UNIVERSITÉ DE LIMOGES

ÉCOLE DOCTORALE SISMI

FACULTÉ DES SCIENCES ET TECHNIQUES

Thèse

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE LIMOGES

Discipline : Mathématiques et ses applications

présentée et soutenue par

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le 22 Février 2021

Méthodes variationnelles pour les équations d' Hamilton-Jacobi et applications

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Résumé

L'objectif de cette thèse est de proposer des méthodes variationnelles pour l'analyse mathématiques et numérique d'une classe d'équations d'HJ. Le caractère métrique de ces équations permet de caractériser l'ensemble des sous-solutions, à savoir, elles sont 1-Lipschitz par rapport à la distance Finslerienne associée au Hamiltonien. De manière équivalente, cela revient à dire que le gradient de ces fonctions appartient à une certaine boule Finslerienne. La solution recherchée est la sous-solution maximale, qui peut être décrite par une formule du type Hopf-Lax, qui résout un problème de maximisation avec contrainte sur le gradient. Nous dérivons un problème dual associé faisant intervenir la variation totale Finslerienne de mesures vectorielles avec contrainte divergente. Nous exploitons la structure de point-selle pour proposer une résolution numérique avec la méthode du Lagrangien augmenté. Cette caractérisation de l'équation d'HJ montre aussi le lien avec des problèmes de transport optimal vers/depuis le bord. Ce lien avec le transport optimal de masse nous amène à généraliser l'approche d'Evans-Gangbo. En effet, nous montrons que la sous-solution maximale de l'équation d'HJ s'obtient en faisant tendre $p \to \infty$ dans une classe de p-Laplaciens de type Finsler avec des obstacles sur le bord. Cela nous permet aussi de construire le flux optimal pour le problème de Beckmann associé. Parmi les applications que l'on regarde, le problème du *Shape from Shading* qui consiste à reconstruire la surface d'un objet en 3D à partir d'une image en nuances de gris de cet objet.

Mots clés: Équations d'Hamilton-Jacobi, transport optimal, dualité de Fenchel-Rockafellar, Lagrangien augmenté, Shape from Shading.

Abstract

In this thesis we propose some variational methods for the mathematical and numerical analysis of a class of HJ equations. Thanks to the metric character of these equations, the set of subsolution corresponds to the set of 1-Lipschitz functions with respect to the Finsler metric associated to the Hamiltonian. Equivalently, it corresponds to the set of functions whose gradient belongs to a Finsler ball. The solution we are looking for is the maximal one, which can be described via a Hopf-Lax formula, solves a maximization problem under gradient constraint. We derive the associated dual problem which involves the Finsler total variation of vector measures under a divergence constraint. We take advantage of this saddle-point structure to use the augmented Lagrangian method for the numerical approximation of HJ equation. This characterization of the HJ equation allows making the link with some optimal transport problems. This link with optimal transport leads us to generalize the Evans-Gangbo approach. In fact, we show that the maximal viscosity subsolution of the HJ equation can be recovered by taking $p \to \infty$ in a class of Finsler p-Laplace problems with boundary obstacles. In addition, this allows us to construct the optimal flow for the associated Beckmann problem. As an application, we use our variational approach for the *Shape from Shading* problem.

Keywords: Hamilton-Jacobi equations, optimal transport, Fenchel-Rockafellar duality, augmented Lagrangian, Shape from Shading.

Remerciements

Je tiens à remercier mon directeur de thèse Nourerddine Igbida pour son soutien, ses précieux conseils, sa patience et de m'avoir donné une certaine liberté d'explorer pleins de domaines passionnants durant ces trois années de thèse. Je tiens à remercier également mon co-directeur Van Thanh Nguyen pour sa disponibilité et son soutien et d'avoir toujours su s'arranger pour discuter malgré le décalage horaire !

J'exprime ma reconnaissance et gratitude à Guillaume Carlier et Julián Toledo d'avoir accepté d'accorder un peu de leur temps pour rapporter ce manuscrit. Leurs questions et remarques m'ont permis de l'améliorer. Je remercie également, Jean-François Aujol, Quentin Mérigot et Francisco Silva qui m'ont fait l'honneur de participer au jury de soutenance.

Ce travail n'aurait pu avoir lieu sans plein de personnes. Je tiens donc à remercier Quentin Mérigot et Filippo Santambrogio de m'avoir encouragé et introduit à ce magnifique domaine de mathématiques et à travers eux tous mes enseignants à Orsay. Merci à Ayman Moussa pour ses nombreux conseils et pour son cours d'analyse réelle ! Je tiens à remercier mes enseignants à Jussieu et Marrakech en particulier, Alexandru Oancea, Hassan El Hamri, Ahmad Amar, Brahim Bahassa, Abdeljalil Ellabane, El Hadi Ait Dads, M'hammed El Kahoui, Salem Nafiri, Mohamed El Rhazi et Youssef H. (merci pour les cours et les matches de foot !). Je voudrais remercier tous les membres d'XLIM, en particulier le département de mathématiques pour l'accueil. Je remercie particulièrement les enseignants avec qui j'ai fait mes premiers pas dans l'enseignement en tant que moniteur puis comme ATER. Je pense à Pascale Sénéchaux, Alain Salinier, Pierre Dusart et Simone Naldi.

Je tiens à remercier Annie Nicolas, Débora Thomas, Marion Massip, Sophie Quille, Yolande Vieceli et Henri Massias pour l'aide apportée au quotidien.

Je remercie Afaf Bouharguane, Jalal Fadili et Abderrahim El Moataz pour les nombreuses discussions et conseils durant mes visites à Bordeaux et à Caen. J'espère que j'aurai l'occasion de collaborer et apprendre de nouvelles choses avec eux.

Je tiens à remercier tou·te·s les thésard·e·s que j'ai croisé·e·s ces trois dernières années. Youssef (mon tout premier contact à Limoges, je me souviendrais toujours de sa chemise noire assez particulière et des nombreuses questions d'algèbre qu'on essayait de résoudre pendant les pauses café), Gaurav avec qui j'ai partagé le bureau (merci pour les échanges et de m'avoir fait découvrir de nombreux endroit en France et aux USA, surtout merci pour la balade à Saint-Pardoux), Laura (merci pour ton aide, surtout en AL1, pour les répétitions et aussi pour les gâteaux !), Tran (merci pour le café vietnamien et la technique de la sieste!), Maxime (merci pour les discussions passionnantes et le trajet partagé pendant le couvre-feu). Merci à Iman, Don, Duc, Leo, Thieu et Maksym pour les discussions passionnantes au quotidien et leur aide. Merci à mes amis libanais Ahmad, Ali, Ali, Joelle, Ghadir et Sahar pour les moments et discussions qui donnent envie de visiter ce beau pays qu'est le Liban. Merci à mes amis de promo du M2 en particulier Amina, Elise, Félix, Farid, Karim et Yanis pour plein de discussions passionnantes et pour tous ces moments inoubliables à Paris. J'en profite pour remercier particulièrement les geeks: Tariq et Pablo pour toutes ses années d'amitié et les discussions sur FreeFem et au-delà. Je n'oublierai surtout pas Ayoub avec qui j'ai partagé ce long chemin depuis Marrakech, donc merci à lui et à ses proches et à sa famille. Merci à mes amis de promo à Marrakech avec qui j'ai gardé le contact et j'ai pu échanger durant toutes ces années, en particulier Anas, Abdo, Iman, Imad, Ismail, Khalid, Omar, Youness et Zizou. Merci à mes amis d'enfance qui liraient ces lignes.

Merci à mes parents, pour tout, et surtout durant ces longues années d'études, je ne peux pas trouver les mots pour dire à quel point je leur suis reconnaissant. Merci à mes frères et surtout à ma sœur FZ qui m'a longuement supporté et dont je suis particulièrement fière d'avoir suivi sa passion pour les sciences et je lui souhaite bonne chance pour sa thèse. Merci à tous les membres de la famille qui m'ont soutenu et encouragé. Merci à Hadda et sa famille pour leur aide durant mes premiers mois en France.

Merci à tous mes amis pongistes en particulier ceux de LSATT. Grâce à eux j'ai pu découvrir le Limousin avec les différentes compétitions et surtout les tournois d'été. Merci pour tous ces moments de convivialité !

Finalement, je remercie ma femme qui doit exister quelque part (je ne sais pas quel théorème mathématique peut affirmer son existence, mais j'ai la foi !) sans qui cette thèse ne se serait pas bien passée (ou pas ?) mais je tiens à la remercier à priori.

À mes parents
À la mémoire de mon grand-père
1930 – 2017

"If you are receptive and humble, mathematics will lead you by the hand."– Paul A.Dirac

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1 General introduction

This thesis is devoted to the mathematical and numerical analysis of a class of Hamilton-Jacobi (HJ) equations with some applications. We will be mainly concerned with equations of first order of the form

$$H(x, \nabla u) = 0 \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set and $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, called the Hamiltonian, satisfying some suitable assumptions that will be recalled later.

These equations have attracted the attention of mathematicians and physicists as they appear in many fields such as classical mechanics, geometry (especially symplectic geometry) with the study of Hamiltonian systems and in optics, more particularly, with Fermat and Huygens's principles (see e.g.[5, 60, 73, 103]). They also arise in the study of Helmhlolz and Schrödinger equations in the framework of WKB[1] method (see e.g.[8]). In addition to this, they have plenty of applications in computer vision and computational geometry, amongst them the Shape from Shading problem which will be the content of Chapter 4, and geodesic extraction which will be discussed in Section 7.4 (see the book of J.Sethian [100] for further examples). Another area of application of HJ equations is optimal control where the dynamical programming principle allows the characterization of the value function as a solution of a subfamily of HJ equations, usually called Hamilton-Jacobi-Bellman (HJB) equations (see e.g.[7, 25, 80]). Lately, they are in the core of the newborn theory of mean field games (see e.g.[74, 75, 76]).

As we will see in the next chapter, the notion of classical solutions, i.e., smooth functions satisfying the equation pointwise, is not a suitable as a notion of solutions, and viscosity theory, introduced by C.Crandall and P.L.Lions in the 1980s (see [9, 31, 32, 80]) provides an adequate framework to study these equations. The omnipresence of HJ equations led to a various methods and techniques to approximate the solutions, we will present some of them in Chapter 2.

[1]Named after Wentzel, Kramer and Brillouin

Outline

This thesis is organised as follows. In Chapter 2, we recall the main notions and tools from the theoretical background on HJ equations, to the duality results and primal-dual algorithms we propose to solve them. In particular, Section 2.1 contains some reminders on viscosity theory, the metric character as well as a brief survey on some existing numerical methods to deal with HJ equations. Section 2.2 concerns some reminders from optimal transport theory which are needed essentially in Chapters 5,6. We recall in Section 2.3 the notion of tangential measure and tangential gradient which plays a major role in optimization on measure spaces. Finally, we recall in Section 2.4 duality results in convex optimization as well as primal-dual algorithms.

Chapter 3 contains our main result proposing a variational formulation for the HJ equation. The starting point is that every Hamiltonian H induces a geodesic distance (actually a quasi-distance) of Finsler type d_σ, where

$$d_\sigma(x,y) = \inf_{\xi\in\Gamma(x,y)} \int_0^1 \sigma(\xi(t),\dot{\xi}(t))\mathrm{d}t,$$

$\Gamma(x,y)$ being the set of Lipschitz curves joining two points $x, y \in \overline{\Omega}$, and σ being the support function of the zero sublevels of H (see (2.6)). This quasi-distance appears to characterize all the subsolutions of the HJ equation, and more generally, it intervene in representing any viscosity solution via Hopf-Lax type formulas. Indeed, every viscosity subsolution u of (1.1) turns to be $1-$Lipschitz with respect to d_σ, i.e., satisfies $u(x) - u(y) \leq d_\sigma(y,x)$ for any $x, y \in \overline{\Omega}$. We prove that this property is equivalent to saying that the gradient of u belongs to some convex ball. More precisely, we show that $\nabla u \in \mathcal{B}_{\sigma^*}$ a.e, where $\mathcal{B}_{\sigma^*}$ is the unit ball of the dual function σ^*. Coupling (1.1) with a Dirichlet boundary condition $u = g$ on $\partial\Omega$, for some "compatible" continuous function $g : \partial\Omega \to \mathbb{R}$, we prove that he maximal viscosity subsolution given by

$$u(x) = \min_{y\in D} d_\sigma(y,x) + g(y),$$

is the unique solution of the problem

$$\max\Big\{ \int_\Omega z(x)\mathrm{d}x,\ \sigma^*(x,\nabla z(x)) \leq 1 \text{ and } z = g \text{ on } \partial\Omega\Big\}. \tag{1.2}$$

To prove duality, we consider a general version of (1.2) by maximizing $\int_\Omega z\mathrm{d}\rho$ for some positive Radon measure ρ instead of $\int_\Omega z\mathrm{d}x$ and we write the problem as an optimization problem involving the sum of two operators (see Section 2.4). We derive a dual problem involving some vector measures under a divergence constraint, with a particular attention to introduce a trace-like operator for these vector fields to handle rigorously their contribution on the boundary. More precisely, we prove the following

Theorem 1.1. *Let $\rho \in \mathcal{M}_b^+(\Omega)$, then*

$$(\mathcal{M}_D): \ \max\left\{ \int_{\Omega_D} u d\rho : \ \sigma^*(x, \nabla u(x)) \le 1 \text{ and } u = g \text{ on } \Gamma_D \right\},$$

and

$$(\mathcal{OF}_D): \inf_{\phi \in \mathcal{DM}^p(\Omega_D)} \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) \mathrm{d}x - \langle \phi \cdot \nu_D, g \rangle : \ -div(\phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\},$$

coincide. Where $\Omega_D = \Omega \setminus D$ and $\mathcal{DM}^p(\Omega_D)$ is the space of $\phi \in L^p(\Omega_D)^N$ whose divergence is a bounded measure on Ω_D.

The proof will be done via an appropriate perturbation technique. This will enable us to solve $(\mathcal{M}_D)$ and $(\mathcal{OF}_D)$ simultaneously by looking for saddle points of a suitable augmented Lagrangian functional L_r:

$$\inf_{(u,q) \in W^{1,\infty}(\Omega_D) \times L^\infty(\Omega_D)^N} \ \sup_{\phi \in \mathcal{DM}^p(\Omega_D)} L_r(u, q; \phi).$$

Keeping in mind that the solution of (1.2) can be recovered by taking $\rho \equiv 1$, we take advantage of this saddle-point structure to use the augmented Lagrangian method to approximate the solution of $(\mathcal{M}_D)$ (an thus the solution of HJ equation) by considering a finite-dimensional optimization problem of the form

$$\inf_{u \in \mathscr{X}_h} \mathcal{F}(u) + \mathcal{G}(\Lambda u) \tag{1.3}$$

for some appropriate functionals $\mathcal{F}, \mathcal{G}$. We apply the ALG2 algorith to solve (1.3) and we provide several numerical examples.

In Chapter 4 we propose a formulation of the so-called Shape from Shading problem within the framework of Chapter 3. SfS is a classical problem in computer vision consisting in the reconstruction of the 3D shape of an object, given a greylevel image of its brightness map. It has been extensively studied, even before the birth of viscosity theory. The PDE formulation of this problem gives rise to a HJ equation. Complementing this equation with a compatible boundary data g, we can recover the maximal viscosity subsolution via (1.2). By assuming that $g \in H^{1/2}(\partial\Omega)$, we derive a slightly different dual problem from the one presented in Chapter 3. More precisely, we prove the following

Theorem 1.2. *The extremal values*

$$(\mathcal{M}): \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u(x) \mathrm{d}x, \ \sigma^*(x, \nabla u(x)) \le 1 \text{ and } u = g \text{ on } \partial\Omega \right\},$$

and

$$(\mathcal{OF}) : \inf_{\phi \in L^2(\Omega)^N} \left\{ \int_\Omega \sigma(x, \phi)\mathrm{d}x - \langle g, \phi \cdot \mathbf{n} \rangle_{H^{1/2}, H^{-1/2}} : -\mathrm{div}(\phi) = 1 \ \text{in} \ \mathcal{D}^{'}(\Omega) \right\},$$

coincide.

As we will see, our variational approach allows handling the degeneracy of the Hamiltonian, which in this case corresponds to points with maximal brightness. More particularly, no regularisation is needed to avoid these points. In contrast with Chapter 3, we opt for Chambolle-Pock's primal-dual algorithm to approximate the solution since it is suitable when working on images and doesn't require solving some linear PDE. More precisely, we write the discrete version of $(\mathcal{M})$ in an inf-sup form

$$(\mathcal{M})_d : \inf_{u \in X} \sup_{\phi \in Y} \mathcal{F}_h(u) + \langle \phi, \nabla_h u \rangle - \mathcal{G}_h(\phi),$$

where $\mathcal{F}_h, \mathcal{G}_h$ are discrete functions to be precised later, and ∇_h is the discrete gradient operator defined via finite differences. In addition, we prove the following result convergence of our discretization

Proposition 1.3. *Assume that the Finsler metric σ associated with the Hamiltionian H is non-degenerate (i.e. $H(x, 0) < 0, \forall x \in \overline{\Omega}$) and that $g = 0$. Let $u_h \in X$ and $\phi_h = (\phi_h^1, \phi_h^2) \in Y$ be a pair of primal-dual solutions to the discrete optimization problem $(\mathcal{M})_d$ and its dual problem. Then $\tilde{u}_h \rightrightarrows u$ and $\tilde{\phi}_h \rightharpoonup \phi$ as the step size $h \to 0$. Moreover, u and ϕ are optimal solutions to $(\mathcal{M})$ and its dual problem, respectively.*

At the end we show several shapes and tests.

Chapter 5 investigates the connection between HJ equation and the Beckmann problem. As one can see, our formulation (1.2) is close to the so-called Kantorovich-Rubinstein problem (see Section 2.2). The main difference is the presence of the boundary constraint $u = g$ on $\partial\Omega$ and the lack of a target measure since $\mu_1 = \mathcal{L}^N \llcorner \Omega$ and $\mu_2 = 0$. In fact, we show that the HJ equation is connected to the following problem

$$\min_{\nu \in \mathcal{M}_b(\partial\Omega)} \max_u \left\{ \int_\Omega u(\mathrm{d}x - \mathrm{d}\nu) - \int_{\partial\Omega} g \, \mathrm{d}\nu \ : \ u \text{ is } 1 - \text{Lipschitz w.r.t } d_\sigma \right\}, \quad (1.4)$$

for which suitable Beckmann and Monge-Kantorovich problems are presented. Again, considering a general version of (1.2),

$$(\mathcal{M}_D) : \max \left\{ \int_\Omega u \, \mathrm{d}\rho \ : \ u(y) - u(x) \le d_\sigma(x, y), \ \forall x, y \in \Omega \text{ and } u = g \text{ on } D \right\},$$

where $\rho \in \mathcal{M}_b(\Omega)$, we prove the following result using perturbation techniques and approximation of degenerate Finsler metrics:

Theorem 1.4. *The optimization problem* $(\mathcal{M}_D)$ *coincides with the following Beckmann-type problem*

$$(\mathcal{BK}): \quad \min_{\phi\in\mathcal{M}_b(\overline{\Omega})^N, \nu\in\mathcal{M}_b(D)} \left\{ \int_{\overline{\Omega}} \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_D g\mathrm{d}\nu : \ -\mathrm{div}(\phi) = \rho - \nu \ in \ \mathcal{D}'(\mathbb{R}^N) \right\}.$$

Observing that

$$\min_{\phi\in\mathcal{M}_b(\overline{\Omega})^N} \left\{ \int_{\overline{\Omega}} \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| : \ -\mathrm{div}(\phi) = \rho - \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\} = W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-),$$

where $W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-)$ is Monge-Kantorovich work between ν and ρ, we get by Kantorovich-Rubinstein dualiy (see Section 2.2) that

$$W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-) = \max_u \left\{ \int_{\overline{\Omega}} ud(\rho - \nu) : \ u \text{ is } 1 - \text{Lipschitz w.r.t } d_\sigma \right\}.$$

At the end, by minimizing over all $\nu \in \mathcal{M}(D)$, we obtain of the following general version of (1.4)

$$(\mathcal{M}_N): \quad \min_{\nu\in\mathcal{M}_b(D)} \max_u \left\{ \int_{\overline{\Omega}} ud(\rho - \nu) + \int_D g\mathrm{d}\nu : \ u \text{ is } 1 - \text{Lipschitz w.r.t } d_\sigma \right\},$$

as well as the following Monge-Kantorovich problem

$$(\mathcal{MK}): \quad \min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}), \nu\in\mathcal{M}_b(D)} \left\{ W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-) + \int_D g\mathrm{d}\nu : \ \pi_1\sharp\gamma = \rho^- + \nu^+, \ \pi_2\sharp\gamma = \rho^+ + \nu^- \right\}.$$

Using optimal transport techniques, we prove the following

Theorem 1.5. *Under the assumptions (H1-H3) (see Section 2.1.2), we have*

$$\max(\mathcal{M}_D) = \min(\mathcal{BK}) = \min(\mathcal{M}_N) = \min(\mathcal{MK}).$$

Finally, we provide some numerical examples showing the potential u and the flow Φ. In particular, we see that the set of degeneracy of d_σ, called the Aubry set (see Section 2.1), plays a role of free-transport region for the optimal transport problems associated to HJ equation.

In chapter 6 we examine a PDE approach à la Evans Gangbo [44] for the following HJ equation with obstacles on the boundary

$$H(x, \nabla u) = 0 \text{ in } \Omega, \ \text{ and } \phi \le u \le \psi \text{ on } \partial\Omega, \tag{1.5}$$

where $\phi, \psi \in C(\partial\Omega)$ satisfying some compatibility condition that will be recalled later. Again, seeking to recover the maximal viscosity subsolution of (1.5), we will consider the following problem

$$\max\Big\{\int_\Omega u\mathrm{d}x : \sigma^*(x,\nabla u) \leq 1 \text{ a.e., } \phi \leq u \leq \psi \text{ on } \partial\Omega\Big\},$$

where σ^* is the dual of the support function of the 0-sublevels of H. To do further analysis, we consider for a Finsler metric H on Ω (not to be confused with the Hamiltonian) and $\rho \in L^2(\Omega)$

$$(\mathcal{KR})_H : \max\Big\{\int_\Omega u\mathrm{d}\rho : H^*(x,\nabla u) \leq 1 \text{ a.e., } \phi \leq u \leq \psi \text{ on } \partial\Omega\Big\},$$

which appears to be the Kantorovich-Rubinstein problem associated to the following mass transport problem

$$(\mathcal{K})_H : \min_{\gamma\in\Pi(\rho^+,\rho^-)}\Big\{\int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y)+\int_{\partial\Omega}\psi(y)\mathrm{d}(\pi_y)_\sharp\gamma-\int_{\partial\Omega}\phi(x)\mathrm{d}(\pi_x)_\sharp\gamma\Big\}.$$

Moreover, duality arguments show that the associated Beckamann's problem reads

$$(\mathcal{B})_H : \min_{\substack{\Phi\in\mathcal{M}^N(\Omega)\\ \nu\in\mathcal{M}(\partial\Omega)}}\Big\{\int_\Omega H(x,\frac{\Phi}{|\Phi|})\mathrm{d}|\Phi|+\int_{\partial\Omega}\psi\mathrm{d}\nu^- -\int_{\partial\Omega}\phi\mathrm{d}\nu^+ : -\mathrm{div}(\Phi) = \rho+\nu \text{ in } \mathcal{D}'(\mathbb{R}^N)\Big\}.$$

The strategy consists in considering a family of Finsler p-Laplace problems with obstacles on the boundary whose solutions will be obtained by minimizing the functional:

$$\mathcal{F}_p(u) := \int_\Omega \frac{H^*(x,\nabla u)^p}{p}\mathrm{d}x - \int_\Omega u\rho\mathrm{d}x,$$

over $\mathcal{W}_{\phi,\psi} = \{u \in W^{1,p}(\Omega) : \phi \leq u \leq \psi \text{ on } \partial\Omega\}$.

We derive suitable estimates to pass to the limit as $p\to\infty$, and thus recover the Kantorovich potential. More precisely, we prove the following

Proposition 1.6. *Let u_p be a minimizer of $\mathcal{F}_p$. Then, up to a subsequence, $u_p \rightrightarrows \mathbf{u}$ on $\overline{\Omega}$. Moreover, $\mathbf{u}$ solves $(\mathcal{KR})_H$.*

Next, we define for $p > N$

$$\Theta_p = H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p).$$

We show that the measures Θ_p and $\Theta_p\cdot\mathbf{n}$ are equibounded in Ω and $\partial\Omega$ respectively, and thus converge to some measures $\Theta \in \mathcal{M}(\overline{\Omega})^N$ and $\theta \in \mathcal{M}(\partial\Omega)$, and we prove that the couple (Θ, θ) solves $(\mathcal{B})_H$. At the end, we relate the Kantorovich potential $\mathbf{u}$ to $(\mathcal{K})_H$. More precisely, we prove the following

Proposition 1.7. *Let $\mathbf{u}$ be the potential constructed in Proposition 1.6. Then $\mathbf{u}$ is a Kantorovich potential for the classical optimal transport problem between $\rho^+ \llcorner \Omega + \theta^+$ and $\rho^- \llcorner \Omega + \theta^-$. Moreover*

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x = \min(\mathcal{K})_H.$$

Finally, in Chapter 7 we present results from some works in progress as well as some perspectives and future works.

Publications

- Augmented Lagrangian method for degenerate Hamilton-Jacobi equations, H.Ennaji, N.Igbida and V.T.Nguyen. (submitted).
- Continuous Lambertian Shape From Shading: A primal-dual algorithm, H.Ennaji, N.Igbida and V.T.Nguyen. (submitted).
- Beckmann-type problem for degenerate Hamilton-Jacobi equations. H.Ennaji, N.Igbida and V.T.Nguyen. (submitted).
- Quasi-convex Hamilton–Jacobi equations via limits of Finsler p-Laplace problems as $p \to \infty$, H.Ennaji, N.Igbida and V.T.Nguyen. (submitted).

2 Preliminaries

In this chapter we recall some notions and tools from PDE theory, optimal transport, optimization and numerical analysis. We start by an overview on HJ equations and their metric character which will play a major role in this thesis. Then, we provide some preliminaries on optimal transport theory. More particularly, the dual formulations of the so-called Monge-Kantorovich problem are recalled. Finally, we recall Fenchel-Rockafellar duality result and we present the two main algorithms that will be used in this manuscript: ALG2 algorithm and the so-called Chambolle-Pock's primal dual algorithm.

2.1 An overview on HJ equations

2.1.1 On viscosity solutions

Given an open bounded domain $\Omega \subset \mathbb{R}^N$ and consider the following Laplace equation

$$-\Delta u := -\text{trace}(D^2 u) = 0 \text{ in } \Omega. \tag{2.1}$$

A priori, one would say that a function $u : \Omega \to \mathbb{R}$ is a solution of (2.1) if $\nabla u, D^2 u$ exist for all $x \in \Omega$ and that (2.1) is satisfied for every $x \in \Omega$. This is known as the notion of classical solution. Due to differentiablity requirement, it is not easy to look for classical solutions, and in practice, one defines a notion of weak solution and then checks for differentiablity. Very often, the weak solutions are defined thanks to integration by parts. Notice that (2.1) is in divergence form as we may write $\Delta u = \text{div}(\nabla u)$. So that a weak solution can be easily defined via integrals. Now considering an equation of the form

$$H(x, \nabla u) = 0 \text{ in } \Omega, \tag{2.2}$$

where $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a continuous convex function. One can easily see that integration by parts is not possible by considering the example

$$|\nabla u| = 1 \text{ in } \Omega, \tag{2.3}$$

which is highly nonlinear in contrast to (2.1). Moreover, under the Dirichlet condition $u = 0$ on $\partial\Omega$, one can prove that no C^1 solutions exists to (2.3) (see figure 2.1).

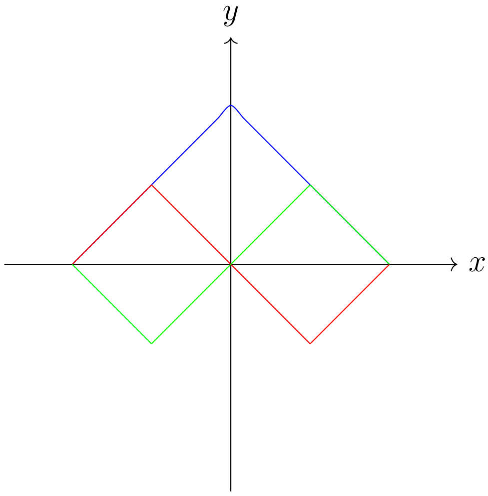

Figure 2.1: Several solutions of (2.3) on $\Omega = (-1, 1)$.

To introduce an appropriate notion of weak solutions one uses the vanishing viscosity method as follows. For $\epsilon > 0$, consider the variant of (2.2):

$$-\epsilon \Delta u_\epsilon + H(x, \nabla u_\epsilon) = 0 \text{ in } \Omega. \tag{2.4}$$

Then we can prove the existence of a solution u_ϵ of (2.4), and thanks to the maximum principle (see e.g.[53]), we get uniqueness. Moreover, standard Bernstein techniques (see e.g.[80, 98, 99]) allow deriving some appropriate estimates to prove the boundedness and equicontinuity of $\{u_\epsilon\}_\epsilon$. Using Ascoli-Arzelà Theorem, we deduce that the sequence $\{u_\epsilon\}_\epsilon$ has a uniform limit u, which is a candidate solution of (2.2). In fact, we can prove the following

Proposition 2.1 ([9, 80]).
- *Let $u_\epsilon \in C(\overline{\Omega}) \cap C^1(\Omega)$ be a classical subsolution of (2.4). If u_ϵ converges to $u \in C(\Omega)$ uniformly on compact subsets of Ω, then*

$$H(x, \nabla w(x)) \leq 0,$$

for any $w \in C^1(\Omega)$ such that $u - w$ reaches a maximum at $x \in \Omega$.

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$$H(x, \nabla w(x)) \geq 0,$$

for any $w \in C^1(\Omega)$ such that $u - w$ reaches a minimum at $x \in \Omega$.

This motivates the following definitions:

Definition 2.2. Given two continuous functions u and w, one says that w is a *strict* supertangent (respectively subtangent) to u at some point $x \in \Omega$ if x is a *strict* local maximizer (respectively minimizer) of $u - w$.

Definition 2.3. Let $u \colon \Omega \to \mathbb{R}$ be a continuous function.

- We say that u is a viscosity subsolution of (2.2) if $H(x, \nabla w(x)) \leq 0$ for any $x \in \Omega$ and any C^1 supertangent function w to u at x.
- We say that u is a viscosity supersolution of (2.2) if $H(x, \nabla \psi(x)) \geq 0$ for any $x \in \Omega$ and any C^1 subtangent function ψ to u at x.
- Finally, u is a viscosity solution of (2.2) if it is both a subsolution and a supersolution.

In sequel, we denote by $\mathcal{S}_H^-(\Omega)$ the family of subsolutions of (2.5). One pertinent property of this family is its stability with respect to uniform convergence, which is not the case for a.e. solutions as we can see by considering a sawtooth function. Moreover, whenever we consider a family $\mathcal{C} \subset \mathcal{S}_H^-(\Omega)$ of locally equibounded functions, then $\inf_{u \in \mathcal{C}} u(x)$ and $\sup_{u \in \mathcal{C}} u(x)$ are still subsolutions to (2.5). We similarly denote by $\mathcal{S}_H^+(\Omega)$ (resp. $\mathcal{S}_H(\Omega)$) the family of supersolutions (resp. solutions) of (2.5). For a detailed exposition on viscosity solutions, we refer the reader to [7, 9, 31, 80] and the references therein.

2.1.2 Metric character of HJ equations

The recent developments of metric formulas related to HJ equations appear essentially in the papers of A.Fathi, F.Camilli, A.Siconolfi [23, 24, 48], for Hamiltonians of Eikonal type, i.e., depending only on the space variable x and the momentum p, and they have played a major role more particularly in the framework of the so-called weak KAM theory. These formulas are of main interest since they turn to provide a characterisation of the set of all subsolutions of HJ equation. In addition, they only depend on the sublevel sets of the Hamiltonian, so the convexity assumption on the Hamiltonian can be weakened by assuming only convexity of sublevel sets. Moreover, as we will see in Chapters 5,6 these metrics appear also as costs in some optimal transport problems associated to our variational approximation of the HJ equation. Other formulas for general Hamiltonians arising in optimal control or Finsler geometry can be found for example in [7, 25, 78, 80].

Let $\Omega \subset \mathbb{R}^N$ be a regular connected open domain. We consider the following HJ equation

$$H(x, \nabla u) = 0, \ x \in \Omega, \tag{2.5}$$

with $H \colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ a continuous Hamiltonian satisfying

[1] Named after Kolmogorovov, Arnold and Moser.

(H1) Coercivity: $Z(x)$ is compact;

(H2) Convexity: $Z(x)$ is convex for any $x \in \Omega$;

(H3) $H(x,0) \leq 0$, i.e., $0 \in Z(x)$ for any $x \in \Omega$.

where for any $x \in \overline{\Omega}$,

$$Z(x) := \{p \in \mathbb{R}^N,\ H(x,p) \leq 0\},$$

is the 0-sublevel set of H.

Unless otherwise specified, the assumptions (H1)-(H3) will hold true for the rest of this manuscript. The solution are to be understood in the sense of viscosity. For $x \in \overline{\Omega}$, we define the support function of the 0-sublevel set $Z(x)$ by

$$\sigma(x,q) := \sup q \cdot Z(x) = \sup\{q \cdot p,\ \ p \in Z(x)\} \quad \text{for } q \in \mathbb{R}^N. \tag{2.6}$$

The assumption (H1)-(H2) ensures that σ is a continuous nonnegative function in $\Omega \times \mathbb{R}^N$, convex and positively homogeneous with respect to q. In addition, under (H2), we will say that the Hamiltonian H is quasiconvex. Due to the assumption (H3), $\sigma(x,q)$ is possible to equal to 0 for $q \neq 0$, which leads to the degeneracy and its dual σ^*, as given below, may take the value $+\infty$. Here, the dual (or polar) σ^* is defined by

$$\sigma^*(x,p) := \sup_q \{p \cdot q : \ \ \sigma(x,q) \leq 1\}.$$

A typical example is the Eikonal equation with

$$H(x,p) = |p| - k(x), \tag{2.7}$$

for a nonnegative continuous function k. In this case, one has $\sigma(x,q) = k(x)|q|$ and $\sigma^*(x,p) = \frac{1}{k(x)}|p|$. We see that $\sigma^*(x,p)$ take the value $+\infty$ for $p \neq 0$, on the zero set of k. We denote by $\Gamma(x,y)$ the set of Lipschitz curves defined on $[0,1]$ joining x, y in $\overline{\Omega}$. We then define the intrinsic distance by

$$d_\sigma(x,y) := \inf_{\zeta \in \Gamma(x,y)} \int_0^1 \sigma(\zeta(t), \dot{\zeta}(t))\mathrm{d}t, \tag{2.8}$$

which is a quasi-distance, *i.e.* satisfying $d_\sigma(x,x) = 0$ and the triangular inequality, but not necessarily symmetric. Some basic properties of d_σ can be summerized in the following

Proposition 2.4. *([48])*

1) d_σ is a quasi-metric, in the sense that for any $x, y \in \overline{\Omega}$ $d_\sigma(x,y) \geq 0$ and $d_\sigma(x,x) = 0$. Moreover, for all $x, y, z \in \overline{\Omega}$ one has $d_\sigma(x,y) \leq d_\sigma(x,z) + d_\sigma(z,y)$.

2) For any $x \in \Omega$ $d_\sigma(x,.) \in \mathcal{S}_H^-(\Omega) \cup \mathcal{S}_H^+(\Omega \setminus \{x\})$.

3) Compatibility condition:

$$v \in \mathcal{S}_H^-(\Omega) \ \textit{ if and only if } \ v(x) - v(y) \leq d_\sigma(y,x) \ \textit{ for any } \ x, y \in \overline{\Omega}. \tag{2.9}$$

Example 2.5. In the case of the Eikonal equation (2.7), we get a Riemannian metric:

$$d_k(x,y) := \inf_{\zeta \in \Gamma(x,y)} \int_0^1 k(\zeta(t))|\dot{\zeta}(t)|\mathrm{d}t.$$

Thanks to the previous proposition, any subsolution of (2.7) is 1-Lipschitz with respect to d_k. Moreover, when $k \equiv 1$, d_1 is nothing by the Euclidean distance, i.e., $d_1(x,y) = |x-y|$.

Example 2.6. Another interesting example can be given from convex geometry. Take a compact, convex set C of $\mathbb{R}^N$ such that $0 \in \mathrm{int}(C)$, then its gauge function reads

$$j_C(p) = \inf\{\lambda \geq 0;\ \lambda p \in C\}.$$

It can be shown that $j_{C^*} = \sigma_C$ where C^* is the polar set of C defined though

$$C^* = \{p \in \mathbb{R}^N :\ p \cdot q \leq 1,\ \forall q \in C\},$$

and σ_C is the support function of C defined through $\sigma_C(p) = \sup_{q \in C} p \cdot q$. Considering the following Hamiltonian $H(x,p) = j_C(p) - 1$, the metric formula (2.8) becomes

$$d_C(x,y) = j_{C^*}(y-x), \text{ for all } x, y \in \overline{\Omega}.$$

Indeed, take $x, y \in \overline{\Omega}$ and let ζ be a Lipschitz curve joining them. We then have

$$\int_0^1 \sigma(\zeta(t), \dot{\zeta}(t))\mathrm{d}t = \int_0^1 j_C^*(\dot{\zeta}(t))\mathrm{d}t = \int_0^1 j_{C^*}(\dot{\zeta}(t))\mathrm{d}t \geq j_{C^*}\left(\int_0^1 \dot{\zeta}(t)\mathrm{d}t\right) = j_{C^*}(y-x),$$

where we have used Jensen's inequality. Taking the inf over ζ, we get

$$d_C(x,y) \geq j_{C^*}(y-x).$$

On the other hand, for $t \in [0,1]$, define $\zeta_{x,y}(t) = x(1-t) + ty$, which satisfies $\zeta_{x,y}(0) = x$ and $\zeta_{x,y}(1) = y$. We get

$$d_C(x,y) \leq \int_0^1 j_{C^*}(\dot{\zeta}_{x,y}(t))\mathrm{d}t = j_{C^*}(y-x),$$

as claimed.

Example 2.7 (see [59]). Consider a Hamiltonian of the form

$$H(x,p) = \langle b(x), p\rangle + \frac{1}{2}|p|^2_{M(x)}, \tag{2.10}$$

where $b(x)$ is called the drift vector field and $M(x) = D(x)D^\dagger(x)$ is a positive definite matrix, D is called the diffusion matrix and $|\cdot|_M$ is the norm associted to the scalar product induced by M. Actually, the Hamiltonian (2.10) is related to the following stochastic differential equation in $\mathbb{R}^N$

$$\mathrm{d}X_t = b(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}M(X_t^\epsilon)\mathrm{d}W_t, \;\; X_0^\epsilon = x_1 \in \mathbb{R}^N.$$

where $\epsilon > 0$ and $(W_t)_{t\geq 0}$ is an N-dimensional Brownian motion. Then, the associated metric formula reads

$$d_M(x,y) = \inf_{\xi\in\Gamma(x,y)} \int_0^1 \sigma_M(\xi(t), \dot{\xi}(t))\mathrm{d}t,$$

with

$$\sigma_M(x,p) = |b(x)|_{M^{-1}(x)}|p|_{M^{-1}(x)} - \langle b(x), p\rangle_{M^{-1}(x)},$$

for any $x \in \Omega, p \in \mathbb{R}^N$. Let us stress that for general Hamiltonians, the support function σ can't be expressed in closed form.

The so called Aubry set is defined as the set where the quasi-metric d_σ degenerates. Prescribing a boundary value on $\partial\Omega$ does not guarantee the uniqueness of viscosity solutions to (2.5) unless $\mathcal{A} = \emptyset$, which is not the case in our situation due to the assumption $(H3)$. The Aubry set $\mathcal{A}$ appears then to be a uniqueness set for (2.5).

Definition 2.8. We define the Aubry set $\mathcal{A}$ as the set of points $x \in \Omega$ such that there exists $(\zeta_n)_n \in \Gamma(x,x)$ with $l(\zeta_n) \geq \delta > 0$ for some $\delta > 0$ and

$$\inf_n \left\{ \int_0^1 \sigma(\zeta_n(t), \dot{\zeta}_n(t))\mathrm{d}t \right\} = 0,$$

where $l(\zeta_n)$ is the Euclidean length of the curve ζ_n.

