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SYSTÈMES DYNAMIQUES NON-RÉGULIERS: APPLICATIONS EN OPTIMISATION ET AUX PROCESSUS DE RAFLES

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To my family,

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Chapter 1

Introduction

Nonsmooth dynamical systems (or NSDS for brevity) are systems with the nonsmoothness appearing in the evolution. In real life, nonsmooth phenomenon occurs frequently even in very simple models, for example, mechanical systems with dry friction or impact, electric circuits with diodes, transistors or relay. Nowadays, engineers and scientists are dealing with more and more complex models that require a high level of precision. So, to get a more adequate and close predictions, they need a better understanding of the mathematical models behind NSDS.

Due to numerous applications of NSDS, the study of these systems is crucial and therefore, their analytical and numerical development is required. Recall that in classical dynamical systems, the trajectories are always supposed to be smooth or differentiable. However, since the trajectories in NSDS are not smooth, the concepts of generalized gradients, generalized subdifferentials, tangent cones, and normal cones play an important role. Fortunately, these concepts were thoroughly studied in nonsmooth analysis, set-valued and variational analysis. We refer, e.g., to the books of J.-P. Aubin and H. Frankowska [12], F. H. Clarke [33], B. S. Mordukhovich [60, 61, 63], J.-P. Penot [81], R. T. Rockafellar and R. J.-B. Wets [85], W. Schirotzek [89]. Day by day, this topic has been enriched by the contributions of many authors and it still attracts considerable attention of mathematicians.

Although NSDS appear in many real-life models (in physics, engineering, biology, etc.), their mathematical formulation seems to be the same. One usually writes NSDS in the form of differential inclusions or evolution variational inequalities. Many aspects of NSDS, such as well-posedness, stability analysis, control analysis, and numerical analysis, have been considered (see, e.g., [1, 3, 23]). In many kinds of NSDS, the so-called "sweeping process" has played an important

role thanks to its broad application in mechanics, physics, engineering, social sciences (see, e.g., [1, 2, 2, 6, 23, 67, 68]). Pioneered by J. J. Moreau [65], sweeping processes has been studied and developed intensively in the last 50 years (for more detail, see Section 1.1 below).

Optimal control of sweeping processes has received great attentions from researchers in recent years. This theory has many applications in the study of crowd motions, robotics engineering, or traffic flows, etc. For the most important results in this direction, one can refer to [24–29, 38–41, 51, 62, 64]. Note that dynamical systems serve as constraints of an optimal control problem. To fully describe the latter, one has to give an *objective function*, which can be an integral function (for Lagrange problems), a real-valued function (for Mayer problems), or the sum of both of them (for Bolza problems).

This dissertation is concerned with three main problems in NSDS. Namely, perturbed sweeping processes, sweeping processes with velocity constraints, and vibro-impact problems are discussed in detail.

Let us briefly introduce the dynamical systems studied in this dissertation and our contributions in the next sections.

1.1 Perturbed Sweeping Processes

Let $T > 0$ be a real number and let $C(t)$, $t \in [0, T]$, be nonempty closed subsets of a real Hilbert space $\mathcal{H}$. For any fixed $x_0 \in C(0)$, the differential inclusion

$$\begin{cases} -\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) \quad \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $\mathcal{N}^{Cl}_{\Omega}(z)$ denotes the Clarke normal cone [33, p. 51] to a closed set Ω at z, is called a *sweeping process*. If $C(t)$ is convex, then the Clarke normal cone coincides with the normal cone in the sense of convex analysis [53, Proposition 2.4.4, p. 52]. An absolutely continuous function $x(\cdot) : [0, T] \to \mathcal{H}$ which satisfies the two conditions in (1.1) is said to be a *solution* of the sweeping process. Note that any absolutely continuous function $x(\cdot) : [0, T] \to \mathcal{H}$ is Fréchet differentiable almost everywhere on $[0, T]$ with respect to the Lebesgue measure (see Proposition 2.8).

The model (1.1) under the assumption that $C(t)$ is *convex* for each $t \in [0, T]$ was introduced by Moreau in [65], where some fundamental results on solution existence and uniqueness were obtained. In [66], the continuity of the solutions has been studied when the convex-valued mapping $C : [0, T] \rightrightarrows \mathcal{H}$ undergoes small

perturbations.

In some subsequent papers, assumptions on the convexity of $C(t)$ have been relaxed. For examples, Colombo and Goncharov [36] obtained existence and uniqueness theorem of the solution under the hypothesis that the sets $C(t)$ are weakly closed and φ-convex. Later, in a more general setting, Bounkhel [19] proved some solution existence and uniqueness results. Namely, the author just assumed the sets $C(t)$ to be prox-regular (see the definition of prox-regularity of a set in Definition 2.37).

Since the function $x(\cdot)$ in (1.1) can be interpreted as the trajectory of a certain mechanical system, which is driven by an external force (the gravitational force, a force generated by an electromagnetic field, a wind, etc.), several authors have studied *perturbed sweeping processes* of the form

$$\begin{cases} -\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t)) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where the perturbation function g is either a single-valued or a multi-valued map satisfying some regularity assumptions. Since $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \{0\}$ if $C(t) = \mathcal{H}$ for all $t \in [0, T]$, then the inclusion in (1.2) reduces to the ordinary differential equation $-\dot{x}(t) = g(t, x(t))$, when g is single-valued. Hence, in that case, (1.2) is a Cauchy problem. In the finite-dimensional setting, where $\mathcal{H} = \mathbb{R}^n$, there are two classical theorems: the Peano theorem [50, Theorem 2.1, p. 10] (for the solution existence of the Cauchy problem) and the Picard-Lindelöf theorem [50, Theorem 1.1, p. 8] (for the existence and uniqueness of the solution of the Cauchy problem). Naturally, one wishes to have some analogues of such theorems for problem (1.2).

Perturbed sweeping processes with the sets $C(t)$, $t \in [0, T]$, being convex or the complement of the interior of a convex set were studied by Castaing, Duc Ha and Valadier [31] and several authors in references therein. For sweeping processes with delay, where $C(t)$, $t \in [0, T]$, are assumed to be compact convex sets, Castaing and Monteiro Marques [30] obtained not only solution existence and uniqueness results but also some topological properties of the solution sets.

Bounkhel and Thibault [21, Corollary 3.5] established new characterizations of r-prox-regular sets in terms of the subdifferentials of the distance functions associated with the sets. Using these characterizations, they proved [21, Theorem 4.2] a solution existence theorem for nonconvex sweeping processes in Hilbert spaces with multi-valued perturbation mappings.

For discontinuous perturbed sweeping processes in the infinite-dimensional

setting, Edmond and Thibault [49, Theorem 3.1] studied solutions in the form of functions of bounded variation, which might be discontinuous. As a corollary, they gave [49, Theorem 5.1] sufficient condition for the existence of absolutely continuous solutions. Later, Nacry [69] improved the results in [49] by considering the perturbation of the normal cone in form of a sum of a single-valued mapping and a set-valued mapping. Weakening the assumption on the movement of the constraint sets in preceding works, Nacry and Thibault [70] obtained the existence and uniqueness results of a solution for the perturbed sweeping process with bounded truncated variation.

The solution existence and uniqueness for the sweeping processes with prox-regular constraint sets $C(t)$ with single-valued perturbations will be investigated systematically in Chapter 3 of this dissertation. Based on the result on the prox-regularity of nonsmooth sublevel sets of Adly, Nacry and Thibault [8, Theorem 4.1], we prove the solution existence as well as the solution uniqueness for a special case when $C(t)$ are sublevel sets under some assumptions. To clarify the applicability of the obtained results, we give some examples having clear mechanical interpretations. Remarkably, the examples can only be solved by invoking the uniqueness of the solution of (1.2).

1.2 Sweeping Processes with Velocity Constraints

Studied firstly by Siddiqi and Manchanda [90] and Bounkhel [19] in some simple forms, *sweeping processes with velocity in a moving set* encompass a class of evolution variational inequalities, which have numerous applications in mechanics and physics (see [7, p. 8] and [45, Section 6.4]). Adopting a more general setting than the ones in [19, 90], Adly, Haddad, and Thibault [7, Theorem 5.1] obtained a result on the solution existence of sweeping processes in separable Hilbert spaces with velocity in a moving bounded convex set. Afterwards, Adly and Le [6, Theorem 1] proved that a similar result can be established for the case where the moving set is unbounded and convex. In addition, by constructing an example (see [6, Example 1]), the authors showed that the sweeping process in question may not have solutions if one of the assumptions of the existence theorem is violated. Vilches and Nguyen [92, Section 5] have improved the result of [6] by weakening the continuity condition of the moving constraint set. The solution existence in [92] has been obtained by applying an existence result on evolution inclusions governed

by time-dependent maximal monotone operators with full domains.

The interested reader is referred to [6, pp. 840–842] for an application of the solution existence results to irregular electrical circuits.

Adly and Haddad [4] have proved the equivalence between sweeping processes with velocity constraints and quasistatic evolution variational inequalities. Focusing on the case of convex constraint sets (the convex case), Jourani and Vilches [54] have established the existence and uniqueness of the solution to the sweeping process in a very general framework by equivalently transforming the problem in question to an ordinary differential equation on a Hilbert space. The obtained results have been applied to quasistatic evolution variational inequalities and nonsmooth electrical circuits [54, Sections 7 and 8]. Let us mention that the authors have also shown [54, p. 5169] that one solution existence result in [19] can be proved by noting that the velocity vector at each time instance is uniquely defined as the projection of the origin of the Hilbert space on the moving constraint set. As a consequence, the corresponding results on the solution existence and uniqueness in [90], which are applicable to the case of convex moving constraint sets, also can be derived in this way.

Recently, Adly and Haddad [5] have obtained existence and uniqueness results for sweeping processes with velocity constraints in the convex case where the constraint set depends on both time and state.

Let $\mathcal{H}$ be a Hilbert space and $C : [0,T] \rightrightarrows \mathcal{H}$ be a set-valued mapping. Let $A_0, A_1 : \mathcal{H} \to \mathcal{H}$ be bounded symmetric linear operators and $f : [0,T] \to \mathcal{H}$ be a continuous mapping. Recall that a linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$. Following [6, 7], we consider the sweeping process

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where $\mathcal{N}^P_{C(t)}(\dot{u}(t))$ is the proximal normal cone (see, e.g., [20, p. 21] and Section 2.5 below) to $C(t)$ at $\dot{u}(t)$. An *absolutely continuous* function $u : [0,T] \to \mathcal{H}$ is said to be a *solution* of (1.3) if it satisfies the differential inclusion and the initial value condition in the formulation of the problem. Since every Lipschitz function $u : [0,T] \to \mathcal{H}$ is absolutely continuous, it is desirable to have sufficient conditions for (1.3) to have a Lipschitz continuous solution.

For concrete examples of sweeping processes with velocity in a moving set we refer to [7, Examples 1 and 2] and [6, Example 1].

The solution existence theorem in [7, Theorem 5.1] for (1.3) was obtained under the following assumptions:

(a) $C(t)$ *is closed convex bounded for every* $t \in [0,T]$;

(b) A_1 *is positive semidefinite, i.e.,* $\langle A_1 x, x\rangle \geq 0$ *for all* $x \in \mathcal{H}$.

For the sweeping process (1.3), the authors of [6] showed that the next two assumptions guarantee the solution existence:

($\widetilde{\text{a}}$) $C(t)$ *is closed convex for every* $t \in [0,T]$;

($\widetilde{\text{b}}$) A_1 *is positive semidefinite and there exist positive constants* α, β *such that* $\langle A_1 x, x\rangle \geq \alpha\|x\|^2 - \beta$ *for all* $x \in C(0)$.

It is worth to emphasize that the settings and results of [6, 7, 54, 92] require the separability of the Hilbert space $\mathcal{H}$.

As far as we know, nonconvex sweeping processes with velocity constraints have only been addressed by Bounkhel [19], who assumed that $A_0 \equiv 0$, A_1 is the identity operator, and the sets $C(t)$ are uniformly prox-regular and contained in a convex compact set for all $t \in [0,T]$.

The sweeping process (1.3) where $C(t)$ is not necessarily convex for every $t \in [0,T]$ will be studied in Chapter 4 of this dissertation. Firstly, by using a result of Yen [94] on the solution sensitivity of parametric variational inequalities, we investigate (1.3) in the case where the set-valued mapping $t \mapsto C(t)$, $t \in [0,T]$, has nonempty closed convex values and is locally Lipschitz-like. Thanks to this approach, the vital requirement of the separability of $\mathcal{H}$ in most of the preceding works is no more required. Note also that a locally Lipschitz-like set-valued mapping with nonempty closed convex values can be not continuous in the Hausdorff distance sense. Secondly, we obtain several solution existence results for the case where $C(t)$ is a finite union of disjoint convex sets.

Assuming that the operator A_0 in (1.3) is coercive and the constraint sets are convex, the authors of [7] have given a condition for the solution uniqueness. Herein, we will prove that (1.3) can have at most one solution if the operator A_1 is coercive. However, the coerciveness of both A_0 and A_1 does not imply the solution uniqueness of (1.3) even in the case of a fixed nonconvex constraint set that is compact, uniformly prox-regular, and connected (see Remark 4.26 below). We think that the solution uniqueness of (1.3) deserves further investigations.

Due to the wide range of applications of (1.3), other properties of the solutions of that problem are also of great interest. In Chapter 5 of this dissertation, we will prove that if some sufficient conditions for the solution existence and uniqueness are satisfied, then the solution is Lipschitz continuous on the initial value. Then, we

will show that the solution set is bounded if some assumptions used in the literature are fulfilled. The solution set is not always closed in the space of continuous vector-valued functions. However, it is a closed subset in an appropriate space. Two sets of sufficient conditions for the convexity of the solution set will be obtained. Interestingly, a sharp outer estimate for the solution set can be established. It is worthy to emphasize that the just-mentioned properties of the solutions of (1.3) are investigated here for the first time. To the best of our knowledge, analogous results are not available in the literature.

1.3 Vibro-impact Problems

Vibro-impact systems are the dynamical multibody systems subjected to perfect non-penetration conditions which generate vibrations and impacts. Because of the impact experiences, the systems involve discontinuities in velocity and the acceleration may contain Dirac masses. Hence, vibro-impact systems cannot be modeled by ordinary differential equations, and one uses *measure differential inclusions* (see, e.g., [59, 76]). More precisely, we consider a mechanical system with a finite number of degrees of freedom, subjected to perfect time-dependent unilateral constraints. Let $I = [0, T]$, $T > 0$, be a bounded time real interval and $d \in \mathbb{N}^*$. Let $g : I \times \mathbb{R}^d \to \mathbb{R}^d$ and $f_i : I \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1, \ldots, m\}$ be some functions and $m \in \mathbb{N}$. We denote by $q \in \mathbb{R}^d$ the representative point of the system in generalized coordinates and define the set of admissible positions at each instant $t \in I$ by

$$C(t) = \{q \in \mathbb{R}^d \mid f_i(t, q) \leq 0 \ \ \forall i \in \{1, \ldots, m\}\}.$$

and the set of active constraints by $J(t, q) = \{i \in \{1, \ldots, m\} \ \mid f_i(t, q) = 0\}$. The vibro-impact system given by g and the functions f_i is formally described by the following second-order differential inclusion in $\mathbb{R}^d$:

$$\ddot{q}(t) - g(t, q(t)) \in -\mathcal{N}^{Cl}_{C(t)}(q(t)),$$

where $\mathcal{N}_{C(t)}(q(t))$ is the Clarke normal cone [33, p. 51] to $C(t)$ at $q(t)$, $t \in I$.

There are many existence results for the vibro-impact problems with time-independent constraints (i.e., when the set of admissible positions does not depend on time: $C(t) = C$ for $t \in [0, T]$). In the single-constraint case, the results have been established by using the position-based algorithm in [77, 79, 80] and by using the velocity-based algorithm in [46, 47, 57–59]. In the multi-constraint case, several

results have been obtained in [13, 71, 72, 74].

For vibro-impact problems with time-dependent constraints (i.e., when the set of admissible positions $C(t)$ depends on time), there are few solution existence theorems. Let us list some important results related to this case that are known in the literature:

Schatzman [88] established an existence result by considering a generalization of the Yosida-type approximation proposed in [78].

Assuming that the set of admissible positions at any instant is defined as a finite intersection of complements of convex sets, Bernicot and Lefebvre-Lepot [15] obtained an existence theorem.

Paoli [73, 75] proposed a time-stepping approximation scheme for the problem and proved its convergence, which gives as a by-product a global existence result when the set of admissible positions at any instant is defined by a finite family of C^2 functions.

The existence of solutions for these second-order differential problems has been studied by Bernicot and Venel [17] in a general and abstract framework. More precisely, the set $C(t)$ of admissible positions is assumed in [17] to be Lipschitz continuous in the Hausdorff distance sense and satisfies an "admissibility" property (see Section 2.3 [17]). The authors also considered a particular case, where the constraints are C^2 functions and have bounded second-order derivatives (see Section 4 in [17]). The assumptions used in this chapter require less regularity on the data of the problem and could be seen as a complementary result of Theorem 3.2 and an improvement of Theorem 4.6 in [17] (see Remark 6.22 for more details).

In Chapter 6 of the dissertation, we will obtain a solution existence theorem for a class of vibro-impact problems where the moving constraints set are the sublevel sets of a family of Lipschitz functions. Herein, we present four explicit conditions for the constraints without requiring any second-order differentiability information on the data involved in the constraints. To prove the convergence of the approximate solutions, we use the time-stepping scheme, which was used in some preceding works. To clarify the applicability of the obtained result, an illustrative example will be given.

1.4 Outline of the Dissertation

The dissertation is organized as follows:

In Chapter 1, we give a short introduction and motivation for studying

of nonsmooth dynamical systems. Our focus is made on three well-known models, namely, the perturbed sweeping process, sweeping process with velocity constraints, and the vibro-impact problem.

In Chapter 2, we describe some notations and necessary background from functional analysis and nonsmooth analysis which are crucial in this dissertation.

Chapter 3 studies nonconvex perturbed sweeping processes. This chapter is based on the joint paper of N. K. Son, N. N. Thieu, and N. D. Yen "*On the solution existence for prox-regular perturbed sweeping processes*", which is available as a preprint [arXiv:2108.07515v1] and was accepted for publication in Journal of Nonlinear and Variational Analysis.

Chapter 4 investigates the solution existence and the solution uniqueness of sweeping processes with velocity constraints. This chapter is based on the paper "*Some classes of nonconvex sweeping processes with velocity constraints*" of S. Adly, N. N. Thieu, and N. D. Yen, which was submitted for publication.

Chapter 5 establishes some fundamental properties of the solutions of sweeping processes with velocity constraints. This chapter is based of the paper "*Solution properties of convex sweeping processes with velocity constraints*" by N. N. Thieu, which is available as a preprint [arXiv:2109.06556v1] and submitted for publication.

Chapter 6 is devoted to a class of vibro-impact problems where the moving constraints set are the sublevel sets of a family of Lipschitz functions. This chapter is based on the paper "*Existence of solutions for a Lipschitzian vibro-impact problem with time-dependent constraints*" of S. Adly and N. N. Thieu, which has been submitted for publication.

Chapter 2

Mathematical Background

The present chapter recalls some notations and results from functional analysis, nonsmooth analysis which are mostly taken from the monographs [22, 33, 43, 56, 86, 87].

2.1 Some Notations and Elementary Concepts

By $\mathbb{N}^*$ we denote the set of positive integers. The notation $[a, b]$ (resp., (a, b)) stands for a closed interval (resp., an open interval) in the real line $\mathbb{R}$. Throughout this paper, let $\mathcal{H}$ be a real Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. The open ball (resp., closed ball) in $\mathcal{H}$ with center x and radius $r > 0$ is denoted by $\mathbb{B}_{\mathcal{H}}(x, r)$ (resp., $\bar{\mathbb{B}}_{\mathcal{H}}(x, r)$). If the space is itself clear by the context, we will omit the subscripts in these notations. The closure, the interior, the boundary, and the closed convex hull of a set $\Omega \subset \mathcal{H}$ are denoted respectively by $\mathrm{cl}(\Omega)$, $\mathrm{int}(\Omega)$, $\partial\Omega$, and $\overline{\mathrm{co}}(\Omega)$. The distance from x to Ω is $d(x, \Omega) := \inf_{y \in \Omega} \|x - y\|$. The *projection* of a point $x \in \mathcal{H}$ onto Ω is defined by $\mathbb{P}_\Omega(x) = \{y \in \Omega \mid d(x, \Omega) = \|x - y\|\}$. For any extended real number $r \in (0, \infty]$, the r-enlargement of Ω, denoted by $U_r(\Omega)$, is defined by $U_r(\Omega) = \{x \in \mathcal{H} \mid d(x, \Omega) < r\}$. The Banach space of continuous functions from $[a, b]$ to $\mathcal{H}$ is denoted by $\mathcal{C}^0([a, b], \mathcal{H})$. The norm is given by $\|x\|_{\mathcal{C}^0} = \max_{t \in [a,b]} \|x(t)\|$.

Definition 2.1. The *Hausdorff distance* between two subsets Ω_1, Ω_2 of $\mathcal{H}$ is given by

$$d_H(\Omega_1, \Omega_2) = \max\{ \sup_{x \in \Omega_1} d(x, \Omega_2), \quad \sup_{y \in \Omega_2} d(y, \Omega_1)\}.$$

Definition 2.2. (See [18, Section 3.3.2, p. 193]) A function $Q : \mathcal{H} \to \mathbb{R}$ is said to be a *quadratic form* on $\mathcal{H}$ if there exists a bilinear symmetric function $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$

such that $Q(x) = B(x,x)$ for all $x \in \mathcal{H}$. A quadratic form Q is said to be *nonnegative* if $Q(x) \geq 0$ for all $x \in \mathcal{H}$.

Proposition 2.3. (See [18, Proposition 3.71, p. 193]) *A quadratic form $Q(\cdot)$ is convex on $\mathcal{H}$ if and only if it is nonnegative.*

Lemma 2.4. (See [8, Lemma 3.2]) *Let $C \subset \mathbb{R}^d$ and $x, y \in C$ with $\|x-y\| < 2\rho$, where $\rho \in (0, +\infty]$. Then, for any $\tau \in [0,1]$ one has $x + \tau(y-x) \in U_\rho(C)$.*

2.2 Vector-valued Functions

Definition 2.5. The *total variation* of a function $f : [a,b] \to \mathcal{H}$ on $[a,b]$ is the nonnegative extended real number

$$\mathrm{Var}(f,[a,b]) = \sup \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\|,$$

where the supremum is taken over all finite partitions $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$. If $\mathrm{Var}(f,[a,b]) < +\infty$, then f is said to be *of bounded variation* on $[a,b]$. The space of all functions of bounded variation from $[0,T]$ to $\mathcal{H}$ is denoted by $\mathrm{BV}([0,T];\mathcal{H})$.

Example 2.6. Let $[a,b] \subset \mathbb{R}$ be an interval.

(i) A increasing or decreasing function $f : [a,b] \to \mathbb{R}$ is of bounded variation.

(ii) The difference of two monotonic functions is a function of bounded variation.

(iii) Suppose that $A_1, \ldots, A_k$ are disjoint intervals of $[a,b]$, whose union is $[a,b]$. The formula $f(x) = \alpha_i$ for $x \in A_i$, where $\alpha_1, \ldots, \alpha_k$ are real numbers, defines a step function, which is a function of bounded variation.

(iv) The function

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x^\beta} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

is of bounded variation on $[0,1]$ if $\alpha > \beta$ and not if $\alpha \leq \beta$; see [56, p. 331].

Definition 2.7. A function $x : [a,b] \to \mathcal{H}$ is said to be *absolutely continuous* on $[a,b]$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sum_{k=1}^{\ell} \|x(b_k) - x(a_k)\| < \varepsilon$ for any finite system of pairwise disjoint subintervals $(a_k, b_k) \subset [a,b]$, $k = 1, \ldots, \ell$, with the total length $\sum_{k=1}^{\ell}(b_k - a_k)$ less than δ.

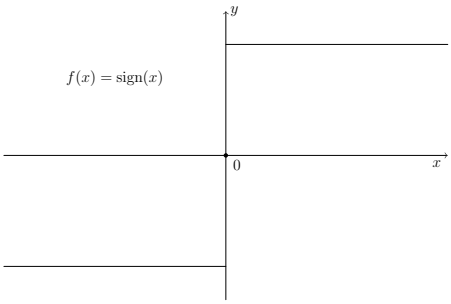


Fig. 2.1: Discontinuous BV function in $[-1, 1]$

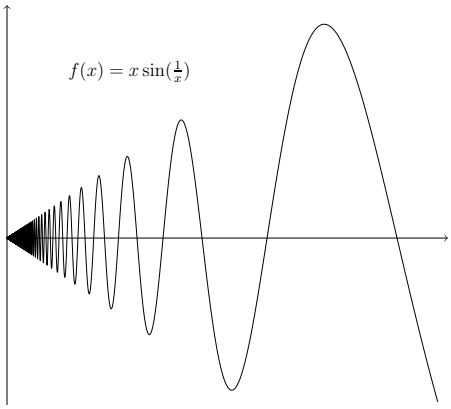


Fig. 2.2: Unbounded variation function in $[0, 1]$

Arguing similarly as [56, Theorem 2, p. 337] (it suffices to replace the absolute value of real numbers by the norm of vectors in $\mathcal{H}$), one can show that *any absolutely continuous function $[a, b] \to \mathcal{H}$ is a function of bounded variation.*

Proofs of the next proposition can be found in the books by Benyamini and Lindenstrauss [14, Corollary 5.12 and Theorem 5.21] or by Diestel and Uhl [43, Corollary 13 of Chapter 3, Theorem 2 on p. 107, and Section 6 of Chapter VII].

Proposition 2.8. *Let $f : [a, b] \to \mathcal{H}$ be absolutely continuous. Then, f is Fréchet differentiable almost everywhere on $[a, b]$ with respect to the Lebesgue measure of the segment.*

Theorem 2.9. (See [11, Theorem 4, p. 13]) *Let $\{x_k(\cdot)\}$ be a sequence of absolutely continuous functions from an interval $I \subset \mathbb{R}$ to a Banach space X satisfying*

(i) *For all $t \in I$, $\{x_k(t)\}_k$ is a relatively compact subset of X;*

(ii) *There exists a positive function $c(\cdot) \in L^1(I, \mathbb{R})$ such that $\|\dot{x}_k(t)\| \leq c(\cdot)$ for almost all $t \in I$.*

Then, there exists a subsequence, still denoted by $\{x_k(\cdot)\}$, converging to an absolutely continuous function $x(\cdot)$ from I to X in the sense that

(i) *$x_k(\cdot)$ converges uniformly to $x(\cdot)$ over compact subsets of I;*

(ii) *$\dot{x}_k(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^1(I, X)$, i.e.,*

$$\lim_{k\to\infty} \int_I \varphi(\tau)\dot{x}_k(\tau)d\tau = \int_I \varphi(\tau)\dot{x}(\tau)d\tau \quad (\forall \varphi \in L^\infty(I, X)).$$

Definition 2.10. A function $f : Y \to \mathcal{H}$ defined on $Y \subset \mathbb{R}^n$ is said to be *(globally) Lipschitz continuous* with modulus $L > 0$ on Y if $\|f(y) - f(y')\| \leq L\|y - y'\|$ for all $y, y' \in Y$. If this inequality holds for all y, y' in some neighborhood of $x \in \mathcal{H}$ then f is said to be *locally Lipschitz* in the neighborhood of x.

Clearly, *any Lipschitz continuous function is an absolutely continuous function.*

2.3 Measures and Integrals

First, we begin with some well-known facts about Riemann integral and Lebesgue integral. We denote by $L^1(\Omega)$ the space of Lebesgue integrable functions from Ω to $\mathbb{R}$. For $p \in (1, \infty)$, let $L^p(\Omega, \mathbb{R})$ denote the space of all measurable functions from Ω to $\mathbb{R}$ satisfying $|f|^p \in L^1(\Omega, \mathbb{R})$.

Proposition 2.11. (See, e.g., [56, Theorem 1, p. 368]) *If f is continuous on $[a, b]$ then its Riemann integral exists and coincides with its Lebesgue integral.*

Remark 2.12. (See [86, Remarks 11.23(c)]) If f and g is Lebesgue integrable on E and if $f(x) \leq g(x)$ for $x \in E$, then $\int_E f(\tau)d\tau \leq \int_E g(\tau)d\tau$.

Theorem 2.13. (See [56, Theorem 6, p. 340]) *If a function $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then its derivative $\dot{f}$ is Lebesgue integrable on $[a, b]$ and*

$$f(x) = f(a) + \int_a^x \dot{f}(\tau)d\tau \quad \forall x \in [a, b].$$

Proposition 2.14. (Hölder's inequality, see, e.g., [22, Theorem 4.6] or [56, p. 385]) *Let $p \in [1, \infty]$, $f \in L^p(\Omega, \mathbb{R})$ and $g \in L^q(\Omega, \mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\int_\Omega |f(\tau)g(\tau)|d\tau \leq \left(\int_\Omega |f(\tau)|^p d\tau\right)^{\frac{1}{p}} + \left(\int_\Omega |g(\tau)|^q d\tau\right)^{\frac{1}{q}}.$$

Proposition 2.15. (See [56, Theorem 8, p. 324]) *If $f \in L^1([a, b], \mathbb{R})$, then $\frac{d}{dx}\left(\int_a^x f(\tau)d\tau\right) = f(x)$ for almost every $x \in [a, b]$.*

We now recall the definition of Bochner integral.

Definition 2.16. (See [43, pp. 44–45]) Let (Ω, Σ, μ) be a finite measurable space and X be a Banach space. A μ-measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions $\{f_k\}$ such that

$$\lim_{k\to\infty} \int_\Omega \|f_k(\omega) - f(\omega)\|_X d\mu = 0.$$

In this case, $\int_E f(\omega)d\mu$ is defined for each $E \in \Sigma$ by $\int_E f(\omega)d\mu = \lim_{k\to\infty} \int_E f_k(\omega)d\mu$, where $\int_E f_k(\omega)d\mu$ is defined in an obvious way.

As noted in [43, p. 45], the limit in Definition 2.16 exists and is independent of the defining sequence $\{f_k\}$.

Next, we give an characterization of Bochner integrable functions.

Proposition 2.17. (See [43, Theorem 2, p. 45]) *A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f(\omega)\|_X d\mu < \infty$.*

The dominated convergence theorem for Bochner integration is stated below.

Theorem 2.18. (Dominated Convergence Theorem, see [43, Theorem 3, p. 45]) *Let $\{f_k\}$ be a sequence of Bochner integrable on Ω. If $\lim_{k\to\infty} f_k = f$ in μ measure, i.e., $\lim_{k\to\infty} \mu\{\omega \in \Omega \mid \|f_k(\omega) - f(\omega)\| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$, and if there exists a real-valued Lebesgue integrable function g on Ω with $\|f_k\| \leq g$ μ-almost everywhere, then f is Bochner integrable on Ω and $\lim_{k\to\infty} \int_E f_k(\omega)d\mu = \int_E f(\omega)d\mu$ for each $E \in \Sigma$.*

Proposition 2.19. (See [43, Theorem 4, p. 46]) *If f is Bochner integrable in μ measure then*

(a) $\lim_{\mu(E)\to 0} \int_E f(\omega)d\mu = 0$;

(b) $\left\| \int_E f(\omega)d\mu \right\| \leq \int_E \|f(\omega)\|d\mu$ *for all* $E \in \Sigma$;

(c) *if $\{E_k\}$ is a sequence of pairwise disjoint sets in Σ and $E := \bigcup_{k=1}^{\infty} E_k$ then*

$$\int_E f(\omega)d\mu = \sum_{k=1}^{\infty} \int_{E_k} f(\omega)d\mu,$$

where the sum on the right-hand side is absolutely convergent.

Next proposition gives an important result on Bochner integrable functions with respect to the Lebesgue measure.

Proposition 2.20. (See [43, Theorem 9, p. 49]) *Let f be Bochner integrable on $[0, 1]$ with respect to Lebesgue measure. Then, one has*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(t) - f(\tau)\|d\tau = 0$$

for almost all $t \in [0,1]$. *Consequently,* $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f(\tau)d\tau = f(t)$ *for almost all* $t \in [0,1]$.

Remark 2.21. In the formulation of Proposition 2.20, one can replace $[0,1]$ by any real interval $[a,b]$. The proof remains the same.

If $1 \leq p < \infty$, the Bochner space $L^p(\Omega, X)$ consists of all μ-measurable functions $f : \Omega \to X$ satisfying

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|_X^p d\mu \right)^{1/p} < \infty$$

(see, e.g., [43, pp. 49–50]). For more details on Bochner integration, we refer to [97, p. 132], [43, Chapter II], and [22, p. 116].

Some useful facts on Bochner integration of absolutely continuous functions will be discussed further in Section 4.2 (see Remark 4.16).

2.4 Sobolev Spaces

Now, we recall the definition and some properties of Sobolev spaces of vector-valued functions. Let Ω be an open subset of $\mathbb{R}$ and X be a Banach space. The space $L^1_{\text{loc}}(\Omega, X)$ of *locally integrable* functions is defined as follows:

$$L^1_{\text{loc}}(\Omega, X) := \left\{ f : \Omega \to X \mid \int_K \|f(\tau)\| d\tau < \infty, \ \forall K \subset \Omega, \ K \text{ is compact} \right\}.$$

Definition 2.22. Let $f \in L^p(\Omega, X)$, where $p \in [1, \infty)$, a function $\tilde{f} \in L^1_{\text{loc}}(\Omega, X)$ is said to be a *weak derivative* of f if

$$\int_\Omega \dot{g}(\tau) f(\tau) d\tau = -\int_\Omega g(\tau) \tilde{f}(\tau) d\tau,$$

for all $g \in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ the space of all real-valued functions that are infinitely differentiable and have compact support in Ω.

The weak derivative of $f \in L^p(\Omega, X)$ is uniquely defined up to a set of measure zero (see [98, Proposition 23.18]).

Definition 2.23. (See, e.g, [32]) Let $p \in [1, +\infty)$, Ω be an open subset of $\mathbb{R}$, and X be a Banach space. The *Sobolev space* $W^{1,p}(\Omega, X)$ is the set of all functions

$f \in L^p(\Omega, X)$ that admit a weak derivative on Ω satisfying $\dot{f} \in L^p(\Omega, X)$. This space is equipped with the norm

$$\|f\|_{W^{1,p}} = \left(\int_\Omega \|f\|^p d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega \|\dot{f}\|^p d\mu\right)^{\frac{1}{p}}.$$

From the above definition, we see that if a sequence $\{f_k\}$ converges strongly to f in $W^{1,p}(\Omega, X)$, then f_k (resp., $\dot{f}_k$) converges strongly to f (resp., $\dot{f}$) in $L^p(\Omega, X)$. It is well known [32, Proposition 1.4.34] that $W^{1,p}(\Omega, X)$ is a Banach space for all $p \in [1, +\infty)$.

Proposition 2.24. (See [32, Theorem 1.4.35]) *Let $p \in [1, \infty)$ and $x \in L^p(\Omega, X)$. The following conditions are equivalent*

(a) $x \in W^{1,p}(\Omega, X)$.

(b) *x is absolutely continuous, differentiable almost everywhere and $\dot{x} \in L^p(\Omega, X)$.*

(c) *there exists a function $y \in L^p(\Omega, X)$ such that for almost every $t_0, t \in \Omega$, one has*

$$x(t) = x(t_0) + \int_{t_0}^{t} y(\tau) d\tau.$$

Remark 2.25. For $\Omega = (0, T)$, if $x : \Omega \to \mathcal{H}$ is an absolutely continuous function, then it is a simple matter to prove that the limits $\lim_{t \to 0^+} x(t)$ and $\lim_{t \to T^-} x(t)$ exist. So, setting $x(0) = \lim_{t \to 0^+} x(t)$ and $x(T) = \lim_{t \to T^-} x(t)$ gives an absolutely continuous function defined on $[0, T]$. Therefore, by Proposition 2.24 one can identify the Sobolev space $W^{1,1}(\Omega, X)$, where $\Omega = (0, T)$, with the space of absolutely continuous functions $u : [0, T] \to \mathcal{H}$ equipped with the norm

$$\|u\|_{W^{1,1}} = \int_0^T \|u(\tau)\| d\tau + \int_0^T \|\dot{u}(\tau)\| d\tau. \tag{2.1}$$

We use this identification and write $W^{1,1}([0, T], \mathcal{H})$ for $W^{1,1}((0, T), \mathcal{H})$.

2.5 Subgradients and Normal Cones

Definition 2.26. For a closed convex set $\Omega \subset \mathcal{H}$, the *normal cone* to Ω at $x \in \mathcal{H}$ in the sense of convex analysis is $\mathcal{N}_\Omega(x) := \{x^* \in \mathcal{H} \mid \langle x^*, y - x \rangle \leq 0, \ \forall y \in \Omega\}$ if $x \in \Omega$ and $\emptyset$ if $x \notin \Omega$.

Definition 2.27. Let f be Lipschitz continuous in a neighborhood of x in $\mathcal{H}$ and let v be any vector in $\mathcal{H}$. *Clarke's generalized directional derivative* of f at x in the direction v, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v) := \limsup_{y\to x,\, t\downarrow 0} \frac{f(y+tv)-f(y)}{t}.$$

Definition 2.28. *The Clarke subgradient* of f at x is a vector $\xi \in \mathcal{H}$ satisfying

$$\langle \xi, v\rangle \le f^0(x;v) \ \text{ for all } \ v \in \mathcal{H}.$$

The set of all the Clarke subgradient of f at x is called *the Clarke subdifferential* of f at x and denoted by $\partial^C f(x)$.

Let Ω be a closed subset of $\mathcal{H}$ and $x \in \Omega$.

Definition 2.29. The set $\mathcal{T}_\Omega^{Cl}(x) := \{v \in \mathcal{H} \mid d^0(x,\Omega;v) = 0\}$ is called *the Clarke tangent cone* to Ω at x. *The Clarke normal cone* to Ω at x is defined by polarity with $\mathcal{T}_\Omega^{Cl}(x)$. Namely, one has $\mathcal{N}_\Omega^{Cl}(x) = \{x^* \in \mathcal{H} \mid \langle x^*, v\rangle \le 0 \quad \forall v \in \mathcal{T}_\Omega^{Cl}(x)\}$.

Definition 2.30. (See, e.g., [20, p. 21]) The *proximal normal cone* $\mathcal{N}_\Omega^P(x)$ to $\Omega \subset \mathcal{H}$ at $x \in \Omega$ is defined by setting

$$\mathcal{N}_\Omega^P(x) = \{\xi \in \mathcal{H} \mid \exists \alpha > 0 \text{ such that } x \in \mathbb{P}_\Omega(x+\alpha\xi)\}.$$

Definition 2.31. A vector $v \in \mathcal{H}$ is a *proximal subgradient* of a function $f : \mathcal{H} \to \mathbb{R}$ at x if there exist a real number $\sigma \ge 0$ and a neighborhood U of x such that

$$\langle v, x'-x\rangle \le f(x') - f(x) + \sigma\|x'-x\|^2,$$

for all $x' \in U$.

Proposition 2.32. *For any $x \in \Omega$, one has*

$$\mathcal{N}_\Omega^P(x) = \{v \in \mathcal{H} \mid \exists t > 0 : d(x+tv,\Omega) = t\|v\|\}.$$

Proof. The proof follows from [20, Proposition 1.7, p. 22]. □

Example 2.33. Let $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \le |x_1|\}$. The proximal normal cone to Ω at $(0,0)$ is $\mathcal{N}_\Omega^P(0,0) = \{0,0\}$.

