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Iterated Monodromy Groups and Transcendental Dynamics
Groupes de monodromie itérée et dynamique transcendante

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Marseille, 31 mai 2021

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Résumé

Les groupes de monodromie itérée relient la dynamique rationnelle et la théorie géométrique des groupes. Dans cette thèse, nous étendons cette connexion à la dynamique transcendante. Nous introduisons les groupes de monodromie itérée pour les fonctions entières post-singulièrement finies et les étudions comme des groupes autosimilaires sur des alphabets infinis. En utilisant l'existence d'araignées périodiques, nous donnons un modèle combinatoire des groupes de monodromie itérée en termes d'automates dendroïdes, généralisant la description pour les polynômes post-singulièrement finis. La classe des applications de la famille exponentielle est discutée en détail, avec une description explicite en termes des suites de tricotage. Nous introduisons un critère de moyennabilité pour les groupes générés par des automates d'activité bornée sur des alphabets infinis, et nous utilisons ce critère pour montrer que le groupe de monodromie itérée d'une fonction entière post-singulièrement finie est moyennable si et seulement si son groupe de monodromie l'est.

Mots-clés. Groupe de monodromie itérée, dynamique transcendante, dynamique holomorphe, automate dendroïde, araignées, moyennabilité.

Abstract

Iterated monodromy groups link rational dynamics and geometric group theory. In this thesis we extend this connection to transcendental dynamics. We introduce iterated monodromy group for post-singularly finite entire functions and study them as self-similar groups with infinite alphabets. Using the existence of periodic spiders, we give a combinatorial model of the iterated monodromy groups in terms of dendroid automata, generalizing the description for post-singularly finite polynomials. We discuss the class of functions in the exponential family, with an explicit description in terms of the kneading sequence. We introduce an amenability criterion for groups generated by bounded activity automata on infinite alphabets, and use the criterion to show that the iterated monodromy group of a post-singularly finite entire function is amenable if and only if its monodromy group is.

Keywords. Iterated monodromy group, transcendental dynamics, holomorphic dynamics, dendroid automaton, spiders, amenability

Résumé substanciel

Nous donnons un aperçu de la thèse.

Dans le Chapitre 1, nous fournissons des informations de base sur la dynamique holomorphe, la dynamique de méthodes de recherche des racines d'un polynôme (en passant), les groupes autosimilaires, les groupes de monodromie itérée et la moyennabilité.

Dans le Chapitre 2, nous commençons par l'étude des groupes de monodromie itérée dans la famille exponentielle. Nous commençons par définir les groupes de monodromie itérée pour les fonctions entières.

Nous montrons que le groupe de monodromie itérée d'une fonction exponentielle peut être explicitement décrit en termes de suite de tricotage.

Théorème (Résultat structurelle pour la famille exponentielle)**.** *Soit f une fonction exponentielle post-singulièrement finie avec suite de tricotage $x_1 \dots x_k \overline{y_1 \dots y_p} \in \mathbb{Z}^\omega$. Alors l'action monodromie itérée de f est conjuguée à l'action de $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ sur $\mathbb{Z}^*$.*

Ici, $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ est le groupe généré par un automate d'activité bornée $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ défini explicitement en termes de la suite de tricotage en analogie forte avec les automates pour les polynômes quadratiques construits dans [BN08].

Nous poursuivons avec une discussion générale sur les groupes $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ pour les suites $x_1 \dots x_k, y_1 \dots y_p \in \mathbb{Z}^*$ avec $x_k \neq y_p$. Nous montrons que pour ces groupes, le graphe de Schreier sur $\mathbb{Z}^\omega$ est une forêt d'arbres avec un nombre dénombrable de bouts. En particulier, nous avons une preuve élémentaire que l'action sur $\mathbb{Z}^\omega$ est récurrente.

Nous concluons le chapitre 2 avec le théorème suivant.

Théorème. *Les groupes $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ sont moyennables mais ils ne sont pas sous-exponentiellement moyennables.*

Dans le Chapitre 3, nous discutons du cœur de la théorie des groupes de la thèse. Nous passons aux groupes généraux autosimilaires agissant sur des alphabets infinis et prouvons notre principal critère de moyennabilité.

Nous voulons montrer la moyennabilité d'une grande classe de groupes générés par des automates d'activité bornée, en vue des groupes de monodromie itérée de fonctions entières. Lorsque nous changeons la cardinalité de l'alphabet X de finie à infinie, le groupe symétrique $\mathrm{Sym}(X)$ passe d'un groupe fini (et donc élémentairement moyennable) à un énorme groupe indénombrable contenant des copies de tous les groupes dénombrables via le théorème de Cayley. Nous devons donc imposer certaines restrictions sur l'action de premier niveau. Nous le faisons en considérant un groupe moyennable $P \subset X$ et le groupe $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$, qui correspondent à des automates d'activité bornée où l'action de chaque état sur X est dans P. Nous montrons le théorème suivant.

Théorème (Moyennabilité pour activité bornée sur des alphabets infinis)**.** *Soit X un ensemble infini dénombrable. Soit P un sous-groupe de* $\mathrm{Sym}(X)$. *Supposons que l'action de P sur X est récurrente. Alors* $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ *est moyennable.*

Le critère de la moyennabilité est basé sur le critère de [JNS16]. Une étape clé de la preuve est de montrer que l'action de $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ sur les mots infinis X^ω est récurrente.

Dans le Chapitre 4, nous discutons de la structure des groupes de monodromie itérée pour les fonctions transcendantes entières générales post-singulièrement finies. Nous étendons la notion d'automates dendroïdes de [Nek09] aux alphabets infinis et fournissons un résultat de structure sur les groupes de monodromie itérée par le théorème suivant.

Théorème (Résultat structurelle pour les fonctions entières générales)**.** *Soit f une fonction entière post-singulièrement finie. Alors le groupe de monodromie itérée de f est un groupe autosimilaire sur un alphabet infini, généré par un automate dendroïde. En particulier, c'est un groupe autosimilaire d'activité bornée.*

Nous montrons ce résultat en utilisant le langage des bisets et des araignées. En particulier, nous étendons la notion d'un ensemble dendroïde de permutations aux alphabets infinis également. En faisant cela, nous montrons que les fonctions entières structurellement finies ont groupes de monodromie élémentairement moyennables. Afin d'obtenir le théorème, nous devons prouver l'existence d'araignées périodiques.

Nous comparons également le graphe de Schreier d'une fonction entière post-singulièrement finie avec la notion classique de complexes de lignes (voir [GO08, Chapitre 7]), afin de montrer que l'action de monodromie est toujours récurrente. Nous pouvons donc appliquer les résultats du Chapitre 3 pour conclure avec le théorème suivant.

Théorème (Moyennabilité de IMGs de fonctions entières)**.** *Soit f une fonction transcendante entière post-singulièrement finie. Alors le groupe de monodromie itérée de f est moyennable si et seulement si le groupe de monodromie de f est moyennable.*

Dans le Chapitre 5, nous concluons cette thèse par une brève perspective sur les travaux futurs.

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1 Introduction

This thesis connects holomorphic dynamics, in particular the dynamics of entire transcendental functions, with self-similar groups in the form of iterated monodromy groups. We investigate iterated monodromy groups of entire functions from a group theoretic perspective; one of our main results is that the iterated monodromy group of an entire function is amenable if and only the monodromy group is. In this introduction we briefly describe the background of the thesis: we introduce holomorphic dynamics in Section 1.1, root finding in Section 1.2, self-similar groups in Section 1.3, iterated monodromy groups in Section 1.4, and amenability in Section 1.5. We give an overview about the structure of the thesis and its main results in Section 1.6.

1.1 Holomorphic Dynamics

Holomorphic dynamics is the study of holomorphic functions under iteration. In one dimension, the four main classes to consider are polynomials $p\colon \mathbb{C} \to \mathbb{C}$, rational maps $r\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, entire transcendental functions $f\colon \mathbb{C} \to \mathbb{C}$ and meromorphic functions $g\colon \mathbb{C} \to \hat{\mathbb{C}}$. We will sketch here some of the basic concepts of holomorphic dynamics; see [Mil06] for a general introduction. We focus on entire functions, including polynomials and entire transcendental functions.

The fundamental sets in the study of the dynamics of a holomorphic function f are the *Fatou set* $F(f)$ (the set of normality of the family of iterates $\{f, f^2, f^3, \ldots\}$), and its complement, the *Julia set* $J(f)$. The Fatou set and the Julia set are both completely invariant under f, so they give a basic dynamical decomposition of the dynamical plane. The connected components of the Fatou set, in short Fatou components, can be classified in terms of their dynamics and is quite easy to understand. However, the dynamics on the Julia set is more “chaotic”.

Another important set for dynamics of an entire function f is the *escaping set* $I(f)$, the set of points which escape to infinity, i.e.

$$I(f) = \left\{ z \in \mathbb{C} \,\middle|\, \lim_{n\to\infty} f^n(z) = \infty \right\}.$$

The escaping set is also completely invariant and thus induces another decomposition $\mathbb{C} = I(f) \,\dot\cup\, (\mathbb{C} \setminus I(f))$. Its advantage is that this decomposition is always non-trivial: $I(f)$ is always a non-empty proper subset of $\mathbb{C}$ [Erë89] (while the Fatou set may or may not be empty). The Julia set is always the boundary of the escaping set, so its dynamics can be studied via the escaping set.

The topology of the escaping set differs between polynomials and transcendental entire functions: for polynomials, the escaping set is open and constitues the unique unbounded connected component of the Fatou set. It is a punctured neighborhood of $\infty \in \hat{\mathbb{C}}$. For transcendental entire functions, the topology of the escaping set is much more delicate because of the essential singularity at ∞. For general entire functions, it is possible to have Fatou components, such as Baker domains, that are in the escaping set. For the entire functions that we will consider in the thesis, the escaping set has no interior, so the Julia set is equal closure of the escaping set.

The dynamics of $I(f)$ can be used as a stepping stone to understand the dynamics of an entire function on the whole complex plane. For example, if f is a polynomial of degree d with connected Julia set, then there is a conformal map $\phi\colon \mathbb{C}\setminus\mathbb{D} \to I(f)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\setminus\mathbb{D} & \xrightarrow{\phi} & I(f) \\ \downarrow{\scriptstyle z^d} & & \downarrow{\scriptstyle f} \\ \mathbb{C}\setminus\mathbb{D} & \xrightarrow{\phi} & I(f) \end{array}$$

So the dynamics on $I(f)$ can be compared with the dynamics of the simpler map z^d. The images under ϕ of radial lines are called *dynamic rays*. A basic question for the dynamics of f is which dynamic rays *land*, i.e., for which angles $\alpha \in \mathbb{R}/\mathbb{Z}$ the limit $\lim_{r\to 1} \phi(re^{\alpha 2\pi i})$ exists. A fundamental result is that (pre-)periodic rays always land at (pre-)periodic points and conversely that every (pre-)periodic point in the Julia set is the landing point of a dynamical ray. Moreover, if the Julia set of f is connected and locally connected, then all dynamic rays land, and the landing points depend continuously on the angles.

Many features of the dynamics of an entire function are controlled by its *singular values*. For example, the Julia set of a polynomial is connected if and only if all of the singular values in $\mathbb{C}$ have bounded orbits. So the dynamics of post-singularly finite maps should be the easiest to understand, and in fact there are many combinatorial tools to do this. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. A *critical value* is the image of a critical point, i.e. $f(c)$ where $f'(c) = 0$. An *asymptotic value* is a limit $\lim_{t\to\infty} f(\gamma(t))$ where $\gamma\colon [0,\infty) \to \mathbb{C}$ is a path with $\lim_{t\to\infty}\gamma(t) = \infty$. The set of singular values is defined as

$$\mathbf{S}(f) = \overline{\{\text{critical and asymptotic values}\}}$$

and the set of post-singular values is $\mathbf{P}(f) = \overline{\bigcup_{n\geq 0} f^n(\mathbf{S}(f))}$. The map f is called *post-singularly finite* if $\mathbf{P}(f)$ is finite. An example of a post-singularly finite polynomial is given by the *Basilica map* $z^2 - 1$. Its Julia set is the boundary of the black set in Figure 1.1.

It is a frequent observation in one-dimensional holomorphic dynamics that most of the dynamical behavior of f is controlled by the dynamics of the singular values

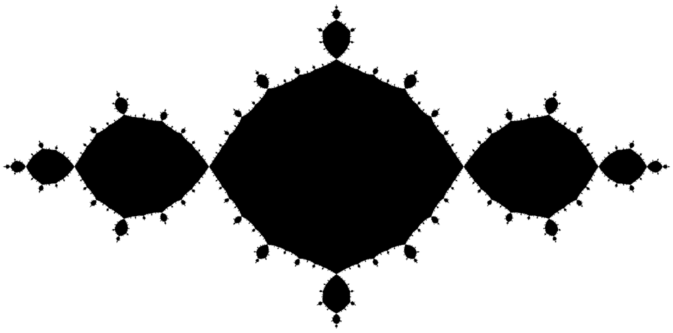


Figure 1.1 – The filled Julia set of $z^2 - 1$ (the "Basilica") in black.

of f. For example, the Julia set of a polynomial is connected if and only if all of the singular values in $\mathbb{C}$ have bounded orbits. So the dynamics of post-singularly finite maps should be the easiest to understand, and in fact there are many combinatorial tools to do this.

For post-singularly finite polynomials, of special interest is the collection of (possibly extended) dynamic rays that land in the post-singular set. Choosing a dynamic ray for each point $z \in \mathbf{P}(f)$, one obtains the notion of a *dynamical spider*: if z is in the Julia set, then one can take a ray that lands at z, while the construction is a bit more complicated if z is in the Fatou set. Spiders are a powerful tool for the classification of polynomials [HS94; BFH92]. Given only simple combinatorial data for the spiders, such as the external angles of all the rays that constitute a spider, it is enough to reconstruct a topological model for the polynomial that is sufficient to specify the actual polynomial uniquely. This is a particular case in which Thurston's fundamental theorem [DH93] can be applied particularly successfully, leading to a very explicit classification of polynomials.

For entire functions of finite order, the escaping set can be also structured via dynamic rays [Rot+11]. This can be used as a basis for the symbolic dynamics of these entire functions, see for example [Mih09]. In particular, the theory of spiders for functions in the exponential family is well developed [SZ03].

It was already observed in [Rot+11] that the topology of connected components of $I(f)$ can be much more complicated than curves that constitue the rays. The appropriate substitute for dynamic rays are "dreadlocks" [BR20]. For a detailed result on was what is possible, see [Rem19]. A discussion of landing proprieties of dreadlocks can be found, for example, in [Pfr19].

1.2 Dynamics of Root finding

A natural source of holomorphic dynamcial systems are root finding algorithms, which are often iterated holomorphic maps. Some of these are described briefly in this section, which is independent of the rest of the thesis (but not of the work of its author).

A prototypical situation is the problem of root finding of polynomials. Given a polynomial $p \in \mathbb{C}[z]$, the fundamental theorem of algebra states that p can be factored as $p(z) = \lambda \prod_{i=1}^{d}(z - \alpha_i)$. The theorem of Abel–Ruffini implies that it is not always possible to find the roots of p in terms of radicals. In practice, iterative methods are used to find the roots numerically.

One such method is Newton's method. For a polynomial p, the Newton map N_p is given by $z \mapsto z - p(z)/p'(z)$. The Newton map is then a rational map $N_p \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and can be studied using the tools of holomorphic dynamics. In the revival of holomorphic dynamics in the 1980's, understanding the global dynamics of Newton maps for cubic polynomials was a motivating problem [Dou86].

Simple roots of the polynomial p become superattracting fixed points of N_p, while multiple roots are attracting fixed points. So N_p has local quadratic convergence for polynomials with only simple roots, and local linear convergence in the general case. In fact, Newton's method is a popular way to polish good approximations of roots that may have been found by other methods.

It was already observed by Smale in the 1980's that the Newton method admits attracting cycles of periods 2 or greater, hence open sets of starting points that fail to converge to roots. Newton's method became a poster child of a "chaotic" dynamical system. This motivated the common preconception that one shouldn't use Newton to find all roots of polynomials globally. However, using the method of holomorphic dynamics, it is possible to use a starting set that is guaranteed to find all roots, see [HSS01] and subsequent refinements.

Other methods, such as the Weierstrass method (also known as Durand–Kerner) and the Ehrlich–Aberth method, work by trying to find all roots at the same time. The Ehrlich–Aberth method is the basis of MPSolve, a standard root-finding software package that has passed the test of time very well.

It was also conjectured by Smale that the Weierstrass method and Ehrlich–Aberth methods admit attracting cycles, an issue that may seem as natural from the point of view of dynamical systems as it seems unlikely from the root finding perspective. Despite the age of the methods, such feature had never been observed numerically, and a frequent assumption in the numerical analysis community was that they "always converge unless good reasons exist for the opposite" (such as preserved symmetries).

However, we established recently [RSS20] that for every degree greater or equal to 3, there is an open set of polynomials with an attracting 4-cycle for the Weierstrass method (in the Jacobi variant).

An interesting dynamical difference of Newton's method to the Weierstrass and Ehrlich-Aberth method can be observed at infinity. For the Newton method, $\infty \in \hat{\mathbb{C}}$ is a repelling fixed point. So it is only possible to go to infinity by doing it in finite time, i.e., landing on a critical point of p and thus a pole of N_p. In contrast, we showed that the Weierstrass method and the Ehrlich–Aberth methods both have orbits that diverge to infinity but are well defined for all time. See [Rei20b]. This dynamical possibility seems to not have been anticipated by the numerical analysis community.

These results were obtained by us during the past few years, partly in collaboration, but do not form part of our thesis.

1.3 Self-Similar groups

We give a short introduction into self-similar groups motivated from automaton groups. For a more in-depth discussion see for example [Nek05].

The study of self-similar groups originated from the study of groups generated by Mealy automata. A Mealy automaton is a finite-state machine that transforms words in an input alphabet into words in an output alphabet, transforming one letter at a time. We will only consider Mealy automata with the same input and output alphabet. A *Mealy automaton* is then determined by its *state set* S and its *alphabet* X and its transition function $\tau\colon S \times X \to X \times S$. We will always assume that the state set is finite, while the alphabet might be infinite. We will write $\tau(q, x)$ also as $(q(x), q_{|x})$ and say that $q(x)$ is the *image* of x under q and $q_{|x}$ is the *section* of q at x.

A classical way to visualize a Mealy automaton is by using a Moore diagram. This diagram has as vertices the set S and for every $q \in S, x \in X$, we insert a edge from q to $q_{|x}$ labeled $x|q(x)$. See for example Figure 1.2 for the Moore diagram of what we will call the *Basilica automaton*. It was already studied in [GŻ02].

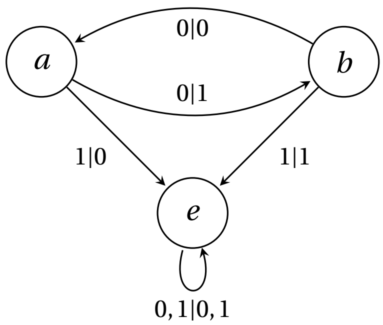


Figure 1.2 – Moore Diagramm of Basilica automaton

The transition function $\tau\colon S \times X \to X \times S$ of a Mealy automaton can be recursively extended to a function $\tau\colon S \times X^* \to X^* \times S$. A way to compute this extension for $q \in S, v \in X^*$ is to start with the concatenation qv and iteratively substitute substrings of the form $q'x$ for $q' \in S, x \in X$ with $q'(x)q'_{|x}$. Each substitution moves the state symbol one letter to the right, and the resulting string is the concatenation of $q(v)$ and $q_{|v}$. See Figure 1.3 for an example.

Each state $q \in S$ determines a mapping $X^* \to X^*$ via $v \mapsto q(v)$. If these mappings are bijections for all $q \in S$, we say that the Mealy automaton is a *group automaton*. It is straightforward to check that it is enough to verify this on words of length 1: if $x \mapsto q(x)$ is a bijection on X for all $q \in S$, an easy induction argument shows that in this case also all extended mappings on X^* are bijections. We see that the Basilica automaton is an example of a group automaton. Here the state e acts as the identity on X^*, so it is an *identity state*. Identity states are also commonly denoted by **1**.

The mappings on X^* given by a state of a Mealy automaton satisfy an additional property, they are *prefix-preserving*: if $v \in X^*$ is a prefix of $w \in X^*$, then $q(v)$ is a prefix of $q(w)$. We can formalize this structure using the *standard X-regular* tree. Its vertices are words in X^*, and for every $v \in X^*, x \in X$ we have an edge from v to vx. The empty word $\emptyset$ is the root of the tree. It is custom to also use X^* as a notation for this tree. Now for $v \in X^*$, the set of descendents of v is vX^*, the set of words that have v as a prefix. Note that the map $w \mapsto vw$ gives a natural identification of X^* and vX^*.

We denote the set of rooted tree automorphisms by $\mathrm{Aut}(X^*)$. In this setting, every state of a group automaton determines an element of $\mathrm{Aut}(X^*)$.

Let $g \in \mathrm{Aut}(X^*)$ be a rooted tree automorphism of X^*. For every $v \in X^*$, the automorphism g restricts to an isomorphism between vX^* and $g(v)X^*$. Composing the isomorphisms $X^* \cong vX^* \xrightarrow{g} g(v)X^* \cong X^*$ we obtain an automorphism $g_{|v} \in \mathrm{Aut}(X^*)$ uniquely determined by $g(vw) = g(v)g_{|v}(w)$ for all $w \in X^*$. We call $g_{|v}$ the *section* of g at v. A automorphism $g \in \mathrm{Aut}(X^*)$ is completely determined by its first level sections $\left(g_{|x}\right)_{x\in X}$ and the action on the first level $x \mapsto g(x)$. This is classically recorded in the *wreath recursion* $\mathrm{Aut}(X^*) \cong \left(\prod_{x\in X} \mathrm{Aut}(X^*)\right) \rtimes \mathrm{Sym}(X)$. We record here classical cocycle formulas for sections:

$$
\begin{aligned}
(g_{|v})_{|w} &= g_{|vw} \\
(gh)_{|v} &= g_{|h(v)} h_{|v} \\
(g^{-1})|v &= (g_{|g^{-1}(v)})
\end{aligned}
$$

We say that $S \subset \mathrm{Aut}(X^*)$ is *self-similar* if it is closed under taking sections. A *self-similar group* is a subgroup of $\mathrm{Aut}(X^*)$ that is self-similar. From the cocycle formulas, it is easy to see that the group generated by a self-similar set is also self-similar. An automorphism $g \in \mathrm{Aut}(X^*)$ is called a *finite state* automorphism if the set of sections $\{g_{|v} : v \in X^*\}$ is finite. From the cocycle formulas, we also see that the set of finite state automorphisms form a group $\mathrm{Aut}^{f.s.}(X^*)$.

For a group automaton given by $\tau : S \times X \to X \times S$, let $\Phi : S \to \mathrm{Aut}(X^*)$ be the map that associates to every state $q \in S$ its mapping $v \mapsto q(v)$. Then $\Phi(S)$ is a self-similar set, more precisely, we have $\Phi(q_{|v}) = (\Phi(q))_{|v}$, so the two notions of sections agree. Since we assume that the state set S is finite, we have that the image $\Phi(S)$ is in $\mathrm{Aut}^{f.s.}(X^*)$. Conversely, given a finite state automorphism $g \in \mathrm{Aut}^{f.s.}(X^*)$, it is possible to construct a group automaton with the set of states equal to the set of sections of g, such that g is the associated automorphism of a state of the group automaton. So finite state automorphisms are precisely the automorphisms that can be described by group automata with finitely many states.

There is an additional stratification of $\mathrm{Aut}^{f.s.}(X^*)$ due to Sidki [Sid00] based on activity growth. We give an overview for finite alphabets X here, see Chapter 2 for the necessary adjustments for infinite alphabets. For $g \in \mathrm{Aut}^{f.s.}(X^*)$, let $\alpha_n(g) \in \mathbb{N}$ be the number of words v of length n for which the section $g_{|v}$ is non-trivial. From an automaton representation of g, it follows that α_n either grows exponentially or polynomially. The finite state automorphisms that have activity growth bounded by a degree n form a subgroup of $\mathrm{Aut}^{f.s.}(X^*)$. We are particularly interested in $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*)$, the group of automorphisms of bounded activity growth.

$$\underline{a0}011 \to 1\underline{b0}11 \to 10\underline{a1}1 \to 100\underline{e1} \to 1101e$$

Figure 1.3 – Computation of $\tau(a, 0011) = (1001, e)$ for the Basilica automaton.

Groups generated by automata have often exotic properties in the perspective of geometric group theory. In particular, the first example of a group of intermediate word growth was the Grigorchuk group [Gri83], an automaton group on a binary alphabet. Our focus will be on amenability, see Section 1.5.

1.4 Iterated Monodromy Groups

Iterated monodromy groups connect holomorphic dynamics and self-similar groups. We discuss here the construction for post-singularly finite polynomials.

Before we discuss iterated monodromy groups, let us recall monodromy groups. For a polynomial f of degree at least two, it is a classical fact from the theory of Riemann surfaces that f restricts to an unbranched covering $f\colon \mathbb{C}\setminus f^{-1}(\mathbf{S}(f)) \to \mathbb{C}\setminus \mathbf{S}(f)$. For a base point $t \in \mathbb{C}\setminus \mathbf{S}(f)$, the fundamental group $\pi_1(\mathbb{C}\setminus \mathbf{S}(f), t)$ acts on $f^{-1}(t)$ by path lifting. This is the *monodromy action*.

If f is post-singularly finite, then the singular set of every iterate is contained in the post-singular set $\mathbf{P}(f)$ of f. In particular, every iterate f^n restricts to an unbranched covering $f\colon \mathbb{C}\setminus f^{-n}(\mathbf{P}(f)) \to \mathbb{C}\setminus \mathbf{P}(f)$. So for a base point $t \in \mathbb{C}\setminus \mathbf{P}(f)$ the fundamental group $\pi_1(\mathbb{C}\setminus \mathbf{S}(f), t)$ acts on all points in the backwards orbit of t at the same time. It is convenient to organize the backwards orbit of t into the *dynamical preimage tree* $\mathscr{T}$. Its vertex set is the disjoint union $\bigsqcup_{n\geq 0} f^{-n}(t)$ such that for a vertex $w \in f^{-n}(t)$, its parent is its image $f(w) \in f^{-n}(t)$. The action of $\pi_1(\mathbb{C}\setminus \mathbf{P}(f), t)$ is called the *iterated monodromy action*, and the resulting permutation group the *iterated monodromy group*.

For a post-singularly finite polynomial of degree d, the dynamical preimage tree is a rooted regular tree of degree d. If we pick a suitable *labeling* of $\mathscr{T}$, i.e., an identification of $\mathscr{T}$ with a standard regular tree X^* for some alphabet X of cardinality d, we can realize the iterated monodromy group of f as a self-similar group of X^*.

There are two major ways to introduce a labeling on $\mathscr{T}$. The first method is to use a dynamical partition and label based on the itinerary with respect to the partition. For example, if $f = z^2 + c$ is quadratic polynomial such that the critical value c is strictly preperiodic, then $J(f)\setminus\{0\}$ has two connected components. We use as alphabet $X = \{L, R\}$. For a base point $t \in J(F)\setminus \mathbf{P}(f)$, we inductively define a labeling as follows: we identify t with the empty word $\emptyset$ and if $w \in f^{-n}(t)$ is labeled via $v \in X^n$, we label the preimage of w that is in the same connected component as c in $J(f) \setminus \{0\}$ by vL and the other one by vR.

