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Note to the Reader

The three chapters of this thesis are self-contained research papers that can be read separately. A general introduction and conclusion frame the document, while summaries in French and in English are available at the end of the manuscript.

Contents

Acknowledgments **1**

General Introduction **5**

Chapter 1. Put–call parity and generalized neo-additive pricing rules **9**
1.1 Introduction . . . 10
1.2 Framework . . . 12
1.3 The FTAP and the CFTAP . . . 12
1.4 NAC and GNAC pricing rules . . . 16
1.5 Interpretation of GNAC pricing rules . . . 18
1.6 Put-Call Parity and the FTAP . . . 20
1.7 Characterization of GNAC pricing rules . . . 22
1.8 Conclusion . . . 23
1.9 Appendix . . . 24

Chapter 2. Convex Asset Pricing **33**
2.1 Introduction . . . 34
2.2 The Model . . . 37
2.3 Viability and no-arbitrage . . . 41
2.4 Viability and the Law of One Price . . . 46
2.5 Payoff Pricing Rule . . . 47
2.6 Complete Viability and Absolute No-Arbitrage . . . 55
2.7 Conclusion . . . 57
2.8 Appendix . . . 58

Chapter 3. Dynamic completeness and Market Frictions **73**
3.1 Introduction . . . 74
3.2 3-period markets with Bid–Ask Spreads . . . 76
3.3 Multi-period Markets With Bid-Ask Spreads . . . 90
3.4 Conclusion . . . 99
3.5 Appendix . . . 101

General Conclusion **111**

General Introduction

This thesis aims to provide market participants (brokers, dealers, asset managers, market makers, and regulators) and economics researchers with new theoretical methods to value securities. We propose a new extension of the fundamental theorem of asset pricing to markets with frictions. First, we propose a simple closed-form pricing expression for security in $2-$period markets with a wide range of frictions but with constant unit security prices. Then, acknowledging that constant unit price is not verified empirically, we extend the Fundamental Theorem of Asset Pricing to $2-$period markets with frictions inducing convex prices. Finally, as a preliminary step towards generalizing our results in multiperiod markets, we characterize dynamic completeness in markets with bid–ask spreads.

The study of securities prices is a significant subject of economics research. Rightly valuing securities is a complicated and costly task requiring to gather and process a colossal amount of information on every asset. It necessitates assessing the odds of various future scenarios, estimating current and future supply and demand, assessing correlations with other markets, and ensuring consistency with relevant marketed securities. Concerning particular securities, for example, derivatives, market participants use more straightforward methods such as replication popularised by Black and Scholes (1973) to reduce the overall pricing costs. Replication is based on the principle, already very present in the economic literature, of the absence of arbitrage opportunity. There is no arbitrage opportunity in a market if it is not possible to make sure profits in an unlimited way. No-arbitrage is naturally satisfied in practice. Therefore its use in models is realistic. Arrow (1953) was the first to acknowledge the importance of arbitrage opportunities in securities' pricing. Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981) formalized the no-arbitrage pricing method in a general framework. Their results are referred to as the Fundamental Theorem of Asset Pricing (FTAP). The FTAP states that when the price functional on portfolios is linear, then the hypothesis of no-arbitrage opportunity is satisfied if, and only if, the prices of state assets are strictly positive. In other words, this theorem characterizes an implicit rule satisfied by prices in an idealized financial market, that is to say, without transaction costs, without constraints on the purchase or sale of an asset, and without taxes.

Due to this idealized market hypothesis, the FTAP does not apply in real-life situations. However, the FTAP raises an exciting question for both theoretical and practical applications: in a more realistic framework, does the non-arbitrage hypothesis also make it possible to identify implicit rules satisfied by prices?

Weakening the hypothesis of linearity of the price functional is necessary to consider the frictions present on the markets. It is the approach of Jouini and Kallal (1995) for the frictions (of the transaction cost type) which make the price functional sub-linear. These authors show that such a price functional does not admit an arbitrage opportunity if, and only if, a positive linear functional supports it. Thus they establish limits to the values that the prices of financial assets can take.

However, we the realism of a sub-linear price functional remains questionable. Sublinearity implies, on the one hand, that the price functional is subadditive. On the other hand that it is positively homogeneous. Reliance on mathematical hypotheses represents a caveat of asset pricing literature for being difficult to test empirically. In recent works, Cerreia-Vioglio, Maccheroni and Marinacci (2015) developed pricing rules relying on a single "technical" hypothesis. They assume the put–call parity (a relation on options made famous by cite BlackScholes73) is satisfied and market participants trade risk-free securities without frictions. Under the assumption that markets are complete, they show that a pricing rule satisfies these properties and monotonicity if, and only if, the pricing rule is a discounted income expectation with respect to a risk-neutral non-additive probability (also called Choquet's expectation). Additionally, **?** demonstrate that a Choquet pricing rule is a no-arbitrage price if, and only if, it is supported by a linear pricing rule. They show that the absence of arbitrage opportunity is equivalent to having a risk-neutral non-additive probability with non-empty core. Together, Cerreia-Vioglio, Maccheroni and Marinacci (2015) and **?** generalize the FTAP to markets with transaction costs on joint purchased where the put-call parity holds. We call this result the Choquet Fundamental Theorem of Asset Pricing (CFTAP). The CFTAP is an elegant generalization of the FTAP based on easily observable economic assumptions such as the put–call parity. It also gives an explicit form to the price rule, which is a natural generalization of the FTAP. Indeed, a rewrite of the FTAP makes it possible to express the pricing rule on financial assets as an expectation with respect to a risk-neutral probability. The CFTAP draw a link with the developments brought in recent years in decision theory. All the consequences of this connection have not been analyzed yet. The first simple observation is that in a market satisfying the assumptions set out above, it is generally incorrect to value an asset using probabilistic reasoning. In addition, the use of observable principles (such as put–call parity) makes its application easier for market participants.

The CFTAP inspired our research project and fed our thinking on several

levels. As a result, we developed three projects. Each chapter of this thesis presents one of them.

In chapter 1, we provide a price functional satisfying the same properties as the CFTAP, but that is more appropriate for empirical studies. We study a particular case when securities prices are payoffs expectation with respect to a generalized neo-additive capacity (GNAC). First, we express prices more straightforwardly with only two parameters and a probability. The limited number of parameters—the price of an asset is a weighted sum of the expected value (a frictionless price), and the maximal and minimal revenues—makes the results more accessible to calibrate and estimate than a Choquet expectation. Then, we highlight the existence of a theoretical connection between asset prices and risk. We show that bid–-ask spreads are proportional to the range of assets' revenues. This result is consistent with empirical evidence suggesting that bid–-ask spreads vary linearly with risk (see Benston and Hagerman (1974), Stoll (1978), Stoll (1985), Amihud and Mendelson (1986)), the range being a simple (albeit imperfect) measure of risk. Eventually, we demonstrate that prices are given by a GNAC if, and only if, in addition to satisfying put–call parity, monotonicity, and cash invariance, there is no friction among assets yielding extreme revenues in the same states of nature.

In Chapter 2, we extend the FTAP to convex price functionals. In the CFTAP and the first chapter of this thesis, prices are positively homogeneous: the ratio of order's price to order's size, the unit price, is a constant function of the size. Market participants cannot observe the form of the pricing function directly. A proxy for the unit price is the temporary market impact. The temporary market impact represents the average price change conditioned on the size and the nature of an order placed in the markets. Large institutions needing to place large orders in the markets are especially attentive to this effect. To reduce their costs associated with short-running (the cost associated with finding a counterparty immediately), imperfect substitution effects (the cost associated with the absence of a perfect substitute for the traded asset), information effect (the cost associated with agents believing that asset is mispriced) they split their orders and keep secret their true size. Several empirical studies were given access to databases enabling them to reconstitute the total order, called metaorder, placed on the markets by large institutions (see Almgren, Thum, Hauptmann and Li (2005), Moro, Vicente, Moyano, Gerig, Farmer, Vaglica, Lillo and Mantegna (2009), Tóth, Lemperiere, Deremble, Lataillade, Kockelkoren and Bouchaud (2011), Bacry, Iuga, Lasnier and Lehalle (2015)). These studies differ quite significantly in the analyzed database, how the price impact is defined and measured, how different assets and periods are collated together in the analysis, and how the fit is performed. Nevertheless, they have all consistently shown that the temporary price impact is concave, approximately increasing as the square root of the order size. Naturally, studies reconstituting

the metaorders underestimate the temporary price impact of large trades of the same nature placed at once on the market. Therefore, the real price impact has a steeper slope confirming the non-constancy of the unit price. Moreover, Kyle (1985), Glosten (1994), Bertsimas and Lo (1998), Biais, Martimort and Rochet (2000) show in different security trading models that the price functional is convex. Therefore, the generalization of the FTAP to convex prices is pertinent. It makes the FTAP applicable to a greater diversity of relevant market structures.

In chapter 3, we characterize dynamic completeness in the presence of bid-ask spreads. This work is a preliminary step towards the generalization of the results of the first two chapters of this thesis to multiperiod security markets. In the frictionless case, in the absence of arbitrage opportunity, the linearity of the price functional incurs no significant difference between the multiperiod markets and the $2-$period markets. However, in markets with frictions, multiperiod markets are not sophistications of $2-$period markets, and the possibility of trade at intermediary creates purely dynamic issues. In particular, the characterization of market completeness, which is not affected by friction in $2-$period market markets, differs in multiperiod markets as trading at intermediary periods (that lowers the minimum number of security necessary for market completeness) is a function of frictions. In this chapter, we study how bid–ask spreads, the principal transaction cost when trading stocks, futures contracts, options, or currency pairs (Kumar (2004)), impact dynamic completeness. We show that bid–ask spreads do not increase the minimum number of traded security necessary for dynamic completeness. Moreover, we demonstrate that some markets are dynamically complete only if market makers' services are costly. Eventually, we characterize dynamic completeness in markets with bid–ask spreads.

Chapter 1

Put–call parity and generalized neo-additive pricing rules

Co-authored with Jean-Philippe Lefort[1]

Abstract

We study price formulas suited for empirical research in financial markets in which put–call parity is satisfied. We find a connection between risk and the bid–ask spread. We further study the compatibility of the model with market frictions, and determine market subsets where the fundamental theorem of asset pricing applies. Finally, we characterize the price formula.

Contents

1.1 Introduction **10**
1.2 Framework **12**
1.3 The FTAP and the CFTAP **12**
1.4 NAC and GNAC pricing rules **16**
1.5 Interpretation of GNAC pricing rules **18**
1.6 Put-Call Parity and the FTAP **20**
1.7 Characterization of GNAC pricing rules **22**
1.8 Conclusion **23**
1.9 Appendix **24**

[1]LEDa, Université Paris-Dauphine

1.1 Introduction

This chapter proposes a new closed-form asset pricing expression when prices satisfy the put–call parity. The corresponding price formula is simpler and easier to calibrate than the existing pricing formula in markets with frictions. Moreover, it explicitly connects price with risk.

This chapter contributes to asset pricing literature in markets with frictions. The principal contributions to this field are from Garman and Ohlson (1981) who proposed a model when prices are linear in the number of shares traded (positively homogeneous); Jouini and Kallal (1995) who generalized the Fundamental Theorem of Asset Pricing (FTAP) (see Ross (1976b) and Harrison and Kreps (1979)) to sublinear prices, and Prisman (1986) who proposed an extension of the FTAP for markets with convex fees such as taxes. Recently, Cerreia-Vioglio, Maccheroni and Marinacci (2015) proposed a new closed-form expression to price securities in financial markets with frictions in which the put–call parity is verified. They show that, under the additional assumptions that there is no friction on frictionless security and prices are monotone, security prices are the Choquet expectation of assets' payoffs with respect to a so-called 'risk-neutral capacity'. Additionally, Bastianello, Chateauneuf and Cornet (2016) demonstrate that a Choquet pricing rule is a no-arbitrage price if, and only if, it is supported by a linear pricing rule. They show that the absence of arbitrage opportunity is equivalent to having a risk-neutral non-additive probability with non-empty core. Together, Cerreia-Vioglio, Maccheroni and Marinacci (2015) and Bastianello, Chateauneuf and Cornet (2016) generalize the FTAP to markets with transaction costs on joint purchased where the put-call parity holds. We call this result the Choquet Fundamental Theorem of Asset Pricing (CFTAP).

We further study the relationship between the Choquet expectation and asset pricing evidenced by the CFTAP. We propose a more straightforward pricing formula by restricting the capacities to be generalized neo-additive capacities (GNAC) (Chateauneuf, Eichberger and Grant (2007) and Eichberger, Grant and Lefort (2012)). Prices can be re-expressed as a function of only two parameters and a probability. The limited number of parameters—the price of an asset is a weighted sum of the expected value (a frictionless price), and the maximal and minimal revenues—makes the price formula more manageable to calibrate and to estimate (see the end of section 1.3) than a Choquet expectation. The expected value parameter measures the effects of friction on pricing. When prices are given by a neo-additive capacity (NAC), the revenue parameters naturally interpret when they are between 0 and 1. However, this additional requirement is compelling because it constrains bid–ask spreads' (viz. the difference between purchase and sell prices) values. We provide more insights on interpreting these parameters in both frameworks in section 1.5.

We highlight a theoretical connection between asset prices and risk. We

show that the bid–ask spread is proportional to the range of an asset's revenues. This is consistent with empirical evidence suggesting that bid–ask spreads vary linearly with risk (see Benston and Hagerman (1974), Stoll (1978), Stoll (1985), Amihud and Mendelson (1986)), the range being a simple (albeit imperfect) measure of risk. Stoll (1978) and Amihud and Mendelson (1986) showed that this relationship is positive—i.e., the higher the risk, the broader the spread. This relationship entails the absence of arbitrage opportunities in the spread. Moreover, it is equivalent to placing a higher emphasis on maximal revenues than minimal revenues.

In section 1.6, we analyze the compatibility of general-capacity price formulas and price formulas given by a GNAC. We represent a general capacity by its associated *Weber set*[2]. We show that there is no friction among a subset of assets if, and only if, the probabilities in the Weber set coincides on specific events. We conclude that subsets of risky frictionless assets might exist even when a general capacity represents prices. Therefore, the FTAP applies to specific parts of a financial market. Naturally, the set of prices given by a GNAC is less flexible. We show that either there is no friction in the market (and the FTAP applies everywhere) or there does not exist a risky *frictionless asset*.

This apparent shortcoming is not particularly concerning because, in practice, risky frictionless assets are unlikely to exist. Moreover, the price formula is compatible with the absence of friction among a subset of assets. Eventually, we demonstrate that prices are given by a GNAC if, and only if, in addition to satisfying put–call parity, cash invariance, and monotonicity, there is no friction among assets that yield extreme revenues in the same states of nature.

We organized this chapter in the following manner. In section 1.2, we present the framework. In section 1.3, we present the FTAP of Harrison and Kreps (1979) and the CFTAP. In section 1.4, we present the price formula given by a NAC and the price formula given by a GNAC and study how bid–ask spreads relate to risk. In section 1.5, we interpret the parameters of a price formula given by a NAC and a price formula given by a GNAC, and we consider situations where price formulas given by a GNAC is better suited for asset pricing. In section 1.6, we discuss the relationship between the CFTAP and the FTAP. Finally, in section 1.7, we characterize the GNAC pricing formula. We gathered the mathematical proofs in the appendix.

[2]Weber sets are very close to rank-dependent probability assignments (see Nehring (1999)) and to Clarke differentials at 0 (see Ghirardato, Maccheroni and Marinacci (2004)) hence our results could be translated in these languages.

1.2 Framework

We consider a two-period $t \in \{0, 1\}$ financial market with trading occurring only on date $t = 0$. The outcome of the second period is uncertain and is represented by a finite set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ comprising m states of nature. At date 0, agents access the market without costs or constraints. They assemble a portfolio among a finite set of primary assets available for trading. They buy (or sell) the right to receive the payoff $X \in \mathbb{R}^\Omega$ (e.g., the right to receive $X(\omega)$ in state of nature ω at date $t = 1$). We assume that the market is complete. In particular, put and call options are available for all assets, and agents can compose a portfolio that gives a frictionless payoff (or cash) $x_{rf} \in \mathbb{R}^\Omega$ which corresponds to the constant unit vector. In our discussion below, $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ is a pricing rule, that is a non-zero map for which $\tilde{\pi}(X)$ $(-\tilde{\pi}(-X))$ represents the amount of resources an agent should spend (or receive) at date 0 when buying (or selling) the payoff X.

1.3 The FTAP and the CFTAP

The absence-of-friction hypothesis, which is at the core of most of the literature on fundamental asset pricing, states that a market is frictionless when splitting orders does not impact the total order price. In other words, the pricing rule is linear: for all payoffs $(X, Y) \in \mathbb{R}^\Omega \times \mathbb{R}^\Omega$ and all $\lambda \in \mathbb{R}$,

$$\tilde{\pi}(\lambda X + Y) = \lambda \tilde{\pi}(X) + \tilde{\pi}(Y).$$

The absence-of-friction hypothesis is not an equilibrium property. For example, if $\tilde{\pi}$ is a negative linear function, then it is optimal for an agent (independently of her preferences) to buy an infinite quantity of assets with positive payoffs, resulting in a sub-optimal portfolio. On the other hand, Harrison and Kreps (1979) demonstrated that *no-arbitrage*, that is, for all $X \in \mathbb{R}^\Omega$,

$$X > 0 \implies \tilde{\pi}(X) > 0,$$

where $X > 0$ implies that $X(\omega) \geqslant 0$ for all $\omega \in \Omega$, with at least one strict inequality, is an essential equilibrium property of frictionless financial markets. They show that absence-of-friction and no-arbitrage are equivalent to the existence of a unique probability such that the price is the expected value of the portfolio's payoffs.

Theorem 1.3.1 (FTAP, Harrison and Kreps (1979))**.** *Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be a non-zero pricing rule. The following statements are equivalent:*

i. $\tilde{\pi}$ is frictionless and has no arbitrage opportunity;

ii. there exists a unique risk-neutral probability μ and a riskless rate $r > -1$

such that

$$\tilde{\pi}(X) = \frac{1}{1+r}\mathbb{E}_{\mu}(X) = \frac{1}{1+r}\sum_{i=1}^{m} X(\omega_i)\mu(\omega_i) \; \forall X \in \mathbb{R}^{\Omega}.$$

This theorem is a fundamental result of financial economics. It provides an explicit formula for pricing assets, supported by equilibrium requirements, and, in a multiple-period market, it demonstrates the existence of a connection between martingale theory and asset pricing. However, its reliance on the absence-of-friction hypothesis is a significant caveat. This hypothesis neglects the transaction costs and fees paid by market participants.

Several pricing models have been developed to generalize the FTAP to incorporate markets' frictions. Recently, Cerreia-Vioglio, Maccheroni and Marinacci (2015) proposed a price formula that incorporates various form of transaction costs. Moreover, their price formula establishes a new link between asset pricing and non-linear expectation theory. In a nutshell, a capacity $\nu : \mathcal{P}(\Omega) \to [0,1]$, also informally referred to as a non-additive probability, satisfies the following properties $\nu(\emptyset) = 0,\ \nu(\Omega) = 1$ (normalization) and $\nu(A) \leqslant \nu(B)$ whenever $A \subseteq B \subseteq \Omega$ (monotonicity). The expected value with respect to a capacity is called the Choquet expected value. It is defined as follows. Consider a vector $X \in \mathbb{R}^{\Omega}$ and a permutation of the states of nature $(\omega_1^*, \omega_2^*, \ldots, \omega_m^*)$ such that $X(\omega_1^*) \geqslant X(\omega_2^*) \geqslant \ldots \geqslant X(\omega_m^*)$. Then, the Choquet expected value of X with respect to the capacity ν is

$$\mathbb{CE}_{\nu}(X) := X(\omega_1^*)\nu(\omega_1^*) + \sum_{i=2}^{m} X(\omega_i^*)[\nu(\{\omega_1^*, \ldots, \omega_i^*\}) - \nu(\{\omega_1^*, \ldots, \omega_{i-1}^*\})].$$

Cerreia-Vioglio, Maccheroni and Marinacci (2015) model relies on the put–call parity (see Stoll (1973)). In a financial market, the put–call parity is satisfied if, for every call options $c_{X,k} := \max(X - kx_{rf}, 0)$ and every put options $p_{X,k} := \max(kx_{rf} - X, 0)$ on the same underlying payoff $X \in \mathbb{R}^{\Omega}$ with strike price $k \in \mathbb{R}$, we have

$$\tilde{\pi}(c_{X,k}) + \tilde{\pi}(-p_{X,k}) = \tilde{\pi}(X) - k\tilde{\pi}(x_{rf}),$$

where $\max(X, Y) \in \mathbb{R}^{\Omega}$ is the vector such that, $\forall \omega \in \Omega$, $\max(X, Y)(\omega) = \max(X(\omega), Y(\omega))$. In other words, the put–call parity force the strategy consisting in buying a call option and selling a put option on the same underlying payoff with identical strike price k to cost the same price as the strategy consisting in buying the underlying payoff and selling k units of the riskless payoff. It is nominally true that when there is no friction, the two strategies earn the same revenues; however, if one strategy were more expensive than the other, then the demand for it would be nil. Thus, at

equilibrium, the two strategies must be equally priced. Furthermore, it is typically assumed that an asset with a higher payoff than another must cost at least the same price, that is, for all $(X, Y) \in \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega}$, $X \geqslant Y$ implies that $\tilde{\pi}(X) \geqslant \tilde{\pi}(Y)$. Therefore, the pricing rule is monotonic. As is usual in asset pricing literature, this model also assumes that risk-free payoff x_{rf} is *frictionless*, that is, for all $X \in \mathbb{R}^{\Omega}$ and for all $k \in \mathbb{R}$, $\tilde{\pi}(X + kx_{rf}) := \tilde{\pi}(X) + k\tilde{\pi}(x_{rf})$. This last property is labeled, cash-invariance. Their main result is the following characterization of these pricing rules.

Theorem 1.3.2 (Cerreia-Vioglio et al. (2015))**.** *Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a non-zero pricing rule. The following statements are equivalent:*

i. $\tilde{\pi}$ satisfies put–call parity, cash-invariance and monotonicity;

ii. there exists a unique risk neutral capacity ν and a unique riskless rate $r > -1$ such that

$$\tilde{\pi}(X) = \frac{1}{1+r}\mathbb{CE}_{\nu}(X), \forall X \in \mathbb{R}^{\Omega}.$$

Cerreia-Vioglio, Maccheroni and Marinacci (2015) show that the capacity is a probability if, and only if, $\tilde{\pi}$ is frictionless. Moreover, they show that bid–ask spreads are positive when the capacity is balanced. However, they do not exclude the arbitrage opportunities created by the presence of frictions on the market. Bastianello, Chateauneuf and Cornet (2016) define an arbitrage opportunity when there is friction as follows. A pricing rule satisfy no buy-and-sell arbitrage opportunity if for any $X_i \in \mathbb{R}^{\Omega}$, $i = 1 \dots n$, and any $Y_j \in \mathbb{R}^{\Omega}$, $j = 1 \dots p$,

$$\sum_{i=1}^{n} X_i = \sum_{j=1}^{p} Y_j \implies \sum_{i=1}^{n} \tilde{\pi}(X_i) \geqslant -\sum_{j=1}^{p} \tilde{\pi}(-Y_j);$$

and

$$\sum_{i=1}^{n} X_i > \sum_{j=1}^{p} Y_j \implies \sum_{i=1}^{n} \tilde{\pi}(X_i) > -\sum_{j=1}^{p} \tilde{\pi}(-Y_j).$$

In words, no buy-and-sell arbitrage opportunity means that if two portfolios earn the same payoffs at date 1 then, even when splitting the portfolios, the total price paid to acquire the payoffs must be higher than the total price received to sell them. They show that a choquet pricing rule is a no arbitrage price (it satisfies the no-buy-and-sell condition) if, and only if, it is supported by a linear pricing rule, that is, if, and only if, its risk-neutral capacity has a non-empty core. Formally, we say that a capacity has a non-empty core when the set core(ν) equal to

$$\{\mu \mid \mu(A) \leqslant \nu(A), \text{ for all } A \in \mathcal{P}(\Omega) \text{ and } \mu : \mathcal{P}(\Omega) \to [0,1] \text{ is a probability}\}.$$

is not empty.

Theorem 1.3.3 (Bastianello, Chateauneuf and Cornet (2016)). *Let* $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ *be a Choquet pricing rule with respect to a capacity* $\nu : \mathcal{P}(\Omega) \to [0,1]$. *The following statements are equivalent:*

(i) $\tilde{\pi}$ *is a no-arbitrage price;*

(ii) there exists a probability $\mu \in core(\nu)$ *with* $\mu(\omega) > 0$ *for all* $\omega \in \Omega$, *that is,*

$$\tilde{\pi}(X) \geqslant \mathbb{E}_{\mu}[X], \text{ for all } X \in \mathbb{R}^{\Omega}_{+}.$$

Together, Theorem 1.3.2 and Theorem 1.3.3 extend in a very elegant manner the FTAP to markets with friction where the put–call parity is satisfied. We call these results the Choquet Fundamental Theorem of Asset Pricing (CFTAP). Reliance on testable hypothesis makes it more valuable to empirical research than previous extension of the FTAP to markets with friction.

Furthermore, Choquet expectations and capacities are at the core of innovation in economic theory, especially in decision theory. Originally, Schmeidler (1989) developed the Choquet expected utility model to generalize the classical expected-utility model and accommodate the Ellsberg paradox and the Allais paradox. Since then, these results have applications in finance (see Chateauneuf, Kast and Lapied (1996), Waegenaere, Kast and Lapied (2003), Chen and Kulperger (2006) and Kast, Lapied and Roubaud (2014)), insurance (see Castagnoli, Maccheroni and Marinacci (2002) and Castagnoli, Maccheroni and Marinacci (2004)), risk measurement (see Denuit, Dhaene, Goovaerts, Kaas and Laeven (2006)) and investment behavior (see Ludwig and Zimper (2006) and Driouchi, Trigeorgis and So (2018)).

Such applications invite the study of connections between the Choquet expectation and asset pricing. We propose to study a particular family of capacities which is among the most convenient and falls between general capacities and probabilities, the family of so-called neo-additive capacities (NACs) or, more precisely, their generalized form, the so-called generalized neo-additive capacities (GNACs). The NACs were developed by Chateauneuf, Eichberger and Grant (2007) to obtain a model of non-linear expected utility more tractable than the Choquet expected utility. Indeed, NACs have fewer parameters needed for calibration than a general capacity, making them more suitable for empirical research. They have applications in asset pricing (see Zimper (2012)), investment behavior (see Ford, Kelsey and Pang (2005)), risk (see Chakravarty and Kelsey (2017)), game theory (see Eichberger and Kelsey (2011), Jungbauer and Ritzberger (2011) and Eichberger and Kelsey (2014)), learning behavior (see Zimper and Ludwig (2009)), health and retirement (see Groneck, Ludwig and Zimper (2016)) and for extending the common knowledge theorem of Aumann (see Dominiak and Lefort (2013)). In the context of asset pricing, the generalized

form of a NAC developed by Eichberger, Grant and Lefort (2012) appears more suitable because it creates no relationship between the *bid–ask spread* and the *power of explanation of a frictionless price*. We introduce more formally the NACs and the GNACs and their associated pricing formulas in the following section. We discuss their interpretation and the previous argument in favor of GNAC pricing rules in section 1.5.

1.4 NAC and GNAC pricing rules

For the sake of our exposition, we assume that the set of null events, that is, the set whose events are "impossible to occur" has only one element, the empty set $\emptyset$. However, the validity of the following results does not rely on this assumption. A NAC is a convex combination of a probability and a parameter which takes values between 0 and 1. More formally, the function ν is a NAC if there exists a probability $p : \mathcal{P}(\Omega) \to [0,1]$ and two reals $\alpha \in [0,1]$ and $\delta \in [0,1]$ satisfying $\min_{E \notin \{\emptyset,\Omega\}} [\alpha + \delta p(E)] \geqslant 0$ and $\max_{E \notin \{\emptyset,\Omega\}} [\alpha + \delta(1 - p(E))] \leqslant 1$ such that

$$\nu(E) = \alpha\delta + (1-\delta)p(E), \forall E \notin \{\emptyset, \Omega\}$$

Chateauneuf, Eichberger and Grant (2007) showed that the Choquet expectation with respect to a NAC is a convex combination of the expected value with the maximal and minimal revenues, that is,

$$\tilde{\pi}(X) = (1-\delta)\mathbb{E}(X \mid p) + \delta(\alpha \max(X) + (1-\alpha)\min(X))$$

where $\max(X)$ (or $\min(X)$) is the maximum (or, respectively, minimum) of the coordinates of X. We say that $\tilde{\pi}$ is a NAC pricing rule if it satisfies this equality. Prices given by a NAC are a combination of a frictionless price and the maximal and the minimal revenues. Eichberger, Grant and Lefort (2012) generalized NACs by letting the parameter α take any real value, and δ take any real value less than 1. The resulting function—which they named a GNAC—is an affine transformation of a probability. The remaining constraints on the parameters make the GNAC normalized and monotone, thus a well-defined capacity. For the sake of discussion, Eichberger, Grant and Lefort (2012) substituted two new parameters, a and b for α and δ. We have reproduced their presentation below. Except for the constraints on the values taken by the parameters, the two formulas are equivalent when $a = \delta\alpha$ and $b = 1 - \delta$. Formally, the function ν is a GNAC if there exists a probability, $p : \mathcal{P}(\Omega) \to [0,1]$, and two reals, a and $b > 0$, satisfying $\min_{E \notin \{\emptyset,\Omega\}} [a + bp(E)] \geqslant 0$ and $\max_{E \notin \{\emptyset,\Omega\}} [a + b(1 - p(E))] \leqslant 1$ such that

$$\nu(E) = a + bp(E), \forall E \notin \{\emptyset, \Omega\}$$

and $\nu(\Omega) = 1$ and $\nu(\emptyset) = 0$. The preceding constraints $\min_{E \notin \{\emptyset,\Omega\}} [a + bp(E)] \geqslant 0$ and $\max_{E \notin \{\emptyset,\Omega\}} [a + b(1 - p(E))] \leqslant 1$, simply ensure that the values of a and b are chosen so that the function ν is monotone. Eichberger, Grant and Lefort (2012) showed that the Choquet expectation with respect to a GNAC is a weighted sum of the expected value with the maximal and the minimal revenues, that is,

$$\tilde{\pi}(X) = b\mathbb{E}(X \mid p) + a \max(X) + (1 - a - b) \min(X).$$

We discuss the interpretation of the parameters of NAC pricing rules and GNAC pricing rules in the following section. Overall, GNAC pricing rules require determining a smaller number of parameters: only $m + 2$ parameters—the values taken by the probability p and the values of a and b—whereas it is necessary to define up to $2^m - 2$ parameters in the general case. As a result, general capacities may provide better accuracy in the pricing of assets, but the additional cost of estimating all the necessary parameters offsets these improvements.

Interestingly, GNAC connects prices with risk through the bid–ask spread—the difference between the price at which one can immediately buy a payoff and the price at which one can immediately sell it. Formally, the bid–ask spread $B : \mathbb{R}^\Omega \to \mathbb{R}$ is

$$B(X) = \tilde{\pi}(X) + \tilde{\pi}(-X),$$

for all $X \in \mathbb{R}^\Omega$. With GNACs, bid–ask spreads are proportional to the range of asset revenues. Indeed, there exists $\lambda \in \mathbb{R}$ such that, for all $X \in \mathbb{R}^\Omega$,

$$B(X) = \lambda \left[\max(X) - \min(X)\right]$$

where $\lambda = 2a + b - 1$, i.e., λ is the difference between the additional weight on the maximal revenue and the additional weight on the minimal revenue. The coefficient λ is the coefficient of proportionality of B. This result is consistent with empirical evidence showing that bid–ask spreads are in direct relationship with risk (see Benston and Hagerman (1974), Stoll (1978), Stoll (1985), Amihud and Mendelson (1986)). Indeed, the range is a simple (albeit imperfect) measure of risk.

Stoll (1978) and Amihud and Mendelson (1986) evidenced that the relationship between the spread and the risk is positive: the higher the risk, the broader the spread. Here, this relationship is natural: the bid–ask spread is necessarily positive, for otherwise, a clear arbitrage opportunity exists. Therefore, we say that there is no arbitrage opportunity in the bid–ask spread when for every $X \in \mathbb{R}^\Omega$,

$$B(X) \geqslant 0.$$

The positivity of the spread has a natural interpretation for GNACs. It results in a positive coefficient of proportionality λ. Hence, the additional weight given to the maximal revenue a is greater than the additional weight given to the minimal revenue $1 - a - b$. For NACs, this condition is even simpler: there is no arbitrage opportunity in the spread if, and only if, $\alpha \geqslant 0.5$. For a general capacity, having no arbitrage opportunity in the bid–ask spread is more demanding. We present the property on the capacity equivalent to having no arbitrage in the spread in the following proposition.

Proposition 1.4.1. Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a Choquet pricing rule with respect to the capacity $\nu : \mathcal{P}(\Omega) \to [0, 1]$. The following statements are equivalent:

i. $\tilde{\pi}$ does not have an arbitrage opportunity in the bid–ask spread;

ii. $\nu(A) + \nu(A^c) \geqslant 1$ for all $A \in \mathcal{P}(\Omega)$.

In particular, when $\tilde{\pi}$ is a GNAC pricing rule then there is no arbitrage opportunity in the bid–ask spread if, and only if, $a \geqslant 1 - a - b$.

Garbade (1982) and Stoll (1985) have shown that Bid–ask spreads are good indicators of the liquidity of the asset: the narrower the bid–ask spread, the more liquid the asset. For a general capacity, the bid–ask spread of riskless payoff is zero and consistent with the perception that riskless assets are the most liquid assets. For GNACs, it is also the case that security with revenues close to being riskless—where their range of revenues is tight—have a small spread. Therefore these securities are more liquid.

Bid–ask spreads are not necessarily proportional to the range for general capacities. However, the following lemma presents the property for proportionality to apply.