Proposition 2.9. *([48],[24])*

1) *The Aubry set $\mathcal{A}$ is a closed subset of Ω.*

2) *If $x \in \mathcal{A}$ then $\mathrm{d}_\sigma(x,.) \in \mathcal{S}_H(\Omega)$. Moreover, $x \notin \mathcal{A}$ if and only if (2.5) admits a strict subsolution around x.*

3) If $g : \mathcal{A} \cup \partial\Omega \to \mathbb{R}$ is a continuous function satisfying the compatibility condition $g(x) - g(y) \le d_\sigma(y, x)$ on $\mathcal{A} \cup \partial\Omega$, then

$$u(x) = \min_{y \in \mathcal{A} \cup \partial\Omega} \{d_\sigma(y, x) + g(y)\}$$

is the unique viscosity solution of the equation (2.5) such that $u = g$ on $\mathcal{A} \cup \partial\Omega$.

2.1.3 Numerical methods to solve HJ equations

We present here some different works and methods concerned with the approximation of HJ equations. This list is non exhaustive and the reader can check [49, 100] for a general presentation of numerical methods for HJ equations, or [46] for a more recent overview.

Optimal control. In [23] the authors propose an optimal control approach to approximate the maximal subsolutions of some degenerate HJ equation. More precisely, they study equations of the form:

$$\begin{cases} H(x, \nabla u) = f \text{ in } \Omega \\ \qquad\quad u = g \text{ on } \partial\Omega \end{cases} \tag{2.11}$$

where $\Omega \subset \mathbb{R}^N$ and f and g are continuous nonnegative functions. They consider a dynamic

$$\dot{v}(t) = p(t) \text{ for } t \in [0, \infty), \;\; v(0) = x, \tag{2.12}$$

where p is a measurable function and they introduce a cost functional

$$J(x, p) = \int_0^T \sigma(v(t), p(t))\mathrm{d}t + g(v(T)) \tag{2.13}$$

where σ is the support function of the sublevel-sets of H. It is well known that the associated dynamic programming equation of this optimal control problem reads

$$\sup_{|p| \le 1} p \cdot \nabla u(x) - \sigma(x, p) = 0$$

which is equivalent to finding sub/super-viscosity solutions to (2.11). Then, they solve the optimal control problem (2.12)-(2.13) via a semidiscrete scheme. More precisely, they choose a step in time $h_\epsilon \in (0, 1)$ for a fixed $\epsilon > 0$ and define the sequence

$$x_{i+1} = x_i + h_\epsilon p_i, i \in \mathbb{N}, \;\; x_0 = x$$

for $x \in \Omega$ and $q_i \in \mathbb{S}^{d-1}$. The discrete cost function becomes

$$J_{h_\epsilon}(x, p_n) = \sum_{i=0}^{n-1} h_\epsilon \sigma_\epsilon(x_i, p_i) + g(x_n)$$

with $n = \inf\{i \in \mathbb{N} : \ x_i \notin \Omega\}$. The value function is

$$u_{h_\epsilon}(x) = \inf_{p_i : n < \infty} J_{h_\epsilon}(x, p_i)$$

which solves by means of the discrete dynamic programming principle, the problem

$$\begin{cases} u_{h_\epsilon}(x) = \inf\limits_{|p|=1} h_\epsilon \sigma_\epsilon(x, p) + u_{h_\epsilon}(x + hp) \ \ x \in \Omega \\ u_{h_\epsilon}(x) = g(x) \ \ x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Finally, they show that for appropriate ϵ, h_ϵ, the approximated solution converges to the maximal solution of (2.11) as $\epsilon, h_\epsilon \to 0$ using stability results. For comprehensive exposition of HJ equations and optimal control one can see for example [25, 80].

Dijkstra type algorithms. One of the famous methods, at especially for Eikonal equations, remains the Fast Sweeping Method (FSM) and the Fast Marching Method (FMM). For the seek of simplicity, we consider the following equation

$$\begin{cases} |\nabla u| = \ f \ \text{ in } \ \Omega \\ \quad\ u = \ 0 \ \text{ on } \ \Gamma \subset \partial\Omega. \end{cases}$$

The FSM is based on an upwind difference discretization solved via Gauss-Seidel iterations with alternating sweeping ordering. More precisely, given a discretization $\{x_{i,j}\}_{ij}$ of Ω, and denoting by $u_{i,j}$ the solution at the grid point $x_{i,j}$, the equation can be rewritten

$$\Big((u_{i,j} - u_{xmin})_+\Big)^2 + \Big((u_{i,j} - u_{ymin})_+\Big)^2 = h^2 f_{i,j}^2 \tag{2.14}$$

where h is the step size and $u_{xmin} = \min(u_{i-1,j}, u_{i+1,j})$, $u_{ymin} = \min(u_{i,j-1}, u_{i,j+1})$. We initialize by $u_{i,j} = 0$ if $(i, j) \in \Gamma$ and ∞ otherwise. We compute the solution $\tilde{u}_{i,j}$ of (2.14) and we update $u_{i,j}$ by taking $\min(u_{i,j}, \tilde{u}_{i,j})$, in such a way we sweep the whole domain following the ordering

1) $i = 1, \cdots, M, j = 1, \cdots, N$

2) $i = M, \cdots, 1, j = 1, \cdots, N$

3) $i = M, \cdots, 1, j = N, \cdots, 1$

4) $i = 1, \cdots, M, j = N, \cdots, 1$

The FSM is easy to implement and converges within few iterations. However, less is known about its convergence and complexity for general HJ equations. As for the FMM, the grid points are divided into three categories: *Accepted* nodes which are points where the value of the solution are already known, *Narrow Band* nodes where the computations take place and finally *Far* nodes which will be computed in the next iterations. The Eikonal equation is written via an upwind finite difference approximation. We then compute the solution $u_{i,j}$ at a grid point (i,j) using the finite-difference scheme on the *Narrow Band*, we take the minimum value of $u_{i,j}$. The grid point (i,j) becomes an *Accepted* point and is removed from the *Narrow Band*. We add the first neighbours of this node to the *Narrow Band* and we continue this way until all points are *Accepted*.

Elliptic approach. In [20], the authors use an elliptic approach to solve the Eikonal equation

$$\begin{cases} |\nabla u| = & 1 \text{ in } \Omega \\ u = & g \text{ on } \partial\Omega \end{cases} \tag{2.15}$$

for a nonnegative continuous function g. To ensure the uniqueness, the authors look for u maximizing $\int_\Omega u\mathrm{d}x$ among functions $v \in H^1(\Omega)$ solving (2.15). The constraint $|\nabla u| = 1$ being nonlinear, they minimize $J(v) = \int_\Omega |\nabla v|^2 \mathrm{d}x - c\int_\Omega v\mathrm{d}x$ for some $c > 0$ instead of $\int_\Omega u\mathrm{d}x$, on $E_g = \{v \in H^1(\Omega) : v \text{ solves } (2.15)\}$. Then they penalize and regularize the problem by minimizing $\tilde{J}(v) = J(v) + \frac{\epsilon_1}{2}\int_\Omega |\Delta v|^2 \mathrm{d}x + \frac{1}{4\epsilon_2}\int_\Omega (|\nabla v|^2 - 1)^2 \mathrm{d}x$ among all $v \in H^2(\Omega)$ with $v = g$ on the boundary. By setting $p = \nabla u$, an equivalent formulation becomes

$$\min_{q\in Q}\left\{\frac{1}{2}\int_\Omega |q|^2 \mathrm{d}x - c\int_\Omega \nabla\phi\cdot q\mathrm{d}x + \frac{1}{4\epsilon_2}\int_\Omega (|q|^2-1)^2\mathrm{d}x + G(q)\right\} \tag{2.16}$$

where $Q = \{q \in L^2(\Omega)^2\}$, and ϕ is the unique solution of Dirichlet problem

$$-\Delta\phi = 1 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega$$

and

$$G(q) = \begin{cases} \frac{\epsilon_1}{2}\int_\Omega |\nabla\cdot q|\mathrm{d}x \text{ if } q \in \{\nabla v : v \in H^2(\Omega) \text{ and } v = g \text{ on } \partial\Omega\} \\ \infty \text{ otherwise.} \end{cases}$$

Finally, they associate a flow to (2.16) and discretize it by an operator-splitting method. Yet, this method seems to be restrictive to the Eikonal equation (2.15) and it is not clear how to adapt it for general Hamiltonians.

2.2 Optimal Transport Theory

All the results of this section can be found in the following classical books [4, 94, 102].

2.2.1 Monge and Kantorovich problem

The classical optimal transport problem goes back to Gaspard Monge with his celebrated paper "Mémoire sur la théorie des déblais et des remblais". Concretely, let us imagine that we have a certain amount of soil which should be moved to fill some holes in the ground, or imagine some bakeries that should supply some coffee shops. All this tasks need be done while minimizing a certain cost, which could be the travelled distance or the effort of transporting some quantity from a position to another *etc.*
So if μ and ν are probability measures on two subsets X and Y of $\mathbb{R}^N$, representing the initial and the target distributions, the modern formulation of Monge problem, consists in minimizing the quantity $\int |x-T(x)|^2 \mathrm{d}\mu(x)$, where T is a Borel Map from X to Y that pushes μ onto ν, *i.e*, ν coincides with the measure obtained by picking every atom at x and putting it at $T(x)$. More precisely, Monge's problem reads

$$\text{(MP)} : \min \int_X |x-T(x)|^2 \mathrm{d}\mu(x), \;\; \mu(T^{-1}(B)) = \nu(B),$$

for any Borelian $B \subset Y$. It is well known that this problem is difficult to solve due to the nonlinear constraint $T_\sharp \mu = \nu$. Moreover, such maps T may not exist as we can see by taking a Dirac mass $\mu = \delta_x$ and another atomless measure ν. Monge's optimal transport problem remained unstudied until around 1940 when Leonid Kantoroivich proposed a "relaxation" of the problem in his paper [70] by allowing mass splitting. More formally, he considered the problem

$$\text{(KP)} : \min \int_{X\times Y} c \mathrm{d}\gamma, \;\; \gamma \in \Pi(\mu,\nu)$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $X\times Y$ with marginals μ and ν respectively, i.e., $(\pi_X)_\sharp\gamma = \mu$ and $(\pi_Y)_\sharp\gamma = \nu$, where π_X, π_Y are the projections from $X \times Y$ onto X and Y respectively. Contrary to (MP), the set $\Pi(\mu,\nu)$ is always nonempty since it contains $\mu \otimes \nu$. Moreover, it's richer than the set of transport maps, in the sense that a transport map T induces a transport plan $\gamma_T = (\mathrm{id} \times T)_\sharp \mu$. In addition, $\Gamma(\mu,\nu)$ is a convex and compact subset of $\mathcal{P}(X\times Y)$ endowed with narrow topology, which helps to get existence results under weak assumptions on the cost function c. All these properties make the analysis of (KP) easier. We have the following

Theorem 2.10. *[94] Let X and Y be compact metric spaces, $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$ and $c : X \times Y \to \mathbb{R}(\mathbb{R} \cup \{\infty\})$ a continuous function (respectively, lower semi-continuous and bounded from bellow). Then (KP) admits a solution.*

We notice that (KP) is a linear optimization problem under convex constraints. So it is natural to derive its dual (see Section 2.4) formulation. This can be achieved (at least formally) by performing a $\min - \max$ interchange.
Let $\gamma \in \mathcal{M}_+(X \times Y)$ and $v \in C_b(X), w \in C_b(Y)$. We have

$$\sup_{v,w} \int_X v \mathrm{d}\mu + \int_Y w \mathrm{d}\nu - \int_{X\times Y} v \oplus w \mathrm{d}\gamma = \begin{cases} 0 \text{ if } \gamma \in \Pi(\mu,\nu), \\ \infty \text{ otherwise,} \end{cases}$$

where $(v \oplus w)(x,y) = v(x) + w(y)$. So we can replace the constraint $\gamma \in \Pi(\mu,\nu)$ in (KP) by the previous sup. This gives

$$\min_\gamma \int c\mathrm{d}\gamma + \sup_{v,w} \int_X v\mathrm{d}\mu + \int_Y w\mathrm{d}\nu - \int_{X\times Y} v \oplus w\mathrm{d}\gamma.$$

By interchanging the inf and sup we get

$$\sup_{v,w} \int_X v\mathrm{d}\mu + \int_Y w\mathrm{d}\nu + \inf_\gamma \int (c - v \oplus w)\mathrm{d}\gamma.$$

Rewriting the inf on γ as a constraint on the potentials v, w, we get:

$$\inf_\gamma \int (c - v \oplus w)\mathrm{d}\gamma = \begin{cases} 0 \text{ if } v \oplus w \leq c \text{ on } X \times Y, \\ -\infty \text{ otherwise.} \end{cases}$$

We define the dual problem (KD) as follows

$$\text{(KD)}: \sup\left\{ \int_X v d\mu + \int_Y w d\nu,\ v \in C_b(X),\ w \in C_b(Y) \textit{ and } v \oplus w \leq c \right\}.$$

Proposition 2.11. *Suppose that X and Y are compact and c is continuous. Then there exists a solution (v, w) to (KD).*

Moreover, if the cost c is a distance on X, then one can obtain the so-called Kantorovich-Rubinstein variant of (KD) as follows. Fix some $x_0 \in X$ and $1 \leq p \leq \infty$. We define the pth moment of a measure $\mu \in \mathcal{P}(X)$ by: $\mathrm{Mom}_p(\mu; x_0) = \frac{1}{p}\int_X c(x,x_0)^p \mathrm{d}\mu(x)$, and

$$\mathcal{P}^p(X) = \{\mu \in \mathcal{P}(X) : \exists x_0 \in X : \mathrm{Mom}(\mu; x_0) < \infty\}.$$

Clearly this definition is independent from x_0: If $\mathrm{Mom}_p(\mu; x_0) < \infty$ for some $x_0 \in X$, then $\mathrm{Mom}_p(\mu; x) < \infty$ for any $x \in X$. This being said, we have the following

Proposition 2.12 (Kantorovich-Rubinstein). *Let* $\mu, \nu \in \mathcal{P}^1(X)$, *then*

$$\min_{\gamma \in \Pi(\mu,\nu)} \int_{X \times X} c(x,y) \mathrm{d}\gamma(x,y) = \max_{v} \left\{ \int_X v \mathrm{d}(\mu - \nu) : \ v : X \to \mathbb{R}, 1 - \textit{Lipschitz w.r.t } c \right\}.$$

2.2.2 Beckmann's transportation problem

Beckmann's problem is a minimal flow type problem. It was proposed by Martin Beckmann [14] in 1952 as a model for transportation. One can think about optimal transference in an urban area represented by a bounded domain $\Omega \subset \mathbb{R}^2$, between two distributions of residents and services. These distributions can be represented by two nonnegative Radon measures ρ_1 and ρ_2, respectively. So, the signed measure $\rho := \rho_2 - \rho_1$ represents the local measure of excess demand. The consumers traffic is given by a traffic flow field, i.e. a vector field $\Phi : \Omega \to \mathbb{R}^N$ whose direction indicates the consumers' travel direction and whose modulus $|\Phi|$ is the intensity of the traffic. The relationship between the excess demand and the traffic flow is obtained from the equilibrium condition:

$$-\operatorname{div}(\Phi) = \rho \quad \text{in } \mathcal{D}'(\Omega).$$

This condition describes some kind of equilibrium: the outflow of consumers equals the excess demand in any subregion $\omega \subset \Omega$, i.e. $\int_{\partial\omega} \Phi \cdot n \, \mathrm{d}s = \rho(\omega)$. Since the measures ρ_1 and ρ_2 have equal masses (a balanced condition between residents and services), and the urban area is assumed to be isolated, i.e., no traffic flow should cross the boundary of the city Ω, the traffic flow Φ is subject to the boundary condition

$$\Phi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ denotes the outward normal vector to the boundary. Assuming the transportation cost per consumer is given by the quantity $|\Phi(x)|$, Beckmann therefore argued that one may define the transportation cost between ρ_1 and ρ_2 as the infimum of the total cost of the traffic $\int_\Omega |\Phi(x)| \, \mathrm{d}x$. So the considered problem reads:

$$(\mathrm{BP}) : \min_{\Phi \in L^1(\Omega)^N} \left\{ \int_\Omega |\Phi(x)| \, \mathrm{d}x : \ -\operatorname{div}(\Phi) = \rho \quad \text{in } \mathcal{D}'(\Omega) \right\}.$$

It can be shown that (BP) is actually equivallent to (KP) in the case of the Euclidean cost, i.e., $c(x,y) = |x - y|$. Again, let us express the divergence constraint on Φ as follows

$$\sup_w - \int_\Omega \Phi \cdot \nabla w \mathrm{d}x + \int_\Omega w \mathrm{d}\rho = \begin{cases} 0 \text{ if } -\operatorname{div}(\Phi) = \rho, \\ \infty \text{ otherwise.} \end{cases}$$

So we may rewrite (BP) as

$$\inf_{\Phi}\left\{\int_{\Omega}|\Phi|\mathrm{d}x+\sup_{w}-\int_{\Omega}\Phi\cdot\nabla w\mathrm{d}x+\int_{\Omega}w\mathrm{d}\rho\right\},$$

and by a formal inf-sup interchange, we get

$$\sup_{w}\int_{\Omega}w\mathrm{d}\rho+\inf_{\Phi}\left\{\int_{\Omega}|\Phi|\mathrm{d}x-\int_{\Omega}\Phi\cdot\nabla w\mathrm{d}x\right\}.$$

Noticing that the infimum on Φ expresses the constraint $|\nabla w|\leq 1$ or alternatively $w\in\mathrm{Lip}(\Omega)$, we get the equivalence with (KP) thanks to Proposition (2.12). This can be proved rigorously via Fenchel-Rockafellar duality. Notice that the existence of a solution to (BP) in $L^1(\Omega)^N$ is not true in general since the latter is nonreflexive. Setting this problem on a reflexive Banach space, say $L^2(\Omega)^N$, the direct method of calculus of variations would provide a minimizer thanks to weak compactness. However, it is convenient to consider this problem in the framework of divergence measure fields, i.e., consider flows $\Phi\in\mathcal{M}_b(\Omega)^N$ with $\mathrm{div}(\Phi)\in\mathcal{M}_b(\Omega)$. Some new variants of Beckmann's problem will be addressed in Chapter 5-6.

2.3 Tangential measure and tangential gradient

Let us recall some facts concerning the notion of tangential gradient which played a main role in this manuscript. To give a glimpse on the necessity to introduce this notion, let us remember that, as an example, Beckmann's problem is an optimisation problem on measure space under divergence constraint. More particularly, the optimal flow satisfies $-\mathrm{div}(\Phi)=\rho\in\mathcal{M}(\overline{\Omega})$. To do further analysis on such a problem and particularly to derive its dual problem we are naturally tempted to integrate by parts in the divergence constraint and write, for some Lipchitz function u

$$\int\nabla u\cdot\upsilon\,\mathrm{d}\gamma=\int u\mathrm{d}\rho.$$

where $\gamma=|\Phi|$ is the total variation of the vectorial measure Φ and $\upsilon=\frac{\Phi}{|\Phi|}$ is the Radon-Nikodym derivative of Φ with respect to $|\Phi|$. Observe that ∇u may not be defined on a $|\Phi|$-positive measure set and thus the previous formula may not have sense. Thanks to [18] it is possible to give a sens to the previous formula as follows. First we can define the tangent space to γ

$$\mathcal{X}_\gamma(x)=\gamma-\mathrm{ess}\bigcup\left\{\upsilon(x):\ \upsilon\in L^1_\gamma(\overline{\Omega},\mathbb{R}^N),\ \mathrm{div}(\upsilon\gamma)\in\mathcal{M}_b(\overline{\Omega})\right\},$$

where the $\gamma-$essential union is defined as a $\gamma-$measurable closed multivalued function such that

- If $v \in L^1_\gamma(\overline{\Omega}, \mathbb{R}^N)$ and $\operatorname{div}(v\gamma) \in \mathcal{M}_b(\overline{\Omega})$, then $v(x) \in \mathcal{X}_\gamma(x)$, for $\gamma-$a.e $x \in \overline{\Omega}$.
- The γ essential union is minimal amongst multivalued functions satisfying the previous properties.

Then, the tangential gradient $\nabla_\gamma(x)$ to a function $u \in C^1(\overline{\Omega})$ with respect to the measure γ is the orthogonal projection of $\nabla u(x)$ onto $\mathcal{X}_\gamma(x)$. Indeed, denoting by $\mathbf{P}_\gamma(x)$ the orthogonal projection on $\mathcal{X}_\gamma(x)$, it has been shown in [17] that the linear operator $u \in C^1(\overline{\Omega}) \to \mathbf{P}_\gamma(x)\nabla u(x) \in L^\infty_\gamma(\overline{\Omega}, \mathbb{R}^N)$ can be uniquely extended to a linear continuous operator

$$\nabla_\gamma : u \in \text{Lip}(\overline{\Omega}) \to \nabla u \in L^\infty_\gamma(\overline{\Omega}, \mathbb{R}^N).$$

Moreover, we have the following useful integration by parts formula

Proposition 2.13 ([17])**.** *Given $\gamma \in \mathcal{M}^+_b(\overline{\Omega})$ and $v \in L^1_\gamma(\overline{\Omega}, \mathbb{R}^N)$ such that $v(x) \in \mathcal{X}_\gamma(x)$ for $\gamma-$a.e x. and div$(\gamma v) := \rho \in \mathcal{M}_b(\overline{\Omega})$. We then have*

$$\int u \mathrm{d}\rho = \int v \nabla_\gamma u \mathrm{d}\gamma,$$

for any $u \in$ Lip$(\overline{\Omega})$.

As it was pointed out in [65, 92], it is possible to adapt this notion in the Finsler setting.

2.4 Fenchel-Rockafellar duality, Primal-Dual methods

Unless otherwise specified, we denote $\mathscr{X}, \mathscr{Y}$ two Banach spaces and by $\mathscr{X}^*, \mathscr{Y}^*$ their topological dual spaces.

2.4.1 Fenchel-Rockafellar duality

Fenchel-Rockafellar duality is one of the main tools in convex optimization. It allows rigorous derivation of the dual problem associated to an optimization problem, and thus reposes on the notion of the dual functional, also called Fenchel-Legendre transform.

Definition 2.14. Let $\mathcal{F} : \mathscr{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, l.s.c. and convex function. We define its dual functional $\mathcal{F}^* : \mathscr{X}^* \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{F}^*(f) = \sup_{u \in \mathscr{X}} \langle f, u \rangle_{\mathscr{X}^*, \mathscr{X}} - \mathcal{F}(u) \quad \text{for any } f \in \mathscr{X}^*.$$

Following the notations of [42], we sometimes denote by $\Gamma_0(\mathscr{X})$ the set of proper, l.s.c and convex functions on $\mathscr{X}$. It can be shown that the Fenchel-Legendre conjugation is an involution, that is, it is its own inverse: $\mathcal{F}^{**} = \mathcal{F}$ for all $\mathcal{F} \in \Gamma_0(\mathscr{X})$.

Many of the problems that we encounter fall into the scope of the following class of optimization problems

$$(\mathrm{P}): \quad \inf_{u\in\mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u) \tag{2.17}$$

where $\mathcal{F} : \mathscr{X} \to (-\infty, +\infty]$, $\mathcal{G} : \mathscr{Y} \to (-\infty, +\infty]$ are proper, l.s.c., convex functions and $\Lambda : \mathscr{X} \to \mathscr{Y}$ is a linear operator. Let us sketch how to derive the dual problem of (P). We start by exploiting the fact that $\mathcal{G}^{**} = \mathcal{G}$ to write

$$\mathcal{G}(\Lambda u) = \sup_{v\in\mathscr{Y}^*} \langle v, \Lambda u\rangle - \mathcal{G}^*(v),$$

and plugging this in (2.17), the problem (P) becomes:

$$\inf_{u\in\mathscr{X}} \sup_{v\in\mathscr{Y}^*} \mathcal{F}(u) + \langle v, \Lambda u\rangle - \mathcal{G}^*(v),$$

and by a formal inf-sup interchange we get

$$\sup_{v\in\mathscr{Y}^*} \inf_{u\in\mathscr{X}} \langle -\Lambda^* v, -u\rangle + \mathcal{F}(u) - \mathcal{G}^*(v) = \sup_{v\in\mathscr{Y}^*} -\mathcal{F}^*(-\Lambda u) - \mathcal{G}^*(v),$$

where Λ^* is the adjoint operator of Λ. So the associated dual problem to (P) reads:

$$(\mathrm{D}): \quad \sup_{v\in\mathscr{Y}^*} -\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v).$$

Using the definitions of $\mathcal{F}^*$, $\mathcal{G}^*$ and Λ^*, it is not difficult to get the following inequality

$$\sup_{v\in\mathscr{Y}^*} \Big(-\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v)\Big) \le \inf_{u\in\mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u),$$

usually called *weak duality*. The reverse inequality, the so-called *strong duality* gives a sufficient condition to justify the previous inf-sup interchange.

Before stating the main result of this section, let us recall the notion of subdifferentiability. Suppose that u is an optimal solution of (P), then the Euler-Lagrange equation would give

$$D\mathcal{F}(u) + \Lambda^* D\mathcal{G}(\Lambda u) = 0.$$

Unfortunately, one of the functions $\mathcal{F}$ or $\mathcal{G}$ may not be differentiable (typically in this manuscript $\mathcal{G}$ will be and indicator function of some convex set), so one need to use a more general notion of derivative.

Definition 2.15. Given a convex functional $\mathcal{F} : \mathscr{X} \to (-\infty, +\infty]$. We say that $p \in \mathscr{X}^*$ is a subgradient of $\mathcal{F}$ at $u \in \mathscr{X}$ if

$$\mathcal{F}(u) + \langle p, w - u\rangle_{\mathscr{X}^*,\mathscr{X}} \leq \mathcal{F}(w), \text{ for all } w \in \mathscr{X}.$$

The set of all subgradients of $\mathcal{F}$ at u is called the subdifferential and is denoted by $\partial\mathcal{F}(u)$. More precisely

$$\partial\mathcal{F}(u) = \{p \in \mathscr{X}^* : \mathcal{F}(u) + \langle p, w - u\rangle_{\mathscr{X}^*,\mathscr{X}} \leq \mathcal{F}(w), \text{ for all } w \in \mathscr{X}\}.$$

Now, let us state the Fenchel-Rockafellar duality result.

Theorem 2.16. *(Strong duality [42, Chap. III]) Assume moreover that there exists $u_0 \in \mathscr{X}$ such that $\mathcal{F}(u_0) < +\infty$, $\mathcal{G}(\Lambda u_0) < +\infty$ and $\mathcal{G}$ is continuous at Λu_0. Then the dual problem admits at least a solution $v \in \mathscr{Y}^*$ and the strong duality holds, i.e.*

$$\max_{v\in\mathscr{Y}^*} -\mathcal{F}^*(-\Lambda^* v) - \mathcal{G}^*(v) = \inf_{u\in\mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Moreover, the pair (u, v) solves the primal-dual problem (P)-(D) if and only if

$$-\Lambda^* u \in \partial\mathcal{F}(u) \;\; and \;\; v \in \partial\mathcal{G}(\Lambda u).$$

Before ending this subsection, let us recall the notion of Moreau's proximal mapping which plays a major role in many algorithms.

Definition 2.17. Assume that $\mathscr{X}$ is a real Hilbert space and let $J \in \Gamma_0(\mathscr{X})$. Then for every $\eta > 0$, the proximal mapping of ηJ is defined though

$$\mathrm{Prox}_{\eta J}(u) = \underset{w\in\mathscr{X}}{\arg\min}\left\{\frac{1}{2}\|w - u\|_X^2 + \eta J(w)\right\}, \text{ for any } u \in \mathscr{X}.$$

Remark 2.18. $\mathrm{Prox}_{\eta J}$ is also denoted by $(\mathrm{id} + \eta\partial J)^{-1}$ and is called the resolvent of ∂J.

Let us recall some basic computations of proximal opertors that will be used later

Proposition 2.19 ([12, 13]). *Let $J \in \Gamma_0(\mathscr{X})$, $u \in \mathscr{X}$ and $\eta > 0$ then*

- $\mathrm{Prox}_{\eta(J+\lambda)}(u) = \mathrm{Prox}_{\eta J}(u)$, *for any $\lambda \in \mathbb{R}$.*
- *For $v \in \mathscr{X}$, then* $\mathrm{Prox}_{\eta(J+\langle v,.\rangle)}(u) = \mathrm{Prox}_{\eta J}(u - \eta v)$.
- *Moreau's identity:*

$$u = \mathrm{Prox}_{\eta J}(u) + \eta\,\mathrm{Prox}_{\eta^{-1}J^*}\left(\frac{u}{\eta}\right).$$

For more details concerning the proximal operator and its properties, we refer the reader to these recent books [12, 13].

2.4.2 Perturbation

As Theorem 2.16 gives only a sufficient condition to prove duality, we will see in Chapter 3 that the qualification constraints are not always easy to check. Yet, one can still prove duality using some perturbation techniques as presented in [42]. Perturbation techniques are more or less known for the optimization community. Besides its employment to prove some regularity results for the incompressible Euler equation (see [2]) or degenerate elliptic equations (see [95]), it is less present in PDE or calculus of variations literature.

Take a function $J \in \Gamma_0(\mathscr{X})$, and consider the following minimization problem

$$(\mathrm{P}) : \inf_{u \in \mathscr{X}} J(u).$$

We introduce a function $T : \mathscr{X} \times \mathscr{Y} \to [-\infty, \infty]$ as well as the following minimization problem

$$(\mathrm{P}_p) : \inf_{u \in \mathscr{X}} T(u, p),$$

such that $T(u, 0) = J(u)$. This being said, the problem (P_0) is just (P). The idea is that we can recover the dual problem of (P) by considering the one of (P_p). Indeed, if T^* is the dual function of T, then

$$(\mathrm{D}) : \inf_{q \in \mathscr{Y}^*} -T^*(0, q),$$

is the dual problem of (P) with respect to T. An interesting situation is when the function J is of the form

$$J(u) = \tilde{J}(u, \Lambda u),$$

with $\tilde{J} : \mathscr{X} \times \mathscr{Y} \to [-\infty, \infty]$ and $\Lambda \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$. In this case, the perturbed function becomes $T(u, p) = \tilde{J}(u, \Lambda u - p)$ for $p \in \mathscr{Y}$. In all this manuscript, we are more concerned with the case where J can be split into the sum of two operators, i.e., problems of the form (2.17). For more details, we refer the reader to [42, Chap. III].

2.4.3 Primal-Dual methods

The main idea behind Primal-Dual methods is that we can recover simultaneously the solutions of (P) and (D) by looking for saddle-points of some appropriate functional. To see this, define the so called Langrangian

$$L(u, v) = \mathcal{F}(u) + \langle v, \Lambda u \rangle - \mathcal{G}^*(v). \tag{2.18}$$

We observe that if $(\mathbf{u}, \phi)$ is a primal-dual solution as in Theorem 2.16, we have

$$\begin{aligned} L(\mathbf{u}, \phi) \leq \sup_{p \in \mathscr{Y}^*} L(\mathbf{u}, p) &= \inf_{q \in X} \mathcal{F}(q) + \mathcal{G}(\Lambda q) \\ &= \sup_{p \in \mathscr{Y}^*} -\mathcal{F}^*(-\Lambda^* p) - \mathcal{G}(-p) = \inf_{q \in \mathscr{X}} L(q, \phi) \leq L(\mathbf{u}, \phi), \end{aligned}$$

so that $(\mathbf{u}, \phi)$ is a saddle-point of L. Conversely, if $(\mathbf{u}, \phi)$ is a saddle-point of L, we easily see that

$$\begin{aligned} L(\mathbf{u}, \phi) = \sup_{p \in \mathscr{Y}^*} L(\mathbf{u}, p) = \mathcal{F}(\mathbf{u}) + \mathcal{G}(\Lambda \mathbf{u}) &= \inf_{q \in \mathscr{X}} L(q, \phi) \\ &= -\mathcal{F}^*(-\Lambda^* \phi) - \mathcal{G}^*(-\phi), \end{aligned}$$

so that $(\mathbf{u}, \phi)$ solve the Primal-Dual problem.

ALG2 algorithm

Consider as before the following optimization problem

$$(\mathrm{P}): \quad \inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

The idea is to introduce a new primal variable $q \in \mathscr{X}$ so the previous problem reads

$$(\mathrm{P}): \quad \inf_{(u,q) \in \mathscr{X} \times \mathscr{Y}, \Lambda u = q} \mathcal{F}(u) + \mathcal{G}(q),$$

and then introduce an Lagrange multiplier $\phi \in \mathscr{Y}^*$ for the constraint $\Lambda u = q$. Then the problem (P) can be written as a saddle-point problem for an augmented Lagrangian (cf [55])

$$L_r(u, q; \phi) = \mathcal{F}(u) + \mathcal{G}(q) + \langle \phi, \Lambda u - q \rangle + \frac{r}{2} |\Lambda u - q|^2, \text{ for } r > 0.$$

Then, as we presented in the begning of the section, (P) can be solved by finding the saddle-points of L_r, *i.e*, by solving

$$\inf_{(u,q) \in \mathscr{X} \times \mathscr{Y}^*} \sup_{\phi \in \mathscr{Y}^*} L_r(u, q; \phi). \tag{2.19}$$

The so called ALG2 algorithm, *a.k.a*, Alternating Direction Method of Multipliers, allows solving (2.19) by alternatively minimizing the augmented Lagrangian L_r in the primal direction and maximizing in the dual one. More explicitly, we initialise with $u_0 \in \mathscr{X}, q_0 \in \mathscr{Y}$ and $\phi_0 \in \mathscr{Y}^*$, and we generate sequences $\{u_i\}_i$, $\{q_i\}_i$ and $\{\phi_i\}$ for $i \in \mathbb{N}^*$, as follows:

- Minimise in u:

$$u_{i+1} \in \arg\min_{u\in\mathscr{X}} L_r(u, q_i; \phi_i) = \arg\min_{u\in\mathscr{X}} \Big\{ \mathcal{F}(u) + \langle \phi_i, \Lambda u\rangle + \frac{r}{2}|\Lambda u - q_i|^2 \Big\}.$$

- Minimise in q:

$$q_{i+1} \in \arg\min_{q\in\mathscr{Y}} L_r(u_{i+1}, q; \phi_i) = \arg\min_{q\in\mathscr{Y}} \Big\{ \mathcal{G}(q) - \langle \phi_i, q\rangle + \frac{r}{2}|\Lambda u_{i+1} - q|^2 \Big\}.$$

- Maximise in ϕ:

$$\phi_{i+1} \in \arg\min_{\phi\in\mathscr{Y}^*} \Big\{ L_r(u, q_i; \phi_i) - \frac{1}{2r}|\phi - \phi_i|^2 \Big\} = \phi_i + r(\Lambda u_{i+1} - q_{i+1}).$$

The convergence can be also established in the case of infinite-dimensional Hilbert spaces. For further details in this direction and about the ALG2, we refer the reader to [41, 51, 55].

Chambolle-Pock's algorithm

As we discussed previously, the solutions of the problems (P)-(D) are exactly the saddle-points of the Lagrangian L defined in (2.18). Now fix some $(u, \phi) \in \mathrm{dom}(\mathcal{F}) \times \mathrm{dom}(\mathcal{G}^*)$ and define the partial Lagrangians:

$$L_u(\phi) = L(u, \phi) \;\text{ and }\; L_\phi(u) = L(u, \phi).$$

Then, (u, ϕ) is a saddle point of L if and only if

$$\phi \;\text{ solves }\; \min_{v\in\mathscr{Y}} -L_u(v) \quad\text{and}\quad u \;\text{ solves }\; \min_{w\in\mathscr{X}} L_\phi(u),$$

this is equivalent to saying that $0 \in \partial(-L_u)(\phi)$ and $0 \in \partial L_\phi(u)$. Thus, we have for $\eta, \tau > 0$:

$$\begin{aligned}
0 \in \partial(-L_u)(\phi) \text{ and } 0 \in \partial L_\phi(u) &\Leftrightarrow \begin{cases} \phi \in \phi + \tau(-\partial L_u)(\phi) \\ u \in u + \eta \partial L_\phi(u), \end{cases} \\
&\Leftrightarrow \begin{cases} \phi = \mathrm{Prox}_{-\tau L_{\bar{u}}}(\phi) \\ u = \mathrm{Prox}_{\eta L_{\bar{\phi}}}(u), \end{cases} \\
&\Leftrightarrow \begin{cases} \phi = \mathrm{Prox}_{\tau \mathcal{G}^*}(\phi + \tau\Lambda\bar{u}) \\ u = \mathrm{Prox}_{\eta\mathcal{F}}(u - \eta\Lambda^*\bar{\phi}). \end{cases}
\end{aligned}$$

where we have used extra variables $(\bar{u}, \bar{\phi}) \in \mathscr{X} \times \mathscr{Y}^*$and standard formulas from calculus of resolvents recalled in Proposition 2.19. Then Chambolle-Pock's algorithm reads:

- Step 1: $u^{n+1} = \mathrm{Prox}_{\eta\mathcal{F}}\big(u^n - \eta\Lambda^*\bar{\phi}^n\big)$.
- Step 2: $\phi^{n+1} = \mathrm{Prox}_{\tau\mathcal{G}^*}(\phi^n + \tau\Lambda u^{n+1})$.
- Step 3: $\bar{\phi}^{n+1} = \phi^{n+1} + \theta(\phi^{n+1} - \phi^n)$, for $\theta \in [0, 1]$.

The variable $\bar{\phi}$ is called an extra gradient variable and Step 3 can be replaced by a similar one for u. The convergence of this algorithm is studied in [26, 27]. More precisely, when $\mathscr{X}, \mathscr{Y}$ are finite dimensional Hilbert spaces, it can be proven that for $\theta = 1$ and $\eta\tau\|\Lambda\|^2 < 1$, where $\|\Lambda\|$ is the operator norm of Λ, then the sequence $\{u^n, \phi^n\}_n$ generated by this algorithm converges to a saddle point of L and thus to a solution of the primal and dual problems (P)-(D). This method is widely applied in image problems. It is worth mentioning that this is a particular case of the so-called Arrow-Hurwicz method. We refer the reader to the papers of A.Chambolle and T.Pock [26, 27] for more details, and to [19] for a recent overview with further applications in image processing.

3 Augmented Lagrangian approach for HJ equation

3.1 Equivalence between HJ and maximization problem

We present the main result showing the correspondence between HJ equation and a maximization problem. To this end, we will consider a general version of (2.5) coupled with Dirichlet boundary condition. Given a closed subset $D \subset \overline{\Omega}$ (typically $D = \partial\Omega$ or $D = \{x\}$ for some $x \in \overline{\Omega}$), we consider the following HJ equation

$$\begin{cases} H(x, \nabla u) = 0 & \Omega \setminus D \\ u = g & D \end{cases} \tag{3.1}$$

where $g\colon D \to \mathbb{R}$ is continuous function satisfying compatibility condition

$$g(x) - g(y) \leq d_\sigma(y, x) \text{ on } D.$$

Then the unique maximal viscosity subsolution of the equation (3.1) such that $u(x) = g(x)$ for any $x \in D$, is given by

$$u(x) = \min_{y \in D} d_\sigma(y, x) + g(y). \tag{3.2}$$

The considerations given in the introduction lead us to look for the maximal subsolutions of the HJ equation, *i.e.*, $H(x, \nabla u) \leq 0$ or equivalently $u(x) - u(y) \leq d_\sigma(y, x)$. We will show that such functions are precisely the ones with gradient in the unit ball of σ^*. More precisely, we prove the following

Proposition 3.1. $\mathcal{S}_H^-(\Omega) = \{u \in W^{1,\infty}(\Omega)$ *and* $\sigma^*(x, \nabla u(x)) \leq 1$ *for a.e* $x \in \Omega\} := \mathcal{B}_{\sigma^*}$.

Then, we transform the problem into a question of maximization of the volume $\int_\Omega u dx$ among the subsolutions u. This leads to the following theorem which is an important step to treat the equation (3.1) via augmented Lagrangian methods.

Theorem 3.2. *The maximal viscosity subsolution of* (3.1)*, given by* (3.2)*, is the unique solution of the problem*

$$\max_{u\in W^{1,\infty}(\Omega)} \Big\{ \int_{\Omega} z(x)\mathrm{d}x,\ \sigma^*(x,\nabla z(x)) \leq 1 \textit{ and } z = g \textit{ on } D\Big\}. \tag{3.3}$$

For the proof of Proposition 3.1, we recall that the result is more or less known in the case where $H(x,0) < 0$ which corresponds to $\mathcal{A} = \emptyset$ (see [66, 68] for example). Here, under the general condition $(H3)$, we need the following Cauchy-Schwartz-like lemma.