This labeling for strictly preperiodic quadratic polynomials is a special case of the construction in [BN08]. There the construction is based on spiders. This allows to also compute the iterated monodromy action and to show that the iterated monodromy action can be computed from the kneading sequence of quadratic polynomials.

Another labeling method is based on bisets. Let f be a post-singularly finite polynomial and $t \in \mathbb{C} \setminus \mathbf{P}(f)$. The *biset* $\mathcal{M}_f$ is the set of homotopy classes of paths in $\mathbb{C} \setminus \mathbf{P}(f)$ from t to a preimage of t. The biset $\mathcal{M}_f$ has commuting $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ left and right actions, see Figure 1.4 for illustration. For a path $p\colon [0,1] \to \mathbb{C} \setminus \mathbf{P}(f)$ from t to a preimage w of t, and g a loop at t, there is a unique lift g^w of g that is a path from w to a preimage w' (potentially equal to w) of t. The left action of g on p is given by the concatenation of $g^w p$, now a path from t to w'. We should note that we concatenate paths in the same way as functions, i.e., "from right to left". The right action of g on p is simply the concatenation pg, this is again a path from t to w'.

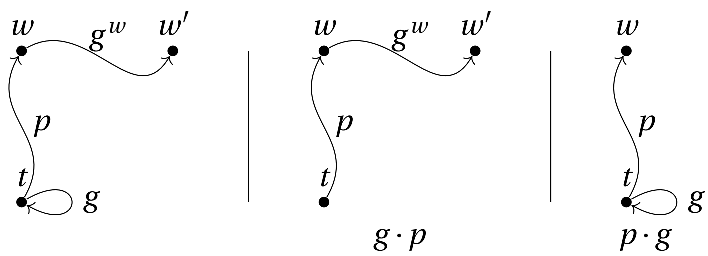


Figure 1.4 – Action on biset

The the action of $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ on the right of $\mathcal{M}_f$ is free, with right orbits corresponding to preimages of t. So the biset $\mathcal{M}_f$ is a permutational bimodule in the sense of [Nek05], and there is a general algebraic machinery to produce a self-similar group. The construction is based on choosing a *basis* X, i.e., a representative system of the right action, so for every preimage w a choice of a path from t to w.

We sketch the construction of the labeling based on the choice of the basis. If X is a basis of $\mathcal{M}_f$ consisting of paths $(p_x)_{x\in X}$, we construct a labeling on $\mathcal{T}$ as follows. The root t is identified with the root $\emptyset$ of X^*. For $w \in f^{-n}(t)$ and $x \in X$, the lift of p_x under f^n starting at w is a path p_x^w from w to a point $w' \in f^{-n-1}(t)$. If w is labeled by $v \in X^n$, then we label w' by xv. Note that we appended x at the beginning of v. In general, w' is not a preimage of w. One can show inductively that this defines a valid labeling: if $u \in f^{-n-1}(t)$ is identified with vy for some $y \in X$, then u is a preimage of w, and so p_x^u is a preimage of p_x^w. So the point labeled xvy is a preimage of the point labeled xv.

Under this labeling, the iterated monodromy action can be completely recovered from the biset.

Based on spiders, [Nek09] shows that every polynomial has an iterate whose iterated monodromy action can be described in terms of dendroid automata. In particular, iterated monodromy groups of polynomials are always isomorphic to self-similar groups generated by bounded automata.

Many classical automata groups arise as iterated monodromy groups of polynomials: the Basilica automaton [GŻ02] is associated to the basilica $z^2 - 1$, and the Fabrykowski–Gupta [FG91] group is associated to a dendroid polynomial of degree 3.

1.5 Amenability

We give a short overview about amenable groups and amenable actions. We will only consider discrete groups. For a general discussion of amenability for locally compact groups see [Pat88; Pie84].

Definition 1.5.1. Let G be a group acting on a set X. We call the action *amenable* if there is an invariant mean, i.e. a linear functional $m\colon l^\infty(X) \to \mathbb{R}$ such that

— $m(1) = 1$;
— $m(f) \geq 0$ for all $f \in l^\infty(X), f \geq 0$;
— $m(f \cdot g) = m(f)$ where $(f \cdot g)(x) = f(g(x))$ for all $f \in l^\infty(X), g \in G$.

We say that the group G is *amenable* if its regular action on itself is amenable.

Amenable groups are closed under taking extensions, subgroups, quotients and direct limits. In particular, a group is amenable if and only if its finitely generated subgroups are amenable.

An amenability criterion that is often used are Følner sequences: for a group G and a finite subset $S \subset G$, a *Følner sequence* for S is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $\lim_{n\to\infty} \frac{|F_n S \delta F_n|}{|F_n|} = 0$. Here $F_N S \delta F_n$ is the symmetric difference between $F_n S = \{gs \colon g \in F_n, s \in S\}$ and F_n. A group is amenable if and only it admits a Følner sequence for every subset $S \subset G$. If G is finitely generated, it is enough to admit a Følner sequence for some generating set S of G. For countable amenable groups, there is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G that is a Følner sequence for all finite subsets $S \subset G$ at the same time.

Finite groups are amenable, as the (normalized) counting measure is an invariant mean. For group $\mathbb{Z}$, it is easy to see that $F_n = \{-n, -n+1, \ldots n-1, n\}$ is a Følner sequence. So $\mathbb{Z}$ is amenable. Using the closure properties of amenability, it follows that every abelian group is amenable.

Finite groups and abelian groups are the basic building blocks of elementary amenable groups. The class of *elementary amenable* groups is the smallest class of groups that contains finite and abelian groups and is closed under taking extensions, subgroups, quotients and direct limits. In fact, every elementary amenable group can by constructed from finite and abelian groups only by taking extensions and direct limits.

Groups of intermediate growth, such as the Grigorchuk group [Gri83], are amenable but not elementary amenable. Similarly to elementary amenable groups we can consider the class of *elementary subexponentially amenable* groups that are obtained from groups of subexponential growth via extensions and direct limits.

The most prominent example of non-amenable groups are non-abelian free groups and hence all groups containing non-abelian free groups. The basilica group is one of the first examples of a group that is amenable [BV05] but not elementary subexponentially amenable [GŻ02].

For groups acting on topological spaces, a powerful amenability criterion is given in [JNS16]. It allows for a recursive proof of amenability for groups of homeomorphisms based on a reduction to a simpler group of germs that is known to be amenable. A key assumption is that every group element has only finitely many germs not in the simpler group of germs, and the orbits of the points with exceptional germs have a recurrent simple random walk on their orbital Schreier graph. The recurrence condition has been replaced by the crystallized notion of extensive amenability in [Jus+16].

The criterion is particularly applicable to the groups generated by automata with polynomial activity growth with recurrent simple random walks on their orbital Schreier graph. For example, the results of [BKN10] that groups generated by bounded automata on finite alphabets, in particular iterated monodromy groups of polynomials, are amenable, have been put in a unified setting.

1.6 Overview over the thesis

The main results of this thesis are presented in three independent chapters that are written as separately publishable papers, and that are already available in preprint form [Rei20c; Rei20a; Reib].

1.6.1 Chapter 1: Introduction

This is the introduction of the thesis where we provide some background.

1.6.2 Chapter 2: Iterated Monodromy Groups of Exponential Maps

In Chapter 2 we start with the study of iterated monodromy groups in the exponential family. We start by defining iterated monodromy groups for entire functions.

For the exponential family, there is a strong analogy to quadratic polynomials, based on the combinatorics of dynamic rays. We will use the dynamically defined labeling on the preimage tree. A key difference between the exponential family and general entire functions is the fact that the monodromy of the exponential function is $\mathbb{Z}$ and in particular abelian. So the dynamical pre-image tree carries an additional $\mathbb{Z}$-symmetry that is given by translations by $2\pi i$. We formalize this extra symmetry into the notion of a regular $\mathbb{Z}$-tree. The resulting self-similar groups are then ZC groups in the sense of [OS10]. We show that the iterated monodromy group of an exponential function can be explicitly described in terms of the kneading sequence.

Theorem (Structure result for the exponential family)**.** *Let f be a post-singularly finite exponential function with kneading sequence $x_1 \dots x_k \overline{y_1 \dots y_p} \in \mathbb{Z}^\omega$. Then the iterated monodromy action of f is conjugate to the action of $\mathscr{K}(x_1 \dots x_k, y_1 \dots y_p)$ on $\mathbb{Z}^*$.*

Here $\mathscr{K}(x_1 \dots x_k, y_1 \dots y_p)$ is the group generated by a bounded activity automaton $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ defined explicitly in terms of the kneading sequence, in strong analogue to the automata for quadratic polynomials constructed in [BN08].

We continue with a general discussion of the groups $\mathscr{K}(x_1 \dots x_k, y_1 \dots y_p)$ for sequences $x_1 \dots x_k, y_1 \dots y_p \in \mathbb{Z}^*$ with $x_k \neq y_p$. We show that for these groups, the Schreier graph on $\mathbb{Z}^\omega$ is a forest of trees with countably many ends. In particular, we have an elementary proof that the action on $\mathbb{Z}^\omega$ is recurrent.

We conclude Chapter 2 with the following theorem.

Theorem. *The groups $\mathscr{K}(x_1 \dots x_k, y_1 \dots y_p)$ are amenable but not elementary subexponentially amenable.*

We use the amenability criterion of Chapter 3 to show amenability and the technique of [Jus18] to conclude that the groups are not elementary subexpoentially amenable.

1.6.3 Chapter 3: Amenability of Bounded Automata Groups on Infinite Alphabets

In Chapter 3 the group theoretical core of the thesis is discussed. We move to general self-similar groups acting on infinite alphabets and prove our main amenability criterion.

We want to show amenability for a large class of groups generated by bounded activity automata, having iterated monodromy groups of entire functions in mind. When we change the cardinality of the alphabet X from finite to countably infinite, the symmetric group $\mathrm{Sym}(X)$ moves from a finite group (and thus elementary amenable) to an enormous uncountable group containing copies of all countable groups via Cayley's theorem. So we have to impose some restrictions on the first level action. We do this by considering an amenable group $P \subset X$ and the group $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$, which correspond to bounded activity automata where the action of every state on X is in P. We show the following theorem.

Theorem (Amenability for bounded activity on infinite alphabets)**.** *Let X be a countably infinite set. Let P be an amenable subgroup of* $\mathrm{Sym}(X)$. *Suppose that the action of P on X is recurrent. Then* $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ *is amenable.*

The amenability criterion is based on the work of [JNS16]. A key step in the proof is to show that the action fo $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ on the right infinite words X^ω is extensively amenable in the sense of [Jus+16]. For this it suffices to show that the random walk on every orbital Schreier graphs is recurrent. We show that we can promote recurrence on the first level to the orbital Schreier graphs of groups generated by bounded activity automorphisms. For this we need to use shortenings where we contract infinite subgraphs to points.

1.6.4 Chapter 4: Iterated Monodromy Groups of Entire Maps and Dendroid Automata

In Chapter 4 we discuss the structure of iterated monodromy groups for general post-singularly finite entire transcendental functions. We extend the notion of dendroid automata from [Nek09] to infinite alphabets and provide a structure result about iterated monodromy groups by the following theorem.

Theorem (Structure result for general entire functions)**.** *Let f be a post-singularly finite entire function. Then the iterated monodromy group of f is a self-similar group on an infinite alphabet, generated by a dendroid automaton. In particular, it is a self-similar group of bounded activity growth.*

We show this result using the language of bisets and spiders. In particular, we extend the notion of a dendroid set of permutations to infinite alphabets as well. By doing this, we show that structurally finite entire functions have elementary amenenable monodromy groups. In order to obtain the theorem, we have to prove the existence of periodic spiders. Since we are working with spiders only up to isotopy, we forgo the construction of dynamical partitions that are set-wise periodic. This greatly simplifies our construction. An alternative approach would use dynamical partitions as developed in [Mih09] based on dynamic rays, or more generally dynamical partitions based on dreadlocks as in [Pfr19].

We also compare the Schreier graph of a post-singularly finite entire function with the classical notion of line complexes (see [GO08, Chapter 7]), in order to show that the monodromy action is always recurrent. So we can apply the results of Chapter 3 to conclude with the following theorem.

Theorem (Amenability of IMGs of entire functions)**.** *Let f be a post-singularly finite entire transcendental function. Then the iterated monodromy group of f is amenable if and only if the monodromy group of f is amenable.*

1.6.5 Chapter 5: Outlook

We conclude this thesis with a brief outlook on further work / applications / research plans.

2 Iterated Monodromy Groups of Exponential Maps

2.1 Introduction

In the iteration theory of rational maps, iterated monodromy groups are self-similar groups associated to post-singularly finite dynamical systems. These groups encode the Julia set of a rational function from the point of view of symbolic dynamics [Nek05]. Conversely, many classical examples of self-similar groups with exotic geometric properties, such as the Fabrykowski-Gupta [FG91] and the Basillica group [GŻ02], arise in a natural way as iterated monodromy groups of rational maps.

Much of the study of symbolic dynamics of quadratic polynomials has been done in terms of dynamic rays, as well as in terms of kneading sequences [BS02; MT88; Thu09], before Iterated Monodromy Groups were introduced as a new and powerful tool [Nek05; BN06]. The relationships between these groups and kneading sequences were developed in [BN08].

This paper is a first in a series of papers that study of iterated monodromy groups of entire functions. Here we focus on a particularly fundamental class of functions, the exponential family, motivated by the well known strong analogy between the combinatorics of quadratic polynomials and exponential maps (see e.g. [Bod+00]). Like polynomials, exponential maps have so far only been studied in terms of rays and kneading sequences (see e.g. [SZ03]) resulting in a complete classification in [LSV08], based on [HSS09].

In this paper, we introduce iterated monodromy groups for exponential maps and compare them to self-similar groups defined just in terms of formal kneading sequences. For an exponential map f, we show that the iterated monodromy action of f is conjugate to the self-similar group action defined by the kneading sequence of f. For all kneading sequences, we show that the obtained group is a left-orderable amenable group that is residually solvable, but not residually finite.

We give a short background in holomorphic dynamics in section 2, with a special focus on the exponential family. Next in section 3 we provide the algebraic and graph theoretic background to define the iterated monodromy group of a post-singularly finite entire function. We give an explicit description of the iterated monodromy group in terms of kneading automata in section 4, see Theorem 2.**B**. The structure of the orbital Schreier graphs is investigated in section 5, where we show in Theorem 2.**C** that every component of the (reduced) orbital Schreier graph is a tree with countably many ends. This result together with the work in [Rei20a] is then used in section 6, where we collect group theoretic properties of the iterated monodromy groups of exponential functions, in particular amenability (see Theorem 2.D).

2.2 Dynamics of Exponential Maps

2.2.1 General entire dynamics

We give a very short introduction into transcendental dynamics relevant to our needs, see [Sch10] for a survey. We start with the definition of a post-singularly finite entire function.

Definition 2.2.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. A *critical value* is the image of a critical point, i.e. $f(c)$ where $f'(c) = 0$. An *asymptotic value* is a limit $\lim_{t\to\infty} f(\gamma(t))$ where $\gamma\colon [0,\infty) \to \mathbb{C}$ is a path with $\lim_{t\to\infty}\gamma(t) = \infty$. The set of singular values is defined as

$$\mathbf{S}(f) = \overline{\{\text{critical and asymptotic values}\}}$$

and the set of post-singular values is $\mathbf{P}(f) = \overline{\bigcup_{n\geq 0} f^n(\mathbf{S}(f))}$. The map f is called *post-singularly finite* if $\mathbf{P}(f)$ is finite.

The following lemma is the basis of our consideration:

Lemma 2.2.2 ([Sch10, Theorem 1.13])**.** *Let f be an entire function. Then f restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(\mathbf{S}(f))$ to $\mathbb{C} \setminus \mathbf{S}(f)$.* △

In fact, an alternative definition of $\mathbf{S}(f)$ is that $\mathbf{S}(f)$ is the smallest closed subset S such that f restricts to an unbranched covering over $\mathbb{C} \setminus S$. As $\mathbf{P}(f)$ is a closed and contains $\mathbf{S}(f)$, we see that that f also restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(\mathbf{P}(f))$ to $\mathbb{C} \setminus \mathbf{P}(f)$. As $\mathbf{P}(f)$ is forward invariant, we have that $\mathbf{P}(f) \subset f^{-1}(\mathbf{P}(f)) \subset f^{-2}(\mathbf{P}(f)) \subset \ldots$ is an increasing chain of closed subsets. From this we can show by induction that f^n restricts to an unbranched covering from $\mathbb{C} \setminus f^{-n}(\mathbf{P}(f)) \to \mathbb{C} \setminus \mathbf{P}(f)$, using the fact that compositions of coverings of manifolds are again coverings.

The *escaping set* $\mathbf{I}(f)$ is the set of points which escape to infinity under the iteration of f, i.e.

$$\mathbf{I}(f) = \{z\colon \lim_{n\to\infty} f^n(z) = \infty\}.$$

Definition 2.2.3. A *dynamic ray* is a maximal injective curve $\gamma\colon (0,\infty) \to \mathbf{I}(f)$ with $\gamma(t) \to \infty$ as $t \to \infty$.

We say that γ *lands* at a if $\gamma(t) \to a$ for $t \to 0$.

We should note that this definition is not the precise standard definition given in [Sch10], however, it is appropriate in the study of post-singularly finite exponential maps as done in [LSV08]. We will only use dynamic rays for exponential maps, so this is not an issue for us.

2.2.2 Combinatorics of exponential maps

The *exponential family* is the family of functions $E_\lambda(z) = \lambda \exp(z)$ for $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The only singular value of $\lambda \exp(z)$ is 0. It is the limiting value along the negative real axis. It is also an omitted value, so for the exponential family, Lemma 2.2.2 specialized to the well-known fact that every function in the exponential family is a covering from $\mathbb{C}$ to $\mathbb{C}^*$.

This is in fact a universal covering, and the group of deck transformations are given by translations of $2\pi i$. In the following, we will often consider collections which form a free orbit under translations with multiplies of $2\pi i$. A prime example is the set of preimages $E_\lambda^{-1}(z)$ of any point $z \in \mathbb{C}^*$. As $\mathbf{S}(E_\lambda(z)) = \{0\}$, we have $\mathbf{P}(E_\lambda(z)) = \left\{E_\lambda^n(0) \colon n \geq 0\right\} = \{0, \lambda, E_\lambda(\lambda), \ldots\}$.

In this subsection, f will always denote a post-singularly finite function in the exponential family. In this setting, 0 is a strictly preperiodic point, as it is an omitted value and has finite forward orbit. We denote the preperiod of 0 as k and the period of 0 as p, so $\mathbf{P}(f) = \{0, f(0), \ldots f^{k+p-1}(0)\}$ with $f^{k+p}(0) = f^k(0)$.

The dynamics of post-singularly finite exponential maps can be studied via dynamical rays, as seen in the following theorem:

Theorem 2.A ([SZ03])**.** *Let $f(z) = \lambda \exp(z)$ be a post-singularly finite function in the exponential family. Then there is a dynamic ray landing at 0 which is preperiodic.* △

We collect some facts about dynamic rays of exponential maps that are all discussed in [SZ03; LSV08].

Fact 2.2.4.

1. Two different dynamic rays do not intersect, but they might land at the same point.
2. The preimage of a dynamic ray is a family of dynamic rays forming a free orbit under translations with multiplies of $2\pi i$.
3. If γ lands at a, then for every $b \in f^{-1}(a)$ there is a unique preimage component of γ landing at b.
4. If γ lands at 0, then all preimage components separate the plane, the connected components of $\mathbb{C} \setminus f^{-1}(a)$ also form a free orbit under translations with multiples of $2\pi i$.

△

Definition 2.2.5. A *ray spider* is a family $\mathbb{S} = (\gamma_a)_{a \in \mathbf{P}(f)}$ such that γ_a is a dynamic ray landing at a for each $a \in \mathbf{P}(f)$.

Remark 2.2.6. In this definition, we do not require any invariance properties.

Our notion of a ray spider is a special case of the general notion of spiders given in [SZ03]. By Theorem 2.A, there exists a ray spider: if γ is a dynamic ray landing at 0, then $\gamma_{f^i(0)} = f^i(\gamma), 0 \leq i < k+p$ is a ray spider. This spider is not necessarily forward invariant, as it might happen that $f^k(\gamma) \neq f^{k+p}(\gamma)$ (the period of the rays may be a multiple of the period of the landing point). This is not an issue in our construction as we will consider the family of pullbacks of a given spider.

Definition 2.2.7. Let $\mathbb{S} = (\gamma_a)_{a\in\mathbf{P}(f)}$ be a ray spider. The *pullback* of $\mathbb{S}$ is the ray spider $(\tilde{\gamma}_a)$ where $\tilde{\gamma}_a$ is the unique preimage of $\gamma_{f(a)}$ landing at a.

The *dynamical partition* associated to $\mathbb{S}$ is the partition of $\mathbb{C}\setminus f^{-1}(\gamma_0)$ into its connected components. We denote the connected component of 0 by $\mathbb{U}_0$ and define $\mathbb{U}_n = \mathbb{U}_0 + 2\pi i n$. Note that the dynamical partition only depends on the ray landing at 0.

The kneading sequence of f is the sequence $(k_n)_{n\in\mathbb{N}}$ so that $f^n(0)\in\mathbb{U}_{k_n}$. The kneading sequence is in fact independent of $\mathbb{S}$, see [LSV08] for a more detailed discussion.

Example 2.2.8. Let $k\in\mathbb{Z}\setminus 0$, and consider $f(z) = 2k\pi i\exp(z)$. For this map, 0 is mapped to $2k\pi i$, which is a fixed point of f. Hence f is post-singularly finite with $\mathbf{P}(f) = \{0, 2k\pi i\}$. Let γ_0 be a dynamic ray landing at 0, and let $\mathbb{U}$ be the associated dynamical partition. Then $0\in\mathbb{U}_0$ by definition of $\mathbb{U}_0$ and $2k\pi i\in\mathbb{U}_k = \mathbb{U}_0 + 2k\pi i$, so the kneading sequence of f is $0\overline{k}$.

2.3 Iterated Monodromy Groups

2.3.1 The dynamical preimage tree $\mathscr{T}$

Let f be a post-singularly finite entire function and $t\in\mathbb{C}\setminus\mathbf{P}(f)$.

Definition 2.3.1. Choose a base point $t\in\mathbb{C}\setminus\mathbf{P}(f)$. Let $L_n := f^{-n}(t)$ be the preimage of t under the n-th iterate of f.

The *dynamical preimage tree* $\mathscr{T}$ is a rooted tree with vertex set $\bigsqcup_{n\geq 0} L_n$ (where $\bigsqcup$ denotes disjoint union) and edges $w\to f(w)$ for $w\in L_{n+1}, f(w)\in L_n$. Its root is t.

The dynamical preimage tree is always a regular rooted tree, i.e. all vertices have the same number of children. For polynomials, this number is the degree of the polynomial. For transcendental entire functions, every vertex has countably infinite many children. We will show in subsection 2.3.3 that for postsingularly finite exponential maps, the dynamical preimage tree has an extra regularity based on the periodicity of the exponential map.

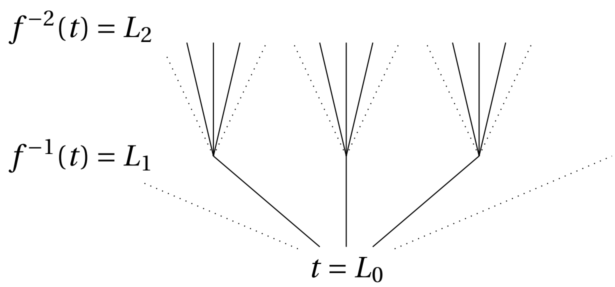


Figure 2.1 – Dynamical preimage tree

2.3.2 Iterated Monodromy Action

Each level of $\mathscr{T}$ is the preimage of t under a covering map, namely $f^n\colon \mathbb{C}\setminus f^{-n}(\mathbf{P}(f)) \to \mathbb{C}\setminus\mathbf{P}(f)$. Hence $\pi_1(\mathbb{C}\setminus\mathbf{P}(f), t)$ acts on L_n via path lifting: if $\gamma\colon [0,1]\to\mathbb{C}\setminus\mathbf{P}(f)$ is a loop based on t and $v\in L_n$ is a n-th preimage of t, then there is a unique lift γ^v making the following diagram commute:

$$\begin{array}{ccc} & & \big(\mathbb{C}\setminus f^{-n}(\mathbf{P}(f)), v\big) \\ & \overset{\gamma^v}{\nearrow} & \big\downarrow f^n \\ ([0,1],0) & \xrightarrow{\ \gamma\ } & \big(\mathbb{C}\setminus\mathbf{P}(f), t\big) \end{array}$$

So $\gamma^v(0) = v$, and $\gamma^v(1)\in L_n$ might be another n-th preimage. We define $[\gamma](v) := \gamma^v(1)$. Using the homotopy lifting properties of coverings, we can see that this defines an action of $\pi_1(\mathbb{C}\setminus\mathbf{P}(f), t)$ on L_n. If $w\in L_{n+1}$ is a child of v, then the following diagram commutes (by uniqueness of lifts):

$$\begin{array}{ccc} & & \big(\mathbb{C}\setminus f^{-n-1}(\mathbf{P}(f)), w\big) \\ & \overset{\gamma^w}{\nearrow} & \big\downarrow f^n \\ & & \big(\mathbb{C}\setminus f^{-n}(\mathbf{P}(f)), v\big) \\ & \overset{\gamma^v}{\nearrow} & \big\downarrow f^n \\ ([0,1],0) & \xrightarrow{\ \gamma\ } & \big(\mathbb{C}\setminus\mathbf{P}(f), t\big) \end{array}$$

By commutativity of the diagram $f(\gamma^w(1)) = \gamma^v(1)$ so $[\gamma](w)$ is also a child of $[\gamma](v)$. This means that actions on the levels are compatible and give rise to an action of $\pi_1(\mathbb{C}\setminus\mathbf{P}(f))$ on $\mathscr{T}$. This is the *iterated monodromy action*.

Definition 2.3.2. Let f be a post-singularly finite entire function, $t\in\mathbb{C}\setminus\mathbf{P}(f)$. Let $\phi\colon \pi_1(\mathbb{C}\setminus\mathbf{P}(f), t)\to\mathrm{Aut}(\mathscr{T})$ be the group homomorphism induced by the iterated monodromy action. The iterated monodromy group of f with base point t is the image of ϕ. By the first factor theorem we have

$$\mathrm{IMG}(f) \cong {}^{\pi_1(\mathbb{C}\setminus\mathbf{P}(f),\, t)}\big/_{\ker\phi}$$

This definition depends a priori on the base point $t\in\mathbb{C}\setminus\mathbf{P}(f)$. For a different base point t', every path from t to t' gives rise to an isomorphism of preimage trees over t and over t', so we can identify the groups up to inner automorphisms. See [Nek05, Proposition 5.1.2] for a detailed discussion in the rational case.