Lemma 1.4.1. Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a Choquet pricing rule with bid–ask spread $B : \mathbb{R}^{\Omega} \to \mathbb{R}$. Then the following assertions are equivalent:

i. The bid–ask spread is proportional to the range of revenues;

ii. $\exists \lambda \in \mathbb{R}$, $\forall E \notin \{\emptyset, \Omega\}$, $B(1_E 0) = \lambda$;

iii. $\nu(E) + \nu(E^c) = k$, $\forall E \notin \{\emptyset, \Omega\}$

where $x_E y \in \mathbb{R}^{\Omega}$ is the vector with coordinates in $E \in \mathcal{P}(\Omega)$ equal to x and coordinates in E^c equal to y.[3]

1.5 Interpretation of GNAC pricing rules

In addition to involving fewer parameters needing calibration, the coefficients of a NAC and a GNAC pricing rule are also easier to interpret than

[3] We refer to such vectors as bets.

a general-capacity pricing rule. For NACs, we interpret the coefficient δ as the power of explanation frictionless pricing in the market. Differently put, δ measures how close the market is to be frictionless. As a result, it provides information on the importance of transaction costs and other frictions on asset pricing. The closer δ is to 1, the less significant the role frictions play. For NACs, revenues bound prices. The following inequalities are satisfied for all $X \in \mathbb{R}^{\Omega}$:

$$\min(X) \leqslant \tilde{\pi}(X) \leqslant \max(X).$$

The parameter α indicates whether the price is close to the maximal bound. When $\delta = 1$ and $\alpha = 1$, the asset price is maximal; when $\delta = 1$ and $\alpha = 0$, the asset price is minimal. The first situation captures agents' extreme confidence that the maximal revenue will be delivered in the future. The second situation captures agents' extreme confidence that the minimal revenue will be delivered in the future. As explained in the previous section, having $\alpha < 0.5$ creates arbitrage opportunities. Therefore, asset prices are never minimal. Other values of α and δ mediate between these extremes.

Interpreting the parameters of a GNAC pricing rule requires more prudence. The coefficient b still provides information on the importance of the role played by friction in the pricing. However, both b and a can take values greater than 1; therefore, for the interpretation to be meaningful, it is preferable to analyze $b/(|a| + b)$ and $|a|/(|a| + b)$. These values can be interpreted similarly to δ and α. The closer $b/(|a| + b)$ to 1, the lesser friction influences the pricing. So, $b/(|a| + b)$ is the power of explanation of a frictionless pricing rule. Similarly, the closer $|a|/(|a| + b)$ is to 1, the greater the effect of frictions on pricing.

For GNACs, revenues do not bound prices. It allows a great deal of pricing flexibility. In particular, GNAC allows calibration of over-confident behavior—when $|a|/(|a| + b)$ is close to 1 and $a > 1$—and under-confident behavior—when $|a|/(|a| + b)$ is close to 1 and $a < 0$. Prices can be disconnected from the revenues of assets, i.e. when prices are either greater than the maximal revenue or smaller than the minimal revenue. Therefore, GNAC pricing rules are relevant to represent both boom and bust scenarios. These situations cannot be represented in a frictionless environment or with a NAC.

Moreover, GNACs allow more flexibility in the calibration of the parameters. NACs impose bounds on the value of the proportionality coefficient of the spread, which has to be smaller than 1. The bounding is even tighter. NACs require λ to be smaller than δ. It creates a strong relationship between the bid–ask spread and the power of explanation of a frictionless pricing rule. For example, it is impossible to have both asset prices explained at 95% by a frictionless pricing rule and a 10% coefficient of proportionality in the bid–ask spread.

1.6 Put-Call Parity and the FTAP

In the previous sections, we argued that GNAC pricing rules are better suited to asset pricing because they incorporate fewer parameters to calibrate, and the parameters are easier to interpret. From this perspective, a frictionless market is ideal with only a probability to calibrate. In this section, we identify subset of the market where the FTAP applies. Intuitively, the FTAP applies to a market subset if splitting orders does not impact the total price. To examine this, we define new objects: *risky frictionless payoffs*. A risky frictionless payoff is an uncertain payoff that presents no friction. Formally, the payoff $X \in \mathbb{R}^\Omega$ is frictionless if, for all $Y \in \mathbb{R}^\Omega$ and all $a \in \mathbb{R}$,

$$\tilde{\pi}(aX + Y) = a\tilde{\pi}(X) + \tilde{\pi}(Y).$$

Likewise, if all bets on a particular event $E \in \mathcal{P}(\Omega)$ are frictionless, then we say that this event is frictionless, that is, the payoffs $\mathbb{1}_E 0$ and $\mathbb{1}_{E^c} 0$ are frictionless. From this perspective, cash invariance is a no-friction property: riskless payoffs are frictionless, and Ω is a frictionless event. Moreover, a frictionless asset has no bid–ask spread. In the following, we determine which capacities' properties are equivalent to the existence of a risky frictionless asset. In particular, we define frictionless sets of payoffs and determine the relationships satisfied by a Choquet pricing rule on this set.

It is well known that Choquet expectations are positively homogeneous, that is, $\forall X \in \mathbb{R}^\Omega$, $\forall k \geqslant 0$, $\tilde{\pi}(kX) = k\tilde{\pi}(X)$, and additive with respect to comonotone vectors (vectors $(X, Y) \in \mathbb{R}^\Omega \times \mathbb{R}^\Omega$ such that for all $\omega, \omega' \in \Omega$, $\omega \neq \omega'$, $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0$). We can associate to a vector $X \in \mathbb{R}^\Omega$ a ranking of the states of nature ρ (that is a bijection between Ω and $\{1, \ldots, m\}$) which associates 1 to the state ω such that $X(\omega)$ is the highest payoff of X, 2 to the second highest and so on. This is useful because the Choquet expectation of $X \in \mathbb{R}^\Omega$ can be regarded as the expectation value of the vector with respect to a probability $\mu_\rho : \mathcal{P}(\Omega) \to [0, 1]$ given by

$$\mu_\rho(E) = \sum_{\omega \in E} [\nu(P_\rho(\omega) \cup \{\omega\}) - \nu(P_\rho(\omega))],$$

for all $E \in \mathcal{P}(\Omega)$ where $P_\rho(\omega) := \{\omega' \in \Omega \mid \rho(\omega') < \rho(\omega)\}$ is the set of predecessors of ω. This representation is particularly helpful when attempting to understand why the Choquet expectation is additive with respect to comonotone vectors. The set of probabilities, μ_ρ, is called the Weber set of ν (Weber (1988)); we denote it $\mathcal{W}(\nu)$. There exists a connection between the absence of friction and Weber sets. Our first results are valid for general capacities and the corresponding Choquet pricing rules. First, we show that an event is frictionless if, and only if, the associated capacity is additive for this event. Moreover, we show that an event is frictionless if, and only if, Weber set's probabilities equal the capacity for this event.

Proposition 1.6.1. Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule with respect to the capacity $\nu : \mathcal{P}(\Omega) \to [0,1]$. Let E be an event in $\mathcal{P}(\Omega)$. The following statements are equivalent:

i. E is frictionless;

ii. $\nu(A) = \nu(A \cap E) + \nu(A \cap E^c)$ for all A in $\mathcal{P}(\Omega)$;

iii. $\mu(E) = \nu(E)$ for all $\mu \in \mathcal{W}(\nu)$.

Furthermore, a payoff is frictionless if, and only if, it can be decomposed as a sum of bets on frictionless events.

Proposition 1.6.2. Let $X = \sum_{i=1}^n x_i \mathbb{1}_{E_i} \in \mathbb{R}^\Omega$ where for all i $x_i > x_{i+1}$, $x_i \in \mathbb{R}$ and $E_i \in \mathcal{P}(\Omega)$. Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule. The following statements are equivalent:

i. X is frictionless;

ii. for all $i \in \{1, \ldots, n\}$, E_i is a frictionless event.

We deduce from Proposition 1.6.2 that the set of frictionless events forms a linear space. We denote it Φ. On Φ, the FTAP applies and any probability of the Weber set of ν can be used to price payoffs: for all $X \in \Phi$,

$$\tilde{\pi}(X) = \mathbb{E}_\mu(X) \quad \text{where } \mu \in \mathcal{W}(\nu).$$

If we can determine that a risky payoff is frictionless, then it is possible to price a large set of payoffs by using the FTAP and any probability within the Weber set of the capacity, and at the same time to price payoffs that are not frictionless with the capacity. From another perspective, we can easily incorporate the existence of frictionless payoffs when calibrating a capacity by letting the corresponding values of the probabilities of the Weber set coincide.

For GNACs, risky frictionless payoffs and frictions are incompatible: GNAC pricing rules require that either the market is frictionless or that there exists no risky frictionless payoff on the market. However, this loss of generality does not turn out to be an argument against GNAC pricing rules. Indeed, it is improbable in practice to encounter a risky frictionless payoff. Our first result is slightly more compelling: it demonstrates that the absence of a bid–ask spread for a risky bet is necessary and sufficient for the absence of bid–ask spreads on the whole market.

Lemma 1.6.1. Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be a GNAC pricing rule with respect to the GNAC $\nu : \mathcal{P}(\Omega) \to [0,1]$ with bid–ask spread $B : \mathbb{R}^\Omega \to \mathbb{R}$. The following assertions are equivalent:

i. $\exists A \in \mathcal{P}(\Omega)$ such that $A \notin \{\emptyset, \Omega\}$ and $\nu(A) + \nu(A^c) = 1$;

ii. $B(X) = 0$ for all $X \in \mathbb{R}^{\Omega}$.

In order to account for the presence of a bid–ask spread on a risky payoff, it is necessary to assume that all other risky payoffs present in the market have a bid–ask spread. In practice, this condition does not seem unrealistically demanding because bid–ask spreads are the most common type of frictions present in financial markets. For Kumar (2004) they represent the principal transaction cost when trading stocks, options, futures and currency pairs. Our second result shows that a GNAC pricing rule is frictionless if, and only if, there exists a frictionless event.

Proposition 1.6.3. Let $\tilde{\pi} : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a GNAC pricing rule and let $E \notin \{\emptyset, \Omega\}$ be an event. The following assertions are equivalent:

i. E is a frictionless event;

ii. $\tilde{\pi}$ is frictionless.

We deduce from Propositions 1.6.2 and 1.6.3 that, if there exists a frictionless risky payoff, then the market is frictionless. Theoretically, this may seem a demanding restriction, but, in practice, it is unrealistic to assume that a risky payoff is frictionless when the market is complete since it implies that this payoff may be added to any other portfolio without friction. In conclusion, for practical valuation matters, the loss of flexibility incurred by a GNAC pricing rule is not problematic when all risky payoffs have a bid–ask spread. Moreover, as we show in the next section, GNAC pricing rules are compatible with the existence of a frictionless subset of payoffs, namely those with matching extreme revenues. GNAC pricing requires the reasonable assumption that purchasing payoffs of this subset jointly do not incur additional transaction costs.

1.7 Characterization of GNAC pricing rules

Put–call parity, cash invariance and monotonicity imply having no friction among comonotone payoffs (Greco (1982)). This section shows that GNAC pricing rules can be characterized by put–call parity, monotonicity, cash invariance and the absence of friction among payoffs yielding extreme revenues in the same states of nature. Since GNACs are a subset of capacities, we only have to demonstrate that a Choquet pricing rule is GNAC if, and only if, there is no friction among payoffs yielding extreme revenues in the same states of nature. The rest of the proof is a consequence of the CF-TAP (section 1.3). We denote $\arg\max X \in \mathcal{P}(\omega)$ (or, $\arg\min X \in \mathcal{P}(\Omega)$) the arguments of the maxima of X that is, the set of states of nature E where the coordinates of X are maximal (or, respectively, minimal). We say that two payoffs $(X, Y) \in \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega}$ have matching extreme revenues if their

maximal and minimal revenues occur in the same states of nature, that is, if

$$\arg\max X \cap \arg\max Y \neq \emptyset; \text{ and}$$
$$\arg\min X \cap \arg\min Y \neq \emptyset.$$

We expand the definition of frictionless payoffs to the absence of friction among payoffs with matching extreme revenues in the following way. There is no friction between payoffs with matching extreme revenues if for every payoffs $(X, Y) \in \mathbb{R}^\Omega \times \mathbb{R}^\Omega$ with matching extreme revenues, we have

$$\tilde{\pi}(X + Y) = \tilde{\pi}(X) + \tilde{\pi}(Y).$$

We show that a Choquet pricing rule is a GNAC if, and only if, there is no friction among matching extreme payoffs.

Proposition 1.7.1. Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be a Choquet pricing rule. The following assertions are equivalent:

i. $\tilde{\pi}$ satisfies no friction among payoffs with matching extreme revenues;

ii. $\tilde{\pi}$ satisfies no friction among bets with matching extreme revenues;

iii. $\tilde{\pi}$ is a GNAC pricing rule.

Therefore, GNAC pricing rules are compatible with markets in which put–call parity is satisfied and there is no friction between matching extreme payoffs. NAC pricing rules are more restrictive. For NACs, the events in which the payoff yields its maximal revenue and its minimal revenue are both overweighted. This adds constraints on the prices of non-matching extreme payoffs. Indeed, Chateauneuf, Eichberger and Grant (2007) showed that NACs imply that there exist $E, F, G, H \neq \emptyset$ with $E \cup F \neq \Omega$, $G \cup H \neq \Omega$ and $E \cap F = \emptyset = G \cap H$ such that

$$\tilde{\pi}(1_{E\cup F}0) \leqslant \tilde{\pi}(1_E 0) + \tilde{\pi}(1_F 0);$$
$$\tilde{\pi}(1_{G\cup H}0) \geqslant \tilde{\pi}(1_G 0) + \tilde{\pi}(1_H 0).$$

Thus the set of prices given by a NAC is not compatible with financial markets in which it is always more expensive to buy assets separately due to frictions. It is also not compatible with a financial market in which buying assets separately is always less expensive.

1.8 Conclusion

We propose a particular case of the pricing function of CFTAP when the capacity used for pricing is an affine transformation of a probability. This

case appears better adapted to empirical applications. It requires defining a smaller number of parameters: this number increases linearly in the uncertainty represented by the number of states of nature, whereas it increases exponentially in the general case. Additionally, we highlight a connection between bid–ask spreads and risk in this case. We show that bid–ask spreads are an increasing function of the range of securities' revenues. Finally, we show that prices are the expected value of securities payoffs with respect to a unique affine transformation of a probability if, and only if, in addition to having the put-call parity, monotonicity, and cash invariance, there is no friction among securities paying their highest payoffs in the same events.

1.9 Appendix

Proof of lemma 1.4.1. We first assume that the bid–ask spread is proportional to a constant. It follows immediately that the bid–ask spread of bets which yield 1 if some event occurs, and 0 if the complementary event occurs, is constant.

Now, we assume that the bid–ask spread of bets of the form $1_E 0$, where E is an event of Ω, is equal to a constant $\lambda \in \mathbb{R}$. We are going to show that the capacity values of complementary events sum to a constant. For all $E \notin \{\emptyset, \Omega\}$ we have

$$B(1_E 0) = \nu(E) + \nu(E^c) - 1.$$

Thus,

$$\nu(E) + \nu(E^c) = \lambda + 1 \text{ for all } E \notin \{\emptyset, \Omega\}.$$

Finally, we assume that the capacity values of complementary events sum to a constant $k \in \mathbb{R}$. We are going to show that the bid–ask spread is proportional to the range of revenues. We let $X \in \mathbb{R}^\Omega$. We denote $x_1, \ldots, x_n$ the n coordinates of X such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. Up to reindexing the states of nature, we assume that the payoff X yields x_i in ω_i for all $i \in \{1, \ldots, n\}$. By definition, the bid–ask spread of X equals

$$\sum_{i=1}^{m} x_i [\nu(\{\omega_j \in \Omega \mid j \leqslant i\}) - \nu(\{\omega_j \in \Omega \mid j < i\}) - \nu(\{\omega_j \in \Omega \mid j \geqslant i\}) + \nu(\{\omega_j \in \Omega \mid j > i\})]$$

which simplifies to

$$x_1[\nu(\{\omega_1\}) + \nu(\{\omega_2, \ldots, \omega_m\}) - 1] - x_m[\nu(\{\omega_1, \ldots, \omega_{m-1}\}) + \nu(\{\omega_m\}) - 1].$$

By applying the above assumption and by substituting $\lambda = k - 1$, we obtain

the desired result

$$B(X) = \lambda(x_1 - x_m).$$

□

Proof of Proposition 1.4.1. We first assume that there is no arbitrage in the bid–ask spread. We are going to show that the capacity values of complementary events sum to a real greater than 1. By assumption, we have

$$\tilde{\pi}(X) \geqslant -\tilde{\pi}(-X), \text{ for all } X \in \mathbb{R}^{\Omega}.$$

In particular, we have

$$\tilde{\pi}(1_A 0) \geqslant -\tilde{\pi}(-1_A 0), \text{ for all } A \in \mathcal{P}(\Omega)$$

which implies

$$\nu(A) + \nu(A^c) \geqslant 1, \text{ for all } A \in \mathcal{P}(\Omega).$$

Now, we assume that the capacity values of complementary events sum to a real greater than 1. We are going to show that there is no arbitrage in the bid–ask spread. We let $X \in \mathbb{R}^{\Omega}$ be a payoff. We denote $x_1, \ldots, x_n$ the n coordinates of X such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. Up to reindexing the states of nature, we assume that the payoff X yields x_i in ω_i for all $i \in \{1, \ldots, n\}$. Then, by definition of a Choquet expectation, we have

$$\tilde{\pi}(X) = \sum_{i=1}^{m} x_i[\nu(\{\omega_j \in \Omega \mid j \leqslant i\}) - \nu(\{\omega_j \in \Omega \mid j < i\})]$$

which, by assumption, is greater than

$$\sum_{i=1}^{m} x_i[1 - \nu(\{\omega_j \in \Omega \mid j > i\}) - (1 - \nu(\{\omega_j \in \Omega \mid j \geqslant i\})].$$

This sum simplifies to

$$\sum_{i=1}^{m} x_i[\nu(\{\omega_j \in \Omega \mid j \geqslant i\}) - \nu(\{\omega_j \in \Omega \mid j > i\})]$$

which is equal to $-\tilde{\pi}(-X)$. We hence obtain the desired result, for all $X \in \mathbb{R}^{\Omega}$,

$$\tilde{\pi}(X) \geqslant -\tilde{\pi}(-X).$$

□

Proof of Proposition 1.6.1. We are going to show that an event E is frictionless if, and only if, the capacity is additive with respect to this event. We first assume that E is a frictionless event. We are going to show that

the capacity is additive with respect to E. By assumption, we have

$$\tilde{\pi}(1_E 0) + \tilde{\pi}(-1_E 0) = \tilde{\pi}(1_E 0 + (-1_E 0))$$

which implies:

$$\nu(E) + \nu(E^c) = 1.$$

Now, we let $A \in \mathcal{P}(\Omega)$ such that $A \cap E \neq \emptyset$ and $A \cap E^c \neq \emptyset$. Then, by assumptions, we have

$$\tilde{\pi}(1_E 0 + 1_A 0) = \tilde{\pi}(1_E 0) + \tilde{\pi}(1_A 0)$$

and

$$\tilde{\pi}(1_{E^c} 0 + 1_A 0) = \tilde{\pi}(1_{E^c} 0) + \tilde{\pi}(1_A 0).$$

It implies

$$\nu(A \cap E) + \nu(E \cup A \cap E^c) = \nu(E) + \nu(A) \tag{1.1}$$

and

$$\nu(A \cap E^c) + \nu(E^c \cup A \cap E) = \nu(E^c) + \nu(A). \tag{1.2}$$

We replace $\nu(E^c)$ by $1 - \nu(E)$, and we combine equations 1.1 and 1.2 to get

$$\nu(A \cap E^c) + \nu(A \cap E) + \nu(E^c \cup A \cap E) + \nu(E \cup A \cap E^c) = 1 + 2\nu(A)$$

We now substitute $\nu(E^c \cup A \cap E)$ with $\tilde{\pi}(1_{E^c \cup A \cap E} 0)$ and $\nu(E \cup A \cap E^c)$ with $\tilde{\pi}(1_{E \cup A \cap E^c} 0)$. By assumption, we get

$$\nu(A \cap E^c) + \nu(A \cap E) + \tilde{\pi}(1_{E^c} 0) + \tilde{\pi}(1_{A \cap E} 0) + \tilde{\pi}(1_E 0) + \tilde{\pi}(1_{A \cap E^c} 0) = 1 + 2\nu(A).$$

Then, again by assumption, we get the desired result

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c).$$

Now we assume that the capacity is additive with respect to an event E. We are going to show that E is frictionless: we are going to show that for all $a \in \mathbb{R}$ and all $X \in \mathbb{R}^\Omega$,

$$\tilde{\pi}(X + a_E 0) = \tilde{\pi}(X) + a\tilde{\pi}(1_E 0). \tag{1.3}$$

We fix $X \in \mathbb{R}^\Omega$ and we denote $x_1, \ldots, x_n$ its n coordinates such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. We denote $A_{2i-1} \cup A_{2i}$ the event in which the payoff yields x_i with $(A_{2i-1} \cup A_{2i}) \cap E = A_{2i-1}$, as in the following table

	x_1	x_2	...	x_n
E	A_1	A_3	...	A_{2n-1}
E^c	A_2	A_4	...	A_{2n}

so that all events in E have an odd subscript and all events in E^c have

an even subscript. Events A_i can be empty. We denote $\mathcal{E}$ the set of even integers in $\{1,\dots,n\}$ and $\mathcal{O}$ the set of odd integers in $\{1,\dots,n\}$ and we fix $i \in \mathcal{O}$. We first show that the equation 1.3 is satisfied for $a > 0$. We denote ρ the ranking associated with X, and μ the corresponding probability in the Weber set. We consider another payoff, $Y = X + a_E 0$, denoting $\rho^\star$ the ranking associated with this payoff, and $\mu^\star$ the corresponding probability in the Weber set. We can now show that $\mu(A_i \cup A_{i+1}) = \mu^\star(A_i \cup A_{i+1})$. By assumption, we can decompose $\nu(\{A_j \mid Y(A_j) \leqslant Y(A_i)\})$ with respect to E, that is with respect to its odd and even events. In other words, we have $\nu(\{A_j \mid Y(A_j) \leqslant Y(A_i)\})$ equal to

$$\nu(\{A_j \mid Y(A_j) \geqslant Y(A_i),\ j \in \mathcal{O}\}) + \nu(\{A_j \mid Y(A_j) \geqslant Y(A_i),\ j \in \mathcal{E}\}). \quad (1.4)$$

Similarly, we can decompose $\nu(\{A_j \mid Y(A_j) > Y(A_i)\})$ with respect to E. It is equal to

$$\nu(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}) + \nu(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{E}\}). \quad (1.5)$$

Since i is odd, we have

$$\nu(\{A_j \mid Y(A_j) \geqslant Y(A_i),\ j \in \mathcal{E}\}) = \nu(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{E}\}). \quad (1.6)$$

By definition, the probability $\mu^\star(A_i \cup A_{i+1})$ is equal to

$$\nu(\{A_j \mid Y(A_j) \geqslant Y(A_i)\}) - \nu(\{A_j \mid Y(A_j) > Y(A_i)\})$$

which, by equations 1.4, 1.5 and 1.6, is equal to

$$\nu(\{A_j \mid Y(A_j) \geqslant Y(A_i),\ j \in \mathcal{O}\}) - \nu(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}).$$

By construction, the equalities

$$\nu(\{A_j \mid Y(A_j) \geqslant Y(A_i),\ j \in \mathcal{O}\}) = \nu(\{A_j \mid X(A_j) \geqslant X(A_i),\ j \in \mathcal{O}\})$$

and

$$\nu(\{A_j \mid Y(A_j) > Y(A_i),\ j \in \mathcal{O}\}) = \nu(\{A_j \mid X(A_j) > X(A_i),\ j \in \mathcal{O}\})$$

are satisfied. Thus, the probability $\mu^\star(A_i \cup A_{i+1})$ is equal to

$$\nu(\{A_j \mid X(A_j) \geqslant X(A_i),\ j \in \mathcal{O}\}) - \nu(\{A_j \mid X(A_j) > X(A_i),\ j \in \mathcal{O}\})$$

which, in turn, by assumption, is equal to $\mu(A_i \cup A_{i+1})$, yielding:

$$\tilde{\pi}(Y) = \tilde{\pi}(X) + a\tilde{\pi}(1_E 0).$$

We also have

$$\tilde{\pi}(X + a_{E^c}0) = \tilde{\pi}(X) + a\tilde{\pi}(1_{E^c}0).$$

We replace $a_{E^c}0$ by $a(\mathbb{1}_\Omega - 1_E0)$ and we use the assumption to replace $\tilde{\pi}(1_{E^c}0)$ by $1 - \tilde{\pi}(1_E0)$ to get:

$$\tilde{\pi}(X + a(\mathbb{1}_\Omega - 1_E0)) = \tilde{\pi}(X) + a(1 - \tilde{\pi}(1_E0)).$$

Hence,

$$\tilde{\pi}(X - a_E0) = \tilde{\pi}(X) - a\tilde{\pi}(1_E0).$$

It follows that, for all $a \in \mathbb{R}$ and all $X \in \mathbb{R}^\Omega$,

$$\tilde{\pi}(Y) = \tilde{\pi}(X) + a\tilde{\pi}(1_E0),$$

that is, E is a frictionless event.

Now, we can show that the capacity is additive with respect to an event E if, and only if, all probability values in the Weber set coincide with the value of the capacity for this event. We first assume that the capacity is additive with respect to an event E. We are going to show that all the probabilities in the Weber set coincide with the value taken by the capacity on E. We fix a probability μ in the Weber set. We consider a vector X associated with this probability, that is, there exists a ranking ρ such that ρ is associated with X and μ is associated with X. We denote $x_1, x_2, \ldots, x_n$ the coordinates of X such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. As shown in the following table, we denote $A_{2i-1} \cup A_{2i}$ the event in which the payoff yields x_i such that $(A_{2i-1} \cup A_{2i}) \cap E = A_{2i-1}$, so that all events in E have an odd subscript.

	x_1	x_2	...	x_n
E	A_1	A_3	...	A_{2n-1}
E^c	A_2	A_4	...	A_{2n}

.

The relationship

$$\nu(\{A_j \mid X(A_j) \geqslant X(A_i)\}) - \nu(\{A_j \mid X(A_j) > X(A_i)\})$$

simplifies to

$$\nu(\{A_j \mid X(A_j) \geqslant X(A_i), j \in \mathcal{O}\}) - \nu(\{A_j \mid X(A_j) > X(A_i), j \in \mathcal{O}\})$$

when i is odd and $\mu(E)$ is equal to

$$\sum_{\substack{i=1 \\ i \in \mathcal{O}}}^{2n} [\nu(\{A_j \mid X(A_j) \geqslant X(A_i), j \in \mathcal{O}\}) - \nu(\{A_j \mid X(A_j) > X(A_i), j \in \mathcal{O}\})]$$

and simplifies to $\nu(E)$.

Now, we assume that all probabilities in the Weber set coincide with the capacity value for an event E, and we will show that the capacity is additive with respect to E. We let E_1, E_2 be two distinct subsets of Ω such that $E = E_1 \cup E_2$ and we consider two events A and B such that $A = B \cup E_1$ and $B \cap E = \emptyset$. We let ρ be a ranking such that $\rho(E_1) > \rho(B) > \rho(E_2) > \rho(\Omega \setminus (E_1 \cup B \cup E_2))$ with the convention that $\rho(A) > \rho(B)$ if $\rho(\omega_i) > \rho(\omega_j)$ for all $\omega_i \in A$ and all $\omega_j \in B$. We let μ be the probability associated with ρ. We have $\mu(E)$ equal to

$$\nu(E_1) + \nu(E_1 \cup B \cup E_2) - \nu(E_1 \cup B)$$

which is, in turn, equal to

$$\nu(A \cap E) + \nu(A \cup E) - \nu(A).$$

We let $\rho^\star$ be a ranking such that

$$\rho^\star(B) > \rho^\star(E_1) > \rho^\star(E_2) > \rho^\star(\Omega \setminus (E_1 \cup B \cup E_2)$$

and we let $\mu^\star$ be the associated probability. We have $\mu^\star(E)$ equal to

$$\nu(B \cup E_1 \cup E_2) - \nu(B) \text{ which is equal to } \nu(A \cup E) - \nu(A \cap E^c)$$

and we get the desired result:

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c), \text{ for all } A \in \mathcal{P}(\Omega).$$

□

Proof of Proposition 1.6.2. We assume that X is a frictionless payoff. We are, therefore, going to show that we can decompose it as a sum of frictionless events, in part, by contradiction. We write X with the following form,

$$X = \sum_{i=1}^{n} x_{iE_i}0.$$

We are going to prove that the E_i are frictionless. We assume that there exist some A_i $i \in \{1, \ldots, n\}$ that are not frictionless. Up to reindexing, we decompose X into two sums. The left sum groups all x_i's on frictionless events and the right one groups x_i's on events with frictions:

$$X = \sum_{i=1}^{k} x_{iE_i}0 + \sum_{i=k+1}^{n} x_{iE_i}0.$$

We have $\tilde{\pi}(X+Y)$ equal to

$$\tilde{\pi}\left(X - \sum_{i=1}^{k} x_i E_i 0 + \sum_{i=1}^{k} x_i E_i 0 + Y\right).$$

By assumption, this is not equal to

$$\tilde{\pi}\left(\sum_{i=k+1}^{n} x_i E_i 0\right) + \tilde{\pi}\left(\sum_{i=1}^{k} x_i E_i 0 + Y\right).$$

By additivity, the preceding equation is equal to

$$\tilde{\pi}\left(\sum_{i=k+1}^{n} x_i E_i 0 + \sum_{i=1}^{k} x_i E_i 0\right) + \tilde{\pi}(Y).$$

We can now recognize $\tilde{\pi}(X) + \tilde{\pi}(Y)$, a contradiction.

Now, we assume that X can be decomposed as a sum of frictionless events. We are going to show that X is frictionless. We have:

$$X = \sum_{i=1}^{n} x_i E_i 0$$

where, for all $i \in \{1, \ldots, n\}$ $x_i \in \mathbb{R}$, the events E_i are frictionless and

$$\sum_{i=1}^{n} 1_{E_i} 0 = \mathbb{1}_\Omega.$$

If we let $Y \in \mathbb{R}^\Omega$, we have $\tilde{\pi}(X+Y)$ equal to

$$\tilde{\pi}\left(\sum_{i=1}^{n} x_i E_i 0 + Y\right).$$

This is, by assumption, equal to:

$$\sum_{i=1}^{n} \tilde{\pi}(x_i E_i 0) + \tilde{\pi}(Y)$$

We get the desired result: $\tilde{\pi}(X) + \tilde{\pi}(Y)$ for all $Y \in \mathbb{R}^\Omega$. □

Proof of lemma 1.6.1. We assume that there exists an event $A \notin \{\emptyset, \Omega\}$ such that $\nu(A) + \nu(A^c) = 1$. We are going to show that the bid–ask spread is nil. From lemma 1.4.1, we have $\lambda = \nu(A) + \nu(A^c) - 1$. Thus $\lambda = 0$ which entails $B(X) = 0$ for all $X \in \mathbb{R}^\Omega$.

Now, we assume that the bid–ask spread is null. By definition, the bid–

ask spread, $B(1_A 0) = 0$ implies $\lambda = 0$ with $\lambda = \nu(A) + \nu(A^c) - 1$. □

Proof of Proposition 1.6.3. We assume that E is a frictionless event; we can show that $\tilde{\pi}$ is frictionless. We consider an event $A \in \mathcal{P}(\Omega)$, we have

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E^c).$$

This implies

$$a + bp(A) = 2a + bp(A).$$

Hence, $a = 0$. Moreover,

$$\nu(E) + \nu(E^c) = b = 1.$$

Thus $a = 0$ and $b = 1$. Therefore, for all $A \in \mathcal{P}(\Omega)$,

$$\nu(A) = p(A).$$

Now, if we assume that $\tilde{\pi}$ is frictionless, then ν is additive. □

Proof of Proposition 1.7.1. First, we assume that the capacity is pairwise additive for payoffs with matching extreme revenues. Then it is, in particular, additive for bets with matching extreme revenues. We will now show that the capacity is a GNAC. To do so, we consider the following property, which we call Property A.

Definition 1.9.1 (Property A, Eichberger, Grant and Lefort (2012))**.** $\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G)$ is satisfied for all events $E, F, G \in \mathcal{P}(\Omega)$ such that $E \cup F \neq \Omega$, $E \cup G \neq \Omega$, $E \cap F = \emptyset = E \cap G$, $F \neq \emptyset$, $G \neq \emptyset$.

Eichberger, Grant and Lefort (2012) showed in Lemma 3 that Property A is satisfied if, and only if, the capacity is a GNAC. We will show that Property A is satisfied. We let $A, B \in \mathcal{P}(\Omega)$, such that $A \cap B \neq \emptyset$ and $A \cup B \neq \Omega$. The bets $1_A 0, 1_B 0 \in \mathbb{R}^\Omega$ have matching extreme revenues. Hence by assumption

$$\tilde{\pi}(1_A 0 + 1_B 0) = \nu(A \cap B) + \nu(A \cup B)$$

which is equal to $\nu(A) + \nu(B)$. Hence, the result is

$$\nu(A \cup B) - \nu(B) = \nu(A) - \nu(A \cap B).$$

We denote $E = A \backslash A \cap B$, $F = A \cap B$ and $G = B$. We get Property A with $F \subset G$:

$$\nu(E \cup G) - \nu(G) = \nu(E \cup F) - \nu(F).$$

Moreover, if we let $F_1, F_2 \subset G$ then

$$\nu(E \cup F_1) - \nu(F_1) = \nu(E \cup F_2) - \nu(F_2).$$

Now, we assume that the capacity is a GNAC then by the definition of a GNAC pricing rule, it is immediate that it is additive among payoffs with matching extreme revenues. □

Chapter 2

Convex Asset Pricing

Co-authored with Victor-Filipe Martins-da-Rocha[1]

Abstract

In order to encompass general financial frictions, we generalize the fundamental theorem of asset pricing to convex price functionals. We identify a new arbitrage condition, called robust no-arbitrage, that characterizes viability and generalizes the well-known no-arbitrage condition used in models with a linear pricing.

Contents

2.1 Introduction **34**
2.2 The Model **37**
2.2.1 Primitives 37
2.2.2 Markets 37
2.2.3 Preliminaries on Convex Analysis 40
2.3 Viability and no-arbitrage **41**
2.3.1 Viability 41
2.3.2 No-Arbitrage 42
2.3.3 Robust No-Arbitrage 43
2.3.4 Geometric Interpretation 44
2.3.5 Graphic Illustration 45
2.4 Viability and the Law of One Price **46**
2.5 Payoff Pricing Rule **47**
2.5.1 Definition and Basic Properties 48

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2.5.2 Proper Pricing Rules and Viability 49
2.5.3 No-Arbitrage and Monotonicity 50
2.5.4 Put-Call Parity and Viability 52
2.5.5 Robust No-Arbitrage and No Asymptotic Free-Lunch 53
2.6 Complete Viability and Absolute No-Arbitrage **55**
2.6.1 Complete Viability 55
2.6.2 Absolute No-Arbitrage 56
2.7 Conclusion **57**
2.8 Appendix **58**
2.8.1 Convex Analysis 58
2.8.2 Proofs 60

2.1 Introduction

We consider a standard two-period financial markets economy under uncertainty where an investor purchases an optimal portfolio to allocate consumption between the two dates and across states of nature. The trading possibilities are described by a price functional that defines the price of each available portfolio. A minimal consistency condition that we expect from an equilibrium price functional is to be viable, in the sense that there exists some consumer with convex, continuous, and strictly increasing preferences who can find an optimal trade. In their seminal contributions, Harrison and Kreps (1979) and Kreps (1981) show that a linear price functional is viable if, and only if, it does not allow for an arbitrage opportunity. A portfolio is an arbitrage opportunity when it is a claim to non-negative consumption tomorrow available for nothing or less today. The Fundamental Theorem of Asset Pricing (FTAP) then states that a linear price functional admits no arbitrage opportunities if, and only if, assets are linearly priced by means of strictly positive states prices, or equivalently strictly positive stochastic discount factors (see for instance Magill and Quinzii (1996) and LeRoy and Werner (2014)).

