Aix*Marseille université
Socialement engagée

NNT/NL : 2022AIXM0006/001ED184

THÈSE DE DOCTORAT

Soutenue à Aix-Marseille Université
le 11 janvier 2022 par

Roman CHERNOV

On core entropy for cosine family
Sur l'entropie du cœur pour la famille cosinus

Discipline
Mathématiques

École doctorale
ED 184 MATHEMATIQUE ET INFORMATIQUE

Laboratoire/Partenaires de recherche
Institut de Mathématiques de Marseille (I2M)

Composition du jury

Anna Miriam BENINI Università di Parma	Rapporteure
Michal MISIUREWICZ University of Indiana (IUPUI)	Rapporteur
Anna ZDUNIK Uniwersytet Warszawski	Examinatrice
Feliks PRZYTYCKI IMPAN Warszawa	Examinateur
Pierre ARNOUX Aix-Marseille Université	Président du jury
Dierk SCHLEICHER Aix-Marseille Université	Directeur de thèse

Affidavit

I, undersigned, Roman CHERNOV, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific direction of Prof. Dierk SCHLEICHER, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and the Aix-Marseille University charter on the fight against plagiarism.

This work has not been submitted previously either in this country or in another country in the same or in a similar version to any other examination body.

Kharkiv (Ukraine), 09/11/2021

R. Chernov

Liste de publications et participation aux conférences

List of all publications :

1. Y. Chen, R. Chernov, M. Flores, M. Fortier Bourque, S. Lee, B. Yang, *Toy Teichmüller spaces of real dimension 2 : the pentagon and the punctured triangle*, Geometriae Dedicata 197 (2018), 193-227.
2. R. Chernov, K. Drach, K. Tatarko, *A sausage body is a unique solution for a reverse isoperimetric problem*, Advances in Mathematics 353 (2019), 431-445.
3. S. Shemyakov, R. Chernov, D. Rumiantsau, D. Schleicher, S. Schmitt, A. Shemyakov, *Finding polynomial roots by dynamical systems – A case study*, Discrete & Continuous Dynamical Systems 40 (12) (2020), 6945-6965.

Participation in conferences :

1. Advancing Bridges in Complex Dynamics (ABCD) in CIRM, Luminy, Marseille ; 20/09/2021 – 24/09/2021 ; talk "On core entropy for cosine family" ; `https://cirmbox.cirm-math.fr/s/ottpMBtzNnH5WdR`

Résumé

L'entropie topologique est connue pour être une mesure de la complexité d'un système dynamique donné par l'itération d'une fonction continue. Pour les fonctions complexes dans tous les cas étudiés, elle s'avère constante, donc dans la dynamique complexe un autre concept est pris en compte, le concept d'entropie du cœur. Pour les polynômes complexes, l'entropie du cœur peut être considérée comme l'entropie topologique restreinte à l'arbre de Hubbard. Dans cette thèse, nous généralisons la notion d'entropie du cœur pour la famille transcendante des fonctions cosinus $\lambda \cos z$ avec $\lambda \in \mathbb{C}$ tel que la fonction ait des orbites critiques bornées.

Nous montrons que dans tout espace de fonctions cosinus avec une combinatoire uniformément bornée, l'entropie du cœur est uniformément bornée et continue. Cependant, dans l'espace global des paramètres complexes, l'entropie du cœur est illimitée même localement : dans un voisinage de chaque paramètre $\lambda \in \mathbb{R}$ tel que $\lambda \geqslant 1$ nous trouvons une séquence de paramètres périodiques tendant vers λ avec l'entropie du cœur tendant à ∞.

Mots clés. Entropie du cœur, cosinus, fonction entière, dynamique transcendante.

Abstract

Topological entropy is known to be a measure of complexity of a dynamical system given by iteration of a continuous map. For complex maps in all studied cases it turns out to be constant, so in complex dynamics another concept is considered, the concept of so-called core entropy. For complex polynomials the core entropy can be viewed as the topological entropy restricted to the Hubbard tree.

In this thesis, we generalize the notion of core entropy for the transcendental family of cosine maps $\lambda \cos z$ with $\lambda \in \mathbb{C}$ such that the map has bounded combinatorics.

We show that in every space of cosine maps with uniformly bounded combinatorics core entropy is uniformly bounded and continuous. However, in the global complex parameter space core entropy is unbounded even locally: in a neighborhood of every parameter $\lambda \in \mathbb{R}$ such that $\lambda \geqslant 1$ we find a sequence of periodic parameters tending to λ with core entropy tending to ∞.

Keywords. Core entropy, cosine map, entire function, transcendental dynamics.

Résumé substanciel

L'entropie topologique est connue pour être une mesure de la complexité d'un système dynamique donnée par l'itération d'une fonction continue. Pour les fonctions complexes dans tous les cas étudiés, elle s'avère constante, donc dans la dynamique complexe un autre concept est pris en compte, le concept d'entropie du cœur. Pour les polynômes complexes, l'entropie du cœur peut être définie comme l'entropie topologique restreinte à l'arbre de Hubbard. Dans cette thèse, nous généralisons la notion d'entropie de noyau pour la famille transcendante des fonctions cosinus $\lambda \cos z$ avec $\lambda \in \mathbb{C}$ telle que la fonction ait des orbites critiques bornées.

Dans le Chapitre 2, nous montrons que dans chaque espace des applications cosinus avec des combinatoires uniformément bornées, l'entropie du cœur est uniformément bornée et continue.

Nous décrivons les propriétés combinatoires de la fonction cosinus dans la Section 2.2. Nous définissons la partition statique du plan complexe donnée par une fonction cosinus et explorons les représentants combinatoires des fonctions cosinus complexes. Ces représentants sont appelés *adresses externes* et ce sont des séquences symboliques obtenues à partir de l'adresse d'une orbite critique relative à la partition statique. Les fonctions cosinus avec des valeurs critiques non échappantes correspondent à des adresses externes uniformément bornées. Nous désignons par $\mathcal{S}_B$ un espace d'adresses externes uniformément bornées .

Nous définissons l'entropie du cœur d'une application cosinus post-critiquement finie dans la Section 2.3 et étendons plus tard ce concept aux espaces d'adresses externes uniformément bornées. Nous obtenons le théorème suivant pour l'entropie du cœur (des énoncés et des bornes précis peuvent être vus par exemple dans l'introduction).

Théorème. *L'entropie du cœur d'une adresse externe uniformément bornée est uniformément bornée.*

Nous prouvons dans la Section 2.6 un résultat de continuité pour l'entropie du cœur. Les principaux outils sont décrits dans les Sections 2.4 et 2.5, ce qui implique le théorème suivant.

Théorème. *L'entropie du cœur est continue sur chaque espace d'adresses externes uniformément bornées $\mathcal{S}_B$.*

Dans le Chapitre 3, nous discutons d'une relation entre les résultats ci-dessus et l'espace global des paramètres complexes. Nous observons le phénomène d'illimité locale de l'entropie du cœur à proximité des paramètres échappents. Dans la Section 3.2, nous utilisons l'expansivité de cosinus et la combinatoire non bornée

aux paramètres échappents. Nous montrons l'existence de paramètres prépériodiques avec des propriétés spécifiques. Ces propriétés sont suffisantes pour obtenir l'entropie du cœur divergente.

Théorème. *Il y a des paramètres échappents* λ_0 *tels que l'entropie du cœur est non bornée dans chaque voisinage de* λ_0.

Remerciements

Foremost, I want to thank to my PhD advisor Dierk Schleicher for his colossal support and for his ambition to make the mathematical community more successful by success of his PhD students. I appreciate that freedom I had during PhD studies and all opportunities to cooperate with other mathematicians. I am grateful to Kostiantyn Drach, who showed me the beauty of complex dynamics and motivated to explore this field, who helped me to pass over many personal and scientific obstacles past years. Many thanks to Bernhard Reinke for his incredibly useful comments and observations during our detailed discussions, for improving my mathematical intuition. Also I want to thank to Dzmitry Dudko for helpful conceptual discussions at the starting period of my PhD project.

Many thanks to members of the extended working group of Dierk Schleicher, in particular to Sergey Shemyakov, Konstantin Bogdanov, Mikhail Hlushchanka and David Pfrang. Interesting conversations and common brainstorming of different problems on retreats and in other meetings improved my mathematical background.

Furthermore, I want to thank to people in Aix-Marseille Université, in particular to my follow-up committee Pierre Arnoux and Laurent Regnier and also to Thierry Coulbois, Axel Gastaldi, Pierre Guillon and Peter Haïssinsky.

I am thankful to the academic and organizing committee of the conference ABCD (Advancing Bridges in Complex Dynamics) which took place at CIRM (Marseille) for letting me present the results of my PhD project.

This thesis was supported by the Advanced Grant HOLOGRAM by the European Research Council.

Last but not least, I want to thank my family, my friends and my love Ira for their support, kindness, inspiration and motivation during all these years.

Contents

Affidavit 2

Liste de publications et participation aux conférences 3

Résumé 4

Abstract 5

Résumé substanciel 6

Remerciements 8

Contents 9

1 Introduction 11

2 Combinatorial Continuity of Core Entropy for Cosine Maps with Uniformly Bounded Combinatorics 14

- 2.1 Introduction and statements of the result . . . 14
- 2.2 Combinatorics of the cosine map . . . 15
  - 2.2.1 Static partition for complex cosine and external addresses . 15
  - 2.2.2 Abstract bounded external addresses . . . 17
  - 2.2.3 Topology and order in spaces of uniformly bounded external addresses . . . 17
  - 2.2.4 Dynamical partition and itineraries . . . 19
  - 2.2.5 From external addresses to kneading sequences . . . 19
- 2.3 Core entropy for post-critically finite cosine . . . 24
  - 2.3.1 Topological entropy . . . 24
  - 2.3.2 (Homotopy) Hubbard trees for post-critically finite cosine . . 25
  - 2.3.3 Definitions of core entropy for pcf maps . . . 26
- 2.4 Graphs with bounded cycles . . . 28
  - 2.4.1 Preliminaries from graph theory . . . 29
  - 2.4.2 Growth rate and spectral determinant . . . 29
  - 2.4.3 Weak covers of graphs . . . 30
- 2.5 Wedges and their properties . . . 30
  - 2.5.1 Wedge as an object encoding combinatorics of a map . . . 30
  - 2.5.2 Definition of a wedge . . . 31

2.5.3 Growth rate on the space of labeled wedges 34
2.5.4 Periodic wedges and their finite models 37
2.6 Continuity of core entropy on spaces of uniformly bounded external addresses 40
2.6.1 Relation between external addresses and wedges 40
2.6.2 Core entropy on $\mathcal{S}_B$ 40

3 Local Unboundedness of Core Entropy at Escaping Parameters 44
3.1 Introduction and statement of the result 44
3.2 Growing entropy 45
3.2.1 Existence of parameters 45
3.2.2 Estimate on core entropy 50

4 Outlook 53
4.1 Questions related to Chapter 2 53
4.1.1 Relation to complex parameter space 53
4.1.2 Hölder-continuity or Universal space is not metrizable 53
4.1.3 Full cosine family 54
4.1.4 Other families of entire transcendental maps 54
4.2 Questions related to Chapter 3 54
4.2.1 Other (periodic) parameter rays 54
4.2.2 Smaller potentials 54
4.2.3 Discontinuity of core entropy on the bifurcation locus 54

1 Introduction

The topological entropy of a dynamical system shows how quickly orbits of different points diverge. In [MP77] Misiurewicz and Przytycki showed that for smooth self-maps of topological degree d on compact manifolds topological entropy is at least $\log d$. Later Lyubich in [L83] proved that for rational functions of degree d on the Riemann sphere the topological entropy is equal to $\log d$. For transcendental entire functions the topological entropy is always turns out to be infinite, as was observed independently in [Wen05] by Wendt and [BFP] by Benini, Fornæss and Peters.

The results above show that topological entropy measured on the Riemann sphere cannot distinguish *holomorphic* maps of given degree, but another concept can be helpful, the concept of core entropy. This notion was introduced by William Thurston motivated by the theory of interval dynamics. For example, for a real quadratic polynomial the topological entropy restricted to the interval containing the critical orbit is not necessarily equal to $\log 2$. The idea of restriction of the topological entropy to an interesting set from dynamical point of view is the concept of core entropy. For post-critically finite complex polynomials usually the Hubbard tree is considered as the "core". Core entropy for quadratic polynomials was studied independently in [Ti1] by Tiozzo and in [DS] by Dudko and Schleicher. It was proved that core entropy can be extended from the set of post-critically finite quadratic polynomials to a continuous function on the parameter space. In [GT] Gao and Tiozzo obtained a similar result for complex polynomials of higher degree.

Invention of homotopy Hubbard trees for transcendental entire functions by Pfrang in [Pfr1] gave a motivation to study the notion of core entropy for transcendental entire maps. The first essential steps were done by Haßler in [H1] for exponential functions. Complex exponential maps can be considered as a transcendental generalization of unicritical polynomials. For example, same as for unicritical polynomials the core entropy of exponential maps is at most $\log 2$.

Although the Hubbard tree is important for understanding the concept of core entropy, it was observed that core entropy can be computed purely combinatorially, without actual using the Hubbard tree (see [MeSch], [G1], [H1] or [Ju] for details).

In this thesis we consider another transcendental family, which is in some sense opposite to the exponential family. It is the one-dimensional cosine family

$$\left\{ g_\lambda(z) = \lambda \frac{e^z + e^{-z}}{2} = \lambda \cosh(z) = \lambda \cos(iz) \mid \lambda \in \mathbb{C}^* \right\}$$

Maps in this family have infinitely many critical points of degree 2 (instead of one "critical" point of infinite degree for exponentials). All critical points are mapped to two critical values $\pm\lambda$ and in one more iteration the latter are mapped to $\cosh\lambda$, so essentially we have one critical orbit. The entire maps $\lambda\cosh(z)$ and $\mu\cos(w)$ are conjugated taking $w = iz$ and $\mu = i\lambda$. We use cosh only for convenience and for underlining analogies with exponential maps (for example, escaping points under iterations of exp and cosh tend to infinity horizontally, while for cos they escape vertically).

Chapter 2 deals with the definition of core entropy for cosine maps, bounds on core entropy and continuity of core entropy.

We define core entropy for cosine maps with non-escaping critical values. As we want to do this combinatorially, we use the symbolic space of uniformly bounded external addresses $\mathcal{S}_B = \{(s_0 s_1 s_2 s_3 \dots) \mid s_j \in B\}$, where B is a bounded alphabet (see Section 2.2 for precise definitions). In Section 2.3 we define core entropy for parameters corresponding to post-critically finite maps using the intuition for Hubbard trees. Equivalently, we can compute core entropy as the exponential growth rate of the number of precritical points of order n or as the largest eigenvalue of the transition matrix for edges of the Hubbard tree.

In Section 2.6 we extend the definition of core entropy to the whole $\mathcal{S}_B$.

The first result of this thesis can be stated as follows. It contrasts with the standard uniform bound $\log 2$ for exponential functions and unicritical polynomials.

Theorem A (Bounded core entropy for bounded alphabet)**.** *Let $\underline{s} = (s_0 s_1 s_2 \dots) \in \mathcal{S}_B, |s_j| \leqslant b$ be a uniformly bounded external address. Then its core entropy is bounded:*

$$h(\underline{s}) \leqslant \log(8b + 6).$$

We do not claim the latter bound to be best possible, but we expect that any better upper bound has to of the kind $\log(k_1 b + k_2)$. It is an outcome of constructions in Section 2.3 and if we allow unbounded combinatorics, then there exist sequences with unbounded core entropy.

The central result of Chapter 2 is the following.

Theorem B (Continuity of core entropy along uniformly bounded addresses)**.** *Core entropy is continuous on the space of uniformly bounded external addresses $\mathcal{S}_B$ for each bounded alphabet B.*

Chapter 3 rises the question of relation between the previous results and the global complex parameter space. One can observe that near escaping parameters the combinatorics cannot be uniformly bounded and this allows to prove local unboundedness of core entropy. We do this for parameters escaping on the real ray $\mathbb{R}^+$.

Theorem C (Locally unbounded core entropy at escaping parameters)**.** *Let $\lambda_0 \in \mathbb{R}^+$ such that $\lambda_0 \geqslant 1$. Then there exists a sequence $\lambda_n \in \mathbb{C}$ with $\lambda_n \to \lambda_0$ such that* *the core entropy of* $g_n(z) = \lambda_n \cosh(z)$ *tends to infinity:*

$$h(g_n) \to \infty.$$

The cosine family thus differs from all families previously investigated in the sense that core entropy is not bounded even locally.

2 Combinatorial Continuity of Core Entropy for Cosine Maps with Uniformly Bounded Combinatorics

2.1 Introduction and statements of the result

Two main topics of this chapter are: "for maps with bounded combinatorics core entropy is bounded" and "on a space of uniformly bounded combinatorics core entropy is continuous". Section 2.2 explains what we mean by maps with bounded combinatorics and what combinatorial spaces we consider. Section 2.3 is focused on describing definitions of core entropy.