2.3.3 $\mathbb{Z}$-regular rooted trees

We use the following definition of rooted trees:

Definition 2.3.3. A *rooted tree* is a tuple $T = (V, E, r)$ such that (V, E) forms a tree (with vertex set V and edge set E) and $r \in V$, which we call the *root* of T. We endow T with the unique orientation so that all vertices are reachable from the root, i.e. for every vertex v, there is directed path from the root to v.

If (v, w) is a directed edge for this orientation, we say that w is a *child* of v and v is the *parent* of w. If v has no children, we call it a *leaf*.

If w is reachable form v, we say that w is a descendant of v and v is an ancestor of w. We denote by T_v the rooted tree which is the induced subgraph on the set of descendants of v together with v as the new root. An end of a rooted tree T is a sequence v_n so that v_0 is the root of T and v_{n+1} is a child of v_n. We denote by ∂T the set of ends of T.

We will mainly consider countable infinite trees without leaves.

In fact, ∂T can be defined without fixing a root of T, one way is by considering equivalence classes of geodesic rays, where two geodesic rays are equivalent if they have a common tail. Given a root r and a geodesic ray γ, there is always a unique geodesic ray starting at r equivalent to γ. Also, ∂T is a totally disconnected Hausdorff space with clopen subset $\partial T_v \subset \partial T$. The topology is also independent of the root. If T is a locally finite tree without leaves, then ∂T is compact.

Definition 2.3.4. A $\mathbb{Z}$*-regular rooted tree* T is a tuple (V, E, r, η), where (V, E, r) is a rooted tree and η is a right $\mathbb{Z}$-action $\eta\colon V \setminus \{r\} \times \mathbb{Z} \to V \setminus \{r\}$ such that for all vertices $v \in V$, the set of its children forms a free orbit under the action.

Note that this implies that the root is fixed by the action, as it is the only vertex without a parent. Also the tree has no leaves, as the empty set is not a free orbit under a $\mathbb{Z}$-action.

An isomorphism between $\mathbb{Z}$-regular rooted trees is a tree isomorphism which preserves the root and commutes with the additional right $\mathbb{Z}$-actions. We denote by $\mathrm{Aut}_{\mathbb{Z}}(T)$ the group of automorphisms of T as a $\mathbb{Z}$-regular rooted tree. Every element of $\mathrm{Aut}_{\mathbb{Z}}(T)$ preserves the root of T and acts by a translation on the first level. We denote by $\rho\colon \mathrm{Aut}_{\mathbb{Z}}(T) \to \mathbb{Z}$ the group homomorphism given by the first level action. The kernel of ρ is the stabilizer of the first level, as every element of $\mathrm{Aut}_{\mathbb{Z}}(T)$ acts by translation, this is also the stabilizer of any vertex on the first level. For a vertex $v \in V$ and a subgroup $G \subset \mathrm{Aut}_{\mathbb{Z}}(T)$ we denote the stabilizer of v in G by $\mathrm{Stab}_G(v)$. We denote the stabilizer of the first level as Stab_G.

Note that $\mathrm{Aut}_{\mathbb{Z}}(T)$ also acts on ∂T. This action is an fact faithful, as every vertex is part of a sequence defining an end.

Example 2.3.5. The standard $\mathbb{Z}$-regular tree has as vertex set $\mathbb{Z}^*$, the set of finite words in $\mathbb{Z}$. Its root is the empty word $\emptyset$. Its edges are all pairs of the form (v, vn) for

$v \in \mathbb{Z}^*, n \in \mathbb{Z}$ (here vn denotes the word v concatenated with the letter n). So for each vertex v, the set of it children are all words obtained by concatenating one letter to it. Also, the set of ends can be identified with the set of right-infinite words, which we denote by $\mathbb{Z}^\omega$.

The right action is given by

$$\eta(vn, m) = v(n+m).$$

So the action is by translation on the last letter. By abuse of notation, we will denote the standard $\mathbb{Z}$-regular tree also by $\mathbb{Z}^*$.

The subgroups of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ were studied in [OS10] under the name of *ZC-groups*. Note that if T is a $\mathbb{Z}$-regular rooted tree and v is a vertex of T, then T_v is also a $\mathbb{Z}$-regular rooted tree. However, in general we have no canonical choice of an isomorphism between T and T_v. This is different for the standard $\mathbb{Z}$-regular tree:

Definition 2.3.6. For $g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$, $v \in \mathbb{Z}^*$ let $g_{|v}$ denote the unique element in $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ such that $g(vw) = g(v)g_{|v}(w)$. We say that $g_{|v}$ is the *section* of g at v.

We will use the following set of easily verifiable cocycle equations:

$$(g_{|v})_{|w} = g_{|vw} \tag{2.3.1}$$

$$(gh)_{|v} = g_{|h(v)}h_{|v} \tag{2.3.2}$$

We say that $g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is of finite activity on level n if the set $\{v \in \mathbb{Z}^n \colon g_{|v} \neq \mathbf{1}\}$ is finite. We define $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ as the group of automorphisms which have finite activity on every level. We will mainly work with subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$. As we work with an infinite alphabet, we have to take care for the wreath recursion.

The wreath recursion for $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is

$$\begin{aligned} \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*) &\cong \left(\prod_{x\in\mathbb{Z}} \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)\right) \rtimes \mathbb{Z} \\ g &\mapsto (x \mapsto g_{|x}, \rho(g)) \end{aligned}$$

Since an automorphism in $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ has only finitely many nontrivial sections on the first level, the wreath recursion for $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is given by

$$\begin{aligned} \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}) &\cong \left(\bigoplus_{x\in\mathbb{Z}} \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)\right) \rtimes \mathbb{Z} \\ g &\mapsto (x \mapsto g_{|x}, \rho(g)) \end{aligned}$$

So we take the direct sum for $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ in the wreath recursion instead of the direct product as for $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$.

We say a subgroup $G \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is self-similar if $g_{|v} \in G$ for all $g \in G$ and $v \in \mathbb{Z}^*$. A subgroup $G \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is self-replicating if for all $v \in \mathbb{Z}^*$ and $g \in G$ there exists an $h \in G$ with $h_{|v} = g$. It is easy to see that is is enough to check this on the first level.

Lemma 2.3.7. *Let f be a post-singularly finite exponential function, $t \in \mathbb{C} \setminus \mathbf{P}(f)$. Then the dynamical preimage tree of f with base point t is a $\mathbb{Z}$-regular tree and* $\mathrm{IMG}(f)$ *is a subgroup of* $\mathrm{Aut}_{\mathbb{Z}}(\mathscr{T})$

Proof. The $\mathbb{Z}$-regular structure is given by translation by multiples of $2\pi i$. As two complex numbers have the same value under the exponential map if and only if they differ by a multiple of $2\pi i$, it is clear that this really defines a $\mathbb{Z}$-regular structure. Also, if w is an n-th preimage of t, and γ is a loop on t, for the lift γ^w, the $2\pi i$ translate of γ^w is also a lift of γ by the $2\pi i$ periodicity of f^n. This shows that the iterated monodromy action commutes which the $\mathbb{Z}$ action given by the $\mathbb{Z}$-regular structure, so $\mathrm{IMG}(f) \subset \mathrm{Aut}_{\mathbb{Z}}(\mathscr{T})$. □

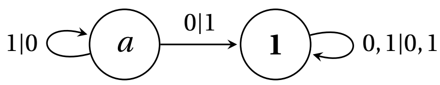
2.4 Combinatorial description

2.4.1 Automata

Definition 2.4.1. An automaton $\mathbf{A}$ is a map $\tau\colon Q \times X \to X \times Q$. We call Q the *state set* and X the *alphabet*. We will write the components of $\tau(a,x)$ often as $(a(x), a|x)$. Here $a(x) \in X$ is called the image of x under a, and $a|x$ is the restriction of a at x.

A *group automaton* is an automaton such that for all $a \in Q$, the map $x \mapsto a(x)$ is a bijection on Q. If the alphabet is $\mathbb{Z}$, that automaton is a $\mathbb{Z}$-automaton if for all $a \in Q$, the map $x \mapsto a(x)$ is a translation on $\mathbb{Z}$, i.e. equal to the map $x \mapsto x + n$ for some $n \in \mathbb{Z}$.

We will only consider automata which have a distinguished identity state $\mathbf{1}$, i.e. a state such that $\tau(\mathbf{1}, x) = (x, \mathbf{1})$ for all $x \in X$. We can draw automata using Moore diagram. As vertices we take the state set Q, and if $\tau(a,x) = (y,b)$, we draw an edge from a to b labeled $x|y$. Here is an example of a Moore diagram, of the so-called binary adding machine.



Definition 2.4.2. Let $\mathbf{A}$ be an automaton given by $\tau\colon Q \times X \to X \times Q$. We extend τ to a map $Q \times X^* \to X^* \times Q$ recursively via

$$\tau(a, xv) = (a(x)a|x(v), a|x|v)$$

If $\mathbf{A}$ is a group automaton, then for each $a \in A$, the extended map $X^* \to X^*$ induces a tree automorphism of the regular X-tree. If $\mathbf{A}$ is a $\mathbb{Z}$-automaton, it is a automorphism preserving the regular $\mathbb{Z}$-tree structure.

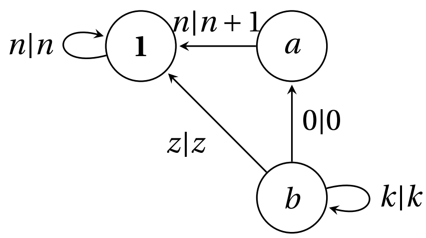
2.4.2 Kneading automata

Definition 2.4.3. Given two words $x_1 \dots x_k, y_1 \dots y_p \in \mathbb{Z}^*$ with $x_k \neq y_p$ the automaton $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ has alphabet $\mathbb{Z}$ and states $a_1 \dots a_k, b_1 \dots b_p$ (and the identity state $\mathbf{1}$) and the following transition function:

$$\begin{aligned}
\tau(a_1, z) &= (z+1, \mathbf{1}) \\
\tau(a_{i+1}, x_i) &= (x_i, a_i) \\
\tau(b_1, x_k) &= (x_k, a_k) \\
\tau(b_1, y_p) &= (y_p, b_p) \\
\tau(b_{i+1}, y_i) &= (y_i, b_i) \\
\tau(q, z) &= (z, \mathbf{1}) \text{ for all other cases.}
\end{aligned}$$

We note that $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ is a $\mathbb{Z}$-automaton, indeed a_1 acts on $\mathbb{Z}$ by the translation by one, and all other states act on $\mathbb{Z}$ as the identity. Figure 2.2 shows a reduced Moore diagram of $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$, where labels with only one letter z are abbreviations for the label $z|z$ and all trivial arrows ending in the identity state have been omitted.

Example 2.4.4. The automaton $\mathbf{K}(0, k)$ with $k \in \mathbb{Z} \setminus \{0\}$ has the following (non-reduced) Moore diagram:



Here n stands for any element of $\mathbb{Z}$, and z for any element of $\mathbb{Z} \setminus \{0, k\}$.

Remark 2.4.5. We see that every non-trivial state has exactly one edge ending in it, so for every non-trivial state there is a unique left-infinite path ending in it. This implies that $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ is a bounded activity automaton in the sense of [Sid04]: For any length m, there are $n+k$ paths of length m ending in a non-trivial state in the Moore diagram, so for any q, the set $\{v \in \mathbb{Z}^m : q|v \neq \mathbf{1}\}$ has cardinality bounded by $n+k$.

We denote by $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ the group of automorphisms of $\mathbb{Z}^*$ generated by $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$.

Theorem 2.B. *Let f be a post-singularly finite exponential function with kneading sequence $x_1 \dots x_k \overline{y_1 \dots y_p} \in \mathbb{Z}^\omega$. Then the iterated monodromy action of f is conjugate to the action of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ on $\mathbb{Z}^*$.*

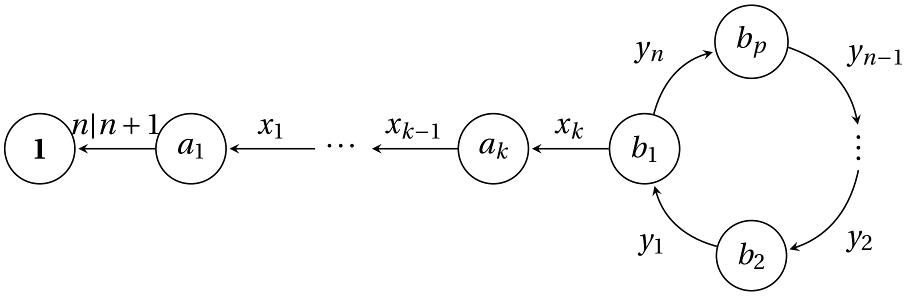


Figure 2.2 – Moore diagram of kneading automata

In particular, for functions of the form $2\pi i k \exp(z)$ with $k \in \mathbb{Z} \setminus \{0\}$, the iterated monodromy action is conjugate to the action of the automata group $\mathcal{K}(0,k)$ discussed in Example 2.4.4.

Proof. We choose a ray spider $\mathbb{S}_0$ for f and consider the sequence $\mathbb{S}_n$, where $\mathbb{S}_{n+1}$ is the pullback of $\mathbb{S}_n$. We denote by $\gamma_{z,n}$ the ray in $\mathbb{S}_n$ landing at z, also let $\mathbb{U}_{*,n}$ be the dynamical partition induced by $\mathbb{S}_n$. Choose a base point $t \in \mathbb{C} \setminus \bigcup_{z \in \mathbf{P}(f)} \gamma_{z,0}$. We recursively define an isomorphism between the dynamical preimage tree $\mathcal{T}$ and the standard $\mathbb{Z}$-tree $\mathbb{Z}^*$. We send the root t to the empty word $\emptyset$. Suppose we already defined the bijection on $L_n \subset \mathcal{T}$, and let $w \in L_n$ be mapped to $v \in \mathbb{Z}^n$. Then for the dynamical partition $\mathbb{U}_{*,n}$, there is exactly one child of w in each component. We send the child lying in $\mathbb{U}_{m,n}$ to vm.

By construction, this defines an isomorphism of $\mathbb{Z}$-trees.

The complement of each ray spider is a simply connected domain. For two points $w_1, w_2 \in \mathbb{C} \setminus \bigcup_{z \in \mathbf{P}(f)} \gamma_{z,n}$ let $g_n(w_1, w_2)$ be a path from w_1 to w_2 crossing no ray of $\mathbb{S}_n$ and let $g_{z,n}(w_1, w_2)$ be a path from w_1 to w_2 crossing only the ray of $\gamma_{z,n}$ once in a positive sense (so that $g_n(w_1, w_2)$ composed with $g_{z,n}(w_1, w_2)$ has winding number 1 around z) and no other ray of $\mathbb{S}_n$. The homotopy classes of g_n are well defined in the fundamental groupoid $\Pi_1(\mathbb{C} \setminus \mathbf{P}(f))$. Let us investigate the lifting behavior of these homotopy classes: let $w, w' \in \mathbb{C} \setminus \bigcup_{z \in \mathbf{P}(f)} \gamma_{z,n}$ and let $v \in f^{-1}(w)$. Let $g_n^v(w, w')$ (or $g_{z,n}^v(w, w')$) denote the lift of $g_n(w, w')$ (respectively $g_{z,n}(w, w')$). Then $g_n^v(w, w')$ is a path in $\mathbb{C}$ meeting no preimage of $\gamma_{z,n}$ for $z \in \mathbf{P}(f)$. Let v' be the preimage of w' in the same component of $\mathbb{U}_{*,n}$ as v. Then $g_n^v(w, w')$ must be homotopic to $g_{n+1}(v, v')$. Similarly, $g_{0,n}^v(w, w)$ is a path which doesn't cross any ray of $\mathbb{S}_{n+1}$, and as $g_{0,n}(w, w)$ has winding number 1 around 0, the lift $g_{0,n}^v(w, w)$ must end in $v + 2\pi i$. Hence $g_{0,n}^v(w, w) \cong g_{n+1}(v, v+1)$ and by composition $g_{0,n}^v(w, w') \cong g_{n+1}(v, v'+1)$. Let $z \in \mathbf{P}(f) \setminus 0$. Then $g_{z,n}^v(w, w')$ crosses no boundary of $\mathbb{U}_{*,n}$, so it must end in w'. If $\tilde{z}$, the preimage of z in the same component of $\mathbb{U}_{*,n}$ is in $\mathbf{P}(f)$, then $g_{z,n}^v(w, w') \cong g_{\tilde{z},n+1}(v, v')$, otherwise $g_{z,n}^v(w, w') \cong g_{n+1}(v, v')$.

Now $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ is freely generated by $(g_{z,0}(t,t))_{z \in \mathbf{P}(f)})$. Numerate $\mathbf{P}(f)$ by $z_1 =$

$0, z_{i+1} = f(z_i), 1 \le i \le n+k-1$. We claim that the group homomorphism given by

$$g_{z_i,0}(t,t) \mapsto a_i, 1 \le i \le k \tag{2.4.1}$$

$$g_{z_i,0}(t,t) \mapsto b_{i-k} k+1 \le i \le n+k \tag{2.4.2}$$

conjugates the iterated monodromy action of f to the action of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$. This follows from the pullback behavior.

□

2.5 Schreier Graphs

For this section, we fix $x_1 \dots x_k, y_1 \dots y_p$ with $x_k \neq y_p$. We will give a combinatorial description of the action of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ on the standard $\mathbb{Z}$-tree $\mathbb{Z}^*$. We will work in this section with the generating set $S := \{a_1, \dots, a_k, b_1, \dots, b_p\}$ of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$.

Definition 2.5.1. Let $n \in \mathbb{N}$. The n-th level *Schreier graph* has vertex set $\mathbb{Z}^n$ and edges $v \to s(v)$ for $v \in \mathbb{Z}^n, s \in S$. The *orbital Schreier graph* Γ_ω has the ends of the standard $\mathbb{Z}$-tree as vertex set (which can be identified with $\mathbb{Z}^\omega$) and also has edges $v \to s(v)$ for $v \in \mathbb{Z}^\omega, s \in S$.

The *reduced Schreier graph* $\overline{\Gamma}_n$ and reduced orbital Schreier graph $\overline{\Gamma}_\omega$ are obtained by deleting all loops of Γ_n respectively Γ_ω.

Let $w_m \in \mathbb{Z}^m$ be the reverse of the length m prefix of $x_1 \dots x_k \overline{y_1 \dots y_p}$. In the Moore diagram in Figure 2.2, we see that w_m is the concatenation of the labels of the unique path p of length m ending in a_1. Let c_m be the starting state of p (so $c_m = a_m$ for $m \le k$, and $c_m = b_{m'}$ for appropriate m' otherwise). Then $c_m|w_m = a_1$ and $s|v \neq a_1$ for all other pairs of a state s and $v \in \mathbb{Z}^m$. As a_1 is the only state which acts non-trivially on the first level, we have

$$\begin{aligned} c_m|w_m(i) &= i+1 \\ s|v(i) &= i \text{ for other pairs.} \end{aligned}$$

Since additionally a_1 only restricts to the identity state, we also have that if $s(v) = w$ with $v \neq w \in \mathbb{Z}^m$ for some state s, then $s(vi) = wi$ for all $i \in \mathbb{Z}$. In fact v and w must differ in exactly one position.

This discussion can be summarized in the following lemma:

Lemma 2.5.2. *The Schreier graph $\overline{\Gamma}_{m+1}$ can be obtained from $\overline{\Gamma}_m$ in the following way: take as vertex set vx where $v \in \mathbb{Z}^m, x \in \mathbb{Z}$. For edges we have the following two construction rules:*

- *$v \to v'$ edge in $\overline{\Gamma}_m$*
  $\Rightarrow vi \to v'i$ is an edge in $\overline{\Gamma}_{m+1}$ for all $i \in \mathbb{Z}$.
- *$w_m i \to w_m(i+1)$ for all $i \in \mathbb{Z}$.*

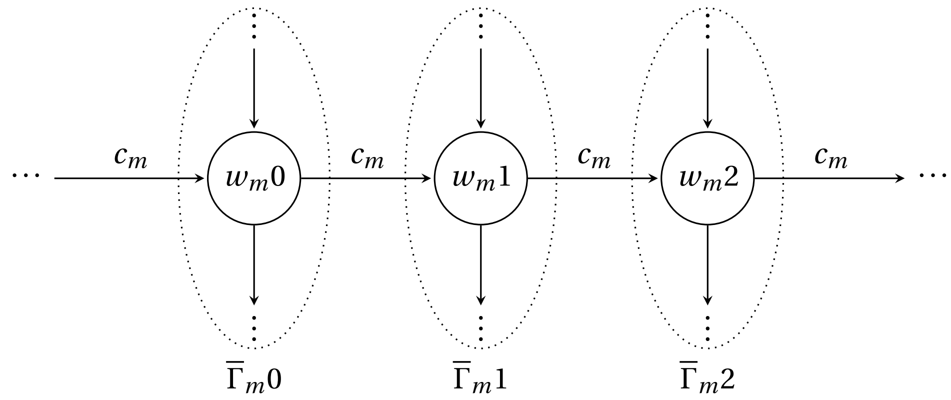


Figure 2.3 – Inductive construction of Schreier graphs

See Figure 2.3 for a visualization of the construction rules. □

Example 2.5.3. We can use this construction to produce the first few $\overline{\Gamma}_m$ for the group $\mathcal{K}(0,1)$. As in Example 2.4.4, we name the generators a and b instead of a_1 and b_1. Note that a acts by translation on the first level, and b acts trivially on the first level, so $\overline{\Gamma}_1$ is just a bi-infinite line. To use the construction rule, we note that $w_1 = 0$, so we obtain $\overline{\Gamma}_2$ as a comb in Figure 2.4. The loops at $1,0$ and $1,1$ are of course not present in the reduced Schreier graph, but we did include them here for they are the loops which "split up" in the further generations: as b restricts to a at $1,0$, we obtain $\overline{\Gamma}_3$ by connecting $\mathbb{Z}$ many copies of $\overline{\Gamma}_2$ by an bi-infinite line going through the copies of $1,0$.

With this inductive description we can prove the following:

Lemma 2.5.4. *For all $m \in \mathbb{N}$, the reduced Schreier graph $\overline{\Gamma}_m$ is a tree with countably (or finitely) many ends.*

Proof. We do induction over m. For $m = 1$, the Schreier graph $\overline{\Gamma}_1$ is a bi-infinite line, so it is in particular a tree with finitely many ends. Now by Lemma 2.5.2, $\overline{\Gamma}_{m+1}$ is the union of countably many copies of $\overline{\Gamma}_m$ and a bi-infinite line intersecting each copy in one point. So it is again a tree. We claim that have the following inductive description of the space of ends:

$$\partial\overline{\Gamma}_{m+1} \cong \mathbb{Z} \times \partial\overline{\Gamma}_m \cup \{-\infty, +\infty\} \tag{2.5.1}$$

Here the right hand space is a compactification of $\mathbb{Z}\times\partial\overline{\Gamma}_m$, where $-\infty$ has the open sets $U_{<n} := \{z \in \mathbb{Z} : z < n\} \times \partial\overline{\Gamma}_m \cup \{-\infty\}$ as neighborhood basis, and similarly $+\infty$ has the open sets $U_{>n} := \{z \in \mathbb{Z} : z > n\} \times \partial\overline{\Gamma}_m \cup \{+\infty\}$ as neighborhood basis. The identification in (2.5.1) works as follows: we take w_m as our root of $\overline{\Gamma}_m$ and w_m0 as the root of $\partial\overline{\Gamma}_{m+1}$. Then we have the following identifications:

- We send $-\infty$ to the end $(w_m(-i))_{i\in\mathbb{N}}$, i.e. we walk the bi-infinite line in the negative direction.
- We send $+\infty$ to the end $(w_m(+i))_{i\in\mathbb{N}}$, i.e. we walk the bi-infinite line in the positive direction.

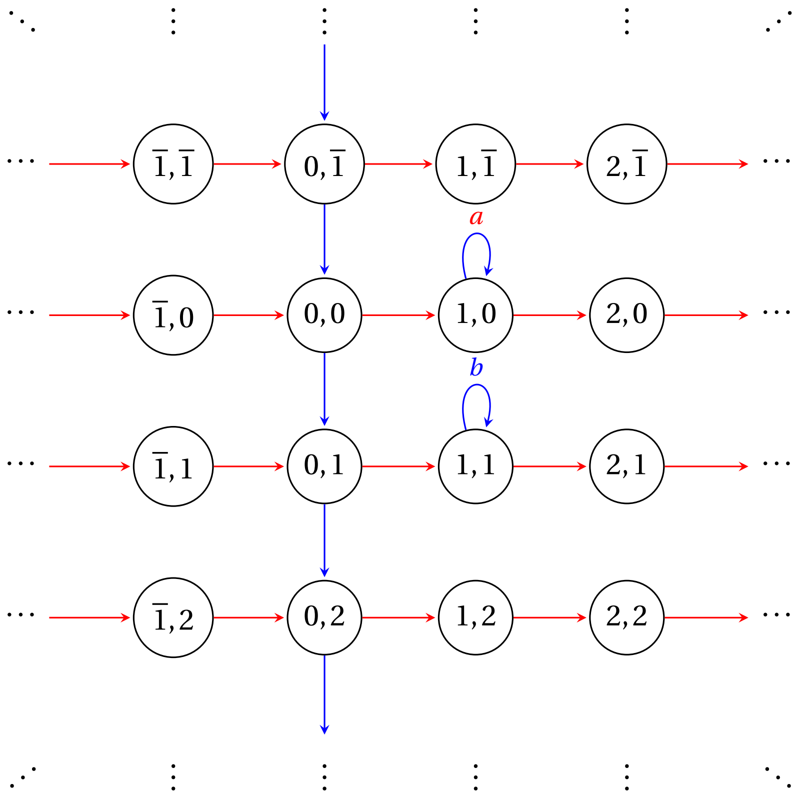


Figure 2.4 – Second-level Schreier-Graph of $\mathcal{K}(0,1)$ (where $\bar{1} = -1$ for notational convenience)

— Given a pair $(z, v) \in \mathbb{Z} \times \partial\overline{\Gamma}_m$, we identify it with the end which is given by the concatenation of the path from $w_m 0$ to $w_m z$ together with the sequence $v_n m$. This means that first walk to the root of the copy of $\overline{\Gamma}_m$ labeled by z, and then go the end defined by the sequence v in this copy.

Using Lemma 2.5.2, it is easy to check that this indeed defines a homeomorphism as given in (2.5.1). Now is a $\mathbb{Z} \times \partial\overline{\Gamma}_m \cup \{-\infty, +\infty\}$ is countable union of countable set, so $\partial\overline{\Gamma}_{m+1}$ is countable. □

Let us fix some notation related to the orbital Schreier graph $\overline{\Gamma}_\omega$. For $u \in \mathbb{Z}^\omega$, let $T_m(u)$ be the induced subgraph of $\overline{\Gamma}_\omega$ on the set $\{u' \in \mathbb{Z}^\omega \colon u_i = u'_i \text{ for all } i > m\}$. We denote the union $\bigcup_{m\in\mathbb{N}} T_m(u)$ by $T(u)$.