The following propositions are specifications of some assertions in [33].

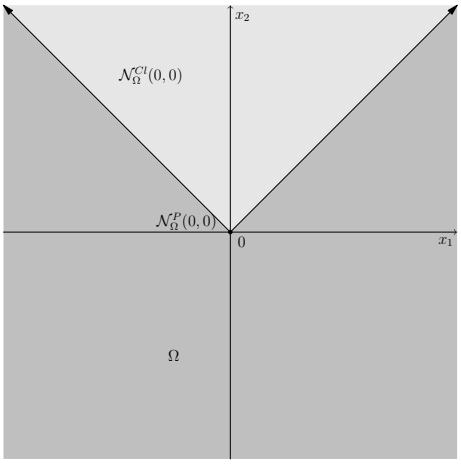


Fig. 2.3: Normal cones of $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq |x_1|\}$ at $(0, 0)$

Proposition 2.34. (See [33, Propositions 2.1.2 and 2.1.5]) *Let f be Lipschitz continuous with modulus L in a neighborhood of x. Then,*

1. *$\partial^C f(x)$ is a nonempty, convex, compact subset of $\mathbb{R}^d$ and $\|\xi\| \leq L$ for every ξ in $\partial^C f(x)$.*
2. *For every v in $\mathbb{R}^d$, one has $f^0(x; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial^C f(x)\}$.*
3. *$\xi \in \partial^C f(x)$ if and only if $f^0(x; v) \geq \langle \xi, v\rangle$ for all v in $\mathbb{R}^d$.*
4. *Let x_k and ξ_k be sequences in $\mathbb{R}^d$ such that $\xi_k \in \partial^C f(x_k)$. Suppose that x_k converges to x, and that ξ is a cluster point of ξ_k. Then one has $\xi \in \partial^C f(x)$.*

Proposition 2.35. (See [33, Proposition 2.2.4]) *If f is Lipschitz continuous near x and $\partial f(x)$ reduces to a singleton $\{\xi\}$, then f is strictly differentiable at x and $\nabla f(x) = \xi$.*

Proposition 2.36. (See [33, Corollary 2 of Theorem 2.4.7]) *Let Ω be given as follows:*

$$\Omega = \{y \in \mathbb{R}^d \mid f_1(y) \leq 0, \ldots, f_m(y) \leq 0\},$$

and let x be such that $f_i(x) = 0$ for $i = 1, \ldots, m$. Then, if each f_i is differentiable at x and if $\nabla f_i(x), i = 1, \ldots, m$, are positively linearly independent, then

$$\mathcal{N}_\Omega^{Cl}(x) = \Big\{ \sum_{i=1}^{m} \lambda_i \nabla f_i(x) \mid \lambda_i \geq 0, \ i = 1, \ldots, m \Big\}.$$

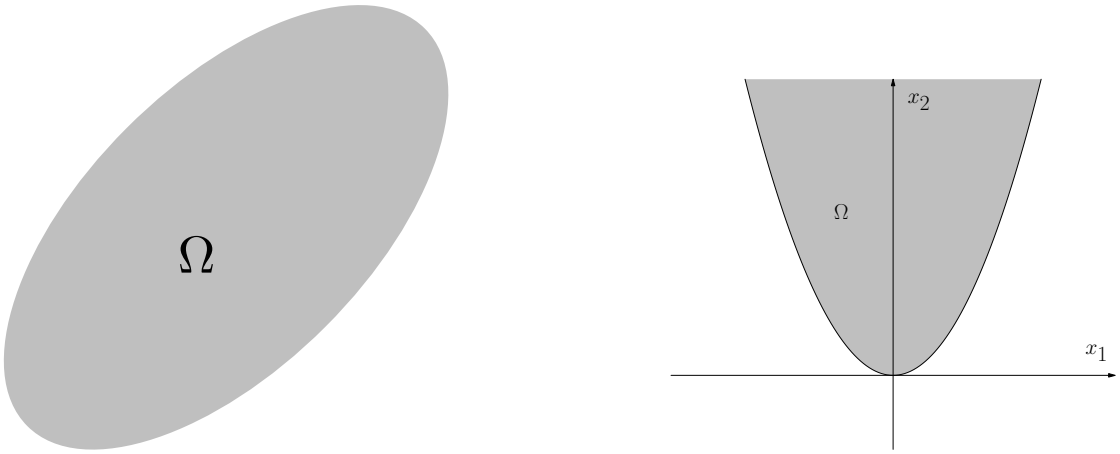


Fig. 2.4: Any convex set is uniformly prox-regular

2.6 Prox-regularity of Sets

Definition 2.37. (See [21]) For some $r > 0$, a nonempty closed set $\Omega \subset \mathcal{H}$ is called *r-prox-regular* if for all $x \in \Omega$, for all $t \in (0, r)$ and for all $\xi \in \mathcal{N}_\Omega^P(x)$ such that $\|\xi\| = 1$, one has $x \in \mathbb{P}_\Omega(x + t\xi)$. One says that Ω is *uniformly prox-regular* if it is r-prox-regular for some constant $r > 0$.

It is a simple matter to verify that every nonempty closed convex set is r-prox-regular for any $r > 0$.

Example 2.38. The complement of an open ball in $\mathcal{H}$ with radius ρ is r-prox-regular, where $r = \rho$. In particular, the complement of an open disk in $\mathbb{R}^2$ is uniformly prox-regular.

More examples of uniformly prox-regular sets will be given and discussed in Chapter 4. The interested reader is referred to [8, 21, 37] for other properties, as well as various characterizations, of uniformly prox-regular sets.

Example 2.39. The two-dimensional set Ω in the Fig. 2.6 is not uniformly prox-regular. Indeed, suppose to the contrary that there is $r > 0$ such that Ω is r-prox-regular. Note that the bisector x_0c divides the angle $\widehat{ax_0b}$ into two angles. The projection on Ω of any point in the angle $\widehat{ax_0c}$, which does not lie in the bisector x_0c, is a singleton consisting of a point belonging to the ray x_0a. Similarly, the projection on Ω of any point in the angle $\widehat{bx_0c}$, which does not lie in the bisector x_0c, is a singleton consisting of a point belonging to the ray x_0b. For any point $x \neq x_0$ in the ray x_0b, the proximal normal cone to Ω at x is the ray from x which is perpendicular to the half-line x_0b and points inward to the angle $\widehat{ax_0b}$. Denote

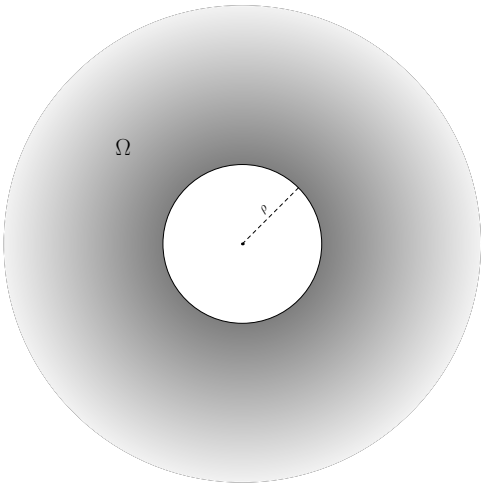


Fig. 2.5: Illustration for Example 2.38 in $\mathbb{R}^2$

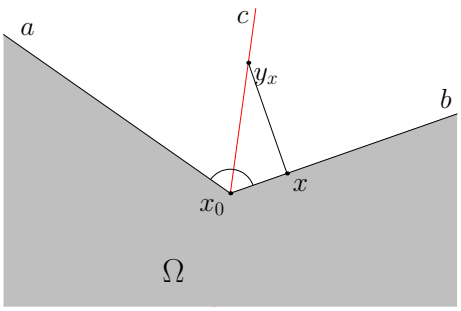
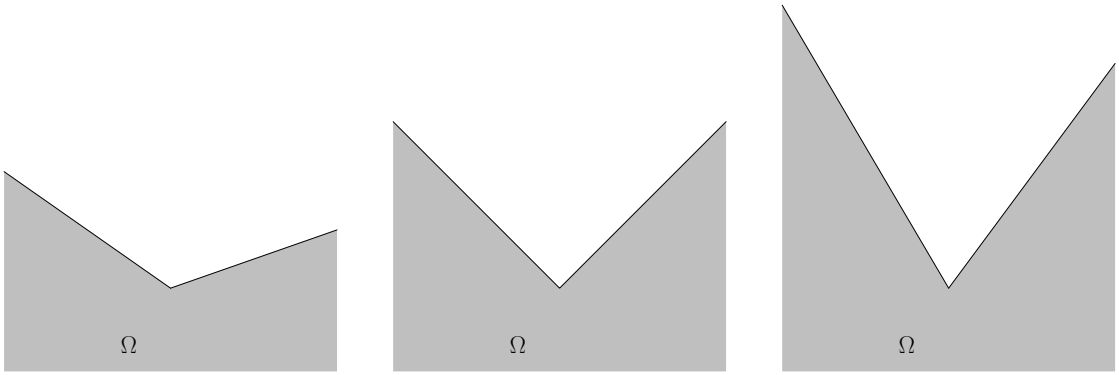


Fig. 2.6: Illustration for Example 2.39



(a) (b) (c)

Fig. 2.7: Examples of not uniformly prox-regular sets in $\mathbb{R}^2$

by y_x the intersection of that ray and the bisector x_0c of the angle $\widehat{ax_0b}$. We see that only the points in the segment $[y_x, x]$ on Ω have x as their projections and the length of the segment $[y_x, x]$ tends to 0 as x approaches x_0. Take any $x \neq x_0$ such that $\|y_x - x\| < \frac{r}{2}$. As the proximal normal cone to Ω at x is a ray, the proximal gradient $\xi \in \mathcal{N}_\Omega^P(x)$ with $\|\xi\| = 1$ is unique. For all $t \in \left(\frac{r}{2}, r\right)$, we have $\|x + t\xi - x\| > \frac{r}{2} > \|y_x - x\|$. So, $x + t\xi$ does not belong to the segment $[y_x, x]$. Therefore, $x \notin \mathbb{P}_\Omega(x + t\xi)$ for all $t \in \left(\frac{r}{2}, r\right)$, which is a contradiction. We have thus proved the above claim.

Arguing as above, we can prove that any set in $\mathbb{R}^2$ having a corner with an acute angle, a right angle, or an obtuse angle is not uniformly prox-regular (see Fig. 2.7).

Lemma 2.40. (See [37, Theorem 3, p. 108]) *Let Ω be a closed subset of $\mathcal{H}$ and $r > 0$. If Ω is r-prox-regular then, for any $x, x' \in \Omega$ and $v \in \mathcal{N}_\Omega^P(x)$, one has*

$$\langle v, x' - x \rangle \leq \frac{1}{2r}\|v\|\|x' - x\|^2.$$

Proposition 2.41. (See [37, Proposition 7, p. 117]) *If a nonempty closed set Ω is uniformly prox-regular, then $\mathcal{N}_\Omega^P(x) = \mathcal{N}_\Omega^{Cl}(x)$. In particular, if Ω is a nonempty closed convex set, then $\mathcal{N}_\Omega^P(x)$ coincides with the normal cone $\mathcal{N}_\Omega(x)$.*

Proposition 2.42. (See [21, Proposition 2.1, p. 313]) *Let Ω be a nonempty closed subset of $\mathcal{H}$ and $x \in \Omega$. The following assertions hold*

(i) $\partial^P d(x, \Omega) = \mathcal{N}_\Omega^P(x) \cap \mathbb{B}$;

(ii) *If Ω is r-prox-regular, then $\partial^P d(x, \Omega)$ is closed convex set in $\mathcal{H}$.*

Proposition 2.43. (See [21, Proposition 4.1, p. 366]) *Let $r > 0$. Assume that $\Omega(t)$ is r-prox-regular for all $t \in [0,T]$. Then, for a given $0 < \delta < r$ the following statement holds: For any $t \in [0,T]$, $x \in \Omega(t) + (r-\delta)\mathbb{B}$, $x_n \to x$, $t_n \to t$ with $t_n \in [0,T]$ and $v_n \in \partial^P d(x_n, \Omega(t_n))$ with $v_n \overset{w}{\rightharpoonup} v$, one has $v \in \partial^P d(x, \Omega(t))$. Here $\overset{w}{\rightharpoonup}$ stands for the weak convergence in $\mathcal{H}$.*

Chapter 3

Prox-Regular Perturbed Sweeping Processes

In this chapter, we study a class of perturbed sweeping processes in the setting adopted by Edmond and Thibault [48, 49]. This chapter has been the object of a publication in Journal of Nonlinear and Variational Analysis. Let $T > 0$ and $I = [0, T]$. Following Edmond and Thibault [48], we consider the next two assumptions.

Assumption (H1). *For each $t \in I$, $C(t)$ is a nonempty closed subset of $\mathcal{H}$ which is r-prox-regular for some constant $r > 0$.*

Assumption (H2). *$C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous function $v : I \to \mathbb{R}$ such that for any $y \in \mathcal{H}$ and $s, t \in I$, one has*

$$\|d(y, C(t)) - d(y, C(s))\| \leq |v(s) - v(t)|.$$

The following result, which is a simplified form of Theorem 5.1 from [49], provides us with an analogue of the Peano theorem [50, Theorem 2.1, p. 10] which works for ordinary differential equations.

Theorem 3.1. (See [49, Theorem 5.1]) *Assume that a family of sets $C(t)$, $t \in I$, in $\mathcal{H}$ satisfies the assumptions* (H1) *and* (H2). *Assume that $G : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a set-valued map with nonempty convex compact values such that*

(a) *For any $x \in \mathcal{H}$, $G(\cdot, x)$ has a measurable selection;*

(b) *For all $t \in I$, $G(t, \cdot)$ is scalarly upper semicontinuous on $\mathcal{H}$;*

(c) *For some compact subset* $K \subset \bar{\mathbb{B}}$ *and for some non-negative function* $\beta(\cdot) \in L^1(I, \mathbb{R})$*, one has for all* $(t, x) \in I \times \mathcal{H}$*,*

$$G(t, x) \subset \beta(t)(1 + \|x\|)K.$$

Assume also that $\mathcal{H}$ *is separable if* $G \not\equiv \{0\}$*. Then, for any* $x_0 \in C(0)$*, the sweeping process*

$$\begin{cases} -\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + G(t, x(t)) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{3.1}$$

has at least one absolutely continuous solution $x(\cdot)$*.*

The next result is an analogue of the Picard-Lindelöf theorem [50, Theorem 1.1, p. 8] from the theory of ordinary differential equations.

Theorem 3.2. (See [48, Theorem 1]) *Assume that a family of sets* $C(t)$*,* $t \in I$*, in* $\mathcal{H}$ *satisfies the assumptions* (H1) *and* (H2)*. Let* $g : I \times \mathcal{H} \to \mathcal{H}$ *be such a separately measurable map on* I *that*

(i) *For every* $\eta > 0$*, there exists a non-negative function* $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ *such that for all* $t \in I$ *and for any* $x, y \in \overline{\mathbb{B}}(0, \eta)$ *one has*

$$\|g(t, x) - g(t, y)\| \leq k_\eta(t)\|x - y\|;$$

(ii) *There exists a non-negative function* $\beta(\cdot) \in L^1(I, \mathbb{R})$ *such that, for all* $t \in I$ *and for all* $x \in \bigcup_{s \in I} C(s)$*, one has* $\|g(t, x)\| \leq \beta(t)(1 + \|x\|)$*.*

Then, for any $x_0 \in C(0)$*, the sweeping process* (1.2) *has one and only one absolutely continuous solution* $x(\cdot)$*. In addition, the solution satisfies the estimate*

$$\|\dot{x}(t) + g(t, x(t))\| \leq (1 + M_{x_0})\beta(t) + |\dot{v}(t)| \quad \text{a.e. } t \in I,$$

where

$$M_{x_0} := \|x_0\| + \exp\left\{2\int_0^T \beta(s)ds\right\} \int_0^T (2\beta(s)(1 + \|x_0\|) + |\dot{v}(s)|)ds.$$

When G is a single-valued mapping, Theorem 3.1 gives sufficient conditions for the existence of solution to problem (1.2). Meanwhile, Theorem 3.2 provides conditions for the existence and uniqueness of solution to problem (1.2). However,

the assumption (c) in Theorem 3.1 is tighter than the assumption (ii) in Theorem 3.2. To justify this fact, let us consider the following example.

Example 3.3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Consider the problem (1.2) with $C(t)$ satisfying the assumptions (H1) and (H2). Let $g : I \times \mathcal{H} \rightrightarrows \mathcal{H}$, $g(t,x) = t\mathbb{P}_{\bar{\mathbb{B}}}(x)$. We see that g is linear with respect to t. In addition, since the projection map onto a closed convex set in Hilbert space is Lipschitz continuous, g satisfies the assumptions (a), (b) of Theorem 3.1 and (i) of Theorem 3.2. Moreover, since $\|g(t,x)\| \leq t$ for all $t \in I$, the assumption (ii) of Theorem 3.2 is also valid. However, the unit ball $\bar{\mathbb{B}}$ in $\mathcal{H}$ is non-compact, so we cannot find any compact set K such that the assumption (c) of Theorem 3.1 holds. So, it is not possible to apply Theorem 3.1 in this case. Nevertheless, for any $x_0 \in C(0)$, Theorem 3.2 assures the solution existence and uniqueness of the problem under consideration.

Remark 3.4. Since the assumptions of the Peano theorem are weaker than those of the Picard-Lindelöf theorem, it would be nice if one can have another version of Theorem 3.1 whose assumption set is weaker than that of Theorem 3.2.

From a result of Edmond and Thibault [48, Proposition 2] it follows that, for every $t \in I$, the mapping $\psi_t : C(0) \to C(t)$ with $\psi_t(x_0) := x(x_0, t)$, where $x(x_0, \cdot)$ denotes the unique solution $x(\cdot)$ of (1.2) with the initial value $x(0) = x_0$, is Lipschitz on any bounded subset of $C(0)$.

3.1 Solution Existence Theorems

Let there be given the functions $f_i : I \times \mathcal{H} \to \mathbb{R}$, $i \in \{1, \ldots, m\}$. Suppose that the set

$$C(t) := \{x \in \mathcal{H} \mid f_i(t,x) \leq 0,\ i \in \{1, \ldots, m\}\}$$

is nonempty for each $t \in I$. Assume that there is an extended real number $\rho \in [0, +\infty]$ satisfying the next four assumptions.

Assumption (A1). *For $x \in \mathcal{H}$ and for all $i \in \{1, \ldots, m\}$, $f_i(\cdot, x)$ is Lipschitz continuous with modulus $L_1 > 0$ on $[0, T]$.*

Assumption (A2). *For each $t \in [0, T]$, and for all $i \in \{1, \ldots, m\}$, $f_i(t, \cdot)$ is locally Lipschitz continuous on $U_\rho(C(t))$.*

Assumption (A3). *There is $\gamma > 0$ such that for all $t \in [0, T]$ and $i \in \{1, \ldots, m\}$, for all $x_1, x_2 \in U_\rho(C(t))$, and for all $\xi_j \in \partial^C f_i(t, \cdot)(x_j)$, $j = 1, 2$,*

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq -\gamma \|x_1 - x_2\|^2.$$

Assumption (A4). *There is $\mu > 0$ with the property that for all $t \in [0,T]$ and $x \in C(t)$ one can find $\overline{v} = v(t,x) \in \mathcal{H}$ with $\|\overline{v}\| = 1$ such that for all $i \in \{1,\ldots,m\}$, for all $\xi \in \partial^C f_i(t,\cdot)(x)$, one has $\langle \xi, \overline{v}\rangle \leq -\mu$.*

Clearly, if $\partial^C f_1(t,\cdot)$ is *monotone* for every $t \in [0,T]$, i.e., $\langle \xi_1 - \xi_2, x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in \mathcal{H}$ and for all $\xi_j \in \partial^C f_i(t,\cdot)(x_j)$, $j = 1,2$, then Assumption (A3) is satisfied with any $\gamma > 0$.

Lemma 3.5. (See [8, Theorem 4.1]) *For all $t \in [0,T]$, the set $C(t)$ is r-prox-regular with $r = \min\{\rho, \frac{\mu}{\gamma}\}$.*

Lemma 3.6. *The set-valued map $C : I \rightrightarrows \mathcal{H}$ is Lipschitz with respect to the Hausdorff distance, with the Lipschitz modulus ϑ, for any $\vartheta \geq \dfrac{L_1}{\mu}$.*

Proof. Fix a real number ϑ such that $\vartheta \geq \mu^{-1}L_1$. Choose a subdivision

$$T_0 = 0 < T_1 < \ldots < T_p = T$$

of $[0,T]$ such that $T_k - T_{k-1} < \vartheta^{-1}\rho$ for $k = 1,\ldots,p$. Fix an index $k \in \{1,\ldots,p\}$ and select any numbers s,t from the segment $I_k := [T_{k-1}, T_k]$. Put $u(s,t) = \vartheta|s-t|$. For any $x \in C(t)$, define $y = x + u(s,t)\overline{v}$. Since $t,s \in I_k$, we have $\|y - x\| = \vartheta|s-t| < \rho$. This proves that $y \in \mathrm{int}(U_\rho(C(t)))$. By [8, Lemma 3.2], for all $\lambda \in [0,1]$ we have

$$x + \lambda(y - x) \in \mathrm{int}(U_\rho(C(t))).$$

Take any $i \in \{1,\ldots,m\}$. By Assumption (A2) and Lebourg's mean value theorem (see, e.g., [33, Theorem 2.3.7, p. 41]) there exists $\lambda \in (0,1)$ such that

$$f_i(t,y) - f_i(t,x) \in \langle \partial_2^C f_i(t, x(\lambda)), u(s,t)\overline{v}\rangle$$

with $x(\lambda) := (1-\lambda)x + \lambda y$. Hence, by Assumptions (A1) and (A4) we have

$$\begin{aligned} f_i(s,y) &= [f_i(s,y) - f_i(t,y)] + f_i(t,x) + [f_i(t,y) - f_i(t,x)] \\ &\leq L_1|s-t| - u(s,t)\mu \\ &= (L_1 - \vartheta\mu)\,|s-t|. \end{aligned}$$

Hence, $f_i(s,y) \leq 0$. Since $i \in \{1,\ldots,m\}$ can be chosen arbitrarily, we have thus shown that the vector $y = x + \vartheta|s-t|\overline{v}$ belongs to $C(s)$. So, $d(x, C(s)) \leq \vartheta|s-t|$ for every $x \in C(t)$. By symmetry, we get $d(x', C(t)) \leq \vartheta|s-t|$ for every $x' \in C(s)$. Consequently, we obtain $d_H(C(t), C(s)) \leq \vartheta|t-s|$.

The proof is complete. □

Theorem 3.7. *Suppose that Assumptions* (A1)–(A4) *are fulfilled. Let* $g : I \times \mathcal{H} \to \mathcal{H}$ *satisfy the three requirements* (a), (b) *and* (c) *in Theorem 3.1. Then, for any* $x_0 \in C(0)$*, the sweeping process*

$$\begin{cases} -\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t)) \quad \text{a.e. } t \in I \\ x(0) = x_0 \end{cases} \tag{3.2}$$

has at least one absolutely continuous solution $x(\cdot)$*.*

Proof. By Lemma 3.5, the set $C(t)$ is r-prox-regular for all $t \in [0, T]$. Moreover, Lemma 3.6 states that

$$d_H(C(t), C(s)) \leq \vartheta |t - s|.$$

For all $y \in \mathcal{H}$, we have that $\|d(y, C(t)) - d(y, C(s))\| \leq d_H(C(t), C(s))$. It follows that $C(t)$ varies in an absolutely continuous way, i.e.,

$$\|d(y, C(t)) - d(y, C(s))\| \leq |v(s) - v(t)|,$$

where $v : I \to \mathbb{R}$, $v(z) = \vartheta z$. By Theorem 3.1, we obtain the desired result. □

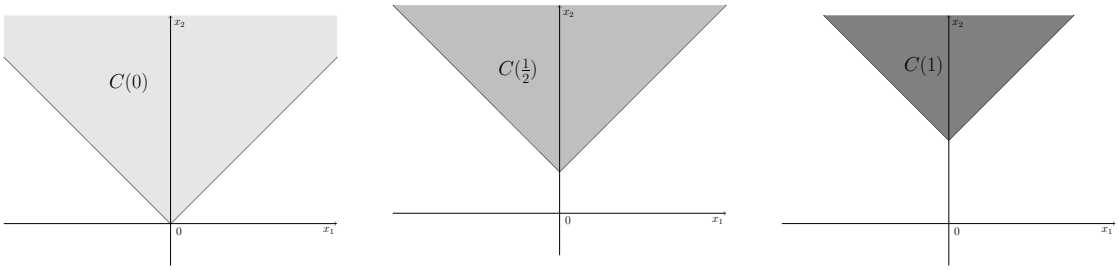
Theorem 3.8. *Suppose that Assumptions* (A1)–(A4) *are fulfilled. Let* $g : I \times \mathcal{H} \to \mathcal{H}$ *be such a separately measurable map on* I *that satisfies the two requirements* (i) *and* (ii) *in Theorem 3.2. Then, for any* $x_0 \in C(0)$*, the sweeping process* (3.2) *has a unique absolutely continuous solution* $x(\cdot)$*.*

Proof. Using Theorem 3.2 instead of Theorem 3.1 and arguing similarly as in the proof of Theorem 3.7, one can obtain the desired result. □

Remark 3.9. The assumptions (A1)–(A4) on the functions f_i, $i \in \{1, \ldots, m\}$, and the family of sets $C(t)$, $t \in I$, do not depend on the choice of x_0 from $C(t)$. Clearly, the requirements (i) and (ii) on $g(t, x)$ in the formulation of Theorem 3.8 also do not depend on the choice of x_0 from $C(t)$.

3.2 Some Illustrative Examples

To illustrate the applicability of Theorem 3.8, we shall provide two examples in dimension 2.



(a) $C(0)$ (b) $C\left(\frac{1}{2}\right)$ (c) $C(1)$

Fig. 3.1: Examples of constraint sets in Example 3.10

Example 3.10. Consider the problem (3.2) with $\mathcal{H} = \mathbb{R}^2$, $m = 1$, $f_1(t, x) = t - x_2 + |x_1|$, and $g(t, x) = 0$ for all $t \in [0, T]$, $x = (x_1, x_2) \in \mathbb{R}^2$. Here, we have

$$C(t) = \{x \in \mathbb{R}^2 \mid -x_2 + |x_1| \leq -t\}. \tag{3.3}$$

Let the initial condition be $x(0) = (0, 0)$. Obviously, $x(0) \in C(0)$ and $f = f_1$ satisfies Assumptions (A1) and (A2). We have

$$\partial^C f_1(t, \cdot)(x) = \begin{cases} \{(1, -1)\} & \text{if } x_1 > 0 \\ [-1, 1] \times \{-1\} & \text{if } x_1 = 0 \\ \{(-1, -1)\} & \text{if } x_1 < 0. \end{cases} \tag{3.4}$$

Since $f_1(t, \cdot)$ is convex, $\partial^C f_1(t, \cdot)$ coincides with the convex subdifferential mapping of $\partial f_1(t, \cdot)$, which is monotone. Hence, for any $t \in [0, T]$, the mapping $\partial^C f_1(t, \cdot)$ is hypermonotone with any $\gamma > 0$. Thus, Assumption (A3) is satisfied. Now, to check Assumption (A4), let us fix any $\mu \in (0, 1]$. Suppose that $t \in [0, T]$ and $x \in C(t)$ are given arbitrarily. For $\overline{v} := (0, 1)$, one has $\langle \xi, \overline{v} \rangle = \xi_2$, where $\xi = (\xi_1, \xi_2) \in \partial^C f_1(t, \cdot)(x)$ can be chosen arbitrarily. Thanks to (3.4), we have $\xi_2 = -1$. Hence,

$$\langle \xi, \overline{v} \rangle = -1 \leq -\mu.$$

We have thus showed that Assumption (A4) is satisfied. Since $g(t, x) \equiv 0$, the requirements (i) and (ii) on g are fulfilled. So, according to Theorem 3.8, (3.2) has a unique absolutely continuous solution $x(\cdot)$. Interestingly, we can give an explicit formula for $x(\cdot)$. Namely, let us show that

$$x_1(t) = 0, \; x_2(t) = t \quad \forall t \in [0, T]. \tag{3.5}$$

Clearly, the trajectory $x(t)$ given by (3.5) satisfies the conditions

$$x(0) = (0, 0) \quad \text{and} \quad -\dot{x}(t) = (0, -1).$$

Since $C(t)$ is convex, the Clarke normal cone to $C(t)$ at any point of $C(t)$ coincides with the normal cone to $C(t)$ at that point in the sense of convex analysis (see [33, Proposition 2.4.4]). So, applying [53, Proposition 2, p. 206] to the set $C(t)$ in (3.3), which is a sublevel set of the continuous convex function $f_1(t, \cdot)$, at the boundary $x(t) = (x_1(t), x_2(t))$, one obtains $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \mathbb{R}_+ \partial f_1(t, \cdot)(x(t))$. Since $x_1(t) \equiv 0$, combining this with (3.4) gives

$$\mathcal{N}^{Cl}_{C(t)}(x(t)) = \mathbb{R}_+([-1, 1] \times \{-1\}).$$

So, $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t))$ for all $t \in [0, T]$. Hence, formula (3.5) describes the unique absolutely continuous solution of the problem in question. The above mathematical model and the solution have the following clear mechanical meanings. In the horizontal coordinate plane $\mathbb{R}^2$, there is a small metal ball standing at the origin of the plane at time $t = 0$. The boundary of $C(0)$ is the union of two orthogonal half-lines. Suppose that the boundary is the frame made from two long sticks of bamboo or wood which are firm enough that they cannot be bend by the metal ball. The set $C(t)$ in (3.3) is the position of $C(0)$ at the time t. The requirement saying that the ball must be inside $C(t)$ at any time t means that it must be in the plane area formed by the frame. The change of $C(t)$ with respect to t corresponds to the movement of the frame along the x_2-axis with the velocity 1. The assumption $g(t, x) \equiv 0$ means that there is no external force acting on the ball. The formula (3.5) of the obtained solution means that the ball always lies in the corner of the frame, when the later moves steadily along the x_2-axis.

Concerning the sweeping problem in Example 3.10, we observe that the role of the normal cone operator $\mathcal{N}^{Cl}_{C(t)}(x(t))$ in the inclusion

$$-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t))$$

is important. Namely, note that the last inclusion implies $x(t) \in C(t)$. Note also that $0 \in \mathcal{N}^{Cl}_{C(t)}(x(t))$ if $x(t) \in C(t)$. So, together with (3.2), it is naturally to

consider the following tighter problem:

$$\begin{cases} -\dot{x}(t) = g(t, x(t)) \quad \text{a.e. } t \in I \\ x(t) \in C(t) \quad \text{for } t \in I \\ x(0) = 0 \end{cases}$$

Since $g(t, x) \equiv 0$, the first and the third conditions of this system imply that $x(t) = 0$ for all $t \in I$. However, for this curve $x(t)$, the second condition of the system is violated. So, *the assertion of Theorem 3.8 may fail to hold if one replaces the inclusion* $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t))$ *by the conditions* $-\dot{x}(t) = g(t, x(t))$ *and* $x(t) \in C(t)$.

Example 3.11. Consider problem (3.2) with the data given in Example 3.10, where the initial point is $x(0) = x_0$ with $x_0 = (x_1^0, x_2^0)$ being an arbitrary point from $C(0)$. The analysis in Example 3.10 shows that the assumptions (A1)–(A4) and the requirements (i) and (ii) on $g(t, x)$ in the formulation of Theorem 3.8 are satisfied. Hence, by Remark 3.9 and Theorem 3.8, the sweeping process (3.2) has a unique absolutely continuous solution $x(\cdot)$. To have an explicit formula for this solution $x(\cdot)$, we first suppose that x_0 belongs to the interior of $C(0)$. This means that $|x_1^0| < x_2^0$. Put $\bar{t}_{x_0} = x_2^0 - |x_1^0|$ and note that $\bar{t}_{x_0} > 0$.

Case 1: $T \leq \bar{t}_{x_0}$. In this case, since $f_1(t, x_0) = t - x_2^0 + |x_1^0| = t - \bar{t}_{x_0} < 0$ for all $t \in [0, T)$, one has $x_0 \in \text{int}(C(t))$ for all $t \in [0, T)$. So, setting $x(t) = x_0$ for $t \in I$, we obtain

$$\mathcal{N}^{Cl}_{C(t)}(x(t)) = \{(0, 0)\}$$

for all $t \in [0, T)$. Therefore, (3.2) is satisfied. Since the solution is unique by Theorem 3.8, the just defined constant trajectory is the unique absolutely continuous solution of the sweeping process under our consideration.

Case 2: $\bar{t}_{x_0} < T$. First, consider the subcase where $\bar{t}_{x_0} \leq 2|x_1^0| + \bar{t}_{x_0} < T$. Let us prove that the unique solution $x(\cdot)$ can be given by the formula

$$x(t) = \begin{cases} x_0 & \text{if } t \in [0, \bar{t}_{x_0}) \\ (x_1^0 - \text{sign}(x_1^0)\frac{t - \bar{t}_{x_0}}{2}, x_2^0 + \frac{t - \bar{t}_{x_0}}{2}) & \text{if } t \in [\bar{t}_{x_0}, 2|x_1^0| + \bar{t}_{x_0}) \\ (0, t) & \text{if } t \in [2|x_1^0| + \bar{t}_{x_0}, T]. \end{cases} \tag{3.6}$$

Note that the function $x(\cdot)$ is absolutely continuous on $[0, T]$ and $x(0) = x_0$. Arguing as in Case 1, we obtain $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t))$ for every $t \in [0, \bar{t}_{x_0})$. For $t \in [\bar{t}_{x_0}, 2|x_1^0| + \bar{t}_{x_0})$, if $x_1^0 \leq 0$ then $t < -2x_1^0 + \bar{t}_{x_0}$. Hence, $x_1(t) = x_1^0 + \frac{t - \bar{t}_{x_0}}{2} < 0$.

Combining this with (3.6) yields

$$f_1(t,x(t)) = t - x_2(t) + |x_1(t)| = t - \left(x_2^0 + \frac{t-\bar{t}_{x_0}}{2}\right) - \left(x_1^0 + \frac{t-\bar{t}_{x_0}}{2}\right) = 0.$$

This means that $x(t) \in \partial C(t)$, $x_1(t) < 0$; so $\partial^C f_1(t,x(t)) = \{(-1,-1)\}$. Thanks to the continuity and convexity of the function $f_1(t,\cdot)$, applying [33, Proposition 2.4.4] we have $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \mathbb{R}_+\{(-1,-1)\}$. It follows that $\dot{x}(t) = (\frac{1}{2},\frac{1}{2}) \in -\mathcal{N}^{Cl}_{C(t)}(x(t))$ for all $t \in (\bar{t}_{x_0}, -2x_1^0 + \bar{t}_{x_0})$. The situation $x_1^0 > 0$ can be treated similarly. Therefore, $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t,x(t))$ for $t \in (\bar{t}_{x_0}, 2|x_1^0| + \bar{t}_{x_0})$. Now, for $t \in [2|x_1^0| + \bar{t}_{x_0}, T]$, one has $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t,x(t))$ by (3.6) and the result given in Example 3.10. Therefore, (3.6) describes the unique absolutely continuous solution $x(\cdot)$ of the problem in question. In the situation where $\bar{t}_{x_0} < T \leq 2|x_1^0| + \bar{t}_{x_0}$, arguing analogously as before, we can show that the formula

$$x(t) = \begin{cases} x_0 & \text{if } t \in [0, \bar{t}_{x_0}) \\ (x_1^0 - \text{sign}(x_1^0)\frac{t-\bar{t}_{x_0}}{2}, x_2^0 + \frac{t-\bar{t}_{x_0}}{2}) & \text{if } t \in [\bar{t}_{x_0}, T] \end{cases} \tag{3.7}$$

describes the unique absolutely continuous solution $x(\cdot)$ of our problem.

Now, suppose that $x(0) \in \partial C(0)$. This means that $x_2^0 - |x_1^0| = 0$. This situation reduces to Case 2 above with $\bar{t}_{x_0} := x_2^0 - |x_1^0| = 0$. So, the unique absolutely continuous solution $x(\cdot)$ of our problem is given by

$$x(t) = \begin{cases} (x_1^0 - \text{sign}(x_1^0)\frac{t}{2}, x_2^0 + \frac{t}{2}) & \text{if } t \in [0, 2|x_1^0|) \\ (0,t) & \text{if } t \in [2|x_1^0|, T] \end{cases} \tag{3.8}$$

whenever $2|x_1^0| < T$, and

$$x(t) = \left(x_1^0 - \text{sign}(x_1^0)\frac{t}{2}, x_2^0 + \frac{t}{2}\right) \quad \text{for } t \in [0,T] \tag{3.9}$$

whenever $2|x_1^0| \geq T$. As in the preceding example, the problem here and the obtained solution can be interpreted respectively as a mechanical problem and a mechanical motion as follows. Suppose that, at time $t = 0$, there is a small metal ball standing at the point $x_0 \in C(0)$ in the horizontal plane $\mathbb{R}^2$. When the set $C(0)$ moves along the x_2-axis with the velocity 1 (see (3.3)), its boundary - a firm frame consisting of two orthogonal half-lines - also moves along the x_2-axis with the velocity 1. The ball cannot overpass the frame. If $x_0 \in \text{int}(C(0))$, $x_1(0) \neq 0$, and $2|x_1^0| + \bar{t}_{x_0} < T$ with $\bar{t}_{x_0} := x_2^0 - |x_1^0|$, then (3.6) shows that the motion of the

ball in the time segment $[0,T]$ and has three phases: (a) Until the time instant $\bar{t}_{x_0}$, the ball stays still; (b) In the time interval $[\bar{t}_{x_0}, 2|x_1^0|+\bar{t}_{x_0})$, the ball goes steadily along one wing of the boundary of $C(0)$ with the speed $\frac{\sqrt{2}}{2}$ (the ball is on the left wing if $x_1(0) < 0$ and it is on the right wing if $x_1(0) > 0$); (c) In the time interval $[2|x_1^0|+\bar{t}_{x_0}, T]$, the ball always lies in the corner of the above-mentioned frame. Similar interpretations can be given for formulas (3.7)–(3.9).

Let the horizontal plane $\mathbb{R}^2$ in the preceding example be replaced by a *vertical* plane $\mathbb{R}^2$, where the x_2-axis is orthogonal to the earth surface and pointing up. Then, the set $C(t)$ given by (3.3) can be interpreted as the position of the set

$$C(0) = \{x \in \mathbb{R}^2 \mid -x_2 + |x_1| \leq 0\}$$

at time t. In other words, in accordance with formula (3.3), the set $C(0)$ is moving up along the x_2-axis with the velocity 1. As before, the boundary of $C(0)$ - a firm frame - also moves along the x_2-axis with the velocity 1. Note that the metal ball in question cannot overpass the frame. Since the ball has the tendency to go down straightly with the acceleration $g_0 = 9.8$, the velocity of its free fall is $-g_0 t$. So the equation of motion of the ball should be $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t))$ for almost everywhere $t \in I$, where $g(t,x) := (0, g_0 t)$. The solution of this mechanical problem is given below.