Lemma 3.3. *For any $q \in \mathbb{R}^N$, and for any $x \in \overline{\Omega}$ we have*

$$\sigma^*(x,q) \leq 1 \Leftrightarrow \langle p,q\rangle \leq \sigma(x,p) \textit{ for any } p \in \mathbb{R}^N.$$

Proof. Assume first that $\sigma^*(x,q) \leq 1$. If $\langle p,q\rangle = 0$, it is obvious. For the case $\langle p,q\rangle > 0$, if $\sigma(x,p) = 0$ then by homogeneity of σ, $\sigma(x,\lambda p) = 0$ for every $\lambda \geq 0$. Consequently, $\sigma^*(x,q) \geq \lambda\langle p,q\rangle \to \infty$ as $\lambda \to \infty$, this contradicts $\sigma^*(x,q) \leq 1$. This implies that $\sigma(x,p) > 0$ and $\langle \frac{p}{\sigma(x,p)}, q\rangle \leq \sigma^*(x,q) \leq 1$, as desired. Conversely, if $\langle p,q\rangle \leq \sigma(x,p)$, by definition of σ^*, we take the sup over all p such that $\sigma(x,p) \leq 1$ in the previous inequality to obtain that $\sigma^*(x,q) \leq 1$. □

Proof of Proposition 3.1. We divide the proof into two parts. Firstly, $\mathcal{S}_H^-(\Omega) \subset \mathcal{B}_{\sigma^*}$. Take a point $x \in \Omega$ such that u is differentiable at x. For every vector $v \in \mathbb{S}^{N-1}$ on the unit sphere, we take $\zeta_h(t) = tx + (1-t)(x-hv)$ for every $h > 0$ and $t \in [0,1]$ and we notice that $\zeta_h \in \Gamma(x-hv,x)$ and for h small enough, ζ_h is close to x. We then have

$$\begin{aligned}
\langle \nabla u(x), v\rangle &\leq \lim_{h\to 0^+} h^{-1}(u(x) - u(x-hv)) \\
&\leq \liminf_{h\to 0^+} h^{-1} d_\sigma(x-hv,x) \\
&\leq \liminf_{h\to 0^+} h^{-1} \int_0^1 \sigma(\zeta_h,\dot{\zeta}_h)\mathrm{d}t = \liminf_{h\to 0^+} \int_0^1 \sigma(\zeta_h,v)\mathrm{d}t \leq \sigma(x,v),
\end{aligned}$$

where we have used the continuity of $x \mapsto \sigma(.,v)$ and Lebesgue theorem. Using the definition of σ^*, we deduce that $\sigma^*(x,\nabla u(x)) \leq 1$ as desired.
Secondly, $\mathcal{B}_{\sigma^*} \subset \mathcal{S}_H^-(\Omega)$. Assume now that $\sigma^*(x,\nabla u(x)) \leq 1$ at any differentiable

point x of u, i.e., $\langle \nabla u(x), p\rangle \leq \sigma(x,p)$ for all $p \in \mathbb{R}^N$. For smooth function u, the argument is simply given by

$$\begin{aligned} u(y) - u(x) &= \int_0^1 \nabla u(\xi(s)) \cdot \dot{\xi}(s) \mathrm{d}s \\ &\leq \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) \mathrm{d}s, \end{aligned}$$

for Lipschitz curves ξ joining x to y in $\overline{\Omega}$. This implies that $u(y) - u(x) \leq d_\sigma(x,y)$. For general Lipschitz function u, one can make use of smooth approximation (see [68, Proposition 5]). □

Proof of Theorem 3.2. Let us see that for any subsolution v (3.1), we have $v \leq u$. Pick any subsolution v of (3.1) satisfying the condition $v = g$ on D. Let $x \in \Omega$, we have for any $y \in D$

$$\begin{aligned} v(x) &= v(x) - v(y) + g(y) \\ &\leq d_\sigma(y,x) + g(y). \end{aligned}$$

This gives that $v \leq u$ in Ω and then $\int_\Omega v(x)\mathrm{d}x \leq \int_\Omega u(x)\mathrm{d}x$. Clearly u is $1-$Lipschitz with respect to d_σ, hence by Proposition (3.1) we have $\sigma^*(x, \nabla u) \leq 1$. Consequently u solves (3.3). If w is another solution, then $\int_\Omega u(x)\mathrm{d}x = \int_\Omega w(x)\mathrm{d}x$ and $w \leq u$ in Ω by the first step. Consequently $u = w$, as desired. □

3.2 HJ and duality results

As we said in the introduction among our main interests in this chapter is to use augmented Lagrangian methods to give a direct algorithm to approximate the solution of the HJ equation.

To this end, we observe that problem (3.3) falls into the scope of the following class of optimization problem (see 2.4)

$$(\mathrm{P}): \inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

where $\mathscr{X}$ and $\mathscr{Y}$ are two Banach spaces with the topological dual spaces $\mathscr{X}^*$ and $\mathscr{Y}^*$, $\mathcal{F} : \mathscr{X} \to (-\infty, +\infty]$, $\mathcal{G} : \mathscr{Y} \to (-\infty, +\infty]$ are proper, l.s.c., convex functions and $\Lambda : \mathscr{X} \to \mathscr{Y}$ is a linear operator, that we will precise later.

As we pointed out in the introduction, because of the degeneracy of the Hamiltonian, we can not directly use Theorem 2.16 to show duality between the maximization problem and its dual problem. The main goal of this section is to show rigorously that the duality still holds true. As a typical example we will consider an HJ equation of Eikonal type, coupled with a zero Dirichlet boundary condtion,

$$\begin{cases} |\nabla u(x)| = k(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, $H(x,p) = |p| - k(x)$ where k is a continuous, nonnegative function on $\overline{\Omega}$. In this case, the problem (3.3) can be rewritten as

$$\sup\left\{ \int_\Omega u \mathrm{d}x : \ |\nabla u| \leq k \ \text{ and } \ u = 0 \ \partial\Omega \right\},$$

or

$$(\mathrm{P}): \ \inf_{u \in V} \{\mathcal{F}(u) + \mathcal{G}(\Lambda u)\}$$

where $V = C^1(\Omega) \cap H_0^1(\Omega)$,

$$\mathcal{F}(u) = -\int_\Omega u \mathrm{d}x, \quad \Lambda u = \nabla u \quad \text{and} \quad \mathcal{G}(q) = \begin{cases} 0 & \text{if } |q| \leq k \\ +\infty & \text{otherwise.} \end{cases}$$

For the case of non-degeneracy, i.e., $k(x) > 0$ on $\overline{\Omega}$, the Fenchel-Rockafellar duality, since the qualification conditions are satisfied (Theorem 2.16, Section 2.4), gives

$$\sup\left\{ \int_\Omega u \mathrm{d}x : |\nabla u| \leq k \text{ and } u = 0 \text{ on } \partial\Omega \right\} = \min_{\phi \in \mathcal{M}_b(\overline{\Omega})^N} \left\{ \int_{\overline{\Omega}} k \ \mathrm{d}|\phi| : -\operatorname{div}(\phi) = \rho \text{ in } \mathcal{D}'(\Omega) \right\}.$$

3.2.1 Duality for HJ equation

To the duality, we consider a more general problem by considering, for a nonnegative Radon measure $\rho \in \mathcal{M}_b^+(\Omega)$ and a closed subset $D \subset \overline{\Omega}$, the problem

$$\max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u \mathrm{d}\rho : \ \sigma^*(x, \nabla u(x)) \leq 1 \text{ and } u = g \ \text{ on } D \right\}.$$

Since the values of u are prescribed on D and the solution of (3.1) is given by the distance to D, we can consider the following problem

$$(\mathcal{M}_D): \ \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_{\Omega_D} u \mathrm{d}\rho : \ \sigma^*(x, \nabla u(x)) \leq 1 \text{ and } u = g \ \text{ on } \Gamma_D \right\},$$

where $\Omega_D := \Omega \setminus D, \Gamma_D := \partial\Omega_D \cap \partial D$ and $\tilde{\Gamma} = \partial\Omega_D \setminus \Gamma_D$. In particular, if $D = \partial\Omega$ then $\Omega_D = \Omega$ and $\Gamma_D = \partial\Omega,\ \tilde{\Gamma} = \emptyset$.

Example 3.4. Consider for example $\Omega = (0,2)^2$ and $D = [1,2] \times [0,2]$. Then, we immediately find $\Omega_D = (0,1) \times (0,2)$ and $\partial\Omega_D = \Gamma_D \cup \tilde{\Gamma}$ with

$$\tilde{\Gamma} = (0,1) \times \{0\} \cup (0,1) \times \{2\} \cup \{0\} \times (0,2), \text{ and } \Gamma_D = \{1\} \times (0,2).$$

As we will see, our dual formulation challenges some kind of trace-like operator for the so called divergence-measure field. To begin with let us sort out formally and briefly our approach, at least in the case where $D = \partial\Omega$. Taking $\mathscr{X}$ to be the Banach space $W^{1,\infty}(\Omega)$ and $\mathscr{Y}$ to be the space $L^\infty(\Omega)^N$, we consider simply

$$\mathcal{F}(u) = \begin{cases} -\int_\Omega u \,\mathrm{d}\rho & \text{if } u \in W^{1,\infty}(\Omega) \text{ and } u = g \text{ on } \partial\Omega \\ \\ +\infty & \text{otherwise,} \end{cases} \quad , \quad \Lambda u = \nabla u,$$

and

$$\mathcal{G}(\eta) = \begin{cases} 0 & \text{if } \sigma^*(x,\eta) \leq 1 \\ \\ +\infty & \text{otherwise,} \end{cases}$$

so that the problem $(\mathcal{M}_D)$ reads as (P). For a formal computation of the dual problem, let us notice that $\mathcal{G}^* : \mathscr{Y}^* \to (-\infty, +\infty]$,

$$\mathcal{G}^*(\Phi) = \int_\Omega \sigma(x,\Phi) \quad \text{for any } \Phi \in \mathscr{Y}^*.$$

The operator $\Lambda^* : \mathscr{Y}^* \to \mathscr{X}^*$ is given by

$$\langle \Lambda^*\Phi, \xi \rangle = \langle \Phi, \nabla\xi \rangle \quad \text{for any } \xi \in \mathscr{X}.$$

Also,

$$\begin{aligned} \mathcal{F}^*(-\Lambda^*\Phi) &= \sup_{u\in\mathscr{X}, u=g \text{ on } \partial\Omega} \int_\Omega -\Phi\cdot\nabla u \,\mathrm{d}x - \mathcal{F}(u) \\ &= \sup_{u\in\mathscr{X}, u=g \text{ on } \partial\Omega} \int_\Omega -\Phi\cdot\nabla u \,\mathrm{d}x + \int_\Omega u \,\mathrm{d}\rho \\ &= \sup_{u\in\mathscr{X}, u=g \text{ on } \partial\Omega} \int_\Omega u\mathrm{div}(\Phi) - \int_{\partial\Omega} \Phi\cdot\nu\, u + \int_\Omega u \,\mathrm{d}\rho \\ &= \begin{cases} -\int_{\partial\Omega} \Phi\cdot\nu\, g & \text{if } -\mathrm{div}(\Phi) = \rho \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, the dual problem reads

$$\inf\Big\{\int_\Omega \sigma(x,\Phi) - \int_{\partial\Omega} \Phi\cdot\nu\, g \ :\ \Phi\in\mathscr{Y}^*,\ -\mathrm{div}(\Phi)=\rho \text{ in } \mathcal{D}'(\Omega)\Big\}. \tag{3.4}$$

Note that the above computation is very formal by the appearance of the trace-like term $\Phi\cdot\nu$ which is not well defined for all Φ.

To handle rigorously the normal trace of the vector-valued dual variable Φ in the dual problem of the type (3.4), we recall the trace-like operator for the so called divergence-measure field (cf. [28, 29, 30]). To this aim, we assume in this section that

$$\Omega_D = \Omega\setminus D \text{ is a regular domain with a deformable Lipschitz boundary } \partial\Omega_D.$$

This is achieved for instance in the case where Ω is a regular domain and $D = \partial\Omega$ or $D=\overline{\omega}$ with a regular domain $\omega\subset\subset\Omega$.

For any $1\le p\le\infty$, we define the set

$$\mathcal{DM}^p(\Omega_D) := \big\{F\in L^p(\Omega_D)^N \ :\ \mathrm{div}\, F =: \mu\in\mathcal{M}_b(\Omega_D)\big\},$$

where $\mu = \mathrm{div}\, F$ is taken in $\mathcal{D}'(\Omega_D)$. See here, that for any $F\in\mathcal{DM}^p(\Omega_D)$, the total variation of $\mathrm{div}\, F$ is given by

$$\begin{aligned}
|\mathrm{div}\, F|(\Omega_D) &:= \sup\Big\{\int \varphi\,\mathrm{d}\mu \ :\ \varphi\in\mathcal{C}_0(\Omega_D),\ |\varphi(x)|\le 1 \text{ for any } x\in\Omega_D\Big\}\\
&= \sup\Big\{\int \varphi\,\mathrm{d}\mu \ :\ \varphi\in\mathcal{C}^1_0(\Omega_D),\ |\varphi(x)|\le 1 \text{ for any } x\in\Omega_D\Big\}\\
&= \sup\Big\{\int F\cdot\nabla\varphi \ :\ \varphi\in\mathcal{C}^1_0(\Omega_D),\ |\varphi(x)|\le 1 \text{ for any } x\in\Omega_D\Big\}.
\end{aligned}$$

In particular, the space $\mathcal{DM}^p(\Omega_D)$ endowed with the norm

$$\|F\|_{\mathcal{DM}^p(\Omega_D)} := \|F\|_{L^p(\Omega_D)} + |\mathrm{div}\, F|(\Omega_D)$$

is a Banach space.

Thanks to [28, 29, 30], for any $1<p\le\infty$, it is possible to define a trace-like operator on the set $\mathcal{DM}^p(\Omega_D)$. Actually, for any $F\in\mathcal{DM}^p(\Omega_D)$, we define $F\cdot\nu$ the normal trace of F on $\partial\Omega_D$, given by $F\cdot\nu\ :\ Lip(\partial\Omega_D)\to\mathbb{R}$ the continuous linear functional such that

$$\langle F\cdot\nu, \xi_{/\partial\Omega_D}\rangle = \int_{\Omega_D}\xi\,\mathrm{div}F + \int_{\Omega_D}\nabla\xi\cdot F, \qquad \text{for any } \xi\in\mathcal{C}^1(\overline{\Omega_D}).$$

Moreover, since Γ_D is a deformable Lipschitz boundary, then the restriction of $F \cdot \nu$ to Γ_D is well defined, this will be denoted by $F \cdot \nu_D$ (cf. [28, 29, 30], see also Remark 3.5).

Remark 3.5.
1. Thanks to [28, 29, 30], for any $F \in \mathcal{DM}^p(\Omega_D)$, with $1 < p \leq \infty$, it is possible to define the normal trace $F \cdot \nu$ locally by using Lipschitz deformation of the boundary. This formulation is very useful in the case where the boundary is partitioned into disjoint deformable Lipschitz patches.

2. In the case where $p = 1$ as well as the case where $L^p(\Omega_D)^N$ is replaced by the space $\mathcal{M}_b(\Omega_D)^N$, the trace may be defined as well, but only as continuous linear form on a subset of $Lip(\partial\Omega_D)$ (cf. [28, 29, 30]).

Now, combining this consideration with the formal computation for (3.4), we introduce the following optimization problem

$$(\mathcal{OF}_D): \inf_{\phi \in \mathcal{DM}^p(\Omega_D)} \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) \mathrm{d}x - \langle \phi \cdot \nu_D, g \rangle : -\mathrm{div}\,(\phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\},$$

where the divergence constraint is understood as follows:

$$\int_{\Omega_D} \nabla\xi \cdot \phi \, \mathrm{d}x - \langle \phi \cdot \nu_D, \xi \rangle = \int_{\overline{\Omega}_D \setminus \Gamma_D} \xi \mathrm{d}\rho, \text{ for any } \xi \in Lip(\overline{\Omega}_D),$$

with $\phi \cdot \nu_D$ being trace-like term on Γ_D as defined above. In other words, we impose that $\phi \cdot \nu_{\tilde{\Gamma}} = 0$ on $\tilde{\Gamma}$.

Our main result in this section is the following duality result.

Theorem 3.6. *Let $\rho \in \mathcal{M}_b^+(\Omega)$, then $(\mathcal{M}_D)$ and $(\mathcal{OF}_D)$ coincide.*

Proof. Consider on $\mathcal{M}_b(\overline{\Omega}_D)$ the following functional $F : \mathcal{M}_b(\overline{\Omega}_D) \mapsto]-\infty, \infty]$ defined by

$$F(h) = \inf_{\phi \in \mathcal{DM}^p(\Omega_D)} \left\{ \int_{\Omega_D} \sigma(x, \phi(x)) \mathrm{d}x + \int_{\Gamma_D} g \mathrm{d}h - \langle \phi \cdot \nu_D, g \rangle : -\mathrm{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\},$$

for any $h \in \mathcal{M}_b(\overline{\Omega}_D)$. Then F is convex and l.s.c. Indeed, take $h_1, h_2 \in \mathcal{M}_b(\overline{\Omega}_D)$ and set $h := th_1 + (1-t)h_2$ for $t \in [0,1]$. Let $\phi_{1,n}, \phi_{2,n} \in \mathcal{DM}^p(\Omega_D)$ be two minimizing sequences of fluxes corresponding to h_1 and h_2 respectively, *i.e.* $-\mathrm{div}(\phi_{i,n}) = \rho + h_i$, in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$ and

$$F(h_i) = \lim_n \int_{\Omega_D} \sigma(x, \phi_{i,n}(x)) \mathrm{d}x + \int_{\Gamma_D} g \, \mathrm{d}h_i - \langle \phi_{i,n} \cdot \nu_D, g \rangle \quad \text{for } i = 1, 2.$$

Set $\phi_n = t\phi_{1,n} + (1-t)\phi_{2,n}$. We clearly see that ϕ_n are admissible for h and

$$\begin{aligned}
F(h) &\leq \liminf_n \int_{\Omega_D} \sigma(x, \phi_n(x))\mathrm{d}x + \int_{\Gamma_D} g\ \mathrm{d}h - \langle \phi_n \cdot \nu_D, g\rangle \\
&= \liminf_n \int_{\Omega_D} \sigma(x, t\phi_{1,n} + (1-t)\phi_{2,n})\mathrm{d}x + \int_{\Gamma_D} g\ \mathrm{d}h - \langle (t\phi_{1,n} + (1-t)\phi_{2,n}) \cdot \nu_D, g\rangle \\
&\leq \lim_n t\Big(\int_{\Omega_D} \sigma(x, \phi_{1,n})\mathrm{d}x + \int_{\Gamma_D} g\ \mathrm{d}h - \langle \phi_{1,n} \cdot \nu_D, g\rangle\Big) \\
&\qquad\qquad + (1-t)\Big(\int_{\Omega_D} \sigma(x, \phi_{2,n})\mathrm{d}x + \int_{\Gamma_D} g\ \mathrm{d}h - \langle \phi_{2,n} \cdot \nu_D, g\rangle\Big) \\
&\leq tF(h_1) + (1-t)F(h_2)
\end{aligned}$$

and this proves convexity. For the lower semicontinuity, take a sequence $h_n \rightharpoonup h$ in $\mathcal{M}_b(\overline{\Omega}_D)$. For every $n \in \mathbb{N}$, we consider a sequence $(\phi_n^k)_{k\in\mathbb{N}}$ of $\mathcal{DM}^p(\Omega_D)$ such that

$$F(h_n) = \lim_{k\to\infty} \int_{\Omega_D} \sigma(x, \phi_n^k(x))\mathrm{d}x + \int_{\Gamma_D} g\ \mathrm{d}h_n - \langle \phi_n^k \cdot \nu_D, g\rangle.$$

We may find some $\psi_n \in L^1(\Omega_D, \mathbb{R}^N)$ such that $-\mathrm{div}(\psi_n) = h - h_n$ in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$, $\|\psi_n\|_{L^1} \to 0$ and $\langle \psi_n \cdot \nu_D, g\rangle \to 0$. In fact, we have $h - h_n \in \mathcal{M}_b(\overline{\Omega}_D) \hookrightarrow W^{-1,p'}(\Omega_D)$ for $p > N$ and $p' := \frac{p}{p-1}$. We consider the following p-Laplace equation

$$\begin{cases} -\Delta_p u_n = h - h_n & \text{in } \Omega_D \\ u_n = 0 & \text{on } \Gamma_D \\ |\nabla u_n|^{p-2}\nabla u_n \cdot \nu_{\tilde{\Gamma}} = 0 & \text{on } \tilde{\Gamma}. \end{cases} \tag{3.5}$$

The system (3.5) admits a unique solution $u_n \in W^{1,p}(\Omega_D)$ such that $u_n = 0$ on Γ_D. Hence, if we set $\psi_n = |\nabla u_n|^{p-2}\nabla u_n$, we see that $\psi_n \in L^{p'}(\Omega_D)$, and then in $L^1(\Omega_D)$. Moreover, we have $-\mathrm{div}(\psi_n) = h - h_n$ in $\mathcal{D}'(\Omega_D)$. Since $h - h_n$ is bounded in $W^{-1,p'}(\Omega_D)$, it is not difficult to prove that u_n is bounded in $W^{1,p}(\Omega_D)$. So, by taking a subsequence if necessary, we have $u_n \rightharpoonup u$ in $W^{1,p}(\Omega_D)$, and uniformly in Ω_D. On the other hand, we have

$$\int_{\Omega_D} |\psi_n|^{p'}\ \mathrm{d}x = \int_{\Omega_D} |\nabla u_n|^p \mathrm{d}x = \langle h - h_n, u_n\rangle_{W^{-1,p'}(\Omega_D), W^{1,p}(\Omega_D)} \underset{n\to\infty}{\longrightarrow} 0.$$

In particular, this implies that $|\psi_n| \underset{n\to\infty}{\longrightarrow} 0$ in $L^1(\Omega_D)$. Moreover, taking $\tilde{g} \in Lip(\Omega_D)$ be such that $\tilde{g} = g$ on Γ_D, we have

$$\langle \psi_n \cdot \nu_D, g\rangle = \int_{\Omega_D} \psi_n \cdot \nabla \tilde{g} \,\mathrm{d}x - \int_{\Omega_D} \tilde{g}\,\mathrm{d}(h - h_n) \quad \underset{n\to\infty}{\longrightarrow} 0.$$

This being said, we clearly have $-\mathrm{div}(\phi_n^k + \psi_n) = \rho + h$, *i.e.* $\phi_n^k + \psi_n$ are admissible fluxes for h. By semicontinuity of the integral, we have

$$\begin{aligned} F(h) &\le \int_{\Omega_D} \sigma(x, (\phi_n^k + \psi_n)(x))\mathrm{d}x + \int_{\Gamma_D} g\,\mathrm{d}h - \langle(\phi_n^k + \psi_n)\cdot \nu_D, g\rangle \\ &\le \int_{\Omega_D} \sigma(x, \phi_n^k(x))\mathrm{d}x + \int_{\Gamma_D} g\,\mathrm{d}h_n - \langle \phi_n^k \cdot \nu_D, g\rangle \\ &\quad + \int_{\Omega_D} \sigma(x, \psi_n(x))\mathrm{d}x + \int_{\Gamma_D} g\,\mathrm{d}(h - h_n) - \langle \psi_n \cdot \nu_D, g\rangle. \end{aligned}$$

Letting $k \to \infty$ we get

$$F(h) \le F(h_n) + \int_{\Omega_D} \sigma(x, \psi_n(x))\mathrm{d}x + \int_{\Gamma_D} g\,\mathrm{d}(h - h_n) - \langle \psi_n \cdot \nu_D, g\rangle.$$

Now, letting $n \to \infty$, and using the fact that $\psi_n \to 0$ in $L^1(\Omega_D)^N$, and $h_n \rightharpoonup h$ in $\mathcal{M}_b(\overline{\Omega}_D)$, as $n \to \infty$, we obtain the lower semicontinuity, i.e.

$$F(h) \le \liminf_n F(h_n).$$

Next let us compute F^*. For any $u \in C(\overline{\Omega}_D)$, we have

$$\begin{aligned} F^*(u) &= \sup_{h\in\mathcal{M}_b(\overline{\Omega}_D)} \int_{\overline{\Omega}_D} u\mathrm{d}h - F(h) \\ &= \sup_{\substack{h\in\mathcal{M}_b(\overline{\Omega}_D)\\ \phi\in\mathcal{DM}^p(\Omega_D)}} \Big\{ \int_{\overline{\Omega}_D} u\mathrm{d}h - \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x - \int_{\Gamma_D} g\,\mathrm{d}h + \langle \phi\cdot\nu_D, g\rangle : \\ &\qquad\qquad -\mathrm{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)\Big\} \\ &= I_1(u) + I_2(u), \end{aligned}$$

where $I_1(u) := -\int_{\Omega_D} u\mathrm{d}\rho$ and

$$I_2(u) := \sup_{\substack{h\in\mathcal{M}_b(\overline{\Omega}_D)\\ \phi\in\mathcal{DM}^p(\Omega_D)}} \Big\{ \int_{\overline{\Omega}_D\setminus\Gamma_D} u\mathrm{d}(\rho+h) - \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x + \int_{\Gamma_D} (u-g)\mathrm{d}h + \langle \phi\cdot\nu_D, g\rangle$$

$$: \ -\mathrm{div}(\phi) = \rho + h \text{ in } \mathcal{D}'(\overline{\Omega}_D\setminus\Gamma_D)\Big\}.$$

Using Lemma 3.7 below, we deduce that, for any $u \in Lip(\Omega_D)$, we have

$$F^*(u) = \begin{cases} -\int_{\Omega_D} u\mathrm{d}\rho & \text{if } \begin{cases}\sigma^*(x,\nabla u)\le 1\\ \text{and } u=g \text{ on } \Gamma_D\end{cases} \\ \\ \infty & \text{otherwise.}\end{cases}$$

Finally, using the fact

$$\inf(\mathcal{OF}_D) = F(0) = F^{**}(0) = \sup_{u\in\mathrm{Lip}(\Omega_D)} -F^*(u) = \max(\mathcal{M}_D),$$

we deduce the result. □

Lemma 3.7. *Let $u \in Lip(\Omega_D)$, we have*

$$\sup_{\substack{h\in\mathcal{M}_b(\overline{\Omega}_D)\\ \phi\in\mathcal{DM}^p(\Omega_D)}} \Big\{ \int_{\overline{\Omega}_D\setminus\Gamma_D} u\mathrm{d}(\rho+h) - \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x + \int_{\Gamma_D} (u-g)\,\mathrm{d}h + \langle \phi\cdot\nu_D, g\rangle$$

$$: \ -\mathrm{div}(\phi) = \rho + h \textit{ in } \mathcal{D}'(\overline{\Omega}_D\setminus\Gamma_D)\Big\}$$

$$= \begin{cases} 0 & if \begin{cases}\sigma^*(x,\nabla u)\le 1\\ and\ u=g \ on\ \Gamma_D\end{cases} \\ \\ \infty & otherwise.\end{cases}$$

Proof. Take u as a test function in the divergence constraint $-\mathrm{div}(\phi) = \rho + h$ in $\mathcal{D}'(\overline{\Omega}_D\setminus\Gamma_D)$, we get

$$\begin{aligned} I(h,\phi) &:= \int_{\overline{\Omega}_D\setminus\Gamma_D} u\mathrm{d}(\rho+h) - \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x + \int_{\Gamma_D} (u-g)\,\mathrm{d}h + \langle \phi\cdot\nu_D, g\rangle \\ &= \int_{\Omega_D} \nabla u\cdot\phi\mathrm{d}x - \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x + \int_{\Gamma_D} (u-g)\,\mathrm{d}h + \langle \phi\cdot\nu_D, g-u\rangle. \end{aligned}$$

If $\sigma^*(x,\nabla u) \leq 1$ and $u = g$ on Γ_D, then following Lemma 3.3, we obtain $\sup I(h,\phi) \leq 0$. Actually, $\sup I(h,\phi) = 0$ in this case by taking $h \equiv -\rho$ and $\phi \equiv 0$. If $u(x_0) \neq g(x_0)$ for some $x_0 \in \Gamma_D$, then we consider Dirac mass at x_0, $h = n\,\mathrm{sign}(u(x_0) - g(x_0))\delta_{x_0}$ for $n \in \mathbb{N}$, and fix $\Phi_0 \in \mathcal{DM}^p(\Omega_D)$ such that $-\mathrm{div}\,\Phi_0 = \rho$ in $\mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)$, we have

$$\sup I(h,\phi) \geq \int_{\Omega_D} \Phi_0 \cdot \nabla u - \int_{\Omega_D} \sigma(x,\Phi_0(x))\mathrm{d}x + n|u(x_0) - g(x_0)| + \langle \Phi_0 \cdot \nu_D, g - u \rangle.$$

Letting $n \to \infty$, we get the result. For the remaining case, i.e. $u = g$ on Γ_D and $\sigma^*(x,\nabla u) > 1$ on a non negligible set, we see first that, for any $u \in Lip(\Omega_D)$, there exists a measurable function that we denote by $q_u : \Omega_D \to \mathbb{R}^N$, such that

$$q_u(x) \cdot \nabla u(x) = \sigma^*(x, \nabla u(x)), \qquad \text{a.e. in } \Omega_D.$$

Indeed, recall that $\sigma^*(x,\nabla u(x)) = \max_p\{\langle p, \nabla u(x)\rangle \ : \ \sigma(x,p) = 1\}$ and the function $x \to \langle p, \nabla u(x)\rangle + I\!I_{[\sigma(x,.)=1]}$ is measurable. Then, q_u is given by the measurable representation in the set

$$\arg\max_p \{\langle p, \nabla u\rangle \ : \ \sigma(x,p) = 1\}.$$

Now, if $u = g$ on Γ_D and $\sigma^*(x,\nabla u) > 1$ in a subset $A \subset \Omega_D$ such that $|A| \neq 0$, we consider $\Phi_{n\epsilon} = n\frac{q_u}{|q_u|}\chi_A * \eta_\epsilon$, where η_ϵ is a sequence of mollifiers. It is clear that there exists $h \in \mathcal{M}_b(\overline{\Omega}_D)$, such that $-\mathrm{div}\,\Phi_{n\epsilon} = \rho + h$. For any n, we have

$$\sup I(h,\phi) \geq \int_{\Omega_D} \Phi_{n\epsilon} \cdot \nabla u - \int_{\Omega_D} \sigma(x,\Phi_{n\epsilon}(x))\mathrm{d}x.$$

Letting $\epsilon \to 0$, we get

$$\sup I(h,\phi) \geq n \int_A \frac{1}{|q_u|}(q_u \cdot \nabla u - \sigma(x,q_u))\mathrm{d}x \geq n \int_A \frac{1}{|q_u|}(\sigma^*(x,\nabla u(x)) - 1)\mathrm{d}x.$$

Then, letting $n \to \infty$, we get the result. $\square$

Remark 3.8. Going over the duality inferred by Theorem 3.6, we have

$$\inf\left\{ \int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x - \langle \phi \cdot \nu_D, g\rangle : \ \phi \in \mathcal{DM}^p(\Omega_D), \ -\mathrm{div}\,(\phi) = \rho \ \text{ in } \ \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \right\}$$

$$= \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_{\Omega_D} u(x)\mathrm{d}\rho, \ \sigma^*(x,\nabla u(x)) \leq 1 \text{ a.e. } x \text{ in } \Omega_D \text{ and } \ u = g \text{ on } \Gamma_D \right\}.$$

It is not clear if the inf is a min . This is closely connected to the regularity of the trace of divergence-measure field. However, one sees that if this is true, i.e. the inf is a min, then the respective extremums u and ϕ satisfy the following PDE:

$$\begin{cases} -\operatorname{div}(\phi) = \rho & \text{in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D) \\ \phi(x)\cdot \nabla u(x) = \sigma(x,\phi(x)) & \text{in } \Omega_D \\ u = g & \text{on } \Gamma_D \\ \sigma^*(x,\nabla u(x)) \leq 1 & \text{in } \Omega_D \\ \phi \cdot \nu_{\tilde{\Gamma}} = 0 & \text{on } \tilde{\Gamma}. \end{cases}$$

3.3 The augmented Lagrangian technique

3.3.1 Formulation of the problem

We set again $\mathscr{X} = W^{1,\infty}(\Omega_D)$ and $\mathscr{Y} = L^\infty(\Omega_D)^N$. For any $u \in \mathscr{X}$ and $\eta \in \mathscr{Y}$, we define

$$\mathcal{F}(u) = \begin{cases} -\int_{\Omega_D} u \,\mathrm{d}\rho & \text{if } u = g \text{ on } \Gamma_D \\ +\infty & \text{otherwise} \end{cases}, \quad \mathcal{G}(\eta) = \begin{cases} 0 & \text{if } \sigma^*(x,\eta) \leq 1 \\ +\infty & \text{otherwise} \end{cases}, \quad \text{and } \Lambda u = \nabla u.$$

Thus, the problem $(\mathcal{M}_D)$ can be rewritten in the form

$$-\inf_{u\in\mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Thanks to Theorem 3.6, we can write

$$-\min\Big\{\mathcal{F}(u)+\mathcal{G}(\Lambda u) : u \in \mathscr{X}\Big\} = -\sup\Big\{-\mathcal{F}^*(-\Lambda^*\phi)-\mathcal{G}^*(\phi) : \phi \in \mathcal{DM}^p(\Omega_D)\Big\}$$

$$= \inf\Big\{\int_{\Omega_D} \sigma(x,\phi(x))\mathrm{d}x - \langle \phi\cdot\nu_D, g\rangle \,:\, \phi \in \mathcal{DM}^p(\Omega_D),\ -\mathrm{div}\,(\phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega}_D \setminus \Gamma_D)\Big\}.$$

Introducing a new primal variable $q \in \mathscr{Y}$ we can write $(\mathcal{M}_D)$ in the following alternative form

$$-\inf_{\substack{(u,q)\in\mathscr{X}\times\mathscr{Y}\\ \Lambda u = q}} \mathcal{F}(u) + \mathcal{G}(q) \tag{3.6}$$

so that $(\mathcal{M}_D)$ and $(\mathcal{OF}_D)$ can be recast in a saddle point form

$$\inf_{(u,q)\in\mathscr{X}\times\mathscr{Y}} \sup_{\phi\in\mathcal{DM}^p(\Omega_D)} L(u,q;\phi)$$

where

$$L(u,q;\phi) = \mathcal{F}(u)+\mathcal{G}(q)+\int_{\Omega_D} \phi\cdot(\Lambda(u)-q)\mathrm{d}x, \qquad \text{for any } (u,q,\phi)\in\mathscr{X}\times\mathscr{Y}\times\mathcal{DM}^p(\Omega_D).$$

More precisely, we have

Proposition 3.9. *u is a solution of $(\mathcal{M}_D)$ if and only if the couple $(u, q := \Lambda u) \in \mathscr{X}\times\mathscr{Y}$ is a solution of*

$$\sup_{\phi\in\mathcal{DM}^p(\Omega_D)} \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} L(u,q;\phi)$$

which is equal to

$$\min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} \sup_{\phi\in\mathcal{DM}^p(\Omega_D)} L(u,q;\phi).$$

Proof. The proof of this result is standard. For completeness let us give the main arguments showing how the duality of the previous section takes part of this result. Using (3.6), it is not difficult to see that

$$\min\Big\{\mathcal{F}(u)+\mathcal{G}(\Lambda u)\ :\ u\in\mathscr{X}\Big\} = \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} \sup_{\phi\in\mathcal{DM}^p(\Omega_D)} L(u,q;\phi).$$

On the other hand, using the definition of $\mathcal{F}^*$, $\mathcal{G}^*$ and Λ^*, one sees that

$$\sup_{\phi\in\mathcal{DM}^p(\Omega_D)} \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} L(u,q;\phi) = \sup\Big\{-\mathcal{F}^*(-\Lambda^*\phi)-\mathcal{G}^*(\phi)\ :\ \phi\in\mathcal{DM}^p(\Omega_D)\Big\}.$$

Thus using Theorem 3.6, the result of the proposition follows. □

For a given $r>0$, we recall that the augmented Lagrangian (cf. [55]) is given by

$$L_r(u,q;\phi) = \mathcal{F}(u)+\mathcal{G}(q)+\langle\phi,\Lambda u-q\rangle+\frac{r}{2}|\Lambda u-q|^2, \qquad \text{for any } (u,q,\phi)\in\mathscr{X}\times\mathscr{Y}\times\mathcal{DM}^p(\Omega_D).$$

In the same way, it can easily be proved that

Proposition 3.10. *Let $r>0$. Then, u is a solution of $(\mathcal{M}_D)$ if and only if the couple $(u, q := \Lambda u)\in\mathscr{X}\times\mathscr{Y}$ is a solution of*

$$\sup_{\phi\in\mathcal{DM}^p(\Omega_D)} \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} L_r(u,q;\phi)$$

which is equal to

$$(\mathscr{S}_r): \quad \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}} \; \sup_{\phi\in\mathcal{DM}^p(\Omega_D)} \; L_r(u,q;\phi).$$

Now for the numerical computation concerning the problem $(\mathcal{M}_D)$, we will focus on the saddle point problem $(\mathscr{S}_r)$. Recall that the addition of the quadratic term $\frac{r}{2}|\Lambda u - q|^2$ has the advantage of improving the convergence of the dual approach (one can see [42, 54]).

3.3.2 Application of ALG2

We approximate the domain Ω_D via a triangulation $\mathcal{T}_h$. For $k \geq 1$, we denote by $\mathbb{P}_k$ the space of polynomials with real coefficients and of degree at most k. We define $\mathscr{X}_h \subset W^{1,\infty}(\Omega_D)$ as the space of continuous functions on $\overline{\Omega}_D$ belonging to $\mathbb{P}_k$ on each triangle. Similarly, $\mathscr{Y}_h$ is the space of vector valued functions belonging to $(\mathbb{P}_{k-1})^d$ on each triangle. Then the problem $(\mathcal{M}_D)$ is discretized by the following finite-dimensional optimization problem:

$$\inf_{u\in\mathscr{X}_h} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Then, as recalled in Section 2.4, for a given $q_0 \in \mathscr{Y}_h$, $\phi_0 \in \mathscr{Y}_h^*$, using ALG2 algorithm we construct a sequence $\{u_i\}_i, \{q_i\}_i, \{\phi_i\}_i$ by optimizing alternatively in u and q, for $i \geq 1$.

Algorithm 1 ALG2 iterations

1st step: $u_{i+1} \in \arg\min_{u\in\mathscr{X}_h} \left\{ \mathcal{F}(u) + \langle \phi_i, \Lambda(u)\rangle + \frac{r}{2}|\Lambda(u) - q_i|^2 \right\}$.
2nd step: $q_{i+1} \in \arg\min_{q\in\mathscr{Y}_h} \left\{ \mathcal{G}(q) - \langle \phi_i, q\rangle + \frac{r}{2}|\Lambda(u_{i+1}) - q|^2 \right\}$.
3rd step: We update the multiplier ϕ by

$$\phi_{i+1} = \phi_i + r(\nabla u_{i+1} - q_{i+1}).$$

The steps of the Algorithm 1 can be detailed as follows:

- The first step amounts to solve a Laplace equation with mixed boundary conditions. Indeed, we have for every $z \in \mathscr{X}_h$ with $z = 0$ on Γ_D

$$r\langle \nabla u_{i+1}, \nabla z\rangle = \langle \rho, z\rangle + \langle (rq_i - \phi_i), \nabla z\rangle$$

which is equivalent to solve

$$-r\Delta u = \rho + \operatorname{div}(\phi - rq) \quad \text{in } \Omega_D,$$

together with the following mixed boundary conditions:

$$\begin{cases} \text{Dirichlet boundary condition } u = g \text{ on } \Gamma_D \\ \text{Homogeneous Neumann boundary condition } (r\nabla u + \phi - rq)\cdot \nu_{\tilde{\Gamma}} = 0 \text{ on } \tilde{\Gamma}. \end{cases}$$

- The second step is a pointwise projection. Indeed, we use P_1 finite element for q and ϕ, we have at each vertex x_k

$$0 \in \partial I\!I_{B^*(x_k,.)}(q_{i+1}(x_k)) - \phi_i(x_k) - r(\nabla u_{i+1}(x_k) - q_{i+1}(x_k)),$$

which is equivalent to perform pointwise projections:

$$q_{i+1}(x_k) = \mathrm{Proj}_{B^*(x_k,.)}\Big(\frac{\phi_i(x_k)}{r} + \nabla u_{i+1}(x_k)\Big)$$

where $B^*(x,.) = \{p \in \mathbb{R}^N : \ \sigma^*(x,p) \leq 1\}$.

3.3.3 Error criterion

Basing on the primal-dual optimality conditions, we use the following stopping criterion

1. $\mathrm{MaxLip} := \sup_x \sigma^*(x, \nabla u(x))$,
2. $\mathrm{Div} := \| - \mathrm{div}(\phi) - \rho\|_{L^2}$,
3. $\mathrm{Dual} := \|\sigma(x,\phi(x)) - \phi(x)\cdot\nabla u\|_{L^2}$,
4. $\mathrm{NBD}_\phi := \Big(\int_{\tilde{\Gamma}}(\phi\cdot\nu)^2\Big)^{1/2}$.