Theorem 2.C. *The connected component of u in $\overline{\Gamma}_\omega$ is $T(u)$. It is a tree with countably many ends.*

Proof. The projection to the prefix of length m is a bijection from the vertex set of $T_m(u)$ to $\mathbb{Z}^m$. It gives rise to graph isomorphism from $T_m(u)$ to $\overline{\Gamma}_m$, as the generating set acts by changing at most one letter at once. So $T(u)$ is an increasing union of trees, hence it is also a tree.

Each end of $T(u)$ either stays in some $T_m(u)$ or leaves all $T_m(u)$. The first kind is a countable union of countable sets, hence we only need to consider ends leaving all $T_m(u)$. Let E_m be the set of edges in $T(u)$ leaving $T_m(u)$. We have a map $E_m \to E_{m-1}$ which sends an edge e leaving $T_m(u)$ to the unique edge leaving $T_{m-1}(u)$ on the geodesic from u to e. It is possible that an edge is send to itself, if it leaves multiple subtrees at once. Now the set of ends leaving all $T_m(u)$ is isomorphic to $\varprojlim E_m$. Now the sets E_m have uniform bounded cardinality. This can be seen as follows: Let w be the m-suffix of u. Then an edge in E_m corresponds to a pair $v \in \mathbb{Z}^m, q \in S \cup S^{-1}$ with $q|v(w) \neq w$, in particular the restriction $q|v$ is not trivial. But $\mathbf{K}(x_1 \dots x_k, y_1 \dots y_p)$ is a bounded activity automaton, so the number of pairs $(v, q) \in \mathbb{Z}^m \times \left(S \cup S^{-1}\right)$ with $q|v \neq \mathbf{1}$ is uniformly bounded, and so are the sets E_m. Hence the inverse limit has finite cardinality, so in total we have countably many ends.

□

2.6 Group theoretic properties

The groups $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ are examples of ZC-groups defined as [OS10]. In particular, they are left-orderable residually solvable groups.

In this section, we will always work with a fixed pair of sequences $x_1 \dots x_k, y_1 \dots y_p$ and we will just write $\mathcal{K}$ instead of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$. We still use $S := \left\{a_1, \dots, a_k, b_1, \dots, b_p\right\}$ as our generating set.

Lemma 2.6.1. *The abelianization of $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ is the free abelian group on $x_1, \dots x_k, y_1 \dots y_p$.*

Proof. We have a family of group homomorphisms

$$\begin{aligned} \overline{\rho}_n \colon \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*) &\to \mathbb{Z} \\ g &\mapsto \sum_{v \in \mathbb{Z}^n} \rho(g_{|v}) \end{aligned}$$

Note that the sum is defined as $g_{|v}$ is trivial for almost all v, so almost all summands are 0. By the cocycle equations 2.3.2 we see that $\overline{\rho}_n$ is indeed a group homomorphism, and for all $g \in \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$, we have $\overline{\rho}_{n+1}(g) = \sum_{v \in \mathbb{Z}} \overline{\rho}_n(g_{|v})$. The transition functions given in the Definition 2.4.3 translate to

$$\begin{aligned} \overline{\rho}_0(a_1) &= 1 \\ \overline{\rho}_0(s) &= 0 \text{ for all } s \in S \setminus (a_0) \\ \overline{\rho}_{n+1}(a_{i+1}) &= \overline{\rho}_n(a_i) \\ \overline{\rho}_{n+1}(b_1) &= \overline{\rho}_n(a_k) + \overline{\rho}_n(b_p) \\ \overline{\rho}_{n+1}(b_{j+1}) &= \overline{\rho}_n(a_j) \end{aligned}$$

If we collect $\overline{\rho}_0, \dots \overline{\rho}_{k+p-1}$ to a group homomorphism $\overline{\rho}\colon \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*) \to \mathbb{Z}^{k+p}$, we can show row by row that $\left(\overline{\rho}(a_1), \dots, \overline{\rho}(a_k), \overline{\rho}(b_1), \dots, \overline{\rho}(b^k)\right) \in \mathbb{Z}^{(k+p)\times(k+p)}$ is the identity matrix. So $\overline{\rho}$ induces an isomorphism between the abelianization of $\mathcal{K}$ and $\mathbb{Z}^{k+p}$. □

Lemma 2.6.2. *$\mathcal{K}$ surjects onto the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$. In particular, $\mathcal{K}$ is of exponential growth.*

Proof. The action on $\mathbb{Z}^2$ gives a map to $\mathbb{Z} \wr \mathbb{Z}$. We see that a_1 and a_2 respectively b_1 if $k = 1$ are mapped to the standard generating set of $\mathbb{Z} \wr \mathbb{Z}$, so we have a surjection. As $\mathbb{Z} \wr \mathbb{Z}$ has exponential growth (see [Par92; BT17] for a detailed discussion), $\mathcal{K}$ also has exponential growth. □

Lemma 2.6.3. *The group $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ is level-transitive and self-replicating. For the derived subgroup $\mathcal{K}' \subset \mathrm{Stab}_{\mathcal{K}}$ we have the following: under the map $\mathrm{Stab}_{\mathcal{K}} \hookrightarrow \bigoplus_{x\in\mathbb{Z}} \mathcal{K}$ induced by the wreath recursion, the image of $\mathcal{K}'$ contains $\bigoplus_{x\in\mathbb{Z}} \mathcal{K}'$ and the composition*

$$\mathcal{K}' \hookrightarrow \mathrm{Stab}_{\mathcal{K}} \hookrightarrow \bigoplus_{x\in\mathbb{Z}} \mathcal{K} \to \mathcal{K} \tag{2.6.1}$$

is surjective, where the last map is the projection map to any summand.

Proof. Note that $a := a_1$ acts just by translations on the first level, and every generator is the section of another generator. This already implies level-transitive and self-replicating. To show that the composition (2.6.1) is surjective, it is easy to see that every generator of $\mathcal{K}$ is a section of a commutator of a generator and a sufficiently large power of a_1. So it is easy to see that $\mathcal{K}'$ surjects geometrically onto $\mathcal{K}$. As a_1 is just the first level shift, and $\mathcal{K}'$ is a normal subgroup of $\mathcal{K}$, to show that $\bigoplus_{x\in\mathbb{Z}} \mathcal{K}' \subset \mathcal{K}'$, it is enough to show that $\mathcal{K}'@0 \subset \mathcal{K}'$. Since $\mathcal{K}$ is self-replicating, it is enough to show that $[s,t]@0 \in \mathcal{K}'$ for every commutator of two generators $s, t \in S$ Now if c and d are the generators which have s and t as sections at z and w, then a straight forward calculation shows $[a^{-z}ca^z, a^{-w}da^w] = [s,t]@0$. □

Lemma 2.6.4. *The groups $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ are not residually finite.*

Proof. By the previous lemma, $\mathrm{Stab}_{\mathcal{K}}$ surjects onto $\mathcal{K}$, and since $\mathcal{K}$ is not abelian (it surjects onto an non-abelian group), neither is $\mathrm{Stab}_{\mathcal{K}}$. Let $x, y \in \mathrm{Stab}_{\mathcal{K}}$ be a non-commuting pair. Suppose $\mathcal{K}$ is residually finite, then there exists a group homomorphism $\phi\colon \mathcal{K} \to F$ to a finite group F such that $\phi([x,y])$ is non-trivial. But F is finite, so $\phi(a_1)$ has finite order. So there is a $n > 0$ With $\phi(a_1^{mn}) = 1$ for all m. Then $\phi([x,y]) = \phi([a_1^{-mn}xa_1^{mn}, y])$. Now under the wreath recursion, x and y have finite support in the direct sum $\bigoplus_{\mathbb{Z}} \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z})$, so for m large enough, the support of $a_1^{-mn}xa_1^{mn}$ and y will be disjoint, hence they commute. So $\phi([x,y]) = \phi([a_1^{-mn}xa_1^{mn}, y])$ is trivial, so we arrive at an contradiction. □

Theorem 2.D. *The groups $\mathcal{K}(x_1 \dots x_k, y_1 \dots y_p)$ are amenable but not elementary subexponentially amenable.*

Proof. We invoke Theorem 3.**B** to show that the groups $\mathcal{K}(x_1 \ldots x_k, y_1 \ldots y_p)$ are amenable. We already observed in Remark 2.4.5 that the groups are generated by bounded activity automata. Hence they are subgroups of $\mathrm{Aut}^{f.s.}_{\mathcal{B}}(\mathbb{Z}^*;\mathbb{Z})$. As the left action of $\mathbb{Z}$ on itself is recurrent, by Theorem 3.**B** the group $\mathrm{Aut}^{f.s.}_{\mathcal{B}}(\mathbb{Z}^*;\mathbb{Z})$ is amenable, and so are the subgroups $\mathcal{K}(x_1 \ldots x_k, y_1 \ldots y_p)$.

The groups have exponential growth by Lemma 2.6.2. Lemma 2.6.3 together with Corollary 3 of [Jus18] imply that the groups are not elementary subexponentially amenable. □

We should note that [Jus18] only deals with finite alphabets. The proof can be easily modified to deal with subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$.

2.7 Outlook

This paper is the beginning of our study of iterated monodromy groups for entire transcendental maps and a stepping stone towards a more general discussion. The regularity of the monodromy of the exponential map simplifies the discussion and has consequences that are special to the exponential case. In particular, the left-order on the dynamical preimage tree heavily uses this regularity. For other entire transcendental functions, we should expect torsion elements in the monodromy group and torsion elements for some iterated monodromy groups of functions in that parameter space.

In an upcoming paper [Reib] we discuss the general structure of iterated monodromy groups of entire maps. In particular, we also apply the results of [Rei20a] to show that the iterated monodromy groups of entire functions are amenable if and only their monodromy group is. For polynomials and the exponential family, the condition is trivially satisfied, as finite groups and abelian groups are amenable. However, there are entire maps with virtually free monodromy groups, so we have to impose this condition.

Moreover, we can also try to generalize from entire functions to meromorphic functions. Here a good starting family would be the functions of the form $M \circ \exp$ including tangent, where M is a Möbius transform. We should think of this as the analogy to the family of bicritical rational maps, see also Appendix D of [Mil00]. In this case, we can also define iterated monodromy group for post-singularly finite maps and show that they are ZC-groups. So the class of ZC-groups, in particular subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ has many examples of self-similar groups coming from complex dynamics. This warrants a further general investigation of ZC-groups.

Outside of this family $M \circ \exp$, we should not expect to have the left-orderability of all IMGs in one parameter space, as it might be a special phenomenon due to the very rigid monodromy groups of exponential maps.

3 Amenability of Bounded Automata Groups on Infinite Alphabets

3.1 Introduction

Self-similar groups provide many examples of “exotic” amenable groups. The Grigorchuk group [Gri83] was the first example of a group of intermediate growth. Groups of intermediate growth are always amenable, but not elementary amenable (see [Cho80]). The basilica group is amenable [BV05], but not elementary subexponentially amenable [GŻ02].

Both the Grigorchuk group and the basilica group are examples of automata groups on a two-letter alphabet of bounded activity growth. They fit into the hierarchy of polynomial activity growth introduced in [Sid00], where both finite and infinite alphabets are considered. Under certain assumptions (which are always satisfied for finite alphabets), these groups do not contain free subgroups (see [Sid04]). This raises the question whether groups by generated by polynomial activity growth automata are in fact amenable.

For finite alphabets, it is shown in [BKN10] that every group generated by a bounded activity automata is amenable.

A large family of such groups are iterated monodromy groups of post-critically finite polynomials [Nek09]. Furthermore, in [AAV13] it is shown that automata groups on finite alphabets of linear activity growths are amenable. The techniques of [BKN10] and [AAV13] have been conceptualized in [JNS16].

In this paper and the forthcoming paper [Reib], we study the iterated monodromy groups of post-singularly finite entire transcendental functions. We expect many similarities of these groups to their polynomial counterparts, so one question in particular is amenability. We can not expect all iterated monodromy groups of post-singularly finite entire functions to be amenable, as there are entire functions with monodromy group given by the free product $C_2 * C_2 * C_2$. Our main result is the following theorem.

Theorem 3.A (Amenability of transcendental IMGs)**.** *Let f be a post-singularly finite entire transcendental function. Then the iterated monodromy group of f is amenable if and only if the monodromy group of f is amenable.*

In this paper we provide the main group theoretic part of the proof of this theorem. We show the following:

Theorem 3.B (Main theorem)**.** *Let X be a countably infinite set. Let P be an amenable subgroup of* $\mathrm{Sym}(X)$. *Suppose that the action of P on X is recurrent. Then* $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ *is amenable.*

In the forthcoming paper [Reib], we show that iterated monodromy groups of post-singularly finite entire transcendental functions are given by bounded activity automata on countably infinite alphabets, so that we can apply Theorem 3.**B** to deduce Theorem 3.**A**.

See Section 2 for a precise definition of $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$; it is roughly the group of bounded activity automata were every first level action is in P.

We note that Theorem 3.**A** is our main motivation for Theorem 3.**B**, but this paper does not logically depend on [Reib].

Overview. In Section 2, we start by introducing self-similar groups on infinite alphabets and related concepts, such as the space of ends. We continue in Section 3 with a discussion of recurrent random walks and how to pass from a recurrent action on the alphabet to a recurrent action of a bounded activity group on the space of ends. This will be a key ingredient to invoke the amenability criterion of [JNS16] in Section 4 to prove Theorem 3.**B**. In Section 5, we briefly discuss the forthcoming paper and further related open questions.

3.2 Regular trees

In this section we introduce of self-similar groups and other relevant concepts and fix the notation. We mostly follow the notation of [Nek05]. See also [Sid00; Sid04] for self-similar groups on infinite alphabets.

Definition 3.2.1. Let X be a countably infinite set. The *standard X-regular tree* has as vertex set X^*, the set of finite words in X. Its root is the empty word $\emptyset$. Its edges are all pairs (v, vx) for $v \in X^*, x \in X$. By abuse of notation, we denote the standard X-regular tree also as X^*, and we denote by $\mathrm{Aut}(X^*)$ the group of rooted tree automorphisms of X^*. We denote the identity of $\mathrm{Aut}(X^*)$ by $\mathbf{1}$.

For $v \in X^*$, let vX^* be the subtree of all descendants of v. If $g \in \mathrm{Aut}(X^*), v \in X^*$, there is a unique $g_{|v} \in \mathrm{Aut}(X^*)$ given by $g(vw) = g(v)g_{|v}(w)$. This is called the *section* of g along v.

A set $S \subset \mathrm{Aut}(X^*)$ is called *self-similar* if it is closed under taking sections, i.e. $g_{|v} \in S$ for all $g \in S, v \in X^*$. We are mainly interested in *self-similar groups*, i.e. subgroups $G \subset \mathrm{Aut}(X^*)$ that are self-similar as sets.

For $g \in \mathrm{Aut}(X^*)$, we denote by $\alpha_n(g) \in \mathbb{N} \cup \infty$ the number of words v of length n for which the section $g_{|v}$ is not trivial. We denote by $\mathrm{Aut}_{\mathrm{fin.}}(X^*)$ the set of automorphisms with finitely many nontrivial sections on every level, i.e., the set of automorphisms $g \in \mathrm{Aut}(X^*)$ with $\alpha_n(g) \in \mathbb{N}$ for all $n \in \mathbb{N}$. If $g \in \mathrm{Aut}_{\mathrm{fin.}}(X^*)$ has a n so that $g_{|v} = \mathbf{1}$ for all $v \in X^n$, we say that that g is *finitary*. If $g \in \mathrm{Aut}_{\mathrm{fin.}}(X^*)$ has a $c \in \mathbb{N}$ so that $\alpha_n(g) \le c$ for all n, we say that g has *bounded activity*.

We denote by $\mathrm{Aut}_{\mathscr{B}}(X^*)$ the set of automorphisms with bounded activity, and by $\mathrm{Aut}_{\mathscr{F}}(X^*)$ the set of finitary automorphisms.

We also have maps $\rho_n\colon \mathrm{Aut}(X^*) \to \mathrm{Sym}(X^n)$, which are induced by the action of $\mathrm{Aut}(X^*)$ on the n-th level of X^*. Let P be a subgroup of $\mathrm{Sym}(X)$. Let $\mathrm{Aut}(X^*;P)$ denote the set of automorphisms such that $\rho_1(g_{|v}) \in P$ for all $v \in X^*$. We denote by $\mathrm{Aut}_{\mathrm{fin.}}(X^*;P), \mathrm{Aut}_{\mathscr{B}}(X^*;P), \mathrm{Aut}_{\mathscr{F}}(X^*;P)$ the intersections of $\mathrm{Aut}_{\mathrm{fin.}}(X^*), \mathrm{Aut}_{\mathscr{B}}(X^*), \mathrm{Aut}_{\mathscr{F}}(X^*)$ with $\mathrm{Aut}(X^*;P)$, respectively.

Since we consider infinite alphabets, let us fix notations for the two versions of wreath products.

Notation 3.2.2. Let A and B be groups, L be a set with an A-left action. The *unrestricted wreath product* $\left(\prod_{l\in L} B\right) \rtimes A$ is denoted by $B \,\mathrm{Wr}_L\, A$, the *restricted wreath product* $\left(\bigoplus_{l\in L} B\right) \rtimes A$ is denoted $B \wr_L A$.

We will mainly work with the restricted wreath product. We denote the right factor embedding $A \to B \wr_L A$ by ι, and by $b@l$ the image of b under the embedding of B into the component indexed by $l \in L$.

If additionally M is a set with a B-left action, we will consider the action of $B \,\mathrm{Wr}_L\, A$ on $L \times M$ given by $((b_l)_{l\in L}, a)(l,m) = (a(l), b_l(m))$.

For a subgroup P of $\mathrm{Sym}(X)$, we denote the n-th iterated restricted wreath product (along X) by P_n. So $P_1 = P$ and $P_{n+1} = P_n \wr_X P$. Note that if P is amenable, then all P_n are amenable. With this in mind we have the following lemma.

Lemma 3.2.3.

$$\begin{array}{rcl} \mathrm{Aut}(X^*;P) & \to & \mathrm{Aut}(X^*;P)\,\mathrm{Wr}_X\, P \\ g & \mapsto & (x \mapsto g_{|x}, \rho_1(g)) \end{array}$$

is an isomorphism of groups. It restricts to isomorphisms

$$\begin{array}{rcl} \mathrm{Aut}_{\mathrm{fin.}}(X^*;P) & \cong & \mathrm{Aut}_{\mathrm{fin.}}(X^*;P) \wr_X P \\ \mathrm{Aut}_{\mathscr{B}}(X^*;P) & \cong & \mathrm{Aut}_{\mathscr{B}}(X^*;P) \wr_X P \\ \mathrm{Aut}_{\mathscr{F}}(X^*;P) & \cong & \mathrm{Aut}_{\mathscr{F}}(X^*;P) \wr_X P \end{array}$$

△

For the first line, see for example [Sid00]. By iteration, we also get isomorphisms

$$\begin{array}{rcl} \mathrm{Aut}_{\mathrm{fin.}}(X^*;P) & \to & \mathrm{Aut}_{\mathrm{fin.}}(X^*;P) \wr_{X^n} P_n \\ g & \mapsto & (v \mapsto g_{|v}, \rho_n(g)) \end{array}$$

and $\mathrm{Aut}_{\mathrm{fin.}}(X^*;P) \cong \mathrm{Aut}_{\mathrm{fin.}}(X^{n*};P_n)$.

3.2.1 Action on the space of ends X^ω

We will also use the action of $\mathrm{Aut}(X^*)$ on the space of ends of X^*. The set of ends of X^* can be identified with X^ω, the set of right infinite words in X. For a word $v \in X^n$, the *open cylinder set* $C(v) = \{vw \colon w \in X^\omega\}$ is the set of right infinite words that have v as a prefix. The open cylinder sets form a basis of the end topology on X^ω. Since X is countably infinite, X^ω is homeomorphic to the Baire space $\mathbb{N}^\mathbb{N}$, in particular X^ω is Hausdorff, but not locally compact. The action of $\mathrm{Aut}(X^*)$ on X^ω is faithful, so we can also think of elements of $\mathrm{Aut}(X^*)$ as homeomorphisms on X^ω.

We will use the language of *germs*: these are equivalence classes of pairs $(g, w) \in \mathrm{Aut}(X^*) \times X^\omega$, where $(g, w) \sim (h, w')$ if $w = w'$ and g and h agree on a neighborhood of w. Since we only consider germs of $\mathrm{Aut}(X^*)$, and $\{C(v)\colon v \text{ is a prefix of } w\}$ forms a neighborhood basis of w, $(g, w) \sim (h, w')$ if and only if $w = w'$, $g(w) = h(w)$ and $g_{|v} = h_{|v}$ for some prefix v of w. Given a subgroup of G of $\mathrm{Aut}(X^*)$, its associated *groupoid of germs* $\mathscr{G}$ has as morphisms germs represented by pairs $(g, w) \in G \times X^\omega$, going from w to $g(w)$, with composition $[(g, w)] \circ [(h, w')] = (gh, w')$ under the condition that $h(w') = w$. The groupoid of germs has as object set X^ω. For an end $w \in X^\omega$, we will be particularly interested in the *isotropy group* $\mathscr{G}_w$, the group of germs going from w to itself. We denote by $\mathscr{T}$ the groupoid of germs of tail equivalences, that is germs of the form (g, w) with $g_{|v}$ trivial for some prefix v of w. Given a groupoid of germs $\mathscr{H}$, we denote by $[[\mathscr{H}]]$ the set of homeomorphisms of X^ω whose germs belong all to $\mathscr{H}$.

If $w \in X^\omega$ can be factored as $w = vu$ with $v \in X^n, u \in X^\omega$, we say that u is the *n-tail* of w. If $w, w' \in X^\omega$ have the same n-tail, we say that w and w' are *n-tail equivalent.* We say that w and w' are tail equivalent (or *cofinal*) if they are n-tail equivalent for some n. The n-tail equivalence class of w is denoted by $T_n(w)$ and $T(w) = \bigcup_{n\in\mathbb{N}} T_n(w)$ is the cofinality class of w.

Lemma 3.2.4. *Let $g \in \mathrm{Aut}_{\mathscr{B}}(X^*)$. There are only finitely many w such that the germ (g, w) is not in $\mathscr{T}$. Moreover $g \in \mathrm{Aut}_{\mathscr{F}}(X^*)$ if and only if $g \in [[\mathscr{T}]]$. If (g, w) is in $\mathscr{T}$, then w and $g(w)$ are cofinal.*

Proof. The ends where the germ of g is not in $\mathscr{T}$ are those where the sections along all prefixes are nontrivial. So they can be identified with the projective limit $\varprojlim \{v \in X^n \colon g_{|v} \neq \mathbf{1}\}$. Since $g \in \mathrm{Aut}_{\mathscr{B}}(X^*)$, the sets in the limit construction have uniformly bounded cardinality. Hence the projective limit is also finite. This proves the first claim.

For the second claim, we already observed that $\mathrm{Aut}_{\mathscr{F}}(X^*) \subset\in [[\mathscr{T}]]$. In the other direction, if g is also in $[[\mathscr{T}]]$, then the projective limit must be empty. As all sets in the limit construction are finite, by Kőnig's lemma one of the sets in the limit construction must be empty. So g is also in $\mathrm{Aut}_{\mathscr{F}}(X^*)$. This proves the first claim. For the last claim, if (g, w) is in $\mathscr{T}$ then w factors as vu with $g_{|v}$ trivial, so $g(w) = g(vu) = g(v)g_{|v}(u) = g(v)u$, so w and $g(w)$ are cofinal. □

In particular, we have shown that $\mathrm{Aut}_{\mathscr{B}}(X^*) \cap [[\mathscr{T}]] = \mathrm{Aut}_{\mathscr{F}}(X^*)$. In fact, the proof shows $\mathrm{Aut}_{\text{fin.}}(X^*) \cap [[\mathscr{T}]] = \mathrm{Aut}_{\mathscr{F}}(X^*)$, as we only need finiteness of every set in the limit construction.

3.2.2 Bounded Automata

Definition 3.2.5. An automorphism $g \in \mathrm{Aut}(X^*)$ is called a *finite state* automorphism if the set of sections $\{g_{|v} : v \in X^*\}$ is finite.

We denote by $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$ the subgroups of finite state automorphisms in $\mathrm{Aut}_{\mathscr{B}}(X^*;P)$. Note that every $g \in \mathrm{Aut}_{\mathscr{F}}(X^*)$ is a finite state automorphism.

A nontrivial automorphism $g \in \mathrm{Aut}_{\mathrm{fin.}}(X^*)$ is called *directed* if there is a word $v \in X^n$ with $g_{|v} = g$ and $g_{|u} \in \mathrm{Aut}_{\mathscr{F}}(X^*)$ for all $u \in X^n, u \neq v$.

Remark 3.2.6. Finite state automorphisms are exactly the automorphisms that can be defined via a finite state automaton, see the discussion in [Sid04, Section 2.2] for infinite alphabets.

A directed automorphism has bounded activity growth. We will use the following structural result about finite state automata of bounded activity growth, see [Sid00, Lemma 17] or [Nek05, Proposition 3.9.11] for finite alphabets, and [Sid04, Section 2.2] for the extension to infinite alphabets.

Lemma 3.2.7. *Let $g \in \mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*)$ be a finite state automorphism. Then there exists a level n such that for all $v \in X^n$, $g_{|v}$ is either directed or finitary.* △

3.3 Random walks

3.3.1 Potential theoretic background

We will use the potential-theoretic setting as in [Woe00, Section I.2]: There are many ways to define networks, it will be convenient for us to start from a conductance function, as we will work mostly in that language:

Definition 3.3.1. Let X be a countably infinite set, let $a: X \times X \to [0,\infty)$ be a symmetric function such that the sum $m(x) := \sum_{y\in X} a(x,y)$ is positive and finite for all $x \in X$. Let $E \subset X \times X$ be the support of a, i.e., $(x,y) \in E$ if and only if $a(x,y) > 0$. We think of (X,E) as a simply undirected graph with possible loops. Consider the function $r: E \to (0,\infty)$ given by $r(e) = 1/a(e)$. If the graph (X,E) is connected, then we call the triple $\mathscr{N} := (X,E,r)$ the associated *network* to X and a. We call $r(e)$ the *resistance* of e, $a(e)$ the *conductance* of e and $m(x)$ the *total conductance* at x. The associated Markov chain on X has transition probabilities $p(x,y) = a(x,y)/m(x)$. We say that $\mathscr{N}$ is recurrent if the associated Markov chain is recurrent for all starting points $x \in X$.

Notation 3.3.2. Let (X,E,r) is a network and Y be a subset of X. We denote by $\chi_Y : X \to \{0,1\}$ the characteristic function of Y. Moreover, the *edge boundary* $\partial^e Y$ is given by set the of edges between Y and $X \setminus Y$, and the *vertex boundary* $\partial^v Y$ is the set of vertices in $X \setminus Y$ that share an edge with an element of Y.

Example 3.3.3. If G is a group acting transitively on X, and λ is a finitely supported symmetric finite measure on X, such that the support of λ generates G, then we can define a network on X with conductances

$$a(x,y) = \sum_{g(x)=y} \lambda(g).$$

In this case the total conductance at every point is equal to the total mass of λ.

