynomial of β, then

$$\mathfrak{C}(\beta) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{d-1} \end{pmatrix}.$$

Let $\vec{E}_1, \vec{E}_2, \vec{e_1}, \ldots, \vec{e_m}, \overline{\vec{e_1}}, \ldots, \overline{\vec{e_m}}$ and $\vec{E}_1^*, \vec{E}_2^*, \vec{e}_1^*, \ldots, \vec{e}_m^*, \overline{\vec{e}_1^*}, \ldots, \overline{\vec{e}_m^*}$ be the eigenbasis and dual basis of $\mathfrak{C}(\beta)$ respectively. The vectors $\vec{e}_j$, $\vec{e}_j^*$ are explicitly given by

$$\vec{e}_j = \begin{pmatrix} 1 \\ \beta_j \\ \vdots \\ \beta_j^{d-2} \\ \beta_j^{d-1} \end{pmatrix}, \quad \vec{e}_j^* = \begin{pmatrix} c_0\beta_j^{d-2} \\ c_1\beta_j^{d-2} + c_0\beta_j^{d-3} \\ \vdots \\ c_{d-2}\beta_j^{d-2} + \cdots + c_1\beta_j + c_0 \\ \beta_j^{d-1} \end{pmatrix},$$

where each $\beta_j = \sigma_j(\beta)$, for $j = 1, \ldots, m$ is a Galois conjugate of β.

Decompose $\vec{\varepsilon}_{n_2+n} = B_0(\tilde{\eta}, n)\vec{E}_1 + B_1(\tilde{\eta}, n)\vec{E}_2 + \sum_{j=1}^m b_j(\tilde{\eta}, n)\vec{e_j} + \overline{b_j(\tilde{\eta}, n)}\,\overline{\vec{e_j}}$ ($\vec{E}_1, \vec{E}_2$ associated to β, β^{-1} respectively). In fact, we can show that $\vec{\varepsilon}_{n_2+n} = \mathfrak{C}(\beta)^n \vec{\varepsilon}_{n_2}$ for all $n \geq 0$ implies $B_0(\tilde{\eta}, n_2) = 0$: since $B_0(\tilde{\eta}, n_2) = \langle \vec{\varepsilon}_{n_2}, \vec{e}_1^* \rangle / \langle \vec{e}_1, \vec{e}_1^* \rangle$, for all $n \geq 0$

$$\begin{aligned} \alpha^n |\langle \vec{\varepsilon}_{n_2}, \vec{e}_1^* \rangle| &= |\langle \mathfrak{C}(\beta)^n \vec{\varepsilon}_{n_2}, \vec{e}_1^* \rangle| \\ &\leq |\langle \vec{\varepsilon}_{n_2+n}, \vec{e}_1^* \rangle| \\ &\leq ||\vec{\varepsilon}_{n_2+n}||_2 ||\vec{e}_1^*||_2 \\ &\leq \sqrt{m}\delta_1 ||\vec{e}_1^*||_2, \end{aligned}$$

which obviously leads to a contradiction as n goes to infinity if $B_0(\tilde{\eta}, n_2) \neq 0$.

The equation for $\vec{\varepsilon}_{n_2+n}$ may be written coordinatewise as

$$\varepsilon_{n_2+n} = B_1(\tilde{\eta}, n_2)\beta^{-n} + 2\sum_{j=1}^m |b_j(\tilde{\eta}, n_2)| \cos(2\pi n\widetilde{\theta}_j - \phi(\widetilde{\theta}_j)),$$

where $\widetilde{\theta_j} = P\theta_j$. From this decomposition and the fact that $\varepsilon_n \pmod 1 = -\sigma_0(\tilde{\eta})\beta^{-n} - \dfrac{2\mathcal{R}_{Pn+j}}{L} \pmod 1$ by equality 2.4, we deduce that $H_j = |b_j(\tilde{\eta}, n_2)|$.

Denote $z(\vec{x}) = 2\sum_{j=1}^{m} |b_j(\tilde{\eta}, n_2)| \cos(2\pi x_j)$. Set $\mathfrak{a} = \delta$, $\mathfrak{b} = 1 - \delta$ for $\delta > 0$ a parameter we will choose later suitably to satisfy the conclusion of the lemma. We face now the task to establish bounds for the integrals of Selberg polynomials, and we do it in the same manner as in [2]. In the calculations below we write $z = z(\vec{x})$, $b_j = b_j(\tilde{\eta}, n_2)$ and all integrals are over $(\mathbb{R}/\mathbb{Z})^m$. Let us start by the Beurling polynomial of degree N:

$$\begin{aligned}
\int \mathcal{B}_N(-z+\mathfrak{a}) d\vec{x} &= \int \mathcal{V}_N(-z+\mathfrak{a}) + \frac{1}{2(N+1)} \Delta_{N+1}(-z+\mathfrak{a}) d\vec{x} \\
&= \int \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) \sin(2\pi k(-z+\mathfrak{a})) d\vec{x} \\
&+ \int \frac{1}{2(N+1)} \left\{ 1 + \sum_{0<|k|\le N+1} \left(1 - \frac{|k|}{N+1}\right) e^{2\pi i k(-z+\mathfrak{a})} \right\} d\vec{x} \\
&= \underbrace{\frac{-1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) \sin(2\pi k \mathfrak{a}) \prod_{j=1}^{m} J_0(4\pi k |b_j|)}_{=(1)} \\
&+ \underbrace{\frac{1}{2(N+1)} \left\{ 1 + \sum_{0<|k|\le N+1} \left(1 - \frac{|k|}{N+1}\right) e^{2\pi i k \mathfrak{a}} \prod_{j=1}^{m} J_0(4\pi k |b_j|) \right\}}_{=(2)},
\end{aligned}$$

where we used Proposition 2.19 in the last equality. Let us denote $\mathfrak{g} = \left(\prod_{j=1}^{m} |b_j|\right)^{1/m}$ and define for each nonnegative integer l the number $k_l = \lfloor \mathfrak{g}^{-1} l \rfloor$. By the AM-GM inequality,

$$\mathfrak{g} \le \frac{1}{m} \sum_{j=1}^{m} |b_j| = \frac{1}{m} \frac{2H}{L} < \frac{\delta_1(\beta)}{2m} < 1.$$

From now on, we will consider N of the form $\lfloor \mathfrak{g}^{-1} T \rfloor$, for each $T \in \mathbb{N}$. We use below the inequality for the Bessel function from Proposition 2.18.

$$\begin{aligned}
|(2)| &\le \frac{1}{2(N+1)} \left\{ 1 + 2 \sum_{k=1}^{N+1} \left(1 - \frac{k}{N+1}\right) \prod_{j=1}^{m} |J_0(4\pi k |b_j|)| \right\} \\
&\le \frac{1}{2(N+1)} \left\{ 1 + 2 \sum_{l=0}^{T-1} \sum_{k=k_l+1}^{k_{l+1}} \left(1 - \frac{k}{N+1}\right) \prod_{j=1}^{m} |J_0(4\pi k |b_j|)| \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2(N+1)}\left\{1+2\sum_{k=1}^{k_1}\left(1-\frac{k}{N+1}\right)\prod_{j=1}^{m}\underbrace{|J_0(4\pi k|b_j|)|}_{\leq 1}\right.\\
&\left.+2\sum_{l=1}^{T-1}\sum_{k=k_l+1}^{k_{l+1}}\left(1-\frac{k}{N+1}\right)\prod_{j=1}^{m}\underbrace{|J_0(4\pi k|b_j|)|}_{\leq 1/\sqrt{2}\pi(k|b_j|)^{1/2}}\right\}\\
&\leq \frac{1}{2(N+1)}\left\{1+2\mathfrak{g}^{-1}+\frac{2}{(\sqrt{2}\pi)^m}\sum_{l=1}^{T-1}\sum_{k=k_l+1}^{k_{l+1}}\underbrace{(k\mathfrak{g})^{-m/2}}_{\leq l^{-m/2}}\right\}\\
&\leq \frac{1}{2(N+1)}\left\{1+2\mathfrak{g}^{-1}+\frac{2}{(\sqrt{2}\pi)^m}\sum_{l=1}^{T-1}l^{-m/2}\underbrace{(k_{l+1}-k_l)}_{\leq \mathfrak{g}^{-1}}\right\}\\
&\leq \frac{1}{2(N+1)}\left\{1+2\mathfrak{g}^{-1}+\frac{2\mathfrak{g}^{-1}}{(\sqrt{2}\pi)^m}O(\sqrt{T})\right\}\\
&= O(1/\sqrt{T}),
\end{aligned}
$$

where the constant implicit in the O sign does not depend on $\tilde{\eta}$ nor n_2. Similarly,

$$
\int \mathcal{B}_N(z-\mathfrak{b})d\vec{x} = \frac{1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k\mathfrak{b})\prod_{j=1}^{m}J_0(4\pi k|b_j|)+O(1/\sqrt{T}).
$$

Having estimated the integral of the Beurling polynomials, we continue with the Selberg ones:

$$
\begin{gathered}
\left|\int \mathcal{B}_N(z-\mathfrak{b})+\mathcal{B}_N(-z+\mathfrak{a})d\vec{x}\right|\\
\leq\\
\left|\frac{1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)(\sin(2\pi k\mathfrak{b})-\sin(2\pi k\mathfrak{a}))\prod_{j=1}^{m}J_0(4\pi k|b_j|)\right|+O(1/\sqrt{T})\\
=\\
\underbrace{\left|\frac{2}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k\delta)\prod_{j=1}^{m}J_0(4\pi k|b_j|)\right|}_{=(3)}+O(1/\sqrt{T}).
\end{gathered}
$$

Fix $\varepsilon>0$ and let us take $\xi<1/2$ such that $\pi\xi/\sin(\pi\xi)\leq 1+\varepsilon$ (recall the definition and property of f in Definition 2.15). Then, for T big enough, we obtain

$$
(3)\leq \frac{2}{N+1}\sum_{k=1}^{N}\left|f\left(\frac{k}{N+1}\right)\sin(2\pi k\delta)\prod_{j=1}^{m}J_0(4\pi k|b_j|)\right|
$$

$$
\begin{aligned}
&\leq \frac{2}{N+1}\left[\sum_{k=1}^{k_1}\left|f\left(\frac{k}{N+1}\right)\right|\underbrace{|\sin(2\pi k\delta)|}_{\leq 2\pi k\delta}\prod_{j=1}^{m}\underbrace{|J_0(4\pi k|b_j|)|}_{\leq 1}\right.\\
&\quad+\sum_{l=1}^{\lfloor \xi T\rfloor}\sum_{k=k_l+1}^{k_{l+1}}\underbrace{\left|f\left(\frac{k}{N+1}\right)\right|}_{\leq |f(l\mathfrak{g}^{-1}/(N+1))|}\underbrace{|\sin(2\pi k\delta)|}_{\leq 1}\prod_{j=1}^{m}\underbrace{|J_0(4\pi k|b_j|)|}_{\leq 1/\sqrt{2}\pi(k|b_j|)^{1/2}}\\
&\quad\left.+\sum_{l=\lfloor \xi T\rfloor+1}^{T-1}\sum_{k=k_l+1}^{k_{l+1}}\left|f\left(\frac{k}{N+1}\right)\right|\underbrace{|\sin(2\pi k\delta)|}_{\leq 1}\prod_{j=1}^{m}\underbrace{|J_0(4\pi k|b_j|)|}_{\leq 1/\sqrt{2}\pi(k|b_j|)^{1/2}}\right]\\
&\leq \frac{4\pi\delta}{N+1}\sum_{k=1}^{k_1}\left(\frac{\pi\xi}{\sin(\pi\xi)}\frac{N+1}{\pi k}+\frac{1}{\pi}\right)k\\
&\quad+\frac{2}{(\sqrt{2}\pi)^m(N+1)}\sum_{l=1}^{\lfloor \xi T\rfloor}\left|f\left(\frac{l\mathfrak{g}^{-1}}{N+1}\right)\right|\sum_{k=k_l+1}^{k_{l+1}}\underbrace{(k\mathfrak{g})^{-m/2}}_{\leq\, l^{-m/2}}\\
&\quad+\frac{2}{(\sqrt{2}\pi)^m(N+1)}\sum_{l=\lfloor \xi T\rfloor+1}^{T-1}\left(\frac{1-\xi}{\sin(\pi(1-\xi))}+\frac{1}{\pi}\right)\sum_{k=k_l+1}^{k_{l+1}}\underbrace{(k\mathfrak{g})^{-m/2}}_{\leq\, l^{-m/2}}\\
&\leq 4(1+\varepsilon)\delta\mathfrak{g}^{-1}+O(1/T)+\frac{2\mathfrak{g}^{-1}}{(\sqrt{2}\pi)^m(N+1)}\sum_{l=1}^{\lfloor \xi T\rfloor}\left(\frac{\pi\xi}{\sin(\pi\xi)}\frac{(N+1)\mathfrak{g}}{\pi l}+\frac{1}{\pi}\right)l^{-m/2}\\
&\quad+O(1/\sqrt{T})\\
&\leq 4(1+\varepsilon)\delta\mathfrak{g}^{-1}+\frac{2(1+\varepsilon)\zeta\left(\dfrac{m+2}{2}\right)}{\pi(\sqrt{2}\pi)^m}+O(1/\sqrt{T}),
\end{aligned}
$$

where ζ denotes the Riemann zeta function. In the same manner,

$$
\left|\int \mathcal{B}_N(-z+\mathfrak{b})+\mathcal{B}_N(z-\mathfrak{a})d\vec{x}\right|\leq 4(1+\varepsilon)\delta\mathfrak{g}^{-1}+\frac{2(1+\varepsilon)\zeta\left(\dfrac{m+2}{2}\right)}{\pi(\sqrt{2}\pi)^m}+O(1/\sqrt{T}).
$$