Linearity of a price functional is not consistent with frictions and transaction costs present in financial markets. If transaction costs are proportional to the volume dealt, then the pricing rules are sublinear (positively homogeneous and subadditive) but not necessarily linear.[2] The extension of the FTAP for sublinear pricing rules has been analyzed by Jouini and Kallal (1995) and Luttmer (1996). Recently, Cerreia-Vioglio, Maccheroni and Marinacci (2015) generalized the FTAP to price functionals that are positively homogeneous but not necessarily subadditive.

[2] A price functional is positively homogeneous if the price $p(\lambda\theta)$ of $\lambda \geqslant 0$ units of some portfolio θ is $\lambda p(\theta)$. It is subadditive if $p(\theta+\eta) \leqslant p(\theta)+p(\eta)$ for any pair θ, η of portfolios.

Positive homogeneity imply that the ratio of order's price to order's size, the unit price, is a constant function of the size. Market participants cannot observe the form of the pricing function directly. A proxy for the unit price is the temporary market impact. The temporary market impact represents the average price change conditioned on the size and the nature of an order placed in the markets. Large institutions needing to place large orders in the markets are especially attentive to this effect. To reduce their costs associated with short-running (the cost associated with finding a counterparty immediately), imperfect substitution effects (the cost associated with the absence of a perfect substitute for the traded asset), information effect (the cost associated with agents believing that asset is mispriced) they split their orders and keep secret their true size. Several empirical studies were given access to databases enabling them to reconstitute the total order, called metaorder, placed on the markets by large institutions (see Almgren, Thum, Hauptmann and Li (2005), Moro, Vicente, Moyano, Gerig, Farmer, Vaglica, Lillo and Mantegna (2009), Tóth, Lemperiere, Deremble, Lataillade, Kockelkoren and Bouchaud (2011), Bacry, Iuga, Lasnier and Lehalle (2015)). These studies differ quite significantly in the analyzed database, how the price impact is defined and measured, how different assets and periods are collated together in the analysis, and how the fit is performed. Nevertheless, they have all consistently shown that the temporary price impact is concave, approximately increasing as the square root of the order size. Naturally, studies reconstituting the metaorders underestimate the temporary price impact of large trades of the same nature placed at once on the market. Therefore, the real price impact has a steeper slope confirming the non-constancy of the unit price. Moreover, Kyle (1985), Glosten (1994), Bertsimas and Lo (1998), Biais, Martimort and Rochet (2000) show in different security trading models that the price functional is convex.

This chapter aims to extend the existing literature by analyzing the validity of the FTAP when the price functional is only assumed to be convex. We start by discussing the appropriate arbitrage concept when the price functional is convex but not necessarily linear. The standard notion of an arbitrage opportunity is related to the strict monotonicity of the investor's preference relation. Formally, recall that a portfolio η is an arbitrage opportunity at some investor's position θ when, replacing the current position θ by the new position $\theta + \eta$, the associated intertemporal incremental consumption is positive, meaning that it is non-negative in any contingency (including the first period) and strictly positive in at least one contingency (including the first period). Suppose we replace the property that "the associated intertemporal incremental consumption is increased" by the weaker property that "the associated intertemporal incremental consumption belongs to an open and convex set Γ containing the positive cone". In that case, we then obtain the concept of a weak arbitrage opportunity at the position θ with respect to the set Γ. The motivation for this definition is

that preferences are assumed to be strictly increasing and continuous. Indeed, by strict monotonicity, if the incremented consumption is positive, the investor strictly prefers the new position $\theta + \eta$. Since his preferences are continuous, he strictly prefers the new position if the corresponding incremented consumption belongs to an open and convex set close enough to the positive cone. The consumption may decrease at some contingencies, but it should sufficiently increase in others to get an overall compensation. We then say that a price functional is a robust no-arbitrage price when there exists a position θ and an open and convex set Γ containing the positive cone such that there are no weak arbitrage opportunities at the position θ with respect to the set Γ. When the price functional is linear, our notion of robust no-arbitrage reduces to the standard concept of no-arbitrage price.

Our first contribution is to show that a convex price functional is viable if, and only if, it is a robust no-arbitrage price. We then generalize the FTAP by showing that a price functional is a robust no-arbitrage price if, and only if, it is supported by strictly positive state prices in the sense that there exists a portfolio $\theta^\star$ such that any incremental price $p(\theta) - p(\theta^\star)$ associated with a different position θ is at least as large as the present value (with respect to the strictly positive state prices) of the incremental payoff.

Following the tradition in financial economics, instead of analyzing the properties of the price functional defined on the set of available portfolios, we may analyze the minimal cost in terms of consumption today to implement some random consumption tomorrow. Such a functional defined on the space of streams at the second period is called the pricing rule associated with a price functional. We analyze the properties that a pricing rule inherits from the price functional. Formally, we show that the pricing rule associated with any convex price functional is naturally convex but also monotone. If the price functional is viable, then the pricing rule is also viable. Furthermore, we show that a pricing rule is viable if, and only if, it is a robust no-arbitrage price. Finally, we extend the FTAP to pricing rules by showing that a pricing rule is a no-arbitrage price if, and only if, strictly positive state prices support it. Our generalization of the FTAP extends the standard result in a linear environment but also the extension proposed by Jouini and Kallal (1995) for sublinear price functionals.

We organized this chapter as follows. First, we present the model in Section 2.2 and introduce some preliminaries on convex analysis. Then, in Section 2.3, we characterize viability when prices are convex. We show that it is equivalent to robust no-arbitrage. Next, in section 2.4 we show that without any loss of generality, we can assume that prices satisfy the law of one price. We analyze the consequences of having a viable price functional on payoff pricing rules in Section 2.5. Finally, we characterize price functional viable for every agent in Section 2.6. We call this concept complete viability and show that markets are completely viable if, and only if, every portfolio satisfies the no-arbitrage condition. We gathered the proofs of this chapter's

results in the appendix.

2.2 The Model

2.2.1 Primitives

There are two periods $t \in \{0, 1\}$. There is a single perishable good at each period which can be consumed. Uncertainty is represented by a finite set Ω with m states of nature that can occur at $t = 1$.

An agent is represented by a triple (e_0, e_1, U) where $e_0 > 0$ represents the agent's non-financial income (such as labor income) at $t = 0$, $e_1 \in \mathbb{R}^{\Omega}_{++}$ is a random variable representing future non-financial income, and $U : C \to \mathbb{R}$ is a utility function defined on the consumption set $C := \mathbb{R}_+ \times \mathbb{R}^{\Omega}_+$ representing the agent's preference relation over consumption plans $c = (c_0, c_1) \in C$. We denote by $\mathcal{A}$ the set of agents (e_0, e_1, U) such that the utility function U satisfies the following properties:

(i) U is strictly increasing, in the sense that for any consumption plans $\tilde{c}, c \in C$,

$$\tilde{c} > c \Longrightarrow U(\tilde{c}) > U(c)$$

with the standard convention that $\tilde{c} > c$ means $\tilde{c} \geqslant c$ and $\tilde{c} \neq c$;

(ii) U is concave;

(iii) U satisfies Inada's condition at the origin, i.e., for any $c \in C$,

$$\lim_{\varepsilon \to 0} \frac{U(\varepsilon, c_1) - U(0, c_1)}{\varepsilon} = \infty$$

and for any $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \frac{U(c_0, (\varepsilon, c_1(-\omega))) - U(c_0, (0, c_1(-\omega)))}{\varepsilon} = \infty$$

where $(x, c_1(-\omega))$ is the vector $(y(s))_{s \in \Omega}$ defined by $y(\omega) := x$ and $y(s) = c_1(s)$ for any $s \neq \omega$.

2.2.2 Markets

Trade occurs at period $t = 0$ and consumption occurs in both periods $t = 0$ and $t = 1$. There is a finite set J of primary assets. We exclude any portfolio restrictions (like short sales constraints or leverage limitations). Therefore, a portfolio of primary assets is a vector $\eta = (\eta_j)_{j \in J} \in \mathbb{R}^J$ where η_j represents the units of asset j in the portfolio.

We represent the price schedule by a function

$$p : \mathbb{R}^J \to \mathbb{R}$$

where $p(\eta)$ represents the cost, paid at $t = 0$, for trading portfolio η. We analyze the decision of an agent trading only at $t = 0$. Therefore, he has no initial portfolio holding inherited from past unmodeled transactions and will liquidate all his positions at the end of period $t = 1$. This implies that for this agent, $p(\eta)$ is also the cost, paid at $t = 0$, for holding portfolio η at the beginning of period $t = 1$.

For each possible state $\omega \in \Omega$ at $t = 1$, we denote by $G(\eta, \omega) \in \mathbb{R}$ the payoff (in units of consumption) of portfolio η. Let $G : \mathbb{R}^J \to \mathbb{R}^\Omega$ be the payoff mapping defined by

$$\forall \eta \in \mathbb{R}^J, \quad G(\eta) := (G(\eta, \omega))_{\omega \in \Omega}.$$

Consider an agent holding the portfolio η at the beginning of period $t = 1$, i.e., after trading in period $t = 0$ and before liquidating all his position at the end of period $t = 1$. If $G(\eta, \omega) > 0$, then the agent is entitled to $G(\eta, \omega)$ units of the consumption good in state ω. If $G(\eta, \omega) < 0$, then the agent is supposed to deliver the amount $|G(\eta, \omega)|$ in state ω.

Example 2.2.1. If there are no taxes and every asset j is short-lived, then

$$G(\eta, \omega) = \sum_{j \in J} \eta_j g_j(\omega)$$

where $g_j(\omega) \in \mathbb{R}$ is the unitary dividend of asset j. In that case, the mapping G is a linear operator. Following Prisman (1986) and Ross (1987), we may allow for taxes by considering the mapping

$$G(\eta, \omega) = \sum_{j \in J} \eta_j g_j(\omega) - T\left(\sum_{j \in J} \eta_j g_j(\omega)\right)$$

where $T : \mathbb{R} \to \mathbb{R}$ is the tax function. Observe that if T is a convex (thus progressive) tax function, then each function $G(\cdot, \omega) : \mathbb{R}^J \to \mathbb{R}$ is concave.

Example 2.2.2. One may also encompass the case where there are no taxes, no dividends at $t = 1$ but assets are long-lived. In that situation, we have

$$G(\eta, \omega) = \sum_{j \in J} -p_{1,j}(-\eta_j, \omega)$$

where $p_{1,j}(\cdot, \omega) : \mathbb{R} \to \mathbb{R}$ is the price functional at date 1 and state ω for trading asset j. Observe that if $p_{1,j}(\cdot, \omega)$ is a convex function, then each function $G(\cdot, \omega) : \mathbb{R}^J \to \mathbb{R}$ is concave.

We assume that there is no cost, no liability, and no gain if the agent does not trade in the financial markets, i.e., $p(0) = 0$ and $G(0) = 0$. We also impose the following convexity properties.

Assumption 2.2.1. The function $p : \mathbb{R}^J \to \mathbb{R}$ is convex and satisfies $p(0) =$

0.[3] Moreover, for every state ω, the payoff function $G(\cdot,\omega) : \mathbb{R}^J \to \mathbb{R}$ is concave and satisfies $G(0) = 0$.

A function $p : \mathbb{R}^J \to \mathbb{R}$ that is convex and satisfies $p(0) = 0$ is called a **price functional**. Convexity of p implies that

$$0 = p((1/2)\theta + (1/2)(-\theta)) \leqslant (1/2)p(\theta) + (1/2)p(-\theta).$$

Therefore, if p is a price functional, then the bid-ask spread of any portfolio is non-negative, i.e.,

$$\forall \theta \in \mathbb{R}^J, \quad p(\theta) \geqslant -p(-\theta).$$

Example 2.2.1. Assume that there is a transaction cost $c_j(|\theta_j|)$ for trading θ_j units of security j. Each function $c_j : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be convex and to satisfy $c_j(0) = 0$. Consider the following price functional

$$p(\theta) = \sum_{j\in J} [p_j\theta_j + c_j(|\theta_j|)] \tag{2.1}$$

where p_j is the (cost-free) unitary price for purchasing or selling each unit of asset j. Observe that when each function c_j is linear, i.e., $c_j(z) = \bar{c}_j z$ for some $\bar{c}_j > 0$, then we get the standard price functional with proportional transaction costs:

$$p(\theta) = \sum_{j\in J} p_j^b[\theta_j]^+ + \sum_{j\in J} p_j^a[\theta_j]^- \tag{2.2}$$

where $p_j^b := p_j + \bar{c}_j$ and $p_j^a := p_j - \bar{c}_j$, that is,

$$p(\theta) = \sum_{j\in J} p_j\theta_j + \sum_{j\in J} c_j|\theta_j| \tag{2.3}$$

and with proportional bid-ask spreads:

$$p(\theta) + p(-\theta) = 2\sum_{j\in J} c_j|\theta_j|. \tag{2.4}$$

A function $p : \mathbb{R}^J \to \mathbb{R}$ is said to be *sub-additive* when

$$\forall \theta, \eta \in \mathbb{R}^J, \quad p(\theta + \eta) \leqslant p(\theta) + p(\eta),$$

and it is said to be *positively homogeneous* when

$$\forall (\lambda, \theta) \in \mathbb{R}_+ \times \mathbb{R}^J, \quad p(\lambda\theta) = \lambda p(\theta).$$

If p is sub-additive and positively homogeneous, then it is said to be *sub-*

[3] Fix a finite set K and recall that a function $f : \mathbb{R}^K \to \mathbb{R}$ is convex when $f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha)f(y)$, for all $x, y \in \mathbb{R}^K$ and all $\alpha \in (0,1)$.

linear. If p is sublinear then it is a price functional (i.e., p is convex and $p(0) = 0$). Observe that the functions defined by (2.2) are sublinear.

2.2.3 Preliminaries on Convex Analysis

Before presenting the concepts of viability and no-arbitrage, we recall the following standard notions of convex analysis.

Fix an arbitrary finite set K and a convex function $f : \mathbb{R}^K \to \mathbb{R}$. We denote by $f0^+ : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$ the recession function of f defined by

$$\forall \eta \in \mathbb{R}^K, \quad (f0^+)(\eta) := \sup\{f(\theta + \eta) - f(\theta) \ : \ \theta \in \mathbb{R}^K\}. \tag{2.5}$$

Remark 2.2.1. If f is sublinear, then $f0^+ = f$. Indeed, by sublinearity, we have $f(\theta + \eta) - f(\theta) \leqslant f(\eta)$ with an equality when $\theta = 0$.

Among the several properties satisfied by the recession function (see Appendix 2.8.1 for details), we will be using the following characterization: for any $\theta, \eta \in \mathbb{R}^K$, we have

$$(f0^+)(\eta) = \sup_{\lambda > 0} \frac{f(\theta + \lambda\eta) - f(\theta)}{\lambda} = \lim_{\lambda \to \infty} \frac{f(\theta + \lambda\eta) - f(\theta)}{\lambda}. \tag{2.6}$$

This implies that a vector η satisfies $(f0^+)(\eta) \leqslant 0$ if, and only if, for any vector $\theta \in \mathbb{R}^K$, the function $\lambda \mapsto f(\theta + \lambda\eta)$ is decreasing on $\mathbb{R}$. If η is such that, for every $\theta \in \mathbb{R}^K$, the function $\lambda \mapsto f(\theta + \lambda\eta)$ is constant on $\mathbb{R}$, then η is called a **direction in which f is constant**. Observe this last property occurs if, and only if, $(f0^+)(\eta) \leqslant 0$ and $(f0^+)(-\eta) \leqslant 0$. In other words, η is a direction in which f is constant when each iso-cost $\{f(\cdot) = f(\theta)\}$ contains the line $\theta + \mathbb{R}\eta$. The following result proves that it is sufficient to find one iso-cost satisfying this property.

Lemma 2.2.1. If there exists a vector $\theta^0 \in \mathbb{R}^K$ and a direction $\eta \in \mathbb{R}^K$ such that the function $\lambda \longmapsto f(\theta^0 + \lambda\eta)$ is constant, then η is a direction in which f is constant, i.e., for every $\theta \in \mathbb{R}^K$, the function $\lambda \longmapsto f(\theta + \lambda\eta)$ is constant.

Given a vector $\theta \in \mathbb{R}^K$ and a direction $\eta \in \mathbb{R}^K$, the function

$$\lambda \longmapsto \frac{f(\theta + \lambda\eta) - f(\theta)}{\lambda}$$

is increasing and we pose

$$f'(\theta; \eta) := \inf_{\lambda > 0} \frac{f(\theta + \lambda\eta) - f(\theta)}{\lambda} = \lim_{\lambda \to 0} \frac{f(\theta + \lambda\eta) - f(\theta)}{\lambda}.$$

The number $f'(\theta; \eta)$ is called the derivative of f at θ in the direction η. The function $\eta \mapsto f'(\theta; \eta)$ is convex and homogeneous of degree 1. We also have

that

$$f'(\theta,\eta) \leqslant \frac{f(\theta+\lambda\eta)-f(\theta)}{\lambda} \leqslant f0^+(\eta)$$

for any $\lambda > 0$.

2.3 Viability and no-arbitrage

An agent $(e_0, e_1, U) \in \mathcal{A}$ chooses a consumption plan $c = (c_0, c_1) \in C$ that satisfies the flow budget constraints at $t = 0$ and at $t = 1$ for any contingency. This means that there exists a portfolio $\theta \in \mathbb{R}^J$ such that

$$c_0 + p(\theta) \leqslant e_0 \quad \text{and} \quad c_1 \leqslant G(\theta) + e_1.$$

Since the utility function U is strictly increasing, optimal consumption plans satisfy the flow budget constraints with equality. This implies that the agent's maximization problem is equivalent to solving

$$\max\{v(\theta) \,:\, \theta \in \Theta\}$$

where Θ is the set of budget feasible portfolios, i.e.,

$$\Theta := \{\theta \in \mathbb{R}^J \,:\, p(\theta) \leqslant e_0 \quad \text{and} \quad G(\theta) \geqslant -e_1\}$$

and $v(\theta) := U(e_0 - p(\theta), e_1 + G(\theta))$ is the indirect utility.

Remark 2.3.1. Since U is strictly increasing, we have the following important property: for any budget feasible portfolio $\theta \in \Theta$ and any portfolio θ', if

$$p(\theta') \leqslant p(\theta) \quad \text{and} \quad G(\theta') \geqslant G(\theta), \tag{2.7}$$

then θ' is also budget feasible, i.e., $\theta' \in \Theta$, and $v(\theta') \geqslant v(\theta)$ with a strict inequality if there is a strict inequality in one of the inequalities in (2.7).

2.3.1 Viability

We recall the concept of viability introduced by Harrison and Kreps (1979) (see also Kreps (1981)).

Definition 2.3.1. A price functional p is said to be **viable** when it is compatible with utility maximization for at least one agent. Formally, there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ such that his maximization problem has a solution, i.e., there exists a portfolio $\theta^\star \in \mathbb{R}^J$ satisfying

$$\theta^\star \in \arg\max\{v(\theta) \,:\, \theta \in \Theta\}. \tag{2.8}$$

Our first result characterizes viability by means of strictly positive state prices (or stochastic discount factor).

Theorem 2.3.1. *A price functional p is viable if, and only if, it is* ***supported by strictly positive state prices*** *in the sense that there exists a strictly positive vector $\mu \in \mathbb{R}^{\Omega}_{++}$ and a portfolio $\theta^\star \in \mathbb{R}^J$ such that*

$$p(\theta) - p(\theta^\star) \geqslant \mu \cdot [G(\theta) - G(\theta^\star)], \quad \textit{for all } \theta \in \mathbb{R}^J. \tag{2.9}$$

Remark 2.3.2. If a price functional p satisfies the asset pricing inequality (2.9), then

$$(p0^+)(\eta) \geqslant p'(\theta^0;\eta) \geqslant \mu \cdot G'(\theta^0;\eta) \geqslant \mu \cdot (G0^+)(\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

If p is sublinear, then $p0^+ = p$. If moreover G is a linear operator, then we have

$$p(\eta) \geqslant \mu \cdot G(\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

If there is no friction and p is linear, the above inequality implies the standard fundamental asset pricing equation

$$p(\eta) = \mu \cdot G(\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

2.3.2 No-Arbitrage

Following the standard literature on asset pricing theory (see among others Cochrane (2005), Skiadas (2009), LeRoy and Werner (2014)), we would like to identify the strongest no-arbitrage condition on prices that is necessary for viability, and that is independent of the agent's characteristics (utility function and endowments). In that respect, Remark 2.3.1 leads us to consider the following natural definition.

Definition 2.3.2. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ is said to be a **no-arbitrage** price when there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that for any direction $\eta \in \mathbb{R}^J$, the conditions

$$G(\theta^0 + \eta) - G(\theta^0) \geqslant 0 \quad \text{and} \quad p(\theta^0 + \eta) \leqslant p(\theta^0)$$

imply

$$G(\theta^0 + \eta) = G(\eta) \quad \text{and} \quad p(\theta^0 + \eta) = p(\theta^0).$$

When the price functional is linear, the above definition coincides with the standard definition of a no-arbitrage price: for any portfolio $\eta \in \mathbb{R}^J$,

$$[G(\eta) \geqslant 0 \quad \text{and} \quad p(\eta) \leqslant 0] \Longrightarrow [G(\eta) = 0 \quad \text{and} \quad p(\eta) = 0].$$

Our no-arbitrage condition is independent of the agent's characteristics and is necessary for viability.

Proposition 2.3.1. If a price functional is viable, then it is a no-arbitrage price.

Remark 2.3.3. Observe that Proposition 2.3.1 can also be proved by means of the characterization result provided in Theorem 2.3.1. Indeed, consider a viable price functional p. It follows from Theorem 2.3.1 that there exists some portfolio $\theta^\star \in \mathbb{R}^J$ and a vector $\mu \in \mathbb{R}^\Omega_{++}$ of strictly positive state prices such that

$$p(\theta) - p(\theta^\star) \geqslant \mu \cdot [G(\theta) - G(\theta^\star)], \quad \text{for all } \theta \in \mathbb{R}^J.$$

Fix now $\eta \in \mathbb{R}^J$ such that $G(\theta^\star + \eta) \geqslant G(\theta^\star)$ and $p(\theta^\star + \lambda\eta) \leqslant p(\theta^\star)$. Observe

$$0 \geqslant p(\theta^\star + \eta) - p(\theta^\star) \geqslant \mu \cdot [G(\theta^\star + \eta) - G(\theta^\star)] \geqslant 0.$$

We then deduce that both terms are equal to zero. Since μ is strictly positive, we also have that $G(\theta^\star + \eta) = G(\theta^\star)$. We have thus proved that p is a no-arbitrage price.

2.3.3 Robust No-Arbitrage

As a necessary condition for viability, the stronger the no-arbitrage condition, the more interesting it is. Ideally, we would like to obtain a necessary and sufficient condition, as is the case when the price functional is linear. We do not know whether a no-arbitrage price (according to Definition 2.3.2) is also a viable price. To obtain this implication, we introduce the following stronger concept of a *robust no-arbitrage price*. This section demonstrates that a price functional is a robust no-arbitrage price if, and only if, it is viable. Under the assumptions of convexity of the price functional and concavity of the payoff functional, we have reasons to believe that the price functional is a no-arbitrage price, if and only if, it is a robust no-arbitrage price. However, we have been unsuccessful in proving this assertion.

Definition 2.3.3. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ is said to be a **robust no-arbitrage price** when there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that for any direction $\eta \in \mathbb{R}^J$, the conditions

$$G'(\theta^0; \eta) \geqslant 0 \quad \text{and} \quad p'(\theta^0; \eta) \leqslant 0$$

imply

$$G'(\theta^0; \eta) = 0 \quad \text{and} \quad p'(\theta^0; \eta) = 0.$$

This is a stronger requirement than no-arbitrage but it coincides with no-arbitrage when p and G are linear.[4]

[4]Indeed, when p is linear, the function $\lambda \mapsto (p(\theta^0 + \lambda\eta) - p(\theta^0))/\lambda$ is constant equal to $p(\eta)$. Similarly, when G is linear, the function $\lambda \mapsto (G(\theta^0 + \lambda\eta) - G(\theta^0))/\lambda$ is constant equal to $G(\eta)$.

Lemma 2.3.1. If a price functional is a robust no-arbitrage price, then it is a no-arbitrage price.

The second important result of this chapter is that robust no-arbitrage is the adequate concept to characterize viability.

Theorem 2.3.2. *A price functional p is viable if, and only if, it is a robust no-arbitrage price.*

The proof of the above result follows from Theorem 2.3.1 and the following characterization of robust no-arbitrage by means of strictly positive state prices.

Theorem 2.3.3. *A price functional p is a robust no-arbitrage if, and only if, it is supported by strictly positive state prices.*

Remark 2.3.4. Equation (2.9) is equivalent to

$$p'(\theta^\star;\eta) \geqslant \mu \cdot G'(\theta^\star;\eta), \quad \text{for all } \eta \in \mathbb{R}^J.$$

2.3.4 Geometric Interpretation

Observe that a price functional p is a no-arbitrage price if, and only if, there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that

$$A(\theta^0, p) \cap C^\star = \emptyset \tag{2.10}$$

where $A(\theta^0, p)$ is the set of all vectors $(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that there exists a portfolio $\eta \in \mathbb{R}^J$ satisfying

$$x_0 \leqslant p(\theta^0) - p(\theta^0 + \eta) \quad \text{and} \quad x_1 \leqslant G(\theta^0 + \eta) - G(\theta^0)$$

and $C^\star = \mathbb{R}_+ \times \mathbb{R}^\Omega_+ \setminus 0$ is the set of positive consumption increments. The stronger concept of robust no-arbitrage corresponds to the following strengthening of (2.10).

Proposition 2.3.2. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ is a robust no-arbitrage price if, and only if, there exist a portfolio θ^0 and an open and convex set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}^\Omega$ such that

$$A(\theta^0, p) \cap \Gamma = \emptyset \quad \text{and} \quad C^\star \subseteq \Gamma. \tag{2.11}$$

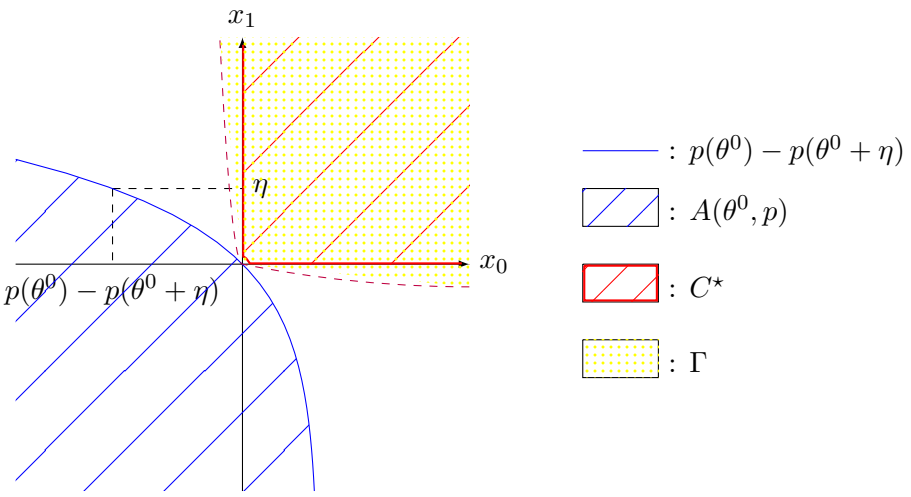
The set Γ can be interpreted as the directions, starting from θ^0, in which the utility function U increases. The fact that Γ is open is related to the continuity of U. Convexity of Γ is related to the concavity of U. Therefore, there is no robust arbitrage opportunity at some portfolio θ^0 only if it is not possible to increase positively consumption by changing of portfolio. Actually, there is no robust arbitrage opportunity at some portfolio θ^0 if,

and only if, there exists no portfolio change arbitrarily close to a portfolio change which increases positively consumption.

2.3.5 Graphic Illustration

We represent the increments in consumption when we replace the initially selected portfolio θ^0 by a portfolio $\theta^0+\eta$. We do so in a simple market model. We assume there is no uncertainty at date 0 regarding the future, and we assume a single asset is traded at date 0. The asset's payoff at date 1 is equal to 1 unit of the consumption good. It is linear in the quantity purchased. In each graph, the x_0 axis represents the increments in consumption at date 0. The x_1 axis represents the increments in consumption at date 1. The set $C^\star$ represents the positive non-zero increments in consumption in both dates. The set $A(\theta^0, p)$ represents the increments that are achievable when selecting a portfolio $\theta^0+\eta$ instead of the portfolio θ^0. Put differently, it represents the increments achievable when modifying the purchased quantity of the asset.

In Graph 1, we illustrate the geometry of a price functional p satisfying robust no-arbitrage at θ^0. The price functional satisfies $A(\theta^0, p)\cap C^\star = \emptyset$ and $p'(\theta^0) > 0$. Therefore, there exists a convex and open set Γ such that $C^\star \subseteq \Gamma$ and $A(\theta^0, p) \cap \Gamma = \emptyset$. Since the price functional is a robust no-arbitrage at θ^0, it also satisfies no-arbitrage at θ^0.



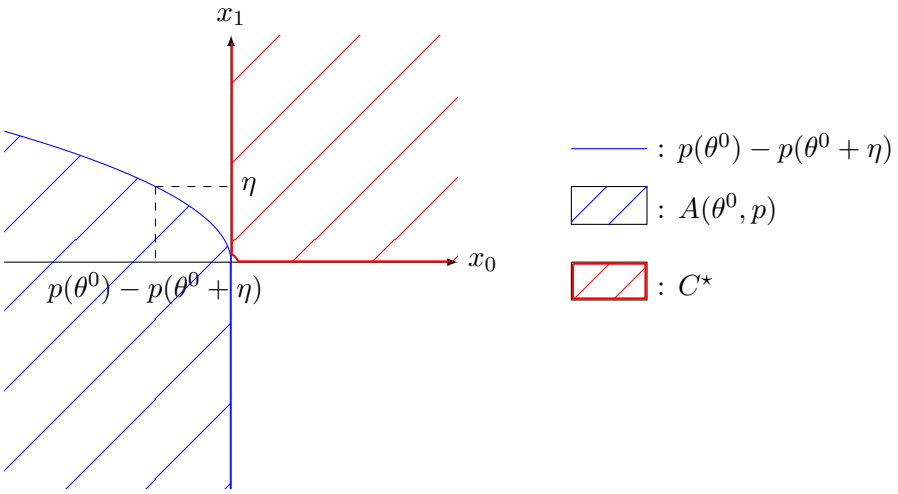
Graph 1: Geometry of robust no-arbitrage with $J = 1$, $S = 1$, an asset which pays 1 in period 1, $p : \mathbb{R} \to \mathbb{R}$ the price functional and $G(\eta) = \eta$ for all $\eta \in \mathbb{R}$.

In Graph 2, we represent a price functional that does not satisfy robust no-arbitrage at θ^0. Indeed, we have $p'(\theta^0) = 0$ and $A(\theta^0, p) \cap C^\star = \emptyset$. Therefore, it is impossible to find an open and convex set Γ such that $C^\star \subseteq \Gamma$. However, the price functional satisfies no-arbitrage at this portfolio. The

price functional p satisfying the equations

$$\begin{cases} p(\theta^0) - p(\theta^0 + \eta) &= 0 \text{ for all } \eta \in]\infty, 0[, \\ p(\theta^0) - p(\theta^0 + \eta) &= -x^2 \text{ for all } \eta \in [0, \infty] \end{cases}$$

is an example of price functional which is a no-arbitrage at θ^0 but not a robust no-arbitrage at this portfolio. However for every portfolio $\eta > 0$, the price functional of Graph 2 is a robust no-arbitrage price at $\theta^0 + \eta$.



Graph 2: Geometry of no-arbitrage with $J = 1$, $S = 1$, an asset which pays 1 in period 1, $p : \mathbb{R} \to \mathbb{R}$ the price functional and $G(\eta) = \eta$ for all $\eta \in \mathbb{R}$.

2.4 Viability and the Law of One Price

In this section, we show that without any loss of generality, we can assume that a viable price functional satisfies the standard law of one price as defined below.

Definition 2.4.1. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ satisfies the **law of one price** when, for any $\theta, \eta \in \mathbb{R}^J$, $G(\eta) = G(\theta)$ implies $p(\eta) = p(\theta)$.

Recall that the agent's maximization problem consists in finding a portfolio $\theta^\star \in \Theta(p)$ such that

$$\theta^\star \in \arg\max\{v(\theta|p) \ : \ \theta \in \Theta(p)\}.$$

If $\theta \in \Theta(p)$, then the agent prefers any portfolio η satisfying $G(\eta) \geqslant G(\theta)$ and $p(\eta) < p(\theta)$. This leads us to introduce the following definition

$$\forall \theta \in \mathbb{R}^J, \quad \tilde{p}(\theta) := \inf\{p(\eta) \ : \ \eta \in \mathbb{R}^J \quad \text{and} \quad G(\eta) \geqslant G(\theta)\}.$$

The function $\tilde{p}$ may a priori take the value $-\infty$. Nonetheless, it is still convex in the sense that its epigraph $\{(\theta, \mu) \in \mathbb{R}^J \times \mathbb{R} : \tilde{p}(\theta) \leqslant \mu\}$ is a convex subset of $\mathbb{R}^J \times \mathbb{R}$.

Lemma 2.4.1. For any price functional p, the associated function $\tilde{p} : \mathbb{R}^J \to [-\infty, \infty)$ is a convex function.

Since the function $\tilde{p}$ is convex and never takes the value $+\infty$, we deduce that either $\tilde{p}(\mathbb{R}^J) \subseteq \mathbb{R}$ or $\tilde{p}(\mathbb{R}^J) = \{-\infty\}$.

Lemma 2.4.2. Either $\tilde{p}$ takes only finite values, or $\tilde{p}$ only takes the value $-\infty$.

We then obtain the following result.

Proposition 2.4.1. If the price functional p is viable, then the associated function $\tilde{p}$ is a price functional satisfying the law of one price.

Actually, a solution to the agent's problem under the price functional p is also a solution to the agent's problem under the price functional $\tilde{p}$.