Theorem A (Bounded core entropy for bounded alphabet)**.** *Let* $\underline{s} = (s_0 s_1 s_2 \dots)$ *with* $|s_j| \leqslant b$ *be a uniformly bounded external address. Then its core entropy is bounded:*

$$h(\underline{s}) \leqslant \log(8b+6)$$

The main ingredient of the proof is to show that the number of precritical points of generation $\leqslant n$ cannot grow faster than a multiple of $(8b+6)^n$. We show this for post-critically finite maps in Theorem 2.3.10 and then extend to other parameters using continuity (which is the second main result). One can observe that it can be done directly without using continuity.

Theorem B (Continuity of core entropy along bounded addresses)**.** *Core entropy is continuous on the space of uniformly bounded external addresses* $\mathcal{S}_B$.

In order to prove Theorem B, we need the three main steps:

Claim 1: the growth rate is continuous on the space of labeled wedges.

Claim 2: for post-critically finite maps the definition of core entropy (in terms of the transition matrix) coincides with the log of the growth rate of the corresponding wedge $h(g_\lambda) = \log r(W)$.

Claim 3: even if the correspondence from labeled wedges to uniformly bounded external addresses can be discontinuous at purely periodic addresses, core entropy is continuous on $\mathcal{S}_B$.

Following [Ti1], we introduce labeled wedges for cosine maps to encode the combinatorics of a map. We do this in Section 2.5. Claim 1 is Theorem 2.5.5. For

Claim 2 we use the notion of periodic labeled wedges and the claim is proved in Theorem 2.5.8.

As one of the last steps, in Lemma 2.6.2 we show that the correspondence from labeled wedges to external addresses is continuous unless the address is purely periodic (in sense of Definition 2.2.7). The latter case is covered by Lemma 2.6.4, which proves Claim 3.

2.2 Combinatorics of the cosine map

2.2.1 Static partition for complex cosine and external addresses

The cosine family $\{g_\lambda = \lambda(e^z + e^{-z})/2\colon \lambda \in \mathbb{C}^*\}$ lies in the class of Eremenko-Lyubich functions, i.e. functions with bounded set of singular values. It means that one can apply a standard construction for a static partition, fundamental domains and external addresses for escaping set.

We can use even more explicit construction following [RoS] or [Sch1] and obtain a partition for the whole plane.

Every map g_λ has the set of critical points $C_{crit} = \{i\pi n, n \in \mathbb{Z}\}$; the critical values are $v_1 = \lambda$ and $v_2 = -\lambda$.

Introduce a set $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A} \coloneqq &\{z \in \mathbb{C}\colon z = tv_1 + (1-t)v_2; t \in [0,1]\}\\ &\cup \{z \in \mathbb{C}\colon \operatorname{Re}(z) = \operatorname{Re}(v_2), \operatorname{Im}(z) \geqslant \operatorname{Im}(v_2)\}\,;\end{aligned}$$

Define the following sets of indices

$$\mathbb{Z}_R \coloneqq \mathbb{Z} \times \{R\}, \qquad \mathbb{Z}_L \coloneqq \mathbb{Z} \times \{L\}, \qquad \mathbb{Z}_S \coloneqq \mathbb{Z} \times \{R, L\}. \tag{2.2.1}$$

For $k \in \mathbb{Z}$ by k_R or k_L we denote (k, R) or (k, L) respectively.

Consider connected components R_j of $g_\lambda^{-1}(\mathbb{C}\backslash\mathcal{A})$, so that

$$g_\lambda\colon R_j \to \mathbb{C}\backslash\mathcal{A}$$

is a conformal isomorphism. These are half-strips of height 2π (Figure 2.1) and we use the following rule for indices: $R_{(0,R)}$ and $R_{(0,L)}$ are the strips containing 0 on its boundary.

There is an obvious discontinuity when λ crosses vertical ray $i\mathbb{R}^+$, because then 0 lies on the boundary of four partition sectors. We can fix the convention for symbols of partition sectors as on Figure 2.2.

Definition 2.2.1 (Static partition)**.** A partition of the plane obtained from $R_j, j \in \mathbb{Z}_S$ (described above) is called a *static partition* for g_λ.

This partition allows us to introduce a symbolic dynamics: for every $z \in \mathbb{C}$ with

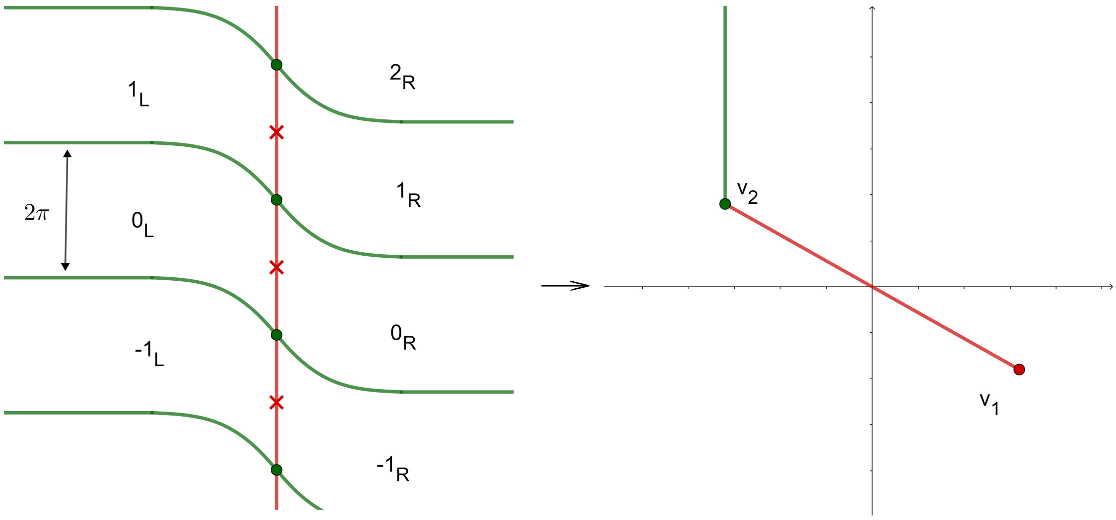

Figure 2.1 – Static partition for g_λ

$g_\lambda^{\circ n}(z) \in \mathbb{C}\backslash\mathcal{A}$ for all $n \in \mathbb{N}$ the *external address* $S(z)$ is the sequence of symbols of the strips containing $z, g_\lambda(z), g_\lambda^{\circ 2}(z) \ldots$.

Remark. In fact, as $\lambda \in \mathbb{C}^*$ one can consider the universal cover of this parameter space $\lambda = \exp\zeta$ with $\zeta \in \mathbb{C}$. In this case there is no discontinuities if we move ζ, but corresponding labels j_L and j_R in the right and in the left half-planes can move far from each other. This corresponds to the way how we introduce abstract external addresses, because we allow the external address of a critical ray landing at zero to have the first symbol not necessarily 0_R or 0_L (details can be seen in the next subsections).

We shall note two important types of symmetries for our family: $2\pi i$-translation symmetry and central symmetry at the origin. We can see an impact of those immediately by looking at the external address of λ (our first critical value). If the address of v_1 is $s_1 s_2 \ldots$, then the address of the second critical value v_2 is $(-s_1)s_2 \ldots$. Here and later by $(-s_1)$ for $s_1 \in \mathbb{Z}_S$ we denote its "opposite" image (for example, the opposite to 1_L is $(-1)_R$, the opposite to 0_R is 0_L and so on).

Every external address can be associated with a dynamic ray [RoS](Theorem 4.1) and they all are cyclically ordered in a neighborhood of ∞. We will formulate this order as a lexicographical order on sequences and use this order in further definitions.

In the next subsection we introduce external addresses purely combinatorially, i.e. as elements of abstract symbolic space, but it will be useful to remember their relation to complex maps.

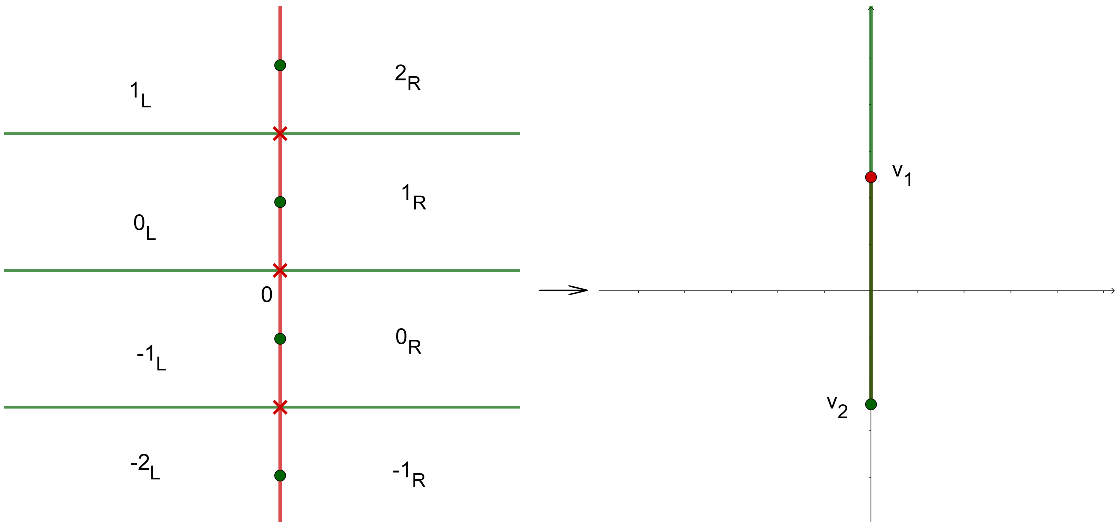

Figure 2.2 – One of possible conventions when $\lambda \in i\mathbb{R}^+$

2.2.2 Abstract bounded external addresses

Definition 2.2.2 (External address)**.** Let $\mathcal{S} := \mathbb{Z}_S^{\mathbb{N}\cup\{0\}} = \{(s_0 s_1 s_2 s_3 \dots)\colon s_k \in \mathbb{Z}_S\}$ be the sequence space over $\mathbb{Z}_S := \mathbb{Z} \times \{R, L\}$ and let $\sigma\colon \mathcal{S} \to \mathcal{S}, (s_0 s_1 s_2 s_3 \dots) \mapsto (s_1 s_2 s_3 \dots)$ be the shift on $\mathcal{S}$. An element $\underline{s} = (s_0 s_1 s_2 s_3 \dots) \in \mathcal{S}$ we call an *external address*.

There is a natural norm on $\mathbb{Z}_S$ induced from $\mathbb{Z}$ by forgetting the letter label, for $j_R, j'_L \in \mathbb{Z}_S$ we have $|j_R| = |j|$ and $|j'_L| = |j'|$.

Definition 2.2.3 (Bounded external address)**.** Let $\underline{s} = (s_1 s_2 s_3 \dots) \in \mathcal{S}$ be an external address. If there is a finite set $B \subset \mathbb{Z}_S$ such that $s_j \in B$ for all $j \in \mathbb{N}$ then we call $\underline{s}$ a *bounded* external address over alphabet B (or B-bounded for shortness). The set of B-bounded addresses we denote as $\mathcal{S}_B$.

In this thesis we will use the alphabet $B = B(b)$ containing *all* symbols whose norm is bounded by some $b \in \mathbb{N}$.

$$B := B(b) = \{t \in \mathbb{Z}_S\colon |t| \leqslant b\}.$$

So, if we write $\underline{s} = (s_0 s_1 s_2 \dots) \in \mathcal{S}_B$ it means that $|s_k| \leqslant b$ for all k.

2.2.3 Topology and order in spaces of uniformly bounded external addresses

The space of B-bounded addresses $\mathcal{S}_B \simeq B^{\mathbb{N}}$ for every finite b is considered with the product topology. This topology is generated by cylinder sets $p_n^{-1}(k) := \{s_1 \dots s_{n-1} k s_{n+1} \dots \mid n \in \mathbb{N}; s_j, k \in B\}$.

Convergence of a sequence in $\mathcal{S}_B$ can be formulated in terms of pointwise convergence: $\underline{s}^{(n)} \to \underline{s}$ if for every n there is $m_0(n)$ so that for all $m > m_0$ we have $s_n^{(m)} = s_n$.

Remark. As we do not fix a particular b in our statements about bounded external addresses, one can consider the universal space $\bigcup \mathcal{S}_B$ when $b \to \infty$. The question of the topology on this set is crucial. If one wants to extend the continuity results from Section 2.6, then this space should be considered with the *final* (it is also called the *inductive*) topology: the finest topology, such that the inclusion maps are continuous. A sequence of addresses is convergent there if there exists an alphabet B such that it is convergent in $\mathcal{S}_B$. Also, one can try to induce the topology the complex plane to this set (or, more precisely, to the subset of realizable addresses). This is a very interesting question on its own. Related to our topic, it leads to unboundedness of the core entropy function. We will discuss this in Chapter 3.

There is a natural lexicographic order on $\mathcal{S}$ (and on $\mathcal{S}_B$), which we can induce from the order on $\mathbb{Z}_S$. The order on $\mathbb{Z}_S$ we define as follows. Denote by $\leqslant_{\mathbb{Z}}$ the standard order for integers (it is an order on each $\mathbb{Z}_L$ and $\mathbb{Z}_R$). For a pair $j_1, j_2 \in \mathbb{Z}_S$ we say $j_1 \leqslant j_2$ if either $j_1, j_2 \in \mathbb{Z}_R$ and $j_1 \leqslant_{\mathbb{Z}} j_2$ or $j_1, j_2 \in \mathbb{Z}_L$ and $j_1 \geqslant j_2$ or $j_1 \in \mathbb{Z}_L$ and $j_2 \in \mathbb{Z}_R$ (see Figure 2.3). We say $j_1 > j_2$ otherwise. In other words, we have just induced the cyclic order near infinity for strips R_j (R_j are defined in subsection 2.2.1), taking the ray $i\mathbb{R}^+$ as a start :

$$\cdots < 2_L < 1_L < 0_L < -1_L < \cdots < -1_R < 0_R < 1_R < 2_R < \cdots .$$

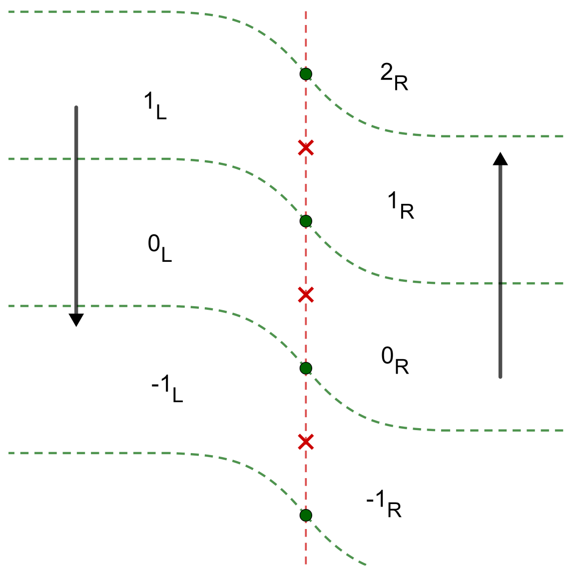

Figure 2.3 – Visualization for the order on the space of external addresses

2.2.4 Dynamical partition and itineraries

For some maps in the family $\{g_\lambda = \lambda(e^z + e^{-z})/2 \colon \lambda \in \mathbb{C}^*\}$ we can introduce a *dynamical partition* using dynamic rays. We want to use a similar construction in our combinatorial setting. For this we want to explain the dynamical picture and use results from [Sch1]. We should underline, that in this chapter we prove continuity results only in *combinatorial* parameter space and many known facts about holomorphic dynamics are used mostly for motivation. Same for this subsection, we want to use some facts about landing behavior of dynamic rays for cosine complex maps. This we do in order to motivate our work with external addresses as with addresses of landing rays. Here we do not claim landing properties for actual holomorphic maps, but only recall some known facts.

For simplicity we assume a map g_λ to be post-critically preperiodic. Then for each critical value v_1, v_2 there exists at least one dynamic ray r_i landing at $v_i, i = 1, 2$. Lift r_1 and r_2 under g_λ and obtain pairs of rays landing at all critical points. We call

$$\mathcal{D} := g_\lambda^{-1}\left(\mathbb{C}\backslash(r_1 \cup r_2)\right)$$

a dynamical partition for g_λ. Each connected component of $\mathcal{D}$ (homeomorphic to a strip in $\mathbb{C}$) we call a *partition sector*. A partition sector lying between critical points $i\pi(j-1)$ and $i\pi j$ we denote by D_j (or simply by the corresponding index j).

For a point $z \in \mathbb{C}$ with $g_\lambda^{\circ n-1}(z) \in \mathcal{D}$ for all $n \in \mathbb{N}$ the sequence of sectors containing $z, g_\lambda(z), g_\lambda^{\circ 2}(z) \ldots$ is called the *itinerary* of z with respect to $\mathcal{D}$:

$$\operatorname{It}(z \mid \mathcal{D})\,.$$

The itinerary of $v_1 = \lambda$ we call the *kneading sequence* of g_λ, $\operatorname{It}(v_1 \mid \mathcal{D})$.

Remark. One can extend the definition of itineraries to a broader set, to points being mapped eventually to the boundary of partition sectors. The details can be seen e.g. in [Pfr1]. We will use only boundary symbols for critical points, for a point $i\pi j$ we use $*_j$.

Remark. A construction of a dynamical partition can be made in a much broader family of entire maps, the details are in [MB].

2.2.5 From external addresses to kneading sequences

Now we want to relate static and dynamical partitions with each other. We do this combinatorially, i.e. in terms of sequences (external addresses and itineraries), but the intuition we use is from complex maps.