It may be checked that $\dfrac{2\zeta\left(\dfrac{m+2}{2}\right)}{\pi(\sqrt{2}\pi)^m}\leq\dfrac{\sqrt{2}\zeta(3/2)}{\pi^2}<0.4$ for all $m\geq 1$. Since $\varepsilon>0$ was arbitrarily chosen, by taking $T\to\infty$ we conclude by means of 2.2 that

$$
\left|\int_{(\mathbb{R}/\mathbb{Z})^m}\mathbb{1}_{J(\delta)}(z(\vec{x}))d\vec{x}-|J(\delta)|\right|\leq 4\delta\mathfrak{g}^{-1}+0.4,
$$

which implies

$$
1-2\delta-0.4-4\delta\mathfrak{g}^{-1}\leq\int_{(\mathbb{R}/\mathbb{Z})^m}\mathbb{1}_{J(\delta)}(z(\vec{x}))d\vec{x}.
$$

To estimate $\mathfrak{g}$, notice that $|b_j|=|\langle\vec{K}_{n_2},\vec{e}_j^{\,*}\rangle|/|\langle\vec{e}_j,\vec{e}_j^{\,*}\rangle|$. The numerator of this fraction is a polynomial in β_j with integer coefficients, of degree at most $d-1$ (in particular non zero, since β_j is of degree d) and height at most K_{n_2} (except for multiplication by a constant only depending on β), and the denominator only depends on β. From Lemma 2.13, we obtain $\mathfrak{g} \geq c|\tilde{\eta}|^{-d}\beta^{-n_2 d}$ for some explicit constant $c = c(\beta)$ depending on $\beta = \alpha^P$ and, by the bound on the period, it may be changed to a dependence on α and L. We may solve the inequality $1/2 < 1-2\delta-0.4-4\delta c^{-1}|\tilde{\eta}\beta^{n_2}|^d$ for some $\delta = \delta(|\tilde{\eta}|, |\sigma_0(\tilde{\eta})|)$, which will satisfy the conclusion. In fact, just notice that

$$\begin{aligned}
\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}(z(\vec{x}))d\vec{x} &= \lim_{N\to\infty} \frac{\#\left\{n \leq N \,|\, |\varepsilon_n(\tilde{\eta}, \beta)| > \delta\right\}}{N} \\
&= \lim_{N\to\infty} \frac{1}{N/P} \# \left\{ n \leq N,\ n \equiv j \pmod{P} \,\middle|\, \left\| \frac{-2\mathcal{R}_n}{L} \right\|_{\mathbb{R}/\mathbb{Z}} > \delta \right\} \\
&= \lim_{N\to\infty} \frac{1}{N/P} \# \left\{ n \leq N,\ n \equiv j \pmod{P} \,\middle|\, \frac{-2\mathcal{R}_n}{L} \pmod{1} \in J(\delta) \right\} \\
&= \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(\frac{-2\mathcal{R}(\vec{x})}{L} \right) d\vec{x}.
\end{aligned}$$

The dependence of δ on $(|\tilde{\eta}|, |\sigma_0(\tilde{\eta})|)$ may be changed to a dependence on $(L, |\eta|, |\sigma_0(\eta)|)$, since $|\eta| \leq |\tilde{\eta}| \leq |\eta|\alpha^P \leq |\eta|\alpha^{L^d}$ and $|\sigma_0(\tilde{\eta})| \leq |\sigma_0(\eta)|$. □

Lemma 2.23 *Let β and $\tilde{\eta}$ as in Lemma 2.22. Suppose $Tr(L\tilde{\eta}\beta^n) = 0 \pmod{L}$ for all $n \geq 0$ and $2H/L \geq \delta_1(\beta)/2$. Then there exists $\delta = \delta(L) > 0$ such that*

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(\frac{-2\mathcal{R}(\vec{x})}{L} \right) d\vec{x} \geq 1/2.$$

Proof. The calculations are analogous. The details are found in the Appendix. □

Lemma 2.24 *Let β and $\tilde{\eta}$ as in Lemma 2.22. Suppose $Tr(L\tilde{\eta}\beta^n) = l \neq 0 \pmod{L}$ for all $n \geq 0$ and $2H < 1$. Then*

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(1/L)} \left(\frac{-2\mathcal{R}(\vec{x}) + l}{L} \right) d\vec{x} \geq 1/2.$$

Proof. From Proposition 2.20 and $2H < 1$ we can deduce $\{\tilde{\eta}\beta^n\} \in [(l-1)/L, (l+1)/L]$ for all $n \geq n_0$, for some $n_0 \geq 0$. Since $l \neq 0$ we have three cases: $J(1/L) \cap [(l-1)/L, (l+1)/L]$ is equal to $[(l-1)/L, l/L]$ (if $l = L-1$) or $[l/L, (l+1)/L]$ (if $l = 1$) or $[(l-1)/L, (l+1)/L]$ (otherwise). In any case,

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(1/L)} \left(\frac{-2\mathcal{R}(\vec{x}) + l}{L} \right) d\vec{x} = \lim_{N\to\infty} \frac{\#\left\{n \leq N | \{\tilde{\eta}\beta^n\} \in J(1/L)\right\}}{N} \geq 1/2.$$

□

Lemma 2.25 *Let β and $\tilde{\eta}$ as in Lemma 2.22. Suppose $Tr(L\tilde{\eta}\beta^n) = l \neq 0 \pmod{L}$ for all $n \geq 0$ and $2H \geq 1$. Then there exists a $\delta = \delta(L) > 0$ such that*

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(\frac{-2\mathcal{R}(\vec{x}) + l}{L} \right) d\vec{x} \geq 1/2.$$

PROOF. The calculations are analogous. The details are found in the Appendix. $\square$

2.3.2 Conclusion

Proof of Theorem 2.1: let $\kappa \in \mathbb{Z}[\alpha]$ be the positive constant appearing in Lemma 2.9 and fix $A, B, C > 1$ for the rest of this section. Define

$$B_\kappa = \kappa B, \quad C_\kappa = |\sigma_0(\kappa)|C.$$

According to Lemmas 2.22, 2.23, 2.24 and 2.25 above (see also the remark below Lemma 2.22), we can find an explicit $\delta = \delta(A, B_\kappa, C_\kappa) > 0$ such that

$$\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)} \left(\frac{-2\mathcal{R}(\vec{x}) + a_j}{L} \right) d\vec{x} \geq 1/2,$$

for all $j = 0, \dots, P-1$. In particular, by Corollary 2.21 there exists $N_0 \geq 1$ such that for all $N \geq N_0$, $|\eta| \in [B_\kappa^{-1}, B_\kappa]$, $|\sigma_0(\eta)| \leq C_\kappa$ and $L \leq A$:

$$\frac{\#\{n \leq N | \{\eta\alpha^n\} \in J(\delta)\}}{N} \geq 1/3.$$

Note that if $\eta = \omega\kappa$, then $|\omega| \in \mathbb{Q}(\alpha) \cap [B^{-1}, B]$ and $|\sigma_0(\omega)| \leq C$. Assume also η is in reduced form, and its denominator L is less or equal to A, which implies the denominator of ω in reduced form is less or equal to A. By Lemma 2.9, considering $R \geq R_0 := \alpha^{N_0+1}$ we obtain

$$\begin{aligned}
\sup_{(x,s) \in \mathfrak{X}_\zeta^{\vec{p}}} |S_R^f((x,s),\omega)| &\leq C_1 R (1 - \lambda\delta^2)^{\log_\alpha(R)/3} \\
&= C_1 R (\alpha^{\log_\alpha(1-\lambda\delta^2)})^{\log_\alpha(R)/3} \\
&= C_1 R^{1+\log_\alpha(1-\lambda\delta^2)/3} \\
&= C_1 R^{\tilde{\gamma}}, \quad \tilde{\gamma} = \tilde{\gamma}(A,B,C) \in (0,1).
\end{aligned}$$

A modulus of continuity for the correlation measure ν_a ($a \in \mathcal{A}$) may be derived from the last inequality using Proposition 2.8: for $\tilde{\gamma} = \tilde{\gamma}(A,B,C) = 1 + \log_\alpha(1-\lambda\delta^2)/3$ we have by Proposition 2.8,

$$\nu_a([\omega - r, \omega + r]) \leq \pi^2 C_1 r^{2(1-\tilde{\gamma})} =: c r^\gamma,$$

for all $0 < r \leq r_0$, for some $r_0 > 0$ and $c > 0$, the last one only depending on the substitution. We have proved Theorem 2.1.

Proof of Theorem 2.2: in view of the uniform dependence of δ on the variables B, C in Lemmas 2.24 and 2.25, similar calculations as the ones we did before lead to an exponent $\gamma = \gamma(A) > 0$ only depending on A: Corollary 2.21 and the existence of some $a_{j^*} \neq 0$ (since

$\mathrm{Tr}(L\kappa\omega\alpha^n) \not\equiv 0 \pmod L$ for some $n \in \mathbb{N}$ by hypothesis), yields

$$\begin{aligned}
\lim_{N\to\infty} \frac{\#\{n \le N | \{\eta\alpha^n\} \in J(\delta)\}}{N} &= \frac{1}{P}\sum_{j=0}^{P-1} \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}\left(\frac{-2\mathcal{R}(\vec{x}) + a_j}{L}\right) d\vec{x} \\
&\ge \frac{1}{P}\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}\left(\frac{-2\mathcal{R}(\vec{x}) + a_{j^*}}{L}\right) d\vec{x} \\
&\ge \frac{1}{2A^d}.
\end{aligned}$$

This means there exists $N_0 \ge 1$ such that for all $N \ge N_0$

$$\frac{\#\{n \le N | \{\eta\alpha^n\} \in J(\delta)\}}{N} \ge \frac{1}{3A^d}.$$

Proceeding in the same way as before, we arrive to the conclusion of Theorem 2.2.

2.4 Proof of Proposition 2.3

Let Q be the minimal polynomial of α and L a positive integer. Consider the set $\mathscr{R}_L$ of $\eta/L \in \mathbb{Q}(\alpha)$, with $\eta \in \mathbb{Z}[\alpha]$, such that

- $\eta = l_0 + \cdots + l_{d-1}\alpha^{d-1} \in \mathbb{Z}[\alpha]$, $\gcd(l_0, \ldots, l_{d-1}, L) = 1$,
- $\forall n \ge 0$ $\mathrm{Tr}(\eta\alpha^n) = 0 \pmod L$.

We will prove these sets are empty except maybe for the divisors of L. With this end, let us note that

$$\begin{aligned}
\mathscr{P}_L := \{&\eta = l_0 + \cdots + l_{d-1}\alpha^{d-1} \in \mathbb{Z}[\alpha] \mid \gcd(l_0, \ldots, l_{d-1}, L) = 1,\ \mathrm{Tr}(\eta\alpha^n) = 0 \pmod L \\
&\text{for all } n \ge 0\} \\
\subseteq \{&\eta \in \mathbb{Z}[\alpha] \mid \mathrm{Tr}(\eta\alpha^n) = 0 \pmod L \text{ for all } n \ge 0\} \\
\subseteq \mathscr{M}_L &:= \{\rho \in \mathbb{Q}(\alpha) \mid \mathrm{Tr}(\rho\alpha^n) = 0 \pmod L \text{ for all } n \ge 0\}.
\end{aligned}$$

Let $\{w_0, \ldots, w_{d-1}\}$ be the dual basis (respect to the trace form) of $\{1, \alpha, \ldots, \alpha^{d-1}\}$. We claim $\mathscr{M}_L$ is a free $\mathbb{Z}$-submodule of $\mathbb{Q}(\alpha)$ with basis $\{Lw_0, \ldots, Lw_{d-1}\}$. The $\mathbb{Z}$-basis of $\mathscr{M}_L$ is found solving

$$\mathrm{Tr}(\alpha^i w_j) = \delta_{ij},\ \text{for } i, j = 0, \ldots, d-1 \iff w_j = \pi_1(V^{-1}e_{j+1}),$$

where δ_{ij} denotes the Kronecker delta, V is the Vandermonde matrix associated to the d Galois conjugates of α, π_1 denotes projection onto the first coordinate and $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$. These equations yield the $\mathbb{Z}$-linear independence of the w_i's (form the $\mathbb{Z}$-linearity of the trace). Alternatively, the w_j's can be calculated as (see [51], Chapter 3, Proposition 2)

$$w_j = \frac{m_j}{Q'(\alpha)},$$

where

$$\frac{Q(X)}{X-\alpha} = m_0 + \cdots + m_{d-1}X^{d-1}, \quad m_j \in \mathbb{Z}[\alpha].$$

Let us observe from this last equation that the denominator of $Q'(\alpha)^{-1}$ expressed in reduced form is a common denominator for all w_j's, a fact that will be used later. Finally, to prove the Lw_j's generate $\mathscr{M}_L$, let $\rho \in \mathscr{M}_L$. Let $\Theta : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{R}^2 \times \mathbb{C}^m \cong \mathbb{R}^d$ be the Minkowski embedding $\Theta(\rho) = (\rho, \sigma_0(\rho), \sigma_1(\rho), \ldots, \sigma_m(\rho))^T$. By definition, for some $p_1, \ldots, p_d \in \mathbb{Z}$ we have

$$\begin{aligned} V\Theta(\rho) &= (Lp_1, \ldots, Lp_d)^T, \\ \iff \Theta(\rho) &= LV^{-1}(p_1, \ldots, p_d)^T = p_1 L\Theta(w_0) + \cdots + p_d L\Theta(w_{d-1}) \\ &\iff \rho = p_1 Lw_0 + \cdots + p_d Lw_{d-1}. \end{aligned}$$

This completes the proof of the claim. Each w_j can be expressed as η_j/E (possibly not in reduced form) with $\eta_j \in \mathbb{Z}[\alpha]$ and E only depending on α (observation last paragraph). Suppose $L \nmid E$, and let $l > 1$ be some factor of L which is not a factor of E. Then, for every $\mathbb{Z}$-linear combination of the Lw_j's, the coefficients of the numerator of this linear combination (written in the canonical basis of the number field) will have l as a common factor, i.e., none of these $\mathbb{Z}$-linear combinations will belong to $\mathscr{P}_L$ (because of the coprimality condition). In consequence, this set is empty and $\mathscr{R}_L$ is empty too. This yields the conclusion of Proposition 2.3.