Example 3.12. Consider problem (3.2) with the data given in Example 3.10 except for $g(t,x) = (0, g_0 t)$, where $g_0 = 9.8$ is the gravitational acceleration. Let the initial condition be $x(0) = (x_1^0, x_2^0)$. As we know from the two examples above, the assumption (A1)–(A4) hold true. Since $g(t,x)$ is independent of the second variable, it is clear that the requirement (i) in Theorem 3.8 is satisfied. In addition, as $g(t,x)$ is a linear function of t, the requirement (ii) in the theorem is satisfied with the choice $\beta(t) = g_0 t$. Hence, by Remark 3.9 and Theorem 3.2, the sweeping process (3.2) has a unique absolutely continuous solution $x(\cdot)$. To provide an explicit formula for this solution $x(\cdot)$, we first consider the situation where $x_0 \in \text{int}(C(0))$. Putting $\bar{t}_{x_0} = x_2^0 - |x_1^0|$, one has $\bar{t}_{x_0} > 0$. Define $\theta^1_{x_0} = \frac{-1+\sqrt{1+2g_0\bar{t}_{x_0}}}{g_0}$ and $\theta^2_{x_0} = \frac{-1+\sqrt{1+2g_0(\bar{t}_{x_0}+2|x_1^0|)}}{g_0}$. It is clear that $0 < \theta^1_{x_0} \leq \theta^2_{x_0}$.

Case 1: $T \leq \theta^1_{x_0}$. Setting

$$x(t) = (x_1^0, x_2^0 - \frac{g_0 t^2}{2}) \quad (\forall t \in I), \tag{3.10}$$

we have $f_1(t, x(t)) = t - x_2^0 + \frac{g_0 t^2}{2} + |x_1^0| < 0$, for any $t \in [0, T)$. Hence, $x(t) \in$

$\text{int}(C(t))$ for all $t \in [0,T)$. So, $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \{(0,0)\}$ for all $t \in [0,T)$. Since $-\dot{x}(t) = (0, g_0 t)$, it follows that the inclusion in (3.2) is satisfied for all $t \in [0,T)$. Therefore, Theorem 3.8 assures that the chosen trajectory is the unique absolutely continuous solution of (3.2).

Case 2: $\theta^1_{x_0} < T$. If $\theta^1_{x_0} \leq \theta^2_{x_0} < T$. then the explicit formula for the solution $x(\cdot)$ is

$$x(t) = \begin{cases} (x^0_1, x^0_2 - \frac{g_0 t^2}{2}) & \text{if } t \in [0, \theta^1_{x_0}) \\ (x^0_1 - \text{sign}(x^0_1)\left(\frac{t-\bar{t}_{x_0}}{2} + \frac{g_0 t^2}{4}\right), x^0_2 + \frac{t-\bar{t}_{x_0}}{2} - \frac{g_0 t^2}{4}) & \text{if } t \in [\theta^1_{x_0}, \theta^2_{x_0}) \\ (0,t) & \text{if } t \in [\theta^2_{x_0}, T]. \end{cases} \quad (3.11)$$

Indeed, the function $x(\cdot)$ is an absolutely continuous on $[0,T]$, $x(0) = x_0$, and a direct verification shows that $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + (0, g_0 t)$ for $t \in [0, \theta^1_{x_0})$. Now, suppose that $x^0_1 \leq 0$. Then we have $x_1(t) = x^0_1 + \frac{t-\bar{t}_{x_0}}{2} + \frac{g_0 t^2}{4} < 0$ for $t \in [\theta^1_{x_0}, \theta^2_{x_0})$. So, for $t \in [\theta^1_{x_0}, \theta^2_{x_0})$, one has

$$f_1(t, x(t)) = t - x^0_2 - \frac{t - \bar{t}_{x_0}}{2} + \frac{g_0 t^2}{4} - x^0_1 - \frac{t - \bar{t}_{x_0}}{2} - \frac{g_0 t^2}{4} = 0.$$

Hence, $x(t) \in \partial C(t)$. Since $x_1(t) < 0$, this implies that $\partial^C f_1(t, x(t)) = \{-1, -1\}$ for every $t \in [\theta^1_{x_0}, \theta^2_{x_0})$. Thanks to the continuity and convexity of $f_1(t, \cdot)$, applying [33, Proposition 2.4.4], we obtain $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \mathbb{R}_+\{(-1,-1)\}$ for $t \in [\theta^1_{x_0}, \theta^2_{x_0})$. Since

$$\dot{x}(t) = \left(\frac{1 + g_0 t}{2}, \frac{1 + g_0 t}{2} - g_0 t\right),$$

one has $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + (0, g_0 t)$ for $t \in (\theta^1_{x_0}, \theta^2_{x_0})$. Thus, for every $t \in (\theta^1_{x_0}, \theta^2_{x_0})$, the inclusion $-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t,x)$ holds. For $t \in (\theta^2_{x_0}, T)$, it is clear that $f_1(t, x(t)) = 0$ and $\dot{x}(t) = (0,1)$. Since $\mathcal{N}^{Cl}_{C(t)}(x(t)) = \mathbb{R}_+([-1,1] \times \{-1\})$, the inclusion

$$-\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t,x)$$

holds for $t \in (\theta^2_{x_0}, T)$. Therefore, the function $x(\cdot)$ given in (3.11) describes the unique absolutely continuous solution of the problem under consideration. The situation $x^0_1 > 0$ can be treated similarly. If $T \leq \theta^2_{x_0}$, arguing analogously, we can

prove that the formula

$$x(t) = \begin{cases} (x_1^0, x_2^0 - \frac{g_0 t^2}{2}) & \text{if } t \in [0, \theta^1_{x_0}) \\ (x_1^0 - \text{sign}(x_1^0)\left(\frac{t-\bar{t}_{x_0}}{2} + \frac{g_0 t^2}{4}\right), x_2^0 + \frac{t-\bar{t}_{x_0}}{2} - \frac{g_0 t^2}{4}) & \text{if } t \in [\theta^1_{x_0}, T] \end{cases} \tag{3.12}$$

describes the unique solution $x(\cdot)$.

Now, suppose that $x(0) \in \partial C(0)$. It is not difficult to show that the unique absolute solution $x(\cdot)$ is described as

$$x(t) = \begin{cases} (x_1^0 - \text{sign}(x_1^0)\left(\frac{t}{2} + \frac{g_0 t^2}{4}\right), x_2^0 + \frac{t}{2} - \frac{g_0 t^2}{4}) & \text{if } t \in [0, \theta^2_{x_0}), \\ (0, t) & \text{if } t \in [\theta^2_{x_0}, T]. \end{cases} \tag{3.13}$$

if $\theta^2_{x_0} < T$, and by the formula

$$x(t) = \left(x_1^0 - \text{sign}(x_1^0)\left(\frac{t}{2} + \frac{g_0 t^2}{4}\right), x_2^0 + \frac{t}{2} - \frac{g_0 t^2}{4}\right) \quad \text{for } t \in [0, T]. \tag{3.14}$$

if $\theta^2_{x_0} \geq T$. The mechanical meanings of the motion modes (3.10)–(3.14) of the metal ball are similar to those explained in Example 3.11.

Remark 3.13. By $\mathcal{R}_T$ we denote the set of end points of the sweeping process (1.2), i.e., the set of all $x(T)$ with $x(\cdot)$ being the unique solution of (3.2) where $x_0 \in C(0)$ is chosen arbitrarily. It is an interesting question that under which conditions on $C(t)$, $t \in [0, T]$, we have $\mathcal{R}_T = C(T)$. The following example shows that even when $C(t)$ is just a linear translation of $C(0)$, we get a negative answer. The system

$$\begin{cases} -\dot{x}(t) \in \mathcal{N}^{Cl}_{C(t)}(x(t)) + g(t, x(t)) \quad \text{a.e. } t \in I, \\ x(T) = x_1 \end{cases} \tag{3.15}$$

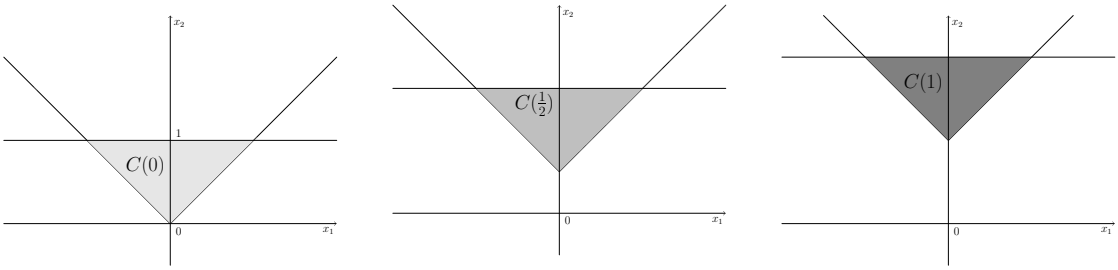
will be used in our analysis.

Example 3.14. Consider problem (3.2) with $\mathcal{H} = \mathbb{R}^2$, $m = 2$,

$$f_1(t, x) = t - x_2 + |x_1|, \quad f_2(t, x) = x_2 - t - 1,$$

and $g(t, x) = 0$ for all $t \in [0, T]$, where $T = 3$, and $x = (x_1, x_2) \in \mathbb{R}^2$. Here, we have

$$C(t) = \{x \in \mathbb{R}^2 \mid -x_2 + |x_1| \leq -t, \ x_2 \leq t + 1\}. \tag{3.16}$$



(a) $C(0)$ (b) $C\left(\frac{1}{2}\right)$ (c) $C(1)$

Fig. 3.2: Examples of constraint sets in Example 3.14

Let the terminal condition be $x(T) = (x_1^1, x_2^1)$. If (3.15) has a solution $x(\cdot)$, then one has $x(0) = x_0$ for some $x_0 = (x_1^0, x_2^0) \in C(0)$. Since the assumptions (A1)–(A4) and the requirements (i) and (ii) on $g(t, x)$ in the formulation of Theorem 3.8 are satisfied, by Remark 3.9 and Theorem 3.8, the sweeping process (3.2) with the chosen x_0 has a unique absolutely continuous solution. Using the formula of $C(t)$ in (3.16), one can easily show that $|x_1^0| \leq 1$. For $\bar{t}_{x_0} := x_2^0 - |x_1^0|$, we have $2|x_1^0| + \bar{t}_{x_0} \leq 2 < T$. Arguing similarly to Example 3.11, we can show that the unique absolutely continuous solution $x(\cdot)$ of (3.2) is given by (3.6) if $x_2^0 = 1$ or if $x_0 \in \text{int}(C(0))$, and by (3.8) if $x_2^0 < 1$ and $x_0 \in \partial C(0)$. In both cases, we have $x(T) = (0, 3)$. So, the following assertions are valid: (i) If $x_1 \neq (0, 3)$, then problem (3.15) has no solution; (ii) If $x_1 = (0, 3)$, then (3.15) have infinite number of solutions; (iii) For any $x_0 \in C(0)$, the unique solution $x(\cdot)$ of (1.2) ends at the point $x(T) = (0, 3)$.

However, if the sweeping process (3.15) in Remark 3.13 is subjected to multi-valued perturbations $g(t, x(t))$, then the above question can be considered as a controllability problem, for which we expect to have a positive solution.

3.3 Conclusions

In this chapter, the solution existence as well as the solution uniqueness for perturbed sweeping processes has been studied under the assumption of the prox-regularity of the constraint sets.

If the perturbation function $g(t, x)$ is multi-valued, then we have deal with multi-valued perturbed sweeping processes in the prox-regular case. For these problems, it is of interest to establish some results on the solution existence, continuous dependence of the solutions, and the reachability of sweeping processes

similar to the ones given in this chapter.

The question of the relaxation of the assumptions of Theorem 3.7 remains open, i.e., in a way that the solution existence of the problem (3.2) is still guaranteed, or not.

Chapter 4

Nonconvex Sweeping Processes with Velocity Constraints

Following Adly, Haddad, and Thibault [7], in this chapter, we will study some classes of sweeping processes with velocity in a moving set. Our main tool is a theorem on the solution sensitivity of parametric variational inequalities. Beside the traditional requirement that the constraint set moves continuously in the Hausdorff distance sense, we intensively use a new assumption on the local Lipschitz-likeness of the constraint set-valued mapping. The obtained results are compared with the existing ones and analyzed by several examples.

Let $\mathcal{H}$ be a Hilbert space and $C : [0,T] \rightrightarrows \mathcal{H}$ be a set-valued mapping. Let $A_0, A_1 : \mathcal{H} \to \mathcal{H}$ be bounded symmetric linear operators and $f : [0,T] \to \mathcal{H}$ be a continuous mapping. Recall that a linear operator $A : \mathcal{H} \to \mathcal{H}$ is said to be symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$. Following [6, 7], we consider the sweeping process

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{P}$$

Firstly, let us discuss some characters of proximal normal cones.

Remark 4.1. Let $x \in \Omega \subset \mathcal{H}$ and $\xi \in \mathcal{N}^P_\Omega(x) \setminus \{0\}$. If α is a positive number such that $x \in \mathbb{P}_\Omega(x + \alpha\xi)$, then $x \in \mathbb{P}_\Omega(x + t\xi)$ for every $t \in (0, \alpha)$. Indeed, by our assumption,

$$d\,(x + \alpha\xi, \Omega) = \|(x + \alpha\xi) - x\| = \alpha\|\xi\|. \tag{4.1}$$

If $x \notin \mathbb{P}_\Omega(x + t\xi)$ for a value $t \in (0, \alpha)$, then the inequality $\|(x + t\xi) - y\| <$

$\|(x+t\xi)-x\|$ holds for some $y \in \Omega$. Therefore,

$$\begin{aligned}\|(x+\alpha\xi)-y\| &\leq \|(x+\alpha\xi)-(x+t\xi)\|+\|(x+t\xi)-y\| \\ &< (\alpha-t)\|\xi\|+\|(x+t\xi)-x\| \\ &= \alpha\|\xi\|.\end{aligned}$$

So, by (4.1) one gets $\|(x+\alpha\xi)-y\| < d(x+\alpha\xi,\Omega)$, which is impossible because $y \in \Omega$.

Remark 4.2. Proximal normal cone is a local structure. Namely, for any $x \in \Omega \subset \mathcal{H}$ and $\rho > 0$, the proximal normal cone to $\Omega \subset \mathcal{H}$ at x coincides with the proximal normal cone to $\Omega \cap \bar{\mathbb{B}}(x,\rho)$ at x, i.e.,

$$\mathcal{N}_{\Omega}^{P}(x) = \mathcal{N}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}^{P}(x). \tag{4.2}$$

Note that both cones in (4.2) contain the element $\xi = 0$. Take any $\xi \in \mathcal{N}_{\Omega}^{P}(x)\backslash\{0\}$. Let $\alpha > 0$ be such that $x \in \mathbb{P}_{\Omega}(x+\alpha\xi)$. Hence, $\|(x+\alpha\xi)-y\| \geq \|(x+\alpha\xi)-x\|$ for all $y \in \Omega$. In particular, the last inequality still holds for all $y \in \Omega\cap\bar{\mathbb{B}}(x,\rho)$. Thus, we have $x \in \mathbb{P}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}(x+\alpha\xi)$, which implies that $\xi \in \mathcal{N}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}^{P}(x)$. So, the inclusion $\mathcal{N}_{\Omega}^{P}(x) \subset \mathcal{N}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}^{P}(x)$ has been proved. Now, let $\xi \in \mathcal{N}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}^{P}(x) \setminus \{0\}$ be given arbitrarily. Let $\alpha > 0$ be such that $x \in \mathbb{P}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}(x+\alpha\xi)$. So, by Remark 4.1, we have $x \in \mathbb{P}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}(x+t\xi)$ for every $t \in (0,\alpha)$. This means that

$$\|(x+t\xi)-y\| \geq \|(x+t\xi)-x\| \quad \big(\forall t \in (0,\alpha),\ \forall y \in \Omega\cap\bar{\mathbb{B}}(x,\rho)\big). \tag{4.3}$$

To prove the inclusion $\xi \in \mathcal{N}_{\Omega}^{P}(x)$ by contradiction, suppose that $\xi \notin \mathcal{N}_{\Omega}^{P}(x)$. Then, by Definition 2.30, $x \notin \mathbb{P}_{\Omega}(x+\beta\xi)$ for every $\beta \in (0,\alpha)$. So, there exists $y_\beta \in \Omega$ such that $\|(x+\beta\xi)-y_\beta\| < \|(x+\beta\xi)-x\|$. Combining the last inequality with (4.3) yields $\|y_\beta - x\| > \rho$. For any $\beta \in (0,\alpha)$ satisfying $\beta < \frac{\rho}{2\|\xi\|}$, we have

$$\|x-y_\beta\| - \|\beta\xi\| \leq \|(x-y_\beta)+\beta\xi\| < \|(x+\beta\xi)-x\| = \beta\|\xi\|.$$

This implies that $\|x-y_\beta\| < 2\beta\|\xi\| < \rho$. Since $\|y_\beta - x\| > \rho$, we have arrived at a contradiction. We have thus shown that $\mathcal{N}_{\Omega\cap\bar{\mathbb{B}}(x,\rho)}^{P}(x) \subset \mathcal{N}_{\Omega}^{P}(x)$. The equality (4.2) has been established.

The local character of proximal normal cone can also be seen through [35, Proposition 1.5] or [20, Proposition 1.7].

Some uniformly prox-regular sets will be discussed in the following examples.

Example 4.3. Let $\mathcal{H} = \mathbb{R}^2$, the set $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq x_1^2\}$ is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular. To prove the r-prox-regularity of Ω with $r = \frac{1}{2}$, observe by the closedness of Ω that the projection of any $u \in \mathbb{R}^2 \backslash \Omega$ on Ω exists and belongs to $\partial\Omega$. Putting $f(x) = \|u - x\|^2$ and $g(x) = -x_1^2 + x_2$, we consider the optimization problem

$$\min\{f(x) \mid g(x) \leq 0\}. \tag{4.4}$$

For all $x \in \mathbb{R}^2$, since $\nabla g(x) = (-2x_1, 1)$ is nonzero, there is some $v \in \mathbb{R}^2$ such that $\langle \nabla g(x), v\rangle < 0$. Applying the Lagrange multiplier rule (see [82, Theorem 1, p. 260] and [34]) to (4.4), one can prove that the problem has a unique solution x_u for each $u \in \mathbb{R}^2 \setminus \Omega$, i.e., $\mathbb{P}_\Omega(u) = \{x_u\}$. Moreover, a careful analysis of the necessary optimality conditions given by the Lagrange multiplier rule shows that, for each $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$, the equality $\bar{x} = \mathbb{P}_\Omega(\bar{u})$ holds for $\bar{u} \in \mathbb{R} \setminus \Omega$ if and only if $\bar{u} = \bar{x} + t\nabla g(\bar{x})$ with $t \in (0, \frac{1}{2})$. Therefore, we have $\mathcal{N}_\Omega^P(\bar{x}) = \mathbb{R}_+ \nabla g(\bar{x})$ for every $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$. For $\bar{x} \in (0,0)$, the equality $\bar{x} = \mathbb{P}_\Omega(\bar{u})$ holds for $\bar{u} \in \mathbb{R} \setminus \Omega$ if and only if $\bar{u} = \bar{x} + t\nabla g(\bar{x}) = (0, t)$ with $t \in (0, +\infty)$. Hence, $\mathcal{N}_\Omega^P((0,0)) = \{0\} \times \mathbb{R}_+$. To find a modulus $r > 0$ for the uniform prox-regularity of Ω, we can argue as follows. Fix a point $\bar{x} \in \partial\Omega \setminus \{(0,0)\}$ and let $\bar{u} = \bar{x} + \tau\nabla g(\bar{x})$ for some $\tau \in (0, \frac{1}{2})$. Since

$$\bar{u} - \bar{x} = \tau\|\nabla g(\bar{x})\| \frac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|} = \tau\sqrt{4\bar{x}_1^2 + 1}\, \frac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|},$$

for $\xi := \frac{\nabla g(\bar{x})}{\|\nabla g(\bar{x})\|}$ one has $\bar{x} \in \mathbb{P}_\Omega(\bar{x} + t\xi)$ if and only if $t := \tau\sqrt{4\bar{x}_1^2 + 1}$ belongs to the interval $(0, \frac{1}{2}\sqrt{4\bar{x}_1^2 + 1})$. Clearly, the infimum of $\frac{1}{2}\sqrt{4\bar{x}_1^2 + 1}$ over the set $\bar{x}_1 \in \mathbb{R} \setminus \{0\}$ is $\frac{1}{2}$. In addition, at $\bar{x} \in (0,0)$, one has $\bar{x} = \mathbb{P}_\Omega(\bar{x} + t(0,1))$ for all $t \in (0, +\infty)$. So, in agreement with Definition 2.37, we can conclude that $r := \frac{1}{2}$ is the best modulus for the uniform prox-regularity of Ω.

From the result established in Example 4.3 we get the following useful examples of uniformly prox-regular sets in spaces of higher dimensions.

Example 4.4. The set $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_2 \leq x_1^2\}$, where $n \geq 3$, is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular.

Example 4.5. The set $\{x = (x_1, x_2, x_3, \ldots) \in \ell^2 \mid x_2 \leq x_1^2\}$ is unbounded, closed, nonconvex, and $\frac{1}{2}$-prox-regular.

Remark 4.6. Let I be a finite index set. The union Ω of disjoint nonempty closed convex subsets $\Omega_i \subset \mathcal{H}, i \in I$, is nonconvex if I has more than one element. If

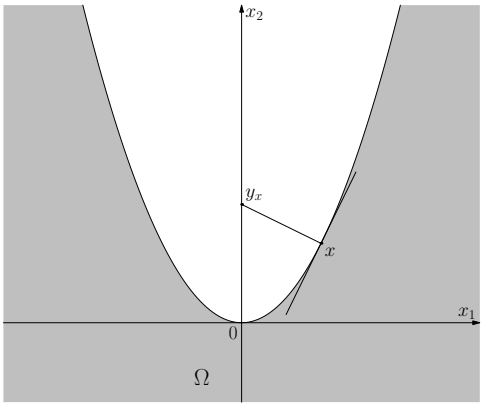


Fig. 4.1: Illustration for Example 4.3

all the numbers $\alpha_{ij} := \inf\{\|x - y\|| \ x \in \Omega_i, \ y \in \Omega_j\}$, with $i, j \in I$ and $i \neq j$, are positive, then Ω is uniformly prox-regular. More precisely, Ω is r-prox-regular, where $r > 0$ is any number satisfying the condition $r \leq \frac{1}{2}\alpha_{ij}$ for all $i, j \in I$ with $i \neq j$. In addition, Ω is not uniformly prox-regular if $\alpha_{ij} = 0$ for a pair $(i, j) \in I \times I$ with $i \neq j$. These assertions can be easily proved by using Definition 2.37 and the fact that the proximal normal cone coincides with the normal cone in the sense of convex analysis if the set under consideration is convex.

4.1 Parametric Variational Inequalities

In this section, we recall some concepts and results relating to parametric variational inequalities.

Let M, Λ be two metric spaces. Let $F : \mathcal{H} \times M \to \mathcal{H}$ be a vector-valued function, and $K : \Lambda \rightrightarrows \mathcal{H}$ be a set-valued map with nonempty closed convex values. For each pair of parameters $(\mu, \lambda) \in M \times \Lambda$, we consider the problem of finding a vector $x \in K(\lambda)$ such that

$$\langle F(x, \mu), y - x\rangle \geq 0 \quad \forall y \in K(\lambda), \tag{4.5}$$

which is a *parametric variational inequality* with a moving constraint set.

The *pseudo-Lipschitz property* of set-valued mappings introduced by Aubin [10, p. 98] is a crucial concept in set-valued and variational analysis. This property is also known under other names: the *Aubin continuity property* [44], the *sub-Lipschitzian property* [84], and the *Lipschitz-like property* [60]. Complete characterizations of the property can be found in [60, 63, 84] and the references therein.

Definition 4.7. (See [60, Definition 1.40] and [63, Definition 3.1]) One says that a set-valued mapping $K : \Lambda \rightrightarrows \mathcal{H}$ is *Lipschitz-like* around a point $(\tilde{\lambda}, \tilde{x})$ in its *graph*, which is the set $\{(\lambda, x) \in \Lambda \times \mathcal{H} \mid x \in K(\lambda)\}$, if there exist a neighborhood V of $\tilde{\lambda}$, a neighborhood W of $\tilde{x}$ and a constant $\kappa > 0$ such that

$$K(\lambda) \cap W \subset K(\lambda') + \kappa d(\lambda, \lambda')\bar{\mathbb{B}}(0, 1), \quad \forall \lambda, \lambda' \in V. \tag{4.6}$$

Remark 4.8. If there exist a neighborhood V of $\tilde{\lambda}$ and a constant $\kappa > 0$ such that

$$K(\lambda) \subset K(\lambda') + \kappa d(\lambda, \lambda')\bar{\mathbb{B}}(0, 1), \quad \forall \lambda, \lambda' \in V, \tag{4.7}$$

then one says that K is *locally Lipschitz* around $\tilde{\lambda}$. If the inclusion in (4.7) holds for some $\kappa > 0$ and for all $\lambda, \lambda' \in \Lambda$, then K is said to be a *Lipschitz* set-valued mapping. It is well known that if K is locally Lipschitz around $\tilde{\lambda}$, then K is Lipschitz-like around $(\tilde{\lambda}, \tilde{x})$ for every $\tilde{x} \in K(\tilde{\lambda})$. In particular, a Lipschitz set-valued mapping is Lipschitz-like around every point in its graph.

Consider the parametric variational inequality (4.5). Let $\bar{x}$ be a solution to it at given parameters $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$. We make two assumptions on the behavior of the function $F(x, \mu)$ around the point $(\bar{x}, \bar{\mu})$. Namely, we assume that there exist a closed convex neighborhood X of $\bar{x}$, a neighborhood U of $\bar{\mu}$, and two positive constants α, l such that

$$\|F(x', \mu') - F(x, \mu)\| \leq l(\|x' - x\| + d(\mu', \mu)), \quad \forall \mu, \mu' \in U,\ x, x' \in X, \tag{4.8}$$

and

$$\langle F(x', \mu) - F(x, \mu), x' - x\rangle \geq \alpha\|x' - x\|^2, \quad \forall \mu \in U,\ x, x' \in X. \tag{4.9}$$

Theorem 4.9. (See [94, Theorem 2.1]) *Assume that $\bar{x}$ is a solution to* (4.5) *with respect to the given parameters $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$, conditions* (4.8) *and* (4.9) *hold, and the set-valued map $K : \Lambda \rightrightarrows \mathcal{H}$ is Lipschitz-like around $(\bar{\lambda}, \bar{x})$. Then, there exist positive constants $\kappa_{\bar{u}}$ and $\kappa_{\bar{\lambda}}$, and neighborhoods $\tilde{U}$ of $\bar{\mu}$ and $\tilde{V}$ of $\bar{\lambda}$ such that*

(i) *For every $(\mu, \lambda) \in \tilde{U} \times \tilde{V}$, there exists a unique solution to* (4.5) *in X, denoted by $x(\mu, \lambda)$;*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in \tilde{U} \times \tilde{V}$, one has*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq \kappa_{\bar{\mu}} d(\mu', \mu) + \kappa_{\bar{\lambda}} d(\lambda', \lambda)^{1/2}. \tag{4.10}$$

4.2 The Case of Convex Constraint Sets

For studying the problem (P), the next two assumptions were used in [6, 7].

Assumption (H1). The constraint sets $C(t)$, $t \in [0,T]$, are nonempty, closed, and convex.

Assumption (H2). The set-valued mapping C is continuous in the Hausdorff distance sense, i.e., there exists a continuous function $g : [0,T] \to \mathbb{R}$ such that

$$d_H(C(s), C(t)) \leq |g(s) - g(t)| \text{ for all } s, t \in [0,T]. \tag{4.11}$$

The results of Adly, Haddad, and Thibault [7] also require the following assumption.

Assumption (H3a). The constraint set $C(0)$ is bounded.

Later, to deal with possibly unbounded constraint sets, Adly and Le [6], have used the next semicoercivity assumption.

Assumption (H3b). There exist positive constants c_1, c_2 such that

$$\langle A_1 x, x\rangle \geq c_1\|x\|^2 - c_2, \quad \forall x \in C(0). \tag{4.12}$$

Remark 4.10. If A_1 is positive semidefinite, then (H3a) implies (H3b). Indeed, if $C(0)$ is bounded, then we can find $\rho > 0$ such that $C(0) \subset \rho\bar{\mathbb{B}}(0,1)$. Choosing $c_1 = 1$ and $c_2 = \rho^2$, we have $\langle A_1 x, x\rangle \geq 0 \geq c_1\|x\|^2 - c_2$ for any $x \in C(0)$. Thus, (H3b) holds true.

Remark 4.11. If (H2a) and (H3b) are satisfied, then exist positive constants $\hat{c}_1, \hat{c}_2$ such that $\langle A_1 x, x\rangle \geq \hat{c}_1\|x\|^2 - \hat{c}_2$ for all $t \in [0,T]$ and $x \in C(t)$. Indeed, let $g : [0,T] \to \mathbb{R}$ be a continuous function satisfying (5.1). Then, for all $t \in [0,T]$, one has $d_H(C(0), C(t)) \leq |g(0) - g(t)| \leq \gamma$, where $\gamma := \max_{t\in[0,T]} |g(0) - g(t)|$. Suppose c_1, c_2 are positive constants such that (4.12) holds. For any $t \in [0,T]$ and for every $y \in C(t)$, since $d(y, C(0)) \leq |g(0) - g(t)| \leq \gamma$, we have for every $\varepsilon > 0$ there exists $x \in C(0)$ with $\|y - x\| \leq \gamma + \varepsilon$. So, $y = x + (\gamma + \varepsilon)w$ for some $w \in \bar{\mathbb{B}}(0,1)$.

Therefore, by (H3b) we get

$$\begin{aligned}
\langle Ay, y\rangle &= A_1\langle x+(\gamma+\varepsilon)w, x+(\gamma+\varepsilon)w\rangle \\
&= \langle A_1x, x\rangle + 2(\gamma+\varepsilon)\langle A_1x, w\rangle + (\gamma+\varepsilon)^2\|w\|^2 \\
&\geq c_1\|x\|^2 - c_2 + 2(\gamma+\varepsilon)\langle A_1x, w\rangle + (\gamma+\varepsilon)^2\|w\|^2 \\
&= c_1\|y-(\gamma+\varepsilon)w\|^2 - c_2 + 2(\gamma+\varepsilon)\langle A_1(y-(\gamma+\varepsilon)w), w\rangle + (\gamma+\varepsilon)^2\|w\|^2 \\
&= c_1\|y\|^2 - c_2 - 2c_1(\gamma+\varepsilon)\langle y, w\rangle + c_1(\gamma+\varepsilon)^2\|w\|^2 + 2(\gamma+\varepsilon)\langle A_1y, w\rangle \\
&\quad -(\gamma+\varepsilon)^2\langle A_1w, w\rangle + (\gamma+\varepsilon)^2\|w\|^2 \\
&\geq c_1\|y\|^2 - c_2 - 2c_1(\gamma+\varepsilon)\|y\| - (c_1+\|A_1\|+1)(\gamma+\varepsilon)^2 \\
&\quad -2(\gamma+\varepsilon)\|A_1\|\|y\| \\
&\geq \Big(\frac{c_1}{2}\|y\|^2 - c_2\Big) + \|y\|\Big(\frac{c_1}{2}\|y\| - 2(\gamma+\varepsilon)(c_1+\|A_1\|)\Big) \\
&\quad -(c_1+\|A_1\|+1)(\gamma+\varepsilon)^2.
\end{aligned} \tag{4.13}$$

If $\|y\| \geq \dfrac{4(\gamma+\varepsilon)}{c_1}(c_1+\|A_1\|)$, then from (4.13) it follows that

$$\langle Ay, y\rangle \geq \frac{c_1}{2}\|y\|^2 - c_2 - (c_1+\|A_1\|+1)(\gamma+\varepsilon)^2. \tag{4.14}$$

On the other hand, if $\|y\| \leq \dfrac{4(\gamma+\varepsilon)}{c_1}(c_1+\|A_1\|)$, then (4.13) implies that

$$\begin{aligned}
\langle Ay, y\rangle &\geq \frac{c_1}{2}\|y\|^2 - c_2 - 2(\gamma+\varepsilon)(c_1+\|A_1\|)\frac{4(\gamma+\varepsilon)}{c_1}(c_1+\|A_1\|) \\
&\quad -(c_1+\|A_1\|+1)(\gamma+\varepsilon)^2 \\
&= \frac{c_1}{2}\|y\|^2 - c_2 - \frac{8(\gamma+\varepsilon)^2}{c_1}(c_1+\|A_1\|)^2 - (c_1+\|A_1\|+1)(\gamma+\varepsilon)^2
\end{aligned} \tag{4.15}$$

By setting $\hat{c}_1 = \dfrac{c_1}{2}$ and $\hat{c}_2 = c_2 + \dfrac{8(\gamma+\varepsilon)^2}{c_1}(c_1+\|A_1\|)^2 + (c_1+\|A_1\|+1)(\gamma+\varepsilon)^2$, from (4.14) and (4.15), we arrive at the claimed result.

Remark 4.12. It can be shown that if the assumptions (H1) and (H2a) are satisfied then the *recession cone* [83, pp. 61–63] of $C(t)$, which is denoted by $0^+C(t)$, is invariant with respect to t, i.e., $0^+C(t) = 0^+C(0)$ for every $t \in [0,T]$.

The solution existence and solution uniqueness results of [7] for sweeping processes with velocity constraints of the form (P) can be stated as follows.

Theorem 4.13. (The moving constraint set is bounded and continuous in the Hausdorff distance sense; see [7, Theorems 5.1 and 5.2]) *Suppose that $\mathcal{H}$ is separable and A_0, A_1 are positive semidefinite. If the assumptions* (H1), (H2a), (H3a) *are satisfied, then* (P) *has at least one Lipschitz solution. If A_0 is coercive, i.e., there*

exists a constant $\alpha_0 > 0$ *such that* $\langle A_0 x, x\rangle \geq \alpha_0 \|x\|^2$ *for all* $x \in \mathcal{H}$*, and* (H1) *is satisfied, then* (P) *has at most one solution.*

The above results of Adly, Haddad, and Thibault have been extended by Adly and Le [6] to the case of possibly unbounded closed convex sets $C(t)$, $t \in [0,T]$. In fact, there is no statement on solution uniqueness of (P) in [6]. However, it is not difficult to see that the proof of Theorem 5.2 in [7] is also valid for the case of unbounded closed convex constraint sets.

Theorem 4.14. (The moving constraint set is continuous in the Hausdorff distance sense; cf. [6, Theorem 1]) *Suppose that* $\mathcal{H}$ *is separable and* A_0, A_1 *are positive semidefinite. If the assumptions* (H1), (H2a), (H3b) *are satisfied, then* (P) *has at least one Lipschitz solution. If* A_0 *is coercive and* (H1) *is satisfied, then* (P) *has at most one solution.*

The separability of $\mathcal{H}$ and the continuity in the Hausdorff distance sense of the set-valued mapping C are vital assumptions in Theorems 4.13 and 4.14, which were proved by Moreau's time discretization techniques and the catching-up algorithm. Besides, as it has been noted in Remark 4.12, if (H1) and (H2a) are satisfied then the recession cone $0^+C(t)$ of $C(t)$ is invariant with respect to t. By using the concept of parametric variational inequality and Theorem 2.1 from [94], which have been recalled in Section 4.1, we now establish a new result on the solution existence and solution uniqueness of (P). Here, $\mathcal{H}$ *can be a non-separable Hilbert space, the constraint set* $C(t)$ *can be unbounded, and the recession cone of* $C(t)$ *can vary when* t *changes in* $[0,T]$.

Theorem 4.15. (The moving constraint set is locally Lipschitz-like) *Let* $\mathcal{H}$ *be a Hilbert space,* $A_0 = 0$*,* $A_1 : \mathcal{H} \to \mathcal{H}$ *a symmetric coercive bounded linear operator, and* $f : [0,T] \to \mathcal{H}$ *a continuous mapping. Assume that* $C : [0,T] \rightrightarrows \mathcal{H}$ *is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then* (P) *has a unique solution* u*, which is a Lipschitz function. Moreover, the unique solution is a continuously differentiable function (provided that one identifies* $\dot{u}(0)$ *with the right derivative of* u *at 0 and* $\dot{u}(T)$ *with the left derivative of* u *at* T*).*

Proof. Since $A_0 = 0$, (P) has the form

$$\begin{cases} A_1\dot{u}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{P$_1$}$$

To apply Theorem 4.9 for (P_1), let us set $M = \mathcal{H}$, $\Lambda = [0,T]$, $F(x,\mu) = A_1x+\mu$ for $(x,\mu) \in \mathcal{H} \times M$, $K(\lambda) = C(\lambda)$ for $\lambda \in \Lambda$. Since A_1 is coercive, there is a constant $\alpha > 0$ such that $\langle A_1x, x\rangle \geq \alpha\|x\|^2$ for all $x \in \mathcal{H}$. Hence, choosing $X = \mathcal{H}$, $U = M$, and $l = \max\{\|A_1\|, 1\}$, we see that the conditions (4.8) and (4.9) are satisfied. For each pair $(\mu,\lambda) \in M \times \Lambda$, by the well-known solution existence theorem for strongly monotone variational inequality (see, e.g., Theorem 4.1 in [52], which has the origin in [55, Theorem 2.1, p. 24]) we know that (4.5) has a unique solution. The latter is denoted by $x(\mu,\lambda)$. For every $\lambda \in \Lambda$, we define a vector $\mu(\lambda) = -f(\lambda)$. Fix a value $\bar{\lambda} = \bar{t} \in [0,T]$ and let $\bar{\mu} = \mu(\bar{\lambda}) = -f(\bar{t})$, $\bar{x} = x(\bar{\mu},\bar{\lambda})$. Since the set-valued mapping $K(\cdot) = C(\cdot)$ is Lipschitz-like around $(\bar{\lambda},\bar{x})$, Theorem 4.9 asserts that there exist positive constants $\kappa_{\bar{u}}$ and $\kappa_{\bar{\lambda}}$, and neighborhoods $\tilde{U}$ of $\bar{\mu}$ and $\tilde{V}$ of $\bar{\lambda}$ such that the inequality (4.10) holds for all $(\mu',\lambda'),(\mu,\lambda) \in \tilde{U} \times \tilde{V}$. As $\tilde{U}$ is a neighborhood of $\bar{\mu} = \mu(\bar{\lambda}) = -f(\bar{t})$, $\mu(\lambda) = -f(\lambda)$, and $f(\cdot)$ is continuous at $\bar{t}$, we can find a neighborhood V_0 of $\bar{t}$ in $[0,T]$ such that $V_0 \subset \tilde{V}$ and $\mu(\lambda) \in \tilde{U}$ for all $\lambda = t$ with $t \in V_0$. Then, by (4.10) one has

$$\begin{aligned}\|x(\mu(t),t) - x(\mu(\bar{t}),\bar{t})\| &\leq \kappa_{\bar{\mu}}\|\mu(t) - \mu(\bar{t})\| + \kappa_{\bar{\lambda}}|t-\bar{t}|^{1/2}\\ &= \kappa_{\bar{\mu}}\|f(t) - f(\bar{t})\| + \kappa_{\bar{\lambda}}|t-\bar{t}|^{1/2}\end{aligned}$$

for every $t \in V_0$. It follows that $\lim\limits_{t\to\bar{t}}\|x(\mu(t),t) - x(\mu(\bar{t}),\bar{t})\| = 0$. Therefore, the formula $z(t) = x(\mu(t),t)$ defines a continuous function $z : [0,T] \to \mathcal{H}$.