For post-critically finite quadratic polynomials the whole combinatorics of a map can be reconstructed from a ray landing at the critical value. Similar to this case, we want to consider each external address in $\underline{s} = (s_0 s_1 s_2 s_3 \ldots) \in \mathcal{S}_B$ as the address of a ray. Unlike for polynomials, we have many critical points and thus we prefer to interpret this address as the address of the ray landing at the critical point 0.

According to this encoding, the address of the first critical value v_1 then should be equal to $\sigma(\underline{s}) = (s_1 s_2 s_3 \dots)$, the address of the second critical value v_2 is equal to $((-s_1)s_2 s_3 \dots)$. Recall, by writing $(-s_1)$ we mean the opposite symbol of s_1 obtained by adding negative sign to the integer part and switching R to L or L to R with the letter part.

We can introduce addresses of (combinatorial) rays landing at all even critical points

$$C_{even} \coloneqq \{j\underline{s} = js_1s_2s_3\dots, j \in \mathbb{Z}_S\};$$

and rays landing at odd critical points

$$C_{odd} \coloneqq \{j\underline{s}' = j(-s_1)s_2s_3\dots, j \in \mathbb{Z}_S.\}$$

Their union, the set of all rays landing at all critical points (we call them *critical rays* for short), is linearly ordered (as a subset of linearly ordered $\mathcal{S}_B$).

For a given address $\underline{s} \in \mathcal{S}_B$ we want to group critical rays into pairs and define a dynamical partition of the space of external addresses.

If $(s_0s_1s_2s_3\dots)$ lands at 0 from the right with $s_0 = l_R$ then $(-l)_L s_1s_2s_3\dots$ lands at zero from the left. Due to $2\pi i$-translation invariance we have $(l+1)_R s_1s_2s_3\dots$ and $(-l+1)_L s_1s_2s_3\dots$ land at critical point $2\pi i$. Similarly, by induction, we get $(l+k)_R s_1s_2s_3\dots$ and $(-l+k)_L s_1s_2s_3\dots$ land at $2\pi ik, k \in \mathbb{Z}$.

In order to simplify the notations, a critical ray $(j_R s_1s_2s_3\dots)$ corresponding to critical point $2\pi ik$ we denote by $r(2k, R)$. We observed that k and j_R are related via $s_0 = l_R$, and for cases when it matters, we write $r(2k, j_R)$, where $j_R = (k+l)_R$. We also use similar notation for left critical rays $r(2k, L)$ or $r(2k, j'_L)$, where $j'_L = (k-l)_L$.

We group even critical rays into pairs of the kind $r(2k, R)$ and $r(2k, L)$ or more explicitly:

$$r(2k, (l+k)_R) \text{ and } r(2k, (-l+k)_L) \tag{2.2.2}$$

In order to recover ray pairs for odd critical points we use the fact that the set of rays $C_{even} \cup C_{odd}$ is linearly ordered. Moreover, for two consecutive rays $r(2k, j_R)$ and $r(2k+2, (j+1)_R)$ from C_{even} there is exactly one ray $j'_R(-s_1)s_2s_3\dots$ from C_{odd} lying between them (j'_R is either equal to $(j+1)_R$ if $s_1 > (-s_1)$ and to j_R if $s_1 < (-s_1)$). This unique ray we call the ray landing at $(2k+1)\pi i$ from right and use the notation $r(2k+1, j'_R)$ or $r(2k+1, R)$. The pairs of critical rays corresponding to $2k+1$ are formed by:

$$\begin{gathered} r(2k+1, (l+k+1)_R) \text{ and } r(2k+1, (-l+k)_L), \text{ if } s_1 > (-s_1); \\ r(2k+1, (l+k)_R) \text{ and } r(2k+1, (-l+k+1)_L), \text{ if } s_1 < (-s_1). \end{gathered} \tag{2.2.3}$$

For the case if $s_0 = l'_L$ we can introduce same notations with a substitution

$(-l)_R$ by l'_L.

Now we are ready to introduce the dynamical partition induced by the address $\underline{s} \in \mathcal{S}_B$ for the space of external addresses.

Definition 2.2.4 (Dynamical partition)**.** Let $\underline{s} \in \mathcal{S}_B$ be a bounded external address and $r(k,R), r(k,L)$ with $k \in \mathbb{Z}$ be addresses of critical rays described above. For $j \in \mathbb{Z}$ we define a *partition sector* D_j as follows:

$$D_j(\underline{s}) \coloneqq \{\underline{t} \in \mathcal{S} \mid r(j-1,R) < \underline{t} < r(j,R)\} \cup \{\underline{t} \in \mathcal{S} \mid r(j,L) < \underline{t} < r(j-1,L)\}$$

The union of all partition sectors is called the *dynamical partition* of $\mathcal{S}$ with respect to $\underline{s}$ and is denoted by $\mathcal{D}(\underline{s})$.

Symbolic sequences with respect to a dynamical partition are called itineraries.

Definition 2.2.5 (Itinerary)**.** For a given external address $\underline{t} \in \mathcal{S}$ such that $\sigma^n(\underline{t}) \in \mathcal{D}(\underline{s})$ for all $n \geqslant 0$ we can denote the *itinerary* of $\underline{t}$ with respect to $\underline{s}$

$$\operatorname{It}(\underline{t} \mid \underline{s}) = (u_0 u_1 u_2 u_3 \dots), u_k \in \mathbb{Z},$$

where u_k is the index of the partition sector D_{u_k} containing $\sigma^{\circ k}(\underline{t})$.

If two addresses $\underline{t}^{(1)}, \underline{t}^{(2)} \in \mathcal{S}$ lie in different partition sectors, then we call them separated. This is related to the fact that there is at least one critical ray pair lying "between" them. This can be formulated in terms of the order on $\mathcal{S}$.

Definition 2.2.6 (Combinatorial separation)**.** For given two addresses $\underline{t}^{(1)}, \underline{t}^{(2)} \in \mathcal{S}$ with $\underline{t}^{(1)} \leqslant \underline{t}^{(2)}$ we say that a pair $r(m,R), r(m,L)$ *separates* $\underline{t}^{(1)}$ and $\underline{t}^{(2)}$ if either

$$\underline{t}^{(1)} < r(m,L) < \underline{t}^{(2)} < r(m,R)$$

or

$$r(m,L) < \underline{t}^{(1)} < r(m,R) < \underline{t}^{(2)}.$$

The definition is symmetric, if $\underline{t}^{(1)} \geqslant \underline{t}^{(2)}$, then we can write same inequalities for pair $\underline{t}^{(2)}, \underline{t}^{(1)}$.

We can consider all critical ray pairs separating two given points, it will be important for the notion of labeled wedges, details are in the section 2.6.

Similarly as for polynomials, the itinerary of the critical value w.r.t. the dynamical partition we call the *kneading sequence* $\nu(\underline{s})$:

$$\nu(\underline{s}) = \operatorname{It}(\sigma(\underline{s}) \mid \underline{s}).$$

The kneading sequence for B-bounded external address is well-defined unless it is purely periodic.

Definition 2.2.7 (Purely periodic external addresses)**.** A B-bounded external address is called *purely periodic* of period p if there exists $p \in \mathbb{N}$ and a critical ray $r \in C_{even} \cup C_{odd}$ so that $\sigma^p(\underline{s}) = r$.

Remark. The meaning of purely periodic external addresses is that the critical orbit eventually hits a critical point (it cannot points of a ray tail because the latter consists of escaping points). Of course, this subfamily of bounded addresses contains addresses with the property $\sigma^p(\underline{s}) = \underline{s}$ (they correspond to the parameters when the critical orbit eventually hits 0), but we want to consider the full subfamily, because tracking exactly zero ray was a convention, not necessity.

We want to define the kneading sequence $(u_1, u_2, u_3, \dots)$ for a purely periodic address. Symbols for the kneading sequence are defined for all n being non-divisible by p. It remains to define u_{pm}, when there is $r(k, R)$ such that $\sigma^{pm}(\underline{s}) = r(k, R)$. In this situation we use a special symbol $*_k$ mentioned before on p. 19, therefore the kneading sequences for purely periodic addresses are defined as well.

The union of $\mathbb{Z}$ with all boundary symbols $\{*_k, k \in \mathbb{Z}\}$ is called the *extended alphabet* $\overline{\mathbb{Z}}$. We consider a natural order $\leqslant_{\overline{\mathbb{Z}}}$ on $\overline{\mathbb{Z}}$ given by $j <_{\overline{\mathbb{Z}}} *_j <_{\overline{\mathbb{Z}}} j+1$ for all $j \in \mathbb{Z}$. We also consider a norm on $\overline{\mathbb{Z}}$ induced from $\mathbb{Z}$, where the norm of an integer is equal to its absolute value and the norm of $*_k$ is equal to $|k|$.

In section 2.6 we require the following useful lemma about purely periodic addresses and we want to prove it here.

Lemma 2.2.8. *Let $\underline{s}$ be a purely periodic sequence of period p with $\sigma^{\circ p}(\underline{s}) = r(j, R)$. Assume $\underline{s}^{(n)} \uparrow \underline{s}$ with $s_0^{(n)}, s_0 \in \mathbb{Z}_R$. Then for every $m \in \mathbb{N}$ there exists n_0 so that for all $n \geqslant n_0$ the first symbol of* $\operatorname{It}\left(\sigma^{\circ mp}(\underline{s}^{(n)}) \mid \underline{s}^{(n)}\right)$ *is equal to $j-1$.*

Proof. Denote by $r^{(n)}(i, R), i \in \mathbb{Z}$ corresponding critical rays of $\underline{s}^{(n)}$ from the right. We want to show that eventually we have:

$$r^{(n)}(j-1, R) < \sigma^{\circ mp}(\underline{s}^{(n)}) < r^{(n)}(j, R).$$

Consider two cases when $j = 2k$ and $j = 2k+1$. For certainty, we assume $s_1 > -s_1$ (the symmetric case is similar). Using 2.2.2 and 2.2.3 we can write the following expression for critical rays $2k-1, 2k$ and $2k+1$ of $\underline{s}$:

$$\begin{aligned} r(2k+1, R) &= r(2k+1, (l+k+1)_R) = (l+k+1)_R(-s_1)s_2s_3\dots; \\ r(2k, R) &= r(2k, (l+k)_R) = (l+k)_R s_1 s_2 s_3 \dots; \\ r(2k-1, R) &= r(2k-1, (l+k)_R) = (l+k)_R(-s_1)s_2s_3\dots. \end{aligned}$$

For the case when $j = 2k$ in order to prove that $\sigma^{\circ mp}(\underline{s}^{(n)}) < r^{(n)}(2k, R)$ we should observe that there exists n_0 so that for all $n \geqslant n_0$ at least first $1 + mp$ entries of $\underline{s}^{(n)}$ and $\underline{s}$ coincide. Denote by $t = t(n)$ the place of the first different symbol between $\underline{s}^{(n)}$ and $\underline{s}$ (note that $t > 1 + mp$).

We have that the entries of $\underline{s}^{(n)}$ and $\underline{s}$ between 1 and $t-1$ coincide, thus the entries of $\sigma^{\circ mp}(\underline{s}^{(n)})$ and $\sigma^{\circ mp}(\underline{s}) = r(2k, R)$ between 1 and $t-1-mp$ are same too. As $\underline{s}^{(n)} < \underline{s}$ we have that $s_t^{(n)} < s_t$. Now look at the corresponding $(t-mp)$-th entries of $\sigma^{\circ mp}(\underline{s}^{(n)})$ and $r^{(n)}(2k, R)$. For the first address it is equal to $s_t^{(n)}$, for the second it is equal to $s_{t-mp}^{(n)} = s_{t-mp} = s_t$ (the first equality is because first $t-1$ entries must coincide and the second is because of periodicity of $\underline{s}$), therefore $\sigma^{\circ mp}(\underline{s}^{(n)}) < r^{(n)}(2k, R)$.

The other part of the inequality $r^{(n)}(2k-1, R) < \sigma^{\circ mp}(\underline{s}^{(n)})$ is easier because the second symbol of $r^{(n)}(2k-1, R)$ is equal to $(-s_1)$, while the second symbol of $\sigma^{\circ mp}(\underline{s}^{(n)})$ is equal to $s_{1+mp}^{(n)} = s_{1+mp} = s_1 > (-s_1)$ (we look at the second symbols because the first ones are equal to $(l+k)_R$).

When $j = 2k+1$ we again consider $n \geqslant n_0$ so that at least $mp+1$ symbols coincide and the first different symbol is on t-th position. The inequality $r^{(n)}(2k, R) < \sigma^{\circ mp}(\underline{s}^{(n)})$ holds since the first symbol of $r^{(n)}(2k, R)$ is equal to $(l+k)_R$, while for $\sigma^{\circ mp}(\underline{s}^{(n)})$ it is equal to $s_{mp}^{(n)} = s_{mp} = s_0 = (l+r+1)_R$.

The inequality $\sigma^{\circ mp}(\underline{s}^{(n)}) < r^{(n)}(2k+1, R)$ holds if we note that first $t-1-mp$ symbols are equal and the symbols on $(t-mp)$-th position are different and equal to $s_t^{(n)}$ and $s_{t-mp}^{(n)}$ respectively. We have $s_t^{(n)} < s_{t-mp}^{(n)}$ as in the previous case.

Therefore we obtain for $j = 2k$ the address of $\sigma^{\circ mp}(\underline{s}^{(n)})$ lies between $r(2k-1, R)$ and $r(2k, R)$; also for $j = 2k+1$ the address of $\sigma^{\circ mp}(\underline{s}^{(n)})$ lies between $r(2k, R)$ and $r(2k+1, R)$, so the first symbol of the itinerary is equal to $j-1$.

$\square$

We can also formulate the same lemma for the case when $\sigma^{\circ p}(\underline{s}) = r(j, L)$.

Lemma 2.2.9. *Let $\underline{s}$ be a purely periodic sequence of period p with $\sigma^{\circ p}(\underline{s}) = r(j, L)$. Assume $\underline{s}^{(n)} \downarrow \underline{s}$ with $s_0^{(n)}, s_0 \in \mathbb{Z}_L$. Then for every $m \in \mathbb{N}$ there exists n_0 so that for all $n \geqslant n_0$ the first symbol of* $\operatorname{It}\left(\sigma^{\circ mp}(\underline{s}^{(n)}) \mid \underline{s}^{(n)}\right)$ *is equal to $j-1$.*

Proof. We use the symmetry at critical points. The partition sectors $D_{j-1}(\underline{s}^{(n)})$ and $D_{j-1}(\underline{s})$ are contained between critical points $*_{j-1}$ and $*_j$, but in the right-half plane $*_j$ lies in the upper part of the boundary (and $\underline{s}^{(n)}$ from below), while in the left-half plane $*_j$ liest in the bottom part of the boundary. We need to change the sign of inequalities for the proof, but the arguments are same. $\square$

Remark. One can obtain a symmetric lemmas stating that if $\underline{s}^{(n)} \downarrow \underline{s}$ with $s_0^{(n)}, s_0 \in \mathbb{Z}_R$ (or $\underline{s}^{(n)} \uparrow \underline{s}$ with $s_0^{(n)}, s_0 \in \mathbb{Z}_L$), then the first symbol of the itineraries of $\sigma^{\circ mp}(\underline{s}^{(n)})$ eventually becomes to be equal to j.

The following lemma gives relation between alphabets of external addresses and corresponding kneading sequences.

Proposition 2.2.10 (Bounded external address implies bounded kneading sequence)**.** *Let $B = \{t \in \mathbb{Z}_S \colon |t| \leqslant b\}$. If $\underline{s} = (s_0 s_1 s_2 \dots) \in \mathcal{S}_B$ (is an external address over alphabet B) then the corresponding kneading sequence $\nu(\underline{s})$ lies in the space of kneading sequences over extended alphabet $\overline{B} \coloneqq \{a \in \overline{\mathbb{Z}} \colon |a| \leqslant 4b+2\}$.*

Proof. Assume $s_0 = j_R \in \mathbb{Z}_R$ (the other case is symmetric) and let $r(2k, j_R)$ be a ray landing at point $*_{2k}$ from the right. Then, from equality 2.2.2, we have $2k = 2(j-l)$. If the second ray landing from the left is $r(2k, j'_L)$, then $2k = 2(j'+l)$. It means if $j_R, j'_L \in B$ (i.e. $|j| \leqslant b$), then $|2k| \leqslant 4b$.

Similarly for odd critical points, applying equality 2.2.3 for $r(2k+1, j_R)$ we have $2k+1 = 2(j-l)$ (or $2(j+l)+1$) and for $r(2k+1, j'_L)$ we have $2k+1 = 2(j+l)+1$ (or $2(j+l)$) . As previously, case distinction is obtained from the value of s_1. Thus if $j \leqslant b$ we obtain $|2k+1| \leqslant 4b+1$.

Finally, in order to reconstruct the n-th entry of the kneading sequence we have to look at "neighboring" critical rays r_1, r_2 of $\sigma^{\circ n}(\underline{t})$. The first entry of at least one ray must coincide with the first entry of $\sigma^{\circ n}(\underline{t})$, therefore the corresponding partition sector contains critical point $*_m$ on its boundary with $|m| \leqslant 4b+1$. The claim follows from an observation that for each m the critical point $*_m$ lies between sectors m an $m+1$, therefore the norm of the entry is bounded by $4b+2$.