In particular, if S is a finite symmetric generating set of G, we can take λ to be the counting measure on S and obtain as the network the *Schreier graph* $\Gamma(G,S,X)$. In the Schreier graph there is an edge from x to y if and only if there is an $s \in S$ with $s(x) = y$. Note that by following the convention of [Woe00], $\Gamma(G,S,X)$ can have loops, but no parallel edges, but if there are multiple generators sending x to y, then the edge (x,y) will have the appropriate higher conductance. By the *uniform random walk* on Schreier graphs we mean the random walk arising from this construction.

It will be also convenient to consider the *reduced Schreier graph* $\tilde{\Gamma}(G,S,X)$ that is obtained from $\Gamma(G,S,X)$ by removing all loops, and assigning constant conductance to each edge. The resulting random walk is the *simple random walk* on the reduced Schreier graph. As S is finite, it follows from [Woe00, Corollary I.3.5] that the uniform random walk is recurrent if and only if the simple random walk is. While the uniform random walk is closer connected to random walks induced by group actions, the simple random walk will have its use in shorting.

If the action of G on X is not transitive, for $x \in X$ the *orbital Schreier graph* of x is the Schreier graph on the orbit of x.

We are mostly interested in the space $\mathcal{D}(\mathcal{N})$ of functions $f\colon X \to \mathbb{R}$ with finite Dirichlet energy $D(f) = \sum_{e\in E} a(e)\big(f(e^+) - f(e^-)\big)^2$. For any choice of base point o, $\mathcal{D}(\mathcal{N})$ is a Hilbert space with norm $\|f\|_{D,o}^2 = D(f) + \|f(o)\|^2$. All choices of o give equivalent norms, so there is a well-defined topology on $\mathcal{D}(\mathcal{N})$, so that f_n converges to f if and only if $\lim_n D(f_n - f) = 0$ and f_n converges to f point-wise.

Let $\mathcal{D}_0(\mathcal{N})$ be the closure of functions with finite support in $\mathcal{D}(\mathcal{N})$. By [Woe00, Theorem I.2.12], the random walk on $\mathcal{N}$ is recurrent if and only if $\chi_X \in \mathcal{D}_0(\mathcal{N})$. We also use $\mathcal{D}_0(\mathcal{N})$ to get the following shorting criterion.

Lemma 3.3.4 ([Woe00, Theorem I.2.19])**.** *Let $X = \bigcup_{i\in I} X_i$ be a partition of X such that $\chi_{X_i} \in \mathcal{D}_0(\mathcal{N})$ for all $i \in I$. Consider the* ***shorted network*** *$\mathcal{N}'$ with vertex set I and conductivity $a'(i,j) = \sum_{x\in X_i, y\in X_j} a(x,y)$ for $i \neq j$, $a'(i,i) = 0$. If $\mathcal{N}'$ is recurrent then so is $\mathcal{N}$.* △

As a special case we want to mention the Nash-Williams criterion [Nas59]:

Lemma 3.3.5. *Let $Y_0 \subset Y_1 \subset \ldots$ be an increasing chain of subsets of X with $Y_i \in \mathcal{D}_0(\mathcal{N})$, such that $\partial^v Y_i \subset Y_{i+1}$, and $\bigcup Y_i = X$. Let $a'_i := \sum_{e\in\partial^e Y_i} a(e)$. If $\sum \frac{1}{a'_i} = \infty$ then $\mathcal{N}$ is recurrent.*

Proof. Let $X_0 = Y_0, X_{n+1} = Y_{n+1} \setminus Y_n$. Then also the characteristic functions of the X_i are in $\mathcal{D}_0(\mathcal{N})$, so we can apply the previous lemma to shorten. The resulting shorted network is the nearest neighbor walk on $\mathbb{N}$ with conductances a'_i, so by [Woe00, Paragraph I.2.16], the shorted network is recurrent, hence $\mathcal{N}$ is also recurrent. □

We will also use the following lemma.

Lemma 3.3.6. *Let $\mathcal{N} = (X, E, r)$ be a network, $Y \subset X$ with $\partial^e Y$ finite. Suppose $\mathcal{N}' = (Y, E', r')$ is a network on Y obtained from $\mathcal{N}$ by restricting to Y and adding and removing finitely many edges and changing finitely many resistances.*

Suppose $\mathcal{N}'$ is a recurrent network. Then χ_Y is in $\mathcal{D}_0(\mathcal{N})$.

Proof. Since $\mathcal{N}'$ is recurrent, χ_Y is in $\mathcal{D}_0(\mathcal{N}')$. So there is a sequence $f_n\colon Y \to \mathbb{R}$ such that $\lim_n D_{\mathcal{N}'}(f_n - \chi_Y) = 0$ and $f_n \to 1$ point-wise on Y.

We extend f_n to X by 0. Then $D_{\mathcal{N}'}(f_n - \chi_Y)$ and $D_{\mathcal{N}}(f_n - \chi_Y)$ differ in only finitely many summands, and these go to 0 by point-wise convergence of the sequence f_n. So we have $\lim_n D_{\mathcal{N}}(f_n - \chi_Y) = 0$ and thus $\chi_Y \in \mathcal{D}_0(\mathcal{N})$. □

3.3.2 Recurrence on orbital Schreier graphs

Definition 3.3.7. Let A be a group, L a left A-set. We say that the action of A on L is recurrent if for all finitely supported symmetric measures λ on A, the random walk on L induced by λ is recurrent for all starting points $l_0 \in L$.

Remark 3.3.8. If A is finitely generated, it is enough to show this for one finitely supported symmetric measure whose support generates A. If S is a finite symmetric generating set of A, it is enough to consider the uniform random walk on the Schreier graph $\Gamma(A,S,L)$ or the simple random walk on $\tilde{\Gamma}(A,S,L)$. See for example [JNS16, Lemma 6] . With this definition it is also clear that recurrent actions are closed under taking subgroups.

Lemma 3.3.9. *Let A, B are groups. Suppose that L a left A-set and M a left B-set such that the actions are both recurrent. Then the action of $B \wr_L A$ on $L \times M$ is also recurrent.*

Proof. Let us first reduce to the case where A and B are both finitely generated and both actions are transitive:

Let $(l, m) \in L \times M, \lambda$ a symmetric finitely supported measure on $B \wr_L A$. Then there are finitely generated subgroups $A' \subset A, B' \subset B$ such that $\operatorname{supp}(\lambda) \subset B' \wr_L A'$. So without loss of generality let A and B be finitely generated. Let L' be the orbit of l under A. Then we have a quotient map $\pi\colon B \wr_L A \to B \wr_{L'} A$ and we can replace λ by $\pi_*(\lambda)$ to assume without loss of generality that the action of A on L is transitive. We can easily replace M with the orbit of m under B.

We can now assume that S and T are finite symmetric generating sets of A and B, respectively, and both actions are transitive. Instead of showing recurrence for arbitrary λ, we can now fix a preferred generating set of $B \wr_L A$ and show recurrence of the simple random walk on the associated reduced Schreier graph .

Fix any base point $l_0 \in L$. We take as our generating set of $B \wr_L A$ the union of $\iota(S)$ and $T@l_0 = \{t@l_0 \colon t \in T\}$, let $\mathcal{N}$ be the resulting network on the reduced Schreier graph with constant resistance. The generating set acts now as follows:

$$\iota(s)(l,m) = (s(l),m)$$
$$t@l_0(l_0,m) = (l_0,t(m))$$
$$t@l_0(l',m) = (l',m) \text{ for } l' \neq l_0.$$

We use the shorting criterion by partitioning $L \times M = \bigcup_{m \in M} L \times m$. For every m, we see that the induced subgraph on $L \times m$ is isomorphic to the reduced Schreier graph $\tilde{\Gamma}(A,S,L)$. Since the action of A on L is recurrent, the simple random walk on $\tilde{\Gamma}(A,S,L)$ is recurrent. Moreover $\partial^e(L \times m)$ is a finite collection of edges at (l_0,m), so by Lemma 3.3.6, we obtain $\chi_{L\times m} \in \mathcal{D}_0(\mathcal{N})$. The shorted network with respect to the partition is isomorphic to the reduced Schreier graph $\tilde{\Gamma}(B,T,L)$, so it is also recurrent. By Lemma 3.3.4, the network $\mathcal{N}$ is then also recurrent. □

Lemma 3.3.10. *Let G be a finitely generated subgroup of $\mathrm{Aut}_{\mathcal{B}}(X^*)$. Assume that the action of G on every finite level is recurrent. Then the action of G on every component of the Schreier graph of the action of G on X^ω is recurrent.*

Proof. Let S be a finite symmetric generating set of G. Let $K > 0$ be a uniform bound on $\alpha_n(s)$ for all $n \in \mathbb{N}, s \in S$. Let Ω be a component of the Schreier graph on X^ω. Let $\mathcal{N}$ the network on associated with the uniform random walk on Ω.

Let E be the set of edges in $\mathcal{N}$ which go between different cofinality classes. By Lemma 3.2.4, E is finite. Since Ω is connected, its vertex set must be contained in finitely many cofinality classes $C_1,\dots C_n$. Choose representatives $w_i \in C_i \cap \Omega$.

We claim that the conductance of $\partial^e T_m(w_i)$ is uniformly bounded by $K|S|$: in fact, if u is the m-tail of w_i, then $\partial^e T_m(w_i)$ (with multiplicities) can be identified with the set $\{(s,v) \in S \times X^m \colon s_{|v}(u) \neq u\}$. This set is contained in $\{(s,v) \in S \times X^m \colon s_{|v} \neq \mathbf{1}\}$, so the bound is clear.

Since Ω is a connected component of the Schreier graph, the conductance of $\partial(T_m(w_i) \cap \Omega)$ is also uniformly bounded by $K|S|$ and $T_m(w_i) \cap \Omega$ has only finitely many components. Each such component may be viewed as a subnetwork of the (recurrent) random walk of G on level m, so the random walk on each component is recurrent by [Woe00, Corollary I.2.15] , and finally by Lemma 3.3.6, their characteristic functions are in $\mathcal{D}_0(\mathcal{N})$.

Let $Y_m := \bigcup_{1\le i\le n} T_m(w_i) \cap \Omega$. Then χ_{Y_m} is the finite sum of the characteristic functions of components of $T_m(w_i) \cap \Omega$, so we obtain $\chi_{Y_m} \in \mathcal{D}_0(\mathcal{N})$. Also, $\partial^e Y_m \subset \bigcup_{1\le i\le n} \partial(T_m(w_i) \cap \Omega)$, so the conductance of $\partial^e Y_m$ is uniformly bounded by $nK|S|$.

We can now take a subsequence Y_{m_i} such that ∂Y_{m_i} is properly contained in $Y_{m_{i+1}}$. By applying Lemma 3.3.5 to the sequence Y_{m_i}, the random walk on $\mathcal{N}$ is recurrent. □

3.4 Amenability of groups generated by bounded activity automata

In this section we will prove the Theorem 3.B. We will use the following criterion.

Theorem 3.C (Theorem 11 in [JNS16])**.** *Let G be a finitely generated group of homeomorphisms of a topological space Y, and $\mathscr{G}$ be its groupoid of germs. Let $\mathscr{H}$ be a groupoid of germs of homeomorphisms of Y. Suppose that the following conditions hold:*

1. *The group $[[\mathscr{H}]] \cap G$ is amenable.*
2. *For every generator $g \in G$ the germ of g at y belongs to $\mathscr{H}$ for all but finitely many $y \in Y$. We say that $y \in Y$ is* singular *if there exists $g \in G$ such that $(g, y) \notin \mathscr{H}$.*
3. *For every singular point $y \in Y$, the action of G on the orbit of y is recurrent.*
4. *The groups of germs $\mathscr{G}_y$ are amenable for all $y \in Y$.*

Then the group G is amenable. △

Remark 3.4.1. This is almost Theorem 11 in [JNS16], but we weakened the condition (1) from $[[\mathscr{H}]]$ being amenable to $[[H]] \cap G$ being amenable. In the last step of the original proof, a certain subgroup K of G is expressed as an extension of a subgroup of $[[\mathscr{H}]]$ and a direct product of (finitely many) isotropy groups $\mathscr{G}_y$, and it remains to show that K is amenable. But $K \subset G$ is in fact an extension of a subgroup of $[[\mathscr{H}]] \cap G$ and a direct product of isotropy groups $\mathscr{G}_y$, so it is amenable also under the weakened condition (1) together with condition (4). The original proof only used the weaker condition.

*Proof of Theorem **3.B**.* In order to show amenability of $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$, it is enough to show amenability of every finitely generated subgroup of $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$. So let $G = \langle S \rangle$ be a finitely generated subgroup of $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$. We will use Theorem 3.C with G acting on X^ω, and $\mathscr{H}$ being the groupoid of tail equivalences $\mathscr{T}$. We will show that each condition of Theorem 3.C is satisfied.

1. The group $[[\mathscr{T}]] \cap G$ is amenable: In fact $[[\mathscr{T}]] \cap \mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P) = [[\mathscr{T}]] \cap \mathrm{Aut}_{\mathscr{B}}(X^*;P) = \mathrm{Aut}_{\mathscr{F}}(X^*;P)$. This follows easily from Lemma 3.2.4. Now $\mathrm{Aut}_{\mathscr{F}}(X^*;P)$ is the direct limit of iterated wreath products of P, so it is amenable, hence $[[\mathscr{T}]] \cap G$ is amenable.
2. This follows directly from Lemma 3.2.4.
3. By inductive application of Lemma 3.3.9, we see that the action of the n-th iterated wreath product of P on X^n is recurrent. The image of G under ρ_n lies in P_n, hence by Remark 3.3.8, the action of G on every level is recurrent. Since $G \subset \mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P) \subset \mathrm{Aut}_{\mathscr{B}}(X^*)$, we get by Lemma 3.3.10 that G acts recurrently on all orbital Schreier graphs.
4. We encapsulate the proof in the following lemma.

□

Lemma 3.4.2. *Let P be an amenable subgroup of* $\mathrm{Sym}(X)$. *Let G be a finitely generated subgroup of* $\mathrm{Aut}^{f.s.}_{\mathscr{B}}(X^*;P)$. *Then the group of germs $\mathscr{G}_w$ is amenable for every $w \in X^\omega$.*

Proof. By replacing X with X^N and P with P_N for sufficiently large $N \in \mathbb{N}$ and possibly enlarging the group G itself, we can use Lemma 3.2.7 to assume without loss of generality the following:

- G has a symmetric generating set S that is self-similar as a set.
- For all $s \in S$ and $x \in X$, the section $s_{|x}$ is either finitary or directed.
- For every directed $s \in S$, there is an $x \in X$ with $s_{|x} = s$. For every $y \neq x$ then $s_{|y}$ is finitary. So every directed generator is directed along a constant path.

See for example [Nek05, Proposition 3.9.11] for more details.

Let $\Omega = \{w \in X^\omega : w \text{ is eventually constant}\}$. Then Ω is invariant under the action of every generator in S, so $X^\omega \setminus \Omega$ is also invariant under the action of G. For every generator, the germs in $X^\omega \setminus \Omega$ are contained in $\mathscr{T}$, so for $w \in X^\omega \setminus \Omega$, the group of germs $\mathscr{G}_w$ is contained in $\mathscr{T}_w = \mathbf{1}$, so it is trivial.

For an end $w \in \Omega$, and a group element $g \in G$, let v_n be the prefix of w of length n and consider the sequence $g_{|v_n}$. We claim that this sequence is eventually constant, and if w is eventually constantly equal to the letter x, then for all $y \in X \setminus \{x\}$, the section $g_{|v_n y}$ is contained in $\mathrm{Aut}_{\mathscr{F}}(X^*;P)$ for n large enough.

This is true for the generating set by direct inspection and the statement follows for every group element $g \in G$ by induction over the word length of g.

We show this by induction for every group element $g \in G$ over the word length of g. The statement is clear for the identity element. For a generator $s \in S$, either $s_{|v_n}$ is trivial for some n, and then the statement is clear, or w is of the form $z(x^\omega)$, with $s_{|z}$ directed along x, and the statement also follows.

Now if $g, h \in G$ both satisfy the statement for all ends and w is a end eventually constantly equal to some letter x, then $w' := h(w)$ is in Ω, with v'_n as the prefix of w' of length n, so that it is eventually constant to some letter x', and

$$(gh)_{|v_n} = g_{|v'_n} h_{|v_n}$$

is eventually constant, and for every $y \neq x$, we have that $y' := h_{|v_n}(y)$ is well defined and different from x, and so

$$(gh)_{|v_n y} = g_{|v'_n y'} h_{|v_n y}$$

is a product of two element of $\mathrm{Aut}_{\mathscr{F}}(X^*;P)$, so it is also in $\mathrm{Aut}_{\mathscr{F}}(X^*;P)$. This finishes the inductive step.

In particular, for $w \in \Omega$ eventually constantly equal to the letter $x \in X$, we get a group homomorphism

$$\begin{aligned} \mathscr{G}_w &\to \operatorname{Aut}_{\mathscr{F}}(X^*;P) \wr_{X\setminus\{x\}} P_x \\ g &\mapsto (y \mapsto g_{|v_n y}, \rho_1(g_{|v_n})) \text{ for } n \text{ large enough.} \end{aligned}$$

Here P_x is the stabilizer of $x \in X$ for the action of P on X. The group homomorphism is injective, and the codomain is amenable, so it follows that the group of germs $\mathscr{G}_w$ is amenable as well. □

3.5 Outlook

We apply the main result of this paper (Theorem 3.**B**) to give a sufficient condition for amenability of iterated monodromy groups of of post-singularly finite entire functions (Theorem 3.**A**), see [Reib]. In the proof of Theorem 3.**B**, we use the version of Theorem 3.**C** from [JNS16], which impose a recurrence condition on the random walk on the orbital Schreier graphs. This recurrence condition was generalized to an extensive amenability condition in [Jus+16]. It is shown in [Jus+16] that every recurrent action is also extensive amenable. In our Theorem 3.**B**, it would be interesting to see whether we could weaken the recurrence condition to a condition about extensive amenability. Another direction to generalize is to step up in the hierarchy of automata with polynomial activity growth. In [AAV13; JNS16], it is shown that the group of automata of linear activity growth acting on a finite alphabet is amenable. Again, a crucial ingredient here is the recurrence of the random walk on the orbital Schreier graphs. It is not clear how this generalizes to infinite alphabets, as it seems that the estimates given in [AAV13; JNS16] to show recurrence used finiteness of the alphabet at an important point.

4 Iterated Monodromy Groups of Entire Maps and Dendroid Automata

4.1 Introduction

Iterated Monodromy Groups are well-established and very useful objects in the dynamics of (post-singularly finite) iterated rational maps, especially polynomials. They have been successfully used in classification problems of polynomials, such as the twisted rabbit problem [BN06]. One important tool to describe iterated monodromy groups of polynomials are dendroid automata [Nek09]; in particular, they are self-similar groups on finite alphabets that act via bounded activity automata in the sense of [Sid00], so they are amenable by [BKN10].

The goal of this paper is to extend this theory to (post-singularly finite) transcendental entire functions. Our first main result is the following.

Theorem 4.A (Structure result)**.** *Let f be a post-singularly finite entire function. Then the iterated monodromy group of f is a self-similar group on an infinite alphabet, generated by a dendroid automaton. In particular, it is a self-similar group of bounded activity growth.*

For a precise description of dendroid automata, see Section 5, where we develop the theory of dendroid automata that act on infinite sets. Another key ingredient are periodic spiders (Section 6).

Here is our second main result.

Theorem 4.B (Amenability of IMGs of entire functions)**.** *Let f be a post-singularly finite entire transcendental function. Then the iterated monodromy group of f is amenable if and only if the monodromy group of f is amenable.*

We should note that the condition on the monodromy group is clearly necessary, as the monodromy group is a quotient of the iterated monodromy group. There are in fact transcendental entire functions with non-amenable monodromy group, such as the free product $C_2 * C_2 * C_2$, so the condition is necessary. We show that compositions of structurally finite entire functions have elementary amenable monodromy groups, so we have a large class of functions with amenable iterated monodromy groups.

This paper is a continuation of our work in [Rei20c], where we introduce iterated monodromy groups for transcendental entire functions in the setting of the exponential family, as well as [Rei20a], where we prove an amenability criterion for groups generated by bounded activity automata on infinite alphabets. We use this criterion to deduce Theorem 4.**B** from Theorem 4.**A**.

Structure of the Paper. In Section 2 we provide the necessary function theoretic background for entire functions. In particular, we introduce Schreier graphs and spiders and compare them to the classical notion of line complexes. In Section 3 we extend the notion of dendroid set of permutations to infinite sets. We show that the monodromy groups of structurally finite entire transcendental maps are elementary amenable. In Section 4 we introduce the language of bisets for entire functions in a non-dynamical setting. We define (non-autonomous) dendroid automata in Section 5 and show how we can pullback spiders to understand the bisets of entire functions. We show in Section 6 how to obtain periodic spiders for entire functions, and we use this to conclude with the proof of the two main theorems.

Convention. We denote the Riemann sphere by $\hat{\mathbb{C}}$. We parametrize paths by closed intervals $I \subset [0,\infty]$. We compose paths in the same fashion as functions, if $p\colon I \to \hat{\mathbb{C}}$ is a path from a to b, and $q\colon I \to \hat{\mathbb{C}}$ is a path from b to c, then qp is the concatenation of p and q and a path from a to c. An *arc* is an injective path.

For a subset $B \subset \hat{\mathbb{C}}$, a path $p\colon I \to \hat{\mathbb{C}}$ is *proper* relative to B if $p^{-1}(B)$ consists precisely of the endpoints of the interval I. A *proper homotopy* relative to B is a homotopy $H\colon I \times [0,1] \to \hat{\mathbb{C}}$ such that each path $H_t = H(-,t)$ is a proper path and the homotopy is constant on endpoints.

4.2 Line graphs and Schreier graphs of entire functions in the Speiser class

We develop the function theory of entire functions with finitely many singular values here. See [BE95] for a more general discussion of singularities of meromorphic functions.

Definition 4.2.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire transcendental function. For $z_0 \in \mathbb{C}$, the *local degree* of f at z_0 or *branch index* of f at z_0 is minimal positive degree $m \geq 1$ appearing in the local power series expansion of f at z_0, i.e., $f(z) = f(z_0) + \alpha(z - z_0)^m +$ (higher order terms). If the local degree is greater than 1, then z_0 is a *critical point*, and $f(z_0)$ is a *critical value*. Note that z_0 is a critical point if and only if $f'(z_0) = 0$.

An *asymptotic value* is a limit $\lim_{t\to\infty} f(\gamma(t))$ where $\gamma\colon [0,\infty) \to \mathbb{C}$ is a path with $\lim_{t\to\infty}\gamma(t) = \infty$. The set of finite singular values is defined as

$$\mathbf{S}(f) = \overline{\{\text{critical and asymptotic values}\}}.$$

The value ∞ is also considered as a singular value, but it is not in the set of finite singular values $\mathbf{S}(f)$. We say that f belongs to the *Speiser class* $\mathscr{S}$ if $\mathbf{S}(f)$ is finite.

We are mostly interested in the topological behaviour of entire functions in the Speiser class. The following lemma will be the basis of our discussion.

Lemma 4.2.2 ([Sch10, Theorem 1.13])**.** *Let f be an entire function. Then f restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(\mathbf{S}(f))$ to $\mathbb{C} \setminus \mathbf{S}(f)$.*

We will mostly consider functions from the Speiser class.

Definition and Lemma 4.C. Let f be an entire function in the Speiser class. For $z \in \mathbf{S}(f)$, let $U \subset \mathbb{C}$ be a simply connected open neighborhood of z that intersects $\mathbf{S}(f)$ only in z. Let V be a connected component of $f^{-1}(U)$. Then V is simply connected and exactly one of the following holds:

- The map f restricts to a biholomorphic map $V \to U$. The unique preimage w of z in V is called a *regular preimage* of z.
- The map f has a unique preimage w of z in V and the map f restricts to an unbranched covering map on $V \setminus \{w\} \to U \setminus \{u\}$ of degree $m > 1$ equal to the local degree of f at w. In this case we say that w is an *algebraic singularity* over z.
- The map f has no preimage of z in V and restricts to an universal covering $V \to U \setminus \{z\}$. In this case we say that V is a *logarithmic tract* over z.

If $U' \subset U$ is another simply connected open neighborhood of z, then every component of $f^{-1}(U)$ contains exactly one component of $f^{-1}(V)$, and the classifications of preimage components agree. In particular, we can compare the classification for any two simply connected open neighborhoods of z by going to a simply connected open neighborhood contained in both of them. A class of compatible logarithmic tracts is identified with a *logarithmic singularity*.

Moreover let $U' \subset \hat{\mathbb{C}}$ be a simply connected open neighborhood of ∞. Then every preimage component of $U' \setminus \{\infty\}$ is a logarithmic tract over infinity.

Proof. Note that by the previous lemma, $V \setminus f^{-1}(z) \to U \setminus \{z\}$ is always an unbranched covering. As $U \setminus \{z\}$ has fundamental group $\mathbb{Z}$, the classification in three different cases follows easily from the classification of connected coverings of $U \setminus \{z\}$. If $U' \subset U$ is another simply connected open neighborhood of z, then $U' \setminus z \hookrightarrow U \setminus z$ is a homotopy equivalence, and they share the same classification.

For discussion of the preimage $f^{-1}(z)$, see for example [For91, Theorem 5.11] for algebraic singularities, and [BE95] for logarithmic singularities. □

A entire function in the Speiser class is called *structurally finite* if it only has finitely many logarithmic singularities and finitely many algebraic singularities. See [Elf34] for the classification of such maps via their Schwarzian derivative.

Our classification of singularities is simplified as we only consider functions in the Speiser class. In particular, we use that $\mathbf{S}(f)$ is discrete, and every point in $\mathbf{S}(f)$ has a simply connected open neighborhood away from the other points in $\mathbf{S}(f)$. See [BE95] for a more general discussion.

Example 4.2.3. We will use the function $f(z) = (1-z)\exp z$ as our running example. As $f'(z) = -z \exp z$, the only critical point of f is 0 of local degree 2, so $f(0) = 1$ is the

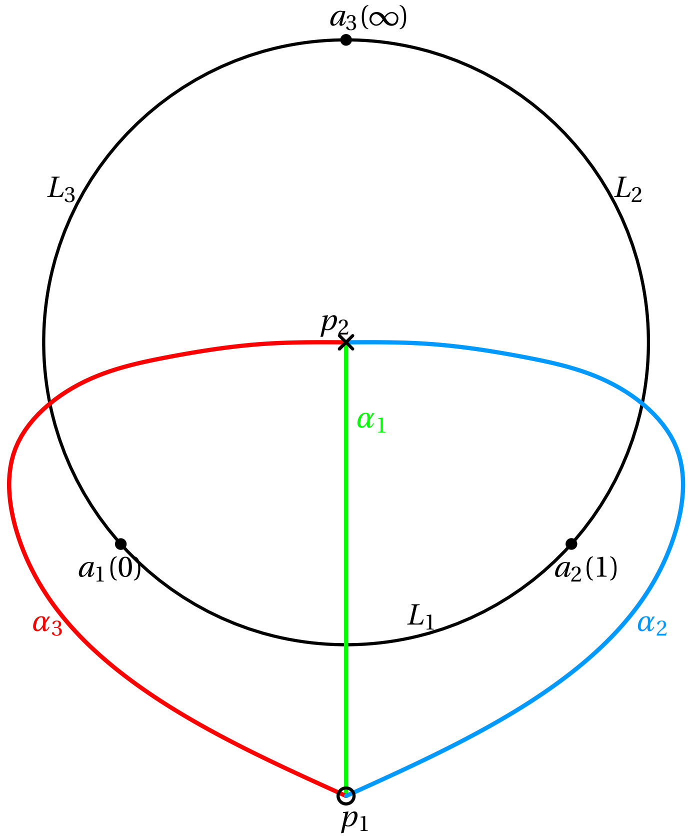


Figure 4.1 – Spine Γ'_s for $f(z) = (1-z)\exp(z)$

only critical value. The path along the negative real axis shows that 0 is an asymptotic value. By the Denjoy–Carleman–Ahlfors theorem (see e.g. [Sch10, Theorem 1.17]), there is exactly one logarithmic singularity over 0 and one logarithmic singularity over ∞. In particular, the function has as finite singular values only 0 and 1 and is structurally finite.