Proposition 2.4.2. If the price functional p is viable, then the associated price functional $\tilde{p}$ is also viable. Formally, if $\theta^\star$ solves the agent's maximization problem under the price functional p, then the same portfolio solves the agent's maximization problem under the price functional $\tilde{p}$, i.e.,

$$\arg\max\{v(\theta|p) \ : \ \theta \in \Theta(p)\} \subseteq \arg\max\{v(\theta|\tilde{p}) \ : \ \theta \in \Theta(\tilde{p})\}.$$

Moreover, we have $p(\theta^\star) = \tilde{p}(\theta^\star)$.

Combining Proposition 2.4.1 and Proposition 2.4.2, we obtain the following result.

Corollary 2.4.1. *If a price functional p is viable, then $\tilde{p}$ is also a price functional that is viable and satisfies the law of one price.*

2.5 Payoff Pricing Rule

Instead of analyzing an agent's optimal portfolio, we may focus directly on the cost at $t = 0$ to implement a specific consumption at $t = 1$. Formally, what is the amount of resources the agent should spend at $t = 0$ in order to implement a specific random consumption plan c_1 at $t = 1$ by trading portfolios? The natural answer is to look for the cheapest portfolio θ satisfying $G(\theta) \geqslant c_1 - e_1$.

2.5.1 Definition and Basic Properties

The set of payoffs which can be obtained by trading portfolios is the subset

$$\Phi := \{x \in \mathbb{R}^\Omega : \exists \theta \in \mathbb{R}^J \text{ such that } x \leqslant G(\theta)\}$$

of $\mathbb{R}^\Omega$. It is is convex and closed[5].

Lemma 2.5.1. The set Φ is convex and closed.

We say that the market is complete when $\Phi = \mathbb{R}^\Omega$.

Definition 2.5.1. The **payoff pricing rule** associated with the price functional p is the function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\forall x \in \Phi, \quad \pi(x) := \inf\{p(\theta) : \theta \in \mathbb{R}^J \quad \text{and} \quad x \leqslant G(\theta)\}. \tag{2.12}$$

Convexity of the payoff pricing rule follows from the convexity of the price functional.

Proposition 2.5.1. The payoff pricing rule associated with a price functional is a convex function from Φ to $\mathbb{R} \cup \{-\infty\}$.

Moreover, the payoff pricing rule associated with a price functional is a monotone function.

Proposition 2.5.2. The payoff pricing rule associated with a price functional is a monotone function from Φ to $\mathbb{R} \cup \{-\infty\}$.[6]

The pricing rule associated with a price functional p coincides with the pricing rule associated with the function $\tilde{p}$.

Lemma 2.5.2. Replacing the price functional p by its associated function $\tilde{p}$ leads to the same pricing rule, i.e.,

$$\pi(x) = \inf\{\tilde{p}(\theta) : x \leqslant G(\theta)\}.$$

Remark 2.5.1. By definition, a price functional p satisfies $p(0) = 0$. This property implies that $\pi(0) \leqslant 0$. However, we do not necessarily have $\pi(0) = 0$, even if p is a no-arbitrage price functional. Indeed, consider the following illustrative example. There is a single state of nature and a single asset. The mapping $G : \mathbb{R} \to \mathbb{R}$ is the identity ($G(\theta) = \theta$, for any $\theta \in \mathbb{R}$). Consider the price functional $p : \mathbb{R} \to \mathbb{R}$ defined by $p(\theta) := |\theta - \xi| - \xi$ for some exogenous

[5]The vector 0 is not necessarily included in Φ, I have to take this into account in section 6

[6]We take the standard convention that $-\infty + z = -\infty$ for any $z \in \mathbb{R} \cup \{-\infty\}$.

$\xi > 0$. This function p is a robust no-arbitrage price functional since we have $p(\theta^0 + \eta) - p(\theta^0) \geqslant \eta$ for any $\eta \geqslant 0$ and any $\theta^0 \geqslant \xi$. Moreover, we have

$$\pi(\theta) = \begin{cases} -\xi & \text{if } \theta \leqslant \xi, \\ \theta - 2\xi & \text{if } \theta > \xi. \end{cases}$$

In particular, we have $\pi(0) = -\xi < 0$ and the bid-ask spread is not necessarily non-negative since $\pi(2\xi) = 0 < -\pi(-2\xi) = \xi$.

If the price functional p satisfies the law of one price and markets are complete, we can follow Cerreia-Vioglio, Maccheroni and Marinacci (2015) and define $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ by posing $\tilde{\pi}(x) := p(\theta)$ for any portfolio θ satisfying $G(\theta) = x$. We then get that our concept of pricing rule corresponds to an inf-convolution of the function $\tilde{\pi}$.

Lemma 2.5.3. Assume that markets are complete and that the price functional p satisfies the law of one price. Let $\tilde{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ be defined by $\tilde{\pi}(x) := p(\theta)$ for any portfolio θ satisfying $G(\theta) = x$. The payoff pricing rule π associated with p satisfies

$$\pi(x) = \inf\{\tilde{\pi}(z) \ : \ z \in \mathbb{R}^\Omega, \quad x \leqslant z\}.$$

2.5.2 Proper Pricing Rules and Viability

If the pricing rule π associated with a price functional only takes finite values, then we say that π is **proper pricing rule**. Viability of the price functional p is a sufficient condition to guarantee that π is a proper pricing rule.

Proposition 2.5.3. If the price functional p is viable, then the associated pricing rule π is proper.

We can extend the definition of viability to pricing rules.

Definition 2.5.2. A function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ is said to be **viable** when there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ and a payoff $x^\star \in \Phi$ satisfying

$$x^\star \in \arg\max\{U(e_0 - \pi(x), e_1 + x) \ : \ x \in X(\pi)\},$$

where $X(\pi)$ is the set of all payoffs $x \in \Phi$ satisfying the flow budget constraints: $\pi(x) \leqslant e_0$ and $x \geqslant -e_1$.

Viability of p implies viability of π.

Proposition 2.5.4. If a price functional p is viable, then the associated pricing rule π is also viable. More precisely, if $\theta^\star$ is optimal under p, i.e.,

$$\theta^\star \in \arg\max\{U(e_0 - p(\theta), e_1 + G(\theta)) \ : \ \theta \in \Theta(p)\},$$

then the associated payoff $G(\theta^\star)$ is optimal under π, i.e.,

$$G(\theta^\star) \in \arg\max\{U(e_0 - \pi(x), e_1 + x) \,:\, x \in X(\pi)\}.$$

2.5.3 No-Arbitrage and Monotonicity

We can slightly modify Theorem 2.3.1 and Theorem 2.3.2 to derive the following equivalence results.

Corollary 2.5.1. *For a given convex function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$, the following properties are equivalent:*

(i) The function π is viable.

*(ii) The function π is a **robust no-arbitrage** price in the sense that there exists a payoff $x_0 \in \Phi$ such that for any direction $y \in \mathbb{R}^\Omega$ for which there exists $\epsilon > 0$ such that $x_0 + \epsilon y \in \Phi$, the conditions*

$$y \geqslant 0 \quad \text{and} \quad \pi'(x_0, y) \leqslant 0$$

imply

$$y = 0 \quad \text{and} \quad \pi'(x_0, y) = 0.$$

*(iii) The function π is **supported by strictly positive state prices** in the sense that there exists a payoff $x_0 \in \Phi$ and a vector $\mu \in \mathbb{R}^\Omega_{++}$ of strictly positive state prices such that*

$$\pi(x) - \pi(x_0) \geqslant \mu \cdot (x - x_0)$$

for all payoffs $x \in \Phi$.

Consider now a proper pricing rule $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ associated with some price functional $p : \mathbb{R}^J \to \mathbb{R}$. It follows from Proposition 2.5.2 that π is automatically monotone. We provide below a strengthening of monotonicity that turns out to be equivalent to viability and robust no-arbitrage when the market is complete.

Definition 2.5.3. A function $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ is **eventually strictly monotone** when, for any $y \in \mathbb{R}^\Omega$, we have

$$y > 0 \Longrightarrow (\pi 0^+)(y) > 0.$$

The above property can be expressed in two different (but equivalent) ways. Recall that for any $x \in \mathbb{R}^\Omega$, we have

$$(\pi 0^+)(y) = \lim_{\lambda \to \infty} \frac{\pi(x + \lambda y) - \pi(x)}{\lambda}.$$

We then deduce that π is eventually strictly monotone if, and only if, for every $y \in \mathbb{R}^\Omega$ and for every $x \in \mathbb{R}^\Omega$, we have

$$y > 0 \Longrightarrow \exists \lambda^\star > 0, \quad \pi(x + \lambda^\star y) > \pi(x).$$

Since the function $\lambda \mapsto \pi(x + \lambda y)$ is convex, we also have that it is strictly increasing on $[\lambda^\star, \infty)$. Recall moreover that

$$(\pi 0^+)(y) = \sup\{\pi(x + y) - \pi(x) \ : \ x \in \mathbb{R}^\Omega\}.$$

This implies that π is eventually strictly monotone if, and only if, for all $y \in \mathbb{R}^\Omega$, we have

$$y > 0 \Longrightarrow \exists x \in \mathbb{R}^\Omega, \quad \pi(x + y) > \pi(x).$$

When markets are complete, robust no-arbitrage is equivalent to eventual strict monotonicity.

Theorem 2.5.1. *A convex function $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ is a robust no-arbitrage price if, and only if, it is eventually strictly monotone.*

Combining Theorem 2.5.1, Theorem 2.3.2, Proposition 2.5.4 and Theorem 2.5.1, we get the following result.

Corollary 2.5.2. *Let p be a price functional and denote by π its corresponding pricing rule. The following properties are satisfied:*

(i) The following conditions are equivalent:

- *(a) the price functional p is viable;*
- *(b) the price functional p is a robust no-arbitrage;*
- *(c) the price functional p is supported by strictly positive states prices.*

(ii) The following conditions are equivalent:

- *(a) the pricing rule π is viable;*
- *(b) the pricing rule π is a robust no-arbitrage price;*
- *(c) the pricing rule π is supported by strictly positive state prices;*

(iii) If p is viable, then π is viable.

(iv) If p is a robust no-arbitrage price, then π is also a robust no-arbitrage price.

(v) If markets are complete then π is viable if, and only if, it is eventually strictly monotone.

When markets are complete, we have the converse of properties (iii) and (iv) if we replace p by the associated $\tilde{p}$ where we recall that $\tilde{p}(\theta) = \inf\{p(\theta + \eta) : \eta \in \mathrm{Ker}(G)\}$.

Proposition 2.5.5. If markets are complete and π is proper and eventually strictly monotone, then $\tilde{p}$ is a no-arbitrage price. Equivalently, if π is proper and viable, then $\tilde{p}$ is viable.

2.5.4 Put-Call Parity and Viability

In a complete market setting, Cerreia-Vioglio, Maccheroni and Marinacci (2015) have characterized non-null pricing rules satisfying *the put-call parity*, *cash invariance*, monotonicity and subadditivity. The corresponding pricing rule is sublinear. Therefore it is convex. It is a Choquet pricing rule generated by a concave nonadditive probability ν, that is, for all $x \in \mathbb{R}^\Omega$,

$$\pi(x) = \int x \, d\nu = \max_{\mu \in core(\nu)} \mathbb{E}_\mu x$$

where the integral is the Choquet integral. We denote x_{rf} the constant unit vector of $\mathbb{R}^\Omega$, it is a risk-free contingent claim.

Definition 2.5.4. A pricing rule $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies cash invariance if for all $x \in \mathbb{R}^\Omega$ an all $k \in \mathbb{R}$,

$$\pi(x + kx_{rf}) = \pi(x) + k\pi(x_{rf}).$$

Given a contingent claim $x \in \mathbb{R}^\Omega$, we denote $c_{x,k}$ the call option (resp. $p_{x,k}$, the put option) on x with strike price $k \geqslant 0$. We have $c_{x,k} = \max(x - k, 0)$ and $p_{x,k} = \min(k - x, 0)$ and the following equation always hold:

$$c_{x,k} - p_{x,k} = x - kx_{rf}.$$

Definition 2.5.5. A pricing rule $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies the put-call parity if for all $x \in \mathbb{R}^\Omega$ an all $k \geqslant 0$,

$$\pi(c_{x,k}) + \pi(-p_{x,k}) = \pi(x) - k\pi(x_{rf}).$$

In the following proposition, we show that when the market is complete, a non-zero sublinear and monotone pricing rule is viable. Hence Choquet pricing rules generated by a concave nonadditive probability are examples of viable pricing rules.

Proposition 2.5.6. When markets are complete, a non-zero pricing rule satisfying sublinearity and monotonicity is viable.

Pricing rules proposed by Cerreia-Vioglio, Maccheroni and Marinacci (2015) are eminently interesting. They have a closed-form expression, and

we demonstrate that they are viable. Moreover, we show in Section 2.5.1 that a pricing rule is systematically monotone when we derive it from a set of primary assets' prices. Hence this assumption can be required without loss of generality. However, the pricing rules proposed by Cerreia-Vioglio, Maccheroni and Marinacci (2015) are not always adapted to describe markets prices in the presence of friction. Prices on financial security markets are not necessarily subbaditive. Depending on the market depth, it may be more profitable to purchase assets separately. Also, the put-call parity is not always satisfied. It is a linearity assumption. Its violation does not result in an arbitrage opportunity. Moreover, put-call parity, together with the other hypotheses of the model, results in positively homogeneous prices. Once more, evidence shows that depending on the market depth, this property is not necessarily satisfied.

Remark 2.5.2. A condition of no-arbitrage in the put-call parity is

$$\pi(c_{x,k} - p_{x,k}) \geqslant -\pi(-(x - kx_{rf})) \quad \forall x \in \mathbb{R}^{\Omega} \quad \text{and} \quad k \in \mathbb{R}.$$

Together with cash invariance, it implies that bid–ask spreads are positive for all contingent claims, that is,

$$\pi(x) \geqslant -\pi(-x) \quad \forall x \in \mathbb{R}^{\Omega}.$$

2.5.5 Robust No-Arbitrage and No Asymptotic Free-Lunch

Our characterization of the viability of a pricing rule is similar to the one of Jouini and Kallal (1999). Jouini and Kallal (1999) consider a 2−period economy in which agents consume in both dates. The set of marketed claims M represents the available contingent claims to consumption. Agents trade these claims at the initial date. Jouini and Kallal (1999) consider the set of marketed claims to be a convex set containing 0 that is not necessarily closed. Moreover, they require that the price of the contingent claim on no consumption is 0 (i.e., $\pi(0) = 0$). In this chapter, we derive the prices of the contingent claims to consumption directly from the prices of the set of portfolios of primary assets (see Section 2.5.1). Hence, the set of marketed claims Φ is a closed convex set containing 0. Furthermore, although not trading on the market is not costly (i.e. $p(0) = 0$), we the price of the contingent claim 0 is less than or equal to 0 (i.e. $\pi(0) \leqslant 0$).

Jouini and Kallal (1999) consider that an economy is viable if there exists at least an agent satisfied with her endowment. There is no condition on the values taken by (c_0, c_1) other than satisfying the budget constraint. Therefore as opposed to our model, short selling and short purchasing are unrestricted. We denote this viability condition *JK-viability*. Stated with this chapter's terminology and formalism, the definition of JK-viability is the following.

Definition 2.5.6. A function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ is said to be **JK-viable** when there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ such that

$$(e_0, e_1) \in \arg\max\{U(c_0, c_1) \,:\, (c_0, c_1) \in C(\pi)\},$$

where $C(\pi)$ is the set of all consumption bundles $(c_0, c_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ satisfying the budget constraint: $c_0 + \pi(c_1 - e_1) \leqslant e_0$ and such that $c_1 - e_1 \in \Phi$.

To compare our results, we consider the case where $M = \Phi$ and $\pi(0) = 0$. Additionally, we require the consumption levels (c_0, c_1) chosen by the agent to be positive. Under these hypotheses, We show that JK-viability and viability are equivalent.

Proposition 2.5.7. Let $\pi : \Phi \to \mathbb{R}$ be a pricing rule such that $\pi(0) = 0$, then π is JK-viable if, and only if, it is viable.

We denote $A(0, \pi)$ the set of vectors $(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that there exists a vector $x \in \Phi$ satisfying

$$x_0 \leqslant -\pi(x) \quad \text{and} \quad x_1 \leqslant x.$$

Jouini and Kallal (1999) show that JK-viability is equivalent to *no asymptotic free-lunch*. We recall the definition of this property with our terminology.

Definition 2.5.7. The function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ is a no asymptotic free-lunch price if $\overline{cone}(A(0, \pi) \setminus C^\star) \cap C^\star = \emptyset$.

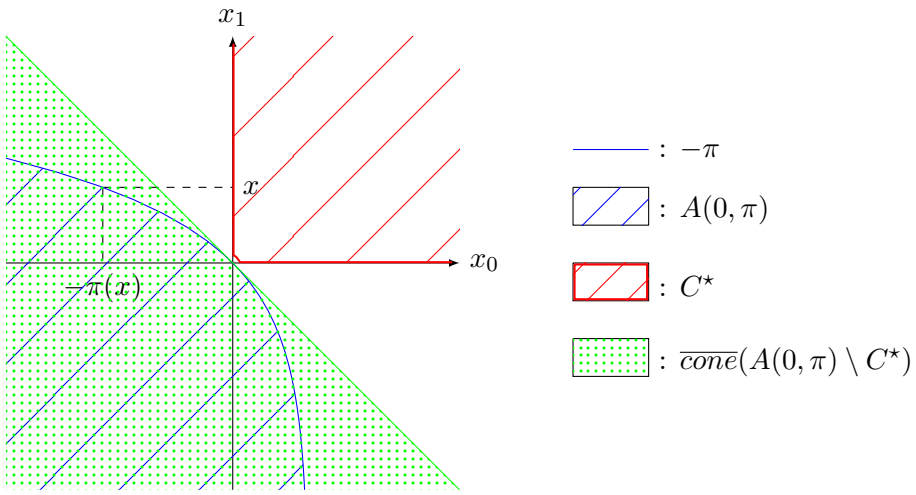
Moreover, a pricing rule is viable if, and only if, it is a robust no-arbitrage price. Therefore by Proposition 2.5.7, a pricing rule is a no asymptotic free-lunch price if, and only if, it is a robust no-arbitrage price.

Proposition 2.5.8. Let $\pi : \Phi \to \mathbb{R}$ be a pricing rule such that $\pi(0) = 0$. The pricing rule π is a no asymptotic free-lunch price if, and only if, it is a robust no-arbitrage price.

Graph 3 illustrates the no-asymptotic free-lunch property when there is no uncertainty.

Similarly to robust no-arbitrage and no-arbitrage, no asymptotic free-lunch represents a strengthening of the *no-free lunch* property of Jouini and Kallal (1995). With our notations, a pricing rule is a no-free lunch price if it is impossible to get arbitrarily close to a positive non-zero net modification in consumption.

Definition 2.5.8. The function $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ is a no free-lunch price if $\overline{A(0, \pi)} \cap C^\star = \emptyset$.



Graph 3: Geometry of no-asymptotic free-lunch with $S = 1$ and $\pi : \mathbb{R} \to \mathbb{R}$ the pricing rule.

Jouini and Kallal (1995) show that a sub-linear pricing rule is JK-viable if, and only if, it is a no-free lunch price. The no-free lunch condition excludes the possibility of obtaining a positive non-zero net modification of consumption. Since we do not assume the set of marketable claims is closed, no free-lunch excludes the possibility of getting arbitrarily close to a positive non-zero net modification of consumption. Additionally, it implies that the set of possible positive non-zero net consumption modifications is not necessarily closed. In our viability characterization, it is unnecessary to exclude the possibility of getting arbitrarily closed to a positive non-zero net modification of consumption. Indeed, we derive the set of marketable claims from the set of portfolios of primary assets. Therefore the set of possible positive non-zero net consumption modifications is closed.

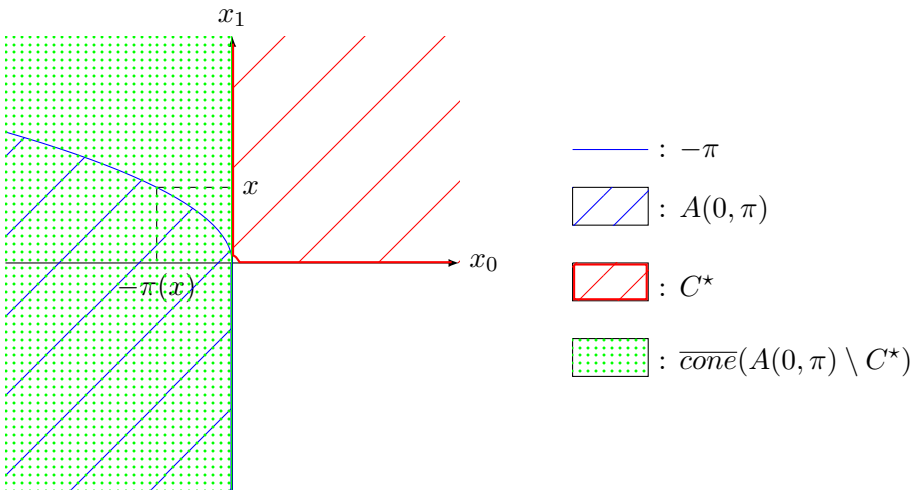
When the pricing rule is convex, Jouini and Kallal (1999) have not proved that no asymptotic free lunch and no free-lunch are equivalent. Graph 4 illustrates a convex pricing rule that is a no-free lunch price but not an asymptotic no free-lunch price. We have

$$\overline{cone}(A(0,\pi) \setminus C^\star) \cap C^\star = \{(0,x) \in \mathbb{R} \times \mathbb{R} : \ x \in]0,\infty[\}.$$

2.6 Complete Viability and Absolute No-Arbitrage

2.6.1 Complete Viability

We introduce complete viability a stronger concept than viability which requires that every portfolio is optimal for at least one agent.



Graph 4: Geometry of no free-lunch with $S = 1$ and $\pi : \mathbb{R} \to \mathbb{R}$ the pricing rule.

Definition 2.6.1. A price functional p is said to be **completely viable** if every portfolio is the solution to the utility maximization problem of at least one agent. Formally, for every $\theta \in \mathbb{R}^J$, there exists $(e_0, e_1, U) \in \mathcal{A}$ such that

$$\theta \in \arg\max\{U(e_0 - p(\tilde{\theta}), e_1 + G(\tilde{\theta})) \; : \; \tilde{\theta} \in \Theta\}. \tag{2.13}$$

It follows from Theorem 2.3.1 that complete viability can also be characterized by means of strictly positive state prices.

Theorem 2.6.1. *A price functional p is completely viable if, and only if, it is **supported by strictly positive state prices everywhere** in the sense that for every portfolio $\theta \in \mathbb{R}^J$ there exists a strictly positive vector $\mu_\theta \in \mathbb{R}^\Omega_{++}$ such that*

$$p(\tilde{\theta}) - p(\theta) \geqslant \mu_\theta[G(\tilde{\theta}) - G(\theta)], \quad \textit{for all } \tilde{\theta} \in \mathbb{R}^J. \tag{2.14}$$

2.6.2 Absolute No-Arbitrage

We introduce absolute no-arbitrage a stronger concept than no-arbitrage which requires that every portfolio satisfy the no-arbitrage conditions.

Definition 2.6.2. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ is said to be an **absolute no-arbitrage price** when for every pair of portfolios $(\theta, \eta) \in \mathbb{R}^J \times \mathbb{R}^J$, the conditions

$$p(\theta + \eta) - p(\theta) \leqslant 0 \quad \text{and} \quad G(\theta + \eta) - G(\theta) \geqslant 0$$

imply

$$p(\theta + \eta) - p(\theta) = 0 \quad \text{and} \quad G(\theta + \eta) - G(\theta) = 0.$$

To demonstrate that complete viability is equivalent to absolute no-arbitrage, we introduce the intermediary concept of absolute robust no-arbitrage which requires that every portfolios satisfy the robust no-arbitrage conditions.

Definition 2.6.3. A price functional $p : \mathbb{R}^J \to \mathbb{R}$ is said to be an **absolute robust no-arbitrage price** when for every pair of portfolios $(\theta, \eta) \in \mathbb{R}^J \times \mathbb{R}^J$, the conditions

$$p'(\theta;\eta) \leqslant 0 \quad \text{and} \quad G'(\theta;\eta) \geqslant 0$$

imply

$$p'(\theta;\eta) = 0 \quad \text{and} \quad G'(\theta;\eta) = 0.$$

It follows from Theorem 2.3.2 that absolute robust no-arbitrage is an adequate concept to characterize complete viability. Hence, every portfolio satisfies the robust no-arbitrage conditions if, and only if, it is the solution to some agent's utility maximization problem.

Theorem 2.6.2. *A price functional is completely viable if, and only if, it is an absolute robust no-arbitrage price.*

We show that the more intuitive concept of absolute no-arbitrage is also adequate to characterize complete viability.

Theorem 2.6.3. *A price functional is completely viable if, and only if, it is an absolute no-arbitrage price.*

2.7 Conclusion

The pricing function proposed in chapter 1 is interesting for empirical studies because it is a closed-form expression with a relatively small number of parameters. However, the absence of friction on riskless security joint with put–call parity imply positively homogeneous prices. Positively homogeneous prices is unrealistic in financial security markets. The abundant empirical literature on market impact have demonstrated that security prices are increasing in the quantity traded. Moreover, several theoretical models have shown that the presence of insiders on the market induce convex prices. Therefore, we study convex prices in this chapter. We demonstrate that markets are viable if, and only if, there is no robust arbitrage opportunity (a strong no arbitrage property). We then demonstrate that the associated pricing rule is necessarily monotone which provides arguments in favour of making this technical assumption in other models. Then, we show that when markets are complete, prices satisfy robust no-arbitrage if, and only if, they are eventually monotone. Eventually, we show that a price functional is viable for every agents if, and only if, every portfolio satisfy the no-arbitrage conditions.

2.8 Appendix

2.8.1 Convex Analysis

Fix some arbitrary integer $n \in \mathbb{N}$ and a convex function $f : \mathbb{R}^n \to \mathbb{R}$. The function f is necessarily continuous on $\mathbb{R}^n$. Fix an arbitrary $x \in \mathbb{R}^n$. For any vector $v \in \mathbb{R}^n$, the limit

$$f'(x;v) = \lim_{\varepsilon \to 0^+} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon}$$

exists in $\mathbb{R}$ and is called the directional derivative of f at x in the direction v.

Let C be a non-empty convex subset of $\mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is **direction of recession** of C when

$$\forall c \in C, \quad \forall \lambda \geqslant 0, \quad c + \lambda v \in C.$$

The set of all directions of recession of C is a convex cone containing the origin, called the **recession cone** of C, and is denoted by 0^+C. Convexity of C implies that

$$0^+C = \{v \in \mathbb{R}^n \ : \ C + v \subseteq C\}.$$

If C is a non-empty convex and closed subset of $\mathbb{R}^n$, then 0^+C is also closed. Moreover, we have

$$0^+C = \{v \in \mathbb{R}^n \ : \ \exists c_0 \in C, \quad \forall \lambda \geqslant 0, \quad c_0 + \lambda v \in C\}$$

and v is a direction of recession if, and only if, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of C and a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 such that

$$v = \lim \lambda_n c_n.$$

A non-empty closed convex set is bounded if, and only if, its recession cone consists of the zero vector alone.

The vector $v \in \mathbb{R}^n$ is a **direction in which the set C is linear** when $c + \lambda v \in C$ for every $c \in C$ and every $\lambda \in \mathbb{R}$. The set of all directions in which C is linear is called the **lineality space** of C and is denoted by $\mathrm{Li}(C)$. We have

$$\mathrm{Li}(C) = 0^+C \cap (-0^+C).$$

The following proposition corresponds to Corollary 8.6.1 in Rockafellar (1970).

Proposition 2.8.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If it is bounded on an affine set, then it is constant on this set.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The recession function $f0^+$:

$\mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(f0^+)(y) := \sup\{f(x+y) - f(x) : x \in \mathbb{R}^n\}.$$

The function $f0^+$ is positively homogeneous and convex. Moreover, it satisfies

$$(f0^+)(y) = \sup_{\lambda>0} \frac{f(x+\lambda y) - f(x)}{\lambda} = \lim_{\lambda\to\infty} \frac{f(x+\lambda y) - f(x)}{\lambda}$$

for any $x \in \mathbb{R}^n$.

The following proposition corresponds to Corollary 8.6.1 in Rockafellar (1970).

Proposition 2.8.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and a vector $y \in \mathbb{R}^n$. The following properties are equivalent:

(i) for every $x \in \mathbb{R}^n$, the function $\lambda \mapsto f(x+\lambda y)$ is constant;

(ii) there exists $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $f(x+\lambda y) \leqslant \alpha$ for every $\lambda \in \mathbb{R}$;

(iii) $(f0^+)(y) \leqslant 0$ and $(f0^+)(-y) \leqslant 0$.

The set of all vectors y such that $(f0^+)(y) \leqslant 0$ is called the **recession cone of** f. This is a convex closed cone containing 0. A vector in the recession cone of f is called a **direction in which** f **recedes.** The set of vectors y such that $(f0^+)(y) \leqslant 0$ and $(f0^+)(-y) \leqslant 0$ is the largest linear subspace contained in the recession cone of f. It is called the **constancy space of** f and vectors in this space are called **directions in which** f **is constant**.

The following proposition corresponds to Theorem 8.7 in Rockafellar (1970).

Proposition 2.8.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. All the nonempty level sets of the form $\{x \in \mathbb{R}^n : f(x) \leqslant \alpha\}$ with $\alpha \in \mathbb{R}$, have the same recession cone and the same lineality space, namely the recession cone and the constancy cone of f, respectively.

The following proposition corresponds to Theorem 8.6 in Rockafellar (1970).

Proposition 2.8.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The following properties are equivalent:

(i) $(f0^+)(y) \leqslant 0$;

(ii) for every $x \in \mathbb{R}^n$, the function $\lambda \mapsto f(x+\lambda y)$ is non-increasing on $\mathbb{R}$;

(iii) there exists $x \in \mathbb{R}^n$ such that function $\lambda \mapsto f(x+\lambda y)$ is non-increasing on $\mathbb{R}$.

2.8.2 Proofs

Proof of Lemma 2.2.1. This result follows directly from (2.6). Indeed, if the function $\lambda \longmapsto f(\theta^0+\lambda\eta)$ is constant, then (2.6) implies that $(f0^+)(\eta) \leqslant 0$ and $(f0^+)(-\eta) \leqslant 0$. Fix now an arbitrary vector $\theta \in \mathbb{R}^K$ and an arbitrary $\lambda \in \mathbb{R}$. If $\lambda > 0$, we use $(f0^+)(\eta) \leqslant 0$, to deduce that $f(\theta+\lambda\eta) \leqslant f(\theta)$. If $\lambda < 0$, we use $(f0^+)(-\eta) \leqslant 0$, to deduce that $f(\theta+(-\lambda)(-\eta)) \leqslant f(\theta)$. We have thus proved that the function $\lambda \mapsto f(\theta+\lambda\eta)$ is convex and attains its maximum at 0. This is possible only if the function $\lambda \mapsto f(\theta+\lambda\eta)$ is constant. □

Proof of Theorem 2.3.1. We first prove that (2.9) is necessary. Assume that there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ and a portfolio $\theta^\star$ such that

$$\theta^\star \in \arg\max\{v(\theta) \,:\, \theta \in \Theta\}.$$

We let $c_0^\star := e_0 - p(\theta^\star)$ and $c_1^\star := e_1 + G(\theta^\star)$ be the corresponding optimal consumption levels. Optimality of $\theta^\star$ implies that

$$(c^\star, \theta^\star) \in \arg\max\{U(c) \,:\, (c,\theta) \in D\}$$

where D is the set of all $((c_0, c_1), \theta) \in (\mathbb{R}\times\mathbb{R}^\Omega)\times\mathbb{R}^J$ satisfying the restrictions

$$0 \leqslant c_0 \leqslant e_0 - p(\theta) \quad \text{and} \quad 0 \leqslant c_1 \leqslant e_1 + G(\theta).$$

Since U satisfies Inada's condition at the origin and endowments e_0 and e_1 are strictly positive, we must have $c_0^\star > 0$ and $c_1^\star \gg 0$. Moreover, the Slater's condition is satisfied.[7] Therefore, there exists a non-zero non-negative vector $(\xi_0, \xi_1) \in \mathbb{R}_+ \times \mathbb{R}_+^\Omega$ of Lagrange multipliers such that

$$(c^\star, \theta^\star) \in \arg\max\{U(c)+\xi_0[e_0-p(\theta)-c_0]+\xi_1\cdot[e_1+G(\theta)-c_1] \,:\, (c,\theta) \in V\} \tag{2.15}$$

where V is the open of set of all $(c,\theta) \in C \times \mathbb{R}^J$ with $c \gg 0$. In particular, we have that the mapping $c_0 \mapsto U(c_0, c_1^\star) - \xi_0 c_0$ attains its maximum at $c_0^\star$. Since U is concave, this implies that

$$\partial_0^+ U(c^\star) \leqslant \xi_0 \leqslant \partial_0^- U(c^\star)$$

where $\partial_0^+ U(c^\star)$ is the right-derivative of the function $c_0 \mapsto U(c_0, c_1^\star)$ at $c_0^\star$ and $\partial_0^- U(c^\star)$ is the left-derivative of the function $c_0 \mapsto U(c_0, c_1^\star)$ at $c_0^\star$. Similarly, for every state $\omega \in \Omega$ at $t = 1$, the mapping $c_1(\omega) \mapsto U(c_0^\star, (c_1(\omega), c_1^\star(-\omega)))-$

[7] Since $(e_0, e_1) \gg 0$ and (p, G) is continuous at 0 with $(p(0), G(0)) = (0,0)$, we can find θ close enough to 0 such that $e_0 - p(\theta) > 0$ and $G(\theta) + e_1 \gg 0$. We can then choose $c_0 := (e_0 - p(\theta))/2$ and $c_1 := (e_1 + G(\theta))/2$.

$\xi_1(\omega)c_1(\omega)$ attains its maximum at $c_1^\star(\omega)$ and we have

$$\partial_\omega^+ U(c^\star) \leqslant \xi_1(\omega) \leqslant \partial_\omega^- U(c^\star)$$

where $\partial_\omega^+ U(c^\star)$ is the right-derivative of the function $c_1(\omega) \mapsto U(c_0^\star, (c_1(\omega), c_1^\star(-\omega)))$ at $c_1^\star(\omega)$ and $\partial_\omega^- U(c^\star)$ is the left-derivative of the function $c_1(\omega) \mapsto U(c_0^\star, (c_1(\omega), c_1^\star(-\omega)))$ at $c_1^\star(\omega)$.

