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Some Results on Turbulent Models

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PhD Thesis

Some Results on Turbulent Models

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"I give you a new commandment: love one another. As I have loved you, so you also should love one another. This is how all will know that you are my disciples, if you have love for one another."

JN 13:34-35

Con xin gửi đến ông bà nội, ông bà ngoại và ba mẹ.

Dành cho Kiều Trang và Hoàng Sang.

To my grandfather, my grandmother and my parents.

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> “Stop judging and you will not be judged. Stop condemning and you will not be condemned. Forgive and you will be forgiven.”
>
> LK 6:37

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Abstract

> "Whoever exalts himself will be humbled; but whoever humbles himself will be exalted."
>
> MT 23:12

Abstract in English

This thesis focuses on studying turbulent models. It consists of four chapters. While the first chapter is for a general introduction, the three remaining chapter are mostly based on three research papers. We will give a summary for each chapter in the following.

In the first part of the thesis, we present a general introduction where it consists of a brief state of the art of the Euler and Navier-Stokes equations, an introduction about turbulent flows with their recently mathematical developments and a review of some α-turbulent models. We explain why we study the problems in Chapters 2 and 3. We also give a summarize of main results obtained in this thesis and a structure of the thesis.

The second part of the thesis performs a new modeling procedure for a 3D turbulent fluid, evolving towards a statistical equilibrium. This will result to add to the equations for the mean field $(\mathbf{v}, p)$ the term $-\alpha\nabla \cdot (\ell(\mathbf{x})D\mathbf{v}_t)$, which is of the Kelvin-Voigt form, where the Prandtl mixing length $\ell(\mathbf{x})$ is not constant and vanishes at the solid walls. We get estimates for mean velocity $\mathbf{v}$ in $L^\infty_t H^1_\mathbf{x} \cap W^{1,2}_t H^{1/2}_\mathbf{x}$, that allow us to prove the existence and uniqueness of a regular-weak solution $(\mathbf{v}, p)$ to the resulting system, for a given fixed eddy viscosity. We then prove a structural compactness result that highlights the robustness of the model. This allows us to consider Reynolds averaged equations and pass to the limit in the quadratic source term in the equation for the turbulent kinetic energy k. This yields the existence of a weak solution to the corresponding Navier-Stokes turbulent kinetic energy system satisfied by $(\mathbf{v}, p, k)$.

The third part of the thesis is devoted to study the rate of convergence of the weak solutions $\mathbf{u}_\alpha$ of α-regularization models to the weak solution $\mathbf{u}$ of the Navier-Stokes equations in the two-dimensional periodic case, as the regularization parameter α goes to zero. More specifically, we will consider the Leray-α, simplified Bardina, and modified Leray-α models. Our aim is to improve known results in terms of convergence rates and also to show estimates valid over long time intervals.

In the fourth part of the thesis we present a derivation of a back-scatter rotational Large Eddy Simulation model, which is the extension of the Baldwin & Lomax model to non-equilibrium problems. The model is particularly designed to mathematically describe a fluid filling a domain with solid walls and consequently the differential operators appearing in the smoothing terms are degenerate at the boundary. After the derivation of the model, we prove some of the mathematical properties coming from the weighted energy estimates and which allow to prove existence and uniqueness of a class of regular weak solutions.

> “This is the time of fulfillment. The kingdom of God is at hand. Repent, and believe in the gospel.”
>
> MK 1:15

Abstract in French – Résumé

Cette thèse porte sur l'étude de quelques modèles utilisés pour la simulation numériques d'écoulements turbulents 3D et 2D, en particulier les modèles sous-maille de type Smagorinsky, les α-modèles, les modèles de Voigt et de Baldwin-Lomax. Elle se compose de quatre chapitres. Le premier chapitre est une introduction générale, les trois autres chapitres sont principalement basés sur trois articles de recherche. Dans ce qui suit, nous donnons un résumé chapitre par chapitre.

Chapitre 1. Dans le premier chapitre de la thèse, nous faisons un état de l'art succin sur les équations d'Euler (E) et de Navier-Stokes (NSE), qui sont

$$\text{(NSE)} \qquad \begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p + \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \end{cases}$$

où

- $\mathbf{v}$ est la vitesse,
- p est la pression,
- $\mathbf{f}$ est la force externe,
- la viscosité cinématique $\nu > 0$ pour (NSE),
- et $\nu = 0$ pour (E);

les écoulements turbulents et les modèles utilisés pour leur simulations, ainsi que quelques développements mathématiques récents sur le sujet. Nous motivons nos choix pour l'étude des modèles considérés dans les chapitres suivants. Enfin, nous énonçons les résultats principaux obtenus et donnons la structure de la thèse.

Chapitre 2. Le deuxième chapitre de la thèse commence par la modélisation d'un fluide turbulent 3D évoluant vers un équilibre statistique. Cela se traduit par l'ajout aux équations dites "NSTKE" satisfaites par le champ moyen $(\mathbf{v}, p)$, du terme $-\alpha\nabla\cdot(\ell(\mathbf{x})D\mathbf{v}_t)$, qui est de la forme de Kelvin-Voigt, où la longueur

de mélange de Prandtl $\ell(\mathbf{x})$ n'est pas constante et s'annule au niveau des parois solides. Nous obtenons des estimations de la vitesse moyenne $\mathbf{v}$ dans $L_t^\infty H_{\mathbf{x}}^1 \cap W_t^{1,2} H_{\mathbf{x}}^{1/2}$, qui nous permettent de prouver l'existence et l'unicité d'une solution faible régulière $(\mathbf{v}, p)$ au système obtenu, pour une viscosité turbulente donnée. Nous montrons ensuite un résultat de compacité structurel qui met en évidence la robustesse du modèle. Cela nous permet de considérer les équations moyennes de Reynolds et de passer à la limite dans le terme source quadratique de l'équation pour l'énergie cinétique turbulente k, d'où l'on déduit l'existence d'une solution faible au système NSTKE correspondant.

Plus précisément, nous étudions les systèmes

(Voigt)

$$\begin{cases} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}} D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases}$$

et

(Voigt-TKE)

$$\begin{cases} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}}(k) D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ k_t + \mathbf{v} \cdot \nabla k - \nabla \cdot (\mu_{\text{turb}}(k) \nabla k) - \nu_{\text{turb}}(k) |D\mathbf{v}|^2 + (\ell + \eta)^{-1} k \sqrt{|k|} = 0, \end{cases}$$

où

- $\mathbf{v}$ est la vitesse moyenne avec $\mathbf{v}_t = \dfrac{\partial \mathbf{v}}{\partial t}$;
- $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ est le stress de déformation;
- p est la pression moyenne modifiée;
- k est l'énergie cinétique turbulente (Turbulent Kinetic Energy (TKE));
- $\nu > 0$ est la viscosité cinématique, ν_{turb} la viscosité des tourbillons (eddy viscosity);
- μ_{turb} est la diffusion des tourbillons et $\eta > 0$ est une petite constante;
- l'échelle de longueur α est donnée par

$$\alpha = \frac{\nu}{u_\star},$$

  ici $u_\star$ est ce qu'on appelle la vitesse de frottement (voir [CRL14]);
- $\mathbf{f}$ est une force externe donnée.

Le résultat principal de ce chapitre est donné par: il existe une solution unique régulière-faible pour (Voigt) et une solution faible pour (Voigt-TKE).

Chapitre 3. Le troisième chapitre de la thèse est consacrée à l'étude de la vitesse de convergence des solutions faibles $\mathbf{u}_\alpha$ des modèles de régularisation α vers la solution faible $\mathbf{u}$ des équations de Navier-Stokes dans le cas périodique de dimension deux, quand le paramètre de régularisation α tend vers zéro. En particulier, nous considérerons les modèles de Leray-α, de Bardina simplifié et de Leray-α modifié. Nous montrons que la convergence est en α^3 dans les espaces d'énergie standard, ce qui améliore substanciellement le taux en $\alpha^2\text{Log}\alpha$ initiallement prouvé par Cao et Titi. Nous montrons également que nos estimations sont valables en temps long.

Plus précisément, nous étudions les systèmes

$$(\alpha\text{-modèles}) \qquad \begin{cases} \partial_t \mathbf{v}_\alpha + N(\mathbf{v}_\alpha) - \nu\Delta\mathbf{v}_\alpha + \nabla p_\alpha = \mathbf{f}, \\ \nabla\cdot\mathbf{v}_\alpha = 0, \end{cases}$$

où

$$N(\mathbf{v}_\alpha) = \begin{cases} (\overline{\mathbf{v}}_\alpha\cdot\nabla)\mathbf{v}_\alpha & \text{Leray-}\alpha, \\ (\mathbf{v}_\alpha\cdot\nabla)\overline{\mathbf{v}}_\alpha & \text{modified Leray-}\alpha, \\ (\overline{\mathbf{v}}_\alpha\cdot\nabla)\overline{\mathbf{v}}_\alpha & \text{simplified Bardina,} \end{cases}$$

et

$$-\alpha^2\Delta\overline{\mathbf{v}}_\alpha + \overline{\mathbf{v}}_\alpha = \mathbf{v}_\alpha,$$

dans le cas des conditions aux limites périodiques. Le résultat principal de ce chapitre est donné par: nous dénotons $\Omega = [0, L]^2$ être un domaine périodique. Suppose que $\mathbf{v}_0 \in \mathcal{P}_\sigma H^1(\Omega)^2$ et $\mathbf{f} \in L^2(\mathbb{R}_+; \mathcal{P}_\sigma L^2(\Omega)^2)$. Alors ça tient $\forall s \geq 0$:

$$\|\mathbf{e}(s)\|^2 + \nu\int_0^s \|\nabla\mathbf{e}\|^2\,dt \leq C_1\alpha^3 \qquad \text{pour tous les } \alpha\text{-modèles},$$

$$\|\nabla\mathbf{e}(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{e}\|^2\,dt \leq \begin{cases} C_2\alpha^2, \\ C_3\alpha^2\left(\log\left(\dfrac{L}{2\pi\alpha}\right)+1\right), \end{cases}$$

pour Leray-α and simplified Bardina, modified Leray-α avec $\mathbf{e} = \mathbf{v} - \mathbf{v}_\alpha$ où $\mathbf{v}$ est la solution de (NSE).

Chapitre 4. Dans le dernier chapitre de la thèse, nous présentons la dérivation d'un modèle de simulation de grands courants de tourbillons en rotation rétrodiffusés, qui est l'extension du modèle de Baldwin & Lomax aux problèmes hors état d'équilibre. D'une certaine manière, ce modèle est la version rotationnelle du modèle de Smagorinsky où le terme sous maille $-\nabla \cdot (\ell^2 |\nabla \mathbf{v}| \nabla \mathbf{v})$ est remplacé par $\nabla \times (\ell^2 |\omega| \omega)$, ω étant la vorticité. Le modèle est particulièrement conçu pour décrire mathématiquement une couche limite, comme un écoulement sur une plaque. Après la dérivation du modèle, nous prouvons certaines propriétés mathématiques provenant des estimations d'énergie pondérées et qui permettent de prouver l'existence et l'unicité d'une classe de solutions faibles régulières au modèle dérivé.

Plus précisément, nous étudions le système
(Baldwin-Lomax)

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{curl}\,(\ell^2(\mathbf{x})\,\overline{\boldsymbol{\omega}}_t) + \operatorname{div}\,(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \operatorname{curl}\,(\kappa\,\ell^2(\mathbf{x}) |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}}) + \nabla \pi = \mathbf{f}, \\ \hfill \operatorname{div} \overline{\mathbf{v}} = 0, \end{cases}$$

où

- $\overline{\mathbf{v}}$ est la vitesse moyenne,
- $\overline{\boldsymbol{\omega}}$ est la vorticité moyenne,
- ℓ est la longueur de mélange,
- π un terme de pression modifié.

Suppose que:

- Ω est borné et de classe C^2;
- $\ell : \overline{\Omega} \to \mathbb{R}^+$ est de classe C^2 et satisfait les deux propriétés suivantes:

$$\ell(\mathbf{x}) \approx \sqrt{d(\mathbf{x}, \partial\Omega)} \qquad \text{pour } \mathbf{x} \text{ proche de } \partial\Omega,$$

  où $d(\mathbf{x}, \partial\Omega)$ la distance de la frontière

$$\forall\, K \subset\subset \Omega,\ \exists\, \ell_K \in \mathbb{R}_+^\star \quad \text{s.t.} \quad \ell(\mathbf{x}) \geq \ell_K > 0 \quad \forall\, \mathbf{x} \in K;$$

- $\mathbf{f} \in L^2(0, T; L^2(\Omega)^3)$ et $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$.

Alor, le résultat principal de ce chapitre est donné par: il existe une solution unique régulière-faible pour (Baldwin-Lomax) avec $\overline{\mathbf{v}}(0) = \overline{\mathbf{v}}_0$ dans Ω and $\overline{\mathbf{v}} = \mathbf{0}$ sur $(0, T) \times \partial\Omega$.

Preface

> "You are the salt of the earth. But if salt loses its taste, with what can it be seasoned? It is no longer good for anything but to be thrown out and trampled underfoot."
>
> MT 5:13

This dissertation concludes four chapters. Chapter 1 gives a general introduction, motivations, results and an organization of the present thesis. The three remaining Chapters 2, 3 and 4 are mostly based on the following research papers, respectively:

1. Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis, with C. Amrouche, L. C. Berselli, and R. Lewandowski, to appear on Nonlinear Analysis TMA, Vol 196, 111790, 2020. Paper available on https://doi.org/10.1016/j.na.2020.111790.

2. Modeling Error of α-Models of Turbulence on a Two Dimensional Torus, with L. C. Berselli, A. A. Dunca, and R. Lewandowski, submitted 2020. Paper available on HAL https://hal.archives-ouvertes.fr/hal-02469048.

3. Rotational forms of Large Eddy Simulation turbulence models: modeling and mathematical theory, with L. C. Berselli, and R. Lewandowski, submitted 2020. Paper available on HAL https://hal.archives-ouvertes.fr/hal-02569244.

Contents

Chapter 1

Introduction

"God has a bigger plan for me than I have for myself."

Anonymous

Contents

1.1 General introduction

Before talking about turbulent models –one of the main objects of the present manuscript– we briefly review history of the equations of fluid motion.These equations are derived from the conservation laws of momentum and mass. Simplest models of these equations are given by the Euler and Navier-Stokes equations (NSE in the sequel). Today, these equations are considered to belong to the list of the most important equations of fluid mechanics. For the convenience of the readers we also recall remarkable results of the Euler and the NSE from the beginning of the last century to the present.

1.1.1 The Euler and Navier-Stokes equations

It seems to be that the notion of pressure was first introduced by Archimedes of Syracuse in his book *Hydrostatics*, which is also known as the first book on mathematical fluid mechanics. His famous quote "*Eureka*!" reminds us to his principle of buoyancy: "*an immersed body is acted upon by a force equal to the weight of water it displaces.*" A long time after Archimedes, Leonardo da Vinci was clearly presented the idea of continuity of the fluid continuum. Note that the word "*turbulence*" (or *turbolenza*) was also first used and introduced by him. Moreover, the initial studies of waves, jets and interacting eddies were also given by Leonardo da Vinci. Later, Galileo Galilei gave us the concept of momentum in physics.

It is known that the conservation of linear momentum was introduced by Sir Issac Newton in 17^{th} century. It is also known as the Newton's second law, i.e.,

$$\text{total applied forces} = \text{mass} \times \text{acceleration}, \tag{1.1.1}$$

which is the essential principle also knows as the "fundamental law of dynamics". In other words (1.1.1) says that the balance of forces F acting on a solid is equal to the product of its mass m by its acceleration a, $F = m \times a$. He also first studied and presented the concept of "viscosity" *defectus lubricitatis*[1]. Then Daniel Bernoulli and Leonhard Euler gave a big step forward in mathematical fluid dynamics that the famous Bernoulli equation (also known as the Bernoulli-Euler equation) of fluid motion was formulated by both of them[2], i.e.,

$$\frac{1}{2}\rho \mathbf{v}^2 + p = \text{constant}, \tag{1.1.2}$$

which first appeared in the Bernoulli's 1738 book *Hydrodynamics*, for more discussion on this equation see [LL87]. The equation (1.1.2) was the first one which says the relation among the velocity $\mathbf{v}$, the pressure p and the density of the fluid ρ in fluid dynamics. Note that we assume that the density of the fluid exists that is also known as the "continumm assumption". This assumption holds for almost all macroscopic phenomena observed in nature. In addition, it also give us the first explanation of the lift of airfoils.

The founder of fluid mechanics as a mathematical discipline seems to be Leonhard Euler. Moreover, it was written by Louis de Lagrange: "Euler did not contribute to fluid mechanics

[1] Note that Newton used the words *defectus lubricitatis* and did not mention the words "viscosity" or "internal friction" as we usually use today, see [Jac91].

[2] Initially derived by Bernoulli and then it is correct by Leonhard Euler.

but created it", see [Tok94]. In fact, Euler gave the complete and correct derivation of (1.1.2) and even invented a hydraulic turbine. The first correct derivation of the mathematical equations of inviscid flows was also given by him in 1755 [Eul55]. More precisely, in the case of ideal homogeneous incompressible fluids these equations are given by the standard Euler equations[3]

$$\begin{cases} \rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1.1.3}$$

where $\partial_t \mathbf{v} = \partial \mathbf{v}/\partial t$ for simplicity, ∇p stands for the pressure forces, $\mathbf{f}$ denotes the external forces (or the body forces) such as gravity and $\nabla\cdot$ the usual divergence operator. It can be seen that the first equation in (1.1.3) has a form as in (1.1.1) (balance of momentum or conservation of momentum) with the mass considered as ρ the density, and the acceleration is expressed by[4]

$$\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v},$$

which is also known as the total or material derivative of $\mathbf{v}$ describing the rate of change at a point moving with fluid locally. The only nonlinear term or the convection term $(\mathbf{v}\cdot\nabla)\mathbf{v}$[5] is one of the main difficulties in mathematical theory when dealing with the Euler equations (also to the NSE below). The second equation in (1.1.3) presents the incompressibility of the fluid. It is also known as the mass continuity equation (or conservation of mass[6]) with its original form given by the following differential form (the spatial or Eulerian form)

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{1.1.4}$$

In the case $\rho = \rho_0 = const$ (in the case of incompressible fluids) is the constant density of the fluid then (1.1.4) yields to the constraint $\nabla\cdot\mathbf{v} = 0$ as in (1.1.3).

[3]It seems that the Euler equations was the second partial differential equations ever written down after the first one-the one dimension wave equation-discovered by d'Alembert in 1746. Later, Euler derived the three dimensions wave equation in 1766, see [Spe08].

[4] In general the material derivative of a vector field $\mathbf{u}$ is defined by

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u}.$$

[5] In the 3D case, the components of the convection term are given by $[(\mathbf{v}\cdot\nabla)\mathbf{v}]_i = \sum_{j=1}^3 v_j\partial_j v_i$ for $\mathbf{v} = (v_1, v_2, v_3)$. Thanks to the divergence constraint sometimes we replace the usual form $(\mathbf{v}\cdot\nabla)\mathbf{v}$ by $\nabla\cdot(\mathbf{v}\otimes\mathbf{v})$ for $\mathbf{v}$ smooth enough, where

$$\mathbf{v}\otimes\mathbf{v} := (v_i v_j)_{1\le i,j\le 3} \quad \text{and} \quad [\nabla\cdot(\mathbf{v}\otimes\mathbf{v})]_i = \sum_{j=1}^3 \partial_j(v_i v_j).$$

[6] Principle of conservation of mass: "the mass of fluid in a material volume V does not change as V moves with the fluid".

Almost two hundred years ago, the equations of viscous flow was first formulated by Claude-Louis Navier in 1822 [Nav22], where the effect of the viscosity is taken into account. The idea of considering the effect of the viscosity came from the Fourier's law[7] which today is known as the heat equation to describe the heat flow, i.e.,

$$\partial_t T - \kappa \Delta T = 0, \tag{1.1.5}$$

where T presents the temperature and $\kappa > 0$ denotes the rate of heat dissipation (also known as diffusivity coefficient) oftenly taken to equal one. In addition, Δ denotes the usual Laplace operator. However, Navier's derivation was based on an incorrect basis molecular model. In fact, he suggested the law of interaction among molecules. Moreover, from the physical point of view, it was recognized that this law was not inconsistent especially for liquids.

More than twenty years later, based on the Cauchy stress principle the same equations were derived by George Gabriel Stokes in 1845 [Sto45]. He was known as the first person who gave the first clear and correct explanation of the appearance of the viscous terms in the Navier-Stokes equations. Note that this derivation is quite similar to that of we use today. The motion of an incompressible homogeneous Newtonian fluid[8] is governed by the Navier-Stokes equations, i.e.,

$$\begin{cases} \rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = \eta\Delta\mathbf{v} - \nabla p + \mathbf{f}, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1.1.6}$$

where $\eta > 0$ denotes the dynamics viscosity and $\eta\Delta\mathbf{v}$ is usually known as the viscosity term. In particular, (1.1.6) "formally"[9] reduces to (1.1.3) as η/ρ goes to zero and to the Stokes equations as η/ρ goes to infinity. It can be seen that the first equation in (1.1.6) is written in the form as in (1.1.1).

It can be seen that both systems (1.1.3) and (1.1.6) are nonlinear due to the convection term (the second term on the left-hand side). They are also nonlocal due to the incompressibility constraint, i.e., $\nabla\cdot\mathbf{v} = 0$. These properties make Euler and Navier-Stokes equations hard to study. For convenience, we usually use the dimensionless form (a form that does not depend

[7] It was established mathematically by Jean-Baptiste Biot in 1804 then experimentally by Jean-Baptiste Joseph Fourier in 1822 [Fou22].

[8] As usually the stress tensor is expressed in the form $\sigma = -\mathbf{I}p + \mathbf{D}$ where $\mathbf{I}$ denotes the identity matrix and $\mathbf{D}$ stands for the deviatoric part of σ. A fluid is called a Newtonian fluid if whose deviatoric part is a linear function of its velocity gradient. In the case of the NSE, $\mathbf{D} = 2\eta D\mathbf{v}$ where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$.

[9] More precisely, it is known that solutions to the NSE tend to that of the Euler equations in some particular sense.

directly on the physical sizes) corresponding to (1.1.6) given by[10] (we do not distinguish the notation)

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \dfrac{1}{Re}\Delta \mathbf{v} - \nabla p + \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1.1.7}$$

where the dimensionless quantity-the so-called Reynolds number-Re given by the following formula

$$Re := \frac{\rho L V}{\eta}. \tag{1.1.8}$$

For simplicity, in the case $V = L = 1$, set $\nu := \frac{\eta}{\rho}$ denotes the kinematic viscosity and then (1.1.8) yields $Re = \nu^{-1}$. It suggests to replace Re^{-1} in the system (1.1.7) by ν as the standard form of the NSE. Here ν (or Re) is the only parameter and also plays an important role in the mathematical theory of the NSE. This number can be described by the ratio of inertial forces to viscous forces and will be discussed in more details below. In other words, it is the ratio of the intensity of the nonlinear effect to the intensity of the viscosity linear effect.

The readers can also find a brief history on the subject of this subsection with more explanation by Isabelle Gallagher[11], by William Layton [Lay08, Section 1.3, Chapter 1], "the first five births of the NSE" by Olivier Darrigol [Dar02], see also [LR16, Chapter 3]. For detailed derivations of the Euler and Navier-Stokes equations we refer the reads to the texts such as [LL87, Ser59, Bat99, CM93, CRL14] and their mathematical theory can be found in classical textbooks [Lio69, Lad69, CF88, Tem95, Tem97, Tem01, Gal00, Soh01, Lio96, FMRT01, MB02, LR02, RR09, Gal11, LR16, RRS16].

1.1.2 Brief state of the art

One of the main foundational questions for every PDE is the well-posedness problem, i.e., in the sense of Jacques Hadamard, that are the existence, uniqueness of solutions and the solution's behavior changes continuously with the initial conditions. Moreover, it is a crucial problem to know whether solutions corresponding to smooth initial data can develop

[10] It can be done by changing of variables in (1.1.6):

$$\mathbf{x}^* := \frac{\mathbf{x}}{L}; \quad t^* := \frac{t}{T}; \quad \mathbf{v}^* := \frac{\mathbf{v}}{V}; \quad p^* := \frac{p}{P\rho}; \quad \mathbf{f}^* := \frac{T^2\mathbf{f}}{L\rho};$$

where V, L, T are the velocity, length and time characteristic, respectively, which are chosen in some particular way, for example $P = U^2$ and $V = L/T$. That leads to a system satisfied by $\mathbf{v}^*$ and p^* and has the same form as (1.1.7). However, for simplicity the notation, we use $(\mathbf{v}, p)$ instead of $(\mathbf{v}^*, p^*)$.

[11] See more details at https://images.math.cnrs.fr/Autour-des-equations-de-Navier-Stokes.html?lang=fr.

singularities in finite time in some particular sense. This subsection aims to summarize well-known results to the Euler and Navier-Stokes equations.

A. Euler equations

In the 2D case the mathematical theory for the Euler equations is completed. For smooth initial data the global classical solutions of the Euler equations is known. The result can be found in [Wol33, Gyu53, Jud63, Kat67, Lad69, Tem76, MB02]. Other results on the lack of uniqueness of weak solution can be found in [Sch93, Shn97a] in the 2D case (on $\mathbb{R}^2$ or on $\mathbb{T}^2$), see also in [DLS09, DLS10, DLS13] for any dimensional bigger than two with solutions satisfied several additional requirements.

However, in general the theory is not completed in the 3D case. Local existence of solution to the Euler equations is investigated by several authors in [Lic25, EM69, Tem76, Tem75, BB74, KL84]. The inviscid limit problem, i.e., the case when $\nu \to 0$, was studied in [Swa71, Kat72, Kat75, Kat84a, Con86, Wan01, DM87, AD04, CW96, TW97, SC98a, SC98b, Mas07, Kel09, Kel08, Kel07, BT07, BS12, BT13, SB14, DN19, CV18, CEIV17, NN18, NN19b, GKLF$^+$19]. Several blow-up criterion results for the 3D Euler equations were provided in [BKM84, CFM96, GT13]. A remarkable result on the existence of global weak solutions to the Euler equations in 3D was given by Weidemann in [Wie11] where the author used the results obtained in [DLS10], see also in [DLS09, DLS13]. Weak solutions to the 3D Euler equations with decreasing energy were also constructed in [Shn97b, Shn98, Shn00, Shn03].

B. Navier-Stokes equations

One of the first results on the existence and uniqueness of classical solutions in the 2D case for the NSE was provided by Leray in [Ler33].

However, the story is totally different in the three dimensions case, for which in general the problem of globally existence of strong solutions is still open. The first results in the 3D case for the NSE are thoses of Oseen and Leray [Ose27, Ler34b]. More precisely, in 1934, Jean Leray [Ler34b] provided the first global existence result of weak solutions (at that time which were called "turbulent solutions"[12] and nowadays in general which are known as Leray-Hopf weak solutions) in $L_t^\infty L_{\mathbf{x}}^2 \cap L_t^2 H_{\mathbf{x}}^1$ to the NSE in the whole space with the initial data in L^2. In addition these solutions satisfy the energy inequality. However, the uniqueness of these solutions is not known so far[13]. For a modern review on this celebrated

[12] The definition of this kind of solution was written in the form which today is known as in the sense of distribution. Note that theory of distribution was seemly initiated by Sergei Sobolev in 1936 (generalized functions) and then was developed and extended by Laurent Schwartz in the late 1940s.

[13] It is believed that the lack of the uniqueness of weak solutions is related to the theory of turbulence.

work we refer the reader to [OP18]. Leray also raised the question of the existence of self-similar[14] solutions of the Navier-Stokes equations in which the answers for this question can be found in [NRv96, Tsa98]. More precisely, the authors showed that self-similar solutions must be trivial under general assumptions such as in $\mathbf{v} \in L^3(\mathbb{R}^3)$ or $\mathbf{v}$ satisfies local energy estimates[15].

Note that the same result was established by Eberhard Hopf in 1951 for smooth bounded domains in [Hop51] subject to the Dirichlet boundary conditions. It has been known that various additional assumptions guarantee the smoothness of Leray-Hopf weak solutions. These assumptions are known as the so-called Prodi-Serrin-Ladyzhenshaya[16] criteria provided by [Pro59, KL57, Ohy60, Lad67, FJR72, Ser62, Ser63, Str88, ISS03, BadVY20a, BC20]. Global results to the NSE were provided in [Soh83, Gig86] in the case of physical boundary. Regularity criteria on the gradient of the velocity $\nabla \mathbf{v}$, or on its components, or on the pressure p, or on the vorticity direction were studied in [BadV95, Ber02, BG02, CT08, BadVB09, BadVY20b].

So far, only local existence and uniqueness results are known for the NSE in 3D. The global existence of smooth solutions is also known for the NSE for small data. One of such results is due to Fujita-Kato in the beginning of 1960s. More precisely, by applying Hilbert space approach with using the theory of fractional power of operator and semigroup of operators Fujita-Kato [KF62, FK64] established an important result on the existence of the classical solutions to the NSE with initial small data in the space $H^{1/2}(\Omega)$[17]. The theory of L^p-strong solutions to the NSE were provided in [Kat84b, GM85]. Partial regularity theorems for suitable weak solutions of the Navier–Stokes equations have been provided by, for example, Scheffer [Sch76], Caffarelli–Kohn–Nirenberg [CKN82] and Lin [Lin98], where the latter one improved the former one. It seems that the global well-posedness result provided by Koch-Tataru in [KT01] is one of the best developments so far in this direction, where the initial

[14] Leray's (backward) self-similar solutions have a form ($\mathbf{f} = \mathbf{0}$)

$$\mathbf{v}(t, \mathbf{x}) = \lambda(t) U(\lambda(t)\mathbf{x}) \quad \text{and} \quad p(t, \mathbf{x}) = \lambda^2(t) P(\lambda(t)\mathbf{x}) \quad \text{where} \quad \lambda(t) = \frac{1}{\sqrt{2a(T-t)}},$$

where $a > 0$, $T \in \mathbb{R}$. For a long time, self-similar solutions are considered to be good candidates for constructing singular solutions of the NSE.

[15] Note that the $L^3(\mathbb{R}^3)$ integrability condition holds in the case that $\mathbf{v}$ satisfies the global energy estimate and might not true if $\mathbf{v}$ satisfies local energy estimates, see more discussion in [Tsa98, Section 4].

[16] It says that if $\mathbf{v}$ is a Leray-Hopf weak solution to the NSE and $\mathbf{v}$ satisfies in addition $\mathbf{v} \in L^r(0, T; L^s(\mathbb{R}^3))$ for $2/r + 3/s = 1$ and $3 \leq s \leq \infty$ then $\mathbf{v}$ is smooth.

[17] Here the authors considered the Cauchy problem in Ω a bounded smooth domain in $\mathbb{R}^3$ with homogeneous Dirichlet boundary conditions. The authors also claimed that the same result could be extended in more general cases such as inhomogeneous Dirichlet boundary conditions, unbounded domain, 2D case, or L^p-theory.

data is considered to be small in the space BMO^{-1}[18]. In 2008, an important ill-posedness result to the NSE was provided by Bourgain-Pavlovic [BP08] in the Besov space $\dot{B}_{\infty}^{-1,\infty}$[19].

Moreover, the existence of weak but not very regular solutions is known, certain kinds of blow-up results are found out, for instance, [HL08, Hou09, HSW12, Tao16, CHK+17], many stability questions are understood, many easier models of these equations have been completely investigated. But the most fundamental properties of these equations are awaiting their discovery. In addition, for the Navier-Stokes equation, in 2000 the question of global existence of smooth solutions and finite time blow up is one of the seven Clay Mathematics Institute "Millenium problems" which is offered with 1 million dollars prize for providing a solution, for which an official statement of the problem was written by Charles L. Fefferman[20]. For interested readers more discussion on this problem can be found in [Lad03, LR16]. Recently, in 2019 based on the "convex integration technique" developed by De Lellis-Székelyhidi in [LS09, LS13], Tristan Buckmaster and Vlad Vicol showed that very weak solutions (so far which are still weaker than Leray-Hopf weak solutions) are not unique in the class of finite energy solutions, see [BV19b]. The nonuniquness of Leray-Hopf weak solutions holds under certain spectral assumption for a linearized Navier-Stokes operator was provided in [JS15]. Although this spectral condition is not known to be true by a rigorous proof, one of the author has proved it numerically in [Gv17]. In addition under suitable assumption on the initial data the author of [Jv14] proved that there exists at least one scale-invariant solution to the 3D NSE which is smooth[21].

1.1.3 Turbulent flows

This subsection is devoted to discuss on turbulent flows. We start with recalling the nature and study of these flows and then we shall present some recently developments on the study of turbulence.

[18] It is a largest critical space-the space which is invariant under the scaling-to the NSE. For simplicity, we assume that $\mathbf{f} = \mathbf{0}$ then the scaling is understood in the sense that if $\mathbf{v}(\mathbf{x},t), p(\mathbf{x},t)$ solves the NSE then $\mathbf{v}_\lambda(\mathbf{x},t) = \lambda\mathbf{v}(\lambda\mathbf{x},\lambda^2 t), p_\lambda(\mathbf{x},t) = \lambda^2(\lambda\mathbf{x},\lambda^2 t)$ is a solution to the NSE with $\mathbf{v}_{0\lambda} = \lambda\mathbf{v}_0(\lambda\mathbf{x})$.

[19] Note that $BMO^{-1} \hookrightarrow \dot{B}_{\infty}^{-1,\infty}$ is a continuous embedding. The ill-posedness is understood in the following sense: Let $\delta > 0$ there exists a solution to the NSE $(\mathbf{v}, p)$ with $\|\mathbf{v}_0\|_{\dot{B}_{\infty}^{-1,\infty}} \leq \delta$ such that $\|\mathbf{v}\|_{\dot{B}_{\infty}^{-1,\infty}} > \frac{1}{\delta}$.

[20] https://www.claymath.org/sites/default/files/navierstokes.pdf. The author considered the cases in the whole space or in the periodic context. It seems that the statement does not provide too much physical meaning, see [Tar06].

[21] More precisely, if the initial velocity $\mathbf{v}_0$ is scale-invariant, i.e., $\mathbf{v}_0 = \lambda\mathbf{v}_0(\lambda\mathbf{x}), \lambda > 0$ and locally Hölder continuous in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ and $\nabla \cdot \mathbf{v}_0 = 0$ then the corresponding Cauchy problem to the NSE has at least one scale-invariant solution which is smooth in $\mathbb{R}^3 \times (0,\infty)$ and locally Hölder continuous in $\mathbb{R}^3 \times (0,\infty) \setminus \{(\mathbf{0}, 0)\}$. Here we recall that the NSE is invariant under the scaling (in the case the external force $\mathbf{f} = 0$ for simplicity) for $\lambda > 0$:

$$\mathbf{v}_\lambda(t,\mathbf{x}) = \lambda\mathbf{v}(\lambda^2 t, \lambda\mathbf{x}); \quad p_\lambda(t,\mathbf{x}) = \lambda^2 p(\lambda^2 t, \lambda\mathbf{x}); \quad \mathbf{v}_{0\lambda}(\mathbf{x}) = \lambda\mathbf{v}_0(\lambda\mathbf{x}).$$

That means if $(\mathbf{v}, p)$ is a solution then so does $(\mathbf{v}_\lambda, p_\lambda)$ for initial data $\mathbf{v}_{0\lambda}$. We say that $\mathbf{v}$ is a scale-invariant solution if $\mathbf{v} = \mathbf{v}_\lambda$ and $p = p_\lambda$ for each $\lambda > 0$.

A. The nature of turbulent flows

Turbulent flows can be easily observed around us which could be the smoke of a hot cup of coffee that we drink every morning, the smoke behind a moving car, the smoke on a chimney, water flow in a river, air flows around a car, an airplane, a train (although it is less easily seen) and so on. Moreover, observing carefully the "smoke" in each example above it can be seen that there are normally three situations: first, the smoke moves almost straightforward; then it fluctuates a little bit; finally, it goes randomly and develops eddies until it disappears in the air. These behaviors can be seen closely by experimenting such as: making a hot cup of coffee, smoking, or blowing out a burning candle and observing their smoke. Therefore, we are also easily creating a turbulent flows. If doing small experiments above several times we can see that the behavior of the smoke seems to be different at each time. It might remind us to the sensitive of the NSE and the well-known effect which is called the "butterfly effect"[22] which was seemly discovered by Edward Norton Lorenz in 1960s[23]. In fact the idea inside is that small changes can have large consequences.

Nowadays, turbulence is the central of many important applications in our real life. In fact there is not precisely definition of turbulence. However, we know some of its characteristics which are: unsteady, chaotic, irregular, random, unpredictable, diffusive, dissipative, rotational and so on. It is known that turbulence occurs at high Reynolds numbers which are defined as in (1.1.8). It seems that the concept of Reynolds number was first introduced by George Stokes in 1850 [Sto09], experimented by Osborne Reynolds[24] in 1883 [Rey83] and named by Arnold Sommerfeld in 1908 [Som08]. The values of Re are corresponded to the characteristics of the fluid for which the behavior of the flow can be normally divided into three situations. For instance, in the case of a pipe water flow which was established by Reynolds: laminar ($Re \leq 2300$), transition ($Re \approx 2300 - 4000$) and turbulent ($Re \geq 4000$), see [Pop00]. In this case the characteristic velocity and characteristic length V and L in (1.1.8) can be the area-averaged axial velocity and the pipe diameter, respectively. Note that this classification depends on Re and maybe is different for other materials.

Let us reconsider an example, see [DLS19], to see the effect of Reynolds number Re on solution $\mathbf{v}$. In the 3D case, for instance, we study (1.1.7) on a domain $\Omega = [0, 2\pi]^3$, $\mathbf{f} = A\sin(x_1)\mathbf{e}_2$ where $A > 0$, $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ and $\mathbf{e}_2 = (0, 1, 0)$. We assume that the

[22] "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

[23] From Wikipedia: "In 1961, Lorenz was running a numerical computer model to redo a weather prediction from the middle of the previous run as a shortcut. He entered the initial condition 0.506 from the printout instead of entering the full precision 0.506127 value. The result was a completely different weather scenario".

[24] In his experiment, Reynolds studied the behavior of water flow in a pipe with another smaller and shorter pipe of dye inside. The dye inside makes it easily to observe. When changing the velocity of the water flow he observed its movement. Later, in 1894, he explained the behavior of this flow by a dimensionless parameter which is now known as the Reynolds number.

density $\rho = \rho_0 = 1$ for simplicity. A stationary solution of (1.1.7) is given by $(\mathbf{v}(\mathbf{x}), p(\mathbf{x})) = \left(\frac{A}{\nu}\sin(x_1)\mathbf{e}_2, 0\right)$. In this case the characteristic velocity and characteristic length could be $V = \frac{A}{\nu}$ and $L = 2\pi$, respectively. Then Reynolds number is given by (1.1.8) with $Re = \frac{2\pi A}{\nu^2}$. This simple example shows that $\mathbf{v}$ is unstable if we increase A (the same as we decrease ν).

B. The study of turbulent flows

The study of turbulent flows can be normally divided into three parts [Pop00]:

P1. Discovery. This part focuses on doing experiment or simulation to provide the qualitative or quantitative information of some particular flows.

P2. Modeling. It aims to provide mathematical models[25] which can be used to predict properties of turbulent flows.

P3. Control. It is the combining of the two parts above.

In this thesis, we will focus on the second part **P2** above. More precisely, in Chapter 2, by using basic turbulence modeling we establish a generalized Navier-Stokes-Voigt model and provide its mathematical analysis. In addition, in Chapter 3 we will study several models such as the Leray-α, modified Leray-α and simplified Bardina which are considered as α-regularization models for the NSE. That means solutions to these models are smoother (and unique) than that of the NSE. Numerical tests on these models are also easier to do than directly on the NSE. In Chapter 4 an extension of the Balwin-Lomax model for turbulent mixing layers is given by both modeling and analyzing.

There are at least two ways to study turbulent flows. The first one is DNS which stands for Direct Numerical Simulation and is used to solve the NSE directly. According to the Kolmogorov laws it requires a huge number of degrees of freedom (DOF) around $\mathcal{O}(Re^{d^2/4})$. It is now still unreasonable for modern computers for high Reynolds numbers. Therefore, in general this approach sometimes is impossible for solving directly the NSE. It is also due to the sensitive of the NSE where a small change in the initial datum can lead to a huge change in the final result. A nature way to reduce the sensitive is trying to compute the "mean" values which is the aim of the second approach as follows.