We will use the notion of line complex (or Speiser graph) for entire functions. They can more generally be used for any surface spread with finitely many singular values, but we restrict our attention to entire functions in the Speiser class. See [GO08, Chapter 7] for a general introduction of line complexes for meromorphic functions.

Definition 4.2.4. Let f be entire transcendental function in the Speiser class, with n finite singular values. Let L be an oriented Jordan curve in $\hat{\mathbb{C}}$ going through all finite singular values and ∞. Then $\mathbf{S}(f) \cup \infty$ separates γ into finitely many arcs $L_1, \ldots L_{n+1}$, where we assume that the L_i are cyclically ordered. We label the set $\mathbf{S}(f) \cup \infty$ with $a_1, \ldots, a_{n+1}$ with $a_{n+1} = \infty$ such that L_i is the arc from a_i to a_{i+1}, with cyclical indices. The line complex or Speiser graph is defined as follows: L separates the plane in two componets H_1 and H_2. Choose points $p_1 \in H_1$ and $p_2 \in H_2$. We think of L as a planar graph with n edges and n vertices and construct the dual graph Γ'_s of L by connecting

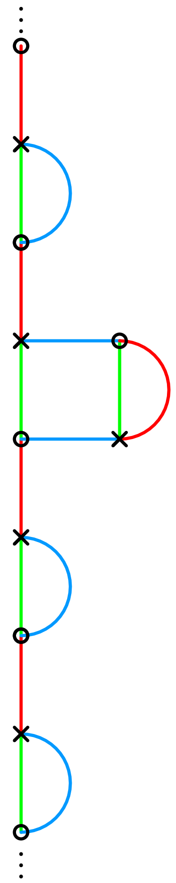


Figure 4.2 – Line complex Γ_s for $f(z) = (1-z)\exp(z)$

p_1 and p_2 via arcs $\alpha_1, \dots, \alpha_{n+1}$ with α_i intersecting L only in one point of L_i, and the α_i intersecting each other only in p_1 and p_2. The *line complex* Γ_s of f with respect to L is the preimage of Γ_s' under f as a planar graph in $\mathbb{C}$.

Example 4.2.5. In our example $f = (1-z)\exp z$, a possible choice for the Jordan curve L is given by the extended real line. See Figure 4.1 for the graph Γ_s' and see Figure 4.2 for the resulting line complex.

Definition 4.2.6 (Spider, Rose graph)**.** A *spider leg* is an injective curve $\gamma\colon [0,\infty) \to \mathbb{C}$ with $\lim_{s\to\infty} \gamma(s) = \infty$. It will also be convenient to think of a spider leg as a closed arc from $[0,\infty]$ to $\hat{\mathbb{C}}$ with $\gamma(\infty) = \infty$. The end point $\gamma(0)$ is also called the *landing point* of γ.

Let A be a finite set of points in $\mathbb{C}$. A *spider* is a family $\mathbb{S} = (\gamma_a)_{a\in A}$ such that γ_a is a spider leg landing at a such that the γ_a are disjoint in $\mathbb{C}$. We can think of $\mathbb{S}$ as a planar tree in $\hat{\mathbb{C}}$.

Let t be a point in $\mathbb{C}$ that is not in the image of any spider leg of $\mathbb{S}$. Taking a dual graph of $\mathbb{S}$ we obtain a *rose graph* Γ'. Its only vertex is t, and for every $a \in A$, we have a loop g_a that intersects $\mathbb{S}$ only once, namely in the interior γ_a. If we think of γ_a as an arc from a to ∞, we choose the orientation on g_a such that the algebraic intersection number $i(g_a, \gamma_a)$ is positive.

Given a set $A \subset \mathbb{C}$ and two spider legs γ, γ' landing at the same point in $a \in A$, we say that they are homotopic relative to A if there are properly homotopic relative to $A \cup \{\infty\} \subset \hat{\mathbb{C}}$ as proper paths in $\hat{\mathbb{C}}$. This means that there is a homotopy $H\colon [0,\infty] \times [0,1] \to \hat{\mathbb{C}}$ from γ to γ' such that $H^{-1}(a) = 0 \times [0,1]$, $H^{-1}(\infty) = \infty \times [0,1], H^{-1}(A \setminus \{a\}) = \emptyset$. So every $H_t = H(-,t)$ is a path from the common landing point of γ, γ' to ∞ intersecting $A \cup \{\infty\}$ at the appropriate endpoints. We are interested in working with spiders up to homotopy. We will use the following lemma to safely pass from considerations up to homotopy to considerations up to isotopy.

Lemma 4.2.7 (Epstein-Zieschang)**.** *Let B be a finite set of point in $\hat{\mathbb{C}}$. Consider $(\hat{\mathbb{C}}, B)$ as a compact marked surface. Let $\gamma_1, \dots, \gamma_n$ be a collection of arcs with endpoints in B such that the following holds:*

- *The arcs intersect B only at the endpoints.*
- *The arcs and their inverses are pairwise nonhomotopic as proper paths relative to B.*
- *The arcs intersect each other at most at their endpoints.*

Let $\gamma'_1, \dots, \gamma'_n$ be another collection of arcs satisfying the same conditions, such that γ_i is homotopic to γ'_i relative to B. Then there is a homeomorphism ϕ isotopic relative to B to the identity with $\phi(\gamma_i) = \gamma'_i$.

For a proof see [Bus10, Theorem A.5].

Lemma 4.2.8 (Lifting properties of entire functions)**.** *Let f be an entire function in the Speiser class. Let $A \subset \mathbb{C}$ be a finite set that contains $\mathbf{S}(f)$. We have the following lifting properties:*

- *For every path $p\colon I \to \mathbb{C} \setminus A$, and every preimage $w \in f^{-1}(p(0))$, there is a unique path $p^w\colon I \to \mathbb{C} \setminus f^{-1}(A)$, such that $p^w(0) = w$ and $f \circ p^w = p$.*
- *For every spider leg γ that is completely disjoint from A , and every preimage w of the landing point of γ, there is a unique lift γ^w of γ landing at w.*
- *For every spider leg γ that intersects A only at the landing point of γ, and every preimage w of the landing point of γ, there as many lifts of γ landing at w as the local degree of f at w.*
- *Let γ, γ' be spider legs that land at the same point of $a \in A$ and are homotopic relative to A via a homotopic $H\colon [0,\infty] \times [0,1] \to \hat{\mathbb{C}}$. Let w be a preimage of a, and $\hat{\gamma}$ a lift of γ landing at w. Then there is a homotopy $\hat{H}\colon [0,\infty] \times [0,1] \to \hat{\mathbb{C}}$ relative to $f^{-1}(A)$ of spider legs landing at w from $\hat{\gamma}$ to a lift of γ'. .*

Proof. The first statement is just the unique path lifting property .

For the second part, the only thing left to show is that $\lim_{t\to\infty} \gamma^w(t) = \infty$. Since $\lim_{t\to\infty} \gamma(t) = \infty$, and f is bounded on every compact subset of $\mathbb{C}$, it follows that γ^w has to leave every compact subset of $\mathbb{C}$.

For the last two statement, let v be the landing point of γ, U be a simply connected open neighborhood of v that intersects $\mathbf{S}(f)$ only in v. Let V be the preimage component of U that contains w. Choose a point on $v' \in \gamma \cap U \setminus \{v\}$. By Lemma 4.C v' has as many preimages in V as the local degree f at w. By the unique path lifting property for every subinterval of $(0,\infty)$, there is a unique lift on $\gamma'\colon (0,\infty) \to \mathbb{C}$ of $\gamma_{|(0,\infty)}$ passing through v'. As before we have $\lim_{t\to\infty} \gamma'(t) = \infty$. From the local behaviour of f at w, we also get $\lim_{t\to 0} \gamma'(0) = 0$. So we can extend γ' to a lift of γ. A similar proof works for homotopies. □

Definition 4.2.9 (Monodromy action)**.** Let f be an entire function in the Speiser class. Let A be a finite subset of $\mathbb{C}$ that contains the singular set of f. Let t be a point in $\mathbb{C} \setminus A$. The monodromy action of $\pi_1(\mathbb{C} \setminus A, t)$ on $f^{-1}(t)$ is defined as follows: for $[g] \in \pi_1(\mathbb{C} \setminus A, t)$ and $w \in f^{-1}(t)$, the action of $[g]$ on w is the endpoint of the lift g^w. By the homotopy lifting property, this is a well defined action. The monodromy group of f is the resulting permutation group on $f^{-1}(t)$.

One should note that the action of g on $f^{-1}(t)$ only depends on the homotopy class in $\pi_1(\mathbb{C} \setminus \mathbf{S}(f), t)$. So the monodromy group of f doesn't depend on the set A. By standard considerations, it only depends on t up inner automorphisms.

Lemma 4.2.10 (Schreier graph)**.** *Let f be an entire function in the Speiser class. Let A be a finite set in $\mathbb{C}$ that contains the singular set of f. Let $\mathbb{S} = (\gamma_a)_{a\in A}$ be a spider and $(g_a)_{a\in A}$ the dual generating set with rose graph Γ' with base point t. Then the preimage Γ of Γ' under f is a locally finite planar graph with vertex set $f^{-1}(t)$ and a topological realization of the* Schreier *graph of the monodromy action of $\pi_1(\mathbb{C} \setminus A, t)$ on $f^{-1}(t)$ with generators $(g_a)_{a\in A}$.*

Moreover, for every $a \in A$, the finite orbits of g_a of the monodromy action are in bijection to finite preimages of a under f, the infinite orbits are g_a are in bijection to logarithmic singularities over a.

In fact, we have the following classification of faces of Γ:

- *Faces with finitely many edges on the boundary contain a unique point w of $f^{-1}(A)$ and are bounded by a loop $x_1 \xrightarrow{g_a^{x_1}} x_2 \xrightarrow{g_a^{x_2}} \cdots \xrightarrow{g_a^{x_k}} x_k = x_1$ given by lifts of g_a along finite g_a orbit for $a = f(w)$.*
- *Faces with infinitely many edges either contain a logarithmic tract over some $a \in A$, and are bounded by an infinite g_a orbit, or they contain a logarithmic tract over ∞.*

Moreover, faces with infinitely many edges have their boundary as deformation retract.

Recall that for a group G with finite generating set S acting on a set X, the Schreier graph $\Gamma(G,S,X)$ has vertex set X and edges $x \to s(x)$ for every $x \in X, s \in S$.

Proof. The fact that the preimage Γ of Γ' is really the topological realization of the Schreier graph of the monodromy action of $\pi_1(\mathbb{C} \setminus A, t)$ on $f^{-1}(t)$ with generators $(g_a)_{a\in A}$ is clear from the definition of the monodromy action.

Every open face of Γ is a component of the preimage of $\mathbb{C} \setminus \Gamma'$, so it is a component of the preimage of one of the components of $\mathbb{C} \setminus \Gamma'$. Now every bounded component of $\mathbb{C} \setminus \Gamma'$ is a simply connected open neighborhood of a single point of A bounded by g_a, and the unique unbounded component is of the form $U \setminus \{\infty\}$ with $U' \subset \hat{\mathbb{C}}$ a simply connected open neighborhood of ∞. So now the classification follows from Lemma 4.C.

What is left to show that every face with infinitely many edges has their boundary as a deformation retract. For every $a \in A$, let $U_a \subset \mathbb{C}$ be the component of $\mathbb{C} \setminus \Gamma'$ that contains a. Denote by $U'_\infty \subset \hat{\mathbb{C}}$ the component of $\hat{\mathbb{C}} \setminus \Gamma'$ containing ∞. Let V be an open face of Γ with infinitely many edges on the boundary. By the classification, we either have $f(V) = U_a \setminus \{a\}$ for some $a \in A$ or or $f(V) = U'_\infty \setminus \{\infty\}$.

As $\partial V \subset \Gamma = f^{-1}(\Gamma')$, we have $f(\overline{V}) \subset \overline{U_a} \setminus \{a\}$ or $f(\overline{V}) \subset \overline{U'_\infty} \setminus \{\infty\}$ In both cases $f(\overline{V}) \subset \mathbb{C} \setminus A \subset \mathbb{C} \setminus \mathbf{S}(f)$ and since $\overline{U_a} \setminus \{a\}$ deformation retracts onto ∂U_a and $\overline{U'_\infty} \setminus \{\infty\}$ deformation retracts on $\partial U'_\infty$, we can use the homotopy lifting principle to obtain a deformation retraction of $\overline{V}$ onto ∂V. □

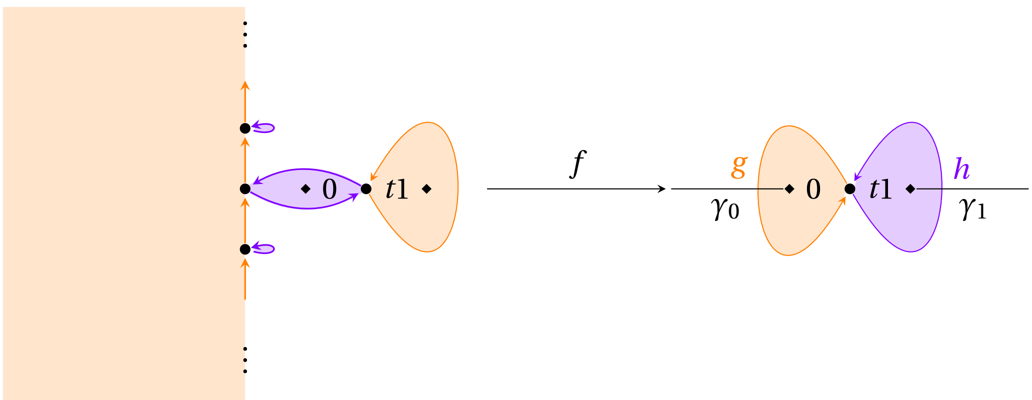


Figure 4.3 – Schreier Graph for $(1-z)\exp(z)$.

Example 4.2.11. In our example $f = (1-z)\exp z$, let us take $A = \mathbf{S}(f) = \{0,1\}$. As f is monotonically decreasing on $[0,1]$, let t be the unique fixed point of f in $[0,1]$. For our spider legs γ_0 and γ_1 we take legs that remain in $\mathbb{R}$. For the resulting rose graph and Schreier graph see Figure 4.3. For ease of notation we called the rose graph generator corresponding to 0 by g and corresponding to 1 by h. We also colored faces of Γ based on their images under f.

Remark 4.2.12 (Isotopy dependence). The rose graph dual to $\mathbb{S}$ is only well-defined up to isotopy relative to $A \cup \{t, \infty\}$, the classes of g_a in $\pi_1(X \setminus A, t)$ are well-defined. If we change $\mathbb{S}$ via an isotopy relative to $A \cup \{t, \infty\}$, the classes of g_a don't change. If we change $\mathbb{S}$ via an isotopy relative to $A \cup \{\infty\}$, we get a conjugated generating set of g_a.

.

We want to compare random walks on line complexes and Schreier graphs.

Lemma 4.2.13 (Line complexes are quasi-isometric to Schreier graphs)**.** *Let f be a function in the Speiser class. Let L be a Jordan curve through $\mathbf{S}(f) \cup \infty$. Then there is a spider $\mathbb{S} = (\gamma_a)_{a \in \mathbf{S}}$ such that associated line complex Γ_s to L and the Schreier graph Γ are quasi-isometric in the following sense:*

The vertex set of Γ is a subset of the vertex set of Γ_s. Every vertex of Γ_s is either a vertex of Γ or connected in Γ_s to a vertex of Γ. For w, w' vertices of Γ, we have $\frac{2}{n} d_\Gamma(w, w') \leq d_{\Gamma_s}(w, w') \leq 2 d_\Gamma(w, w')$, where n is the cardinality of $\mathbf{S}(f)$.

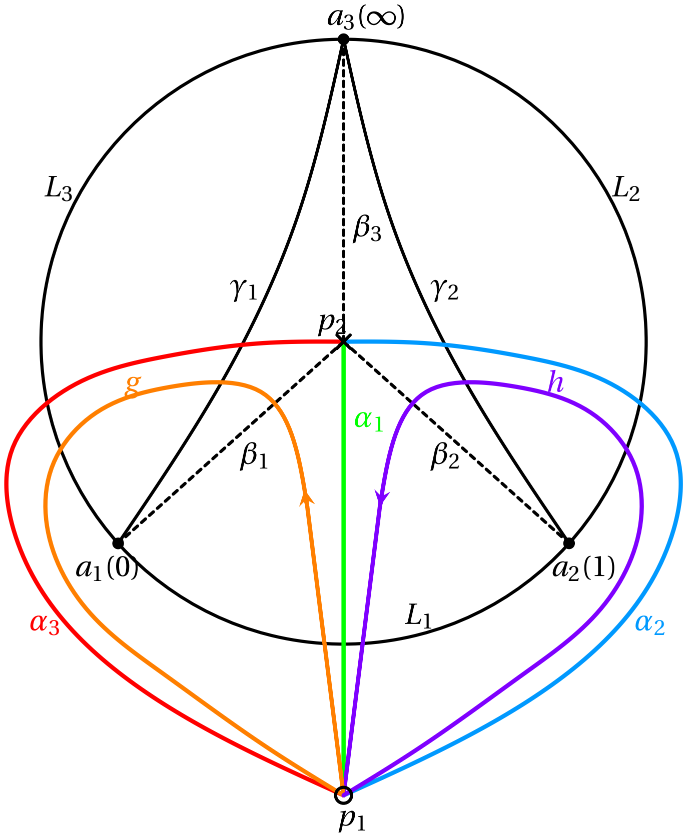


Figure 4.4 – Spine Γ'_s and rose graph Γ' for $f(z) = (1-z)\exp(z)$

Proof. We illustrate the constructions used in the proof in Figure 4.4 and Figure 4.5 for our example function $(1-z)\exp z$. We keep the notation of Definition 4.2.4. In particular $a_1, \ldots, a_{n+1}$ are the singular values of f in cyclic order on L, including $a_{n+1} = \infty$. We introduce a spider that lies completely in H_2. As H_2 is a Jordan domain, there is a unique isotopy class of an arc γ_i for $1 \leq i \leq n$ from a_i to ∞ with the interior of γ_i in H_2. It is possible to realize the isotopy classes via a spider $\mathbb{S} = (\gamma_i)_{1 \leq i \leq n}$ such that the γ_i only meet in ∞. For example, use the hyperbolic metric on H_2 and let γ_a be the geodesic from a_i to ∞.

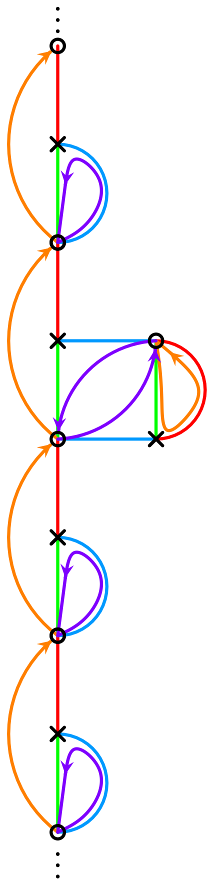


Figure 4.5 – Line complex Γ_s and Schreier graph Γ for $f(z) = (1-z)\exp(z)$

Similarly, for $1 \le i \le n+1$ it is possible to connect a_i to p_2 via an arc β_i in H_2 that does intersects Γ_s' only in p_2. Then the cyclic order of arcs leaving p_2 is $\beta_1, \alpha_1, \beta_2, \alpha_2, \dots, \beta_{n+1}, \alpha_{n+1}$. We think of the α_i as oriented paths from p_1 to p_2, so α_i^{-1} is a path from p_2 to p_1. By convention $\alpha_0 = \alpha_n$. We recall that we concatenate paths in the same way as functions, i.e., "from right to left".

Let g_i be the dual generating set to $\mathbb{S}$. Then g_i is homotopic to $\alpha_{i-1}^{-1}\alpha_i$: as the γ_j are homotopic to $\beta_n\beta_j$, we see that by the cyclic ordering of the arcs at p_2, the concatenation $\alpha_{i-1}^{-1}\alpha_i$ has a positive transversal intersection in p_2 with $\beta_{n+1}\beta_i$, and removable intersections in p_2 with $\beta_{n+1}\beta_j$ for $j \neq i$. So in fact g_i and $\alpha_{i-1}^{-1}\alpha_i$ are homotopic.

Now let Γ be the Schreier graph with respect to the generators g_i. Then Γ has as vertex set $f^{-1}(p_1)$, and the line complex Γ_s has vertex set $f^{-1}(p_1) \cup f^{-1}(p_2)$. Every element of $f^{-1}(p_2)$ is connected via an edge to an element of $f^{-1}(p_1)$, this proves the first statement.

For the inequality $d_{\Gamma_s}(w, w') \le 2d_\Gamma(w, w')$ for $w, w' \in f^{-1}(p_1)$, let w, w' be connected by a path q in Γ of combinatorial length m. Then q is a lift h^w for some $h = h_1 \dots h_m$

with $m = d_\Gamma(w, w')$, $h_j = g_{i_j}$. For h_j let $\tilde{h}_j$ be the concatenation $\alpha_{i_j-1}^{-1}\alpha_{i_j}$. Then $\tilde{h}_j$ is homotopic to h_j in $\mathbb{C} \setminus \mathbf{S}(f)$, so $\tilde{h} = \tilde{h}_1 \dots \tilde{h}_m$ is homotopic to h in $\mathbb{C} \setminus \mathbf{S}(f)$. So $\tilde{h}^w$ is also a path from w to w'. Since $\tilde{h}^w$ corresponds to a combinatorial path in Γ_s of length $2m$, we have $d_{\Gamma_s}(w, w') \leq 2m$.

For the inequality $\frac{2}{n} d_\Gamma(w, w') \leq d_{\Gamma_s}(w, w')$ for $w, w' \in f^{-1}(p_1)$, let w, w' be connected by a path q in $\Gamma_s$of combinatorial length m. Then q is a lift h^w for some $h = h_1 \dots h_m$ with $m = d_{\Gamma_s}(w, w')$, $h_j = \alpha_{i_j}^{\epsilon_j}$. As Γ_s is bipartite, we have $\epsilon_j = (-1)^j$ and m even. Let $m = 2m'$ and $h = h'_1 \dots h'_{m'}$ with $h'_j = \alpha_{l_j}^{-1}\alpha_{k_j}$. By convention, we use α_0^{-1} instead of α_n^{-1}. Then h'_j is homotopic to $\tilde{h}_j = g_{l_j+1} g_{l_j+2} \dots g_{k_j}$ if $l_j y k_j$, and $\tilde{h}_j = (g_{k_j+1} g_{k_j+2} \dots g_{l_j})^{-1}$ if $l_j > k_j$. In particular, every lift of h'_j is a combinatorial path of length $\leq n$ in Γ. So $\tilde{h} = \tilde{h}_1 \dots \tilde{h}'_m$ is homotopic to h in $\mathbb{C} \setminus \mathbf{S}(f)$. So $\tilde{h}^w$ is also a path from w to w'. Since $\tilde{h}^w$ corresponds to a combinatorial path in Γ_s of length $\leq nm'$, we have $d_\Gamma(w, w') \leq nm'$. So the inequality $\frac{2}{n} d_\Gamma(w, w') \leq d_{\Gamma_s}(w, w')$ follows. □

Lemma 4.2.14 (Recurrence of monodromy action)**.** *Let f be a function in the Speiser class. Then the monodromy action of f is recurrent.*

Proof. In order to show that the monodromy action of f is recurrent, it is enough to show that for one generating set of $\pi_1(X \setminus \mathbf{S}(f))$ the random walk on the associated Schreier graph is recurrent. Let L be a Jordan curve as in Lemma 4.2.4. Then by the previous lemma, the associated line complex Γ_s is quasi-isometric to a Schreier graph Γ of f. Now Γ_s and Γ are both regular graphs, so in particular they have bounded geometry. So we can apply [Woe00, Theorem I.3.10] to see that the simple random walk on Γ is recurrent if and only if the simple random walk on Γ_s is recurrent. Now the fact that the simple random walk on a line complex of an entire function is recurrent is well-known, see [Doy84; Mer03]. In fact, the random walk on an extended line complex for an entire function is recurrent, and the line complex is a subnetwork of the extended line complex, so by [Woe00, Corollary I.2.15], the simple random walk on Γ_s is also recurrent. □

4.3 Dendroid permutations

We extend the notion of a family of dendroid permutations from [Nek09, Section 2] to infinite sets.

Definition 4.3.1 (Dendroid permutation)**.** Let X be a set, let $a_i \in \mathrm{Sym}(X), i \in I$ be a family of permutations. The *cycle diagram* $D((a_i)_{i \in I})$ is a 2-dimensional CW-complex built as follows: The 0-skeleton is the discrete set X. For every $x \in X, i \in I$, we insert a 1-cell $x \xrightarrow{a_i} a_i(x)$. So the 1-skeleton is the Schreier graph of $(a_i)_{i \in I}$ on X. For every $i \in I$ and every finite orbit $x_1 \xrightarrow{a_i} x_2 \xrightarrow{a_i} \cdots \xrightarrow{a_i} x_{k+1} = x_1$ of a_i, glue in a 2-cell along the loop $x_1 \xrightarrow{a_i} \cdots \xrightarrow{a_i} x_1$. We say that the family $(a_i)_{i \in I}$ is a *dendroid set of permutations* if $D((a_i)_{i \in I})$ is contractable.

Lemma 4.3.2. *Let X be a set, let $a_i \in \mathrm{Sym}(X), i \in I$ be a family of permutations. We consider the Schreier graph of $X, (a_i)_{i\in I}$ and do the following modification: for every finite orbit a_i, remove exactly one edge in the orbit. Call the resulting graph $\hat{\Gamma}$. Then $\hat{\Gamma}$ is a deformation retract of the cycle diagram $D((a_i)_{i\in I})$. In particular $\hat{\Gamma}$ is a tree if and only if family $(a_i)_{i\in I}$ is a dendroid set of permutations.*

Proof. We can define the deformation retraction cell-wise. For every 2-cell, we numerate the bounding loop $x_1 \xrightarrow{a_i} x_2 \xrightarrow{a_1} \cdots \xrightarrow{a_i} x_{k+1} = x_1$ such that we remove the last edge $x_k \xrightarrow{a_i} x_1$ in our construction of $\hat{\Gamma}$. So topologically it is equivalent to define a deformation retraction of $\overline{\mathbb{D}}$ onto a proper closed interval of S^1 (or a point in S^1 if $k = 1$), it is clear that this is possible.