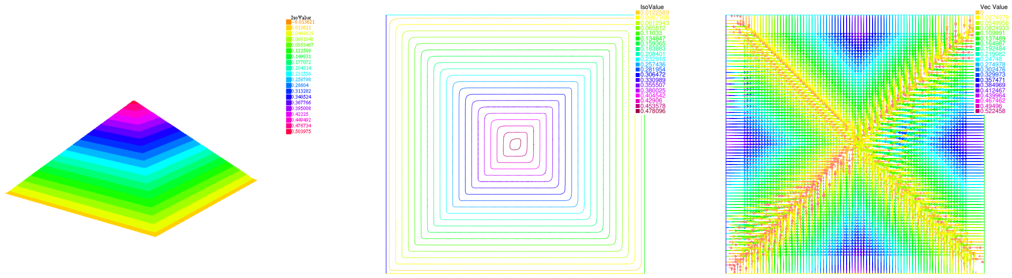
We expect MaxLip $\leq$ 1 and Div, Dual, NBD$_\phi$ to be small. In addition, we compute $\|u - u_{exact}\|$ for different norms where u is the computed solution and u_{exact} is the exact solution, whenever the latter is easily found. Let us mention that NBD$_\phi$ will concern only the Test 2 where we prescribe data on a closed set D, so that $\tilde{\Gamma} \neq \emptyset$.

To implement the algorithm we use FreeFem++ [58], which is particularly adapted to solve the Laplace equation in the first step of ALG2. We use P_2 finite element for u and P_1 finite element for ϕ and q (see e.g. [15]). All the tests are executed on a macOs Mojave 10.14.4.

3.4 Numerical experiments

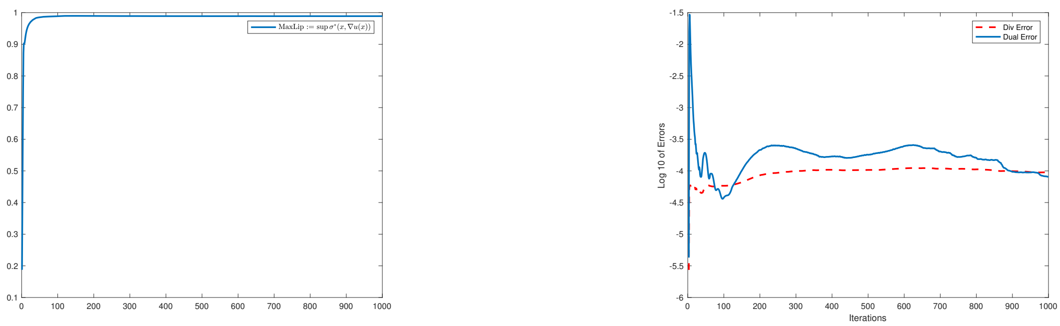
3.4.1 Test 1:

We first examine the case $|\nabla u| = f(x, y) \equiv 1$ in $\Omega = (0, 1)^2$ and $u = 0$ on $\partial\Omega$.



(a) (b) (c)

Figure 3.1: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ



(a) Feasibility of u. (b) Divergence and Dual errors

Figure 3.2: Error criterion for 1000 iterations with $N_h = 120$

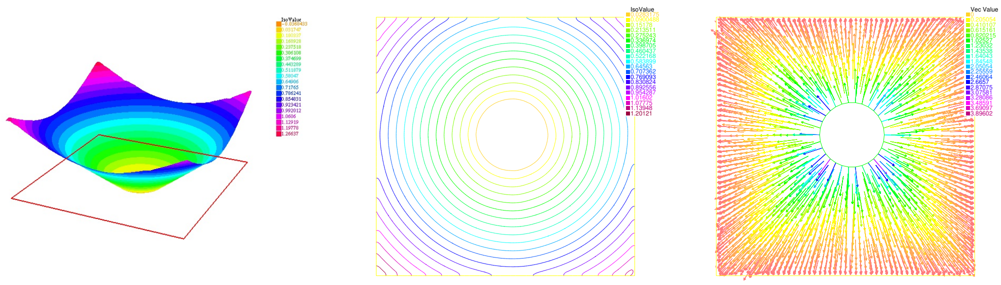
The exact solution is $u_{exact}(x, y) = d((x, y), \partial\Omega) = \min(x, 1 - x, y, 1 - y)$. We measure $\|u - u_{exact}\|$ in different norms, with different mesh sizes N_h and for 1000 iterations.

N_h	Time execution	$\|u - u_{exact}\|_{L^2}$	$\|u - u_{exact}\|_{L^1}$	$\|u - u_{exact}\|_{L^\infty}$
30	55.3254s	1.19311e-3	1.2476e-3	9.71803e-4
60	212.838s	3.68023e-4	3.82991e-4	3.48313e-4
120	855.136s	1.64122e-4	1.7198e-4	1.66483e-4

This test shows us that ALG2 iteration converges to the accurate solution u as the mesh is refined.

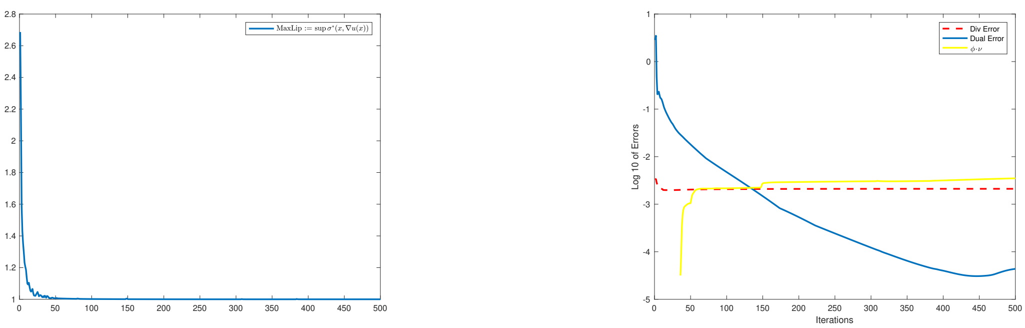
3.4.2 Test 2:

We consider the same equation $|\nabla u| = 1$ on $\Omega = (-1,1)^2 \setminus D$ where D is the euclidean ball centered at $(0.05, 0.09)$ and of radius 0.25, and we set u to be equal to 0 on D.



(a) (b) (c)

Figure 3.3: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ

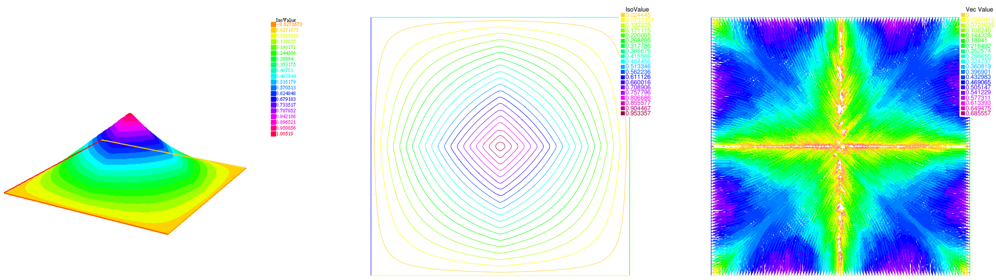


(a) Feasibility of u. (b) Divergence, Dual and NBD$_\phi$ errors

Figure 3.4: Error criterion for 500 iterations with $N_h = 50$

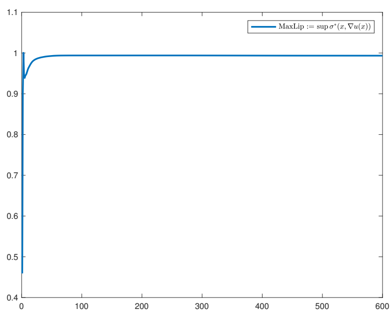
3.4.3 Test 3:

Always with the Eikonal case, take $f(x, y) = \sqrt{(1-|x|)^2 + (1-|y|)^2}$. The exact solution in $\Omega = (-1, 1)^2$ is $u_{exact}(x, y) = (1 - |x|).(1 - |y|)$.

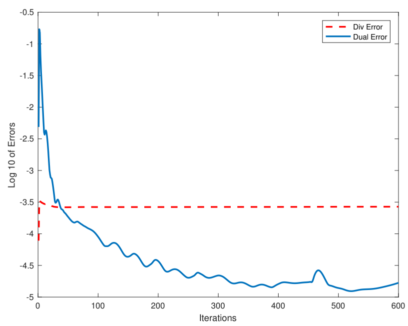


(a) (b) (c)

Figure 3.5: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ



(a) Feasibility of u.



(b) Divergence and Dual errors.

Figure 3.6: Error criterion for 600 iterations with $N_h = 120$

N_h	Time execution	$\|\|u - u_{exact}\|\|_{L^2}$	$\|\|u - u_{exact}\|\|_{L^1}$	$\|\|u - u_{exact}\|\|_{L^\infty}$
30	39.1138s	2.27077e-3	2.54665e-3	2.47988e-3
60	158.138s	6.49636e-4	7.41279e-4	1.57558e-3
120	647.024s	2.99273e-4	3.22747e-4	6.54593e-4
240	2610.03s	6.97158e-05	7.83619e-05	2.807e-4

3.4.4 Riemannian case

We take $\sigma((x,y),v) = \sqrt{\beta_1 v_1^2 + \beta_2 v_2^2}$, with $\beta_1, \beta_2 > 0$. It is not difficult to see that $\sigma^*(q) = \sqrt{\frac{q_1^2}{\beta_1} + \frac{q_2^2}{\beta_2}}$. As in [16], the projection onto the unit ball of σ^*, $B^* = \{q, \sigma^*(q) \leq 1\}$ is given by

$$\mathrm{Proj}_{B^*}(q) = \begin{cases} q & \text{if } q \in B^* \\ \left(\frac{\beta_1 q_1}{\beta_1+\zeta}, \frac{\beta_2 q_2}{\beta_2+\zeta}\right) & \text{otherwise,} \end{cases}$$

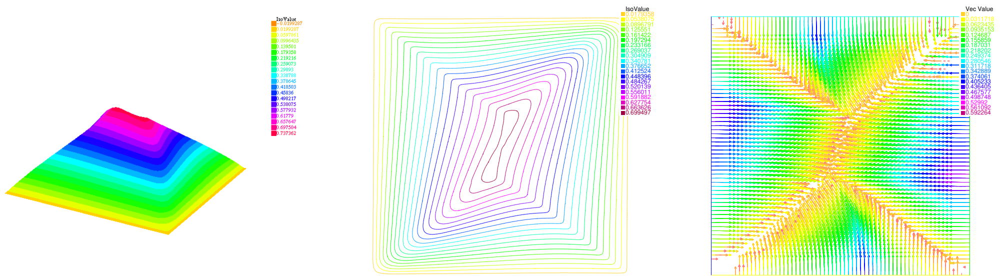
where ζ is the zero of the function

$$F(\zeta) = 1 - \left(\frac{\beta_1 q_1}{(\beta_1+\zeta)^2} + \frac{\beta_2 q_1}{(\beta_2+\zeta)^2}\right),$$

which can be computed with a dichotomy algorithm.

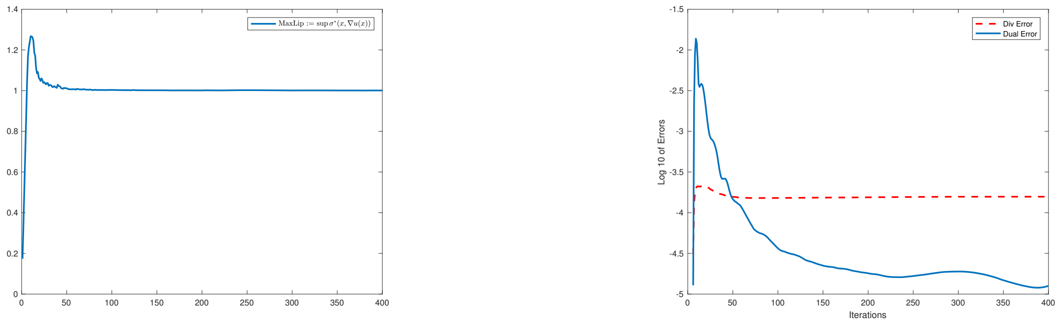
For this test, we take

$$\beta_1 = \frac{1}{e^{-2\left(\sqrt{2(x-0.5)^2+2(x-0.5)(y-0.5)+(y-0.5)^2)}\right)}} \quad \text{and} \quad \beta_2 = \frac{2}{e^{-2\left(\sqrt{2(x-0.5)^2+2(x-0.5)(y-0.5)+(y-0.5)^2)}\right)}}.$$



(a) (b) (c)

Figure 3.7: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ



(a) Feasibility of u.

(b) Divergence and Dual errors.

Figure 3.8: Error criterion for 400 iterations with $N_h = 64$

3.4.5 Anisotropic Eikonal equation:

One interesting case is the so called anisotropic Eikonal equation. Consider a symmetric positive definite matrix M modelling the anisotropy, and define the following equation

$$H(x, \nabla u) = \sqrt{\nabla u^{\dagger} M \nabla u} - 1 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

In this case B^* is an ellipse and the projection can be computed as in the Riemannian case [72]:

$$\mathrm{Proj}_{B^*}(q) = \begin{cases} q & \text{if } q \in B^* \\ (\zeta M + I_n)^{-1} q & \text{otherwise,} \end{cases}$$

where ζ is the unique positive root of the function

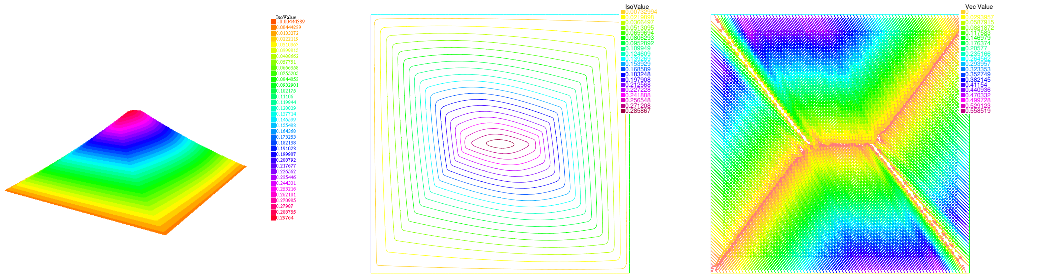
$$F(\zeta) = \bar{q}_{\zeta}^{\dagger} M \bar{q}_{\zeta} - 1 \text{ with } \bar{q}_{\zeta} = (\zeta M + I_n)^{-1} q,$$

which can be found with a dichotomy method. We perform a test as in [81, Example 2] by taking

$$M = \begin{pmatrix} l_1(x,y) & -l_3(x,y) \\ -l_3(x,y) & l_2(x,y) \end{pmatrix}$$

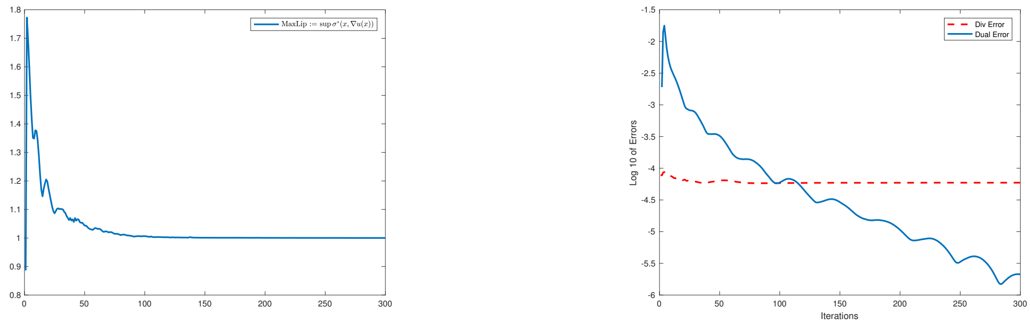
with

$$\begin{aligned} l_1(x,y) &= \frac{1}{e^{-2\left(\sqrt{2.(x-0.5)^2+2.(x-0.5).(y-0.5)+(y-0.5)^2}\right)}} \\ l_2(x,y) &= 2 l_1(x,y) \\ l_3(x,y) &= l_1(x,y). \end{aligned}$$



(a) (b) (c)

Figure 3.9: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ



(a) Feasibility of u. (b) Divergence and Dual errors.

Figure 3.10: Error criterion for 300 iterations with $N_h = 80$.

3.4.6 Polyhedral case

Consider k vectors $p_1, \cdots, p_k$ and define for any $v \in \mathbb{R}^N$ the following Finsler metric

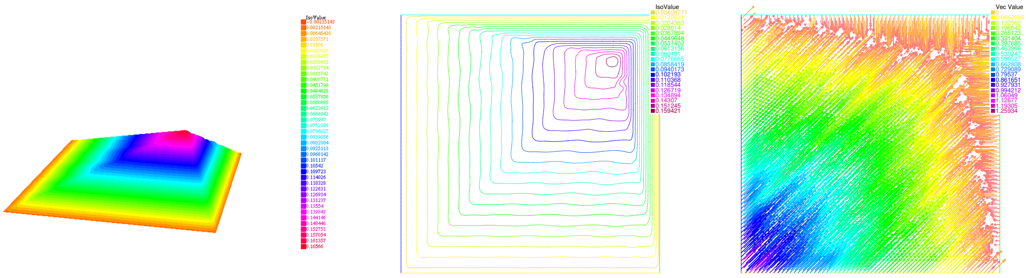
$$\sigma(v) = \max_{1 \leq i \leq k} < v, p_i >$$

usually called a crystalline norm. We can easily check that the unit ball B^* of σ^* is nothing but the convex hull of the vectors $p_1, \cdots, p_k$:

$$B^* = conv(p_1, \cdots, p_k).$$

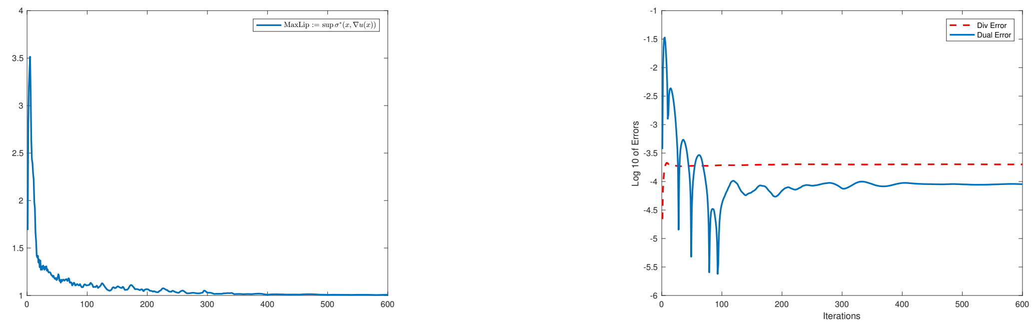
The projection onto B^* can be performed easily (see [16, 66]). We start by determining the vertices $s_1, \cdots, s_k$ of B^* and the corresponding outward normal vectors ν_i to the edges of B^*. Afterwards, if $v \notin B^*$, we distinguish to cases: either $v \in [s_i, s_{i+1}] + \mathbb{R}^+ \nu_i$, and in this case we project v onto the segment $[s_i, s_{i+1}]$, or it belongs to a sector $s_i + \mathbb{R}^+\nu_i + \mathbb{R}^+\nu_{i+1}$ and in this case its projection is s_i.

We perform a test with $p_1 = (1, -1), p_2 = (1, -0.8), p_3 = (-0.8, 1), p_4 = (-1, 1), p_5 = (-1, -1)$. We take $N_h = 64$ and 600 iterations.



(a) (b) (c)

Figure 3.11: Left to right: $3D$ plot of the solution u, contour plot of u, the flux ϕ



(a) Feasibility of u.

(b) Divergence and Dual errors.

Figure 3.12: Error criterion for 600 iterations with $N_h = 64$.

4 Primal-Dual Algorithm for Shape from Shading

4.1 Introduction

Shape from Shading (SfS) consists in reconstructing the $3D$ shape of an object from its given $2D$ image brightness. The shape of a surface $u(x_1, x_2)$ is related to the image brightness $I(x_1, x_2)$ by the Horn image irradiance equation:

$$\mathcal{R}(n(x_1, x_2)) = I(x_1, x_2), \tag{4.1}$$

where $I(x_1, x_2)$ is the brightness greylevel measured in the image at point (x_1, x_2); $\mathcal{R}(n(x_1, x_2))$ is the reflectance map and $n(x_1, x_2)$ is the unit normal at point $(x_1, x_2, u(x_1, x_2))$ given by

$$n(x_1, x_2) = \frac{1}{\sqrt{1 + |\nabla u(x_1, x_2)|^2}}(-\partial_{x_1} u(x_1, x_2), -\partial_{x_2} u(x_1, x_2), 1).$$

In (4.1), the irradiance function $I(x_1, x_2)$ is known since it is measured at each pixel of the brightness image, for example, in terms of greylevel in the interval $[0, 1]$. The implicit unknown is the surface $u(x_1, x_2)$, which has to be reconstructed.

In the case of Lambertian and the surface illuminated by a simple distant light source of direction $\ell = (w, r) = (w_1, w_2, r) \in \mathbb{R}^3$, one has $\mathcal{R}(n(x_1, x_2)) = n(x_1, x_2) \cdot (w_1, w_2, r)$ and, by (4.1),

$$I\sqrt{1 + |\nabla u|^2} + \nabla u \cdot w - r = 0. \tag{4.2}$$

This equation falls into the scope of Hamilton-Jacobi equations

$$H(x, \nabla u) = 0 \text{ in } \Omega, \tag{4.3}$$

where the Hamiltonian H is defined by $H(x,p) = I\sqrt{1+|p|^2}+p\cdot w-r$. In particular, if the object is vertically enlightened, i.e., $\ell = (0,0,1)$, one obtains the standard Eikonal equation

$$|\nabla u(x_1,x_2)| = \sqrt{\frac{1}{I^2(x_1,x_2)} - 1}. \tag{4.4}$$

As pointed out in [39] (see also [104]), there are three major families of numerical methods allowing the resolution of the SfS problem. Namely, PDE methods (cf [6, 47, 79, 91, 93]), optimization methods (cf [34, 61]) and approximating the image irradiance equation (cf [57, 88, 90]).

Here, we are particularly interested in the study of the PDE formulation in terms of HJ equations (4.3) as it was presented in the previous chapter. As we know, the theory of viscosity solutions [31, 32, 80] provides a suitable framework to study equations of the form (4.3), and applications of the viscosity theory to the SfS problem go back to the works of Lions, Rouy and Tourin [79, 93] and recently in the work of Prados, Camilli and Faugeras [91]. Several difficulties arise while dealing with the SfS problem, namely compatibility of boundary conditions and the degeneracy of the Hamiltonian. As we recalled in Chapter 2,it is well known that for (4.3) coupled with the boundary condition $u = g$ on $\partial\Omega$, to admit a solution one needs to check that $g(x) - g(y) \leq d_\sigma(y,x)$ for all $x, y \in \partial\Omega$, where d_σ is the intrinsic distance associated to the Hamiltonian (see (2.8)). In addition, imposing only boundary conditions is not sufficient to ensure the uniqueness of solution to the HJ equations (4.3). It turns out that the set of degeneracy of the distance d_σ, called the Aubry set, plays the role of a uniqueness set for (4.3) (see e.g. [48]). In the case of Eikonal equation (4.4), the Aubry set $\mathcal{A}$ can be taken as the zero set $[k=0]$ of $k = \sqrt{I^{-1/2} - 1}$. In other words, it corresponds to the points with maximal intensity I, i.e., $I(x_1,x_2) = 1$ so that the right hand side in (4.4) vanishes. Most of the authors (cf [23, 91, 93] for example) choose to regularize the equation to avoid these points. We will not encounter this difficulty in our approach since we do not need to deal with the inverse of possibly vanishing functions. We only need to perform projections onto euclidean balls whose radii may be equal to zero.

Our approach is based on the results of Chapter 3, we characterize the maximal viscosity subsolution of (4.3) in terms of a maximization problem. We then associate a dual problem and exploit the saddle-point structure to approximate the solution of (4.3) using the Chambolle–Pock (CP) algorithm. Our approach lies between the PDE and optimization methods since we start by characterizing the maximal viscosity subsolution of the HJ equation thanks to the intrinsic metric of the Hamiltonian and we end up with an optimization problem under gradient constraint. Moreover, the convergence of discretization is also studied in detail.

4.2 Maximization problem and duality in continuous setting

As we discussed in the introduction, we will consider the PDE formulation of the SfS problem in terms of HJ equation. Following our setting in Section 3.1, we consider the following HJ equation where $D \subset \overline{\Omega}$

$$\begin{cases} H(x, \nabla u) = 0 & \text{in } \Omega \setminus D \\ u = g & \text{on } D \end{cases} \tag{4.5}$$

where $g\colon D \to \mathbb{R}$ is a continuous function satisfying the compatibility condition

$$g(x) - g(y) \leq d_\sigma(y, x) \quad \text{for any } x, y \in D.$$

Thanks to Theorem 3.2, the following result allows us to approach the SfS problem via a maximization problem.

Theorem 4.1. *The unique maximal viscosity subsolution of the equation* (4.5) *can be recovered via the following maximization problem*

$$(\mathcal{M}) : \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u(x)\mathrm{d}x,\ \sigma^*(x, \nabla u(x)) \leq 1 \textit{ and } u = g \textit{ on } D \right\}.$$

We propose here a different proof of duality than the one presented in Chapter 3. For simplicity, we will state it for the case $D = \partial\Omega$ (which is essentially the case for other numerical computations). In addition, we assume that $g \in H^{1/2}(\partial\Omega)$.

Theorem 4.2. *We have*

$$\max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u(x)\mathrm{d}x,\ \sigma^*(x, \nabla u(x)) \leq 1 \textit{ and } u = g \textit{ on } \partial\Omega \right\}$$
$$= \inf_{\phi \in L^2(\Omega)^N} \left\{ \int_\Omega \sigma(x, \phi)\mathrm{d}x - \langle g, \phi \cdot \mathbf{n} \rangle_{H^{1/2}, H^{-1/2}} :\ -\mathrm{div}(\phi) = 1 \ \textit{in}\ \mathcal{D}'(\Omega) \right\} := (\mathcal{OF}).$$

Proof. To prove the duality between $(\mathcal{M})$ and $(\mathcal{OF})$, we use a perturbation technique as follows. Define on $L^2(\Omega)^N$ the following functional

$$E(p) := -\sup \left\{ \int_\Omega u\mathrm{d}x : u \in \mathrm{Lip}(\Omega),\ \sigma^*(x, \nabla u(x) - p(x)) \leq 1,\ u = g \text{ on } \partial\Omega \right\}.$$

Then, one can check that E is convex and lower semicontinuous. To compute E^* we start by observing that since $u = g$ on $\partial\Omega$, we can assume thanks to trace lifting Theorem that $g = \gamma_0(w)$ for some w in $H^1(\Omega)$, and $u = \xi + w$ with $\xi \in H^1_0(\Omega) \cap W^{1,\infty}(\Omega)$.

We then have for any $\phi \in L^2(\Omega)^N$:

$$
\begin{aligned}
E^*(\phi) &= \sup_{p \in L^2(\Omega)^N} \int_\Omega \phi \cdot p \,\mathrm{d}x - E(p) \\
&= \sup_{p \in L^2(\Omega)^N, \xi \in H_0^1(\Omega)} \left\{ \int_\Omega \phi \cdot p \,\mathrm{d}x + \int_\Omega \xi \mathrm{d}\rho + \int_\Omega w \mathrm{d}x : \sigma^*(x, \nabla(\xi + w) - p) \leq 1 \right\}
\end{aligned}
$$

Set $q = \nabla(\xi + w) - p$ we get $p = \nabla(\xi + w) - q$, we then have

$$
\begin{aligned}
E^*(\phi) &= \sup_{q \in L^2(\Omega)^N, \xi \in H_0^1(\Omega)} \left\{ \int_\Omega \phi \cdot (\nabla(\xi + w) - q) \mathrm{d}x + \int_\Omega \xi \mathrm{d}x + \int_\Omega w \mathrm{d}x : \sigma^*(x, q(x)) \leq 1 \right\} \\
&= \sup_{\substack{\xi \in H_0^1(\Omega) \\ q \in L^2(\Omega)^N,\ \sigma^*(x,q(x)) \leq 1}} \left\{ \int_\Omega \phi \cdot \nabla \xi \mathrm{d}x + \int_\Omega \xi \mathrm{d}x + \int_\Omega \phi \cdot \nabla w \mathrm{d}x + \int_\Omega w \mathrm{d}x - \int_\Omega \phi \cdot q \mathrm{d}x \right\}.
\end{aligned}
$$

The last quantity is finite if we impose that $\int_\Omega \phi \cdot \nabla \xi \mathrm{d}x + \int_\Omega \xi \mathrm{d}x = 0$ for all $\xi \in H_0^1(\Omega)$, which means that $-\mathrm{div}(-\phi) = 1$ and consequently $\phi \in H_{\mathrm{div}}(\Omega)$, where

$$
H_{\mathrm{div}}(\Omega) = \{\phi \in L^2(\Omega)^N,\ \mathrm{div}(\phi) \in L^2(\Omega)\}.
$$

Thus the normal trace of ϕ is well-defined and $\phi.\mathbf{n} \in H^{-1/2}(\partial\Omega)$. Taking $\nu = -\phi.\mathbf{n}$, then $-\mathrm{div}(-\phi) = 1 - \nu$ in $\mathcal{D}'(\mathbb{R}^N)$ and therefore, for such a ϕ, integrating by parts we get

$$
\begin{aligned}
E^*(-\phi) &= \sup_{q \in L^2(\Omega)^N} \left\{ \int_\Omega \phi \cdot q \,\mathrm{d}x : \sigma^*(x, q(x)) \leq 1 \right\} + \langle w, \nu \rangle_{H^{1/2}, H^{-1/2}} \\
&= \int_\Omega \sigma(x, \phi) \mathrm{d}x - \langle g, \phi.\mathbf{n} \rangle_{H^{1/2}, H^{-1/2}}.
\end{aligned}
$$

Finally,

$$
\max \mathcal{M} = -E(0) = -E^{**}(0) = -\sup -E^*(-\phi) = \inf E^*(-\phi) = \inf(\mathcal{OF}_D),
$$

as desired. □

Remark 4.3. When the object is enlightened vertically, i.e., $\ell = (0, 0, 1)$ and $u = 0$ on $\partial\Omega$, the SfS problem amounts to solve the following Eikonal equation

$$\begin{cases} |\nabla u| = k \text{ in } & \Omega \\ u = 0 \text{ on } & \partial\Omega \end{cases} \tag{4.6}$$

where $k(x) = \sqrt{I^{-2}(x) - 1}$. In this case, the duality in Theorem 4.2 reads as

$$\max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u \mathrm{d}x : |\nabla u| \le k,\ u = 0 \text{ on } \partial\Omega \right\} = \inf_{\phi \in L^2(\Omega)^N} \left\{ \int_\Omega k(x)|\phi| \mathrm{d}x : -\mathrm{div}(\phi) = 1 \text{ in } \mathcal{D}'(\Omega) \right\}.$$

As for the Aubry set $\mathcal{A}$, it can be taken as the zero set $[k = 0]$ of $k = \sqrt{I^{-2} - 1}$, which corresponds to the points with maximal intensity I, i.e., $I(x) = 1$ so that $k(x)$ vanishes. As we will see in the next section, dealing with nonempty Aubry set does not represent an obstacle in our approach. Contrary to the works (*e.g* [23, 91, 93]) where the authors approximate the degenerate HJ equation via non-degenerate one (typically, by considering (4.6) with $k_\epsilon = \max(k, \epsilon)$ for $\epsilon > 0$), the only step where we deal with degeneracy points is the projection onto a ball of radius k, which may be equal to zero.

Remark 4.4 (Boundary conditions). It is well known that a natural choice for boundary conditions is the Dirichlet boundary condition. As pointed out in [39], the images we will consider contain an occluding boundary (see Fig 4.1) which will be taken as the boundary $\partial\Omega$. Particularly, assuming that the object is placed on a flat table suggests taking $u = 0$ on $\partial\Omega$ or more generally, if the height g of the surface on which is placed is known one can take $u = g$ on $\partial\Omega$.

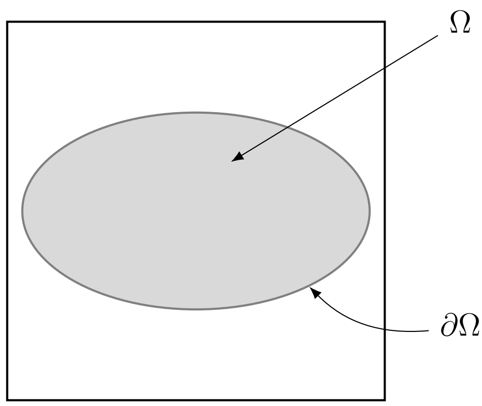


Figure 4.1: An object and its occluding boundary

4.3 Discretization

The main result in this section will be the convergence of primal-dual solutions of the discretized (finite-dimensional) problems to the ones of the original problems in continuous setting.

4.3.1 Discretization of the domain and operators

Let $\Omega \subset \mathbb{R}^d$ be an image domain, which can be taken as $\Omega = [0,1]^2$. Following [26], we discretize the domain Ω using a regular grid $m \times n$: $\{(ih, jh) : 1 \leq i \leq m, 1 \leq j \leq n\}$ for a fixed $h > 0$. We denote by $D_d = \{(i,j) : (ih, jh) \in D\}$ the indexes whose spatial positions belong to D and by $u_{i,j}$ the values of u at (ih, jh). The space $X = \mathbb{R}^{m \times n}$ is equipped with a scalar product and an associated norm as follows:

$$\langle u, v \rangle = h^2 \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j} v_{i,j} \quad \text{and} \quad \|u\| = \sqrt{\langle u, u \rangle}.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, we define the components of the discrete gradient operator via finite differences:

$$(\nabla_h u)^1_{i,j} = \begin{cases} \frac{u_{i+1,j} - u_{i,j}}{h} & \text{if } i < m \\ 0 & \text{if } i = m \end{cases}, \qquad (\nabla_h u)^2_{i,j} = \begin{cases} \frac{u_{i,j+1} - u_{i,j}}{h} & \text{if } j < n \\ 0 & \text{if } j = n. \end{cases}$$

Then the discrete gradient $\nabla_h : X \longrightarrow Y = \mathbb{R}^{m \times n \times 2}$ given by $(\nabla_h u)_{i,j} = \big((\nabla_h u)^1_{i,j}, (\nabla_h u)^2_{i,j}\big)$. Similar to the continuous setting, we define a discrete divergence operator $\operatorname{div}_h : Y \to X$, which is the minus of the adjoint of ∇_h, given by $\operatorname{div}_h = -\nabla_h^*$. That is, $\langle -\operatorname{div}_h \phi, u \rangle_X = \langle \phi, \nabla_h u \rangle_Y$ for any $\phi = (\phi^1, \phi^2) \in Y$ and $u \in X$. It follows that div is explicitly given by

$$(\operatorname{div}_h \phi)_{i,j} = \begin{cases} \frac{\phi^1_{i,j}}{h} & \text{if } i = 1 \\ \frac{\phi^1_{i,j} - \phi^1_{i-1,j}}{h} & \text{if } 1 < i < m \\ \frac{-\phi^1_{m-1,j}}{h} & \text{if } i = m \end{cases} + \begin{cases} \frac{\phi^2_{i,j}}{h} & \text{if } j = 1 \\ \frac{\phi^2_{i,j} - \phi^2_{i,j-1}}{h} & \text{if } 1 < j < n \\ \frac{-\phi^2_{i,n-1}}{h} & \text{if } j = n. \end{cases}$$

Proposition 4.5. *([26, 27]) Under the above-mentioned definitions and notations, one has that*

- *The adjoint operator of ∇_h is $\nabla_h^* = -\operatorname{div}_h$.*
- *Its norm satisfies: $\|\nabla_h\|^2 = \|\operatorname{div}_h\|^2 \leq 8/h^2$.*

4.3.2 Discretization of the optimization problem

Based on the discrete gradient and divergence operators, we propose a discrete version of $(\mathcal{M})$ as follows

$$(\mathcal{M})_d : \quad \min_{\substack{u \in X \\ u_{i,j} = g_{i,j} \ \forall (i,j) \in D_d}} \Big\{ -h^2 \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j} + \mathbb{I}_{B_{\sigma^*}}(\nabla_h u) \Big\}$$

where $B_{\sigma^*} := \{v \in Y : \sigma^*(ih, jh, v_{i,j}) \leq 1, \forall (i,j)\}$ the unit ball w.r.t. σ^*, and $\mathbb{I}_{B_{\sigma^*}}$ is the indicator function in the sense of convex analysis, that is,

$$\mathbb{I}_{B_{\sigma^*}}(v) = \begin{cases} 0 & \text{if } v \in B_{\sigma^*} \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, the discrete version $(\mathcal{M})_d$ can be written as

$$\min_{u \in X} \mathcal{F}_h(u) + \mathcal{G}_h(\nabla_h u),$$

where

$$\mathcal{F}_h(u) = \begin{cases} -h^2 \sum_{i=1}^m \sum_{j=1}^n u_{i,j} & \text{if } u_{i,j} = g_{i,j} \ \forall (i,j) \in D_d \\ +\infty & \text{otherwise} \end{cases}, \text{ and } \mathcal{G}_h = \mathbb{I}_{B_{\sigma^*}}.$$

Let $u^* \in X^*$, we then have

$$\begin{aligned} \mathcal{F}_h^*(u^*) = \sup_{u \in X} \langle u, u^* \rangle_X - \mathcal{F}_h(u) &= \sup_{\substack{u \in X \\ u_{i,j} = g_{i,j} \ \forall (i,j) \in D_d}} h^2 \sum_{i=1}^m \sum_{j=1}^n u_{i,j} u^*_{i,j} + h^2 \sum_{i=1}^m \sum_{j=1}^n u_{i,j} \\ &= \sup_{\substack{u \in X \\ u_{i,j} = g_{i,j} \ \forall (i,j) \in D_d}} h^2 \sum_{i=1}^m \sum_{j=1}^n u_{i,j} (u^*_{i,j} + 1) \\ &= \begin{cases} h^2 \sum_{(i,j) \in D_d} g_{i,j} \big(u^*_{i,j} + 1\big) & \text{if } -u^*_{i,j} = 1 \text{ for } (i,j) \notin D_d \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that

$$\mathcal{F}_h^*(\mathrm{div}_h \phi) = \begin{cases} h^2 \sum_{(i,j) \in D_d} g_{i,j} ((\mathrm{div}_h \phi)_{i,j} + 1) & \text{if } (-\mathrm{div}_h \phi)_{i,j} = 1 \text{ for } (i,j) \notin D_d \\ +\infty & \text{otherwise.} \end{cases}$$

On the other hand, we have for $q = (q^1, q^2) \in Y^*$

$$\mathcal{G}_h^*(q) = \sup_{p = (p^1, p^2) \in Y} \langle p, q \rangle_Y - \mathcal{G}_h(p) = \sup_{p \in B_{\sigma^*}} h^2 \sum_{i=1}^m \sum_{j=1}^n \big(p^1_{i,j} q^1_{i,j} + p^2_{i,j} q^2_{i,j}\big) = h^2 \sum_{i=1}^m \sum_{j=1}^n \sigma(ih, jh, q_{i,j}).$$

Consequently, the corresponding discrete dual problem is given by

$$
\begin{aligned}
(\mathcal{OF})_d:\ &\max_{\phi\in Y}\Big\{-\mathcal{F}_h^*(\mathrm{div}_h\phi)-\mathcal{G}_h^*(\phi)\Big\}\\
&=-\min_{\substack{\phi\in Y\\ (-\mathrm{div}_h\phi)_{i,j}=1 \text{ for } (i,j)\notin D_d}} h^2\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma(ih,jh,\phi_{i,j})+\sum_{(i,j)\in D_d} g_{i,j}((\mathrm{div}_h\phi)_{i,j}+1)\right\}.
\end{aligned}
\tag{4.7}
$$

In the case of Eikonal equations $|\nabla u(x)| = k(x)$, the primal-dual relations can be explicitly written as

$$
\begin{aligned}
&\min_{\substack{u\in X\\ u_{i,j}=g_{i,j}\ \forall (i,j)\in D_d}}\Big\{-h^2\sum_{i=1}^{m}\sum_{j=1}^{n}u_{i,j}+\mathbb{I}_{B(0,k_{i,j})}(\nabla_h u_{i,j})\Big\}\\
&=-\min_{\substack{\phi\in Y\\ (-\mathrm{div}_h\phi)_{i,j}=1 \text{ for } (i,j)\notin D_d}} h^2\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}k_{i,j}\|\phi_{i,j}\|+\sum_{(i,j)\in D_d} g_{i,j}((\mathrm{div}_h\phi)_{i,j}+1)\right\},
\end{aligned}
$$

where $\mathbb{I}_{B(0,k_{i,j})}$ is the indicator function of the Euclidean ball with center 0 and radius $k_{i,j}$, the latter being the value of k at (ih, jh).