Since the function U is concave and strictly increasing, we deduce that $\xi_0 > 0$ and $\xi_1(\omega) > 0$ for all $\omega \in \Omega$. From (2.15), we also deduce that

$$\xi_0[p(\theta) - p(\theta^\star)] \geqslant \xi_1 \cdot [G(\theta) - G(\theta^\star)], \quad \text{for all } \theta \in \mathbb{R}^J.$$

Posing $\mu := (1/\xi_0)\xi_1$, we get the desired result.

Now, we prove that (2.9) is sufficient. Let $\theta^\star \in \mathbb{R}^J$ and μ satisfying (2.9). Fix an arbitrary strictly concave, strictly increasing and differentiable function $u : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x) - u(0)}{x} = \infty \quad \text{and} \quad \lim_{x \to \infty} u'(x) = 0.$$

We can take, for instance, $u(x) := \sqrt{x}$. Fix an arbitrary $\beta \in (0, 1)$ and an arbitrary strictly positive vector $P \in \mathbb{R}^\Omega_{++}$ satisfying $\sum_{\omega \in \Omega} P(\omega) = 1$ (i.e., P is a probability measure on Ω with full support). Choose $e_0 > \max\{0, p(\theta^\star)\}$ and $e_1 \gg 0$ such that

$$\frac{\beta P(\omega) u'(c_1^\star(\omega))}{u'(c_0^\star)} = \mu(\omega), \quad \text{for all } \omega \in \Omega \tag{2.16}$$

where

$$c_0^\star := e_0 - p(\theta^\star) > 0 \quad \text{and} \quad c_1^\star := e_1 + G(\theta^\star) \gg 0.$$

The existence of $e_1 \gg 0$ satisfying the above conditions follows from the fact $u' : (0, \infty) \to (0, \infty)$ is a one-to-one function. We pose

$$U(c_0, c_1) := u(c_0) + \beta \sum_{\omega \in \Omega} P(\omega) u(c_1(\omega)).$$

Observe that $\theta^\star \in \Theta$. We claim that $\theta^\star$ is optimal. Indeed, let $\theta \in \Theta$. By concavity of u, we have

$$v(\theta) - v(\theta^\star) \leqslant u'(c_0^\star)\,[-p(\theta) + p(\theta^\star)] + \beta \sum_{\omega \in \Omega} P(\omega) u'(c_1^\star(\omega))\,[G(\theta, \omega) - G(\theta^\star, \omega)].$$

Inequality (2.9) combined with (2.16) then implies the desired result. □

Proof of Proposition 2.3.1. Consider a price functional p that is viable. This means that there exist an agent $(e_0, e_1, U) \in \mathcal{A}$ and an optimal portfolio

$\theta^\star$ in the sense that

$$\theta^\star \in \arg\max\{v(\theta) \,:\, \theta \in \Theta\}.$$

We claim that the no-arbitrage condition is satisfied for $\theta^0 = \theta^\star$. Indeed, fix an arbitrary direction $\eta \in \mathbb{R}^J$ such that $G(\theta^\star + \eta) \geqslant G(\theta^\star)$ and $p(\theta^\star + \eta) \leqslant p(\theta^\star)$. Observe that $\theta^\star + \eta \in \Theta$ and

$$v(\theta^\star+\eta) = U(e_0-p(\theta^\star+\eta), e_1+G(\theta^\star+\eta)) \geqslant U(e_0-p(\theta^\star), e_1+G(\theta^\star)) = v(\theta^\star).$$

Optimality of $\theta^\star$ implies that we must have an equality. Since $(c_0, c_1) \mapsto U(c_0, c_1)$ is strictly increasing, this implies that $G(\theta^\star + \eta) = G(\theta^\star)$ and $p(\theta^\star + \eta) = p(\theta^\star)$. □

Proof of Lemma 2.3.1. Consider a price functional $p : \mathbb{R}^J \to \mathbb{R}$ and assume it is a robust no-arbitrage price with respect to a portfolio θ^0. We claim that p is a no-arbitrage price with respect to the same portfolio. To prove this, fix an arbitrary direction η such that $G(\theta^0 + \eta) \geqslant G(\theta^0)$ and $p(\theta^0 + \eta) \leqslant p(\theta^0)$. Recall that the function $\lambda \mapsto (p(\theta^0 + \lambda\eta) - p(\theta^0))/\lambda$ is increasing. We then deduce that $p'(\theta^0, \eta) \leqslant p(\theta^0 + \eta) - p(\theta^0) \leqslant 0$. Similarly, the function $\lambda \mapsto (G(\theta^0 + \lambda\eta) - G(\theta^0))/\lambda$ is decreasing and we deduce that $G'(\theta^0, \eta) \geqslant G(\theta^0 + \eta) - G(\theta^0) \geqslant 0$. Since p is a robust no-arbitrage price, we deduce that $G'(\theta^0; \eta) = 0$ and $p'(\theta^0; \eta) = 0$. Applying again the fact that the function $\lambda \mapsto (p(\theta^0 + \lambda\eta) - p(\theta^0))/\lambda$ is increasing and the function $\lambda \mapsto (G(\theta^0 + \lambda\eta) - G(\theta^0))/\lambda$ is decreasing, we deduce that $p(\theta^0 + \eta) - p(\theta^0) \geqslant p'(\theta^0, \eta) = 0$ and $G(\theta^0 + \eta) - G(\theta^0) \leqslant G'(\theta^0, \eta) = 0$ This implies the desired result: $p(\theta^0 + \eta) = p(\theta^0)$ and $G(\theta^0 + \eta) = G(\theta^0)$. □

Proof of Theorem 2.3.3. We first prove sufficiency of (2.9). Let $p : \mathbb{R}^J \to \mathbb{R}$ be a price functional for which there exists a portfolio $\theta^\star$ satisfying (2.9). We claim that p is a robust no-arbitrage price with respect to portfolio $\theta^\star$. Indeed, fix an arbitrary direction $\eta \in \mathbb{R}^J$ satisfying

$$G'(\theta^\star; \eta) \geqslant 0 \quad \text{and} \quad p'(\theta^\star; \eta) \leqslant 0.$$

Observe that for any $\lambda > 0$, we have

$$\frac{p(\theta^\star + \lambda\eta) - p(\theta^\star)}{\lambda} \geqslant \mu \cdot \frac{G(\theta^\star + \lambda\eta) - G(\theta^\star)}{\lambda}.$$

Passing to the limit when λ tends to zero, we deduce that

$$p'(\theta^\star; \eta) \geqslant \mu \cdot G'(\theta^\star; \eta).$$

Since $G'(\theta^\star; \eta) \geqslant 0$ and $p'(\theta^\star; \eta) \leqslant 0$, we must have $\mu \cdot G'(\theta^\star; \eta) = 0$ and $p'(\theta^\star; \eta) = 0$. Since μ is strictly positive, we can deduce that $G'(\theta^\star; \eta) = 0$. We have thus proved that p is a robust no-arbitrage price.

To prove that (2.9) is a necessary condition, we fix a price functional $p : \mathbb{R}^J \to \mathbb{R}$ that is a robust no-arbitrage price. Let A be the subset of $\mathbb{R} \times \mathbb{R}^\Omega$ of all vectors (a_0, a_1) such that there exists $\eta \in \mathbb{R}^J$ satisfying

$$a_0 \leqslant -p'(\theta^\star; \eta) \quad \text{and} \quad a_1 \leqslant G'(\theta^\star; \eta).$$

Since the mapping $\eta \mapsto p'(\theta^\star; \eta)$ is convex (and therefore continuous), we deduce that the set A is convex and closed. We let B be the subset of $\mathbb{R} \times \mathbb{R}^\Omega$ of all vectors (b_0, b_1) such that

$$0 \leqslant b_0, \quad 0 \leqslant b_1 \quad \text{and} \quad b_0 + \sum_{\omega \in \Omega} b_1(\omega) = 1.$$

The set B is compact and convex. Since p is a robust no-arbitrage, we must have $A \cap B = \emptyset$. Applying the Strict Convex Separation Theorem, there exists a non-zero $(\xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ and $\alpha < \beta$ such that

$$\forall \eta \in \mathbb{R}^J, \quad -\xi_0 p'(\theta^\star; \eta) + \xi_1 \cdot G'(\theta^\star; \eta) \leqslant \alpha < \beta \leqslant \xi_0 b_0 + \xi_1 \cdot b_1, \quad \forall (b_0, b_1) \in B.$$

Choosing $\eta = 0$, we deduce that $\xi_0 > 0$ and $\xi_1 \gg 0$. Observe that the mapping $\eta \mapsto -\xi_0 p'(\theta^\star; \eta) + \xi_1 \cdot G'(\theta^\star; \eta)$ is positively homogeneous. In particular, for any $\lambda > 0$ and any $\eta \in \mathbb{R}^J$, we have

$$\lambda \left[-\xi_0 p'(\theta^\star; \eta) + \xi_1 \cdot G'(\theta^\star; \eta) \right] \leqslant \alpha.$$

Passing to the limit when λ tends to infinite, we deduce that

$$-\xi_0 p'(\theta^\star; \eta) + \xi_1 \cdot G'(\theta^\star; \eta) \leqslant 0, \quad \text{for all } \eta \in \mathbb{R}^J.$$

If we pose $\mu := (1/\xi_0)\xi_1$, we get the desired result. □

Proof of Proposition 2.3.2. We first assume that p is a robust no-arbitrage price. It follows from Theorem 2.3.2 that p is a viable price. That is, there exist an agent $(e_0, e_1, U) \in \mathcal{A}$ and a portfolio $\theta^\star \in \mathbb{R}^J$ such that $\theta^\star$ is optimal, i.e.,

$$\theta^\star \in \arg\max\{v(\theta|p) \ : \ \theta \in \Theta(p)\}.$$

Recall that $v(\theta|p) = U(e_0 - p(\theta), e_1 + G(\theta))$. We pose $c^\star = (c_0^\star, c_1^\star)$ where $c_0^\star := e_0 - p(\theta^\star)$ and $c_1^\star := e_1 + G(\theta^\star)$. Since U satisfies the Inada's condition at the origin, we must have $c^\star \gg 0$. Let Γ be the set directions where U increases starting form $c^\star$, i.e.,

$$\Gamma := \{(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^\Omega \ : \quad c^\star + x \gg 0 \quad \text{and} \quad U(c^\star + x) > U(c^\star)\}.$$

Since U is concave, the set Γ is convex. Since U is continuous, the set Γ is

open. Since U is strictly increasing, the set Γ satisfies

$$C^\star \subseteq \Gamma.$$

Optimality of $\theta^\star$ implies that $A(\theta^\star, p) \cap \Gamma = \emptyset$.

We assume now that p satisfies the properties of Proposition 2.3.2. We let A be the set of all $(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that there exists $\eta \in \mathbb{R}^J$ satisfying

$$x_0 \leqslant p(\theta^\star) - p(\theta^\star + \eta) \quad \text{and} \quad x_1 \leqslant G(\theta^\star + \eta) - G(\theta^\star).$$

Since p is convex and G is concave, the set A is convex. Since p satisfies the properties of Proposition 2.3.2, we have

$$A \cap \Gamma = \emptyset.$$

Applying the standard Convex Separation Theorem, there exists a non-zero $(\xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^\Omega$ such that

$$\xi_0 x_0 + \xi_1 \cdot x_1 \leqslant \xi_0 \gamma_0 + \xi_1 \cdot \gamma_1,$$

for all $x = (x_0, x_1) \in A$ and all $\gamma = (\gamma_0, \gamma_1) \in \Gamma$. Choosing $x = (0, 0)$, we derive that $\xi_0 \geqslant 0$ and $\xi_1 \geqslant 0$. Actually, since Γ is open and contains $C^\star$, we deduce that $\xi_0 > 0$ and $\xi_1 \gg 0$. Fix now an arbitrary $\eta \in \mathbb{R}^J$. Choosing $x_0 = p(\theta^\star) - p(\theta^\star + \eta)$ and $x_1 = G(\theta^\star + \eta) - G(\theta^\star + \eta)$, we deduce that

$$p(\theta^\star + \eta) - p(\theta^\star) \geqslant \mu \cdot [G(\theta^\star + \eta) - G(\theta^\star)]$$

where $\mu := (1/\xi_0)\xi_1$. This implies that p is a robust no-arbitrage price. □

Proof of Lemma 2.4.1. Following Theorem 4.2 in Rockafellar (1970), it is sufficient to prove that

$$\tilde{p}(\lambda\theta + \lambda'\theta') < \lambda\alpha + \lambda'\alpha', \quad \text{for all } \lambda, \lambda' \geqslant 0 \text{ with } \lambda + \lambda' = 1$$

whenever $\tilde{p}(\theta) < \alpha$ and $\tilde{p}(\theta') < \alpha'$. Since α is not a lower bound of $\{p(\eta) : \eta \in \mathbb{R}^J \text{ and } G(\eta) \geqslant G(\theta)\}$, there exists $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that $\alpha > p(\eta)$. Similarly, there exists $\eta' \in \mathbb{R}^J$ with $G(\eta') \geqslant G(\theta')$ such that $\alpha' > p(\eta')$. Convexity of p implies that

$$p(\lambda\eta + \lambda'\eta') \leqslant \lambda p(\eta) + \lambda' p(\eta') < \lambda\alpha + \lambda'\alpha'.$$

Concavity of G implies $G(\lambda\eta + \lambda'\eta') \geqslant G(\lambda\theta + \lambda'\theta')$. We then get

$$\tilde{p}(\lambda\theta + \lambda'\theta') \leqslant p(\lambda\eta + \lambda'\eta') < \lambda\alpha + \lambda'\alpha'.$$

This proves that the epigraph of $\tilde{p}$ is convex. □

Proof of Lemma 2.4.2. Assume that there exists $\theta^0 \in \mathbb{R}^J$ such that $\tilde{p}(\theta^0) = -\infty$. Fix an arbitrary $\theta \in \mathbb{R}^J$ and let $\theta_1 := 2\theta - \theta^0$. Fix an arbitrary real number $\alpha_0 \in \mathbb{R}$ and let $\alpha_1 > \tilde{p}(\theta_1)$.[8] Since $\theta = (1/2)\theta^0 + (1/2)\theta_1$, convexity of $\tilde{p}$ implies that

$$\tilde{p}(\theta) \leqslant (1/2)\alpha_0 + (1/2)\alpha_1.$$

Since α_0 is arbitrary, we deduce that $\tilde{p}(\theta) = -\infty$. We have thus proved that either $\tilde{p}(\mathbb{R}^J) \subseteq \mathbb{R}$ or $\tilde{p}(\mathbb{R}^J) = \{-\infty\}$. □

Proof of Proposition 2.4.1. Assume that the price functional p is viable. This means that there exists an optimal portfolio $\theta^\star \in \arg\max\{v(\theta|p) : \theta \in \Theta(p)\}$. Observe that we must have $\tilde{p}(\theta^\star) = p(\theta^\star)$ by optimality of $\theta^\star$. This implies that $\tilde{p}(\theta^\star) > -\infty$. We then deduce that $\tilde{p}(\mathbb{R}^J) \subseteq \mathbb{R}$. Since we already know that $\tilde{p}$ is convex (Lemma 2.4.1), we conclude that $\tilde{p}$ is a price functional. To prove that $\tilde{p}$ satisfies the law of one price, we fix two portfolios $\theta, \theta' \in \mathbb{R}^J$ satisfying $G(\theta) = G(\theta')$. Observe that for any $\eta \in \mathbb{R}^J$, the conditions $G(\eta) \geqslant G(\theta)$ and $G(\eta) \geqslant G(\theta')$ are equivalent. We then get that $\tilde{p}(\theta) = \tilde{p}(\theta')$. □

Proof of Proposition 2.4.2. Let $\theta^\star \in \arg\max\{v(\theta|p) : \theta \in \Theta(p)\}$. Since $\tilde{p} \leqslant p$, we have $\Theta(p) \subseteq \Theta(\tilde{p})$ and $v(\theta|p) \leqslant v(\theta|\tilde{p})$ for any $\theta \in \Theta(p)$. In particular, we have $\theta^\star \in \Theta(\tilde{p})$. To prove that $\theta^\star \in \arg\max\{v(\theta|\tilde{p}) : \theta \in \Theta(\tilde{p})\}$, we fix an arbitrary $\theta \in \Theta(\tilde{p})$ and show that $v(\theta|\tilde{p}) \leqslant v(\theta^\star|\tilde{p})$. Actually, we have $v(\theta|\tilde{p}) \leqslant v(\theta^\star|p)$. Indeed, fix an arbitrarily small $\varepsilon > 0$. Continuity of U implies that there exists $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that

$$v(\theta|\tilde{p}) - \varepsilon \leqslant U(e_0 - p(\eta), e_1 + G(\theta)) \leqslant v(\eta|p) \leqslant v(\theta^\star|p).$$

Since this inequality is valid for any $\varepsilon > 0$, we deduce that $v(\theta|\tilde{p}) \leqslant v(\theta^\star|p)$. Since $\tilde{p} \leqslant p$, we also have $v(\theta^\star|p) \leqslant v(\theta^\star|\tilde{p})$. We have thus proved that

$$\forall \theta \in \Theta(\tilde{p}), \quad v(\theta|\tilde{p}) \leqslant v(\theta^\star|p) \leqslant v(\theta^\star|\tilde{p})$$

which implies that $\theta^\star$ is optimal under the price function $\tilde{p}$. Moreover, replacing θ by $\theta^\star$ in the above inequality, we deduce that $v(\theta^\star|\tilde{p}) = v(\theta^\star|p)$, and therefore $\tilde{p}(\theta^\star) = p(\theta^\star)$. □

Proof of Lemma 2.5.1. By definition, Φ is closed. We now show that Φ is convex. Let $(x, y) \in \Phi \times \Phi$, then there exists $(\theta, \theta') \in \mathbb{R}^J \times \mathbb{R}^J$ such that $G(\theta) \geqslant x$ and $G(\theta') \geqslant y$. By concavity of G, we have for all $\lambda \in (0, 1)$,

$$G(\lambda\theta + (1-\lambda)\theta') \geqslant \lambda\lambda G(\theta) + (1-\lambda)G(\theta') \geqslant \lambda x + (1-\lambda)y.$$

Hence, $\lambda x + (1-\lambda)y \in \Phi$. □

[8] The fact that $\tilde{p}(\theta_1) < \infty$ is crucial for this argument.

Proof of Proposition 2.5.1. We are going to show that $\pi : \Phi \to \mathbb{R} \cup \{-\infty\}$ is a convex function. To do so, we are going to show that $\text{epi}(\pi)$ is a convex subset of $\mathbb{R}^\Omega$. We have

$$\text{epi}(\pi) = \{(x, \mu) \in \Phi \times \mathbb{R} : \ \mu \geqslant \pi(x)\}.$$

Let $((x, \mu), (x', \mu')) \in \text{epi}(\pi) \times \text{epi}(\pi)$, then there exist $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant \pi(x)$ and $p(\theta) \leqslant \mu$ and $\theta' \in \mathbb{R}^J$ such that $G(\theta') \geqslant \pi(x')$ and $p(\theta') \leqslant \mu'$. Convexity of p implies that for all $\lambda \in (0, 1)$,

$$p(\lambda\theta + (1-\lambda)\theta') \leqslant \lambda p(\theta) + (1-\lambda)p(\theta') \leqslant \lambda\mu + (1-\lambda)\mu'.$$

Since $G(\lambda\theta + (1-\lambda)\theta') \geqslant \lambda\mu + (1-\lambda)\mu'$, we have

$$\pi(\lambda x + (1-\lambda)x') \leqslant p(\lambda\theta + (1-\lambda)\theta') \leqslant \lambda\mu + (1-\lambda)\mu'.$$

Hence $\text{epi}(\pi)$ is a convex subset of $\mathbb{R}^\Omega$. □

Proof of Proposition 2.5.2. Let x, y be two payoffs in Φ such that $x \geqslant y$. Fix an arbitrary portfolio $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant x$. We also have $G(\theta) \geqslant y$ and we deduce that $\pi(y) \leqslant p(\theta)$. We have thus proved that $\pi(y)$ is a lower bound of the set $\{p(\theta) : \ G(\theta) \geqslant x\}$. This implies that $\pi(y) \leqslant \pi(x)$. □

Proof of Lemma 2.5.2. Let $\tilde{\pi}(x)$ be the pricing rule associated with $\tilde{p}$, i.e.,

$$\tilde{\pi}(x) := \inf\{\tilde{p}(\theta) : \ x \leqslant G(\theta)\}.$$

Since $\tilde{p} \leqslant p$, we have $\tilde{\pi} \leqslant \pi$. To prove the converse inequality, we fix $x \in \Phi$ and start by analyzing the case where $\tilde{\pi}(x) \in \mathbb{R}$. Fix some arbitrary $\varepsilon > 0$. By definition of $\tilde{\pi}(x)$, there must exist $\theta \in \mathbb{R}^J$ with $G(\theta) \geqslant x$ such that $\tilde{\pi}(x) + \varepsilon \geqslant \tilde{p}(\theta)$. By definition of $\tilde{p}(\theta)$, there exists $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that $\tilde{p}(\theta) + \varepsilon \geqslant p(\eta)$.We have thus proved that $\tilde{\pi}(x) + 2\varepsilon \geqslant p(\eta)$ where $G(\eta) \geqslant x$. It then follows that $\tilde{\pi}(x) + 2\varepsilon \geqslant \pi(x)$. Since this is true for any $\varepsilon > 0$, passing to the limit when ε tends to zero, we get the desired result. Now, we analyze the case where $\tilde{\pi}(x) = -\infty$. Fix an arbitrary $M > 0$. There must exist $\theta \in \mathbb{R}^J$ with $G(\theta) \geqslant x$ such that $\tilde{p}(\theta) \leqslant -M$. By definition of $\tilde{p}(\theta)$, there must exist $\eta \in \mathbb{R}^J$ with $G(\eta) \geqslant G(\theta)$ such that $\tilde{p}(\theta) \leqslant p(\eta) \leqslant -M/2$. We have thus proved that for any $M > 0$, there exists $\eta \in \mathbb{R}^J$ such that $G(\eta) \geqslant x$ and $p(\eta) \leqslant -M/2$. This necessarily implies that $\pi(x) = -\infty$. □

Proof of Lemma 2.5.3. Fix $z \in \mathbb{R}^\Omega$ such that $x \leqslant z$. Since markets are complete, there exists a portfolio $\theta \in \mathbb{R}^J$ such that $z = G(\theta)$. By definition of $\tilde{\pi}$, we have $\tilde{\pi}(z) = p(\theta)$. This implies that $\pi(x) \leqslant \tilde{\pi}(z)$ and we proved that

$$\pi(x) \leqslant \rho(x) := \inf\{\tilde{\pi}(z) : \ z \in \mathbb{R}^\Omega \quad \text{and} \quad x \leqslant z\}.$$

Reciprocally, let $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant x$. By posing $z := G(\theta)$, we get that $p(\theta) = \tilde{\pi}(z)$, and consequently, $\rho(x) \leqslant p(\theta)$. We have thus proved that $\rho(x)$ is a lower bound of the set $\{p(\theta) : x \leqslant G(\theta)\}$. This, in turn, implies that $\rho(x) \leqslant \pi(x)$. □

Proof of Proposition 2.5.3. Assume that the price functional p is viable and let $\theta^\star \in \arg\max\{v(\theta|p) : \theta \in \Theta(p)\}$. We claim that $\pi(G(\theta^\star)) = p(\theta^\star)$. By construction, we have $\pi(G(\theta^\star)) \leqslant p(\theta^\star)$. Assume, by way of contradiction, that $\pi(G(\theta^\star)) < p(\theta^\star)$. Then $p(\theta^\star)$ is not a lower bound of the set $\{p(\theta) : \theta \in \mathbb{R}^J \text{ and } G(\theta^\star) \leqslant G(\theta)\}$. Therefore, there exists $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant G(\theta^\star)$ and $p(\theta) < p(\theta^\star)$. This contradicts the optimality of $\theta^\star$. □

Proof of Proposition 2.5.4. Let $x^\star := G(\theta^\star)$. We are going to show that $\pi(x^\star) = p(\theta^\star)$. Assume that the price functional p is viable and let $\theta^\star \in \arg\max\{v(\theta|p) : \theta \in \Theta(p)\}$. By construction, we have $\pi(G(\theta^\star)) \leqslant p(\theta^\star)$. Assume, by way of contradiction, that $\pi(G(\theta^\star)) < p(\theta^\star)$. Then $p(\theta^\star)$ is not a lower bound of the set $\{p(\theta) : \theta \in \mathbb{R}^J \text{ and } G(\theta^\star) \leqslant G(\theta)\}$. Therefore, there exists $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant G(\theta^\star)$ and $p(\theta) < p(\theta^\star)$. This contradicts the optimality of $\theta^\star$. Hence, $\pi(x^\star) = p(\theta^\star)$.

Fix an arbitrary payoff $x \in \Phi$ satisfying the budget restrictions: $\pi(x) \leqslant e_0$ and $x \geqslant -e_1$.

We first assume that $\pi(x) < e_0$. Choose $\varepsilon > 0$ small enough such that $\pi(x) + \varepsilon \leqslant e_0$. By definition of $\pi(x)$, there must exist some portfolio $\theta \in \mathbb{R}^J$ with $x \leqslant G(\theta)$ such that $p(\theta) \leqslant \pi(x) + \varepsilon$. It then follows that $\theta \in \Theta(p)$. Since $\theta^\star$ is optimal, we deduce that

$$U(e_0 - p(\theta), e_1 + G(\theta)) \leqslant U(e_0 - p(\theta^\star), e_1 + G(\theta^\star)) = U(e_0 - \pi(x^\star), e_1 + x^\star).$$

This implies that

$$U(e_0 - \pi(x) - \varepsilon, e_1 + x) \leqslant U(e_0 - \pi(x^\star), e_1 + x^\star).$$

Passing to the limit when ε tends to zero, we get the desired result.

We now assume that $\pi(x) = e_0$. Convexity of π implies that for any $\lambda \in [0, 1)$, we have

$$\pi(\lambda x) \leqslant \lambda\pi(x) + (1 - \lambda)\pi(0) \leqslant \lambda\pi(x) < e_0.$$

We can then apply the previous argument to show that

$$U(e_0 - \pi(\lambda x), e_1 + \lambda x) \leqslant U(e_0 - \pi(x^\star), e_1 + x^\star).$$

Passing to the limit when λ tends to 1, we get the desired result. □

Proof of Theorem 2.5.1. Fix a convex function $\pi : \mathbb{R}^\Omega \to \mathbb{R}$. We first

assume that π is a robust no-arbitrage price. It follows from Corollary 2.5.1 that there exist a payoff $x_0 \in \mathbb{R}^\Omega$ and a strictly positive vector $\mu \in \mathbb{R}^\Omega_{++}$ satisfying

$$\pi(x) - \pi(x_0) \geqslant \mu \cdot (x - x_0), \quad \text{for all } x \in \mathbb{R}^\Omega.$$

Fix an arbitrary $y \in \mathbb{R}^\Omega$ with $y > 0$. For any $\lambda > 0$, we have

$$\pi(x_0 + \lambda y) - \pi(x_0) \geqslant \lambda \mu \cdot y.$$

Passing to the limit when λ tends to infinite, we get that $(\pi 0^+)(y) \geqslant \mu \cdot y$. Since $y > 0$ and $\mu \gg 0$, we deduce the desired result: $(\pi 0^+)(y) > 0$.

Now, we assume that π is eventually strictly monotone. To prove that π is a robust no-arbitrage price, it is sufficient to show that π is viable. Fix an arbitrary agent $(e_0, e_1, U) \in \mathcal{A}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of payoffs such that $x_n \in X(\pi)$ for each n and

$$\lim \pi(x_n) = \sup\{U(e_0 - \pi(x), e_1 + x) \ : \ x \in X(\pi)\},$$

where we recall that $X(\pi)$ is the set of all payoffs $x \in \mathbb{R}^\Omega$ satisfying the flow budget constraints: $\pi(x) \leqslant e_0$ and $x \geqslant -e_1$. We claim that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Assume, by way of contradiction, that $\lim \|x_n\| = \infty$. We pose

$$y_n := \frac{1}{\|x_n\|} x_n.$$

Observe that $\|y_n\| = 1$ for each n. Passing to a subsequence if necessary, we can assume that $(y_n)_{n\in\mathbb{N}}$ converges to some y. Recall that $y_n \geqslant -e_1$ for each n. Passing to the limit, we deduce that $y \geqslant 0$. Since $\|y\| = 1$, we actually have that $y > 0$. Moreover, $x_n \in \Gamma := \{x \in \mathbb{R}^\Omega \ : \ \pi(x) \leqslant e_0\}$. We then deduce that y belongs to the recession cone $0^+\Gamma$. If follows from Proposition 2.8.2 that y belongs to the recession cone of π, i.e., $(\pi 0^+)(y) \leqslant 0$. This contradicts the fact that π is eventually strictly monotone.

We have thus proved that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence if necessary, we can assume that there exists $x^\star \in \mathbb{R}^\Omega$ such that $\lim x_n = x^\star$. Continuity of the function π (recall that π is convex) and the utility function U implies that $x^\star$ solves the agent's maximization problem. In particular, π is viable. □

Proof of Proposition 2.5.5. Fix a price functional $p : \mathbb{R}^J \to \mathbb{R}$ and recall that the associated pricing rule $\pi : \mathbb{R}^\Omega \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\pi(x) := \inf\{p(\theta) \ : \ \theta \in \mathbb{R}^J \quad \text{and} \quad G(\theta) \geqslant x\}.$$

We assume that π is proper (i.e., $\pi(\mathbb{R}^\Omega) \subseteq \mathbb{R}$) and eventually strictly monotone. It follows from Lemma 2.5.2 that

$$\pi(x) = \inf\{\tilde{p}(\theta) \ : \ x \leqslant G(\theta)\}.$$

In particular, we have $\tilde{p}(\theta) \neq -\infty$ for all θ and $\tilde{p}$ is a price functional. To prove that $\tilde{p}$ is a robust no-arbitrage price functional, we fix a portfolio $\theta^0 \in \mathbb{R}^J$ and let $\eta \in \mathbb{R}^J$ be such that

$$\tilde{p}'(\theta;\eta) \leqslant 0 \quad \text{and} \quad G(\eta) \geqslant 0.$$

We let $y := G(\eta)$ and claim that $y = 0$. To prove this, we fix an arbitrary payoff $x \in \mathbb{R}^\Omega$ and let $\theta \in \mathbb{R}^J$ be a portfolio satisfying $G(\theta) \geqslant x$. We have

$$\pi(x+y) \leqslant \tilde{p}(\theta+\eta) \leqslant \tilde{p}(\theta)$$

where the first inequality follows from the definition of $\pi(x+y)$ and the second inequality follows from the property: $\tilde{p}'(\theta;\eta) \leqslant 0$. Since $\pi(x+y) \leqslant \tilde{p}(\theta)$ for any $\theta \in \mathbb{R}^J$ satisfying $G(\theta) \geqslant x$, we deduce that $\pi(x+y) \leqslant \pi(x)$. This inequality is valid for any $x \in \mathbb{R}^\Omega$. We then deduce that $(\pi 0^+)(y) \leqslant 0$. Since π is eventually strictly monotone, we must have $y = G(\eta) = 0$. Since $\tilde{p}$ satisfies the law of one price, we deduce that $\tilde{p}'(\theta;\eta) = 0$. □

Proof of Proposition 2.5.6. Recall that π is eventually strictly monotone if, and only if, for all $y \in \mathbb{R}^\Omega$, we have

$$y > 0 \Longrightarrow \exists x \in \mathbb{R}^\Omega, \quad \pi(x+y) > \pi(x).$$

By way of contradiction, assume that $y > 0$ and $\forall x \in \mathbb{R}^\Omega$, $\pi(x+y) \leqslant \pi(y)$. Moreover, for all $x \geqslant 0$, monotonicity implies $\pi(x+y) \geqslant \pi(y)$. Hence for all $x \geqslant 0$, $\pi(x+y) = \pi(x)$. In particular, as π is positively homogeneous it implies that $\pi(y) = 0$ for all $y > 0$ which in turn imply $\pi(x) = 0$ for all $x \geqslant 0$. Let $z \geqslant 0$, monotonicity implies that $\pi(-z) \leqslant 0$, also by convexity we have

$$\pi(-0.5z + 0.5z) \leqslant 0.5\pi(-z) + 0.5\pi(z).$$

Hence, $\pi(-z) = 0$. As z was chosen arbitrarily, we have $\pi(x) = 0$ for all $x < 0$. For all $y \in \mathbb{R}^\Omega$, there exist $x > 0$ and $z < 0$ such that $x > y > z$, hence by monotonicity $\pi(y) = 0$. It contradicts the hypothesis that π is non-zero. Hence, π is eventually strictly monotone and according to Theorem 2.5.1 it is viable. □

Proof of Proposition 2.5.7. Let π be a JK-viable pricing rule, then there exists an agent $(e_0, e_1, U) \in \mathcal{A}$ such that (e_0, e_1) is the solution of his maximisation problem:

$$0 \in \arg\max(U(e_0 - \pi(x), e_1 + x) : \ x \in X(\pi)).$$

Hence, π is viable. Now, let π be a viable pricing rule, then there exists an agent $a = (e_0, e_1, U) \in \mathcal{A}$ and a payoff $x^\star \in \Phi$ such that

$$x^\star \in \arg\max(U(e_0 - \pi(x), e_1 + x) : \ x \in X(\pi)).$$

Introducing a new agent (e_0', e_1', U) with $e_0' = e_0 - \pi(x^\star)$ and $e_1' = e_1 + x^\star$, we have

$$(e_0', e_1') \in \arg\max\{U(c_0, c_1) \,:\, (c_0, c_1) \in C(\pi)\}.$$

Hence, π is JK-viable. □

Proof of Theorem 2.6.1. The result is a direct consequence of Theorem 2.3.1. □

Proof of Theorem 2.6.3. We first assume that the price functional p is completely viable. From Theorem 2.6.1 it implies that p is supported by strictly positive state price everywhere. This means that for any $\theta \in \mathbb{R}^J$, there exists $\mu_\theta \in \mathbb{R}^\Omega_{++}$ such that

$$p(\tilde{\theta}) - p(\theta) \geqslant \mu_\theta[G(\tilde{\theta}) - G(\theta)] \quad \text{for all } \tilde{\theta} \in \mathbb{R}^J.$$

Let $\eta \in \mathbb{R}^J$ such that $p(\theta + \eta) - p(\theta) \leqslant 0$ and $G(\theta + \eta) - G(\theta) \geqslant 0$. Then by assumption we have

$$0 \geqslant p(\theta + \eta) - p(\theta) \geqslant \mu_\theta[G(\theta + \eta) - G(\theta)] \geqslant 0.$$

Thus,

$$p(\theta + \eta) - p(\theta) = 0 \quad \text{and} \quad \mu_\theta[G(\theta + \eta) - G(\theta)] = 0.$$

Since μ_θ is strictly positive, we have $G(\theta+\eta)-G(\theta) = 0$. The price functional p is an absolute no-arbitrage price.