As a graph is a tree if and only if it is contractible, $\hat{\Gamma}$ is a tree if and only if it is contractible. So the statement follows via the homotopy equivalence between $\hat{\Gamma}$ and $D((a_i)_{i\in I})$. □

From the previous lemma we see that orbits in dendroid set of permutations must either be disjoint or intersect in at most one point, as we otherwise could construct a cycle in $\hat{\Gamma}$. Also, if a dendroid set of permutations fixes a point $x \in X$, then it is an isolated vertex of $\hat{\Gamma}$, so in fact $X = \{x\}$.

Lemma 4.3.3. *Let $a_i \in \mathrm{Sym}(X), i \in I$ be a dendroid set of permutations. Suppose that I is finite and for every $i \in I$, a_i has only finitely many orbits that are nontrivial, i.e., a_i has only finitely many orbits that consist of more than one point. Then the group generated by the a_i is elementary amenable.*

Proof. Let $X_i = X \setminus \mathrm{Fix}(a_i)$. If X consists of one point then there is nothing to show. Otherwise we have $X = \bigcup_{i\in I} X_i$. Let $J \subset I$ be the subset of indices $j \in I$ such that a_j has an infinite orbit. As we assume that every a_i has only finitely many nontrivial orbits, $j \in J$ if and only if X_j is infinite. For $\phi\colon J \to \mathbb{Z}$, let

$$H_\phi := \left\{ g \in \mathrm{Sym}(X) \colon \text{ for all } j \in J, g(x) = a_j^{\phi(j)}(x) \text{ for all but finitely many } x \in X_j \right\}.$$

Then we have the following:

— For $\phi, \psi\colon J \to \mathbb{Z}$, we have $H_\phi H_\psi \subset H_{\phi+\psi}$ and $(H_\phi)^{-1} = H_{-\phi}$. This is straightforward to check.

— For $\phi \neq \psi\colon J \to \mathbb{Z}$, we have $H_\phi \cap H_\psi = \emptyset$. As $\phi \neq \psi$, there is a $j \in J$ with $\phi(j) \neq \psi(j)$. As a_j has an infinite orbit, $a_j^{\phi(j)}$ and $a_j^{\psi(j)}$ differ on an infinite subset of X_j. So now element $g \in \mathrm{Sym}(X)$ cannot cofinally agree to both $a_j^{\phi(j)}$ and $a_j^{\psi(j)}$, so H_ϕ and H_ψ have to be disjoint.

— For $i \in I \setminus J$, $a_i \in H_0$, for $j \in J$, $a_j \in H_{\delta_j}$ where δ_j is the Kronecker delta on I: For $i, j \in I$, we have that the intersection of an a_i orbit and an a_j orbit is at most one point. As we have only finitely many nontrivial a_i orbits and nontrivial a_j orbits, a_i moves only finitely many points in X_j. From this the statement easily follows.

Let G be the group generated by the a_i. Then the above shows that we have a group homomorphism $\Phi\colon G \to \mathbb{Z}^J$ uniquely determined by $g \in H_{\Phi(g)}$ for $g \in G$. The kernel is a subgroup of H_0. Now every element of H_0 has finite support: Every element of H_0 has finite support on X_j for $j \in J$. As $X = \bigcup_{i\in I} X_i$ and X_i is finite for $i \in I \setminus J$, we obtain finite support on X. In particular $\ker \Phi \subset H_0$ is locally finite. So in particular, G is the extension of an abelian group by a locally finite group. So it is elementary amenable. □

Lemma 4.3.4. *Let f be an entire function in the Speiser class. Let A be a finite set in $\mathbb{C}$ that contains the singular set of f. Let $\mathbb{S} = (\gamma_a)_{a\in A}$ be a spider. Let t be a point in $\mathbb{C} \setminus \mathbb{S}$. Let $g_a \in \pi_1(\mathbb{C} \setminus A, t)$ be the generating set dual to $\mathbb{S}$. Let $\Phi\colon \pi_1(\mathbb{C} \setminus A, t) \to \mathrm{Sym}(f^{-1}(t))$ be the group homomorphism induced by the monodromy action. Then $\left(\Phi(g_a)_{a\in A}\right)$ is a dendroid set of permutations.*

Proof. We will show that we can obtain $D(\Phi(g_a)_{a\in A})$ as a deformation retract of $\mathbb{C}$. As $\mathbb{C}$ is contractible, this will show that $D(\Phi(g_a)_{a\in A}$ is contractible. Using Lemma 4.2.10, we see that the Schreier graph Γ together with the faces with finitely many bounding edges is homeomorphic to $D(\Phi(g_a)_{a\in A})$. Also by Lemma 4.2.10, the faces of Γ with infinitely many bounding edges can be deformation retracted onto their boundary. So we can contract $\mathbb{C}$ onto $D(\Phi(g_a)_{a\in A})$. □

Our running example is an example of a structurally finite maps. We obtain from Lemma 4.3.3 and Lemma 4.3.4 the following corollary:

Corollary 4.3.5. Monodromy groups of structurally finite entire maps are elementary amenable.

From the proof of Lemma 4.3.3 it is also easy to see that monodromy groups of structurally finite entire maps can be realized as subgroups of Houghton's family of groups [Hou79].

4.4 Marked entire maps and Bisets

In this section we introduce the language of bisets used for entire maps. See [Nek05, Section 2] for a general introduction to bisets in the context of self-similar groups on finite alphabets, and [BD17] for an introduction for bisets to rational Thurston theory. We follow the acting convention from [Nek05]. The proof of most statements in this section are essentially as in the case for polynomials and finite alphabets. We compare the formalism of bisets to the definition of iterated monodromy groups for entire functions given in [Rei20c].

4.4.1 Bisets and non-autonomous automata

Let G and H be groups. A *G-H-biset* ${}_G\mathscr{M}_H$ is a set $\mathscr{M}$ together with a left G-action and a right H-action such that the actions commute, i.e. we have $g\cdot(m\cdot h) = (g\cdot m)\cdot h$ for all $g \in G, m \in \mathscr{M}, h \in H$.

If ${}_G\mathcal{M}_H$ and ${}_H\mathcal{N}_K$ are bisets, the *tensor product* ${}_G\mathcal{M}\otimes\mathcal{N}_K$ is $\mathcal{M}\times\mathcal{N}/\sim$ where $\sim$ is the equivalence relation generated by $(m\cdot h,n)\sim(m,h\cdot n)$. We will denote the element of $\mathcal{M}\otimes\mathcal{N}$ represented by (m,n) also as $m\otimes n$. The tensor product ${}_G\mathcal{M}\otimes\mathcal{N}_K$ is again a G-K-bisets via $g\cdot(m\otimes n)=(g\cdot m)\otimes n$ and similar for the K-right action.

For a biset ${}_G\mathcal{M}_H$, we can consider the set of H-orbits $\mathcal{M}/H$ with the induced G-left action on H-orbits. Given two bisets ${}_G\mathcal{M}_H$ and ${}_H\mathcal{N}_K$, we have a natural map

$$\begin{array}{rcl}\mathcal{M}\otimes\mathcal{N}/K & \to & \mathcal{M}/H\\ m\otimes n\cdot K & \mapsto & m\cdot H\end{array}$$

We say that the biset ${}_G\mathcal{M}_H$ is *right free* if the right action of H is free on $\mathcal{M}$. In this case, we call a representative system $X\subset\mathcal{M}$ of $\mathcal{M}/H$ also a basis of $\mathcal{M}$. A basis X has then the property that every element $m\in\mathcal{M}$ can be written as $x\cdot h$ for a unique pair $x\in X,h\in H$. If the basis X is fixed, we will denote the unique factorization of this form for $g\cdot x$ as $g(x)\cdot g_{|x}$. We note that $g(x)$ is then the representative of the H orbit of x.

If ${}_G\mathcal{M}_H$ is right free with basis X and ${}_H\mathcal{N}_K$ is right free with basis Y, by [Nek05, Proposition 2.3.2] we have that ${}_G\mathcal{M}\otimes\mathcal{N}_K$ is again right free with basis $X\times Y$, in the sense that every element of $\mathcal{M}\otimes\mathcal{N}$ can be written as $x\otimes y\cdot k$ for a unique triple $x\in X,y\in Y,k\in K$.

We will need different notions of comparing bisets. We follow the naming convention of [BD17]: given a pair of group isomorphisms $\Phi\colon G\to G'$, $\Psi\colon H\to H'$, a Φ-Ψ-*congruence* between a ${}_G\mathcal{M}_H$ and ${}_{G'}\mathcal{N}_{H'}$ is a bijection $\Xi\colon{}_G\mathcal{M}_H\to{}_{G'}\mathcal{N}_{H'}$ with $\Xi(g\cdot m\cdot h)=\Phi(g)\cdot\Xi(m)\cdot\Psi(h)$. If $G=H,G'=H',\Psi=\Phi$, then we say that Ξ is a Φ-*conjugacy*. If $G=G',H=H'$ and $\Phi=\mathrm{id}_G,\Psi=\mathrm{id}_H$, then we say that Ξ is an *isomorphism* of G-H-bisets.

Definition 4.4.1. For a set A, let A_+ be the disjoint union of A together with the singleton $\{\mathbf{1}\}$. An (non-autonomous) *automaton* is a map $\tau\colon A_+\times X\to X\times B_+$ such that $\tau(\mathbf{1},x)=(x,\mathbf{1})$ for all $x\in X$. We call A the *input state set*, B the *output state set* and X the *alphabet* of $\mathbf{A}$. If $\tau(a,x)=(y,b)$, we also write $a_{|x}=b,a(x)=y$. We say that a restricts to b at x. If for every $a\in A$, the map $x\mapsto a(x)$ is bijective, we call $\mathbf{A}$ a *group automaton*.

The automata that we consider are non-autonomous in the sense that they have in general two different state sets. They are also called “time-varying automata” or “piecewise automata”.

Lemma 4.4.2. *Let $\tau\colon A_+\times X\to X\times B_+$ be a group automaton. Let F_A and F_B be the free groups on A and B respectively. Then we can associate a biset ${}_{F_A}\mathcal{M}_{F_B}$ that is right free with basis X and the action described by $\tau(a,x)=(y,b)$ if and only if $a\cdot x=y\cdot b$ for all $a\in A_+,b\in B_+,x,y\in X$.*

This is standard, we give a proof for completeness.

Proof. We can take as underlying set of ${}_{F_A}\mathcal{M}_{F_B}$ the set $X\times F_B$. The right action of F_B is given by $(x,g)\cdot h=(x,gh)$ for $g,h\in F_B$. For $a\in A$, we define a mapping $a\cdot -\colon X\times F_B\to$

$X \times F_B$ via $a \cdot (x, g) = (a(x), a_{|x} g)$. As $x \mapsto a(x)$ is a bijection, it is clear that that $a \cdot -$ is a bijection, with inverse given by $a^{-1} \cdot (x, g) = (a^{-1}(x), (a_{|a^{-1}(x)})^{-1} g)$. By the universal property of F_A, we now have a left action of F_A on $X \times F_B$. It is straightforward to see that the left action of F_A commutes with the right action of F_B, so we indeed have a biset ${}_{F_A}\mathcal{M}_{F_B}$.

It is clear that the right action is free with $X \times \{\mathbf{1}\} \cong X$ a basis of ${}_{F_A}\mathcal{M}_{F_B}$. Moreover, as $a \cdot (x, \mathbf{1}) = (a(x), a_{|x}) = (a(x), \mathbf{1}) \cdot a_{|x}$, the biset has the described action. □

4.4.2 Bisets of marked entire functions

Definition 4.4.3. A *marked entire map* is a map $f : (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ where f is an entire function (in the Speiser class), $A, B \subset \mathbb{C}$ are finite sets, $s \in \mathbb{C} \setminus A$, $t \in \mathbb{C} \setminus B$, $f(A) \subset B$, the singular set of f is contained in B.

Definition 4.4.4. Let $f : (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ be a marked entire map. The *biset* $\mathcal{M}_f$ of f is the set of homotopy classes of paths from a to an element of $f^{-1}(b)$ in $\mathbb{C} \setminus A$. The group $\pi_1(C \setminus A, s)$ acts on $\mathcal{M}_f$ on the right by precomposition of loops, and $\pi_1(\mathbb{C} \setminus B, t)$ acts on the left via postcomposition with lifts.

Lemma 4.4.5. *Let $f : (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ be a marked entire map. The biset $\mathcal{M}$ of f is right free, and $\mathcal{M} / \pi_1(\mathbb{C} \setminus A, s)$ is a left-$\pi_1(\mathbb{C} \setminus B, t)$ set isomorphic to the set $f^{-1}(t)$ with the monodromy action.*

Proof. The proof is analogous to the case for polynomials. See for example [Nek05, Proposition 5.1.1]. □

Lemma 4.4.6. *Let $f : (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ and $g : (\mathbb{C}, B, t) \to (\mathbb{C}, C, u)$ be marked entire maps. Then the composition $g \circ f : (\mathbb{C}, A, s) \to (\mathbb{C}, C, u)$ is a marked entire map and the biset of $g \circ f$ is isomorphic to $M_g \otimes M_f$, the tensor product of M_g and M_f over the $\pi_1(C \setminus B, t)$ action. Moreover the mapping $M_g \otimes M_f / \pi_1(\mathbb{C} \setminus A, s) \to M_g / \pi_1(\mathbb{C} \setminus B, t)$ corresponds to the map f on $f^{-1}(g^{-1}(u))$ to $g^{-1}(u)$.*

Proof. The isomorphism is given as follows: an element of M_g is represented by a path p from t to an element of $g^{-1}(u)$. An element of M_f is represented by a path q from s to an element of $f^{-1}(t)$, say z. Let p^z be the lift p with respect to f starting at z. We send $p \otimes q$ to the concatenation $p^z \cdot$. It is straightforward to check that this gives an isomorphism, compare for example [Nek09, Proposition 5.5].

From the construction the second statement also follows. □

4.4.3 Self-Similar groups and Iterated Monodromy Actions

We recall definitions surrounding self-similar groups and automata groups on infinite alphabets. See also [Sid00], but we use the language of bisets as our starting point similar to [Nek09].

Definition 4.4.7. Let X be a set. The X-regular tree has as vertex set X^*, the set of finite words in X. The edges are of the form $v \to vx$, the root of the tree is the empty word. By abuse of notation, we denote the X-regular tree by X^*. Let $\operatorname{Aut}(X^*)$ be the set of automorphisms of X^* as a rooted tree. For an element $g \in \operatorname{Aut}(X^*)$ and a word $v \in X^*$, there is a unique element $g_{|v} \in \operatorname{Aut}(X^*)$ with $g(vw) = g(v)g_{|v}(w)$ for all $w \in X^*$. This is called the *section* of g at v. A subgroup $G \subset \operatorname{Aut}(X^*)$ is called self-similar if it is closed under taking sections.

An automorphism $g \in \operatorname{Aut}(X^*)$ is a *finite state* automorphism if the set of sections $\{g_{|v} \colon v \in X^*\}$ is finite. An automorphism $g \in \operatorname{Aut}(X^*)$ has *bounded activity* if there is a $C > 0$ such that for every $n \in \mathbb{N}$, the cardinality of $\{v \in X^n \colon g_{|v} \text{ non-trivial}\}$ is bounded by n.

Lemma 4.4.8. *Let G be a group, and ${}_G\mathcal{M}_G$ a right-free biset. Let X be a basis of $\mathcal{M}$. Then X^n is a basis of $\mathcal{M}^{\otimes n}$ and for every $g \in G$, the map $v \mapsto g(v)$ is a rooted tree automorphism of X^*. There is a group homomorphism $\Phi \colon G \to \operatorname{Aut}(X^*)$ compatible with taking sections in the following sense: $\Phi(g)_{|v} = \Phi(g_{|v})$. In particular, the image is self-similar.*

Proof. This is standard. See for example [Nek05, Proposition 2.3.3]. □

Definition 4.4.9 (Iterated Monodromy Group)**.** Let $f \colon (\mathbb{C}, A, s) \to (\mathbb{C}, A, s)$ be a marked entire map. Choose a basis X of $\mathcal{M}_f$. Then X^n is a basis of $\mathcal{M}_f^{\otimes n}$, and we have a bijection $X^n \cong \mathcal{M}_f^{\otimes n}/\pi_1(\mathbb{C} \setminus A, s) \cong f^{-n}(s)$. For $g \in \pi_1(\mathbb{C} \setminus A, s)$, the mapping $v \mapsto g(v)$ on X^n agrees with the monodromy action of g under f^n on $\cong f^{-n}(s)$. The *iterated monodromy group* of f is the image of Φ as in Lemma 4.4.8 for the biset of f.

Lemma 4.4.10. *Let $f \colon (\mathbb{C}, A, s) \to (\mathbb{C}, A, s)$ be a marked entire map. The iterated monodromy group of f and f^n are isomorphic for every $n \geq 1$.*

Proof. The monodromy action of f^m determines the monodromy action of f^k for $k < m$. So The monodromy action of all iterates of f^n determine the monodromy actions of all iterates of f. From this the isomorphism follows. □

4.5 Dendroid automata and pullbacks of spiders

In this section we define dendroid automata. This is a generalization of the notion introduced in [Nek09] to infinite alphabets.

Definition 4.5.1. Let $\tau \colon A_+ \times X \to X \times B_+$ be a group automaton. We call it a *dendroid automaton* if the conditions are satisfied:

— $(x \mapsto a(x))_{a \in A}$ is a dendroid set of permutations.
— For all $b \in B$ there are unique $a \in A, x \in X$ with $a_{|x} = b$.
— For all $a \in A$, all restrictions of a along a infinite orbit of a are trivial, and for every finite orbit of a, all restrictions but at most one along the orbit are trivial.

Example 4.5.2. Let $A = \{g, h\}$ and $X = \mathbb{Z} \cup \{*\}$. Consider the following mapping $\tau\colon A_+ \times X \to X \times A_+$ given by

$$\begin{aligned}
\tau(g,z) &= (z+1,\mathbf{1})\\
\tau(g,*) &= (*,h)\\
\tau(h,*) &= (0,g)\\
\tau(h,0) &= (*,\mathbf{1})\\
\tau(q,z) &= (z,\mathbf{1}) \text{ for all other cases.}
\end{aligned}$$

Then τ describes a dendroid automaton. In order to see this, it will be convenient to consider the dual Moore graph of τ: it has as vertex set X and for every $x \in X, q \in A$ we have an edge from x to $q(x)$ labeled by $q_{|x}$ and colored according to $q = g$ or $q = h$. See Figure 4.6. In fact, we will see later that the almost the same figure also encodes the biset of $(1-z)\exp z$.

We can contract the cycle diagram $D(g,h)$ by first contracting to the infinite g orbit. So we see that the permutations induced by g and h on X indeed form a dendroid set of permutations. The other two criteria are also easily checked: g appears as a restriction only for $\tau(b,*)$ and h appreas as a restriction only for $\tau(a,*)$. Now $*$ is on a 2-orbit for h and a 1-orbit for g, so the third criterion is also satisfied.

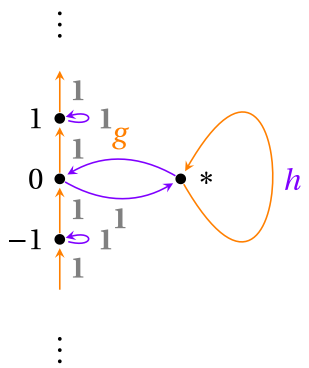


Figure 4.6 – Dual Moore diagram for Example 4.5.2

Lemma 4.5.3. *Let A be a finite set, X be a countably infinite set. Let $\tau\, A_+ \times X \to X \times A_+$ be a dendroid automaton. Then for every $a \in A$, a acts on X^* by Lemma 4.4.8 as a finite state automorphism of bounded activity.*

Proof. Every section of a is given by an element of A_+, so as A is finite, a is a finite state automorphism. For every element in $b \in A$ and $n \in \mathbb{N}$, there is a unique word $v \in X^n$ and $a \in A$ with $a_{|v} = b$. So for every $a \in A$, $\{v \in X^n \colon a_{|v} \text{ non-trivial}\}$ is bounded by the cardinality of A, so a also acts with bounded actifity. □

Lemma 4.5.4 (Dendroid model of bisets of entire maps)**.** *Let $f\colon (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ be a marked entire map. Let $\mathbb{S} = (\gamma_b)_{b\in B}$ be a spider for B such that $f(s)$ and t are not on any spider leg of $\mathbb{S}$. For every $a \in A$, choose a spider leg γ'_a that is the lift of $\gamma_{f(a)}$ landing at a. We call the resulting spider $\mathbb{S}' = (\gamma_B)_{B\in B}$ a pullback spider of $\mathbb{S}$. Let $\Phi_A\colon F_A \to \pi_1(\mathbb{C} \setminus A, s)$ and $\Phi_B\colon F_B \to \pi_1(\mathbb{C} \setminus B, t)$ be the isomorphisms sending a to g'_a and b to g_b, respectively. Let X be the set $f^{-1}(b)$.*

Then for every $x \in X, b \in B$ the lift of the generator g_b starting in x intersects the spider $\mathbb{S}'$ in at most one leg.

Let $\tau\colon B_+ \times X \to X \times A_+$ given as follows: $b(x)$ is the endpoint of lift of the generator g_b starting in x, and $b_{|x} = a$ if the lift of the generator g_b starting at x intersects the spider leg γ'_a. If lift of the generator g_b starting at x does not intersect $\mathbb{S}'$, we set $b_{|x} = \mathbf{1}$.

Then the following holds:

- *τ is a dendroid automaton.*
- *The biset associated to τ is Φ_B-Φ_A-congruent to the biset associated to f.*

Proof. Since we assume that neither $f(s)$ nor t lie on a spider leg of $\mathbb{S}$, it follows that s and every preimage of t do not lie on a spider leg of $\mathbb{S}'$. Since $\mathbb{C}\setminus\mathbb{S}'$ is simply connected, for every $x \in f^{-1}(t)$ there is a unique homotopy class p_x of a path from s to x that has a representative that does not intersect $\mathbb{S}'$. So the classes $(p_x)_{x\in X}$ form a basis of $\mathcal{M}_f$.

For $b \in B$, $x \in X$, let us consider $g_b \cdot p_x$. This is the path p_x composed with the lift g_b^x. Now g_b intersects $\mathbb{S}$ only in one point in the interior of γ_b. As all legs of $\mathbb{S}'$ are lifts of legs in $\mathbb{S}$, the lift g_b^x intersects the spider $\mathbb{S}'$ in at most one leg. If g_b^x does not intersect a leg of $\mathbb{S}'$, then $g_b \cdot p_x$ is a path in $\mathbb{C}\setminus\mathbb{S}'$ from s to $g_b(x)$, so it is homotopic to $p_{g_b(x)}$. Otherwise, let γ'_a be the spider leg that intersects g_b^x. As the intersection of g_b and γ_b is positive, so is the intersection of g_b^x and γ'_a. It follows that $g_b \cdot p_x$ only crosses $\mathbb{S}'$ once positively in γ'_a, so it is homotopic to $p_{g_b(x)} \cdot g'_a$.

From this we see that restriction behavior of $\mathcal{M}_f$ is the same as the description of τ, so the biset of τ is indeed Φ_B-Φ_A-congruent to the biset associated to f, via $x \cdot h \mapsto p_x \cdot \Phi_A(h)$ for $x \in X, h \in F_A$.

As the action of $\pi_1(\mathbb{C}\setminus B, t)$ on $\mathcal{M}_f/\pi_1(\mathbb{C}\setminus A, s)$ is identified with the monodromy action of $\pi_1(\mathbb{C}\setminus B, t)$ on $f^{-1}(t)$, we know by Lemma 4.3.4 that the family of permutations induced by B on X is dendroid.

By Lemma 4.2.10, we have an identification of finite g_b orbits with preimages of b under f, and an identification of infinite g_b orbits with logarithmic singularities over b. We have nontrivial restrictions only along the orbits identified with some $a \in A$, and there exactly once along the edge that intersects γ'_a. This shows that τ is has the correct restriction behaviour. □

Example 4.5.5. We work through this construction for our example $f(z) = (1-z)\exp(z)$ with $A = 0, 1$. We already know that $\mathbf{S}(f) = A$, and as $f(0) = 1, f(1) = 0$, we have that A is forward invariant, in fact A is the post-singular set of f. f is monotonically decreasing on $[0, 1]$, let t be the unique fixed point of f in $[0, 1]$. For our spider legs γ_0 and γ_1 we take legs that remain in $\mathbb{R}$. Then γ'_1 is the lift of γ_0 landing at 1, we see that γ'_1 is the same as γ_1 up to parametrization. There are two preimages of γ_1 landing at 0. Let γ'_0 be the preimage that lies in the upper half plane. Note that γ_0' is homotopic to γ_0 relative to A. So we have a fixed generating set for both sides of the biset. Consider Figure 4.7. We see that we indeed obtain the same dendroid automaton as in Example 4.5.2. .

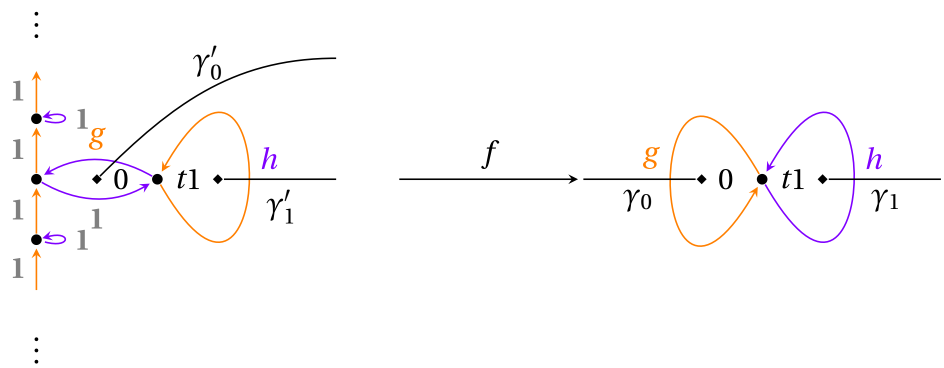


Figure 4.7 – Labeled Schreier Graph for $(1-z)\exp(z)$.

We can use the pullback of spiders to give a better understanding of the monodromy group of the composition of two marked entire functions. We use the notion of product automata:

Definition 4.5.6. Let $\tau_1\colon C_+ \times X \to X \times B_+$ and $\tau_2\colon B_+ \times Y \to Y \times A_+$ be two automata. The *product automaton* $\tau_1 \otimes \tau_2$ has input state set C, output state set A and alphabet $X \times Y$ and transition function

$$(c, x, y) \longmapsto \big(c(x), c_{|x}(y), (c_{|x})_{|y}\big)$$

for $c \in C_+, x \in X, y \in Y$.

We note that the product of two group automata is again a group automaton, and it is straightforward to check that the associated biset of a product automaton is isomorphic to the tensor product of the associated bisets of the two group automata.