To end this subsection, let us recall that (see Theorem 2.16) a pair $(u,\phi)\in X\times Y$ solves the primal and dual problems $(\mathcal{M})_d$ and $(\mathcal{OF})_d$ if and only if

$$\mathrm{div}_h(\phi)\in\partial\mathcal{F}_h(u) \text{ and } \phi\in\partial\mathcal{G}_h(\nabla_h u),$$

or equivalently, they satisfy the following system

$$
\begin{cases}
-(\mathrm{div}(\phi))_{i,j}=1 & \text{for } \ (i,j)\notin D_d\\[2ex]
\phi_{i,j}\cdot\nabla_h u_{i,j}=\sigma(ih,jh,\phi_{i,j}) & \text{for all } (i,j)\\[2ex]
u_{i,j}=g_{i,j} & \text{for every } \ (i,j)\in D_d.
\end{cases}
\tag{4.8}
$$

4.3.3 The convergence of discretization

In this subsection, we will show a result on the convergence of discretization, *i.e*, where the solutions of the discrete optimization and its discrete dual problem converge to the ones of the corresponding problems in continuous setting. First, let us describe how to interpolate elements of X and Y.

We know the values of $u_h\in X$ at the vertices (i,j), $(i,j+1)$, $(i+1,j+1)$, $(i+1,j)$ of a small square (see Fig 4.2). We interpolate $u_h\in X$ by piecewise affine functions on the sub-triangles, i.e., taking $\tilde{u}_h\in L^2(\Omega)$ as an affine function on the sub-triangles and coincides with u_h on all the vertices. Then $\tilde{u}_h$ is Lispchitz function and its gradient is, by the definition of $\tilde{u}_h$, given by

$$\nabla \tilde{u}_h(x, y) = \left(\frac{u_h(i+1, j) - u_h(i, j)}{h}, \frac{u_h(i, j+1) - u_h(i, j)}{h} \right) \tag{4.9}$$

on the sub-triangle of the vertices $(i, j), (i, j+1), (i+1, j)$; and

$$\nabla \tilde{u}_h(x, y) = \left(\frac{u_h(i+1, j+1) - u_h(i, j+1)}{h}, \frac{u_h(i+1, j+1) - u_h(i+1, j)}{h} \right) \tag{4.10}$$

on the sub-triangle of the vertices $(i, j+1), (i+1, j+1), (i+1, j)$.
Let $\tilde{\phi}_h \in L^2(\Omega)^2$ be an interpolation of $\phi_h \in Y$ such that $\int_\Omega \sigma(x, \tilde{\phi}_h) \mathrm{d}x = h^2 \sum_{i=1}^m \sum_{j=1}^n \sigma(ih, jh, (\phi_h)_{i,j})$.

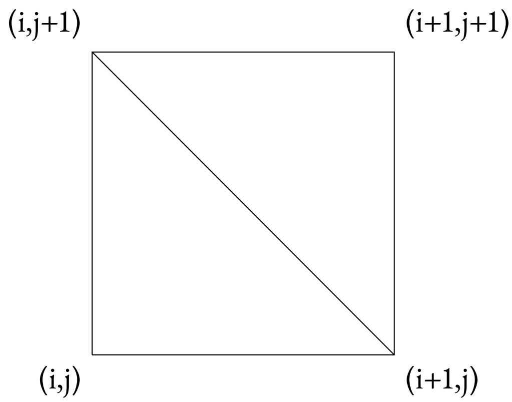


Figure 4.2: Sub-triangles

Proposition 4.6 (Convergence of discretization)**.** *Assume that the Finsler metric σ associated with the Hamiltionian H is non-degenerate (i.e. $H(x, 0) < 0, \forall x \in \overline{\Omega}$) and that $g = 0$. Let $u_h \in X$ and $\phi_h = (\phi_h^1, \phi_h^2) \in Y$ be a pair of primal-dual solutions to the discrete optimization problem $(\mathcal{M})_d$ and its dual problem (4.7). Then $\tilde{u}_h \rightrightarrows u$ and $\tilde{\phi}_h \rightharpoonup \phi$ as the step size $h \to 0$. Moreover, u and ϕ are optimal solutions to $(\mathcal{M})$ and its dual problem, respectively.*

Proof. Since u_h is feasible for the discrete optimization problem $(\mathcal{M})_d$, its discrete gradient $\nabla_h u_h \in B_{\sigma^*}$ is bounded for all small $h > 0$. In other words, the sequences $\left\{ \frac{u_h(i+1,j) - u_h(i,j)}{h} \right\}$ and $\left\{ \frac{u_h(i,j+1) - u_h(i,j)}{h} \right\}$ are bounded for $h > 0$ and $i = 1, ..., m, j = 1, ..., n$. Following (4.9) and (4.10), the sequence $\{\tilde{u}_h\}$ is equi-Lispchitz. Combining with the fact that $u_h = g$ on D_d, by Ascoli-Arzelà's Theorem, up to a subsequence, $\tilde{u}_h$

converges uniformly to some Lipschitz function u on $\overline{\Omega}$ as the step size $h \to 0$. By the optimality of u_h and ϕ_h, we have

$$\mathcal{F}_h(u_h) + \mathcal{G}_h(\nabla_h u_h) = -\mathcal{F}_h^*(\mathrm{div}_h \phi_h) - \mathcal{G}_h^*(\phi_h).$$

More concretely,

$$h^2 \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j} = h^2 \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma(ih, jh, (\phi_h)_{i,j})$$

or equivalently

$$\int_\Omega \tilde{u}_h \mathrm{d}x = \int_\Omega \sigma(x, \tilde{\phi}_h) \mathrm{d}x.$$

Since σ is non-degenerate and $\tilde{u}_h$ is bounded, $\tilde{\phi}_h$ is also bounded in $L^1(\Omega)$. Hence, $\tilde{\phi}_h \rightharpoonup \phi$ weakly* in $\mathcal{M}_b(\overline{\Omega})$. Using the lower-semicontinuity of the integrand (see [3, Theorem 2.38]), we deduce that

$$\int_\Omega \sigma(x, \phi) \mathrm{d}x \leq \lim_{h \to 0} \int_\Omega \sigma(x, \tilde{\phi}_h) \mathrm{d}x = \lim_{h \to 0} \int_\Omega \tilde{u}_h \mathrm{d}x = \int_\Omega u \mathrm{d}x.$$

By the duality result in the continuous setting given in Section 2 (Theorem 4.2), we deduce that u and ϕ are primal-dual optimal solutions. □

Remark 4.7. In the case where σ is a degenerate Finsler metric, there is no boundedness on $\tilde{\phi}_h$ and we cannot thus pass to the limit for $\tilde{\phi}_h$. However, we still have the uniform convergence of $\tilde{u}_h$ to some Lipschitz function u and moreover u is actually an optimal solution to the maximization problem $(\mathcal{M})$.

4.4 Numerical resolution

In this section we focus on the case where the light direction is vertical, *i.e.*, $\ell = (0, 0, 1)$.

4.4.1 Saddle-point structure

As we pointed out in Section 4.3, the discrete version $(\mathcal{M})_d$ of $(\mathcal{M})$ can be rewritten in the form

$$\inf_{u \in X} \mathcal{F}_h(u) + \mathcal{G}_h(\nabla_h u) \tag{4.11}$$

or in an inf-sup form as

$$\inf_{u \in X} \sup_{\phi \in Y} \mathcal{F}_h(u) + \langle \phi, \nabla_h u \rangle - \mathcal{G}_h^*(\phi)$$

where

$$\mathcal{F}_h(u) = \begin{cases} -h^2 \sum_{i=1}^m \sum_{j=1}^n u_{i,j} & \text{if } u_{i,j} = g_{i,j} \ \forall (i,j) \in D_d \\ +\infty & \text{otherwise} \end{cases}, \text{ and } \mathcal{G}_h = \mathbb{I}_{B_{\sigma^*}}.$$

Both the functions $\mathcal{F}_h$ and $\mathcal{G}_h$ are lower-semicontinuous, convex and they are "proximable", *i.e.* we can compute their proximal operators (see Defintion 2.17):

$$\begin{aligned} \operatorname{Prox}_{\tau \mathcal{F}_h}(u) &= \operatorname*{argmin}_{v \in X} \frac{1}{2}\|u - v\|^2 + \tau \mathcal{F}_h(v) \\ \operatorname{Prox}_{\eta \mathcal{G}_h}(\psi) &= \operatorname*{argmin}_{\phi \in Y} \frac{1}{2}\|\psi - \phi\|^2 + \eta \mathcal{G}_h(\phi) \end{aligned}$$

where $\tau, \eta > 0$. Then the Chambolle–Pock algorithm [27] can be applied to (4.11):

Algorithm 2 Chambolle–Pock iterations

1st step. Initialization: choose $\eta, \tau > 0, \theta \in [0,1], u^0$ and take $\phi^0 = \nabla_h u^0$, $\bar{u}^0 = u^0$.
2nd step. For $k \leq \text{Iter}_{max}$ do

$$\begin{aligned} \phi^{k+1} &= \operatorname{Prox}_{\eta \mathcal{G}_h^*}(\phi^k + \eta \nabla_h(\bar{u}^k)); \\ u^{k+1} &= \operatorname{Prox}_{\tau \mathcal{F}_h}(u^k - \tau \nabla_h^*(\phi^{k+1})); \\ \bar{u}^{k+1} &= u^{k+1} + \theta(u^{k+1} - u^k). \end{aligned}$$

As we pointed out in Section 2.4.3, if $\theta = 1$ and $\eta\tau\|\nabla_h\|^2 < 1$, the sequence $\{u^k\}$ converges to an optimal solution of (4.11). Contrary to the ALG2 algorithm, the Chambolle–Pock algorithm does not require to solve a Laplace equation at each iteration, we only need to perform some algebraic operations, namely the multiplication by apply the gradient and the divergence in each iteration. Moreover, it easy to implement on Matlab which allows working on images easily contrary to the ALG2 algorithm which was implemented using FreeFem++ to solve linear PDEs.

In order to compute $\operatorname{Prox}_{\eta \mathcal{G}_h^*}$ we make use of the celebrated Moreau identity

$$\phi = \operatorname{Prox}_{\eta \mathcal{G}_h^*}(\phi) + \eta \operatorname{Prox}_{\eta^{-1}\mathcal{G}_h}(\phi/\eta), \ \forall \phi \in Y.$$

Moreover, $\operatorname{Prox}_{\eta^{-1}\mathcal{G}_h}$ is nothing but the projection onto $B(0, k_{i,j})$. Indeed

$$\begin{aligned}
\mathrm{Prox}_{\eta^{-1}\mathcal{G}_h}(\psi) &= \arg\min_{q\in Y} \frac{1}{2}|q-\psi|^2 + \frac{1}{\eta}\mathcal{G}_h(q) \\
&= \arg\min_{q_{i,j}\in B(0,k_{i,j})} \frac{1}{2}|q-\psi|^2 \\
&= \mathrm{Proj}_{B(0,k_{i,j})}(\psi_{i,j}).
\end{aligned}$$

Consequently,

$$\left(\mathrm{Prox}_{\eta\mathcal{G}_h^*}(\psi)\right)_{i,j} = \psi_{i,j} - \eta\mathrm{Proj}_{B(0,k_{i,j})}(\psi_{i,j}/\eta).$$

Let us now compute the proximal operator of $\mathcal{F}_h$. We have

$$\mathrm{Prox}_{\tau\mathcal{F}_h}(u) = \underset{v\in X}{\mathrm{argmin}}\frac{1}{2}\|v-u\|^2+\tau\mathcal{F}_h(v) = \underset{v=g \text{ on } D_d}{\mathrm{argmin}}\ \frac{1}{2}\|v-u\|^2-\tau h^2\sum_{i=1}^{m}\sum_{j=1}^{n} v_{i,j}.$$

Writing the first-order optimality condition we get

$$(\mathrm{Prox}_{\gamma\mathcal{F}_h}(u))_{i,j}-u_{i,j}-\tau h^2 = 0 \Leftrightarrow (\mathrm{Prox}_{\gamma\mathcal{F}_h}(u))_{i,j} = u_{i,j}+\tau h^2, \forall i=1,...,m, j=1,...,n.$$

So in practice, we update u_{n+1} via the previous formula and we then set its values to g on the Dirichlet domain.

For applications in image, as usual, one can always assume that $h = 1$ since it only scales the domain. The details of the 2nd step in Algorithm 1 are then given by

- compute ϕ^{k+1}:

$$\begin{aligned}
\bar{\phi}^{k+1} &= \phi^k + \eta\nabla_h\bar{u}^k \\
\phi_{i,j}^{k+1} &= \bar{\phi}_{i,j}^{k+1} - \eta\mathrm{Proj}_{B(0,k_{i,j})}(\bar{\phi}_{i,j}^{k+1}/\eta),\ 1\le i\le m, 1\le j\le n;
\end{aligned}$$

- compute u^{k+1}:

$$\begin{aligned}
v^{k+1} &= u^k + \tau\mathrm{div}_h(\phi^{k+1}) \\
u_{i,j}^{k+1} &= v_{i,j}^{k+1} + \tau,\ 1\le i\le m, 1\le j\le n.
\end{aligned}$$

Remark 4.8. Another way to formulate the problem $(\mathcal{M})$ (in the continuous setting) is to take

$$\mathcal{F}(u) = -\int_\Omega u\mathrm{d}x, \text{ and } \mathcal{G}(q,v) = \begin{cases} 0 & \text{if } |q|\le k \text{ and } v=g \text{ on } \partial\Omega \\ \infty & \text{otherwise,} \end{cases}$$

for all $u \in W^{1,\infty}(\Omega)$, and $(q, v) \in L^\infty(\Omega)^d \times L^2(\partial\Omega)$. In this case, the problem $(\mathcal{M})$ can be rewritten as

$$\inf_u \mathcal{F}(u) + \mathcal{G}(K(u))$$

where $K = (\nabla, \gamma_0)$, and γ_0 is the trace operator on the boundary. This being said, at the second step of the Algorithm 2 we need to compute γ_0^* which turns requiring to solve a PDE. Indeed, we define

$$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$$

through $\gamma_0(u) = u_{|\partial\Omega}$ for every $u \in H^1(\Omega)$. By definition, for any $(u, v) \in H^1(\Omega) \times L^2(\partial\Omega)$

$$\langle \gamma_0 u, v \rangle_{L^2(\partial\Omega)} = \langle u, \gamma_0^* v \rangle_{H^1(\Omega)}.$$

This means that

$$\int_{\partial\Omega} uv \mathrm{d}S = \int_\Omega u(\gamma_0^* v)\mathrm{d}x + \int_\Omega \nabla u \nabla(\gamma_0^* v)\mathrm{d}x$$

for any $u \in H^1(\Omega)$. In other words $\gamma_0^* v$ solves the following PDE

$$-\Delta z + z = 0 \text{ in } \Omega \text{ and } \partial_n z = v \text{ on } \partial\Omega.$$

Thus we opt for the first formulation in order to avoid additional costs to the computations.

4.4.2 Error criterion

As usual, we can check the optimality conditions (4.8) associated to $(\mathcal{M})_d$ and $(\mathcal{OF})_d$. Namely we check the following conditions:

- Divergence error: $\| - \mathrm{div}_h(\phi) - 1\|_2$.
- Dual error: $\|\sigma(x, \phi) - \nabla_h u \cdot \phi\|_1$.
- Lip error: $\sup_{i,j} \sigma^*(ih, jh, \nabla_h u_{i,j})$.

We expect Divergence error and Dual error to be small. Note that for vertical light direction, the support function σ is easy to compute. More particularly, one has for every $p \in \mathbb{R}^d$, $\sigma(x, p) = k(x)|p|$ where $|p|$ is the euclidean norm of p. Thus, for the Lip error, we can check the value $\sup_{i,j}(\|\nabla_h u_{i,j}\| - k_{i,j})$ and expect it to be close to zero.

4.4.3 Numerical examples

We test for some commonly used images: Mozart and vase images taken from [104] and Basilica and vaso images taken from [50].

In these cases, the shapes are reconstructed by solving the Eikonal equation $|\nabla u(x, y)| = k(x, y)$ in 2D with $g \equiv 0$, *i.e*, with homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The algorithm was implemented in Matlab and executed on a 2,3 GHz CPU running macOs Catalina system.

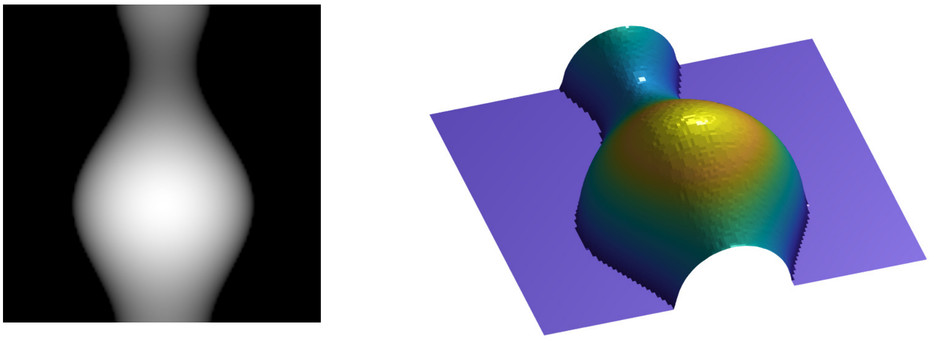
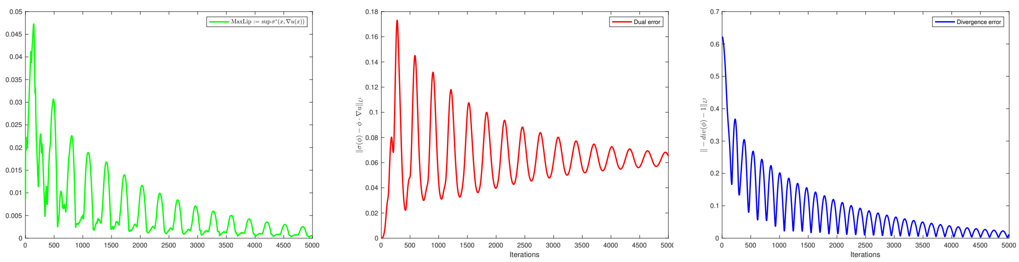


Figure 4.3: Left to right: Initial image, the reconstructed shape



(a) Lip Error (b) Dual Error (c) Divergence Error

Figure 4.4: Error criterion for 5000 iterations and $\tau = 0.01$ and $\eta = 8/\tau$

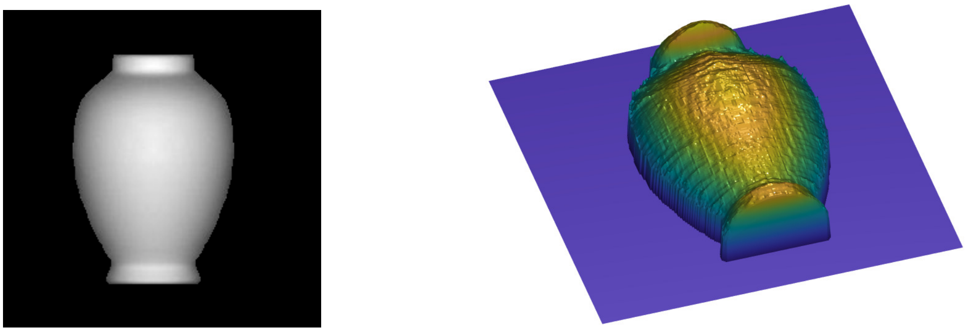
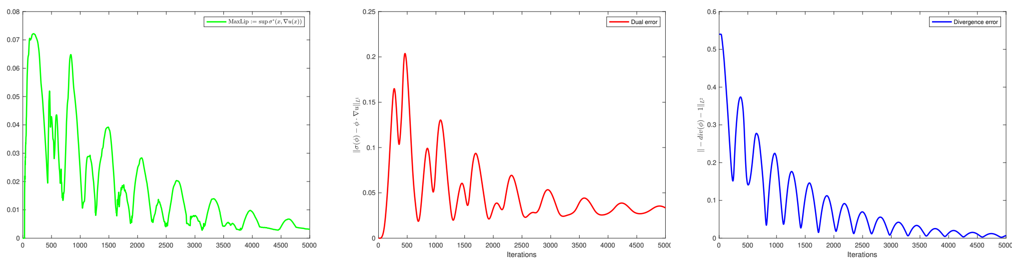
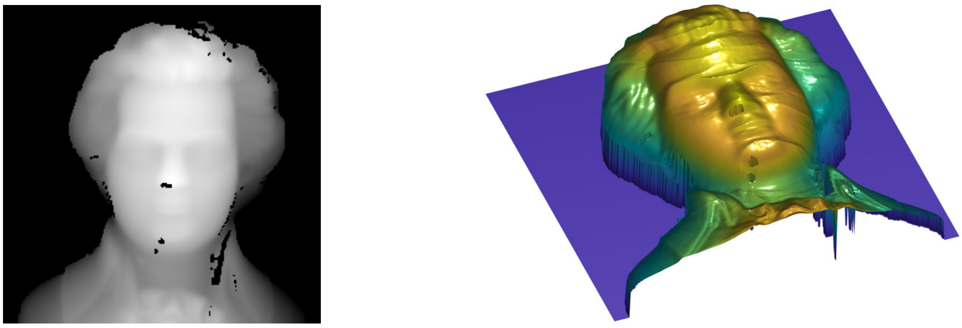


Figure 4.5: Left to right: Initial image, the reconstructed shape



(a) Lip Error (b) Dual Error (c) Divergence Error

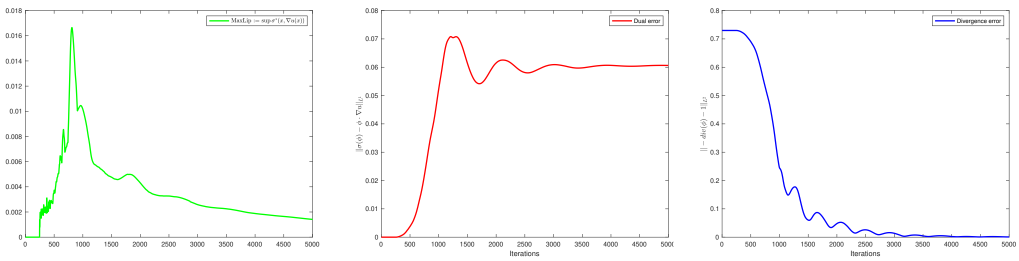
Figure 4.6: Error criterion for 5000 iterations and $\tau = 0.01$ and $\eta = 8/\tau$



(a)

(b)

Figure 4.7: Left to right: Initial image, the reconstructed shape

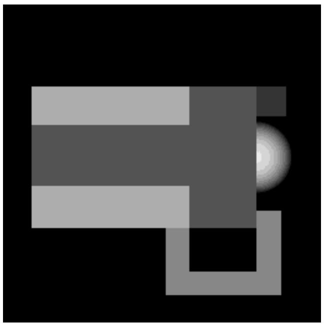


(a) Lip Error

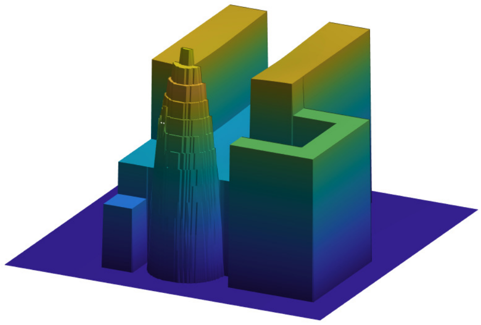
(b) Dual Error

(c) Divergence Error

Figure 4.8: Error criterion for 5000 iterations and $\tau = 0.001$ and $\eta = 8/\tau$

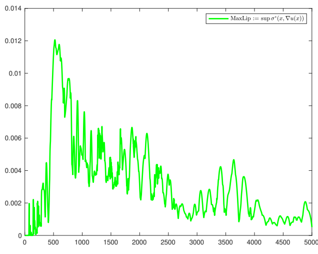


(a)

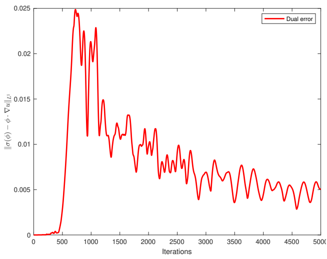


(b)

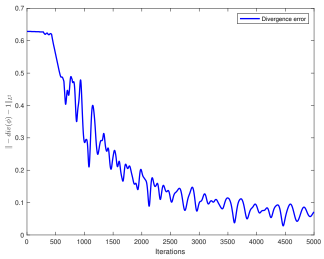
Figure 4.9: Left to right: Initial image, the reconstructed shape



(a) Lip Error



(b) Dual Error



(c) Divergence Error

Figure 4.10: Error criterion for 5000 iterations and $\tau = 0.001$ and $\eta = 8/\tau$

Shape	Execution Time
vase	20.10s
vaso	76.03s
Mozart	76.82s
Basilica	79.31s

Let us mention that for most shapes, only a hundred of iterations is enough to reconstruct a reasonable solution. We took 5000 iterations in order to check that the error criteria are getting smaller.

4.5 Comments and extensions

Let us mention that our strategy works (at least theoretically) to a general light direction. In this case one has to solve a PDE of the form

$$\begin{cases} H(x, \nabla u) = 0 \text{ in } & \Omega \\ u = 0 \text{ on } & \partial\Omega \end{cases} \tag{4.12}$$

Similarly, the maximal viscosity subsolution of (4.12) can be recovered via the following maximization problem

$$(\mathcal{M})_g : \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega u dx : \ \sigma^*(x, \nabla u) \leq 1, \ u = 0 \text{ on } \partial\Omega \right\}.$$

Contrary to the case where the object is enlightened vertically, *i.e.*, $\ell = (0, 0, 1)$, where the Prox_{G^*} in Algorithm 1 involved projections on euclidean balls, in this general case we need to be able to project on

$$Z(x) = \{p \in \mathbb{R}^d : \ \sigma^*(x, p) \leq 1\},$$

for every $x \in \Omega$. To this end, one can use the following result.

Theorem 4.9. *[13] Let $Z = \{p \in \mathbb{R}^d : \ H(p) \leq c\}$ where $c \in \mathbb{R}$ and $H : \mathbb{R}^d \to (-\infty, \infty]$ is a convex, proper function. If there exists $\bar{p} \in \mathbb{R}^d$ such that $H(\bar{p}) < c$ then*

$$\text{Proj}_Z(p) = \begin{cases} \text{Proj}_{\text{dom}(H)}(p) \text{ if } H(\text{Proj}_{\text{dom}(H)}(p)) \leq c \\ \text{Prox}_{\bar{\eta} H}(p) \text{ otherwise}. \end{cases} \tag{4.13}$$

where η is any positive root of the equation $f(\eta) := H(\text{Prox}_{\eta H}(p)) - c = 0$.

Taking $\ell = (1, 0, 1)$ as light direction in (4.2), we obtain the following Hamiltonian

$$H(x, \nabla u) = I\sqrt{1 + |\nabla u|} + \partial_{x_1} u - 1,$$

for which $Z(x_1, x_2) = \{p \in \mathbb{R}^2 : \ I(x_1, x_2)\sqrt{1 + |\nabla u(x_1, x_2)|} + \partial_{x_1} u(x_1, x_2) \leq 1\}$. A priori there is no closed form for the support function σ of Z and its dual σ^*, so in future works, we are planing to investigate and propose methods to perform efficiently projections of the form (4.13), and to adapt our techniques for non-Lambertian surfaces.

5 Beckmann problem and HJ equations

5.1 Introduction

Our main interest here lies in the study of the connection between HJ equations of the type

$$\begin{cases} H(x, \nabla u(x)) = 0 & \text{in } \Omega \\ u = g & \text{on } D \subset \overline{\Omega}. \end{cases} \tag{5.1}$$

and the Beckmann problem as well as the Monge-Kantorovich problem in the case where H is a Hamiltonian satisfying the assumptions (H1)-(H3) recalled in 2.1. Recall that Beckmann's problem is a divergence PDE-constrained optimization problem. It is important in the study of transportation activities. As to the Monge-Kantorovich problem, it consists in finding the best way to push forward between two given measures (usually called goods and consumers) related to some cost function. It is a linear optimization problem which appears in the study of optimal transportation and allocation of resources.

Following our presentation in Section 2.2, consider two nonnegative Radon measures ρ_1 and ρ_2 representing the distributions of resident and services, and let $\Phi : \Omega \to \mathbb{R}^N$ be the traffic flow satisfying the equilibrium condition:

$$-\operatorname{div}(\Phi) = \rho := \rho_2 - \rho_1 \quad \text{in } \mathcal{D}'(\Omega),$$

together with the no-flux boundary condition, i.e., $\Phi \cdot \mathbf{n} = 0$ on $\partial\Omega$. Now, given a non-degenerate Finsler metric F, and assume that the transportation cost per costumer is $F(x, \Phi(x))$, then Beckmann's problem reads

$$\inf_{\Phi \in L^1(\Omega)^N} \left\{ \int_\Omega F(x, \Phi(x))\, \mathrm{d}x \ : \ -\operatorname{div}(\Phi) = \rho_2 - \rho_1 \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\}.$$

Moreover, thanks to Thereom 2.16 we have duality (see e.g.[16, 65, 67, 94] for details) with

$$\begin{aligned} &\min\left\{\iint_{\overline{\Omega}\times\overline{\Omega}} d_F(x,y)\,\mathrm{d}\mu : \pi_1\#\rho=\rho_1, \pi_2\#\rho=\rho_2\right\} \\ &=\max\left\{\int_{\overline{\Omega}} u\,\mathrm{d}(\rho_2-\rho_1)\ :\ u(y)-u(x)\leq d_F(x,y) \text{ for all } x\in\overline{\Omega}\right\}, \end{aligned} \tag{5.2}$$

where $d_F(x,y) := \inf_{\substack{\varphi\in \mathrm{Lip}([0,1],\overline{\Omega})\\ \varphi(0)=x,\,\varphi(1)=1}} \int_0^1 F(\varphi(t),\varphi'(t))\mathrm{d}t$. In the next section we show how to provide similar results from our formulation to the solution of (5.1).

5.2 Hamilton-Jacobi equation and Beckmann's problem

The connection between HJ equation coupled with a Dirichlet condition (5.1) and Beckmann's problem is not straightforward.

In the case where $H(x,p)=|p|-k(x)$ (i.e., the Eikonal equation $|\nabla u(x)|=k(x)$) complemented with Dirichlet boundary condition $u=g$ on $\partial\Omega$, the viscosity solution of (5.1) can be characterized through the following optimization problem

$$\max\left\{\int_{\Omega} u\,\mathrm{d}x\ :\ u(x)-u(y)\leq d_k(y,x) \text{ and } u=g \text{ on } \partial\Omega\right\},$$

where d_k is given by

$$d_k(x,y)=\inf\left\{\int_0^1 k(\varphi(t))\,|\varphi'(t)|\,\mathrm{d}t\ :\ \varphi\in\mathrm{Lip}([0,1],\overline{\Omega}),\ \varphi(0)=x,\ \varphi(1)=1\right\}.$$

Thanks to Kantorovich duality, it is meaningful to consider this problem as some kind of push-forward of $\rho_1 := \mathcal{L}^N \llcorner \Omega$. The offset is clearly connected to the lack of a measure ρ_2 which can fit out the problem with the balanced property. Actually, the linked Beckmann and Monge-Kantorovich problems aim to find moreover the optimal ρ_2 concentrated on $\partial\Omega$ which will consume $\rho_1 := \chi_\Omega$. More precisely, we will prove that the problem is connected to

$$\min_{\nu\in\mathcal{M}_b(\partial\Omega)}\max_u\left\{\int_\Omega u(\mathrm{d}x-\mathrm{d}\nu)-\int_{\partial\Omega} g\,\mathrm{d}\nu\ :\ u \text{ is } 1-\text{Lipschitz w.r.t } d_k\right\},$$

which provides the following modified Beckmann problem

$$\min_{\nu\in\mathcal{M}_b(\partial\Omega)}\inf_{\Phi\in L^1(\Omega)^N}\left\{\int_\Omega k(x)\,|\Phi|(x)\,\mathrm{d}x+\int_{\partial\Omega} g\,\mathrm{d}\nu:\ -\operatorname{div}(\Phi)=1-\nu\text{ in }\mathcal{D}'(\mathbb{R}^N)\right\}.$$

as well as to the following modified Monge-Kantorovich problem

$$\min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\nu\in\mathcal{M}_b(\partial\Omega)}\left\{\int_{\overline{\Omega}\times\overline{\Omega}} d_k(x,y)\mathrm{d}\gamma(x,y)+\int_{\partial\Omega} g\mathrm{d}\nu:\ \pi_1\sharp\gamma=\nu^+,\ \pi_2\sharp\gamma=1+\nu^-\right\}.$$

For the general case of degenerate Hamiltonian H, we will also obtain the corresponding results by means of the intrinsic distance d_σ associated to the Hamiltonian H (see (2.8)).

The main result of this chapter is the rigorous treatment for the case of degenerate Hamiltonian H and its degenerate intrinsic metric d_σ, as well as non-zero Dirichlet condition. Moreover, we also illustrate numerical examples.

Again, recall that we are considering the following HJ equation

$$\begin{cases} H(x,\nabla u)=0 & \text{in }\ \Omega\\ u=g & \text{on }\ D\end{cases}\tag{5.3}$$

a closed subset $D\subset\overline{\Omega}$ (typically $D=\partial\Omega$ or $D=\overline{\omega}$ for some $\omega\subset\subset\Omega$) and $g\colon D\to\mathbb{R}$ is a continuous function satisfying the compatibility condition

$$g(x)-g(y)\le d_\sigma(y,x)\qquad\text{for any } x,y\in D.$$

Thanks to Proposition 2.9, the unique maximal viscosity subsolution of the equation (5.3) can be recovered via the following maximization problem

$$\max\left\{\int_\Omega u\,\mathrm{d}x\ :\ u(x)-u(y)\le d_\sigma(y,x),\ \forall x,y\in\Omega\text{ and }u=g\text{ on }D\right\},\tag{5.4}$$

where $d_\sigma(.,.)$ is the intrinsic distance associated to the Hamiltonian defined by (2.8).

5.2.1 Main results

In order to prove the connection between (5.4) and a Beckmann-type problem, we will consider a slightly more general variant of (5.4) by considering for $\rho\in\mathcal{M}_b(\Omega)$, the following maximization problem

$$(\mathcal{M}_D):\ \max\left\{\int_\Omega u\,\mathrm{d}\rho\ :\ u(y)-u(x)\le d_\sigma(x,y),\ \forall x,y\in\Omega\text{ and }u=g\text{ on }D\right\}.$$

We can clearly see that the solution of (5.4) can be recovered by taking $\rho \equiv 1$.

The connection with the Beckmann problem is given in the following theorem.

Theorem 5.1. *The optimization problem $(\mathcal{M}_D)$ coincides with the following Beckmann-type problem*

$$(\mathcal{BK}) : \qquad \min_{\substack{\phi\in\mathcal{M}_b(\overline{\Omega})^N \\ \nu\in\mathcal{M}_b(D)}} \left\{ \int_{\overline{\Omega}} \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_D g\mathrm{d}\nu : \ -\operatorname{div}(\phi) = \rho - \nu \textit{ in } \mathcal{D}'(\mathbb{R}^N) \right\}.$$

Moreover, u and (ϕ, ν) are optimal solutions to $(\mathcal{M}_D)$ and $(\mathcal{BK})$, respectively, if and only if

$$\begin{cases} -\operatorname{div}(\phi) = \rho - \nu & \textit{in } \mathcal{D}'(\mathbb{R}^N) \\ \phi(x)\cdot\nabla_{|\phi|}u(x) = \sigma\Big(x, \frac{\phi}{|\phi|}(x)\Big) & |\phi| - \textit{a.e. } x \\ u = g & \textit{on } D, \end{cases}$$

where $\nabla_{|\phi|}u$ denotes the tangential gradient with respect to $|\phi|$, the total variation of ϕ (cf. [18]).

In particular, we have the following.

Corollary 5.2. *Let u be the maximal viscosity subsolution to (3.1) and (ϕ, ν) an optimal solution to $(\mathcal{BK})$ with $\rho = 1$, then*

$$\begin{cases} -\operatorname{div}(\phi) = 1 - \nu & \textit{in } \mathcal{D}'(\mathbb{R}^N) \\ \phi(x)\cdot\nabla_{|\phi|}u(x) = \sigma\Big(x, \frac{\phi}{|\phi|}(x)\Big) & |\phi| - a.e.\ x \\ u = g & \textit{on } D. \end{cases}$$

For the case of non-degenerate Finsler metric σ (i.e., $H(x,0) < 0$) we know that, by (5.2), the minimal value of $(\mathcal{BK})$ is the same with e have (see [36] for the particular case $\sigma = |.|$)

$$\min_{\phi\in\mathcal{M}_b(\Omega)^N} \left\{ \int_\Omega \sigma(x, \frac{\phi}{|\phi|}(x))d|\phi| : \ -\operatorname{div}(\phi) = \rho - \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\} = W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-)$$

the so called Monge-Kantorovich work between ν and ρ given by

$$W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-) = \min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})} \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_\sigma(x,y)\mathrm{d}\gamma(x,y) : \ \pi_1\sharp\gamma = \rho^- + \nu^+,\ \pi_2\sharp\gamma = \rho^+ + \nu^- \right\}.$$

Moreover, recall that the Robinstein-Kantorovich duality implies that

$$W_{d_\sigma}(\rho^- + \nu^+, \rho^+ + \nu^-) = \max_u \Big\{ \int_{\overline{\Omega}} ud(\rho - \nu) : \ u \text{ is } 1 - \text{Lipschitz w.r.t } d_\sigma \Big\}.$$

So, minimizing over all $\nu \in \mathcal{M}(D)$ leads to the study of the following problem

$$(\mathcal{M}_N) : \min_{\nu \in \mathcal{M}_b(D)} \max_u \Big\{ \int_{\overline{\Omega}} u\mathrm{d}(\rho - \nu) + \int_D g\mathrm{d}\nu : \ u \text{ is } 1 - \text{Lipschitz w.r.t } d_\sigma \Big\},$$

as well as Monge-Kantorovich problem

$$(\mathcal{MK}) : \min_{\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}), \nu \in \mathcal{M}_b(D)} \Big\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_\sigma(x,y)\mathrm{d}\gamma(x,y) + \int_D g\mathrm{d}\nu : \ \pi_1\sharp\gamma = \rho^- + \nu^+, \ \pi_2\sharp\gamma = \rho^+ + \nu^- \Big\}.$$

The following theorem ensures that the above-mentioned relations still hold true for the case of degenerate HJ equation, i.e., $H(x,0) \le 0$.

Theorem 5.3. *Under the assumptions (H1-H3), we have*

$$\max(\mathcal{M}_D) = \min(\mathcal{BK}) = \min(\mathcal{M}_N) = \min(\mathcal{MK}).$$

As a typical example we will consider an HJ equation of Eikonal type, coupled with a zero Dirichlet boundary condtion,

$$\begin{cases} |\nabla u(x)| = k(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, $H(x,p) = |p| - k(x)$ where k is a continuous, nonnegative function on $\overline{\Omega}$. In this case, the problem $(\mathcal{M}_D)$ can be rewritten as

$$\sup\Big\{ \int_\Omega u\mathrm{d}x : \ |\nabla u| \le k \ \text{ and } \ u = 0 \ \partial\Omega \Big\},$$

or

$$(\mathcal{P}) : \inf_{u\in V}\{\mathcal{F}(u) + \mathcal{G}(\Lambda u)\}$$

where $V = C^1(\Omega) \cap H_0^1(\Omega)$,

$$\mathcal{F}(u) = -\int_\Omega u\mathrm{d}x, \quad \Lambda u = \nabla u \quad \text{and} \quad \mathcal{G}(q) = \begin{cases} 0 & \text{if } |q| \le k \\ +\infty & \text{otherwise.} \end{cases}$$

For the case of non-degeneracy, i.e., $k(x) > 0$ on $\overline{\Omega}$, the Fenchel-Rockafellar duality, since the qualification conditions are satisfied (see e.g. [42, Theorem III 4.1]), gives

$$\sup\left\{\int_\Omega u\mathrm{d}x : |\nabla u| \le k \text{ and } u = 0 \text{ on } \partial\Omega\right\} = \min_{\phi\in\mathcal{M}_b(\overline{\Omega})^N}\left\{\int_{\overline{\Omega}} k\,\mathrm{d}|\phi| : -\operatorname{div}(\phi) = \rho \text{ in } \mathcal{D}'(\Omega)\right\}.$$

However, dealing with general degenerate Hamiltionians at least two difficulties arise. Firstly, the qualification conditions are not satisfied to apply directly the Fenchel-Rockafellar duality. Secondly, in this setting the problem $(\mathcal{BK})$ is not coercive, it follows that the existence of an optimal solution to $(\mathcal{BK})$ is not trivial. These two issues will be bypassed *via* approximation and optimal transport techniques.