2.5 Dependence of r_0

The aim of this section is to prove Proposition 2.29, which provides a lower bound for r_0 appearing in Theorems 2.1 and 2.2. To accomplish this we will find an upper bound for the error in the approximation of the integrals of Corollary 2.21. We are only able to do this in the case $\deg(\alpha) = 4$. First, let us recall the notion of type of a real number: for a real number θ we say its *type* is at most $\tau \geq 1$ if there exists a positive constant $c(\theta)$ such that for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ we have

$$|q\theta - p| \geq \frac{c(\theta)}{q^\tau}.$$

The set of real numbers satisfying an inequality like the one above for some $\tau \geq 1$ is called the set of *Diophantine numbers*.

We use a general bound for linear forms in logarithms to show that θ_1 is Diophantine. We summarize this calculation in the next

Lemma 2.26 *Let $\alpha_1 = e^{2\pi i\theta_1}$ be an algebraic number on the unit circle which is not a root of unity. Then θ_1 is Diophantine and its type is bounded from above by some explicit constant τ_{α_1}.*

Proof. Let us denote by p_q the nearest integer to $q\theta_1$, i.e., $||q\theta_1||_{\mathbb{R}/\mathbb{Z}} = |q\theta_1 - p_q|$. Note that

$$\begin{aligned} |\alpha_1^q - 1| &= |e^{2\pi i q\theta_1} - e^{2\pi i p_q}| \\ &\le 2\pi|q\theta_1 - p_q|. \end{aligned}$$

Now we use Theorem 2.2 from [24] to deduce that for every $q \in \mathbb{Z} \setminus \{0\}$ holds $\log|\alpha_1^q - 1| \ge -\tau_{\alpha_1}\log(eq)$, for some explicit constant $\tau_{\alpha_1} > 1$ depending only on α_1, which finally implies

$$|q\theta_1 - p_q| \ge \frac{e^{-\tau_{\alpha_1}}}{2\pi q^{\tau_{\alpha_1}}} =: \frac{c_{\alpha_1}}{q^{\tau_{\alpha_1}}}.$$

In other words, the type of θ_1 is $\le \tau_{\alpha_1}$. □

As we have already mentioned, our aim is to understand how fast is the approximation of the integrals appearing in Corollary 2.21 by Birkhoff sums. The classic way to do this is to use the Koksma-Hlawka inequality for integrals of dimension greater than one. Unfortunately, the integrand of the integrals in Corollary 2.21 are given by the functions

$$F(x_1, \ldots, x_m) = \mathbb{1}_J\left(\frac{-2\mathcal{R}(x_1, \ldots, x_m) + a_j}{L}\right),$$

$j = 0, \ldots, P-1$, which we do not expect to have bounded *variation in the sense of Hardy and Krause* (see [50]). So we restrict ourselves to dimension 1 ($m = 1$, so $\deg(\alpha) = 4$), where we can prove that the variation of this function is bounded explicitly. This reduces ourselves to the classic result of Koksma which we recall next. We start with the definition of *discrepancy.*

Definition 2.27 *Let $N \ge 1$ and $(u_n)_{n\ge1} \subset [0,1]$. The discrepancy of the sequence u_n is defined by*

$$D_N(u_n) = \sup_{0\le\mathfrak{a}<\mathfrak{b}\le1}\left|\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{[\mathfrak{a},\mathfrak{b}]}(u_n) - (\mathfrak{b} - \mathfrak{a})\right|.$$

By allowing only sets of the form $[0, \mathfrak{b}]$ in the definition of D_N, we obtain the star-discrepancy D_N^*. It follows the obvious inequality $D_N^*(u_n) \le D_N(u_n)$.

It is known that the discrepancy of the sequence $(n\widetilde{\theta_1})_{n\ge1}$, where $\widetilde{\theta_1} = P\theta_1$, is bounded as

$$D_N(n\widetilde{\theta_1}) \le DN^{-1/\tau},$$

where τ is any strict upper bound for the type of $\widetilde{\theta_1}$ and D is an absolute constant. This fact is derived from the Erdős-Turan inequality (see [50]). The inequality above holds if we replace the upper bound for the type of $\widetilde{\theta_1}$ by an upper bound for the type of θ_1, since $\widetilde{\theta_1} = P\theta_1$. Now we recall the Koksma inequality.

Theorem 2.28 (see [50]) *Let $F : [0,1] \longrightarrow \mathbb{R}$ be a function of bounded variation $V(F) < +\infty$. For any sequence $(u_n)_{n\ge1} \subseteq [0,1]$ and $N \ge 1$, holds*

$$\left|\frac{1}{N}\sum_{n=1}^{N}F(u_n) - \int F(x)dx\right| \le V(F)D_N^*(u_n).$$

Now we use the Koksma inequality for every function $F_j(x) = \mathbb{1}_{J(2\delta)}\left(\frac{-2\mathcal{R}(x)+a_j}{L}\right)$ ($j = 0,\dots,P-1$), where $\delta = \delta(A,B,C)$ is the one obtained from Lemmas 2.22 to 2.25 for fixed $A,B,C>1$. Note by replacing δ for 2δ in these lemmas, we can conclude that the integral of each F_j is greater or equal to $2/5$: it is enough to look at the inequalities at the end of the proofs (in the case of Lemma 2.24 this is not true, but changing δ for $\delta/2$ of course yields the same result).

Also, it is not difficult to see that the variation of F_j is less or equal to $8H$ (recall H is defined in the paragraph below the proof of Corollary 2.21). Define $n_0 := \lceil \log_\alpha(|\sigma_0(\eta)|/\delta)\rceil$ and $N_j = \#\{1 \le n \le N \mid n \equiv j \pmod P\} \ge \lfloor N/P \rfloor$. This allows us to affirm that for any $N > n_0$ and $j = 0,\dots,P-1$

$$\int_{\mathbb{R}/\mathbb{Z}} \mathbb{1}_{J(2\delta)}\left(\frac{-2\mathcal{R}(x)+a_j}{L}\right)dx - \frac{8DH}{\lfloor N/P\rfloor^{1/\tau}} - \frac{n_0}{\lfloor N/P\rfloor}$$
$$\le$$
$$\frac{\#\left\{n_0 < n \le N,\ n\equiv j \pmod P \,\middle|\, \frac{-2\mathcal{R}_n+a_j}{L} \pmod 1 \in J(2\delta)\right\}}{\lfloor N/P\rfloor}$$
$$\le$$
$$\frac{\#\left\{n_0 < n \le N,\ n\equiv j \pmod P \,\middle|\, \frac{-2\mathcal{R}_n+a_j}{L} - \sigma_0(\eta)\alpha^{-n} \pmod 1 \in J(\delta)\right\}}{\lfloor N/P\rfloor}$$
$$\le$$
$$\frac{\#\{n_0 < n \le N,\ n\equiv j \pmod P | \{\eta\alpha^n\} \in J(\delta)\}}{\lfloor N/P\rfloor}$$
$$\le$$
$$\frac{\#\{1 \le n \le N,\ n\equiv j \pmod P | \{\eta\alpha^n\} \in J(\delta)\}}{\lfloor N/P\rfloor}.$$

Let us add these inequalities for $j = 0,\dots,P-1$ and multiply by $\lfloor N/P\rfloor/N$ both sides, yielding

$$\frac{1}{P}\sum_{j=0}^{P-1}\int_{\mathbb{R}/\mathbb{Z}} \mathbb{1}_{J(2\delta)}\left(\frac{-2\mathcal{R}(x)+a_j}{L}\right)dx - \frac{1}{N/P} - \frac{8DH}{(N/P)^{1/\tau}} - \frac{n_0}{N/P}$$
$$\le$$
$$\frac{\#\{1\le n\le N | \{\eta\alpha^n\}\in J(\delta)\}}{N}.$$

Since each integral term in the left-hand side is greater than $2/5$ we conclude the right-hand side is greater than $1/3$ as soon as $N \ge N_0$, with N_0 defined by any integer solution to

$$\frac{8DH}{(N_0/P)^{1/\tau}} + \frac{n_0}{N_0/P} + \frac{P}{N_0} < 1/15.$$

In particular, we may choose any $N_0 \ge P\max(45n_0, (360DH)^\tau, 45)$. In this manner, for $\eta = \omega\kappa = \frac{1}{L}(l_0 + \dots + l_{d-1}\alpha^{d-1}) \in \mathbb{Q}(\alpha)$, we conclude that we can take $r_0 = r_0(\omega)$ in

Theorem 2.1 as

$$r_0 = \frac{c_\alpha}{\alpha^{P\max(\lceil \log_\alpha(|\sigma_0(\eta)|/\delta)\rceil, H^\tau)}},$$

for certain explicit constant $c_\alpha > 0$, deduced from the relation $R_0 = \alpha^{N_0+C_2+1}$ and Lemmas 2.8, 2.9. With this we have proved

Proposition 2.29 *Let $A, B, C > 1$ and $\omega \in \mathbb{Q}(\alpha)\setminus\{0\}$ satisfying the conditions of Theorem 2.1. Suppose α is a Salem number of degree equal 4 and let $\alpha_1 = e^{2\pi i\theta_1}$ be the Galois conjugate on the upper half of the unit circle. Then θ_1 is Diophantine and if τ is an upper bound for its type, there exists a constant $c_\alpha > 0$ only depending on α such that $r_0(\omega)$ appearing in Theorem 2.1 satisfies*

$$r_0(\omega) > \frac{c_\alpha}{\alpha^{A^4\max(\lceil \log_\alpha(C/\delta)\rceil, H^\tau)}},$$

where $H = \sqrt{U(\theta_1)^2 + V(\theta_1)^2}$ (see Corollary 2.21 for definition of U and V) and $\delta = \delta(A,B,C) > 0$ comes from any of Lemmas 2.22 to 2.25.

Almost identical calculations lead to an expression bounding $r_0(\omega)$ in the case of Theorem 2.2, so we omit them.

2.6 Appendix

In this section we give the details of the proof of Lemmas 2.23 and 2.25.

Proof of Lemma 2.23: let $\mathfrak{a} = \delta$, $\mathfrak{b} = 1-\delta$ for $\delta > 0$ a parameter we will choose later suitably to satisfy the conclusion of the lemma. Let $z(\vec{x}) = -2\mathcal{R}(\vec{x})/L$. Then

$$\begin{aligned}
\int \mathcal{B}_N(-z+\mathfrak{a})d\vec{x} &= \int \mathcal{V}_N(-z+\mathfrak{a}) + \frac{1}{2(N+1)}\Delta_{N+1}(-z+\mathfrak{a})d\vec{x} \\
&= \int \frac{1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k(-z+\mathfrak{a}))d\vec{x} \\
&+ \int \frac{1}{2(N+1)}\left\{1 + \sum_{0<|k|\le N+1}\left(1 - \frac{|k|}{N+1}\right)e^{2\pi i k(-z+\mathfrak{a})}\right\}d\vec{x} \\
&= \underbrace{\frac{-1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k\mathfrak{a})\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)}_{=(1)} \\
&+ \underbrace{\frac{1}{2(N+1)}\left\{1 + \sum_{0<|k|\le N+1}\left(1 - \frac{|k|}{N+1}\right)e^{2\pi i k\mathfrak{a}}\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)\right\}}_{=(2)}.
\end{aligned}$$

From the hypothesis follows that there exists $j^* \in \{1,\ldots,m\}$ such that $2H_{j^*}/L \ge \delta_1(\beta)/2m$. Set $\mathfrak{g} = H_{j^*}/L$ and note $\mathfrak{g}^{-1} \le 4m\delta_1(\beta)^{-1} = \mathfrak{h}^{-1}$. Define for each nonnegative integer l the number $k_l = \lfloor \mathfrak{h}^{-1} l \rfloor$. From now on, we will consider N of the form $\lfloor \mathfrak{h}^{-1} T \rfloor$, for each $T \in \mathbb{N}$. We use below the inequality for the Bessel function from Proposition 2.18.