We now assume that the price functional p is an absolute no-arbitrage price, we are going to show that it is an absolute robust no-arbitrage price. Fix $\theta \in \mathbb{R}^J$, by convexity the function

$$\lambda \to \frac{p(\theta + \lambda\eta) - p(\theta)}{\lambda}$$

is increasing. Similarly, by concavity the function

$$\lambda \to \frac{G(\theta + \lambda\eta) - G(\theta)}{\lambda}$$

is decreasing. Let $\eta \in \mathbb{R}^J$ such that

$$p'(\theta; \eta) \leqslant 0 \quad \text{and} \quad G'(\theta; \eta) \geqslant 0.$$

We have

$$0 \geqslant p'(\theta; \eta) \geqslant p(\theta) - p(\theta - \eta)$$

and

$$G(\theta) - G(\theta - \eta) \geqslant G'(\theta; \eta) \geqslant 0.$$

Applying absolute no-arbitrage, we get

$$p(\theta) - p(\theta - \eta) = 0 \quad \text{and} \quad G(\theta) - G(\theta - \eta) = 0$$

which in turn imply

$$p'(\theta; \eta) = 0 \quad \text{and} \quad G'(\theta; \eta) = 0.$$

We get the desired result. It follows from Theorem 2.6.2 that p is completely viable. □

Chapter 3

Dynamic completeness and Market Frictions

Abstract[1]

proposes the first characterization of dynamic completeness in markets with frictions. In frictionless markets with no available arbitrage opportunity, the fundamental theorem of asset pricing states that dynamic completeness is equivalent to having a unique normalized vector of strictly positive event prices under which every investment makes zero profit. First, we show that it is also equivalent to the weaker condition that a supporting event price vector with a zero first-period price does not exist. Then, we demonstrate that there is no arbitrage opportunity in multi-period security markets with bid–ask spreads if, and only if, frictionless no-arbitrage markets support them. Eventually, we prove that the absence of a supporting event price vector with a zero first-period price also characterizes dynamic completeness in markets with bid–ask spreads. On the other hand, we show that having a unique normalized vector of strictly positive event prices supporting these markets is unnecessary for dynamic completeness.

Contents

3.1 Introduction **74**

3.2 3-period markets with Bid–Ask Spreads **76**

3.2.1 The Information Structure 76

3.2.2 The Market Structure 77

3.2.3 Dynamic Completeness and the Set of Available Payoffs 78

[1]The author wishes to express her warmest thanks to Françoise Forges, Jean-Philippe Lefort and Victor-Filipe Martins-da-Rocha for the many helpful discussions, comments and suggestions.

3.2.4 Trading Strategies 79
3.2.5 No-arbitrage 82
3.2.6 Characterization of Dynamic Completeness . . . 83
3.2.7 Geometric Representation of Dynamic Completeness 85
3.2.8 Examples 86
3.3 Multi-period Markets With Bid-Ask Spreads **90**
3.3.1 Uncertainty And Information 90
3.3.2 Securities, Portfolios And Payoffs 91
3.3.3 Dynamic Completeness and the Set of Available Payoff Streams 93
3.3.4 Trading Strategy 94
3.3.5 No-arbitrage 96
3.3.6 Characterization of Dynamic Completeness . . . 97
3.4 Conclusion **99**
3.5 Appendix **101**
3.5.1 Payoff Matrix of Example 3.2.1 101
3.5.2 Proofs 102

3.1 Introduction

This article proposes the first characterization of dynamic completeness in the presence of market frictions. We demonstrate that markets with bid–ask spreads are dynamically complete if, and only if, every frictionless supporting markets have a non-zero event-0 price.

Dynamic completeness is an eminently desirable property of financial security markets that requires that every contract or security be traded (possibly by replicating them). It ensures that market participants perfectly transfer risk and smooth their consumption intertemporally. Over the past century, financial markets have produced a multitude of innovative products, including many new forms of derivatives, alternative risk transfer products, exchange-traded funds, and variants of tax-deductible equity, to increase risk-sharing opportunities (Van Horne (1985)) and move financial markets towards dynamic completeness (see Allen and Gale (1994) and Tufano (2003)). For example, option contracts reduce significantly the number of securities necessary for dynamic completeness (see Ross (1976a), Breeden and Litzenberger (1978), Friesen (1979), Green and Jarrow (1987), Nachman (1988), Baptista (2003)). The remaining sources of dynamic incompleteness are explained by informational problems such as moral hazard, adverse selection, unforecastable events, or the existence of too many events (see Laffont (1989)), short-selling restrictions (see Raab and Schwager (1993)), transaction costs (see Merton (1989) and Ross (1989)), taxes or fees. However,

these frictions do not necessarily result in markets incompleteness. For example, Raab and Schwager (1993) provides a sufficient condition for market completeness in 2−period security markets with short-selling restrictions.

This paper demonstrates that the principal transaction cost when trading stocks, futures contracts, options, or currency pairs (see Kumar (2004)), the bid–ask spread, does not necessarily result in dynamically incomplete markets. Actually, in some cases, suppressing bid–ask spreads makes the markets dynamically incomplete (see example 3.2.1). Bid–ask spreads represent the remuneration of market makers, key participants of security markets who provide bid and ask offers for securities resulting in a significant improvement of markets' liquidity. The size of bid–ask spreads has been explained by the extent of the competition between market makers (see Tinic and West (1972) Stoll (1978), Ho and Stoll (1983) and Biais, Martimort and Rochet (2000)), by inventory consideration (see Tinic (1972), Garman (1976), Amihud and Mendelson (1980) and Ho and Stoll (1981)), by adverse selection arising from asymmetric information (see Bagehot (1971), Copeland and Galai (1983), Glosten and Milgrom (1985), Kyle (1985) and Glosten (1989)), by the ability of market makers and investors to find counterparties (see Demsetz (1968) and Duffie, Gârleanu and Pedersen (2005)), by the distribution of securities holdings (see Lagos and Rocheteau (2009)), and by the extent of the deployment of algorithmic trading (see Hendershott, Jones and Menkveld (2011)). Additionally, Cohen, Maier, Schwartz and Whitcomb (1981) and Martins-da-Rocha and Vailakis (2010) prove the existence of bid–ask spreads at equilibrium in financial security markets models.

The characterization of dynamic completeness is well-known in frictionless markets (see Magill and Quinzii (1996) or LeRoy and Werner (2014)). In a standard frictionless economy with no arbitrage opportunity available, dynamic completeness is equivalent to having a unique normalized vector of strictly positive event prices under which every investment makes a zero profit. First, we notice that a payoff stream can be generated equivalently by a portfolio strategy (which records the holding in each security at the end of each period) or by a trading strategy (which records the orders passed in each event). However, trading strategies outperform portfolio strategies in the analysis of financial markets in the presence of frictions because they permit the use of more straightforward mathematical methods involving positive spanning. We demonstrate that the set of payoff streams that a trading strategy can generate is equal to the positive span of the payoff matrix. Therefore, we propose a new characterization of the absence of arbitrage opportunity using trading strategies in multi-period security markets with bid-ask spreads. We show that no-arbitrage is equivalent to the existence of supporting frictionless markets with no-arbitrage opportunity. Then, we demonstrate the equivalence between the uniqueness of the vector of strictly positive event prices supporting the market and the weaker condition that

a supporting event price vector with a zero first-period price does not exist, in frictionless markets with no arbitrage opportunity. Finally, we show that the uniqueness of the vector of event prices supporting the economy is not necessary for dynamic completeness in the presence of friction. On the other hand, we prove that the absence of supporting event prices vector with zero initial event price remains equivalent to dynamic completeness in markets with bid-ask spreads.

This paper is organized as follows. For the sake of exposition, we first present our results in a particular case of a 3−period security market with bid–ask spread in Section 3.2. We introduce the concept of trading strategies and present our characterization of dynamic completeness in this setting. We also determine the minimal number of traded securities necessary for dynamic completeness. Our results are illustrated graphically both in section 3.2.6 and 3.2.7. In section 3.2.8, we provide examples of security markets that are dynamically complete with bid–ask spreads. In Section 3.3, we extend the results of Section 3.2 to the general case of multi-period security markets with bid–ask spreads. We provide examples of applications of the results in the conclusion.

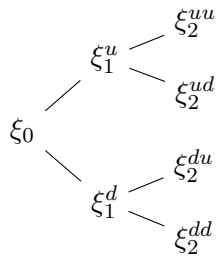
3.2 3-period markets with Bid–Ask Spreads

This section presents the characterization of dynamic completeness in a particular case of 3-period security markets with bid–ask spreads. We do not present first the results in a 2−period security markets as is usually the case because the presence of bid–ask spread has no influence on completeness in these markets (see Remark 3.2.1) while it is no longer the case in multi-period security markets as the prices at which trades take place impacts future payoffs.

Throughout the section, we assume there is no restriction to short selling and no limitation to the quantity of security purchased and sold at the initial period. We additionally assume agents can infinitely split their orders, and they share the same information structure. Hence incompleteness cannot result from the presence of one of these frictions. We present the general multiperiod case in Section 3.3.

3.2.1 The Information Structure

We represent uncertainty about the future by a set of events that can happen at each period. We denote ξ_t an event happening at period t. We denote it ξ when the precision is unnecessary. At period 0, agents do not know which events will realize in the future. At period 1, they know that only a subset of events may happen at period 2. We represent the unfolding of events in the following event tree.

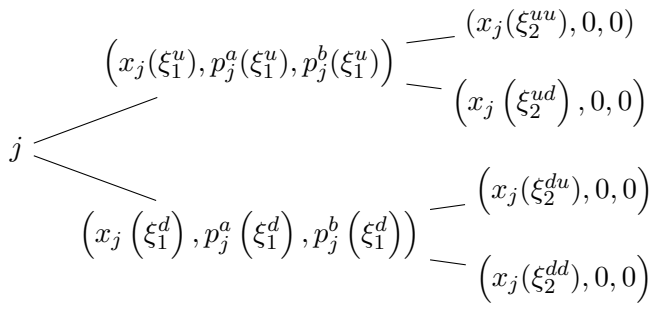


It should be interpreted in the following manner, if ξ_1^u realizes then only events ξ_2^{uu} and ξ^{ud} can happen at period 2. We denote Ξ_0 the set containing only the event ξ_0, Ξ_1 the set of events $\{\xi_1^u, \xi_1^d\}$ and Ξ_2 the set of events $\{\xi_2^{uu}, \xi_2^{ud}, \xi_2^{du}, \xi_2^{dd}\}$.

3.2.2 The Market Structure

We consider markets in which J securities are traded at period 0 and period 1. To each security j, corresponds a dividend stream represented by a vector $x_j \in \mathbb{R}^6$. We denote $x_j(\xi)$ the dividend paid by security j in event $\xi \in \Xi_1 \cup \Xi_2$ and $x(\xi) \in \mathbb{R}^J$ the dividends paid by each securities in event $\xi \in \Xi_1 \cup \Xi_2$.

There are two types of actors participating in the markets: investors and market makers. Market makers actively quote two-sided markets in a particular security, providing bids and asks. The bid–ask spread, the difference between the buy price and the selling price they propose, compensate them for their services, and the risk they bear for holding securities during several periods. Investors purchase and sell securities to a market maker. They buy at the ask price and sell at the bid price (a market maker buys at the bid price and sells at the ask price). We adopt their perspective in the following. The ask price of security j, denoted $p_j^a(\xi)$, represents the amount spent by an investor to purchase this security in the event $\xi \in \Xi_1 \cup \Xi_2$. Its bid price, $p_j^b(\xi)$, represents the amount received by the investor when he sells security j in event $\xi \in \Xi_1 \cup \Xi_2$. We have period 2 prices for practicality. We set them equal to 0, that is $p_j^a(\xi_2) = p_j^b(\xi_2) = 0$ for all $\xi_2 \in \Xi_2$ and all security j. The spread between the ask and the bid price is the bid–ask spread. We denote $p^a(\xi) \in \mathbb{R}^J$ the vector of securities ask prices and $p^b(\xi) \in \mathbb{R}^J$ the vector of securities bid prices in an event $\xi \in \Xi_1 \cup \Xi_2$. We present the unfolding of the dividends and prices of x_j in a tree by associating a triplet $(x_j(\xi), p_j^a(\xi), p_j^b(\xi))$ to each non-initial node $\xi \in \Xi_1 \cup \Xi_2$.



At each period, agents constitute a portfolio of securities. We denote $h(\xi) \in \mathbb{R}^J$ the portfolio held in an event $\xi \in \Xi_0 \cup \Xi_1 \cup \Xi_2$. The coordinates of $h(\xi)$ can be either positive, negative, or zero. Positivity of the jth coordinate of $h(\xi)$ means that the agent owns security j. Negativity means that she has sold j and owes its dividend to its owner. The triplet $h = (h_0, h_1, h_2)$ is a trading strategy where h_t, $t = 0, 1, 2$, is a vector taking coordinates $h(\xi_t)$ for all $\xi_t \in \Xi_t$.

3.2.3 Dynamic Completeness and the Set of Available Payoffs

A trading strategy's payoff in a particular event represents the net amount received by the agent after trading in the markets at this period. She first receives the dividends of the portfolio she had constituted at the previous period. Next, she trades on the markets. These two components enter the payoff she receives. Formally, the payoff in an event $\xi_1 \in \Xi_1$ of a trading strategy h is denoted $z(h, p^a, p^b)(\xi_1)$. It is equal to

$$\overbrace{x(\xi_1)h(\xi_0)}^{\text{dividends}} - \overbrace{p^b(\xi_1)\min(h(\xi_1) - h(\xi_0)), 0)}^{\text{sales revenue}} - \overbrace{p^a(\xi_1)\max(h(\xi_1) - h(\xi_0), 0)}^{\text{purchases cost}} \tag{3.1}$$

where for $(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k$, $z = \max(x, y)$ means $z_i = \max(x_i, y_i)$ for every $i = 1, \ldots, k$ and $z = \min(x, y)$ means $z_i = \min(x_i, y_i)$ for every $i = 1, \ldots, k$. Since there is no market opened at period 2, the payoff of a trading strategy in an event $\xi_2 \in \Xi_2$ represents solely the difference between the dividends received and due. Formally, the payoff in an event $\xi_2 \in \Xi_2$ of a trading strategy h is denoted $z(h, p^a, p^b)(\xi_2)$. It is equal to $x(\xi_2)h(\xi_2^-)$ where ξ_2^- denotes the immediate predecessor of event ξ_2.

The set of payoff streams that are replicated by portfolio strategies is the set $\mathcal{M}(p)$ equal to

$$\left\{z \in \mathbb{R}^6 \mid \exists h \text{ s.t. } z(\xi_t) = z(h, p^a, p^b)(\xi_t) \text{ for all } t = 0, 1, 2\right\}.$$

Markets are dynamically complete if every payoff stream can be replicated. Formally, markets are dynamically complete if $\mathcal{M}(p) = \mathbb{R}^6$. In frictionless markets, we call this set the asset span because it is equal to the span (in

the mathematical sense) of a set of payoff streams. In our context, this set is not a span due to bid–ask spreads. Therefore, we call it the set of available payoff streams. In the following section, we provide an equivalent definition for the set of available payoff streams.

3.2.4 Trading Strategies

First, we introduce the concept of trading strategies. A trading strategy records the unfolding of market orders placed in each event. We denote $b^a(\xi) \in \mathbb{R}_+^J$ the ask orders placed in event $\xi \in \Xi_0 \cup \Xi_1$ and $b^b(\xi) \in \mathbb{R}_+^J$ the bid orders placed in event $\xi \in \Xi_0 \cup \Xi_1$. We emphasize the fact that orders exclusively admit non-negative[2] value as opposed to portfolios that equally admit negative values. Since a portfolio is equal to the sum of ask and bid orders placed at the previous periods, we can recover the orders placed on the market from a portfolios strategy and vice-versa (see also Proposition 3.2.1). We notice that to a given portfolio strategy h, we can associate the orders placed in the markets in the following manner

$$b^a(\xi_0) = \overbrace{\max(h(\xi_0), 0)}^{\text{ask orders placed at } t=0}, \quad b^b(\xi_0) = \overbrace{-\min(h(\xi_0), 0)}^{\text{bid orders placed at } t=0}$$

and

$$b^a(\xi_1) = \overbrace{\max(h(\xi_1) - h(\xi_0), 0)}^{\text{ask orders placed in event } \xi_1}, \quad b^b(\xi_1) = \overbrace{-\min(h(\xi_1) - h(\xi_0), 0)}^{\text{bid orders placed in event } \xi_1}$$

for all $\xi_1 \in \Xi_1$. At period 0, an investor can place $2J$ different types of orders (the factor 2 stands for the ask and bid orders). At period 1, an investor can place $4J$ different order types (since there are 2 events). Therefore, an investor can place a total of $6J$ different market orders, and a trading strategy is a vector b of $\mathbb{R}_+^{6J}$ equal to

$$\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \\ b^a(\xi_1^u) \\ b^b(\xi_1^u) \\ b^a(\xi_1^d) \\ b^b(\xi_1^d) \end{pmatrix}.$$

Each market order endows its issuer with a particular payment stream between periods 0 and 2. The payment stream of an ask order placed in event ξ_t, $0 \leqslant t \leqslant T-1$ on security j corresponds to the payment carried out

[2]We implement the following convention: positive means strictly superior to zero, non-negative means superior or equal to 0, non-positive means inferior or equal to 0 and negative means strictly inferior to 0.

in event ξ_t by the buyer to purchase the security, the dividends he receives in the successor events and 0 otherwise. Similarly, the payment stream of a bid order placed ξ_t, $0 \leqslant t \leqslant T-1$ on j is equal to the payment received by the seller of j in this event, the dividends paid to the buyer in the successor events, and zero otherwise. For example, the payment stream of a buy order placed on security j in event ξ_0 is a vector $\hat{\phi}_j^a(\xi_0)$ equal to

$$\begin{pmatrix} -p_j^a(\xi_0) \\ x_j(\xi_1^u) \\ x_j(\xi_1^d) \\ x_j(\xi_2^{uu}) \\ x_j\left(\xi_2^{ud}\right) \\ x_j(\xi_2^{du}) \\ x_j(\xi_2^{dd}) \end{pmatrix} \begin{matrix} \text{payment in event } \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{matrix} .$$

Similarly, the payment stream of a sell order placed on security j in event ξ_1^u is a vector $\hat{\phi}_j^b(\xi_1^u)$ equal to

$$\begin{pmatrix} 0 \\ p_j^b(\xi_1^u) \\ 0 \\ -x_j(\xi_2^{uu}) \\ -x_j\left(\xi_2^{ud}\right) \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \text{payment in event } \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{matrix} .$$

We regroup the payment streams of bid and ask orders issued in event ξ_0 in a $7 \times 2J$ matrix $\phi(\xi_0)$ equal to

$$\begin{matrix} \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{matrix} \begin{matrix} \begin{matrix} \hat{\phi}_1^a(\xi_0) & \dots & \hat{\phi}_J^a(\xi_0) & \hat{\phi}_1^b(\xi_0) & \dots & \hat{\phi}_J^b(\xi_0) \end{matrix} \\ \begin{pmatrix} -p_1^a(\xi_0) & \dots & -p_J^a(\xi_0) & p_1^b(\xi_0) & \dots & p_J^b(\xi_0) \\ x_1(\xi_1^u) & \dots & x_J(\xi_1^u) & -x_1(\xi_1^u) & \dots & -x_J(\xi_1^u) \\ x_1(\xi_1^d) & \dots & x_J(\xi_1^d) & -x_1(\xi_1^d) & \dots & -x_J(\xi_1^d) \\ x_1(\xi_2^{uu}) & \dots & x_J(\xi_2^{uu}) & -x_1(\xi_2^{uu}) & \dots & -x_J(\xi_2^{uu}) \\ x_1\left(\xi_2^{ud}\right) & \dots & x_J\left(\xi_2^{ud}\right) & -x_1\left(\xi_2^{ud}\right) & \dots & -x_J\left(\xi_2^{ud}\right) \\ x_1(\xi_2^{du}) & \dots & x_J(\xi_2^{du}) & -x_1(\xi_2^{du}) & \dots & -x_J(\xi_2^{du}) \\ x_1(\xi_2^{dd}) & \dots & x_J(\xi_2^{dd}) & -x_1(\xi_2^{dd}) & \dots & -x_J(\xi_2^{dd}) \end{pmatrix} \end{matrix} .$$

Similarly, we regroup the payment streams of bid and ask orders issued

in event $\xi_1^u \in \Xi_1$ in a $7 \times 2J$ matrix $\hat{\phi}(\xi_1^u)$ equal to

$$\begin{array}{c} \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{\phi}_1^a(\xi_1^u) & \dots & \hat{\phi}_J^a(\xi_1^u) & \hat{\phi}_1^b(\xi_1^u) & \dots & \hat{\phi}_J^b(\xi_1^u) \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ -p_1^a(\xi_1^u) & \dots & -p_J^a(\xi_1^u) & p_1^b(\xi_1^u) & \dots & p_J^b(\xi_1^u) \\ 0 & \dots & 0 & 0 & \dots & 0 \\ x_1(\xi_2^{uu}) & \dots & x_J(\xi_2^{uu}) & -x_1(\xi_2^{uu}) & \dots & -x_J(\xi_2^{uu}) \\ x_1\left(\xi_2^{ud}\right) & \dots & x_J\left(\xi_2^{ud}\right) & -x_1\left(\xi_2^{ud}\right) & \dots & -x_J\left(\xi_2^{ud}\right) \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

And, we regroup the payment streams of bid and ask orders issued in event $\xi_1^d \in \Xi_1$ in a $7 \times 2J$ matrix $\hat{\phi}(\xi_1^d)$ equal to

$$\begin{array}{c} \\ \xi_0 \\ \xi_1^u \\ \xi_1^d \\ \xi_2^{uu} \\ \xi_2^{ud} \\ \xi_2^{du} \\ \xi_2^{dd} \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{\phi}_x^a\left(\xi_1^d\right) & \dots & \hat{\phi}_J^a\left(\xi_1^d\right) & \hat{\phi}_x^b\left(\xi_1^d\right) & \dots & \hat{\phi}_J^b\left(\xi_1^d\right) \end{array} \\ \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ -p_1^a\left(\xi_1^d\right) & \dots & -p_J^a\left(\xi_1^d\right) & p_1^b\left(\xi_1^d\right) & \dots & p_J^b\left(\xi_1^d\right) \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ x_1(\xi_2^{du}) & \dots & x_J(\xi_2^{du}) & -x_1(\xi_2^{du}) & \dots & -x_J(\xi_2^{du}) \\ x_1(\xi_2^{dd}) & \dots & x_J(\xi_2^{dd}) & -x_1(\xi_2^{dd}) & \dots & -x_J(\xi_2^{dd}) \end{pmatrix} \end{array}.$$

The payment matrix $\hat{\mathcal{P}}$ is a $7 \times 6J$ matrix whose columns represent the payments across all the events of a one unit trade order placed on a security at a non-terminal event

$$\left(\hat{\phi}(\xi_0) \quad \hat{\phi}(\xi_1^u) \quad \hat{\phi}(\xi_1^d)\right).$$

A trading strategy $b \in \mathbb{R}_+^6$ generates the payment stream $\hat{z} \in \mathbb{R}^7$ when $\hat{z} = \hat{\mathcal{P}}b$. We call payment positive span the set of payments that a trading strategy can generate. We denote it $\hat{\mathcal{B}}(p^a, p^b)$. We have

$$\hat{\mathcal{B}}(p^a, p^b) = \{\hat{z} \in \mathbb{R}^7 \mid \hat{z} = \hat{\mathcal{P}}b \text{ for some } b \in \mathbb{R}_+^{6J}\}.$$

The payment matrix is different from the *payoff matrix* which represents the payments across all *future* events of a one-unit trade order placed on a security at a non-terminal event. It is the concatenation of the payoff matrix with the period 0 payments of every order. We denote $\phi(\xi_t)$ the sub-matrix formed by selecting every rows of the matrix $\hat{\phi}(\xi_t)$ except the first. It represents the payoff streams of bid and ask orders placed in the non-terminal event ξ_t. We denote $\mathcal{P}$ the payoff matrix. It is a $6 \times 6J$ equal to

$$\left(\phi(\xi_0) \quad \phi(\xi_1^u) \quad \phi(\xi_1^d)\right).$$

We denote Φ the set of columns of the payoff matrix $\mathcal{P}$.

Remark 3.2.1. In 2−period markets the payoff matrix only depends on securities dividends. Therefore the presence of a bid–ask spread does not modify the characterization of completeness. 2−period markets are complete if, and only if, the rank of the payoff matrix is equal to the number of states of nature, here 4.

A trading strategy $b \in \mathbb{R}^6_+$ generates the payoff stream $z \in \mathbb{R}^6$ when $z = \mathcal{P}b$. We call payoff positive span the set of payoff streams that can be generated by a trading strategy. We denote it $\mathcal{B}(p^a, p^b)$. We have

$$\mathcal{B}(p^a, p^b) = \{z \in \mathbb{R}^6 \mid z = \mathcal{P}b \text{ for some } b \in \mathbb{R}^{6J}_+\}.$$

Since a trading strategy exclusively admits non-negative coordinates, $\hat{\mathcal{B}}(p^a, p^b)$ is *the positive span* of the columns of the payment matrix $\hat{\mathcal{P}}$ and $\mathcal{B}(p^a, p^b)$ is *the positive span* of the columns of the payoff matrix $\mathcal{P}$ (hence their name). Indeed, Davis (1954) defines the positive span of a finite set of vectors $V = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ has the set p-span(V) equal to

$$\text{p-span}(V) := \{\lambda_1 v_1 + \ldots \lambda_k v_k \mid \lambda_i \geqslant 0 \text{ for all } i = 1, \ldots, k\}.$$

We say that a finite set $V \subset \mathbb{R}^n$ positively span $\mathbb{R}^n$ if p-span$(V) = \mathbb{R}^n$.

Remark 3.2.2. To generalize frictionless methods to markets with bid–ask spreads, it is also possible to separate a portfolio strategy between the ask portfolio strategy, consisting of the ask orders placed in the markets, and the bid portfolio strategy, consisting of the bid orders placed in the markets. However, the monotonicity of these strategies with time complicates the study since the set of payoff streams generated by a bid–ask portfolio strategy is not a positive span.

We demonstrate in the following proposition that a payoff stream is generated by a trading strategy if, and only if, a portfolio strategy exists that replicates it. Therefore, the payoff positive span is equal to the set of available payoff streams.

Proposition 3.2.1. The set of available payoff streams is equal to the payoff positive span.

Proposition 3.2.1 shows that markets are dynamically complete if for every payoff stream there exists a trading strategy that generates it. Formally, markets are dynamically complete if $\mathcal{B}(p^a, p^b) = \mathbb{R}^6$.

3.2.5 No-arbitrage

Dynamic completeness is characterized in the frictionless case under a mild equilibrium property, the absence of arbitrage opportunity. To extend this characterization to markets with frictions, we define an arbitrage

opportunity for a trading strategy instead of a portfolio strategy. We denote $\mathbb{R}^n_{++}$ the set of vectors with strictly positive coordinates and we denote $C^\star = \mathbb{R}_+ \times \mathbb{R}^6_+ \setminus 0$ the set of positive payment streams. There exists an arbitrage opportunity in the markets if there exists a trading strategy $b \in \mathbb{R}^{6J}_+$ generating a non-negative payment stream with at least one strictly positive payment, that is such that $\hat{\mathcal{P}} b > 0$. Additionally, a vector $\mu \in \mathbb{R}^7$ is said to support the markets when for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. We show in Theorem 3.2.1 that there is no arbitrage opportunity in the markets if, and only if, a vector of strictly positive event prices supports the markets.

Theorem 3.2.1. *There is no arbitrage opportunity in multi-period markets with bid-ask spreads if, and only if, a vector of strictly positive event prices supports the markets.*

When there are frictions, supporting strictly positive event prices represents the existence of underlying no-arbitrage frictionless markets supporting the markets. The absence of arbitrage opportunity implies that strictly positive event prices support securities prices, in the sense that prices are greater than the weighted sum of expected payoffs for event prices μ, that is, we have

$$\mu_{\xi_0} p^a_j(\xi_0) \geqslant \sum_{\xi \in \Xi_1 \cup \Xi_2} \mu_\xi x_j(\xi) \geqslant \mu_{\xi_0} p^b_j(\xi_0)$$

$$\mu_{\xi^u_1} p^a_j(\xi^u_1) \geqslant \mu_{\xi^{uu}} x_j(\xi^{uu}) + \mu_{\xi^{ud}} x_j(\xi^{ud}) \geqslant \mu_{\xi^u_1} p^b_j(\xi^u_1),$$

and

$$\mu_{\xi^d_1} p^a_j(\xi^d_1) \geqslant \mu_{\xi^{du}} x_j(\xi^{du}) + \mu_{\xi^{dd}} x_j(\xi^{dd}) \geqslant \mu_{\xi^d_1} p^b_j(\xi^d_1)$$

for every security $j \in J$. Therefore trading strategies permit a straightforward generalization of the fundamental theorem of asset pricing to markets with bid–ask spreads. When markets are frictionless, the payoffs stream of an ask order is the opposite of the payoff stream of the bid order on the same security in the same event. Hence, the vector of event prices supports the market with an equality sign and Theorem 3.2.1 coincides with the fundamental theorem of asset pricing (Harrison and Kreps (1979), Magill and Quinzii (1996)) expressed for trading strategies instead of portfolio strategies.

3.2.6 Characterization of Dynamic Completeness

In the absence of bid–ask spread the market presented in the previous section is dynamically complete only if at least 2 securities are traded. Proposition 3.2.2 shows that the presence of a bid–ask spread does not increase the number of traded securities necessary for dynamic completeness.

Proposition 3.2.2. Markets are dynamically complete only if at least 2 securities are traded.

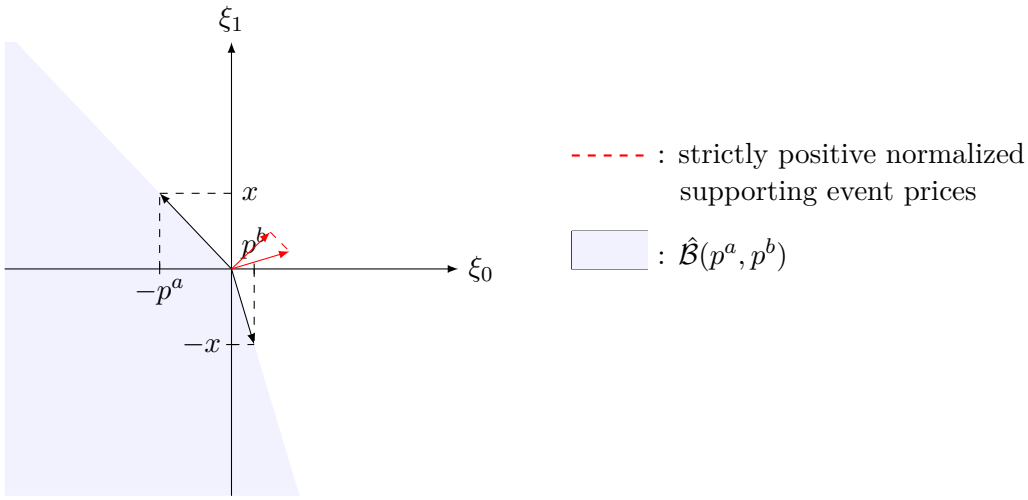


Figure 3.1: 2-period dynamically complete security market with no uncertainty and a single traded security x.

As for frictionless markets, having 2 securities traded each period is not sufficient for markets to be dynamically complete. Dynamic completeness also depends on the values of each security dividend and price. In frictionless markets, under no-arbitrage, markets are dynamically complete if, and only if, a unique normalized strictly positive event prices vector supports the market. We show in the following theorem that the uniqueness of this vector is equivalent to having exclusively *non-zero period*-0 *event prices* supporting the market. A vector of event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ is said to have a non-zero period-0 price if $\mu_{\xi_0} \neq 0$.

Proposition 3.2.3. When markets are frictionless and admit no arbitrage opportunity, a unique normalized vector of strictly positive event prices supports the market if, and only if, every non-zero supporting event prices have a non-zero period-0 price.

Figure 3.1 represents a dynamically complete (every payoff at period 1 can be traded at period 0) 2$-$period security market with no uncertainty and a single traded security with a bid–ask spread. The ξ_0-axis represents the payment received at period 0, and the ξ_1-axis represents the payoff received at period 1. The black vectors represent the payoff streams of bid and ask orders placed at periods 0 and 1. As illustrated in this figure, dynamic completeness does not require that a unique vector of strictly positive event prices supports the market when there are frictions. However, we demonstrate in Theorem 3.2.2 that every supporting event prices vector must have a non-zero period-0 price.

Theorem 3.2.2. *The following conditions are equivalent*

i. Markets are dynamically complete;

ii. for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$;

iii. every event price vector supporting the markets has a non-zero period-0 price.

We illustrate Theorem 3.2.2 in a simple 3$-$period market with no uncertainty in Figure 3.2 in Section 3.2.7. We provide an example of dynamically complete markets with bid–ask spreads in Section 3.2.8.

3.2.7 Geometric Representation of Dynamic Completeness

We consider a 3$-$period security market with no arbitrage opportunity, one security traded at each period, and no uncertainty about the future. Figure 3.2 represents the payoff streams of bid and ask orders placed at periods 0 and 1. It depicts the case in which the security's dividends x is equal to $(0.5, 1)$, its ask price at period 1 is equal to 0.5, and its bid price, $p_j^b(\xi_1)$, is not specified. The ξ_1-axis represents the payoff received at period 1, and the ξ_2-axis represents the payoff received at period 2. The vectors represent the payoff streams of the security. In addition, we represent in the following graph the unfolding of the dividends and prices of x.

$$j \text{ — } (0.5, 0.5, p_j^b(\xi_1)) \text{ — } (1, 0, 0)$$

The blue set represents the positive span of the payoff streams of bid and ask orders placed at period 0 and ask orders placed at period 1. We remark that markets are dynamically complete if, and only if, the payoff stream of a bid order placed at period 1 is not included in this set. Additionally, by no-arbitrage, p^b cannot be strictly greater than p^a. Since the second coordinate of $\phi^b(\xi_1)$ is fixed (it is equal to the dividend of j at period 2), it implies that markets are dynamically complete if, and only if, $\phi^b(\xi_1)$ takes a value on the red line. Hence, markets are dynamic complete if, and only if, $-0.5 < p^b(\xi_1) \leqslant p^a(\xi_1)$. When negative values of $p^a(\xi_1)$ and $p^b(\xi_1)$ are not economically meaningful we can directly conclude that as for frictionless markets, dynamic completeness is equivalent to having the security's price at period 1 is different from the dividend received in this event $p^b(\xi_1) \neq -x(\xi_1)$ and $p^a(\xi_1) \neq x(\xi_1)$. More generally, it follows from Theorem 3.2.2 that a 3-period market with no uncertainty and a single security j is dynamically complete if, and only if, for every $q \in \mathbb{R}^2$ such that $q^\top x$ is equal to zero, the product of q with the ask orders payoff streams and the product of q with the bid orders' payoff stream placed at period 1 have strictly opposite signs.