Before ending this section, let us mention that thanks to the duality result Chapter 3, we have the following

Corollary 5.4. *The extremal values* $(\mathcal{BK})$ *and*

$$(\mathcal{OF}_D): \quad \inf_{\phi\in\mathcal{DM}^p(\Omega)}\left\{\int_\Omega \sigma(x,\phi(x))\mathrm{d}x - \langle\phi\cdot\mathbf{n}, g\rangle : \; -\mathrm{div}(\phi) = \rho \;\; in \;\; \mathcal{D}'(\Omega)\right\},$$

coincide.

The formulation of the problem $(\mathcal{OF}_D)$ as well as the definition of $\mathcal{DM}^p(\Omega)$ and further comments are recalled in Remark 5.8.

5.3 Proofs

5.3.1 Preparatory results

Let $\nu \in \mathcal{M}_b(D)$ satisfy $\rho(\Omega) = \nu(D)$. We define two functionals $\mathcal{T} : L^1(\Omega)^N \mapsto \mathbb{R}\cup\{+\infty\}$ and $\mathcal{E} : \mathrm{Lip}(\Omega) \mapsto \mathbb{R}\cup\{-\infty\}$ defined by:

$$\mathcal{T}: \quad \phi \mapsto \mathcal{T}(\phi) = \begin{cases} \int_\Omega \sigma(x,\phi(x))\mathrm{d}x & \text{if} - \operatorname{div}(\phi) = \rho - \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{E} :, \quad u \mapsto \mathcal{E}(u) = \begin{cases} \int_{\overline{\Omega}} u\mathrm{d}(\rho-\nu) & \text{if } \sigma^*(x,\nabla u(x)) \le 1 \text{ a.e. } x\in\Omega \\ -\infty & \text{otherwise.} \end{cases}$$

Lemma 5.5. *Assume that σ is a degenerate Finsler metric. Let $\mathcal{T}, \mathcal{E}$ be defined as above and $\rho(\Omega) = \nu(D)$. Then*

$$\inf_{\phi\in L^1(\Omega)^N} \mathcal{T}(\phi) = \sup_{u\in\mathrm{Lip}(\Omega)} \mathcal{E}(u).$$

Proof. The proof will be divided into two steps. We first prove for the case of non-degenerate Finsler metric σ, *i.e.*, there exist two positive constants K_1, K_2 such that

$$K_1|p| \leq \sigma(x,p) \leq K_2|p| \text{ for any } x \in \overline{\Omega}, p \in \mathbb{R}^N.$$

In this setting, due to the non-degeneracy of σ, the qualification conditions are satisfied and the result follows directly from the Fenchel-Rockafellar duality (see e.g. [42, Theorem III 4.1]). For the general case, we check at once that $\sup \mathcal{E} \leq \inf \mathcal{T}$ by taking u as a test function in the divergence constraint $-\operatorname{div}(\phi) = \rho - \nu$ in $\mathcal{D}'(\mathbb{R}^N)$. Therefore, it remains to prove that

$$\inf \mathcal{T} \leq \sup \mathcal{E}. \tag{5.5}$$

We now proceed by an approximation via non-degenerate Finsler metrics. For $n \in \mathbb{N}^*$ and $x \in \overline{\Omega}$, define

$$\sigma_n(x,p) := \max(\sigma(x,p), \frac{|p|}{n}) \text{ for every } p \in \mathbb{R}^N,$$

which establishes a sequence of non-degenerate Finsler metrics σ_n on $\overline{\Omega}$ satisfying

$$\frac{|p|}{n} \leq \sigma_n(x,p) \leq K|p| \text{ for some constant } K > 0.$$

Thanks to [35, Thereom 5.1], we have that $d_{\sigma_n} \to d_\sigma$ in the space of Finsler distances endowed with the topology induced by uniform convergence on compact subsets of $\overline{\Omega} \times \overline{\Omega}$. To prove the inverse inequality (5.5), let us introduce for $n \in \mathbb{N}^*$ the functionals

$$\mathcal{T}_n : L^1(\Omega)^N \mapsto \mathbb{R} \cup \{+\infty\}, \qquad \phi \mapsto \begin{cases} \int_\Omega \sigma_n(x,\phi(x))\mathrm{d}x & \text{if } -\operatorname{div}(\phi) = \rho - \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{E}_n : \operatorname{Lip}(\Omega) \mapsto \mathbb{R} \cup \{-\infty\}, \quad u \mapsto \begin{cases} \int_{\overline{\Omega}} u\mathrm{d}(\rho - \nu) & \text{if } \sigma_n^*(x, \nabla u(x)) \leq 1 \text{ a.e. } x \in \Omega \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from the non-degeneracy of σ_n and the first step of the proof that $\inf \mathcal{T}_n = \sup \mathcal{E}_n$. We are now in a position to show that $\sup \mathcal{E}_n \to \sup \mathcal{E}$ as $n \to \infty$. Let u_n be a maximizer for $\mathcal{E}_n$, i.e., $\sup \mathcal{E}_n = \int_{\overline{\Omega}} u_n \mathrm{d}(\rho - \nu)$ and $\sigma_n^*(x, \nabla u_n(x)) \leq 1$ a.e. in Ω. Fix $x_0 \in \Omega$. Since $\rho(\Omega) = \nu(D)$, we see that $u_n := u_n - u_n(x_0)$ is still a maximizer. Thus we can assume that $u_n(x_0) = 0$ for any n. Since

$$u_n(y) - u_n(x) \leq d_{\sigma_n}(x,y) \leq K|x-y| \text{ in } \Omega \times \Omega,$$

$\{u_n\}$ is equi-Lipschitz continuous. By Ascoli-Arzelà's theorem, there exists a Lipschitz function u such that, up to a subsequence, $u_n \rightrightarrows u$ uniformly in Ω. Since $d_{\sigma_n} \to d_\sigma$ as $n \to \infty$ we deduce that u is admissible for $\mathcal{E}$, i.e. $u(y) - u(x) \leq d_\sigma(x, y)$ in $\Omega \times \Omega$. Consequently

$$\lim_n(\sup \mathcal{E}_n) = \lim_n \int_{\overline{\Omega}} u_n \mathrm{d}(\rho - \nu) = \int_{\overline{\Omega}} u \mathrm{d}(\rho - \nu) \leq \sup \mathcal{E},$$

and

$$\inf \mathcal{T} \leq \lim_n(\inf \mathcal{T}_n) \leq \sup \mathcal{E}$$

as claimed in (5.5). □

Before ending up this subsection, we recall the notion of disintegration of measures which will be useful in the proof of existence of optimal solution to the Beckmann-type problem.

Theorem 5.6. *(cf. [94]) Let X, Y be locally compact metric spaces and $\pi \colon X \to Y$ a Borel map. For any $\eta \in \mathcal{M}_b^+(X)$ there exist a family of probability measures $(\eta_y)_{y \in Y}$ on X concentrated on $\pi^{-1}(\{y\})$ such that for any test function $u \in C(X)$, the mapping $y \mapsto \int_X u \mathrm{d}\eta_y$ is Borel measurable and*

$$\int_X u(x) \mathrm{d}\eta(x) = \int_Y \int_X u(x) \mathrm{d}\eta_y(x) \mathrm{d}\pi_\sharp \eta(y).$$

5.3.2 Proofs of the main results

We get started with the proof of Theorem 5.1 by the following result.

Proposition 5.7. *We have*

$$\max(\mathcal{M}_D) = \inf(\mathcal{BK}) = \inf \widetilde{(\mathcal{BK})},$$

where

$$\widetilde{(\mathcal{BK})} : \inf_{\phi \in L^1(\Omega)^N, \nu \in \mathcal{M}_b(D)} \left\{ \int_\Omega \sigma(x, \phi(x)) \mathrm{d}x + \int_D g \mathrm{d}\nu : -\mathrm{div}(\phi) = \rho - \nu \textit{ in } \mathcal{D}'(\mathbb{R}^N) \right\}.$$

Proof. First observe that

$$\max(\mathcal{M}_D) \leq \inf(\mathcal{BK}) \leq \inf \widetilde{(\mathcal{BK})}. \tag{5.6}$$

Indeed, take u satisfying $\sigma^*(x, \nabla u(x)) \leq 1$ a.e. x in Ω, $u = g$ on D as a test function in the divergence constraint $-\operatorname{div}(\phi) = \rho - \nu$ in $\mathcal{D}'(\mathbb{R}^N)$, we get

$$\int_\Omega u \mathrm{d}\rho = \int_\Omega \frac{\phi}{|\phi|} \cdot \nabla u \mathrm{d}|\phi| + \int_D g \mathrm{d}\nu \leq \int_\Omega \sigma\Big(x, \frac{\phi}{|\phi|}(x)\Big) \mathrm{d}|\phi| + \int_D g \mathrm{d}\nu.$$

This implies (5.6).

So, it is sufficient to show the duality between $(\mathcal{M}_D)$ and $(\widetilde{\mathcal{BK}})$, *i.e.* $\max(\mathcal{M}_D) = \inf\,(\widetilde{\mathcal{BK}})$. We use a perturbation technique as in [36]. Define on $C(D)$ the following functional

$$F : v \in C(D) \mapsto -\max_u \Big\{ \int_\Omega u \mathrm{d}\rho : u \in \operatorname{Lip}(\Omega),\, \sigma^*(x, \nabla u(x)) \leq 1,\ u+v = g \text{ on } D \Big\},$$

which is well-defined. Let us show that F is convex and l.s.c.. Consider $v_1, v_2 \in C(D)$ and set $v = tv_1 + (1-t)v_2$ for $t \in [0,1]$. Let $u_1, u_2 \in \operatorname{Lip}(\Omega)$ be two maximizers corresponding to v_1 and v_2 respectively, *i.e.* $\sigma^*(x, \nabla u_i(x)) \leq 1,\ u_i + v_i = g$ on D and

$$F(v_i) = -\int_\Omega u_i \mathrm{d}\rho \quad \text{for } i = 1, 2.$$

Define $u = tu_1 + (1-t)u_2$. It is evident that $u + v = g$ on D. And using the homogeneity of σ^*, we obtain $\sigma^*(x, \nabla u) \leq t\sigma^*(x, \nabla u_1) + (1-t)\sigma^*(x, \nabla u_2) \leq 1$ so that u is admissible for v. Finally, we get

$$F(v) \leq -\int_\Omega u \mathrm{d}\rho = t\Big(-\int_\Omega u_1 \mathrm{d}\rho\Big) + (1-t)\Big(-\int_\Omega u_2 \mathrm{d}\rho\Big) = tF(v_1) + (1-t)F(v_2),$$

which proves the convexity. For the lower semicontinuity, take a sequence $v_n \rightrightarrows v$ uniformly on D. For every $n \in \mathbb{N}$, consider a maximizer u_n corresponding to v_n such that $u_n + v_n = g$ on D, u_n are 1-Lipschitz w.r.t. d_σ (i.e., $u_n(y) - u_n(x) \leq d_\sigma(x, y)$ or equivalently, $\sigma^*(x, \nabla u_n(x)) \leq 1$ a.e. $x \in \Omega$), and

$$F(v_n) = -\int_\Omega u_n \mathrm{d}\rho.$$

Since $u_n(y) - u_n(x) \leq d_\sigma(x, y) \leq K|x - y|$, the functions u_n are equi-Lipschitz in the Euclidean distance. Since the sequence $\{v_n\}$ is convergent, it is bounded on D, and so is the sequence $\{u_n\}$. By Ascoli-Arzelà 's theorem, there exists a Lipschitz function u such that $u_n \rightrightarrows u$ uniformly in Ω as $n \to \infty$. It is clear that $u + v = g$ on D and

$u(y)-u(x) \leq d_\sigma(x,y)$, *i.e.* u is admissible for v. The lower semicontinuity is completed by

$$F(v) \leq -\int_\Omega u\mathrm{d}\rho = \lim_{n\to\infty} -\int_\Omega u_n \mathrm{d}\rho = \liminf_{n\to\infty} F(v_n).$$

Since F is convex and l.s.c., we have $F = F^{**}$, in particular $F(0) = F^{**}(0)$. Let us finish the proof by computing F^* and F^{**}. For any $\nu \in \mathcal{M}_b(D)$, we see that

$$\begin{aligned} F^*(\nu) &= \sup_{v\in\mathcal{C}(D)} \int_D v\mathrm{d}\nu - F(v) \\ &= \sup_{v\in\mathcal{C}(D),\, u\in\mathrm{Lip}(\Omega)} \left\{ \int_D v\mathrm{d}\nu + \int_\Omega u\mathrm{d}\rho : \ \sigma^*(x,\nabla u(x))) \leq 1,\ \ u+v=g \ \text{ on } \ D \right\} \\ &= \sup_{u\in\mathrm{Lip}(\Omega)} \left\{ \int_\Omega u\mathrm{d}\rho + \int_D (g-u)\mathrm{d}\nu : \ \sigma^*(x,\nabla u(x))) \leq 1,\ g-u\in C(D) \right\}. \end{aligned}$$

For any constant $c \in \mathbb{R}$ and a Lipschitz extension $\tilde{g}$ of g, one can see that $u := \tilde{g} + c$ is an admissible test function in the definition of $F^*(\nu)$ and

$$\int_\Omega u\mathrm{d}\rho + \int_D (g-u)\mathrm{d}\nu = c(\rho(\Omega) - \nu(D)) + \int_\Omega \tilde{g}\,\mathrm{d}\rho.$$

So, if $\rho(\Omega) \neq \nu(D)$, then $F^*(\nu) = +\infty$. This implies that

$$F^*(\nu) = \begin{cases} \sup_{u\in\mathrm{Lip}(\Omega)} \left\{ \int_\Omega u\mathrm{d}\rho + \int_D (g-u)\mathrm{d}\nu : \ \sigma^*(x,\nabla u(x))) \leq 1 \right\} & \text{if } \nu(D) = \rho(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Following Lemma 5.5, for any $\nu \in \mathcal{M}_b(D)$ such that $\nu(D) = \rho(\Omega)$, we have

$$\begin{aligned} F^*(\nu) &= \sup_{u\in\mathrm{Lip}(\Omega)} \left\{ \int_\Omega u\mathrm{d}\rho + \int_D (g-u)\mathrm{d}\nu : \quad \sigma^*(x,\nabla u(x))) \leq 1 \right\} \\ &= \int_D g\mathrm{d}\nu + \sup_{u\in\mathrm{Lip}(\Omega)} \left\{ \int_{\overline{\Omega}} u\mathrm{d}(\rho-\nu) : \quad \sigma^*(x,\nabla u(x))) \leq 1 \right\} \\ &= \int_D g\mathrm{d}\nu + \inf_{\phi\in L^1(\Omega)^N} \left\{ \int_\Omega \sigma(x,\phi(x))\mathrm{d}x : \ -\mathrm{div}(\phi) = \rho-\nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\} \\ &= \inf_{\phi\in L^1(\Omega)^N} \left\{ \int_\Omega \sigma(x,\phi(x))\,\mathrm{d}x + \int_D g\mathrm{d}\nu : \ -\mathrm{div}(\phi) = \rho-\nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\}. \end{aligned}$$

Consequently,

$$\max(\mathcal{M}_D) = -F(0) = -F^{**}(0) = -\sup_{\nu \in \mathcal{M}_b(D)} -F^*(\nu) = \inf \widetilde{(\mathcal{BK})}.$$

□

Remark 5.8. Following our approach in Chapter 3, it is possible to show that the optimal ν in $(\mathcal{BK})$ is somehow related to the trace of the optimal flow ϕ. For completeness let us recall briefly some of the main ingredients we used, and for a simpler presentation, consider the case where $D = \partial\Omega$. Then we prove in [56] the duality between $(\mathcal{M}_D)$ and

$$(\mathcal{OF}_D): \inf_{\phi \in \mathcal{DM}^p(\Omega)} \left\{ \int_\Omega \sigma(x, \phi(x))dx - \langle \phi \cdot \mathbf{n}, g \rangle : -\mathrm{div}\,(\phi) = \rho \text{ in } \mathcal{D}'(\Omega) \right\},$$

where we define for $1 \le p \le \infty$,

$$\mathcal{DM}^p(\Omega) := \left\{ \phi \in L^p(\Omega)^N \ : \ \mathrm{div}\, \phi =: \mu \in \mathcal{M}_b(\Omega) \right\},$$

endowed with the graph norm

$$\|\phi\|_{\mathcal{DM}^p(\Omega)} := \|\phi\|_{L^p(\Omega)} + |\mathrm{div}\, \phi|(\Omega).$$

The main interest of introducing such a space is to give a sense to the trace term $\phi \cdot \mathbf{n}$ which is not always defined for a general measure field ϕ. In particular, for any measure field $\phi \in \mathcal{DM}^p(\Omega)$, one can define a trace $\phi \cdot \mathbf{n}$ on $\partial\Omega$ as a linear form on $\mathrm{Lip}(\partial\Omega)$ such that

$$\langle \phi \cdot \mathbf{n}, \xi_{/\partial\Omega} \rangle = \int_\Omega \xi \, \mathrm{div}\phi + \int_\Omega \nabla\xi \cdot \phi, \qquad \text{for any } \xi \in \mathcal{C}^1(\overline{\Omega}).$$

One can see that, at least formally, ν plays the role of $-\phi \cdot \mathbf{n}$ in $(\mathcal{OF}_D)$.

Our aim now is to use the optimal mass transportation interpretation to prove that the inf in $(\mathcal{BK})$ is actually a min, *i.e.* the existence of optimal solution to the Beckmann-type problem $(\mathcal{BK})$.

Proposition 5.9. *There exist* $\nu \in \mathcal{M}_b(D)$ *and a vector measure* Φ *such that* $-\mathrm{div}(\Phi) = \rho - \nu$ *in* $\mathcal{D}'(\mathbb{R}^N)$ *and*

$$\inf(\mathcal{BK}) \le \int_{\overline{\Omega}} \sigma\left(x, \frac{\Phi}{|\Phi|}\right) \mathrm{d}|\Phi| + \int_D g \mathrm{d}\nu \le \min(\mathcal{MK}).$$

In particular we see that $\min(\mathcal{BK}) = \max(\mathcal{M}_D)$.

Proof. Take (γ, ν) be a solution of $(\mathcal{MK})$ and define a vector measure Φ_γ through

$$< \Phi_\gamma, v >= \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 v(\xi(t))\dot{\xi}(t)\mathrm{d}t\mathrm{d}\gamma(x,y) \quad \forall v \in C(\overline{\Omega})^N,$$

with ξ being a geodesic joining x and y with respect to d_σ. Let us check the feasibility of Φ_γ for $(\mathcal{BK})$, *i.e.*

$$-\operatorname{div}(\Phi_\gamma) = \rho - \nu \text{ in } \mathcal{D}'(\mathbb{R}^N). \tag{5.7}$$

For any $w \in C^1(\overline{\Omega})$, by definition, we have

$$\begin{aligned} < \Phi_\gamma, \nabla w > &= \int_{\Omega\times\Omega} \int_0^1 \frac{dw(\xi(t))}{dt} \mathrm{d}t\mathrm{d}\gamma(x,y) \\ &= \int_{\Omega\times\Omega} (w(y) - w(x))\mathrm{d}\gamma(x,y) \\ &= \int_{\overline{\Omega}} w\mathrm{d}(\rho - \nu), \end{aligned}$$

which gives (5.7). The next task is to show that

$$\int_{\overline{\Omega}} \sigma(x, \frac{\Phi_\gamma}{|\Phi_\gamma|}(x))\mathrm{d}|\Phi_\gamma| + \int_D g\mathrm{d}\nu \le \int_{\overline{\Omega}\times\overline{\Omega}} d_\sigma(x,y)\mathrm{d}\gamma(x,y) + \int_D g\mathrm{d}\nu.$$

To do so, for each $t \in [0,1]$, define vector measure E_t by setting $E_t(v) := \int_{\overline{\Omega}\times\overline{\Omega}} v(\xi(t))\dot{\xi}(t)\mathrm{d}\gamma(x,y)$ for $v \in C(\overline{\Omega})^N$. We get $\Phi_\gamma = \int\limits_0^1 E_t \mathrm{d}t$ and, by Jensen's inequality,

$$\int_{\overline{\Omega}} \sigma(x, \frac{\Phi_\gamma}{|\Phi_\gamma|}(x))\mathrm{d}|\Phi_\gamma| \le \int_0^1 \int_{\overline{\Omega}} \sigma(x, \frac{E_t}{|E_t|}(x))\mathrm{d}|E_t|\mathrm{d}t. \tag{5.8}$$

Now define $\pi_t : (x,y) \in \overline{\Omega} \times \overline{\Omega} \mapsto \xi(t)$ for $t \in [0,1]$, where ξ is as before, a geodesic joining x and y. Consider $\eta_t = (\pi_t)_\sharp\gamma$. Using disintegration theorem (see Theorem 5.6) for γ with respect to π_t, we can find probability measures γ_z supported on $\pi_t^{-1}(\{z\})$ such that

$$\int_{\overline{\Omega}\times\overline{\Omega}} u(x,y)\mathrm{d}\gamma(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} u(x,y)\mathrm{d}\gamma_z(x,y)\mathrm{d}\eta_t(z). \tag{5.9}$$

We check at once that $E_t << \eta_t$ with the density $\frac{dE_t}{d\eta_t}(z) = \int_{\overline{\Omega}\times\overline{\Omega}} \dot{\xi}(t)\mathrm{d}\gamma_z(x,y)$, which follows from the fact that, for any test function v,

$$\int_{\overline{\Omega}} v(z)\mathrm{d}E_t = \int_{\overline{\Omega}} v(\xi(t))\dot{\xi}(t)\mathrm{d}\gamma(x,y)$$

$$
\begin{aligned}
&= \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} v(\xi(t))\dot{\xi}(t)\mathrm{d}\gamma_z(x,y)\mathrm{d}\eta_t(z) \\
&= \int_{\overline{\Omega}} v(z) \int_{\overline{\Omega}\times\overline{\Omega}} \dot{\xi}(t)\mathrm{d}\gamma_z(x,y)\mathrm{d}\eta_t(z).
\end{aligned}
$$

On the other hand, by (5.9) and Jensen's inequality,

$$
\begin{aligned}
\int_{\overline{\Omega}\times\overline{\Omega}} \sigma(\xi(t),\dot{\xi}(t))\mathrm{d}\gamma(x,y) &= \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} \sigma(\xi(t),\dot{\xi}(t))\mathrm{d}\gamma_z(x,y)\mathrm{d}\eta_t(z) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} \sigma(z,\dot{\xi}(t))\mathrm{d}\gamma_z(x,y)\mathrm{d}\eta_t(z) \\
&\geq \int_{\overline{\Omega}} \sigma\Big(z, \int_{\overline{\Omega}\times\overline{\Omega}} \dot{\xi}(t)\mathrm{d}\gamma_z(x,y)\Big)\mathrm{d}\eta_t(z) \\
&= \int_{\overline{\Omega}} \sigma(z, \frac{E_t}{|E_t|}(z))\mathrm{d}|E_t|(z).
\end{aligned}
\tag{5.10}
$$

Finally, we observe that since ξ is a geodesic

$$
\begin{aligned}
\int_{\overline{\Omega}\times\overline{\Omega}} d_\sigma(x,y)\mathrm{d}\gamma(x,y) + \int_D g\mathrm{d}\nu &= \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \sigma(\xi(t),\dot{\xi}(t))\mathrm{d}\gamma(x,y)dt + \int_D g\mathrm{d}\nu \\
&\geq \int_0^1 \int_{\overline{\Omega}} \sigma(z, \frac{E_t}{|E_t|}(z))\mathrm{d}|E_t|(z) + \int_D g\mathrm{d}\nu \ \ \text{(by (5.10))} \\
&\geq \int_{\overline{\Omega}} \sigma(x, \frac{\Phi_\gamma}{|\Phi_\gamma|}(x))\mathrm{d}|\Phi_\gamma| + \int_D g\mathrm{d}\nu \ \ \text{(by (5.8))}.
\end{aligned}
$$

Consequently

$$
\inf(\mathcal{BK}) \leq \int_{\overline{\Omega}} \sigma(x, \frac{\Phi_\gamma}{|\Phi_\gamma|}(x))\mathrm{d}|\Phi_\gamma| + \int_D g\mathrm{d}\nu \leq \int_{\overline{\Omega}\times\overline{\Omega}} d_\sigma(x,y)\mathrm{d}\gamma(x,y) + \int_D g\mathrm{d}\nu = \min(\mathcal{MK}).
$$

□

Proof of Theorem 5.3. By Kantorovich duality, one has $\max(\mathcal{M}_N) = \min(\mathcal{MK})$. Combining this with Propositions 5.7 and 5.9, we conclude that the Beckmann-type problem $(\mathcal{BK})$ admits an optimal solution and $\max(\mathcal{M}_D) = \min(\mathcal{BK}) = \max(\mathcal{M}_N) = \min(\mathcal{MK})$.

□

Proof of Theorem 5.1. The proof of the duality between $\max(\mathcal{M}_D)$ and $\inf(\mathcal{BK})$ is followed from Proposition 5.7 while the existence of optimal solution to the Beckmann-type problem $(\mathcal{BK})$ is a consequence of Proposition 5.9 (see also Theorem 5.3). Let us now show the optimality conditions. Indeed, u and (ϕ, ν) are optimal solutions for $(\mathcal{M}_D)$ and $(\mathcal{BK})$, respectively, if and only if

$$\int_\Omega \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_D g\mathrm{d}\nu = \int_\Omega u\mathrm{d}\rho,$$

or

$$\int_\Omega \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| = \int_{\bar\Omega} u\mathrm{d}(\rho - \nu) = \int_\Omega \phi(x) \cdot \nabla_{|\phi|} u(x)\mathrm{d}|\phi|.$$

This is equivalent to

$$\sigma(x, \frac{\phi}{|\phi|}(x)) = \phi(x) \cdot \nabla_{|\phi|} u(x) \quad \text{for } |\phi|\text{-a.e. } x,$$

as desired.

□

5.4 Numerical results

As we pointed out in the previous sections, the maximization problem $(\mathcal{M}_D)$ is linked to the Monge-Kantorovich type problem $(\mathcal{MK})$. The measure ρ needs not to satisfy the standard mass balance condition. However, transporting a part of the mass from/to the Dirichlet region D is allowed. In addition, taking $\rho = \mathcal{L}^N_{|\Omega}$, $(\mathcal{M}_D)$ allows recovering the solution of HJ equation. This was the content of Chapter 3. Here, we will focus essentially on the solution of $(\mathcal{BK})$.

5.4.1 Formulation of the problem

We set $\mathscr{X} = W^{1,\infty}(\Omega)$ and $\mathscr{Y} = L^\infty(\Omega)^N \times \mathcal{C}(D)$, with

$$\mathcal{F}(u) = -\int_{\bar\Omega} u \,\mathrm{d}\rho, \qquad \Lambda u = (\nabla u, u_{|D}), \qquad \text{for any } u \in \mathscr{X}$$

and

$$G(\eta, h) = \begin{cases} 0 & \text{if } \sigma^*(x, \eta) \le 1 \text{ and } h = g \text{ on } D \\ +\infty & \text{otherwise,} \end{cases} \qquad \text{for any } (\eta, h) \in \mathscr{Y}.$$

Thus, we can rewrite the problem $(\mathcal{M}_D)$ in the form

$$-\inf_{u \in \mathscr{X}} \mathcal{F}(u) + \mathcal{G}(\Lambda u).$$

Thanks to Theorem (5.3) and Proposition (5.7), we have

$$-\min\Big\{\mathcal{F}(u)+\mathcal{G}(\Lambda u) : u\in\mathscr{X}\Big\} = -\sup\Big\{-\mathcal{F}^*(-\Lambda^*(\phi,\nu))-\mathcal{G}^*(\phi,\nu)\phi\in L^1(\Omega)^N,\nu\in\mathcal{M}_b(D)\Big\}$$

$$=\min_{\phi\in L^1(\Omega)^N,\nu\in\mathcal{M}_b(D)}\Big\{\int_\Omega\sigma(x,\phi(x))\mathrm{d}x+\int_D g\mathrm{d}\nu:\ -\mathrm{div}\,(\phi)=\rho-\nu\ \text{in}\ \mathcal{D}'(\mathbb{R}^N)\Big\}.$$

Introducing a new primal variable $(p,q)\in\mathscr{Y}$, $(\mathcal{M}_D)$ reads

$$-\inf_{\substack{(u,(p,q))\in\mathscr{X}\times\mathscr{Y}\\ \Lambda u=(p,q)}}\mathcal{F}(u)+\mathcal{G}(p,q).$$

This allows us to rewrite $(\mathcal{M}_D)$ and $(\mathcal{BK})$ in a saddle point form

$$(\mathscr{S}):\ \min_{(u,(p,q))\in\mathscr{X}\times\mathscr{Y}}\ \sup_{\phi\in L^1(\Omega)^N,\nu\in\mathcal{M}_b(D)} L(u,(p,q);\phi,\nu)$$

where

$$L(u,(p,q);\phi,\nu)=\mathcal{F}(u)+\mathcal{G}(p,q)+\int_\Omega\phi\cdot(\nabla u-p)\,\mathrm{d}x+\int_D\nu\cdot(u-q)\mathrm{d}x.$$

As usual, it is convenient to consider the following augmented Lagrangian

$$L_r(u,(p,q);\phi,\nu)=L(u,(p,q);\phi,\nu)+\frac{r}{2}|\nabla u-p|^2+\frac{r}{2}|u_{|D}-q|^2,\ r>0,$$

which has the same saddle points as L. Thus, the problem we will focus on is

$$(\mathscr{S}_r):\ \min_{(u,q)\in\mathscr{X}\times\mathscr{Y}}\ \sup_{\phi\in L^1(\Omega)^N,\nu\in\mathcal{M}_b(D)} L_r(u,(p,q);\phi,\nu).$$

The main difference with the above formulation and the one in Section 3.3 is the presence of an additional Lagrange multiplier ν corresponding to the boundary condition $u=g$ on D. The resolution of $(\mathcal{M}_D)$ was used via ALG2 algorithm and was implemented using finite element method. Up to our knowledge, it is not straightforward to define finite element functions ν on a closed subset D (typically $D=\partial\Omega$). In the next section we explain how to tackle this difficulty.

5.4.2 Practical implementation

As we pointed out in the previous subsection, it is not clear how to solve the saddle point problem $(\mathscr{S}_r)$ via ALG2 algorithm due to the presence of the measure ν on the Dirichlet region. However, thanks to Corollary 5.4 and Remark 5.8 which shows that ν is somehow linked to the trace of the optimal flow ϕ on D, we can restrict ourselves to a formulation involving only the potential u and the flow ϕ. More precisely, for any $u \in \mathscr{X}$ and $\eta \in Z = L^\infty(\Omega)$, we take

$$\mathcal{F}(u) = \begin{cases} -\int_\Omega u \, \mathrm{d}\rho & \text{if } u = g \text{ on } D \\ +\infty & \text{otherwise} \end{cases}, \quad \mathcal{G}(\eta) = \begin{cases} 0 & \text{if } \sigma^*(x,\eta) \leq 1 \\ +\infty & \text{otherwise} \end{cases}, \quad \text{and } \Lambda u = \nabla u.$$

Thus, following our approach in [56], we can focus on the following saddle point problem

$$\inf_{(u,q)\in \mathscr{X}\times Z} \sup_{\phi \in \mathcal{DM}^p(\Omega\backslash D)} L(u,q;\phi)$$

where

$$L(u,q;\phi) = \mathcal{F}(u) + \mathcal{G}(q) + \int_\Omega \phi\cdot(\Lambda(u)-q)dx, \quad \text{for any } (u,q,\phi) \in \mathscr{X}\times Z\times \mathcal{DM}^p(\Omega\backslash D).$$

Hence, the augmented Lagrangian reads

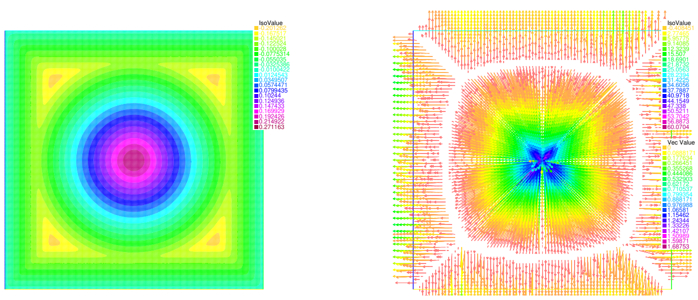
$$L_r(u,q;\phi) = \mathcal{F}(u) + \mathcal{G}(q) + \langle \phi, \Lambda u - q\rangle + \frac{r}{2}|\Lambda u - q|^2, \quad \text{for any } (u,q,\phi) \in \mathscr{X}\times Z\times \mathcal{DM}^p(\Omega\backslash D).$$

5.4.3 Some examples

We take $\Omega = [0,1]\times[0,1]$ and $g = 0$ on $\partial\Omega$. The first three examples are performed with a Finsler distance d_σ of Riemannian type

$$d_\sigma(x,y) = \inf_{\substack{\zeta \in \mathrm{Lip}([0,1];\overline{\Omega}) \\ \zeta(0)=x, \zeta(1)=y}} \int_0^1 k(\zeta(t))|\dot{\zeta}(t)|\mathrm{d}t.$$

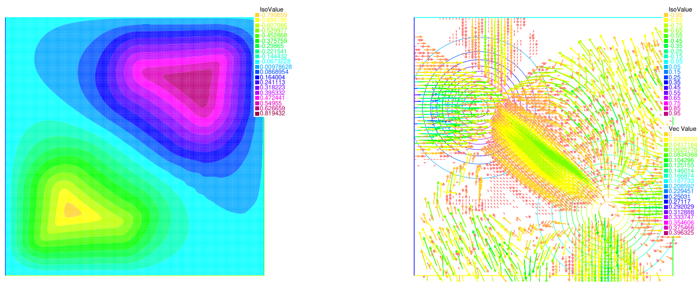
For the first test we take $k(x,y) = 1$ and $\rho^+ = 2$, $\rho^- = \delta_{(0.5,0.5)}$.



(a) (b)

Figure 5.1: (A): the potential u, (B): the flow ϕ.

In the second test we take $\rho^+ = 4\chi_{[(x-0.3)^2+(y-0.2)^2<0.03]}$ and $\rho^- = 4\chi_{[(x-0.7)^2+(y-0.8)^2<0.03]}$ and $k(x,y) = 5 - 3e^{-2*((x-0.5)^2+(y-0.5)^2)}$.



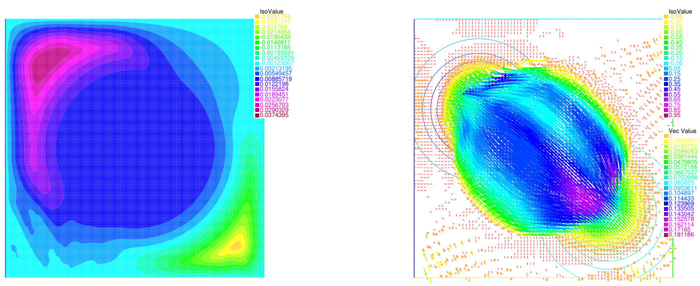
(a) (b)

Figure 5.2: (A): the potential u, (B): the flow ϕ.

In the third test we take two Gaussian densities $\rho^+ = e^{-40*((x-0.75)^2+(y-0.3)^2)}$ and $\rho^- = e^{-40*((x-0.3)^2+(y-0.65)^2)}$. We change the distance d_σ by taking a degenerate k. In particular, we choose

$$k(x,y) = \sqrt{(1-2x)^2(y-y^2)^2 + (1-2y)^2(x-x^2)^2}\chi_{\mathcal{B}}$$

where $\mathcal{B} = \{(x,y) \in [0,1]^2 : \sqrt{(x-0.5)^2+(y-0.5)^2} > 0.25\}$. We clearly see that the flux is concentrated essentially on the region where k vanishes, *i.e*, on $\mathcal{B}^c$, which in terms of optimal transport with respect to d_σ represents a free transport region.



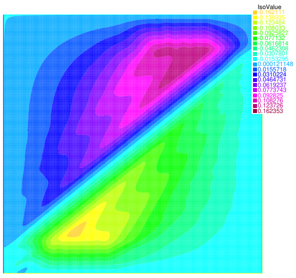
(a) (b)

Figure 5.3: (A): the potential u, (B): the flow ϕ privileging the zero set of k .

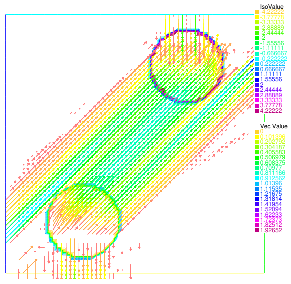
In the last test we consider a Finsler metric of cystalline type, namely

$$\sigma(v) = \max_{i=1,\cdots,5} v \cdot d_i$$

with $d_1 = (1,-1)$, $d_2 = (1,-0.8)$, $d_3 = (-0.8,1)$, $d_4 = (-1,1)$, $d_5 = (-1,-1)$.



(a)



(b)

Figure 5.4: (A): the potential u, (B): the flow ϕ .

5.5 Comments and extentions

A natural extension one can think of is the HJ equation with double obstacles on the boundary. More precisely one can consider the equation

$$H(x, \nabla u) = 0 \text{ on } \Omega,\ g_1 \leq u \leq g_2 \text{ on } \partial\Omega$$

where $g_i : \partial\Omega \to \mathbb{R}$ are continuous functions satisfying the compatibility condition $g_1(x) - g_2(y) \leq d_\sigma(y, x)$ for any $x, y \in \partial\Omega$.

In order to establish the link between this problem and a Bekmann-type problem, we consider as previously the following maximization problem

$$(\mathcal{M}_D)_o : \max\left\{ \int_\Omega u \,\mathrm{d}\rho \ : \ u \in W^{1,\infty}(\Omega),\ \sigma^*(x, \nabla u) \leq 1 \text{ and } g_1 \leq u \leq g_2 \text{ on } \partial\Omega \right\}.$$

Similarly, we can state the following result.

Theorem 5.10. *The optimization problem* $(\mathcal{M}_D)_o$ *coincides with the following Beckmann-type problem, denoted by* $(\mathcal{BK})_o$

$$\min_{\substack{\phi \in \mathcal{M}_b(\overline{\Omega})^N \\ \nu \in \mathcal{M}_b(\partial\Omega)}} \left\{ \int_\Omega \sigma(x, \frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_{\partial\Omega} g_2 \mathrm{d}\nu^+ - \int_{\partial\Omega} g_1 \mathrm{d}\nu^- : -\operatorname{div}(\phi) = \rho - \nu \ \mathit{in}\ \mathcal{D}'(\mathbb{R}^N) \right\}.$$

Moreover, u and (ϕ,ν) are optimal solutions to $(\mathcal{M}_D)_o$ and $(\mathcal{BK})_o$, respectively if and only if

$$\begin{cases} -\operatorname{div}(\phi) = \rho - \nu & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \phi(x)\cdot\nabla_{|\phi|}u(x) = \sigma\Big(x, \frac{\phi}{|\phi|}(x)\Big) & \text{for } |\phi| - a.e.\ x \\ g_1 \le u \le g_2 \text{ on } \partial\Omega & \text{and } u = g_1 \text{ for } \nu^- - \text{ a.e. } x \text{ and } u = g_2 \text{ for } \nu^+ - \text{ a.e. } x. \end{cases}$$

Sketch of proof. We define on $C(\partial\Omega)\times C(\partial\Omega)$ the following functional on $C(\partial\Omega)\times C(\partial\Omega)$ by, for $(v,w)\in C(\partial\Omega)\times C(\partial\Omega)$,

$$E(v,w) = -\sup\Big\{\int_\Omega u\mathrm{d}\rho : u\in \mathrm{Lip}(\Omega),\ \sigma^*(x,\nabla u(x))\le 1,\ g_1\le u+v,\ u+w\le g_2 \text{ on } \partial\Omega\Big\}.$$

Most of the arguments of Section 5.3 can be reproduced to show that E is convex and lower semicontinuous, which gives $E(0,0)=E^{**}(0,0)$. It follows that $\max(\mathcal{M}_D)_o = \min(\mathcal{BK})_o$. We now turn to the optimality conditions. First observe that, for any feasible u and (ϕ,ν), we get

$$\begin{aligned} \int_{\overline{\Omega}} u\mathrm{d}\rho &= \int_\Omega \nabla_{|\phi|}u(x)\frac{\phi}{|\phi|}(x)\mathrm{d}|\phi| + \int_{\overline{\Omega}} u\mathrm{d}\nu \\ &\le \int_\Omega \sigma(x,\frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_D g_2\mathrm{d}\nu^+ - \int_{\partial\Omega} g_1\mathrm{d}\nu^-, \end{aligned} \tag{5.11}$$

where we have used $\nabla_{|\phi|}u(x)\phi(x) \le \sigma(x,\frac{\phi}{|\phi|}(x))$ for $|\phi|-$ a.e. x by the fact that u is 1-Lipschitz w.r.t. d_σ.