The other way is the statistical approach. It aims to compute the mean (averaged or filtered) value of the velocity instead of the true one. The idea is to compute the averaged velocity (in some particular sense, however, the most naturally being the ensemble averages) rather than compute it at each point in space and time. Then the question is what is the equation satisfied by the mean velocity? It is natural to apply the mean operator to the NSE to

[25] From the numerical point of view it is not able to perform simulating directly to the NSE.

find the system we are looking for. However, this system is not closed. It is due to the nonlinear term in the NSE. More precisely, let us denote the mean operator by the bar "$-$" which could be the sum averaged, time-averaging or probability expectation for example. Therefore, we have the Reynolds decomposition of the velocity $\mathbf{v}$ and the pressure p are given by

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' \quad \text{and} \quad p = \overline{p} + p'. \tag{1.1.9}$$

The mean operator is assumed to be a linear operator and satisfies Reynolds rules which are

$$\overline{\partial \mathbf{v}} = \partial \overline{\mathbf{v}} \quad \text{and} \quad \overline{\overline{\mathbf{v}}} = \overline{\mathbf{v}}, \tag{1.1.10}$$

where ∂ denotes any first order differential operator.

B1. Turbulent incompressible flows

In this part we will focus on turbulent incompressible flows. The main idea is to find the system satisfied by mean quantities $(\overline{\mathbf{v}}, \overline{p})$ and then using further assumptions such as Boussinesq approximations to close the mean equations. Applying the mean operator (1.1.9) to the NSE and using the Reynolds rules (1.1.8) we obtain from (1.1.7) the system satisfied by the mean velocity, i.e., the mean Navier-Stokes equations:

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} = -\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} = 0, \end{cases} \tag{1.1.11}$$

where the Reynolds stress given by

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \overline{\mathbf{v}' \otimes \mathbf{v}'}. \tag{1.1.12}$$

In the above computations we have used the divergence free constraint and

$$(\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \overline{(\mathbf{v} \cdot \nabla)\mathbf{v}} = -\nabla \cdot (\overline{\mathbf{v} \otimes \mathbf{v}} - \overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) = -\nabla \cdot \left(\overline{\mathbf{v}' \otimes \mathbf{v}'}\right).$$

It can be seen that (1.1.11) is not closed. That is we do not know how to deal with the Reynolds stress term in the right-hand side. It leads us to the question how to close the system. To do that it is natural to assume that the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ can be expressed by only mean quantities. The most popular way to close the system is to use the Boussinesq assumption. It seems that he is the first one who works on the closure problem. By introducing the concept of eddy viscosity (also known as turbulent viscosity) ν_{turb} he proposed in 1877 [Bou77, Bou03][26] that

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_{\mathrm{turb}} D\overline{\mathbf{v}} + \frac{2}{3} k \,\mathrm{Id}, \tag{1.1.13}$$

[26] The word "Boussinesq approximation" was seemly first used by Rayleigh in 1916 [Ray16].

where Id is the identity matrix, $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ presents the deformation stress tensor of the velocity (or the symmetric part of the velocity) and $k = \frac{1}{2}tr(\boldsymbol{\sigma}^{(\mathrm{R})}) = \frac{1}{2}\overline{|\mathbf{v}'|^2}$ denotes the turbulent kinetic energy, where tr stands for the trace operator. Putting (1.1.13) into (1.1.11) with using the fact $\nabla \cdot \overline{\mathbf{v}} = 0$ we obtain

$$\left\{\begin{aligned} \partial_t\overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \nabla \cdot ((2\nu + \nu_{\mathrm{turb}})D\overline{\mathbf{v}}) + \nabla(\overline{p} + (2/3)k) &= \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} &= 0. \end{aligned}\right. \tag{1.1.14}$$

Another assumption about the Reynolds stress that will be considered in the next chapter is given by

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\alpha\ell\, D\partial_t\overline{\mathbf{v}} - \nu_{\mathrm{turb}}D\overline{\mathbf{v}} + \frac{2}{3}k\,\mathrm{Id}. \tag{1.1.15}$$

where α is a length scale and ℓ is the Prandtl mixing length [Pra10], which can be considered as the mean distance traveled by a small ball of fluid before disappearing because of the turbulent mixing. Similarity, putting (1.1.15) into (1.1.11) with using the fact $\nabla \cdot \overline{\mathbf{v}} = 0$ we obtain the generalized Navier-Stokes-Voigt equations:

$$\left\{\begin{aligned} \partial_t\overline{\mathbf{v}} - \alpha\nabla \cdot (\ell\, D\partial_t\overline{\mathbf{v}}) + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \nabla \cdot ((2\nu + \nu_{\mathrm{turb}})D\overline{\mathbf{v}}) + \nabla(\overline{p} + (2/3)k) &= \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} &= 0, \end{aligned}\right. \tag{1.1.16}$$

which will be considered as the main subject in Chapter 2. Moreover, the system coupled by (1.1.16) with the equation satisfied by the turbulent kinetic energy k is also investigated.

An alternative form for the convection term in the NSE (1.1.7) is given by the following well-known identity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla|\mathbf{v}|^2 + \boldsymbol{\omega} \times \mathbf{v},$$

where the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. It allows us to rewrite the NSE (1.1.7) in a rotational form

$$\left\{\begin{aligned} \partial_t\mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} - \nu\Delta\mathbf{v} + \nabla\pi &= \mathbf{f}, \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned}\right. \tag{1.1.17}$$

where $\pi = p + \frac{1}{2}|\mathbf{v}|^2$ the modified pressure. Applying the mean operator (1.1.9) to (1.1.17) which yields (similar as above):

$$\left\{\begin{aligned} \partial_t\overline{\mathbf{v}} + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \overline{\boldsymbol{\omega}' \times \mathbf{v}'} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{\pi} &= \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} &= 0. \end{aligned}\right. \tag{1.1.18}$$

where $\boldsymbol{\omega}' = \nabla \times \mathbf{v}'$ the fluctuation of the vorticity. As a result, a problem raises to "close" (1.1.18). In order to do that the modeling in Chapter 4 leads us to consider $\mathbf{A}^{(\mathrm{R})}$ the "*rotational Reynolds stress*" with

$$\left\{\begin{aligned} &\mathrm{curl}\ \mathbf{A}^{(\mathrm{R})} = \overline{\boldsymbol{\omega}' \times \mathbf{v}'}, \\ &\mathrm{div}\,\mathbf{A}^{(\mathrm{R})} = 0. \end{aligned}\right. \tag{1.1.19}$$

In addition, by using basic turbulence modeling in Chapter 4 we are led to use

$$\mathbf{A}^{(\mathrm{R})} = \ell^2\, \partial_t \overline{\boldsymbol{\omega}} + \nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}} + \nabla(-\Delta)^{-1}(\nabla \nu_{\mathrm{turb}} \cdot \overline{\boldsymbol{\omega}}). \tag{1.1.20}$$

Putting (1.1.20) into (1.1.18) with $\nu_{\mathrm{turb}} = \kappa \ell^2 |\overline{\boldsymbol{\omega}}|$ we obtain

$$\begin{cases} \partial_t \overline{\mathbf{v}} + \mathrm{curl}\ (\ell^2\, \partial_t \overline{\boldsymbol{\omega}}) + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \mathrm{curl}\ (\kappa \ell^2 |\overline{\boldsymbol{\omega}}|\, \overline{\boldsymbol{\omega}}) + \nabla \overline{\pi} = \mathbf{f}, \\ \mathrm{div}\, \overline{\mathbf{v}} = 0, \end{cases} \tag{1.1.21}$$

which will be modeled and analyzed in details in Chapter 4.

All forms (1.1.13), (1.1.15) and (1.1.20) lead us to the question raised by determination of the eddy viscosity ν_{turb}. That is one of the challenges in turbulence modeling. In addition we also need to find the equation which is satisfied by k and also seek the formula for ℓ. Sometimes the system (1.1.11) with $\boldsymbol{\sigma}^{(\mathrm{R})}$ given by (1.1.13) is called the eddy viscosity model. An example for the eddy viscosity was suggested by Prandtl [Pra52]

$$\nu_{\mathrm{turb}} = C \ell^2 |D \overline{\mathbf{v}}|, \tag{1.1.22}$$

where C is a dimensionless constant. Combining (1.1.14)-(1.1.22) leads to the Smagorinsky's model, one of the popular (and also the first introduced) turbulent models. In the case of a flow over a plate $\Omega = \mathbb{R}^2 \times \{z > 0\}$, Obukhov [Obu46] proposed

$$\ell = \ell(z) = \kappa z, \tag{1.1.23}$$

where $\kappa \in [0.35, 0.42]$ is the Von Kármán constant. A more complicated formula can be found in Van Driest [VD56]:

$$\ell = \ell(z) = \kappa z (1 - e^{-z/A}), \tag{1.1.24}$$

where A depends on the oscillations of the plate and ν.

B1.1. The $(k - \mathcal{E})$ model

As mentioned above, in order to find the system (1.1.11) which is satisfied by mean values $\overline{\mathbf{v}}$ and $\overline{p}$ we need to model the turbulent viscosity ν_{turb} and turbulent kinetic energy k when using the Boussinesq assumption (1.1.13). We start with finding an equation for k. The authors in [CRL14, Theorem 4.2] provided an equation which is satisfied by k. That is

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot (\overline{e' \mathbf{v}'}) = -\boldsymbol{\sigma}^{(\mathrm{R})} : \nabla \overline{\mathbf{v}} - \mathcal{E}, \tag{1.1.25}$$

where

$$\begin{cases} e := \dfrac{1}{2} |\mathbf{v}'|^2 = \overline{e} + e' = k + e', \\ \epsilon := 2\nu |D\mathbf{v}|^2 \Rightarrow \overline{\epsilon} = 2\nu \overline{|D\mathbf{v}|^2} = 2\nu |D\overline{\mathbf{v}}|^2 + 2\nu \overline{|D\mathbf{v}'|^2} = 2\nu |D\overline{\mathbf{v}}|^2 + \overline{\epsilon'}, \\ \mathcal{E} := \overline{\epsilon'} = 2\nu \overline{|D\mathbf{v}'|^2}. \end{cases}$$

We call ϵ the dissipation and $\mathcal{E}$ the mean dissipation of the fluctuation. Replacing $\boldsymbol{\sigma}^{(\mathrm{R})}$ in (1.1.25) by (1.1.13) yields

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k + \nabla \cdot (\overline{e'\mathbf{v}'}) = \nu_{\text{turb}} |D\overline{\mathbf{v}}|^2 - \mathcal{E}, \tag{1.1.26}$$

since $k\mathrm{Id} : \nabla\overline{\mathbf{v}} = k\nabla \cdot \overline{\mathbf{v}} = 0$ and $D\overline{\mathbf{v}} : \nabla\overline{\mathbf{v}} = |D\overline{\mathbf{v}}|^2$. Assume that there exists a turbulent diffusion coefficient $\mu_{\text{turb}} > 0$ with[27]

$$\mu_{\text{turb}} := \mu_{\text{turb}}(k, \mathcal{E}) = c_k \frac{k^2}{\mathcal{E}}, \tag{1.1.27}$$

where c_k is a dimensionless constant such that

$$\nabla \cdot (\overline{e'\mathbf{v}'}) = -\nabla \cdot (\mu_{\text{turb}} \nabla k).$$

Combining the previous equation with (1.1.26) yields

$$\partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot \left(c_k \frac{k^2}{\mathcal{E}} \nabla k \right) = \nu_{\text{turb}} |D\overline{\mathbf{v}}|^2 - \mathcal{E}. \tag{1.1.28}$$

We also need to find an equation for $\mathcal{E}$. That can be done by following the analysis in [CRL14, equation 4.125] we come with an equation for $\mathcal{E}$

$$\partial_t \mathcal{E} + \overline{\mathbf{v}} \cdot \nabla \mathcal{E} - \nabla \cdot \left(c_\epsilon \frac{k^2}{\mathcal{E}} \nabla \mathcal{E} \right) = c_\eta |D\overline{\mathbf{v}}|^2 - (c_{\epsilon 2} - c_\gamma) \frac{\mathcal{E}^2}{k}, \tag{1.1.29}$$

where $c_\epsilon, c_\eta, c_{\epsilon 2}$ and c_γ are dimensionless constants. By employing dimensional analysis it is suggested to consider for some dimensionless constant c_ν

$$\nu_{\text{turb}} = c_\nu \ell \sqrt{|k|} \quad \text{with} \quad \ell = \frac{k\sqrt{|k|}}{\mathcal{E}} \quad \Rightarrow \quad \nu_{\text{turb}} = c_\nu \frac{k^2}{\mathcal{E}}. \tag{1.1.30}$$

Therefore, combining (1.1.14)-(1.1.28)-(1.1.29)-(1.1.30) yields

$$\begin{cases} \partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \nabla \cdot \left(\left(2\nu + c_\nu \frac{k^2}{\mathcal{E}} \right) D\overline{\mathbf{v}} \right) + \nabla(\overline{p} + (2/3)k) = \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} = 0, \\ \partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot \left(c_k \frac{k^2}{\mathcal{E}} \nabla k \right) - c_\nu \frac{k^2}{\mathcal{E}} |D\overline{\mathbf{v}}|^2 + \mathcal{E} = 0, \\ \partial_t \mathcal{E} + \overline{\mathbf{v}} \cdot \nabla \mathcal{E} - \nabla \cdot \left(c_\epsilon \frac{k^2}{\mathcal{E}} \nabla \mathcal{E} \right) - c_\eta |D\overline{\mathbf{v}}|^2 + (c_{\epsilon 2} - c_\gamma) \frac{\mathcal{E}^2}{k} = 0, \end{cases} \tag{1.1.31}$$

which is known as the $(k - \mathcal{E})$ model. This model was first developed by Launder and Spalding [Lau72]. For many turbulent flows in engineering applications as well as in oceanography the $(k - \mathcal{E})$ model is also known to provide reliable predictions of mean properties, see Mohammadi-Pironneau [MP94]. Further, the dimensionless constants above are not universal and can be taken as in [MP94] for instance:

$$c_\nu = c_k = 0,09; \quad c_\epsilon = 0,07; \quad c_\eta = 0,063; \quad c_{\epsilon 2} - c_\gamma = 1,92.$$

[27] In fact, formula (1.1.27) comes from dimensional analysis.

Remark 1.1.1. *We should point out that in the modeling above of the* $(k - \mathcal{E})$ *model, assumptions on the fluid are needed. We will summarize these assumptions here, for more details see [CRL14, 1-3], [MP94, 1-6]:*

1. *The Boussinesq assumption holds;*
2. *Transport of scalar fields by fluctuating vector fields yields turbulent diffusion;*
3. *The eddy viscosity and the turbulent coefficient are all functions of* k *and* $\mathcal{E}$ *and can be derived by dimensional analysis;*
4. *Additional symmetry properties of turbulent flows hold as well as isotropy of the fluctuations;*
5. *Turbulence is ergodic;*
6. *Turbulent flows are Gaussian, which means that* $\mathbf{v}$ *has a Gaussian distribution.*

B1.2. The NSTKE model

In the case $\ell = k\sqrt{|k|}/\mathcal{E}$ in (1.1.31) gives us a simplified version of the $(k - \mathcal{E})$ model which is known as the Navier-Stokes turbulent kinetic energy (NSTKE) model given by

$$
\begin{cases}
\partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} - \nabla \cdot ((2\nu + \nu_{\text{turb}}(k)) D\overline{\mathbf{v}}) + \nabla(\overline{p} - (2/3)k) = \overline{\mathbf{f}}, \\
\nabla \cdot \overline{\mathbf{v}} = 0, \\
\partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot (\mu_{\text{turb}}(k) \nabla k) + \dfrac{k\sqrt{|k|}}{\ell} - \nu_{\text{turb}}(k)|D\overline{\mathbf{v}}|^2 = 0,
\end{cases}
\tag{1.1.32}
$$

and is mathematically studied and simulated in more details in [Lew97a, CRL14]. A modified version of the NSTKE in which an additional term (the backscatter term) $-\alpha\nabla\cdot(\ell D\partial_t\overline{\mathbf{v}})$ will be added into the first equation in (1.1.32) for some constant $\alpha > 0$. This version follows by employing the Reynolds stress of the form (1.1.15) instead of (1.1.13). We will explain why we should take into account the form (1.1.15) in some particular case rather than the usual one, see more details in the modeling part in Chapter 2. This model will be presented in more details in Chapter 2.

1.1.4 Recently developments on the study of turbulence

This part aims to present recently mathematical theory on the study of turbulence. We start with the NSE, which is known as describing sufficiently accuracy incompressible homogeneous Newtonian fluids. Let us consider the NSE on $\Omega = [0, 2\pi]^3$ in the periodic case with the body force $\mathbf{f} = \mathbf{0}$ for simplicity

$$
\begin{cases}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = 0, \\
\nabla \cdot \mathbf{v} = 0.
\end{cases}
\tag{1.1.33}
$$

It is well-known that sufficient regular solutions satisfy the energy balance law[28], i.e.,

$$\frac{d}{dt}e(t) = -\nu \int_\Omega |\nabla \mathbf{v}(t)|^2 \, d\mathbf{x} \quad \forall t > 0, \tag{1.1.34}$$

where the total kinetic energy given by

$$e(t) := \frac{1}{2}\int_\Omega |\mathbf{v}(t)|^2 \, d\mathbf{x} \quad \forall t > 0.$$

It can be seen from (1.1.34) that it could be formally expected that the energy dissipation rate vanishes as $\nu \to 0$. However, both physical and numerical observations show that it is not true. More precisely, the dissipation rate is finite and positive, which is known as "anomalous dissipation". In the early of 1940s, Kolmogorov [Kol41] pioneered the study of statistical theory of turbulent flows. That means the components v_1, v_2, v_3 of the velocity vector field $\mathbf{v}$ can be assumed to be random variables. Moreover, Kolmogorov's theory predict that the energy dissipation is strictly positive and does not depend on ν as $\nu \to 0$ which is tested in [Fri95, Chapter 5]. He introduced the concept of energy cascade[29] that improved Richardson's theory[30] in 1922 [Ric07, page 66]. Moreover, under suitable similarity and isotropy assumptions, Kolmogorov derived his famous $-5/3$ law[31] that there exists an inertial range of wavenumber $[k_1, k_2]$ such that the energy spectrum $E(k)$ satisfies (see Figure 1.1 below)

$$E(k) = C\mathcal{E}^{2/3}k^{-5/3} \quad \forall k \in [k_1, k_2], \tag{1.1.35}$$

where C is a dimensionless constant. In addition the authors in [CRL14] used the similarity assumption that there exists $[k_1, k_2]$ such that

$$[k_1, k_2] \subset \left[\frac{2\pi}{\ell}, \frac{2\pi}{\lambda_0}\right], \tag{1.1.36}$$

[28] Taking L^2-scalar product in (1.1.33) by $\mathbf{v}$ and using the facts $((\mathbf{v}\cdot\nabla)\mathbf{v}, \mathbf{v}) = (\nabla p, \mathbf{v}) = 0$.

[29] We know from [CRL14, Chapter 5]: "the energy of large eddies is transferred to smaller eddies, the energy of which is transferred to even smaller eddies, and so on up to a final eddy size λ_0, known as the Kolmogorov scale, with an associated time scale τ_0. Both λ_0 and τ_0 are functions of the viscosity ν and the turbulent dissipation $\mathcal{E}$. Dimensional analysis therefore yields

$$\lambda_0 = \nu^{3/4}\mathcal{E}^{-1/4} \quad \text{and} \quad \tau_0 = \nu^{1/2}\mathcal{E}^{-1/2}."$$

This is also understood as the transfer of energy from the low wavenumbers to the high wavenumbers.

[30] "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity-in the molecular sense."

[31] In fact Kolmogorov did not derive the $-5/3$ law but the $2/3$ law in [Kol41]. That is there exists an inertial range $[r_1, r_2]$, where $0 < r_1 < r_2$, such that

$$\overline{|\mathbf{v}(x+r) - \mathbf{v}(r)|^2} \approx \mathcal{E}^{2/3}k^{2/3} \quad \forall r \in [r_1, r_2].$$

However, the $-5/3$ law can be derived by using the principles in his work. Therefore, this law is always attributed to Kolmogorov.

where ℓ the Prandtl mixing length and $\lambda_0 = \nu^{3/4}\mathcal{E}^{-1/4}$ the Kolmogorov scale and the scale separation assumption[32] to rederived the $-5/3$ law. Then at least formally by (1.1.36) k_2 might goes to ∞ as $\nu \to 0$. The latter limit reminds us to the Euler equations.

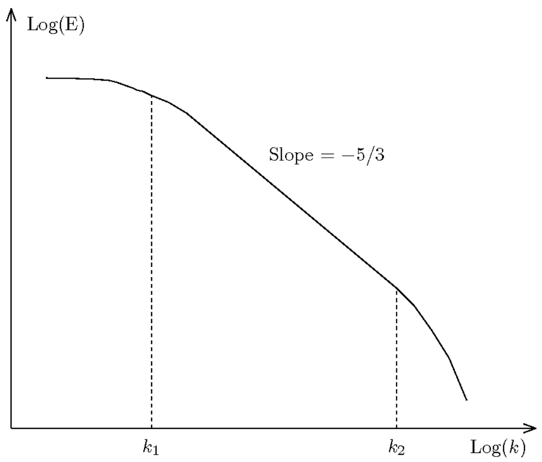

Figure 1.1: The $-5/3$ law (energy spectrum log-log curve).

As already mentioned, the NSE (1.1.33) is formally reduced to the Euler equations as $\nu \to 0$, i.e.,

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{1.1.37}$$

Therefore, the energy balance law of (1.1.37) is given by

$$\frac{d}{dt}e(t) = 0 \quad \forall t > 0, \tag{1.1.38}$$

which implies that the total kinetic energy is preserved if the solutions to (1.1.37) are smooth enough. In his famous paper in 1949 [Ons49], Onsager proposed that the "anomalnous dissipation" for weak solutions to the Euler equations might occur as a consequence of the energy cascade. Based on his idea we can write his prediction down in the modern PDE language as follows:

Onsager's conjecture. *Let $(\mathbf{v}, p)$ be a weak solution to the Euler equations (1.1.3) with*

$$|\mathbf{v}(t,\mathbf{x}) - \mathbf{v}(t,\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}|^\alpha \quad \forall \mathbf{x},\mathbf{y} \in [0,2\pi]^3, t \ge 0,$$

where C does not depend on $\mathbf{x}, \mathbf{y}, t$. Then

(a) *If $\alpha > 1/3$ then $E(t)$ is necessarily a constant;*

[32] That is the mixing length ℓ is locally assumed as a constant and $\lambda_0 << \ell$. Note that for high Reynolds number flows, that is, turbulent flows, this assumption is usually satisfied.

(b) If $\alpha < 1/3$ then there are solutions for which $E(t)$ is strictly decreasing.

It is well-known that the positive part (a) was proved in [Eyi94, CET94, ES06, CCFS08]. However, the negative part (b) is much more complicated and just recently is completed in a series of efforts. It is partially solved in [Buc14, Buc15, BDLIS15, DS17, DLS10, DLS09, DLS13, DLS14, Ise13, Sch93, Shn97a, Shn00] and is finally completed in [Ise18], see also an improvement of the latter one in [BdLSV19]. In the case of bounded domain the solution of part (a) in the conjecture is provided in [BT18, DN18, NN19a, BTW19]. We prefer the readers to [DLS19] for a discussion on the connection between turbulence and geometry. For more discussion on the conjecture we prefer the reader to a survey on convex integration and phenomenologies in turbulence in [BV19a].

1.1.5 Some α-regularization models

A part of this thesis deals with some regularization models to the NSE. For convenience, we say a little bit about these models.

On one hand one of the main aims of α-regularization models is to regularize the nonlinear term in the NSE. This term is usually the source of difficult issues. On the other hand these models are used to provide numerical simulations of turbulence flows. As mentioned in the previous part, sometimes it is impossible to perform a DNS on the NSE. It is due to the Kolmogorov's theory which predicts that simulating incompressible turbulent flows by using the NSE requiring $N \approx \mathcal{O}(Re^{d^2/4})$ degrees of freedom where $d = 2, 3$. This number is too large, for example $N \approx 10^{18}$ to simulate some realistic flows such as geophysical flows by using the NSE. Nowadays, even modern computers are still not powerful enough to do this job.

As mentioned before, the convection term in the NSE will be regularized by a term $N(\mathbf{v}_\alpha)$. More precisely, α-regularization models are given by

$$\begin{cases} \partial_t \mathbf{v}_\alpha + N(\mathbf{v}_\alpha) - \nu\Delta\mathbf{v}_\alpha + \nabla p_\alpha = \mathbf{f}, \\ \nabla \cdot \mathbf{v}_\alpha = 0, \end{cases} \tag{1.1.39}$$

where the regularized convection term given by

$$N(\mathbf{v}_\alpha) = \begin{cases} -\overline{\mathbf{v}}_\alpha \times (\nabla \times \mathbf{v}_\alpha) & \text{Navier-Stokes-}\alpha \text{ model}, \\ (\overline{\mathbf{v}}_\alpha \cdot \nabla)\mathbf{v}_\alpha & \text{Leray-}\alpha \text{ model}, \\ (\mathbf{v}_\alpha \cdot \nabla)\overline{\mathbf{v}}_\alpha & \text{Modified Leray-}\alpha \text{ model}, \\ (\overline{\mathbf{v}}_\alpha \cdot \nabla)\overline{\mathbf{v}}_\alpha & \text{"Simplified Bardina" model}, \end{cases} \tag{1.1.40}$$

here the differential filtered $\overline{\mathbf{v}}_\alpha$ is described by the solution of the following equations

$$-\alpha^2\Delta\overline{\mathbf{v}}_\alpha + \overline{\mathbf{v}}_\alpha = \mathbf{v}_\alpha,$$

subjected to the periodic boundary conditions. It seems that Leray was the first author who has contributed on this direction. In his paper in 1934 he replaced the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ by $(\overline{\mathbf{v}}\cdot\nabla)\mathbf{v}$ where $\overline{\mathbf{v}} = \mathbf{v}\star\rho_\alpha$ where $\star$ denotes the standard convolution and ρ_α be a mollifier, see more details in [Ler34b, page 207]. The main reason of this regularization is that Leray did not succeeded to provide regular solutions to the NSE for all time. The models in (1.1.40) are established and studied by Foias-Holm-Titi [FHT01, FHT02], Cheskidov-Holm-Olson-Titi [CHOT05], Ilyin-Lunasin-Titi [ILT06] and Layton-Lewandowski [LL03, LL06], Cao-Lunasin-Titi [CLT06], respectively. These models are called α-models since these "formally" reduce to the NSE as α tends to zero. After Leray'paper in 1934, the Navier-Stokes-α model (NS-α in the sequel) (also known as the viscous Cammassa-Holm model with its inviscid 1D form introduced in [CH93]) is the first one in the family of α-models. Moreover, the inviscid NS-α –with $\nu = 0$ in the NS-α model (1.1.40)– model was introduced in [HMR98] as a natural generalization of the 1D Cammassa-Holm model. A series of papers by Chen-Foias-Holm-Olson-Titi [CFH+98, CFH+99a, CFH+99b] provided a relation between solutions to the NS-α model and turbulence. In fact the authors were able to establish explicit steady analytical solution of the NS-α which is compared successfully with numerical tests in the case of turbulent flows in pipes and channels.

Note that the Bardina closure model of turbulence was first introduced by Bardina-Ferziger-Reynolds in [BFR80] to perform simulations of the atmosphere. A simplified version of the Bardina's model, was modeled and studied in [LL03, LL06], then in [LB18] the case of whole space. The convection term in this model is designed by $N(\mathbf{u}_\alpha) = \nabla\cdot(\overline{\mathbf{u}_\alpha\otimes\mathbf{u}_\alpha})$. Then almost at the same time Cao-Lunasin-Titi proposed a variant of this model [CLT06], which is the one we will consider and that we still call "Simplified Bardina model" for simplicity. In fact in lectures it is sometimes called the zeroth ADM (Approximate deconvolution modeling) model.

The turbulence models above belong to the class of Large Eddy Simulation models (LES). Note that in the 3D case global solutions to these models are more regular than that of the NSE and are unique as well. It is not in the case of the NSE where in general the global existence of strong solutions is not known so far and only global weak solutions are known. In addition, it has been proved that the solutions of α-models reduce to that of the NSE as the regularization parameter α goes to zero, see for instance in [FHT02, LL06]. More precisely, the authors provided that regular solutions of Navier-Stokes-α and simplified Bardina models converge to a weak solution to the NSE as $\alpha \to 0$.

1.2 Motivations of the thesis

In this section we will explain to the readers the motivations of the study in this thesis. On one hand we provide modeling of new turbulent models and then using analysis tools to prove existence and uniqueness of regular weak solutions in Chapters 2 and 4. On the other hand we estimate the error of some α-regularization models to the NSE in Chapter 3 with giving explicitly the rate of convergence.

1.2.1 Generalized Navier-Stokes-Voigt models

By basic turbulence modeling we take into account the backscatter term in the NSE. It yields an additional term which has a form $-\nabla \cdot (\ell D \partial_t \overline{\mathbf{v}})$. It is known as the Kelvin-Voigt form. The model we get is considered as a generalization of the Navier-Stokes-Voigt equations (NSVE). The latter was first formulated and mathematical studied by Oskolkov, see [Osk73, Osk80], who proved the existence and uniqueness of weak and strong solutions in some particular sense. Several mathematical problems which are related to the NSVE have been studied by Kalantarov-Titi [KT09], Levant-Ramos-Titi [LRT10] and Ramos-Titi [RT10]. The authors provided a relation between the NSVE and turbulence modeling. In addition, in [LT10] Larios-Titi showed the connection between the NSVE and the simplified Bardina's model, which is considered as a Large Eddy Simulation model. In addition an interpretation of the NSVE in terms of approximate deconvolution models has been studied by Berselli-Kim-Rebholz in [BKR16]. The main aim of Chapter 2 is to provide the mathematical theory for the NSVE which as our knowledge was not modeled and studied so far. In addition we also make a connection between the NSVE and the turbulent kinetic energy equation-the equation satisfied by k. More precisely, the existence result to the Navier-Stokes-Voigt turbulent kinetic energy system is provided.

1.2.2 Modeling error of α-turbulent models

As mentioned in the previous part, it is known that $\mathbf{v}_\alpha \to \mathbf{v}$ as $\alpha \to 0$ in some particular sense, where $\mathbf{v}$ and $\mathbf{v}_\alpha$ are solutions to the NSE and α-models in (1.1.40). Therefore, a natural question comes in mind how fast $\mathbf{v}_\alpha$ goes to $\mathbf{v}$ as α tends to zero, where $\mathbf{v}$ denotes solutions to the NSE. Can we explicitly find the rate for the convergence? That is the main motivation of the study in Chapter 3. More precisely, by employing special properties of the Stokes operator and also of the nonlinear term in the 2D periodic case we provide several results to measure the rate of convergence of $\mathbf{v}_\alpha$ to $\mathbf{v}$ as α goes to zero. The rate of convergence is investigated for the last three models above in (1.1.40). We do not study the case of the Navier-Stokes-α since in the 2D case the rotational convection term is not well-defined. The study in this direction comes back to Cao and Titi [CT09] (for four α-models above) in the 2D case and Chen-Guenther-Kim-Thomann-Waymire [CGK+08] (for the Navier-Stokes-α-model), Dunca [Dun18] (for four α-models above) in the 3D case. All authors provided the rate of convergence in the periodic setting. The aim of Chapter 3 is to improve the results provided by Cao-Titi [CT09] in the 2D case. Moreover, we also give remarks on the 3D case where the rate of convergence might not be obtained in the case of existence of finite time blow up.

1.2.3 Rotational forms for LES turbulent models

We apply the mean operator to the rotational NSE which yields a systems satisfied by the mean quantities. However, as usual the rotational term raises a problem of closing the system. The basic turbulence modeling suggests us to consider a new form for the Reynolds stress which has a rotational form. As a result, the obtained system is added a rotational back-scatter term $\nabla \times (\ell^2 \partial_t \overline{\boldsymbol{\omega}})$. Moreover, the eddy viscosity term is suggested to consider in the form $\nabla \times (\kappa \ell^2 |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}})$. Both new terms yield a new LES turbulent model which is modeled and analyzed in Chapter 4.

1.3 Contributions of the thesis

This section aims to summarize the main results of the thesis which will be presented in more details in Chapters 2, 3 and 4 below.

1.3.1 Results in Chapter 2

We will investigate the following systems in Chapter 2: first the generalized Navier-Stokes-Voigt equations (GNSVE)

$$\left\{ \begin{aligned} \partial_t \overline{\mathbf{v}} - \alpha \nabla \cdot (\ell(\mathbf{x}) D \partial_t \overline{\mathbf{v}}) + \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} - \nabla \cdot (\nu_{\text{turb}} D \overline{\mathbf{v}}) + \nabla \pi = \mathbf{f}, \\ \nabla \cdot \overline{\mathbf{v}} = 0, \end{aligned} \right. \tag{1.3.41}$$

and then the Navier-Stokes-Voigt turbulent kinetic energy (NSVTKE) system

$$\left\{ \begin{aligned} \partial_t \overline{\mathbf{v}} - \alpha \nabla \cdot (\ell(\mathbf{x}) D \partial_t \overline{\mathbf{v}}) + \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} - \nabla \cdot (\nu_{\text{turb}}(k) D \overline{\mathbf{v}}) + \nabla \pi = \mathbf{f}, \\ \nabla \cdot \overline{\mathbf{v}} = 0, \\ \partial_t k + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot (\mu_{\text{turb}}(k) \nabla k) - \nu_{\text{turb}}(k) |D \overline{\mathbf{v}}|^2 + (\ell(\mathbf{x}) + \eta)^{-1} k \sqrt{|k|} = 0, \end{aligned} \right. \tag{1.3.42}$$

for some modified mean pressure π and some constants $\alpha, \eta > 0$. Both systems are supplemented by the homogeneous Dirichlet boundary conditions. From basic turbulence modeling we employ a new form of Boussinesq assumption (1.1.15) which allows us to model the systems (1.3.41). The existence and uniqueness of regular weak solutions are provided for the GNSVE (1.3.41). For the NSVTKE (1.3.42), we also provide an existence result but the uniqueness of this kind of weak solutions is not known so far. The proofs are mostly based on the standard Galerkin method, using some compactness results. The main results in Chapter 2 are summarized as follows:

Theorem 1.3.1. *Let $\mathbf{v}_0 \in V$, $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$ and $\nu_{\text{turb}} \in L^\infty([0,\infty[\times\Omega)$ such that $\nu_{\text{turb}} \geq 0$ a.e. in $[0,\infty[\times\Omega$ where Ω is a bounded in $\mathbb{R}^3$. Then, there exists a unique regular-weak solution of the initial boundary value problem (1.3.41) in $[0,T]$, which satisfies for all $t \geq 0$ the energy equality:*

$$E(t)(\alpha,\ell) + \int_0^t \|(2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v}(s)\|^2\, ds = E(0)(\alpha,\ell) + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle\, ds,$$

where $E(t)(\alpha,\ell) := \dfrac{1}{2}\left(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell} D\mathbf{v}(t)\|^2\right)$[33].

Theorem 1.3.2. *Let be given $\mathbf{v}_0 \in V$, $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$ and $0 \leq k_0 \in L^1(\Omega)$. Assume that ν_{turb} and μ_{turb} are given by (2.5.53) and (2.5.54). Then, there exists a weak solution $(\mathbf{v},k)$ to (1.3.42) such that:*

$$\mathbf{v} \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}),$$

and

$$k \in L^\infty(0,T;L^1(\Omega)), \quad k \in \bigcap_{1<p<5/4} L^p(0,T;W^{1,p}(\Omega)).$$

Moreover, $k \geq 0$ a.e. in $(0,T)\times\Omega$.

1.3.2 Results in Chapter 3

Chapter 3 is devoted to study the rate of convergence of weak solutions $\mathbf{v}_\alpha$ to α-models (1.1.40) to that of $\mathbf{v}$ to the NSE. The analysis is investigated in the 2D periodic case. More precisely, we improve the results provided by Cao and Titi in [CT09]. Let $\mathbf{e} := \mathbf{v} - \mathbf{v}_\alpha$ stands for the error. The main results of Chapter 3 are presented as follows:

Theorem 1.3.3. *Let $\Omega = [0,L]^2$ be a periodic domain. Assume that $\mathbf{v}_0 \in \mathcal{P}_\sigma H^1(\Omega)^2$ and $\mathbf{f} \in L^2(\mathbb{R}_+;\mathcal{P}_\sigma L^2(\Omega)^2)$. Then, it holds $\forall s \geq 0$:*

$$\|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2\, dt \leq C_1\alpha^3 \qquad \text{for all } \alpha\text{-models},$$

$$\|\nabla\mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{e}\|^2\, dt \leq \begin{cases} C_2\alpha^2, \\ C_3\alpha^2\left(\log\left(\dfrac{L}{2\pi\alpha}\right)+1\right), \end{cases}$$

for Leray-α and simplified Bardina, modified Leray-α models, respectively, where C_i for $i = 1,2,3$ are time-independent constants and only depending on ν and the initial data, and $\mathcal{P}_\sigma$ denotes the Leray projection.

[33] Note that $\|\cdot\|$ always denote the usual $L^2(\Omega)$ norm.

1.3.3 Results in Chapter 4

In Chapter 4 we will provide the modeling and analysis of the following LES turbulent model

$$\begin{cases} \partial_t \overline{\mathbf{v}} + \operatorname{curl}\,(\ell^2\, \partial_t \overline{\boldsymbol{\omega}}) + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \operatorname{curl}\,(\kappa \ell^2 |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}}) + \nabla \overline{\pi} = \mathbf{f}, \\ \operatorname{div} \overline{\mathbf{v}} = 0. \end{cases} \tag{1.3.43}$$

The main mathematical result is provided in the following, under suitable assumptions on the domain Ω and the mixing length ℓ.

Theorem 1.3.4. *Assume that* $\mathbf{f} \in L^2(0,T; L^2(\Omega)^3)$ *and* $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$. *Then, system (1.3.43) with* $\overline{\mathbf{v}}(0) = \overline{\mathbf{v}}_0$ *in* Ω *and* $\overline{\mathbf{v}} = \mathbf{0}$ *on* $(0,T) \times \partial\Omega$ *has a unique regular-weak solution.*

Note that the proof of Theorem 1.3.4 is more complicated than that of Theorem 1.3.1. More precisely, we first provide the analysis for the ϵ-regularized system then take the limit as $\epsilon \to 0$.

1.4 Organization of the thesis

The thesis is divided into four chapters. Chapter 1 is for the introduction. Chapters 2 to 4 are for presenting the contributions of the thesis. The structure of each chapter will be described below:

Chapter 1 is organized as follows: We start with a general introduction in Section 1.1 where we present the history of the Euler equations and the NSE, a brief state of the art of these equations, an introduction about turbulent flows with its recent developments and some α-turbulent models. Section 1.2 explains why we study the problems in Chapters 2 and 3. Then the results in these chapters will be summarized in Section 1.3. Last, Section 1.4 provides a general structure of the present thesis.

Chapter 2 is organized as follows: Section 2.2 is devoted to modeling and to explain the motivations for the systems of PDE we study. Then, in Section 2.3 we use functional analysis and interpolation theory to provide estimates in various spaces, especially in $H^{1/2}(\Omega)$. The proof of the existence and uniqueness results for the generalized Navier-Stokes-Voigt equations (2.4.27) and then also for model (2.1.1) is developed in Section 2.4. Finally, the compactness result and analysis of the NSTKE-Voigt system is performed in Section 2.5.

Chapter 3 is organized as follows: In Section 3.2 we set the mathematical framework. In Section 3.3 we derive from energy balances uniform-in-time energy (type) estimates for weak solutions of the NSE and for all α-models as well. This is the main step before investigating the rates of convergence in Section 3.4, where we prove the estimates (3.1.11)-(3.1.12).

Section 3.5 is devoted to the study of the convergence rate for the pressure in which the proof of (3.1.13) is provided. In Section 3.6, we make some additional remarks about the 3D case for which the situation is quite different.

Chapter 4 is organized as follows: In Section 4.2 we set the mathematical framework that we use in the whole chapter. Sections 4.3 provide the turbulence modeling where Subsections 4.3.1 and 4.3.2 are devoted to modeling and to explain the motivations for the systems (4.1.4) and (4.1.5). The analysis of the obtained model from the previous section is presented in Section 4.4 where the proofs of the main weighted estimate (4.4.31) and Theorem 4.1.1 are provided in Subsections 4.4.1 and 4.4.2, respectively.