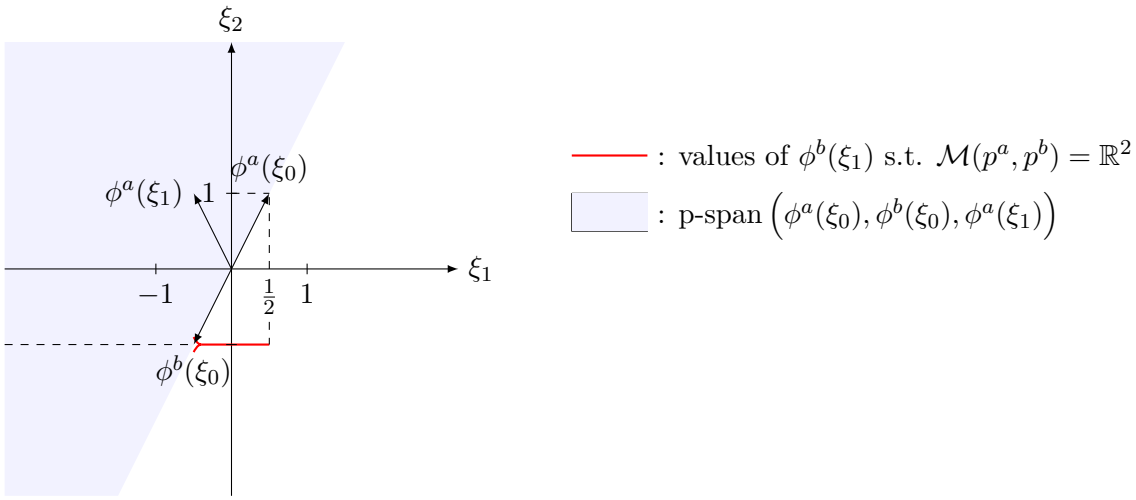


Figure 3.2: Geometry of a 3-period market with no uncertainty, no arbitrage opportunity, and a single security $x_j = (0.5, 1)$ (that is, $S = 1$ and $J = 1$) with ask price equal to 0.5 at period 1.

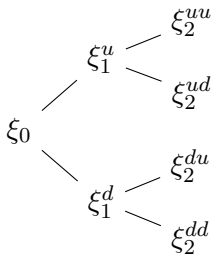
Proposition 3.2.4. 3−period markets with no uncertainty and a single security are dynamically complete if, and only if, for every payoff stream $q \in \mathbb{R}^2$ such that $q^\top \phi^a(\xi_0) = 0$, $q^\top \phi^a(\xi_1)$ and $q^\top \phi^b(\xi_1)$ have strictly opposite sign.

In the following subsection, we present examples of 3−period dynamically complete markets with bid–ask spreads.

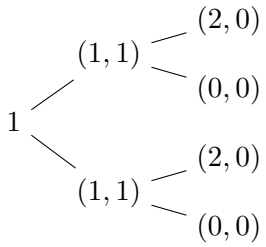
3.2.8 Examples

The presence of a bid–ask spread does not necessarily result in dynamic incompleteness. In the following example, we present 3−period frictionless and dynamically incomplete security markets that become dynamically complete when a market maker charges a transaction cost to compensate her services.

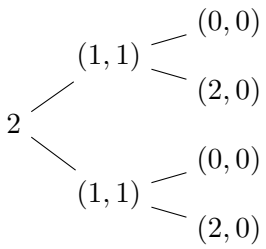
Example 3.2.1. We consider the following 3-period market with uncertainty at period 0 regarding the outcome of future periods.



There are two securities available for trading at period 0 and period 1. Their dividends are equal to $(x_1, x_2) \in \mathbb{R}^6 \times \mathbb{R}^6$. We initially assume the securities' prices present no bid–ask spread. The following graph presents the unfolding of security 1 dividends and prices.



We present the payoffs of security 2 similarly.

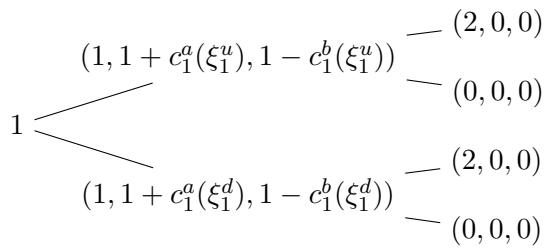


The one-period matrix in event ξ_0 is equal to

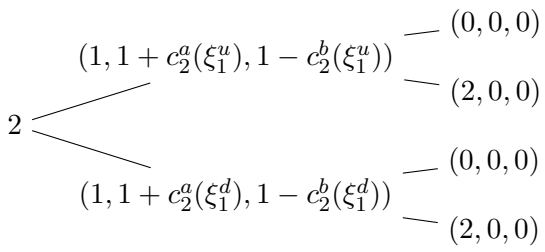
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

It is of rank 1 therefore markets are dynamically incomplete.

Now, we assume securities dividends are unchanged, but the services of the market makers are costly. In each event $\xi_1 \in \Xi_1$, they charge a transaction cost $0 < c_j^a(\xi_1) < p_j(\xi_1)$ on ask orders on j and a transaction cost $0 < c_j^b(\xi_1) < p_j(\xi_1)$ on bid orders on j. Hence, the new ask price of security j is equal to $p_j^a(\xi_1) = p_j(\xi_1) + c_j^a(\xi_1)$ and its new bid price in each event ξ_1 is equal to $p_j^b(\xi_1) = p_j(\xi_1) - c_j^b(\xi_1)$. We present the payoffs of j in a tree in which a triplet $\left(x_j(\xi), p_j^a(\xi), p_j^b(\xi)\right)$ representing the dividend of j, its ask price and its bid price in event ξ is associated to each non-initial node.



We present the payoffs of security 2 similarly.



The payoff matrix is presented in Appendix 3.5.1. We are going to show that markets are dynamically complete. Assume by contradiction that there exists a non-zero $z \in \mathbb{R}^k$ such that $z\phi \leqslant 0$ for all $\phi \in \Phi$. Denote z_i the ith coordinate of z. We have $zx_j \leqslant 0$ and $-zx_j \leqslant 0$ which imply

$$z_1 + z_2 + 2z_3 + 2z_5 = 0 \quad \text{and} \quad z_1 + z_2 + 2z_4 + 2z_6 = 0.$$

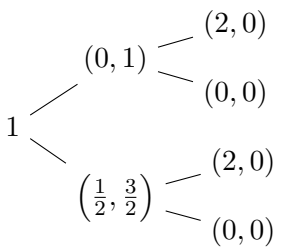
Inequalities $z\phi_1^a(\xi_1^u) \leqslant 0$ and $z\phi_1^b(\xi_1^u) \leqslant 0$ imply

$$-(1 + c_1^a(\xi_1^u))z_1 + 2z_3 \leqslant 0 \quad \text{and} \quad (1 - c_1^b(\xi_1^u))z_1 - 2z_3 \leqslant 0.$$

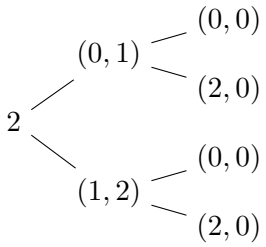
Therefore, we have $z_1 \geqslant 0$ and $z_3 \geqslant 0$. Similarly, we obtain from the other inequalities $z_i \geqslant 0$ for $i = 2, 4, 5, 6$. Hence $z = 0$, a contradiction. We conclude that there does not exist a non-zero z such that $z\phi \leqslant 0$ for all $\phi \in \Phi$. Markets are dynamically complete.

Now, we present an example of 3−period security markets which becomes dynamically incomplete when the market maker services become costly.

Example 3.2.2. We assume there are 2 traded securities available for trading. Their dividends are equal to $(x_1, x_2) \in \mathbb{R}^6 \times \mathbb{R}^6$. We initially assume there is no bid–ask spread, securities can be purchased and sold at a same price $p_j(\xi)$ in every event $\xi \in \Xi_0 \cup \Xi_1 \cup \Xi_2$. We represent the unfolding of the dividends and prices of j in a tree by associating a couple $(x_j(\xi), p_j(\xi))$ to each event $\xi \in \Xi_1 \cup \Xi_2$.



We present the payoffs of security 2 similarly.



The one-period payoff matrix in event ξ_0 is equal to

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

The one-period payoff matrices in events $\xi_1 \in \Xi_1$ are equal to

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

They are all of rank 2 therefore markets are dynamically complete.

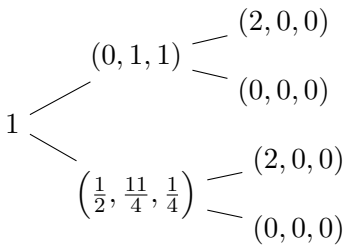
Now, we assume the market makers services are costly in event ξ_1^d, they charge a transaction cost $p_j(\xi_1^d) > c_j(\xi_1^d) > 0$ on security j such that

$$p_1^a(\xi_1^d) + x_1(\xi_1^d) = p_2^a(\xi_1^d) + x_2(\xi_1^d)$$

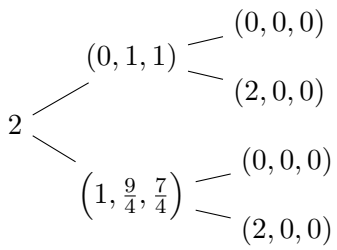
where $p_1^a(\xi_1^d) = p_1(\xi_1^d) + c_1(\xi_1^d)$ and $p_2^a(\xi_1^d) = p_2(\xi_1^d) + c_2(\xi_1^d)$. We have

$$c_1(\xi_1^d) - c_2(\xi_1^d) = 1.$$

We take for example $c_1(\xi_1^d) = \frac{5}{4}$ and $c_1(\xi_1^d) = \frac{1}{4}$. We present the unfolding of the dividends and prices of x_1 in the following tree.



We present them similarly for security 2.



The payoff matrix is equal to

$$
\begin{array}{c}
 \\
\xi_1^u \\
\xi_1^d \\
\xi_2^{uu} \\
\xi_2^{ud} \\
\xi_2^{du} \\
\xi_2^{dd}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
x_1 & x_2 & -x_1 & -x_2 & \phi_1^a(\xi_1^u) & \phi_2^a(\xi_1^u) & \phi_1^b(\xi_1^u) & \phi_2^b(\xi_1^u) & \phi_1^a\left(\xi_1^d\right) & \phi_2^a\left(\xi_1^d\right) & \phi_1^b\left(\xi_1^d\right) & \phi_2^b\left(\xi_1^d\right)
\end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & -\frac{1}{2} & -1 & 0 & 0 & 0 & 0 & -\frac{11}{4} & -\frac{9}{4} & \frac{1}{4} & \frac{7}{4} \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2
\end{pmatrix}
\end{array}.
$$

Let $z = (-3, 1, -\frac{3}{2}, -\frac{3}{2}, \frac{5}{4}, 1)$. The product of z with any of the columns of the payoff matrix is negative. Hence, this payoff does not belong to the set of available payoff streams. It cannot be replicated by a dynamic trading strategy using the securities 1 and 2. The bid–ask spread makes the markets dynamically incomplete.

3.3 Multi-period Markets With Bid-Ask Spreads

This section presents the characterization of dynamic completeness in general multi-period security markets with frictions creating a bid–ask spread. We assume there is no restriction to short-selling and no limitation to the quantity of security purchased and sold at the initial period. We also assume agents can infinitely split their orders, and they share the same information structure. Hence incompleteness does not result from the presence of one of these frictions.

3.3.1 Uncertainty And Information

The future is uncertain. We use the same notations as LeRoy and Werner (2014). Uncertainty is specified by a set of states S. Each of the states represents a description of the economic environment for all periods $t = 0, 1, \ldots, T$. At period 0 agents do not know which state will be realized. However, as time passes, they obtain more and more information about the state. At period T they discover the actual state. Formally, the information of agents at period t is described by a partition F_t of the set of states S (a partition F_t of S represents a collection of subsets of S such that each state s belongs to exactly one element of F_t). The period-0 partition is the

trivial partition $F_0 = \{S\}$. The period$-T$ partition is the total partition $F_T = \{\{s\} : s \in S\}$. The partition F_{t+1} is finer than partition F_t; that is, the element of period$-(t+1)$ partition to which it belongs. The $(T+1)$-tuple of partitions $\{F_0, F_1, \ldots, F_T\}$ is the information filtration F. The partitions are assumed to be common across agents; that is, all agents possess the same information. The number k_t denotes the number of elements in the filtration F_t.

For better exposition, we represent the information filtration as an event tree with each element of partition F_t being a period$-t$ event denoted ξ_t. An event is a node of the event tree (see Figure 3.3). The event $\xi_0 = F_0$ represents the root node. We denote ξ_t^{++} the set of successors of the event ξ_t. It is equal to the set of events $\xi_\tau \subset \xi_t$ with $\xi_\tau \in F_\tau$ for $\tau > t$. The immediate successors of ξ_t are the events $\xi_{t+1} \subset \xi_t$ with $\xi_{t+1} \in F_{t+1}$. The number of immediate successor of event ξ_t is denoted $k(\xi_t)$. The predecessor of the event ξ_t are the events $\xi_\tau \supset \xi_t$ with $\xi_\tau \in F_\tau$ for $\tau < t$. The unique immediate predecessor of ξ_t is the event $\xi_{t-1} \subset \xi_t$. It is denoted ξ_t^-. The set of all events at all future periods $t = 1, \ldots, T$ is denoted Ξ, and $k = \#(\Xi)$ represents the number of future events, that is events in Ξ. Therefore there is a total of $k+1$ events including ξ_0.

3.3.2 Securities, Portfolios And Payoffs

We consider security markets with J securities traded at each period up to period T-1. Each security is characterized by the dividends it pays at each period, a vector x_j of $\mathbb{R}^k$. The dividend matrix $X = \begin{pmatrix} x_1 & \ldots & x_J \end{pmatrix}$ represents the dividend streams of the J securities traded in the markets. The dividend of a security j in an event $\xi \in \Xi$ is denoted $x_j(\xi)$. We gather the dividends on every security in an event $\xi \in \Xi$, in a single row vector $x(\xi) = (x_1(\xi), \ldots, x_J(\xi))$.

We consider a market with two types of actors investors and market makers. An investor represents any party that trades on a financial security market. A market maker is a party who actively quotes two-sided markets in a particular security, providing bids and asks. The bid–ask spread compensates the market maker services. Investors purchase and sell securities to a market maker. They buy at the ask price and sell at the bid price (a market maker buys at the bid price and sells at the ask price). We adopt their perspective in the following. The ask price of security j in an event $\xi \in \Xi$ is denoted $p_j^a(\xi) \in \mathbb{R}^J$ and its bid price is denoted $p_j^b(\xi) \in \mathbb{R}^J$. For notational convenience, we have period$-T$ bid and ask prices $p_j^b(\xi_T) \in \mathbb{R}^J$ and $p_j^a(\xi_T) \in \mathbb{R}^J$ even though trading does not take place at period T. These prices are all set equal to zero. We denote $p^a(\xi) \in \mathbb{R}^J$ the row vector with coordinates equal to the J securities' ask price in the event ξ. Similarly, we

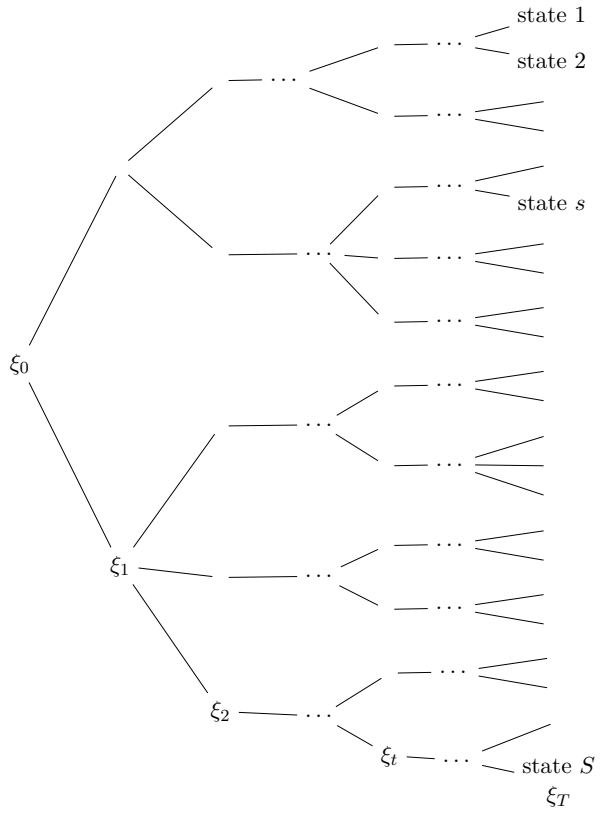


Figure 3.3: Example of an event tree with T periods and S states of the nature.

denote $p^b(\xi) \in \mathbb{R}^J$ the row vector with coordinates equal to the J securities' bid price in the event ξ.

At each non-terminal period, investors constitute a portfolio of securities by trading on the markets. A portfolio of securities in an event $\xi \in \xi_0 \cup \Xi$ is represented as a vector $h(\xi) \in \mathbb{R}^J$. The jth coordinate of $h(\xi)$ is denoted $h_j(\xi)$, it represents the holding of security j. If it is positive, then the portfolio holder is entitled to receive the dividends of security j in the successor events. If it is negative, she is entitled to pay the dividends of j to her counterpart in the successor events. We denote h_t the portfolios held in every event of period t. The $T-$uplet $(h_0, h_1, \ldots, h_{T-1})$ is called a portfolio strategy.

3.3.3 Dynamic Completeness and the Set of Available Payoff Streams

The payoff of a portfolio strategy h in event ξ_t is denoted $z(h, p^a, p^b)(\xi_t)$. An investor first receives the dividends of the portfolio she had constituted in the previous period. Then she trades on the markets. Hence her payoff is equal to

$$\overbrace{x(\xi_t)h(\xi_t^-)}^{\text{dividends}} - \overbrace{p^b(\xi_t)\min\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right)}^{\text{sales revenue}} - \overbrace{p^a(\xi_t)\max\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right)}^{\text{purchases cost}}. \tag{3.2}$$

Therefore the payoff equals the magnitude of the payment at ξ_t to the investor (or, if negative, from the investor). We denote $z_t(h, p^a, p^b)$ the vector of payoffs $z(h, p^a, p^b)(\xi_t)$ in all period$-t$ events. We say that a portfolio strategy h replicates a payoff stream $z \in \mathbb{R}^k$ if $z(\xi) = z(h, p^a, p^b)(\xi)$ for all $\xi \in \Xi$.

Markets with bid–ask spreads are dynamically complete if it is possible to construct for every payoff stream a portfolio strategy that replicates it. The set of payoff streams available via trades on securities is the set $\mathcal{M}(p^a, p^b)$ equal to

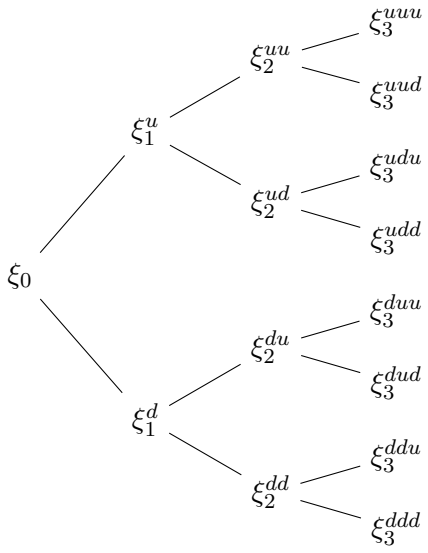
$$\left\{z \in \mathbb{R}^k \mid \exists h \text{ s. t. } z(\xi) = z(h, p^a, p^b)(\xi) \text{ for all } \xi \in \Xi\right\}.$$

In frictionless security markets, this set is called the asset span since it is equal to the span of the columns of the market matrix. In markets with bid–ask spreads this set is not a span. We, therefore, call it the set of available payoff streams. Markets with bid–ask spreads are dynamically complete if $\mathcal{M}(p^a, p^b) = \mathbb{R}^k$, otherwise they are dynamically incomplete. We use the notation $\mathcal{M}(p^a, p^b)$ to emphasize the fact that the presence of a bid–ask spread affects dynamic completeness.

3.3.4 Trading Strategy

First, we introduce the concept of trading strategies. A trading strategy records the unfolding of market orders placed in each event. We denote $b^a(\xi) \in \mathbb{R}^J_+$ the ask orders placed in a non-terminal event ξ and $b^b(\xi) \in \mathbb{R}^J_+$ the bid orders placed in a non-terminal event ξ. We emphasize that orders exclusively admit non-negative values as opposed to portfolios that equally admit negative values. An order placed on a security entitles its issuer to a stream of payoffs in the following periods. Previous to defining the payoff stream of an order, we introduce the necessary notations. We denote $\mathbb{1}_{\xi_t} \in \mathbb{R}^{k+1}$ the vector with coefficient 1 for the coordinate corresponding to the event ξ_t and 0 in all other events and we denote $\mathbb{1}_{\xi_t^{++}} \in \mathbb{R}^{k+1}$ the vectors with coefficient 1 in all coordinates corresponding to an event $\xi_\tau \subset \xi_t$ and 0 in all other events, that is $\mathbb{1}_{\xi_t^{++}} \in \mathbb{R}^{k+1}$ takes the value 1 in all successor events of event ξ_t and 0 otherwise. We illustrate this notation in the following example.

Example 3.3.1. Consider the following 4-period market



We have $\mathbb{1}_{\xi_1^{u++}} = \begin{pmatrix} 0\\0\\0\\1\\1\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0\\ \vdots \end{pmatrix}$ and $\mathbb{1}_{\xi_1^{d++}} = \begin{pmatrix} 0\\0\\0\\0\\0\\1\\1\\0\\0\\0\\0\\1\\1\\1\\1\\ \vdots \end{pmatrix}$

We define the function

$$\begin{array}{rccc} f & : & \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} & \to & \mathbb{R}^{k+1} \\ & & ((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)) & \mapsto & (x_1 y_1, x_2 y_2, \ldots, x_k y_k) \end{array}$$

which associates the product of coordinates to two vectors. We denote $x_j\left(\xi_t^{++}\right)$ the dividends of a unit of security j purchased (or sold) in an event ξ_t, $0 < t < T$. We have $x_j\left(\xi_t^{++}\right) = f\left(x_j, \mathbb{1}_{\xi_t{}^{++}}\right)$.

The payment stream of an ask order placed on security j in an event $\xi_t \in F_t$, $0 \leqslant t < T$, is represented by the vector $\hat{\phi}_j^a(\xi_t)$ with coordinates equal to the ask price in event ξ_t, the dividends associated with the holding of the security in successors of event ξ_t and zero in all other events, that is,

$$\hat{\phi}_j^a(\xi_t) = -p_j^a(\xi_t)\mathbb{1}_{\xi_t} + x_j\left(\xi_t^{++}\right).$$

The payment stream of a bid order placed on security j in an event $\xi_t \in F_t$, $0 \leqslant t < T$, is represented by the vector $\hat{\phi}_j^b(\xi_t)$ with coordinates equal to the bid price in event ξ_t, the dividends due in successors of event ξ_t and zero in all other events, that is,

$$\hat{\phi}_j^b(\xi_t) = p_j^b(\xi_t)\mathbb{1}_{\xi_t} - x_j\left(\xi_t^{++}\right).$$

The payment streams of bid and ask orders placed in event ξ_t on the J securities is a $k+1 \times 2J$ matrix denoted $\hat{\phi}(\xi_t)$ with entries $\hat{\phi}_j^a(\xi_t)$, and $\hat{\phi}_j^b(\xi_t)$ for all $1 \leqslant j \leqslant J$ that is,

$$\hat{\phi}(\xi_t) = \begin{pmatrix} \hat{\phi}_1^a(\xi_t) & \ldots & \hat{\phi}_J^a(\xi_t) & \hat{\phi}_1^b(\xi_t) & \ldots & \hat{\phi}_J^b(\xi_t) \end{pmatrix}.$$

We denote $\hat{\Phi}$ the set of all the payment streams. We have $\#\hat{\Phi} = 2J(k+1-S)$. Therefore, a trading strategy is a vector of $\mathbb{R}_+^{2J(k+1-S)}$. Its coordinates represent the size of the buy and bid orders placed on the markets in each trading event. The payment matrix $\hat{\mathcal{P}}$ is a $k+1 \times 2J(k+1-S)$ matrix with entries $\hat{\phi} \in \hat{\Phi}$ such that $\hat{\mathcal{P}}$ is equal to

$$\begin{pmatrix} \hat{\phi}(\xi_0) & \hat{\phi}(\xi_1(1)) & \ldots & \hat{\phi}(\xi_1(k_1)) & \ldots & \hat{\phi}(\xi_{T-1}(1)) & \ldots & \hat{\phi}(\xi_{T-1}(k_{T-1})) \end{pmatrix}.$$

The first J columns of the payment matrix represent the payment streams of ask orders placed at period 0. The columns $J+1$ to $2J$ of the payment matrix represent the payment streams of bid orders placed at the period 0. The successor columns represent the payment streams of ask orders placed at period $1 \leqslant t \leqslant T-1$ in some event at period t. We presented examples of payment matrix in Section 3.2.

The payment positive span $\hat{\mathcal{B}}(p^a, p^b)$ is the set of payment streams that

can be generated by a trading strategy, that is

$$\hat{\mathcal{B}}(p^a, p^b) = \left\{ \hat{z} \in \mathbb{R}^k \mid \hat{z} = \hat{\mathcal{P}} b \text{ for some } b \in \mathbb{R}_+^{2J(k+1-S)} \right\}.$$

Since trading strategies exclusively accept non-negative coordinates, the payment positive span is the positive span in the mathematical sense (see Section 3.2 for a definition) of the set of payment streams.

The payment matrix is different from the *payoff matrix* which represents the payments across all *future* events of a one-unit trade order placed on a security at a non-terminal event. It is the concatenation of the payoff matrix with the period 0 payments of every order. We denote $\phi(\xi_t)$ the submatrix formed by selecting every rows of the matrix $\hat{\phi}(\xi_t)$ except the first. It represents the payoff streams of bid and ask orders placed in the non-terminal event ξ_t. We denote $\mathcal{P}$ the payoff matrix. It is a $k \times 2J(k+1-S)$ equal to

$$\Big(\phi(\xi_0) \quad \phi(\xi_1(1)) \quad \dots \quad \phi(\xi_1(k_1)) \quad \dots \quad \phi(\xi_{T-1}(1)) \quad \dots \quad \phi(\xi_{T-1}(k_{T-1})) \Big).$$

We denote Φ the set of columns of the payoff matrix $\mathcal{P}$.

The payoff positive span $\mathcal{B}(p^a, p^b)$ is the set of payoff streams that can be realized by a trading strategy, that is

$$\mathcal{B}(p^a, p^b) = \left\{ z \in \mathbb{R}^k \mid z = \mathcal{P} b \text{ for some } b \in \mathbb{R}_+^{2J(k+1-S)} \right\}.$$

Since trading strategies only take positive values, the payoff positive span is equal to the positive span (see Section 3.2 for a definition) of the set of columns of the payoff matrix that is,

$$\mathcal{B}(p^a, p^b) = \text{p-span}\,(\Phi).$$

A trading strategy generates a payoff stream if, and only if, a portfolio replicates it as well. Put differently, the set of available payoff streams is equal to the payoff positive span.

Proposition 3.3.1. The set of available payoff streams is equal to the payoff positive span.

Proposition 3.3.1 implies that the set of available payoff streams is equal to the payoff positive span. Therefore, a market is dynamically complete when $\mathcal{B}(p^a, p^b) = \mathbb{R}^k$ and trading strategies can be used to characterize dynamic completeness.

3.3.5 No-arbitrage

Dynamic completeness is characterized in the frictionless case under a mild equilibrium property, the absence of arbitrage opportunity. To ex-

tend this characterization to markets with frictions, we define an arbitrage opportunity for a trading strategy instead of a portfolio strategy. We denote $\mathbb{R}^n_{++}$ the set of vectors with strictly positive coordinates and we denote $C^\star = \mathbb{R}_+ \times \mathbb{R}^k_+ \setminus 0$ the set of positive payment streams. There exists an arbitrage opportunity in the markets if there exists a trading strategy $b \in \mathbb{R}^{2J(k+1-S)}_+$ generating a non-negative payment stream with at least one strictly positive payment, that is such that $\hat{\mathcal{P}} b > 0$. Additionally, a vector $\mu \in \mathbb{R}^{k+1}$ is said to support the markets when for every $\hat{z} \in \hat{\mathcal{B}}(p^a, p^b)$, we have $\hat{z}^\top \mu \leqslant 0$. We show in Theorem 3.3.1 that there is no arbitrage opportunity in the markets if, and only if, a vector of strictly positive event prices supports the markets.

Theorem 3.3.1. *There is no arbitrage opportunity in multi-period markets with bid-ask spreads if, and only if, a vector of strictly positive event prices supports the market.*

When there are frictions, supporting strictly positive event prices represents the existence of underlying no-arbitrage frictionless markets supporting the markets. The absence of arbitrage opportunity implies that strictly positive event prices support securities prices, in the sense that prices are greater than the weighted sum of expected payoffs for event prices μ, that is, we have

$$\mu_{\xi_t} p^a_j(\xi_t) \geqslant \sum_{\xi \in \xi_t^{++}} \mu_\xi x_j(\xi) \geqslant \mu_{\xi_t} p^b_j(\xi_t)$$

for every non-terminal event ξ_t and every security $j \in J$. Therefore trading strategies permit a straightforward generalization of the fundamental theorem of asset pricing to markets with bid–ask spreads. When markets are frictionless, the payoff stream of an ask order is the opposite of the payoff stream of the bid order on this same security in the same event. Hence, the vector of event prices supports the market with an equality sign, and Theorem 3.3.1 coincides with the fundamental theorem of asset pricing (Harrison and Kreps (1979), Magill and Quinzii (1996)) expressed for trading strategies instead of portfolio strategies.

3.3.6 Characterization of Dynamic Completeness

Proposition 3.3.1 highlights the link between dynamic completeness and the positive span of payoff streams. Moreover, it allows us to use mathematical knowledge on positive spans to characterize dynamic completeness. To begin with, we show that a number of $\left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$ traded securities is necessary for markets to be dynamically complete.

Proposition 3.3.2. Markets with bid–ask spreads are dynamically complete only if at least $\left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$ securities are traded.

Example 3.3.2. Consider a market with k future events and only one security $x \in \mathbb{R}^k$ available for trading. According to Proposition 3.3.2 markets with k events are dynamically complete only of at least $\frac{k+1}{2(k+1-S)}$ securities are traded. Hence, markets are dynamically complete only if the total number of events exceeds twice the number of states, that is $k+1 \geqslant 2S$.

The bound on the necessary number of traded security does not imply that a greater number of traded securities is necessary to satisfy dynamic completeness (see Section 3.2 for an example) in the presence of bid–ask spreads. However, the informativeness of this first result must be nuanced. Indeed, this bound is not informative whenever $S \leqslant \frac{k+1}{2}$ as it merely implies that at least one security must be traded. In any case, providing that this number of security is traded on the market is insufficient to ensure dynamic completeness. It also depends on the values of the securities prices and dividends. In frictionless markets, dynamic completeness is equivalent to having a unique normalized strictly positive vector of supporting event prices under no-arbitrage. We show in the following theorem that the uniqueness of this vector is equivalent to having exclusively *non-zero period*-0 *event prices* supporting the market. A vector of event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ is said to have a non-zero period-0 price if $\mu_{\xi_0} \neq 0$.

Proposition 3.3.3. When markets are frictionless and admit no arbitrage opportunity, a unique normalized vector of strictly positive event prices supports the market if, and only if, every non-zero supporting event prices have a non-zero period-0 price.

We demonstrate in Theorem 3.3.2 that every supporting event prices vector must have a non-zero period-0 price.

Theorem 3.3.2. *The following propositions are equivalent:*

i. Markets are dynamically complete;

ii. for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ whose product with q is positive;

iii. every event price vector supporting the markets has a non-zero period-0 price.

In particular, Theorem 3.3.2 implies that it is necessary and sufficient to find a payoff stream with a non-positive product with every order payoff stream positive to demonstrate that markets are dynamically incomplete.

3.4 Conclusion

Market makers are essential actors of financial markets. To compensate for the risk they bear by holding securities for several periods, they charge positive transaction costs on trades creating frictions called bid–ask spreads. Bid–ask spreads represent the principal transaction costs borne by investors when trading on financial markets. We show that markets may be dynamically complete even in the presence of bid–ask spreads. Moreover, in some cases, removing bid–ask spreads will result in dynamically incomplete markets (see Example 3.2.1). Finally, we demonstrate that dynamic completeness is equivalent to the absence of frictionless supporting markets with a zero period-0 price.

Applications of our results concern the regulation of securities pricing in security exchanges, particularly the size of acceptable bid–ask spreads. Competition between market makers prevents bid–ask spreads from being excessively large; nonetheless, the question remains whether a complementary regulation is necessary to achieve specific goals such as completeness (see Duffie and Rahi (1995)). Another potential application of these results concerns replacing post-trade intermediaries on security exchanges with distributed ledgers technologies (DLTs). These technologies provide the possibility of disposing of intermediaries in trades and are expected to reduce transaction costs. Glosten (1994) shows that DLTs provide as much liquidity as can be expected in extreme adverse selection environments. Nonetheless, he nuances the expectation that DLTs necessarily cut transaction costs by demonstrating that the spread in small trades in electronic limit-order is positive. In contrast, it is possible to imagine a competitive pricing model with zero small-trade spread (see Glosten (1989)). DLTs' proponents must demonstrate that they outclass the current market organization to gain financial regulators' support. One of the various questions they should answer is: regarding risk-sharing, will securities exchanges benefit from switching to DLTs?