$$
\begin{aligned}
|(2)| &\leq \frac{1}{2(N+1)} \left\{ 1 + 2 \sum_{k=1}^{N+1} \left(1 - \frac{k}{N+1}\right) \prod_{j=1}^{m} \left| J_0 \left(\frac{4\pi k H_j}{L} \right) \right| \right\} \\
&\leq \frac{1}{2(N+1)} \left\{ 1 + 2 \sum_{l=0}^{T-1} \sum_{k=k_l+1}^{k_{l+1}} \left(1 - \frac{k}{N+1}\right) \prod_{j=1}^{m} \left| J_0 \left(\frac{4\pi k H_j}{L} \right) \right| \right\} \\
&\leq \frac{1}{2(N+1)} \left\{ 1 + 2 \sum_{k=1}^{k_1} \left(1 - \frac{k}{N+1}\right) \prod_{j=1}^{m} \underbrace{\left| J_0 \left(\frac{4\pi k H_j}{L} \right) \right|}_{\leq 1} \right. \\
&\qquad \left. + 2 \sum_{l=1}^{T-1} \sum_{k=k_l+1}^{k_{l+1}} \left(1 - \frac{k}{N+1}\right) \underbrace{\prod_{j=1}^{m} \left| J_0 \left(\frac{4\pi k H_j}{L} \right) \right|}_{\leq 1/\sqrt{2}\pi (k\mathfrak{g})^{1/2}} \right\} \\
&\leq \frac{1}{2(N+1)} \left\{ 1 + 2\mathfrak{h}^{-1} + \frac{\sqrt{2}}{\pi} \sum_{l=1}^{T-1} \sum_{k=k_l+1}^{k_{l+1}} \underbrace{(k\mathfrak{g})^{-1/2}}_{\leq l^{-1/2}} \right\} \\
&\leq \frac{1}{2(N+1)} \left\{ 1 + 2\mathfrak{h}^{-1} + \frac{\sqrt{2}}{\pi} \sum_{l=1}^{T-1} l^{-1/2} \underbrace{(k_{l+1} - k_l)}_{\leq \mathfrak{h}^{-1}} \right\} \\
&\leq \frac{1}{2(N+1)} \left\{ 1 + 2\mathfrak{h}^{-1} + \frac{\sqrt{2}\mathfrak{h}^{-1}}{\pi} O(\sqrt{T}) \right\} \\
&= O(1/\sqrt{T})
\end{aligned}
$$

Similarly,

$$
\int \mathcal{B}_N(z - \mathfrak{b}) d\vec{x} = \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) \sin(2\pi k \mathfrak{b}) \prod_{j=1}^{m} J_0 \left(\frac{4\pi k H_j}{L} \right) + O(1/\sqrt{T}).
$$

Having estimated the integral of the Beurling polynomials, we continue with the Selberg ones:

$$\left|\int \mathcal{B}_N(z-\mathfrak{b})+\mathcal{B}_N(-z+\mathfrak{a})d\vec{x}\right|$$

$$\leq$$

$$\left|\frac{1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)(\sin(2\pi k\mathfrak{b})-\sin(2\pi k\mathfrak{a}))\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)\right| + O(1/\sqrt{T})$$

$$=$$

$$\underbrace{\left|\frac{2}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k\delta)\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)\right|}_{=(3)} + O(1/\sqrt{T}).$$

Fix $\varepsilon > 0$ and let us take $\xi < 1/2$ such that $\pi\xi/\sin(\pi\xi) \leq 1+\varepsilon$ (recall the definition and property of f in Definition 2.15). Then, for T big enough, we obtain

$$\begin{aligned}
(3) &\leq \frac{2}{N+1}\sum_{k=1}^{N}\left| f\left(\frac{k}{N+1}\right)\sin(2\pi k\delta)\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)\right| \\
&\leq \frac{2}{N+1}\left[\sum_{k=1}^{k_1}\left|f\left(\frac{k}{N+1}\right)\right|\underbrace{|\sin(2\pi k\delta)|}_{\leq 2\pi k\delta}\prod_{j=1}^{m}\underbrace{\left|J_0\left(\frac{4\pi k H_j}{L}\right)\right|}_{\leq 1}\right. \\
&\quad + \sum_{l=1}^{\lfloor \xi T\rfloor}\sum_{k=k_l+1}^{k_{l+1}}\underbrace{\left|f\left(\frac{k}{N+1}\right)\right|}_{\leq |f(l\mathfrak{h}^{-1}/(N+1))|}\underbrace{|\sin(2\pi k\delta)|}_{\leq 1}\underbrace{\prod_{j=1}^{m}\left|J_0\left(\frac{4\pi k H_j}{L}\right)\right|}_{\leq 1/\sqrt{2}\pi(k\mathfrak{g})^{1/2}} \\
&\quad \left. + \sum_{l=\lfloor \xi T\rfloor+1}^{T-1}\sum_{k=k_l+1}^{k_{l+1}}\left|f\left(\frac{k}{N+1}\right)\right|\underbrace{|\sin(2\pi k\delta)|}_{\leq 1}\underbrace{\prod_{j=1}^{m}\left|J_0\left(\frac{4\pi k H_j}{L}\right)\right|}_{\leq 1/\sqrt{2}\pi(k\mathfrak{g})^{1/2}}\right] \\
&\leq \frac{4\pi\delta}{N+1}\sum_{k=1}^{k_1}\left(\frac{\pi\xi}{\sin(\pi\xi)}\frac{N+1}{\pi k}+\frac{1}{\pi}\right)k \\
&\quad + \frac{\sqrt{2}}{\pi(N+1)}\sum_{l=1}^{\lfloor \xi T\rfloor}\left|f\left(\frac{l\mathfrak{h}^{-1}}{N+1}\right)\right|\sum_{k=k_l+1}^{k_{l+1}}\underbrace{(k\mathfrak{g})^{-1/2}}_{\leq l^{-1/2}} \\
&\quad + \frac{\sqrt{2}}{\pi(N+1)}\sum_{l=\lfloor \xi T\rfloor+1}^{T-1}\left(\frac{1-\xi}{\sin(\pi(1-\xi))}+\frac{1}{\pi}\right)\sum_{k=k_l+1}^{k_{l+1}}\underbrace{(k\mathfrak{g})^{-1/2}}_{\leq l^{-1/2}}
\end{aligned}$$

$$\leq 4(1+\varepsilon)\delta\mathfrak{h}^{-1} + O(1/T) + \frac{\sqrt{2}\mathfrak{h}^{-1}}{\pi(N+1)}\sum_{l=1}^{\lfloor \xi T\rfloor}\left(\frac{\pi\xi}{\sin(\pi\xi)}\frac{(N+1)\mathfrak{h}}{\pi l} + \frac{1}{\pi}\right) l^{-1/2}$$

$$+ O(1/\sqrt{T})$$

$$\leq 4(1+\varepsilon)\delta\mathfrak{h}^{-1} + \frac{\sqrt{2}(1+\varepsilon)\zeta\left(\frac{3}{2}\right)}{\pi^2} + O(1/\sqrt{T}),$$

where ζ denotes the Riemann zeta function. In the same manner,

$$\left|\int \mathcal{B}_N(-z+\mathfrak{b}) + \mathcal{B}_N(z-\mathfrak{a})d\vec{x}\right| \leq 4(1+\varepsilon)\delta\mathfrak{h}^{-1} + \frac{\sqrt{2}(1+\varepsilon)\zeta\left(\frac{3}{2}\right)}{\pi^2} + O(1/\sqrt{T}).$$

It may be checked that $\frac{\sqrt{2}\zeta(3/2)}{\pi^2} < 0.4$. Since $\varepsilon > 0$ was arbitrarily chosen, by taking $T \to \infty$ we conclude by means of 2.2 that

$$\left|\int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}(z(\vec{x}))d\vec{x} - |J(\delta)|\right| \leq 4\delta\mathfrak{h}^{-1} + 0.4,$$

which implies

$$1 - 2\delta - 0.4 - 4\delta\mathfrak{h}^{-1} \leq \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}(z(\vec{x}))d\vec{x}.$$

Then, by solving $1/2 < 1 - 2\delta - 0.4 - 16\delta m\delta_1(\beta)^{-1}$, we obtain the desired $\delta = \delta(L)$.

Proof of Lemma 2.25: let $\mathfrak{a} = \delta$, $\mathfrak{b} = 1-\delta$ for $\delta > 0$ a parameter we will choose later suitably to satisfy the conclusion of the lemma. Let $z(\vec{x}) = \frac{-2\mathcal{R}(\vec{x}) + l}{L}$. As before, we have

$$\int \mathcal{B}_N(-z+\mathfrak{a})d\vec{x} = \underbrace{\frac{-1}{N+1}\sum_{k=1}^{N} f\left(\frac{k}{N+1}\right)\sin(2\pi k\mathfrak{a})\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)}_{=(1)}$$

$$+ \underbrace{\frac{1}{2(N+1)}\left\{1 + \sum_{0<|k|\leq N+1}\left(1 - \frac{|k|}{N+1}\right)e^{2\pi i k\mathfrak{a}}\prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right)\right\}}_{=(2)}.$$

From the hypothesis follows that there exists $j^* \in \{1,\ldots,m\}$ such that $H_{j^*} \geq 1/2m$. Set $\mathfrak{g} = H_{j^*}/L$ and note $\mathfrak{g}^{-1} \leq 2mL = \mathfrak{h}^{-1}$. Define for each nonnegative integer l the number $k_l = \lfloor \mathfrak{h}^{-1} l\rfloor$. From now on, we will consider N of the form $\lfloor \mathfrak{h}^{-1}T\rfloor$, for each $T \in \mathbb{N}$. The same calculations of the proof above lead to $(2) = O(1/\sqrt{T})$. We conclude in the same manner that

$$\left| \int \mathcal{B}_N(z - \mathfrak{b}) + \mathcal{B}_N(-z + \mathfrak{a}) d\vec{x} \right|$$

$$\leq$$

$$\left| \frac{1}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) (\sin(2\pi k \mathfrak{b}) - \sin(2\pi k \mathfrak{a})) \prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right) \right| + O(1/\sqrt{T})$$

$$=$$

$$\underbrace{\left| \frac{2}{N+1} \sum_{k=1}^{N} f\left(\frac{k}{N+1}\right) \sin(2\pi k \delta) \prod_{j=1}^{m} J_0\left(\frac{4\pi k H_j}{L}\right) \right|}_{=(3)} + O(1/\sqrt{T}).$$

And once again, the same calculations as in the proof above lead to

$$\left| \int_{(\mathbb{R}/\mathbb{Z})^m} \mathbb{1}_{J(\delta)}(z(\vec{x})) d\vec{x} - |J(\delta)| \right| \leq 4\delta \mathfrak{h}^{-1} + 0.4.$$

Then, by solving $1/2 < 1 - 2\delta - 0.4 - 8\delta m L$, we obtain the desired $\delta = \delta(L)$.

Chapter 3

Lyapunov exponents of the spectral cocycle for topological factors of the Thue-Morse substitution

The content of this chapter will be submitted for publication.

This paper continues the work of [18, 21] around the spectral cocycle defined explicitly in [21]. The study of this extension of the Rauzy-Veech cocycle has given fruitful results as effective rates of weak mixing for several classes of systems such as interval exchange transformations [4], linear flows on translations surfaces of genera higher or equal than 2 [41] and special suspension flows over more general S-adic systems (including substitutions) [19, 21, 22]. Other consequences are the nature of the spectral measures involved, specifically, sufficient and necessary conditions for purely singular spectrum [13, 21, 20].

One of the main motivations of this article is to explore the behavior of the spectral cocycle under conjugacy and semi-conjugacy of the substitution subshifts involved. We have restricted our study to the Thue-Morse substitution. A special property of the Thue-Morse substitution is that we may represent its twisted Birkhoff sums as a product, namely

$$p_n(\omega) = \prod_{j=0}^{n-1} 2\sin(\pi\{\omega 2^j\}).$$

Lot of attention has been given to the undertanding of the asymptotics of L^p norms of p_n: part of the proof of a prime number theorem for the sum-of-digits function, due to Maduit and Rivat in [55], uses the asymptotic

$$||p_n||_1 \sim 2^{n\delta}, \text{ with } \delta = 0.40325...$$

A survey with the link between these products and the Thue-Morse sequence is found in [62]. We will be interested in the asymptotics of p_n for fixed ω as n goes to infinity. Apparently contradictory with the latter result, the growth is sub-exponential for generic ω. This will imply, in particular, that the top Lyapunov exponent of the spectral cocycle for the Thue-Morse substitution vanishes, which is a known fact (see [5]). In fact, this will be the case for all factors of the Thue-Morse subshift, which is our main result.

Theorem 3.1 *For every topological factor of the Thue-Morse subshift coming from an aperiodic primitive substitution ζ we have $\chi_\zeta^+(\omega) = 0$ almost surely.*

We organize the paper as follows: in Section 2 we recall several notions around substitutions and dynamics, in particular, those related to the Thue-Morse sequence. In addition, we recall the definition of the spectral cocycle, the twisted Birkhoff sum and define its generalization: the twisted correlation. To finish, we state a result about asymptotic laws of expanding maps of the unit interval which will be essential for our estimates.

Section 3 begins with two general results about the top Lyapunov exponent of the spectral cocycle. It continues with fine estimates for the twisted Birkhoff sums in the Thue-Morse case (in particular, showing its sub-exponential behavior) and also for its twisted correlations. For simplicity, the bounds are only shown here in the simplest case. The general case is presented in the Appendix. Finally, we prove Theorem 3.1 relying on the estimates found before.

3.1 Background

3.1.1 Substitutions

For the basic notions on substitutions we follow [61]. Let us start by fixing a positive integer $m \geq 2$ and a finite alphabet $\mathcal{A} = \{1, \ldots, m\}$. A **substitution** on the alphabet $\mathcal{A}$ is a map $\zeta : \mathcal{A} \longrightarrow \mathcal{A}^+$, where $\mathcal{A}^+$ denote the set of finite (nonempty) words on $\mathcal{A}$. By concatenation, it is natural to extend a substitution to $\mathcal{A}^+$, to $\mathcal{A}^{\mathbb{N}}$ (one-sided sequences) or $\mathcal{A}^{\mathbb{Z}}$ (two-sided sequences). In particular, the iterates $\zeta^n(a) = \zeta(\zeta^{n-1}(a))$ for $a \in \mathcal{A}$, are well defined.