Summing up all the above, we can assert that, for every $t \in [0,T]$, the variational inequality (4.5) with the chosen function F, the set-valued mapping K, where $(\mu,\lambda) := (-f(t),t))$, has the unique solution $z(t)$, and the function $z(\cdot)$ is continuous on $[0,T]$. In particular, for every $t \in [0,T]$, one has

$$\langle A_1z(t) - f(t), y - z(t)\rangle \geq 0 \quad \forall y \in C(t). \tag{4.16}$$

Since the set $C(t)$ is closed convex for every $t \in [0,T]$, by [20, Example 1.4, p. 24] we know that, for any $x \in \mathcal{H}$, the proximal normal cone $\mathcal{N}^P_{C(t)}(x)$ coincides with the normal cone of $C(t)$ at x in the sense of convex analysis. So, the condition (4.16) yields

$$A_1z(t) - f(t) \in -\mathcal{N}^P_{C(t)}(z(t)). \tag{4.17}$$

Conversely, since the inclusion $A_1z - f(t) \in -\mathcal{N}^P_{C(t)}(z)$ is equivalent to the condition

$$\langle A_1z - f(t), y - z\rangle \geq 0 \quad \forall y \in C(t),$$

one has $A_1\dot{u}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t))$ if and only if $\dot{u}(t) = z(t)$. Since $z(\cdot)$ is continuous on $[0,T]$, the norm $\|z(t)\|$ is bounded for every $t \in [0,T]$. So, the Lebesgue integral $\int_0^T \|z(\tau)\|d\tau$ exists. By Proposition 2.17, z is Bochner integrable over the interval $[0,T]$ with respect to the Lebesgue measure. Setting

$$u(t) = u_0 + \int_0^t z(\tau)d\tau \quad (\forall t \in [0,T]), \tag{4.18}$$

we have $\dot{u}(t) = z(t)$ for all $t \in [0,T]$. Indeed, applying Proposition 2.20 and the arguments in its proof (recalling by Proposition 2.11 that the Lebesgue integral of a continuous real-valued function coincides with the Riemann integral), for all $t \in (0,T)$, the limit $\lim\limits_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right]$ exists and it is equal to $z(t)$. So, from the relation $\lim\limits_{h\to 0}\frac{u(t+h)-u(t)}{h} = \lim\limits_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right]$ it follows that, for all $t \in (0,T)$, the derivative $\dot{u}(t)$ exists and one has $\dot{u}(t) = z(t)$. Moreover, for any $t, s \in [0,T]$ with $s \leq t$,

$$\begin{aligned}\|u(t) - u(s)\| = \|\int_0^t z(\tau)d\tau - \int_0^s z(\tau)d\tau\| &\leq \int_s^t \|z(\tau)\|d\tau \\ &\leq \max\{\|z(\tau)\| \mid \tau \in [0,T]\}(t-s).\end{aligned}$$

Thus, this function u is Lipschitz continuous with the modulus $L = \max\limits_{\tau\in[0,T]}\|z(\tau)\|$. The fulfillment of (4.17) for all $t \in [0,T]$ and the equality $u(0) = u_0$ assure that u is a Lipschitz solution of (P_1). It remains to prove that $u(\cdot)$ is the unique solution of (P_1). Arguing by contradiction, suppose that (P_1) has another solution $v(\cdot)$ for which there is $\bar{t} \in [0,T]$ such that $v(\bar{t}) \neq u(\bar{t})$. Set $w(t) = v(t) - u(t)$ for all $t \in [0,T]$. Clearly, w is absolutely continuous on $[0,T]$ and $w(0) = 0$. Since $\dot{v}(t) = z(t)$ for almost every $t \in [0,T]$, we have $\dot{w}(t) = \dot{v}(t) - \dot{u}(t) = 0$ for almost every $t \in [0,T]$. As $w(\bar{t}) \neq 0$, there exists $x^* \in \mathcal{H}$ such that $\langle x^*, w(\bar{t})\rangle > 0$. Consider the function $\varphi(t) := \langle x^*, w(t)\rangle$. Note that φ is absolutely continuous on $[0,T]$, $\varphi(0) = 0$, and $\dot{\varphi}(t) = \langle x^*, \dot{w}(t)\rangle = 0$ for almost every $t \in [0,T]$. Applying Theorem (2.13) for the scalar function φ, one has $\varphi(t) = \varphi(0) + \int_0^t \dot{\varphi}(\tau)d\tau = 0$ for each $t \in [0,T]$. In particular, $\varphi(\bar{t}) = 0$. Hence, one gets $\langle x^*, w(\bar{t})\rangle = 0$, which is a contradiction. We have thus established the solution uniqueness of (P_1). So, formula (4.18) defines the unique solution of (P_1), which is a Lipschitz function on $[0,T]$. Moreover, the unique solution is a continuously differentiable function.

The proof is complete. □

Remark 4.16. By the arguments in the final part of the above proof, we obtain the following useful facts on the Bochner integration:

(a) If $z : [0,T] \to X$, where X is a Banach space, is a continuous function, then the formula $u(t) = u_0 + \int_0^t z(\tau)d\tau$ defines a continuously differentiable function $u : [0,T] \to X$ and we have $\dot{u}(t) = z(t)$ for all $t \in [0,T]$.

(b) Let $u, v : [0,T] \to X$, where X is a reflexive Banach space, be absolutely continuous functions. If $u(0) = v(0)$ and $\dot{u}(t) = \dot{v}(t)$ for a.e. $t \in [0,T]$, then $u(t) = v(t)$ for all $t \in [0,T]$.

(c) (See the proof of Theorem 2 on p. 107 in [43]) Let $u : [0,T] \to X$, where X is a reflexive Banach space, be an absolutely continuous function. Then,

$$u(t) = u_0 + \int_0^t \dot{u}(\tau)d\tau \quad (\forall t \in [0,T]).$$

(d) If $z : [0,T] \to X$, where X is a Banach space, is a Bochner integrable function with respect to the Lebesgue measure, then the formula $u(t) = u_0 + \int_0^t z(\tau)d\tau$ defines a function $u : [0,T] \to X$, which is Fréchet differentiable a.e. on $[0,T]$ and we have $\dot{u}(t) = z(t)$ for a.e. $t \in [0,T]$.

To prove (c), it suffices to put $v(t) = u_0 + \int_0^t \dot{u}(\tau)d\tau$ for $t \in [0,T]$, and apply the assertion (b). The fact that the function $\dot{u}(\cdot)$ is Bochner integrable on $[0,T]$ is shown with detailed explanations in the proof of [43, Theorem 2, p. 107]. The assertion (d) follows from [43, Theorem 9, p. 49] which asserts that, under the assumptions made, $\lim_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} z(\tau)d\tau\right] = z(t)$.

For any Hilbert space $\mathcal{H}$ of dimension greater or equal 2, there exist set-valued mappings $C : \mathbb{R} \rightrightarrows \mathcal{H}$ with nonempty closed convex values, Lipschitz-like around every point in their graphs, which are *not* continuous in the Hausdorff distance sense on any interval $[a,b] \subset \mathbb{R}$, where $a < b$. The forthcoming example justifies our observation.

Example 4.17. Let $\mathcal{H} = \mathbb{R}^2$, $\Lambda = \mathbb{R}$, $K(\lambda) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \lambda x_1\}$ for all $\lambda \in \mathbb{R}$. Given any $\bar{\lambda} \in \Lambda$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in K(\bar{\lambda})$, we will show that K is Lipschitz-like around $(\bar{\lambda}, \bar{x})$ by a direct verification based on Definition 4.7. First, suppose that $\bar{x} = (0,0)$. Take any $\rho > 0$ and choose $V = \Lambda$, $W = \bar{\mathbb{B}}(0,\rho)$, $\kappa = \sqrt{\frac{\rho}{2}}$.

To verify condition (4.6), fix arbitrary elements $\lambda, \lambda' \in V$. Since

$$K(\lambda) \cap W = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \lambda x_1,\ \|x\| \leq \rho\right\},$$

we have for every $x = (x_1, x_2) \in K(\lambda) \cap W$ the following

$$\begin{aligned} d(x, K(\lambda')) &\leq d\left(\left(\sqrt{\tfrac{\rho}{1+\lambda^2}}, \lambda\sqrt{\tfrac{\rho}{1+\lambda^2}}\right), K(\lambda')\right) \\ &= \sqrt{\tfrac{\rho}{(1+\lambda^2)(1+(\lambda')^2)}}|\lambda' - \lambda| \\ &\leq \sqrt{\tfrac{\rho}{2}}|\lambda' - \lambda|. \end{aligned}$$

It follows that $K(\lambda) \cap W \subset K(\lambda') + \kappa|\lambda' - \lambda|\bar{\mathbb{B}}(0,1)$. The Lipschitz-likeness of K around $(\bar{\lambda}, \bar{x})$ has been proved. Now, suppose that $\bar{x} \neq (0,0)$. Take any $\rho \in (0,1)$ and choose $V = \bar{\mathbb{B}}(\bar{\lambda}, \frac{\rho}{\|\bar{x}\|})$, $W = \bar{\mathbb{B}}(\bar{x}, \rho)$. To define the constant κ, let us consider the expression $\Delta'(\lambda) := (1+\lambda^2)\rho - \bar{x}_1^2(\lambda - \bar{\lambda})^2$, where $\lambda \in \bar{\mathbb{B}}\left(\bar{\lambda}, \frac{\rho}{\|\bar{x}\|}\right)$. Since

$$\begin{aligned} \Delta' &= (1+\lambda^2)\rho - \bar{x}_1^2(\lambda - \bar{\lambda})^2 \\ &\geq (1+\lambda^2)\rho - \|\bar{x}\|^2\tfrac{\rho^2}{\|\bar{x}\|^2} \geq \lambda^2\rho, \end{aligned}$$

one has $\Delta'(\lambda) \geq 0$ for all $\lambda \in \bar{\mathbb{B}}\left(\bar{\lambda}, \frac{\rho}{\|\bar{x}\|}\right)$. So, the number

$$\mu := \max\left\{\frac{1}{2}\left[|(1+\lambda\bar{\lambda})\bar{x}_1| + \sqrt{(1+\lambda^2)\rho - \bar{x}_1^2(\lambda - \bar{\lambda})^2}\right] \mid \lambda \in \bar{\mathbb{B}}\left(\bar{\lambda}, \frac{\rho}{\|\bar{x}\|}\right)\right\} \tag{4.19}$$

is well defined, and we have $\mu \geq 0$. Let $\kappa = \max\{\mu, 1\}$. To verify (4.6), let $\lambda, \lambda' \in V$ be given arbitrarily. Since

$$d(\bar{x}, K(\lambda)) = \frac{|\bar{x}_2 - \lambda\bar{x}_1|}{1+\lambda^2}\|(\lambda, -1)\| = \frac{|\bar{\lambda}\bar{x}_1 - \lambda\bar{x}_1|}{\sqrt{1+\lambda^2}} \begin{aligned} &= \frac{|\bar{x}_1|}{\sqrt{1+\lambda^2}}|\lambda - \bar{\lambda}| \\ &\leq \|\bar{x}\||\lambda - \bar{\lambda}| \\ &\leq \rho, \end{aligned}$$

one has $K(\lambda) \cap W \neq \emptyset$. Clearly, $K(\lambda) \cap W$ is a line segment with the end-points

$$\tilde{x} := \left(\frac{(1+\lambda\bar{\lambda})\bar{x}_1 - \sqrt{\Delta'(\lambda)}}{1+\lambda^2}, \lambda\frac{(1+\lambda\bar{\lambda})\bar{x}_1 - \sqrt{\Delta'(\lambda)}}{1+\lambda^2}\right)$$

and

$$\hat{x} := \left(\frac{(1+\lambda\bar{\lambda})\bar{x}_1 + \sqrt{\Delta'(\lambda)}}{1+\lambda^2}, \lambda\frac{(1+\lambda\bar{\lambda})\bar{x}_1 + \sqrt{\Delta'(\lambda)}}{1+\lambda^2}\right),$$

which may coincide. Note that

$$\begin{aligned} d(\tilde{x}, K(\lambda')) = \frac{|\tilde{x}_2 - \lambda'\tilde{x}_1|}{1+(\lambda')^2}\|(\lambda', -1)\| &= \frac{|\lambda\tilde{x}_1 - \lambda'\tilde{x}_1|}{\sqrt{1+(\lambda')^2}} \\ &= \frac{|\tilde{x}_1|}{\sqrt{1+(\lambda')^2}}|\lambda' - \lambda| \\ &= \frac{|(1+\lambda\bar{\lambda})\bar{x}_1 - \sqrt{\Delta'(\lambda)}|}{(1+\lambda^2)\sqrt{1+(\lambda')^2}}|\lambda' - \lambda| \\ &\leq \frac{|(1+\lambda\bar{\lambda})\bar{x}_1| + \sqrt{\Delta'(\lambda)}}{2}|\lambda' - \lambda|. \end{aligned}$$

Similarly,

$$d(\hat{x}, K(\lambda')) \leq \frac{|(1+\lambda\bar{\lambda})\bar{x}_1| + \sqrt{\Delta'(\lambda)}}{2}|\lambda' - \lambda|.$$

Since $d(x, K(\lambda')) \leq \max\{d(\tilde{x}, K(\lambda')), d(\hat{x}, K(\lambda'))\}$ for every $x = (x_1, x_2) \in K(\lambda) \cap W$, it follows from the above estimates, (4.19), and the formula $\kappa = \max\{\mu, 1\}$, that $d(x, K(\lambda')) \leq \kappa|\lambda' - \lambda|$. This proves that $K(\lambda) \cap W \subset K(\lambda') + \kappa|\lambda' - \lambda|\bar{\mathbb{B}}(0, 1)$. So, the set-valued mapping K is Lipschitz-like around every point $(\bar{\lambda}, \bar{x}) \in \operatorname{gph} K$.

It is well known that any Hilbert space $\mathcal{H}$ of dimension greater or equal 2 admits the representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ are orthogonal subspaces, and $\dim(\mathcal{H}_0) = 2$. Fixing a coordinate system in $\mathcal{H}_0$, we can identify $\mathcal{H}_0$ with $\mathbb{R}^2$. Define a set-valued mapping $C : \mathbb{R} \rightrightarrows \mathcal{H}$ by setting $C(t) = K(t) \oplus \mathcal{H}_1$ for all $t \in \mathbb{R}$. Then, from the above analysis it follows that C has nonempty closed convex values, and C is Lipschitz-like around every point in its graphs. For any interval $[a, b] \subset \mathbb{R}$, where $a < b$, C is not continuous in the Hausdorff distance sense on $[a, b]$. Indeed, one has $0^+C(t) = C(t)$ for every $t \in [a, b]$ and $C(t) \neq C(t')$ for any $t, t' \in [a, b]$ with $t' \neq t$. Hence the condition $0^+C(t) = 0^+C(0)$ for every $t \in [a, b]$, which is necessary for the continuity of C in the Hausdorff distance sense on $[a, b]$, is violated.

The next example is designed to show how Theorem 4.15 can be used for solving concrete problems.

Example 4.18. Consider the sweeping process (P) with $\mathcal{H} = \mathbb{R}^2$, $T = 1$, $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(t) = \begin{pmatrix} 1+\sqrt{t} \\ t\sqrt{t} \end{pmatrix}$, and $u_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let $C(t) = K(t)$ with K being the set-valued mapping defined in Example 4.17. For each $t \in [0, 1]$, since $C(t)$ is the straight line $tx_1 - x_2 = 0$, one has $\mathcal{N}^P_{C(t)}(\dot{u}(t)) = \mathbb{R}\begin{pmatrix} t \\ -1 \end{pmatrix}$. Then, (P)

becomes

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{pmatrix} - \begin{pmatrix} 1+\sqrt{t} \\ t\sqrt{t} \end{pmatrix} \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,1], \\ u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases} \tag{4.20}$$

As shown in Example 4.17, C is Lipschitz-like around every point in its graph. So, all the assumptions of Theorem 4.15 are satisfied and, by that theorem, problem (4.20) has a unique solution $u(\cdot) : [0,1] \to \mathbb{R}^2$, which is a continuously differentiable function. To find an explicit formula for $u(t)$, we observe from the proof of Theorem 4.15 that $\dot{u}(t) = z(t)$ for all $t \in [0,1]$, where $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$ is the unique solution of the parametric variational inequality

$$\langle A_1 z(t) - f(t), y - z(t) \rangle \geq 0 \quad \forall y \in C(t).$$

The latter is equivalent to $A_1 z(t) - f(t) \in -\mathcal{N}^P_{C(t)}(z(t))$. This means that there exists $\beta \in \mathbb{R}$ satisfying

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} - \begin{pmatrix} 1+\sqrt{t} \\ t\sqrt{t} \end{pmatrix} = \beta \begin{pmatrix} t \\ -1 \end{pmatrix}.$$

Therefore, $z_1(t) = \beta t + \sqrt{t} + 1$ and $z_2(t) = t\sqrt{t} - \beta$. The condition $z(t) \in C(t)$ forces $z_2(t) = t z_1(t)$. Hence, $\beta = \dfrac{-t}{1+t^2}$. Thus,

$$z(t) = \begin{pmatrix} 1 + \sqrt{t} - \dfrac{t^2}{1+t^2} \\ t\sqrt{t} + \dfrac{t}{1+t^2} \end{pmatrix} = \begin{pmatrix} \sqrt{t} + \dfrac{1}{1+t^2} \\ t\sqrt{t} + \dfrac{t}{1+t^2} \end{pmatrix}$$

for all $t \in [0,1]$. Using Remark 4.16(c), we have $u(t) = u_0 + \displaystyle\int_0^t z(\tau) d\tau$ for each $t \in [0,1]$. Therefore,

$$u(t) = \begin{pmatrix} \dfrac{2}{3} t\sqrt{t} + \arctan t \\ \dfrac{2}{5} t^2\sqrt{t} + \dfrac{1}{2}\ln(1+t^2) \end{pmatrix} \qquad (t \in [0,1]),$$

a continuously differentiable function on $[0,1]$, is the unique solution of (4.20).

The solution uniqueness result established in Theorem 4.15 is new, because the

operator $A_0 = 0$ is positive semidefinite, but not coercive. Thus, in some sense, our result complements those given in Theorem 4.13 and 4.14. A natural question arises: *Whether the coerciveness of A_1 also guarantees the solution uniqueness of* (P) *in the case where $A_0 \neq 0$?* The following theorem, whose proof is based on some ideas of [7], solves this question in the affirmative.

Theorem 4.19. *If $C(t)$ is nonempty and convex for every $t \in [0,T]$, A_1 is coercive, and A_0 is positive semidefinite, then* (P) *can have at most one solution.*

Proof. Suppose that $u(\cdot)$ and $v(\cdot)$ are two solutions of (P), where $C(t)$ is nonempty and convex for every $t \in [0,T]$, A_1 is coercive, and A_0 is positive semidefinite. Then $u, v : [0,T] \to \mathcal{H}$ are absolutely continuous functions, $u(0) = v(0) = u_0$,

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t) \tag{4.21}$$

and

$$\langle A_1\dot{v}(t) + A_0v(t) - f(t), \dot{v}(t) - z\rangle \leq 0 \quad \forall z \in C(t) \tag{4.22}$$

for a.e. $t \in [0,T]$. Since $\dot{u}(t)$ and $\dot{v}(t)$ belong to $C(t)$ for almost every $t \in [0,T]$, substituting $z = \dot{v}(t)$ to the inequality in (4.21) and $z = \dot{u}(t)$ to the inequality in (4.22) yields

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - \dot{v}(t)\rangle \leq 0$$

and

$$\langle A_1\dot{v}(t) + A_0v(t) - f(t), \dot{v}(t) - \dot{u}(t)\rangle \leq 0$$

for almost every $t \in [0,T]$. Adding the last inequalities side by side, one gets

$$\langle A_1(\dot{u}(t) - \dot{v}(t)) + A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0 \tag{4.23}$$

for almost every $t \in [0,T]$. Since A_1 is coercive, there is a number $\alpha_1 > 0$ such that $\langle A_1x, x\rangle \geq \alpha_1\|x\|^2$ for all $x \in \mathcal{H}$. Thus, (4.23) implies that

$$\alpha_1\|\dot{u}(t) - \dot{v}(t)\|^2 + \langle A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0 \quad \text{a.e. } t \in [0,T]. \tag{4.24}$$

Taking the Lebesgue integral of both sides of (4.24) and applying Remark 2.12, we obtain

$$\int_0^T \alpha_1\|\dot{u}(\tau) - \dot{v}(\tau)\|^2 d\tau + \int_0^T \langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau)\rangle d\tau \leq 0. \tag{4.25}$$

Since $\frac{d}{d\tau}\langle A_0(u(\tau)-v(\tau)), u(\tau)-v(\tau)\rangle = 2\langle A_0(u(\tau)-v(\tau)), \dot{u}(\tau)-\dot{v}(\tau)\rangle$ at every point τ where both derivatives $\dot{u}(\tau), \dot{v}(\tau)$ exist, using Theorem 2.13 and noting that $u(0) = v(0)$, one has

$$\langle A_0(u(T)-v(T)), u(T)-v(T)\rangle = 2\int_0^T \langle A_0(u(\tau)-v(\tau)), \dot{u}(\tau)-\dot{v}(\tau)\rangle d\tau.$$

Thus, (4.25) is equivalent to

$$\int_0^T \alpha_1\|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau + \frac{1}{2}\langle A_0(u(T)-v(T)), u(T)-v(T)\rangle \leq 0.$$

Since A_0 is positive semidefinite, the latter implies

$$\int_0^T \|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau \leq 0. \tag{4.26}$$

As $\int_0^T \|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau \geq 0$, by (4.26) we have $\int_0^T \|\dot{u}(\tau)-\dot{v}(\tau)\|^2 d\tau = 0$. Hence, by [56, Corollary of Theorem 5, pp. 299–300], $\dot{u}(t) = \dot{v}(t)$ almost everywhere on $[0,T]$. So, thanks to Remark 4.16(b), we obtain $u(t) = v(t)$ for all $t \in [0,T]$. Thus, (P) can have at most one solution. □

4.3 The Case of Nonconvex Constraint Sets

Using the results in Section 4.2, we will establish some facts about solution existence for sweeping processes with nonconvex constraint sets. The obtained results differ from those of Bounkhel [19]. Note that the union of convex sets are not convex in general. Let $I = \{1, \ldots, m\}$ be a finite index set with $m \geq 2$. Let $C_i : [0,T] \rightrightarrows \mathcal{H}$, $i \in I$, be set-valued mappings with nonempty closed convex values such that, for any $t \in [0,T]$ and $i, j \in I$ with $i \neq j$, $C_i(t)$ does not intersect $C_j(t)$. Then, the set $C(t) := \bigcup_{i\in I} C_i(t)$ is closed and nonconvex for every $t \in [0,T]$. The uniform prox-regularity of such kind of sets has been discussed in Remark 4.6. In this section, we will study (P) with $C : [0,T] \rightrightarrows \mathcal{H}$ being the just defined set-valued mapping. To do so, for each $i \in I$, we consider the problem

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C_i(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{P_{C_i}}$$

The following theorems establish the solution existence for three classes of

nonconvex sweeping processes with velocity constraints. *The key point here is that the problems in question admit multiple solutions.*

Theorem 4.20. (The moving constraint set is bounded and continuous in the Hausdorff distance sense) *Suppose $\mathcal{H}$ be separable and A_0, A_1 are positive semidefinite. If every set-valued mapping C_i, $i \in I$, satisfies the assumptions* (H1), (H2a), *and* (H3a), *then* (P) *has an uncountable number of Lipschitz solutions, among them there are m solutions $u^{(i)}$, $i \in I$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$.*

Proof. Let $i \in I$ be chosen arbitrarily. Since C_i satisfies the conditions (H1), (H2a), and (H3a), under the assumptions made, (P_{C_i}) has a Lipschitz solution $u^{(i)}(\cdot)$ by Theorem 4.13. If $\dot{u}_i(t) \in C_i(t)$, then the condition $C_i(t) \cap C_j(t) = \emptyset$ for $j \in I \setminus \{i\}$ and the closedness of $C_j(t)$, $j \in I \setminus \{i\}$, assure that there is a number $\rho_i(t) > 0$ satisfying $C_j(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right) = \emptyset$ for all $j \in I \setminus \{i\}$. So, one gets

$$C(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right) = C_i(t) \cap \bar{\mathbb{B}}\left(\dot{u}^{(i)}(t), \rho_i(t)\right).$$

Therefore, thanks to Remark 4.2 and the fact that the inclusion $\dot{u}^{(i)}(t) \in C_i(t)$ holds for almost every $t \in [0,T]$, we have

$$\begin{aligned} \mathcal{N}^P_{C(t)}(\dot{u}^{(i)}(t)) = \mathcal{N}^P_{C(t)\cap\bar{\mathbb{B}}\left(\dot{u}^{(i)}(t),\rho_i(t)\right)}(\dot{u}^{(i)}(t)) &= \mathcal{N}^P_{C_i(t)\cap\bar{\mathbb{B}}\left(\dot{u}^{(i)}(t),\rho_i(t)\right)}(\dot{u}^{(i)}(t)) \\ &= \mathcal{N}^P_{C_i(t)}(\dot{u}^{(i)}(t)) \end{aligned}$$

for almost every $t \in [0,T]$. Since $u^{(i)}(\cdot)$ is a Lipschitz solution of (P_{C_i}), this yields

$$\begin{cases} A_1\dot{u}^{(i)}(t) + A_0u^{(i)}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}^{(i)}(t)) & \text{a.e. } t \in [0,T], \\ u^{(i)}(0) = u_0. \end{cases}$$

Hence, $u^{(i)}(\cdot)$ is a Lipschitz solution of (P).

Next, fix a pair $(i,j) \in I \times I$ with $i \neq j$, and let $u^{(i)}$ be a Lipschitz solution of (P_{C_i}), $u^{(j)}$ be a Lipschitz solution of (P_{C_j}). Then both functions $u^{(i)}$ and $u^{(j)}$ are Lipschitz solutions of (P). These functions are distinct. Indeed, if $u^{(i)}(t) = u^{(j)}(t)$ for all $t \in [0,T]$ then, since the inclusions $\dot{u}^{(i)}(t) \in C_i(t)$ and $\dot{u}^{(j)}(t) \in C_j(t)$ hold for a.e. $t \in [0,T]$, we find $\bar{t} \in (0,T)$ such that the derivatives $\dot{u}^{(i)}(\bar{t})$ and $\dot{u}^{(j)}(\bar{t})$ exist, $\dot{u}^{(i)}(\bar{t}) \in C_i(\bar{t})$ and $\dot{u}^{(j)}(\bar{t}) \in C_j(\bar{t})$. This is impossible because $\dot{u}^{(i)}(\bar{t}) = \dot{u}^{(j)}(\bar{t})$ and $C_i(\bar{t}) \cap C_j(\bar{t}) = \emptyset$. We have proved the existence of pairwise distinct Lipschitz solutions $u^{(1)}, \ldots, u^{(m)}$ of (P), for which one has $\dot{u}^{(i)}(t) \in C_i(t)$ for every $i \in I$ and for almost every $t \in [0,T]$.

Let $\tau \in (0,T)$ be arbitrarily chosen. By Theorem 4.13, the problem

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C_1(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,\tau], \\ u(0) = u_0, \end{cases} \tag{4.27}$$

has a Lipschitz solution, which we denote by $u_{1,\tau}(\cdot)$. Similarly, the problem

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}^P_{C_2(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [\tau,T], \\ u(\tau) = u_{1,\tau}(\tau), \end{cases} \tag{4.28}$$

has a Lipschitz solution, which is denoted by $u_{2,\tau}(\cdot)$. Setting

$$u_\tau(t) = \begin{cases} u_{1,\tau}(t) & \text{if } t \in [0,\tau], \\ u_{2,\tau}(t) & \text{if } t \in (\tau,T], \end{cases}$$

we see that u_τ is Lipschitz continuous function satisfying $u_\tau(0) = u_0$. As noted at the beginning of this proof, if $z \in C_1(t)$ (resp., $z \in C_2(t)$), then $\mathcal{N}^P_{C_1(t)}(z) = \mathcal{N}^P_{C(t)}(z)$ (resp., $\mathcal{N}^P_{C_2(t)}(z) = \mathcal{N}^P_{C(t)}(z)$). Therefore, from (4.27) and (4.28) it follows that $A_1\dot{u}_\tau(t) + A_0u_\tau(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}_\tau(t))$ for almost every $t \in [0,T]$. Hence, u_τ is a Lipschitz solution of (P). Now, take any $\tau_1, \tau_2 \in (0,T)$ with $\tau_1 < \tau_2$. Since $u_{\tau_1}(\tau_1) = u_{\tau_2}(\tau_1)$, arguing similarly as in the above proof of the pairwise distinctness of the solutions $u^{(1)}, \ldots, u^{(m)}$ of (P), we can show that the restrictions of u_{τ_1} and u_{τ_2} on $[\tau_1, \tau_2]$ are two different functions. So, the family $\{u_\tau \mid \tau \in (0,T)\}$ consists of pairwise distinct Lipschitz functions. Hence, by the uncountability of $(0,T)$ we can assert that (P) has an uncountable number of Lipschitz solutions. □

Theorem 4.21. (The moving constraint set is continuous in the Hausdorff distance sense) *Suppose $\mathcal{H}$ is separable and A_0, A_1 are positive semidefinite. If every set-valued mapping C_i, $i \in I$, satisfies the assumptions* (H1), (H2a), *and* (H3b), *then* (P) *has an uncountable number of Lipschitz solutions, among them there are m solutions $u^{(i)}$, $i \in I$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$.*

Proof. Using the same arguments as the ones in the proof of Theorem 4.20 and applying Theorem 4.14 instead of Theorem 4.13, we then obtain the desired results. □

Theorem 4.22. (The moving constraint set is locally Lipschitz-like) *Suppose that $\mathcal{H}$ is a Hilbert space, $A_0 = 0$, $A_1 : \mathcal{H} \to \mathcal{H}$ is a symmetric coercive bounded linear*

operator, and $f : [0,T] \to \mathcal{H}$ *is a continuous mapping. Assume that, for* $i \in I$*, the set-valued mapping* C_i *has nonempty closed convex values and is Lipschitz-like around every point in its graph. Then* (P) *has an uncountable number of Lipschitz solutions, among them there are* m *continuously differentiable solutions* $u^{(i)}$*,* $i \in I$*, with* $\dot{u}^{(i)}(t) \in C_i(t)$ *for almost every* $t \in [0,T]$*.*

Proof. It suffices to follow the proof scheme of Theorem 4.20 and use Theorem 4.15 instead of Theorem 4.13. □

4.4 Illustrative Examples

In general, problem (P) does not have a unique solution even in the case where $C(t)$ is convex; see [7, Example 1]. For the convex case, Adly, Haddad, and Thibault [7, Theorem 5.2] (see Theorem 4.13 in Section 4.2) have proved that if A_0 is coercive, then (P) can have at most one solution. By constructing an example, we now show that this condition is not enough to obtain the solution uniqueness in the case where $C(t)$ is r-prox-regular and connected for each $t \in [0,T]$.

Example 4.23. Consider problem (P) with $T = 1$, $\mathcal{H} = \mathbb{R}^2$, $A_0 = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f(t) \equiv 0$, $u_0 = (0,0)$, and $C(t) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid (1+t)^2 \le x_1^2 + x_2^2 \le 9\}$. Clearly, A_0 and A_1 are coercive, $C(t)$ is an annulus, which is r-prox-regular with $r = 1$ and connected for each $t \in [0,T]$. As the condition (5.1) is fulfilled with $g(t) := t$ and $C(0)$ is bounded, the assumptions (H2a) and (H3a) are satisfied. Since $C(t)$, $t \in [0,T]$, are nonempty and closed, the assumption (H1) is partially satisfied. Nevertheless, here Theorem 4.13 cannot be applied, because the set-valued mapping C has nonconvex values. So, the solution existence of (P) is under question. Let $u_1(t) = \left(\frac{1}{2}(1+t)^2 - \frac{1}{2}, 0\right)$ for $t \in [0,T]$. We see that

$$\dot{u}_1(t) = (1+t, 0) \in C(t)$$

and $\mathcal{N}^P_{C(t)}(\dot{u}_1(t)) = \mathbb{R}_- \times \{0\}$ for $t \in [0,T]$. Since

$$A_1\dot{u}_1(t) + A_0u(t) - f(t) = \begin{pmatrix} 1+t \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(1+t)^2 - \frac{1}{2} \\ 0 \end{pmatrix} \in -\mathcal{N}^P_{C(t)}(\dot{u}_1(t))$$

for all $t \in [0,T]$ and $u_1(0) = (0,0)$, u_1 is a continuously differentiable solution

of (P). Now, let

$$u_2(t) = \frac{1}{2\sqrt{2}}\left((1+t)^2 - 1, (1+t)^2 - 1\right) \quad (\forall t \in [0,T]).$$

We have $u_2(0) = (0,0)$, $\dot{u}_2(t) = \frac{1}{\sqrt{2}}(1+t, 1+t) \in C(t)$ and

$$\mathcal{N}^P_{C(t)}(\dot{u}_2(t)) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2 \leq 0\}.$$

Then,

$$A_1\dot{u}_1(t) + A_0u(t) + f(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+t) \\ \frac{1}{\sqrt{2}}(1+t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2\sqrt{2}}(1+t)^2 - \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}}(1+t)^2 - \frac{1}{2\sqrt{2}} \end{pmatrix} \in -\mathcal{N}^P_{C(t)}(\dot{u}_2(t)).$$

Therefore, $u_2(\cdot)$ is also a continuously differentiable solution of (P). So, (P) has multiple solutions.

The next two examples will shed light on the assertions about solution uniqueness in Theorem 4.14 and Theorem 4.15. It turns out that the convexity assumption on the sets $C(t)$, $t \in [0,T]$, cannot be replaced by uniform prox-regularity and connectedness.

Example 4.24. Let T, $\mathcal{H}$, A_0, A_1, and f be as in the preceding example. Let $C(t) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid (1+t)^2 \leq x_1^2 + x_2^2\}$ for all $t \in [0,T]$. Then, $C(t)$ is unbounded, r-prox-regular with $r = 1$ and connected for each $t \in [0,T]$. The assumptions (H2a) and (H3b) are fulfilled. Since the assumption (H1) is just partially satisfied, Theorem 4.14 cannot be used. Set $u(t) = (\frac{1}{2}t^2 + t)a$ for $t \in [0,T]$, where a is any point in $\partial C(0)$. By a direct verification, we can show that u is a continuously differentiable solution of (P). So, (P) has multiple solutions.

Example 4.25. Let T, $\mathcal{H}$, A_1, f, and $C(\cdot)$ be the same as in Example 4.23. The fulfillment of (5.1) with $g(t) := t$ shows that C is a Lipschitz set-valued mapping. Hence, as noticed in Remark 4.8, C is Lipschitz-like around every point in its graph. Choosing $A_0 = 0$, we see that, except for the required convexity of each $C(t)$, all other assumptions of Theorem 4.15 are satisfied. It is easy to verify that the formula $u(t) = (\frac{1}{2}t^2 + t)a$, where $a \in \mathbb{R}^2$ and $\|a\| = 1$, defines a continuously differentiable solution of (P). So, (P) has multiple solutions.

Remark 4.26. In Examples 4.23 and 4.25, if the formula of $C(t)$ is changed to

$$C(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2 \leq 9\right\},$$

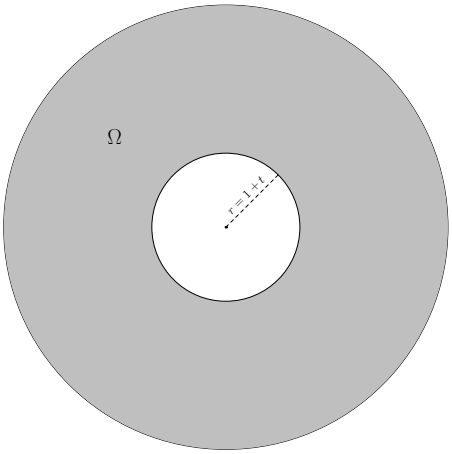


Fig. 4.2: Illustration for Example 4.23

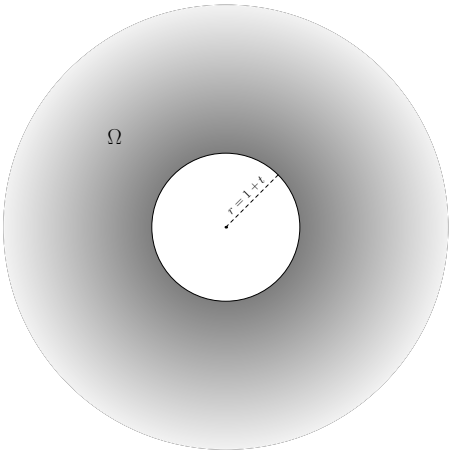


Fig. 4.3: Illustration for Example 4.24

then one has a problem with a fixed constraint set. The formula $u(t) = ta$, where $a \in \mathbb{R}^2$ and $\|a\| = 1$, defines a continuously differentiable solution of the problem (P). So, (P) can have multiple solutions even in the case of a fixed nonconvex constraint set, which is compact, uniformly prox-regular, and connected. This observation is also valid for Example 4.24, if the constraint set is kept fixed, i.e., one takes

$$C(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2\right\}$$

for all $t \in [0, T]$.

If a person uses a motorbike to go on a road starting from A on a time interval $[0, T]$ then, roughly speaking, at every time instant he/she can choose one level of velocity from the set $\{0, 1, 2, 3\}$ of the motorcycle gear levels. Different choices of the velocity level $\dot{u}(t)$ for various disjoint segments of $[0, T]$ generate different path length functions $u(t)$. Here one has $u(0) = 0$. The following example will put this very common daily nonconvex sweeping process with velocity constraints in an abstract setting.

Example 4.27. Consider problem (P) with A_1, A_0, f, u_0 being given arbitrarily, and $C(t) = \{v^1, \ldots v^m\}$ for all $t \in [0, T]$, where $m \geq 2$ and v^i, $i \in I := \{1, \ldots, m\}$, are pairwise distinct points in $\mathcal{H}$. By Remark 4.6, we know that C is uniformly prox-regular. Let $\tau_0 = 0 < \tau_1 < \cdots < \tau_k = T$ be a partition of the interval $[0, T]$. Let $\dot{u}(t)$ be a step function that takes just one value from $\{v^1, \ldots v^m\}$ on each interval (τ_j, τ_{j+1}), $j = 0, \ldots, k-1$. The formula $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ gives a

Lipschitz function defined on $[0,T]$. It is obvious that, for any $z \in \{v^1, \dots v^m\}$ and $t \in [0,T]$, one has $\mathcal{N}^P_{C(t)}(z) = \mathcal{H}$. Hence, the two conditions in the formulation of (P) are satisfied. Thus, $u(t)$ is a solution of (P). We have shown that (P) has uncountable number of Lipschitz solutions.

The next example can serve as an illustration for Theorem 4.22.