□

2.3 Core entropy for post-critically finite cosine

2.3.1 Topological entropy

Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous function. A subset $E \subset X$ is called (n, ϵ)-separated if for all $x, y \in E, x \neq y$ and for $0 \leqslant j < n$ we have $d\left(f^{\circ j}(x), f^{\circ j}(y)\right) > \epsilon$.

Definition 2.3.1 (Topological entropy)**.** Let $K \subset X$ be a compact subset of X. Denote by $r_n(\epsilon, K)$ the largest cardinality of an (n, ϵ)-separated set of K. The topological entropy of f with respect to K is defined as

$$h_{top}(f, K) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K)$$

The topological entropy of f with respect to the whole space X we denote as $h_{top}(f) \coloneqq h_{top}(f, X)$.

Topological entropy is preserved under topological conjugation.

Remark. One can define the topological entropy in terms of (n, ϵ)-spans, but the definitions are equivalent (details are in e.g. [dMvS]).

There is an equivalent definition for the topological entropy in terms of open covers. Let $\mathcal{U}$ be an open cover of X, denote by $N(\mathcal{U})$ the smallest cardinality of a subcover of $\mathcal{U}$ (it is always finite by compactness of X). If $\mathcal{U}, \mathcal{V}$ are two covers of X, then $\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{U}\}$ their common refinement. Let $\mathcal{U}^n = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \ldots \vee f^{-n}(\mathcal{U})$, where $f^{-k}(\mathcal{U}) = \{f^{-k}(U) \mid U \in \mathcal{U}\}$.

Definition 2.3.2 (Topological entropy, alternative)**.** The topological entropy of f with respect to cover $\mathcal{U}$ is defined as:

$$h(f,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}^n)$$

The topological entropy of f is defined as the supremum over all covers of X:

$$h(f) = \sup_{\mathcal{U}} h(f,\mathcal{U}).$$

The equivalence of the definitions can be obtained taking $\mathcal{U}$ as an open cover with all elements of diameter at most ϵ and Lebesgue number 2δ, because in this case we have:

$$r_n(\epsilon, X) \leqslant N(\mathcal{U}^n) \leqslant r_n(\delta, X).$$

The following lemma states a standard and quite useful equality for the topological entropy, the proof can be found e.g. in ([dMvS], Lemma7.2).

Lemma 2.3.3 (Entropy of iterates)**.** *If $f\colon X \to X$ is a continuous map of a compact metric space X then*

$$h(f^{\circ m}) = m \cdot h(f).$$

In the next subsection we want to work with the topological entropy on a graph (to be more precise, on the Hubbard tree) and for this we introduce some notions and results from [LM93].

Definition 2.3.4 (Horseshoe)**.** Let f be a continuous map of a graph into itself. An interval I and its s subintervals $J_1, \dots, J_s$ with disjoint interiors are called an s-horseshoe for f if every J_j maps onto the whole I. We denote this s-horseshoe by $(I; J_1, \dots, J_s)$.

Theorem 2.3.5 ([LM93], Theorem A)**.** *If a continuous map f of a graph into itself has an s-horseshoe then $h(f) \geqslant \log s$.*

The topological entropy is an important notion in the context of core entropy, because one can define core entropy as a restriction of the topological entropy of a map to its Hubbard tree (the “core” of the map). Hubbard trees for maps in our family are discussed in the next section.

2.3.2 (Homotopy) Hubbard trees for post-critically finite cosine

Hubbard trees is a well-studied object for post-critically finite polynomials. It can be viewed as a minimal compact invariant tree in $\mathbb{C}$ containing all critical orbits (see details e.g. in [BFH] and [Poi1]).

For some transcendental entire maps (for example, post-singularly finite exponential maps) a Hubbard tree fails to be invariant, see the details in [PRS] and it is more natural to consider so-called Homotopy Hubbard Trees, which are introduced in [Pfr1].

Definition 2.3.6 (Homotopy Hubbard Tree)**.** Let f be a post-singularly finite entire function. A *Homotopy Hubbard Tree* for f is a finite embedded tree $H \subset \mathbb{C}$ such that:

— the post-singular set $P(f)$ is contained in H and all endpoints of H are contained in $P(f)$;

— H is forward invariant up to homotopy relative to $P(f)$;

— the induced self-map of H is expansive.

The following theorem states existence and uniqueness of the Homotopy Hubbard Tree for a given psf-map.

Theorem 2.3.7 ([Pfr1], Theorem B)**.** *Let f be a post-singularly finite entire function. Then f has a Homotopy Hubbard Tree and this tree is unique up to homotopy relative to the post-singular set $P(f)$.*

In [Pfr1] it was also observed that a logarithmic singularity (i.e. existence of finite asymptotic values) is the only obstacle for a Homotopy Hubbard Tree to be actually invariant (i.e. to be the Hubbard Tree). As cosine maps do not have a logarithmic singularity, we expect for post-singularly finite cosine maps to have actual Hubbard Trees. In the thesis the actual invariance is not an issue, because notions in the next subsection work even for the case of trees defined up to homotopy relative to the post-critical set.

2.3.3 Definitions of core entropy for pcf maps

In this section we want to define core entropy for post-critically finite cosine maps.

Let $g_\lambda = \lambda(e^z + e^{-z})/2$ be a pcf cosine map and H be its (Homotopy) Hubbard Tree.

Definition A (On the (Homotopy) Hubbard Tree)**.** Core entropy of g_λ is defined as the topological entropy with respect to the (Homotopy) Hubbard tree:

$$h(g_\lambda) = h_{top}(g_\lambda, H)$$

Note that even for the case of homotopy trees, the definition does not depend on a particular tree in the class, because of the following Lemma from [Pfr1].

Lemma 2.3.8 ([Pfr1], Lemma 4.13)**.** *Let $F, \tilde{F}\colon H \to H$ be induced self-maps of H. There exists a homeomorphism $\theta\colon H \to H$ that restricts to a conjugation of F and $\tilde{F}$ on the set of marked points. More precisely, $\theta\,|_{V_F}\colon V_F \to V_{\tilde{F}}$ is a bijection satisfying*

$$\theta^{-1} \circ \tilde{F} \circ \theta(v) = F(v) \text{ for all } v \in V_F.$$

Conjugacy of different induced maps on the set of vertices of the Hubbard tree is sufficient to obtain the same entropy, because on each edge connecting two neighboring vertices the induced map is injective.

The next definition is partially related to the result of [MS] stating that for $f\colon I \to I$ piecewise monotone maps of the interval the topological entropy is equal to the exponential growth rate of turning points of $f^{\circ n}$ (which are the points where the function changes its monotone behavior from growth to decay and vise versa). If we interpret turning points of $f^{\circ n}$ as critical points (or precritical for f), then the motivation for the next definition is obvious. Recall, that a point z is precritical of order n if $f^{\circ n}(z)$ is a critical point.

Definition B (Count of precritical points)**.** Core entropy of g_λ is defined as

$$h(g_\lambda) = \limsup_{n\to\infty} \frac{1}{n} \log N(n),$$

where $N(n)$ is the number of precritical points of order $\leqslant n$ on H.

If the interval joining two points on the Hubbard tree contains one precritical point of order n and no other precritical points of smaller order, then each of $n-1$ forward images of the interval is contained in one sector of the dynamical partition and n-th image is mapped into few sectors. This information about separation is preserved for homotopy trees as well.

For the third definition we can construct the transition matrix for the edges of H in the following way. The set of edges of H is finite, thus by numbering the edges, the map f on H can be described as a matrix A with entries 0 or 1, such that the j-th column is showing where j-th edge is mapped.

Definition C (Growth rate of transition matrix)**.** Let A be the transition matrix for the edges of H. Then core entropy of g_λ is defined as

$$h(g_\lambda) = \log \xi,$$

where ξ is the largest eigenvalue of A.

Remark. In fact, both definitions B and C can be generalized purely in combinatorial terms (avoiding explicit use of the Hubbard Tree). Definition C is generalized in [G1] for polynomials (not necessarily pcf) using *Thurston's entropy algorithm.* In [H1] there is a similar algorithm for generalization of Definition B for exponential maps.

The following theorem is quite standard in the context of the topological entropy on graphs.

Theorem 2.3.9 (Equivalence of definitions)**.** *Let g_λ be a post-critically finite cosine map. Then the values of core entropy computed with respect to definitions A, B and C are equal.*

Proof. Equivalence of definitions A and C can be obtained from results of [G1]. Post-critical case is covered there, therefore we have that the topological entropy on the Hubbard tree of a polynomial is equal to the output of Thurston's entropy algorithm (for pcf maps it coincides with the largest eigenvalue of the transition matrix). Hubbard trees of postricitally finite cosine maps are also finite and the induced maps are of finite degree.

Equivalence of definitions A and B is a consequence from [A97].

□

The following theorem gives an upper bound for core entropy for pcf-maps. In next sections we will generalize this result for the whole space of uniformly bounded combinatorics $\mathcal{S}_B$.

Theorem 2.3.10 (Bounded core entropy for bounded combinatorics)**.** *Let* $B = \{t \in \mathbb{Z}_S \colon |t| \leqslant b\}$ *and let* g_λ *be a postcritically finite cosine map, so that its corresponding external address* $\underline{s} \in \mathcal{S}_B$*. Then its core entropy is bounded*

$$h(g_\lambda) \leqslant \log(8b + 6).$$

Proof. Proposition 2.2.10 shows an upper bound ($4b$+2) for the number of different symbols of the kneading sequence corresponding to a given external address. This means, that the points of the postcritical set can separated by critical points $i\pi k$ only with $k \leqslant 4b + 2$, thus there are $8b + 5$ such critical points. If we denote the number of edges of H by T, then we can trivially estimate the number of precritical points of order $\leqslant n$:

$$N(n) \leqslant T \cdot (8b + 6)^n.$$

The reason for this bound is inductive: precritical points of order 1 can subdivide T initial points into at most $T(8b + 6)$ intervals, precritical points of order 2 can subdivide $T(8b + 6)$ intervals into at most $T(8b + 6)^2$ intervals and so on. Using Definition B, the statement follows. □

Remark. The bound if the latter theorem is not best possible, but we can observe that any upper bound has to be at least the logarithm of a linear function on b. For example, core entropy of parameters with periodic kneading sequences $(\overline{0*_b})$ is equal to $\log(b - 1)$.

2.4 Graphs with bounded cycles

The following section is a collection of tools required for Section 2.5 about labeled wedges for cosine maps.

2.4.1 Preliminaries from graph theory

In this and in the next section we work with graphs as purely combinatorial objects. By a *graph* G we mean a pair (V, E), where V is a set of *vertices* (finite or countable) and E is a set of pairs of vertices, which called *edges*. As we mostly interested in *directed* graphs, we assume that elements of E are ordered pairs; every edge $e \in E$ has a source $s(e) \in V$ and a target $t(e) \in V$.

A finite sequence of edges $(e_1, e_2, \dots, e_n)$ with $t(e_j) = s(e_{j+1})$ for every j from 1 to $(n-1)$ is called a (finite) *path*. In case if a path has additionally $t(e_n) = s(e_1)$ then we call it *closed*.

A *length* of a path is called its combinatorial length, i.e. the number of edges it consists of.

A closed path is called a *simple cycle* if it has no self-intersections, i.e. the only repeated vertices are the first and the last vertices. A *multi-cycle* is a finite union of simple cycles with disjoint vertex support (i.e. set of visited vertices).

For a vertex $v \in V$ we define its *outgoing degree* as the cardinality of all edges having their source at v; the corresponding set of outgoing edges we denote as $Out(v)$. An outgoing degree for the whole graph is defined as a supremum of outgoing degrees over all vertices.

Definition 2.4.1 (Graph with bounded cycles)**.** A graph G has bounded cycles if it has:
— bounded outgoing degree;
— for every n bounded number of simple cycles of length n.

2.4.2 Growth rate and spectral determinant

Definition 2.4.2 (Growth rate)**.** Let Γ be a graph with bounded cycles, $C(\Gamma, n)$ be the number of closed paths in it. We define the growth rate of Γ as

$$r = \limsup_n (C(\Gamma), n))^{\frac{1}{n}}$$

Also for Γ we denote the *spectral determinant* in the following way:

$$P(t) = \sum_{\gamma \text{ multicycle}} (-1)^{C(\gamma)} t^{l(\gamma)} \tag{2.4.1}$$

where $l(\gamma)$ is the length of the multicycle, $C(\gamma)$ is the number of connected components of γ.

Let $K(\Gamma, n)$ be the number of multicycles of length n in Γ and

$$\sigma(\Gamma) = \limsup_n (K(\Gamma, n))^{\frac{1}{n}}$$

Theorem 2.4.3 ([Ti1], Theorem 2.3). *Suppose $\sigma(\Gamma) \leqslant 1$; then the formula 2.4.1 defines a holomorphic function $P(z)$ in the unit disk $|z| < 1$, and moreover the function P(z) is non-zero in the disk $|z| < r^{-1}$; if $r > 1$, we also have $P(r^{-1}) = 0$.*

Lemma 2.4.4 ([Ti1], Lemma 2.5). *If Γ is a finite graph, then its growth rate equals the largest real eigenvalue of its adjacency matrix.*

2.4.3 Weak covers of graphs

Let Γ_1, Γ_2 be two locally finite graphs. A graph map $\pi\colon \Gamma_1 \to \Gamma_2$ maps vertices to vertices and edges to edges so that if vertices u, v are linked in Γ_1 then $\pi(u), \pi(v)$ are linked in Γ_2.

A graph map π is called a weak cover of graphs if it is surjective on the set of vertices and the induced map $Out(v) \to Out(\pi(vi))$ is bijective for every vertex v.

Lemma 2.4.5 ([Ti1], Lemma 5.3). *Let $\pi\colon \Gamma_1 \to \Gamma_2$ be a weak cover of graphs with bounded cycles, and $S \neq \emptyset$ a finite set of vertices of Γ_1.*

1. *Suppose that every closed path in Γ_1 passes through S. Then for each n we have the estimate*
$$C(\Gamma_1, n) \leqslant n \cdot \#S \cdot C(\Gamma_2, n)$$
which implies
$$r(\Gamma_1) \leqslant r(\Gamma_2)$$
2. *Suppose that G is a set of closed paths in Γ_2 such that each $\gamma \in G$ crosses at least one vertex w with the property that: the set $S_w := \pi^{-1}(w) \cap S$ is non-empty, and any lift of γ from an element of S_w ends in S_w. Then there exists $L \geqslant 0$, which depends on S, such that for each n we have*
$$\#\{\gamma \in G : l(\gamma) = n\} \leqslant n \sum_{k=0}^{L} C(\Gamma_1, n+k).$$

Lemma 2.4.6 ([Ti1], Lemma 5.4). *Let Γ_1, Γ_2 be finite graphs, and $\pi\colon \Gamma_1 \to \Gamma_2$ a weak cover. Then the growth rate of Γ_1 equals the growth rate of Γ_2*

2.5 Wedges and their properties

2.5.1 Wedge as an object encoding combinatorics of a map

A formal definition of a wedge will be given in the next subsection; here we would like to give a conceptual explanation for this object. As it was explained previously in Definition C, for post-critically finite maps core entropy may be computed from the transition matrix: if we write all possible pairs of points of post-critical set and how they map to each other, then the leading eigenvalue of this matrix should give us core entropy. In case if post-critical set is infinite, we cannot directly apply the same idea, but it can be useful to consider all possible pairs of points of $P(g_\lambda)$ and to encode how they map to each other.

Every map $g_\lambda(z) = \lambda \cosh(z)$ has two critical values $\pm\lambda$, but essentially there is only one critical orbit since both critical values jump to $\lambda \cosh(\lambda)$. If there is a Hubbard tree H, we can denote an interval joining $g_\lambda^{\circ i}(0)$ and $g_\lambda^{\circ j}(0)$ in H by an unordered pair of integers (i, j) (here we assume $i, j \geqslant 2$, for the remaining case we use 1^+ to denote λ and 1^- to denote $-\lambda$). As critical points of g_λ are $i\pi k, k \in \mathbb{Z}$ and we want to use positive integers to denote points of the critical orbit, we denote critical points by $*_k$-symbol. For example, a critical point located in -3π we denote as $*_{-3}$.

If an interval (i, j) on H does not contain any critical point, then g_λ maps it to the interval $(i + 1, j + 1)$. In case if this interval contains some critical points, we can track each subinterval separately.

For example, if an interval $(4, 5)$ (i.e. the one joining $g_\lambda^{\circ 4}(0)$ and $g_\lambda^{\circ 5}(0)$) is separated by three critical points, say $*_5, *_6$ and $*_7$, then we have the following:

— first subinterval between points $g_\lambda^{\circ 4}(0)$ and $*_5$ will be mapped to $(1^-, 5)$. It is so, because the forth critical value is mapped to the fifth; critical point $*_5$ (which is odd) is mapped to 1^-. We write $(1^-, 5)$ but not $(5, 1^-)$ because of our convention to use unordered pairs of integers.

— Second subinterval between critical points $*_5$ and $*_6$ is mapped to $(1^-, 1^+)$. Intervals $(1^-, 1^+)$ and $(1^+, 1^-)$ are considered as ordered pairs as only exceptions (it will be important later).

— Third subinterval joining $*_6$ and $*_7$ is mapped to $(1^+, 1^-)$.