For a word $w \in \mathcal{A}^+$ denote its length by $|w|$ and by $|w|_a$ the number of symbols a found in w. The **substitution matrix** associated to a substitution ζ is the $m \times m$ matrix with integer entries defined by $M_\zeta(a, b) = |\zeta(b)|_a$. A substitution is called **primitive** if its susbtitution matrix is primitive. If there exist $q \in \mathbb{N}, q \geq 2$ such that $|\zeta(a)| = q$ for all $a \in \mathcal{A}$, we say ζ is a **constant length substitution**.

The **substitution subshift** associated to ζ is the set X_ζ of sequences $(x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ exist $a \in \mathcal{A}$ and $n \in \mathbb{N}$ such that $x_i \ldots x_{i+k}$ is a subword of some $\zeta^n(a)$. A classical result is that the $\mathbb{Z}$-action by the **left-shift** $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ on this subshift is minimal and uniquely ergodic when ζ is primitive. From now on we only consider primitive and **aperiodic** substitutions, which means in the primitive case that the subshift is not finite.

Finally, let us recall the definition of conjugacy and semi-conjugacy. We say two subshifts $(X,T),(Y,T')$ are **topologically conjugate** if there is an homeomorphism $\pi : X \longrightarrow Y$ such that $\pi \circ T = T' \circ \pi$. If π is only onto, then we say X and Y are **topologically semi-conjugate**, or that Y is a **factor** of X.

3.1.2 Thue-Morse sequence

Let $\zeta_{TM} : 0 \mapsto 01, 1 \mapsto 10$ be the Thue-Morse substitution. This substitution has two fixed points, namely, $\mathbf{u} = \zeta_{TM}^{\infty}(0)$ and $\mathbf{v} = \zeta_{TM}^{\infty}(1)$. The former one is called the Thue-Morse sequence. Note $\mathbf{v}$ is equal to $\mathbf{u}$ after changing the symbols 0's for 1's and 1's for 0's. Other characterization of this sequence which we will use along the whole text is the next arithmetic property valid for all integers $k \geq 0$:

$$\mathbf{u}_k = s_2(k) \pmod 2,$$

where $s_2(k)$ is equal to the sum of digits in the binary expansion of k, i.e., if $k = k_0 + \cdots + k_{n-1}2^{n-1}$ with $k_j \in \{0,1\}$, then $s_2(k) = k_0 + \cdots + k_{n-1}$.

3.1.3 Conjugacy and factor list of the Thue-Morse subshift

Here we recall some results around conjugate and factor subshifts of the Thue-Morse subshift. Two general results are the next ones.

Theorem 3.2 ([33]) *Subshift topological factors of substitution systems are topologically conjugate to substitution subshifts.*

Theorem 3.3 ([34]) *There exist finitely many subshift topological factors of a substitution subshift up to topological conjugacy.*

Deciding when two substitution systems are topologically conjugate or semi-conjugate (one is a factor of the other) is a problem recently solved (the conjugacy problem follows from the second one by coalesence of substitution subshifts).

Theorem 3.4 ([35]) *For two uniformly recurrent substitution subshifts (X,T) and (Y,T'), it is decidable wheter they are semi-conjugate. Moreover, if (Y,T') is aperiodic, then there exists a computable constant r such that for any factor $\pi : X \longrightarrow Y$ there exist $k \in \mathbb{Z}$ and a factor $\pi' : X \longrightarrow Y$ of radius less than r, such that $\pi = (T')^k \circ \pi'$.*

It is tempting to think we may list all factors from a given substitution system, but the radius r in the last result depends strongly on the factor system. On the other hand, in the constant length case, this is partially solved: it is possible to give the list of all factors of constant length (see [27, 35]). Examples of conjugate substitution systems to the Thue-Morse subshift which are not constant length are given in [30].

Theorem 3.5 ([30]) *There exist infinitely many non-constant length, primitive, injective substitutions with Perron-Frobenius eigenvalue equal to 2, conjugate to the Thue-Morse substitution.*

Example Consider the substitutions ζ_1 and ζ_2 defined below. Both are conjugate to the Thue-Morse substitution. The substitution ζ_3 defines a topological factor.

$$\begin{aligned}
&\zeta_1 : 0 \mapsto 01, 1 \mapsto 20, 2 \mapsto 10;\\
&\zeta_2 : 0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1;\\
&\zeta_3 : 0 \mapsto 01,\ 1 \mapsto 00.
\end{aligned}$$

3.1.4 Mixing coefficients and bounded law of iterated logarithm

A major technical tool needed to obtain the bounds of Section 3.3 is a bounded iterated logarithm law for uniformly expanding systems, in the case of observables of unbounded variation.

Following [28], the result comes from the study of some mixing coefficients, definition we recall next. For a random variable X, set $X^{(0)} = X - \mathbb{E}(X)$. For a random vector $(X_1, \ldots, X_k)$ and a σ-algebra $\mathcal{F}$, define

$$\phi(\mathcal{F}, X_1, \ldots, X_k) = \sup_{(x_1,\ldots,x_k)\in\mathbb{R}^k} \left\| \mathbb{E}\left(\prod_{j=1}^{k} \left(\mathbb{1}_{\{X_j \le x_j\}}\right)^{(0)} \middle| \mathcal{F} \right)^{(0)} \right\|_\infty .$$

Let S be a bijective bimeasurable map. For a process $\mathbf{X} = (X_j)_{j\ge 0}$ with $X_i = S^i \circ X_0$ and $\sigma(X_0) \subset \mathcal{F}$. Define

$$\phi_{k,\mathbf{X}}(n) = \max_{1\le l\le k} \sup_{n\le i_1<\cdots<i_l} \phi(\mathcal{F}, X_{i_1}, \ldots, X_{i_l}).$$

The main importance of these coefficients is that a suitable control on them implies a bounded law of iterated logarithm if S is a uniformly expanding map (definition in [28]) and f is in the closure of $\mathrm{Mon}_2(M,\nu)$ (denoted below by $\mathrm{Mon}_2^c(M,\nu)$), where $\mathrm{Mon}_2(M,\nu)$ is the set of f such that $\nu(|f|^2) \le M^2$ and f is the finite sum of functions monotonic in some interval and null elsewhere.

Theorem 3.6 ([28]) *Let S be a uniformly expanding map of the unit interval with an absolutely continuous invariant measure ν. Then, for any $M > 0$ and $f \in \mathrm{Mon}_2^c(M,\nu)$, there exists a nonnegative constant A such that*

$$\sum_{n=1} \frac{1}{n} \nu \left(\max_{1\le k\le n} \left| \sum_{i=0}^{k-1} (f \circ S^i - \nu(f)) \right| \ge A\sqrt{n \log\log(n)} \right) < \infty.$$

This convergence implies a bounded law of iterated logarithm: for almost every ω,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n \log\log(n)}} \left| \sum_{i=0}^{n-1} (f \circ S^i(\omega) - \nu(f)) \right| \leq A.$$

3.1.5 Solutions of $s_2(k+a) - s_2(k) = d$

Here we briefly recall some results from [38]. Consider the equation $s_2(k+a) - s_2(k) = d$, for $a \in \mathbb{N}$ and $d \in \mathbb{Z}$. Denote by $\mathcal{S}_{a,d}$ its solution set. Note that if $d > \lceil \log_2(a) \rceil$, then $\mathcal{S}_{a,d} = \emptyset$. Also, $d < -n$ implies $\mathcal{S}_{a,d} \cap \{0, \ldots, 2^n - 1\} = \emptyset$. These will be trivial cases of the next lemma.

To change between words and numbers represented by digits consider the next notation: $\underline{k}_2$ denotes the word associated to the digits of k in base 2, i.e., if $k = k_0 + \cdots + k_{n-1}2^{n-1}$ then, $\underline{k}_2 = k_0 \ldots k_{n-1} \in \{0,1\}^*$. Similarly, for a word $w = w_0 \ldots w_{n-1} \in \{0,1\}^*$, denote by $\overline{w}^2$ the number $w_0 + \cdots + w_{n-1}2^{n-1}$.

Lemma 3.7 (see [38]) *There exists a finite set of words $\mathcal{P}_{a,d} = \{\mathfrak{p}_a^d(1), \ldots, \mathfrak{p}_a^d(s)\} \subset \{0,1\}^*$, such that*

$$k \in \mathcal{S}_{a,d} \iff \underline{k}_2 \in \bigcup_{i=1}^{s} [\mathfrak{p}_a^d(i)].$$

Denote by $||\mathcal{P}_{a,d}||$ the length of the longest word of $\mathcal{P}_{a,d}$.

Lemma 3.8 *For all $a \in \mathbb{N}$ and $d \in \mathbb{Z}$, if the set $\mathcal{P}_{a,d}$ is nonempty, it satisfies*

- $\#\mathcal{P}_{a,d} \leq 2a$.
- $||\mathcal{P}_{a,d}|| \leq 2\lceil \log_2(a) \rceil + |d|$.

Proof. Both claims come direct from the proof of Lemma 3.7: to prove the first, note a has at most $\lceil \log_2(a) \rceil$ digits in base 2, which defines $2^{\lceil \log_2(a) \rceil} \leq 2a$ possible prefixes for the words in $\mathcal{P}_{a,d}$. For the second point, again since a has at most $\lceil \log_2(a) \rceil$ digits in base 2, the longest word has a prefix of length at most $\lceil \log_2(a) \rceil$ followed by a suffix from some $\mathcal{P}_{1,d'}$, and $|d'| \leq \lceil \log_2(a) \rceil + |d|$. In summary, $||\mathcal{P}_{a,d}|| \leq 2\lceil \log_2(a) \rceil + |d|$. □

Remark By considering all suffixes to complete shorter words, we may suppose all words from $\mathcal{P}_{a,d}$ have the same length (of the longest word), i.e., $||\mathcal{P}_{a,d}|| = 2\lceil \log_2(a) \rceil + |d|$. Of course the cardinal of $\mathcal{P}_{a,d}$ will be bigger, but each suffix is of length at most $2\lceil \log_2(a) \rceil$. In summary, we may suppose $\#\mathcal{P}_{a,d} \leq 2a^3$.

3.1.6 Spectral cocycle and top Lyapunov exponents

As usual, we use the notation $e(x) := e^{2\pi i x}$. Following [21], let ζ be a substitution over $\mathcal{A} = \{1, \ldots, m\}$ and let $\zeta(a) = w_1 \ldots w_{k_a}$. The **spectral cocycle** is the cocycle over $(\mathbb{R}/\mathbb{Z})^m$

with $(a,b) \in \mathcal{A}^2$ entry given by

$$\mathscr{C}_\zeta(\xi,1)(a,b) = \sum_{j=1}^{k_a} \delta_{w_j,b}\, e(\xi_{w_1} + \cdots + \xi_{w_{j-1}}), \quad \xi \in (\mathbb{R}/\mathbb{Z})^m$$
$$\mathscr{C}_\zeta(\xi,n) = \mathscr{C}_\zeta((M_\zeta^T)^{n-1}\xi,1)\ldots\mathscr{C}_\zeta(\xi,1)$$

(the original definition actually uses a minus sign in the exponentials, but it is just a convention and it will not affect the calculations). The spectral cocycle is suitable for studying the spectrum of suspension flows over substitution subshifts: if $\vec{p}$ is the positive vector defining the suspension and $\omega \in \mathbb{R}$, we will set $\xi = \omega\vec{p} \pmod{\mathbb{Z}^d}$. We refer the interested reader to [21] for some results that justify this decomposition.

If we want to study the $\mathbb{Z}$-action, we can consider $\vec{p} = \vec{1} = (1,\ldots,1)^T$ and the decomposition $\xi = \omega\vec{p}$. If the substitution is of constant length q, the endomorphism given by the substitution matrix acts on $\omega\vec{p}$ in $(\mathbb{R}/\mathbb{Z})^d$ in the same way the q-times map $x \mapsto qx \pmod 1$ acts on ω in $\mathbb{R}/\mathbb{Z}$. In the latter case we will refer to the cocycle over $\mathbb{R}/\mathbb{Z}$ as the **restricted** spectral cocycle.

Example Let $\zeta_{TM} : 0 \mapsto 01, 1 \mapsto 10$ be the Thue-Morse substitution. Then, for $(\xi_0,\xi_1)^T = \omega\vec{1}$,

$$\mathscr{C}_{\zeta_{TM}}(\omega,1) = \begin{pmatrix} 1 & e(\omega) \\ e(\omega) & 1 \end{pmatrix},$$
$$\mathscr{C}_{\zeta_{TM}}(\omega,n) = \begin{pmatrix} 1 & e(2^{n-1}\omega) \\ e(2^{n-1}\omega) & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & e(\omega) \\ e(\omega) & 1 \end{pmatrix}.$$

The pointwise **top Lyapunov exponent** of the restricted spectral cocycle $\mathscr{C}_\zeta$, is defined for $\omega \in [0,1)$ as

$$\chi_\zeta^+(\omega) = \limsup_{n\to\infty} \frac{1}{n} \log||\mathscr{C}_\zeta(\omega\vec{1},n)||.$$

In the case of a constant length substitution, since the q-times map ($q \in \mathbb{N}, q \geq 2$) is an ergodic transformation for the Lebesgue measure (denoted by Leb), the Furstenberg-Kesten theorem implies that $\chi_\zeta^+(\omega)$ is almost everywhere constant, value we will denote by χ_ζ^+.