Lemma 4.5.7. *Let $f\colon (\mathbb{C}, A, s) \to (\mathbb{C}, B, t)$ and $g\colon (\mathbb{C}, B, t) \to (\mathbb{C}, C, u)$ be marked entire maps. Let $\mathbb{S} = (\gamma_c)_{c\in C}$ a spider such that neither $g(f(s)), g(t)$ nor u lie on $\mathbb{S}$. Let $\mathbb{S}' = (\gamma_b')_{b\in b}$ be a pullback spider of $\mathbb{S}$ under g as in Lemma 4.5.4, with resulting automaton $\tau_1\colon C_+ \times X \to X \times B_+$. Let $\mathbb{S}'' = (\gamma_a'')_{a\in A}$ be a pullback spider of $\mathbb{S}'$ under f as in Lemma 4.5.4. with resulting automaton $\tau_2\colon B_+ \times Y \to Y \times A_+$. Then the biset of $g \circ f$ is isomorphic to the biset of the product automaton $\tau_1 \otimes \tau_2$.*

Moreover, if the monodromy group of g is $P \subset \mathrm{Sym}(g^{-1}(u))$, and the monodromy group of f is $Q \subset \mathrm{Sym}(f^{-1}(t))$, then the monodromy group of $g \circ f$ is isomorphic to a subgroup of the restricted wreath product $Q \wr_X P$. In particular, if the monodromy groups of f and g are (elementary) amenable, so is the monodromy group of $g \circ f$.

Proof. The first part is a direct consequence from Lemma 4.5.4 and Lemma 4.4.6. For the second part, it is clear that the monodromy group is isomorphic to a subgroup of the unrestricted wreath product. By construction of the automaton τ_1, we see that every state has only finitely many letters in X where it restricts nontrivially. From this we see that we actually have to be in the restricted wreath product. As the restricted wreath product preservers (elementary) amenability, we get the claim about (elementary) amenability as well. □

We see that together with Corollary 4.3.5, finite compositions of structurally finite entire transcendental functions have elementary amenable monodrony groups.

Corollary 4.5.8. Let f be a map in the Speiser class. For every $n \geq 1$, the monodromy group of f is amenable if and only if the monodromy group of f^n is amenable.

Proof. If the monodromy group of f is amenable, then an inductive application of the previous lemma shows that the monodromy group of f^n is amenable. In the other direction, the monodromy group of f is a quotient of the monodromy group of f^n. As amenability is preserved by taking quotients, we obtain the result . □

4.6 Periodic spiders

Now we actually do dynamics with entire functions. First recall standard definitions from holomorphic dynamics. See [Mil06] for an introduction.

Definition 4.6.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. The *Fatou set* $F(f)$ is the set of normality of f, it is the largest open subset in $\mathbb{C}$ such that the family of iterates $f, f^2, \ldots$ forms a normal family. The *Julia set* $J(f)$ is the complement of $F(f)$.

A periodic point w of f of period n is called *superattracting* if $(f^n)'(w) = 0$, i.e., if a critical point lies on the forward orbit of w. A periodic point is called *repelling* if $|(f^n)'(w)| > 1$. Superattracting periodic points are always in the Fatou set, repelling periodic points are in the Julia set.

We will use the following dynamical facts about post-singularly finite entire functions. See for example [Sch10] for a general overview of dynamics of entire functions, and [Pfr19, Section 2] for dynamics of post-singularly finite entire functions.

Lemma 4.6.2 (Böttcher coordinates)**.** *Let f be a post-singularly finite transcendental entire function. Then every periodic point is either superattracting or repelling. Let a be a periodic point in $F(f) \cap \mathbf{P}(f)$ of period k. Let U be the component of the Fatou set containing a. Then there is a conformal map $\Phi\colon U \to \mathbb{D}$ and an $n > 1$ such that the following diagram commutes.*
$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathbb{D} \\ \downarrow{\scriptstyle f^k} & & \downarrow{\scriptstyle z^n} \\ U & \xrightarrow{\Phi} & \mathbb{D} \end{array}$$
(Here z^n is shorthand for the map $z \mapsto z^n$ on $\mathbb{D}$).

Proof. For the first claim see [Pfr19, Corollary 2.13]. For a general reference for Böttcher coordinates see [Mil06, Chapter 9], in the case of post-singularly finite transcendental entire functions, see for example [Pfr19, Proposition 2.34]. □

Definition 4.6.3 (Internal rays)**.** In the notation of the previous lemma, for $\theta \in \mathbb{R}/\mathbb{Z}$ the preimage of the radius $R_\theta := \{re^{2\pi i\theta} : r \in [0,1)\}$ under Φ is called the *internal ray* r_θ in U of angle θ.

Lemma 4.6.4 (Periodic internal rays land)**.** *Every periodic ray lands at a repelling periodic point of f. For a given periodic Fatou component, different periodic rays land at different periodic points.*

Proof. See the discussion up to [Pfr19, Proposition 2.37] for the first statement, and [Pfr19, Proposition 2.41] for the second. □

Remark 4.6.5. In the course of the proof of our main theorem it will convenient to pass to iterates of f. For a finite set of periodic points of f and periodic legs, let m be the least common multiple of all periods. Then f^m fixes all the periodic points and periodic legs. Also, for a finite set of preperiodic points, choosing m highly divisible enough makes sure , that point is mapped to a fixed point after one iteration. As we have seen in Corollary 4.5.8 , the monodromy group of f is amenable if and only if the monodromy group of f^m is amenable, and by Lemma 4.4.10 that the iterated monodromy groups of f and f^m are naturally isomorphic, so there is no loss in our theorems in doing this step.

We import [Mih10] here:

Lemma 4.6.6. *Let f be a post-singularly finite entire transcendental function. Let $A \subset \mathbb{C}$ be a finite forward invariant set that properly contains $\mathbf{P}(f)$. Let z_0 be a fixed point in $A \cap J(f)$. Let γ be a leg, i.e., an arc from z_0 to ∞ meeting A only in z_0. Let $\mathcal{L}$ be the leg pullback map at z_0. Then there exists $n < m \in \mathbb{N}$ with $\mathcal{L}^n(\gamma)$ homotopic to $\mathcal{L}^m(\gamma)$ relative to A.*

Proof. This is a special case of [Mih10, Theorem 3.3]. In fact $U := \mathbb{C} \setminus A$ is an admissible expansion domain in the sense of [Mih10, Definition 3.1] , so we can indeed apply the cited result. □

Lemma 4.6.7. *Let f be a post-singularly finite transcendental entire function. Then there exists a natural number $n \geq 1$, a fixed point t of f^n, and a spider $\mathbb{S} = (\gamma_a)_{a \in \mathbf{P}(f)}$, such that $\mathbb{S}$ is isotopic to a pullback spider of $\mathbb{S}$ under f^n relative to $\mathbf{P}(f) \cup \{t\}$.*

Proof. We will repeatedly use Remark 4.6.5, that is, we will pass to a higher iterate of f to repeatedly pass from periodic to fixed objects. A transcendental entire function has infinitely many periodic points [Ber91]. By Remark 4.6.5, we can assume without loss of generality that f has a fixed point t distinct from $\mathbf{P}(f)$, every periodic point of $\mathbf{P}(f)$ is fixed by f and every preperiodic point of $\mathbf{P}(f)$ is mapped to a fixed point of f.

Let $a_1,\dots,a_k$ the periodic points in $\mathbf{P}(f)\cap F(f)$, let $b_1,\dots,b_l$ be the periodic points in $\mathbf{P}(f)\cap J(f)$, and let $c_1,\dots,c_m$ be the set of strictly preperiodic points in $\mathbf{P}(f)$.

For every a_i, there are infinitely many repelling periodic points on the boundary of the Fatou component with center a_i that are connected to a_i via a periodic internal ray by Lemma 4.6.4. So we can choose repelling periodic points a_i' with internal rays r_i connecting a_i to a_i' such that all a_i' are distinct and disjoint from $\mathbf{P}(f)$. By passing to an even higher iterate if needed, we can assume that the rays r_i are in fact fixed by f.

From now on, let $A := \{a_1',\dots,a_k',b_1,\dots,b_l\}$ and $B := \{a_1',\dots,a_k'\}\cup\mathbf{P}(f)\cup\{t\}$ and $\mathbf{P}_{\text{per}}(f) := \{a_1,\dots,a_k,b_1,\dots,b_l\}$. Then B is forward invariant and properly contains $\mathbf{P}(f)$, and $A\subset B$ consists of repelling periodic points. Since the rays $r_1,\dots,r_k$ are pairwise disjoint arcs, they do not separate the plane. So we can choose a collection of disjoint spider legs $(\hat\gamma_a)_{a\in A}$ that meet the rays $r_1,\dots,r_k$ and the set B only possibly at the endpoints of the spider legs.

By Lemma 4.6.6, the homotopic classes relative to B of the pullbacks $\mathscr{L}^n(\hat\gamma_a)$ are eventually periodic. By Lemma 4.2.7, this also means that the isotopic classes are eventually periodic.

Since the rays $r_1,\dots,r_k$ are forward invariant under f, the pullbacks $\mathscr{L}^n(\hat\gamma_a)$ meet the rays also only possibly at the end points of the spider legs. Also, for fixed n, $\mathscr{L}^n(\hat\gamma_a)$ and $\mathscr{L}^n(\hat\gamma_b)$ are disjoint for $a\neq b$. So by replacing $\hat\gamma_a$ by $\mathscr{L}^n(\hat\gamma_a)$ for n large enough and again passing to a high enough iterate of f, we can assume that $\hat\gamma_a$ is isotopic to its pullback $\mathscr{L}(\hat\gamma_a)$ relative to B.

We can now define spider legs $(\tilde\gamma_a)_{a\in\mathbf{P}_{\text{per}}(f)}$: for the repelling fixed points $b_1,\dots b_l$, we take $\tilde\gamma_{b_j}=\hat\gamma_{b_j}$, for the superattracting fixed points, we take $\tilde\gamma_{a_i}$ as the concatenation of the internal ray r_i and $\hat\gamma_{a_i'}$. Note that $(\tilde\gamma_a)$ are pairwise disjoint.

We finally can define our invariant spider $\mathbb{S}=(\gamma_p)_{p\in\mathbf{P}(f)}$: for $b_1,\dots b_t$, we take $\gamma_{b_j}=\mathscr{L}(\hat\gamma_{b_j})$, for the super attracting periodic points, we take γ_{a_i} as the concatenation of the internal ray r_i and $\mathscr{L}(\hat\gamma_{a_i'})$. For the preperiodic points $c_1,\dots,c_k$, we chose for γ_{c_i} a lift of $\tilde\gamma_{f(c_i)}$ in the sense of Lemma 4.2.8.

Every spider leg of $\mathbb{S}$ is a pullback of a spider leg of $(\tilde\gamma_a)_{a\in\mathbf{P}_{\text{per}}(f)}$ landing at different points. So in fact the legs of $\mathbb{S}$ are disjoint. Also note that for every $p\in\mathbf{P}_{\text{per}}(f)$, we have that $\tilde\gamma_p$ and γ_p are isotopic relative to $\mathbf{P}(f)\cup\{t\}$. For b_j this is clear as $\tilde\gamma_{b_j}=\hat\gamma_{b_j}$ and $\gamma_{b_j}=\mathscr{L}(\hat\gamma_{b_j})$ are isotopic relative to $B\supset\mathbf{P}(f)\cup\{t\}$. For a_i we have that $\tilde\gamma_{a_i}$ and γ_{a_i} are the concatenation of r_i and $\hat\gamma_{a_i'}$ or $\mathscr{L}(\hat\gamma_{a_i'})$, respectively, and we can apply Lemma 4.2.7 to promote the isotopy between $\hat\gamma_{a_i'}$ and $\mathscr{L}(\hat\gamma_{a_i'})$ relative to B to an isotopy between $\tilde\gamma_{a_i}$ and γ_{a_i}.

Note that by construction, every leg of $\mathbb{S}$ landing at a periodic point is homotopic to a spider leg of $\tilde\gamma$. For $q\in\mathbf{P}_{\text{per}}(f)$, let H_q be a homotopy between $\tilde\gamma_q$ and γ_q relative to B. For every $p\in\mathbf{P}(f)$, let H_p' be the lift of $H_{f(p)}$ at p starting at γ_p in the sense of Lemma 4.2.8. Call the spider leg at the end of the homotopy γ_p'. Then γ_p' is a lift of $\gamma_{f(p)}$, so $\mathbb{S}'=(\gamma_p')_{p\in\mathbf{P}(f)}$ is a pullback spider that is homotopic to $\mathbb{S}$ legwise, so by Lemma 4.2.7 it is isotopic. This proves the theorem.

□

Remark 4.6.8. The construction of periodic spiders should be compared to the construction of dynamical partitions in [Mih09] and [Pfr19]. For a large class of entire functions defined in [Rot+11], which in particular contains entire functions of finite order, it is possible to realize isotopy classes of periodic spiders via spiders that are periodic as arcs and not just up to isotopy, using dynamic rays. For general entire functions dynamic rays might not exists, but there is the notion of dreadlocks [BR20], that has been used in [Pfr19] to define dynamical partitions for general entire functions. We are interested in spiders for the combinatorial understanding of the biset of entire functions, so considering them up to isotopy is good enough for us. Also, this notion is flexible enough for generalization to topological entire maps in the sense of [HSS09].

*Proof of Theorem 4.**A**.* By Lemma 4.4.10 and Lemma 4.6.7 and passing to an iterate if needed, we have a fixed point t of f and a spider $\mathbb{S}$ for $\mathbf{P}(f)$ that is isotopic to some pullback spider $\mathbb{S}'$ under f relative to $\mathbf{P}(f) \cup \{t\}$. Note that since $\mathbb{S}$ and $\mathbb{S}'$ are isotopic, the associated generating sets of $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ are the same. By Lemma 4.5.4, we have that the biset of f is isomorphic to a biset of an autonomous dendroid automaton. . So by Lemma 4.5.3, we are done. □

*Proof of Theorem 4.**B**.* If the iterated monodromy group of f is amenable, then so it is its monodromy group, as it is a quotient of the iterated monodromy group. So it suffices to show that if the monodromy group is amenable, then so is the iterated monodromy group.

We apply the criterion of [Rei20a]. The first level action of the self-similar group in the previous proof is the monodromy action of some iterate f^n of f. So it is the monodromy action of an entire function (in the Speiser class), so by 4.2.14, the first level action is recurrent. By assumption the monodromy action of f is amenable, so by and Corollary 4.5.8, so is the monodromy action of f^n. So the monodromy group of the first level of the self-similar group in the previous proof is amenable. So by Theorem 3.**B**, the iterated monodromy group is also amenable. □

Via the results of Corollary 4.3.5 and Lemma 4.5.7, we see that compositions of structurally finite entire transcendental functions have amenable monodrony groups, so by Theorem 4.**B**, post-singularly finite functions in this class also have amenable monodromy group. As our running example $(1-z)\exp z$ is structurally finite, its iterated monodromy group is amenable.

4.7 Outlook

We describe iterated monodromy groups of post-singularly finite entire transcendental functions via dendroid automata. The natural question arises when we try to go in the other direction: which dendroid automata can appear as the combinatorial description of entire functions. For polynomials, there is an answer given in [Nek05, Theorem 6.10.8]. This is done in two steps: first, the class of considered functions is extended from complex polynomials to "topological polynomials", and dendroid automata are also considered for topological polynomials. Then, a criterion in [BFH92] is used to see which topological polynomials are equivalent to actual complex polynomials. The classification result is based on Thurston's classification of rational maps [DH93]. For entire functions, an analogous classification result is so far only available for exponential functions [HSS09].

The definition of bisets of post-singularly finite entire functions can be also extended to topological models of post-singularly finite entire functions. While topological models do not need to have periodic spiders, the methods of Section 5 still apply. We use this in [Reia] to define a combinatorial topology of bisets of topological models of entire functions, and approximate post-singularly finite entire functions via post-singularly polynomials with the same dynamics on the post-singular set. From this construction we show that in the space of marked groups in the sense of [Gri84], iterated monodromy groups of post-singularly finite structurally finite entire functions are in the closure of iterated monodromy groups of post-singularly finite polynomials.

5 Outlook

In the thesis, we constructed iterated monodromy groups for entire functions and gave a model via dendroid automata. The methods we used in particular in Chapter 4 are mostly topological in nature. We hope to extend this results to larger classes of maps.

5.1 Transcendental Thurston Theory

Classical Thurston theory of rational maps studies topological models of post-singularly finite rational maps. A Thurston map is a continuous map $f\colon (\hat{\mathbb{C}}, A) \to (\hat{\mathbb{C}}, A)$ such that A is a finite set with $f(A) \subset A$, the map f restricts to a covering $\hat{\mathbb{C}} \setminus f^{-1}(A) \to \hat{\mathbb{C}} \setminus A$ and f can be written as a composition $h_1 \circ g \circ h_2$, where the h_i are orientation preserving homeomorphisms of $\hat{\mathbb{C}}$ and g is an rational map. While this is not the standard definition of a Thurston map, it is useful as it can be easily generalized to transcendental Thurston maps.

Two Thurston maps $f\colon (\hat{\mathbb{C}}, A) \to (\hat{\mathbb{C}}, A)$ and $g\colon (\hat{\mathbb{C}}, B) \to (\hat{\mathbb{C}}, B)$ are called *Thurston equivalent* if there are (orientation preserving) homeomorphisms $h_1, h_2\colon (\hat{\mathbb{C}}, A) \to (\hat{\mathbb{C}}, B)$ such that h_1 is isotopic to h_2 relative A, $h_i(A) = B$ and the following diagram commutes:

$$\begin{array}{ccc} (\hat{\mathbb{C}}, A) & \xrightarrow{h_1} & (\hat{\mathbb{C}}, B) \\ \big\downarrow f & & \big\downarrow g \\ (\hat{\mathbb{C}}, A) & \xrightarrow{h_2} & (\hat{\mathbb{C}}, B) \end{array}$$

A Thurston map is called *realizable* if it is Thurston equivalent to a rational map. Thurston's characterization [DH93] shows that a Thurston map is realizable if and only it does not have a so-called *Thurston multi-curve obstruction.* For polynomials, the situation simplifies. For polynomials, a polynomial Thurston map is realized if and only if it doesn't have a Levy obstruction, see [BFH92]. This is used in [Nek09] to give a realization criterion in terms of the biset.

The proof of the classification is based on the action on the Teichmüller space $\mathcal{T}_A = \mathcal{T}(\hat{\mathbb{C}}, A)$. A Thurston map $f\colon (\hat{\mathbb{C}}, A) \to (\hat{\mathbb{C}}, A)$ induces the *Thurston pullback map* $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$, and a Thurston map is realized if and only if its pullback map has a fixed point.

A transcendental Thurston map is a map $f\colon (\mathbb{C}, A) \to (\mathbb{C}, A)$ such that $A \subset \mathbb{C}$ is finite with $f(A) \subset A$, the map f restricts to a covering $\mathbb{C} \setminus f^{-1}(A) \to \mathbb{C} \setminus A$ and f can be written as a composition $h_1 \circ g \circ h_2$, where the h_i are orientation preserving homeomorphisms of $\mathbb{C}$ and g is an entire transcendental map. A transcendental Thurston map is a topological model of a post-singularly finite entire function, and we have similar notions of Thurston equivalence and realizability. It it also possible to define bisets and iterated monodromy groups for them.

For classical Thurston maps, the isotopy class of f is encoded in the biset of f, see [Kam01]. If we look at Thurston maps with a fixed marked set $A \subset \mathbb{C}$, we can consider the set of Thurston maps with marked set A up to isotopy relative to A to obtain the *mapping class biset*. The mapping class group of homeomorphisms of $(\mathbb{C}, A)$ acts on the mapping class biset via pre- and postcomposition. Two Thurston maps (with the same marked set) are Thurston equivalent if they are conjugated in the mapping class biset. This allows us to answer questions about Thurston equivalence via algebraic computations in biset. A celebrated result in this direction is the solution of the *twisted rabbit problem* [BN06]. More generally, [BD17] shows that the questions whether two Thurston maps are equivalent is algorithmically decidable.

There is also the positive Thurston criterion [Thu20]. One should note that hyperbolic polynomials cannot have Thurston obstructions, and are always realized.

In the setting of transcendental dynamics, so far the only analog of the Thurston theorem has been shown in the case of the exponential family [HSS09], and a full combinatorial classification of post-singularly finite exponential maps has been achieved in [LSV08].

So while there isn't yet a full analog of Thurston theorem for transcendental maps, the approach of Thurston theory for entire functions via biset might be interesting. Similar to the twisted rabbit problem, it should be possible to solve the classification problem for a very concrete class of hyperbolic entire functions: maps in the cosine family $z \mapsto a\cos z + b\sin z$ with a super-attracting fixed point and an attracting two-cycle. By analogy with hyperbolic polynomials, one should expect that all Thurston maps in this class are realized, and a possible line of attack is an explicit computation in the associated mapping class biset.

5.2 Polynomial approximation of Thurston maps

An approach to understand the dynamics of entire transcendental functions is to approximate them via polynomials and investigate how the dynamics transfer from polynomials to entire functions. See for example [Bod+00] for the exponential family, and [KK99] for a general statement on the Hausdorff convergence of Julia sets.

On the level of transcendental Thurston maps, the formalism of bisets allows for a new kind of combinatorial convergence: using the description of bisets in terms of labeled Schreier graphs, we can consider combinatorial convergences on pointed labeled Schreier graphs. In the forthcoming [Reia] we show that this is combinatorial convergence implies locally uniform convergence of the associated Thurston pullback maps in Teichmüller space. This in particular implies that the property of being realized is open in the combinatorial topology, and we can approximate post-singularly finite entire functions via post-singularly finite polynomials preserving the dynamics on the post-singular set.

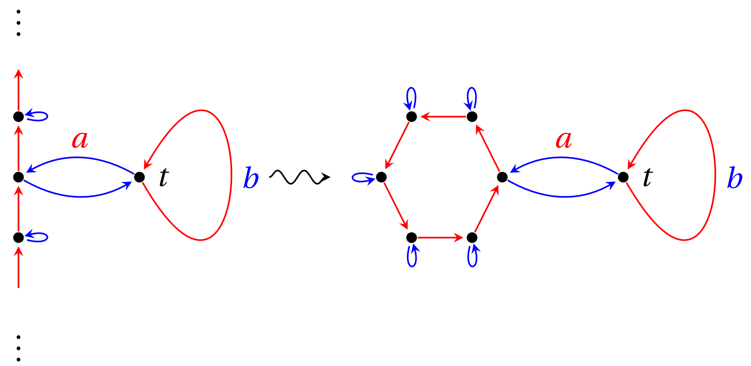


Figure 5.1 – Combinatorial approximation of $(1-z)\exp z$ by a polynomial

On the level of (compositions of) structurally finite maps, we can use this to show that in the space of marked groups (in the sense of [Gri84]), iterated monodromy groups of structurally finite entire functions are in the closure of iterated monodromy groups of polynomials. As polynomials are residually finite, IMGs of structurally finite entire functions are locally embeddable in the class of finite groups.

5.3 Landing properties of dreadlocks / rays

For post-singularly finite rational functions, the Schreier graphs of higher and higher iterates look closer and closer to the Julia sets. A way to make this precise is to use the notion of contracting self-similar groups as developed in [Nek05]. In this setting it is possible to construct a symbolic Julia set out of the iterated monodromy action, and show that the dynamics on the symbolic Julia set and the actual Julia set are conjugate. The comparison of the symbolical Julia set and the actual Julia set shows in particular that the Julia set of post-singularly finite rational functions are connected and locally connected (a well known result that is usually proved using contracting properties of the hyperbolic metric).

For polynomials, a classical way to describe the Julia set is by using laminations. We discuss for simplicity only post-singularly finite polynomials. In this case, there are Böttcher coordinates $\Phi\colon I(f) \to \mathbb{C} \setminus \mathbb{D}$ conjugating f to z^d, and the inverse map extends on the boundary to a quotient $S^1 \to J(f)$. We compare the Julia set of f to the Julia set of z^d and see what additional identifications we have to carry out. Using an appropriate basis for the biset (see for example [BD18, Section 5]), it is possible to compute this quotient map in terms of the biset of f.

For post-singularly entire transcendental functions, the Julia set is much bigger than the Julia set for polynomials. In particular, the escaping set $I(f)$ is now a subset of the

Julia set. Points in the interior the same dynamic ray (or the same dreadlock) share the same combinatorics; we cannot expect to distinguish them just by using symbolic dynamics.

However, for certain entire functions, called *docile* in [ARS21], it is possible to do a similar comparison for the Julia set of f to the Julia set of a dynamically simpler function in the same parameter space. One such class of functions are *strongly subhyperbolic* entire maps, as shown in [Mih09]. A post-singularly finite entire function is strongly subhyperbolic if no asymptotic values are in the Julia set of f and the local degree of points in the Julia set is uniformly bounded. Strongly subhyperbolic maps have an expanding orbifold metric near the Julia set. A natural question is if the landing relations for docile entire can be also understood by the biset. The class of strongly subhyperbolic maps seems suitable to ensure that the involved self-similar group is contracting in the right sense.

5.4 Geometric group theory of self-similar groups on infinite alphabets

The main focus of this thesis are iterated monodromy groups of entire functions, realized as self-similar groups of bounded activity growth on infinite alphabets. In particular, we showed an amenability criterion for groups generated by automata of bounded activity growth. It is clear that the condition of amenability of the monodromy group arising from the first level action is necessary. For our proof we also needed the recurrence of the first level action in order to deduce that action on every orbital Schreier graph of the ends X^ω is recurrent, and thus extensively amenable. It would be interesting to see if it is possible to show extensively amenability directly.

For finite alphabets, it is shown in [AAV13] that the action on the end of the trees is recurrent for groups generated by automata of linear activity growth. It is not clear if one can similarly deduce the recurrence for infinite alphabets.

We studied in particular amenability of iterated monodromy groups of entire functions. A group theoretical property of particular interest for iterated monodromy groups of rational maps is word growth. They are examples of iterated monodromy groups of polynomials with intermediate growth, such as z^2+i [BP06] and the Fabrykowski–Gupta [FG91] group.

For entire transcendental functions, the word growth of the monodromy group of every iterate forms a lower bound for the word growth of the iterated monodromy group. In contrast to rational maps, where the monodromy group of every iterate is finite, these bounds are often non-trivial. For example, we have shown that the second iterate of a function in the exponential family has as monodromy group $\mathbb{Z} \wr \mathbb{Z}$, and in particular exponential word growth, so every function in the exponential family has exponential word growth for the iterated monodromy group as well. Apart from the exponential family, most structurally finite entire transcendental functions have a monodromy group of exponential growth, so their iterated monodromy group has exponential growth as well.

For general entire functions, the growth of the monodromy group might be a more interesting object to study than the growth of the iterated monodromy group. As the iterated monodromy group of a post-critically finite rational map can be realized as the monodromy group of an meromorphic function (compare [LM97]), we expect the existence of entire transcendental functions with monodromy groups of intermediate growth. However, it is less clear if it is possible to have an entire transcendental function such that the monodromy group of the second iterate has subexponential growth.

An other topic are algorithmic properties of iterated monodromy groups of entire functions. For example, we expect that the word problem for IMGs can be solved by working with automata representatives, provided one has a sensible representation of the monodromy action. In particular, compositions of structurally finite entire maps might be an approachable class to consider.

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