— Forth subinterval from $*_7$ to $g_\lambda^{\circ 5}(0)$ is mapped to $(1^-, 6)$.

This example can make a construction of a wedge more natural and more clear.

Note that at this moment we distinguish all iterates of the orbit. In case of (pre)-periodic parameters, we can introduce an equivalence relation on a wedge and obtain so-called (pre)-periodic wedge. Details can be found in subsection 2.5.4.

2.5.2 Definition of a wedge

Denote $\widetilde{\Sigma} = \{(i, j) \in \mathbb{N}^2, 2 \leqslant i < j\}$ the set of ordered pairs and add to this set the pairs of kind $(1^+, j)$ and $(1^-, j)$ for $j \in \mathbb{N}$. The obtained set we call the *wedge for cosine map* Σ (or just the *wedge* for short, Figure 2.4):

$$\Sigma = \widetilde{\Sigma} \cup \{(1^+, j), j \geqslant 2\} \cup \{(1^-, j), j \geqslant 2\} \cup \{(1^+, 1^-)\} \cup \{(1^-, 1^+)\}$$

For a point $v = (i, j)$ on the wedge the entry i is called the *height* of v and j is its *width*. If one of the coordinates is 1^+ or 1^- then the corresponding width or height is assumed to be equal to 1.

Consider $A = \{*_{-s}, *_{(-s+1)}, \ldots, *_{(s-1)}, *_s\}$. We say $*_k$ is even (odd) if k is even

$$
\begin{array}{ccccc}
 & & & & \dots \\
 & & & (3\ \ 4) & \dots \\
 & & (2\ \ 3) & (2\ \ 4) & \dots \\
(1^+\ 1^-) & (1^+\ 2) & (1^+\ 3) & (1^+\ 4) & \dots \\
(1^-\ 1^+) & (1^-\ 2) & (1^-\ 3) & (1^-\ 4) & \dots
\end{array}
$$

Figure 2.4 – Wedge for cosine map

(odd, respectively). A *labeling* of the wedge is an assignment for every vertex $(i, j) \in \Sigma$ a label $\emptyset$ or $(\alpha_1, \dots, \alpha_r) \in A^r$, where α_j are consecutive symbols of A (going in the usual or the opposite direction).

The wedge with a labeling is called a *labeled wedge* (Figure 2.5). A vertex with the label equal to $\emptyset$ is called *non-separated* and otherwise it is called *separated*.

Note that we assume $1^{\pm} + 1 = 2$ in further definitions.

Also note that the vertices $(1^+, 1^-)$ and $(1^-, 1^+)$ are always separated (the corresponding label must contain at least $*_0$). Moreover, their labels contain the same set of points of A, but they are written in opposite direction w.r.t. each other.

Now we can associate a directed graph Γ to a labeled wedge W as follows. The vertices of Γ are the points of Σ and the edges are defined in the following way:

1. if the vertex (i, j) is non-separated then the only outgoing edge is

$$(i, j) \to (i + 1, j + 1)$$

and this edge we call of *upward* type.

2. if the vertex (i, j) is separated with a label $(\alpha_1, \dots, \alpha_r)$ then there are $r + 1$

outgoing edges:

$$\begin{aligned}
&(i, j) \to (1^{\alpha_1}, i+1)\\
&(i, j) \to (1^{\alpha_1}, 1^{\alpha_2})\\
&(i, j) \to (1^{\alpha_2}, 1^{\alpha_3})\\
&\qquad \dots\\
&(i, j) \to (1^{\alpha_r}, j+1)
\end{aligned}$$

where $1^{\alpha_j} = 1^+$ if α_j is even and 1^- if α_j is odd. Edges directing to $\{1^\pm, 1^\mp\}$ are called *central.* Among the edges remained an edge is called *forward* if it increases the width and *backward* if it does not. Note that if $i \geqslant 2$ then we have exactly one backward and one forward edge for a separated vertex.

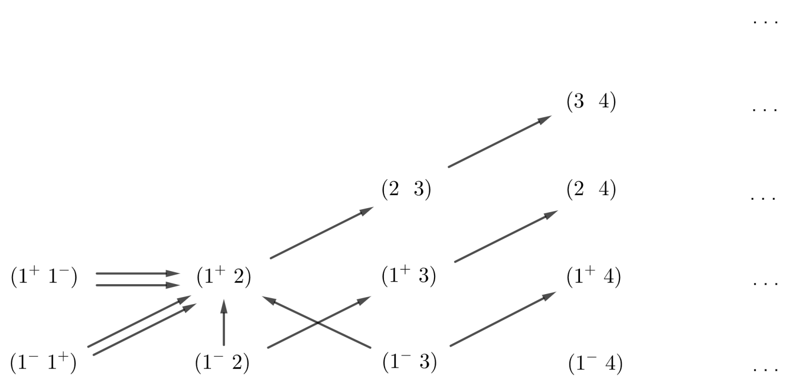

Figure 2.5 – Example of a labeled wedge

Remark. Definitions of a labeled wedge and its associated graph were equivalent for polynomials in [Ti1] and [GT]: given a labeled wedge, we could construct a graph by the procedure above and vise versa, for every vertex of a graph we could recover the corresponding label (for quadratic polynomials point 0 was only critical point; and for polynomials of higher degree the information was encoded by layers of edges). It is not same in our case. For example, if a pair $(3, 5)$ has a label $(*_0, *_{-1})$, then there must be three outgoing edges to $(1^+, 4)$,$(1^+, 1^-)$ and $(1^-, 6)$; but having only three such edges we do not know, which label this should correspond to. This example motivates us to color outgoing edges of a separated vertex if we want to have equivalent objects (but usually we go in one direction: from a labeled wedge to its associated graph).

The topology on the space of labeled wedges is given by the following definition of convergence for convergence.

Definition 2.5.1 (Convergence of wedges)**.** Let (W_n) be a sequence of labeled wedges. We say this sequence converges to a labeled wedge W if for every finite subset of vertices $S \subset \Sigma$ there exists n_0, such that for all $n \geqslant n_0$ the labels of W_n and W coincide on S.

Remark. For a given labeled wedge we always have a bound on the number of outgoing edges for every vertex: $\#A+1$. Later in subsection 2.6.1, when we discuss the construction of the labeled wedge for a given external address $\underline{s} \in \mathcal{S}_B$, number $\#A$ will correspond to the number of symbols of the alphabet for the kneading sequence (being uniformly bounded on $\mathcal{S}_B$ in Proposition 2.2.10).

2.5.3 Growth rate on the space of labeled wedges

We need the following combinatorial statement being similar to [Ti1] Proposition 4.2.

Proposition 2.5.2 (Location of closed paths)**.** *Let Γ be the graph associated to a labeled wedge W. Then the following hold:*

1. *each vertex along any closed path of length n has height at most n;*
2. *each vertex along any closed path of length n has width at most $2n$;*
3. *for every diagonal $D_k^{\pm} = \{1^{\pm}, k+1\} \cup \{(i, i+k) \in \Sigma : i \geqslant 2\}$ there exists at most one separated vertex, such that it is contained in the support of at least one closed path.*

Proof.

1. Upward edges always increase the height of a vertex, thus a closed path must contain at least one edge of other type (backward, forward or central). As the target of this edge has height 1, we obtain there must be at least one vertex of height 1 along the closed path. Since every edge increases the height of at most 1, the claim follows.
2. Forward and upward edges always increase the width of a vertex, along each closed path there must be at least a backward or central edge. By the previous point, the source of such edge has height $\leqslant n$, hence its target has width $\leqslant n+1$. The claim follows by the fact that each remaining $n-1$ edge increases the width by at most 1.
3. If a vertex $v = (i, j)$ of the height $i \geqslant 2$ is the target of some edge, then it must be the upward edge from $v_1 = (i-1, j-1)$, thus v_1 is non-separated. v_1 is also a target of another edge and again it it non-separated and so on. By induction, we obtain all vertices on this diagonal with lower height are non-separated, thus we cannot have other separated points on the closed path.

□

Lemma 2.5.3 (Multicycles are determined by only backward and central edges)**.** *Let γ be a multicycle of finite length on the associated graph Γ of a labeled wedge W; B be the set of backward edges of γ; C be the set of central edges of γ ($0 \leqslant |C| \leqslant 2$) and F be a set of edges following the central ones (if they exists; $|F| = |C|$). Then all remaining edges can be reconstructed from B, C and F.*

Proof. The reconstruction process goes in algorithmic way similarly for all edges of B, C and F.

Let e be an edge from our initial data (i.e. $e \in B \cup C \cup F$). Follow by e and consider its target. If the obtained vertex is a source of another known edge (from B, C or F), then follow by it and consider a new target vertex and so on. If the target is not a source of any edge from B, C and F, then we shall follow by the forward edge (if the current vertex of W is separated) or the upward edge (if the vertex is not).

We do this algorithm until we come back to e (meaning that a simple cycle containing e is over) and then we pick another edge from B, C and F, such that we have not yet considered.

As every simple cycle must contain at least one backward or central edge, eventually we cover all cycles of the given multicycle. □

Remark. A reason to include F to the initial data is the following. Vertices $(1^+, 1^-)$ and $(1^-, 1^+)$ are special: these are only vertices with no outgoing backward edges and with two outgoing forward edges. Therefore if we come to one of such points, then we cannot apply the algorithm directly due to multiple choice of outgoing forward edge. This is why our data includes F (the set of edges *following* the central, do not mix with the set of forward edges of γ).

Proposition 2.5.4 (Bound on the number of multicycles)**.** *Let W be a labeled wedge over alphabet A ($\#A = S$) and Γ be the associated graph. Then the number of multicycles of length n of Γ is bounded by*

$$K(\Gamma, n) \leqslant 4S^2(4n+1)^{\sqrt{4n}+2}$$

Proof. Note that items (1), (2) of Proposition 2.5.2 imply that multicycles of length n are located on a finite part of W with height n and length $2n$.

Lemma 2.5.3 says what data for defining a multicycle is necessary: the set of backward edges B, the set of central edges C and the set F of edges following after central (if C is non-empty). In our upper bound we count all possible configurations of B, C and F, even those which cannot define a multicycle.

Note that a subset of backward edges is always defined by the set of source vertices, while for a central edge we need to know both the source and the color. It is so, because from every separated vertex (except $(1^\pm, 1^\mp)$) there is exactly one outgoing backward edge; while there can be from 0 to $S-1$ outgoing central edges. We will use this for counting configurations of B and C.

Number of possibilities for F is always bounded by 4 because there are only two possible targets of central vertices $(1^{\pm}, 1^{\mp})$ and from each there are two possibilities to pick a forward edge.

Denote by V a set of all possible vertices such that can be a source for a backward or a central edge of a multicycle of a length n. The estimate goes as follows. We pick b backward edges, c central edges, pick colors for central edges and pick edges following the central. We assume all choices to be made independently, therefore we multiply them. Then sum up over all possible values of b and c and obtain that the number of multicycles is bounded by

$$\sum_{b=0}^{B_{\max}} \sum_{c=0}^{C_{\max}} \binom{|V|}{b} \cdot \binom{|V|}{c} \cdot S^c \cdot 4 \tag{2.5.1}$$

Note that $C_{\max}$ is always bounded by 2 because along every multicycle there cannot be more than 2 central edges. Also recall the following combinatorial inequality:

$$\sum_{b=0}^{B_{\max}} \binom{|V|}{b} \leqslant (|V|+1)^{B_{\max}} \tag{2.5.2}$$

The inequality above is true because every choice of b elements $(0 \leqslant b \leqslant B_{\max})$ from $|V|$ can be viewed as a correspondence from $B_{\max}$ elements to $|V|$ possibilities to be chosen plus one possibility to be not chosen.

Using inequality 2.5.2 for estimate 2.5.1 we obtain

$$\sum_{b=0}^{B_{\max}} \sum_{c=0}^{C_{\max}} \binom{|V|}{b} \cdot \binom{|V|}{c} \cdot S^c \cdot 4 \leqslant 4S^2 \cdot (|V|+1)^{B_{\max}+2}$$

It remains to estimate $|V|$ and $B_{\max}$. For V we use item (3) of Proposition 2.5.2; it says that every diagonal cannot contain more then one separated vertex such that it is contained in the support of a closed path. In particular, it is applicable to backward or central edges of a multicycle, therefore $|V|$ is bounded by the number of diagonals intersecting a multicycle. As it was said in the beginning of the proof, maximal possible width is bounded by $2n$, therefore there are not more than $4n$ diagonals. Thus, $|V| \leqslant 4n$.

A concluding claim is the following: $B_{\max} \leqslant \sqrt{4n}$. Let $e_1, \ldots, e_{B_{\max}}$ be the backward edges along multicycle ordered by increasing height (recall be the height of an edge we mean the height of its source). Let $h_1, \ldots, h_{B_{\max}}$ be the corresponding heights. Note the target of each e_j is either $(1^+, h_j)$ or $(1^-, h_j)$, thus there cannot be more than two edges of the same height; it implies that $j \leqslant 2h_j$.

$$\frac{{B_{\max}}^2}{2} < \frac{B_{\max}(B_{\max}+1)}{2} = \sum_{j=1}^{B_{\max}} j \leqslant \sum_{j=1}^{B_{\max}} 2h_j \leqslant 2n$$

The last inequality in the chain above holds because the sum of all heights cannot be greater than the length of the whole multicycle.

Combining inequalities above we get the required estimate:

$$4S^2(4n+1)^{\sqrt{4n}+2}$$

$\square$

The following theorem shows that the growth rate is continuous on the space of labeled wedges (over uniformly bounded alphabet).

Theorem 2.5.5 (Continuity of the growth rate on the space of labeled wedges)**.** *If a sequence of labeled wedges W_n (over alphabet A) converges to W then the growth rate of W_n converges to the growth rate of W.*

Proof. In Proposition 2.5.2 items (1) and (2) give that all multicycles of length m lie in a finite part of a wedge (of height m and width $2m$). It means that for every k the coefficient of t^k in the spectral determinant $P_n(t)$ converges to the corresponding coefficient of $P(t)$. Proposition 2.5.4 gives an upper bound for the number of multicycles:

$$\sigma(W_n) \leqslant \limsup_m \left(4S^2(4m+1)^{\sqrt{4m}+2}\right)^{\frac{1}{m}} = \\ = \limsup_m \exp\left(\frac{\sqrt{4m}+2}{m}\log\left(4S^2(4m+1)\right)\right) \leqslant 1. \quad (2.5.3)$$

Thus applying the Theorem 2.4.3 we obtain that $P_n \to P$ uniformly on compacts of the unit disk. By Rouché's theorem, smallest real positive root converges to the smallest real positive root, therefore $r(W_n) \to r(W)$. $\square$

We are almost ready to relate labeled wedges with external addresses, but we need to look at the special case of periodic wedges.

2.5.4 Periodic wedges and their finite models

For labeled wedges corresponding to post-critically finite maps g_λ we should be able to identify some points on the wedge. We can deal with the following equivalence relation on $\{1^-, 1^+, 2, 3, \ldots\}$: for given $p \geqslant 1, q \geqslant 0$ we say that $i \equiv_{p,q} j$ if either $\min\{i,j\} \leqslant q$ and $i = j$ or $\min\{i,j\} \geqslant q+1$ and $i \equiv j \mod p$.

For the case when $q = 0$ we have two different equivalence relations $k(p+1) \equiv_{p,0+} 1^+ \not\equiv_{p,0+} 1^-$ and $k(p+1) \equiv_{p,0-} 1^- \not\equiv_{p,0-} 1^+$.

Such equivalence relations induce equivalence relations on the set of ordered pairs $(i,j) \equiv_{p,q} (k,l)$ iff $i \equiv_{p,q} k$ and $j \equiv_{p,q} l$. On the set unordered pairs (which is the wedge without labeling) the relation is the following: $\{i,j\}$ is equivalent to $\{k,l\}$ if either $(i,j) \equiv_{p,q} (k,l)$ or $(i,j) \equiv_{p,q} (l,k)$.

Definition 2.5.6. We call a labeled wedge periodic of period p and pre-period q if the following holds:

1. Any two pairs (i,j) and (k,l) being equivalent under $\equiv_{p,q}$ have the same label;
2. if $i \equiv_{p,q} j$ then (i,j) is non-separated.

A pair (i,j) s.t. $i \equiv_{p,q} j$ is called *diagonal.*

For a labeled wedge of period p and pre-period q we can construct a finite graph Γ^F. The set of vertices is the set of classes of non-diagonal vertices W under relation $\equiv_{p,q}$. The set of edges Γ^F is obtained from the labeling of W, same as edges of Γ (but now we connect classes of vertices).

The following proposition is an important step towards the proof of Claim 2.

Proposition 2.5.7. *Let W be a periodic labeled wedge with associated (infinite) graph Γ. Then their growth rates are equal $r(\Gamma) = r(\Gamma^F)$.*

For the proof we introduce an intermediate graph $\Gamma^{(2)}$ and call it finite 2-cover of Γ^F.

Vertices of $\Gamma^{(2)}$ is the set of $\equiv_{p,q}$-classes non-diagonal *ordered* pairs of integers and the edges are inherited from Γ in the usual way. For example, if a vertex $(2,6)$ of a labeled wedge W of period 5 and pre-period 2 is separated with a label $(*_0, *_1)$, then there are three edges directing to the vertices $(3,1^+)$, $(1^+,1^-)$ and $(1^-,2)$. The last vertex is $(1^-,2)$ because it is equivalent to $(1^-,7)$ on W under $\equiv_{5,2}$. The graph $\Gamma^{(2)}$ is introduced mainly in order to remember the information about backward and forward edges of Γ : backward edges on Γ correspond to edges on $\Gamma^{(2)}$ directed to vertices of the kind (i,j) s.t. $i > j$.