3.1.7 Twisted Birkhoff sums and twisted correlations

In this subsection we define a special kind sum appearing in the calculations of the top Lyapunov exponents for factors of Thue-Morse, but whose study might be interesting by itself. First, we recall the notion of twisted Birkhoff sum:

Definition 3.9 *Let (X,μ,T) be a dynamical system, f a measurable function and $\omega \in [0,1)$. The **twisted Birkhoff sum** $S_N^f(\omega,x)$ (at time N, of parameter ω) of f at a point $x \in X$ is defined by*

$$S_N^f(\omega,x) = \sum_{k=0}^{N-1} f(T^k x) e(k\omega).$$

This type of sum (or integral in the continuous case) is of major importance in several recent works like [53, 66, 22, 4] since uniform (on $x \in X$) upper bounds allow to obtain lower bounds for the dimension of spectral measures of the dynamical systems involved: substitutions, interval exchange transformations, suspensions over S-adic systems (including translation surfaces) and higher dimensional tilings.

A generalization of this kind of sum is the next concept.

Definition 3.10 *Let (X, μ, T) be a dynamical system, f a measurable function. Let $a_1 < \cdots < a_t$ positive integers and $\omega \in [0, 1)$. The* ***twisted correlation*** $C_N^f(a_1, \ldots, a_t, \omega, x)$ *(at time N, of parameters $a_1, \ldots, a_t$ and ω) of f at a point $x \in X$ is defined by*

$$C_N^f(a_1, \ldots, a_t, \omega, x) = \sum_{k=0}^{N-1} f(T^k x) f(T^{k+a_1} x) \ldots f(T^{k+a_t} x) e(k\omega).$$

We will study this concept in the case of the Thue-Morse subshift, for the functions $f = \mathbb{1}_{[0]}, \mathbb{1}_{[1]}$. The twisted correlations of these functions will be denoted by $C_N^0(a_1, \ldots, a_t, \omega, \mathbf{x})$, $C_N^1(a_1, \ldots, a_t, \omega, \mathbf{x})$ respectively, for $\mathbf{x} \in X_{\zeta_{TM}}$. In the case we were considering the fixed points of Thue-Morse, $\mathbf{u}, \mathbf{v}$, we have the simple relation

$$\begin{aligned} C_N^0(a_1, \ldots, a_t, \omega, \mathbf{u}) &= C_N^1(a_1, \ldots, a_t, \omega, \mathbf{v}) \\ C_N^1(a_1, \ldots, a_t, \omega, \mathbf{u}) &= C_N^0(a_1, \ldots, a_t, \omega, \mathbf{v}). \end{aligned}$$

Remark The notation $C_N^f(a_1, \ldots, a_t, \omega, \mathbf{u})$ is consistent because, although $\mathbf{u}$ does not belong to the subshift $X_{\zeta_{TM}}$ ($\mathbf{u}$ is a one-sided sequence), the sum only takes in account the right-hand side of a point $\mathbf{x} \in X_{\zeta_{TM}}$, so we only need to consider points in $X_{\zeta_{TM}}$ with the same "future" as $\mathbf{u}$ and $\mathbf{v}$.

It turns out that better algebraic properties emerge if we consider the function $f = \mathbb{1}_{[0]} - \mathbb{1}_{[1]}$, and the twisted correlation of the former functions may be bounded from above using a finite combination of twisted correlations of the latter function. For this last function we denote its twisted correlations by $C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{x})$, and for the Thue-Morse sequence is given by

$$C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{u}) = \sum_{k=0}^{N-1} (-1)^{s_2(k)+s_2(k+a_1)+\ldots+s_2(k+a_t)} e(k\omega). \tag{3.1}$$

Of course,

$$C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{u}) = (-1)^{t+1} C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{v})$$

Where do these sums come up? They appear naturally when studying the twisted Birkhoff sums of (topological) factors of the Thue-Morse subshift: by the Curtis-Hedlund-Lyndon theorem, every factor π comes from a *sliding block code* $\widehat{\pi} : \mathcal{L}_{2R+1} \to \mathcal{B}$. By Theorem 3.2 factors of substitution systems are conjugate to substitution subshifts. Denoting by σ the factor substitution, $\sigma^{\infty}(a) = \mathbf{w}$ one of its fixed points and $\mathbf{x} \in \pi^{-1}(\{\mathbf{w}\})$, we have the next relation for $f = \mathbb{1}_{[b]}$:

$$S^f_{|\sigma^n(a)|}(\omega, \mathbf{w}) = \sum_{w:\widehat{\pi}(w)=b} \sum_{k=0}^{|\sigma^n(a)|-1} \mathbb{1}_{[w_{-R}\dots w_R]_{-R}}(T^k\mathbf{x})e(k\omega)$$
$$= \sum_{w:\widehat{\pi}(w)=b} \sum_{k=0}^{|\sigma^n(a)|-1} \mathbb{1}_{[w_{-R}]}(T^{k-R}\mathbf{x}) \cdot \ldots \cdot \mathbb{1}_{[w_R]}(T^{k+R}\mathbf{x})e(k\omega)$$

Note in the last equality there are some terms depending on negative coordinates, but the number of terms is bounded (in fact computable according to Theorem 3.4) as n goes to infinity, so it will no interfer at finding asymptotic bounds. We obtain the twisted Birkhoff sum at a fixed point in the factor as a sum of different twisted correlations in the Thue-Morse subshift.

3.2 Results

3.2.1 Top Lyapunov exponent of a general substitution

A simple observation leads to a basic fact about the top Lyapunov exponent of an arbitrary substitution. Here and in the proof of Proposition 3.14, we use the following simple fact: consider $f(x) = \log(|2\cos(\pi x)|)$, $g(x) = \log(|2\sin(\pi x)|)$. It is an exercise to check $f, g \in L^1([0,1], \mathrm{Leb})$ and both have mean equal to zero.

Proposition 3.11 *Let ζ be an arbitrary substitution. Then, $\chi^+_\zeta(\omega) \geq 0$ for Lebesgue almost every $\omega \in [0,1)$.*

Proof. Note that in any line indexed by a, all exponentials $1, \ldots, e((|\zeta^n(a)| - 1)\omega)$ appear as a summand in some entry of this line. From this fact we deduce rapidly,

$$\begin{aligned}||\mathscr{C}_\zeta(\omega, n)||_\infty &\geq \sum_{b\in\mathcal{A}} |\mathscr{C}_\zeta(\omega, n)(a, b)| \\ &\geq |1 + \ldots + e((|\zeta^n(a)| - 1)\omega)| \\ &= |2\cos(\pi\{|\zeta^n(a)|\omega\})|/|e(\omega) - 1|.\end{aligned}$$

A uniform distribution result may serve to conclude. Alternatively, an elementary argument is the next one: denote as above $f(x) = \log(|2\cos(\pi x)|)$. We will conclude the proof by means of the next Borel-Cantelli based argument. Let $\varepsilon > 0$ and $\mathfrak{G}_n(\varepsilon) = \mathfrak{G}_n = \{\omega \in [0,1) | f(\{|\zeta^n(a)|\omega\}) \geq n\varepsilon\}$. Since the Lebesgue measure is invariant for all q-times maps,

$$\begin{aligned}\mathrm{Leb}(\mathfrak{G}_n) &= \mathrm{Leb}\left(\{\omega \in [0,1) \,|\, f(\{|\zeta^n(a)|\omega\}) \geq n\varepsilon\}\right) \\ &= \mathrm{Leb}\left(\{\omega \in [0,1) \,|\, |f(\omega)| \geq n\varepsilon\}\right) \\ &= \sum_{k\geq n} \mathrm{Leb}\left(\left\{\omega \in [0,1) \,|\, k \leq \frac{|f(\omega)|}{\varepsilon} < k+1\right\}\right).\end{aligned}$$

Then,

$$\begin{aligned}\sum_{n\geq 1}\mathrm{Leb}(\mathfrak{G}_n) &= \sum_{n\geq 1}\sum_{k\geq n}\mathrm{Leb}\left(\left\{\omega\in[0,1)\,|\,k\leq \frac{|f(\omega)|}{\varepsilon}<k+1\right\}\right)\\ &= \sum_{k\geq 1}k\,\mathrm{Leb}\left(\left\{\omega\in[0,1)\,|\,k\leq \frac{|f(\omega)|}{\varepsilon}<k+1\right\}\right)\\ &\leq \int_0^1 \frac{|f(x)|}{\varepsilon}dx<+\infty.\end{aligned}$$

From Borel-Cantelli lemma we deduce that for any $\varepsilon>0$, $\mathrm{Leb}\left(\liminf_n \mathfrak{G}_n(\varepsilon)\right)=1$. Considering $\mathfrak{G}=\bigcap_i \liminf_n \mathfrak{G}_n(1/i)$ we have $\mathrm{Leb}(\mathfrak{G})=1$ and for any $\omega\in\mathfrak{G}$, $\lim_n \dfrac{f(\{|\zeta^n(a)|\omega\})}{n}=0$. □

In personal communication, B. Solomyak suggested us that the same is true for the general spectral cocycle in the natural situation that the toral endomorphism is ergodic. The top Lyapunov exponent in this case is defined by

$$\chi_\zeta^+(\xi)=\limsup_{n\to\infty}\frac{1}{n}\log||\mathscr{C}_\zeta(\xi,n)||.$$

Note that we use the same notation for both the restricted spectral cocycle and the general spectral, we only change the variable $\omega\in[0,1)$ to $\xi\in(\mathbb{R}/\mathbb{Z})^m$. Let us recall a characterization of χ_ζ^+.

Proposition 3.12 *Let ζ be a primitive substitution such that M_ζ does not have eigenvalues that are roots of unity. Then $\chi_\zeta^+(\xi)$ is almost everywhere equal to a constant $\chi_\zeta^+(\mathscr{C}_\zeta)$ and*

$$\chi_\zeta^+(\mathscr{C}_\zeta)=\lim_{n\to\infty}\frac{1}{n}\int_{(\mathbb{R}/\mathbb{Z})^m}\log\left(||\mathscr{C}_\zeta(\xi,n)||\right)d\lambda_m(\xi)=\inf_n\frac{1}{n}\int_{(\mathbb{R}/\mathbb{Z})^m}\log\left(||\mathscr{C}_\zeta(\xi,n)||\right)d\lambda_m(\xi),$$

where λ_m is the Haar measure on $(\mathbb{R}/\mathbb{Z})^m$.

Proof. This is just an application of the Furstenberg-Kesten Theorem. □

From an equidistribution result due to B. Host one can deduce that the top Lyapunov exponent of the general spectral cocycle is a nonnegative number, since all integrals are bounded from below by $\chi_\zeta^+(\omega)$ in the case the characteristic polynomial is irreducible: this follows from an argument of [20]. If we only know there are no eigenvalues roots of unity, we still have the result since the logarithmic Mahler measure (in several variables) of an integer polynomial is nonnegative:

Theorem 3.13 *Let ζ be a primitive substitution such that M_ζ does not have eigenvalues that are roots of unity. Then, $\chi_\zeta^+(\mathscr{C}_\zeta)\geq 0$.*

Proof. Indeed, using the Frobenius norm $||\cdot||_F$,

$$||\mathscr{C}_\zeta(\xi,n)||_F^2=\sum_{a,b\in\mathcal{A}}|p_{a,b}^{(n)}(e(\xi_1),\dots,e(\xi_m))|^2,$$

for some polynomials $p_{a,b}^{(n)} \in \mathbb{Z}[X_1, \dots, X_m]$ (in fact the coefficients are equal to 0 or 1, by definition of the spectral cocycle). Then, for any $n \geq 1$,

$$\begin{aligned}
\frac{1}{n}\int_{(\mathbb{R}/\mathbb{Z})^m} \log\left(||\mathscr{C}_\zeta(\xi,n)||_F\right) d\lambda_m(\xi) &= \frac{1}{2n}\int_{(\mathbb{R}/\mathbb{Z})^m} \log\left(\sum_{a,b\in\mathcal{A}} |p_{a,b}^{(n)}(e(\xi_1),\dots,e(\xi_m))|^2\right) d\lambda_m(\xi) \\
&\geq \frac{1}{nm^2}\sum_{a,b\in\mathcal{A}}\int_{(\mathbb{R}/\mathbb{Z})^m} \log\left(\left|mp_{a,b}^{(n)}(e(\xi_1),\dots,e(\xi_m))\right|\right) d\lambda_m(\xi) \\
&= \frac{1}{nm^2}\sum_{a,b\in\mathcal{A}} \mathfrak{m}\left(mp_{a,b}^{(n)}\right) \\
&\geq 0,
\end{aligned}$$

since the logarithmic Mahler measure of any polynomial in $\mathbb{Z}[X_1, \dots, X_m]$ is nonnegative. The proof follows from Proposition 3.12. □

3.2.2 The Thue-Morse top Lyapunov exponent

The fact that the top Lyapunov exponent for Thue-Morse is equal to zero is not a new result (see [5]), but we want to make an explicit calculation to motivate the calculus for the factors. Set

$$\begin{aligned}
X_n(\omega) &= \sum_{k=0}^{2^n-1} \mathbb{1}_{[0]}(T^k\mathbf{u})e(k\omega) \\
Y_n(\omega) &= \sum_{k=0}^{2^n-1} \mathbb{1}_{[1]}(T^k\mathbf{u})e(k\omega).
\end{aligned}$$

In particular, we have

$$X_n(\omega) + Y_n(\omega) = \sum_{k=0}^{2^n-1} e(k\omega) = \frac{e(2^n\omega)-1}{e(\omega)-1}.$$

Note $|X_n(\omega) + Y_n(\omega)| \leq 2/|e(\omega)-1|$. On the other hand,

$$\begin{aligned}
X_n(\omega) - Y_n(\omega) &= \sum_{k=0}^{2^n-1} (-1)^{s_2(k)} e(k\omega) \\
&= \sum_{k=0}^{2^n-1} e(s_2(k)/2 + k\omega) \\
&= \sum_{k_0=0}^{1} \cdots \sum_{k_{n-1}=0}^{1} e(k_0/2 + \cdots + k_{n-1}/2 + \omega(k_0 + k_1 2 + \cdots + k_{n-1}2^{n-1})) \\
&= \prod_{j=0}^{n-1} \sum_{k_j=0}^{1} e(k_j/2 + \omega k_j 2^j)
\end{aligned}$$

$$= \prod_{j=0}^{n-1} 1 + e(1/2 + \omega 2^j)$$

$$= (-2i)^n e(\omega(2^n - 1)/2) \prod_{j=0}^{n-1} \sin(\pi\omega 2^j)$$

In particular,

$$|X_n(\omega) - Y_n(\omega)| = \prod_{j=0}^{n-1} 2\sin(\pi\{\omega 2^j\})$$

To estimate both X_n and Y_n, consider the inequalities

$$\begin{gathered} \max(|X_n(\omega) + Y_n(\omega)|, |X_n(\omega) - Y_n(\omega)|) \\ \leq \\ |X_n(\omega)| + |Y_n(\omega)| \\ \leq \\ |X_n(\omega) + Y_n(\omega)| + |X_n(\omega) - Y_n(\omega)|. \end{gathered}$$

Set $f(x) = 2\sin(\pi x)$. Since $T : [0,1) \longrightarrow [0,1)$ defined by $Tx = 2x \pmod 1$ is ergodic for the Lebesgue measure, we have that for almost every ω

$$\begin{aligned} \frac{1}{n} \log \prod_{j=0}^{n-1} f(T^n\omega) &= \frac{1}{n} \sum_{j=0}^{n-1} \log(f(T^n\omega)) \\ &\to \int_0^1 \log(f(x))dx = 0. \end{aligned}$$

It is not difficult to see the same calculations are valid for the twisted Birkhoff sums at $\mathbf{v} = \zeta_{TM}^{\infty}(1)$. With this we conclude that almost surely $\chi^+_{\zeta_{TM}}(\omega) \leq 0$ and equality follows from Lemma 3.11.