Chapter 2

Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis

"Nothing that enters one from outside can defile that person; but the things that come out from within are what defile."

MK 7:15

Contents

This chapter is devoted to present the results which have been published in [ABLN20].

Abstract: We perform a new modeling procedure for a 3D turbulent fluid, evolving towards a statistical equilibrium. This will result to add to the equations for the mean field $(\mathbf{v}, p)$ the term $-\alpha\nabla \cdot (\ell(\mathbf{x})D\mathbf{v}_t)$, which is of the Kelvin-Voigt form, where the Prandtl mixing

length $\ell = \ell(\mathbf{x})$ is not constant and vanishes at the solid walls. We get estimates for mean velocity $\mathbf{v}$ in $L_t^\infty H_{\mathbf{x}}^1 \cap W_t^{1,2} H_{\mathbf{x}}^{1/2}$, that allow us to prove the existence and uniqueness of a regular-weak solutions $(\mathbf{v}, p)$ to the resulting system, for a given fixed eddy viscosity. We then prove a structural compactness result that highlights the robustness of the model. This allows us to consider Reynolds averaged equations and pass to the limit in the quadratic source term in the equation for the turbulent kinetic energy k. This yields the existence of a weak solution to the corresponding Navier-Stokes turbulent kinetic energy system satisfied by $(\mathbf{v}, p, k)$.

Key words: Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.
2010 MSC: 76D05, 35Q35, 76F65, 76D03, 35Q30.

2.1 Introduction

The purpose of this chapter is to model incompressible turbulent flows as generalized viscoelastic materials involving the Prandtl mixing length ℓ (see e.g. Prandtl in [Pra10]), to show the existence and uniqueness of regular-weak solutions to the resulting system of Partial Differential Equations (PDE),

$$\left\{ \begin{aligned} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}} D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{aligned} \right. \tag{2.1.1}$$

for a given turbulent viscosity (eddy viscosity) ν_{turb}, when the motion takes place in a bounded smooth domain Ω and Dirichlet conditions are given at the boundary. We then study the existence weak solutions to the corresponding NSTKE[1] system,

$$\left\{ \begin{aligned} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}}(k) D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ k_t + \mathbf{v} \cdot \nabla k - \nabla \cdot (\mu_{\text{turb}}(k) \nabla k) - \nu_{\text{turb}}(k) |D\mathbf{v}|^2 + (\ell + \eta)^{-1} k \sqrt{|k|} = 0, \end{aligned} \right. \tag{2.1.2}$$

where, to fix the notation,

- $\mathbf{v}$ is the mean velocity[2], $\mathbf{v}_t = \dfrac{\partial \mathbf{v}}{\partial t}$;
- $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ is the deformation stress;
- p is the modified mean pressure;

[1] Here, RANS = Reynolds Averaged Navier-Stokes and NSTKE = Navier-Stokes-Turbulent-Kinetic-Energy. NSTKE model is a specific RANS model.

[2] Usually, the mean (or averaged) velocity is denoted by $\overline{\mathbf{v}}$. Throughout the paper we omit the over-line for simplicity, except in Section 2.2, devoted to turbulence modelling.

- k is the Turbulent Kinetic Energy (TKE);
- $\nu > 0$ is the kinematic viscosity, ν_{turb} the eddy viscosity;
- μ_{turb} is the eddy diffusion and $\eta > 0$ is a small constant;
- the length scale α is that of the boundary layer, given by the relation

$$\alpha = \frac{\nu}{u_\star}, \tag{2.1.3}$$

here $u_\star$ is the so called friction velocity (see [CRL14]);
- $\mathbf{f}$ is a given source term.

As usual, the systems are set in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. The mixing length $\ell = \ell(\mathbf{x}) \geq 0$ is defined over Ω and, according to well known physical laws (see (2.2.12) and (2.2.13) below), $0 \leq \ell \in C^1(\overline{\Omega})$ and vanishes at the boundary $\Gamma = \partial\Omega$ as follows:

$$\ell(\mathbf{x}) \simeq d(\mathbf{x}, \Gamma) = \rho(\mathbf{x}), \quad \text{when} \quad \mathbf{x} \to \Gamma,\, \mathbf{x} \in \Omega, \tag{2.1.4}$$

where $d(\mathbf{x}, \Gamma)$ denotes the distance of the point $\mathbf{x}$ from the boundary.

Model (2.1.1) is close to viscoelastic materials models, given by the Kelvin-Voigt relation:

$$\boldsymbol{\sigma} = E\,\boldsymbol{\varepsilon} + \eta\,\boldsymbol{\varepsilon}_t, \tag{2.1.5}$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor and $\boldsymbol{\varepsilon}$ the strain-rate tensor. In this case, E is the modulus of elasticity and η the viscosity (see for instance Germain [Ger62] or Gurtin [Gur81]). In fluid mechanics, $\boldsymbol{\varepsilon} = D\mathbf{v}$, and this model is used to describe some non Newtonian fluids, such as lubricants. For such flows, the law (2.1.5) becomes

$$\boldsymbol{\sigma} = -p\,\mathrm{Id} + \nu D\mathbf{v} + \gamma^2 D\mathbf{v}_t \qquad \nu,\, \gamma \in \mathbb{R}^+,$$

that yields the incompressible Navier-Stokes-Voigt equations:

$$\begin{cases} \mathbf{v}_t - \gamma^2 \Delta \mathbf{v}_t + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \hfill \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.1.6}$$

Mathematical investigations about system (2.1.6) were first carried out by Oskolkov, see [Osk73, Osk80], who proved the existence and uniqueness of weak and strong solutions in some particular sense. Then, several mathematical problems concerning (2.1.6) have been studied by Titi *et al.* [KT09, LRT10, RT10], making a clear relation between Navier-Stokes-Voigt and turbulence modeling. In addition, in [LT10] Larios-Titi showed the connection between the Navier-Stokes-Voigt equations and the simplified Bardina's model introduced in [LL06], designed as a Large Eddy Simulation model. In addition an interpretation of the Navier-Stokes-Voigt equations in terms of approximate deconvolution models is also given in [BKR16].

In this chapter we connect the Prandtl-Smagorinsky's model with the Turbulent Kinetic Energy (TKE) model, to calculate the eddy viscosity ν_{turb}. To make it clear, let $\boldsymbol{\sigma}^{(\mathrm{R})}$ denote the Reynolds stress. We will show how, combining the energy inequality with the equation satisfied by k (without any closure assumption), we are led to set –in certain specific regimes, such as the convergence to stable statistical states see (2.2.20)– the following constitutive law

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\alpha\ell\, D\mathbf{v}_t - \nu_{\text{turb}} D\mathbf{v} + \frac{2}{3}k\,\mathrm{Id},$$

instead of the usual one

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_{\text{turb}} D\mathbf{v} + \frac{2}{3}k\,\mathrm{Id}, \tag{2.1.7}$$

associated with the classical Boussinesq assumption. This yields the PDE system (2.1.1) including the term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$, and then also the NSTKE system (2.1.2) after having performed the usual closure procedure about k, where $\nu_{\text{turb}} = \nu_{\text{turb}}(k) = \ell\sqrt{k}$.

Turning to the analysis of the systems, we observe that –according to assumption (2.1.4) about the mixing length ℓ– the additional generalized Kelvin-Voigt term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$ enforces for the equations a natural functional structure in $H^{1/2}(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1/2}$, cf. Lions & Magenes [LM72], which is a critical scaling-invariant space for the Navier-Stokes equations. In particular, we obtain for the velocity sharp estimates in $W^{1,2}(0,T; H^{1/2}(\Omega)^3)$, as well as in $L^\infty(0,T; H^1_0(\Omega)^3)$. We are then able to prove the existence and uniqueness of regular-weak solution to (2.1.1) (see Theorem 2.4.1 and the generalisation in Theorem 2.4.2).

However, we believe that the most interesting result of this paper is the compactness result we prove in Lemma 2.5.1. We consider an eddy viscosities sequence $(\nu^n_{\text{turb}})_{N\in\mathbb{N}}$ which is bounded in $L^\infty([0,\infty[\times\Omega)$ and in addition converges *a.e.* to ν_{turb} in $[0,\infty[\times\Omega$ as $N\to\infty$. We also show that the corresponding regular-weak sequence of solution $(\mathbf{v}^n)_{N\in\mathbb{N}}$ converges, in some sense, to the regular-weak solution $\mathbf{v}$ of the limit problem with ν_{turb} as eddy viscosity. Moreover, we get the convergence of the energies, that is $\nu^n_{\text{turb}}|D\mathbf{v}^n|^2 \to \nu_{\text{turb}}|D\mathbf{v}|^2$ in the sense of the measures.

This compactness result allows us to prove the existence of a solution to the NSTKE-Voigt system (2.1.2) (see Theorem 2.5.1 below). We stress that the usual system coupling $\mathbf{v}$, p. and k only yields a variational inequality for k when passing to the limit in the equations, because of the lack of strong convergence of the energies (see [CRL14, Lew97a]). This observation makes Theorem 2.5.1 a relevant and original result.

Plan of the chapter. This chapter is organized as follows: Section 2.2 is devoted to modeling and to explain the motivations for the systems of PDE we study. Then, in Section 2.3 we use functional analysis and interpolation theory to provide estimates in various spaces, especially in $H^{1/2}(\Omega)$. The proof of the existence and uniqueness results for the generalized Navier-Stokes-Voigt equations (2.4.27) and then also to the model (2.1.1) is developed in Section 2.4. Finally, the compactness result and analysis of the NSTKE-Voigt system is performed in Section 2.5.

2.2 Kelvin-Voigt modeling for turbulent flows

In this section (and only in this section) $\mathbf{v}$ and p denote the velocity and pressure of the fluid respectively (and not the mean fields unlike in the rest of the paper). Hence, the couple $(\mathbf{v}, p)$ solves the Navier-Stokes equations (NSE),

$$\begin{cases} \mathbf{v}_t + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \tag{2.2.8}$$

We first recall some results about basic turbulence modeling to derive the equation for the mean $\overline{\mathbf{v}}$ and to define the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$. Then, we show how –when simultaneously using the Prandtl-Smagorinsky, the turbulent kinetic energy models, and the equation satisfied by the TKE– we get the additional term $-\alpha \nabla \cdot (\ell D \overline{\mathbf{v}}_t)$ in the equation for $\overline{\mathbf{v}}$. This occurs in specific regimes, such as the convergence to a statistical equilibrium (see Remark 2.2.2).

We wish to mention that a very close modeling process has been previously performed in Rong, Layton, and Zhao [RLZ19]. The latter paper gave us some inspiration for the modelling procedure we develop here. One main difference is that we study the TKE equations, while in their paper, Rong, Layton, and Zhao considered a rotational structure, without involving the equation for the turbulent kinetic energy. Moreover, they were looking at back-scatter terms, so that our point of view and interpretation are –at the very end– rather different.

2.2.1 Recalls of basic turbulence modeling

According to the Reynolds decomposition, $\mathbf{v}$ and p are decomposed as the sum of their mean and fluctuation (cf. [BIL06, CRL14])

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad \text{and} \quad p = \overline{p} + p',$$

where the averaging filter is linear, commutes with any differential operator (namely $D\overline{\psi} = \overline{D\psi}$), and it is idempotent (that is $\overline{\overline{\psi}} = \overline{\psi}$). From these assumptions, one gets the relation

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \boldsymbol{\sigma}^{(\mathrm{R})},$$

where the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ is given by

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \overline{\mathbf{v}' \otimes \mathbf{v}'}.$$

Therefore, applying the mean operator to the NSE (4.1.1) yields

$$\begin{cases} \overline{\mathbf{v}}_t + \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \nabla \overline{p} = \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} = 0. \end{cases} \tag{2.2.9}$$

To "close" (2.2.9), one must express $\boldsymbol{\sigma}^{(\mathrm{R})}$ in terms of mean quantities. As we already said in the introduction, the Boussinesq assumption [Bou77] yields

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_{\mathrm{turb}} D\overline{\mathbf{v}} + \frac{2}{3} k \, \mathrm{Id},$$

where we recall that ν_{turb} is the eddy viscosity, $k = \frac{1}{2}\overline{|\mathbf{v}'|^2}$ the turbulent kinetic energy (TKE), and $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ the deformation tensor.

One main challenge in turbulence modelling is the determination of ν_{turb}. In what follows, we combine the Prandtl-Smagorinsky's model

$$\nu_{\mathrm{turb}} = \ell \sqrt{\alpha \ell} |D\overline{\mathbf{v}}|, \tag{2.2.10}$$

where the boundary layer length scale α is given by (2.1.3), and the NSTKE model with ν_{turb} is given by

$$\nu_{\mathrm{turb}} = \ell \sqrt{k}. \tag{2.2.11}$$

Dimensionless constants may be involved in the above equations. We have set them equal to 1 for the sake of simplicity.

Observe that both models involve the Prandtl mixing length ℓ. In the case of a flow over a plate $\Omega = \mathbb{R}^2 \times \{z > 0\}$, one finds in Obukhov [Obu46] the following law

$$\ell = \ell(z) = \kappa z, \tag{2.2.12}$$

where $\kappa \in [0.35, 0.42]$ is the Von Kármán constant. A more sophisticated formula (however very popular especially for the use in the computation of the turbulent channel flow) can be found in Van Driest [VD56]:

$$\ell = \ell(z) = \kappa z(1 - e^{-z/A}), \tag{2.2.13}$$

where A depends on the oscillations of the plate and ν. Alternative formulas are provided in [LPMP18]. In all cases, ℓ satisfies the law (2.1.4) and in particular it vanishes at the solid boundary of the domain.

2.2.2 Modelling process

We start from the natural energy inequality[3] deduced from the equation (2.2.9) by integration by parts, at any time positive time t (assuming as usual that solutions are smooth enough to carry on all the calculations)

$$\langle \overline{\mathbf{v}}_t, \overline{\mathbf{v}} \rangle + \nu \|\nabla \overline{\mathbf{v}}(t)\|^2 + \langle \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}(t) \rangle \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t) \rangle. \tag{2.2.14}$$

We aim to formulate the contribution of the term

$$\mathscr{T}(t) = \langle \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}(t) \rangle,$$

by means of mean quantities. This which will be deduced by using the equation satisfied by k (see [CRL14, Sec. 4.4.1])

$$\partial_t k + \nabla \cdot (\overline{\mathbf{v}}\, k + \overline{e' \mathbf{v}'}) = -\boldsymbol{\sigma}^{(\mathrm{R})} : \nabla \overline{\mathbf{v}} - \varepsilon + \overline{\mathbf{f}' \cdot \mathbf{v}'},$$

where $e = k + e' = \frac{1}{2}|\mathbf{v}'|^2$ denotes the kinetic energy of the fluctuations, and ε is the turbulent dissipation,

$$\varepsilon := \nu \overline{|D\mathbf{v}'|^2}.$$

Integrating formally the equation satisfied by k in space –leaving apart eventual boundary condition issues– leads to

$$\frac{d}{dt} \int_\Omega k(t) = \mathscr{T}(t) - \int_\Omega \varepsilon(t) + \overline{\langle \mathbf{f}', \mathbf{v}' \rangle},$$

that we insert in the inequality (2.2.14) to obtain

$$\langle \overline{\mathbf{v}}_t, \overline{\mathbf{v}} \rangle + \frac{d}{dt} \int_\Omega k + \nu \|\nabla \overline{\mathbf{v}}(t)\|^2 + \int_\Omega \varepsilon(t) \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t) \rangle + \overline{\langle \mathbf{f}'(t), \mathbf{v}'(t) \rangle}. \tag{2.2.15}$$

In order to eliminate the term $\frac{d}{dt} \int_\Omega k$ from (2.2.15), we enforce equality between the Prandtl-Smagorinsky's model (2.2.10) and the NSTKE one (2.2.11), which leads to the closure equality

$$k = \alpha \ell |D\overline{\mathbf{v}}|^2. \tag{2.2.16}$$

Then, by using (2.2.16), we get the formal identity

$$\frac{d}{dt} \int_\Omega k = \alpha \frac{d}{dt} \int_\Omega \ell |D\overline{\mathbf{v}}|^2 = -2 \langle \alpha \nabla \cdot (\ell D\overline{\mathbf{v}}_t), \overline{\mathbf{v}} \rangle. \tag{2.2.17}$$

Finally, we combine (2.2.15) with (2.2.17), which leads to the inequality

$$\langle \overline{\mathbf{v}}_t - 2\alpha \nabla \cdot (\ell D\overline{\mathbf{v}}_t) - \nu \Delta \overline{\mathbf{v}}, \overline{\mathbf{v}} \rangle + \int_\Omega \varepsilon(t) \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t) \rangle + \overline{\langle \mathbf{f}'(t), \mathbf{v}'(t) \rangle}. \tag{2.2.18}$$

[3] We use $\| \cdot \|$ for the L^2-norm in this section and $\langle\, .\, ,\, .\, \rangle$ for the associated scalar product.

Comparing (2.2.14) with (2.2.18), suggests to put (up to a redefinition of the parameter α)

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\alpha\ell D\overline{\mathbf{v}}_t - \nu_{\mathrm{turb}} D\overline{\mathbf{v}} + \frac{2}{3} k\,\mathrm{Id},$$

and yields the following energy inequality

$$\frac{1}{2}\frac{d}{dt}(\|\overline{\mathbf{v}}(t)\|^2 + \alpha\|\sqrt{\ell}D\overline{\mathbf{v}}\|^2) + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 \le \langle\overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t)\rangle. \tag{2.2.19}$$

Comparing inequalities (2.2.19) and (2.2.18), we see that all this makes sense when:

$$\|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 + \overline{\langle\mathbf{f}'(t), \mathbf{v}'(t)\rangle} \le \|\sqrt{\varepsilon(t)}\|^2, \tag{2.2.20}$$

and in this case the system satisfied by $\overline{\mathbf{v}}$ becomes

$$\begin{cases} \overline{\mathbf{v}}_t - \alpha\nabla\cdot(\ell D\overline{\mathbf{v}}_t) + \nabla\cdot(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}) - \nu\Delta\overline{\mathbf{v}} - \nabla\cdot(\nu_{\mathrm{turb}}D\overline{\mathbf{v}}) + \nabla\left(\overline{p} + \frac{2}{3}k\right) = \overline{\mathbf{f}}, \\ \nabla\cdot\overline{\mathbf{v}} = 0. \end{cases}$$

Remark 2.2.1. *When $\ell(\mathbf{x})$ is constant and equal to 2α (to set ideas), and as $\nabla\cdot\overline{\mathbf{v}}_t = 0$, we have $\alpha\nabla\cdot(\ell(\mathbf{x})D\overline{\mathbf{v}}_t) = \alpha^2\Delta\overline{\mathbf{v}}_t$. Therefore, we get in this case the usual (constant coefficients) Kelvin-Voigt term involved in Equation (2.1.6). Observe that in our model the linear differential operator $-\nabla\cdot(\ell(\mathbf{x})\,D)$ turns out to be degenerate at the boundary, hence different mathematical tools need to the invoked.*

Remark 2.2.2. *Condition (2.2.20) asks for some comments. To see if it can be justified, let us take a constant source term $\mathbf{f}(t) = \mathbf{f}$, without turbulent fluctuation, which means $\mathbf{f}' = \mathbf{0}$. In this case relation (2.2.20) simplifies to*

$$\|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 \le \|\sqrt{\varepsilon(t)}\|^2. \tag{2.2.21}$$

The usual closed equation for k is

$$k_t + \overline{\mathbf{v}}\cdot\nabla k - \nabla\cdot(\mu_{\mathrm{turb}}\nabla k) = \nu_{\mathrm{turb}}|D\overline{\mathbf{v}}|^2 - \varepsilon,$$

which gives (while ignoring possible boundary conditions)

$$\frac{d}{dt}\int_\Omega k = \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 - \|\sqrt{\varepsilon(t)}\|^2.$$

Therefore, (2.2.21) indicates a decrease of TKE, which means a decrease of the turbulence, towards a laminar state, or a stable statistical equilibrium, such as a grid turbulence.

2.3 Functional setting and estimate

The analysis of system (2.1.1) yields immediately standard a priori estimates in $L_t^\infty L_{\mathbf{x}}^2$ and $L_t^2 H_{\mathbf{x}}^1$, taking the solution itself as test function. The question is whether the Voigt term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$ provides additional regularity as in the case with $0 < l =$ const., see Larios and Titi [LT10]. The issue is the degeneration of the mixing length ℓ at the boundary, according to (2.3.22) below. The purpose of this section is to derive from the interpolation theory a general estimate, that will later on enable us to show that the term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$ yields additional $W^{1,2}(0,T;H^{1/2})$ and $L^\infty(0,T;H^1)$ regularity.

2.3.1 Framework and preliminaries

As usual in mathematical fluid dynamics, we use the following spaces,

$$\begin{aligned}
\mathcal{V} &= \left\{\boldsymbol{\varphi}\in\mathcal{D}(\Omega)^3,\ \nabla\cdot\boldsymbol{\varphi}=0\ \text{ in }\Omega\right\},\\
H &= \left\{\mathbf{v}\in L^2(\Omega)^3,\ \nabla\cdot\mathbf{v}=0\ \text{ in }\Omega,\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on }\Gamma\right\},\\
V &= \left\{\mathbf{v}\in H_0^1(\Omega)^3,\ \nabla\cdot\mathbf{v}=0\ \text{ in }\Omega\right\},
\end{aligned}$$

where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ with the topology used in distribution theory and we recall that $\mathcal{V}$ is dense in H and V for their respective topologies, see Girault and Raviart [GR86]. Here $L^2(\Omega)$ and $H_0^1(\Omega)$ stand for the usual Lebesgue and Sobolev spaces.

Throughout the rest of the paper, the mixing length $\ell = \ell(\mathbf{x}) \in C^1(\overline{\Omega})$ is such that

$$\begin{cases}\forall\, K\subset\Omega, \quad K \text{ compact}, \quad \inf_K \ell > 0,\\ \ell(\mathbf{x}) \simeq d(\mathbf{x},\Gamma) = \rho(\mathbf{x}), \quad \text{when} \quad \mathbf{x}\to\Gamma, \text{ for } \mathbf{x}\in\Omega.\end{cases} \tag{2.3.22}$$

According to the classical interpolation theory we recall that

$$H^{1/2}(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1/2},$$

and also, recalling the behavior of ℓ at the boundary from (2.3.22), we can introduce the Lions-Magenes space

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2} = \left\{u\in H^{1/2}(\Omega),\ \text{ s.t. }\ \ell^{-1/2}u\in L^2(\Omega)\right\},$$

cf. [LM72, Ch. 1]. In the following we will consider the following Hilbert space

$$V_{1/2} = \left\{\mathbf{v}\in H^{1/2}(\Omega)^3\,;\ \nabla\cdot\mathbf{v}=0 \text{ in }\Omega \quad\text{and}\quad \mathbf{v}\cdot\mathbf{n}=0\text{ on }\Gamma\right\}$$

equipped with the norm of $H^{1/2}(\Omega)^3$.
Finally, recall that when Ω is connected, the differential operator $D = \frac{\nabla+\nabla^t}{2}$ is well defined over $H^s(\Omega)^3$ whatever $s\geq 0$. In addition (see Nečas and I. Hlaváček [NH80]),

$$\mathrm{K} := \operatorname{Ker} D = \left\{\mathbf{v}\in H^s(\Omega)^3 \ \text{ s.t. } \exists\,(\mathbf{a},\mathbf{b})\in\mathbb{R}^3\times\mathbb{R}^3;\ \mathbf{v}(\mathbf{x}) = \mathbf{b}\times\mathbf{x}+\mathbf{a}\right\},$$

and we recall the following Korn inequality

$$\exists\, c(\Omega) : \ \forall\, \mathbf{v} \in H^1(\Omega)^3, \quad \|\mathbf{v}\|_{H^1(\Omega)^3/\mathrm{K}} \leq C\|D\mathbf{v}\|_{L^2(\Omega)^9}, \tag{2.3.23}$$

where for any given Banach space B and any closed subspace $E \subset B$, B/E denotes the quotient space. Moreover, for any $\mathbf{v} \in H^1_0(\Omega)^3$, we have $\|\mathbf{v}\|_{H^1(\Omega)^3} \leq C\|D\mathbf{v}\|_{L^2(\Omega)^9}$, because in this case the kernel K is reduced to $\mathbf{0}$.

2.3.2 Main general estimate

We deduce now the most relevant inequality, which will be used to prove a priori estimates for the generalized Voigt model, when using the solution itself as test function.

Theorem 2.3.1. *Let $\mathbf{v} \in \mathcal{D}'(\Omega)^3$ be such that $\sqrt{\ell}D\mathbf{v} \in L^2(\Omega)^9$. Then $\mathbf{v} \in H^{1/2}(\Omega)^3$ and there exists a constant $C = C(\Omega)$ such that*

$$\|\mathbf{v}\|_{H^{1/2}(\Omega)^3/\mathrm{K}} \leq C\|\sqrt{\ell}D\mathbf{v}\|_{L^2(\Omega)^9}. \tag{2.3.24}$$

In particular,

$$W = \left\{\mathbf{v} \in H;\ \sqrt{\ell}D\mathbf{v} \in L^2(\Omega)^9\right\} \hookrightarrow V_{1/2}, \tag{2.3.25}$$

with continuous embedding.

Proof. We argue in two steps.

Step 1. Let $\mathbf{v} \in \mathcal{D}'(\Omega)^3$ such that $\sqrt{\ell}D\mathbf{v} \in L^2(\Omega)^9$ and $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^9$. As $D\mathbf{v} \in L^2_{\mathrm{loc}}(\Omega)^3$, then we have

$$|\langle D\mathbf{v}, \boldsymbol{\varphi}\rangle| = \left|\int_\Omega \sqrt{\ell}\, D\mathbf{v} : \frac{\boldsymbol{\varphi}}{\sqrt{\ell}}\right| \leq C\|\sqrt{\ell}\, D\mathbf{v}\|_{L^2(\Omega)^9}\|\boldsymbol{\varphi}\|_{H^{1/2}_{00}(\Omega)^9}.$$

Because of the density of $\mathcal{D}(\Omega)$ in $H^{1/2}_{00}(\Omega)$, this shows that $D\mathbf{v} \in \left[H^{1/2}_{00}(\Omega)^9\right]'$ with the estimate

$$\|D\mathbf{v}\|_{\left[H^{1/2}_{00}(\Omega)^9\right]'} \leq C\|\sqrt{\ell}\, D\mathbf{v}\|_{L^2(\Omega)^9}. \tag{2.3.26}$$

Step 2. According to [ACGK06, CMM18], we have

$$\forall\, \mathbf{v} \in L^2(\Omega)^3, \quad \|\mathbf{v}\|_{L^2(\Omega)^3/\mathrm{K}} \leq C\|D\mathbf{v}\|_{H^{-1}(\Omega)^9}.$$

Therefore, we deduce from classical interpolation theorems and from the following identities (see in [LM72]),

$$[H^1(\Omega)^3/\mathrm{K}, L^2(\Omega)^3/\mathrm{K}]_{1/2} = H^{1/2}(\Omega)^3/\mathrm{K}, \quad \text{and} \quad [L^2(\Omega), H^{-1}(\Omega)]_{1/2} = [H^{1/2}_{00}(\Omega)]',$$

the inequality

$$\|\mathbf{v}\|_{H^{1/2}(\Omega)^3/\mathrm{K}} \leq C\|D\mathbf{v}\|_{\left[H^{1/2}_{00}(\Omega)^9\right]'}.$$

Hence the estimate (2.3.24) follows by using (2.3.26) and obviously also the embedding (2.3.25). $\square$

2.4 Well-posedness for the generalized Navier-Stokes-Voigt equations

In this section we start with the analysis of system (2.1.1) without eddy viscosity, that means $\nu_{\text{turb}} = 0$. This is done both for simplicity of presentation and to highlight the role of the generalized Voigt term. The resulting system, called *generalized Navier-Stokes-Voigt equations*, is the following:

$$\begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } (0,T)\times\Omega, \\ \mathbf{v}|_\Gamma = 0 & \text{on } (0,T)\times\Gamma, \\ \mathbf{v}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega, \end{cases} \tag{2.4.27}$$

which is set in $Q_T = (0,T)\times\Omega$, where Ω is a given Lipschitz bounded domain in $\mathbb{R}^3$ with its boundary $\Gamma = \partial\Omega$, T a fixed positive time[4], and ℓ satisfies (2.1.4). The main results of this section are the existence and uniqueness of regular-weak solutions (see Definition 2.4.1 below), when the initial velocity $\mathbf{v}_0 \in V$.

Throughout the rest of the paper, the L^2-norm of a given u is simply denoted by $\|u\|$, while $\|\cdot\|_p$ and $\|\cdot\|_{s,p}$ denote the standard $L^p(\Omega)$ and $W^{s,p}(\Omega)$ norms, respectively.

2.4.1 Strong solutions

This aim of this subsection is to prove that given a positive finite time T, any strong (classical) solution $\mathbf{v}$ of (2.4.27) has natural bounds in $L^\infty(0,T;V)\cap W^{1,2}(0,T;V_{1/2})$ derived from energy balances, showing that the term $-\alpha\nabla\cdot(\ell(\mathbf{x})\,D\mathbf{v}_t)$ –despite the degeneracy at the boundary– brings a strong regularizing effect on the system. In particular, the generalized Voigt term provides stronger a priori estimate when compared to the usual (non regularized) Navier-Stokes equations, since it allows to show bounds in critical scaling-invariant spaces *à la* Kato-Fujita. These estimates are essential for proving the existence result of the next subsection.

[4]Remind that when $\nabla\cdot\mathbf{v} = 0$, then $\nabla\cdot(\mathbf{v}\otimes\mathbf{v}) = (\mathbf{v}\cdot\nabla)\,\mathbf{v}$. We use either of these forms without necessarily warning, depending on the situation.

Following [Ler34b, Lew19], when considering $\mathbf{v}_0 \in V \cap C(\overline{\Omega})^3$, we say that $(\mathbf{v}, p)$ is a strong solution to (2.4.27) over $Q_T = [0, T] \times \Omega$, if

- $\forall\, \tau < T$, $\mathbf{v} \in C^2(Q_\tau)^3$, $p \in C^1(Q_\tau)$, and they satisfy the relations ((2.4.27), i), ii)) in the classical sense in $Q_\tau = [0, \tau] \times \Omega$,
- $\mathbf{v}(t, \cdot) \in C(\overline{\Omega})^3$ for all $t < T$, and $\mathbf{v}(t, \cdot) = 0$ on Γ,
- $\mathbf{v}(t, \cdot)$ uniformly converges to $\mathbf{v}_0$ as $t \to 0^+$.

Remark 2.4.1. *We frequently talk about the velocity* $\mathbf{v}$ *as a strong solution, without mentioning the pressure* p*. This means that we have implicitly projected the system over divergence-free vector fields by the Leray projector, which eliminates the pressure. The pressure can be recovered via the De Rham procedure (see e.g. Temam [Tem01]).*

Remark 2.4.2. *We say that a strong solution* $\mathbf{v}$ *of* (2.4.27) *has a singularity at a given time* $0 < T^\star < \infty$ *if* $\|\mathbf{v}(t)\|_\infty \to \infty$ *as* $t \to T^\star$*,* $t < T^\star$*. At this stage, we are not able to show that any strong solution has no singularity. We do not even know if there exist strong solutions, which remains an open problem.*

The estimates we get are based on the following non standard version of Gronwall's Lemma, the proof of which is carried out, *e.g.*, in Emmrich [Emm99].

Lemma 2.4.1. *Let* $\lambda \in L^1(0, T)$*, with* $\lambda(t) \geq 0$ *for almost all* $t \in [0, T]$*, let* $g \in C([0, T])$ *be a non-decreasing function, and let* $f \in L^\infty(0, T)$*, such that* $\forall\, t \in [0, T]$*, it holds*

$$f(t) \leq g(t) + \int_0^t \lambda(s) f(s)\, ds.$$

Then, we have

$$f(t) \leq g(t) \exp\left(\int_0^t \lambda(s)\, ds \right).$$

In this subsection we assume that $\mathbf{f}(t) = \mathbf{f} \in C(\overline{\Omega})$ does not depend on t, and we denote by F either $\|\mathbf{f}\|_{-1,2}^2$ or $\|\mathbf{f}\|^2$ so far non risk of confusion occurs, and C denotes any constant (normally $C_P\|\mathbf{f}\| \leq \|\mathbf{f}\|_{-1,2}$, C_P being the Poincaré's constant). Among many choices for the functional spaces of the source term (see also the discussion in the next subsection, where different choices are considered), this one has the advantage that it yields a clear and neat bound of the growth of the r.h.s in the estimates for statistical equilibrium.

The main result of this subsection is the following.

Lemma 2.4.2. *Let* $\mathbf{v}$ *be a strong solution of* (2.4.27) *in* $Q_T = [0,T] \times \Omega$*, for a* $T > 0$*. Then, the following estimates hold true for all* $s \in [0,T[$*:*

$$\|\mathbf{v}(s)\|^2_{1/2,2} + \nu \int_0^s \|\nabla \mathbf{v}(t)\|^2\,dt \le C\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right), \tag{2.4.28}$$

$$\nu\|\nabla \mathbf{v}(s)\|^2 \le \left(\nu\|\nabla \mathbf{v}_0\|^2 + Fs\right)\exp\left\{\frac{C}{\alpha\nu^2}\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right)\right\}, \tag{2.4.29}$$

and

$$\begin{aligned}C\alpha\int_0^s \|\mathbf{v}_t(t)\|^2_{1/2,2}\,dt + \alpha\int_0^s \|\sqrt{\ell}D\mathbf{v}_t(t)\|^2\,dt &\le Fs + \nu\|\nabla \mathbf{v}_0\|^2 \\ + \frac{C}{\alpha\nu^2}\left(\nu\|\nabla \mathbf{v}_0\|^2 + FT\right)&\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right)\exp\left\{\frac{C}{\alpha\nu^2}\left(\frac{FT}{\nu} + E(0)(\alpha,\ell)\right)\right\},\end{aligned} \tag{2.4.30}$$

where $2E(0)(\alpha,\ell) = \|\mathbf{v}_0\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}_0\|^2$*. In particular,* $\mathbf{v}$ *has natural bounds in* $L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2})$ *and* $\sqrt{\ell}D\mathbf{v}_t \in L^2(0,T;L^2(\Omega)^9)$*.*

Proof. We take the dot product of ((2.4.27), i)) by $\mathbf{v}$. We integrate by parts and we use the identity $\langle(\mathbf{v}\cdot\nabla)\,\mathbf{v},\mathbf{v}\rangle = 0$. These calculations are justified because $\mathbf{v}$ is a strong solution, and this gives for all $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2) + \nu\|\nabla \mathbf{v}(t)\|^2 = \langle \mathbf{f}(t),\mathbf{v}(t)\rangle \le \frac{F}{2\nu} + \frac{\nu}{2}\|\nabla \mathbf{v}(t)\|^2,$$

hence (2.4.28) follows after integration in time by using (2.3.24), the fact that the norm of $V_{1/2}$ is that inherited from $H^{1/2}(\Omega)^3$, and $H^{1/2}(\Omega)^3 \hookrightarrow L^2(\Omega)^3$ with continuous dense injection.

We next take the dot product of ((2.4.27), i)) by $\mathbf{v}_t$. In this case the non-linear term brings a contribution (in this new energy budget) given by

$$\|\mathbf{v}_t(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{v}(t)\|^2 = \langle \mathbf{f}(t),\mathbf{v}_t(t)\rangle - \langle(\mathbf{v}\cdot\nabla)\,\mathbf{v},\mathbf{v}_t\rangle(t),$$

As we can estimate

$$|\langle \mathbf{f}(t),\mathbf{v}_t(t)\rangle| \le \frac{F}{2} + \frac{1}{2}\|\mathbf{v}_t(t)\|^2,$$

we obtain by using (2.3.24), keeping half of the contribution of the term $\alpha\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2$,

$$\frac{1}{2}\|\mathbf{v}_t(t)\|^2 + C\alpha\|\mathbf{v}_t(t)\|^2_{1/2,2} + \frac{\alpha}{2}\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{v}(t)\|^2 \le \frac{F}{2} + |\langle(\mathbf{v}\cdot\nabla)\,\mathbf{v},\mathbf{v}_t\rangle(t)|. \tag{2.4.31}$$

To deal with the nonlinear term, we use standard interpolation inequalities. The key of the process is the continuous Sobolev embedding $H^{1/2}(\Omega)^3 \hookrightarrow L^3(\Omega)^3$, which is the limit case. Therefore, we have

$$\begin{aligned}|\langle(\mathbf{v}\cdot\nabla)\,\mathbf{v},\mathbf{v}_t\rangle(t)| &\le \|\mathbf{v}(t)\|_6\|\nabla \mathbf{v}(t)\|\,\|\mathbf{v}_t(t)\|_3 \\ &\le C\|\nabla \mathbf{v}(t)\|^2\,\|\mathbf{v}_t(t)\|_{1/2,2} \\ &\le \frac{1}{2C\alpha}\|\nabla \mathbf{v}(t)\|^4 + \frac{C\alpha}{2}\|\mathbf{v}_t(t)\|^2_{1/2,2},\end{aligned}$$

so that (2.4.31) becomes

$$\|\mathbf{v}_t(t)\|^2 + C\alpha\|\mathbf{v}_t(t)\|^2_{1/2,2} + \alpha\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2 + \nu\frac{d}{dt}\|\nabla\mathbf{v}(t)\|^2 \leq F + \frac{1}{C\alpha}\|\nabla\mathbf{v}(t)\|^4. \tag{2.4.32}$$

In particular it follows from the above estimate that

$$\nu\frac{d}{dt}\|\nabla\mathbf{v}(t)\|^2 \leq F + \frac{1}{C\alpha}\|\nabla\mathbf{v}(t)\|^4,$$

that we integrate over $[0,s]$, $s \in [0,T]$, so that

$$\nu\|\nabla\mathbf{v}(s)\|^2 \leq \nu\|\nabla\mathbf{v}_0\|^2 + Fs + \frac{1}{C\alpha}\int_0^s \|\nabla\mathbf{v}(t)\|^4\,dt.$$

Lemma 2.4.1 is then applied on every time interval $[0,\tau]$ for $\tau < T$, with

$$f(t) = \nu\|\nabla\mathbf{v}(t)\|^2 \quad \text{and} \quad \lambda(t) = \frac{1}{C\alpha\nu}\|\nabla\mathbf{v}(t)\|^2,$$

which are both are in $L^1(0,\tau)\cap L^\infty(0,\tau)$ and $g(t) = \nu\|\nabla\mathbf{v}_0\|^2 + Ft$ which is a non decreasing function. These lead to

$$\nu\|\nabla\mathbf{v}(s)\|^2 \leq (\nu\|\nabla\mathbf{v}_0\|^2 + Fs)\exp\left\{\frac{1}{C\alpha\nu}\int_0^s \|\nabla\mathbf{v}(t)\|^2\,dt\right\},$$

and yields (2.4.29) by using (2.4.28). Therefore, the inequality (2.4.30) is deduced from (2.4.32) combined with (2.4.28)-(2.4.29). □

2.4.2 Existence and uniqueness of regular-weak solutions

We start by giving the definition of a "regular-weak solution" to the generalized Navier-Stokes-Voigt system (2.4.27). This definition is based on Lemma 2.4.2. We say "weak solution" since it is given by a weak formulation, "regular" since, because of Lemma 2.4.2, we will search for a solution in $L^\infty(0,T;V)\cap W^{1,2}(0,T;V_{1/2})$. This space is considerably smaller than that involved in "standard" Leray-Hopf weak solutions to the Navier-Stokes equations (4.1.1) that are just in $L^\infty(0,T;H)\cap L^2(0,T;V)$. As we shall see, regular weak solutions are unique and satisfy the energy equality, a fact which is still not known about Leray-Hopf weak solutions to the NSE.