Proof of Proposition 2.5.7. As diagonal vertices are always non-separated, any non-trivial closed path cannot pass through any diagonal vertex of Γ. Consider the subgraph Γ^{ND} obtained by excluding all diagonal vertices and note that their growth rates are equal:

$$r\left(\Gamma^{ND}\right) = r(\Gamma)$$

We have the following chain of maps

$$\Gamma^{ND} \to \Gamma^{(2)} \to \Gamma^F.$$

As they are defined by quotienting with respect to equivalence relations, they are weak covers of graphs. Since $\Gamma^{(2)}$ and Γ^F are both finite, Lemma 2.4.6 is applicable, from which we obtain an equality of the growth rate $r(\Gamma^{(2)}) = r(\Gamma^F)$.

We can get an inequality $r(\Gamma^{ND}) \leqslant r(\Gamma^{(2)})$ from Lemma 2.4.5 when find a finite set S through which every closed path goes. We claim that the set of vertices with the height 1 and the width $\leqslant p+q+1$ has this property.

Firstly, we should note that vertices along any closed path in Γ^{ND} cannot have the height larger then $p+q$. For each diagonal $D_k^{\pm} = \{1^{\pm}, k+1\} \cup \{(i, i+k) \in \Sigma : i \geqslant 2\}$ either

1. all elements of height $\leqslant p+q$ are non-separated (thus, by periodicity, all elements of $D_k^{\pm}$ are non-separated) or
2. there is a separated element of height $\leqslant p+q$. If we denote by $i_0 \leqslant p+q$ the smallest height of a separated vertex then no other element of $D_k^{\pm}$ of greater height can lie on any closed path. It is so because all other elements are non-separated (from Prop.2.5.2 (3)) and the path will never closed up once it contains a vertex of the diagonal of the height greater then i_0.

Every non-trivial closed path in Γ contains backward and/or central edges and they are directed to the vertices of height 1. Combining with the previous argument, we obtain that their width cannot be greater then $p+q+1$, so every closed path passes through S. This concludes the proof of the first estimate $r(\Gamma^{ND}) \leqslant r(\Gamma^{(2)})$.

In order to prove the other inequality, we shall estimate the number of closed paths in $\Gamma^{(2)}$. A closed path in $\Gamma^{(2)}$ can either contain a backward or central edge, or not. Denote by G the family of all closed paths in $\Gamma^{(2)}$ containing at least one edge with the target whose height it greater or equal then the width (i.e. with at least one backward or central edge or $(2,1^{\pm})$). Then for G Lemma 2.4.5 (2) can be applied, there exists $L \geqslant 0$ s.t.

$$\#\{\gamma \in G : l(\gamma) = n\} \leqslant n \sum_{k=0}^{L} C\left(\Gamma^{ND}, n+k\right).$$

The remaining closed paths can be estimated from above by the number of vertices of $\Gamma^{(2)}$, because each of them can be recovered from its starting point. Indeed, if the initial vertex (i,j) is separated, then we follow the forward direction $(1^{\pm}, j+1)$; if (i,j) is not separated, then we follow upward $(i+1, j+1)$.

Therefore we have the following estimate on the number of closed path in $\Gamma^{(2)}$:

$$C\left(\Gamma^{(2)}, n\right) = \#\{\text{paths of length } n \text{ in G}\} + \#\{\text{remaining path of length } n\} \leqslant$$
$$n \sum_{k=0}^{L} C\left(\Gamma^{ND}, n+k\right) + \#V\left(\Gamma^{(2)}\right)$$

From the estimate above we obtain $r(\Gamma^{(2)}) \leqslant r(\Gamma^{ND})$. □

For a post-critically finite map we can construct an associated labeled wedge from the corresponding Hubbard tree as it was done in subsection 2.5.1.

The following theorem shows that core entropy for post-critically finite maps is equal to the logarithm of the growth rate of associated labeled wedge.

Theorem 2.5.8 (Core entropy for post-critically finite maps)**.** *Let g_λ be a post-critically finite cosine map, H be its Hubbard tree, W be corresponding labeled* *wedge with the associated graph* Γ. *Then*

$$h(g_\lambda) = \log r\left(\Gamma\right).$$

Proof. Proposition 2.5.7 gives that the growth rates of Γ and Γ^F coincide. The transition matrix of H is exactly the adjacency matrix of the finite model Γ^F. Lemma 2.4.4 gives that $r(\Gamma^F)$ is equal to the largest eigenvalue of the adjacency matrix, core entropy is equal to the logarithm of it. $\square$

2.6 Continuity of core entropy on spaces of uniformly bounded external addresses

2.6.1 Relation between external addresses and wedges

Now we want to explain how to construct the associated wedge to a given external address $\underline{s} = (s_0 s_1 s_2 s_3 \dots) \in \mathcal{S}_B$. This goes very similar to the procedure of recovering kneading sequences in subsection 2.2.4. As mentioned previously, we consider $\underline{v}^{(1)} := \sigma\left(\underline{s}\right) = (s_1 s_2 s_3 \dots)$ as the address of the first critical value and $\underline{v}^{(2)} := ((-s_1), s_2 s_3 \dots)$ as the address of the second critical value.

Each of sets $C_{odd} = \{(j s_1 s_2 s_3 \dots), j \in \mathbb{Z}_S\}$ and $C_{even} = \{(j(-s_1) s_2 s_3 \dots), j \in \mathbb{Z}_S\}$ has a natural division into pairs of the kind $r(k, R)$ and $r(k, L)$ (this division depends on s_0, the explicit formulas are 2.2.2 and 2.2.3).

Now we want to define a labeling of the wedge corresponding to $\underline{s}$. We say that the label of vertex $(1^-, 1^+)$ contains $*_m$ if and only if the ray pair $r(m, R), r(m, L)$ separates $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ (the definition of separation is given in 2.2.6). Vertices of the kind $(1^-, n)$ are labeled with all points $*_m$ so that $r(m, R), r(m, L)$ separate $\underline{v}^{(2)}$ and $\sigma^{\circ(n)}(\underline{s})$; vertices $(1^+, n)$ are labeled with all $*_m$ so that $r(m, R), r(m, L)$ separate $\underline{v}^{(1)}$ and $\sigma^{\circ(n)}(\underline{s})$. And (i, j) with $2 \leqslant i < j$ are labeled with all points separating $\sigma^{\circ(i)}(\underline{s})$ and $\sigma^{\circ(j)}(\underline{s})$. The order of the critical points coincides with the direction of moving from $\sigma^{\circ(i)}(\underline{s})$ and $\sigma^{\circ(j)}(\underline{s})$, therefore it is either coincides with $\leqslant_{\mathbb{Z}}$ or it is opposite to $\leqslant_{\mathbb{Z}}$.

Note that periodic addresses (of period p and pre-period q) correspond to periodic labeled wedges (of period p and pre-period q). It it so because according to Definition 2.5.6 of periodic wedge, for two pairs $(i, j), (k, l)$ with $(i, j) \equiv_{p,q} (k, l)$ we have $\sigma^{\circ i}(\underline{s}) = \sigma^{\circ k}(\underline{s})$ and $\sigma^{\circ j}(\underline{s}) = \sigma^{\circ l}(\underline{s})$, therefore labels must coincide. And for the second condition, if $i \equiv_{p,q} j$ then $\sigma^{\circ i}(\underline{s}) = \sigma^{\circ j}(\underline{s})$, thus (i, j) is non-separated.

2.6.2 Core entropy on $\mathcal{S}_B$

Motivated by Theorem 2.5.8, we define core entropy on $\mathcal{S}_B$.

Definition 2.6.1 (Core entropy on $\mathcal{S}_B$)**.** Core entropy of $\underline{s} \in \mathcal{S}_B$ is equal to the logarithm of the growth rate of the corresponding labeled wedge W:

$$h(\underline{s}) = \log r(W)$$

We want to prove that core entropy is continuous on $\mathcal{S}_B$. In order to do this, we need a few additional lemmas.

Lemma 2.6.2. *Let $\underline{s}^{(n)}$ be a sequence of B-bounded addresses and W_n be a sequence of corresponding wedges. If $\underline{s}^{(n)} \to \underline{s}$ and $\underline{s}$ is not purely periodic (in sense of Definition 2.2.7), then $W_n \to W$ where W is the corresponding wedge to $\underline{s}$.*

Proof. We want to show that for each vertex $(i, j) \in \Sigma$ there exists $n_0 = n_0(j)$ so that for all $n \geqslant n_0$ the corresponding labels of W_n and W coincide on (i, j).

Due to the fact that $\underline{s}$ is not purely periodic we have that the first symbol of the itinerary of $\sigma^{\circ j}(\underline{s})$ is not equal to any star, but equal to a whole number (same for $\sigma^{\circ i}(\underline{s})$). Thus there exists two consecutive critical rays r_1, r_2 with $r_1 < \sigma^{\circ j}(\underline{s}) < r_2$. There exists t_0 so that the strict difference of addresses in both inequalities can be seen in first t_0 symbols. Take n_0 so that at least $j + t_0$ symbols of $\underline{s}$ and $\underline{s}^{(n)}$ coincide (n_0 exists because of convergence of the addresses). Then $\sigma^{\circ j}(\underline{s}^{(n)})$ lies between corresponding rays r_1^n, r_2^n (and first entries of r_1 and r_1^n, as well as r_2 and r_2^n, coincide). Same can be done for $\sigma^{\circ i}(\underline{s}^{(n)})$.

It means for $n \geqslant n_0$ a pair separates $\sigma^{\circ i}(\underline{s}^{(n)})$ and $\sigma^{\circ j}(\underline{s}^{(n)})$ if and only if it separates $\sigma^{\circ i}(\underline{s})$ and $\sigma^{\circ j}(\underline{s})$.

□

The case of purely periodic sequences we treat separately with lemmas below.

Lemma 2.6.3. *Let $\underline{s}$ be a purely periodic B-bounded address of period p. Then there exist monotone limits of wedges:*

$$\lim_{\underline{s}^{(n)} \uparrow \underline{s}} W(\underline{s}^{(n)}) = W^-(\underline{s}) \qquad \lim_{\underline{s}^{(n)} \downarrow \underline{s}} W(\underline{s}^{(n)}) = W^+(\underline{s}).$$

Moreover, each of $W^-(\underline{s}), W(\underline{s})$ and $W^+(\underline{s})$ is a periodic wedge of period p.

Proof. We assume that we are in the case when $\sigma^{\circ p}(\underline{s}) = r(k, R)$ (the case $r(k, L)$ is basically same).

Existence of monotone limits we want to show by checking that the labeling eventually stabilizes for points (i, j) in all three cases: when none of i, j is divisible by p, when they both divisible by p and when exactly one of i, j is divisible by p.

Note that the equality $\sigma^{\circ i}(\underline{s}) = r(k, R)$ is possible only if $i \equiv 0 \mod p$, therefore as in Lemma 2.6.2 we have same continuity for labels of points (i, j) when none of i, j divisible by p.

If both i, j are divisible by p, then from Lemma 2.2.8 there exists $n_0 = n_0(j)$ so that for all $n \geqslant n_0$ we have that the first entry of itineraries $\sigma^{\circ i}(\underline{s}^{(n)})$ and $\sigma^{\circ j}(\underline{s}^{(n)})$ coincide (and equal to $k-1$ if we have an address from the right and k for addresses from the left). Similarly if $\underline{s}^{(n)} \downarrow \underline{s}$, then the first symbol of itineraries $\sigma^{\circ i}(\underline{s}^{(n)})$ and $\sigma^{\circ j}(\underline{s}^{(n)})$ are k for addresses from the right and $k-1$ for addresses from the left. In both cases we obtain that if i, j are divisible by p then labels of (i, j) eventually become empty (this shows that some of diagonal vertices eventually become to be non-separated, it is required for the proof that limiting wedges are periodic).

Now consider the case when exactly one number i or j is divisible by p, assume i for simplicity of notations. Denote by $m \in \mathbb{Z}$ the first symbol of the itinerary of $\sigma^{\circ j}(\underline{s})$ (it is not a star because j is not divisible by p and same for $\sigma^{\circ j}(\underline{s}^{(n)})$ for large n). From Lemma 2.2.8 we get that the first symbol of the itinerary of $\sigma^{\circ i}(\underline{s}^{(n)})$ eventually stabilizes to $k-1$ (for addresses from the right and to k for addresses from the left), thus the labels for (i, j) eventually stabilize too.

It remains to show that all three wedges $W^{-}(\underline{s}), W(\underline{s})$ and $W^{+}(\underline{s})$ are purely periodic of period p. $W(\underline{s})$ is purely periodic by construction. For each $W^{-}(\underline{s})$ and $W^{+}(\underline{s})$ we know that from convergence of addresses for every $m \in \mathbb{N}$ there exists n_0 so that the first symbol of itineraries $\sigma^{\circ i}(\underline{s}^{(n)})$ and $\sigma^{\circ i+mp}(\underline{s}^{(n)})$ become to be equal, and same for $\sigma^{\circ j}(\underline{s}^{(n)})$ and $\sigma^{\circ j+mp}(\underline{s}^{(n)})$, therefore labels of (i, j) and $(i+m_1p, j+m_2p)$ eventually becomes to be equal. We showed before, that the diagonal vertices of the kind (m_1p, m_2p) are non-separated, but the same is true for all other diagonal vertices $(i, i+mp)$ (because the itineraries stabilize again). Therefore $W^{-}(\underline{s})$ and $W^{+}(\underline{s})$ are periodic. Note that the equivalence relation $\equiv_{p,0+}$ corresponds to the case when $\sigma^{\circ p}(\underline{s})$ equals to the address of an even critical ray and $\equiv_{p,0-}$ corresponds to an odd critical ray.

□

Lemma 2.6.4. *Let $\underline{s}$ be a purely periodic B-bounded address of period p. Then the finite models of $W^{-}(\underline{s}), W(\underline{s})$ and $W^{+}(\underline{s})$ are isomorphic. As a consequence, their growth rates are equal.*

Proof. From the construction in Lemmas 2.6.2 and 2.6.3 we observe that the labels of (i, j) of the periodic wedges $W^{-}(\underline{s}), W(\underline{s})$ and $W^{+}(\underline{s})$ coincide when none of (i, j) divisible by p or both (i, j) are divisible by p.

Let $i = mp$ be divisible by p (the case for j is same) and the first symbol of the itinerary of $\sigma^{\circ mp}(\underline{s})$ is equal to $*_k$ (it means that either $\sigma^{\circ p}(\underline{s}) = r(k, R)$ or $\sigma^{\circ p}(\underline{s}) = r(k, L)$, consider the case of $r(k, R)$). Denote by $l \in \mathbb{Z}$ the first symbol of the itinerary of $\sigma^{\circ j}(\underline{s})$ (it cannot be equal to any star since j is not divisible by p). If $l \geqslant k+1$ then the label of (mp, j) for wedge $W^{-}(\underline{s})$ is equal to $\{*_k, *_{k+1}, \ldots, *_{l-1}\}$ and for the wedges $W(\underline{s})$ and $W^{+}(\underline{s})$ it is equal to $\{*_{k+1}, \ldots, *_{l-1}\}$ (or empty set when $l = k+1$).

On the associated graphs there are $l-k+1$ (or $l-k$ respectively) outgoing edges but they differ only in first two edges. Consider the case when $l = k+1$, then there are two outgoing edges from (mp, j) for $W^{-}(\underline{s})$ (they are directed to $(1^{\pm}, mp+1)$ and $(1^{\pm}, j+1)$, the sign depends whether k is even or odd) and only one outgoing edge for $W(\underline{s}), W^{+}(\underline{s})$ going to $(mp+1, j+1)$. But the vertex $(1^{\pm}, mp+1)$ is diagonal and the vertices $(1^{\pm}, j+1)$ and $(mp+1, j+1)$ are equivalent under $\equiv_{p,0\pm}$, therefore the finite models are isomorphic.

The case when $l > k+1$ adds to all $W^-(\underline{s}), W(\underline{s}), W^+(\underline{s})$ several more outgoing central edges directed to vertices $(1^{\pm}, 1^{\mp})$, but the argument is same: one of the edges of $W^-(\underline{s})$ goes to a diagonal vertex, while the next edges are correspondent to each other.

When $l \leqslant k$ we observe that the label of (mp, j) for wedges $W^-(\underline{s}), W(\underline{s})$ are $\{*_{k-1}, \dots, *_{l+1}$ and for $W^+(\underline{s})$ it is $*_k, *_{k-1}, \dots, *_{l+1}$. The argument for isomorphism between $W^-(\underline{s}), W(\underline{s}), W^+(\underline{s})$ is same. $\square$

Now we are ready to prove the main theorem.

Theorem 2.6.5. *Core entropy is continuous in the space of uniformly bounded sequences.*

Proof. For points which are not purely periodic Lemma 2.6.2 proves the continuity of the transition between external addresses to the space of labeled wedges and Theorem 2.5.5 gives the continuity for wedges. The case of purely periodic sequences is covered by Lemma 2.6.4. $\square$

By continuity, we can extend Theorem 2.3.10 about a bound on core entropy to the whole space $\mathcal{S}_B$.