On the other hand, u and (ϕ,ν) are optimal for $(\mathcal{M}_D)_o$ and $(\mathcal{BK})_o$ respectively, if and only if

$$\int_\Omega u\mathrm{d}\rho = \int_\Omega \sigma(x,\frac{\phi}{|\phi|}(x))\mathrm{d}|\phi| + \int_{\partial\Omega} g_2\mathrm{d}\nu^+ - \int_{\partial\Omega} g_1\mathrm{d}\nu^-,$$

i.e. the equality holds in (5.11), which is equivalent to the system of optimality conditions as desired. □

Let us mention that problems of the form $(\mathcal{M}_D)_o$ and $(\mathcal{BK})_o$ can arise when studying optimal transport problems with some import/export costs on the boundary. In the case where the transport cost is given by the Euclidean distance $c(x,y)=|x-y|$ (in our case when considering the Eikonal equation $|\nabla u|=1$) we refer the reader to [82]. In the next chapter, we will discuss a PDE approach for $(\mathcal{M}_D)_o$.

6 HJ equation and Finsler p-Laplace approximations

6.1 Introduction and remainders

6.1.1 A short survey on OHJ equation

This chapter is concerned with a p-Laplace approach à la Evans-Gangbo [44] to a general class of the transport problems studied in Chapter 5. As we pointed out in 5.5, we shall consider the following HJ equation with double obstacles on the boundary:

$$H(x, \nabla u) = 0 \text{ on } \Omega, \ \phi \leq u \leq \psi \text{ on } \partial\Omega, \tag{6.1}$$

where H is a nondegenerate convex Hamiltonian and ϕ, ψ are continuous functions satisfying

$$\phi(x) - \psi(y) \leq d_H(y, x), \ \forall x, y \in \partial\Omega, \tag{6.2}$$

where d_H is the intrinsic metric associated to the Hamiltonian H. Let us recall that (6.2) is a necessary and sufficient condition for the existence of subsolution.

Before presenting in details our problem, let us say few words about some existing works (essentially close to our setting) dealing with HJ equation with obstacles.

Metric approaches:

In [23], Camilli et al. studied systems of first order HJ equation with implicit obstacles. Amongst their main results, a representation formula for the following OHJ equation

$$\max\{H(x, \nabla u), u - \varphi\} = 0 \ \text{ in } \ \Omega, \tag{6.3}$$

coupled with Dirichlet boundary condition

$$u = g \ \text{ on } \ \partial\Omega, \tag{6.4}$$

where $\varphi : \overline{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \cup \mathcal{A} \to \mathbb{R}$ are continuous functions. More precisely, they show that when φ, g satisfy (2.9), then the unique solution of (6.3)-(6.4) such that $u = \min\{\varphi, g\}$ on $\partial\Omega \cup \mathcal{A}$ is given by

$$u(x) = \min\left\{ \min_{\partial\Omega\cup\mathcal{A}} \{d_H(y,x) + g(y)\}, \min_{\overline{\Omega}} \{d_H(y,x) + \varphi(y)\} \right\}.$$

We observe here that the equation (6.3) can be rewritten in the form

$$F(x, u, \nabla u) = 0 \text{ in } \Omega,$$

where $F(x,t,p) = \max\{H(x,p), t - \varphi\}$. The 0-sublevels of F are convex, and thus (6.3)-(6.4) can be treated using the techniques of [78, 80]. In contrast with (6.3), N.Igbida proposes in [64] a metric formula to handle the obstacle in the following OHJ equation

$$\begin{cases} \min\{H(x, \nabla u), u - \varphi\} = 0 \text{ in } \Omega \\ \qquad\qquad\qquad\qquad u = \varphi \text{ in } \partial\Omega \end{cases} \tag{6.5}$$

where φ is a continuous function, not necessarily satisfying (2.9). Note that in this case, the Hamiltonian is of the form $F(x,t,p) = \min\{H(x,p), t - \varphi\}$, and its 0-sublevels are not convex in general. So that the metric machinery in [24, 48, 78, 80] doesn't allow providing representation formulas of such equations. We will come back to this result in details in Chapter 7.

Penalization techniques:

Penalization techniques remain a natural approach to deal with obstacle problems for general PDEs. For a HJ equation of the form

$$\max\{F(x, u, \nabla u), u - \varphi\} = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega, \tag{6.6}$$

it is convenient to consider the following penalized PDE

$$-\epsilon\Delta u_\epsilon + F(x, u_\epsilon, \nabla u_\epsilon) + \alpha_\epsilon(u_\epsilon - \varphi) = 0 \text{ in } \Omega, \text{ and } u_\epsilon = 0 \text{ on } \partial\Omega, \tag{6.7}$$

where α_ϵ is a smooth function approximating the monotone graph

$$\alpha(s) = \begin{cases} 0 \text{ if } s < 0 \\ [0, \infty[\text{ if } s = 0 \\ \emptyset \text{ if } s > 0. \end{cases}$$

Then, the standard techniques in [71, 80] allow deriving appropriate estimates on u_ϵ and by passing to the limit as $\epsilon \to 0$, to recover a viscosity solution of (6.6). In this framework, we can mention for example the work of Gagnetti et al.[21] where the authors considered the following obstacle problem

$$\max\{u + H(x, \nabla u), u - \varphi\} = 0 \text{ in } \Omega, \tag{6.8}$$

coupled with homogeneous Dirichlet boundary condition, where $\varphi : \Omega \to \mathbb{R}$ is a smooth function satisfying $\varphi \geq 0$ on $\partial\Omega$ and $H : \mathbb{R}^n \times \overline{\Omega} \to \mathbb{R}$ is a smooth coercive Hamiltonian. Moreover, they assume the existence of a strict subsolution of (6.8), i.e, a function $\mathbf{w} \in C^1(\overline{\Omega}) \times C^2(\Omega)$ such that $\mathbf{w} = 0$ on $\partial\Omega$, $\mathbf{w} \leq \varphi$ and

$$\mathbf{w} + H(x, \nabla \mathbf{w}) < 0 \text{ in } \overline{\Omega}.$$

They studied a version of (6.7) with $F(x, u, \nabla u) = u + H(x, \nabla u)$ and a convex penalization function α_ϵ. They make use of the adjoint method [2] (see [45]) to derive some useful estimates, with a particular attention to the term $\partial_\epsilon \alpha_\epsilon(s)$. This allows them to show that u_ϵ converges uniformly towards a solution u of (6.8) and

$$\|u_\epsilon - u\|_{L^\infty} \leq C\epsilon^{\frac{1}{2}},$$

where C is a positive constant not depending on ϵ. They also extend their approach to the case of coupled systems. We were able to prove similar results for

$$\min\{u + H(x, \nabla u), u - \varphi\} = 0 \text{ in } \Omega, \tag{6.9}$$

by considering the following penalized PDE

$$-\epsilon\Delta u_\epsilon + u_\epsilon + H(x, \nabla u_\epsilon) + f\alpha_\epsilon(u_\epsilon - \varphi) = f \text{ in } \Omega, \text{ and } u_\epsilon = 0 \text{ on } \partial\Omega,$$

where f is a positive Lipschitz function and $\alpha_\epsilon(s) = \alpha(s/\epsilon)$ with α a smooth function verifying

$$\alpha^{'} \geq 0, \ \alpha(s) = 0 \text{ for } s \leq 0 \text{ and } \alpha \to 1 \text{ as } t \to \infty.$$

However, it is not clear to us how to obtain similar results both for (6.8)-(6.9) without a zero order term, i.e, when the Hamiltonians is of the form $F(x, u, \nabla u) = H(x, \nabla u)$.

To end this section, let us mention that in addition to optimal stopping problems, obstacle HJ equations arise typically in the vanishing viscosity limit of HJ equation with incompatible boundary conditions (see e.g [80, Chapter 6]).

[2] The method consists essentially in considering a linearization of (6.7) and then study the adjoint of the linearization operator.

In this chapter, we are more concerned with HJ equation with double obstacles on the boundary (6.1). We will come back to obstacle problems of the form (6.5) in the next chapter.

Recall that our starting point is a problem of the form

$$H(x, \nabla u) = 0 \text{ on } \Omega,\ \phi \leq u \leq \psi \text{ on } \partial\Omega.$$

Thanks to Theorem 3.2, the maximal viscosity subsolution of (6.1) can be recovered via the following maximization problem

$$\max\Big\{\int_\Omega u\mathrm{d}x\,:\ \sigma^*(x, \nabla u) \leq 1 \text{ a.e.},\ \phi \leq u \leq \psi \text{ on } \partial\Omega\Big\}, \tag{6.10}$$

where σ^* is the dual of the support function of the 0-sublevel sets of the Hamiltonian H. Considering $\int_\Omega u\mathrm{d}\rho$ in (6.10) instead of $\int_\Omega u\mathrm{d}x$ for some $\rho \in L^2(\Omega)$, this problem can easily be linked to a mass transport problem with boundary costs. Our aim is to construct a solution of this (6.10) as well as the optimal flow of the associated Beckmann problem using the Evans-Gagbo machinery. Before presenting the problem and the main ingredients, let us give a broad overview on this p-Laplace approach.

6.1.2 Reminders on the Evans-Gangbo approach

Given a Borel measure ρ with $\int_\Omega \mathrm{d}\rho = 0$, and consider the Kantorovich problem

$$\min_{\gamma\in\mathcal{M}^+(\Omega\times\Omega)}\Big\{\int_{\Omega\times\Omega}|x-y|\mathrm{d}\gamma(x,y)\,:\ (\pi_x)_\sharp\gamma = \rho^+, (\pi_y)_\sharp\gamma = \rho^-\Big\}, \tag{6.11}$$

as well as the associated Kantorovich–Rubinstein problem

$$\max\Big\{\int_\Omega u\mathrm{d}\rho\,:\ u \in \mathrm{Lip}_1(\Omega)\Big\}. \tag{6.12}$$

In the case where $\rho \in L^1(\Omega)$, the optimal solution of (6.12), usually called Kantorovich potential, can be obtained via PDE method in the celebrated paper of Evans-Gangbo [44]. More precisely, they prove that when $\mathrm{supp}(\rho^+)\cap\mathrm{supp}(\rho^-) = \emptyset$, the solution u of (6.12) is the uniform limit of u_p, where u_p is the solution of

$$\begin{cases} -\Delta_p u_p &= \rho \quad \text{in } B(0,R) \\ u_p &= 0 \quad \text{on } \partial B(0,R), \end{cases}$$

for some large $R > 0$. Moreover, there exists a function $\sigma \in L^1(\Omega)$ such that (σ, u) solves the following Monge–Kantorovich system

$$\begin{cases} -\mathrm{div}(\sigma\nabla u) = \rho & \text{in } \Omega \\ |\nabla u| \leq 1 & \text{in } \Omega \\ |\nabla u| = 1 & \sigma - \text{ a.e.} \end{cases}$$

and the vector measure $\Phi = \sigma\nabla u$ solves Beckmann problem

$$\inf_{\Phi\in\mathcal{M}(\Omega)^N} \left\{ \int_\Omega \mathrm{d}|\Phi| \,:\, -\mathrm{div}(\Phi) = \rho \text{ in } \mathcal{D}'(\overline{\Omega}) \right\}. \tag{6.13}$$

Moreover, all the optimal values coincide

$$\min(6.11) = \inf(6.13) = \max(6.12).$$

Variants of (6.11) with boundary costs were addressed in [82] where the boundary costs can be seen as some import/export taxes. Similar results were obtained in [40] with some weighted Euclidean distance as a cost. The use of PDE techniques à la Evans–Gangbo in the Finsler framework was addressed recently in [65]. It is well known that Finsler metrics generalise the Riemannian ones and are of main interest in the study of optimal transport and minimal flow problems since they allow considering anisotropy, obstacles...

In this chapter we consider a Finsler variants of (6.12)-(6.13) with the presence of boundary costs. We consider some suitable variational p-Laplace problems to provide a solutions to these problems.

6.1.3 Reminder on Finsler metrics

Let Ω be a bounded open subset of $\mathbb{R}^N$, a Finsler metric is a continuous function $H : \overline{\Omega}\times\mathbb{R}^N \to [0,\infty)$ such that $H(x,.)$ is convex, and positively $1-$homogeneous in the second variable, that is, $H(x,tp) = tH(x,p)$ for every $t \geq 0$.
We define the dual of a Finsler metric H (which is also a Finsler metric) by

$$H^*(x,q) = \sup_{H(x,p)\leq 1} \langle p, q\rangle = \sup_{p\neq 0} \frac{\langle p, q\rangle}{H(x,p)}.$$

Throughout this chapter, we assume that H is a non-degenerate Finsler metric, that is, there exist $a, b > 0$ such that

$$a|p| \leq H(x,p) \leq b|p| \tag{6.14}$$

for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$. Similarly, we have

$$\tilde{a}|q| \leq H^*(x,q) \leq \tilde{b}|q| \tag{6.15}$$

for some $\tilde{a}, \tilde{b} > 0$.

Note that a Finsler metric is not symmetric in general. Moreover, we have the Cauchy–Schwarz like inequality

$$\langle p, q \rangle \leq H(x,p)H^*(x,q). \tag{6.16}$$

Every Finsler metric induces a Finsler distance via the so called length (or action) functional. The action of a Lipchitz curve $\xi \in \mathrm{Lip}([0,1];\overline{\Omega})$ is defined through

$$A_H(\xi) = \int_0^1 H(\xi(s), \dot{\xi}(s))\mathrm{d}s. \tag{6.17}$$

The induced distance d_H by the action functional (6.17) reads as

$$d_H(x,y) = \inf_{\xi \in \Gamma(x,y)} A_H(\xi)$$

where $\Gamma(x,y) = \{\xi \in \mathrm{Lip}([0,1];\overline{\Omega}) : \ \xi(0) = x, \ \xi(1) = y\}$ and $\mathrm{Lip}([0,1];\overline{\Omega})$ is the set of Lipschitz continuous functions $u : [0,1] \to \overline{\Omega}$.

Assuming that $H^*(x,.)$ is differentiable on $\mathbb{R}^n \setminus \{0\}$, we have thanks to Euler's homogeneous function theorem (see e.g [85])

$$\partial_\xi H^*(x,p) \cdot p = H^*(x,p) \ \text{ for any } \ p \in \mathbb{R}^N, \tag{6.18}$$

and by convexity of H^*, we have

$$\partial_\xi H^*(x,p) \cdot q \leq H^*(x,q) \ \text{ for any } \ p, q \in \mathbb{R}^N,$$

so that, by (6.15)

$$|\partial_\xi H^*(x,p) \cdot q| \leq \tilde{b}|\zeta| \ \text{ for any } \ p, q \in \mathbb{R}^N. \tag{6.19}$$

Finally, we have

$$H(x, \partial_\xi H^*(x,p)) = 1 \ \text{ for any } p \in \mathbb{R}^N.$$

For details and additional properties we refer to [97].

6.1.4 Presentation of the problem

Given $\phi, \psi \in C(\partial\Omega)$ satisfying

$$\phi(x) - \psi(y) \le d_H(y,x) \text{ for all } x, y \in \partial\Omega, \tag{6.20}$$

we consider the following variant of Kantorovich-Rubinstein problem

$$(\mathcal{KR})_H : \ \max\Big\{ \int_\Omega u \mathrm{d}\rho : \ H^*(x, \nabla u) \le 1 \text{ a.e., } \phi \le u \le \psi \text{ on } \partial\Omega \Big\}.$$

Using perturbation techniques as in Chapter 3 (see also [43, Theorem 5.12]), we can derive the following variant of Beckmann's problem

$$(\mathcal{B})_H : \min_{\Phi \in \mathcal{M}^N(\Omega), \nu \in \mathcal{M}(\partial\Omega)} \Big\{ \int_\Omega H(x, \frac{\Phi}{|\Phi|}) \mathrm{d}|\Phi| + \int_{\partial\Omega} \psi \mathrm{d}\nu^- - \int_{\partial\Omega} \phi \mathrm{d}\nu^+ : \ -\mathrm{div}(\Phi) = \rho + \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \Big\},$$

as well as the associated Kantorovich problem

$$(\mathcal{K})_H : \min_{\gamma \in \Pi(\rho^+, \rho^-)} \Big\{ \int_{\overline{\Omega} \times \overline{\Omega}} d_H(x,y) \mathrm{d}\gamma(x,y) + \int_{\partial\Omega} \psi(y) \mathrm{d}(\pi_y)_\sharp \gamma - \int_{\partial\Omega} \phi(x) \mathrm{d}(\pi_x)_\sharp \gamma \Big\},$$

where $\Pi(\rho^+, \rho^-) = \{\gamma \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) : \ (\pi_x)_\sharp \gamma \llcorner \Omega = \rho^+, (\pi_y)_\sharp \gamma \llcorner \Omega = \rho^-\}$. The existence of optimal solution to $(\mathcal{K})_H$ can be obtained using the direct method of calculus of variations. Moreover, all the extremal values coincide:

$$\min(\mathcal{B})_H = \min(\mathcal{K})_H = \max(\mathcal{KR})_H.$$

Our aim is to show, using the Evans-Gangbo approach, that the solution of the HJ equation (6.1) can be obtained as the limit as $p \to \infty$, of the solution of the following Finsler (also called anisotropic) $p-$Laplace problem

$$\begin{cases} -\mathrm{div}(H^*(x, \nabla u)^{p-1} \partial_\xi H^*(x, \nabla u)) = \rho & \text{in } \Omega \\ H^*(x, \nabla u)^{p-1} \partial_\xi H^*(x, \nabla u) \cdot \mathbf{n} \ge 0 & \text{on } \{u = \phi\} \\ H^*(x, \nabla u)^{p-1} \partial_\xi H^*(x, \nabla u) \cdot \mathbf{n} \le 0 & \text{on } \{u = \psi\} \\ H^*(x, \nabla u)^{p-1} \partial_\xi H^*(x, \nabla u) \cdot \mathbf{n} = 0 & \text{in } \{\phi < u < \psi\} \\ \phi \le u \le \psi & \text{on } \partial\Omega, \end{cases} \tag{6.21}$$

with $\rho \equiv 1$, where $\mathbf{n}$ is the exterior normal to the boundary $\partial\Omega$ and $\partial_\xi H^*$ stands for the derivative of H^* with respect to the second variable. The chapter is organised as follows. In Section 2 we provide a solution to (6.21) by studying a family of variational Finsler p-Laplace problems, and by passing to the limit as $p \to \infty$, we construct a solution $\mathbf{u}$ of $(\mathcal{KR})_H$, and hence, the one of (6.1). In Section 3 we construct a flow Φ to $(\mathcal{B})_H$ and we provide, under suitable assumptions, a solution to the following Monge-Kantorovich system

$$
\begin{cases}
-\mathrm{div}(\omega\partial_\xi H^*(x,\nabla u)) = \rho & \text{in } \Omega \\
\partial_\xi H^*(x,\nabla u)\cdot \mathbf{n} \geq 0 & \text{on } \{u=\phi\} \\
\partial_\xi H^*(x,\nabla u)\cdot \mathbf{n} \leq 0 & \text{on } \{u=\psi\} \\
\partial_\xi H^*(x,\nabla u)\cdot \mathbf{n} = 0 & \text{in } \{\phi<u<\psi\} \\
\phi \leq u \leq \psi & \text{on } \partial\Omega \\
H^*(x,\nabla u) \leq 1 & \text{in } \Omega \\
H^*(x,\nabla u) = 1 & \omega - \text{a.e.}
\end{cases}
\tag{6.22}
$$

In Section 4 we make the link between $\mathbf{u}$ and $(\mathcal{K})_H$.

Remark 6.1.

- Note that given a positive continuous function $k:\overline{\Omega}\to\mathbb{R}$, defining a Finsler metric $H(x,p)=k(x)|p|$ for every $(x,p)\in\overline{\Omega}\times\mathbb{R}^N$, we easily see that

$$
H^*(x,q) = \frac{|q|}{k(x)}
$$

and the systems (6.22)-(6.21) becomes the ones studied in [40].

- Let us recall that we can define Finsler metrics via the so called Minkowski functional. Indeed, given a convex, closed and bounded set $K\subset\mathbb{R}^N$ containing the origin in its interior, we define the Minkowski functional of K (also called gauge function) by

$$
\mathbf{g}_K(p) = \inf\{t>0:\ t^{-1}p\in K\},
$$

we can easily check that $\mathbf{g}_K$ is a Finsler metric. Moreover, considering $H^*(x,p)=\mathbf{g}_K(p)$ and $\phi=\psi$, we recover the Monge-Kantorovich system studied in [33].

6.2 Finsler p-Laplace problem

We consider, for $p>N$, the following minimization problem

$$
\min_{u\in\mathcal{W}_{\phi,\psi}} \mathcal{F}_p(u) := \int_\Omega \frac{H^*(x,\nabla u)^p}{p}\mathrm{d}x - \int_\Omega u\rho\mathrm{d}x \tag{6.23}
$$

where $\rho\in L^2(\Omega)$ and $\mathcal{W}_{\phi,\psi}=\{u\in W^{1,p}(\Omega):\ \phi\leq u\leq\psi \text{ on } \partial\Omega\}$. Observe that $\mathcal{W}_{\phi,\psi}$ is a closed, convex subset of $W^{1,p}(\Omega)$. The functional $\mathcal{F}_p$ is coercive, strictly convex and lower semicontinuous on $\mathcal{W}_{\phi,\psi}$. Therefore $\mathcal{F}_p$ admits a unique minimizer u_p on $\mathcal{W}_{\phi,\psi}$. Observe that when $H^*(x,.)\in C^1(\mathbb{R}^N\setminus\{0\})$, u_p is a weak solution of (6.21).

The following result shows that we can obtain a Kantorovich potential to $(\mathcal{KR})_H$ from the minimizers of $\mathcal{F}_p$.

Proposition 6.2. *Let u_p be a minimizer of $\mathcal{F}_p$. Then, up to a subsequence, $u_p \rightrightarrows \mathbf{u}$ on $\overline{\Omega}$. Moreover, $\mathbf{u}$ solves $(\mathcal{KR})_H$.*

Proof. We divide the proof into two steps. First, we show the convergence of u_p. To do so, we need to derive some estimate on u_p independent of p. Define $v(x) = \min_{y\in\partial\Omega} \psi(y) + d_H(y,x)$. Regarding the compatibility condition (6.20), we have $\phi \leq v \leq \psi$ on $\partial\Omega$. It is not difficult to see that v is $1-$Lipschitz with respect to d_H and equivalently (see e.g. [56, Proposition 2.1]), we have that $H^*(x, \nabla v(x)) \leq 1$ a.e.. Using the fact that u_p is a minimizer of $\mathcal{F}_p$, we have

$$\int_\Omega \frac{H^*(x,\nabla u_p)^p}{p}\mathrm{d}x - \int_\Omega u_p\rho\mathrm{d}x \leq \int_\Omega \frac{H^*(x,\nabla v)^p}{p}\mathrm{d}x - \int_\Omega v\rho\mathrm{d}x \leq \frac{|\Omega|}{p} - \int_\Omega v\rho\mathrm{d}x. \tag{6.24}$$

Thanks to Theorem 2.E in [101], there is a Morrey-type inequality independent of p

$$\|u\|_{L^\infty(\Omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega)} \text{ for any } u \in W_0^{1,p}(\Omega), p > N,$$

where the constant C_Ω does not depend on p and u. Observing that we can apply the above inequality to $(u_p - \max_{\partial\Omega}\psi)^+$ and $(u_p - \min_{\partial\Omega}\phi)^-$ which are in $W_0^{1,p}(\Omega)$ to obtain

$$\|u_p^+\|_{L^\infty(\Omega)} \leq C_\Omega\|\nabla u_p\|_{L^p(\Omega)} + |\max_{\partial\Omega}\psi|,$$

and

$$\|u_p^-\|_{L^\infty(\Omega)} \leq C_\Omega\|\nabla u_p\|_{L^p(\Omega)} + |\min_{\partial\Omega}\phi|.$$

So

$$\|u_p\|_{L^\infty(\Omega)} \leq C_1\|\nabla u_p\|_{L^p(\Omega)} + C_2.$$

From (6.24) and the preceding inequality we deduce that

$$\int_\Omega \frac{H^*(x,\nabla u_p)^p}{p}\mathrm{d}x \leq \frac{|\Omega|}{p} - \int_\Omega v\rho\mathrm{d}x + \int_\Omega u_p\rho\mathrm{d}x \leq C_3(1 + \|\nabla u_p\|_{L^p(\Omega)}),$$

where C_3 is a positive constant not depending on p. Combining this with (6.14), we get

$$\|H^*(x,\nabla u_p)\|^p_{L^p(\Omega)} \leq C_4 p(1 + \|H^*(x,\nabla u_p)\|_{L^p(\Omega)}) \tag{6.25}$$

which implies that

$$\|H^*(x,\nabla u_p)\|_{L^p(\Omega)} \leq (C_5 p)^{\frac{1}{p-1}} \tag{6.26}$$

for some constant C_5 independent from p. Again, by (6.14), we get

$$\|\nabla u_p\|_{L^p(\Omega)} \leq C_6. \tag{6.27}$$

Now take some $N < m \leq p$. Then by Hölder's inequality

$$\|\nabla u_p\|_{L^m(\Omega)} \leq |\Omega|^{\frac{p-m}{pm}} \|\nabla u_p\|_{L^p(\Omega)}. \tag{6.28}$$

Thanks to (6.27), (6.28) and the Morrey-Sobolev embedding from $W^{1,m}(\Omega)$ to Hölder spaces,

$$|u_p(x) - u_p(y)| \leq C_7 |x-y|^{1-\alpha}$$

with $\alpha = \frac{N}{m}$. By Ascoli-Arzelà's theorem, up to a subsequence, $u_p \rightrightarrows \mathbf{u}$ on $\overline{\Omega}$ for some continuous function $\mathbf{u}$ satisfying $\phi \leq \mathbf{u} \leq \psi$ on $\partial\Omega$. Observe that $\mathbf{u} \in W^{1,\infty}(\Omega)$ thanks to (6.27) and (6.28).

Second, we are now in a position to show that $\mathbf{u}$ solves $(\mathcal{KR})_H$. To do so, we take any $v \in \mathcal{W}_{\phi,\psi}$ such that $H^*(x, \nabla v(x)) \leq 1$ a.e.. Using the optimality of u_p we see that

$$-\int_\Omega u_p \rho \mathrm{d}x \leq \mathcal{F}_p(u_p) \leq \mathcal{F}_p(v) \leq \frac{|\Omega|}{p} - \int_\Omega v\rho \mathrm{d}x.$$

Taking the limit up to a subsequence, we get

$$\sup \left\{ \int_\Omega v\rho \mathrm{d}x : \; H^*(x, \nabla v) \leq 1, \text{ a.e., } \phi \leq v \leq \psi \text{ on } \partial\Omega \right\} \leq \int_\Omega \mathbf{u}\rho \mathrm{d}x.$$

It remains to show that $\mathbf{u}$ is $1-$Lipschitz with respect to d_H, that is, $H^*(x, \nabla\mathbf{u}(x)) \leq 1$ a.e.. Recall that $\phi \leq \mathbf{u} \leq \psi$ on $\partial\Omega$. Again, using (6.26), we consider $N < m \leq p$ and we use Hölder's inequality to get

$$\|H^*(x, \nabla u_p)\|_{L^m(\Omega)} \leq (C_5 p)^{\frac{1}{p-1}} |\Omega|^{\frac{p-m}{pm}}.$$

Since $u_p \rightrightarrows \mathbf{u}$ uniformly on $\overline{\Omega}$, we can assume that up to a subsequence $u_p \rightharpoonup \mathbf{u}$ weakly in $W^{1,m}(\Omega)$, and particularly, $\nabla u_p \rightharpoonup \nabla\mathbf{u}$ in $L^m(\Omega, \mathbb{R}^N)$. Mazur's lemma (see [42] for example) ensures the existence of a convex combination of ∇u_{p_k} converging in norm toward $\nabla\mathbf{u}$. More precisely, there exists $\{U_i\}$ such that

$$U_i = \sum_{k=i}^{n_i} \alpha_k^i \nabla u_{p_k}$$

where $\sum_{k=i}^{n_i} \alpha_i^k = 1$, and $\alpha_k^i \geq 0,\ i \leq k \leq n_i$ and $\|U_i - \nabla \mathbf{u}\|_{L^m(\Omega)} \to 0$ as $i \to +\infty$. Since H^* is continuous, we have

$$\begin{aligned}
\|H^*(x, \nabla \mathbf{u})\|_{L^m(\Omega)} &\leq \liminf_{i\to\infty} \|H^*(x, \sum_{k=i}^{n_i} \alpha_k^i \nabla u_{p_k})\|_{L^m(\Omega)} \\
&\leq \liminf_{i\to\infty} \sum_{k=i}^{n_i} \alpha_k^i \|H^*(x, \nabla u_{p_k})\|_{L^m(\Omega)} \\
&\leq \liminf_{i\to\infty} \sum_{k=i}^{n_i} \alpha_k^i (C_5 p_k)^{\frac{1}{p_k-1}} |\Omega|^{\frac{p_k-m}{mp_k}} = |\Omega|^{\frac{1}{m}},
\end{aligned}$$

which completes the proof by taking $m \to \infty$. □

6.3 Construction of the optimal flow for $(\mathcal{B})_H$

In the sequel, we assume that (6.20) is strict, that is,

$$\phi(x) - \psi(y) < d_H(y, x) \text{ for all } x, y \in \partial\Omega.$$

The next lemma is important to construct a solution of $(\mathcal{B})_H$, we follow the main ideas of [82, Thereom 3.4].

Lemma 6.3. *Assume that $H^*(x, .) \in C^1(\mathbb{R}^N \setminus \{0\})$, and define, for $p > N$*

$$\Theta_p = H^*(x, \nabla u_p)^{p-1} \partial_\xi H^*(x, \nabla u_p).$$

Then, the distribution defined through

$$\langle \Theta_p \cdot \mathbf{n}, \eta \rangle = \int_\Omega \Theta_p \nabla \eta \mathrm{d}x - \int_\Omega \eta \rho \mathrm{d}x,\ \eta \in \mathcal{D}(\mathbb{R}^N) \tag{6.29}$$

is a Radon measure concentrated on $\partial\Omega$. Moreover, we have

$$\int_{\partial\Omega} \eta \mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot \nabla \eta \mathrm{d}x - \int_\Omega \eta \rho \mathrm{d}x \textit{ for all } \eta \in W^{1,p}(\Omega). \tag{6.30}$$

Proof. We follow the main argumants of Mazón et al. in [82]. If u_p is a minimizer of $\mathcal{F}_p$, then we clearly have $-\mathrm{div}(\Theta_p) = \rho$ in $\mathcal{D}'(\Omega)$. It follows that the $\Theta_p \cdot \mathbf{n}$ defined by (6.29) is a distribution supported on $\partial\Omega$. Let us show moreover that

$$\mathrm{supp}(\Theta_p \cdot \mathbf{n}) \subset \{x \in \partial\Omega :\ u_p(x) = \phi(x)\} \cup \{x \in \partial\Omega :\ u_p(x) = \psi(x)\}.$$

Take a test function $\eta \in C^\infty(\overline{\Omega})$ whose support is disjoint from $\{x \in \partial\Omega : u_p(x) = \phi(x)\} \cup \{x \in \partial\Omega : u_p(x) = \psi(x)\}$. There exists some $\epsilon > 0$ so that $u_p + t\eta$ remains admissible for (6.23) for $|t| < \epsilon$, *i.e.*, $\phi \leq u_p + t\eta \leq \psi$. By optimality of u_p, we get the variational inequality

$$\int_\Omega \Theta_p \cdot \nabla(v - u_p)\mathrm{d}x \geq \int_\Omega (v - u_p)\rho \mathrm{d}x \text{ for all } v \in \mathcal{W}_{\phi,\psi}.$$

In particular, for $v = u_p + t\eta$, we get

$$t \int_\Omega \Theta_p \cdot \nabla\eta \mathrm{d}x \geq t \int_\Omega \eta\rho \mathrm{d}x.$$

This holds for positive and negative t, such that $|t| \leq \epsilon$. Consequently

$$\int_\Omega \Theta_p \cdot \nabla\eta \mathrm{d}x = \int_\Omega \eta\rho \mathrm{d}x.$$

In other words, $\langle \Theta_p \cdot \mathbf{n}, \eta \rangle = 0$ and $\mathrm{supp}(\Theta_p \cdot \mathbf{n}) \subset \{u_p = \phi\} \cup \{u_p = \psi\}$. We are now in a position to show that $\Theta_p \cdot \mathbf{n}$ is actually a Radon measure. Indeed, since the inequality (6.20) is strict, the two compact sets $\{x \in \partial\Omega : u_p(x) = \phi(x)\}$ and $\{x \in \partial\Omega : u_p(x) = \psi(x)\}$ are disjoint. There exist $\eta_1, \eta_2 \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\eta_1(x) = \begin{cases} 1 \text{ on } \{u_p = \phi\}, \\ 0 \text{ on } \{u_p = \psi\}, \end{cases} \quad \text{and } \eta_2(x) = \begin{cases} 1 \text{ on } \{u_p = \psi\}, \\ 0 \text{ on } \{u_p = \phi\}. \end{cases}$$

Then we can write $\Theta_p \cdot \mathbf{n} = D_1 + D_2$, where D_1, D_2 are distributions given by

$$\langle D_1, \eta \rangle = \langle \Theta_p \cdot \mathbf{n}, \eta\eta_1 \rangle \text{ and } \langle D_2, \eta \rangle = \langle \Theta_p \cdot \mathbf{n}, \eta\eta_2 \rangle.$$

This being said, for any positive test function η, we have that $\mathrm{supp}(\eta\eta_1) \cap \{u_p = \psi\} = \emptyset$, and for $0 \leq t < \epsilon$ we have $u_p + t(\eta\eta_1) \in \mathcal{W}_{\phi,\psi}$. Consequently

$$t \int_\Omega \Theta_p \cdot \nabla(\eta\eta_1)\mathrm{d}x \geq t \int_\Omega (\eta\eta_1)\rho \mathrm{d}x,$$

i.e,

$$\langle D_1, \eta \rangle \geq 0. \tag{6.31}$$

On the other hand, for any positive test function η, we have that $\mathrm{supp}(\eta\eta_2) \cap \{u_p = \phi\} = \emptyset$ and for $-\epsilon < t \leq 0$, we have that $u_p + t(\eta\eta_2) \in \mathcal{W}_{\phi,\psi}$. Consequently

$$t \int_\Omega \Theta_p \cdot \nabla(\eta\eta_2)\mathrm{d}x \geq t \int_\Omega (\eta\eta_2)\rho \mathrm{d}x.$$

In other words,

$$\langle D_2, \eta \rangle \leq 0. \tag{6.32}$$

In conclusion, D_1 and $-D_2$ are positive distributions. Hence, they are positive Radon measures. It follows that the distribution $\Theta_p \cdot \mathbf{n}$ is a Radon measure on $\partial\Omega$. Moreover, (6.31) and (6.32) give

$$\mathrm{supp}((\Theta_p \cdot \mathbf{n})^+) \subset \{u_p = \phi\} \text{ and } \mathrm{supp}((\Theta_p \cdot \mathbf{n})^-) \subset \{u_p = \psi\}.$$

□

Proposition 6.4. *Under the previous assumptions, there exist two Radon measures* Θ *and* θ *in* Ω *and* $\partial\Omega$ *respectively such that up to a subsequence* $\Theta_p \rightharpoonup \Theta$ *and* $\Theta_p \cdot \mathbf{n} \rightharpoonup \theta$ *in the sense of measures.*

Proof. First, we consider as before $v(x) = \min_{y\in\partial\Omega} \psi(y) + d_H(y, x)$. We plug u_p and v in (6.30) to get

$$\int_{\partial\Omega} (u_p - v)\mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot \nabla(u_p - v)\mathrm{d}x - \int_\Omega (u_p - v)\rho \mathrm{d}x.$$

In other words

$$\int_\Omega (u_p - v)\rho \mathrm{d}x = \int_\Omega \Theta_p \cdot \nabla(u_p - v)\mathrm{d}x + \int_{\{u_p=\psi\}} (\psi - v)\mathrm{d}(\Theta_p \cdot \mathbf{n})^- - \int_{\{u_p=\phi\}} (\phi - v)\mathrm{d}(\Theta_p \cdot \mathbf{n})^+.$$

We see that $\phi < v \leq \psi$ on $\partial\Omega$ so that $\psi - v \geq 0$ and $\phi - v < 0$, thus $\phi - v < -C_1$ for some positive constant C_1. So we obtain

$$\int_\Omega \Theta_p \cdot \nabla u_p \mathrm{d}x + C_1 \int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^+ \leq \int_\Omega (u_p - v)\rho \mathrm{d}x + \int_\Omega \Theta_p \cdot \nabla v \mathrm{d}x. \tag{6.33}$$

Since H^* is a Finsler metric, we have by Euler's homogeneous function theorem (see *e.g.* [85]) that $\partial_\xi H^*(x, \xi) \cdot \xi = H^*(x, \xi)$ for any $\xi \in \mathbb{R}^N$. Thus

$$\int_\Omega \Theta_p \cdot \nabla u_p \mathrm{d}x = \int_\Omega H^*(x, \nabla u_p)^{p-1} \partial_\xi H^*(x, \nabla u_p) \cdot \nabla u_p \mathrm{d}x = \int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x.$$

Using this fact in (6.33), we get

$$\int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x + C_1 \int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^+ \leq C_2 + \int_\Omega \Theta_p \cdot \nabla v \mathrm{d}x,$$

where $C_2 > 0$ is independent from p. On the other hand, thanks to (6.16) we have

$$\begin{aligned}\int_\Omega \Theta_p \cdot \nabla v \mathrm{d}x &\leq \int_\Omega H(x, \Theta_p) H^*(x, \nabla v) \mathrm{d}x \\ &= \int_\Omega H(x, H^*(x, \nabla u_p)^{p-1} \partial_\xi H^*(x, \nabla u_p)) H^*(x, \nabla v) \mathrm{d}x \\ &= \int_\Omega H^*(x, \nabla u_p)^{p-1} H(x, \partial_\xi H^*(x, \nabla u_p)) H^*(x, \nabla v) \mathrm{d}x \\ &= \int_\Omega H^*(x, \nabla u_p)^{p-1} H^*(x, \nabla v) \mathrm{d}x,\end{aligned}$$

where we have used the homogeneity of H and the fact that $H(x, \partial_\xi H^*(x, \xi)) = 1$ for any $\xi \in \mathbb{R}^N \backslash \{0\}$. Using Hölder and Young's inequalities and the fact that $H^*(x, \nabla v) \leq 1$ a.e, we get

$$\begin{aligned}\int_\Omega H^*(x, \nabla u_p)^{p-1} H^*(x, \nabla v) \mathrm{d}x &\leq \Big(\int_\Omega H^*(x, \nabla u_p)^{(p-1)p'} \mathrm{d}x \Big)^{\frac{1}{p'}} |\Omega|^{\frac{1}{p}} \\ &= \frac{p-1}{p} \int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x + \frac{1}{p} |\Omega|.\end{aligned}$$

We deduce that

$$\frac{1}{p} \int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x + C_1 \int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^+ \leq C_2 + \frac{1}{p} |\Omega|.$$

Since $p > N$, it follows from (6.25) and (6.27) that the first term of the preceding is bounded and therefore

$$\int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^+ \leq C_3, \tag{6.34}$$

for some positive constant C_3. Defining $w(x) = \max_{y \in \partial\Omega} \phi(y) - d_H(x, y)$, we see that $\phi \leq w < \psi$ and following the same lines we get that

$$\int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^- \leq C_4. \tag{6.35}$$

As a consequence, we deduce that $\Theta_p \cdot \mathbf{n}$ is bounded in $\mathcal{M}(\partial\Omega)$ and thus there exists $\theta \in \mathcal{M}(\partial\Omega)$ such that $\Theta_p \cdot \mathbf{n} \rightharpoonup \theta$ weakly*.