Definition 2.4.1. *We say that a function $\mathbf{v} \in L^\infty(0,T;V)\cap W^{1,2}(0,T;V_{1/2})$ is a regular-weak solution of the initial boundary value problem (2.4.27) if it holds true that*

$$\frac{d}{dt}\Big[(\mathbf{v},\boldsymbol{\phi}) + \alpha(\ell D\mathbf{v}, D\boldsymbol{\phi})\Big] + \nu(\nabla\mathbf{v},\nabla\boldsymbol{\phi}) + ((\mathbf{v}\cdot\nabla)\,\mathbf{v},\boldsymbol{\phi}) = \langle\mathbf{f},\boldsymbol{\phi}\rangle \quad \forall\boldsymbol{\phi}\in V,$$

in the sense of $\mathcal{D}'(0,T)$ and the initial datum is attained at least in the sense of $V_{1/2}$, that is

$$\lim_{t\to 0^+}\|\mathbf{v}(t)-\mathbf{v}_0\|_{V_{1/2}} = 0.$$

The main theorem we prove is the following one, showing the well-posedness of the system, globally in time. To fix the ideas and for the simplicity, we stay in a usual weak solutions framework by taking the source term $\mathbf{f} = \mathbf{f}(t)$ in the space[5] $L^2(0,T;H^{-1/2}(\Omega)^3)$. However, many variants can be considered, starting with $\mathbf{f} \in L^2(0,T;V'_{1/2})$, or $\mathbf{f}(t) = \mathbf{f} \in L^2(\Omega)^3$ following the previous subsection, which does not change too much. An interesting case would be $\mathbf{f} \in L^2_{uloc}(\mathbb{R}^+;V'_{1/2})$, for which additional work remains to be done in the context of the long-time behavior (see [BL19]).

Theorem 2.4.1. *Let be given $\mathbf{v}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$. Then, there exists a unique regular-weak solution of the initial boundary value problem (2.4.27) in $[0,T]$, which satisfies the energy (of the model) equality for all $t \geq 0$,*

$$E(t)(\alpha,\ell) + \nu \int_0^t \|\nabla \mathbf{v}(s)\|^2\, ds = E(0)(\alpha,\ell) + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle\, ds. \tag{2.4.33}$$

where $E(t)(\alpha,\ell) := \frac{1}{2}\left(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2\right)$.

Proof. The proof follows by a standard Faedo-Galerkin approximation with suitable a-priori estimates, compactness argument, and interpolation results. It is divided into the following four steps:

1) Construction of approximate solutions, locally in time;

2) Uniform estimates;

3) Passing to the limit in the equations;

4) Energy balance and uniqueness.

Step 1. Construction of approximate solutions, locally in time. Let $\{\boldsymbol{\psi}_n\}_n \subset \mathcal{V}$ be a Hilbert basis of V which we can suppose, without lack of generality, to be orthonormal in H as well as orthogonal in V. We look for approximate Galerkin functions

$$\mathbf{v}^n(t,x) = \sum_{j=1}^n c_{jn}(t)\,\boldsymbol{\psi}_j(x) \qquad \text{for } n \in \mathbb{N},$$

which has to solve the generalized Navier-Stokes-Voigt equations projected over $\mathbf{W}_n = \text{Span}(\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_n)$, that is to solve

$$\frac{d}{dt}\left[(\mathbf{v}^n,\boldsymbol{\psi}_m) + \alpha(\ell D\mathbf{v}^n, D\boldsymbol{\psi}_m)\right] + \nu(\nabla\mathbf{v}^n,\nabla\boldsymbol{\psi}_m) + ((\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n,\boldsymbol{\psi}_m) = \langle \mathbf{f},\boldsymbol{\psi}_m\rangle,$$
$$(\mathbf{v}^n(0),\boldsymbol{\psi}_m) = (\mathbf{v}_0,\boldsymbol{\psi}_m),$$

[5] Recall that $H^{-1/2}(\Omega) = [H^{1/2}_0(\Omega)]'$ and be aware that $H^{-1/2}(\Omega) \subsetneq [H^{1/2}_{00}(\Omega)]'$ with strict inclusion, see Lions-Magenes [LM72].

for $m = 1, \dots, n$. The above problem is a Cauchy problem for a system of n-ordinary differential equations in the coefficients $c_{nm}(t)$. We define the following quantities for $j, l, m = 1, \dots, n$:

$$\alpha_{jm} := \alpha(\ell D\boldsymbol{\psi}_j, D\boldsymbol{\psi}_m), \qquad \beta_{jm} := \nu(\nabla\boldsymbol{\psi}_j, \nabla\boldsymbol{\psi}_m),$$
$$\gamma_{jlm} := ((\boldsymbol{\psi}_j \cdot \nabla)\,\boldsymbol{\psi}_l, \boldsymbol{\psi}_m), \qquad f_m(t) := \langle \mathbf{f}(t), \boldsymbol{\psi}_m\rangle.$$

We have a non-homogeneous system of ordinary differential equations with constant coefficients (which we write with the convention of summation over repeated indices)

$$(\delta_{jm} + \alpha_{jm})c'_{jn}(t) + \beta_{jm}c_{jn}(t) + \gamma_{jlm}c_{jn}(t)c_{ln}(t) = f_m(t), \qquad m = 1, \dots, n,$$

where δ_{ij} denotes the standard Kronecker delta. The initial condition is $c_{jn}(0) = (\mathbf{v}_0, \boldsymbol{\psi}_j)$. The above system is not in “normal form” and in order to obtain a system for which we can apply the Cauchy-Lipschitz Theorem, we have to show that the matrix $(\delta_{jm} + \alpha_{jm})$ can be inverted. Hence, since we work in a finite dimensional spaces it is enough to show that its kernel contains only the null vector. So let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ be such that

$$(\delta_{jm} + \alpha_{jm})\,\xi_j = 0.$$

Multiplying the above equation by ξ_m and summing also over $m = 1, \dots, n$ leads to

$$0 = \|\boldsymbol{\xi}\|^2 + \alpha(\ell\phi, \phi) = \|\boldsymbol{\xi}\|^2 + \alpha(\sqrt{\ell}\phi, \sqrt{\ell}\phi) \geq \|\boldsymbol{\xi}\|^2 \quad \text{with} \quad \phi := \sum_{j=1}^{n} \xi_j D\boldsymbol{\psi}_j,$$

where the last inequality holds true due to the facts that $\alpha > 0$ and $\ell(\mathbf{x}) \geq 0$. Hence, this implies that $\boldsymbol{\xi} \equiv \mathbf{0}$, and consequently the matrix $(\delta_{jm} + \alpha_{jm})$ can be inverted and we denoting by $\mathcal{A} = \mathcal{A}_{jm}$ its inverse. This allows us, after multiplication by $\mathcal{A}$, to rewrite the Galerkin system of ODEs for the coefficients c_{jn} as follows

$$c'_{jn}(t) + \beta'_{jm}c_{jn}(t) + \gamma'_{jlm}c_{jn}(t)c_{ln}(t) = f'_m(t),$$

for appropriate $\beta'_{jm}, \gamma'_{jlm}, f'_m(t)$ and to apply the basic theory of ordinary differential equations. Note that the coefficient from the right-hand side $f'_m(t) = A_{km}\langle \mathbf{f}(t), \boldsymbol{\psi}_k\rangle$ is not continuous but just $L^2(0,T)$, hence one has to resort to an extension of the Cauchy-Lipschitz theorem, with absolutely continuous functions, under Carathéodory hypotheses (see Walter [Wal98]).

Since the system for the coefficients $c_{jn}(t)$ is nonlinear (quadratic) we obtain that there exists a unique solution $c_{jn}(t) \in W^{1,2}(0, T_n)$, for some $0 < T_n \leq T$.

Step 2. Estimates. By taking $\mathbf{v}^n$ as test function, one gets the identity

$$\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{v}^n(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n(t)\|^2\Big) + \nu\|\nabla\mathbf{v}^n(t)\|^2 = \langle \mathbf{f}, \mathbf{v}^n\rangle, \tag{2.4.34}$$

from which it follows

$$\frac{d}{dt}\left(\|\mathbf{v}^n(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n(t)\|^2\right) + \nu\|\nabla\mathbf{v}^n(t)\|^2 \leq \frac{C_P}{\nu}\|\mathbf{f}\|^2_{-1/2,2},$$

where $C_P = C_P(\Omega)$ is the Poincaré-type constant such that

$$\|u\|^2_{1/2,2} \leq C_P\|\nabla u\|^2 \qquad \forall\, u \in H^1_0(\Omega).$$

Hence, integrating over $(0,t)$ for $t < T_n$ we get

$$E^n(t)(\alpha,\ell) + \nu\int_0^t \|\nabla\mathbf{v}^n(s)\|^2\,ds \leq E^n(0)(\alpha,\ell) + \frac{C_P}{\nu}\int_0^t \|\mathbf{f}(s)\|^2_{-1/2,2}\,ds. \tag{2.4.35}$$

where $E^n(t)(\alpha,\ell) := \|\mathbf{v}^n(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n(t)\|^2$. Next, we observe that since $\mathbf{v}^n(0) \to \mathbf{v}_0$ in V and $0 \leq \ell \in C(\overline{\Omega})$, then it holds

$$\alpha\|\sqrt{\ell}D\mathbf{v}^n(0)\|^2 \leq \alpha\max_{x\in\overline{\Omega}}\ell(\mathbf{x})\,\|\nabla\mathbf{v}^n(0)\|^2 \leq \alpha\max_{\mathbf{x}\in\overline{\Omega}}\ell(\mathbf{x})\,\|\nabla\mathbf{v}_0\|^2,$$

which holds $\mathbf{v}^n(0)$ being the orthogonal projection of $\mathbf{v}_0$. This shows that, under the given assumptions on $\mathbf{v}_0$ and $\mathbf{f}$, the r.h.s of (2.4.35) can be bounded independently of $n \in \mathbb{N}$ and consequently, a standard continuation argument proves in fact that $T_n = T$. Moreover, it also holds

$$\mathbf{v}^n \in L^\infty(0,T;H) \cap L^2(0,T;V) \quad \text{and} \quad \sqrt{\ell}D\mathbf{v}^n \in L^\infty(0,T;L^2(\Omega)^9), \tag{2.4.36}$$

with norms bounded uniformly in $n \in \mathbb{N}$. Therefore, according to Theorem 2.3.1, we also obtain

$$\|\mathbf{v}^n(t)\|^2 + \|\mathbf{v}^n(t)\|^2_{V_{1/2}} + \int_0^t \|\nabla\mathbf{v}^n(s)\|^2\,ds \leq C\left[\int_0^t \|\mathbf{f}(s)\|^2_{-1/2,2}ds + \|\mathbf{v}_0\|^2_{1,2}\right], \tag{2.4.37}$$

for a constant C depending on ν, α, ℓ and Ω. In addition, this inequality proves that

$$\mathbf{v}^n \in L^\infty(0,T;V_{1/2}),$$

with bounds independent of $n \in \mathbb{N}$.

In order to give a proper meaning to the time derivative, we now use as test function $\mathbf{v}^n_t$, which is allowed, since it vanishes at the boundary and it is divergence-free. We get

$$\|\mathbf{v}^n_t(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n_t(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{v}^n(t)\|^2 = (\mathbf{f},\mathbf{v}^n_t) - ((\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n,\mathbf{v}^n_t). \tag{2.4.38}$$

We estimate the r.h.s of (2.4.38), thanks to the Cauchy-Schwarz, Hölder, Young and Sobolev inequalities, which give that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|(\mathbf{f},\mathbf{v}^n_t)| \leq C_\epsilon\|\mathbf{f}\|^2_{-1/2,2} + \epsilon\|\mathbf{v}^n_t\|^2_{V_{1/2}},$$

and

$$|((\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n,\mathbf{v}^n_t)| \leq \|\mathbf{v}^n\|_6\|\nabla\mathbf{v}^n\|\|\mathbf{v}^n_t\|_3 \leq C\|\nabla\mathbf{v}^n\|^2\|\mathbf{v}^n_t\|_{V_{1/2}} \leq C_\epsilon\|\nabla\mathbf{v}^n\|^4 + \epsilon\|\mathbf{v}^n_t\|^2_{V_{1/2}}.$$

By the above inequalities we can absorb terms in the l.h.s, to obtain

$$C\|\mathbf{v}_t^n(t)\|_{V_{1/2}}^2+\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{v}^n(t)\|^2\leq C_\epsilon\left[\|\mathbf{f}(t)\|_{-1/2,2}^2+\|\nabla\mathbf{v}^n(t)\|^4\right],$$

for some $C_\epsilon=C(\ell,\alpha,\Omega)$. Integrating over $[0,s]$ for $s\in[0,T]$, one obtains, using the fact that $\mathbf{v}^n(0)$ is the projection of $\mathbf{v}_0$

$$C\int_0^s\|\mathbf{v}_t^n(t)\|_{V_{1/2}}^2\,dt+\frac{\nu}{2}\|\nabla\mathbf{v}^n(s)\|^2\leq\frac{\nu}{2}\|\nabla\mathbf{v}(0)\|^2+C_\epsilon\int_0^s\|\mathbf{f}(t)\|_{-1/2,2}^2\,dt+C_\epsilon\int_0^s\|\nabla\mathbf{v}^n(t)\|^4\,dt, \tag{2.4.39}$$

and in particular,

$$\frac{\nu}{2}\|\nabla\mathbf{v}^n(s)\|^2\leq\frac{\nu}{2}\|\nabla\mathbf{v}_0\|^2+C_\epsilon\int_0^s\|\mathbf{f}(t)\|_{-1/2,2}^2\,dt+C_\epsilon\int_0^s\|\nabla\mathbf{v}^n(t)\|^4\,dt.$$

We apply the Gronwall's lemma 2.4.1 to get

$$\frac{\nu}{2}\|\nabla\mathbf{v}^n(s)\|^2\leq\left(\frac{\nu}{2}\|\nabla\mathbf{v}_0\|^2+C_\epsilon\int_0^s\|\mathbf{f}(t)\|_{-1/2,2}^2\,dt\right)\exp\left\{C_\epsilon\int_0^s\|\nabla\mathbf{v}^n(t)\|^2\,dt\right\}, \tag{2.4.40}$$

and the r.h.s of (2.4.40) is bounded uniformly in n due the a priori estimate (2.4.36). This proves that

$$\mathbf{v}^n\in L^\infty(0,T;V), \tag{2.4.41}$$

from which we also deduce by using (2.4.39) that

$$\mathbf{v}_t^n\in L^2(0,T;V_{1/2}),\quad\text{and therefore by (2.4.36)}\quad\mathbf{v}^n\in W^{1,2}(0,T;V_{1/2}), \tag{2.4.42}$$

with uniform bounds in $n\in\mathbb{N}$. Beside estimates in $V_{1/2}$, it is important to stress that with the same calculations, starting from (2.4.38) as in the proof of Lemma 2.4.2, we also have

$$\sqrt{\ell}D\mathbf{v}_t^n\in L^2(0,T;L^2(\Omega)^9),$$

again with uniform bound in $n\in\mathbb{N}$.

Step 3. Passing to the limit in the approximate equations. By the uniform bounds above and standard compactness results there exists $\mathbf{v}\in W^{1,2}(0,T;V_{1/2})\cap L^\infty(0,T;V)$ and a sub-sequence (relabelled as $\mathbf{v}^n$) such that

$$\begin{cases}\mathbf{v}^n\overset{*}{\rightharpoonup}\mathbf{v} & \text{in}\quad L^\infty(0,T;V),\\ \sqrt{\ell}D\mathbf{v}^n\overset{*}{\rightharpoonup}\sqrt{\ell}D\mathbf{v} & \text{in}\quad L^\infty(0,T;L^2(\Omega)^9),\\ \mathbf{v}^n\rightharpoonup\mathbf{v} & \text{in}\quad L^p(0,T;V)\quad\text{for all } 1<p<\infty,\\ \mathbf{v}_t^n\rightharpoonup\mathbf{v}_t & \text{in}\quad L^2(0,T;V_{1/2}),\\ \sqrt{\ell}D\mathbf{v}_t^n\rightharpoonup\sqrt{\ell}D\mathbf{v}_t & \text{in}\quad L^2(0,T;L^2(\Omega)^9),\end{cases} \tag{2.4.43}$$

To get strong convergence in appropriate spaces, we use the Aubin-Lions compactness lemma (see Temam [Tem01]) with the triple

$$V \hookrightarrow V_{3/4} \hookrightarrow V_{1/2},$$

where $V_{3/4} = [V, H]_{3/4}$, each embedding being dense and continuous. Moreover, since Ω is bounded by the Rellich-Kondrachov Theorem, these embeddings are also compact. Therefore, the sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ is (pre)compact in $L^2(0, T; V_{3/4})$ and (up to a sub-sequence)

$$\mathbf{v}^n \to \mathbf{v} \quad \text{in} \quad L^2(0, T; V_{3/4}),$$

which implies in particular strong convergence in $L^2(0, T; L^4(\Omega)^3)$. By standard results this allows to pass to the limit in the weak formulation, showing that indeed $\mathbf{v}$ is a regular-weak solution. We skip the details. It remains to check the initial data. The weak convergence implies that for $0 \le t \le T$

$$\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{v}(s)\|^2\,ds \le \|\mathbf{v}(0)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(0)\|^2 + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s)\rangle\,ds.$$

Observe that the above inequality is obtained from (2.4.34), after integration in time and passing to the limit. The inequality comes from the fact that $\nabla\mathbf{v}^n \rightharpoonup \nabla\mathbf{v}$ in $L^2(0, T; L^2(\Omega)^9)$, and we have to consider the inferior limit of the norms of $\nabla\mathbf{v}^n$. In addition, we observe that since $\nabla\mathbf{v}^n(0) \to \nabla\mathbf{v}_0$ in $L^2(\Omega)$, we can suppose –up to a further sub-sequence– that $\nabla\mathbf{v}^n(0, \mathbf{x}) \to \nabla\mathbf{v}_0(\mathbf{x})$ a.e. $\mathbf{x} \in \Omega$, hence using the boundedness of ℓ and Lebesgue dominated convergence, we have

$$\|\sqrt{\ell}D\mathbf{v}^n(0)\|^2 \to \|\sqrt{\ell}D\mathbf{v}_0\|^2,$$

showing also that the initial datum is assumed strongly at the initial time.

Step 4. Energy balance and uniqueness. We start with the energy balance (2.4.33). To this end one has first to justify the use of $\mathbf{v}$ as test function. From the results above, we deduce that $\mathbf{v}\otimes\mathbf{v} \in L^\infty(0, T; L^3(\Omega)^9)$ which yields in particular $(\mathbf{v}\cdot\nabla)\mathbf{v} \in L^2(0, T; V')$ and $\langle(\mathbf{v}\cdot\nabla)\mathbf{v}, \mathbf{v}\rangle = 0$ according to standard results. From there, the relevant point is to check that for any $s \in [0, T]$:

$$\int_0^s (\ell D\mathbf{v}_t, D\mathbf{v})\,dt = \frac{1}{2}\|\sqrt{\ell}D\mathbf{v}(s)\|^2 - \frac{1}{2}\|\sqrt{\ell}D\mathbf{v}_0\|^2, \tag{2.4.44}$$

since all other terms are well-behaved due to the available regularity of $\mathbf{v}$. However, $\sqrt{\ell}D\mathbf{v}, \sqrt{\ell}D\mathbf{v}_t \in L^2(0, T; L^2(\Omega)^9)$. Therefore, by identifying $L^2(\Omega)^9$ with its dual space, we naturally have

$$\langle \ell D\mathbf{v}_t, D\mathbf{v}\rangle = \langle\sqrt{\ell}D\mathbf{v}_t, \sqrt{\ell}D\mathbf{v}\rangle = \frac{1}{2}\frac{d}{dt}\|\sqrt{\ell}D\mathbf{v}\|^2,$$

hence (2.4.44) and then (2.4.33) follows.

Moreover, this result allows us also to prove uniqueness of regular-weak solutions. In fact, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions corresponding to the same initial datum and same external force, taking the difference and testing (by the above argument this is fully justified) with $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$ one obtains the following differential equality for the difference for any $t \in [0, T]$:

$$\|\mathbf{V}(t)\|^2 + \alpha \|\sqrt{\ell} D\mathbf{V}(t)\|^2 + \nu \int_0^t \|\nabla \mathbf{V}\|^2 \, ds = - \int_0^t \int_\Omega (\mathbf{V} \cdot \nabla) \; \mathbf{v}_2 \cdot \mathbf{V} \, d\mathbf{x} ds.$$

Hence, by the usual Sobolev inequalities

$$\|\mathbf{V}(t)\|^2 + \alpha \|\sqrt{\ell} D\mathbf{V}(t)\|^2 + \nu \int_0^t \|\nabla \mathbf{V}\|^2 \, ds \leq \frac{\nu}{2} \int_0^t \|\nabla \mathbf{V}\|^2 \, ds + \frac{C}{\nu} \int_0^t \|\nabla \mathbf{v}_2\|^4 \|\mathbf{V}\|^2 \, ds,$$

and since $\mathbf{V}(0) = \mathbf{0}$ the Gronwall's lemma shows that $\mathbf{V} \equiv \mathbf{0}$, due to the fact that

$$\nabla \mathbf{v}_2 \in L^\infty(0, T; L^2(\Omega)^9) \subset L^4(0, T; L^2(\Omega)^9).$$

□

Remark 2.4.3. *The pressure is not involved in Definition 2.4.1. However, let* $(\mathbf{v}_0, \mathbf{f})$ *be given as in Theorem 2.4.1 and let* $\mathbf{v}$ *be the corresponding regular-weak solution. Then, by the De Rham theorem, we easily deduce the existence of* $p \in \mathcal{D}'(0, T; L^2(\Omega)/\mathbb{R})$ *such that* $(\mathbf{v}, p)$ *satisfies system (2.4.27) in the sense of the distributions. The regularity of the pressure is probably even better than that, but this point, which is inessential for the results of this paper, remains to be investigated.*

Remark 2.4.4. *Definition 2.4.1 is equivalent to the following: The field* $\mathbf{v}$ *is a regular-weak solution to (2.4.27) if:*

1. $\mathbf{v} \in W^{1,2}(0, T; V_{1/2}) \cap L^\infty(0, T; V)$, $\sqrt{\ell} D\mathbf{v}_t \in L^2(Q_T)^9$,
2. *for all* $\mathbf{w} \in L^2(0, T; V)$, $\forall\, s < T$:

$$\int_0^s (\mathbf{v}_t, \mathbf{w}) \, dt + \alpha \int_0^s (\sqrt{\ell} D\mathbf{v}_t, \sqrt{\ell} D\mathbf{w}) \, dt - \int_0^s \int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x} dt + \nu \int_0^s \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x} dt = \int_0^s \langle \mathbf{f}, \mathbf{w} \rangle \, dt,$$

3. $\lim_{t \to 0^+} \|\mathbf{v}(t) - \mathbf{v}_0\|_{V_{1/2}} = 0$.

Once the above results of existence and uniqueness have been proved for the generalized Navier-Stokes-Voigt equations, it is straightforward to prove the same also for the model (2.1.1) with an additional turbulent viscosity ν_{turb} which is non-negative and bounded. We do not reproduce here the proof, which follows the same steps of Theorem 2.4.1 since the additional term with the turbulent viscosity ν_{turb} neither improves the a priori estimates neither creates additional problems when taking weak limits of the approximate solutions.

Theorem 2.4.2. *Let $\nu_{\text{turb}} \in L^\infty([0,\infty[\times\Omega)$ such that $\nu_{\text{turb}} \geq 0$ a.e. in $[0,\infty[\times\Omega$ and let and $\mathbf{v}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$ be given. Then, the system*

(2.4.45)
$$\begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\text{turb}}D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } (0,T)\times\Omega, \\ \mathbf{v}|_\Gamma = 0 & \text{on } (0,T)\times\Gamma, \\ \mathbf{v}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases}$$

has a unique regular-weak solution that satisfies the energy balance[6] *(equality)*

$$E(t)(\alpha,\ell) + \int_0^t \|(2\nu+\nu_{\text{turb}})^{1/2}D\mathbf{v}(s)\|^2\,ds = E(0)(\alpha,\ell) + \int_0^t \langle \mathbf{f}(s),\mathbf{v}(s)\rangle\,ds,$$

where again $E(t)(\alpha,\ell) = \dfrac{1}{2}\left(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2\right)$.

2.5 Turbulent Voigt model involving the TKE

In this section we consider the generalized Voigt model with an additional turbulent viscosity $\nu_{\text{turb}} = \nu_{\text{turb}}(k)$, coupled with the evolution equation for the turbulent kinetic energy k. In particular, we prove a compactness result which allows then to prove in an easy way existence of weak solutions.

2.5.1 A compactness Lemma

We consider a family of models as in (2.4.45), associated with different realizations of the turbulent viscosity and study the behavior of the solutions, under rather mild conditions on the given additional viscosities.

To this end let be given a family of turbulent viscosities $(\nu^n_{\text{turb}})_{n\in\mathbb{N}}$ such that

$$\forall\, n \geq 0, \qquad \nu^n_{\text{turb}} \in L^\infty([0,\infty[\times\Omega), \quad \nu^n_{\text{turb}} \geq 0 \ \text{ a.e. in } [0,\infty[\times\Omega.$$

Let $\mathbf{v}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$. Let $(\mathbf{v}^n, p^n)$ finally denote the distributional solution to

(2.5.46)
$$\begin{cases} \mathbf{v}^n_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}^n_t) + (\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n - \nu\Delta\mathbf{v}^n - \nabla\cdot(\nu^n_{\text{turb}}D\mathbf{v}^n) + \nabla p^n = \mathbf{f} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v}^n = 0 & \text{in } (0,T)\times\Omega, \\ \mathbf{v}^n|_\Gamma = 0 & \text{on } (0,T)\times\Gamma, \\ \mathbf{v}^n_{|t=0} = \mathbf{v}_0 & \text{in } \Omega, \end{cases}$$

and such that $\mathbf{v}^n \in W^{1,2}(0,T;V_{1/2}) \cap L^\infty(0,T;V)$, is a regular-weak solution to (2.5.46).

[6] Remind that since $\nabla\cdot\mathbf{v} = 0$, then $\Delta\mathbf{v} = 2\nabla\cdot D\mathbf{v}$. Therefore, $\langle -\nu\Delta\mathbf{v} + \nabla\cdot(\nu_{\text{turb}}D\mathbf{v}),\mathbf{w}\rangle = ((2\nu + \nu_{\text{turb}})D\mathbf{v}, D\mathbf{w})$.

Remark 2.5.1. *In this section $\mathbf{v}^n$ denotes the velocity associated to the turbulent viscosity ν^n_{turb}, namely (2.5.46), and not a Galerkin approximation as in the previous section. We keep the same notation for the simplicity, so far no risk of confusion occurs.*

Concerning the behavior as $n \to +\infty$ of the solutions $\mathbf{v}^n$ we have the following lemma.

Lemma 2.5.1. *Assume that the sequence $(\nu^n_{\text{turb}})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,\infty[\times\Omega)$ and converges almost everywhere to ν_{turb} in $Q_\infty = [0,\infty[\times\Omega$.*
Then, it follows that:

1) The sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ weakly converges in $W^{1,2}(0,T;V_{1/2}) \cap L^p(0,T;V)$, for all $p < \infty$, to the regular-weak solution $\mathbf{v}$ of the limit problem

(2.5.47)
$$\begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\text{turb}}D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } (0,T)\times\Omega, \\ \mathbf{v}|_\Gamma = 0 & \text{on } (0,T)\times\Gamma, \\ \mathbf{v}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases}$$

2) The sequence $(\nu^n_{\text{turb}}|D\mathbf{v}^n|^2)_{n\in\mathbb{N}}$ converges in $L^1(Q_T)$ and the sense of measures to $\nu_{\text{turb}}|D\mathbf{v}|^2$ in Q_T, which means that

$$\forall\varphi \in C(\overline{Q}_T), \quad \int_0^T\int_\Omega \nu^n_{\text{turb}}|D\mathbf{v}^n|^2\varphi\, d\mathbf{x}dt \xrightarrow[n\to\infty]{} \int_0^T\int_\Omega \nu_{\text{turb}}|D\mathbf{v}|^2\varphi\, d\mathbf{x}dt. \tag{2.5.48}$$

Proof. In order to simplify the notation we extract sub-sequences, without changing the notation. However, by the uniqueness result of Theorem 2.4.1, we finally get convergence for the whole sequence because of the uniqueness of solutions to the limit problem.

1) The estimates (2.4.37), (2.4.39), (2.4.40) and (2.4.42) derived for the Galerkin approximations are the same for the present sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$. Therefore, $(\mathbf{v}^n)_{n\in\mathbb{N}}$ is weakly pre-compact in $W^{1,2}(0,T;V_{1/2}) \cap L^p(0,T;V)$, for all $p < \infty$, as well as pre-compact in $L^2(0,T;V_{3/4})$ by Aubin-Lions Lemma. Therefore, it weakly converges (up to a subsequence) to some $\mathbf{v}$ in $W^{1,2}(0,T;V_{1/2}) \cap L^p(0,T;V)$, strongly in $L^2(0,T;V_{3/4})$. In particular, the strong convergence holds in $L^2(0,T;L^4(\Omega))$, then in $L^4(0,T;L^4(\Omega))$ by usual arguments. Passing to the limit in the equations satisfied by $\mathbf{v}^n$ is then straightforward (cf. Theorem 2.4.1), except in the eddy viscosity term. To this end, let be given $\mathbf{w} \in L^2(0,T;V)$, we can write

$$\langle -\nabla\cdot(\nu^n_{\text{turb}}D\mathbf{v}^n), \mathbf{w}\rangle = \int_0^T\int_\Omega \nu^n_{\text{turb}}D\mathbf{v}^n : D\mathbf{w}\, d\mathbf{x}ds = \int_0^T\int_\Omega D\mathbf{v}^n : \nu^n_{\text{turb}}D\mathbf{w}\, d\mathbf{x}ds.$$

Next, as $(\nu_{\text{turb}}^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$, we have the following estimate

$$\text{for a.e } (t,\mathbf{x}) \in Q_T \qquad |\nu_{\text{turb}}^n(t,\mathbf{x})D\mathbf{w}(t,\mathbf{x})| \le \sup_{n\in\mathbb{N}} \|\nu_{\text{turb}}^n\|_{L^\infty_{t,\mathbf{x}}} |D\mathbf{w}(t,\mathbf{x})| \in L^2(Q_T).$$

On the other hand, according to the *a.e* convergence of ν_{turb}^n, it follows also

$$\nu_{\text{turb}}^n D\mathbf{w} \to \nu_{\text{turb}} D\mathbf{w} \quad \text{ a.e in } Q_T.$$

Then, by the Lebesgue dominated convergence theorem, one has the strong convergence

$$\nu_{\text{turb}}^n D\mathbf{w} \to \nu_{\text{turb}} D\mathbf{w} \quad \text{ in } L^2(Q_T).$$

From the weak convergence in $L^2(0,T;V)$ we have

$$D\mathbf{v}^n \rightharpoonup D\mathbf{v} \quad \text{ in } L^2(Q_T).$$

The convergence of the eddy viscosity term follows then from

$$\begin{aligned}\langle -\nabla\cdot(\nu_{\text{turb}}^n D\mathbf{v}^n), \mathbf{w}\rangle = \int_0^T \int_\Omega D\mathbf{v}^n : \nu_{\text{turb}}^n D\mathbf{w}\, d\mathbf{x}ds \xrightarrow[n\to\infty]{} & \int_0^T \int_\Omega D\mathbf{v} : \nu_{\text{turb}} D\mathbf{w}\, d\mathbf{x}ds \\ &= \langle -\nabla\cdot(\nu_{\text{turb}} D\mathbf{v}), \mathbf{w}\rangle,\end{aligned}$$

As a consequence, $\mathbf{v}$ is indeed a regular-weak solution to (2.5.47) on $[0,T]$, for all positive T, and by the uniqueness of the solution, all the sequence does converge.

2) We split the proof of this property into three steps:

i) Weak convergence of the sequence $((2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n)_{N\in\mathbb{N}}$ to $(2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v}$ in $L^2(Q_T)$;

ii) Strong convergence of the same sequence by the "energy method";

iii) Proof of the convergence in measures from (2.5.48).

i) This step is very similar to point 1). In fact, we already proved that the sequence $((2\nu+\nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n)_{N\in\mathbb{N}}$ is bounded in $L^2(Q_T)^9$, uniformly in $N \in \mathbb{N}$. Moreover, we already know that $D\mathbf{v}^n \rightharpoonup D\mathbf{v}$ in $L^2(Q_T)$. Let us define

$$\mathbf{A}_n := (2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n \qquad \text{and} \qquad \mathbf{A} := (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v}.$$

We aim to prove that $\mathbf{A}_n \rightharpoonup \mathbf{A}$ in $L^2(Q_T)^9$. To do so, let us fix $\mathbf{w} \in L^2(0,T;V)$. By the hypothesis of a.e. convergence on $(\nu_{\text{turb}}^n)_{n\in\mathbb{N}}$ it follows that

$$(2\nu + \nu_{\text{turb}}^n)^{1/2}\, D\mathbf{w} \xrightarrow[n\to\infty]{} (2\nu + \nu_{\text{turb}})^{1/2}\, D\mathbf{w} \quad \text{ a.e. in } Q_T.$$

Moreover, one has also

$$\left|(2\nu + \nu^n_{\text{turb}}(t,\mathbf{x}))^{1/2}\, D\mathbf{w}(t,\mathbf{x})\right| \le \left(2\nu + \sup_n \|\nu^n_{\text{turb}}\|_{L^\infty_{t,\mathbf{x}}}\right)^{1/2} |D\mathbf{w}(t,\mathbf{x})| \in L^2(Q_T).$$

Therefore, again by Lebesgue's theorem we obtain the strong convergence

$$(2\nu + \nu^n_{\text{turb}})^{1/2}\, D\mathbf{w} \xrightarrow[n\to\infty]{} (2\nu + \nu_{\text{turb}})^{1/2}\, D\mathbf{w} \quad \text{in } L^2(Q_T),$$

hence as before weak L^2-convergence of $D\mathbf{v}^n$ implies

$$\begin{aligned}\int_0^T \int_\Omega (2\nu + \nu^n_{\text{turb}})^{1/2} D\mathbf{v}^n : D\mathbf{w}\, d\mathbf{x}dt &= \int_0^T \int_\Omega (2\nu + \nu^n_{\text{turb}})^{1/2}\, D\mathbf{w} : D\mathbf{v}^n\, d\mathbf{x}dt \\ \xrightarrow[n\to\infty]{} \int_0^T \int_\Omega (2\nu + \nu_{\text{turb}})^{1/2}\, D\mathbf{w} : D\mathbf{v}\, d\mathbf{x}dt &= \int_0^T \int_\Omega (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v} : D\mathbf{w}\, d\mathbf{x}dt,\end{aligned}$$

yielding the desired weak convergence.

ii) *Energy method.* We now prove the strong L^2-convergence of the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ to $\mathbf{A}$. To do so, we use the energy method (see [CRL14, Lew97a]), based on the energy (equality) balance (2.4.33) satisfied by both $\mathbf{v}^n$ and $\mathbf{v}$, with dissipative terms from eddy viscosity terms

$$\int\int_{Q_t} \nu^n_{\text{turb}} |D\mathbf{v}^n|^2 \qquad \text{and} \qquad \int\int_{Q_t} \nu_{\text{turb}} |D\mathbf{v}|^2,$$

in the corresponding equations. Observe that at this stage is very important to have the energy balance satisfied with an equality, instead of the inequality, as it holds for the (un-regularized) Navier-Stokes equations. For an overview and recent results on the possible energy equality for incompressible fluids, see [BC20].

According to the notations introduced in Theorem 2.4.1, we rewrite the energy balances as follows, for all $t < T$,

$$\begin{aligned} E(t)(\alpha,\ell) + \int_0^t \int_\Omega |\mathbf{A}|^2\, d\mathbf{x}ds &= \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, ds + E(0)(\alpha,\ell), \\ E^n(t)(\alpha,\ell) + \int_0^t \int_\Omega |\mathbf{A}_n|^2\, d\mathbf{x}ds &= \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, ds + E(0)(\alpha,\ell), \end{aligned} \tag{2.5.49}$$

A critical tool in the energy method is that of integrating over $[0,T]$, with the respect to the time variable, both the equations in (2.5.49) and to perform then an integration by parts. This yields the following two equalities

$$\begin{aligned} \int_0^T E(t)(\alpha,\ell)\, dt + \int_0^T \int_\Omega (T-t)|\mathbf{A}|^2\, d\mathbf{x}dt &= \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, dsdt + TE(0)(\alpha,\ell), \\ \int_0^T E^n(t)(\alpha,\ell)\, dt + \int_0^T \int_\Omega (T-t)|\mathbf{A}_n|^2\, d\mathbf{x}dt &= \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, dsdt + TE(0)(\alpha,\ell). \end{aligned} \tag{2.5.50}$$

Arguing with the usual compactness tools as developed in the proof of Theorems 2.4.1 and 2.4.2 we obtain that

$$\int_0^T \|\mathbf{v}^n(t)\|^2 dt \xrightarrow[n\to\infty]{} \int_0^T \|\mathbf{v}(t)\|^2\, dt,$$
$$\int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, dsdt \xrightarrow[n\to\infty]{} \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, dsdt$$
$$TE^n(0)(\alpha,\ell) \xrightarrow[n\to\infty]{} TE(0)(\alpha,\ell).$$

Therefore, by using the integrated energy equalities (2.5.50), we also get by comparison

$$\int_0^T \int_\Omega \left[\alpha\ell|D\mathbf{v}^n|^2 + (T-t)|\mathbf{A}_n|^2\right]\, d\mathbf{x}dt \xrightarrow[n\to\infty]{} \int_0^T \int_\Omega \left[\alpha\ell|D\mathbf{v}|^2 + (T-t)|\mathbf{A}|^2\right]\, d\mathbf{x}dt.$$

Let now $\mathbf{B}_n, \mathbf{B}$ be defined as follows

$$\mathbf{B}_n := (\alpha\ell + (T-t)(2\nu + \nu^n_{\text{turb}}))^{1/2} D\mathbf{v}^n \quad \text{and} \quad \mathbf{B} := (\alpha\ell + (T-t)(2\nu + \nu_{\text{turb}}))^{1/2} D\mathbf{v}.$$

The information coming from the convergence of the integrated energy equalities can be rewritten as follows

$$\|\mathbf{B}_n\|^2 \xrightarrow[n\to\infty]{} \|\mathbf{B}\|^2.$$

Next, the assumptions on the function $\ell(\mathbf{x})$ imply also that, for a given $\mathbf{w} \in L^2(0,T;V)$,

$$\begin{aligned}&\left|(\alpha\ell(\mathbf{x}) + (T-t)(2\nu + \nu^n_{\text{turb}}(t,\mathbf{x})))^{1/2}\, D\mathbf{w}(t,\mathbf{x})\right| \\ &\leq \left(\alpha \max_{\mathbf{x}\in\Omega} \ell + T(2\nu + \sup_n \|\nu^n_{\text{turb}}\|_{L^\infty_{t,\mathbf{x}}})\right)^{1/2} |D\mathbf{w}(t,\mathbf{x})| \in L^2(Q_T).\end{aligned}$$

and, by using the same argument as before, this implies the strong L^2-convergence of $(\alpha\ell(\mathbf{x}) + (T-t)(2\nu + \nu^n_{\text{turb}}(t,\mathbf{x})))^{1/2}\, D\mathbf{w}(t,\mathbf{x})$. The weak $L^2(Q_T)$-convergence of $D\mathbf{v}^n$ to $D\mathbf{v}$ implies the weak $L^2(Q_T)$ convergence $\mathbf{B}_n \rightharpoonup \mathbf{B}$. This together with the convergence of the norms $\|\mathbf{B}_n\|$ implies that

$$\mathbf{B}_n \xrightarrow[n\to\infty]{} \mathbf{B} \quad \text{in } L^2(Q_T)^9.$$

Next, we observe that, for all $t < T$ and for all $\mathbf{x} \in \Omega$ we can write

$$\begin{aligned}\mathbf{A}_n &= (2\nu + \nu^n_{\text{turb}}))^{1/2} D\mathbf{v}^n = (\alpha\ell + (T-t)(2\nu + \nu^n_{\text{turb}}))^{1/2} D\mathbf{v}^n \frac{(2\nu + \nu^n_{\text{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu^n_{\text{turb}}))^{1/2}} \\ &= \mathbf{B}_n \frac{(2\nu + \nu^n_{\text{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu^n_{\text{turb}}))^{1/2}}\end{aligned}$$

and for all $T' < T$ it holds

$$\frac{(2\nu + \nu^n_{\text{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu^n_{\text{turb}}))^{1/2}} \xrightarrow[n\to\infty]{} \frac{(2\nu + \nu_{\text{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\text{turb}}))^{1/2}} \qquad \text{a.e. in } Q_{T'}.$$

The above point-wise convergence and the L^2-strong convergence of $\mathbf{B}_n$ gives then

$$\mathbf{A}_n \to \mathbf{B}\frac{(2\nu+\nu_{\text{turb}}))^{1/2}}{(\alpha\ell+(T-t)(2\nu+\nu_{\text{turb}}))^{1/2}} = \mathbf{A} \qquad \text{in } L^2(Q_{T'}).$$

As T can be any positive time, this concludes this step.

iii) *Proof of* (2.5.48). The same argument used in the above step can be also used to show that

$$\sqrt{\nu_{\text{turb}}^n} D\mathbf{v}^n \to \sqrt{\nu_{\text{turb}}} D\mathbf{v} \qquad \text{in } L^2(Q_T),$$

simply writing

$$\sqrt{\nu_{\text{turb}}^n} D\mathbf{v}^n = (\alpha\ell+(T-t)(2\nu+\nu_{\text{turb}}^n))^{1/2} D\mathbf{v}^n \frac{\sqrt{\nu_{\text{turb}}^n}}{(\alpha\ell+(T-t)(2\nu+\nu_{\text{turb}}^n))^{1/2}}$$

as the right-hand side converges in $L^2(Q_T)$ as $n\to+\infty$ to

$$\mathbf{B}\frac{\sqrt{\nu_{\text{turb}}^n}}{(\alpha\ell+(T-t)(2\nu+\nu_{\text{turb}}^n))^{1/2}} = \sqrt{\nu_{\text{turb}}} D\mathbf{v}.$$

This shows that

$$\int_{Q_T} \nu_{\text{turb}}^n(t,\mathbf{x})|D\mathbf{v}^n(t,\mathbf{x})|^2\, d\mathbf{x}dt \to \int_{Q_T} \nu_{\text{turb}}(t,\mathbf{x})|D\mathbf{v}(t,\mathbf{x})|^2\, d\mathbf{x}dt. \tag{2.5.51}$$

On one hand, we deduce from the above that the sequence $(\sqrt{\nu_{\text{turb}}^n}D\mathbf{v}^n)_{n\in\mathbb{N}}$ weakly converges to $\sqrt{\nu_{\text{turb}}}D\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^9)$. On the other hand, the corresponding sequence of the norms also converges by (2.5.51). Hence $(\sqrt{\nu_{\text{turb}}^n}D\mathbf{v}^n)_{n\in\mathbb{N}}$ strongly converges to $\sqrt{\nu_{\text{turb}}}D\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^9)$, and the conclusion of part 2 of Lemma 2.5.1 easily follows. □

2.5.2 Application to the NSTKE-Voigt model

We now apply the existence result for the Generalized Voigt model, together with the compactness lemma, to study the Voigt model coupled with the equation of the turbulent kinetic energy. The NSTKE-Voigt model is in fact obtained by coupling the turbulent Navier-Stokes-Voigt equation to the equation for the TKE, following the law (4.3.22), which gives the following initial boundary value problem

$$\begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\text{turb}}(k)D\mathbf{v}) + \nabla p = \mathbf{f}, & (i)\\ \nabla\cdot\mathbf{v} = 0, & (ii)\\ \mathbf{v}|_\Gamma = 0, & (iii)\\ \mathbf{v}_{|t=0} = \mathbf{v}_0, & (iv)\\ k_t + \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu_{\text{turb}}(k)\nabla k) = \nu_{\text{turb}}(k)|D\mathbf{v}|^2 - (\ell+\eta)^{-1}k\sqrt{|k|}, & (v)\\ k|_\Gamma = 0, & (vi)\\ k_{|t=0} = k_0. & (vii) \end{cases} \tag{2.5.52}$$

This system calls for two comments:

1) According to Lemma 2.5.1, we know how to deal with bounded eddy viscosities and the argument does not work with general viscosities. This is why we cannot take the unbounded law (4.3.22). We replace it, as often done with this class of problems, by

$$\nu_{\text{turb}}(k) = \ell\, T_{M'}(\sqrt{|k|}), \qquad M' \in \mathbb{N} \tag{2.5.53}$$

where T_M is the usual "truncation function" at height M, for a given large $M \in \mathbb{N}$, which is defined by

$$T_M(x) := \begin{cases} x & \text{if } |x| \le M, \\ M\dfrac{x}{|x|} & \text{if } |x| > M, \end{cases}$$

for all $x \in \mathbb{R}$. The eddy viscosity (2.5.53) will fit with that considered in Lemma 2.5.1. Similarly, we assume that the diffusion coefficient for the turbulent kinetic energy satisfies

$$\mu_{\text{turb}}(k) = C\ell\, T_{M''}(\sqrt{|k|}), \tag{2.5.54}$$

for some dimensionless constant C and another $M'' \in \mathbb{N}$.