Theorem 2.6.6. *For each $\underline{s} \in \mathcal{S}_B$ we have*

$$h(\underline{s}) \leqslant \log(8b+6).$$

Proof. Every address in $\mathcal{S}_B$ can be approximated by (pre-)periodic ones, so the statement follows from Theorems 2.3.10 and 2.6.5. $\square$

3 Local Unboundedness of Core Entropy at Escaping Parameters

3.1 Introduction and statement of the result

Core entropy of a post-critically finite complex polynomial can be defined as topological entropy restricted to the Hubbard tree H of the polynomial. It can be computed as the largest eigenvalue of the transition matrix of edges of H or, equivalently, as the exponential growth rate of precritical points on H. More details can be seen e.g. in [G1], [Ju], [DS] or in Chapter 2. Another interpretation of core entropy can be viewed in the context of biaccessibility dimention of the Julia set, details are in [MeSch] and [DS]. One can get a motivation to explore the core entropy of transcendental maps due to existence of Homotopy Hubbard Trees proven in [Pfr1]. A generalization of the notion of core entropy was done for the family of exponential maps in [H1]. We want to focus on a (complex-)one-dimensional family of cosine maps

$$\left\{ g_\lambda(z) = \lambda \frac{e^z + e^{-z}}{2} = \lambda \cosh(z) = \lambda \cos(iz) \mid \lambda \in \mathbb{C}^* \right\}$$

In the previous chapter several definitions of core entropy are discussed for this family. One of the results gives an upper bound for core entropy depending on the uniform bound for the combinatorics of a map. Moreover, it was proven that core entropy is continuous along sequences with uniformly bounded combinatorics. A natural question to ask is how this relates to the complex parameter space.

In this chapter we show that core entropy can be unbounded even locally in the complex parameter space (in contrast to the continuity result mentioned above). We prove that in a neighborhood of every real parameter $\lambda \in \mathbb{R}^+$ with $\lambda \geqslant 1$ there exists a sequence of periodic parameters with the diverging core entropy.

Theorem C. *Let $\lambda_0 \in \mathbb{R}^+$ such that $\lambda_0 \geqslant 1$. Then there exists a sequence λ_n with $|\lambda_n - \lambda_0| \to 0$ and core entropy of $g_n(z) = \lambda_n \cosh(z)$ tends to infinity:*

$$h(g_n) \to \infty.$$

An important step for the proof of the main theorem is to show existence of parameters λ_n with prescribed combinatorics: first n points of the orbit of λ_n are close to the corresponding points of the orbit of λ_0 and thus grow at exponential rate, but the $(n+1)$-st iterate of λ_n "jumps" to value $\lambda_n + 2\pi i k_n$ with k_n very big. This construction together with control on k_n are sufficient to make core entropy tending to infinity.

3.2 Growing entropy

3.2.1 Existence of parameters

In this subsection we show existence of parameters $\lambda_n \to \lambda_0$ with the property $g_{\lambda_n}^{\circ n+1}(\lambda_n) = \lambda_n + 2\pi i k_n$. A rough explanation of the idea of our construction is the following (the precise statements are given afterwards). First, we introduce the "parameter functions" $C^n(\lambda) = g_\lambda^{\circ n}(\lambda)$ and observe that for large n they are expansive near $\lambda_0 \in \mathbb{R}^+$ with $\lambda_0 \geqslant 1$. From dynamically point of view, the parameter functions C^n are formally defined on the parameter space, but they have some common features with usual cosine maps defined on the dynamical space.

Lemma 3.2.2 shows that if C^n is invertible on a neighborhood U and $C^n(U)$ is small, then we can "enlarge" the image passing to the next parameter function C^{n+1}. This is used in Lemma 3.2.3 showing that whatever small neighborhood U_0 of λ_0 we start, the image $C^n(U_0)$ contain a disk of radius $\pi/\sqrt{2}$ for a large n. Finally, Lemma 3.2.4 shows that the next image $C^{n+1}(U_0)$ spreads over a big portion of the complex plane, in particular it intersects a definite part near the imaginary line and this gives the existence of parameters with the property $C^{n+1}(\lambda) = \lambda + 2\pi i k$.

We start with the following simple lemma.

Lemma 3.2.1. *Let $U \subset \mathbb{C}$ be a simply connected domain so that $0 \notin \overline{U}$. Then there is a constant $G > 0$ depending on U so that for all $\lambda_1, \lambda_2 \in U$*

$$|\log \lambda_1 - \log \lambda_2| \leqslant G|\lambda_1 - \lambda_2|.$$

Proof. We have

$$|\log \lambda_1 - \log \lambda_2| \leqslant |\lambda_1 - \lambda_2| \cdot \max_{z \in \overline{U}} |\log(z)|' = |\lambda_1 - \lambda_2| \max_{z \in \overline{U}} \frac{1}{|z|}$$

Take $G := 1/|z_0|$ where $z_0 \in \overline{U}$ with the smallest absolute value, then the statement follows. □

Note that if U is a neighborhood of some λ_0 and the diameter of U goes to zero, then $G \to 1/\lambda_0$.

For next propositions we need to define the "parameter functions":

$$C^n(\lambda) := g_\lambda^{\circ n}(\lambda).$$

Thus $C^0(\lambda) = id$, $C^1 = \lambda \cosh \lambda$, $C^2 = \lambda \cosh(\lambda \cosh \lambda)$ etc. and $C^{n+1}(\lambda) = \lambda \cosh(C^n(\lambda))$. By induction, one can observe that for real $\lambda_0 \in \mathbb{R}^+$ the derivative $(C^n)'(\lambda_0)$ is purely real and non-zero, thus the function C^n is locally biholomorphic; moreover for big n the derivative has a large absolute value, therefore C^n is expansive near λ_0.

REMARK. Although each parameter function C^n is not the n-th iterate of some function in the canonical sense (at least, $C^{n+1} \neq C^1 \circ C^n$), it still has some properties similar to the iterated cosh. For example, item 3 of Lemma 3.2.2 shows that for small domains a "corrected" equality is true: $C^{n+1} = C^{n,n+1} \circ C^n$ with a well-defined correction term $C^{n,n+1}$. This is a reason why we used superscript instead of subscript in the notation C^n.

The following lemma gives some estimates for iterated cosh similarly to the estimates for iterated exp in Lemma 2 in [BBS]. The idea can be stated as: if the biholomorphic image $C^n(U)$ is small and convex, then C^{n+1} is biholomorphic on U too.

Here and further we assume that U is a neighborhood of a real $\lambda_0 \geqslant 1$.

Lemma 3.2.2. *Suppose that $C^n\colon U \to V$ a biholomorphism with $|\lambda| \geqslant 0.9$ and $|(C^n)'(\lambda)| \geqslant 2$ for all $\lambda \in U$. Also suppose that $\mathrm{Re}(V) > \xi \geqslant 3$, V is convex and is contained in a disk of radius $\pi/\sqrt{2}$. Then*

1. *$C^{n+1}\colon U \to C^{n+1}(U)$ is a biholomorphism;*
2. *$|(C^{n+1})'| > 2|(C^n)'|$ on U;*
3. *the map $C^{n,n+1}\colon V \to C^{n+1}(U)$ given by $C^{n,n+1} := C^{n+1} \circ (C^n)^{-1}$ is a biholomorhism with $|(C^{n,n+1})'| \geqslant e^\xi/8$.*

Proof. 1. It is sufficient to prove that C^{n+1} is injective on U. Assume for some $\lambda_1, \lambda_2 \in U$ we have the following equalities :

$$\begin{gathered} C^{n+1}(\lambda_1) = C^{n+1}(\lambda_2) \\ \lambda_1 \cosh C^n(\lambda_1) = \lambda_2 \cosh C^n(\lambda_2) \\ \log\cosh C^n(\lambda_1) - \log\cosh C^n(\lambda_2) = \log\lambda_2 - \log\lambda_1 \end{gathered}$$

The last equality is obtained by taking an appropriate branch of the logarithm, which exists due to biholomorphicity of C^n (the equations are equivalent except for the choice of the logarithm branch in the last equality).

We want to prove the following chain of inequalities for $\lambda_1, \lambda_2 \in U$:

$$\begin{aligned} |\log\cosh C^n(\lambda_2) - \log\cosh C^n(\lambda_1)| \geqslant 0.9^2 |C^n(\lambda_2) - C^n(\lambda_1)| \geqslant \\ \geqslant 0.9^2 \cdot 2|\lambda_2 - \lambda_1| \geqslant 0.9^3 \cdot 2 \cdot |\log\lambda_2 - \log\lambda_1|, \quad (3.2.1) \end{aligned}$$

then it implies that the equality $C^{n+1}(\lambda_1) = C^{n+1}(\lambda_2)$ is true for $\lambda_1, \lambda_2 \in U$ only when $\lambda_1 = \lambda_2$.

Denote $z_j := C^n(\lambda_j)$, for $j = 1, 2$. For the first inequality we can observe that

$$\log\cosh z_2 - \log\cosh z_1 = \int_{z_1}^{z_2} (\log\cosh z)'\, dz =$$
$$= \int_{z_1}^{z_2} \tanh(z) dz = \int_{z_1}^{z_2} \left(1 - \frac{2e^{-2z}}{1+e^{-2z}}\right) dz.$$

A simple computation shows that $2e^{-2z}/(1+e^{-2z}) < 0.1$ if $\mathrm{Re} z > \xi \geqslant 3$, thus $|\tanh(z)| > 0.9$. Moreover, for the same reason we obtain that the argument of $\tanh(z)$ is close to 1, to be more precise $|\cos\arg\tanh(z)| \geqslant 1/\sqrt{(1+0.01)} > 0.9$, this proves the first inequality in chain (3.2.1).
The second inequality can be obtained if we use biholomorphicity of C^n, convexity of V and the bound for the derivative:

$$|\lambda_2 - \lambda_1| = |(C^n)^{-1}(z_2) - (C^n)^{-1}(z_1)| \leqslant \sup_{z\in V} |\left((C^n)^{-1}\right)'(z)| \cdot |z_2 - z_1| \leqslant \frac{1}{2}|z_2 - z_1|.$$

And the last part of chain (3.2.1) is Lemma 3.2.1, $|\log\lambda_1 - \log\lambda_1| \leqslant G|\lambda_1 - \lambda_2|$ with $G = 10/9$, because $|\lambda| \geqslant 0.9$ for all $\lambda \in U$.

2. We have

$$\left(C^{n+1}\right)'(\lambda) = \cosh C^n(\lambda) + \lambda \sinh C^n(\lambda) \cdot (C^n)'(\lambda) =$$
$$= \frac{e^{C^n(\lambda)}}{2}\left(1 + e^{-2C^n(\lambda)} + \lambda\left(1 - e^{-2C^n(\lambda)}\right)(C^n)'(\lambda)\right).$$

Since $|e^{-2C^n(\lambda)}| \leqslant e^{-2\xi} < 1/9 \leqslant 1 - (0.8/|\lambda|)$, we obtain that

$$|\lambda\left(1 - e^{-2C^n(\lambda)}\right)| > 0.8.$$

For the same reason we have $|1 + e^{-2C^n(\lambda)}| < 1.1$. Thus we can write:

$$|\left(C^{n+1}\right)'(\lambda)| > \frac{e^{\xi}}{2}|0.8(C^n)'(\lambda) - 1.1| > 2|(C^n)'(\lambda)|,$$

where the last inequality holds since $|(C^n)'(\lambda)| \geqslant 2$ and $\xi \geqslant 3$.

3. $C^{n,n+1}\colon V \to C^{n+1}(U)$ is a biholomorphism as a composition of two biholomorphic maps, it remains to show the estimate for the derivative. Recall $C^{n,n+1}(z) = \lambda(z)\cosh(z)$ with $\lambda(z) = (C^n)^{-1}(z)$.

$$\left(C^{n,n+1}\right)' = \lambda\sinh z + \lambda'\cosh z = e^z\left(\lambda\frac{1-e^{-2z}}{2} + \lambda'\frac{1+e^{-2z}}{2}\right)$$

Similarly as in previous estimates, we have $|\lambda(1-e^{-2z})| > 0.8$, $|1+e^{-2z}| < 1.1$.

Also, $|\lambda'(z)| \leqslant 1/2$ since $|C^n| \geqslant 2$. So we can write

$$|\lambda \frac{1-e^{-2z}}{2} + \lambda' \frac{1+e^{-2z}}{2}| > |\frac{0.8}{2} - \frac{1.1}{2 \cdot 2}| = \frac{1}{8},$$

and therefore $|(C^{n,n+1})'| > e^{\xi}/8$.

□

Using the previous lemma we can prove that whatever small neighborhood of λ_0 we take, we can find a sufficiently large n so that $C^n(U)$ is big due to expansivity of C^n.

Lemma 3.2.3. *For every λ_0 as above, and every neighborhood U_0, there is an $N_0 \in \mathbb{N}$ (depending on U_0) so that for all $n \geqslant N_0$, there is a neighborhood $U_n \subset U_0$ of λ_0 so that $C^n(U_n)$ is a disk of radius $\pi/\sqrt{2}$, and C^{n+1} is injective on U_n.*

Proof. Note that for sufficiently small U_0 we can always get $|(C^n)(\lambda_0))'| > 2$. In particular this means that C^n is locally biholomorphic. Suppose we have a neighborhood $\widetilde{U}_n$ so that $C^n\colon \widetilde{U}_n \to C^n(\widetilde{U}_n)$ is biholomorphic and $C^n(\widetilde{U}_n)$ contains a disk of some radius. Denote by r_n the largest possible radius and by $V_n \subset C^n(\widetilde{U}_n)$ the corresponding disk. If $r_n > \pi/\sqrt{2}$, then we are done, taking $U_n \subset \widetilde{U}_n$ as the preimage of V_n.

If $r_n \leqslant \pi/\sqrt{2}$, then we want to apply Lemma 3.2.1. By restricting the radius of V_n we can assure that $\lambda \geqslant 0.9$. For every $n \geqslant 3$ and $\lambda \geqslant 0.9$ we have $\mathrm{Re}(V_n) > \xi \geqslant 4$ and V_n is convex. Therefore we can use the statements of the lemma. In particular, C^{n+1} is biholomorphic on $\widetilde{U}_n$ and the absolute value of derivative of $C^{n,n+1}$ is bounded by $e^{\xi}/8$. By the Koebe 1/4-theorem, the image contains a disk around $C^{n+1}(\lambda_0)$ of radius $r_{n+1} \geqslant r_n e^{\xi}/32 > r_n$.

Repeating this procedure finite number of steps we obtain that for some N_0 we have $r_{N_0} > \pi/\sqrt{2}$ and same for other $n \geqslant N_0$.

□

Lemma 3.2.4 (Critical point image)**.** *The neighborhood U_n from Lemma 3.2.3 contains parameters λ_n so that $C^{n+1}(\lambda_n) = \lambda_n + 2\pi i k_n$ with $k_n \in \mathbb{Z}$ and $|k_n| > C^n(\lambda_0)$.*

Proof. The image $C^n(U_n)$ is a disk of radius $\pi/\sqrt{2}$ and it contains a square of side π with the center at $z_0 \coloneqq C^n(\lambda_0)$, denote the square by Q_n. Consider $f\colon Q_n \to \mathbb{C}$ given by $f(z) \coloneqq \lambda(z)(\cosh(z) - 1) = C^{n,n+1}(z) - \lambda(z)$, where $\lambda(z) = (C^n)^{-1}(z)$. In order to understand the image of Q_n under f we can compose f to cosh and then subtract 1 and multiply by $\lambda(z)$ (see Figure 3.1).

The boundary of $\cosh(Q_n)$ consists of two imaginary intervals joining $\cosh(z_0 - \pi/2 \pm i\pi/2)$ with $\cosh(z_0 + \pi/2 \pm i\pi/2)$ and two arcs being half-ellipses. Note that both elliptic arcs are almost circular ones of radii $r_\pm = 1/2 \exp(z_0 \pm \pi/2)$ since

$$\cosh\left(z_0 \pm \frac{\pi}{2} + it\right) = \frac{1}{2}\exp\left(z_0 \pm \frac{\pi}{2}\right)(\exp(it) + \exp(-2z_0\pi - it)).$$

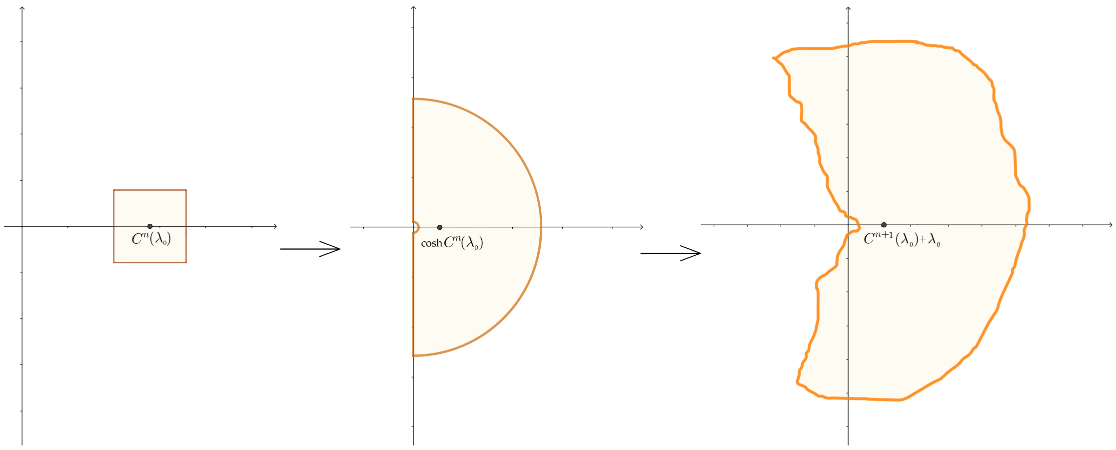

Figure 3.1 – Schematic visualization of f

Similarly, as in previous lemmas, $|\exp(-2z_0\pi - it)| \leqslant 0.1$, thus $\cosh(Q_n)$ contains a half-annulus between radii $1.1r_-$ and $0.9r_+$. Note that $0.9r_+ - 1.1r_- \gg 4\pi$. Subtraction of 1 moves the image to the left, but this still contains two imaginary intervals of of length bigger then 4π.