Example 4.28. Consider problem (P) where $\mathcal{H} = \mathbb{R}^2$, $A_0 = 0$, $A_1 \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix, $f : [0,T] \to \mathbb{R}^2$ is a continuous function,

$$C_1(t) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq e^{-x_1+t}\right\},$$

$C_2(t) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$, and $C(t) = C_1(t) \cup C_2(t)$ for $t \in [0,T]$. According to Remark 4.6, $C(t)$ is not uniformly prox-regular for any $t \in [0,T]$. Meanwhile, each mapping C_i, $i \in \{1,2\}$, is Lipschitz-like around every point in its graph. To verify this property for C_1, one can apply a suitable implicit function theorem for set-valued mappings (for instance, [84, Theorem 3.2] and [95, Theorem 3.3]). Since all the assumptions of Theorem 4.22 are satisfied, we can assert that (P) has an uncountable number of Lipschitz solutions, among them there are two continuously differentiable solutions $u^{(i)}$, $i \in \{1,2\}$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$.

To verify the local Lipschitz-likeness of an implicit set-valued mapping defined by a generalized inequality system in infinite-dimensional Hilbert spaces or Banach spaces, one can use, e.g., some results in [42, 93].

Interestingly, Theorem 4.21 can be applied to the sweeping process considered in Example 4.28.

Example 4.29. Let $\mathcal{H}$, A_0, A_1, $f(\cdot)$, and $C_1(\cdot)$, $C_2(\cdot)$, and $C(\cdot)$ be the same as in Example 4.28. To show that every set-valued mapping C_i, $i \in \{1,2\}$, satisfies the assumptions (H1), (H2a), and (H3b), it suffices to verify the continuity of C_1 in the Hausdorff distance sense. To do so, take any $t, s \in [0,T]$ with $s < t$. Then, one has $C_1(t) \subset C_1(s)$. Given any $y = (y_1, y_2) \in C_1(s)$, we define $x = (x_1, x_2)$, where $x_1 = y_1 + t - s$ and $x_2 = y_2$. Since $e^{-x_1+t} = e^{-(y_1+t-s)+t} = e^{-y_1+s} \leq y_2 = x_2$, we get $x \in C_1(t)$. As $\|x - y\| = t - s$, it follows that $d_H(C_1(s), C_1(t)) \leq |t - s|$ for all $t, s \in [0,T]$. Therefore, by Theorem 4.21, (P) has an uncountable number of Lipschitz solutions, among them there are 2 solutions $u^{(i)}$, $i \in \{1,2\}$, with $\dot{u}^{(i)}(t) \in C_i(t)$ for almost every $t \in [0,T]$. Note that, to apply Theorem 4.21 for this sweeping process, as A_0 one can choose an arbitrary symmetric positive semidefinite 2×2 matrix (i.e., it is not necessary to put $A_0 = 0$).

4.5 Open Questions

Several open questions related to the results given in Sections 4.2–4.4 will be formulated in this section.

4.5.1 An Iteration Scheme

Let $\mathcal{H}$ be a Hilbert space, $A_0 : \mathcal{H} \to \mathcal{H}$ a symmetric positive semidefinite bounded linear operator, $A_1 : \mathcal{H} \to \mathcal{H}$ a symmetric coercive bounded linear operator, and $f : [0,T] \to \mathcal{H}$ a continuous mapping. Assume that $C : [0,T] \rightrightarrows \mathcal{H}$ is a set-valued mapping with nonempty closed convex values, which is Lipschitz-like around every point in its graph. Then, according to Theorem 4.19, the sweeping process (P) can have at most one solution. If $A_0 = 0$, by Theorem 4.15 we know that (P) has a unique solution, which is a continuously differentiable function. The first open question is about the case where A_0 is a nonzero operator.

(Q1) *If $A_0 \neq 0$, then the above assumptions are sufficient for the solution existence of* (P)*?*

If **(Q1)** can be solved in the affirmative, then it is of interest to have an iteration scheme to find the unique solution of (P). Based on Theorem 4.15, we can propose such a scheme. At the initial step $k = 0$, one solves the problem (P_1) and denotes the unique solution by $u^{(0)}$. Clearly, $u^{(0)}$ is a *rough* approximate solution of (P), because the operator $A_0 \neq 0$ had no role in creating the function. If u is the exact solution of (P), which is to be found, and $u^{(k)}$ is an approximate solution of (P) at a step $k \in \{0, 1, 2, \dots\}$, then

$$A_1\dot{u}(t) + A_0u(t) - f(t) \approx A_1\dot{u}(t) + A_0u^{(k)}(t) - f(t) \quad \text{a.e. } t \in [0,T].$$

Hence, setting $\tilde{f}_{k+1}(t) = -A_0u^{(k)}(t) + f(t)$ for all $t \in [0,T]$, we have

$$A_1\dot{u}(t) + A_0u(t) - f(t) \approx A_1\dot{u}(t) - \tilde{f}_k(t) \quad \text{a.e. } t \in [0,T].$$

So, the approximate problem of (P) at step $k+1$ is

$$\begin{cases} A_1\dot{u}(t) - \tilde{f}_{k+1}(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{$P_{1,k+1}$}$$

Since $\tilde{f}_{k+1} : [0,T] \to \mathcal{H}$ is a continuous function, problem ($P_{1,k+1}$) is of the form (P_1). Therefore, by Theorem 4.15, it has a unique solution, which is denoted

by $u^{(k+1)}$. The just described iteration scheme yields a sequence of continuously differentiable functions $\{u^{(k)}\}_{k\in\mathbb{N}}$. The second open question is as follows.

(Q2) *Whether the sequence* $\{u_k\}_{k\in\mathbb{N}}$ *converges to a solution of* (P)*?*

4.5.2 A Regularization Method

It is appealing to study the problem (P_1) in the setting of Theorem 4.15 with A_1 being only a symmetric positive semidefinite bounded linear operator. Let us denote the problem by (P_0) and its solution set by S_0.

(Q3) *Can we obtain a solution existence result for the problem* (P_0)*?*

If $S_0 \neq \emptyset$, then it would be reasonable to try to get a solution by the Tikhonov regularization method, which has been successfully applied for monotone variational inequalities (see, e.g., [91, Theorem 2.3]). For each $\varepsilon > 0$, the operator $A_1 + \varepsilon\mathrm{Id}$, where Id denotes the identity function, is coercive. Therefore, by Theorem 4.15, the regularized problem

$$\begin{cases} (A_1 + \varepsilon\mathrm{Id})\dot{u}(t) - f(t) \in -\mathcal{N}^P_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases}$$

of (P_0) has a unique solution, which is denoted by u_ε. The following questions deserve further considerations:

(Q4) *If* $S_0 \neq \emptyset$*, then the solution* u_ε *of the regularized problem converges in* $C^0([0,T],\mathcal{H})$ *to a solution of the original problem as* $\varepsilon \to 0^+$*?*

(Q5) *If* $S_0 \neq \emptyset$*, then the limit of* u_ε *as* $\varepsilon \to 0^+$*, if exists, is a solution of* (P_0) *whose derivative has the smallest* $L^2([0,T],\mathcal{H})$ *norm?*

4.5.3 Problems Having a Fixed Connected Uniformly Prox-Regular Constraint Set

Several examples of sweeping processes with uniformly prox-regular constraint sets have been given in Section 4.4. In Example 4.27, despite of the fact that the constraint set is fixed and finite, (P) has multiple solutions for any choice of A_1, A_0, and f. In addition, from Remark 4.26 where the constraint set of the problem under consideration is fixed and both operators A_0, A_1 are coercive, we see that the solution uniqueness cannot be guaranteed. Thus, the next questions seem to be interesting.

(Q6) *Under which conditions, can we obtain the solution existence for* (P) *when the constraint set is fixed, uniformly prox-regular, and connected?*

(Q7) *Under which conditions, can we obtain the solution uniqueness for* (P) *when the constraint set is fixed, uniformly prox-regular, and connected?*

4.6 Conclusions

In this chapter, we have established the solution existence for some classes of sweeping processes in Hilbert spaces with velocity constraints where the constraint sets can be either convex or nonconvex as well. For the convex case, a new result on the solution uniqueness has been obtained. For the nonconvex case, we have proved that there are many classes of problems having an uncountable number of solutions.

Using a theorem on the solution sensitivity of parametric variational inequalities, we have proposed a new approach to the solution existence and solution uniqueness of sweeping processes with velocity constraints. Among other things, being locally Lipschitz-like, the constraint set mapping needs not to be continuous in the Hausdorff distance sense. An example has been given to show the advantage of the new results. Other illustrative examples, where the focus was made on uniform prox-regularity of the constraint sets, have been presented.

Seven open problems deserving further investigations have been formulated.

Chapter 5

Solution Properties of Convex Sweeping Processes with Velocity Constraints

Some properties of solutions of convex sweeping processes with velocity constraints are studied in this chapter. Namely, the solution sensitivity with respect to the initial value, the boundedness, the closedness, and the convexity of the solution set are discussed in detail.

Using the same notation as in preceding chapter, we let $A_0, A_1 : \mathcal{H} \to \mathcal{H}$ be positive semi-definite, bounded symmetric linear operators and $f : [0,T] \to \mathcal{H}$ be a continuous mapping. In this chapter, we will only consider the case where $C(t)$, $t \in [0,T]$ is convex, which implies that the proximal cone in the formulation of (P) can be substituted by the normal cone in the sense of convex analysis. We then recall the problem (P)

$$\begin{cases} A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{P}$$

We denote by $\mathrm{Sol}(P, u_0)$ the solution set of (P) with the initial value u_0. For the reader's convenience, before investigating the solution properties for problem (P), we restate the assumptions that were used in preceding chapter and present some new ones, which will also be discussed.

Assumption (H1). *The constraint sets $C(t)$, $t \in [0,T]$, are nonempty, closed, and convex.*

Assumption (H1a). *The constraint sets $C(t)$, $t \in [0,T]$, are nonempty and*

convex.

Assumption (H2a). *The set-valued mapping C is continuous in the Hausdorff distance sense, i.e., there exists a continuous function $g : [0,T] \to \mathbb{R}$ such that*

$$d_H(C(s), C(t)) \leq |g(s) - g(t)| \quad \forall s, t \in [0,T]. \tag{5.1}$$

Assumption (H2b). *C is Lipschitz-like around every point in its graph.*

Assumption (H3a). *The constraint set $C(0)$ is bounded.*

Assumption (H3b). *There exist positive constants c_1, c_2 such that*

$$\langle A_1 x, x\rangle \geq c_1\|x\|^2 - c_2, \quad \forall x \in C(0).$$

Assumption (H3c). *There exist positive constants c_1, c_2 such that*

$$\langle A_1 x, x\rangle \geq c_1\|x\|^2 - c_2, \quad \forall t \in [0,T],\ \forall x \in C(t).$$

The following theorem can be deduced from the proof of [7, Theorem 5.2].

Theorem 5.1. (Cf. [7, Theorem 5.2]) *If A_0 is coercive and $C(t)$ is nonempty and convex for every $t \in [0,T]$, then* (P) *has at most one solution.*

5.1 Solution Sensitivity with respect to the Initial Value

In this section, we investigate the solution sensitivity of (P) with respect to the initial value when the solution is unique. The following theorem takes account of the case where the operator A_0 is coercive.

Theorem 5.2. *If the assumption* (H1a) *is satisfied,* $\mathrm{Sol}(P, u_0)$ *is nonempty for every $u_0 \in C(0)$, and A_0 is coercive with the modulus of coercivity α_0, then the mapping $\varphi : C(0) \to \mathcal{C}^0([0,T], \mathcal{H})$, $u_0 \mapsto u(u_0, \cdot)$, where $u(u_0, \cdot)$ denotes the unique solution of* (P)*, is Lipschitz continuous with the modulus $\sqrt{\frac{\|A_0\|}{\alpha_0}}$.*

Proof. Let $x_0, y_0 \in C(0)$ be given arbitrarily. Then, by our assumptions and Theorem 5.1, the sweeping process (P) has a unique solution $x(\cdot)$ with the initial value x_0 (resp., a unique solution $y(\cdot)$ with the initial value y_0). Since $C(t)$ is

convex, the inclusion

$$A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t)) \tag{5.2}$$

in the formulation of (P) can be rewritten equivalently as

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t).$$

As $\mathcal{N}_{C(t)}(\dot{u}(t)) = \emptyset$ if $\dot{u}(t) \notin C(t)$, the fulfillment of (5.2) for almost every $t \in [0,T]$ implies that $\dot{u}(t) \in C(t)$ for almost every $t \in [0,T]$. Therefore, the inclusions $\dot{x}(t) \in C(t)$ and $\dot{y}(t) \in C(t)$ hold for almost every $t \in [0,T]$. So, we have

$$\begin{cases} \langle A_1\dot{x}(t) + A_0x(t) - f(t), \dot{x}(t) - \dot{y}(t)\rangle \leq 0, \\ \langle -A_1\dot{y}(t) - A_0y(t) + f(t), \dot{x}(t) - \dot{y}(t)\rangle \leq 0 \end{cases} \tag{5.3}$$

for almost every $t \in [0,T]$. Adding the inequalities in (5.3) side by side yields

$$\langle A_1(\dot{x}(t) - \dot{y}(t)), \dot{x}(t) - \dot{y}(t)\rangle + \langle A_0(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle \leq 0 \quad \text{a.e. } t \in [0,T].$$

Since A_1 is positive semi-definite, this implies that

$$\langle A_0(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle \leq 0 \quad \text{a.e. } t \in [0,T]. \tag{5.4}$$

Taking the Lebesgue integral on both sides of the inequality in (5.4) and applying Remark 2.12, we obtain

$$\int_0^t \langle A_0(x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau)\rangle d\tau \leq 0 \quad (\forall t \in [0,T]). \tag{5.5}$$

As $\dfrac{d}{d\tau}\langle A_0(x(\tau) - y(\tau)), x(\tau) - y(\tau)\rangle = 2\langle A_0(x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau)\rangle$ at every point τ where both derivatives $\dot{x}(\tau), \dot{y}(\tau)$ exist, by Theorem 2.13 one has

$$\begin{aligned} \int_0^t \langle A_0(x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau)\rangle d\tau \quad &= \frac{1}{2}[\langle A_0(x(t) - y(t)), x(t) - y(t)\rangle \\ &\quad - \langle A_0(x(0) - y(0)), x(0) - y(0)\rangle]. \end{aligned} \tag{5.6}$$

Then, from (5.5) it follows that $\langle A_0(x(t)-y(t)), x(t)-y(t)\rangle - \langle A_0(x_0-y_0), x_0-y_0\rangle \leq 0$. Hence, by the coerciveness of A_0, we get

$$\alpha_0\|x(t) - y(t)\|^2 \leq \langle A_0(x(t) - y(t)), x(t) - y(t)\rangle \leq \langle A_0(x_0 - y_0), x_0 - y_0\rangle$$

$$\leq \|A_0\|\|x_0 - y_0\|^2.$$

Therefore, $\|x(t) - y(t)\| \leq \sqrt{\frac{\|A_0\|}{\alpha_0}}\|x_0 - y_0\|$ for all $t \in [0, T]$. So, the inequality

$$\|x - y\|_{\mathcal{C}^0} \leq \sqrt{\frac{\|A_0\|}{\alpha_0}}\|x_0 - y_0\|$$

holds for any $x_0, y_0 \in C(0)$. We have thus proved that the mapping φ is Lipschitz continuous on $C(0)$ with the modulus $\sqrt{\frac{\|A_0\|}{\alpha_0}}$. □

According to Theorem 4.19, the nonemptiness and convexity of $C(t)$ together with the coerciveness of A_1 can also guarantee the solution uniqueness for (P) if such a solution exists. A natural question arises: *Could we get a similar result as the one in Theorem 5.2 for the case under consideration?* The next theorem gives a complete answer to this question.

Theorem 5.3. *If the assumption* (H1a) *is fulfilled,* $\mathrm{Sol}(P, u_0)$ *is nonempty for every* $u_0 \in C(0)$*, and* A_1 *is coercive with the modulus of coercivity* α_1*, then the mapping* $\varphi : C(0) \to \mathcal{C}^0([0, T], \mathcal{H})$*,* $u_0 \mapsto u(u_0, \cdot)$*, where* $u(u_0, \cdot)$ *denotes the unique solution of* (P)*, is Lipschitz continuous with the modulus* $\sqrt{\frac{T\|A_0\|}{2\alpha_1}} + 1$*.*

Proof. For any $x_0, y_0 \in C(0)$, the assumptions made and Theorem 4.19 assure that (P) has a unique solution $x(\cdot)$ (resp., $y(\cdot)$) with the initial value x_0 (resp., y_0). Then, arguing similarly as in the proof of Theorem 5.2, we have

$$\langle A_1\dot{x}(t) + A_0x(t) - f(t), \dot{x}(t) - \dot{y}(t)\rangle \leq 0$$

and

$$\langle A_1\dot{y}(t) + A_0y(t) - f(t), \dot{y}(t) - \dot{x}(t)\rangle \leq 0$$

for almost every $t \in [0, T]$. Adding the last inequalities side by side, one obtains

$$\langle A_1(\dot{x}(t) - \dot{y}(t)), \dot{x}(t) - \dot{y}(t)\rangle + \langle A_0(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle \leq 0 \tag{5.7}$$

for almost every $t \in [0, T]$. Combining the coerciveness of A_0 with (5.7) yields

$$\alpha_1\|\dot{x}(t) - \dot{y}(t)\|^2 \leq -\langle A_0(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle \quad \text{a.e. } t \in [0, T]. \tag{5.8}$$

Since the function $t \mapsto -\langle A_0(x(t) - y(t)), \dot{x}(t) - \dot{y}(t)\rangle$ is integrable (in the Lebesgue sense), from (5.8) we can deduce that the function $t \mapsto \alpha_1\|\dot{x}(t) - \dot{y}(t)\|^2$ is also

integrable. Integrating both sides of the inequality in (5.8), we obtain

$$\int_0^t \alpha_1 \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 d\tau \leq -\int_0^t \langle A_0(x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau)\rangle d\tau. \tag{5.9}$$

At every point τ where both derivatives $\dot{x}(\tau), \dot{y}(\tau)$ exist, we have

$$\frac{d}{d\tau}\langle A_0(x(\tau) - y(\tau)), x(\tau) - y(\tau)\rangle = 2\langle A_0(x(\tau) - y(\tau)), \dot{x}(\tau) - \dot{y}(\tau)\rangle.$$

Hence, as noted in the preceding proof, by Theorem 2.13 we have (5.6). Consequently, from (5.9) it follows that

$$\begin{aligned}\int_0^t \alpha_1 \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 d\tau \leq -\frac{1}{2}[&\langle A_0(x(t) - y(t)), x(t) - y(t)\rangle \\ &- \langle A_0(x(0) - y(0)), x(0) - y(0)\rangle].\end{aligned}$$

Since A_0 is positive semidefinite, the latter implies

$$\int_0^t \alpha_1 \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 d\tau \leq \frac{1}{2}\langle A_0(x(0) - y(0)), x(0) - y(0)\rangle \leq \frac{\|A_0\|}{2}\|x_0 - y_0\|^2.$$

So, we have

$$\int_0^t \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 d\tau \leq \frac{\|A_0\|}{2\alpha_1}\|x_0 - y_0\|^2. \tag{5.10}$$

In addition, for each $t \in [0, T]$ one has

$$\begin{aligned}\|x(t) - y(t)\| &= \left\| \left(x_0 + \int_0^t \dot{x}(\tau)d\tau\right) - \left(y_0 + \int_0^t \dot{y}(\tau)d\tau\right) \right\| \\ &\leq \|x_0 - y_0\| + \int_0^t \|\dot{x}(\tau) - \dot{y}(\tau)\|d\tau.\end{aligned} \tag{5.11}$$

The inequality shows that the function $t \mapsto \|\dot{x}(t) - \dot{y}(t)\|$ belongs to the space $L^2([0, T], \mathbb{R})$. Therefore, setting $\beta(t) = 1$ for $t \in [0, T]$ and using the Hölder's inequality (see Proposition 2.14) for functions from $L^2([0, T], \mathbb{R})$, we have

$$\int_0^t (\beta(\tau)\|\dot{x}(\tau) - \dot{y}(\tau)\|)d\tau \leq \left(\int_0^t \beta(\tau)^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|\dot{x}(\tau) - \dot{y}(\tau)\|^2 d\tau\right)^{\frac{1}{2}}.$$

Then, combining this with (5.10) yields

$$\int_0^t \|\dot{x}(\tau) - \dot{y}(\tau)\|d\tau \leq \sqrt{t}\sqrt{\frac{\|A_0\|}{2\alpha_1}}\|x_0 - y_0\| \leq \sqrt{T}\sqrt{\frac{\|A_0\|}{2\alpha_1}}\|x_0 - y_0\|$$

for every $t \in [0,T]$. Hence, thanks to (5.11), we get

$$\|x(t)-y(t)\| \leq \|x_0-y_0\| + \sqrt{\frac{T\|A_0\|}{2\alpha_1}}\|x_0-y_0\| = \left(\sqrt{\frac{T\|A_0\|}{2\alpha_1}}+1\right)\|x_0-y_0\|$$

for all $t \in [0,T]$. This implies that the mapping φ defined in the statement of the theorem is Lipschitz continuous on $C(0)$ with the modulus $\sqrt{\frac{T\|A_0\|}{2\alpha_1}}+1$. □

5.2 Boundedness of the Solution Set

Noting that the Sobolev space $W^{1,1}([0,T],\mathcal{H})$ is the space of all absolutely continuous functions with its derivative in $L^1([0,T],\mathcal{H})$ (see Proposition 2.24), we can view the solution set of (P) as a subset of $W^{1,1}([0,T],\mathcal{H})$. Of course, at the same time, it is a subset of $\mathcal{C}^0([0,T],\mathcal{H})$.

If (P) has a unique solution then, under suitable conditions, we have established the solution sensitivity with respect to the initial value. When the solution uniqueness is not guaranteed, the solution set of (P) may be unbounded. Let us consider an example.

Example 5.4. Let $\mathcal{H} = \mathbb{R}^2$, $A_0 = A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $u_0 = (0,0)$, $f(t) = (0,t)$, and $C(t) = \mathbb{R} \times \{0\}$ for all $t \in [0,T]$. For every $\lambda \in \mathbb{R}$, we define a function by setting $u^{(\lambda)}(t) = (\lambda t, 0)$ for all $t \in [0,T]$. Clearly, $u^{(\lambda)}(0) = (0,0)$ and $\dot{u}^{(\lambda)}(t) = (\lambda, 0) \in C(t)$ for all $t \in [0,T]$. In addition,

$$\begin{aligned} A_1\dot{u}^{(\lambda)}(t) + A_0u^{(\lambda)}(t) - f(t) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{u}_1^{(\lambda)}(t) \\ \dot{u}_2^{(\lambda)}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1^{(\lambda)}(t) \\ u_2^{(\lambda)}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -t \end{pmatrix}. \end{aligned}$$

Since $\mathcal{N}_{C(t)}(\dot{u}^{(\lambda)}(t)) = \{0\} \times \mathbb{R}$, this yields $A_1\dot{u}^{(\lambda)}(t) + A_0u^{(\lambda)}(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}^{(\lambda)}(t))$ for all $t \in [0,T]$. Thus, for any $\lambda \in \mathbb{R}$, $u^{(\lambda)}$ is a solution of (P). As $\|u^{(\lambda)}\|_{\mathcal{C}^0} = |\lambda|T$, the solutions of (P) form an unbounded subset of $\mathcal{C}^0([0,T],\mathcal{H})$.

Our aim in this section is to establish some sets of conditions ensuring that the solution set of (P) is bounded.

Theorem 5.5. *If $C(t)$ is nonempty for all $t \in [0,T]$ and the assumptions* (H2a), (H3a) *are satisfied then, for any $u_0 \in C(0)$, the solution set* $\mathrm{Sol}(P,u_0)$ *is bounded in both spaces $\mathcal{C}^0([0,T],\mathcal{H})$ and $W^{1,1}([0,T],\mathcal{H})$.*

Proof. Let $u_0 \in C(0)$ be given arbitrarily. If $\mathrm{Sol}(P,u_0)$ is empty, then it is bounded. Suppose that $\mathrm{Sol}(P,u_0) \neq \emptyset$ and u is an element from $\mathrm{Sol}(P,u_0)$. As $C(0)$ is bounded, we can find $\rho_0 > 0$ such that $C(0) \subset \rho_0 \bar{\mathbb{B}}(0,1)$. Let $g : [0,T] \to \mathbb{R}$ be a continuous function satisfying (5.1). Thus, for all $t \in [0,T]$ one has $C(t) \subset C(0) + |g(0) - g(t)|$. This implies that for all $t \in [0,T]$ one has $C(t) \subset \rho \bar{\mathbb{B}}(0,1)$, where $\rho := \rho_0 + \max\{|g(0) - g(s)|| \ s \in [0,T]\}$. Since $\dot{u}(t) \in C(t)$ for almost every $t \in [0,T]$, one has $\|\dot{u}(t)\| \leq \rho$ for almost every $t \in [0,T]$. For any $t \in [0,T]$, we put $\Omega_1(t) = \{s \in [0,t] \mid \|\dot{u}(s)\| \leq \rho\}$ and $\Omega_2(t) = \{s \in [0,t] \mid \|\dot{u}(s)\| > \rho\}$. Then, the sets $\Omega_1(t)$ and $\Omega_2(t)$ are measurable, and $\mu(\Omega_2(t)) = 0$ with μ being the Lebesgue measure on $\mathbb{R}$. So, by Remark 4.16(c) and Proposition 2.19, we have

$$\begin{aligned}
\|u(t)\| = \|u_0 + \int_0^t \dot{u}(\tau)d\tau\| &= \|u_0 + \int_{\Omega_1(t)} \dot{u}(\tau)d\tau + \int_{\Omega_2(t)} \dot{u}(\tau)d\tau\| \\
&\leq \|u_0\| + \int_{\Omega_1(t)} \|\dot{u}(\tau)\|d\tau + \int_{\Omega_2(t)} \|\dot{u}(\tau)\|d\tau \\
&\leq \|u_0\| + \rho\mu(\Omega_1(t)) \\
&\leq \|u_0\| + \rho T.
\end{aligned}$$

Thus, $\|u\|_{\mathcal{C}^0} \leq \|u_0\| + \rho T$. This establishes the boundedness of $\mathrm{Sol}(P,u_0)$ in $\mathcal{C}^0([0,T],\mathcal{H})$. Since $\|u(t)\| \leq \|u_0\| + \rho T$ for all $t \in [0,T]$, $\|\dot{u}(t)\| \leq \rho$ for a.e. $t \in [0,T]$, and $u \in \mathrm{Sol}(P,u_0)$ was chosen arbitrarily, by (2.1) we can assert that $\mathrm{Sol}(P,u_0)$ is a bounded subset of the Sobolev space $W^{1,1}([0,T],\mathcal{H})$. □

To deal with the case where the sets $C(t)$, $t \in [0,T]$, can be unbounded, we will need the following technical lemma. Since we still have not found any reference containing this statement, a detailed proof is given here.

Lemma 5.6. *Let f be a Lebesgue integrable, real-valued function defined on $[0,T]$. If*

$$f(t) \leq a + b\int_0^t f(\tau)d\tau \quad \text{a.e. } t \in [0,T] \tag{5.12}$$

for some constants a, b with $b \neq 0$, then $\displaystyle\int_0^t f(\tau)d\tau \leq \frac{a}{b}(\exp(bt) - 1)$ for all $t \in [0,T]$.

Proof. Let f be a Lebesgue integrable function on $[0,T]$ satisfying (5.12).

Multiplying both sides of the inequality in (5.12) by $\exp(-bt)$ yields

$$\exp(-bt)f(t) - b\exp(-bt)\int_0^t f(\tau)d\tau \leq a\exp(-bt) \quad \text{a.e. } t \in [0,T]. \tag{5.13}$$

By Proposition 2.15 one has

$$\frac{d}{ds}\left(\exp(-bs)\int_0^s f(\tau)d\tau\right) = \exp(-bs)f(s) - b\exp(-bs)\int_0^s f(\tau)d\tau.$$

Thus, taking the Lebesgue integral on both sides of the inequality in (5.13) and applying Remark 2.12, we obtain

$$\int_0^t \frac{d}{ds}\left(\exp(-bs)\int_0^s f(\tau)d\tau\right)ds \leq \int_0^t a\exp(-bs)ds \quad \forall t \in [0,T].$$

It follows that

$$\exp(-bt)\int_0^t f(\tau)d\tau \leq \frac{a}{b}(1-\exp(-bt)) \quad \forall t \in [0,T].$$

Hence, we get

$$\int_0^t f(\tau)d\tau \leq \frac{a}{b}(\exp(bt)-1) \quad \forall t \in [0,T].$$

The proof is complete. □

Theorem 5.7. *If the assumptions* (H1a), (H2a) *and* (H3b) *are satisfied then, for any $u_0 \in C(0)$, the solution set* $\mathrm{Sol}(P,u_0)$ *is bounded in both spaces* $\mathcal{C}^0([0,T],\mathcal{H})$ *and* $W^{1,1}([0,T],\mathcal{H})$.

Proof. Given any $u_0 \in C(0)$. If $\mathrm{Sol}(P,u_0)$ is empty, then it is bounded. Suppose that $\mathrm{Sol}(P,u_0)$ is nonempty. Take any $u \in \mathrm{Sol}(P,u_0)$ and let $\varepsilon > 0$ be given arbitrarily. Since $C(t)$ is nonempty, for any $t \in [0,T]$ there exists $z_t \in C(t)$ satisfying $\|u_0 - z_t\| < d(u_0, C(t)) + \varepsilon$. By (H2a), we have

$$\begin{aligned}\|z_t\| - \|u_0\| \leq \|u_0 - z_t\| &< d(u_0,C(t)) + \varepsilon \leq d_H(C(0),C(t)) + \varepsilon \\ &\leq |g(0) - g(t)| + \varepsilon.\end{aligned}$$

Then, setting $\beta := \|u_0\| + \max\limits_{\tau\in[0,T]}|g(0)-g(\tau)| + \varepsilon$, we get $\|z_t\| < \beta$. So, for every $t \in [0,T]$ one can find some $z_t \in C(t)$ such that $\|z_t\| < \beta$. As $u \in \mathrm{Sol}(P,u_0)$, by (H1a) one has for almost every $t \in [0,T]$ that

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t).$$

Substituting $z = z_t$ into the above inequality yields $\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z_t\rangle \leq 0$ for almost every $t \in [0,T]$. Thus,

$$\langle A_1\dot{u}(t), \dot{u}(t)\rangle - \langle A_1\dot{u}(t), z_t\rangle + \langle A_0u(t) - f(t), \dot{u}(t)\rangle - \langle A_0u(t) - f(t), z_t\rangle \leq 0. \quad (5.14)$$

Using the assumptions (H2a), (H3b), and Remark 4.16, we can find positive constants $\hat{c}_1, \hat{c}_2$ such that $\langle A_1x, x\rangle \geq \hat{c}_1\|x\|^2 - \hat{c}_2$ for all $t \in [0,T]$ and $x \in C(t)$. Then, (5.14) implies that

$$\hat{c}_1\|\dot{u}(t)\|^2 - \hat{c}_2 - \langle A_1\dot{u}(t), z_t\rangle + \langle A_0u(t) - f(t), \dot{u}(t)\rangle - \langle A_0u(t) - f(t), z_t\rangle \leq 0$$

for a.e. $t \in [0,T]$. So, one has

$$\hat{c}_1\|\dot{u}(t)\|^2 - \hat{c}_2 - \beta\|A_1\|\|\dot{u}(t)\| - (\|A_0\|\|u(t)\| + \|f\|_{\mathcal{C}^0})\|\dot{u}(t)\| - \beta(\|A_0\|\|u(t)\| + \|f\|_{\mathcal{C}^0}) \leq 0$$

for a.e. $t \in [0,T]$. For each $t \in [0,T]$, setting $a_1(t) = \beta\|A_1\| + \|A_0\|\|u(t)\| + \|f\|_{\mathcal{C}^0}$ and

$$a_2(t) = \beta(\|A_0\|\|u(t)\| + \|f\|_{\mathcal{C}^0}) + \hat{c}_2,$$

we get

$$\hat{c}_1\|\dot{u}(t)\|^2 - a_1(t)\|\dot{u}(t)\| - a_2(t) \leq 0 \quad \text{a.e. } t \in [0,T]. \quad (5.15)$$

As $\hat{c}_1 > 0$ and $a_2(t) > 0$ for $t \in [0,T]$, the quadratic polynomial

$$q(x) := \hat{c}_1x^2 - a_1(t)x - a_2(t)$$

has two roots with different signs. Hence, (5.15) holds if and only if

$$\|\dot{u}(t)\| \leq \frac{a_1(t) + \sqrt{a_1(t)^2 - 4\hat{c}_1a_2(t)}}{2\hat{c}_1} \quad \text{a.e. } t \in [0,T].$$

Since $\sqrt{a_1(t)^2 - 4\hat{c}_1a_2(t)} \leq a_1(t)$, this yields $\|\dot{u}(t)\| \leq \dfrac{a_1(t)}{\hat{c}_1}$ for a.e. $t \in [0,T]$. Therefore,

$$\|\dot{u}(t)\| \leq \frac{\beta\|A_1\| + \|A_0\|\|u(t)\| + \|f\|_{\mathcal{C}^0}}{\hat{c}_1}$$

for a.e. $t \in [0,T]$. Then one has

$$\|\dot{u}(t)\| \leq \gamma(1 + \|u(t)\|) \quad \text{a.e. } t \in [0,T], \quad (5.16)$$

where $\gamma := \max\left\{\dfrac{\beta\|A_1\|+\|f\|_{\mathcal{C}^0}}{\hat{c}_1}, \dfrac{\|A_0\|}{\hat{c}_1}\right\}$. Since

$$\|u(t)\| = \|u_0 + \int_0^t \dot{u}(\tau)d\tau\| \leq \|u_0\| + \int_0^t \|\dot{u}(\tau)\|d\tau \tag{5.17}$$

(see Remark 4.16(c) and Proposition 2.19(ii)), from (5.16) it follows that

$$\|\dot{u}(t)\| \leq \gamma(1 + \|u_0\|) + \gamma \int_0^t \|\dot{u}(\tau)\|d\tau \quad \text{a.e. } t \in [0,T].$$

So, applying Lemma 5.6 for $f(t) := \|\dot{u}(t)\|$, $a := \gamma(1 + \|u_0\|)$, and $b := \gamma$ gives

$$\int_0^t \|\dot{u}(\tau)\|d\tau \leq (1 + \|u_0\|)(\exp(\gamma t) - 1) \leq (1 + \|u_0\|)(\exp(\gamma T) - 1) \quad \forall t \in [0,T].$$

Combining this with (5.17) yields

$$\|u(t)\| \leq \|u_0\| + (1 + \|u_0\|)(\exp(\gamma T) - 1) \quad \forall t \in [0,T]. \tag{5.18}$$

It follows that $\|u\|_{\mathcal{C}^0} \leq \|u_0\| + (1 + \|u_0\|)(\exp(\gamma T) - 1)$. So, $\mathrm{Sol}(P, u_0)$ is a bounded subset of $\mathcal{C}^0([0,T], \mathcal{H})$. Finally, using the estimates (5.16), (5.18), and formula (2.1), we can find a constant $\rho > 0$ such that $\|u\|_{W^{1,1}} \leq \rho$ for any $u \in \mathrm{Sol}(P, u_0)$. The proof is complete. □

Theorem 5.8. *If the assumptions* (H1a), (H2b) *and* (H3c) *are satisfied then, for any $u_0 \in C(0)$, the solution set* $\mathrm{Sol}(P, u_0)$ *is bounded in both spaces* $\mathcal{C}^0([0,T], \mathcal{H})$ *and* $W^{1,1}([0,T], \mathcal{H})$.

Proof. For each $t \in [0,T]$, pick a point $x_t \in C(t)$. As C is Lipschitz-like around (t, x_t), there exist an open neighborhood V_t of t in the induced topology of $[0,T] \subset \mathbb{R}$, a neighborhood W_t of x_t in $\mathcal{H}$, and a constant $\kappa_t > 0$ such that

$$C(t') \cap W_t \subset C(t'') + \kappa_t |t' - t''| \bar{\mathbb{B}}(0,1) \quad \forall t', t'' \in V_t. \tag{5.19}$$

Since $[0,T] = \bigcup_{t \in [0,T]} V_t$, the compactness of $[0,T]$ implies the existence of $t_1, \ldots, t_k$ in $[0,T]$ such that $[0,T] = \bigcup_{i=1}^{k} V_{t_i}$. For each $i \in \{1, \ldots, k\}$, we have $x_{t_i} \in W_{t_i}$. So, thanks to (5.19), for every $t \in V_{t_i}$ we can find $z_t^{(i)} \in C(t)$ and $\xi_t^{(i)} \in \bar{\mathbb{B}}(0,1)$ satisfying $x_{t_i} = z_t^{(i)} + \kappa_{t_i}|t - t_i|\xi_t^{(i)}$. Then,

$$\|z_t^{(i)}\| \leq \|x_{t_i}\| + \kappa_{t_i}|t - t_i| \leq \|x_{t_i}\| + \kappa_{t_i} T. \tag{5.20}$$

Setting $\beta = \max\{\|x_{t_i}\|+\kappa_{t_i}T \mid i \in \{1,\dots,k\}\}$, we have $\beta > 0$. For each $t \in [0,T]$, there is some $i \in \{1,\dots,k\}$ such that $t \in V_{t_i}$ and, by (5.20), the element $z_t^{(i)} \in C(t)$ satisfies the estimate $\|z_t^{(i)}\| \leq \beta$. Therefore, for every $t \in [0,T]$, there exists at least one point of the form $z_t^{(i)}$ such that $z_t^{(i)} \in C(t)$ and $\|z_t^{(i)}\| \leq \beta$.

Let $u_0 \in C(0)$ be given arbitrarily. Since $\mathrm{Sol}(P,u_0)$ bounded if it is empty, it suffices to consider the case $\mathrm{Sol}(P,u_0) \neq \emptyset$. Take any $u \in \mathrm{Sol}(P,u_0)$. By (H1a) we deduce for almost every $t \in [0,T]$ that $\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0$ for all $z \in C(t)$. Substituting $z = z_t^{(i)}$ into the last inequality yields $\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z_t^{(i)}\rangle \leq 0$ for almost every $t \in [0,T]$. Using the assumption (H3c) and repeating the final part of the proof of Theorem 5.5 (starting from inequality (5.14)), we can show that the solution set $\mathrm{Sol}(P,u_0)$ is bounded in both spaces $\mathcal{C}^0([0,T],\mathcal{H})$ and $W^{1,1}([0,T],\mathcal{H})$. □

Remark 5.9. The boundedness of $\mathrm{Sol}(P,u_0)$ in Theorem 5.8 is also valid if instead of the assumption (H2b) one requires that C is inner semicontinuous at every point in its graph, i.e., for every $(t,x) \in [0,T] \times \mathcal{H}$ with $x \in C(t)$, if $U \subset \mathcal{H}$ is an open set containing x, then there exists a neighborhood V of t in $[0,T]$ such that $C(t') \cap U \neq \emptyset$ for all $t' \in V$. Indeed, for each $t \in [0,T]$, select a point $x_t \in C(t)$. The inner semicontinuity of C at (t,x_t) assures that there is an open neighborhood V_t of t in the induced topology of $[0,T]$ such that $C(t') \cap \mathbb{B}(x_t,1) \neq \emptyset$ for every $t' \in V_t$. By the compactness of $[0,T]$, from the open covering $\{V_t\}_{t\in[0,T]}$ of the segment we can extract a finite subcover $V_{t_1},\dots,V_{t_k}$. So, for each $t \in [0,T]$, there exists an index $i \in \{1,\dots,k\}$ such that $t \in V_{t_i}$. Since $C(t) \cap \mathbb{B}(x_{t_i},1) \neq \emptyset$, there is a vector $z_t^{(i)} \in C(t) \cap \mathbb{B}(x_{t_i},1)$. Then one has $\|z_t^{(i)}\| \leq \beta$, where $\beta := \max\{\|x_i\|+1 \mid i \in \{1,\dots,k\}\}$. Consequently, for each $t \in [0,T]$, there exists at least one point of the form $z_t^{(i)}$ such that $z_t^{(i)} \in C(t)$ and $\|z_t^{(i)}\| \leq \beta$. Then, as noted above, the usage of (H3c) and the repetition of the final part of the proof of Theorem 5.5 yield the desired assertion.