In fact, we will show finer estimates for the growth of the product $p_n(\omega) := \prod_{j=0}^{n-1} 2\sin(\pi\{\omega 2^j\})$.

Proposition 3.14 *There exists a positive constant B such that for almost all ω, there is a positive integer $n_0(\omega)$ such that for all $n \geq n_0(\omega)$,*

$$\max(p_n(\omega), p_n^{-1}(\omega)) \leq e^{B\sqrt{n \log\log(n)}},$$

where $p_n(\omega) = \prod_{j=0}^{n-1} 2\sin(\pi\{\omega 2^j\})$.

Proof. Fix $\varepsilon > 0$. Consider the function of the unit interval $f(x) = \log(2\sin(\pi x))$. One can check that $\mathrm{Leb}(f) = 0$ and $\mathrm{Leb}(|f|^2) \leq 2^2$. Set f_1, f_2 as

$$\begin{aligned} f_1(x) &= 2\log(2\sin(\pi x)) \text{ if } x \in (0, 1/2] \text{ and } f_1(x) = 0 \text{ otherwise}, \\ f_2(x) &= 2\log(2\sin(\pi x)) \text{ if } x \in [1/2, 1) \text{ and } f_2(x) = 0 \text{ otherwise}. \end{aligned}$$

Then, $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$ and $f \in \mathrm{Mon}_2(2, \mathrm{Leb})$. We may apply Theorem 3.6 to the doubling map S and the function f: we get for almost every ω there exists $n_0(\omega) \geq 1$ such that for every $n \geq n_0(\omega)$,

$$\left|\sum_{j=0}^{n-1} f \circ S^j(\omega)\right| \leq \underbrace{(A+\varepsilon)}_{=:B} \sqrt{n \log\log(n)}.$$

Take exponential to obtain the desired inequality, by noticing that

$$e^{|x|} = e^{\max(x,-x)} = \max(e^x, e^{-x}).$$

$\square$

3.2.3 Upper bounds for twisted correlations

In this subsection we prove upper bounds for the twisted correlations on the Thue-Morse system. First, we consider the twisted correlation at the fixed point $\mathbf{u} = \zeta_{TM}^{\infty}(0)$ of the function $f = \mathbb{1}_{[0]} - \mathbb{1}_{[1]}$. This sum was denoted $C_{2^n}^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{u})$ and is defined in 3.1. We start by stating the main result.

Theorem 3.15 *Let $1 \leq a_1 < \cdots < a_t$ positive integers and $f = \mathbb{1}_{[0]} - \mathbb{1}_{[1]}$. Then,*

- *if t is even, there exists $B > 0$, depending only on f, such that for almost every ω,*

$$C_{2^n}^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{u}) = O_{a_t,\omega}\left(n^t e^{B\sqrt{n \log\log(n)}}\right).$$

- *if t is odd, for every $\varepsilon > 0$ and almost every ω,*

$$C_{2^n}^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{u}) = O_{a_t,\omega}\left(n^{t+1+\varepsilon}\right).$$

Remark The notation $O_{a_t,\omega}$ means that the implicit constant only depends on a_t and ω. In the proof it might seem it depends on all $a_1, \ldots, a_t$, t and ω, but this could be turned into a dependence only on a_t and ω since $t \leq a_t$ and $1 \leq a_1 < \cdots < a_t$.

For simplicity, we will only prove the case $t = 1$ here. For the general case, see the Appendix.

Since the geometric sum with term $e(k\omega)$ is bounded for $\omega \neq 0$, we will work instead with $\widetilde{C}_{2^n}(a, \omega, \mathbf{u}) := \sum_{k=0}^{2^n-1}[1 + (-1)^{s_2(k+a)-s_2(k)}]e(k\omega)$: any upper bound for this latter sum, implies the same bound for $C_{2^n}^{\pm}(a, \omega, \mathbf{u})$ by changing the implicit constant in $O_{a_t,\omega}$.

Let us recall that if $d > \lceil \log_2(a) \rceil$, then $\mathcal{S}_{a,d} = \emptyset$; and $d < -n$ implies $\mathcal{S}_{a,d} \cap \{0, \ldots, 2^n - 1\} = \emptyset$. We recall also a notation introduced for a word $w = w_0 \ldots w_{n-1} \in \{0,1\}^*$: denote by $\overline{w}^2$ the integer $w_0 + \cdots + w_{n-1}2^{n-1}$.

$$
\begin{aligned}
\widetilde{C}_{2^n}(a,\omega,\mathbf{u}) &= 2\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\ \sum_{\substack{k\in\mathcal{S}_{a,-2d}\\ 0\le k\le 2^n-1}} e(k\omega)\\
&= 2\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\ \sum_{\substack{\underline{k}_2\in\bigcup_i [\mathfrak{p}_{a,-2d}(i)]\\ 0\le k\le 2^n-1}} e(k\omega)\\
&= 2\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\ \sum_{i=1}^{\#\mathcal{P}_{a,-2d}}\ \sum_{k=0}^{2^{n-||\mathcal{P}_{a,-2d}||}-1} e\left(\left(2^{||\mathcal{P}_{a,-2d}||}k+\overline{\mathfrak{p}_{a,-2d}(i)}^2\right)\omega\right)\\
&= 2\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\ \sum_{i=1}^{\#\mathcal{P}_{a,-2d}} e\left(\overline{\mathfrak{p}_{a,-2d}(i)}^2\omega\right)\sum_{k=0}^{2^{n-||\mathcal{P}_{a,-2d}||}-1} e\left(2^{||\mathcal{P}_{a,-2d}||}k\omega\right)\\
&= 2(e(2^n\omega)-1)\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\ \sum_{i=1}^{\#\mathcal{P}_{a,-2d}} \frac{e\left(\overline{\mathfrak{p}_{a,-2d}(i)}^2\omega\right)}{e\left(2^{||\mathcal{P}_{a,-2d}||}\omega\right)-1}
\end{aligned}
$$

Applying the first remark below Lemma 3.7,

$$
\begin{aligned}
|\widetilde{C}_{2^n}(a,\omega,\mathbf{u})| &\le 2a^3\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}\frac{1}{\left|\sin\left(\pi 2^{||\mathcal{P}_{a,-2d}||+1}\omega\right)\right|}\\
&\le \pi a^3\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}||2^{||\mathcal{P}_{a,-2d}||+1}\omega||_{\mathbb{R}/\mathbb{Z}}^{-1}
\end{aligned}
$$

Following the second remark,

$$
|\widetilde{C}_{2^n}(a,\omega,\mathbf{u})| \le \pi a^3\sum_{d=-\lceil \log_2(a)/2\rceil}^{\lfloor n/2\rfloor}||2^{2\lceil\log_2(a)\rceil+2d+1}\omega||_{\mathbb{R}/\mathbb{Z}}^{-1}
$$

Now we recall a well-known consequence of Borel-Cantelli lemma.

Lemma 3.16 (see [8], Lemma 6.2.6) *Let $\varepsilon>0$. For almost every $x\in\mathbb{R}$ there exist $d_0(x)\in\mathbb{N}$ such that*

$$
||2^n x||_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{n^{1+\varepsilon}},
$$

holds for every $n\ge d_0(x)$.

Then,

$$
\begin{aligned}
|\widetilde{C}_{2^n}(a,\omega,\mathbf{u})| &= O_{a,\omega}\left(\sum_{d=d_0(\omega)}^{\lfloor n/2\rfloor}(2\lceil\log_2(a)\rceil+2d+1)^{1+\varepsilon}\right)\\
&= O_{a,\omega}(n^{2+\varepsilon})
\end{aligned}
$$

This finishes the proof for the case $t = 1$. We have already seen the case $t = 0$, which is the case of twisted Birkhoff sums in last section.

To finish this section, we will pass to uniform bounds for the twisted correlations for all points of the subshift. We relay on the classic prefix-suffix decomposition:

Proposition 3.17 *Let ζ be a primitive substitution with Perron-Frobenius eigenvalue θ. For any $\mathbf{x} \in X_\zeta$ and $N \geq 1$ there exists $n \geq 1$ such that*

$$\mathbf{x}_{[0,N-1]} = p_0\zeta(p_1)\ldots\zeta^n(p_n)\zeta^n(s_n)\ldots\zeta(s_1)s_0,$$

where p_i, s_i are respectively proper prefixes and suffixes of words of $\{\zeta(a)|a \in \mathcal{A}\}$. Moreover, there exists $C > 0$ such that $|\log(N) - n\log(\theta)| \leq C$.

Corollary 3.18 *Let $C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{x})$ be the twisted correlation over $\mathbf{x} \in X_{\zeta_{TM}}$. Then*

- *if t is even, there exists $B > 0$ independent of ω and t such that for almost every ω,*
$$C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{x}) = O_{a_t,\omega}\left(\log(N)^{t+1}e^{B\sqrt{\log(N)\log\log\log(N)}}\right).$$
- *if t is odd, for every $\varepsilon > 0$ and almost every ω,*
$$C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{x}) = O_{a_t,\omega}\left(\log(N)^{t+2+\varepsilon}\right).$$

Proof. Following the decomposition from Proposition 3.17, the bound follows simply from triangle inequality:

$$\begin{aligned}
&|C_N^{\pm}(a_1, \ldots, a_t, \omega, \mathbf{x})| \\
&\leq \sum_{j=0}^{n} \delta_{p_j,0}|C^{\pm}_{|\zeta^j(p_j)|}(a_1, \ldots, a_t, \omega, \mathbf{u})| + \delta_{p_j,1}|C^{\pm}_{|\zeta^j(p_j)|}(a_1, \ldots, a_t, \omega, \mathbf{v})| \\
&+ \sum_{j=0}^{n} \delta_{s_j,0}|C^{\pm}_{|\zeta^j(s_j)|}(a_1, \ldots, a_t, \omega, \mathbf{u})| + \delta_{s_j,1}|C^{\pm}_{|\zeta^j(s_j)|}(a_1, \ldots, a_t, \omega, \mathbf{v})| \\
&= \begin{cases} \sum_{j=0}^{n} O_{a_t,\omega}\left(j^t e^{B\sqrt{j\log\log(j)}}\right) & \text{if } t \text{ is even,} \\ \sum_{j=0}^{n} O_{a_t,\omega}\left(j^{t+1+\varepsilon}\right) & \text{if } t \text{ is odd.} \end{cases} \\
&\leq \begin{cases} (n+1)O_{a_t,\omega}\left(n^t e^{B\sqrt{n\log\log(n)}}\right) & \text{if } t \text{ is even,} \\ (n+1)O_{a_t,\omega}\left(n^{t+1+\varepsilon}\right) & \text{if } t \text{ is odd.} \end{cases}
\end{aligned}$$

By using the second claim of Proposition 3.17, we finish the proof of the corollary. □

3.3 Top Lyapunov exponents of factors

In this section we will prove Theorem 3.1. The proof relies on the bounds for the twisted correlations found in subsection 3.2.3: we will see that the coefficients of the spectral cocycle matrices of the semi-conjugate substitutions, correspond to linear combinations of twisted correlations from the Thue-Morse sequence.