Multiplication by $\lambda(z) + 1$ does not distort $\cosh(Q_n) + 1$ too much. Every point is multiplied by a number with absolute value $|\lambda| > 0.9$ and a small argument $\arg\lambda$. Note that if z in the upper half plane then the argument is positive and if z in the lower half plane then the argument is negative.

This means that $f(Q_n)$ contains a purely imaginary interval of length greater than 2π.

Moreover, f is injective on Q_n, this can be obtained from injectivity of $C^{n,n+1}$ on Q_n. Indeed, Q_n is simply connected and the derivative is bounded from below:

$$|f'(z)| = |\left(C^{n,n+1}\right)' + \lambda'(z)| > \frac{e^{\xi}}{8} - \frac{1}{2} > 0.$$

This implies that there exists a point z_n such that $f(z_n) = 2\pi i k_n$ with $|k_n| > 1.1r_- \gg z_0$. Using $z_n = C^n(\lambda_n)$ we can rewrite the equality $f(z_n) = 2\pi i k_n$ also as $C^{n+1}(\lambda_n) = \lambda_n + 2\pi i k_n$ with $|k_n| > C^n(\lambda_0)$ and this is exactly the statement we wanted to prove. □

Remark. We know that the sequence $C^n(\lambda_0)$ grows as an iterated exponential and in the construction above we obtain that $|k_n|$ is of order $C^{n+1}(\lambda_0)$. As we do not want to deal with additional constants, we "sacrifice" one iterate and get that $|k_n| > z_0 = C^n(\lambda_0)$.

3.2.2 Estimate on core entropy

In this subsection we want to estimate core entropy for cosine maps defined by the preperiodic parameters λ_n from Lemma 3.2.4 (be "preperiodic parameters" we mean those cosine maps whose critical values lie on a preperiod). The latter lemma guaranties the property $C^{n+1}(\lambda_n) = \lambda_n + 2\pi i k_n$, or equivalently,

$$g_{\lambda_n}^{\circ(n+1)}(\lambda_n) = \lambda_n + 2\pi i k_n$$

.

Since such maps are post-critically finite, their (Homotopy) Hubbard trees are well-defined (see [Pfr1] for details) and we can use any of the "classical" definitions of core entropy, for example, the following.

Definition 3.2.5 (Core entropy for pcf cosine maps)**.** Let g be a post-critically finite cosine map, H be its (Homotopy) Hubbard Tree and $N(n)$ be the number of precritical points on H of order $\leqslant n$. Then core entropy of g is defined as

$$h(g) = \limsup_{n\to\infty} \frac{1}{n} \log N(n).$$

The following proposition gives an explicit lower bound on core entropy for precritical parameters from Lemma 3.2.4.

Proposition 3.2.6 (Bound on core entropy)**.** *Let $N(m)$ be the number of precritical points of order $\leqslant m$ on the Hubbard tree of $g_n(z) = \lambda_n \cosh(z)$, where λ_n is so that $g_n^{\circ(n+2)}(0) = \lambda_n + 2\pi i k_n$. Then the following estimate holds*

$$N\left(m(n+2)\right) > (2|k_n| - 2)^{m+1}.$$

As a consequence, a lower bound on core entropy is:

$$h(g_n) \geqslant \frac{\log(2|k_n| - 2)}{n+2}.$$

Proof. For two points a, b on the Hubbard tree H_n we denote by $[a, b]$ the arc on H_n connecting a and b. Let $I := [\lambda_n, \lambda_n + 2\pi i k_n] \subset H$. Obviously, the number of precritical points of order $\leqslant m$ on the whole H_n is bounded from below by the number of precritical points on I. Every dynamical partition for cosine is always $2\pi i$-periodic, thus I contains $2|k_n| - 1$ critical points lying between λ_n and $\lambda_n + 2\pi i k_n$, so $N(0) \geqslant 2|k_n| - 1$. For simplicity of notations, assume $k_n > 0$.

Consider a subinterval of I lying between two consecutive critical points, $I_l := [\pi i l, \pi i(l + 1)]$. We claim that every such subinterval (there are $2k_n - 2$ of them on I) contains at least $2k_n - 1$ precritical points of order $\leqslant n+2$. The key observation is that the union of $n+2$ forward-iterates of I_l contains I:

$$\bigcup_{j=1}^{n+2} g_n^{\circ j}\left([\pi i l, \pi i(l+1)]\right) \supset I.$$

Indeed, the image of I_l is $[-\lambda_n, \lambda_n]$, the next image contains $[\lambda_n, g_n(\lambda_n)]$ (because $[-\lambda_n, \lambda_n]$ contains 0 being mapped to λ_n and $\pm\lambda_n$ are mapped to $g_n(\lambda_n)$). Next iterates of the latter interval are of the kind $[g_n^{j-1}(\lambda_n), g_n^j(\lambda_n)]$, $j = 2, \ldots, n+2$. Their union contains I.

The observation above gives the fact that each I_l contains points being mapped to the points of I in at most $n+2$ iterates, i.e. the number of precritical points of order $\leqslant n+2$ on I_l is bounded from below by $2k_n - 1$. This means that that the number of precritical points of order $\leqslant n+2$ is bounded by $(2k_n - 2)(2_k - 1) > (2k_n - 2)^2$.

Essentially it was the inductive step, because the number of precritical points of order $\leqslant 2(n+2)$ on I_l can be bounded from above by the number of precritical points of order $\leqslant (n+2)$ on I, thus we immediately obtain

$$N(2(n+2)) > (2k_n - 2)^3$$

Repeating this procedure m times we get the required inequality for $N(m(n+2))$. Plugging the estimate to the definition of core entropy, we immediately obtain

$$h(g_n) = \limsup_{m\to\infty} \frac{1}{m} \log N(m) \geqslant \lim_{m\to\infty} \frac{1}{m(n+2)} \log N(m(n+2)) \geqslant \frac{\log(2|k_n| - 2)}{n+2}.$$

□

Now we can prove the main theorem.

Theorem 3.2.7. *Let $\lambda_0 \in \mathbb{R}^+$ such that $\lambda_0 \geqslant 1$. Then there exists a sequence λ_n with $|\lambda_n - \lambda_0| \to 0$ and core entropy of $g_n(z) = \lambda_n \cosh(z)$ tends to infinity:*

$$h(g_n) \to \infty.$$

Proof. Lemmas 3.2.3 and 3.2.4 imply that for every U_0 there exist $N_0(U_0)$ so that for all $m \geqslant N_0$ there are periodic parameters λ_m with the property $g_m^{\circ m+1}(\lambda_m) = \lambda_m + 2\pi i k_m$. Let $U_0 := D(\lambda_0, 10^{-n})$ to be the disk of radius 10^{-n} centered at λ_0. Then (probably, after taking subsequences) there is a sequence $\lambda_n \to \lambda_0$ with $g_n^{\circ n+1}(\lambda_n) = \lambda_n + 2\pi i k_n$.

Proposition 3.2.6 gives a lower bound on core entropy of g_n

$$h(g_n) \geqslant \frac{\log(2|k_n| - 2)}{n+2}.$$

Divergence of the latter fraction can be obtained from the inequality $|k_n| >$

$C^n(\lambda_0)$. Indeed, the denominator grows linearly, while $C^n(\lambda_0)$ grows as an iterated exponential. In particular, there exists an α so that for large n we have $\log(2|k_n|-2) > \log(2C^n(\lambda_0)-2) > \alpha C^{n-1}(\lambda_0) \gg \alpha e^{n-1}$ and this finalizes the proof. $\square$

Remark. We believe that that for parameters g_n core entropy is equal exactly to $\log(2|k_n|)/(n+2)$, but this requires more work. A reason why $\log(2|k_n|-2)$ (instead of $\log(2|k_n|)$) was in the numerator in Proposition 3.2.6 is because at each step of our computation we discarded two "end-subintervals" (the ones containing of the endpoints of I_l). Our estimate is still sufficient to obtain diverging core entropy.

Remark. If we want to compute core entropy of g_n in terms of transition matrix A_n, then we obtain the same lower bound. This is because Proposition 3.2.6 gives existence of an interval covering itself in $n+2$ iterates with degree $\geqslant 2|k_n|-2$. This means that a lower bound on the largest eigenvalue of $(A_n)^{n+2}$ is $2|k_n|-2$.

4 Outlook

4.1 Questions related to Chapter 2

4.1.1 Relation to complex parameter space

Even if we have continuity of core entropy in every $\mathcal{S}_B$, we have not proved similar results in the global complex parameter space $\mathbb{C}^*$ or at least on the subspace of parameters corresponding non-escaping critical values (the combinatorics is not uniformly bounded there, while core entropy is well-defined). This question is partially discussed in Chapter 3 and a surprising discontinuity result is obtained.

4.1.2 Hölder-continuity or Universal space is not metrizable

In [Ti1] it was proved that for quadratic polynomials core entropy not only continuous, but also Hölder-continuous (near parameters with positive core entropy). The latter notion depends very much on the metric considered. In our case, every $\mathcal{S}_B$ is metrizable and one standard metric can be defined as follows. For two external addresses $\underline{x} = (x_0x_1x_2\dots)$ and $\underline{y} = (y_0y_1y_2\dots)$ we introduce $\delta(x_j, y_j)$ being equal to 1 if $x_j = y_j$ and 0 otherwise. Then the distance between x and y is equal to

$$d(x,y) = \sum_{j=0}^{\infty} \frac{1}{2^j}\delta(x_j, y_j).$$

This metric can be obtained from the discrete metric on B, when we assume all symbols are equidistant from each other. We can also obtain another geometrically intuitive metric on $\mathcal{S}_B$ if the distance between symbols $a_1, a_2 \in B$ is $|a_2 - a_1|$:

$$d(x,y) = \sum_{j=0}^{\infty} \frac{1}{(2b+1)^j}|y_j - x_j|.$$

The reason to put $(2b+1)^j$ in the denominator is because the metric should respect the lexicographical order on $\mathcal{S}_B$. The metric degenerates when $b \to \infty$, but this is not surprising: the universal space $\cup\mathcal{S}_B$ with the inductive topology (discussed in Remark on p. 18) is not first countable and therefore is not metrizable (and the question of Hölder-continuity is not even defined).

4.1.3 Full cosine family

In [GT] labeled wedges were introduced for polynomials of higher degrees. The labeled wedges consist of several layers corresponding to different critical orbits. In context of cosine maps one can consider the full cosine family $\{ae^z+b^{-z} \mid a, b \in \mathbb{C}^*\}$ and introduce labeled wedges (with two layers). The proof of continuity in spaces with uniformly bounded combinatorics requires more work, but we believe the statement is true.

4.1.4 Other families of entire transcendental maps

There are more possible generalizations if we allow critical points of higher degree or asymptotic values. Letting a map to have an asymptotic value is a separate interesting case partially studied in [H1] for the exponential family.

4.2 Questions related to Chapter 3

4.2.1 Other (periodic) parameter rays

One reason to work with the particular parameter ray $\mathbb{R}^+$ is simplicity of computations. For parameters λ_0 with sufficiently high potential escaping on another fixed (or periodic) ray one can formulate statements similar to Lemmas 3.2.2, 3.2.3, 3.2.4, because we still have essentially horizontal escape for λ_0 and large expansion of parameter functions C^n near λ_0.

4.2.2 Smaller potentials

Periodic parameters with growing core entropy can be possibly found for λ_0 of a small potential (for example, for $\mathbb{R}^+$ we mean λ_0 with $0 < \lambda_0 < 1$). The main idea is that for each λ_0 we can "wait" for a finite number of iterations m_0 until it gets a sufficiently fast rate of escape and then use ideas of the construction above. In this case the estimate on the core entropy may change to the kind

$$h(g_n) \geqslant \frac{\log(2|k_n| - 2)}{m_0 + n + 2},$$

but the diverging numerator should remain.

4.2.3 Discontinuity of core entropy on the bifurcation locus

Speculating further, one can assume that core entropy cannot be continuous in the bifurcation locus of cosine parameters. The reason for this is that parameters escaping on rays are dense in the bifurcation locus and every parameter escaping on rays can be approximated by pcf parameters with growing entropy. The estimates of core entropy in our method depend on the potential and this can be an obstacle for proving of discontinuity of core entropy on the bifurcation locus, so the question remains open.

Bibliography

[A97] L. Alsedà, J. Guaschi, J. Los, F. Mañosas, P. Mumbru, *Canonical representatives for patterns of tree maps*, Topology 36(5) (1997), 1123–1153.

[BBS] M. Bailesteanu, V. Balan, D. Schleicher, *Hausdorff Dimension of Exponential Parameter Rays and Their Endpoints*, Nonlinearity 21 (2007), no. 1, 113–120.

[BR1] A. Benini, L. Rempe, *A landing theorem for entire functions with bounded post-singular sets*, Geom. and Funk. An. 30 (2020), 1465–1530.

[BFH] B. Bielefeld, Y. Fisher, J. Hubbard, *The classification of critically preperiodic polynomials as dynamical systems*, J. Amer. Math. Soc. 5 (1992), no. 4, 721–762.

[BFP] A.M. Benini, J.E. Fornæss, H. Peters, *Entropy of transcendental entire functions*, arXiv:1808.06360[math.DS].

[DS] D. Dudko, D. Schleicher, *Core Entropy of Quadratic Polynomials*, Arnold Math. J. 6 (2020), 333–385.

[G1] Y. Gao, *On Thurston's core entropy algorithm*, Trans. Amer. Math. Soc. 373 (2020), 747–776.

[GT] Y. Gao, G. Tiozzo, *The core entropy for polynomials of higher degree*, arXiv:1703.08703 [math.DS].

[H1] M. Haßler, *Core entropy of unicritical maps*, Bachelor thesis, Jacobs University Bremen, 2020.

[Ju] W. Jung, *Core entropy and biaccessibility of quadratic polynomials*, arXiv:1401.4792[math.DS].

[LM93] J. Llibre, M. Misiurewicz, *Horseshoes, entropy and periods for graph maps*, Topology, 32(3) (1993) 649–664.

[L1] M. Lyubich, *Conformal Geometry and Dynamics of Quadratic Polynomials, vol I-II*, Book in preparation, `http://www.math.stonybrook.edu/~mlyubich/book.pdf`

[L83] M. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems 3 (1983), no. 3, 351–385.

[MB] H. Mihaljević-Brandt, *Topological Dynamics of Transcendental Entire Functions*, PhD-thesis, `https://livrepository.liverpool.ac.uk/3069540/`

[dMvS] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, 25 (1993), 606 p.

[MeSch] P. Meerkamp, D. Schleicher, *Hausdorff dimension and biaccessibility for polynomial Julia sets*, Proceedings of the Am. Math. Soc. 141 (2013), no. 2, 533–542.

[MS] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math., 67 (1980), 45–63.

[MP77] M. Misiurewicz, F. Przytycki, *Topological entropy and degree of smooth mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), no. 6, 573–574.

[Pfr1] D. Pfrang, *Homotopy Hubbard Trees for Post-Singularly Finite Transcendental Entire Maps*, PhD-thesis, `https://opus.jacobs-university.de/files/893/phd20190805_Pfrang.pdf`, 2019.

[Poi1] A. Poirier, *Hubbard trees*, Fund. Math. 208 (2010), no. 3, 193–248.

[PRS] D. Pfrang, M. Rothgang, D. Schleicher, *Homotopy Hubbard Trees for post-singularly finite exponential maps*, arXiv:1812.11831 [math.DS]

[RoS] G. Rottenfußer, D. Schleicher, *Escaping Points of the Cosine Family*, Transcendental Dynamics and Complex Analysis, 2008, 396–424.

[RRRS] G. Rottenfußer, J. Rückert, L. Rempe, D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) 173 (2011), no. 1, 77–125.

[Sch1] D. Schleicher, *The Dynamical Fine Structure of Iterated Cosine Maps and a Dimension Paradox*, Duke Math. Journal 136 (2007), no. 2, 343–356.

[Ti1] G. Tiozzo, *Continuity of core entropy of quadratic polynomials*, Invent. Math. 203 (2016), no. 3, 891–921.

[Wen05] M. Wendt, *Zufällige Juliamengen und invariante Maße mit maximaler Entropie*, Ph.D. thesis, University of Kiel, `https://macau.unikiel.de/receive/dissertation_diss_00001412` (German), 2005.