Remark 5.10. If a set-valued mapping is Lipschitz-like around a point in its graph then it is inner semicontinuous at that point (see, e.g., [93, Proposition 3.1]). On the other hand, there exist locally Lipschitz-like mappings which are not continuous in the Hausdorff distance sense (see Example 4.17). Clearly, if the mapping $C : [0,T] \rightrightarrows \mathcal{H}$ is continuous in the Hausdorff distance sense, then it is inner semicontinuous at every point in its graph. Example 4.17 shows that the converse is not true in general.

Remark 5.11. The continuity in the Hausdorff distance sense of $C(\cdot)$ together with the assumption (H3b) implies (H3c) (see Remark 4.16). However, a similar

implication may not hold under the inner semicontinuity of $C(\cdot)$ at every point in its graph or even under the Lipschitz-likeness of $C(\cdot)$ around every point in its graph.

5.3 Closedness of the Solution Set

First, let us show that the closedness of $\mathrm{Sol}(P, u_0)$ may not available even for very simple problems in finite dimensions.

Proposition 5.12. *The solution set of* (P) *may not be closed in* $\mathcal{C}^0([0,T],\mathcal{H})$.

Proof. We will prove the proposition by constructing a suitable example. Let $\mathcal{H} = \mathbb{R}$, $A_0 = 0$, $A_1 = 0$, $u_0 = 0$, $f(t) \equiv 0$, and $C(t) = \mathbb{R}$ for all $t \in [0,T]$. Then, an absolutely continuous function $u : [0,T] \to \mathbb{R}$ is a solution of (P) if and only if

$$\begin{cases} 0 \in \mathcal{N}_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0,T], \\ u(0) = 0. \end{cases}$$

Since $C(t) = \mathbb{R}$ for all $t \in [0,T]$, $\mathcal{N}_{C(t)}(\dot{u}(t)) = \{0\}$ for any t where $\dot{u}(t)$ exists. So, any absolutely continuous function $u : [0,T] \to \mathbb{R}$ with $u(0) = 0$ is a solution of (P). For $k \in \mathbb{N}$, let

$$x_k(t) = \begin{cases} t^2 \sin(\frac{1}{t^2}) & \text{if } t \in (\frac{1}{k}, T] \\ \frac{t}{k} \sin(k^2) & \text{if } t \in [0, \frac{1}{k}]. \end{cases}$$

and

$$x(t) = \begin{cases} t^2 \sin(\frac{1}{t^2}) & \text{if } t \in (0,T] \\ 0 & \text{if } t = 0. \end{cases}$$

Clearly, $x_k(\cdot)$ is a Lipschitz function for each $k \in \mathbb{N}$. Since $x_k(0) = 0$, $x_k(\cdot)$ is a solution of (P) for every $k \in \mathbb{N}$. In addition, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \sup_{t\in[0,T]} |x(t) - x_k(t)| &= \sup_{0<t\le\frac{1}{k}} \left| t^2 \sin\left(\frac{1}{t^2}\right) - \frac{t}{k}\sin(k^2) \right| \\ &\le \sup_{0<t\le\frac{1}{k}} \left| t^2 \sin\left(\frac{1}{t^2}\right) \right| + \sup_{0<t\le\frac{1}{k}} \left| \frac{t}{k}\sin(k^2) \right| \\ &\le \sup_{0<t\le\frac{1}{k}} t^2 + \sup_{0<t\le\frac{1}{k}} \frac{t}{k} \\ &= \frac{2}{k^2}. \end{aligned}$$

Therefore, x_k strongly converges to x in $\mathcal{C}^0([0,T],\mathbb{R})$ as $k \to \infty$. However, since $x(\cdot)$ is not of bounded variation (see Example 2.6(iv)), it is not absolutely continuous. Hence, x is not a solution of (P). We have thus shown that $\mathrm{Sol}(P,u_0)$ is non-closed in $\mathcal{C}^0([0,T],\mathcal{H})$. □

We now present a lemma on the relation between strong convergence of sequence of functions in $L^1([0,T],\mathcal{H})$ and its pointwise convergence.

Lemma 5.13. *Let $\{x_n\}$ be a sequence in $L^1([0,T],\mathcal{H})$ and let $x \in L^1([0,T],\mathcal{H})$ be such that x_n converges strongly to x in $L^1([0,T],\mathcal{H})$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}(t)$ converges to $x(t)$ almost everywhere on $[0,T]$.*

Proof. Since $\{x_n\}$ is a strongly convergent sequence, it is a Cauchy sequence. Hence, for every positive integer k we can find a positive integer n_k such that

$$\|x_m - x_q\|_{L^1} \leq \frac{1}{2^k} \quad (\forall m \geq n_k, \forall q \geq n_k).$$

Without loss of generality we may assume that $n_{k_1} < n_{k_2}$ whenever $k_1 < k_2$. Clearly, the above choice of $\{n_k\}$ implies that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ having the property

$$\|x_{n_{k+1}} - x_{n_k}\|_{L^1} \leq \frac{1}{2^k} \quad \forall k \geq 1. \tag{5.21}$$

Define

$$y_m(t) = \sum_{k=1}^{m} \|x_{n_{k+1}}(t) - x_{n_k}(t)\|. \tag{5.22}$$

For all $t \in [0,T]$, by (5.22) and (5.21) we have

$$|y_m(t)| = \sum_{k=1}^{m} \|x_{n_{k+1}}(t) - x_{n_k}(t)\| \leq \sum_{k=1}^{m} \frac{1}{2^k} \leq 1.$$

Thus, $|y_m(t)| \leq 1$ for every $t \in [0,T]$. Since $x_n \in L^1([0,T],\mathcal{H})$ is measurable for all $n \in \mathbb{N}$, the function $y_m : [0,T] \to \mathbb{R}$ is also measurable for all $m \in \mathbb{N}$. As $\{y_m\}$ is a increasing sequence of real-valued functions, by the monotone convergence theorem [22, Theorem 4.1] one can assert that $y_m(t)$ converges to a function $y(t)$ almost everywhere on $[0,T]$. Since $|y(t)| \leq 1$ for all $t \in [0,T]$, we see that $y \in L^1([0,T],\mathbb{R})$. On the other hand, for $i > j \geq 2$, we have

$$\|x_{n_i}(t) - x_{n_j}(t)\| \leq \|x_{n_i}(t) - x_{n_{i-1}}(t)\| + \ldots + \|x_{n_{j+1}}(t) - x_{n_j}(t)\| \leq y(t) - y_{n_{j-1}}(t). \tag{5.23}$$

It follows that, for almost every $t \in [0,T]$, $\{x_{n_k}(t)\}$ is a Cauchy sequence in $\mathcal{H}$ and it converges to a finite limit, say, $\tilde{x}(t)$. From (5.23), letting i tend to infinity, we obtain

$$\|\tilde{x}(t) - x_{n_j}(t)\| \leq y(t) - y_{n_{j-1}}(t) \leq y(t)$$

for almost every $t \in [0,T]$ and for any $j \geq 2$. Hence, one has $\tilde{x} \in L^1([0,T],\mathcal{H})$. Since $\|x_{n_k}(t) - \tilde{x}(t)\|^2 \to 0$ and $\|x_{n_k}(t) - \tilde{x}(t)\| \leq y(t)$ almost everywhere on $[0,T]$, using the Dominated Convergence Theorem 2.18, we can deduce that $\|x_{n_k} - \tilde{x}\|_1 \to 0$. Since x_n converges strongly to x in $L^1([0,T],\mathcal{H})$ and $L^1([0,T],\mathcal{H})$ is a subspace of $L^1([0,T],\mathcal{H})$, x_n converges strongly to x in $L^1([0,T],\mathcal{H})$. By the uniqueness of limit, we have $\tilde{x} = x$. Therefore, we have shown that $x_{n_k}(t)$ converges to $x(t)$ almost everywhere on $[0,T]$.

The proof is complete. □

Remark 5.14. In the formulation of Lemma 5.13, one can replace $L^1([0,T],\mathcal{H})$ by any Bochner space $L^p(\Omega,X)$ with $1 \leq p < \infty$. The proof remains the same, provided that one writes $L^p(\Omega,X)$ instead of $L^1([0,T],\mathcal{H})$ and $L^p([0,T],\mathbb{R})$ instead of $L^1([0,T],\mathbb{R})$.

Next, we will prove that the solution set of (P) is closed if it is regarded as a subset of an appropriate space. More precisely, the following theorem confirms that the Sobolev space $W^{1,1}([0,T],\mathcal{H})$ is such a space. (This result can be explained by the well known fact that the norm of $W^{1,1}([0,T],\mathcal{H})$ is finer than the one of $\mathcal{C}^0([0,T],\mathcal{H})$.)

Theorem 5.15. *If the assumption* (H1) *is satisfied then, for any $u_0 \in C(0)$, the solution set* $\mathrm{Sol}(P,u_0)$ *is closed in* $W^{1,1}([0,T],\mathcal{H})$.

Proof. Let $u_0 \in C(0)$ be given. Suppose that $\{u_k\} \subset \mathrm{Sol}(P,u_0)$ is a sequence converging strongly in $W^{1,1}([0,T],\mathcal{H})$ to u as $k \to \infty$. Then, u is an absolutely continuous function. To prove that u satisfies the initial condition in (P), we can argue as follows. Since the norm in $W^{1,1}([0,T],\mathcal{H})$ is given by (2.1), we have

$$\lim_{k\to\infty} \int_0^T \|u_k(\tau) - u(\tau)\| d\tau = 0 \tag{5.24}$$

and

$$\lim_{k\to\infty} \int_0^T \|\dot{u}_k(\tau) - \dot{u}(\tau)\| d\tau = 0. \tag{5.25}$$

Note that $u_k(t) = u_k(0) + \int_0^t \dot{u}_k(\tau)d\tau$ and $u(t) = u(0) + \int_0^t \dot{u}(\tau)d\tau$ for every $t \in [0,T]$ and for all $k \in \mathbb{N}$ (see [9, Remark 3.4(c)]). Hence, from (5.24), (5.25),

and Proposition 2.19 it follows that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \int_0^T \|u_k(\tau) - u(\tau)\| d\tau \\
&= \lim_{k\to\infty} \left[\int_0^T \left\| u_k(0) - u(0) + \int_0^\tau (\dot{u}_k(s) - \dot{u}(s)) ds \right\| d\tau \right] \\
&\geq \liminf_{k\to\infty} \left[\int_0^T \left(\|u_k(0) - u(0)\| - \left\| \int_0^\tau (\dot{u}_k(s) - \dot{u}(s)) ds \right\| \right) d\tau \right] \\
&\geq \liminf_{k\to\infty} \left[\int_0^T \left(\|u_k(0) - u(0)\| - \int_0^T \|\dot{u}_k(s) - \dot{u}(s)\| ds \right) d\tau \right] \\
&= \liminf_{k\to\infty} \left[T\|u_0 - u(0)\| - T \int_0^T \|\dot{u}_k(s) - \dot{u}(s)\| ds \right] \\
&= T\|u_0 - u(0)\|.
\end{aligned}$$

So, $u(0) = u_0$.

It remains to prove that u satisfies the differential inclusion in (P).

Setting $\mathcal{C} = \{\varphi \in L^1([0,T], \mathcal{H}) \mid \varphi(t) \in C(t) \text{ a.e. } t \in [0,T]\}$, we will prove that $\mathcal{C}$ is closed in $L^1([0,T], \mathcal{H})$. Let $\{\varphi_m\} \subset D$ be a sequence converging strongly in $L^1([0,T], \mathcal{H})$ to a function ψ. Thanks to Lemma 5.13, we can find a subsequence $\{\varphi_{m_j}\}$ of $\{\varphi_m\}$ such that $\varphi_{m_j}(t)$ converges to $\psi(t)$ for almost every $t \in [0,T]$. Since $\varphi_{m_j}(t) \in C(t)$ a.e. $t \in [0,T]$ and $C(t)$ is closed, we have $\psi(t) \in C(t)$ a.e. $t \in [0,T]$. Hence, one has $\psi \in \mathcal{C}$. This shows that $\mathcal{C}$ is closed in $L^1([0,T], \mathcal{H})$.

Since $\{u_k\} \subset \mathrm{Sol}(P, u_0)$, we have $\dot{u}_k \in \mathcal{C}$ for all $k \in \mathbb{N}$. From (5.25) it follows that $\dot{u} \in \mathcal{C}$. So, $\dot{u}(t) \in C(t)$ for almost every $t \in [0,T]$. As $C(t)$ is convex for all $t \in [0,T]$, the inclusion $A_1\dot{u}_k(t) + A_0u_k(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}_k(t))$ is equivalent to

$$\langle A_1\dot{u}_k(t) + A_0u_k(t) - f(t), \dot{u}_k(t) - z\rangle \leq 0 \quad \forall z \in C(t). \tag{5.26}$$

For each $k \in \mathbb{N}$, (5.26) holds for a.e. $t \in [0,T]$. Thus, there exists a subset $D_k \subset [0,T]$ having zero Lebesgue measure that (5.26) holds for every t in $[0,T]\setminus D_k$. Putting $D = \bigcup_{k\in\mathbb{N}} D_k$, we see that D is a set of zero Lebesgue measure and (5.26) holds for all $k \in \mathbb{N}$ and for every t in $[0,T] \setminus D$. For each t from $[0,T] \setminus D$, passing the inequality in (5.26) to the limit yields

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t).$$

Thus, for almost every $t \in [0,T]$, one has $A_1\dot{u}(t) + A_0u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t))$.

We have thus proved that $u \in \mathrm{Sol}(P, u_0)$ and, therefore, established the desired

closedness of $\mathrm{Sol}(P, u_0)$ in $W^{1,1}([0,T], \mathcal{H})$. □

5.4 Convexity of the Solution Set

As the normal cone in the sense of convex analysis to a convex set can be presented in a variational way, sweeping processes and variational inequalities are closely related. So, the convexity of the solution set of a sweeping process may have some connections with that property of the solution set of a variational inequality.

Theorem 5.16. *If the assumption* (H1) *is fulfilled and* $A_0 = 0$*, then* $\mathrm{Sol}(P, u_0)$ *is convex for every* $u_0 \in C(0)$*.*

Proof. Let $u_0 \in C(0)$ be taken arbitrarily. It suffices to consider the case where $\mathrm{Sol}(P, u_0)$ is nonempty. Under the assumption (H1) and the condition $A_0 = 0$, an absolutely continuous function u belongs to $\mathrm{Sol}(P, u_0)$ if and only if $u(0) = u_0$ and

$$\langle A_1 \dot{u}(t) - f(t), y - \dot{u}(t) \rangle \geq 0 \quad \forall y \in C(t)$$

for a.e. $t \in [0,T]$. The latter means that $z(t) := \dot{u}(t)$ is a solution of the variational inequality

$$\langle F(z,t), y - z \rangle \geq 0 \quad \forall y \in C(t) \tag{5.27}$$

for a.e. $t \in [0,T]$, where $F(z,t) := A_1 z - f(t)$. By the assumed positive semidefiniteness of A_1, one has

$$\langle F(z',t) - F(z,t), z' - z \rangle = \langle A_1(z' - z), z' - z \rangle \geq 0$$

for every $z, z' \in \mathcal{H}$. Hence, $F(\cdot, t) : \mathcal{H} \to \mathcal{H}$ is a monotone operator. Moreover, since the linear operator A_1 is bounded, $F(\cdot, t)$ is continuous. Therefore, applying Minty's lemma [55, Lemma 1.5] for the monotone variational inequality (5.27), we can assert that the solution set of (5.27) is closed an convex for every $t \in [0,T]$. Consequently, if u, v are two elements of $\mathrm{Sol}(P, u_0)$ and $\lambda \in (0,1)$ is given arbitrarily, $(1-\lambda)\dot{u}(t) + \lambda \dot{v}(t)$ is a solution of (5.27) for almost every $t \in [0,T]$. Since $t \mapsto (1-\lambda)\dot{u}(t) + \lambda\dot{v}(t)$ is Bochner integrable (see [32, Proposition 1.4.17]), the formula $w(t) := u_0 + \displaystyle\int_0^t [(1-\lambda)\dot{u}(\tau) + \lambda \dot{v}(\tau)]\, d\tau$ defines an absolutely continuous function. Clearly, $w(0) = u_0$. In addition, we have $\dot{w}(t) = (1-\lambda)\dot{u}(t) + \lambda\dot{v}(t)$ for a.e. $t \in [0,T]$ (see, e.g., [9, Remark 3.4(d)]). So, $w(t)$ is a solution of (5.27) for

a.e. $t \in [0,T]$. This implies that

$$A_1\dot{w}(t) + A_0 w(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{w}(t)) \quad \text{a.e. } t \in [0,T].$$

Hence, $w \in \mathrm{Sol}(P,u_0)$. The convexity of $\mathrm{Sol}(P,u_0)$ has been proved. □

The kernel of the operator $A_0 : \mathcal{H} \to \mathcal{H}$ plays an important role in the forthcoming results. Recall that $\ker A_0 := \{x \in \mathcal{H} \mid A_0 x = 0\}$. Note that the quadratic form $\varphi(y) := \langle A_0 y, y\rangle$ is Fréchet differentiable on $\mathcal{H}$ because A_0 is bounded (see, e.g., [96, Proposition 2.1]). Since $\langle A_0 y, y\rangle \geq 0$ for all $y \in \mathcal{H}$, a vector $x \in \mathcal{H}$ satisfies the equality $\langle A_0 x, x\rangle = 0$ if and only if x is a solution of the optimization problem $\min\{\varphi(y) \mid y \in \mathcal{H}\}$. If x is a solution of the latter, then by the Fermat rule one has $\nabla\varphi(x) = 0$, i.e., $A_0 x = 0$. Conversely, if $A_0 x = 0$ then $\varphi(x) = 0$. Therefore, we have

$$\{x \in \mathcal{H} \mid \langle A_0 x, x\rangle = 0\} = \ker\, A_0. \tag{5.28}$$

Under a mild assumption, using one solution u of (P), we can construct a closed convex set $\mathcal{K}$ in $W^{1,1}([0,T],\mathcal{H})$, such that the solution set $\mathrm{Sol}(P,u_0)$ is contained in $u+\mathcal{K}$. Thus, the closed convex set $u+\mathcal{K}$ is an outer estimate for $\mathrm{Sol}(P,u_0)$. The estimate is sharp, because in some cases it holds as an equality (see Theorem 5.18 below).

Theorem 5.17. *Suppose that* (H1) *is satisfied. For any* $u_0 \in C(0)$*, if* $\mathrm{Sol}(P,u_0)$ *is nonempty and* u *is a selected solution of* (P)*, then*

$$\mathrm{Sol}(P,u_0) \subset u + \mathcal{K}, \tag{5.29}$$

where

$$\mathcal{K} := \big\{y \in W^{1,1}([0,T],\mathcal{H}) \mid y(0) = 0,\ \dot{y}(t) \in (C(t) - \dot{u}(t)) \cap \ker\, A_0 \ \text{ a.e. } t \in [0,T]\big\} \tag{5.30}$$

is a closed convex set.

Proof. Select a solution u of (P). Let $v \in \mathrm{Sol}(P,u_0)$ be chosen arbitrarily. Since (H1) is fulfilled, we have

$$\begin{cases} \langle A_1\dot{u}(t) + A_0 u(t) - f(t), \dot{u}(t) - z\rangle \leq 0 \quad \forall z \in C(t), \\ \langle A_1\dot{v}(t) + A_0 v(t) - f(t), \dot{v}(t) - z\rangle \leq 0 \quad \forall z \in C(t) \end{cases}$$

for a.e. $t \in [0,T]$. As $\dot{u}(t)$ and $\dot{v}(t)$ belong to $C(t)$ for almost every $t \in [0,T]$, the latter implies that

$$\langle A_1\dot{u}(t) + A_0u(t) - f(t), \dot{u}(t) - \dot{v}(t)\rangle \leq 0$$

and

$$\langle A_1\dot{v}(t) + A_0v(t) - f(t), \dot{v}(t) - \dot{u}(t)\rangle \leq 0$$

for a.e. $t \in [0,T]$. From the last inequalities one gets

$$\langle A_1(\dot{u}(t) - \dot{v}(t)) + A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0$$

for a.e. $t \in [0,T]$. As A_1 is positive semidefinite, it follows that

$$\langle A_0(u(t) - v(t)), \dot{u}(t) - \dot{v}(t)\rangle \leq 0$$

for a.e. $t \in [0,T]$. Integrating both sides of the last inequality and applying Remark 2.12 yield

$$\int_0^t \langle A_0(u(\tau) - v(\tau)), \dot{u}(\tau) - \dot{v}(\tau)\rangle d\tau \leq 0 \quad \forall t \in [0,T].$$

As it has been noted in the proof of Theorem 5.2, this implies

$$\langle A_0(u(t) - v(t)), u(t) - v(t)\rangle - \langle A_0(u(0) - v(0)), u(0) - v(0)\rangle \leq 0 \quad \forall t \in [0,T].$$

Since $u(0) = v(0)$, the latter means that $\langle A_0(u(t) - v(t)), u(t) - v(t)\rangle \leq 0$ for all $t \in [0,T]$. So, by the positive semidefiniteness of A_0, we obtain

$$\langle A_0(u(t) - v(t)), u(t) - v(t)\rangle = 0 \quad \forall t \in [0,T].$$

Therefore, setting $x(t) := v(t) - u(t)$, $t \in [0,T]$, by (5.28) we have $x(t) \in \ker A_0$ for all $t \in [0,T]$. It is clear that $x(0) = v(0) - u(0) = 0$ and

$$\dot{x}(t) = \dot{v}(t) - \dot{u}(t) \in C(t) - \dot{u}(t)$$

for a.e. $t \in [0,T]$. Since $x(\cdot)$ is an absolutely continuous function, from the condition $A_0x(t) = 0$ for all $t \in [0,T]$ we deduce that $A_0\dot{x}(t) = 0$ for a.e. $t \in [0,T]$. Hence, $\dot{x} \in \mathcal{K}$. We have thus shown that (5.29) is valid. The convexity and closedness of $\mathcal{K}$ can be easily verified by using the convexity and closedness of $C(t)$

for all $t \in [0,T]$. □

In the next theorem, we investigate the convexity of the solution set in the case where $A_0 \neq 0$.

Theorem 5.18. *Suppose that* (H1) *is satisfied,* $A_1 = 0$*, and* $f(t) \perp \ker A_0$ *(i.e.,* $\langle f(t), x\rangle = 0$ *for every* $x \in \ker A_0$*) for all* $t \in [0,T]$*. Then,* $\mathrm{Sol}(P, u_0)$ *is convex for every* $u_0 \in C(0)$*.*

Proof. Let $u_0 \in C(0)$ be given arbitrarily and u be a solution of (P). By Theorem 5.29, the inclusion (5.29), where the set $\mathcal{K}$ is defined in (5.30), holds. Take any $x \in \mathcal{K}$. Then, the function v defined by setting $v(t) = u(t) + x(t)$, $t \in [0,T]$, is a solution of (P). Indeed, for almost every $t \in [0,T]$, one has

$$\dot{v}(t) = \dot{u}(t) + \dot{x}(t) \in \dot{u}(t) + (C(t) - \dot{u}(t)) = C(t).$$

Note that $v(0) = u(0) + x(0) = u_0$. Since $\dot{x}(t) \in \ker A_0$ for a.e. $t \in [0,T]$, $x(0) = 0$, and the linear operator A_0 is bounded, by [32, Proposition 1.4.22] we have

$$A_0 x(t) = A_0 \left(x(0) + \int_0^t \dot{x}(\tau) d\tau \right) = A_0 \int_0^t \dot{x}(\tau) d\tau = \int_0^t A_0 \dot{x}(\tau) d\tau = 0 \quad (5.31)$$

for all $t \in [0,T]$. By Ω we denote the set of all $t \in [0,T]$ where the derivatives $\dot{u}(t)$, $\dot{x}(t)$ exist, the inclusion $A_0 u(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{u}(t))$ is satisfied, and

$$\dot{x}(t) \in (C(t) - \dot{u}(t)) \cap \ker A_0.$$

By our assumptions, Ω is a subset of full measure of $[0,T]$. For any $t \in \Omega$ and for any $z \in C(t)$, by (5.31) we have

$$\begin{aligned} \langle A_0 v(t) - f(t), z - \dot{v}(t)\rangle &= \langle A_0(u(t) + x(t)) - f(t), z - (\dot{u}(t) + \dot{x}(t))\rangle \\ &= \langle A_0 u(t) - f(t), z - (\dot{u}(t) + \dot{x}(t))\\ &= \langle A_0 u(t) - f(t), z - \dot{u}(t)\rangle - \langle A_0 u(t), \dot{x}(t)\rangle + \langle f(t), \dot{x}(t)\rangle \\ &= \langle A_0 u(t) - f(t), z - \dot{u}(t)\rangle - \langle u(t), A_0 \dot{x}(t)\rangle + \langle f(t), \dot{x}(t)\rangle. \end{aligned}$$

Since $\dot{x}(t) \in \ker A_0$ and $f(t) \perp \ker A_0$, it follows that $\langle u(t), A_0 \dot{x}(t)\rangle = 0$ and $\langle f(t), \dot{x}(t)\rangle = 0$. Therefore,

$$\langle A_0 v(t) - f(t), z - \dot{v}(t)\rangle = \langle A_0 u(t) - f(t), z - \dot{u}(t)\rangle. \quad (5.32)$$

As $u \in \mathrm{Sol}(P, u_0)$, the right hand side of (5.32) is nonnegative. Hence, from (5.32)

we can deduce that $\langle A_0 v(t) - f(t), z - \dot{v}(t)\rangle \geq 0$. Since $z \in C(t)$ is can be chosen arbitrarily, we get

$$\langle A_0 v(t) - f(t), z - \dot{v}(t)\rangle \geq 0 \quad \forall z \in C(t)$$

for all $t \in \Omega$. Equivalently, $A_0 v(t) - f(t) \in -\mathcal{N}_{C(t)}(\dot{v}(t))$ for all $t \in \Omega$. It follows that v is a solution of (P). So, we have proved that $u + \mathcal{K} \subset \mathrm{Sol}(P, u_0)$. Combining this with (5.29) yields $\mathrm{Sol}(P, u_0) = u + \mathcal{K}$. Hence, the desired convexity of $\mathrm{Sol}(P, u_0)$ follows from the convexity of the set $u + \mathcal{K}$. □

In connection with Theorems 5.16–5.18, we would like to raise the following open questions.

Question 1. *We wonder if the assumptions $A_1 = 0$ and $f(t) \perp \ker A_0$ for all $t \in [0, T]$ could be dropped in the formulation of Theorem 5.18? In other words, does estimate* (5.29) *hold as an equality just under the assumption* (H1)*?*

Question 2. *Is there any example showing that, under the assumption* (H1)*, the solution set of* (P) *could be nonconvex?*

5.5 Conclusions

In this chapter, we have obtained several new results on the solution sensitivity with respect to the initial value, as well as the closedness, the boundedness, and the convexity of the solution set for sweeping processes with convex velocity constraints. In addition, an outer estimate for the solution set is also given. Hoping for further in-depth studies on the solution set, we have proposed two open questions.

Chapter 6

A Lipschitzian Vibro-impact Problem with Time-dependent Constraints

In this chapter, we study a mechanical system with a finite number of degrees of freedom, subjected to perfect time-dependent unilateral constraints, in which the constraints are not necessarily convex nor smooth. The dynamics is described in a form of a second-order measure differential inclusion.

Let $I = [0, T]$, $T > 0$, be a bounded time real interval and $d \in \mathbb{N}$. Let $g : I \times \mathbb{R}^d \to \mathbb{R}^d$ and $f_i : I \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1, \dots, m\}$ be some functions and $m \in \mathbb{N}$. We denote by $q \in \mathbb{R}^d$ the representative point of the system in generalized coordinates and define the set of admissible positions at each instant $t \in I$ by

$$C(t) = \{q \in \mathbb{R}^d \mid f_i(t, q) \leq 0 \ \ \forall i \in \{1, \dots, m\}\}.$$

and the set of active constraints by $J(t, q) = \{i \in \{1, \dots, m\} \ \mid f_i(t, q) = 0\}$. The vibro-impact system given by g and the functions f_i is formally described by the following second-order differential inclusion in $\mathbb{R}^d$:

$$\ddot{q}(t) - g(t, q(t)) \in -\mathcal{N}^{Cl}_{C(t)}(q(t)). \tag{6.1}$$

Denote by $\nabla f_i(t, \cdot)(q)$ the derivative of $f_i(t, q)$ with respect to the second variable q and by $\partial f_i(\cdot, q)$ the derivative of f_i with respect to the first variable t. In what follows, given a set $\Omega \subset \mathbb{R}^d$, we denote its interior and boundary respectively by $\text{int}(\Omega)$ and $\partial\Omega$.

Since $\mathcal{N}^{Cl}_{C(t)}(q) = \emptyset$ if $q(t) \notin C(t)$, if q is a solution of (6.1), then $q(t)$ must

belong to $C(t)$ for all $t \in I$. If $q(t) \in \text{int}(C(t))$ for all $t \in I$, then $\mathcal{N}^{Cl}_{C(t)}(q(t)) = \{0\}$ for all $t \in I$, so (6.1) becomes $\ddot{q} = g(t,q)$, which is an ordinary differential equation.

If $q(t) \in \text{int}(C(t))$ for all $t \in (t_0,t_1) \cup (t_1,t_2)$, $q(t_1) \in \partial C(t_1)$, then

$$\dot{q}(t_1^-) \in -\mathcal{T}(t_1,q(t_1)) \ \text{ and } \ \dot{q}(t_1^+) \in \mathcal{T}(t_1,q(t_1)), \tag{6.2}$$

where

$$\mathcal{T}(t,q) := \{v \in \mathbb{R}^d \ \mid \ \partial f_i(\cdot,q)(t) + \langle \nabla f_i(t,\cdot)(q), v\rangle \leq 0 \quad \forall i \in J(t,q) \ \}.$$

Observe that the set $\mathcal{T}(t,q)$ is polyhedral convex for each pair (t,q). In particular, $\mathcal{T}(t,q)$ is convex and closed. The inclusion (6.2) will be proved in Subsection 4.2.

Note that the function $\dot{q}$ may be discontinuous at some $t \in I$ if $J(t,q(t))$ is nonempty. Therefore, in general, we cannot find a solution q of (6.1) for which, there exists a differentiable derivative $\dot{q}$. Hence, we look for a solution q of (6.1) whose derivative $\dot{q}$ is of *bounded variation*. The latter implies that $\dot{q}$ is differentiable almost everywhere on I. Then, $\ddot{q}$ can be understood as a *Stieltjes measure*. Therefore, (6.1) can be extended in the distributional sense:

$$\begin{cases} \dot{q} \in BV([0,T];\mathbb{R}^d) \\ d\dot{q} - g(\cdot,q(\cdot))dt \in -\mathcal{N}^{Cl}_{C(\cdot)}(q(\cdot))dt, \end{cases}$$

where $BV([0,T];\mathbb{R}^d)$ stands for the space of all functions of bounded variation from $[0,T]$ to $\mathbb{R}^d$. More precisely, the second inclusion is taken in the Radon measure space $\mathcal{M}(0,T;\mathbb{R}^d)$, which is the dual space of the space of all continuous functions from $[0,T]$ to $\mathbb{R}^d$, denoted by $C([0,T],\mathbb{R}^d)$. For $\varphi \in C(I,\mathbb{R}^d)$ and for $\xi(\cdot) \in -\mathcal{N}^{Cl}_{C(\cdot)}(q(\cdot))$,

$$\begin{aligned} d\dot{q}: \quad & C(I,\mathbb{R}^d) \to \mathbb{R}; \\ & \langle d\dot{q}, \varphi\rangle = \int_I \varphi d\dot{q}, \\ g(\cdot,q(\cdot))dt: \quad & C(I,\mathbb{R}^d) \to \mathbb{R}; \\ & \langle g(\cdot,q(\cdot))dt, \varphi\rangle = \int_I \langle g(t,q(t)),\varphi(t)\rangle dt, \\ \xi(\cdot)dt: \quad & C(I,\mathbb{R}^d) \to \mathbb{R}; \end{aligned}$$

$$\langle \xi(\cdot)dt, \varphi\rangle = \int_I \langle \xi(t), \varphi(t)\rangle dt.$$

Since the relation (6.2) does not uniquely define $\dot{q}(t^+)$, we will follow [68] to impose *the impact law*

$$\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-)),$$

where $\mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-))$ is the nearest point of $\dot{q}(t^-)$ in $\mathcal{T}(t,q(t))$.

To sum up, we are interested in investigating the next problem.

Problem $(\mathcal{P})$**.** Let $(q_0,p_0) \in C(0)\times\mathcal{T}(0,q_0)$. Find $q:[0,T]\to\mathbb{R}^d$, with $T>0$, such that

(P1) q is absolutely continuous on $[0,T]$, $\dot{q}\in \mathrm{BV}(0,T;\mathbb{R}^d)$;

(P2) $q(t)\in C(t)$ for all $t\in[0,T]$;

(P3) $d\dot{q} - g(\cdot,q(\cdot))dt \in -\mathcal{N}^{Cl}_{C(\cdot)}(q(\cdot))dt$;

(P4) $\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-))$ for all $t\in[0,T]$;

(P5) $q(0)=q_0$ and $\dot{q}(0)=p_0$.

Let

$$C = \{(t,q)\in[0,T]\times\mathbb{R}^d \mid q\in C(t)\}.$$

We now propose some regularity assumptions.

Assumption A1. There exists an extended real $\rho\in(0,+\infty]$ such that

(i) for all $i\in\{1,\dots,m\}$, f_i is differentiable on $U_\rho(C)$ and its derivative $\nabla f_i(\cdot,\cdot): U_\rho(C)\to\mathbb{R}$ is Lipschitz continuous with rank L;

(ii) there is $\gamma>0$ such that for all $t\in[0,T]$ and $i\in\{1,\dots,m\}$, for all $q_1,q_2\in U_\rho(C(t))$,

$$\langle \nabla f_i(t,\cdot)(q_1) - \nabla f_i(t,\cdot)(q_2), q_1-q_2\rangle \geq -\gamma\|q_1-q_2\|^2.$$

(iii) for all $t\in[0,T]$ and for all $i\in\{1,\dots,m\}$, one has $\|\nabla f_i(t,\cdot)(q)\|\leq L$ for all $q\in U_\rho(C(t))$.

Assumption A2. There is $\mu>0$ with the property that for all $t\in[0,T]$ and $q\in C(t)$ there exists $v=v(t,q)\in\mathbb{R}^d$ with $\|v\|=1$ such that for all $i\in\{1,\dots,m\}$, one has

$$\langle \nabla f_i(t,\cdot)(q), v\rangle \leq -\mu. \tag{6.3}$$

Remark 6.1. From assumption A1(i), it follows that

(i) For each $i \in \{1, \ldots, m\}$, for all $t, t' \in [0, T]$ and $q, q' \in \mathbb{R}^d$,

$$|\partial f_i(\cdot, q)(t) - \partial f_i(\cdot, q')(t')| \leq L(|t - t'| + \|q - q'\|).$$

(ii) for each $i \in \{1, \ldots, m\}$, for all $t, t' \in [0, T]$, $q, q' \in U_\rho(C(t))$,

$$\|\nabla f_i(t, \cdot)(q) - \nabla f_i(t', \cdot)(q')\| \leq L(|t - t'| + \|q - q'\|).$$

Remark 6.2. From assumptions A1 and A2, it follows that for all $i \in \{1, \ldots, m\}$, $\mu \leq \|\nabla f_i(t, \cdot)(q)\| \leq L$ for all $t \in [0, T]$ and $|\partial f_i(\cdot, q)(t)| \leq L$ for all $q \in U_\rho(C(t))$. In particular, $\nabla f_i(t, \cdot)(q) \neq 0$ for all $i \in \{1, \ldots, m\}$.

We are going to present some characterizations of the set of admissible positions $C(t)$ and the Clarke's normal cone $\mathcal{N}^{Cl}_{C(t)}(q)$. Thanks to assumptions A1 and A2, the following proposition is valid.

Proposition 6.3. (See [8, Theorem 3.1]) *Suppose that assumptions* A1(i)-(ii) *and* A2 *holds, then, for all* $t \in [0, T]$, *the set* $C(t)$ *is* r*-prox-regular with* $r = \min\{\rho, \frac{\mu}{\gamma}\}$.

Proposition 6.4. *Under assumptions* A1(i) *and* A2, $C(\cdot)$ *is* ϑ*-Lipschitzian on* $[0, T]$, *with* $\vartheta \geq \dfrac{L}{\mu}$.