As for Θ_p, we have

$$\int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x = \int_\Omega \Theta_p \cdot \nabla u_p \mathrm{d}x = \int_{\partial\Omega} u_p \mathrm{d}(\Theta_p \cdot \mathbf{n}) + \int_\Omega u_p \rho \mathrm{d}x.$$

Keeping in mind (6.34) and (6.35), Hölder's inequality gives

$$\int_\Omega H^*(x, \nabla u_p)^{p-1} \mathrm{d}x \leq C_5. \tag{6.36}$$

Since $\partial_\xi H^*(x,\xi)\cdot\zeta \leq H^*(x,\zeta)$ and taking into account (6.14), we deduce that

$$|\partial_\xi H^*(x, \nabla u_p)| \leq \tilde{b},$$

and consequently in view of (6.36)

$$\int_\Omega |\Theta_p| \mathrm{d}x \leq C_6.$$

Thus, there exists $\Theta \in \mathcal{M}(\overline{\Omega})^N$ such that up to a subsequence $\Theta_p \rightharpoonup \Theta$. □

Theorem 6.5. *Let* $\mathbf{u}$ *and* Θ *as in Propositions 6.2 and 6.4 and set* $\omega = H(x, \Theta)$*. Assume moreover that*

$$H^*(x, \nabla_\omega \mathbf{u}) \leq 1 \quad \omega \text{ a.e.}. \tag{6.37}$$

Then $(\omega \partial_\xi H^*(x, \nabla_\omega \mathbf{u}), \mathbf{u})$ *solves* (6.22)*.*

Proof. First, let us recall that thanks to Lemma 6.3 and Proposition 6.4 we have

$$-\mathrm{div}(\Theta) = \rho + \theta \text{ in } \mathcal{D}'(\overline{\Omega}).$$

Since $\frac{\mathrm{d}\Theta}{\mathrm{d}\omega} \in L^1_\omega(\overline{\Omega})^N$ we also have

$$-\mathrm{div}(\omega \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}) = \rho + \theta \text{ in } \mathcal{D}'(\overline{\Omega}).$$

So $\frac{\mathrm{d}\Theta}{\mathrm{d}\omega}(x) \in \mathcal{X}_\omega(x)$ for $\omega-$a.e. x (see the Section 2.3 for the definition of $\mathcal{X}_\omega$) . Since $\Theta_p \rightharpoonup \Theta$ (cf. Proposition 6.4), we have by Reshetnyak's lower semicontinuity theorem (see [3, Thereom 2.38]):

$$\int_\Omega H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta| \leq \liminf_p \int_\Omega H(x, \frac{\Theta_p}{|\Theta_p|}) \mathrm{d}|\Theta_p|.$$

Using Hölder's inequality combined with (6.18)-(6.19), we get

$$\begin{aligned}\int_\Omega H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta| &\leq \liminf_p \int_\Omega H\Big(x, H^*(x, \nabla u_p)^{p-1} \partial_\xi H^*(x, \nabla u_p)\Big) \mathrm{d}x \\ &= \liminf_p \int_\Omega H^*(x, \nabla u_p)^{p-1} H(x, \partial_\xi H^*(x, \nabla u_p)) \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_p \Big(\int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x \Big)^{\frac{p-1}{p}} \\
&= \liminf_p \Big(\int_\Omega H^*(x, \nabla u_p)^{p-1}\, \partial_\xi H^*(x, \nabla u_p) \cdot \nabla u_p \mathrm{d}x \Big)^{\frac{p-1}{p}} \\
&= \liminf_p \Big(\int_\Omega \nabla u_p \mathrm{d}\Theta_p \Big)^{\frac{p-1}{p}} \\
&= \liminf_p \Big(\int_\Omega u_p \rho \mathrm{d}x + \int_{\partial\Omega} u_p \mathrm{d}(\Theta_p \cdot \mathbf{n}) \Big)^{\frac{p-1}{p}} \\
&= \int_\Omega \mathbf{u} \rho \mathrm{d}x + \int_{\partial\Omega} \phi \mathrm{d}\theta^+ - \int_{\partial\Omega} \psi \mathrm{d}\theta^- .
\end{aligned}
$$

Finally, using the integration by parts formula recalled in Proposition 2.13, we get

$$
\int_\Omega \mathbf{u} \rho \mathrm{d}x + \int_{\partial\Omega} \phi \mathrm{d}\theta^+ - \int_{\partial\Omega} \psi \mathrm{d}\theta^- = \int_\Omega \nabla_\omega \mathbf{u} \mathrm{d}\Theta,
$$

so that

$$
\int_\Omega H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta| \leq \int_\Omega \nabla_\omega \mathbf{u} \mathrm{d}\Theta.
$$

Keeping in mind (6.37), we get

$$
\int_\Omega \nabla_\omega \mathbf{u} \mathrm{d}\Theta \leq \int_\Omega H^*(x, \nabla_\omega \mathbf{u}) H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta| \leq \int_\Omega H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta|.
$$

Hence,

$$
\int_\Omega H(x, \frac{\Theta}{|\Theta|}) \mathrm{d}|\Theta| = \int_\Omega \nabla_\omega \mathbf{u} \mathrm{d}\Theta. \tag{6.38}
$$

From (6.38), we deduce that

$$
\nabla_\omega \mathbf{u} \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = 1 \; \omega - \text{ a.e.}
$$

Since $H(x, \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}) = 1 \; \omega -$ a.e, we have that $H^*(x, \nabla_\omega \mathbf{u}) = 1 \;\; \omega -$ a.e. This implies by definition of H^* that

$$
\frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = H(x, \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}) \partial_\xi H^*(x, \nabla_\omega \mathbf{u}) = \partial_\xi H^*(x, \nabla_\omega \mathbf{u}) \quad \omega - \text{ a.e.}
$$

Finally,

$$
-\mathrm{div}(\omega \partial_\xi H^*(x, \nabla_\omega \mathbf{u})) = \rho \; \text{ in } \Omega,
$$

and reproducing the arguments of Lemma 6.3, we recover the constraints on $\partial_\xi H^*(x, \nabla_\omega \mathbf{u}) \cdot \mathbf{n}$ in (6.22). □

In the case where $\omega \ll \mathcal{L}^N$, we have the following

Corollary 6.6. *Let* $\mathbf{u}$ *and* Θ *as in Propositions 6.2 and 6.4 and set* $\omega = H(x, \Theta)$. *If* $\omega \ll \mathcal{L}^N$, *then*

$$\omega(x) > 0 \Rightarrow H^*(x, \nabla \mathbf{u}) = 1,$$

and

$$\int_\Omega \omega(x) \partial_\xi H^*(x, \nabla \mathbf{u}) \cdot \nabla \eta \mathrm{d}x = \int_\Omega \eta \rho \mathrm{d}x, \ \textit{for all } \eta \in \mathcal{D}(\Omega).$$

In particular,

$$-\operatorname{div}(\omega \partial_\xi H^*(x, \nabla \mathbf{u})) = \rho \ \textit{in } \Omega.$$

Proposition 6.7. *The couple* (Θ, θ) *solves* $(\mathcal{B})_H$.

Proof. First, take any admissible potential $v \in C^1(\overline{\Omega})$ for $(\mathcal{KR})_H$ and an admissible couple of flows (Ψ, ν) for $(\mathcal{B})_H$. Since $H^*(x, \nabla v) \leq 1$ for a.e.x, we have

$$\begin{aligned}
\int_\Omega H(x, \frac{\Psi}{|\Psi|}) \mathrm{d}|\Psi| &\geq \int_\Omega H(x, \frac{\Psi}{|\Psi|}) H^*(x, \nabla v) \mathrm{d}|\Psi| \\
&\geq \int_\Omega \frac{\Psi}{|\Psi|} \nabla v \mathrm{d}|\Psi| \\
&= \int_\Omega \nabla v \mathrm{d}\Psi \\
&= \int_{\overline{\Omega}} v \mathrm{d}(\rho + \nu) \\
&\geq \int_\Omega v \mathrm{d}\rho + \int_{\partial\Omega} \phi \mathrm{d}\nu^+ - \int_{\partial\Omega} \psi \mathrm{d}\nu^-
\end{aligned}$$

and consequently

$$\int_\Omega H(x, \frac{\Psi}{|\Psi|}) \mathrm{d}|\Psi| + \int_{\partial\Omega} \psi \mathrm{d}\nu^- - \int_{\partial\Omega} \phi \mathrm{d}\nu^+ \geq \int_\Omega v \mathrm{d}\rho$$

using Lemma 6.11 and taking the infimum on (Ψ, ν) we deduce that $\min(\mathcal{B})_H \geq \sup(\mathcal{KR})_H$. By Theorem 6.5 and Corollary 6.6 we have

$$\int_\Omega H(x, \frac{\Phi}{|\Phi|}) \mathrm{d}|\Phi| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ = \int_\Omega \mathbf{u} \mathrm{d}\rho.$$

Hence (Θ, θ) solves $(\mathcal{B})_H$. □

Remark 6.8. It is possible to solve (6.22) without the assumption (6.37) as in [65, Theorem 3.10]. Yet, one needs to define the ω-tangential gradient of $\mathbf{u}$ with respect to H (see [92] for details).

The link between HJ equation and Beckmann's problem is already established in Chapter 5. Thanks to Proposition 6.2, the maximal viscosity subsolution of (6.1) is obtained by taking the limit as $p \to \infty$ in the minimizers of $\mathcal{F}_p$ with $\rho \equiv 1$. Thus, combining this fact with Proposition 6.7, we have the following

Corollary 6.9. *Assume that $\rho \equiv 1$, and let $\mathbf{u}$ the maximal viscosity subsolution of (6.1) and (Θ, θ) an optimal solution to $(\mathcal{B})_H$. Then*

$$\begin{cases} -\operatorname{div}(\Theta) = 1 + \theta & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \Theta(x) \cdot \nabla_{|\Theta|}\mathbf{u}(x) = H\Big(x, \frac{\Theta}{|\Theta|}(x)\Big) & \text{for } |\Theta| - \text{a.e. } x \\ \phi \leq \mathbf{u} \leq \psi \text{ on } \partial\Omega & \text{and } \mathbf{u} = \psi \text{ for } \theta^- \text{ a.e. } x \text{ and } \mathbf{u} = \phi \text{ for } \theta^+ \text{ a.e. } x. \end{cases}$$

6.4 Back to Monge–Kantorovich problem

Proposition 6.10. *Let $\mathbf{u}$ be the potential constructed in Proposition 6.2. Then $\mathbf{u}$ is a Kantorovich potentional for the classical optimal transport problem between $\rho^+ \llcorner \Omega + \theta^+$ and $\rho^- \llcorner \Omega + \theta^-$. Moreover*

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x = \min(\mathcal{K})_H.$$

Proof. In the definition of $\Theta_p \cdot \mathbf{n}$ in (6.29), we take as a test function $\eta = \mathbf{u}$ to get

$$\int_{\partial\Omega} \mathbf{u}\mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot \nabla \mathbf{u} \mathrm{d}x - \int_\Omega \mathbf{u}\rho \mathrm{d}x.$$

Thanks to Proposition 6.4, passing to the limit $p \to \infty$ (up to a subsequence) we get

$$\int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_\Omega \Theta \cdot \nabla \mathbf{u} \mathrm{d}x - \int_\Omega \mathbf{u}\rho \mathrm{d}x. \tag{6.39}$$

Since $\mathbf{u}$ is $1-$Lipschitz with respect to d_H, we may find thanks to Lemma 6.11, a sequence of smooth functions w_ϵ converging uniformly to $\mathbf{u}$ and enjoying the property of being $1-$Lipschitz with respect to d_H. By definition of $\Theta_p \cdot \mathbf{n}$, we get

$$\int_{\partial\Omega} (\mathbf{u} - w_\epsilon)\mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot (\nabla \mathbf{u} - \nabla w_\epsilon)\mathrm{d}x - \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x.$$

Taking $p \to \infty$ (again, up to a subsequence) and keeping in mind (6.39), we get

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u} \mathrm{d}\theta = \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x + \int_{\partial\Omega} (\mathbf{u} - w_\epsilon)\mathrm{d}\theta + \int_\Omega \Theta \cdot \nabla w_\epsilon \mathrm{d}x = A_\epsilon + B_\epsilon, \tag{6.40}$$

with $A_\epsilon = \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x + \int_{\partial\Omega} (\mathbf{u} - w_\epsilon)\mathrm{d}\theta$ and $B_\epsilon = \int_\Omega \Theta \cdot \nabla w_\epsilon \mathrm{d}x$. Since w_ϵ convergences uniformly to $\mathbf{u}$, we have that $A_\epsilon \to 0$ as $\epsilon \to 0$. We claim that

$$B_\epsilon \to \int_\Omega H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta|$$

as $\epsilon \to 0$. We first observe that

$$\begin{aligned}
\int_\Omega \mathbf{u}\rho \mathrm{d}x &= \lim_{\epsilon\to 0} \int_\Omega w_\epsilon \rho \mathrm{d}x \\
&= \lim_{\epsilon\to 0} \int_\Omega \nabla w_\epsilon \frac{\Theta}{|\Theta|}\mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ \\
&\leq \lim_{\epsilon\to 0} \int_\Omega H^*(x, \nabla w_\epsilon) H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ \\
&\leq \int_\Omega H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+
\end{aligned}$$

where in the last inequality we have used Lemma 6.11.

Again we proceed as in the proof of Theorem 6.5: since $\Theta_p \rightharpoonup \Theta$, we have by Reshetnyak's lower semicontinuity theorem:

$$\begin{aligned}
\int_\Omega H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta| &\leq \liminf_p \int_\Omega H(x, \frac{\Theta_p}{|\Theta_p|})\mathrm{d}|\Theta_p| \\
&= \liminf_p \int_\Omega H\Big(x, (H^*(x, \nabla u_p))^{p-1} \partial_\xi H^*(x, \nabla u_p)\Big)\mathrm{d}x \\
&= \liminf_p \int_\Omega H^*(x, \nabla u_p)^{p-1} H(x, \partial_\xi H^*(x, \nabla u_p))\mathrm{d}x \\
&\leq \liminf_p \Big(\int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x\Big)^{\frac{p-1}{p}} \\
&= \liminf_p \Big(\int_\Omega H^*(x, \nabla u_p)^{p-1}\, \partial_\xi H^*(x, \nabla u_p) \cdot \nabla u_p \mathrm{d}x\Big)^{\frac{p-1}{p}} \\
&= \liminf_p \Big(\int_\Omega \nabla u_p \mathrm{d}\Theta_p\Big)^{\frac{p-1}{p}} \\
&= \int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u} \mathrm{d}\theta
\end{aligned}$$

$$= \lim_{\epsilon \to 0} \int_\Omega w_\epsilon \rho \mathrm{d}x + \int_{\partial\Omega} w_\epsilon \mathrm{d}\theta$$

where we have used Hölder's inequality and Euler's theorem combined with the fact that $H^*(x, \partial_\xi H(x,\xi)) = 1$ for any $\xi \in \mathbb{R}^N \setminus \{0\}$. Coming back to (6.40) we get

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_\Omega H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta|.$$

To conclude, let us observe that taking $v \in W^{1,\infty}(\Omega)$ such that $H^*(x, \nabla v(x)) \leq 1$, we have

$$\begin{aligned} \int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta &= \int_\Omega H(x, \frac{\Theta}{|\Theta|})\mathrm{d}|\Theta| \\ &\geq \int_\Omega \frac{\Theta}{|\Theta|} \cdot \nabla v \mathrm{d}|\Theta| \\ &= \int_\Omega \nabla v \mathrm{d}\Theta = \int_\Omega v\rho \mathrm{d}x + \int_{\partial\Omega} v \mathrm{d}\theta. \end{aligned}$$

We have thanks to Proposition 6.2 and the classical Kantorovich duality

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y) d\gamma(x,y),$$

where γ is a solution of

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y) : (\pi_x)_\sharp \gamma = \rho^+ \llcorner \Omega + \theta^+, (\pi_y)_\sharp \gamma = \rho^- \llcorner \Omega + \theta^- \right\}.$$

Since $(\pi_x)_\sharp \gamma \llcorner \partial\Omega = \theta^+$ and $(\pi_y)_\sharp \gamma \llcorner \partial\Omega = \theta^-$ we deduce that

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x = \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y) d\gamma(x,y) + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ = \min(\mathcal{K})_H.$$

□

To end this section let us recall the following useful approximation result [65, 67, 68].

Lemma 6.11. *Let $w \in W^{1,\infty}(\Omega)$ such that $H^*(x, \nabla w(x)) \leq 1$ for a.e. $x \in \Omega$. Then, there exists $w_\epsilon \in C^1(\overline{\Omega})$ such that $w_\epsilon \rightrightarrows w$ in compact subsets of Ω and*

$$H^*(x, \nabla w_\epsilon(x)) \leq 1.$$

7 Perspectives and future work

We end this dissertation by presenting results from ongoing works and some possible extensions and open question that are of interest to us.

7.1 Quasivariational Approach for obstacle HJ equation

In this section we present briefly some results from an ongoing work with N.Igbida concerning obstacle HJ (OHJ for short) equation with application to the formation of lakes and dunes. We suggest to use an evolutionary quasivariational inequality to solve numerically the OHJ equation by exploiting similar characterisations of the appropriate solution as in Chapter 3.

Given a continuous function $\mathfrak{g} : \Omega \to \mathbb{R}^+$, we consider the following OHJ equation

$$\begin{cases} u \geq \mathfrak{g} & \text{in } \Omega \\ H(x, \nabla u) = 0 & \text{in } [u > \mathfrak{g}] \end{cases} \tag{7.1}$$

This equation can be recast in the form

$$F(x, u, \nabla u) = 0 \text{ in } \Omega,$$

where $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$F(x, t, p) = \min(H(x, p), t - \mathfrak{g}(x)), \text{ for } (x, t, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

As one can see, the 0 sublevel sets of F are not convex in general, so producing a metric associated to F is not straightforward. To handle the obstacle in (7.1), N.Igbida proposed in [64] an inf-sup formula as follows. For $\xi \in \mathrm{Lip}([0, 1]; \overline{\Omega})$, we define

$$\Lambda_\xi(s_1, s_2) = \int_{s_1}^{s_2} \sigma(\xi(t), \dot{\xi}(t)) \mathrm{d}t,$$

for $0 \leq s_1 \leq s_2 \leq 1$, where σ is the support function of the 0 sublevels of H. Then, the $\mathfrak{g}-$action functional of the curve ξ reads

$$A_{\mathfrak{g}}(\xi) = \max_{t\in[0,1]} \mathfrak{g}(\xi(t)) + \Lambda_\xi(t,1).$$

For $x, y \in \overline{\Omega}$, we define the minimum $\mathfrak{g}-$action from x to y through

$$S_{\mathfrak{g}}(x,y) = \inf_{\xi\in\Gamma(x,y)} A_{\mathfrak{g}}(\xi). \tag{7.2}$$

and

$$\mathcal{I}_{\mathfrak{g}}(x,y) = S_{\mathfrak{g}}(x,y) - \mathfrak{g}(x).$$

Then, it is shown in [64] that $S_{\mathfrak{g}}$ is the maximal viscosity subsolution of (7.1). Moreover, when the obstacle $\mathfrak{g}$ satisfies the compatibility condition

$$\mathfrak{g}(x) - \mathfrak{g}(y) \leq d_\sigma(y,x), \text{ for any } x, y \in \overline{\Omega}, \tag{7.3}$$

where d_σ is the associated distance to H, then $\mathcal{I}_{\mathfrak{g}} = d_\sigma$. Which shows that the quasi-distance $\mathcal{I}_{\mathfrak{g}}$ is consistent with the case where the obstacle is a subsolution for HJ equation.

To describe the morphology of a sandpile or a lake on an arbitrary landscape, we consider the following obstacle Eikonal equation

$$\begin{cases} |\nabla u| = k \text{ on } [u > \mathfrak{g}] \\ \quad u \geq \mathfrak{g} \text{ in } \Omega \\ \quad u = \mathfrak{g} \text{ on } D. \end{cases} \tag{7.4}$$

where D is a closed subset of $\overline{\Omega}$, where k is related to the so-called repose angle of the granular material. To describe the equilibrium state we consider a Radon measure ρ modelling the source. One could imagine $\rho = \sum_{i=1}^m \delta_{x_i}$ where δ_{x_i} are Dirac masses at points $x_i \in \overline{\Omega}$, to model the case of punctual distribution of sand or water. This being said, we consider by analogy with the problem $(\mathcal{M}_D)$ in Chapter 3, the following maximization problem

$$\max\left\{ \int u d\rho : \ u \in K_{\mathfrak{g}}^D \right\}, \tag{7.5}$$

where

$$K_{\mathfrak{g}}^D = \left\{ u \in W^{1,\infty}(\Omega) : \ u \geq \mathfrak{g}, \ u = \mathfrak{g} \text{ on } D \text{ and } |\nabla u| \leq k \text{ in } [u > \mathfrak{g}] \right\},$$

is the set of the so-called admissible profiles. Then, thanks to (7.2), N.Igbida provides a solution to (7.5). More precisely, he proves the following

Theorem 7.1. *Assume that $\rho \in \mathcal{M}_b^+(\Omega)$ and $k \in C(\overline{\Omega})$, $k \geq 0$, $\mathfrak{g} \in W^{1,\infty}(\overline{\Omega})$, $\mathfrak{g} \geq 0$. Define*

$$S_{\mathfrak{g}}(D, x) = \min_{y \in D} S_{\mathfrak{g}}(y, x) \text{ for any } x \in \overline{\Omega}.$$

Then

1) $S_{\mathfrak{g}}(D, .) \in K_{\mathfrak{g}}^D$.
2) $S_{\mathfrak{g}}(D, .)$ *is a solution of* (7.5).
3) u *is a solution of* (7.5) *if and only if* $u \in K_{\mathfrak{g}}^D$ *and*

$$u = S_{\mathfrak{g}}(D, .)\ \rho - a.e. \text{ in } \Omega.$$

It is not clear how to treat (7.5) as in Chapter 3 since the dynamic takes place only on the region $[u > \mathfrak{g}]$. More precisely, the main difficulty arising in (7.5) compared to what we have developed in the previous chapters, is the lack of convexity of $K_{\mathfrak{g}}^D$.

Few works investigated problems of the form (7.4). In the case of lakes, i.e., $k = 0$, Doferman and Evans [38] considered the following PDE

$$\begin{cases} \partial_t u_\epsilon \in \operatorname{div}\left(\dfrac{\mathcal{H}(u_\epsilon - \mathfrak{g})}{\epsilon} \nabla u_\epsilon\right) + \rho \text{ in } \mathbb{R}^2 \times \{t > 0\} \\ \qquad\qquad\qquad\qquad u_\epsilon = \mathfrak{g} \text{ in } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$

where $\mathcal{H}$ is the Heaviside function. Then, the limit $\mathbf{u} = \lim_{\epsilon \to 0} u_\epsilon$ satisfies (7.4). The authors justify their results by rigorous proofs as well as asymptotics and numerical simulations.

In [10, 11], Barrett and Prigozhin use a quasivariational inequality to model growing sandpiles, i.e., $k \geq k_0 > 0$ with an obstacle $\mathfrak{g}$ satisfying (7.3). To do so, they define, for $v \in C(\overline{\Omega})$, the following operator

$$M(v) = \begin{cases} k \text{ on } [v > \mathfrak{g}] \\ \max(k, |\nabla \mathfrak{g}|) \text{ elsewhere,} \end{cases}$$

and $K(z) = \{z \in W^{1,\infty}(\Omega) :\ z \geq \mathfrak{g} \text{ and } |\nabla z| \leq M(v)\}$. Then, the proposed quasivariational inequality consists in finding $u(x,t)$ such that $u(x, 0) = \mathfrak{g}(x)$ and for a.a. $t \in (0, T)$, $u \in K(u)$ solves

$$\int_\Omega (\partial_t u - \rho)(v - u) \mathrm{d}x \geq 0 \text{ for all } v \in K(u). \tag{7.6}$$

As one can see, $M(v)$ is discontinuous in general, so the authors study a regularized version of (7.6) with a continuous gradient constraint $M_\epsilon(v)$. Let us mention that they also study the case of lake. However, they considerd a small k_0 in their formulations and numerical simulations instead of $k_0 = 0$.

Our strategy is close to the one in [10, 11] in the sense that, instead of considering (7.5), we regard the following problem

$$\max\left\{\int z\mathrm{d}\rho,\ z \in K(\mathbf{u})\right\}, \tag{7.7}$$

where $\mathbf{u} = S_g(D,.)$ is the solution of (7.5) given in Theorem 7.1. First, let us observe that (7.7) is related to the Eikonal equation

$$\begin{cases} |\nabla u| = M(\mathbf{u}) \text{ in } \Omega \setminus D \\ \quad u = \mathfrak{g} \text{ on } D. \end{cases} \tag{7.8}$$

Due to discontinuity of M, the appropriate notion of solutions is the so-called Monge solutions introduced by Newcomb and Su [86]. So our starting point is to investigate the existence of a unique Monge solution of (7.8), and show by analogy with Theorem 3.2, that it solves (7.7). In this case, we conjecture that (7.5) and (7.7) are equivalent. This suggests considering the following quasivariational equation

$$\rho \in \partial_t u + \partial \mathbb{I}_{K(z)}(u), \tag{7.9}$$

where $\partial \mathbb{I}_{K(z)}$ is the subdifferential of $\mathbb{I}_{K(z)}$ in $L^2(\Omega)$. To approximate numerically the solution of (7.9), we use an Euler implicit scheme. We denote by δt the time step and u^n the solution at time $t = n\delta t$ with $n \in \mathbb{N}$. We initialize by u_0 and we generate a sequence $\{u^n\}_n$ as follows:

$$u^n + \delta t \rho^n := w^n \in u^{n+1} + \partial \mathbb{I}_{K(u^n)}(u^{n+1}),$$

or equivalently

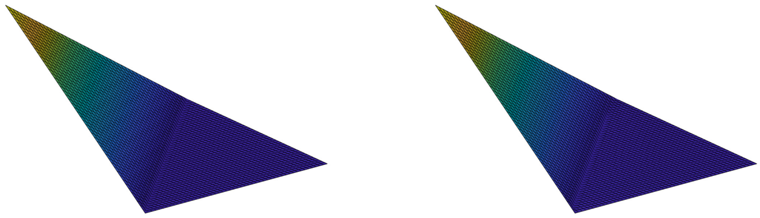
$$w^n \in \big(\mathrm{id} + \partial \mathbb{I}_{K(u^n)}\big)(u^{n+1}).$$

This amounts to saying that

$$u^{n+1} = \underset{z \in K(u^n)}{\arg\min}\, \mathcal{F}(z)$$

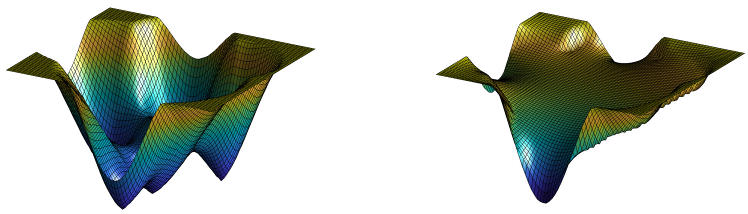
with $\mathcal{F}(z) = \frac{1}{2}\|z - w^n\|^2$.

Minimizing the functional $\mathcal{F}$ over $K(u^n)$ can be done efficiently by Chambolle-Pock's algorithm as in Section 4.3. We therefore present some preliminary results for the case $k \equiv 0$.



(a) (b)

Figure 7.1: (A): the landscape height function $\mathfrak{g}$, (B): the constructed solution u .



(a) (b)

Figure 7.2: (A): the landscape height function $\mathfrak{g}$, (B): the constructed solution u .

In the figure 7.1, the geometry of the obstacle given by a simple function $\mathfrak{g}(x, y) = \min(0, y - x))$ doesn't allow collecting water. On the other hand, we see in figure 7.2[1]. that water fills the lake. These preliminary results are encouraging and tend to support our conjectures.

To conclude this section, let us say that besides the equivalence between (7.5) and (7.7) there are many interesting questions to deal with:

- To prove the convergence of the approximate solution by the Euler scheme in time and the convergence as $t \to \infty$ of the solution of (7.9) to the solution of (7.7).
- Up to our knowledge, the regularity of the the free boundary $\partial[u = \mathfrak{g}]$ of the OHJ is less studied (if not non-existent !) and the techniques in the classical textbooks (see e.g.[52, 71]) cannot be applied directly for OHJ equations. We are planing to address this question in future work.

[1]This obstacle was created from an image taken from `https://www.numerical-tours.com/matlab/`

7.2 Hughes model for crowd motion

Amongst our future works is to adapt the methods and techniques of this thesis to so-called Hughes model for crowd motion (see [62, 63]). Let us say few words about this model, the difficulties arising in its mathematical and numerical study, and how we are planing to tackle them.

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We denote by $\mu = \mu(x,t)$ the density of some population where $x \in \Omega$ and $t \geq 0$ representing the time variable. Then μ satisfies the continuity equation

$$\partial_t \mu + \mathrm{div}(\mu v) = 0, \ \mu(x,0) = \mu_0(x) \geq 0, \tag{7.10}$$

where $v(x,t)$ is the velocity field which can be related to μ via $v(x,t) = (1 - \mu(x,t))\nabla u(x)$, and u is the potential modelling the common sens of the task. Assuming that the individuals try to avoid regions with high densities while heading their destination, the potential u will satisfy the following Eikonal equation

$$\begin{cases} |\nabla u| = k(\mu) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.11}$$

with $k(\mu) = (1-\mu)$. So that the model reads

$$\begin{cases} \partial_t \mu + \mathrm{div}(\mu k(\mu)\nabla u) = 0, \\ |\nabla u| = k(\mu). \end{cases} \tag{7.12}$$

coupled with homogeneous boundary conditions $\mu(x,t)_{|\partial\Omega} = 0$ and $u(x,t)_{|\partial\Omega} = 0$ and initial condition $\mu(x,0) = \mu_0(x) \geq 0$. The first observation is that in (7.11), $|\nabla u|$ may blow up as μ becomes close to 1. In addition, due to the nonlinearity in (7.10), the entropic solution of such equation is not unique in general. This makes the mathematical and numerical treatment of (7.12) difficult. In the one dimensional case, the authors in [37] propose a "double" regularization of (7.11), in the sense that they add a viscosity term as in (2.4) in addition to the regularization of the right hand side in (7.11) to avoid discontinuities of ∇u. More precisely, they solve (7.11) with $k_\delta(\mu) = (\delta + k(\mu))^2$ instead of $k(\mu)$ for $\delta > 0$. It turn out that our variational approach presented in Chapter 3 seems to give good results as the main step in our algorithm to solve (7.11) will consist in projecting onto the euclidean ball centred at 0 and of a "large" radius. Moreover, discontinuous right hand side in the Eikonal equation seems not to affect the method. Following [87], we intend to solve (7.12) iteratively as follows. Assuming that μ is known at time t_n, we compute the potential u_n by solving (7.11) using the augmented Lagrangian method. Then, we update the velocity field v_n and we plug it in (7.10) to compute the new density μ_{n+1}.

7.3 HJ on networks

We are also interested in adapting our approach to the framework of networks or more generally, metric random walk spaces in the spirit of [83, 84]. HJ equations on networks are of main interest particularly in modelling vehicular traffic flows, social networks and data transmission *etc*. Contrary to linear PDE, the theory of HJ equation on networks is under development and we refer the reader to these recent works [1, 22, 69, 96] and the references therein more details on the topic. Our starting point will be to consider HJ equations on topological networks, i.e., graphs embedded in the Euclidean space. To make this section self-contained we recall some notions and definitions from [96].

Let $\mathcal{V} = \{v_i\}_{i\in I} \subset \mathbb{R}^N$ be a collection of pairwise distinct points called vertices, $\mathcal{C} = \{\gamma_j\}_{j\in I}$ a collection of smooth, non-intersecting curves defined through

$$\gamma_j : [0, l_j] \to \mathbb{R}^N, l_j > 0, \text{ for all } j \in J,$$

and $\mathcal{E} = \{e_j\}_{j\in I}$ be the collection of edges such that

- $e_j = \gamma_j((0, l_j))$ and $\bar{e}_j = \gamma_j([0, l_j])$,
- $\gamma_j(0), \gamma_j(l_j) \in \mathcal{V}$ for every $j \in J$,
- $\text{card}(e_j \cap \mathcal{V}) = 2$ for every $j \in J$,
- $\bar{e}_j \cap \bar{e}_k \subset \mathcal{V}$ and $\text{card}(\bar{e}_j \cap \bar{e}_k) \leq 1$.
- Every two vertices $p, q \in \mathcal{V}$ can be connected by a finite sequence of edges. In other words, there exists $\{e_j\}_{j=1}^N$ such that $p \in \bar{e}_1, q \in \bar{e}_N$ and $\text{card}(\bar{e}_j \cap \bar{e}_{j+1}) = 1$.

Given $x_i \in \mathcal{V}$, $\text{Inc}_i = \{j \in J : x_i \in \bar{e}_j\}$ is the set of indices of edges having x_i as an endpoint.

Then $\mathcal{N} = \bigcup_{j\in J} \bar{e}_j$ is called a topological network. We say that $u : \overline{\mathcal{N}} \to \mathbb{R}$ is continuous if its restriction u_j to $[0, l_j]$ is continuous for each $j \in J$.

Definition 7.2 (Differentiation).

- If $x \in e_j$, the differential along e_j is defined through
$$\partial_j u(x) = \frac{\partial}{\partial x} u_j(\gamma_j^{-1}(x)).$$
- If $x = v_i$, then
$$\partial_j u(x) = \frac{\partial}{\partial x} u_j(\gamma_j^{-1}(x)), \text{ for } j \in \text{Inc}_i.$$

On a topological network, a Hamiltonian $H : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$ is regarded as a collection $(H_j)_{j\in J}$ of continuous functions $H_j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$ satisfying analogous assumptions to (H1)-(H3) stated Chapter 2-Section 2.1.2. In addition, to ensure the continuity of H on the vertices and take into account the fact that $\mathcal{N}$ is not oriented, we assume

- $H_j(\gamma_j^{-1}(v_i), p) = H_k(\gamma_k^{-1}(v_i), p)$, for any $p \in \mathbb{R}, i \in I, j, k \in \mathrm{Inc}_i$
- $H_j(\gamma_j^{-1}(v_i), p) = H_j(\gamma_j^{-1}(v_i), -p)$, for any $p \in \mathbb{R}, i \in I, j \in \mathrm{Inc}_i$.

Given $g : \overline{\mathcal{N}} \to \mathbb{R}$ be a continuous function and consider the following Dirichlet problem

$$\begin{cases} H(x, \partial u(x)) = 0 & \text{in } \mathcal{N} \\ u = g & \text{on } \partial\mathcal{N}. \end{cases} \tag{7.13}$$

Then, an appropriate notion of viscosity solution to equation of type (7.13) can be introduced (again, see e.g.[22, 96]). Moreover, for $x, y \in \overline{\mathcal{N}}$, we can define similarly to the standard setting, the associated metric to H:

$$d_\sigma(x, y) = \inf_{\eta \in \Gamma(x,y)} \int_0^1 \sigma(\eta(t), \dot{\eta}(t))\mathrm{d}t,$$

where

$$\Gamma(x, y) = \Big\{\eta : [0, 1] \to \mathcal{N} \text{ piecewise differnetiable curves such that: } \eta(0) = x, \eta(1) = y.\Big\},$$

and

$$\sigma_j(x, p) = \sup_{q \in Z_j(\gamma_j^{-1}(x))} p \cdot q.$$

we have the following

Theorem 7.3 ([96]). *Assume that g satisfies the following compatibility condition*

$$g(x) - g(y) \le d_\sigma(y, x), \textit{ for any } x, y \in \partial\mathcal{N},$$

then the unique viscosity solution to (5.1) *is given by*

$$u(x) = \inf_{y \in \partial\mathcal{N}} \{g(y) + d_\sigma(y, x)\}. \tag{7.14}$$

Reproducing the same arguments as in Chapter 3, the maximal viscosity solution given by the formula (7.14) is the unique solution of the following maximization problem

$$(\mathcal{M}) : \max \Big\{ \int_{\mathcal{N}} u\mathrm{d}x : \sigma^*(x, \nabla u(x)) \le 1 \text{ a.e and } u = 0 \text{ on } \partial\mathcal{N} \Big\}$$

where $\int_{\mathcal{N}} u\mathrm{d}x = \sum_{j \in J} \int_0^{l_j} u_j(s)\mathrm{d}x$. Many interesting questions remain to explore, both from a mathematical and numerical perspectives. In particular, the dual problem of $(\mathcal{M})$ and the optimal transport problems linked to it are worth exploring.

7.4 Geodesic extraction

In geometry, the shortest paths or geodesics play a central role. Consider for example a Riemannian manifold $(\mathbf{M}, \mathfrak{m})$, then for any $x, y \in \mathbf{M}$, the geodesic distance is defined by

$$d_{\mathbf{M}}(x,y) = \min_{\xi\in\Gamma(x,y)} \int_0^1 \mathfrak{m}\Big((\dot\xi(\mathrm{t}), \dot\xi(\mathrm{t}))\Big)^{\frac{1}{2}} \mathrm{d}t,$$

where $\Gamma(x, y)$ is the set of smooth curves joining x and y. Then $\xi \in \Gamma(x, y)$ is a geodesic if $d_{\mathbf{M}}(x,y) = \int_0^1 \Big(\mathfrak{m}(\dot\xi(t), \dot\xi(t))\Big)^{\frac{1}{2}} \mathrm{d}t$. We know that for any $D \subset M$, $d_{\mathbf{M}}(., D)$ is the unique viscosity solution of the Eikonal equation

$$\|\nabla u\|_{\mathfrak{m}^{-1}} = 1 \ \text{ in } \mathbf{M} \setminus D, \ \text{ and } \ u = 0 \ \text{ on } \ D, \tag{7.15}$$

where $\|.\|_{\mathfrak{m}^{-1}}$ is the dual norm associated to $\mathfrak{m}$. Once the distance computed, a gradient descent allows extracting the geodesic ξ between a given $x \in \mathbf{M}$ and D. More precisely, the geodesic ξ solves

$$\dot\xi(t) = -\frac{\mathfrak{m}(\xi(t))^{-1}\nabla d_{\mathbf{M}}(\xi(t), D)}{\|\mathfrak{m}(\xi(t))^{-1}\nabla d_{\mathbf{M}}(\xi(t), D)\|_{\mathfrak{m}}}, \ \text{ with } \xi(0) = x.$$

This kind of questions appear in many applications such as meshing, image analysis and shape detection of tumors from medical images, or tractography and neural fiber tracking in neuroscience (see e.g.[77, 89] and the references therein). We are planing to apply our method to address these questions, compare with existing results and possibly consider general forms of (7.15) on Finsler manifolds.

Acronyms

HJ	Hamilton-Jacobi
OHJ	Obstacle Hamilton-Jacobi
SfS	Shape from Shading
MK	Monge-Kantorovich
BK	Beckmann

Glossary

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^N$, $\mathbb{S}^N$	The N-dimensional Euclidean space and sphere.
$\mathscr{X}$, $\mathscr{Y}$	Two Banach or Hilbert spaces.
$\lvert . \rvert$ or $\lVert . \rVert$	The Euclidean norm.
$\lim, \liminf, \limsup$	Limit, inferior limit, superior limit.
Ω	A nonempty bounded domain of $\mathbb{R}^d$.
∇	Gradient operator.
div, $\nabla\cdot$	Divergence operator.
Δ, Δ_p	Laplace and p-Laplace operators.
$C(\Omega)$, $C_b(\Omega)$, $C_c(\Omega)$	Spaces of continuous, bounded continuous and compactly supported continuous functions on Ω.
$C^\infty(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$	Spaces of infinitely differentiable functions, infinitely differentiable with compact support and distributions on Ω.
$\mathcal{P}(\Omega)$, $\mathcal{M}(\Omega)$, $\mathcal{M}_+(\Omega)$, $\mathcal{M}_b(\Omega)$	Spaces of probability measures, finite measures, positive finite measures and vector valued measures on Ω.
$\mathcal{S}_H^-$, $\mathcal{S}_H^+$, $\mathcal{S}_H$	The sets of viscosity subsolutions, supersolutions and solutions of HJ equation.
Lip(Ω)	The set of Lipschitz functions on Ω.
χ_A	Characteristic function of a set A: $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.
$\mathbb{I}_C$	Indicator function of a set C in the sense of convex analysis, i.e., equals 0 on C and ∞ on C^c.
$\Gamma_0(\mathscr{X})$	The set of propoer, convex and l.s.c functions on $\mathscr{X}$.
δ_x	Dirac mass at x.
$\lvert C\rvert$, $\mathcal{L}^N(C)$	Lebesgue measure of a set $C \subset \mathbb{R}^N$.
$\mu \ll \nu$	The measure μ is absolutely continuous w.r.t ν.
$\mu \llcorner C$	The restriction of the measure μ to a set C.
$\mu_n \rightharpoonup \mu$	The μ_n converges to μ in the sense of set measures.
$\lvert\Phi\rvert$	The total variation of a vector measure Φ.
$\frac{\Phi}{\lvert\Phi\rvert}$	The Radon-Nikodym derivative of Φ w.r.t $\lvert\Phi\rvert$.
$M^\dagger$	The transpose of a matrix M.

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