The sketch of the proof has already been shown, but we write down explicitly and rigorously. Let (X_σ, T) be an aperiodic topological factor of $(X_{\zeta_{TM}}, T)$ arising from the aperiodic primitive substitution σ. By Theorem 3.4, the factor map $\pi : X_{\zeta_{TM}} \longrightarrow X_\sigma$ is defined by a sliding block code $\widehat{\pi}$ of radius r, where r is a computable constant depending on X_σ. Consider a point $\mathbf{w}$ starting with the block $\sigma^n(a)$, for a letter $a \in \mathcal{A}$. Choose any $\mathbf{x} \in \pi^{-1}(\{\mathbf{w}\})$. Then, setting $f = \mathbb{1}_{[b]}$, we have

$$
\begin{aligned}
|\mathscr{C}_\sigma(\omega, n)(a,b)| &= |S^f_{|\sigma^n(a)|}(\omega, \mathbf{w})| \\
&\leq \sum_{w:\widehat{\pi}(w)=b} \left| \sum_{k=0}^{|\sigma^n(a)|-1} \mathbb{1}_{[w_{-r}]}(T^{k-r}\mathbf{x}) \cdot \ldots \cdot \mathbb{1}_{[w_r]}(T^{k+r}\mathbf{x}) e(k\omega) \right| \\
&\leq \sum_{w:\widehat{\pi}(w)=b} \left| \sum_{k=0}^{|\sigma^n(a)|-(r+1)} \mathbb{1}_{[w_{-r}]}(T^{k}\mathbf{x}) \cdot \ldots \cdot \mathbb{1}_{[w_r]}(T^{k+2r}\mathbf{x}) e(k\omega) \right| + \Delta_r
\end{aligned}
$$

In the last sum, there are terms depending on negative coordinates of $\mathbf{x}$, but at most on the first r negative coordinates, which we leave apart in the term Δ_r since they make no difference in the asymptotic as n goes to infinity.

Consider the next simple relations for $F(\mathbf{x}) = \mathbb{1}_{[0]}(\mathbf{x}) - \mathbb{1}_{[1]}(\mathbf{x})$:

$$
\begin{aligned}
\mathbb{1}_{[0]}(\mathbf{x}) &= \frac{1}{2}\left[1 + F(\mathbf{x})\right], \\
\mathbb{1}_{[1]}(\mathbf{x}) &= \frac{1}{2}\left[1 - F(\mathbf{x})\right].
\end{aligned}
$$

By definition $C^{\pm}_N(a_1, \ldots, a_t, \omega, \mathbf{x}) = C^F_N(a_1, \ldots, a_t, \omega, \mathbf{x})$. Replacing in the sum above each factor $\mathbb{1}_{[0]}(\mathbf{x}), \mathbb{1}_{[1]}(\mathbf{x})$ by $\frac{1}{2}[1 + F(\mathbf{x})]$, $\frac{1}{2}[1 - F(\mathbf{x})]$ respectively, we are left with a sum of different twisted correlations (on different sets of parameters a_i) of the function F. By the last remark, we are able to estimate each of them using Corollary 3.18. The formulas from this corollary implies rapidly that $\chi^+_\sigma(\omega) \leq 0$ almost surely, and Proposition 3.11 again yields the equality.

3.4 Appendix

3.4.1 Upper bounds for twisted correlations: general case.

Let us begin with the case when t is odd. For every $j = 1, \ldots, t$, consider the set $\mathcal{D}_j = \{-n, \ldots, \lceil \log_2(a_j) \rceil\}$ and the decomposition $\mathcal{D}_j = \mathcal{D}_j^0 \cup \mathcal{D}_j^1$ in even and odd elements respectively. Since the geometric series of the exponentials is bounded for $\omega \neq 0$, we consider equivalently

$$\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u}) = C^{\pm}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u}) + \sum_{k=0}^{2^n-1} e(k\omega)$$

to deduce the upper bound.

$$\begin{aligned}
\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u}) &= \sum_{k=0}^{2^n-1} \left(1+(-1)^{s_2(k)+s_2(k+a_1)+\cdots+s_2(k+a_t)}\right) e(k\omega) \\
&= \sum_{k=0}^{2^n-1} \left(1+\prod_{j=1}^{t}(-1)^{s_2(k+a_j)-s_2(k)}\right) e(k\omega) \\
&= 2 \sum_{\substack{d_j\in\mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t\equiv_2 0}} \sum_{\substack{k\in\cap_j \mathcal{S}_{a_j,d_j} \\ 0\le k\le 2^n-1}} e(k\omega).
\end{aligned}$$

The first sum in the last equality sums over all elements of each $\mathcal{D}_1^{c_1},\ldots,\mathcal{D}_t^{c_t}$, for all combinations of variables $c_1,\ldots,c_t\in\{0,1\}$ satisfying $c_1+\cdots+c_t\equiv_2 0$ (congruence (mod 2)). Before continue, we will need a generalization of Lemma 3.7.

Corollary 3.19 *Let $1\le a_1<\ldots<a_t$ positive integers and $d_1,\ldots d_t\in\mathbb{Z}$. Then,*

$$k\in\bigcap_j \mathcal{S}_{a_j,d_j} \iff \underline{k}_2\in\bigcup_i [\mathfrak{p}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}(i)],$$

for some words $\mathfrak{p}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}(i)\in\bigcap_j \mathcal{P}_{a_j,d_j} =: \mathcal{P}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}$.

This can be checked rapidly since all of these words correspond to words of some $\mathcal{P}_{a_j,d_j}$ which has the longest words among all these sets, or is empty for the choice of $a_1,\ldots,a_t,d_1,\ldots,d_t$. The same remark as the one below Lemma 3.7 applies: we may assume all words have the same length and correspondly we deduce the inequalities of the next

Corollary 3.20 *By completing shorter words of $\mathcal{P}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}$, if this set is nonempty we have*

- $\#\mathcal{P}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}\le 2a_t^3$
- $||\mathcal{P}^{d_1,\ldots,d_t}_{a_1,\ldots,a_t}|| = 2\lceil\log_2(a_t)\rceil+|d_{j^*}|$, *for some $1\le j^*\le t$.*

Applying the first corollary, we obtain

$$\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u}) = 2 \sum_{\substack{d_j\in\mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t\equiv_2 0}} \sum_{\substack{k\in\cap_j \mathcal{S}_{a_j,d_j} \\ 0\le k\le 2^n-1}} e(k\omega).$$

$$= 2 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}} \sum_{k=0}^{2^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}-1} e\left(\left(2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}k + \overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\right)\omega\right)$$

$$= 2(e(2^n\omega)-1) \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}} \frac{e\left(\overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\omega\right)}{e\left(2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}\omega\right)-1}$$

Taking absolute value,

$$\begin{aligned}
&|\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})| \\
&\leq 2a_t^3 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \frac{1}{\left|\sin\left(\pi 2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||+1}\omega\right)\right|} \\
&\leq \pi a_t^3 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} ||2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||+1}\omega||_{\mathbb{R}/\mathbb{Z}}^{-1}.
\end{aligned}$$

According to our second corollary, $||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}|| = 2\lceil\log_2(a_t)\rceil + |d_{j^*}|$ for some j^*, yielding

$$|\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega)| \leq 2\pi a_t^3 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} ||2^{2\lceil\log_2(a_t)\rceil+|d_{j^*}|}\omega||_{\mathbb{R}/\mathbb{Z}}^{-1}$$

To estimate each term in this sum we argue as in subsection 3.2.3: by Lemma 3.16, for almost every ω there exist $d_0(\omega)$ such that

$$||2^n\omega||_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{n^{1+\varepsilon}},$$

for all $n \geq d_0(\omega)$. In particular,

$$\begin{aligned}
|\widetilde{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})| &\leq O_{a_t,\omega}(1) + 2\pi a_t^3 \sum_{l=1}^{t} \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0 \\ j^*=l \\ |d_{j^*}| \geq d_0(\omega)}} (2\lceil\log_2(a_t)\rceil + |d_l| + 1)^{1+\varepsilon} \\
&= O_{a_t,\omega}(n^{t+1+\varepsilon})
\end{aligned}$$

The even case is worked in a similar way: consider

$$\widehat{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u}) = C_{2^n}^{\pm}(a_1,\ldots,a_t,\omega,\mathbf{u}) + \sum_{k=0}^{2^n-1}(-1)^{s_2(k)}e(k\omega).$$

Since $p_n(\omega) = O_\omega(e^{B\sqrt{n\log\log(n)}})$ by Proposition 3.14, it is enough to prove the required upper bound for $\widehat{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})$.

$$\begin{aligned}
&\widehat{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})\\
&=\sum_{k=0}^{2^n-1}\left((-1)^{s_2(k)}+(-1)^{s_2(k)+s_2(k+a_1)+\cdots+s_2(k+a_t)}\right)e(k\omega)\\
&=\sum_{k=0}^{2^n-1}\left(1+\prod_{j=1}^{t}(-1)^{s_2(k+a_j)-s_2(k)}\right)e\left(\frac{s_2(k)}{2}+k\omega\right)\\
&=2\sum_{\substack{d_j\in\mathcal{D}_j^{c_j}\\ c_1+\cdots+c_t\equiv_2 0}}\ \sum_{\substack{k\in\cap_j\mathcal{S}_{a_j,d_j}\\ 0\le k\le 2^n-1}} e\left(\frac{s_2(k)}{2}+k\omega\right)\\
&=2\sum_{\substack{d_j\in\mathcal{D}_j^{c_j}\\ c_1+\cdots+c_t\equiv_2 0}}\ \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}}\ \sum_{k=0}^{2^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}-1}\mathfrak{A}_k,
\end{aligned}$$

where

$$\mathfrak{A}_k=e\left(\frac{s_2\left(2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}k+\overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\right)}{2}+\left(2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}k+\overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\right)\omega\right).$$

Let us factorize the terms independent of k in the exponentials $\mathfrak{A}_k$, and continue from the last line.

$$\begin{aligned}
&\widehat{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})\\
&=2\sum_{\substack{d_j\in\mathcal{D}_j^{c_j}\\ c_1+\cdots+c_t\equiv_2 0}}\ \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}}\underbrace{e\left(\frac{s_2\left(\overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\right)}{2}+\overline{\mathfrak{p}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}^2\omega\right)}_{=\mathfrak{B}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)}\\
&\qquad\sum_{k=0}^{2^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}-1}e\left(\frac{s_2\left(2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}k\right)}{2}+k2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}\omega\right)\\
&=2\sum_{\substack{d_j\in\mathcal{D}_j^{c_j}\\ c_1+\cdots+c_t\equiv_2 0}}\ \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}}\mathfrak{B}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)\sum_{k=0}^{2^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}-1}e\left(\frac{s_2\left(k\right)}{2}+k2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}\omega\right)
\end{aligned}$$

$$= 2 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}} \mathfrak{B}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}(i)(-\sqrt{-1})^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||} e(\omega(2^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}-1)/2)$$

$$\prod_{j=0}^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||-1} 2\sin(\pi\omega 2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||+j})$$

Taking absolute value,

$$\begin{aligned}
&|\widehat{C}_{2^n}(a_1,\ldots,a_t,\omega,\mathbf{u})| \\
&\leq 2 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \sum_{i=1}^{\#\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}} \left| \prod_{j=0}^{n-||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||-1} 2\sin(\pi\omega 2^{||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||+j}) \right| \\
&\leq 4a_t^3 \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0}} \prod_{j=||\mathcal{P}_{a_1,\ldots,a_t}^{d_1,\ldots,d_t}||}^{n-1} 2\sin(\pi\{\omega 2^j\}) \\
&= 4a_t^3 \sum_{l=1}^{t} \sum_{\substack{d_j \in \mathcal{D}_j^{c_j} \\ c_1+\cdots+c_t \equiv_2 0 \\ j^*=l}} \prod_{j=2\lceil \log_2(a_t)\rceil+|d_l|}^{n-1} 2\sin(\pi\{\omega 2^j\}).
\end{aligned}$$

From Proposition 3.14 there exists a universal $B > 0$, such that for almost all ω, $p_n(\omega) = \prod_{j=0}^{n-1} 2\sin(\pi\{\omega 2^j\}) \leq e^{B\sqrt{n\log\log(n)}}$, for all $n \geq n_0(\omega)$, for some $n_0(\omega)$. In particular,

$$\begin{aligned}
|\sum_{j=l}^{n-1} \log(2\sin(\pi\{\omega 2^j\}))| &\leq |\sum_{j=0}^{n-1} \log(2\sin(\pi\{\omega 2^j\}))| + |\sum_{j=0}^{l-1} \log(2\sin(\pi\{\omega 2^j\}))| \\
&\leq B\sqrt{n\log\log(n)} + \\
&\quad \max\left(\max_{1\leq k\leq n_0(\omega)} |\log(2\sin(\pi\{\omega 2^k\}))|, B\sqrt{l\log\log(l)}\right) \\
&\leq C(\omega)B\sqrt{n\log\log(n)}
\end{aligned}$$

This finish the proof of Proposition 3.15.

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Résumé

Cette thèse étudie le spectre des systèmes associés aux substitutions, en particulier le spectre continu. Nous avons basé l'analyse sur l'étude du cocycle spectral et des sommes (et intégrales) de Birkhoff tordues. Ces outils ont été utilisés récemment dans de nombreux travaux pour assurer des taux quantitatifs de mélange faible et singularité du spectre dans des contextes tels que les sous-décalages substitutifs, les systèmes S-adiques, les surfaces de translations, les pavages substitutifs déterministes et aléatoires et les transformations d'échange d'intervalles.

Les premiers résultats sont obtenus dans le cas des flots de suspension sur les substitutions de type Salem. Nous prouvons des décroissances de type Hölder pour les mesures de corrélation sur les paramètres spectraux appartenant au corp algébrique engendré par le nombre de Salem. La preuve est basée sur une analyse fine de la distribution modulo 1 de la suite $(\eta\alpha^n)_{n\geq 0}$, où $\eta \in \mathbb{Q}(\alpha)$ et α est le nombre de Salem correspondant.

La deuxième série de résultats est liée à la substitution de Thue-Morse. Nous étudions le comportement des exposants de Lyapunov maximaux du cocycle spectral associé à la substitution de Thue-Morse et à ses facteurs topologiques. Nous prouvons que pour tous les facteurs topologiques, l'exposant de Lyapunov maximal est nul, et nous donnons également le comportement sous-exponentiel des sommes de Birkhoff tordues.