2) Usually, the dissipation term in the r.h.s of the equation for k is considered as $\varepsilon := \ell^{-1}k\sqrt{|k|}$. Unfortunately, due to the degeneration of ℓ at the boundary Γ, there could be further issues when passing to the limit in this term. As a precaution, we have approximated it by $\varepsilon = (\ell+\eta)^{-1}k\sqrt{|k|}$ where $\eta > 0$ is a small parameter. We have not studied the behavior of the solutions when $\eta \to 0$, yet.

The main existence result we are able to prove for the NSTKE-Voigt system is the following.

Theorem 2.5.1. *Let be given $\mathbf{v}_0 \in V$, $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$ and $0 \le k_0 \in L^1(\Omega)$. Assume that ν_{turb} and μ_{turb} are given by (2.5.53) and (2.5.54). Then, there exists $(\mathbf{v}, k)$ such that:*

1. *The vector field $\mathbf{v}$ verifies*

$$\mathbf{v} \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}),$$

and it is a regular-weak solution to the subsystem [(2.5.52)-(i)-(ii)-(iii)-(iv))],

2. *The scalar field k verifies*

$$k \in L^\infty(0,T;L^1(\Omega)), \quad k \in \bigcap_{1<p<5/4} L^p(0,T;W^{1,p}(\Omega)) = K_{5/4},$$

and it is solution of the subsystem [(2.5.52)-(v)-(vi)-(vii))] in the sense of the distribution in Q_T. Moreover, $k \ge 0$ a.e. in Q_T.

Proof. System (2.5.52) is very close to that studied for in [CRL14, Chapter 8]. Therefore, we only indicate the changes in the proof of existence, without giving full details, which can be easily filled by the reader. The main difference is given by the result of the compactness Lemma 2.5.1, which is essential to the proof. The further (compared to the previously studied systems) regularity enforced by the generalized Voigt term is the key to prove the existence results for the full NSTKE model.

One technical issue is due to the quadratic source term $\nu_{\text{turb}}(k)|D\mathbf{v}|^2$ in the TKE equation, which is *a priori* in $L^1(Q_T)$ and not better and hence we have to deal with a generalized Navier-Stokes-Voigt system with a right-hand side in L^1, for which the theory cannot be directly handled. To overcome this fact, we truncate this term as well as the initial data, leading to the following regularized (truncated) system:

(2.5.55)
$$\begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\text{turb}}(k)D\mathbf{v}) + \nabla p = \mathbf{f}, & (i) \\ \nabla\cdot\mathbf{v} = 0, & (ii) \\ \mathbf{v}|_\Gamma = 0, & (iii) \\ \mathbf{v}_{|t=0} = \mathbf{v}_0, & (iv) \\ k_t + \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu_{\text{turb}}(k)\nabla k) = T_M(\nu_{\text{turb}}(k)|D\mathbf{v}|^2) - (\ell+\eta)^{-1}k\sqrt{|k|}, & (v) \\ k|_\Gamma = 0, & (vi) \\ k_{|t=0} = T_M(k_0). & (vii) \end{cases}$$

The proof of existence for the approximate (truncated) system is now done by means of suitable applications of the Schauder fixed-point theorem. As first step we fix $\widetilde{k} \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, let $\widetilde{\mathbf{v}} = \mathbf{v}(\widetilde{k})$ be the unique regular-weak solution to

$$\begin{cases} \widetilde{\mathbf{v}}_t - \alpha\nabla\cdot(\ell\, D\widetilde{\mathbf{v}}_t) + (\widetilde{\mathbf{v}}\cdot\nabla)\,\widetilde{\mathbf{v}} - \nu\Delta\widetilde{\mathbf{v}} - \nabla\cdot(\nu_{\text{turb}}(\widetilde{k})D\widetilde{\mathbf{v}}) + \nabla p = \mathbf{f}, \\ \nabla\cdot\widetilde{\mathbf{v}} = 0, \\ \widetilde{\mathbf{v}}|_\Gamma = 0, \\ \widetilde{\mathbf{v}}_{|t=0} = \mathbf{v}_0, \end{cases}$$

which is the subsystem [(2.5.55)-(i)-(ii)-(iii)-(iv))] with $\nu_{\text{turb}}(k)$ is replaced by $\nu_{\text{turb}}(\widetilde{k})$. The existence of $\widetilde{\mathbf{v}}$ follows from Theorems 2.4.1-2.4.2. The next step is to analyze the equation for k, considering

$$\begin{cases} k_t + \widetilde{\mathbf{v}}\cdot\nabla k - \nabla\cdot(\mu_{\text{turb}}(k)\nabla k) = T_M(\nu_{\text{turb}}(k)|D\widetilde{\mathbf{v}}|^2) - (\ell+\eta)^{-1}k\sqrt{|k|}, \\ k|_\Gamma = 0, \\ k_{|t=0} = T_M(k_0), \end{cases} \tag{2.5.56}$$

which is a non linear parabolic equation with both coefficients and a source term smooth enough to allow application of the standard variational theory. The existence of a weak solution $k \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ to Problem (2.5.56), it is easily proved and, in addition, it follows $k_t \in L^2(0,T;H^{-1}(\Omega))$.

The solution of full system (2.5.55) –with $M \in \mathbb{N}$ fixed–can be obtained by finding a fixed point of the map

$$\widetilde{k} \to k,$$

by means of the Schauder fixed point theorem as in [CRL14], as follows by the boundedness in appropriate spaces. Observe that this will not ensure uniqueness of the solution. To solve the full system (2.5.52) (with truncated viscosities but with the L^1-right-hand-side) we iteratively construct the sequence

$$(k^n)_{n\in\mathbb{N}},$$

in the following way: we start from $k^0 \equiv 0$ and $\mathbf{v}^0$ the corresponding solution of the [(2.5.52)-(i)-(ii)-(iii)-(iv))], with $\nu_{\text{turb}} = \nu_{\text{turb}}(0)$. Then, we iteratively construct the sequence of solutions along the following Picard iterative scheme, which is suitable by the uniqueness result: For $n \geq 1$, let be given

$$(\mathbf{v}^{n-1}, k^{n-1}) \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}) \times L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).$$

Then, the couple

$$(\mathbf{v}^n, k^n) \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}) \times L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

is defined as the solution of the following system

(2.5.57)
$$\begin{cases} \mathbf{v}_t^n - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t^n) + (\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n - \nu\Delta\mathbf{v}^n - \nabla\cdot(\nu_{\text{turb}}(k^n)D\mathbf{v}^n) + \nabla p^n = \mathbf{f}, \\ \nabla\cdot\mathbf{v}^n = 0, \\ \mathbf{v}^n|_\Gamma = 0, \\ \mathbf{v}^n_{|t=0} = \mathbf{v}_0, \\ k_t^n + \mathbf{v}^{n-1}\cdot\nabla k^n - \nabla\cdot(\mu_{\text{turb}}(k^n)\nabla k^n) = T_n(\nu_{\text{turb}}(k^n)|D\mathbf{v}^{n-1}|^2) - (\ell+\eta)^{-1}k^n\sqrt{|k^n|}, \\ k^n|_\Gamma = 0, \\ k^n_{|t=0} = T_n(k_0). \end{cases}$$

By using the estimates from [CRL14, Chapter 8] that, up to a sub-sequence,

$$\begin{cases} k^n \rightharpoonup k & \text{in} \quad L^q(0,T;W_0^{1,q}) \qquad \text{for all } 1 \leq q < 5/4, \\ k_t^n \rightharpoonup k_t & \text{in} \quad L^q(0,T;W^{-1,q}) \qquad \text{for all } 1 \leq q < 5/4, \\ k^n \to k & \text{in} \quad L^q(Q_T) \qquad \text{for all } 1 \leq q < 29/14 \quad \text{and a.e. in } Q_T. \end{cases} \tag{2.5.58}$$

Next, we observe that $x \mapsto \ell\, T_{M'}\big(\sqrt{|x|}\big)$ is a continuous function over $\mathbb{R}$, we have that $\nu_{\text{turb}}^n = \nu_{\text{turb}}(k^n) \to \nu_{\text{turb}} = \nu_{\text{turb}}(k)$ a.e. in Q_T. Next, since $\ell \in C^1(\overline{\Omega})$, we also have

$$0 \leq \nu_{\text{turb}} \leq M'\|\ell\|_\infty,$$

showing that $(\nu_{\mathrm{turb}}^n)_{n\in\mathbb{N}}$ verifies all the requirements of Lemma 2.5.1, by (2.5.53) and (2.5.58). Therefore, $\mathbf{v}^n \to \mathbf{v} = \mathbf{v}(k)$, the corresponding regular-weak solution to the subsystem [(2.5.57), (i), (ii), (iii), (iv))]. Notice that thanks to the uniqueness of the limit, all the sequence does converges.

Passing to the limit in the equation for k follows what is done in [CRL14, Chapter 8], except about the quadratic source term, that needs to be reconsidered. Always by Lemma 2.5.1, $\nu_{\mathrm{turb}}(k^n)|D\mathbf{v}^n|^2 \to \nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2$ in $L^1(Q_T)$ and in the sense of measures. Finally, according to Lemma 2.5.2, we deduce that $T_n(\nu_{\mathrm{turb}}(k^n)|D\mathbf{v}^n|^2) \to \nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2$ in $L^1(Q_T)$ up to a subsequence, the conclusion being from this point straightforward (cf. Lemma 2.5.2), which ends the proof. □

Lemma 2.5.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(Q_T)$ that converges to f in $L^1(Q_T)$. Then from the sequence $(T_n(f_n))_{n\in\mathbb{N}}$ we can extract a subsequence that still converges to f in $L^1(Q_T)$.*

Proof. Since $f_n \to f$ in $L^1(Q_T)$, we can find a sub-sequence (still denoted by f_n) and a function $F \in L^1(Q_T)$ such that:

(i) $f_n(t,\mathbf{x}) \to f(t,\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in Q_T$,

(ii) $|f_n(t,\mathbf{x})| \le F(t,\mathbf{x})$ for all $n \in \mathbb{N}$ and for a.e. $(t,\mathbf{x}) \in Q_T$.

Hence, the sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ satisfies the following:

(1) The sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ converges to $f(t,\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in Q_T$ and $f(t,\mathbf{x})$ is a.e. finite. This follows since a.e. for fixed $(t_0,\mathbf{x}_0)$ it holds $f_n(t_0,\mathbf{x}_0) < f(t_0,\mathbf{x}_0) + 1$ for all $n > n_0(t_0,\mathbf{x}_0) \in \mathbb{N}$, hence $g_n(t_0,\mathbf{x}_0) = f_n(t_0,x_0)$ for all $n > \max\{n_0, f(t_0,\mathbf{x}_0)+1\}$;

(2) The sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ is uniformly equi-absolutely integrable, that is for all $\epsilon > 0$ there exists $\delta > 0$ such that for all measurable $H \subset Q_T$ such that $\mathrm{meas}(H) < \delta$, it holds

$$\sup_{n\in\mathbb{N}} \int_H |T_n(f_n(t,\mathbf{x}))|\,d\mathbf{x} \le \sup_{n\in\mathbb{N}} \int_H |f_n(t,\mathbf{x})|\,d\mathbf{x} \le \int_H |F(t,\mathbf{x})|\,d\mathbf{x} < \epsilon,$$

the last inequality being valid since $F \in L^1(Q_T)$.

The hypotheses of Vitali convergence theorem are satisfied, hence $g_n \to g$ in $L^1(Q_T)$. Therefore, the proof is complete. □

Chapter 3

Modeling Error of α-Models of Turbulence on a Two-Dimensional Torus

> "One does not live on bread alone, but by every word that comes forth from the mouth of God."
>
> MT 4:4B

Contents

This chapter is mostly based on the research paper [BDLN20].

Abstract: This chapter is devoted to study the rate of convergence of the weak solutions $\mathbf{u}_\alpha$ of α-regularization models to the weak solution $\mathbf{u}$ of the Navier-Stokes equations in the two-dimensional periodic case, as the regularization parameter α goes to zero. More specifically, we will consider the Leray-α, the simplified Bardina, and the modified Leray-α models. Our aim is to improve known results in terms of convergence rates and also to show estimates valid over long time intervals.

Key words: Rate of convergence, α-turbulence models, Navier-Stokes equations.

2010 MSC: 35Q30, 35Q35, 65M15, 76F65, 76D05.

3.1 Introduction

In this work we study the rates of convergence of weak solutions of several two dimensional α-models of turbulence to the weak solution of the Navier-Stokes equations (NSE), with periodic boundary conditions. We work mainly in two space dimensions, even if some remarks concerning the three dimensional case are given in Section 3.6. The turbulence models we study belong to the class of Large Eddy Simulation models (LES), used to carry out numerical simulations of turbulence flows, that cannot be performed by the NSE. In fact, according to Kolmogorov laws, it would require $\mathcal{O}(Re^{d^2/4})$ degrees of freedom where $d = 2, 3$, which is still inaccessible to modern computers, for higher (real-life) Reynolds numbers [BIL06, CRL14]. The motivation to consider the 2D case is because this setting is appropriate to analyse models that simulate layers of shallow water in stratified flows, such as those occurring in the ocean or in the atmosphere [CMP97, Lew97b].

Let $L > 0$ denotes a given length scale, $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ for $t > 0$ and $\mathbf{x} \in \Omega$, where $\Omega = [0, L]^2$ be a periodic domain, denote the velocity and the pressure of an incompressible fluid, respectively, which satisfies the NSE,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } (0, +\infty) \times \Omega, \tag{3.1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } (0, +\infty) \times \Omega, \tag{3.1.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega, \tag{3.1.3}$$

where the constant $\nu > 0$ denotes the kinematic viscosity, $\mathbf{u}_0$ and $\mathbf{f}$ are given as the initial velocity and the external force in the same order. The α-models aim at regularizing the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and are given by the following general abstract form

$$\partial_t \mathbf{u}_\alpha + N(\mathbf{u}_\alpha) - \nu \Delta \mathbf{u}_\alpha + \nabla p_\alpha = \mathbf{f}, \qquad \text{in } (0, +\infty) \times \Omega, \tag{3.1.4}$$

$$\nabla \cdot \mathbf{u}_\alpha = 0, \qquad \text{in } (0, +\infty) \times \Omega, \tag{3.1.5}$$

$$\mathbf{u}_\alpha|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega, \tag{3.1.6}$$

where, for $\alpha > 0$ the fields $\mathbf{u}_\alpha$ and p_α are the filtered velocity and pressure, respectively, at frequencies of order $1/\alpha$. The α-models under study herein are: the Leray-α, the simplified Bardina and the modified Leray-α models, given by

$$N(\mathbf{u}_\alpha) = \begin{cases} (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha & \text{Leray-}\alpha \text{ model (L-}\alpha\text{)}, \\ (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{Simplified Bardina model (SB)}, \\ (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{Modified Leray-}\alpha \text{ model (ML-}\alpha\text{)}, \end{cases} \tag{3.1.7}$$

and the bar operator is given by solving the following equation

$$\overline{\mathbf{v}} - \alpha^2 \Delta \overline{\mathbf{v}} = \mathbf{v} \quad \text{in } \Omega, \tag{3.1.8}$$

in the setting of periodic functions with zero mean value.

The first model $N_\alpha(\mathbf{v}) = (\overline{\mathbf{v}} \cdot \nabla)\mathbf{v}$ is due to J. Leray [Ler34a, Ler34b], who considered the problem in the whole space $\mathbb{R}^3$, and where $\overline{\mathbf{v}} = \mathbf{v} * \rho_\alpha$, for a standard mollifier ρ_α. Note that in the whole space it is also possible to explicitly write a Kernel G_α such that $\overline{\mathbf{v}} = \mathbf{v} * G_\alpha$ for the Helmholtz filter [LB18]. The class of α-models has been the subject of many investigations in the last two decades, see for instance [CLT06, CFH+99a, CFH+99b, CFH+98, CHOT05, FHT01, FHT02, HT88, ILT06, LL03, LL06]. It is known that the Cauchy problem has global, unique, and regular solutions, with $\mathbf{u}_\alpha$ at least in $L^\infty_t H^1_{\mathbf{x}} \cap L^2_t H^2_{\mathbf{x}}$. These solutions converge to solutions of the NSE as $\alpha \to 0$. which means that $\mathbf{u}_\alpha \to \mathbf{u}$, $p_\alpha \to p$, where $(\mathbf{u}, p)$ is the corresponding weak solution of the NSE, under suitable assumptions about the data.

In this paper we will study the rate of convergence as $\alpha \to 0$, namely the norm of

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_\alpha, \tag{3.1.9}$$

in various spaces such as $L^\infty_t L^2_{\mathbf{x}}$, $L^2_t H^1_{\mathbf{x}}$, $L^\infty_t H^1_{\mathbf{x}}$ and $L^2_t H^2_{\mathbf{x}}$. This study is motivated by the results in Cao and Titi [CT09], in which the authors proved that for all 2D α-models (3.1.4)-(3.1.7) (also for the Navier-Stokes-α model, in which the nonlinear term is given by $N(\mathbf{u}_\alpha) = -\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha)$[1]), the following $L^\infty_t L^2_{\mathbf{x}}$ estimate holds true on a given time interval $[0, T]$

$$\sup_{t \in [0,T]} \|\overline{\mathbf{u}}_\alpha(t) - \mathbf{u}(t)\|^2 \leq C\alpha^2 \left(T \left(1 + \log \left(\frac{L}{2\pi\alpha} \right) \right) + 1 \right) \qquad \forall\, \alpha \leq \frac{L}{2\pi}, \tag{3.1.10}$$

[1] The nonlinear term of the Navier-Stokes-α model does not seem to be well-defined in the 2D case. Therefore, this model is not considered in this paper.

where C is a generic constant and when no risk of confusion occurs, $\|\cdot\|$ stands for the usual L^2-norm. To prove the convergence rate (3.1.10) it is assumed that

$$\mathbf{u}_0 \in \mathcal{D}(-\mathcal{P}_\sigma\Delta) \qquad \text{and} \qquad \mathbf{f} \in L^2([0,T];\mathcal{P}_\sigma L^2(\Omega)^2),$$

here $\mathcal{P}_\sigma$ being the so-called Helmholtz-Leray projector. The logarithmic factor that appears in (3.1.10) comes from the application of an inequality initially proved by Brézis and Gallouët in [BG80]. Cao-Titi's result raises two questions:

i) Is it possible to improve the $O(\alpha^2\log(1/\alpha))$ rate, and what about the convergence rate in stronger norms?

ii) Is it possible to prove an estimate global in time?

In this paper we positively answer to both these questions by showing that when

$$\mathbf{u}_0 \in \mathcal{P}_\sigma H^1(\Omega)^2 \qquad \text{and} \qquad \mathbf{f} \in L^2(\mathbb{R}_+;\mathcal{P}_\sigma L^2(\Omega)^2),$$

we get an estimate uniform in time of order $O(\alpha^3)$ in the $L^\infty_t L^2_{\mathbf{x}} \cap L^2_t H^1_{\mathbf{x}}$ norms for the error. More specifically we prove that for all α-models (3.1.4)-(3.1.7), it holds

$$\|\mathbf{e}(s)\|^2 + \nu\int_0^s \|\nabla\mathbf{e}\|^2\,dt \leq C\alpha^3 \qquad \forall\, s \geq 0, \tag{3.1.11}$$

where C is a time-independent constant, see Theorem 3.4.1 below. We also get a uniform in time estimate in the $L^\infty_t H^1_{\mathbf{x}} \cap L^2_t H^2_{\mathbf{x}}$ norms of order $O(\alpha^2)$ for the L-α model, and in $O(\alpha^2\log(1/\alpha))$ for SB and ML-α models, namely for all $s \geq 0$, we will prove

$$\|\nabla\mathbf{e}(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{e}\|^2\,dt \leq \begin{cases} C\alpha^2 & \text{for L-}\alpha, \\ C\alpha^2\left(\log\left(\dfrac{L}{2\pi\alpha}\right)+1\right) & \text{for SB and ML-}\alpha, \end{cases} \tag{3.1.12}$$

see Theorem 3.4.2 below. Estimates (3.1.11) and (3.1.12) are the main results in the present work.

Thanks to (3.1.11)-(3.1.12), we are also able to study the rates of convergence of the pressures, by showing (see Theorem 3.5.1 below) that

$$\int_0^s \|\nabla q\|^2\,dt \leq \begin{cases} C\alpha^{5/2} & \text{for L-}\alpha\ , \\ C\alpha^2\left(\log\left(\dfrac{L}{2\pi\alpha}\right)+1\right) & \text{for SB and ML-}\alpha, \end{cases} \tag{3.1.13}$$

where C is independent of the time and $q := p_\alpha - p$. Note that the estimates (3.1.11)-(3.1.13) are presented under a small condition of α, that is $\alpha \leq L/2\pi$.

Plan of the chapter. The chapter is organized as follows: In Section 3.2 we set the mathematical framework. In Section 3.3 we derive from energy balances uniform-in-time energy(type) estimates for weak solutions of the NSE and for all α-models as well. This is the main step before investigating the rates of convergence in Section 3.4, where we prove the estimates (3.1.11)-(3.1.12). Section 3.5 is devoted to the study of the convergence rate for the pressure, in which the proof of (3.1.13) is provided. In Section 3.6, we make some additional remarks about the 3D case for which the situation is quite different and not in the focus of the present chapter.

3.2 Mathematical framework

In this section we set the functional spaces we are working with. We show basic properties of the Helmholtz filter, then we carry out the Leray projection of the NSE and Leray-α on divergence-free fields spaces. The section ends with a brief state of the art about the NSE and all α-models as well.

3.2.1 Function spaces

Let $\Omega := [0, L]^2$ be a periodic domain. For $1 \le p \le \infty$ and $m \in \mathbb{N}$, let $L^p(\Omega)$ and $H^m(\Omega)$ denote the standard Lebesgue and Sobolev spaces on Ω, respectively. The $L^p(\Omega)$-norm is denoted by $\|\cdot\|_p$ for all $1 \le p \le \infty$, except for the case $p = 2$ where $\|\cdot\| \equiv \|\cdot\|_2$. Boldface symbols are used for vectors, matrices, or space of vectors. We denote by Π the set of all trigonometric polynomials on Ω with spatial zero mean, i.e.,

$$\int_\Omega \phi(\mathbf{x})\, d\mathbf{x} = 0, \quad \forall\, \phi \in \Pi.$$

Let us define

$$\mathbf{\Lambda} := \left\{ \boldsymbol{\varphi} \in \Pi^2 \,:\, \nabla \cdot \boldsymbol{\varphi} = 0 \right\}.$$

As usual when studying the NSE we define the following standard Hilbert functional spaces

$$\mathbf{H} := \text{the closure of } \mathbf{\Lambda} \text{ in } L^2(\Omega)^2,$$
$$\mathbf{V} := \text{the closure of } \mathbf{\Lambda} \text{ in } H^1(\Omega)^2.$$

Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the standard inner product and norm on $\mathbf{H}$, that are

$$(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} \quad \text{and} \quad \|\mathbf{u}\|^2 := \int_\Omega |\mathbf{u}|^2\, d\mathbf{x}.$$

The inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{V}}$ and the corresponding norm $\|\mathbf{u}\|_{\mathbf{V}}$ on $\mathbf{V}$ are defined as follows

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{V}} := \|\nabla \mathbf{u}\|.$$

In the sequel, we use the symbol $\mathcal{P}_\sigma$ to denote the Helmholtz-Leray orthogonal projection operator of $L^2(\Omega)^2$ onto $\mathbf{H}$. We next consider an orthonormal basis $\{\boldsymbol{\varphi}_j\}_{j\in\mathbb{N}}$, of $\mathbf{H}$ consisting of eigenfunctions of the Laplace operator

$$-\Delta \,:\, H^2(\Omega)^2 \cap \mathbf{V} \longrightarrow \mathbf{H},$$

and for $m \geq 1$, $\mathbf{H}_m := \mathrm{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_m\}$.

Let $A = -\mathcal{P}_\sigma\Delta$ be the so-called Stokes operator, with domain $\mathcal{D}(A) := H^2(\Omega)^2 \cap \mathbf{V}$. Then, it is well-known (cf. [CT09, FHT02]) that:

$$A\mathbf{u} = -\mathcal{P}_\sigma\Delta\mathbf{u} = -\Delta\mathbf{u} \qquad \forall\, \mathbf{u} \in \mathcal{D}(A).$$

Let $\lambda_1 > 0$ be the first eigenvalue of A, i.e., $A\boldsymbol{\varphi}_1 = \lambda_1\boldsymbol{\varphi}_1$, and the above setting leads to $\lambda_1 = (2\pi/L)^2$. By virtue of the Poincaré inequality we have

$$\lambda_1\|\mathbf{u}\|^2 \leq \|\nabla\mathbf{u}\|^2 \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{3.2.14}$$

$$\lambda_1\|\nabla\mathbf{u}\|^2 \leq \|A\mathbf{u}\|^2 = \|\Delta\mathbf{u}\|^2 \qquad \forall\, \mathbf{u} \in \mathcal{D}(A). \tag{3.2.15}$$

Then, it follows by (3.2.14)-(3.2.15) that there exist positive dimensionless constants c_1, c_2 such that

$$c_1\|A\mathbf{u}\| \leq \|\mathbf{u}\|_{H^2(\Omega)} \leq c_2\|A\mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathcal{D}(A).$$

In the following, we will make an intensive use of the 2D-Ladyžhenskaya inequality [Lad69]:

$$\|\mathbf{u}\|_4 \leq C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2} \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{3.2.16}$$

where C is a non-negative dimensionless constant.

3.2.2 On the Helmholtz filter

The filter operator used to construct the turbulence models is the differential filter associated with the Helmholtz filter, see Germano [Ger86], or [BL12, DJ04, LL08]. Given a cut length $\alpha > 0$ (which will be called the filter radius), for each $\mathbf{u} \in \mathbf{H}$, then $\overline{\mathbf{u}} \in H^2(\Omega)^2 \cap \mathbf{V}$ is the unique solution of the equation (3.1.8). By a direct calculation from (3.1.8) we deduce

$$\|\mathbf{u} - \overline{\mathbf{u}}\| = \alpha^2\|\Delta\overline{\mathbf{u}}\| \qquad \forall\, \mathbf{u} \in \mathbf{H}.$$

Moreover, we already know that the filter satisfies the following inequality, see [Dun18]:

$$\|\overline{\mathbf{u}}\| + \alpha\|\nabla\overline{\mathbf{u}}\| + \alpha^2\|\Delta\overline{\mathbf{u}}\| \leq C\|\mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathbf{H}, \tag{3.2.17}$$

where C is a Sobolev constant. It follows that

$$\|\nabla\mathbf{u} - \nabla\overline{\mathbf{u}}\| = \alpha^2\|\nabla\Delta\overline{\mathbf{u}}\| \leq C\alpha\|\Delta\mathbf{u}\| \qquad \forall\, \mathbf{u} \in \mathcal{D}(A). \tag{3.2.18}$$

3.2.3 On the Leray projection operator

Throughout the rest of the paper we assume

$$\mathbf{u}_0 \in \mathbf{V} \quad \text{and} \quad \mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H}). \tag{3.2.19}$$

In order to eliminate the pressure from the equations, we apply the Helmholtz-Leray orthogonal projection $\mathcal{P}_\sigma : L^2(\mathbb{T}_2)^2 \to \mathbf{H}$ on divergence-free fields to both the NSE and α-models. We get the following functional equations:

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \mathcal{P}_\sigma[(\mathbf{u} \cdot \nabla)\mathbf{u}] - \nu\Delta\mathbf{u} & = \mathbf{f}, \\ \mathbf{u}|_{t=0} & = \mathbf{u}_0, \end{cases} \tag{3.2.20}$$

as well as

$$\begin{cases} \dfrac{d\mathbf{u}_\alpha}{dt} + \mathcal{P}_\sigma[N(\mathbf{u}_\alpha)] - \nu\Delta\mathbf{u}_\alpha & = \mathbf{f}, \\ \mathbf{u}_\alpha|_{t=0} & = \mathbf{u}_0, \end{cases} \tag{3.2.21}$$

where we used the facts that $\mathcal{P}_\sigma\mathbf{f} \equiv \mathbf{f}$ since $\mathbf{f} \in \mathbf{H}$, $\mathcal{P}_\sigma\Delta\mathbf{u} = \Delta\mathbf{u}$ due to the periodic setting, and $\mathcal{P}_\sigma(\nabla p) = \mathcal{P}_\sigma(\nabla p_\alpha) = 0$. Once the velocity is calculated, the pressures p and p_α are solutions of the following Poisson equations

$$-\Delta p = \nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}) \qquad \text{and} \qquad -\Delta p_\alpha = \nabla \cdot (N(\mathbf{u}_\alpha)).$$

From now when speaking of solutions to the NSE and to α-models we will only consider the velocities, and the pressures can be associated by solving the above equations.

Remark 1. *Thanks to the Leray-Helmholtz decomposition and for simplicity we assume that* $\mathbf{f}$ *is divergence free. If not the case, the gradient part of* $\mathbf{f}$ *can be added to the pressure (to obtain a modified pressure) and* $\mathcal{P}_\sigma\mathbf{f}$ *will replace* $\mathbf{f}$*.*

Remark 2. *A common property of all α-models considered in the present paper is that these models "formally" reduce to the NSE when $\alpha = 0$. It can be seen directly from the equality (3.1.8).*

3.2.4 Brief state of the art

It is well-known that in the 2D case, there exists a unique solution of the NSE, global in time, without formation of singularities, see for example Temam [Tem95, Tem01]. Nevertheless, this does not resolve the computational issues of the shallow waters or of stratified flows.

The proof of the existence and uniqueness of solution of the α-models given by (3.1.7) can be established by using the standard Galerkin method. The L-α model was implemented computationally by Cheskidov-Holm-Olson-Titi [CHOT05]. Ilyin-Lunasin-Titi introduced and studied the ML-α model in the 3D periodic case, see [ILT06] and it was tested numerically in [GKT08]. However, the global existence and uniqueness for 2D can be proved in a similar way.

The Bardina closure model of turbulence was first introduced by Bardina-Ferziger-Reynolds in [BFR80] to perform simulations of the atmosphere. A simplified version of the Bardina's model, was modeled and studied in [LL03, LL06], then in [LB18] the whole space case was studied. This model is designed by $N(\mathbf{u}_\alpha) = \overline{\nabla \cdot (\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha)}$. Cao-Lunasin-Titi proposed a variant of this model [CLT06], which is the one we consider in this paper and that we still call "Simplified Bardina model" (SB).

3.3 A priori estimates

3.3.1 General orientation

As the data are given as in (3.2.19) it is well-known that both the NSE (3.1.1)-(3.1.3) and the α-model (3.1.5)-(3.1.6) (for any nonlinearity $N(\mathbf{u}_\alpha)$ as those given in (3.1.7)) admit unique solutions $\mathbf{u}$ and $\mathbf{u}_\alpha$, respectively, such that

$$\mathbf{u}, \mathbf{u}_\alpha \in L^\infty(\mathbb{R}_+; \mathbf{V}) \cap L^2(\mathbb{R}_+; H^2(\Omega)^2 \cap \mathbf{V}).$$

To shorten the notation in the following we set

$$\mathcal{F} := \|\mathbf{f}\|^2_{L^2(\mathbb{R}_+;\mathbf{H})}.$$

In this section, we detail the $L^2(\mathbb{R}_+; H^2(\Omega)^2 \cap \mathbf{V})$ estimates to get precise constants, for the various models. The analysis is based on 2D energy inequalities, using the Ladyženskaya inequality (3.2.16) and the following identities

$$(\mathcal{P}_\sigma((\mathbf{u}\cdot\nabla)\mathbf{u}), \mathbf{u}) = (\mathcal{P}_\sigma((\mathbf{u}\cdot\nabla)\mathbf{u}), \Delta\mathbf{u}) = 0 \qquad \forall\, \mathbf{u} \in \mathcal{D}(A), \tag{3.3.22}$$

which are well-known [Tem95] and on the extension to the L-α and the SB models,

$$(\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \mathbf{u}_\alpha) = (\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \Delta\overline{\mathbf{u}}_\alpha) = 0 \qquad \forall\, \mathbf{u}_\alpha \in \mathbf{V}.$$

However, the nonlinearity in the ML-α model is less favorable, since we only have

$$(\mathcal{P}_\sigma N(\mathbf{u}_\alpha), \overline{\mathbf{u}}_\alpha) = 0.$$

3.3.2 Estimates for the NSE

We recall the basic estimate for weak solutions to the two dimensional NSE.

Lemma 3.3.1 (NSE). *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}$ of the NSE satisfies*

$$\|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2 \, dt \leq \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1} =: C_{N1} \qquad \forall\, s \geq 0, \tag{3.3.23}$$

$$\|\nabla\mathbf{u}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}\|^2 \, dt \leq \|\nabla\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu} =: C_{N2} \qquad \forall\, s \geq 0. \tag{3.3.24}$$

Remark 3. *Estimate (3.3.23) in the previous theorem can be obtained more generally when $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{V}')$ where $\mathbf{V}'$ denotes the dual space of $\mathbf{V}$. We use the condition $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$ for both estimates (3.3.23) and (3.3.24) for shortness.*

Proof. Proofs are well-known but we reproduce them to keep precise track of the constants and to see differences with the other models. We argue step by step, first proving (3.3.23).

Step 1. $L^\infty_t L^2_{\mathbf{x}} \cap L^2_t H^1_{\mathbf{x}}$ estimate of u. Take the scalar product of the NSE (3.2.20) with $\mathbf{u}$ and use the identity $(\mathcal{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{u}) = 0$, which lead to the following estimate

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla\mathbf{u}\|^2 \leq \|\mathbf{f}\|\|\mathbf{u}\|. \tag{3.3.25}$$

Using Poincaré and Young inequalities on the r.h.s (right-hand side) of (3.3.25) yields:

$$\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla\mathbf{u}\|^2 \leq \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.3.26}$$

Integrating (3.3.26) on $[0, s]$ for $s \geq 0$, one has

$$\|\mathbf{u}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}\|^2 \, dt \leq \|\mathbf{u}_0\|^2 + \frac{1}{\nu\lambda_1}\int_0^s \|\mathbf{f}\|^2 \, dt. \tag{3.3.27}$$

Finally, the estimate (3.3.23) follows by (3.3.27) since $s \geq 0$ can be chosen arbitrary.

Step 2. $L^\infty_t H^1_{\mathbf{x}} \cap L^2_t H^2_{\mathbf{x}}$ estimate of u. In order to prove the estimate (3.3.24), we take $-\Delta\mathbf{u}$ as a test function for the NSE (3.2.20). As we already have said, in the 2D case periodic the nonlinear term vanishes, cf. (3.3.22). By the Young inequality the term corresponding to the body force can be estimated by

$$(\mathbf{f}, -\Delta\mathbf{u}) \leq \frac{1}{2\nu}\|\mathbf{f}\|^2 + \frac{\nu}{2}\|\Delta\mathbf{u}\|^2,$$

and the rest of the proof follows as for the first estimate. Thus, the proof is complete. $\square$

3.3.3 Estimates for the Leray-α model

We now prove a uniform estimate for weak solutions to the Leray-α model.

Lemma 3.3.2 (L-α). *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}_\alpha$ of the L-α satisfies the following energy-type estimates*

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2\, dt \le \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1} \qquad \forall\, s \ge 0, \tag{3.3.28}$$

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2\, dt \le \frac{CC_{L1}^2}{\nu^4}\left(\|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1}\right) + \frac{2\mathcal{F}}{\nu} =: C_L \quad \forall\, s \ge 0, \tag{3.3.29}$$

where C_{L1} is given in (3.3.32).

Proof. For the L-α model, we recall the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha \quad \text{where} \quad \overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha = \mathbf{u}_\alpha.$$

We argue in three steps, with an intermediate step to estimate $\overline{u}_\alpha$ uniformly in time.