Proof. Fix a real number ϑ such that $\vartheta \geq \mu^{-1}L$. Choose a subdivision

$$0 < T_1 < \ldots < T_p = T$$

of $[0, T]$ such that $T_k - T_{k-1} < \dfrac{1}{\vartheta}\rho$. Fix any k and select $s, t \in I_k := [T_{k-1}, T_k]$. Then, take any $i \in \{1, \ldots, m\}$. Put $u(s, t) = \vartheta|s - t|$. For any $x \in C(t)$, define $y := x + u(s, t)v$. Since $t, s \in I_k$, we have $\|y - x\| = \vartheta|s - t| < \rho$. This proves that $y \in \text{int}(U_\rho(C(t)))$. By Lemma 2.4, for all $\lambda \in [0, 1]$ we have

$$x(\lambda) = x + \lambda(y - x) \in \text{int}U_\rho(C(t)).$$

Now we consider the expression $f_i(t, x + u(s, t)v) - f_i(t, x)$. Since $f_i(s, \cdot)$ is differentiable on $U_\rho(C(t))$, by the mean value theorem there exists $\lambda \in (0, 1)$ such that

$$f_i(t, x + u(s, t)v) - f_i(t, x) = \langle \nabla f_i(t, \cdot)(x_\lambda), u(s, t)v \rangle$$

with $x_\lambda = \lambda x + (1-\lambda)(x + u(s,t)v)$. Hence, by Remark 6.1, we have

$$\begin{aligned} f_i(s, x + u(s,t)v) &= [f_i(s, x + u(s,t)v) - f_i(t, x + u(s,t)v)] + f_i(t,x) \\ &\quad + [f_i(t, x + u(s,t)v) - f_i(t,x)] \\ &\leq L|s-t| + f_i(t,x) + \langle \nabla f_i(t,\cdot)(x_\lambda), u(s,t)v\rangle. \end{aligned}$$

By (6.3) and the inclusion $x \in C(t)$ we get

$$f_i(s, x + u(s,t)v) \leq L|s-t| - u(s,t)\mu = (L - \vartheta\mu)\,|s-t| \leq 0,$$

where the inequality is valid due to the choice of ϑ. Since $i \in \{1, \ldots, m\}$ can be chosen arbitrarily, this implies that the vector $x + u(s,t)v = x + \vartheta|s-t|v$ belongs to $C(s)$. Hence, $x \in C(s) + \vartheta|s-t|(-v)$. It follows that

$$C(t) \subset C(s) + \vartheta|s-t|(-v) \subset C(s) + \vartheta|s-t|\mathbb{B}.$$

Thus, $C(\cdot)$ is ϑ-Lipschitzian on $[T_{k-1}, T_k]$. So, we can infer that $C(\cdot)$ is ϑ-Lipschitzian on $[0, T]$. □

6.1 An Existence Result for the Vibro-impact Problem

The approximate solutions will be constructed by the following time-discretization scheme. Let N be a positive natural number and $h = T/N$, we define $t_n = nh$ for all $0 \leq n \leq N$ and

1. $Q_{-1} = q_0 - hp_0$, $Q_0 = q_0$,
2. for all $n \in \{0, \ldots, N\}$,

$$G_n = \int_{t_n}^{t_{n+1}} g(s, Q_n)ds$$

and

$$V_n = 2Q_n - Q_{n-1} + h^2 G_n, \quad Q_{n+1} \in \operatorname*{argmin}_{x \in C(t_{n+1})} \|V_n - x\|. \tag{6.4}$$

In this scheme, we use the approximation

$$\ddot{q}(x) \approx \frac{q(x+h) - 2q(x) + q(x-h)}{h^2}.$$

Clearly, this leads to (6.4). We define the discrete velocities as

$$P_n = \frac{Q_{n+1} - Q_n}{h} \quad \text{for all } \ n \in \{-1, \dots, N\}.$$

The sequence of approximate solutions q_N is given by

$$q_N(t) = Q_n + (t - t_n)\frac{Q_{n+1} - Q_n}{h} \quad \forall t \in [t_n, t_{n+1}],\ \forall n \in \{0, \dots, N-1\}$$

and

$$p_N(t) = P_n = \frac{Q_{n+1} - Q_n}{h} \quad \forall t \in [t_n, t_{n+1}],\ \forall n \in \{0, \dots, N-1\}.$$

For the existence of a solution to our problem we will need the following assumptions:

Assumption A3. For all $q \in \mathbb{R}^d$, $g(\cdot, q)$ is measurable on $[0, T]$ and for all $t \in [0, T]$, $g(t, \cdot)$ is continuous on $\mathbb{R}^d$. Moreover, there exist $L_g > 0$ and $F \in L^1(0, T; \mathbb{R})$ such that for almost every $t \in [0, T]$ one has

$$\|g(t, q) - g(t, \tilde{q})\| \leq L_g \|q - \tilde{q}\| \quad \forall (q, \tilde{q}) \in (\mathbb{R}^d)^2 \text{ s.t. } (t, q) \in U_\rho(C), (t, \tilde{q}) \in U_\rho(C),$$
$$\|g(t, q)\| \leq F(t) \quad \forall q \in \mathbb{R}^d \text{ s.t. } (t, q) \in U_\rho(C).$$

Assumption A4. For all $t \in [0, T]$, $q \in U_\rho(C(t))$, and for all $j, k \in J(t, q)$ and $j \neq k$, one has

$$\langle \nabla f_j(t, \cdot)(q), \nabla f_k(t, \cdot)(q) \rangle \geq 0.$$

Proposition 6.5. *Under assumptions* A1(i) *and* A2, *for any $t \in I$ and $q \in C(t)$, the Clarke normal cone to $C(t)$ at q can be computed by the formula*

$$\mathcal{N}^{Cl}_{C(t)}(q) = \begin{cases} \{0\} & \text{if } q \in \operatorname{int}(C(t)) \\ \left\{ w \in \mathbb{R}^d \ \middle| \ w = \sum\limits_{i \in J(t,q)} \lambda_i \nabla f_i(t, \cdot)(q), \ \ \lambda_i \geq 0 \right\} & \text{if } q \in \partial C(t). \end{cases}$$

Proof. If $q \in \operatorname{int}(C(t))$, then the Clarke tangent cone is equal to the whole space $\mathbb{R}^d$. Therefore, $\mathcal{N}^{Cl}_{C(t)}(q) = \{0\}$. Now we consider the case q is on the boundary $\partial C(t)$ of $C(t)$. Then $J(t, q) \neq \emptyset$. From Assumption A2 it follows that $\{\nabla f_i(t, \cdot)(q) \mid i \in J(t, q)\}$ is positively linearly independent. Hence, by Proposition 2.36 we obtain the desired formula for $\mathcal{N}^{Cl}_{C(t)}(q)$. □

From Proposition 6.5 we can deduce the next formula for computing the

corresponding Clarke tangent cone:

$$\mathcal{T}^{Cl}_{C(t)}(q) = \{v \in \mathbb{R}^d \mid \langle \nabla f_i(t,\cdot)(q), v\rangle \leq 0, \forall i \in J(t,q)\}. \tag{6.5}$$

Lemma 6.6. *Let $t \in [0,T]$, $q \in C(t)$ and $v = v(t,q)$ be the vector existed by assumption* A2. *There exist $\rho' > 0$, $\tau \in (0,\rho']$ and $\theta \in (0,\rho']$ such that for all $t' \in I$, $|t'-t| \leq \tau$, and for all q' from the open ball $\mathbb{B}(q,\theta)$ centered at q with radius θ,*

$$\langle \nabla f_i(t',\cdot)(q'), v\rangle \leq -\frac{\mu}{3}, \quad \forall i \in \{1,\ldots,m\}.$$

Proof. Let $q \in C(t)$, v be defined in A2. For all $t' \in I$, $q' \in \mathbb{R}^d$ such that $\|q'-q\| \leq \rho$, and for any $i \in \{1,\ldots,m\}$, by Remark 6.1(ii) we have

$$\begin{aligned}\langle \nabla f_i(t',\cdot)(q') - \nabla f_i(t,\cdot)(q), v\rangle &\leq \|\nabla f_i(t',\cdot)(q') - \nabla f_i(t,\cdot)(q)\|\|v\| \\ &\leq L(|t-t'|+\|q-q'\|).\end{aligned}$$

Hence,

$$\langle \nabla f_i(t',\cdot)(q'), v\rangle \leq -\mu + L(|t-t'|+\|q-q'\|).$$

Choose $\tau = \theta = \min\{\mu/3L, \rho\}$. Then we have $\langle \nabla f_i(t',\cdot)(q'), v\rangle \leq -\frac{\mu}{3}$. □

Our main result is the next theorem.

Theorem 6.7. *Suppose that assumptions* A1–A3 *hold. Let $(q_0,p_0) \in C(0) \times \mathcal{T}(0,q_0)$. Then, there is a subsequence of $\{q_N\}$, still denoted by $\{q_N\}$, of the approximate solutions which converges uniformly on $[0,T]$ to a limit q satisfying* (P1)–(P3). *Furthermore, if assumption* A4 *holds, then q also satisfies* (P4) *and* (P5)*, and it is a solution of problem $(\mathcal{P})$ on $[0,T]$.*

To make the proof of this theorem easier for understanding, we present it in the forthcoming three subsections.

6.1.1 Convergence of the Approximate Solutions

In this subsection, we shall prove that the discrete sequence $\{q_N\}$ constructed in the latter section converges to a limit, which will later be verified to be a solution of problem $(\mathcal{P})$. More precisely, we will prove that $\{p_N\}$ is uniformly bounded and it has bounded variation in Propositions 6.12 and 6.13.

Lemma 6.8. *For all $n \in \{0, \dots, N-1\}$, one has*

$$P_{n-1} - P_n + hG_n \in \mathcal{N}^{Cl}_{C(t_{n+1})}(Q_{n+1}). \tag{6.6}$$

Proof. By definition of the scheme, for all $x \in C(t_{n+1})$, we have

$$\begin{aligned}\|V_n - Q_{n+1}\|^2 &\leq \|V_n - x\|^2 \\ &= \|V_n - Q_{n+1}\|^2 + 2\langle V_n - Q_{n+1}, Q_{n+1} - x\rangle + \|Q_{n+1} - x\|^2\end{aligned}$$

Hence,

$$2\langle V_n - Q_{n+1}, x - Q_{n+1}\rangle \leq \|Q_{n+1} - x\|^2.$$

By definition, $V_n - Q_{n+1} = h(P_{n-1} - P_n + hG_n)$, so

$$\langle P_{n-1} - P_n + hG_n, x - Q_{n+1}\rangle \leq \frac{1}{2h}\|Q_{n+1} - x\|^2, \quad \forall x \in C(t_{n+1}). \tag{6.7}$$

If $Q_{n+1} \in \mathrm{int}(C(t_{n+1}))$, we can choose $\varepsilon > 0$ sufficiently small so that $x_1 = Q_{n+1} + \varepsilon E$ and $x_2 = Q_{n+1} - \varepsilon E$ belong to $C(t_{n+1})$, where $E = (1, \dots, 1) \in \mathbb{R}^d$. Then we have

$$P_{n-1} - P_n + hG_n = 0.$$

Otherwise $J(t_{n+1}, Q_{n+1}) \neq \emptyset$. We know that by (6.5), the Clarke's tangent cone of $C(t_{n+1})$ at Q_{n+1} is

$$\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}) = \{w \in \mathbb{R}^d \mid \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w\rangle \leq 0, \forall i \in J(t_{n+1}, Q_{n+1})\}.$$

So we need to show that

$$\langle P_{n-1} - P_n + hG_n, w\rangle \leq 0, \quad \forall w \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}).$$

Indeed, by assumption A2, $\mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})) \neq \emptyset$. Note that

$$\mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})) = \{w \in \mathbb{R}^d \mid \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w\rangle < 0, \forall i \in J(t_{n+1}, Q_{n+1})\}.$$

Take any $\bar{w} \in \mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}))$. We will prove that $Q_{n+1} + s\bar{w} \in C(t_{n+1})$ for $s > 0$ sufficiently small. For any $s \geq 0$, there exists $q_\lambda := Q_{n+1} + \lambda s\bar{w}$ with $\lambda \in (0, 1)$, such that

$$f_i(t_{n+1}, Q_{n+1} + s\bar{w}) - f_i(t_{n+1}, Q_{n+1}) = \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1} + \lambda s\bar{w}), s\bar{w}\rangle.$$

For s small enough such that $\|s\bar{w}\| \leq \rho$, we have $Q_{n+1} + s\bar{w} \in U_\rho(C(t_{n+1}))$. By Remark 6.1(ii),

$$\|\nabla f_i(t_{n+1}, \cdot)(Q_{n+1} + \lambda s\bar{w}) - \nabla f_i(t_{n+1}, \cdot)(Q_{n+1})\| \leq \lambda sL\|\bar{w}\|.$$

Then, $\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1} + \lambda s\bar{w}) - \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), s\bar{w}\rangle \leq \lambda L s^2 \|\bar{w}\|^2$. Hence,

$$\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1} + \lambda s\bar{w}), s\bar{w}\rangle \leq \lambda L s^2\|\bar{w}\| + s\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), \bar{w}\rangle.$$

Since $\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), \bar{w}\rangle < 0$, we can choose s small enough such that $f_i(t_{n+1}, Q_{n+1} + s\bar{w}) \leq 0$. This implies that $Q_{n+1} + s\bar{w} \in C(t_{n+1})$. Now we choose $x = Q_{n+1} + s\bar{w}$ such that $x \in C(t_{n+1})$, by (6.7) we get

$$\langle P_{n-1} - P_n + hG_n, s\bar{w}\rangle \leq \frac{1}{2h}\|s\bar{w}\|^2.$$

Letting $s \to 0$, one has

$$\langle P_{n-1} - P_n + hG_n, \bar{w}\rangle \leq 0.$$

By assumption A2, there exits a unit vector $v(t_{n+1}, Q_{n+1}) \in \mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}))$. Therefore, for all $v \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$, the sequence $\{v_k\}_{k\in\mathbb{N}^*}$, which is defined by

$$v_k = v + \frac{1}{k}v(t_{n+1}, Q_{n+1})$$

for all $k \geq 1$, converges to v. We also see that $v_k \in \mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}))$ for all $k \geq 1$. So, $\mathrm{int}(\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}))$ is dense in $\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$. This leads to

$$\langle P_{n-1} - P_n + hG_n, w\rangle \leq 0, \quad \forall w \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1}),$$

which implies that $P_{n-1} - P_n + hG_n \in \mathcal{N}^{Cl}_{C(t_{n+1})}(Q_{n+1})$. □

Remark 6.9. One can reformulate (6.6) as follows: For all $n \in \{0, \ldots, N-1\}$, there exist nonnegative real numbers λ_i^n, $i = 1, \ldots, m$ such that $\lambda_i^n = 0$ for all $i \notin J(t_{n+1}, Q_{n+1})$, and

$$P_n - P_{n-1} - hG_n = -\sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}). \tag{6.8}$$

Lemma 6.10. *For each $i \in J(t_{n+1}, Q_{n+1})$ and $\|P_n\| \leq \dfrac{\rho N}{2T}$, one has*

$$L + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n\rangle \geq -\gamma h\|P_n\|^2. \tag{6.9}$$

Proof. For all $i \in J(t_{n+1}, Q_{n+1})$, we have $f_i(t_{n+1}, Q_{n+1}) = 0 \geq f_i(t_n, Q_n)$. Thus,

$$\begin{aligned} 0 &\geq f_i(t_n, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n) + f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &\geq -hL - h\langle \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), P_n \rangle, \end{aligned}$$

where $q^n_{\alpha_i} = \alpha_i Q_n + (1-\alpha_i)Q_{n+1}$ for some $\alpha_i \in (0,1)$. It follows that

$$\begin{aligned} L + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \rangle &\geq \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), P_n \rangle \\ &= \left\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), \frac{Q_{n+1} - q^n_{\alpha_i}}{\alpha_i h} \right\rangle \\ &\geq \frac{1}{h} \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q^n_{\alpha_i}), Q_{n+1} - q^n_{\alpha_i} \rangle \end{aligned}$$

Since $\|P_n\| \leq \dfrac{\rho N}{2T}$, by Lemma 2.4 we know that $q^n_{\alpha_i} \in U_\rho(C(t_{n+1}))$. By assumption A1(ii), we obtain (6.9). □

Lemma 6.11. *Let $N > N^0$, where $N^0 = \max\left\{\dfrac{T}{2}, \dfrac{6TL}{\mu\theta}\right\}$. Then, for all $n \in \{0, \ldots, N-1\}$, we have*

$$\|P_n\| \leq 2\|P_{n-1}\| + 2h\|G_n\| + \frac{6L}{\mu}.$$

Proof. Let $w = \dfrac{6L}{\mu} v(t_n, Q_n)$, where $v(t_n, Q_n)$ is the unit vector defined in assumption A2 for $(t, x) = (t_n, Q_n)$, i.e., for all $i \in \{1, \ldots, m\}$, one has $\langle \nabla f_i(t_n, \cdot)(Q_n), v(t_n, Q_n) \rangle \leq -\mu$. Then,

$$Q_n + hw \in C(t_{n+1}).$$

Indeed, by Remark 6.2 and mean value theorem, we have

$$f_i(t_{n+1}, Q_n + hw) \leq f_i(t_n, Q_n + hw) + L|t_{n+1} - t_n|.$$

By mean value theorem, there exists $q^n_\alpha = \alpha Q_n + (1-\alpha)(Q_n + hw)$ with $\alpha \in (0,1)$, such that

$$f_i(t_n, Q_n + hw) - f_i(t_n, Q_n) = \langle \nabla f_i(t_n, \cdot)(q^n_\alpha), hw \rangle.$$

Since $N \geq \dfrac{6TL}{\mu\theta}$, $q_\alpha^n \in B(Q_n, \theta)$. By Lemma 6.6, we have

$$\langle \nabla f_i(t_n, \cdot)(q_\alpha^n), w\rangle \leq \frac{-\mu}{3}\frac{6L}{\mu} = -2L.$$

Therefore, for all $i \in \{1, \ldots, m\}$,

$$f_i(t_{n+1}, Q_n + hw) \leq f_i(t_n, Q_n) + \langle \nabla f_i(t_n, \cdot)(q_\alpha^n), hw\rangle + hL \leq 0.$$

We have proved that $Q_n + hw \in C(t_{n+1})$. As $Q_{n+1} \in \operatorname{argmin}_{x\in C(t_{n+1})}\|V_n - x\|$, it follows that

$$\|2Q_n - Q_{n-1} + h^2G_n - Q_{n+1}\| \leq \|2Q_n - Q_{n-1} + h^2G_n - Q_n - hw\|.$$

Thus, $\|P_{n-1} - P_n + hG_n\| \leq \|P_{n-1} - w + hG_n\|$. So, we get $\|P_n\| \leq 2\|P_{n-1}\| + 2h\|G_n\| + \|w\|$, which yields the conclusion. □

Proposition 6.12. *There exist $N^1 > N^0$ and $\kappa > 0$ such that*

$$\|P_n\| \leq \kappa \quad \forall n \in \{0, \ldots, N-1\}, \quad \forall N > N^1.$$

Proof. We now define two real sequences $\{\kappa_k\}_{k\in\mathbb{N}}$ and $\{\tau_k\}_{k\in\mathbb{N}^*}$ by setting $\kappa_0 = \|p_0\| + 1$,

$$\begin{aligned}\kappa_k &= \kappa_{k-1} + \frac{12L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)} \\ &= \kappa_0 + k\left(\frac{12L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)}\right) \ \forall k \geq 1\end{aligned}$$

and

$$\tau_k = \frac{\min\{\tau, \theta\}}{2\kappa_k} = \frac{\min\{\tau, \theta\}}{2\kappa_0 + 2k\left(\frac{2L}{\mu} + \|F\|_{L^1(0,T;\mathbb{R}^d)}\right)} \qquad \forall k \geq 1.$$

It is easy to see that the series $\sum\limits_{k=1}^{\infty} \tau_k$ is a divergent sum, hence, there exists $k_0 \geq 1$ such that $\sum_{k=1}^{k_0} \tau_k > T$. Let $\kappa = \kappa_{k_0}$. Define

$$\bar{\kappa} = 2\kappa + 2\|F\|_{L^1(0,T;\mathbb{R}^d)} + \frac{6L}{\mu}$$

and

$$N^1 = \max\left(N^0, \frac{2T\bar{\kappa}}{\rho}, \frac{2T\bar{\kappa}}{\theta}, \frac{2T}{\tau}, \frac{2\gamma\bar{\kappa}^2T}{L}\right).$$

We now prove that for all $N > N^1$ and we can construct a finite family of real numbers $(\tau_k^N)_{1\leq k\leq k_0}$ such that $\tau_0^N = 0 < \tau_1^N < \cdots < \tau_{k_0^N}^N = T$ with $1 \leq k_0^N \leq k_0$ and for all $k \in \{1, \ldots, k_0^N\}$, in each interval $[\tau_{k-1}^N, \tau_k^N)$, one has

$$\|P_n\| \leq \kappa_k \quad \forall n \in \{0, \ldots, N-1\}.$$

Consider the interval $[0, \tau_1]$ instead of $[0, T]$. From assumption A2, we can define a vector $w_0 = \frac{6L}{\mu} v(t_0, Q_0)$. Note that $\|P_{-1}\| = \|p_0\| \leq \kappa_0 \leq \kappa$, by Lemma 6.11 we have $\|P_0\| \leq \bar{\kappa}$. Since $0 < h = \dfrac{T}{N} \leq \dfrac{\theta}{2\bar{\kappa}}$,

$$\|Q_1 - Q_0\| = h\|P_0\| \leq \frac{\theta}{2} < \theta.$$

Moreover, $|t_1 - t_0| \leq h \leq \tau/2 < \tau$, we have $(t_1, Q_1) \in \mathbb{B}(t_0, \tau) \times \mathbb{B}(Q_0, \theta)$. We will prove that $w_0 - P_0 \in \mathcal{T}^{Cl}_{C(t_1)}(Q_1)$. Indeed, for all $i \in J(t_1, Q_1)$, by Lemma 6.10 one has

$$\begin{aligned}
\langle \nabla f_i(t_1, \cdot)(Q_1), w_0 - P_0 \rangle &= \langle \nabla f_i(t_1, \cdot)(Q_1), w_0 \rangle + L - (L + \langle \nabla f_i(t_1, \cdot)(Q_1), P_0 \rangle) \\
&\leq \frac{-\mu\|w_0\|}{3} + L + \gamma h \|P_0\|^2 \\
&\leq -2L + L + \gamma h \bar{\kappa}^2 \leq -\frac{L}{2}.
\end{aligned}$$

From the latter inequality, it follows that $w_0 - P_0 \in \mathcal{T}^{Cl}_{C(t_1)}(Q_1)$. Since $P_{-1} - P_0 + hG_0 \in \mathcal{N}^{Cl}_{C(t_0)}(Q(0))$, we get

$$\langle (P_{-1} - w_0) - (P_0 - w_0) + hG_0, w_0 - P_0 \rangle \leq 0.$$

This yields $\langle P_{-1} - w_0 + hG_0, w_0 - P_0 \rangle \leq -\|P_0 - w_0\|^2$, which implies that

$$\|P_0 - w_0\| \leq \|P_{-1} - w_0\| + h\|G_0\|.$$

Hence,

$$\|P_0\| \leq \|P_{-1}\| + \frac{12L}{\mu} + h\|G_0\| \leq \kappa_1 \leq \kappa.$$

Next, we will prove by induction that

$$\|P_n - w_0\| \leq \|P_{-1} - w_0\| + h\sum_{\ell=0}^{n} \|G_\ell\| \quad \forall n \in \{0, \ldots, N-1\}.$$

Indeed, let $n \in \{0, \dots, N-1\}$. Suppose that

$$\|P_k - w_0\| \leq \|P_{-1} - w_0\| + h \sum_{\ell=0}^{k} \|G_\ell\| \quad \forall k \in \{0, \dots, n-1\}.$$

Then,

$$\|P_k\| \leq 2\|w_0\| + \|P_{-1}\| + h \sum_{\ell=0}^{k} \|G_\ell\| \leq \kappa_1 \quad \text{for all} \quad k \in \{0, \dots, n-1\}$$

and by Lemma 6.11 we infer that $\|P_n\| \leq \bar{\kappa}$. Since $0 < h \leq \frac{\theta}{2\bar{\kappa}}$,

$$\|Q_{n+1} - Q_n\| = h\|P_n\| \leq \frac{\theta}{2} < \theta.$$

Moreover, as $|t_{n+1} - t_n| \leq h < \tau$, we have $(t_{n+1}, Q_{n+1}) \in B(t_n, \tau) \times B(Q_n, \theta)$. For all $i \in J(t_{n+1}, Q_{n+1})$, by Lemma 6.10 one has

$$\begin{aligned}
\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w_0 - P_n \rangle &= \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w_0 \rangle + L - (L + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \rangle) \\
&\leq \frac{-\mu\|w_0\|}{3} + L + \gamma h \|P_n\|^2 \\
&\leq -2L + L + \gamma h \bar{\kappa}^2 \leq -\frac{L}{2}.
\end{aligned}$$

It follows that $w_0 - P_n \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$. Therefore,

$$\langle (P_{n-1} - w_0) - (P_n - w_0) + hG_n, w_0 - P_n \rangle \leq 0.$$

This yields

$$\|P_n - w_0\| \leq \|P_{n-1} - w_0\| + h\|G_n\| \leq \|P_{-1} - w_0\| + h \sum_{l=0}^{n} \|G_\ell\|.$$

Hence,

$$\|P_n\| \leq \|P_{-1}\| + \frac{12L}{\mu} + h \sum_{l=0}^{n} \|G_\ell\| \leq \kappa_1.$$

We have shown that $\|P_n\| \leq \kappa_1$ for all $n \in \{0, \dots, N\}$ on the interval $[0, \tau_1]$. Putting $\tau_0^N = 0$, we define $\tau_1^N = \min\{\tau_0^N + \tau_1, T\}$. If $\tau_0^N + \tau_1 < T$, we have $\tau_1^N - \tau_0^N = \tau_1$. If $T > \tau_1^N$, then $k_0 > 1$, $(t_{N+1}, Q_{N+1}) \in C$ and $\|P_{N+1}\| \leq \kappa_1 \leq \kappa$.

Assume now that $\tau_0^N + \tau_1 < T$. By Lemma 6.6 and assumption A2, we can define

a vector $w_1 = \dfrac{6L}{\mu} v(t_{N+1}, Q_{N+1})$. For the sack of simplicity, we will recount the index from 0 instead of $N+1$. By the same argument, we can prove that $\|P_n\| \leq \kappa_2$ for all $n \in \{0, \ldots, N-1\}$ on the interval $[\tau_1^N, \tau_1^N + \tau_2]$. We now can divide the interval $[0, T]$ into subintervals $[\tau_i^N, \tau_i^N + \tau_{i+1}]$ for $i \in \{1, \ldots, k_0\}$. Repeating the same argument for finitely many steps, we get the desired result. □

Proposition 6.13. *There exists $\kappa' > 0$ such that, for all $N > N^1$, we have*

$$\sum_{n=0}^{N-1} \|P_n - P_{n-1}\| \leq \kappa'.$$

Proof. We decompose $[0, T]$ into the subintervals $[\tau_k^N, \tau_{k+1}^N]$, $k \in \{0, \ldots, k_0^h - 1\}$, which were defined in the proof of Proposition 6.12. Consider the interval $[\tau_0^N, \tau_1^N]$. We have shown that

$$w_0 - P_n \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$$

for all $n \in \{0, \ldots, N-1\}$. We now prove that the closed ball $\bar{\mathbb{B}}\left(w_0 - P_n, \frac{1}{2}\right) \subset \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$. Indeed, let $a \in \bar{\mathbb{B}}\left(w_0 - P_n, \frac{1}{2}\right)$. Then, $\|a - (w_0 - P_n)\| \leq \frac{1}{2}$. As in the proof of Proposition 6.12, one has $\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w_0 - P_n \rangle \leq -\frac{L}{2}$ for all $n \in \{0, \ldots, N-1\}$. Then,

$$\begin{aligned}
\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), a \rangle &= \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), a - (w_0 - P_n) \rangle \\
&\quad + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), w_0 - P_n \rangle \\
&\leq \|\nabla f_i(t_{n+1}, \cdot)(Q_{n+1})\| \|a - (w_0 - P_n)\| - \frac{L}{2} \\
&\leq 0.
\end{aligned}$$

This proves that $a \in \mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$. Since the tangent cone $\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$ is closed and convex [33, p. 51], for every $x \in \mathbb{R}^d$, by [59, Lemma 4.3, p. 22] we have

$$\|x - \mathbb{P}_{\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})}(x)\| \leq \|x - w_0 + P_n\|^2 - \|\mathbb{P}_{\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})}(x) - w_0 + P_n\|^2.$$

Applying this with $x = P_{n-1} - P_n + hG_n$, we get

$$\|P_{n-1} - P_n + hG_n - \bar{P}\| \leq \|P_{n-1} - P_n + hG_n - w_0 + P_n\|^2 - \|\tilde{P} - w_0 + P_n\|^2,$$

where $\bar{P} = \mathbb{P}_{\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})}(P_{n-1} - P_n + hG_n)$. It follows that

$$\|P_{n-1} - P_n + hG_n - \bar{P}\| \leq \|P_{n-1} + hG_n - w_0\|^2 - \|\bar{P} - w_0 + P_n\|^2.$$

Recall that $P_{n-1} - P_n + hG_n \in \mathcal{N}^{Cl}_{C(t_{n+1})}(Q_{n+1})$ (see Lemma 6.8). Since $\mathcal{N}^{Cl}_{C(t_{n+1})}(Q_{n+1})$ is the dual cone of $\mathcal{T}^{Cl}_{C(t_{n+1})}(Q_{n+1})$, $\bar{P} = 0$. We get

$$\begin{aligned}
\|P_{n-1} - P_n\| &= \|P_{n-1} - P_n + hG_n - hG_n\| \\
&\leq h\|G_n\| + \|P_{n-1} - P_n + hG_n\| \\
&= h\|G_n\| + \|(P_{n-1} - P_n + hG_n) - \bar{P}\| \\
&\leq h\|G_n\| + \|P_{n-1} - P_n + hG_n\|^2 - \|P_n - w_0\|^2 \\
&\leq h\|G_n\| + \|P_{n-1} - w_0\|^2 - \|P_n - w_0\|^2 + h^2\|G_n\|^2 + 2h\langle G_n, P_{n-1} - w_0\rangle \\
&\leq h\|G_n\| + \|P_{n-1} - w_0\|^2 - \|P_n - w_0\|^2 + h^2\|G_n\|^2 + 2h\|G_n\|\|P_{n-1} - w_0\| \\
&= (1 + h\|G_n\| + 2\|P_{n-1} - w_0\|)\, h\|G_n\| + \|P_{n-1} - w_0\|^2 - \|P_n - w_0\|^2 \\
&\leq (1 + h\|G_n\| + 2\|P_{n-1}\| + 2\|w_0\|)\, h\|G_n\| + \|P_{n-1} - w_0\|^2 - \|P_n - w_0\|^2.
\end{aligned}$$

It follows that

$$\|P_{n-1} - P_n\| \leq h\left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu}\right)\|G_n\| + \|P_{n-1} - w_0\|^2 - \|P_n - w_0\|^2$$

for $n = 0, \ldots, N-1$. Adding these inequalities, we get

$$\begin{aligned}
\sum_{n=0}^{N-1}\|P_{n-1} - P_n\| &\leq \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu}\right)\sum_{n=0}^{N-1} h\|G_n\| + \|P_0 - w_0\|^2 \\
&\quad - \|P_N - w_0\|^2 \\
&\leq T\left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\kappa + \frac{12L}{\mu}\right)\|F\|_{L^1(0,T;\mathbb{R}^d)} + 2\left(\kappa + \frac{6L}{\mu}\right)^2.
\end{aligned}$$

Similarly, we can obtain the same result for all the subintervals $[\tau_i^N, \tau_{i+1}^N]$ where $i \in \{1, \ldots, k_0\}$. Since the number of the subintervals $[\tau_i^N, \tau_{i+1}^N]$ is finite, the proof is complete. □

From Propositions 6.12 and 6.13 we can infer that the sequence $\{q_N\}$ is uniformly Lipschitz continuous and that the sequence $\{p_N\}$ is uniformly bounded in $L^\infty(0,T;\mathbb{R}^d)$ and in $BV([0,T];\mathbb{R}^d)$. For any $t \in [0,T]$, it is clear that $q_N(t)$ is bounded for all N. Moreover, since p_N is the derivative of q_N, by Proposition 2.9, there exists a subsequence of $\{q_N\}$, still denoted by $\{q_N\}$, converging uniformly to an absolutely continuous function q over $[0,T]$. In addition, by [59, Theorem 2.1], we can extract subsequences of $\{p_N\}$, still denoted by $\{p_N\}$ and find

$p \in BV([0,T];\mathbb{R}^d)$ such that

$$\begin{aligned} p_N &\to p \quad \text{pointwise in } [0,T], \\ dp_N &\rightharpoonup dp \quad \text{weakly* in } \mathcal{M}(0,T;\mathbb{R}^d). \end{aligned}$$

6.1.2 Properties of the Limit Trajectory

In this subsection, we will prove that the limit trajectory q satisfies the properties (P1)–(P3).

The definitions of q_N and p_N imply that

$$q_N(t) = q_0 + \int_0^t p_N(s)ds \ \ \forall t \in [0,T] \ \ \forall n > N^1.$$

Passing to the limit as $N \to +\infty$, by dominated convergence theorem [22, Theorem 4.2,p. 90] we obtain

$$q(t) = q_0 + \int_0^t p(s)ds \ \ \forall t \in [0,T]. \tag{6.10}$$

Hence $\dot{q} = p \in BV([0,T];\mathbb{R}^d)$ which implies that q is Lipschitz continuous with rank κ on $[0,T]$.

Proposition 6.14. *For all $t \in [0,T]$, $q(t) \in C(t)$.*

Proof. Indeed, for all $t \in [0,T]$ and for all $N > N^1$, there exists $n \in \{0,\dots N-1\}$ such that $t \in [t_n, t_{n+1}]$. Then, for all $i \in \{1,\dots,m\}$,

$$\begin{aligned} f_i(t,q(t)) - f_i(t_n,q_N(t_n)) &= f_i(t,q(t)) - f_i(t,q_N(t_n)) + f_i(t,q_N(t_n)) - f_i(t_n,q_N(t_n)) \\ &\le L\|q(t) - q_N(t_n)\| + L|t_n - t| \\ &\le L\|q(t) - q_N(t_n)\| + hL \\ &\le L(\|q(t) - q(t_n)\| + \|q(t_n) - q_N(t_n)\|) + hL. \end{aligned}$$

Since q is Lipschitz continuous with modulus κ, we have

$$\begin{aligned} f_i(t,q(t)) - f_i(t_n,q_N(t_n)) &\le L(\kappa(t-t_n) + \sup\{\|q(s) - q_N(s)\|_{\mathbb{R}^d} |\ s \in [0,T]\}) + hL \\ &\le L(\kappa h + \|q - q_N\|_{C([0,T];\mathbb{R}^d)}) + hL. \end{aligned} \tag{6.11}$$

Since $\{q_N\}$ converges uniformly to q on $[0,T]$, $f_i(t_n,q_N(t_n)) = f_i(t_n,Q_n) \le 0$, and (6.11) holds for all $N > N^1$, we can conclude that $f_i(t,q(t)) \le 0$.

The proof is complete. □

We are going to show that the limit trajectory satisfies property (P3). By the definition of p_N, the Stieltjes measure $d\dot{q}_N = dp_N$ is a sum of Dirac's measures

$$dp_N(t) = \sum_{n=0}^{N-1}(P_n - P_{n-1})\delta(t - t_n).$$

Define

$$g_N(t) = \sum_{n=0}^{N-1} hG_n\delta(t-t_n) - \sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n(\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t_n,\cdot)(q(t_n))\delta(t-t_n),$$

and

$$U_N(t) = \sum_{n=0}^{N-1}\sum_{i=1}^{m}\delta(t-t_n)\lambda_i^n\nabla f_i(t,\cdot)(q(t)),$$

where the constants λ_i^n are given in Remark 6.9. Then, (6.8) can be rewritten as

$$dp_N(t) = -U_N(t) + g_N(t). \tag{6.12}$$

Lemma 6.15. *For all $i \in \{1,\dots,m\}$ and for all $N > N^1$ we have*

$$\sum_{n=0}^{N-1}|\lambda_i^n| \leq \frac{1}{\mu}\left(\kappa' + \|F\|_{L^1(0,T;\mathbb{R})}\right).$$

Proof. Let $i \in \{1,\dots,m\}$, $n \in \{0,\dots,N-1\}$. By (6.8) we have

$$\left\|\sum_{i=1}^{m}\lambda_i^n\nabla f_i(t_{n+1},\cdot)(Q_{n+1})\right\| \leq \|P_n - P_{n-1}\| + h\|G_n\|$$

By assumption A1(ii), for fixed n, there exists v such that

$$\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\rangle \leq -\mu.$$

Hence,

$$\begin{aligned}\langle P_n - P_{n-1} + hG_n, v\rangle &= \left\langle\sum_{i=1}^{m}\lambda_i^n\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\right\rangle \\ &= \sum_{i=1}^{m}\lambda_i^n\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}), v\rangle \\ &\leq \sum_{i=1}^{m}\lambda_i^n(-\mu).\end{aligned}$$

For every fixed i, we have

$$\lambda_i^n \leq \sum_{i=1}^{m} \lambda_i^n \leq \frac{1}{\mu} \left(\|P_n - P_{n-1}\| + \|hG_n\| \right).$$

Hence,

$$\sum_{n=0}^{N-1} |\lambda_i^n| = \sum_{n=0}^{N-1} \lambda_i^n \leq \frac{1}{\mu} \sum_{n=0}^{N-1} (\|P_n - P_{n-1}\| + h\|G_n\|) \leq \frac{1}{\mu} (\kappa' + \|F\|_{L^1(0,T;\mathbb{R})}).$$

The proof is complete. □

Let $\Lambda_i^N(t) = \sum_{n=0}^{N-1} \lambda_i^n \delta(t - t_n)$. By the above lemma, Λ_i^N is uniformly bounded, then there exists a subsequence of $\{\Lambda_i^N\}$ converging weakly* to nonnegative measure Λ_i in $\mathcal{M}(0,T;\mathbb{R})$. Therefore, U_N has a subsequence which converges weakly* to U in $\mathcal{M}(0,T;\mathbb{R}^d)$ with $U(t) = \sum_{i=1}^{m} \Lambda_i(t) \nabla f_i(t,\cdot)(q(t))$. Since $\nabla f_i(t,\cdot)(q(t)) \in \mathcal{N}_{C(t)}^{Cl}(q(t))$, we obtain $U \in \mathcal{N}_{C(\cdot)}^{Cl}(q(\cdot))dt$.

Lemma 6.16. *The sequence $\{g_N\}$ converges weakly* to $g(\cdot,q)dt$ in $\mathcal{M}(0,T;\mathbb{R}^d)$, where $g(\cdot,q)dt$ is the measure of density $g(\cdot,q)$ with respect to Lebesgue's measure on $[0,T]$.*

Proof. Let $\varphi \in C([0,T];\mathbb{R}^d)$. By the definition of g_N, we have

$$\begin{aligned}
\langle g_N, \varphi \rangle &= \sum_{n=0}^{N-1} h \langle G_n, \varphi(t_n) \rangle + \sum_{n=0}^{N-1} \sum_{i=1}^{m} \lambda_i^n \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\
&\quad - \nabla f_i(t_n,\cdot)(q(t_n)), \varphi(t_n) \rangle \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle g(s,Q_n), \varphi(t_n) \rangle ds + \sum_{n=0}^{N-1} \sum_{i=1}^{m} \lambda_i^n \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\
&\quad - \nabla f_i(t_n,\cdot)(q(t_n)), \varphi(t_n) \rangle \\
&= \int_0^T \langle g(s,q(s)), \varphi(s) \rangle ds + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle g(s,Q_n) - g(s,q(s)), \varphi(s) \rangle ds \\
&\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle g(s,Q_n), \varphi(t_n) - \varphi(s) \rangle ds \\
&\quad + \sum_{n=0}^{N-1} \sum_{i=1}^{m} \lambda_i^n \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t_n,\cdot)(q(t_n)), \varphi(t_n) \rangle.
\end{aligned} \tag{6.13}$$

Moreover, for all $n \in \{0, \dots, N-1\}$, we have $(t_n, q(t_n)) \in C$ and

$$\begin{aligned}\|Q_{n+1} - q(t_n)\| &\leq \|Q_{n+1} - Q_n\| + \|q_N(t_n) - q(t_n)\| \\ &\leq \kappa h + \|q - q_N\|_{C([0,T];\mathbb{R}^d)}.\end{aligned}$$

Let $\varepsilon_n := \|Q_{n+1} - q(t_n)\|$. From Remark 6.1 and Lemma 6.15 it follows that

$$\begin{aligned}&\left\|\sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n\langle\nabla f_i(t_{n+1},\cdot)(Q_{n+1}) - \nabla f_i(t_n,\cdot)(q(t_n)), \varphi(t_n)\rangle\right\| \\ &\leq \sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n L(h+\varepsilon_n)\|\varphi(t_n)\| \\ &\leq \sum_{n=0}^{N-1}\sum_{i=1}^{m}\lambda_i^n L((\kappa+1)h + \|q-q_N\|_{C([0,T];\mathbb{R}^d)})\|\varphi\|_{C([0,T];\mathbb{R}^d)} \\ &\leq \frac{mL}{\mu}((\kappa+1)h + \|q-q_N\|_{C([0,T];\mathbb{R}^d)})\|\varphi\|_{C([0,T];\mathbb{R}^d)}(\mathrm{Var}(p_N,[0,T]) + \|F\|_{L^1(0,T;\mathbb{R}^d)})\end{aligned}$$

In addition,

$$\begin{aligned}\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\langle g(s,Q_n) - g(s,q(s)), \varphi(s)\rangle ds\right| &\leq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} L_g\|Q_n - q(s)\|\|\varphi(s)\| ds \\ &\leq L_g(\kappa h + \|q-q_N\|_{C([0,T],\mathbb{R}^d)})\int_0^T\|\varphi(s)\| ds.\end{aligned}$$

We also have

$$\begin{aligned}\left|\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\langle g(s,Q_n), \varphi(t_n) - \varphi(s)\rangle ds\right| &\leq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\|g(s,Q_n)\|\|\varphi(t_n) - \varphi(s)\| ds \\ &\leq \omega_\varphi(h)\|F\|_{L^1([0,T];\mathbb{R}^d)},\end{aligned}$$

where ω_φ denotes the modulus of continuity of φ. Therefore, letting N to ∞ in (6.13) we get

$$\langle g_N, \varphi\rangle \to \int_0^T\langle g(s,q(s)), \varphi(s)\rangle ds.$$

The proof is complete. □

Passing (6.12) to the limit yields $dp - g(\cdot,q)dt \in -\mathcal{N}^{Cl}_{C(\cdot)}(q(\cdot))dt$.