Equally, recent literature on asset pricing in markets with frictions provides closed-form expression to pricing rules. However, a trade-off for the increased precision is that some authors assume markets are complete. It is the case in particular, in Cerreia-Vioglio, Maccheroni and Marinacci (2015) and Araujo, Chateauneuf and Faro (2012) which are two of the most significant models of the field. Indeed, Araujo, Chateauneuf and Faro (2012) assume the pricing rule is the super-replication price of some underlying incomplete security market. It amounts to assume completeness in the traditional sense that it is possible to trade every payoff stream. It is difficult to improve these models by extending them to multiperiod settings as preliminary theoretical questions must be addressed. For example, Araujo, Chateauneuf, Faro and Holanda (2019) determine which properties of Araujo, Chateauneuf and Faro (2012) and Cerreia-Vioglio, Maccheroni

and Marinacci (2015) are stable in 3−period security markets. Naturally, another question relates to the assumption that markets are complete. In 2−period security markets, completeness depends exclusively on the rank of the dividend matrix. It is unaffected by the presence of friction. In multi-period security markets, prices impact the available payoff streams through trading happening at intermediary periods. Therefore non-linearities modify the characterization of dynamic completeness. Hence, any attempt to extend these results to multiperiod settings necessitates characterizing dynamic completeness and determining how compelling it is in the presence of friction. Araujo, Chateauneuf, Faro and Holanda (2019) do not address this question. They adopt a non-standard definition of completeness: the payoffs received at the intermediary period are not part of it. It is as if agents initially ignore that markets re-open at the intermediary period. A significant improvement to our contribution is to characterize dynamic completeness when it is explicit that the traded quantity impacts the unitary price. Indeed, in security markets, market makers provide ask and bid offers for specific quantities. Therefore, prices are not linear in quantity purchased or sold. For example, in Kyle (1985), Glosten (1994) and Biais, Martimort and Rochet (2000), prices are convex and increasing in the traded quantity. This will be the subject of future research.

3.5 Appendix

3.5.1 Payoff Matrix of Example 3.2.1

The payoff matrix of Example 3.2.1 is the following:

$$
\begin{array}{c}
\\
\xi_{11}^a \\
\xi_{12}^a \\
\xi_{21}^a \\
\xi_{22}^a \\
\xi_{11}^b \\
\xi_{12}^b \\
\xi_{21}^b \\
\xi_{22}^b
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
x_1 & x_2 & -x_1 & -x_2 & \phi_1^a(\xi_1^a) & \phi_2^a(\xi_1^a) & \phi_1^b(\xi_1^a) & \phi_2^b(\xi_1^a) & \phi_1^a\left(\xi_1^d\right) & \phi_2^a\left(\xi_1^d\right)
\end{array} \\
\left(\begin{array}{cccccccccc}
1 & 1 & -1 & -1 & -(1+c_1^a(\xi_1^a)) & -(1+c_2^a(\xi_1^a)) & 1-c_1^b(\xi_1^a) & 1-c_2^b(\xi_1^a) & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -(1+c_1^a(\xi_1^d)) & -(1+c_2^a(\xi_1^d)) \\
2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2 \\
2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0
\end{array}\right)
\begin{array}{c}
\\
\xi_{11}^b \\
\xi_{12}^b \\
\xi_{21}^b \\
\xi_{22}^b \\
\xi_{11}^d \\
\xi_{12}^d \\
\xi_{21}^d \\
\xi_{22}^d
\end{array}
\begin{array}{c}
\begin{array}{cc}
\phi_1^b\left(\xi_1^d\right) & \phi_2^b\left(\xi_1^d\right)
\end{array} \\
\left(\begin{array}{cc}
0 & 0 \\
0 & 0 \\
-2 & 0 \\
0 & -2 \\
1-c_1^b(\xi_1^d) & 1-c_2^b(\xi_1^d) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right)
\end{array}.
\end{array}
$$

3.5.2 Proofs

Proof of Proposition 3.2.1. Fix a vector $z \in \mathcal{M}(p^a, p^b)$. We are going to show $z \in \mathcal{B}(p^a, p^b)$. By assumption, there exists a portfolio strategy h such that $z(\xi) = z(h, p^a, p^b)(\xi)$ for all $\xi \in \Xi$. Recall from Equation 3.1 that $z(h, p^a, p^b)(\xi_1)$ is equal to

$$x(\xi_1)h(\xi_0) - p^b(\xi_1)\min(h(\xi_1) - h(\xi_0)), 0) - p^a(\xi_1)\max(h(\xi_1) - h(\xi_0), 0)$$

for all $\xi_1 \in \Xi_1$ and

$$z(h, p^a, p^b)(\xi_2) = x(\xi_2)h(\xi_2^b)$$

for all $\xi_2 \in \Xi_2$. Let $b \in \mathbb{R}^{6J}$ be a trading strategy associated with h that is, such that

$$\begin{cases} b^a(\xi_0) &= \max\left(h(\xi_0), 0\right) \\ b^b(\xi_0) &= -\min\left(h(\xi_0), 0\right) \end{cases}$$

and

$$\begin{cases} b^a(\xi_1) &= \max\left(h(\xi_1) - h(\xi_0), 0\right) \\ b^b(\xi_1) &= -\min\left(h(\xi_1) - h\left(\xi_0^b\right), 0\right) \end{cases}$$

for all $\xi_1 \in \Xi_2$. Note that $b^a(\xi_2)$ and $b^b(\xi_2)$ are not defined since there is no trading taking place at time 2. We have $h(\xi_0) = b^a(\xi_0) - b^b(\xi_0)$ and

$$h(\xi_1) = h(\xi_0) + b^a(\xi_1) - b^b(\xi_1)$$

for every $\xi_1 \in \Xi_1$. Hence,

$$z(\xi_1) = x(\xi_1)(b^a(\xi_0) - b^b(\xi_0)) - p^a(\xi_1)b^a(\xi_1) + p^b(\xi_1)b^b(\xi_1)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)\left(b^a(\xi_0) - b^b(\xi_0) + b^a(\xi_1) - b^b(\xi_1)\right)$$

for all $\xi_2 \in \Xi_2$. Hence, we have

$$z = \phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{\xi_1 \in \Xi_1} \phi(\xi_1)\begin{pmatrix} b^a(\xi_1) \\ b^b(\xi_1) \end{pmatrix}.$$

Hence, $z = \mathcal{P}b$ and $z \in \mathcal{B}(p^a, p^b)$.

Now fix a vector $z \in \mathcal{B}(p^a, p^b)$. We are going to show that z belongs to $\mathcal{M}(p^a, p^b)$. By assumption, there exists $b \in \mathbb{R}_+^{6J}$ such that $z = \mathcal{P}b$. It is equivalent to the existence of vectors $b^a(\xi) \in \mathbb{R}_+^J$ and $b^b(\xi_t) \in \mathbb{R}_+^J$ for all $\xi \in \xi_0 \cup \Xi_1$ such that

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{\xi_1 \in \Xi_1} \phi(\xi_1)\begin{pmatrix} b^a(\xi_1) \\ b^b(\xi_1) \end{pmatrix} = z.$$

It implies the following equations

$$z(\xi_1) = x(\xi_1)(b^a(\xi_0) - b^b(\xi_0)) - p^a(\xi_1)b^a(\xi_1) + p^b(\xi_1)b^b(\xi_1)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)\left(b^a(\xi_0) - b^b(\xi_0) + b^a(\xi_1) - b^b(\xi_1)\right)$$

for all $\xi_2 \in \Xi_2$. We let h be a portfolio strategy such that $h(\xi_0) = b^a(\xi_0) - b^b(\xi_0)$ and

$$\begin{cases} \max\left(h(\xi_1) - h(\xi_0), 0\right) &= b^a(\xi_1) \\ \min\left(h(\xi_1) - h(\xi_0), 0\right) &= -b^b(\xi_1) \end{cases}$$

for every $\xi_1 \in \Xi_1$. We obtain $z(\xi_1)$ equal to

$$d\left(h\left(\xi_0\right)\right) - p^b(\xi_1)\min\left(h(\xi_1) - h\left(\xi_0\right), 0\right) - p^a(\xi_1)\max\left(h(\xi_1) - h\left(\xi_0\right), 0\right)$$

for all $\xi_1 \in \Xi_1$ and

$$z(\xi_2) = x(\xi_2)h(\xi_1)$$

for all terminal event $\xi_2 \in \Xi_2$. Therefore $z \in \mathcal{M}(p^a, p^b)$. □

Proof of Theorem 3.2.1. First we assume there is no arbitrage opportunity, we are going to show that there exists a vector of strictly positive event prices such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. No-arbitrage implies there exists no trading strategy $b \in \mathbb{R}_+^{6J}$ such that $\hat{\mathcal{P}}b \in C^\star$. Therefore, we have $\hat{\mathcal{B}}(p^a, p^b) \cap C^\star = \emptyset$. In particular, let $\Delta = \{\mu \in \mathbb{R}_+^7 \mid \sum_{i=0}^6 \mu_i = 1\}$, we have $\hat{\mathcal{B}}(p^a, p^b) \cap \Delta = \emptyset$. The set $\hat{\mathcal{B}}(p^a, p^b)$ is the positive span of the columns of the payoff matrix. Hence, it is a closed convex set. Additionally, Δ is compact. Therefore the theorem of strict separation of convex applies and there exists $\mu \in \mathbb{R}^7$ such that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < \inf_{z \in \Delta} z^\top \mu.$$

Suppose that $\mu_\xi \leqslant 0$ for some event $\xi \in \Xi_0 \cup \Xi_1 \cup \Xi_2$. Consider $\mu' \in \Delta$ such that $\mu'_\xi = 1$ and $\mu'_{\xi'} = 0$ for every $\xi' \neq \xi$. Then, ${\mu'}^\top \mu \leqslant 0$ so that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < 0,$$

contradicting $z^\top \mu = 0$ for $z = 0$. It remains to show that $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Suppose there exists $z' \in \hat{\mathcal{B}}(p^a, p^b)$ such that $z^\top \mu' > 0$. Since $\hat{\mathcal{B}}(p^a, p^b)$ is a positive span, there exists $\alpha \in \mathbb{R}_+$ such that $\alpha z' \in \hat{\mathcal{B}}(p^a, p^b)$ and $(\alpha z')^\top \mu > \min_{z \in \Delta} z^\top \mu$, a contradiction.

Now, assume there exists $\mu \in \mathbb{R}_{++}^7$ such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we

have $z^\top \mu \leqslant 0$. We are going to show that there is no-arbitrage opportunity. Assume by contradiction there exists a trading strategy $b \in \mathbb{R}_+^{6J}$ such that $\hat{\mathcal{P}} b \in C^\star$. Denote $\tilde{z}$ the payoff stream of this trading strategy. By assumption, we have ${z'}^\top \mu \leqslant 0$ with $\mu \in \mathbb{R}_{++}^7$ implying $z' = 0$. Therefore there is no arbitrage opportunity. □

Proof of Proposition 3.2.2. By Proposition 3.2.1 markets are dynamically complete if $\mathcal{B}(p^a, p^b) = \mathbb{R}^6$. Then, p-span$(\Phi) = \mathbb{R}^6$. Hence as a consequence of Corollary 2.4 of Regis (2015) which states that any positive spanning set of $\mathbb{R}^6$ contains a basis of $\mathbb{R}^6$, the payoff matrix must have at least 7 columns. It implies that at least $J \geqslant \frac{7}{2(6+1-4)}$ securities must be traded. Since J takes only integer values, markets are dynamically complete only if $J \geqslant 2$. □

Proof of Proposition 3.2.3. First, we assume that there exists a unique normalized $\nu \in \mathbb{R}_{++}^7$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. We are going to show that every supporting event prices have a non-zero period-0 price. Assume by contradiction that there exist $\mu = \begin{pmatrix} 0 \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\tilde{\mu} \in \mathbb{R}^6$ such that $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\epsilon > 0$ be such that $\nu + \epsilon\mu \in \mathbb{R}_{++}^7$, then $(\nu + \epsilon\mu)^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$ contradicting the uniqueness of ν.

Now, we assume that every $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$ are such that $\mu_{\xi_0} \neq 0$. We are going to show that there exists a unique normalized $\nu \in \mathbb{R}_{++}^7$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ with $\nu_{\xi_0} \in \mathbb{R}_+^*$ and $\tilde{\nu} \in \mathbb{R}_{++}^6$. Suppose by contradiction that there exists $\nu' = \begin{pmatrix} \nu'_{\xi_0} \\ \tilde{\nu}' \end{pmatrix}$ with $\nu'_{\xi_0} \in \mathbb{R}_+^*$ and ${\nu_{\xi_0}}' \in \mathbb{R}_{++}^6$ such that $\nu' \neq \lambda\nu$ for every $\lambda \in \mathbb{R}_+$ and ${\nu'}^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$. Let $\alpha \in \mathbb{R}$ be such that $\nu_{\xi_0} = \alpha\nu'_{\xi_0}$ and $(\nu - \alpha\nu')^\top z = 0$ implying $\nu = \alpha\nu'$, a contradiction.

□

Proof of Theorem 3.2.2. The proof of $i.$ equivalent to $ii.$ follows from the characterization of positive spanning sets of Davis (1954). For the sake of clarity, we present it in our context. Assume that markets are dynamically complete. We are going to show that for every non-zero $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. Assume by contradiction that there exists a payoff stream $q \in \mathbb{R}^6$ such that $q^\top \phi \leqslant 0$ for every $\phi \in \Phi$. Denote ϕ_i, $i = 1, \ldots, 6J$, the ith element of Φ and dynamic completeness implies that there exists a trading strategy $b \in \mathbb{R}_+^{6J}$ such that $z = \mathcal{P} b$, that

is

$$z = \sum_{i=1}^{6J} b_i \phi_i$$

where b_i is the ith coordinate of b. Moreover, $z^\top z > 0$ that is,

$$\sum_{i=1}^{6J} (b_i \phi_i)^\top z > 0$$

implies that at least one element of the sum is positive. Hence there exists an order payoff stream ϕ_i such that $b_i \phi_i^\top z > 0$, a contradiction. Therefore, for every non-zero $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$.

Then assume that for every payoff stream $q \in \mathbb{R}^6$ there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that markets are dynamically complete. Assume by contradiction that markets are dynamically incomplete. It implies that there exists a payoff stream $z \notin \mathcal{B}(p^a, p^b)$, that is $\mathcal{B}(p^a, p^b) \cap z = \emptyset$. Therefore according to Rockafellar (1970) Theorem 11.3, there exists a hyperplane containing the origin that properly separates $\mathcal{B}(p^a, p^b)$ and z. Denote q' its normal vector at the origin then for either $q = q'$ or $q = -q'$, we have $q^\top \phi \leqslant 0$ for every $\phi \in \Phi$, a contradiction.

Now, we are going to show that *ii.* is equivalent to *iii.*. First, assume that for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. Let $\nu_{\xi_0} \in \mathbb{R}$, $\tilde{\nu} \in \mathbb{R}^6$ and let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ be such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. By assumption, there exists $\phi \in \Phi$ such that $\phi^\top \tilde{\nu} > 0$. It implies $\nu_{\xi_0} \neq 0$.

Then, we assume that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^6$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. We are going to show that for every payoff stream $q \in \mathbb{R}^6$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. Assume by contradiction that there exists $\tilde{\nu} \in \mathbb{R}^6$, such that $\phi^\top \tilde{\nu} \leqslant 0$ for every $\phi \in \Phi$. Therefore, we have $\nu = \begin{pmatrix} 0 \\ \tilde{\nu} \end{pmatrix} \in \mathbb{R}^7$ such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, a contradiction.

□

Proof of Proposition 3.3.1. Fix a vector $z \in \mathcal{M}(p^a, p^b)$. We are going to show that $z \in \mathcal{B}(p^a, p^b)$. By assumption, there exists a portfolio strategy

h such that $z_t = z_t(h, p^a, p^b)$ for all $1 \leqslant t \leqslant T$. Recall from Equation 3.2 that $z(h, p^a, p^b)(\xi_t)$ is equal to

$$d\left(h\left(\xi_t^-\right)\right) - p^b(\xi_t)\min\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right) - p^a(\xi_t)\max\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right).$$

that is, period t payoff in event ξ_t is equal to the dividend received from holding the portfolio $h(\xi_t^b)$ at the beginning of period t plus the gain earned from trading taking place at period t. Let $b \in \mathbb{R}^{2J(k+1-S)}$ be a trading strategy such that

$$\begin{cases} b^a(\xi_0) &= \max\left(h(\xi_0), 0\right) \\ b^b(\xi_0) &= -\min\left(h(\xi_0), 0\right) \\ b^a(\xi_t) &= \max\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right) \text{ for all } 0 < t < T \\ b^b(\xi_t) &= -\min\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right) \text{ for all } 0 < t < T \end{cases}.$$

We denote $E(\xi_t)$ the set of predecessor of ξ_t (see Section 3.3.1). We have

$$h(\xi_t) = \sum_{\xi_\tau \in E(\xi_t)} \left(b^a(\xi_\tau) - b^b(\xi_\tau)\right)$$

for every $0 \leqslant t \leqslant T-1$. Hence,

$$z(h, p^a, p^b)(\xi_t) = x(\xi_t) \sum_{\xi_\tau \in E(\xi_t)} \left(b^a(\xi_\tau) - b^b(\xi_\tau)\right) - p^a(\xi_t)b^a(\xi_t) + p^b(\xi_t)b^b(\xi_t)$$

for all $\xi_t \in F_t$ and all $0 < t < T$ and

$$z(h, p^a, p^b)(\xi_T) = x(\xi_T) \sum_{\xi_\tau \in E(\xi_t)} \left(b^a(\xi_\tau) - b^b(\xi_\tau)\right)$$

for all $\xi_T \in F_T$ where $b^a(\xi_t)$ represents the quantities of securities purchased in event ξ_t, $b^b(\xi_t)$ represents the quantities of securities sold in event ξ_t and $\sum_{\xi_\tau \in E(\xi_t)} \left(b^a(\xi_\tau) - b^b(\xi_\tau)\right)$ are the cumulative quantities of securities traded up to time t. Note that b_T is not defined since there is no trading taking place at time T. Hence, we have

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0) \end{pmatrix} + \sum_{t=1}^{T-1} \sum_{\xi_t \in F_t} \phi(\xi_t)\begin{pmatrix} b^a(\xi_t) \\ b^b(\xi_t) \end{pmatrix} = z$$

which is equal to $\mathcal{P}b = z$. Therefore, $z \in \mathcal{B}(p^a, p^b)$.

Now fix a vector $z \in \mathcal{B}(p^a, p^b)$. We are going to show that z belongs to $\mathcal{M}(p^a, p^b)$. By assumption, there exists $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $z = \mathcal{P}b$. It is equivalent to the existence of row vectors $b^a(\xi_t) \in \mathbb{R}^J$ and $b^b(\xi_t) \in \mathbb{R}^J$

for all $\xi_t \in F_t$ and all $0 \leqslant t \leqslant T-1$ such that

$$\phi(\xi_0)\begin{pmatrix} b^a(\xi_0) \\ b^b(\xi_0)\end{pmatrix} + \sum_{t=1}^{T-1}\sum_{\xi_t \in F_t} \phi(\xi_t)\begin{pmatrix} b^a(\xi_t) \\ b^b(\xi_t)\end{pmatrix} = z.$$

It implies the following equality

$$x(\xi_t)\sum_{\xi_\tau \in E(\xi_t)}\Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big) - p^a(\xi_t)b^a(\xi_t) + p^b(\xi_t)b^b(\xi_t) = z(\xi_t)$$

for all non-terminal event ξ_t. We let h be a portfolio strategy such that

$$\begin{cases} h(\xi_t^-) & = & \sum_{\xi_\tau \in E(\xi_t)}\Big(b^a(\xi_\tau) - b^b(\xi_\tau)\Big) \\ \max\Big(h(\xi_t) - h(\xi_t^-), 0\Big) & = & b^a(\xi_t) \\ \min\Big(h(\xi_t) - h(\xi_t^-), 0\Big) & = & -b^b(\xi_t). \end{cases}$$

We obtain $z(\xi_t)$ equal to

$$d\left(h\left(\xi_t^-\right)\right) - p^b(\xi_t)\min\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right) - p^a(\xi_t)\max\left(h(\xi_t) - h\left(\xi_t^-\right), 0\right)$$

for all non-terminal event ξ_t and

$$z(\xi_T) = x(\xi_T)h(\xi_T^b)$$

for all terminal event ξ_T. Therefore $z \in \mathcal{M}(p^a, p^b)$. □

Proof of Theorem 3.3.1. First we assume there is no arbitrage opportunity, we are going to show that there exists a vector of strictly positive event prices such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. No-arbitrage implies there exists no trading strategy $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $\hat{\mathcal{P}}b \in C^\star$. Therefore, we have $\hat{\mathcal{B}}(p^a, p^b) \cap C^\star = \emptyset$. In particular, let $\Delta = \{\mu \in \mathbb{R}_+^{k+1} \mid \sum_{i=0}^k \mu_i = 1\}$, we have $\hat{\mathcal{B}}(p^a, p^b) \cap \Delta = \emptyset$. The set $\hat{\mathcal{B}}(p^a, p^b)$ is the positive span of the columns of the payoff matrix. Hence, it is a closed convex set. Additionally, Δ is compact. Therefore the theorem of strict separation of convex applies and there exists $\mu \in \mathbb{R}^{k+1}$ such that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < \inf_{z \in \Delta} z^\top \mu.$$

Suppose that $\mu_\xi \leqslant 0$ for some event $\xi \in \Xi \cup \xi_0$. Consider $\mu' \in \Delta$ such that $\mu'_\xi = 1$ and $\mu'_{\xi'} = 0$ for every $\xi' \neq \xi$. Then, ${\mu'}^\top \mu \leqslant 0$ so that

$$\sup_{z \in \hat{\mathcal{B}}(p^a, p^b)} z^\top \mu < 0,$$

contradicting $z^\top \mu = 0$ for $z = 0$. It remains to show that $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Suppose there exists $z' \in \hat{\mathcal{B}}(p^a, p^b)$ such that $z^\top \mu' > 0$. Since $\hat{\mathcal{B}}(p^a, p^b)$ is a positive span, there exists $\alpha \in \mathbb{R}_+$ such that $\alpha z' \in \hat{\mathcal{B}}(p^a, p^b)$ and $(\alpha z')^\top \mu > \min_{z \in \Delta} z^\top \mu$, a contradiction.

Now, assume there exists $\mu \in \mathbb{R}^{k+1}_{++}$ such that for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, we have $z^\top \mu \leqslant 0$. We are going to show that there is no-arbitrage opportunity. Assume by contradiction there exists a trading strategy $b \in \mathbb{R}_+^{2J(k+1-S)}$ such that $\hat{\mathcal{P}} b \in C^\star$. Denote $\tilde{z}$ the payoff stream of this trading strategy. By assumption, we have ${z'}^\top \mu \leqslant 0$ with $\mu \in \mathbb{R}^{k+1}_{++}$ implying $z' = 0$. Therefore there is no arbitrage opportunity. □

Proof of Proposition 3.3.2. By definition, markets are dynamically completeness if $\mathcal{M}(p^a, p^b) = \mathbb{R}^k$. Then, p-span$(\phi) = \mathbb{R}^k$. Hence as a consequence of Corrolary 2.4 of Regis (2015) which states that any positive spanning set of $\mathbb{R}^k$ contains a basis of $\mathbb{R}^k$, the book order's payoffs matrix must have at least $k+1$ columns. It implies that at least $J \geqslant \frac{k+1}{2(k+1-S)}$ securities must be traded. Since J takes only integer values, markets are dynamically complete only if $J \geqslant \left\lceil \frac{k+1}{2(k+1-S)} \right\rceil$. □

Proof of Proposition 3.3.3. First, we assume that there exists a unique normalized $\nu \in \mathbb{R}^{k+1}_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. We are going to show that every supporting event prices have a non-zero period-0 price. Assume by contradiction that there exist $\mu = \begin{pmatrix} 0 \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^7 \setminus \{0\}$ with $\tilde{\mu} \in \mathbb{R}^k$ such that $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\epsilon > 0$ be such that $\nu + \epsilon\mu \in \mathbb{R}^{k+1}_{++}$, then $(\nu + \epsilon\mu)^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$ contradicting the uniqueness of ν.

Now, we assume that every $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1} \setminus \{0\}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$ are such that $\mu_{\xi_0} \neq 0$. We are going to show that there exists a unique normalized $\nu \in \mathbb{R}^{k+1}_{++}$ such that $\nu^\top z = 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. Let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ with $\nu_{\xi_0} \in \mathbb{R}^*_+$ and $\tilde{\nu} \in \mathbb{R}^k_{++}$. Suppose by contradiction that there exists $\nu' = \begin{pmatrix} \nu'_{\xi_0} \\ \tilde{\nu'} \end{pmatrix}$ with $\nu'_{\xi_0} \in \mathbb{R}^*_+$ and ${\nu_{\xi_0}}' \in \mathbb{R}^k_{++}$ such that $\nu' \neq \lambda\nu$ for every $\lambda \in \mathbb{R}_+$ and ${\nu'}^\top z = 0$ for every $z \in \mathcal{B}(p^a, p^b)$. Let $\alpha \in \mathbb{R}$ be such that $\nu_{\xi_0} = \alpha\nu'_{\xi_0}$ and $(\nu - \alpha\nu')^\top z = 0$ implying $\nu = \alpha\nu'$, a contradiction.

□

Proof of Theorem 3.3.2. The proof of $i.$ equivalent to $ii.$ follows from the characterization of positive spanning sets of Davis (1954). For the sake of clarity, we present it in our context. Assume that markets are dynamically

complete. We are going to show that for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$. Assume by contradiction that there exists a payoff stream $q \in \mathbb{R}^k$ such that $q\phi \leqslant 0$ for every $\phi \in \Phi$. Let $b \in \mathbb{R}_+^{2J(k+1-S)}$ be a trading strategy such that $z = \mathcal{P}b$, that is

$$z = \sum_{i=1}^{2J(k+1-S)} b_i \phi_i.$$

Moreover, $z^\top z > 0$ that is,

$$\sum_{i=1}^{2J(k+1-S)} b_i \phi_i^\top z > 0$$

implies that at least one element of the sum is positive. Hence there exists an order payoff stream ϕ_i such that $b_i \phi_i^\top z > 0$, a contradiction. Therefore, for every non-zero payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$.

Now assume that for every payoff stream $q \in \mathbb{R}^k$ there exists an order payoff stream $\phi \in \Phi$ such that $q\phi > 0$. We are going to show that markets are dynamically complete. Assume by contradiction that markets are dynamically incomplete. It implies that $z \notin$ p-span(Φ), that is p-span$(\Phi) \cap z = \emptyset$. Therefore according to Rockafellar (1970) Theorem 11.3, there exists a hyperplane with vector normal at the origin q' that properly separates p-span(ϕ) from the rest of $\mathbb{R}^k$, that is such that for either $q = q'$ or $q = -q'$, $q^\top \phi \leqslant 0$ for all $\phi \in \Phi$, a contradiction.

Now, we are going to show that *ii.* is equivalent to *iii.*. First, assume that for every payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. We are going to show that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. Let $\nu_{\xi_0} \in \mathbb{R}$, $\tilde{\nu} \in \mathbb{R}^k$ and let $\nu = \begin{pmatrix} \nu_{\xi_0} \\ \tilde{\nu} \end{pmatrix}$ be such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$. By assumption, there exists $\phi \in \Phi$ such that $\phi^\top \tilde{\nu} > 0$. It implies $\nu_{\xi_0} \neq 0$.

Then, we assume that every event prices $\mu = \begin{pmatrix} \mu_{\xi_0} \\ \tilde{\mu} \end{pmatrix} \in \mathbb{R}^{k+1}$ with $\mu_{\xi_0} \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^k$ satisfying $z^\top \mu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, are such that $\mu_{\xi_0} \neq 0$. We are going to show that for every payoff stream $q \in \mathbb{R}^k$, there exists an order payoff stream $\phi \in \Phi$ such that $q^\top \phi > 0$. Assume by contradiction that there exists $\tilde{\nu} \in \mathbb{R}^k$, such that $\phi^\top \tilde{\nu} \leqslant 0$ for every $\phi \in \Phi$. Therefore, we have $\nu = \begin{pmatrix} 0 \\ \tilde{\nu} \end{pmatrix} \in \mathbb{R}^{k+1}$ such that $z^\top \nu \leqslant 0$ for every $z \in \hat{\mathcal{B}}(p^a, p^b)$, a contradiction.

□

General Conclusion

This thesis contributes to a major research axis in economics: improving the consideration of frictions (transaction costs, taxes, restrictions on trades) in financial asset valuation models. Moreover, it relates to considering these frictions in the fundamental theorem of asset pricing, one of the core results of financial economics.

The first chapter studies a special case of the generalization of the fundamental theorem of asset pricing proposed by Cerreia-Vioglio, Maccheroni and Marinacci (2015) and **?** to markets with frictions. It proposes an explicit and straightforward formula for valuing financial assets on the markets verifying the put–call parity, and presenting frictions of the transaction costs type. We show that together no friction on riskless security, no friction among securities paying their highest payoffs in the same events, monotone prices, and put–call parity are equivalent to prices equal to the expected value of securities' payoff with respect to a unique affine transformation of a probability. Furthermore, we establish a relationship between the most studied and highest transaction costs present on the financial markets, market makers' remuneration, the bid-ask spread, and risk. We show that the bid–ask spread of a financial asset, i.e., the difference between its buy price and its sell price, is proportional to its risk. We show that bid–ask spreads are increasing functions of the range of securities' revenues.

The assumptions made by Cerreia-Vioglio, Maccheroni and Marinacci (2015) and **?** to generalize the fundamental theorem of asset pricing force asset prices to be constant with the quantity purchased. This property is not empirically verified. The abundant empirical literature on market impact has demonstrated that security prices are increasing in quantity traded. Moreover, several theoretical models have shown that the presence of insiders on the market induces convex prices. However, the absence of friction on riskless security joint with put–call parity implies positively homogeneous prices. Positively homogeneous prices are unrealistic in financial security markets. Based on this observation, in the second chapter of this thesis, we characterize the concept of absence of arbitrage opportunity in the presence of frictions making prices convex, and we extend the fundamental theorem of asset pricing to this type of market. We demonstrate that markets are viable if, and only if, there is no robust arbitrage opportunity (a strong

no-arbitrage property). We then demonstrate that the associated pricing rule is necessarily monotone which provides arguments in favor of making this technical assumption in other models. Eventually, we show that when markets are complete, prices satisfy robust no-arbitrage if, and only if, they are eventually monotone.

The results presented in the first two chapters are in two-period financial markets. In the absence of friction, two-period models are reasonable simplifications of multiple-period models. However, the possibility of participating in the market at intermediate periods creates questions specific to these so-called dynamic models in the presence of frictions. The last chapter is devoted to one of these questions. In the presence of bid-ask spreads, we determine the conditions allowing dynamic completeness, that is to say, the conditions allowing its participants to transfer their risk exposures fully and perfectly smooth their consumption. We show that bid-ask spreads do not increase the minimum number of assets necessary for dynamic completeness and that in some cases, on the contrary, they make the market dynamically complete.

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RÉSUMÉ

Cette thèse contribue à un axe de recherche majeur en économie : améliorer la prise en compte des frictions (coûts de transaction, taxes, restrictions sur les échanges) dans les modèles de valorisation des actifs financiers. Il s'agit de considérer ces frictions dans le théorème fondamental l'évaluation des actifs financiers, l'un des résultats fondateurs de l'économie financière.
Le premier chapitre propose une formule explicite et simple de valorisation des actifs financiers sur les marchés vérifiant la parité put-call et présentant des frictions de type coûts de transaction. Nous établissons une relation entre les coûts de transaction les plus élevés des marchés financiers, la rémunération des market makers, appelée bid-ask spread, et le risque. Nous montrons que le bid-ask spread d'un actif financier, c'est-à-dire la différence entre son prix d'achat et son prix de vente, est proportionnel à son risque.
Les hypothèses faites dans le premier chapitre contraignent les prix des actifs à être constants avec la quantité achetée. Cette propriété n'est pas vérifiée empiriquement. Au contraire, de nombreuses études menées sur les marchés financiers ont montré que les prix augmentent avec la quantité achetée. De plus, les modèles stratégiques standards d'achat-vente montrent que les prix sont convexes dans la quantité achetée. Partant de ce constat, dans le deuxième chapitre de cette thèse, nous caractérisons le concept d'absence d'opportunité d'arbitrage en présence de frictions rendant les prix convexes, et nous étendons le théorème fondamental de la valorisation des actifs à ce type de marché.
Les résultats présentés dans les deux premiers chapitres portent sur des marchés financiers à deux périodes. En l'absence de friction, les modèles à deux périodes sont des simplifications raisonnables des modèles à périodes multiples. Cependant, les frictions et la possibilité de participer au marché à des périodes intermédiaires posent des questions spécifiques à ces modèles dits dynamiques. Le dernier chapitre est consacré à l'une de ces questions. En présence de frictions de type bid-ask spread, nous déterminons les conditions permettant la complétude dynamique, c'est-à-dire les conditions permettant à ses participants de transférer leurs expositions au risque pleinement et de parfaitement lisser leur consommation. Nous montrons que les bid-ask spreads n'augmentent pas le nombre minimum d'actifs nécessaires à la complétude dynamique et que dans certains cas, au contraire, ils rendent le marché dynamiquement complet.

MOTS CLÉS

Théorème fondamental de l'evaluation des actifs financiers, friction, coûts de transaction, parité call-put, taxe, bid-ask spread, prix Choquet, prix convexes, arbitrage, viabilité, market maker, marché complet, marché incomplet, efficience, innovation financière, partage du risque, market spanning, capacité néo-additive.

ABSTRACT

This thesis contributes to a major research axis in economics: improving the consideration of frictions (transaction costs, taxes, restrictions on trades) in financial asset valuation models. It relates to considering these frictions in the fundamental theorem of asset pricing, one of the founding results of financial economics.
The first chapter proposes an explicit and straightforward formula for valuing financial assets on the markets verifying the put-call parity and presenting frictions of the transaction costs type. We establish a relationship between the most studied and highest transaction costs present on the financial markets, market makers' remuneration, called the bid-ask spread, and risk. We show that the bid-ask spread of a financial asset, i.e., the difference between its buy price and its sell price, is proportional to its risk.
The assumptions made in the first chapter force asset prices to be constant with the quantity purchased. This property is not empirically verified. On the contrary, many studies have shown that prices increase with the quantity purchased on the financial markets. Also, standard strategic buy-sell models show that prices are convex in the purchased quantity. Based on this observation, in the second chapter of this thesis, we characterize the concept of absence of arbitrage opportunity in the presence of frictions making prices convex, and we extend the fundamental theorem of asset pricing to this type of market.
The results presented in the first two chapters are in two-period financial markets. In the absence of friction, two-period models are reasonable simplifications of multiple-period models. However, the possibility of participating in the market at intermediate periods creates questions specific to these so-called dynamic models in the presence of friction. The last chapter is devoted to one of these questions. In the presence of bid-ask spread type frictions, we determine the conditions allowing dynamic completeness, that is to say, the conditions allowing its participants to transfer their risk exposures fully and perfectly smooth their consumption. We show that bid-ask spreads do not increase the minimum number of assets necessary for dynamic completeness and that in some cases, on the contrary, they make the market dynamically complete.

KEYWORDS

Fundamental theorem of asset pricing, friction, transaction cost, taxe, bid-ask spread, Choquet pricing, neo-additive capacity, put-call parity, convex asset pricing, arbitrage, viability, market maker, complete market, incomplete market, efficiency, financial innovation, risk sharing, market spanning.