Step 1. $L_t^\infty L_{\mathbf{x}}^2 \cap L_t^2 H_{\mathbf{x}}^1$ estimate of $\mathbf{u}_\alpha$. Taking $\mathbf{u}_\alpha$ as a test function in the L-α model (3.2.21) gives

$$\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu\|\nabla \mathbf{u}_\alpha\|^2 \le \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2.$$

Since $(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \mathbf{u}_\alpha) = 0$, see [CT09], this leads to (3.3.28).

Step 2. $L_t^\infty H_{\mathbf{x}}^1 \cap L_t^2 H_{\mathbf{x}}^2$ estimate of $\overline{\mathbf{u}}_\alpha$. Testing (3.2.21) by $-\Delta \overline{\mathbf{u}}_\alpha$ and replacing $\mathbf{u}_\alpha$ by $\overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ yield

$$\frac{d}{dt}\left(\|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2\right) + \nu\|\Delta \overline{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2\|\nabla\Delta\overline{\mathbf{u}}_\alpha\|^2 \le \frac{\|\mathbf{f}\|^2}{\nu}. \tag{3.3.30}$$

Here, the vanishing of the nonlinear term has been used, i.e.,

$$(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], -\Delta\overline{\mathbf{u}}_\alpha) = ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha), -\Delta\overline{\mathbf{u}}_\alpha) = 0,$$

which is a consequence of (3.3.22). Therefore, by (3.3.30) for all $s \ge 0$

$$\|\nabla \overline{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2\|\Delta\overline{\mathbf{u}}_\alpha(s)\|^2 + \nu\int_0^s \left(\|\Delta\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2\|\nabla\Delta\overline{\mathbf{u}}_\alpha\|^2\right) dt \le C_{L1}, \tag{3.3.31}$$

where C_{L1} is given by

$$\|\nabla\overline{\mathbf{u}}_0\|^2 + \alpha^2\|\Delta\overline{\mathbf{u}}_0\|^2 + \frac{\mathcal{F}}{\nu} \le (1+\lambda_1)\|\nabla\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu} =: C_{L1}, \tag{3.3.32}$$

here the inequalities

$$\|\nabla\overline{\mathbf{u}}_0\| \le \|\nabla\mathbf{u}_0\| \qquad \text{and} \qquad \alpha^2\|\Delta\overline{\mathbf{u}}_0\|^2 \le \|\mathbf{u}_0\|^2,$$

given by (3.2.17) and the Poincaré inequality has been applied.

Step 3. $L^\infty_t H^1_{\mathbf{x}} \cap L^2_t H^2_{\mathbf{x}}$ estimate of $\mathbf{u}_\alpha$. We test (3.2.21) by $-\Delta \mathbf{u}_\alpha$ which leads now to the following equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu\|\Delta \mathbf{u}_\alpha\|^2 = (\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \Delta \mathbf{u}_\alpha) + (\mathbf{f}, -\Delta \mathbf{u}_\alpha). \tag{3.3.33}$$

The first term on the r.h.s of (3.3.33) can be estimated by:

$$\begin{aligned}(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha], \Delta \mathbf{u}_\alpha) &\leq C\|\overline{\mathbf{u}}_\alpha\|_4\|\nabla \mathbf{u}_\alpha\|_4\|\Delta \mathbf{u}_\alpha\| \\ &\leq C\|\nabla \overline{\mathbf{u}}_\alpha\|\|\nabla \mathbf{u}_\alpha\|^{1/2}\|\Delta \mathbf{u}_\alpha\|^{3/2} \\ &\leq \frac{C}{\nu^3}\|\nabla \overline{\mathbf{u}}_\alpha\|^4\|\nabla \mathbf{u}_\alpha\|^2 + \frac{\nu}{4}\|\Delta \mathbf{u}_\alpha\|^2. \end{aligned}\tag{3.3.34}$$

Here we used the Hölder, 2D-Ladyžhenskaya (3.2.16), Sobolev, and Young inequalities, respectively. From (3.3.33)-(3.3.34) one obtains

$$\frac{d}{dt}\|\nabla \mathbf{u}_\alpha\|^2 + \nu\|\Delta \mathbf{u}_\alpha\|^2 \leq \frac{2}{\nu}\|\mathbf{f}\|^2 + \frac{C}{\nu^3}\|\nabla \overline{\mathbf{u}}_\alpha\|^4\|\nabla \mathbf{u}_\alpha\|^2,$$

which yields

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\Delta \mathbf{u}_\alpha\|^2\, dt \leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu^3}\int_0^s \|\nabla \overline{\mathbf{u}}_\alpha\|^4\|\nabla \mathbf{u}_\alpha\|^2\, dt \qquad \forall\, s \geq 0. \tag{3.3.35}$$

Finally, both estimates (3.3.28) and (3.3.31) are applied in (3.3.35) to get (3.3.29), which ends the proof. □

3.3.4 Estimates for the Simplified Bardina model

In this section we prove a uniform estimate for weak solutions to the simplified Bardina model.

Lemma 3.3.3 (SB). *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}_\alpha$ of the SB model satisfies*

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\nabla \mathbf{u}_\alpha\|^2\, dt \leq \|\mathbf{u}_0\|^2 + \frac{\mathcal{F}}{\nu\lambda_1} \qquad \forall\, s \geq 0, \tag{3.3.36}$$

$$\|\nabla \mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\Delta \mathbf{u}_\alpha\|^2\, dt \leq \frac{CC_{S1}^2}{\nu^2\lambda_1} + \frac{2\mathcal{F}}{\nu} =: C_S \qquad \forall\, s \geq 0, \tag{3.3.37}$$

where C is a positive constant and C_{S1} is given by (3.3.38).

Proof. We recall that for this model, the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha,$$

and we will prove the global-in-time estimate in two steps, starting with bounds on $\overline{u}_\alpha$.

Step 1. $L_t^\infty H_{\mathbf{x}}^1 \cap L_t^2 H_{\mathbf{x}}^2$ estimate of $\overline{\mathbf{u}}_\alpha$. Taking $-\Delta\overline{\mathbf{u}}_\alpha$ as a test function in (3.2.21) and using the fact $\mathbf{u}_\alpha = \overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha$ give us

$$\frac{d}{dt}\left(\|\nabla\overline{\mathbf{u}}_\alpha\|^2 + \alpha^2\|\Delta\overline{\mathbf{u}}_\alpha\|^2\right) + \nu\|\Delta\overline{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2\|\nabla\Delta\overline{\mathbf{u}}_\alpha\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|^2,$$

where the identity $(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], -\Delta\overline{\mathbf{u}}_\alpha) = 0$, has been used. Thus, we get

(3.3.38)
$$\|\nabla\overline{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2\|\Delta\overline{\mathbf{u}}_\alpha(s)\|^2 + \nu\int_0^s \left(\|\Delta\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2\|\nabla\Delta\overline{\mathbf{u}}_\alpha\|^2\right) dt \leq C_{S1} \qquad \forall\, s\geq 0,$$

where $C_{S1} := C_{L1}$ as given in (3.3.32).

Step 2. $L_t^\infty H_{\mathbf{x}}^1 \cap L_t^2 H_{\mathbf{x}}^2$ estimate of $\mathbf{u}_\alpha$. Taking $\mathbf{u}_\alpha$ as test function in (3.2.21) we obtain

$$\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu\|\nabla\mathbf{u}_\alpha\|^2 \leq \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2.$$

Since $(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], \mathbf{u}_\alpha) = 0$, this leads to (3.3.36). Taking $-\Delta\mathbf{u}_\alpha$ as test function in (3.2.21) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 = (\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) - (\mathbf{f}, \Delta\mathbf{u}_\alpha). \tag{3.3.39}$$

The nonlinear term on the r.h.s of (3.3.39) is estimated by:

$$\begin{aligned}
(\mathcal{P}_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) &\leq C\|\overline{\mathbf{u}}_\alpha\|_4\|\nabla\overline{\mathbf{u}}_\alpha\|_4\|\Delta\mathbf{u}_\alpha\| \\
&\leq C\|\overline{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\overline{\mathbf{u}}_\alpha\|\|\Delta\overline{\mathbf{u}}_\alpha\|^{1/2}\|\Delta\mathbf{u}_\alpha\| \\
&\leq \frac{C}{\nu}\|\overline{\mathbf{u}}_\alpha\|\|\nabla\overline{\mathbf{u}}_\alpha\|^2\|\Delta\overline{\mathbf{u}}_\alpha\| + \frac{\nu}{4}\|\Delta\mathbf{u}_\alpha\|^2.
\end{aligned} \tag{3.3.40}$$

In the above inequalities the Hölder, 2D-Ladyžhenskaya, and Young inequalities have been applied, respectively. The estimates (3.3.39)-(3.3.40) lead to

$$\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 \leq \frac{2}{\nu}\|\mathbf{f}\|^2 + \frac{C}{\nu}\|\overline{\mathbf{u}}_\alpha\|\|\nabla\overline{\mathbf{u}}_\alpha\|^2\|\Delta\overline{\mathbf{u}}_\alpha\|.$$

and by using (3.3.38) we get

$$\begin{aligned}
\|\nabla\mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{u}_\alpha\|^2\, dt &\leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu}\int_0^s \|\overline{\mathbf{u}}_\alpha\|\|\nabla\overline{\mathbf{u}}_\alpha\|^2\|\Delta\overline{\mathbf{u}}_\alpha\|\, dt \\
&\leq \frac{2\mathcal{F}}{\nu} + \frac{CC_S}{\nu\lambda_1}\int_0^s \|\Delta\overline{\mathbf{u}}_\alpha\|^2\, dt \\
&\leq \frac{2\mathcal{F}}{\nu} + \frac{CC_S^2}{\nu^2\lambda_1} \qquad \forall\, s\geq 0.
\end{aligned}$$

Therefore, the proof is complete. □

3.3.5 Estimates for the Modified Leray-α model

In this section we prove a uniform estimate for weak solutions to the modified Leray-α model

Lemma 3.3.4 (ML-α). *Let $\mathbf{u}_0 \in \mathbf{V}$ and let $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. Then, the unique weak solution $\mathbf{u}_\alpha$ of the ML-α model satisfies*

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 \, dt \leq C_{M2} \qquad \forall\, s \geq 0, \tag{3.3.41}$$

$$\|\nabla \mathbf{u}_\alpha(t)\|^2 + \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \leq \frac{C_{M4}}{\nu^4} + \frac{2\mathcal{F}}{\nu} =: C_M \qquad \forall\, s \geq 0, \tag{3.3.42}$$

where $C_{M4} := C\, C_{M1} C_{M2} C_{M3}$ with C is a positive constant, while for $i = 1, 2, 3$, the constants C_{Mi} are given by (3.3.45), (3.3.49) and (3.3.54), respectively.

Proof. The nonlinear term of this model is given by

$$N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha.$$

This case requires more care than the previous ones, since the cancellations are less favorable. We prove it into three steps, starting with a $L^\infty_t L^2_{\mathbf{x}} \cap L^2 H^1_{\mathbf{x}}$ estimate of $\overline{\mathbf{u}}_\alpha$, then a $L^\infty_t L^2_{\mathbf{x}} \cap L^2 H^1_{\mathbf{x}}$ estimate of $\mathbf{u}_\alpha$, to finally get the conclusion.

Step 1. $L^\infty_t L^2_{\mathbf{x}} \cap L^2_t H^1_{\mathbf{x}}$ estimate of $\overline{\mathbf{u}}_\alpha$. Taking $\overline{\mathbf{u}}_\alpha$ as test function in (3.2.21) and replacing $\mathbf{u}_\alpha$ by $\overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ we obtain

$$\frac{d}{dt}\left(\|\overline{\mathbf{u}}_\alpha\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2\right) + \nu \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + 2\nu\alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2 \leq \frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2. \tag{3.3.43}$$

Here the fact $(\mathcal{P}_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \overline{\mathbf{u}}_\alpha) = 0$ and the Poincaré inequality have been used on the r.h.s. Then one gets from (3.3.43)

$$\|\overline{\mathbf{u}}_\alpha(s)\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_\alpha(s)\|^2 + \nu \int_0^s \left(\|\nabla \overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2 \|\Delta \overline{\mathbf{u}}_\alpha\|^2\right) ds \leq C_{M1} \qquad \forall s \geq 0, \tag{3.3.44}$$

where as in (3.3.32) above C_{M1} is given by

$$\|\overline{\mathbf{u}}_0\|^2 + \alpha^2 \|\nabla \overline{\mathbf{u}}_0\|^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} \leq (1 + \lambda_1)\|\mathbf{u}_0\|^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} =: C_{M1}. \tag{3.3.45}$$

Step 2. $L^\infty_t L^2_{\mathbf{x}} \cap L^2_t H^1_{\mathbf{x}}$ estimate of $\mathbf{u}_\alpha$. Taking $\mathbf{u}_\alpha$ as test function in (3.2.21) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu \|\nabla \mathbf{u}_\alpha\|^2 = -(\mathcal{P}_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \mathbf{u}_\alpha) + (\mathbf{f}, \mathbf{u}_\alpha). \tag{3.3.46}$$

The nonlinear term on the r.h.s of (3.3.46) can be now estimated by

$$\begin{aligned}
-(\mathcal{P}_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \mathbf{u}_\alpha) &\leq C \|\mathbf{u}_\alpha\|_4^2 \|\nabla \overline{\mathbf{u}}_\alpha\| \\
&\leq C \|\mathbf{u}_\alpha\| \|\nabla \mathbf{u}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\| \\
&\leq \frac{C}{\nu}\|\mathbf{u}_\alpha\|^2 \|\nabla \overline{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_\alpha\|^2.
\end{aligned}$$

Here we used the Hölder, 2D-Ladyženskaya, and Young inequalities, respectively. Using the Young inequality for the other term on the r.h.s of (3.3.46) gives

$$\frac{d}{dt}\|\mathbf{u}_\alpha\|^2 + \nu\|\nabla\mathbf{u}_\alpha\|^2 \le \frac{2}{\lambda_1\nu}\|\mathbf{f}\|^2 + \frac{C}{\nu}\|\mathbf{u}_\alpha\|^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2. \tag{3.3.47}$$

Using the estimate (3.3.44) leads to

$$\begin{aligned}\int_0^s \|\mathbf{u}_\alpha\|^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2\,dt &= \int_0^s \left(\|\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2 + \alpha^4\|\Delta\overline{\mathbf{u}}_\alpha\|^2\right)\|\nabla\overline{\mathbf{u}}_\alpha\|^2\,dt \\ &\le \frac{4C_{M1}^2}{\nu} \qquad \forall\, s \ge 0.\end{aligned} \tag{3.3.48}$$

Here, we also used the following identity

$$\|\mathbf{u}_\alpha\|^2 = \|\overline{\mathbf{u}}_\alpha\|^2 + 2\alpha^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2 + \alpha^4\|\Delta\overline{\mathbf{u}}_\alpha\|^2.$$

Therefore, by (3.3.47)-(3.3.48) we get (3.3.41) with

$$\|\mathbf{u}_\alpha(s)\|^2 + \nu\int_0^s \|\nabla\mathbf{u}_\alpha\|^2\,dt \le \frac{2\mathcal{F}}{\nu\lambda_1} + \frac{4CC_{M1}^2}{\nu^2} + \|\mathbf{u}_0\|^2 =: C_{M2} \qquad \forall\, s \ge 0. \tag{3.3.49}$$

Step 3. $L_t^\infty H_{\mathbf{x}}^1 \cap L_t^2 H_{\mathbf{x}}^2$ estimate of $\mathbf{u}_\alpha$. We take $-\Delta\mathbf{u}_\alpha$ as test function in (3.2.21) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 = (\mathcal{P}_\sigma[(\mathbf{u}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) - (\mathbf{f}, \Delta\mathbf{u}_\alpha). \tag{3.3.50}$$

The nonlinear term can be estimated as follows

$$\begin{aligned}(\mathcal{P}_\sigma[(\mathbf{u}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) &\le C\|\mathbf{u}_\alpha\|_4\|\nabla\overline{\mathbf{u}}_\alpha\|_4\|\Delta\mathbf{u}_\alpha\|_2 \\ &\le C\|\mathbf{u}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha\|^{1/2}\|\nabla\overline{\mathbf{u}}_\alpha\|^{1/2}\|\Delta\mathbf{u}_\alpha\|^{3/2} \\ &\le \frac{C}{\nu^3}\|\mathbf{u}_\alpha\|^2\|\nabla\mathbf{u}_\alpha\|^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2 + \frac{\nu}{4}\|\Delta\mathbf{u}_\alpha\|^2,\end{aligned} \tag{3.3.51}$$

by using the Hölder, 2D-Ladyženskaya, Sobolev, and Young inequalities, respectively. From (3.3.50)-(3.3.51) we obtain:

$$\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 + \nu\|\Delta\mathbf{u}_\alpha\|^2 \le \frac{C}{\nu^3}\|\mathbf{u}_\alpha\|^2\|\nabla\overline{\mathbf{u}}_\alpha\|^2\|\nabla\mathbf{u}_\alpha\|^2 + \frac{2}{\nu}\|\mathbf{f}\|^2, \tag{3.3.52}$$

and in particular

$$\frac{d}{dt}\|\nabla\mathbf{u}_\alpha\|^2 \le \frac{CC_{M2}}{\nu^3}\|\nabla\overline{\mathbf{u}}_\alpha\|^2\|\nabla\mathbf{u}_\alpha\|^2 + \frac{2}{\nu}\|\mathbf{f}\|^2. \tag{3.3.53}$$

Hence, by (3.3.53) we obtain

$$\|\nabla\mathbf{u}_\alpha(s)\|^2 \le \left(\|\nabla\mathbf{u}_0\|^2 + \frac{2\mathcal{F}}{\nu}\right)\exp\left\{\frac{CC_{M2}}{\nu^4}\right\} =: C_{M3} \qquad \forall\, s \ge 0. \tag{3.3.54}$$

Together with (3.3.52) and (3.3.54) one obtains (3.3.42). Thus, the proof is complete also for this model. $\square$

3.4 The rate of convergence of $\mathbf{u}_\alpha$ to $\mathbf{u}$

In this section, we study the rate of convergence –in terms of α– of the weak solutions $\mathbf{u}_\alpha$ of the three α-models to the weak solution $\mathbf{u}$ of the NSE (in some suitable norms) as α tends to zero. We recall that, throughout this section the vector $\mathbf{e}$, defined as in (3.1.9), denotes the error between $\mathbf{u}$ and $\mathbf{u}_\alpha$ which are the weak solutions of the NSE (3.2.20) and of one of the α-models (3.2.21), respectively.

3.4.1 Error estimate in $L^\infty_t L^2_{\mathbf{x}} \cap L^2 H^1_{\mathbf{x}}$

The first main result in this section is given by the following theorem:

Theorem 3.4.1. *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. *Then*

$$\|\mathbf{e}(s)\|^2 + \nu \int_0^s \|\nabla \mathbf{e}\|^2 \, dt \leq C_r \alpha^3 \qquad \forall\, s \geq 0, \tag{3.4.55}$$

where C_r is given explicitly in (3.4.63), (3.4.65) and (3.4.66) for the L-α, SB and ML-α models.

Proof. As the three models share some common features, in a first step we consider these common ones, and in a second step we treat them separately to prove some specific estimates.

Step 1. Common features. We subtract (3.2.21) from (3.2.20) and by multiplying by $\mathbf{e}$ and integrating by parts we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{e}\|^2 + \nu \|\nabla \mathbf{e}\|^2 = (-\mathcal{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \mathcal{P}_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}). \tag{3.4.56}$$

We add and subtract on the r.h.s of (3.4.56) the term $((\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha, \mathbf{e})$ and then rewrite it in the following form:

$$\begin{aligned}
R &= (-\mathcal{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \mathcal{P}_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}) \\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), \mathcal{P}_\sigma \, \mathbf{e}) \\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), \mathbf{e}) \\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e}) =: I_1 + I_2,
\end{aligned} \tag{3.4.57}$$

We will deal with the two terms (a common one and a residual term) on the r.h.s of (3.4.57) separately. Replacing $\mathbf{u}_\alpha$ by $\mathbf{u} - \mathbf{e}$, the first term in (3.4.57) is rewritten as follows:

$$\begin{aligned}
I_1 = (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)(\mathbf{u}-\mathbf{e}), \mathbf{e}) \\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}, \mathbf{e}) \\
&= ((-\mathbf{e}\cdot\nabla)\mathbf{u}, \mathbf{e}) \\
&= ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}),
\end{aligned}$$

where $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}, \mathbf{e}) = 0$ has been used and the result is then estimated by

$$
\begin{aligned}
I_1 = ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}) &\le C\|\mathbf{e}\|_4\|\nabla\mathbf{e}\|\|\mathbf{u}\|_4 \\
&\le C\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{3/2}\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2} \\
&\le \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\mathbf{e}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}\|^2. \qquad (3.4.58)
\end{aligned}
$$

The first inequality from above is due to the Hölder inequality with the pairing (1/4, 1/2, 1/4), the second one is obtained by applying the 2D-Ladyženskaya inequality and the last one comes from using the Young inequality with the pairing (1/4, 3/4).

The residual term $I_2 = (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e})$ will be estimated for each model separately.

Step 2. Analysis specific for the various models.

L-α model. For this model the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. The residual term is written as follows

$$
I_2 = (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) = -(((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}).
$$

The Hölder, 2D-Ladyženskaya, (3.1.8), (3.2.18), Sobolev, Poincaré, and Young inequalities are then used to get the following estimates:

$$
\begin{aligned}
I_2 &\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4\|\nabla\mathbf{u}_\alpha\|\|\mathbf{e}\|_4 \\
&\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha - \nabla\overline{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\mathbf{u}_\alpha\|\,\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2} \\
&\le \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}\|\Delta\overline{\mathbf{u}}_\alpha\|^{1/2}\,\|\Delta\mathbf{u}_\alpha\|^{1/2}\,\|\nabla\mathbf{e}\| \\
&\le \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}\|\Delta\mathbf{u}_\alpha\|\,\|\nabla\mathbf{e}\| \\
&\le \frac{CC_L\alpha^3}{\nu\lambda_1}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}\|^2. \qquad (3.4.59)
\end{aligned}
$$

Notice that $\|\nabla\mathbf{u}_\alpha(t)\|$ in the above estimate is uniformly bounded by $C_L^{1/2}$ where C_L given by Lemma 3.3.2. Collecting estimates (3.4.58) and (3.4.59) we obtain

$$
\frac{d}{dt}\|\mathbf{e}\|^2 + \nu\|\nabla\mathbf{e}\|^2 \le \frac{CC_L\alpha^3}{\nu\lambda_1}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\mathbf{e}\|^2. \qquad (3.4.60)
$$

We are now going to apply the Gronwall's lemma for (3.4.60). Although the argument is standard we still provide the details for this model, while for the other ones the details will be skipped. Let us define

$$
A(s) := -\frac{C}{\nu^3}\int_0^s \|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\,dt \qquad \forall\, s \ge 0,
$$

where C is given in (3.4.60). Multiplying both sides of (3.4.60) by $\exp\{A(t)\}$ yields

$$\|\mathbf{e}(s)\|^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1}\exp\{-A(s)\}\int_0^s \|\Delta\mathbf{u}_\alpha\|^2\,dt \qquad \forall\, s \geq 0, \tag{3.4.61}$$

where we has used the facts that $A(s) \leq 0$ and $\mathbf{e}_0 = \mathbf{0}$. Thus, let us combine (3.4.61) with Lemmas 3.3.1 and 3.3.2 to prove uniform bounds for the modulus of $A(s)$ and for the integral from the r.h.s. to obtain

$$\|\mathbf{e}(s)\|^2 \leq \frac{CC_L^2}{\nu^2\lambda_1}\exp\left\{\frac{CC_{N1}^2}{\nu^4}\right\}\alpha^3 =: E_L\alpha^3 \qquad \forall\, s \geq 0, \tag{3.4.62}$$

where C_L and C_{N1} are given by Lemmas 3.3.2 and 3.3.1, respectively. Finally, we combine (3.4.60) and (3.4.62) to get (3.4.55), with C_r given by

$$C_r := C\left(\frac{C_L^2}{\nu^2\lambda_1} + \frac{C_{N1}^2 E_L}{\nu^4}\right). \tag{3.4.63}$$

SB model. In this case the residual term is given by

$$\begin{aligned} I_2 &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= -(((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) - ((\overline{\mathbf{u}}_\alpha\cdot\nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha), \mathbf{e}) \\ &= R_1 + R_2. \end{aligned} \tag{3.4.64}$$

The term R_1 on the r.h.s of (3.4.64) can be handled as (3.4.59) in the L-α model. The second term R_2 can be estimated as in the ML-α below, observing that $\|\overline{\mathbf{u}}\| \leq \|\mathbf{u}\|$. Therefore, the constant C_r in this case has the following form

$$C_r := C\left(\frac{C_S^2}{\nu^2\lambda_1} + \frac{C_{N1}^2 E_S}{\nu^4}\right) \quad \text{where} \quad E_S := \frac{CC_S^2}{\nu^2\lambda_1}\exp\left\{\frac{C_{N1}^2}{\nu^4}\right\}. \tag{3.4.65}$$

ML-α model. In this case the residual term is rewritten as

$$\begin{aligned} I_2 &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= ((\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}, \mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha), \end{aligned}$$

and is handled precisely as in the L-α case. Then, the proof for this case follows by that of the L-α model, with C_r given by

$$C_r := C\left(\frac{C_M^2}{\nu^2\lambda_1} + \frac{C_{N1}^2 E_M}{\nu^4}\right) \quad \text{where} \quad E_M := \frac{CC_M^2}{\nu^2\lambda_1}\exp\left\{\frac{C_{N1}^2}{\nu^4}\right\}. \tag{3.4.66}$$

□

From Theorem 3.4.1 we have immediately the following results:

Corollary 3.4.1. *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and let* $\mathbf{f} \in L^2(\mathbb{R}_+;\mathbf{H})$. *Then, it follows*

$$\|\overline{\mathbf{e}}(s)\|^2 + \nu \int_0^s \|\nabla \overline{\mathbf{e}}\|^2 \, dt \le C_r \alpha^3 \qquad \forall\, s \ge 0,$$

where $\overline{\mathbf{e}} = \overline{\mathbf{u}} - \overline{\mathbf{u}}_\alpha$ *and* C_r *is given by Theorem 3.4.1 for each* α*-model.*

Proof. The proof follows directly by Theorem 3.4.1 and (3.2.17). □

Corollary 3.4.2. *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and let* $\mathbf{f} \in L^2(\mathbb{R}_+;\mathbf{H})$. *Then, it follows*

$$\|(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)\|^2 + \nu \int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt \le C_{cor}(\alpha^3 + \alpha^2) \qquad \forall\, s \ge 0, \tag{3.4.67}$$

where C_{cor} *is given by (3.4.70).*

Proof. The triangle inequality, Theorem 3.4.1, Lemma 3.3.1, relation (3.2.17), and Poincaré inequality yield for all $s \ge 0$

$$\begin{aligned}
\|\mathbf{u} - \overline{\mathbf{u}}_\alpha(s)\|^2 &\le 2\left(\|(\mathbf{u} - \mathbf{u}_\alpha)(s)\|^2 + \|(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)(s)\|^2\right) \\
&\le 2C_r\alpha^3 + 2\alpha^4\|\Delta \overline{\mathbf{u}}_\alpha(s)\|^2 \\
&\le 2C_r\alpha^3 + 2C\alpha^2\|\mathbf{u}_\alpha(s)\|^2 \\
&\le 2C_r\alpha^3 + 2C\frac{C_E}{\lambda_1}\alpha^2.
\end{aligned} \tag{3.4.68}$$

Here, for each α-model C_E is given by C_L, C_S or C_M in Lemmas 3.3.2, 3.3.3 and 3.3.4, respectively. Moreover, C_r is given by Theorem 3.4.1. Similarly, we have

$$\begin{aligned}
\nu \int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt &\le 2\nu \left(\int_0^s \|\nabla(\mathbf{u} - \mathbf{u}_\alpha)\|^2 \, dt + \int_0^s \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2 \, dt\right) \\
&\le 2C_r\alpha^3 + 2C\alpha^2\nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \\
&\le 2C_r\alpha^3 + 2CC_E\alpha^2 \qquad \forall\, s \ge 0.
\end{aligned} \tag{3.4.69}$$

Thus, (3.4.67) follows by (3.4.68) and (3.4.69) with the constant C given by

$$C_{cor} = 2\max\{C_r, CC_E, CC_E/\lambda_1\}. \tag{3.4.70}$$

□

3.4.2 Error estimate in $L_t^\infty H_{\mathbf{x}}^1 \cap L_t^2 H_{\mathbf{x}}^2$

We now prove convergence rates in stronger norms, at the price of weaker rates. Throughout the rest of the paper, we assume $\alpha < L/2\pi$. Before going on to state the results, we start with a technical result, see [CT09, Prop. 4.2], that follows from a well-known result due to Brézis and Gallouët [BG80].

Lemma 3.4.1. *Let $0 \leq \alpha < \lambda_1^{-1/2} = L/2\pi$, and let $\mathbf{u}_\alpha$ be the weak solutions of any of α-models considered here. Then, there exist K_1 and K_2 such that*

$$\|\overline{\mathbf{u}}_\alpha(t)\|_\infty^2 \leq K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \qquad \forall\, t \geq 0. \tag{3.4.71}$$

Proof. For the proof apply the same argument as in [CT09, Prop. 4.2], with the only difference that here due to the global estimates (we derived previously) we can work on arbitrary time intervals. Note that in [CT09, Prop. 4.2] it is required $\mathbf{u}_0 \in \mathcal{D}(A)$. However, thanks to the estimates which are presented in Section 3.3, it is enough to take $\mathbf{u}_0 \in \mathbf{V}$. □

We are now in order to state the next main result in this section.

Theorem 3.4.2. *Let $\alpha < L/2\pi$, $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$, and let us define*

$$D(s) := \|\nabla \mathbf{e}(s)\|^2 + \nu \int_0^s \|\Delta \mathbf{e}\|^2\, dt \qquad \forall\, s \geq 0.$$

Then, the following estimates hold true:
1. For the L-α model

$$D(s) \leq C_R\, \alpha^2,$$

where C_R is given by (3.4.81).
2. For the SB model

$$D(s) \leq C_R\, \alpha^2 \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_S\right),$$

where C_R is given by (3.4.83).
3. For the ML-α model

$$D(s) \leq C_R\, \alpha^2 \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_M\right),$$

where C_R is given by (3.4.88).
Here, the constants C_S, C_M, K_1 and K_2 are given by Lemmas 3.3.4, 3.3.3 and 3.4.1, respectively.

Proof. As before we first prove estimates valid for all models and then we pass to consider the specific ones.

Step 1. Common features.

Subtracting (3.2.21) from (3.2.20) and taking $-\Delta \mathbf{e}$ as a test function yields:

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{e}\|^2 + \nu \|\Delta \mathbf{e}\|^2 = (-\mathcal{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \mathcal{P}_\sigma[N(\mathbf{u}_\alpha)], -\Delta \mathbf{e}). \tag{3.4.72}$$

Adding and subtracting the term $((\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e})$ to the r.h.s of (3.4.72):

$$RHS = (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), -\Delta\mathbf{e}). \tag{3.4.73}$$

By recalling the definition $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$, the first term on the r.h.s of (3.4.73) can be split as follows:

$$\begin{aligned}
I_1 &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e})\\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + ((\mathbf{u}-\mathbf{e})\cdot\nabla)(\mathbf{u}-\mathbf{e}), -\Delta\mathbf{e})\\
&= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{e} - (\mathbf{e}\cdot\nabla)\mathbf{u} + (\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e})\\
&= ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) + (\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) =: I_{11} + I_{12},
\end{aligned} \tag{3.4.74}$$

where the vanishing of the term $((\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e})$ has been used. The first term on the r.h.s of (3.4.74) is bounded by

$$\begin{aligned}
I_{11} = ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) &\le C\|\mathbf{u}\|_4\|\nabla\mathbf{e}\|_4\|\Delta\mathbf{e}\|\\
&\le C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2}\|\Delta\mathbf{e}\|^{3/2}\\
&\le \frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\nabla\mathbf{e}\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2.
\end{aligned} \tag{3.4.75}$$

In (3.4.75), the Hölder, 2D-Ladyžhenskaya, and Young inequalities have been applied. Similarly, the other term on the r.h.s of (3.4.74) can be handled as follows:

$$\begin{aligned}
I_{12} = ((\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) &\le C\|\mathbf{e}\|_4\|\nabla\mathbf{u}\|_4\|\Delta\mathbf{e}\|\\
&\le C\|\mathbf{e}\|^{1/2}\|\nabla\mathbf{e}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\Delta\mathbf{u}\|^{1/2}\|\Delta\mathbf{e}\|\\
&\le \frac{C}{\lambda_1^{1/2}}\|\nabla\mathbf{e}\|\|\nabla\mathbf{u}\|^{1/2}\|\Delta\mathbf{u}\|^{1/2}\|\Delta\mathbf{e}\|\\
&\le \frac{C}{\nu\lambda_1}\|\nabla\mathbf{e}\|^2\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\| + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2.
\end{aligned} \tag{3.4.76}$$

Using (3.4.75)-(3.4.76) the quantity I_1 in (3.4.74) can be bounded by

$$I_1 \le \left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\right)\|\nabla\mathbf{e}\|^2 + \frac{\nu}{3}\|\Delta\mathbf{e}\|^2. \tag{3.4.77}$$

In the following, we will estimate the second term I_2 from the r.h.s of (3.4.73), separately for each α-model.

Step 2. Analysis specific for the various models.

L-α model. The nonlinear term is given in this case by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. Therefore, the residual term I_2 can be estimated as follows

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\
&= ((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\
&\leq C\|\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|_4 \|\nabla\mathbf{u}_\alpha\|_4 \|\Delta\mathbf{e}\| \\
&\leq C\|\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha\|^{1/2} \|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|^{1/2} \|\nabla\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{u}_\alpha\|^{1/2} \|\Delta\mathbf{e}\| \\
&\leq C\alpha \|\nabla\mathbf{u}_\alpha\| \|\Delta\mathbf{u}_\alpha\| \|\Delta\mathbf{e}\| \\
&\leq \frac{C}{\nu}\alpha^2 \|\nabla\mathbf{u}_\alpha\|^2 \|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2 \\
&\leq \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{6}\|\Delta\mathbf{e}\|^2,
\end{aligned} \tag{3.4.78}
$$

where the Hölder, 2D-Ladyžhenskaya, (3.1.8)-(3.2.17), and Young inequalities have been applied. Moreover, C_L is given by Lemma 3.3.2. Using estimates (3.4.72)-(3.4.78) leads to

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\mathbf{e}\|^2 + \nu\|\Delta\mathbf{e}\|^2 \leq &\left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\right)\|\nabla\mathbf{e}\|^2 \\
&+ \frac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha\|^2,
\end{aligned} \tag{3.4.79}
$$

and we can rewrite it as follows

$$
y'(t) - g(t)y(t) \leq h(t) \qquad \forall\, t \geq 0,
$$

where for all $t \geq 0$

$$
\begin{cases}
y(t) := \|\nabla\mathbf{e}(t)\|^2, \\
g(t) := \dfrac{C}{\nu^3}\|\mathbf{u}(t)\|^2\|\nabla\mathbf{u}(t)\|^2 + \dfrac{C}{\nu\lambda_1}\|\nabla\mathbf{u}(t)\|\|\Delta\mathbf{u}(t)\|, \\
h(t) := \dfrac{CC_L\alpha^2}{\nu}\|\Delta\mathbf{u}_\alpha(t)\|^2.
\end{cases}
$$

Therefore, since $\nabla\mathbf{e}(0) = \mathbf{0}$, an application of the Gronwall's lemma gives

$$
\|\nabla\mathbf{e}(s)\|^2 \leq \frac{CC_L}{\nu^2}\exp\left\{\frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2}C_{N2}^{1/2}}{\nu^2\lambda_1}\right\}\alpha^2 =: R_L\alpha^2 \qquad \forall\, s \geq 0. \tag{3.4.80}
$$

Finally, combining (3.4.79) and (3.4.80) yields

$$
\|\nabla\mathbf{e}(s)\|^2 + \nu\int_0^s \|\Delta\mathbf{e}\|^2\,dt \leq C_R\,\alpha^2 \qquad \forall\, s \geq 0,
$$

where

$$
C_R := \left(\frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2}C_{N2}^{1/2}}{\nu^2\lambda_1}\right)R_L + \frac{CC_L^2}{\nu^2}. \tag{3.4.81}
$$

SB model. In this case the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$ and adding and subtracting the term $(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ lead us to

$$\begin{aligned} I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\ &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\ &=: I_{21} + I_{22}. \end{aligned} \tag{3.4.82}$$

Here, the first term on the r.h.s of (3.4.82) can be handled as follows

$$\begin{aligned} I_{21} &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\ &= (((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \Delta\mathbf{e}), \end{aligned}$$

which is similar to (3.4.78) in the L-α model. The other term can be rewritten as follows

$$\begin{aligned} I_{22} &= (-(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\ &= ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}), \end{aligned}$$

which turns out to be similar to (3.4.86) in the ML-α model below. Therefore, the constant C_R in this case is similar as in the ML-α model and has the form

$$C_R := \left(\frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2} C_{N2}^{1/2}}{\nu^2 \lambda_1} \right) R_S + \frac{C C_S}{\nu^2}. \tag{3.4.83}$$

Here C_S is given by Lemma 3.3.3 and

$$R_S := \frac{C C_S}{\nu^2} \exp\left\{ \frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2} C_{N2}^{1/2}}{\nu^2 \lambda_1} \right\}.$$

Thus, the proof is complete.

ML-α model. The nonlinear term is given now by $N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$ and the residual term can be rewritten as follows

$$\begin{aligned} I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta\mathbf{e}) \\ &= ((\mathbf{u}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\ &= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) + ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e}) \\ &=: I_{21} + I_{22}. \end{aligned} \tag{3.4.84}$$

The first term from the r.h.s of (3.4.84) can be estimated by

$$\begin{aligned}
I_{21} &= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e})\\
&\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4\|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|_4\|\Delta\mathbf{e}\|_2\\
&\le C\|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|^{1/2}\|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|\,\|\Delta\overline{\mathbf{u}}_\alpha - \Delta\mathbf{u}_\alpha\|^{1/2}\|\Delta\mathbf{e}\|\\
&\le C\alpha\|\Delta\mathbf{u}_\alpha\|\|\nabla\mathbf{u}_\alpha\|\|\Delta\mathbf{e}\|\\
&\le \frac{C}{\nu}\alpha^2\|\nabla\mathbf{u}_\alpha\|^2\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2\\
&\le \frac{CC_M}{\nu}\alpha^2\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2, \qquad (3.4.85)
\end{aligned}$$

where C_M is given by Lemma 3.3.4. Next, we bound the second term on the r.h.s of (3.4.84) as follows (recall in the all section we are in the case $\alpha < L/2\pi$):

$$\begin{aligned}
I_{22} &= ((\overline{\mathbf{u}}_\alpha\cdot\nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta\mathbf{e})\\
&\le C\|\overline{\mathbf{u}}_\alpha\|_\infty\|\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha\|\|\Delta\mathbf{e}\|\\
&\le C\alpha\|\overline{\mathbf{u}}_\alpha\|_\infty\|\Delta\mathbf{u}_\alpha\|\|\Delta\mathbf{e}\|\\
&\le \frac{C\alpha^2}{\nu}\|\overline{\mathbf{u}}_\alpha\|_\infty^2\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2\\
&\le \frac{C\alpha^2}{\nu}\left(K_1\log\left(\frac{L}{2\pi\alpha}\right) + K_2\right)\|\Delta\mathbf{u}_\alpha\|^2 + \frac{\nu}{12}\|\Delta\mathbf{e}\|^2. \qquad (3.4.86)
\end{aligned}$$

Here, K_1 and K_2 are given in Lemma 3.4.1. In (3.4.85)-(3.4.86), we have used the inequalities Hölder, 2D-Ladyzhenshaya, Young and formula (3.4.71) in Lemma 3.4.1. Putting (3.4.77) and (3.4.85)-(3.4.86) into the r.h.s of (3.4.72), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla\mathbf{e}\|^2 + \nu\|\Delta\mathbf{e}\|^2 \le{}& \left(\frac{C}{\nu^3}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + \frac{C}{\nu\lambda_1}\|\nabla\mathbf{u}\|\|\Delta\mathbf{u}\|\right)\|\nabla\mathbf{e}\|^2\\
&+ \frac{C\alpha^2}{\nu}\left(K_1\log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_M\right)\|\Delta\mathbf{u}_\alpha\|^2. \qquad (3.4.87)
\end{aligned}$$

Since inequality (3.4.87) shares a similar structure with (3.4.79) then the rest of the proof follows by that of the L-α model. The constant C_R in this case is given by

$$C_R := \left(\frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2}C_{N2}^{1/2}}{\nu^2\lambda_1}\right)R_M + \frac{CC_M}{\nu^2}. \qquad (3.4.88)$$

Here

$$R_M := \frac{CC_M}{\nu^2}\exp\left\{\frac{C_{N1}^2}{\nu^4} + \frac{C_{N1}^{1/2}C_{N2}^{1/2}}{\nu^2\lambda_1}\right\}.$$

Thus, the proof is complete. □

From the Theorem 3.4.2 we can easily deduce the following corollaries for related errors.