6.1.3 Checking the Impact Law and the Initial Data

In this subsection, we will prove that the limit trajectory satisfies the impact law (P4) and the initial data (P5).

Lemma 6.17. *If $J(t,q) \neq \emptyset$ then $\dot{q}(t^+) \in \mathcal{T}(t,q(t))$.*

Proof. Let $t \in I$ be chosen arbitrarily. Consider an index i such that $f_i(t,q(t)) = 0$. We have

$$\begin{aligned} 0 &\geq f_i(t+\varepsilon, q(t+\varepsilon)) - f_i(t,q(t)) \\ &= \varepsilon \partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), q(t+\varepsilon) - q(t)\rangle + o(\varepsilon). \end{aligned}$$

Dividing both sides by ε and letting $\varepsilon \to 0$, we obtain

$$\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), \dot{q}(t^+)\rangle \leq 0.$$

We have shown that $\dot{q}(t^+) \in \mathcal{T}(t,q(t))$.

Similarly, we can prove that $\dot{q}(t^-) \in -\mathcal{T}(t,q(t))$. □

Lemma 6.18. *For each $i \in J(t_{n+1}, Q_{n+1})$ and $\|P_n\| \leq \dfrac{\rho N}{2T}$, one has*

$$\partial f_i(\cdot, Q_{n+1})(t_{n+1}) + \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), P_n\rangle \geq -h(L + L\|P_n\| + \gamma\|P_n\|^2).$$

Proof. For all $i \in J(t_{n+1}, Q_{n+1})$, $f_i(t_{n+1}, Q_{n+1}) = 0 \geq f_i(t_n, Q_n)$. Thus,

$$\begin{aligned} 0 &\geq f_i(t_n, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n) + f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}) \\ &= -h\partial f_i(\cdot, Q_n)(t_\alpha^n) - h\langle \nabla f_i(t_{n+1},\cdot)(q_\beta^n), P_n\rangle, \end{aligned}$$

where $t_\alpha^n = \alpha t_n + (1-\alpha)t_{n+1}$ and $q_\beta^n = \beta Q_n + (1-\beta)Q_{n+1}$ for some $\alpha, \beta \in (0,1)$, satisfying

$$\langle \partial f_i(\cdot, Q_n)(t_\alpha^n), t_n - t_{n+1}\rangle = f_i(t_n, Q_n) - f_i(t_{n+1}, Q_n),$$

and

$$\langle \nabla f_i(t_{n+1},\cdot)(q_\beta^n), Q_n - Q_{n+1}\rangle = f_i(t_{n+1}, Q_n) - f_i(t_{n+1}, Q_{n+1}).$$

Hence,

$$\begin{aligned} \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}), P_n\rangle \geq -\partial f_i(\cdot, Q_n)(t_\alpha^n) + \langle \nabla f_i(t_{n+1},\cdot)(Q_{n+1}) \\ - \nabla f_i(t_{n+1},\cdot)(q_\beta^n), P_n\rangle \end{aligned}$$

$$\geq -\partial f_i(\cdot, Q_n)(t_\alpha^n) + \frac{1}{\beta h}\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q_\beta^n), Q_{n+1} - q_\beta^n\rangle.$$

Since $h\|P_n\| \leq \frac{\rho}{2}$, by Lemma 2.4 we know that $q_\beta^n \in U_\rho(C(t_{n+1}))$. Therefore, by Remark 6.1(i),

$$\begin{aligned}\|\partial f_i(\cdot, Q_{n+1})(t_{n+1}) - \partial f_i(\cdot, Q_n)(t_\alpha^n)\| &\geq -L(|t_{n+1} - t_\alpha| + \|Q_{n+1} - Q_n\|)\\ &= -Lh(\alpha + \|P_n\|)\\ &\geq -Lh(1 + \|P_n\|).\end{aligned}$$

Then, by assumption A1(ii), one has

$$\begin{aligned}\frac{1}{\beta h}\langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}) - \nabla f_i(t_{n+1}, \cdot)(q_\beta^n), Q_{n+1} - q_\beta^n\rangle &\geq -\frac{\gamma}{\beta h}\|Q_{n+1} - q_\beta^n\|^2\\ &= -\gamma\beta h\|P_n\|^2\\ &\geq -\gamma h\|P_n\|^2\end{aligned}$$

Hence,

$$\partial f_i(\cdot, Q_{n+1})(t_{n+1}) + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n\rangle \geq -h(L + L\|P_n\| + \gamma\|P_n\|^2).$$

The proof is complete. □

Proposition 6.19. *For all $t \in (0, T)$, one has $\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q)}(\dot{q}(t^-))$.*

Proof. **Step 1:** We consider the case that $J(t, q(t)) = \emptyset$. Since f_i are continuous for all $i \in \{1, \ldots, m\}$, we may define $\rho_t \in (0, \min(\rho, t, T - t))$ such that, for all $i \in \{1, \ldots, m\}$ we have

$$f_i(s, y) \leq \frac{1}{2} f_i(t, q(t)) < 0 \ \ \forall s \in [t - \rho_t, t + \rho_t], y \in \bar{\mathbb{B}}(q(t), \rho_t)$$

and we define $N_t > \max\left\{N^1, \frac{4T(\kappa+1)}{\rho_t}\right\}$ such that $\|q - q_N\|_{C([0,T];\mathbb{R}^d)} \leq \frac{\rho_t}{4}$ for all $N > N_t$. Then, for all $\tilde{\rho} \in (0, \rho_t]$ and for all $N > N_t$, we define

$$n_- = \left\lfloor \frac{t - \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor + 1, \quad n_+ = \left\lfloor \frac{t + \frac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor.$$

It follows that

$$\begin{aligned} 2h < (n_- - 1)h \leq t - \frac{\tilde{\rho}}{4(\kappa+1)} < hn_- < \ldots < hn_+ \\ \leq t + \frac{\tilde{\rho}}{4(\kappa+1)} < (n_+ + 1)h < T - 2h \end{aligned}$$

and

$$P_{n_- - 1} = p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right), \qquad P_{n_+} = p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right).$$

By relation (6.8) we have

$$P_{n_+} - P_{n_- - 1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{n=n_-}^{n_+}\sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}).$$

Moreover, for all $n \in \{n_-, \ldots, n_+\}$ we have $t_n = nh \in \left[t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)}\right]$ and

$$\begin{aligned} |t_{n+1} - t| &\leq \frac{\tilde{\rho}}{4(\kappa+1)} + h \leq \frac{\rho_t}{2(\kappa+1)} < \rho_t, \\ \|Q_{n+1} - q(t)\| &\leq \|Q_{n+1} - q_N(t)\| + \|q_N(t) - q(t)\| \\ &\leq \kappa|t_{n+1} - t| + \|q - q_N\|_{C([0,T];\mathbb{R}^d)} < \rho_t. \end{aligned}$$

It follows that $f_i(t_{n+1}, Q_{n+1}) < 0$ and $\lambda_i^n = 0$ for all $i \in \{1, \ldots, m\}$ and for all $n \in \{n_-, \ldots, n_+\}$. Thus,

$$\begin{aligned} \left\| p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right) - p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right) \right\| &= \left\| \sum_{n=n_-}^{n_+} hG_n \right\| \\ &\leq \int_{t_{n_-}}^{t_{n_+ + 1}} F(s)ds \\ &\leq \int_{t - \frac{\tilde{\rho}}{4(\kappa+1)}}^{t + \frac{\tilde{\rho}}{4(\kappa+1)} + h} F(s)ds. \end{aligned}$$

Letting N to infinity, we obtain that $\|p(t^+) - p(t^-)\| = 0$. This means that

$$\dot{q}(t^-) = p(t^-) = p(t^+) = \dot{q}(t^+).$$

Step 2: Now, let $t \in (0, T)$ be such that $J(t, q(t)) \neq \emptyset$. Consider the case if $J(t, q(t)) = \{1, \ldots, m\}$, we let $\rho_t = \frac{1}{2}\min(\rho, t, T - t)$. Otherwise, using the continuity of the mappings $f_i, i \in \{1, \ldots, m\}$ we may define ρ_t in $(0, \min(\rho, t, T - t))$

such that, for all $i \in \{1, \dots, m\} \setminus J(t, q(t))$ we have

$$f_i(s, y) \leq \frac{1}{2} f_i(t, q(t)) < 0 \ \ \forall s \in [t - \rho_t, t + \rho_t], y \in \bar{B}(q(t), \rho_t).$$

Then, by the uniform convergence of (q_N) to q on $[0, T]$, we can define

$$N_t > \max\left(N^1, \frac{4T(\kappa + 1)}{\rho_t}\right)$$

such that $\|q - q_N\|_{C([0,T];\mathbb{R}^d)} \leq \frac{\rho_t}{4}$ for all $N > N_t$. We will show that for all $N > N_t$ and for all $t_n \in \left[t - \frac{\rho_t}{4(\kappa+1)}, t + \frac{\rho_t}{4(\kappa+1)}\right]$, $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$. Indeed, let $N > N_t$ and $t_n \in \left[t - \frac{\rho_t}{4(\kappa+1)}, t + \frac{\rho_t}{4(\kappa+1)}\right]$. We have

$$\begin{aligned} |t_{n+1} - t| &\leq \frac{\rho_t}{4(\kappa + 1)} + h \leq \frac{\rho_t}{2(\kappa + 1)} < \rho_t, \\ \|Q_{n+1} - q(t)\| &\leq \|Q_{n+1} - q_N(t)\| + \|q_N(t) - q(t)\| \\ &\leq \kappa |t_{n+1} - t| + \|q - q_N\|_{C([0,T];\mathbb{R}^d)} < \rho_t. \end{aligned}$$

In addition, we have

$$f_i(t_{n+1}, Q_{n+1}) < 0 \ \ \forall i \notin J(t, q(t)).$$

Therefore, $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$. Represent $J(t, q(t))$ as $J(t, q(t)) = J_1(t, q(t)) \cup J_2(t, q(t))$ with

$$\begin{aligned} J_1(t, q(t)) = \Big\{ & i \in J(t, q(t)) \mid \exists \rho_i \in (0, \rho_t], \exists N_i > N_t, \forall N > N_i, \\ & \forall t_n \in \left[t - \frac{\rho_i}{4(\kappa+1)}, t + \frac{\rho_i}{4(\kappa+1)}\right] \cap [0, T], f_i(t_{n+1}, Q_{n+1}) < 0 \Big\} \end{aligned}$$

and

$$\begin{aligned} J_2(t, q(t)) = \Big\{ & i \in J(t, q(t)) \mid \forall \rho_i \in (0, \rho_t], \forall N_i > N_t, \exists N > N_i, \\ & \exists t_n \in \left[t - \frac{\rho_i}{4(\kappa+1)}, t + \frac{\rho_i}{4(\kappa+1)}\right] \cap [0, T], f_i(t_{n+1}, Q_{n+1}) = 0 \Big\}. \end{aligned}$$

Since $J_1(t, q(t))$ is a finite set, we may define

$$\begin{cases} \tilde{\rho}_t = \min\{\rho_i \mid i \in J_1(t, q(t))\}, \tilde{N}_t = \max\{N_i \mid i \in J_1(t, q(t))\} & \text{if } J_1(t, q(t)) \neq \emptyset \\ \tilde{\rho}_t = \rho_t, \tilde{N}_t = N_t & \text{if } J_1(t, q(t)) = \emptyset. \end{cases}$$

Now let $\tilde{\rho} \in (0, \tilde{\rho}_t]$ and $N > \tilde{N}_t$. As before, we define

$$n_- = \left\lfloor \frac{t - \dfrac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor + 1, \quad n_+ = \left\lfloor \frac{t + \dfrac{\tilde{\rho}}{4(\kappa+1)}}{h} \right\rfloor$$

which implies that

$$\begin{aligned} 2h < (n_- - 1)h \leq t - \frac{\tilde{\rho}}{4(\kappa+1)} < n_- h < \ldots < n_+ h \\ \leq t + \frac{\tilde{\rho}}{4(\kappa+1)} < (n_+ + 1)h < T - 2h \end{aligned}$$

and

$$P_{n_- - 1} = p_N\left(t - \frac{\tilde{\rho}}{4(\kappa+1)}\right) \quad P_{n_+} = p_N\left(t + \frac{\tilde{\rho}}{4(\kappa+1)}\right).$$

Thus,

$$P_{n_+} - P_{n_- - 1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{n=n_-}^{n_+} \sum_{i=1}^{m} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}).$$

Since $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$, $i \notin J(t_{n+1}, Q_{n+1})$ implies that $i \in J_1(t, q(t))$. Thus,

$$P_{n_+} - P_{n_- - 1} = \sum_{n=n_-}^{n_+} hG_n - \sum_{i \in J_2(t, q(t))} \sum_{n=n_-}^{n_+} \lambda_i^n \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}). \tag{6.14}$$

If $J_2(t, q(t)) = \emptyset$ using the same arguments as in Step 1, we can obtain that $\dot{q}(t^+) = \dot{q}(t^-)$. Moreover, since $q(s) \in C(s)$ for all $s \in [0, T]$, $\dot{q}(t^+) \in \mathcal{T}(t, q(t))$. It follows that $\dot{q}(t^+) = \dot{q}(t^-) \in \mathcal{T}(t, q(t))$ and therefore we have

$$\dot{q}(t^-) = \dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t, q(t))}(\dot{q}(t^-)).$$

For the case where $J_2(t,q(t)) \neq \emptyset$, we rewrite (6.14) as follows

$$\begin{aligned} &p_N\left(t+\frac{\tilde{\rho}}{4(\kappa+1)}\right)-p_N\left(t-\frac{\tilde{\rho}}{4(\kappa+1)}\right)\\ &=-\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\lambda_i^n\nabla f_i(t,\cdot)(q(t))+\sum_{n=n_-}^{n_+}hG_n\\ &\quad-\sum_{i\in J_2(t,q(t))}\textstyle\sum_{n=n_-}^{n_+}\lambda_i^n\Big(\nabla f_i(t_{n+1},\cdot)(Q_{n+1})-\nabla f_i(t,\cdot)(q(t))\Big). \end{aligned} \tag{6.15}$$

Before continuing the proof, we prove the following two technical lemmas.

Lemma 6.20. *We have*

$$p(t^+)-p(t^-)\in-\sum_{i\in J_2(t,q(t))}\mathbb{R}_+\nabla f_i(t,\cdot)(q(t)).$$

Proof. We can estimate the last two terms of (6.15) as follows

$$\left\|\sum_{n=n_-}^{n_+}hG_n\right\|\leq\int_{t_{n_-}}^{t_{n_++1}}F(s)ds\leq\int_{t-\frac{\tilde{\rho}}{4(\kappa+1)}}^{t+\frac{\tilde{\rho}}{4(\kappa+1)}+h}F(s)ds$$

and, let $\Delta_i^n(t)=\lambda_i^n(\nabla f_i(t_{n+1},\cdot)(Q_{n+1})-\nabla f_i(t,\cdot)(q(t)))$, using Lemma 6.15 and Remark 6.1(ii) we have

$$\begin{aligned} \left\|\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\Delta_i^n(t)\right\|&\leq\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\|\Delta_i^n(t)\|\\ &\leq\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\lambda_i^nL(|t_{n+1}-t|+\|Q_{n+1}-q(t)\|)\\ &\leq\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\lambda_i^nL\left(\left(h+\frac{\tilde{\rho}}{4(\kappa+1)}\right)+\|q-q_N\|_{C([0,T];\mathbb{R}^d)}\right)\\ &\leq L\left(\left(h+\frac{\tilde{\rho}}{4(\kappa+1)}\right)+\|q-q_N\|_{C([0,T];\mathbb{R}^d)}\right)\\ &\quad\times\frac{m}{\mu}\Big(\mathrm{Var}(p_N,[0,T])+\|F\|_{L^1(0,T;\mathbb{R}^d)}\Big). \end{aligned}$$

From (6.15), it follows that

$$\begin{aligned} \lim_{\tilde{\rho}\to0^+}\lim_{N\to\infty}\ &\Big\|p_N\left(t+\tfrac{\tilde{\rho}}{4(\kappa+1)}\right)-p_N\left(t-\tfrac{\tilde{\rho}}{4(\kappa+1)}\right)\\ &+\textstyle\sum_{i\in J_2(t,q(t))}\sum_{n=n_-}^{n_+}\lambda_i^n\nabla f_i(t,\cdot)(q(t))\Big\|=0. \end{aligned} \tag{6.16}$$

We now will prove that the set $S := \sum_{i\in J_2(t,q(t))} \mathbb{R}_+\nabla f_i(t,\cdot)(q(t))$ is a closed subset of $\mathbb{R}$. Indeed, let $\{x_n\}$, with $x_n = \sum_{i\in J_2(t,q(t))} x_{i,n}\nabla f_i(t,\cdot)(q(t))$, be a sequence in S converging to some x^*. By assumption A2, there exists $v = v(t,q(t))$ such that $\|v\|= 1$ and

$$\langle x_n, v\rangle = \left\langle \sum_{i\in J_2(t,q(t))} x_{i,n}\nabla f_i(t,\cdot)(q(t)), v \right\rangle = \sum_{i\in J_2(t,q(t))} x_{i,n}\langle \nabla f_i(t,\cdot)(q(t)), v\rangle \leq (-\mu) \sum_{i\in J_2(t,q(t))} x_{i,n}.$$

From this it follows that

$$0 \leq x_{i,n} \leq \sum_{i\in J_2(t,q(t))} x_{i,n} \leq \frac{1}{\mu}\langle x_n, -v\rangle \leq \frac{1}{\mu}\|x_n\|.$$

Since $\{x_n\}$ is a convergent sequence, there exists $l > 0$ such that for each $i \in J_2(t,q(t))$ we have $0 \leq x_{i,n} < l$ for all n. Hence, there exists a subsequence of $\{x_{i,n}\}$, denoted by $\{x_{i,n'}\}$ and nonnegative number x_i^* such that for all $i \in J_2(t,q(t))$

$$x_{i,n'} \xrightarrow{n'\to+\infty} x_i^*.$$

Since the sequence $\{x_n\}$ converges to x^*, the sequence $\{x_{n'}\}$ also converges to x^*. We have

$$\left\| x_{n'} - \sum_{i\in J_2(t,q(t))} x_i^*\nabla f_i(t,\cdot)(q(t)) \right\| \leq \sum_{i\in J_2(t,q(t))} |x_{i,n'} - x_i^*|\|\nabla f_i(t,\cdot)(q(t))\|.$$

From this we obtain the limit

$$x^* = \sum_{i\in J_2(t,q(t))} x_i^*\nabla f_i(t,\cdot)(q(t)) \in S.$$

We have shown that $\sum_{k\in J(t,q)} \mathbb{R}_+\nabla f_i(t,\cdot)(q)$ is closed. Hence, by (6.16) we get the desired result. □

Lemma 6.21. *For all $i \in J_2(t,q(t))$, one has*

$$\partial f_i(\cdot,q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), \dot{q}(t^+)\rangle = 0.$$

Proof. By Lemma 6.17, $\dot{q}(t^+) \in \mathcal{T}(t, q(t))$. Hence, for each $i \in J_2(t, q(t))$,

$$\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), \dot{q}(t^+) \rangle \leq 0.$$

We only need to prove that

$$\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), \dot{q}(t^+) \rangle \geq 0; \ \forall i \in J_2(t, q(t)).$$

Let $i \in J_2(t, q(t))$ and $\tilde{\rho} \in (0, \tilde{\rho}_t]$. By the definition of $J_2(t, q(t))$, there exists a subsequence $\{N_\alpha\}_{\alpha \in \mathbb{N}}$ strictly increasing to infinity such that, for all $\alpha \in \mathbb{N}$ we have $N_\alpha > \tilde{N}_t$. Let $h_\alpha = T/N_\alpha$, then there exists $nh_\alpha \in \left[t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)}\right]$ such that $f_i(t_{n+1}, Q_{n+1}) = 0$, i.e. $i \in J(t_{n+1}, Q_{n+1})$. We define

$$n_\alpha = \max\left\{ n \in \mathbb{N} \mid nh_\alpha \in \left[t - \frac{\tilde{\rho}}{4(\kappa+1)}, t + \frac{\tilde{\rho}}{4(\kappa+1)}\right] \ \text{ and } \ i \in J(t_{n+1}, Q_{n+1}) \right\}.$$

By Lemma 6.18 we have

$$\partial f_i(\cdot, Q_{n+1})(t_{n+1}) + \langle \nabla f_i(t_{n+1}, \cdot)(Q_{n+1}), P_n \rangle \geq -h(L + L\|P_n\| + \gamma \|P_n\|^2).$$

It follows that

$$\begin{aligned} &\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), P_{n_+} \rangle \\ &\geq -h_\alpha(1 + \kappa + \gamma\kappa^2) + \left(\partial f_i(\cdot, q(t))(t) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1})\right) \\ &\quad + \langle \nabla f_i(t, \cdot)(q(t)), P_{n_+} - P_{n_\alpha} \rangle + \langle \nabla f_i(t, \cdot)(q(t)) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1}), P_{n_\alpha} \rangle. \end{aligned} \tag{6.17}$$

We can estimate the second and fourth terms of the right-hand side of (6.17) as follows

$$\begin{aligned} \partial f_i(\cdot, q(t))(t) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1}) &\geq -L(|t - t_{n_\alpha+1}| + \|Q_{n_\alpha+1} - q(t)\|) \\ &\geq -L\left(\frac{\tilde{\rho}}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right) \end{aligned}$$

and

$$\begin{aligned} &\langle \nabla f_i(t, \cdot)(q(t)) - \nabla f_i(t_{n_\alpha+1}, \cdot)(Q_{n_\alpha+1}), P_{n_\alpha} \rangle \\ &\geq -L(|t - t_{n_\alpha+1}| + \|Q_{n_\alpha+1} - q(t)\|)\|P_{n_\alpha}\| \\ &\geq -L\kappa\left(\frac{\tilde{\rho}}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right). \end{aligned}$$

If $n_\alpha = n_+$, the third term of the right-hand side of (6.17) vanishes. Otherwise,

we rewrite it as follows

$$\begin{aligned}\langle \nabla f_i(t,\cdot)(q(t)), P_{n_+} - P_{n_\alpha}\rangle &= \left\langle \nabla f_i(t,\cdot)(q(t)), \sum_{n=n_\alpha+1}^{n_+} hG_n \right\rangle + \langle \nabla f_i(t,\cdot)(q(t)), e_1\rangle \\ &\geq -L\int_{t-\frac{\tilde\rho}{4(\kappa+1)}}^{t+\frac{\tilde\rho}{4(\kappa+1)}} F(s)ds + \langle \nabla f_i(t,\cdot)(q(t)), e_2\rangle \\ &\quad + \langle \nabla f_i(t,\cdot)(q(t)), e_1 - e_2\rangle\end{aligned}$$

where

$$\begin{aligned}e_1 &= \sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n \nabla f_j(t_{n+1},\cdot)(Q_{n+1}),\\ e_2 &= \sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n \nabla f_j(t,\cdot)(q(t)).\end{aligned}$$

Since $i \notin J(t_{n+1}, Q_{n+1})$ for all $n \in \{n_\alpha+1, \dots, n_+\}$ by the definition of n_α and the inclusion $J(t_{n+1}, Q_{n+1}) \subset J(t, q(t))$, assumption A4 implies that the second term of the right-hand side of this last inequality is non-negative. Furthermore, the last term can be estimate as

$$\begin{aligned}\langle \nabla f_i(t,\cdot)(q(t)), e_1 - e_2\rangle &\geq -\sum_{n=n_\alpha+1}^{n_+} \sum_{j\in J(t_{n+1},Q_{n+1})} \lambda_j^n L^2(|t-t_{n+1}|+\|Q_{n+1}-q(t)\|) \\ &\geq -L^2 m\left(\frac{\tilde r}{4(\kappa+1)} + h_\alpha + \|q - q_{N_\alpha}\|_{C([0,T];\mathbb{R}^d)}\right) \\ &\quad \times (\mathrm{Var}(p_N,[0,T]) + \|F\|_{L^1(0,T;\mathbb{R}^d)}).\end{aligned}$$

Recalling that $P_{n_+} = p_N\left(t + \frac{\tilde\rho}{4(\kappa+1)}\right)$, then, passing the right-hand side of (6.17) to the limit, we obtain

$$\begin{aligned}\lim_{\tilde\rho\to 0^+}\lim_{N_\alpha\to\infty} \partial f_i(\cdot,q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), P_{n_+}\rangle &= \partial f_i(\cdot,q(t))(t) \\ &\quad + \langle \nabla f_i(t,\cdot)(q(t)), p(t^+)\rangle \\ &\geq 0.\end{aligned}$$

This means that $\partial f_i(\cdot,q(t))(t) + \langle \nabla f_i(t,\cdot)(q(t)), \dot q(t^+)\rangle \geq 0$. □

We now continue the proof of Proposition 6.19. We have $\dot{q}(t^+) \in \mathcal{T}(t, q(t))$ and

$$\dot{q}(t^+) - \dot{q}(t^-) \in - \sum_{i \in J_2(t,q(t))} \mathbb{R}_+ \nabla f_i(t, \cdot)(q(t)).$$

Hence there exist non-negative real numbers $\bar{\lambda}_i$, for $i \in J_2(t, q(t))$, such that

$$\dot{q}(t^+) - \dot{q}(t^-) = - \sum_{i \in J_2(t,q(t))} \bar{\lambda}_i \nabla f_i(t, \cdot)(q(t))$$

for all $w \in \mathcal{T}(t, q(t))$

$$\langle \dot{q}(t^-) - \dot{q}(t^+), w - \dot{q}(t^+) \rangle = \sum_{i \in J_2(t,q(t))} \bar{\lambda}_i \langle \nabla f_i(t, \cdot)(q(t)), w - \dot{q}(t^+) \rangle.$$

But, using the previous proposition, for all $w \in \mathcal{T}(t, q(t))$ and for all $i \in J_2(t, q(t))$, we have

$$\begin{aligned} \langle \nabla f_i(t, \cdot)(q(t)), w - \dot{q}(t^+) \rangle &= (\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), w \rangle) \\ &\quad - (\partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), \dot{q}(t^+) \rangle) \\ &= \partial f_i(\cdot, q(t))(t) + \langle \nabla f_i(t, \cdot)(q(t)), w \rangle) \\ &\leq 0. \end{aligned}$$

Hence

$$\langle \dot{q}(t^-) - \dot{q}(t^+), w - \dot{q}(t^+) \rangle \leq 0 \ \ \forall w \in \mathcal{T}(t, q(t)).$$

As $\mathcal{T}(t, q(t))$ is a closed convex subset of $\mathbb{R}^d$, the above is equivalent to

$$\dot{q}(t^+) = \mathbb{P}_{\mathcal{T}(t,q(t))}(\dot{q}(t^-)).$$

The proof is complete. □

Finally we observe that the limit trajectory satisfies the initial data. Indeed, with (6.10) we have immediately $q(0) = q_0$. Moreover, $p_0 \in \mathcal{T}(0, q_0)$ we can prove that $\dot{q}(0^+) = p_0 = \mathbb{P}_{\mathcal{T}(0,q_0)}(p_0)$ by the same kind of computations. Indeed, if $t = t_0 = 0$, we may define $\rho_{t_0} \in (0, \min(\rho, T))$ such that

$$J(s, y) \subset J(t_0, q(t_0)) \ \ \forall s \in [t_0 - \rho_{t_0}, t_0 + \rho_{t_0}] \cap [0, T] \ \ \forall y \in \bar{\mathbb{B}}(q(t_0), \rho_{t_0})$$

and we define N_{t_0} (respectively, $\tilde{\rho}_{t_0}$ and $\tilde{N}_{t_0}$ if $J(t_0, q(t_0)) \neq \emptyset$). in the same way as previously. Then, for all $\tilde{\rho} \in (0, \rho_{t_0}]$ and for all $N > h_{t_0}$ (respectively, for all

$\tilde{\rho} \in (0, \tilde{\rho}_{t_0}]$ and for all $N > \tilde{N}_{t_0}$ if $J(t_0, q(t_0)) \neq \emptyset$) we define

$$n_- = 0, \quad n_+ = \left[\frac{t_0 + \dfrac{\tilde{\rho}}{4(\kappa+1)}}{h} \right].$$

We get

$$P_{n_- - 1} = P_{-1} = p_0, \quad P_{n_+} = p_N \left(t_0 + \frac{\tilde{\rho}}{4(\kappa+1)} h \right)$$

Using the same computation as above, we obtain $\dot{q}(0^+) = p_0$.

Remark 6.22. A similar existence result was proved in [17, Theorem 4.6]. Let us mention that our proof does not require any second-order information or boundedness on the constraints f_i such that $(A3)$ and $(A4)$ used in [17]. In fact, the boundedness conditions on $|\nabla^2 f_i(t,\cdot)(q)|$ and $|\partial^2 f_i(\cdot,q)(t)| + |\partial(\nabla f_i(\cdot,\cdot)(q))(t)|$ used in [17] are not necessary in our analysis. Moreover, the condition (R_q) used in [17] is replaced here by the weak uniform Slater condition A2. Our existence result is more specific to constraints inequalities, uses less regularity assumptions on the constraints f_i and could be seen as complementary to [17, Theorem 3.2]. In fact, Theorem 3.2 in [17] gives a global existence result for second-order differential inclusions involving a general abstract prox-regular and Lipschitz continuous set $C(t)$. When applying this result to the particular case of finite inequality constraints

$$C(t) = \{q \in \mathbb{R}^d \mid f_i(t,q) \leq 0 \ \ \forall i \in \{1, \ldots, m\}\}, \tag{6.18}$$

two main questions arise: under which conditions on the data f_i the set $C(t)$ is Lipschitz continuous? and is prox-regular? It is well known that the sublevel of prox-regular functions may fail to be prox-regular and also the prox-regularity of sets is not stable under intersection (see [8] for more details). Our aim here is to give some verifiable and practical conditions on the data f_i to satisfy both the prox-regularity and Lipschitz continuity properties of the set $C(t)$ in (6.18). An other way to obtain Theorem 6.7 is to assume A1–A3 to prove via Propositions 6.3 and 6.4 the prox-regularity and the Lipschitz continuity of the set $C(t)$ given in (6.18) and then apply the general Theorem 3.2 in [17]. For the convenience of the reader, we prefer to give a direct and self-contained proof specific to constraints inequalities based on the time-stepping algorithm. We mention that this technique for proving existence result for nonsmooth second-order differential inclusion problems was

also used in [15, 16, 73]. The following example shows that the assumptions $(A3)$ and $(A4)$ in [17] could not be satisfied.

6.2 Example

Let $t \in [0,1]$ and for $i \in \{1,2\}$, $f_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_1(t,(x,y)) = \begin{cases} -y - t & \text{if } x \leq 0 \\ -\frac{1}{4}x^2 - y - t & \text{if } 0 \leq x \leq 1 \\ -\frac{1}{2}x + \frac{1}{4} - y - t & \text{if } x \geq 1, \end{cases}$$

and

$$f_2(t,(x,y)) = \begin{cases} -y - t & \text{if } x \geq 4 \\ -\frac{1}{4}(4-x)^2 - y - t & \text{if } 3 \leq x \leq 4 \\ \frac{1}{2}(x-4) + \frac{1}{4} - y - t & \text{if } x \leq 3. \end{cases}$$

Consider the problem $\mathcal{P}$ with the set

$$C(t) = \{q = (x,y) \in \mathbb{R}^2 \mid f_i(t,q) \leq 0, i \in \{1,2\}\}$$

and $g(t,q) = 0$.

Observe that $f_i(\cdot,\cdot)$, $i \in \{1,2\}$ are differentiable and theirs derivatives are Lipschitz continuous with rank $L = \frac{\sqrt{5}}{2}$. This shows that the assumption A1(i) holds. Note that $\partial f_1(\cdot,q)(t) = \partial f_2(\cdot,q)(t) = -1$ and

$$\nabla f_1(t,\cdot)(x,y) = \begin{cases} (0,-1) & \text{if } x \leq 0 \\ (-\frac{1}{2}x,-1) & \text{if } 0 \leq x \leq 1 \\ (-\frac{1}{2},-1) & \text{if } x \geq 1, \end{cases}$$

and

$$\nabla f_2(t,\cdot)((x,y)) = \begin{cases} (0,-1) & \text{if } x \geq 4 \\ (\frac{1}{2}(4-x),-1) & \text{if } 3 \leq x \leq 4 \\ (\frac{1}{2},-1) & \text{if } x \leq 3. \end{cases}$$

Assumption A1(ii) is always true for $v = (0, 1)$ and $\mu = 1$. We also have $\|f_i(t, \cdot)(x, y)\| \leq L$ and therefore, assumption A1(iii) holds. Assumption A2 is satisfied with the choice of $\gamma = \frac{1}{2}$. If $J(t, q) = \{1, 2\}$ we have

$$\langle \nabla f_1(t, \cdot)(q), \nabla f_2(t, \cdot)(q) \rangle = -\frac{1}{2}\frac{1}{2} + (-1)(-1) = \frac{3}{4} \geq 0.$$

Hence, assumption A4 holds. We have showed that assumptions A1–A4 are satisfied for the above problem. By Theorem 6.7, the problem has a solution.

Note that the second order derivative with respect to the second variable q of f_1 (of f_2) does not exists at $q = (0, y)$ (at $q = (4, y)$, respectively) for any $y \in \mathbb{R}$. Hence $f_1, f_2 \notin C^2([0, 1] \times \mathbb{R}^2; \mathbb{R})$. This shows that the assumptions proposed in [15, 17, 73] cannot be applied to ensure the existence solution for this example.

6.3 Conclusions

In this paper, we have presented some regularity conditions for the data to ensure the existence of solution for a class of vibro-impact problems. These conditions neither require the second-order differentiability nor convexity of constraint functions. Some assumptions relate to the uniformly prox-regularity of the set of admissible positions. We also give an example to illustrate the applicability of the provided assumptions.

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Titre thèse français: Systèmes Dynamiques Non-réguliers: Applications en Optimisation et aux Processus de Rafles

Résumé : Dans cette thèse, nous étudions quelques classes de systèmes dynamiques non-réguliers. Plus précisément, les processus de rafles perturbés, les processus de rafles avec contraintes de vitesse ainsi que les problèmes de vibro-impact sur un ensemble de contraintes non-convexe dépendant du temps.

Le premier sujet porte sur l'existence et l'unicité de solutions pour les processus de rafles perturbés non-convexes. Dans le cadre adopté par Edmond et Thibault [Mathematical Programming 104 (2005), 347–373], nous étudions une classe de processus de rafles perturbés. Sous des hypothèses appropriées, nous obtenons deux théorèmes d'existence de solutions pour les processus de rafles perturbés, les ensembles de contraintes étant des ensembles de sous-niveaux prox-réguliers. Les résultats sont appliqués à l'analyse du comportement de certains procédés de rafles en mécanique unilatérale.

Le deuxième sujet porte sur certaines classes de processus de rafles avec vitesse dans un ensemble en mouvement. En plus de l'existence et l'unicité de la solution pour le cas d'un ensemble de contraintes convexe en mouvement, des résultats sur l'existence de la solution et la multiplicité de la solution où l'ensemble de contraintes est une union finie d'ensembles convexes disjoints sont également obtenus. Notre outil principal est un théorème sur la sensibilité des solutions des inéquations variationnelles paramétriques. Outre l'exigence traditionnelle selon laquelle l'ensemble de contraintes se déplace continuellement dans le sens de la distance de Hausdorff, nous utilisons intensivement une nouvelle hypothèse de type Lipschitz des multi-applications à valeurs dans l'ensemble de contraintes. Les résultats obtenus sont comparés à ceux existants et analysés à l'aide de plusieurs exemples. De plus, certaines propriétés de solutions de processus de rafles convexe avec des contraintes de vitesse sont également étudiées. En effet, la sensibilité des solutions par rapport à la valeur initiale, la limitation, la fermeture et la convexité de l'ensemble de solutions sont discutées en détail.

Le troisième sujet porte sur un problème de vibro-impact, qui est décrit sous la forme d'inclusion différentielle à mesure de second ordre. Grâce à une discrétisation de notre problème par l'algorithme de pas de temps, on construit une suite de solutions approchées qui converge vers une solution du problème considéré.

Mots clés : Processus de rafles, problème de vibro-impact, contrainte dépendante du temps, contrainte de vitesse, prox-régularité, ensemble de sous-niveaux, Propriétés de type Lipschitz des multi-applications.

Titre thèse anglais: Nonsmooth Dynamical Systems: Applications in Optimization and Sweeping Processes

Abstract: In this dissertation, we study some classes of nonsmooth dynamical systems. Namely, perturbed sweeping processes, sweeping processes with velocity constraints, and vibro-impact problems are investigated.

The first topic is on the solution existence and uniqueness of nonconvex perturbed sweeping processes. In the setting adopted by Edmond and Thibault [Mathematical Programming 104 (2005), 347–373], we study a class of perturbed sweeping processes. Under suitable assumptions, we obtain two solution existence theorems for perturbed sweeping processes with the constraint sets being prox-regular sublevel sets. The results are applied to analyzing the behavior of some concrete mechanical sweeping processes.

The second topic is on some classes of sweeping processes with velocity in a moving set. In addition to the solution existence and the solution uniqueness for the case of a moving convex constraint set, some results on the solution existence and the solution multiplicity where the constraint set is a finite union of disjoint convex sets are also obtained. Our main tool is a theorem on the solution sensitivity of parametric variational inequalities. Beside the traditional requirement that the constraint set moves continuously in the Hausdorff distance sense, we intensively use a new assumption on the local Lipschitz-likeness of the constraint set-valued mapping. The obtained results are compared with the existing ones and analyzed by several examples. In addition, some properties of solutions of convex sweeping processes with velocity constraints are also studied. Namely, the solution sensitivity with respect to the initial value, the boundedness, the closedness, and the convexity of the solution set are discussed in detail.

The third topic is on a vibro-impact problem, which is described in the form of second-order measure differential inclusion. We are able to discretize our problem by the time-stepping algorithm and construct a sequence of approximate solutions which is proved to converge to a solution of the problem in consideration.

Keywords: Sweeping process, vibro-impact problem, time-dependent constraint, velocity constraint, prox-regularity, sublevel set, Lipschitz-likeness