Corollary 3.4.3. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. *Then*

$$\|\nabla \overline{\mathbf{e}}(s)\|^2 + \nu \int_0^s \|\Delta \overline{\mathbf{e}}\|^2 \, dt \le D(s) \le C_R h(\alpha) \quad \forall\, s \ge 0,$$

where $\overline{\mathbf{e}} = \overline{\mathbf{u}} - \overline{\mathbf{u}}_\alpha$, C_R *is given as in Theorem 3.4.2 and* $h(\alpha)$ *is given by Corollary 3.4.4 below.*

Proof. The proof follows directly by Theorem 3.4.2 and (3.2.17). □

Corollary 3.4.4. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$, *and let us define*

$$E(s) := \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)\|^2 + \nu \int_0^s \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt \qquad \forall\, s \ge 0.$$

Then, it follows

$$E(s) \le 2C_R\, h(\alpha) + 2CC_E \alpha^2 \qquad \forall\, s \ge 0, \tag{3.4.89}$$

where

$$h(\alpha) := \begin{cases} \alpha^2 & \text{for the L-}\alpha\text{ model,} \\ \alpha^2 \left(K_1 \log(L/2\pi\alpha) + K_2 + C_S\right) & \text{for the SB model,} \\ \alpha^2 \left(K_1 \log(L/2\pi\alpha) + K_2 + C_M\right) & \text{for the ML-}\alpha\text{ model.} \end{cases}$$

Proof. The proof shares the same idea with Corollary 3.4.2. We start with

$$\begin{aligned} \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)\|^2 &\le 2(\|\nabla(\mathbf{u} - \mathbf{u}_\alpha)(s)\|^2 + \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)(s)\|^2) \\ &\le 2C_R\, h(\alpha) + 2\alpha^4 \|\nabla \Delta \overline{\mathbf{u}}_\alpha(s)\|^2 \\ &\le 2C_R\, h(\alpha) + 2C\alpha^2 \|\nabla \mathbf{u}_\alpha(s)\|^2 \\ &\le 2C_R\, h(\alpha) + 2CC_E\alpha^2 \qquad \forall\, s \ge 0, \end{aligned} \tag{3.4.90}$$

where (3.2.17) has been used in the third inequality. The constant C_E is defined as in Corollary 3.4.2. Similarly, for all $s \ge 0$

$$\begin{aligned} I &= \nu \int_0^s \|\Delta \mathbf{u} - \Delta \overline{\mathbf{u}}_\alpha\|^2 \, dt \\ &\le 2\nu \int_0^s \|\Delta \mathbf{u} - \Delta \mathbf{u}_\alpha\|^2 \, dt + 2\nu \int_0^s \|\Delta \mathbf{u}_\alpha - \Delta \overline{\mathbf{u}}_\alpha\|^2 \, dt \\ &\le 2C_R h(\alpha) + 2\nu\alpha^4 \int_0^s \|\Delta\Delta \overline{\mathbf{u}}_\alpha\|^2 \, dt \\ &\le 2C_R h(\alpha) + 2\alpha^2 \nu \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \\ &\le 2C_R h(\alpha) + 2CC_E\alpha^2. \end{aligned} \tag{3.4.91}$$

Therefore, (3.4.89) follows by combining (3.4.90) and (3.4.91). □

3.5 The rate of convergence of p_α to p

In this section we focus on the order of the error of the pressure, by using the results from the previous sections. Let p and p_α be the pressures associated to the weak solutions $\mathbf{u}$ and $\mathbf{u}_\alpha$ of the NSE (3.1.1)-(3.1.3) and all α-models (3.1.4)-(3.1.6), respectively. It will be shown that the difference

$$q := p - p_\alpha$$

is bounded in terms of the parameter α, uniformly in time, in a suitable norm.

Theorem 3.5.1. *Let $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. We define*

$$I(s) := \int_0^s \|\nabla q\|^2 \, dt \qquad \forall\, s \geq 0.$$

Then, the following estimates hold true:
1. For for the L-α model

$$I(s) \leq C_1 \alpha^{5/2} + C_2 \alpha^3 \qquad \forall\, s \geq 0,$$

where C_1 and C_2 are given by (3.5.96).

2. For for the SB model

$$I(s) \leq C_1 \alpha^3 + C_2 \alpha^{5/2} \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right)^{1/2} + C_3 \alpha^2 \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right) \quad \forall\, s \geq 0,$$

where C_1, C_2 and C_3 are given by (3.5.100).

3. For for the ML-α model

$$I(s) \leq C_1 \alpha^4 + C_2 \alpha^3 + C_3 (\alpha^{5/2} + \alpha^2) \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right) \quad \forall\, s \geq 0,$$

where C_1, C_2 and C_3 are given by (3.5.105).

Proof. It follows subtracting from the NSE (3.1.1)-(3.1.3) the α-model (3.1.4)-(3.1.6) that

$$-\Delta q = \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)] =: \nabla \cdot \mathbf{g}. \tag{3.5.92}$$

We are assuming that p and p_α are periodic and with zero average. The vanishing of the mean values of p and p_α ensure their uniqueness (up to an arbitrary function of time). Multiplying (3.5.92) by q and integrating on Ω the Cauchy-Schwarz inequality yields

$$\|\nabla q\|^2 \leq \|\mathbf{g}\|^2 = \int_\Omega |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x}. \tag{3.5.93}$$

In order to estimate the error of the pressure we are led to bound the r.h.s of (3.5.93). Replacing $\mathbf{e}$ by $\mathbf{u}-\mathbf{u}_\alpha$, adding and subtracting the term $(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha$ gives

$$\begin{aligned}
\|\mathbf{g}\|^2 &= \int_\Omega |(\mathbf{u}\cdot\nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2\,d\mathbf{x}\\
&= \int_\Omega |(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2\,d\mathbf{x}\\
&= \int_\Omega |-(\mathbf{e}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{e} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2\,d\mathbf{x}\\
&\le 4\int_\Omega \left(|(\mathbf{e}\cdot\nabla)\mathbf{u}|^2 + |(\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}|^2 + |(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2\right)\,d\mathbf{x}. \qquad (3.5.94)
\end{aligned}$$

By (3.5.94) one has for all $s\ge 0$:

$$I(s) = \int_0^s \|\mathbf{g}\|^2\,dt \le 4(I_1+I_2+I_3)(s). \qquad (3.5.95)$$

The estimate is given for each α-model separately.

L-α model. In this case we have $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha$. Each term on the r.h.s of (3.5.95) will be estimated below. First,

$$\begin{aligned}
I_1(s) &:= \int_0^s\int_\Omega |(\mathbf{e}\cdot\nabla)\mathbf{u}|^2\,d\mathbf{x}\,dt\\
&\le \int_0^s \|\mathbf{e}\|_4^2\|\nabla\mathbf{u}\|_4^2\,dt\\
&\le \int_0^s \|\mathbf{e}\|\,\|\nabla\mathbf{e}\|\,\|\nabla\mathbf{u}\|\,\|\Delta\mathbf{u}\|\,dt\\
&\le C_r^{1/2}C_{N2}^{1/2}\,\alpha^{3/2}\left(\int_0^s\|\nabla\mathbf{e}\|^2\,dt\right)^{1/2}\left(\int_0^s\|\Delta\mathbf{u}\|^2\,dt\right)^{1/2}\\
&\le \frac{C_rC_{N2}}{\nu}\alpha^3 \qquad \forall\, s\ge 0,
\end{aligned}$$

where we have used the Hölder and 2D-Ladyžhenskaya inequalities, Lemmas 3.3.1, 3.4.1, and 3.4.2. Next, we have

$$\begin{aligned}
I_2(s) &:= \int_0^s\int_\Omega |(\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}|^2\,d\mathbf{x}\,dt\\
&\le \int_0^s \|\mathbf{u}_\alpha\|\,\|\nabla\mathbf{u}_\alpha\|\,\|\nabla\mathbf{e}\|\,\|\Delta\mathbf{e}\|\,dt\\
&\le \frac{C_L}{\lambda^{1/2}}\left(\int_0^s\|\nabla\mathbf{e}\|^2\,dt\right)^{1/2}\left(\int_0^s\|\Delta\mathbf{e}\|^2\,dt\right)^{1/2}\\
&\le \frac{C_LC_r^{1/2}C_R^{1/2}}{\nu}\alpha^{5/2} \qquad \forall\, s\ge 0,
\end{aligned}$$

here we have used the Hölder and 2D-Ladyžhenskaya inequalities, Lemma 3.3.2, Theorems 3.4.1 and 3.4.2, respectively. Finally

$$\begin{aligned} I_3(s) &:= \int_0^s \int_\Omega |((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha|^2 \, d\mathbf{x}\, dt \\ &\leq \int_0^s \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\| \, \|\nabla \mathbf{u}_\alpha\| \, \|\Delta \mathbf{u}_\alpha\| \, dt \\ &\leq 2CC_L\alpha^3 \int_0^s \|\Delta\overline{\mathbf{u}}_\alpha\| \|\Delta\mathbf{u}_\alpha\| \, dt \\ &\leq \frac{2CC_L^{3/2}}{\nu}\alpha^3 \qquad \forall\, s \geq 0, \end{aligned}$$

here in addition we have used (3.1.8), (3.4.90), and Lemma 3.3.2. Thus the proof of the convergence rate for this model follows by collecting the previous estimates

$$I(s) \leq \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu}\alpha^{5/2} + \left(\frac{C_r C_{N2}}{\nu} + \frac{2CC_L^{3/2}}{\nu}\right)\alpha^3 \qquad \forall\, s \geq 0. \tag{3.5.96}$$

SB model. For this model we have for all $s \geq 0$

$$\begin{aligned} I(s) &= \int_0^s \int_\Omega |(\mathbf{u}\cdot\nabla)\mathbf{u} - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha|^2 \, d\mathbf{x}dt \\ &\leq 4(I_1 + I_2 + I_3)(s). \end{aligned} \tag{3.5.97}$$

One has used the fact that $\mathbf{u}_\alpha = \mathbf{u} - \mathbf{e}$ in the second term inside the integral. Similarly, by using Corollary 3.4.1 and Lemma 3.3.3 we get

$$\begin{aligned} I_1(s) &:= \int_0^s \int_\Omega |((\mathbf{u} - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}|^2 \, d\mathbf{x}dt \\ &\leq \int_0^s \|\mathbf{u} - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \, \|\nabla\mathbf{u}\| \, \|\Delta\mathbf{u}\| \, dt \\ &\leq \frac{C_{N2}^{1/2}}{\nu} C_{cor}^{1/2}(\alpha^2 + \alpha^3)^{1/2} \left(\nu\int_0^s \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt\right)^{1/2} \left(\nu\int_0^s \|\Delta\mathbf{u}\|^2 \, dt\right)^{1/2} \\ &\leq \frac{C_{N2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) \qquad \forall\, s \geq 0. \end{aligned} \tag{3.5.98}$$

We deal with the second integral by using Lemma 3.3.3 and Theorems 3.4.1 and 3.4.2 to show

$$\begin{aligned} I_2(s) &:= \int_0^s \int_\Omega |(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{e}|^2 \, d\mathbf{x}dt \\ &\leq \int_0^s \|\overline{\mathbf{u}}_\alpha\| \|\nabla\overline{\mathbf{u}}_\alpha\| \|\nabla\mathbf{e}\| \|\Delta\mathbf{e}\| \, dt \\ &\leq \frac{CC_S}{\nu\lambda_1^{1/2}} \left(\nu\int_0^s \|\nabla\mathbf{e}\|^2 \, dt\right)^{1/2} \left(\nu\int_0^s \|\Delta\mathbf{e}\|^2 \, dt\right)^{1/2} \\ &\leq \frac{CC_S}{\nu\lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_S\right)^{1/2} \qquad \text{since } \alpha \leq \frac{L}{2\pi}. \end{aligned} \tag{3.5.99}$$

Similarly, the last term can be estimated for all $s \geq 0$ by

$$\begin{aligned}
I_3(s) &:= \int_0^s \int_\Omega |(\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u} - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x}dt \\
&\leq \int_0^s \|\overline{\mathbf{u}}_\alpha\| \|\nabla \overline{\mathbf{u}}_\alpha\| \|\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\| \, dt \\
&\leq \frac{C_S^{1/2}}{\nu\lambda_1^{1/2}} (2C_R\, h(\alpha) + 2CC_S\alpha^2)^{1/2} \left(\nu \int_0^s \|\nabla \mathbf{u}_\alpha\|^2 \, dt\right)^{1/2} \left(\nu \int_0^s \|\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)\|^2 \, dt\right)^{1/2} \\
&\leq \frac{C_S}{\nu\lambda_1^{1/2}} (2C_R\, h(\alpha) + 2CC_S\alpha^2) \\
&= \frac{CC_S}{\nu\lambda_1^{1/2}} \left[C_R\, \alpha^2 \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_S\right) + C_S\alpha^2\right] \quad \text{since } \alpha \leq \frac{L}{2\pi}.
\end{aligned}$$

Therefore, by the above estimates we get

(3.5.100)
$$\begin{aligned}
I(s) \leq \frac{C_{N2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) &+ \frac{CC_S}{\nu\lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_S\right)^{1/2} \\
&+ \frac{CC_S}{\nu\lambda_1^{1/2}} \left[C_R \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_S\right) + C_S\right] \alpha^2.
\end{aligned}$$

Thus the proof for this model is completed.

ML-α model. For this model I_1 is estimated as in the L-α above. We start with I_2 by

$$\begin{aligned}
I_2(s) &\leq \int_0^s \|\mathbf{u}_\alpha\| \, \|\nabla \mathbf{u}_\alpha\| \, \|\nabla \mathbf{e}\| \, \|\Delta \mathbf{e}\| \, dt \\
&\leq \frac{C_M}{\lambda_1^{1/2}} \left(\int_0^s \|\nabla \mathbf{e}\|^2 \, dt\right)^{1/2} \left(\int_0^s \|\Delta \mathbf{e}\|^2 \, dt\right)^{1/2} \\
&\leq \frac{C_M C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_M\right)^{1/2}, \qquad (3.5.101)
\end{aligned}$$

for all $s \geq 0$. One has used the results Lemma 3.3.4, Theorems 3.4.1 and 3.4.2. The term I_3 is bounded by

$$I_3(s) := \int_0^s \int_\Omega |(\mathbf{u}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x}\, dt \leq 2(I_{31} + I_{32})(s). \qquad (3.5.102)$$

By (3.4.90) and Lemma 3.3.4 yield

$$\begin{aligned}
I_{31}(s) &:= \int_0^s \int_\Omega |((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x}\, dt \\
&\leq \int_0^s \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\|_4^2 \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|_4^2 \, dt \\
&\leq \int_0^s \|\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha\| \, \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2 \|\Delta(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\| \, dt \\
&\leq \frac{CC_M^2}{\nu} \alpha^4 \qquad \forall\, s \geq 0. \qquad (3.5.103)
\end{aligned}$$

The other term can be estimated for all $s \geq 0$ by

$$
\begin{aligned}
I_{32}(s) &:= \int_0^s \int_\Omega |(\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, dt \\
&\leq \int_0^s \|\overline{\mathbf{u}}_\alpha\|_\infty^2 \|\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\|^2 \, dt \\
&\leq \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2 \int_0^s \|\Delta \mathbf{u}_\alpha\|^2 \, dt \\
&\leq \frac{C_M}{\nu} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2, \qquad (3.5.104)
\end{aligned}
$$

here Lemma 3.4.1 and (3.2.18) have been applied. Therefore, by (3.5.101)-(3.5.104)

$$
\begin{aligned}
I(s) \leq \frac{C_r C_{N2}}{\nu} \alpha^3 &+ \frac{C_M C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_M \right)^{1/2} \\
&+ \frac{C C_M^2}{\nu} \alpha^4 + \frac{C_M}{\nu} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2, \qquad (3.5.105)
\end{aligned}
$$

which concludes the proof. $\square$

The Poincaré inequality and Theorem 3.5.1 directly give us the following consequence:

Corollary 3.5.1. *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$. *Then*

$$
\int_0^s \|q\|^2 \, dt \leq \frac{1}{\lambda_1} \int_0^s \|\nabla q\|^2 \, dt \leq g(\alpha) \qquad \forall \, s \geq 0,
$$

where

$$
g(\alpha) := \begin{cases} C_1 \alpha^{5/2} + C_2 \alpha^3, \\ C_1 \alpha^3 + C_2 \alpha^{5/2} \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right)^{1/2} + C_3 \alpha^2 \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right), \\ C_1 \alpha^4 + C_2 \alpha^3 + C_3(\alpha^{5/2} + \alpha^2) \left(\log\left(\frac{L}{2\pi\alpha}\right) + 1 \right), \end{cases}
$$

for the L-α, SB and ML-α model, respectively. The constants C_i for $i = 1, 2, 3$ are given as in Theorem 3.5.1.

3.6 The 3D case: a few additional remarks

This section is devoted to give a few remarks in the 3D case. First, known results on the rate of convergence in the 3D case are recalled. Then the possibility of the irregularity on the convergence rate is also provided.

3.6.1 Known results

The problem in the periodic box $\Omega = [0, L]^3$ is rather different since the solutions of the NSE are not known to be globally smooth. Moreover, the available estimates for the convective term are different from those employed in the previous sections.

One of the first results on the rate of convergence in the 3D case is given by Chen, Guenther, Kim, Thomann, and Waymire [CGK+08]. The authors proved the following estimate

$$\int_0^T \|\mathbf{u} - \mathbf{u}_\alpha\| \, dt \leq C(T)\alpha, \tag{3.6.106}$$

where (and in the sequel) $\|\cdot\|$ denotes the $L^2(\Omega)$ norm, $\mathbf{u}$ and $\mathbf{u}_\alpha$ are the weak solutions of the NSE and Navier-Stokes-α model (known as the viscous Camassa-Holm equations), respectively. Their analysis is carried out in the 3D periodic setting and assumes a small data condition of Besov type (such that existence and uniqueness of weak solutions $\mathbf{u}$ of the 3D NSE is ensured).

Another result concerning the convergence rate has been obtained by the author of [Dun18]. For both 2D and 3D cases it is provided for all α-models herein (also for the Navier-Stokes-α model) that

$$\sup_{t\in[0,T]} \|\mathbf{e}(t)\|^2 + \int_0^T \|\nabla \mathbf{e}\|^2 \, dt \leq C(T)\alpha^2. \tag{3.6.107}$$

Here as previous parts $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$. The result is obtained under assumptions on the data $\mathbf{u}_\alpha(0, \cdot) = \mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2([0, T]; \mathbf{H})$. In addition, an extra assumption is made on the weak solution of the 3D NSE such that $\mathbf{u} \in L^4([0, T]; H^1(\Omega)^3)$. The latter condition ensures existence and uniqueness of the weak solutions. Note that the logarithmic term in (3.1.10) is removed.

3.6.2 Possible of the irregularity on the convergence rate

The well-known Leray-Serrin-Prodi (LSP) uniqueness assumption for the 3D NSE, see Leray [Ler34b], Prodi [Pro59] and Serrin [Ser63], is given by

$$\mathbf{u} \in L^r([0, T]; L^s(\Omega)^3) \quad \text{where} \quad \frac{3}{s} + \frac{2}{r} = 1, \ s \geq 3. \tag{3.6.108}$$

It is also known, see for example Galdi [Gal00, Def. 2.1 and Theorem 4.2], that weak solutions satisfying LSP condition are unique and regular in the set of all Leray-Hopf weak solutions.

As mentioned above, the author of [Dun18] required $\mathbf{u} \in L^4([0, T]; H^1(\Omega)^3)$ to provide (3.6.107). Then the standard Sobolev embedding implies that $\mathbf{u} \in L^4([0, T]; L^6(\Omega)^3)$ which satisfies (3.6.108) with $r = 4$ and $s = 6$. For more details, $C(T)$ in (3.6.107) is given by the form

$$C(T) = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4 \, ds \right\}, \tag{3.6.109}$$

where C is the Sobolev constant and $C_1 = C_1(\mathbf{u}_0, \mathbf{f}, \nu)$. It follows that the error is uniformly bounded in time, i.e.,

$$\sup_{t\geq 0} \|\mathbf{e}(t)\|^2 + \nu \int_0^\infty \|\nabla \mathbf{e}\|^2 \, dt \leq C_\infty \alpha^2,$$

where

$$C_\infty = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^\infty \|\nabla \mathbf{u}\|^4 \, ds \right\}.$$

The story is totally different if the weak solutions of the 3D NSE do not satisfy (3.6.108). Assume that a weak solution $\mathbf{u}$ of the NSE is regular up to a time $T_* < \infty$ and cannot be smoothly extended, we say that $\mathbf{u}$ becomes irregular at the time T_* (or that T_* is an epoch of irregularity). Assume that T_* is the first time that $\mathbf{u}$ becomes irregular, see Galdi [Gal00, Def. 6.1], then it is well-known that the $H^1(\Omega)$-norm of $\mathbf{u}$, $\|\nabla \mathbf{u}(t)\|^2$ will blow-up as t approaches T_* from below, see for instance [Gal00, Theorem 6.4], Leray [Ler34b] and Scheffer [Sch76]. More specifically, there exists $\epsilon = \epsilon_{T_*} > 0$ small enough such that

$$\|\nabla \mathbf{u}(t)\| \geq \frac{C\nu^{3/4}}{(T_* - t)^{1/4}} \qquad \forall\, t \in (T_* - \epsilon, T_*), \tag{3.6.110}$$

where $C > 0$ is only depending on Ω. Using (3.6.110) the constant $C(T)$ in (3.6.109) with $T_* - \epsilon < T < T_*$ will blow-up in the following way

$$\begin{aligned} C(T) &= C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T \|\nabla \mathbf{u}\|^4 \, ds \right\} \\ &\geq C_1 \exp\left\{ \frac{C}{\nu^3} \int_{T_*-\epsilon}^T \|\nabla \mathbf{u}\|^4 \, ds \right\} \\ &\geq C_1 \exp\left\{ C \int_{T_*-\epsilon}^T \frac{1}{T_* - s} \, ds \right\} \\ &= C_1 \frac{\epsilon^C}{(T_* - T)^C}, \end{aligned}$$

which shows the effect of being T_* an epoch of irregularity on the convergence rate.

3.7 Conclusions

In this work, after assuming the not so restrictive assumptions on the data $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(\mathbb{R}_+; \mathbf{H})$, we provided the rate of convergence, as $\alpha \to 0^+$, of $\mathbf{u}_\alpha$ to $\mathbf{u}$ as well as of p_α to p. In addition our argument is tied up to the periodic case mostly because of special properties of the Stokes operator A and of the convective term in this setting.

The extension of the results to other boundary conditions such as the Dirichlet boundary conditions is left as future works. Moreover, it is more complicated to extend the results to the Euler equations (perfect fluids without viscosity).

In the 3D case extra-assumptions for the uniqueness of solution of the NSE are probably necessary to be assumed, to obtain rates of convergence. As mentioned in the introduction, we do not present the case of the Navier-Stokes-α here where the nonlinear term is not well-defined in the 2D case. However, the results for this model should similar to that of the L-α model (provided above).

It seems to be the case that all results herein can be established when the periodic domain $\Omega = [0, L]^2$ is replaced by the whole space $\mathbb{R}^2$, following the approach developed in [LB18]. However, the existence and uniqueness of weak solutions of all α-models herein needs to be studied carefully. Also this issue will be investigated in a forthcoming work.

Chapter 4

Rotational forms of Large Eddy Simulation turbulence models: modeling and mathematical theory

"No one has greater love than this, to lay down one's life for one's friends."

JN 15:13

Contents

This chapter is mostly based on the research paper [BLN20].

Abstract: In this chapter we present a derivation of a back-scatter rotational Large Eddy Simulation model, which is the extension of the Baldwin and Lomax model to non-equilibrium problems. The model is particularly designed to mathematically describe a fluid filling a domain with solid walls and consequently the differential operators appearing in the smoothing terms are degenerate at the boundary. After the derivation of the model, we prove some of the mathematical properties coming from the weighted energy estimates and which allow to prove existence and uniqueness of a class of regular weak solutions.

Key words: Fluid mechanics; Turbulence models; Rotational Large Eddy Simulation models; Navier-Stokes equations.

2010 MSC: 76D05, 35Q30, 76F65, 76D03, 35Q35.

4.1 Introduction

The aim of this paper is twofold: From one side we are deriving in a consistent way a rotational Large Eddy Simulation model, capable of taking into account of back-scatter of energy; from another side we are also showing, by using rather elaborate functional analysis tools, the existence of weak solutions for the models we propose.

Recall that, the motion of a turbulent incompressible flow in a 3D domain Ω can be simulated by using a turbulence model such as the following eddy viscosity model[1]:

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \operatorname{div}\big((2\nu + \nu_{\text{turb}})\, \mathbf{D}\overline{\mathbf{v}}\big) + \nabla \overline{p} = \mathbf{f}, \\ \operatorname{div} \overline{\mathbf{v}} = 0, \end{cases} \tag{4.1.1}$$

where $\overline{\mathbf{v}}_t = \partial_t \overline{\mathbf{v}}$ for simplicity, $\overline{\mathbf{v}} = \overline{\mathbf{v}}(t, \mathbf{x}) = (\overline{v}_1(t, \mathbf{x}), \overline{v}_2(t, \mathbf{x}), \overline{v}_3(t, \mathbf{x}))$ is the mean velocity of the fluid, $p = \overline{p}(t, \mathbf{x})$ the mean pressure, $\nu > 0$ the kinematic viscosity, $\nu_{\text{turb}} \geq 0$ the eddy viscosity (also known as the turbulent viscosity), $\mathbf{f} = \mathbf{f}(t, \mathbf{x}) = (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), f_3(t, \mathbf{x}))$ the external source term, $\mathbf{D}\overline{\mathbf{v}} = \frac{1}{2}(\nabla \overline{\mathbf{v}} + \nabla \overline{\mathbf{v}}^T)$ the deformation stress of the mean velocity, and "div" stands for the divergence operator.

In the whole chapter we will consider the problem with homogeneous Dirichlet boundary conditions, i.e.,

$$\overline{\mathbf{v}} = \mathbf{0} \quad \text{on} \quad (0, T) \times \partial\Omega,$$

and this poses certain technical problems, which are not present in the case of homogeneous turbulence treated in the whole space or in the space-periodic setting. However, numerical simulations would require the use of wall laws (see [CRL14]).

One basic problem in turbulence modeling is the determination of the eddy viscosity ν_{turb}, for which there are many options (see a comprehensive presentation of this question in [BIL06, CRL14]). One of the most popular models (and one among the first introduced) is the Smagorinsky one [Sma60] for which the eddy viscosity is given by

$$\nu_{\text{turb}} = \kappa\, \ell^2 |\mathbf{D}\overline{\mathbf{v}}|,$$

[1] Thanks to the divergence free constraint $\operatorname{div} \overline{\mathbf{v}} = 0$, we have $\operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) = (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}}$ where $\overline{\mathbf{v}} \otimes \overline{\mathbf{v}} = (\overline{v}_i \overline{v}_j)$ for $1 \leq i, j \leq 3$. Therefore, these both forms are used throughout this chapter without any confusion, but also the "rotational form" will be used.

where κ is the von Kármán dimensionless constant (the value of which is about 0.41) and ℓ is the Prandtl mixing length (see [Pra10]). The peculiarity of the modeling and of the equations derived is the degeneracy of the differential operators by means of the function $\ell(\mathbf{x})$, which is vanishing at the boundary. The models we study can be interpreted as obtained with the application of a differential filter with radius vanishing near to the boundary; hence, the model is not over-smoothing the boundary layer. The analysis of wall-laws or of other boundary conditions requires tools not developed yet for this problem.

In the case of a flow over a plate, identified by the plane $(x, y, 0)$ and the domain $\Omega = \mathbb{R}^2 \times \{z > 0\}$, one finds in Obukhov [Obu46] the following law

$$\ell = \ell(z) = \kappa z.$$

Considering a bi-layer model for a turbulent boundary layer over a plate, Baldwin & Lomax [BL78] suggested –from heuristic arguments– to use in the inner part of boundary layer the following formula

$$\nu_{\text{turb}} = \kappa\, \ell^2(z)|\overline{\boldsymbol{\omega}}|, \tag{4.1.2}$$

where $\overline{\boldsymbol{\omega}} = \operatorname{curl} \overline{\mathbf{v}}$ denotes the mean vorticity, while the function ℓ (not a constant now) is determined by the Van Driest formula [VD56],

$$\ell(z) := \kappa\, z\, (1 - \mathrm{e}^{-z/A});$$

here A depends on the oscillations of the plate and on the kinematic viscosity ν, while $z \geq 0$ is again the distance from the plate. As it is well-known, the Smagorinsky model is *over-diffusive*, and model (4.1.2) looks to be a very interesting alternative, leading by (4.1.1) to the system

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} - \operatorname{div}(\kappa\, \ell^2(z)|\overline{\boldsymbol{\omega}}|\, \mathbf{D}\overline{\mathbf{v}}) + \nabla \overline{p} = \mathbf{f}, \\ \hfill \operatorname{div} \overline{\mathbf{v}} = 0. \end{cases} \tag{4.1.3}$$

However, the eddy viscosity term $-\operatorname{div}(\kappa\, \ell^2(z)|\overline{\boldsymbol{\omega}}|\, \mathbf{D}\overline{\mathbf{v}})$ in (4.1.3) does not follow the rotational structure of formula (4.1.2). In [RLZ19], the authors suggest a purely rotational form $\operatorname{curl}\, (\kappa\, \ell^2(\mathbf{x})|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}})$ –which is consistent with (4.1.2)– yielding the following system

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{div}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \operatorname{curl}\, (\kappa\, \ell^2(\mathbf{x})|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) + \nabla \pi = \mathbf{f}, \\ \hfill \operatorname{div} \overline{\mathbf{v}} = 0, \end{cases} \tag{4.1.4}$$

for some modified pressure term π.

In addition to being over-diffusive, the Smagorinsky model (but this limitation is also shared by non adaptive eddy viscosity models) is not capable of taking into account phenomena of back-scatter of energy. Consequently, system (4.1.4) seems of interest limited to (statistically) stationary or equilibrium flows. A first complete existence theory for the Baldwin & Lomax model in the steady case has been recently given in [BB20].

In order to consider more complex physical settings, a variant has been proposed in [RLZ19] including a non-smoothing dispersive term, in the same spirit as in Voigt models (also written as Voight sometimes). The mathematical theory in this case needs to handle degenerate operators and weighted estimates. For this reason, in [ABLN20] we have modeled a back-scatter term of a Voigt form such as $-\alpha\,\mathrm{div}\,(\ell(\mathbf{x})\,\mathbf{D}\overline{\mathbf{v}}_t)$, where $\alpha > 0$ denotes the length scale, for turbulence evolving towards a statistical equilibrium, where $\ell(\mathbf{x})$ is a smooth positive function, vanishing only at the boundary of the domain and with a prescribed rate. In [ABLN20] we also studied the properties of the corresponding PDE system, in conjunction with the equation satisfied by the turbulent kinetic energy (TKE in the following). In [RLZ19], the authors suggested instead a back-scatter term under rotational form, such as $\mathrm{curl}\,(\ell^2(\mathbf{x})\,\overline{\boldsymbol{\omega}}_t)$, obtaining the following system:

$$
\begin{cases}
\overline{\mathbf{v}}_t + \mathrm{curl}\,(\ell^2(\mathbf{x})\,\overline{\boldsymbol{\omega}}_t) + \mathrm{div}\,(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu\Delta\overline{\mathbf{v}} + \mathrm{curl}\,(\kappa\,\ell^2(\mathbf{x})|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) + \nabla\pi = \mathbf{f}, \\
\mathrm{div}\,\overline{\mathbf{v}} = 0,
\end{cases} \tag{4.1.5}
$$

for some modified pressure term π.

In this paper we show:

1. How to derive systems (4.1.4) and (4.1.5) from a standard turbulence modeling procedure,
2. Existence and uniqueness results of classes of weak solutions for these systems supplemented with smooth enough initial data and Dirichlet homogeneous boundary conditions, under certain reasonable mathematical assumptions.

The main mathematical result we prove is the following.

Theorem 4.1.1. *Assume that:*

- *the domain Ω is bounded and of class C^2 (not necessarily with a flat boundary);*
- *the function $\ell : \overline{\Omega} \to \mathbb{R}^+$ is of class C^2 and satisfies the two following properties:*

$$
\ell(\mathbf{x}) \approx \sqrt{d(\mathbf{x}, \partial\Omega)} \qquad \text{for } \mathbf{x} \text{ close to } \partial\Omega, \tag{4.1.6}
$$

where $d(\mathbf{x}, \partial\Omega)$ denotes the distance from the boundary and

$$
\forall\ K \subset\subset \Omega,\ \exists\, \ell_K \in \mathbb{R}^\star_+ \quad s.t. \quad \ell(\mathbf{x}) \geq \ell_K > 0 \quad \forall \mathbf{x} \in K; \tag{4.1.7}
$$

- $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$ *and* $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$[2].

Then, System (4.1.5) *with* $\overline{\mathbf{v}}(0) = \overline{\mathbf{v}}_0$ *in* Ω *and* $\overline{\mathbf{v}} = \mathbf{0}$ *on* $(0,T)\times\partial\Omega$ *has a unique "regular weak" solution.*

Theorem 4.1.1 is a consequence of the weighted estimate (4.4.31) below, which is the main mathematical result of this paper and of a proper application of monotonicity techniques, coupled with localization of the test functions.

Plan of the chapter. Chapter 4 is organized as follows: In Section 4.2 we set the mathematical framework that we use in the whole chapter. Sections 4.3 provide the turbulence modeling where Subsections 4.3.1 and 4.3.2 are devoted to modeling and to explain the motivations for the systems (4.1.4) and (4.1.5). The analysis of the obtained model from the previous section is presented in Section 4.4 where the proofs of the main weighted estimate (4.4.31) and Theorem 4.1.1 are provided in Subsections 4.4.1 and 4.4.2, respectively.

4.2 Functional setting

In the sequel $\Omega \subset \mathbb{R}^3$ will be a smooth and bounded open set, as usual we write $\mathbf{x} = (x_1, x_2, x_3)$ for all $\mathbf{x} \in \mathbb{R}^3$. In particular, we assume that the boundary $\partial\Omega$ is of class $C^{0,1}$, such that the normal unit vector $\mathbf{n}$ at the boundary is well defined and other relevant properties hold true. We also define the distance $d(\mathbf{x}, A)$ of a point from a closed set $A \subset \mathbb{R}^3$ as follows

$$d(\mathbf{x}, A) := \inf_{\mathbf{y}\in A} |\mathbf{x} - \mathbf{y}|,$$

and we denote by $d(\mathbf{x})$ the distance of $\mathbf{x}$ from the boundary of the domain Ω

$$d(\mathbf{x}) := d(\mathbf{x}, \partial\Omega) \qquad \forall\, \mathbf{x} \in \Omega. \tag{4.2.8}$$

For our analysis we will use the customary Lebesgue $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$ of integer index $k \in \mathbb{N}$ and with $1 \le p \le \infty$. The $L^2(\Omega)$-norm will be denoted by $\|\,.\,\|$ for simplicity. We use boldface for vectors, matrices and tensors. We recall that $L^p_0(\Omega)$ denotes the subspace of $L^p(\Omega)$ with zero mean value, while $W^{1,p}_0(\Omega)$ is the closure of the smooth and compactly supported functions with respect to the $\|\,.\,\|_{1,p}$-norm. As usual we denote $H^1_0(\Omega) = W^{1,2}_0(\Omega)$. In addition, if Ω is bounded and if $1 < p < \infty$, the following two relevant inequalities hold true:

1) the Poincaré inequality

$$\exists\, C_P(p,\Omega) > 0 : \qquad \|\mathbf{u}\|_p \le C_P \|\nabla \mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega)^3; \tag{4.2.9}$$

[2] The divergence-free spaces $W^{1,p}_{0,\sigma}(\Omega)$ are defined below by (4.2.12).

2) the Korn inequality

$$\exists\, C_K(p,\Omega) > 0 : \qquad \|\nabla \mathbf{u}\|_p \le C_K \|\mathbf{D}\mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W_0^{1,p}(\Omega)^3. \tag{4.2.10}$$

The Korn inequality allows to control the full gradient in $L^p(\Omega)$ by its symmetric part, for functions which are zero at the boundary (see for instance in Malek, Nečas, Rokyta, and Ružička [MNRR96]). Classical results (see Bourguignon and Brezis [BB74]) concern controlling the full gradient with curl & divergence. The following inequality holds true: For all $s \ge 1$ and $1 < p < \infty$, there exists a constant $C = C(s,p,\Omega)$ such that,

$$\|\mathbf{u}\|_{s,p} \le C\Big[\|\mathrm{div}\,\mathbf{u}\|_{s-1,p} + \|\mathrm{curl}\ \mathbf{u}\|_{s-1,p} + \|\mathbf{u}\cdot\mathbf{n}\|_{s-1/p,p,\partial\Omega} + \|\mathbf{u}\|_{s-1,p}\Big],$$

for all $\mathbf{u} \in W^{s,p}(\Omega)^3$, where $\|\,.\,\|_{s-1/p,p,\partial\Omega}$ is the trace norm as explained below. This same result has been later improved by von Wahl [vW92] obtaining, under geometric conditions on the domain, the following estimate without lower order terms: Let Ω be such that $b_1(\Omega) = b_2(\Omega) = 0$, where $b_i(\Omega)$ denotes the i-th Betti number, that is the dimension of the i-th homology group $H^i(\Omega,\mathbb{Z})$. Then, there exists $C = C(p,\Omega)$ such that

$$\|\nabla \mathbf{u}\|_p \le C\big(\|\mathrm{div}\,\mathbf{u}\|_p + \|\mathrm{curl}\ \mathbf{u}\|_p\big), \tag{4.2.11}$$

for all $\mathbf{u} \in W^{1,p}(\Omega)^3$ satisfying either $(\mathbf{u}\cdot\mathbf{n})_{|\partial\Omega} = 0$ or $(\mathbf{u}\times\mathbf{n})_{|\partial\Omega} = \mathbf{0}$. As usual in fluid mechanics, when working with incompressible fluids, it is natural to incorporate the divergence-free constraint directly in the function spaces. These spaces are built upon completing the space of free divergence smooth vector fields with compact support, denoted as $\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega)^3$, in an appropriate topology. For $1 < p < \infty$ we define

$$\begin{cases} L^p_\sigma(\Omega) := \overline{\Big\{\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega)^3\Big\}}^{\|\,\cdot\,\|_p}, \\ W^{1,p}_{0,\sigma}(\Omega) := \overline{\Big\{\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega)^3\Big\}}^{\|\,\cdot\,\|_{1,p}}. \end{cases} \tag{4.2.12}$$

A basic tool in mathematical fluid mechanics is the construction of a continuous right inverse of the divergence operator with zero Dirichlet boundary conditions. An explicit construction is due to the Bogovskiĭ and it is reviewed in Galdi [Gal11, Chapter 3]. The following results holds true.

Proposition 4.2.1. *Let $\omega \subset \mathbb{R}^3$ be a bounded Lipchitz domain and let $f \in L^p_0(\omega)$. Then, there exists at least one $\mathbf{u} = \mathrm{Bog}_\omega(f) \in W^{1,p}_0(\omega)^3$ which solves the boundary value problem*

$$\begin{cases} \mathrm{div}\,\mathbf{u} = f & \text{in } \omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega. \end{cases}$$

Among other spaces, the operator Bog_ω is linear and continuous from $L^p_0(\omega)$ to $W^{1,p}_0(\omega)^3$, for all $p \in (1,\infty)$.

4.3 Modeling

In this part we perform the modeling leading to the model (4.1.4) in Subsection 4.3.1 and the rotational back-scatter model (4.1.5) in Subsection 4.3.2.

4.3.1 On the Baldwin & Lomax model

We start by recalling some facts about the Baldwin & Lomax model which will be used later on. Let $\Omega \subset \mathbb{R}^3$ denote the flow domain. We decompose any field $\psi = \psi(t, \mathbf{x})$ with $(t, \mathbf{x}) \in \mathbb{R}_+ \times \Omega$, as the sum of its mean (denoted by a bar) and its fluctuation,

$$\psi = \overline{\psi} + \psi',$$

as suggested by Reynolds [BIL06, CRL14]. The bar operator denotes any linear statistical filter that does not need to be specified, beside that we assume it verifies at least the Reynolds rules:

$$\partial\overline{\psi} = \overline{\partial\psi} \qquad \text{and} \qquad \overline{\overline{\psi}} = \overline{\psi}, \tag{4.3.13}$$

for any linear differential operator ∂.

Let us start by considering the following rotational form of the Navier-Stokes equations (NSE in the sequel),

$$\begin{cases} \mathbf{v}_t + \boldsymbol{\omega} \times \mathbf{v} - \nu\Delta\mathbf{v} + \nabla\left(p + \dfrac{|\mathbf{v}|^2}{2}\right) = \mathbf{f}, \\ \operatorname{div} \mathbf{v} = 0, \end{cases} \tag{4.3.14}$$

where $(\mathbf{v}, p)$ denotes the pair of the velocity and the pressure, and the alternative form of the convective term follows by using the well-known identity

$$(\mathbf{v} \cdot \nabla)\, \mathbf{v} = \frac{1}{2}\nabla|\mathbf{v}|^2 + \boldsymbol{\omega} \times \mathbf{v},$$

where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$. We apply the bar operator to (4.3.14). By using the Reynolds rules (4.3.13), one obtains to the following system

$$\begin{cases} \overline{\mathbf{v}}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \overline{\boldsymbol{\omega}' \times \mathbf{v}'} - \nu\Delta\overline{\mathbf{v}} + \nabla\overline{q} = \mathbf{f}, \\ \operatorname{div} \overline{\mathbf{v}} = 0, \end{cases} \tag{4.3.15}$$

where the force is chosen such that $\mathbf{f} = \overline{\mathbf{f}}$ for simplicity. The Bernoulli pressure and the fluctuation of the vorticity are given, respectively, by

$$q = p + \frac{|\mathbf{v}|^2}{2} \qquad \text{and} \qquad \boldsymbol{\omega}' = \operatorname{curl} \mathbf{v}'.$$

This leads to the issue of modeling the turbulent flux term $\overline{\boldsymbol{\omega}' \times \mathbf{v}'}$ only by mean (averaged) quantities. According to the Helmholtz-Hodge theorem, under reasonable regularity and decay assumptions, there exists a unique vector field $\mathbf{A}^{(\mathrm{R})}$ such that

$$\begin{cases} \text{curl } \mathbf{A}^{(\mathrm{R})} = \overline{\boldsymbol{\omega}' \times \mathbf{v}'}, \\ \text{div } \mathbf{A}^{(\mathrm{R})} = 0, \end{cases} \tag{4.3.16}$$

and in what follows we call $\mathbf{A}^{(\mathrm{R})}$ the "*rotational Reynolds stress.*" As usual in turbulent modeling, the fundamental question is how to express $\mathbf{A}^{(\mathrm{R})}$ in terms of averaged quantities. It is natural to assume that $\mathbf{A}^{(\mathrm{R})}$ is a function of the mean vorticity $\overline{\boldsymbol{\omega}}$. Following the standard Reynolds-stress modeling-procedure and respecting the divergence free constraint $\text{div } \mathbf{A}^{(\mathrm{R})} = 0$, we are led to set

$$\mathbf{A}^{(\mathrm{R})} = \nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}} + \nabla \psi, \tag{4.3.17}$$

for some scalar function ψ which will be specified later on. Notice that from the Reynolds rules combined with the Schwarz theorem, we have $\text{div}\, \overline{\boldsymbol{\omega}} = 0$. Therefore, taking the divergence of (4.3.17) and using $\text{div } \mathbf{A}^{(\mathrm{R})} = 0$ yields a Poisson equation for ψ:

$$-\Delta \psi = \text{div } (\nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}}),$$

hence, since $\overline{\boldsymbol{\omega}}$ is divergence-free,

$$\psi = (-\Delta)^{-1} (\nabla \nu_{\mathrm{turb}} \cdot \overline{\boldsymbol{\omega}}).$$

In conclusion, the closure assumption for the rotational Reynolds stress can be expressed as follows

$$\mathbf{A}^{(\mathrm{R})} = \nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}} + \nabla (-\Delta)^{-1} (\nabla \nu_{\mathrm{turb}} \cdot \overline{\boldsymbol{\omega}}). \tag{4.3.18}$$

Taking the curl of (4.3.18) gives

$$\text{curl } \mathbf{A}^{(\mathrm{R})} = \text{curl } (\nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}}).$$

Therefore, according to the Baldwin & Lomax model if $\nu_{\mathrm{turb}} = \kappa\, \ell^2(\mathbf{x}) |\overline{\boldsymbol{\omega}}|$, and by noting that

$$\begin{cases} \dfrac{1}{2} \overline{|\mathbf{v}|^2} = \dfrac{1}{2} |\overline{\mathbf{v}}|^2 + k, \\ \text{div } (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) = \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \nabla \left(\dfrac{|\overline{\mathbf{v}}|^2}{2} \right), \end{cases}$$

(where $k = \frac{1}{2} \overline{|\mathbf{v}'|^2}$ denotes the turbulent kinetic energy) we get as closure equations from (4.3.15) the following system:

$$\begin{cases} \overline{\mathbf{v}}_t + \text{div } (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \text{curl } (\kappa\, \ell^2(\mathbf{x}) |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}}) + \nabla (\overline{p} + k) = \mathbf{f}, \\ \text{div}\, \overline{\mathbf{v}} = 0, \end{cases}$$

which yields to the system (4.1.4) by setting the modified pressure $\pi = \overline{p} + k$, and where we recall that $\overline{\boldsymbol{\omega}} = \text{curl } \overline{\mathbf{v}}$. In a vorticity/velocity formulation it is also relevant to consider the rotational form of the convective term, hence

$$\begin{cases} \overline{\mathbf{v}}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \text{curl } (\kappa\, \ell^2(\mathbf{x}) |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}}) + \nabla \overline{q} = \mathbf{f}, \\ \text{div}\, \overline{\mathbf{v}} = 0. \end{cases}$$

4.3.2 Introduction of the rotational back-scatter term

Following a modeling similar to that already employed in [ABLN20], we show in this section how to derive the following model

$$\begin{cases} \overline{\mathbf{v}}_t + \text{curl } (\ell^2\, \overline{\boldsymbol{\omega}}_t) + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \text{curl } (\kappa \ell^2 |\overline{\boldsymbol{\omega}}|\, \overline{\boldsymbol{\omega}}) + \nabla \overline{q} = \mathbf{f}, \\ \text{div}\, \overline{\mathbf{v}} = 0, \end{cases} \tag{4.3.19}$$

for a turbulent flow evolving towards a statistical equilibrium.

Equation (4.3.15) combined with (4.3.16) becomes:

$$\begin{cases} \overline{\mathbf{v}}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \text{curl } \mathbf{A}^{(\mathrm{R})} + \nabla \overline{q} = \mathbf{f}, \\ \text{div}\, \overline{\mathbf{v}} = 0. \end{cases} \tag{4.3.20}$$

According to Leray's result [Ler34b], we know any turbulent solution (smooth enough to carry on all the calculations) to (4.3.20) satisfies the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|\overline{\mathbf{v}}(t)\|^2 + \nu \|\nabla \overline{\mathbf{v}}(t)\|^2 + \langle \text{curl } \mathbf{A}^{(\mathrm{R})}, \overline{\mathbf{v}}(t) \rangle \leq \langle \mathbf{f}(t), \overline{\mathbf{v}}(t) \rangle, \tag{4.3.21}$$

in the sense of distributions over $(0, T)$, provided that the boundary conditions do not bring additional terms (such as occurs i) with the no-slip boundary condition; ii) when $\Omega = \mathbb{R}^3$, or iii) in the space periodic case, for instance). Let us set

$$\mathscr{I}(t) := \langle \text{curl } \mathbf{A}^{(\mathrm{R})}, \overline{\mathbf{v}}(t) \rangle.$$

The aim of what follows is to study the contribution of this term in the energy inequality (4.3.21). To do so, we use the well-known formula

$$\nu_{\text{turb}} = C_k \ell \sqrt{k}, \tag{4.3.22}$$

relating eddy viscosity ν_{turb} and turbulent kinetic energy k, see [CRL14]. Then, we combine (4.3.22) with $\nu_{\text{turb}} = \kappa \ell^2 |\overline{\boldsymbol{\omega}}|$, leading to the closure equation for k

$$k = \frac{\ell^2}{2} |\overline{\boldsymbol{\omega}}|^2 = \frac{\ell^2}{2} |\text{curl } \overline{\mathbf{v}}|^2. \tag{4.3.23}$$

We assume now that the production of turbulent kinetic energy is mainly due to small scales eddies, which are in a statistical equilibrium and that no-stratification occurs. By a straightforward generalization of what is done in [CRL14, Section 4.4.1], we get the following equation for k:

$$k_t + \overline{\mathbf{v}} \cdot \nabla k + \operatorname{div}\left(\overline{e'\mathbf{v}'}\right) = \mathbf{A}^{(\mathrm{R})} \cdot \overline{\boldsymbol{\omega}} - \varepsilon + \overline{\mathbf{f}' \cdot \mathbf{v}'}, \tag{4.3.24}$$

where the rotational turbulent dissipation is given in this case by $\varepsilon = \nu\overline{|\boldsymbol{\omega}'|^2}$, and e' denotes the fluctuation of the kinetic energy of the fluctuation $e = \frac{1}{2}|\mathbf{v}'|^2$. The combination of (4.3.23) and (4.3.24) gives the formal following energy equality:

$$\frac{d}{dt}\int_\Omega k(t) = \int_\Omega \ell^2\, \overline{\boldsymbol{\omega}}_t \cdot \overline{\boldsymbol{\omega}} = \mathscr{I}(t) - \int_\Omega \varepsilon(t) + \overline{\langle \mathbf{f}', \mathbf{v}' \rangle}. \tag{4.3.25}$$

From (4.3.21) and (4.3.25) it follows the following inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|\overline{\mathbf{v}}(t)\|^2 + \|\ell\, \overline{\boldsymbol{\omega}}(t)\|^2\right) + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \|\sqrt{\varepsilon}(t)\|^2 \leq \langle \mathbf{f}(t), \overline{\mathbf{v}}(t)\rangle + \overline{\langle \mathbf{f}', \mathbf{v}' \rangle}. \tag{4.3.26}$$

The energy inequality (4.3.26) suggests to add the term $\ell^2\, \overline{\boldsymbol{\omega}}_t$ to the rotational Reynolds stress in formula (4.3.18), leading to the following expression for the no-equilibrium rotational Reynolds stress

$$\mathbf{A}^{(\mathrm{R})} = \ell^2\, \overline{\boldsymbol{\omega}}_t + \nu_{\mathrm{turb}}\, \overline{\boldsymbol{\omega}} + \nabla(-\Delta)^{-1}(\nabla\nu_{\mathrm{turb}} \cdot \overline{\boldsymbol{\omega}}). \tag{4.3.27}$$

When we plug (4.3.27) into (4.3.20) to get the following energy inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|\overline{\mathbf{v}}(t)\|^2 + \|\ell\, \overline{\boldsymbol{\omega}}(t)\|^2\right) + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \|\sqrt{\nu_{\mathrm{turb}}}\, \overline{\boldsymbol{\omega}}(t)\|^2 \leq \langle \mathbf{f}(t), \overline{\mathbf{v}}(t)\rangle. \tag{4.3.28}$$

We compare (4.3.26) and (4.3.28), which is consistent when the following compatibility condition is satisfied:

$$\|\sqrt{\nu_{\mathrm{turb}}}\, \overline{\boldsymbol{\omega}}(t)\|^2 + \overline{\langle \mathbf{f}', \mathbf{v}' \rangle} \leq \|\sqrt{\varepsilon}(t)\|^2, \tag{4.3.29}$$

which we assume to be held near statistical equilibium. Hence, (4.3.19) follows by combining (4.3.20) and (4.3.27). Finally, (4.3.19) yields the model (4.1.5) by setting the modified pressure $\pi = \overline{p} + k$. An example which satisfies (4.3.29) is given by the following remark.

Remark 4.3.1. *The assumption in condition (4.3.29) can be justified as in [ABLN20, Remark 2.2]. More precisely, (for a time averaging filter) this condition holds true when source term is constant $\mathbf{f}(t, \mathbf{x}) = \mathbf{f}(\mathbf{x})$, without turbulent fluctuation, i.e., $\mathbf{f}' = \mathbf{0}$. It implies a decrease of TKE, which means a decrease of the turbulence, towards a laminar state, or a stable statistical equilibrium, such as a grid turbulence.*

4.4 Analysis of the model

In this part of the chapter we perform the mathematical analysis of the back-scatter rotational model, by using established methods of analysis for non-Newtonian fluids. We first present the main weighted estimate in Subsection 4.4.1 and then proving the existence and uniqueness results in Subsection 4.4.2, as stated in Theorem 4.1.1.

4.4.1 Main estimate

In this section we show a bound involving the weighted-curl and weighted-gradient, which does not follow directly from the classical tools combining weighted estimates and harmonic analysis. As employed in [BB20] it can be shown that for fields in $W^{1,p}_{0,\sigma}(\Omega)$ one can prove the weighted estimate

$$\int_\Omega |\nabla \mathbf{v}(\mathbf{x})|^p \, w(\mathbf{x}) \, \mathrm{d}\mathbf{x} \leq C(w, \Omega, p) \int_\Omega |\mathrm{curl}\ \mathbf{v}(\mathbf{x})|^p \, w(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{4.4.30}$$

provided that the *weight* function $w \in L^1_{loc}(\mathbb{R}^3)$, which is s.t. $w \geq 0$ a.e., belongs to the Muckenhoupt class A_p, for $1 < p < \infty$, that is there exists C such that

$$\sup_{Q \subset \mathbb{R}^n} \left(\fint_Q w(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right) \left(\fint_Q w(\mathbf{x})^{1/(1-p)} \, \mathrm{d}\mathbf{x} \right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^3$ (see also in Stein [Ste93]). It is well-known that the powers of the distance function $w(\mathbf{x}) = \big(d(\mathbf{x}, \partial\Omega)\big)^\alpha$ are Muckenhoupt weights of class A_p if and only if $-1 < \alpha < p-1$, hence in the relevant cases we could not infer the required estimates if $\ell(\mathbf{x}) = \big(d(\mathbf{x}, \partial\Omega)\big)^\alpha$, for $\alpha \geq p - 1$.

In our case, we can prove a crucial estimate, close to (4.4.30), in a different and direct way, by using the special Hilbert structure when $p = 2$. Our estimate, displayed in the following lemma, is based on elementary direct computations and plays a fundamental role on the analysis of the rotational back-scatter system (4.1.5).

Lemma 4.4.1. *Assume that the function ℓ is such that $\ell^2 \in W^{2,\infty}(\Omega)$ and let $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)$. Then, there exists a positive constant $C(\ell) = C(\|D^2\ell^2\|_\infty)$ such that*

$$\int_\Omega \ell^2 |\nabla \mathbf{v}|^2 \mathrm{d}\mathbf{x} \leq \int_\Omega \ell^2 |\mathrm{curl}\ \mathbf{v}|^2 \mathrm{d}\mathbf{x} + C(\ell) \int_\Omega |\mathbf{v}|^2 \mathrm{d}\mathbf{x}. \tag{4.4.31}$$

Proof. We start from the well-known vector-calculus identity

$$-\Delta \mathbf{v} = \mathrm{curl}\ (\mathrm{curl}\ \mathbf{v}) - \nabla(\mathrm{div}\, \mathbf{v}) = \mathrm{curl}\ (\mathrm{curl}\ \mathbf{v}), \tag{4.4.32}$$

that holds for any divergence free vector field $\mathbf{v}$. Then multiplying (4.4.32) by $\ell^2\mathbf{v}$ and integrating by parts on Ω we obtain[3]:

$$\int_\Omega \nabla\mathbf{v} : \nabla(\ell^2\mathbf{v})\, d\mathbf{x} = \int_\Omega (\text{curl } \mathbf{v}) \cdot (\text{curl } (\ell^2\mathbf{v}))\, \mathrm{d}\mathbf{x}, \tag{4.4.33}$$

where the fact that $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$ has been used. We argue in two steps, considering separately both sides of (4.4.33) one after another.

Step 1. The left-hand side (l.h.s) of (4.4.33) can be rewritten as follows

$$\begin{aligned}
\int_\Omega \nabla\mathbf{v} : \nabla(\ell^2\mathbf{v})\, \mathrm{d}\mathbf{x} &= \int_\Omega \sum_{i,j=1}^{3} \partial_j v_i\, \partial_j(\ell^2 v_i)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_\Omega \sum_{i,j=1}^{3} \partial_j v_i^2 \partial_j \ell^2\, \mathrm{d}\mathbf{x} + \int_\Omega \sum_{i,j=1}^{3} (\partial_j v_i)^2 \ell^2\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_\Omega \sum_{j=1}^{3} \partial_j \left(\sum_{i=1}^{3} v_i^2\right) \partial_j \ell^2\, \mathrm{d}\mathbf{x} + \int_\Omega \ell^2 |\nabla v|^2\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_\Omega \sum_{j=1}^{3} \partial_j |\mathbf{v}|^2 \partial_j \ell^2\, \mathrm{d}\mathbf{x} + \int_\Omega \ell^2 |\nabla \mathbf{v}|^2\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_\Omega \nabla|\mathbf{v}|^2 \cdot \nabla\ell^2\, \mathrm{d}\mathbf{x} + \int_\Omega \ell^2 |\nabla \mathbf{v}|^2\, \mathrm{d}\mathbf{x} \\
&= -\frac{1}{2}\int_\Omega |\mathbf{v}|^2 \Delta\ell^2\, \mathrm{d}\mathbf{x} + \int_\Omega \ell^2 |\nabla \mathbf{v}|^2\, \mathrm{d}\mathbf{x},
\end{aligned} \tag{4.4.34}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ and in the last equality in (4.4.34) we used integration by parts possible again since $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$.

Step 2. The right-hand side (r.h.s) of (4.4.33) can be rewritten as

$$\int_\Omega (\text{curl } \mathbf{v}) \cdot (\text{curl } (\ell^2\mathbf{v}))\, \mathrm{d}\mathbf{x} = \int_\Omega \ell^2 |\text{curl } \mathbf{v}|^2\, \mathrm{d}\mathbf{x} + \int_\Omega (\text{curl } \mathbf{v}) \cdot \big((\nabla\ell^2) \times \mathbf{v}\big)\, \mathrm{d}\mathbf{x}, \tag{4.4.35}$$

where the identity curl $(\ell^2\mathbf{v}) = \ell^2$curl $\mathbf{v} + (\nabla\ell^2) \times \mathbf{v}$ has been used. Combining (4.4.34) and (4.4.35) yields

$$\begin{aligned}
\int_\Omega \ell^2 |\nabla\mathbf{v}|^2\, \mathrm{d}\mathbf{x} &= \int_\Omega \ell^2 |\text{curl } \mathbf{v}|^2\, \mathrm{d}\mathbf{x} + \int_\Omega (\text{curl } \mathbf{v}) \cdot \big((\nabla\ell^2) \times \mathbf{v}\big)\, \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{v}|^2 \Delta\ell^2\, \mathrm{d}\mathbf{x}, \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{4.4.36}$$

The difficulty in the r.h.s of (4.4.36) is due to the integral I_2, that we will consider in the following. Let δ_{ij} denotes the Kronecker tensor,

$$\delta_{ij} = 1 \ \text{ if } \ i = j, \quad \delta_{ij} = 0 \ \text{ if } \ i \neq j. \tag{4.4.37}$$

Let ε_{ijk} denotes the Levi-Civita tensor, that is fully characterised by:

[3] We denote $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$ for any two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ for $1 \le i, j \le 3$.

$$\varepsilon_{123} = 1,\ \varepsilon_{ijk} \text{ is antisymmetric against the indices.}$$

In particular the vector cross product is expressed through the Levi-Civita tensor by the equation

$$(\mathbf{a} \times \mathbf{b})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_j b_k,$$

and the following relation holds (see [CRL14])

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}. \tag{4.4.38}$$

Using these tools, we rewrite component by component the integrand in I_2 as follows

$$\begin{aligned}
(\mathrm{curl}\ \mathbf{v}) \cdot \big((\nabla \ell^2) \times \mathbf{v}\big) &= \sum_{i,j,k,m,p=1}^{3} \epsilon_{ijk} (\partial_j v_k)\, \epsilon_{imp} (\partial_m \ell^2) v_p \\
&= \sum_{j,k,m,p=1}^{3} (\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km})(\partial_j v_k)\, (\partial_m \ell^2) v_p \\
&= \sum_{j,k=1}^{3} [(\partial_j v_k)\, (\partial_j \ell^2) v_k - (\partial_j v_k)\, (\partial_k \ell^2) v_j] \\
&= \frac{1}{2} \nabla |\mathbf{v}|^2 \cdot \nabla \ell^2 - \sum_{j,k=1}^{3} (\partial_j v_k)\, (\partial_k \ell^2) v_j.
\end{aligned} \tag{4.4.39}$$

As $\mathbf{v}$ and ℓ vanishe at the boundary and $\mathbf{v}$ is divergence-free, we deduce from (4.4.39)

(4.4.40)

$$\begin{aligned}
I_2 = \int_\Omega (\mathrm{curl}\ \mathbf{v}) \cdot \big((\nabla \ell^2) \times \mathbf{v}\big)\, \mathrm{d}\mathbf{x} &= \frac{1}{2} \sum_{j=1}^{3} \int_\Omega \nabla |\mathbf{v}|^2 \cdot \nabla \ell^2\, \mathrm{d}\mathbf{x} - \sum_{j,k=1}^{3} \int_\Omega (\partial_j v_k)\, (\partial_k \ell^2) v_j\, \mathrm{d}\mathbf{x} \\
&= -\frac{1}{2} \int_\Omega |\mathbf{v}|^2\, \Delta \ell^2\, \mathrm{d}\mathbf{x} + \sum_{j,k=1}^{3} \int_\Omega v_j v_k\, \partial_{jk} \ell^2\, \mathrm{d}\mathbf{x},
\end{aligned}$$

which leads to

$$I_2 \leq \left| \int_\Omega (\mathrm{curl}\ \mathbf{v}) \cdot \big((\nabla \ell^2) \times \mathbf{v}\big)\, \mathrm{d}\mathbf{x} \right| \leq C \|D^2 \ell^2\|_\infty \|\mathbf{v}\|^2.$$

In addition, the other integral I_3 on the r.h.s of (4.4.36) is bounded by

$$I_3 = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \Delta \ell^2\, \mathrm{d}\mathbf{x} \leq C \|D^2 \ell^2\|_\infty \|\mathbf{v}\|^2.$$

We get the estimate (4.4.31) by combining the two estimates above and (4.4.36), which concludes the proof. □

4.4.2 Existence and uniqueness results

Throughout this section, we assume that assumptions (4.1.6) and (4.1.7) in Theorem 4.1.1 hold, that is $\ell(\mathbf{x}) = O(\sqrt{d(\mathbf{x}, \partial\Omega)})$ near the boundary and ℓ is strictly positive inside the domain. Moreover, we also assume that $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$ and $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$. Finally, recall that $\overline{\boldsymbol{\omega}} = \operatorname{curl} \overline{\mathbf{v}}$. As in (4.2.8) we write $d(\mathbf{x}, \partial\Omega) = d(\mathbf{x})$.

Without loss of generality, and according to the modeling introduced in [ABLN20] motivated by dimensional analysis, we consider now the back-scatter Baldwin & Lomax model with the following explicit expression for the length ℓ

$$\ell(\mathbf{x}) = \sqrt{d_0\, d(\mathbf{x})} \quad \text{for some length } d_0 > 0, \tag{4.4.41}$$

which is consistent with assumptions (4.1.6) and (4.1.7) and $\ell^2(\mathbf{x}) = d_0\, d(\mathbf{x}) \in C^2(\overline{\Omega})$, if the boundary of the domain $\partial\Omega$ is at least of class C^2 (cf. the assumptions in Lemma 4.4.1 and see also Gilbarg and Trudinger [GT01, Chapter 14]). Consequently, we now study the existence and uniqueness problems for the following model

$$\begin{cases} \overline{\mathbf{v}}_t + \operatorname{curl}\,(d_0\, d\, \overline{\boldsymbol{\omega}}_t) + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \nu\Delta\overline{\mathbf{v}} + \operatorname{curl}\,(d_0\, d\, |\overline{\boldsymbol{\omega}}|\, \overline{\boldsymbol{\omega}}) + \nabla\overline{p} = \mathbf{f}, \\ \hfill \operatorname{div} \overline{\mathbf{v}} = 0, \end{cases} \tag{4.4.42}$$

where $\overline{p}$ is some modified pressure and, again for simplicity and without loss of generality, we suppose from now on that $d_0 = 1$. Recall that the above system is supplemented by the Dirichlet boundary conditions $\overline{\mathbf{v}} = \mathbf{0}$ on $(0,T) \times \partial\Omega$ and the initial datum $\overline{\mathbf{v}}(0) = \overline{\mathbf{v}}_0$ in Ω.

In order to prove existence of weak solutions we observe that the basic a-priori estimate is obtained by testing with $\overline{\mathbf{v}}$ itself and obtaining (after integration by parts, if solutions are smooth to perform all computations) the following energy inequality for all $s \in (0,T)$

$$\begin{aligned} \|\overline{\mathbf{v}}(s)\|^2 + \|\sqrt{d}\,\overline{\boldsymbol{\omega}}(s)\|^2 + \nu \int_0^s \|\nabla\overline{\mathbf{v}}\|^2\,\mathrm{d}t + 2\int_0^s \int_\Omega d\,|\overline{\boldsymbol{\omega}}|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t& \\ \leq \|\overline{\mathbf{v}}_0\|^2 + \|\sqrt{d}\,\overline{\boldsymbol{\omega}}_0\|^2 + \frac{C}{\nu}\int_0^s \int_\Omega \|\mathbf{f}\|^2\mathrm{d}\mathbf{x}\mathrm{d}t.& \end{aligned}$$

Here, the vanishing contribution of the rotational convection term has been used (after modifying the pressure) and the dimensionless constant C comes from applying the Poincaré and Young inequalities. It follows by using (4.2.9) and a natural $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$ assumption, that

$$\begin{aligned} \overline{\mathbf{v}} &\in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^1_0(\Omega)^3), \\ d^{1/2}\overline{\boldsymbol{\omega}} &\in L^\infty(0,T;L^2(\Omega)^3), \\ d^{1/3}\overline{\boldsymbol{\omega}} &\in L^3(0,T;L^3(\Omega)^3). \end{aligned}$$

Hence, from one side we have for the mean velocity $\overline{\mathbf{v}}$ the same estimates valid for the Leray-Hopf weak solutions of the Navier-Stokes equations; On the other side we have further estimates on the mean vorticity which are weighted by the distance from the boundary, hence not enough to directly apply standard methods. We observe that both (dispersive/back-scatter and dissipative/eddy viscosity) the additional degenerate terms pose some mathematical difficulties: If in the system (4.4.42) one would have been given the following smoothing term

$$\operatorname{curl}\,(|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}),$$

then the a-priori estimate, and the divergence-free constraint with (4.2.11) will imply directly that $\overline{\mathbf{v}} \in L^3(0,T;W^{1,3}_{0,\sigma}(\Omega))$, allowing us to apply the same tools valid for the Smagorinsky model as in [Lio69]. Here the estimates degenerate at the boundary (being the mean vorticity weighted by the distance function d) and this prevents from using the solution itself as a legitimate test function.

Next, if the dispersive term would have been given by

$$\operatorname{curl}\,(\overline{\boldsymbol{\omega}}_t) = -\Delta\overline{\mathbf{v}}_t,$$

(where the equality is valid for divergence-free functions) the same well-known tools valid for the Voigt model can be used as in [CLT06]. We note in particular that in problem (4.4.42) the presence of this dispersive term does not allow us to prove by comparison the classical regularity in negative spaces for the time derivative $\overline{\mathbf{v}}_t$ (as needed by Aubin-Lions type compactness results); In addition it is also not easy to prove from the weak formulation that the solution is weakly continuous in $L^2(\Omega)$ as required by the compactness results à la Hopf (or in the refined form of Landes and Mustonen [LM87]).

Each term poses some questions which can be separately handled, but the combination of the effects of both weighted terms requires to have a precise interplay between some local (in space) estimates on a double approximated system.

For these reasons we first ϵ-regularize the system by a hyper-dissipative term and we then approximate it by a Galerkin procedure. We first pass to the limit in the Galerkin system and then pass to the limit in the smoothed system, by using further regularity on the time derivative which is obtained in a way similar to [ABLN20].

4.4.3 The approximate system: existence and further regularity

For simplicity we assume from now on that $\mathbf{f} = \mathbf{0}$, but the introduction of an external force $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$ can be done with minor changes. Moreover, throughout the section, we assume $\ell = \sqrt{d}$, but observe that assumptions (4.1.6) and (4.1.7) would be enough.

In order to apply the standard Galerkin method and monotonicity, we approximate the system (4.4.42) by the following one

(4.4.43)
$$\begin{cases} \overline{\mathbf{v}}^\epsilon_t + \operatorname{curl}\,(d\,\overline{\boldsymbol{\omega}}^\epsilon_t) + \overline{\boldsymbol{\omega}}^\epsilon \times \overline{\mathbf{v}}^\epsilon - \nu\Delta\overline{\mathbf{v}}^\epsilon + \operatorname{curl}\,(d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon) - \epsilon\operatorname{div}(|\mathbf{D}\overline{\mathbf{v}}^\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon) + \nabla\overline{p}^\epsilon = \mathbf{0}, \\ \operatorname{div}\overline{\mathbf{v}}^\epsilon = 0, \end{cases}$$

and we study it with homogeneous Dirichlet boundary conditions.

The above system falls within the standard class of monotone problems as those considered in Lions [Lio69] and Ladyžhenskaya [Lad69] in the analysis of the Smagorinsky model. Here, in addition to the standard Smagorinsky model, we have two perturbation terms, which can be easily handled.

We have the following result.

Theorem 4.4.1. *Let be given $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$, then there exists a unique weak solution $\overline{\mathbf{v}}^\epsilon$ to system (4.4.43) with $\overline{\mathbf{v}}_0$ as initial datum and with homogeneous Dirichlet boundary conditions. This means that*

$$\overline{\mathbf{v}}^\epsilon \in L^\infty(0,T;L^2_\sigma(\Omega)^3) \cap L^3(0,T;W^{1,3}_{0,\sigma}(\Omega)),$$

is such that

$$\int_0^T\int_\Omega (\overline{\boldsymbol{\omega}}^\epsilon \times \overline{\mathbf{v}}^\epsilon)\cdot\boldsymbol{\phi} + \nu\nabla\overline{\mathbf{v}}^\epsilon : \nabla\boldsymbol{\phi} + d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon\cdot\operatorname{curl}\boldsymbol{\phi} + \epsilon\,|\mathbf{D}\overline{\mathbf{v}}^\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon : \mathbf{D}\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t$$
$$= \int_0^T\int_\Omega \overline{\mathbf{v}}^\epsilon\cdot\boldsymbol{\phi}_t + d\,\overline{\boldsymbol{\omega}}^\epsilon\cdot\operatorname{curl}\,\boldsymbol{\phi}_t\,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \overline{\mathbf{v}}_0\cdot\boldsymbol{\phi}(0) + d\,\overline{\boldsymbol{\omega}}_0\cdot\operatorname{curl}\,\boldsymbol{\phi}(0)\,\mathrm{d}\mathbf{x}, \tag{4.4.44}$$

for all $\boldsymbol{\phi} \in C^\infty_{0,\sigma}([0,T)\times\Omega)^3$.

Proof. Testing by divergence free test vector fields as is custom, we do not consider the pressure term that can be recovered through the usual ways. The proof is based on an application of the Galerkin method to prove existence of approximate solutions. Denoting by $\overline{\mathbf{v}}^{\epsilon,m} \in V_m$ for all $t \in (0,T)$ a finite dimensional approximation to $\overline{\mathbf{v}}^\epsilon$ one has the following energy estimate for all $s \in [0,T]$

$$\|\overline{\mathbf{v}}^{\epsilon,m}(s)\|^2 + \|\sqrt{d}\,\overline{\boldsymbol{\omega}}^{\epsilon,m}(s)\|^2 + 2\nu\int_0^s \|\nabla\overline{\mathbf{v}}^{\epsilon,m}\|^2\,\mathrm{d}t + 2\epsilon\int_0^s \|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m}\|_3^3\,\mathrm{d}t$$
$$+ 2\int_0^s\int_\Omega d\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t \le \|\overline{\mathbf{v}}_0\|^2 + \|\sqrt{d}\,\overline{\boldsymbol{\omega}}_0\|^2,$$

which shows, by using (4.2.9)-(4.2.10) that

$$\overline{\mathbf{v}}^{\epsilon,m} \in L^\infty(0,T;L^2_\sigma(\Omega)^3) \cap L^3(0,T;W^{1,3}_{0,\sigma}(\Omega)),$$

with estimates depending on $\epsilon > 0$, but independent of $m \in \mathbb{N}$.

Next, testing with $\overline{\mathbf{v}}_t$, we can see that the contribution of the rotational convective term can be estimated as follows:

$$
\begin{aligned}
\Big|\int_\Omega (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}}) \cdot \overline{\mathbf{v}}_t \, \mathrm{d}\mathbf{x}\Big| &\leq \|\overline{\mathbf{v}}_t\| \, \|\overline{\mathbf{v}}\|_6 \|\overline{\boldsymbol{\omega}}\|_3 \\
&\leq \|\overline{\mathbf{v}}_t\| \, \|\overline{\mathbf{v}}\|_6 \|\nabla \overline{\mathbf{v}}\|_3 \\
&\leq C\|\overline{\mathbf{v}}_t\| \, \|\nabla \overline{\mathbf{v}}\| \, \|\mathbf{D}\overline{\mathbf{v}}\|_3 \\
&\leq \frac{1}{2}\|\overline{\mathbf{v}}_t\|^2 + C\|\nabla \overline{\mathbf{v}}\|^2 \, \|\mathbf{D}\overline{\mathbf{v}}\|_3^2,
\end{aligned}
$$

for smooth enough $\overline{\mathbf{v}}$, where we used the Korn inequality (4.2.10).

By using $\overline{\mathbf{v}}_t^{\epsilon,m}$ as test function and the previous estimates (where $\mathbf{v}$ is replaced by $\overline{\mathbf{v}}^{\epsilon,m}$) we then obtain the following differential inequality

$$
\begin{aligned}
\frac{1}{2}\big(\|\overline{\mathbf{v}}_t^{\epsilon,m}(s)\|^2 + \|\sqrt{d}\,\overline{\boldsymbol{\omega}}_t^{\epsilon,m}(s)\|^2\big) + \frac{d}{dt}\frac{\nu}{2}\|\nabla \overline{\mathbf{v}}^{\epsilon,m}\|^2 + \frac{d}{dt}\frac{\epsilon}{3}\|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m}\|_3^3 \\
+ \frac{d}{3dt}\int_\Omega d\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|^3 \mathrm{d}\mathbf{x} \leq C\|\nabla \overline{\mathbf{v}}^{\epsilon,m}\|^2 \, \|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m}\|_3^2.
\end{aligned}
$$

An application of the Gronwall lemma –possible since $\overline{\mathbf{v}}^{\epsilon,m} \in L^3(0,T; W^{1,3}_{0,\sigma}(\Omega))$– shows that

$$
\overline{\mathbf{v}}^{\epsilon,m} \in L^\infty(0,T; W^{1,3}_{0,\sigma}(\Omega)) \qquad \text{and} \qquad \overline{\mathbf{v}}_t^{\epsilon,m} \in L^2(0,T; L^2(\Omega)^3),
$$

again uniformly in $m \in \mathbb{N}$. The above estimates with Aubin-Lions compactness lemma (cf. [Lio69]) are enough to infer that, for each fixed $\epsilon > 0$, there exists

$$
\overline{\mathbf{v}}^\epsilon \in L^\infty(0,T; W^{1,3}_{0,\sigma}(\Omega)) \cap H^1(0,T; L^2_\sigma(\Omega)),
$$

such that when $m \to +\infty$

$$
\begin{aligned}
\overline{\mathbf{v}}^{\epsilon,m} &\overset{*}{\rightharpoonup} \overline{\mathbf{v}}^\epsilon && \text{in } L^\infty(0,T; W^{1,3}_{0,\sigma}(\Omega)), \\
\overline{\mathbf{v}}^{\epsilon,m} &\rightharpoonup \overline{\mathbf{v}}^\epsilon && \text{in } L^3(0,T; W^{1,3}_{0,\sigma}(\Omega)), \\
\overline{\mathbf{v}}_t^{\epsilon,m} &\rightharpoonup \overline{\mathbf{v}}_t^\epsilon && \text{in } L^2(0,T; L^2_\sigma(\Omega)^3), \\
\sqrt{d}\,\overline{\boldsymbol{\omega}}_t^{\epsilon,m} &\rightharpoonup \sqrt{d}\,\overline{\boldsymbol{\omega}}_t^\epsilon && \text{in } L^2(0,T; L^2(\Omega)^3), \\
|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m}|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m} &\rightharpoonup \chi_1 && \text{in } L^{3/2}(0,T; L^{3/2}(\Omega)^9), \\
|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m}|\mathbf{D}\overline{\mathbf{v}}^{\epsilon,m} &\overset{*}{\rightharpoonup} \chi_1 && \text{in } L^\infty(0,T; L^{3/2}(\Omega)^9), \\
d^{2/3}\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|\overline{\boldsymbol{\omega}}^{\epsilon,m} &\rightharpoonup \chi_2 && \text{in } L^{3/2}(0,T; L^{3/2}(\Omega)^3), \\
d^{2/3}\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|\overline{\boldsymbol{\omega}}^{\epsilon,m} &\overset{*}{\rightharpoonup} \chi_2 && \text{in } L^\infty(0,T; L^{3/2}(\Omega)^3), \\
\overline{\mathbf{v}}^{\epsilon,m} &\to \overline{\mathbf{v}}^\epsilon && \text{in } L^2(0,T; L^q_\sigma(\Omega)^3) \qquad \forall\, q < \infty.
\end{aligned}
$$

The above convergences are enough to pass to the limit in the approximate equations, except in the monotone terms.

In particular, for the Baldwin & Lomax term, it follows that

$$\begin{aligned}&\int_0^T\int_\Omega d\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|\overline{\boldsymbol{\omega}}^{\epsilon,m}\cdot \text{curl }\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &=\int_0^T\int_\Omega d^{2/3}\,|\overline{\boldsymbol{\omega}}^{\epsilon,m}|\overline{\boldsymbol{\omega}}^{\epsilon,m}\cdot d^{1/3}\text{curl }\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t \quad\overset{m\to+\infty}{\to}\quad \int_0^T\int_\Omega \chi_2\cdot d^{1/3}\text{curl }\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_0^T\int_\Omega d^{1/3}\chi_2\cdot \text{curl }\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

for all smooth functions $\boldsymbol{\phi}$ with compact support. Hence, one gets (the trick of distributing powers of the distance function on the integrands will be used several times in the sequel)

$$\begin{aligned}&\int_0^T\int_\Omega (\overline{\boldsymbol{\omega}}\times\overline{\mathbf{v}})\cdot\boldsymbol{\phi}+\nu\nabla\overline{\mathbf{v}}:\nabla\boldsymbol{\phi}+\epsilon\chi_1:\mathbf{D}\boldsymbol{\phi}+d^{1/3}\chi_2\cdot\text{curl}\,\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\quad=\int_0^T\int_\Omega \overline{\mathbf{v}}\cdot\boldsymbol{\phi}_t+d\,\overline{\boldsymbol{\omega}}\cdot\text{curl }\boldsymbol{\phi}_t\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_\Omega \overline{\mathbf{v}}_0\cdot\boldsymbol{\phi}(0)+d\,\overline{\boldsymbol{\omega}}_0\cdot\text{curl }\boldsymbol{\phi}(0)\,\mathrm{d}\mathbf{x},\end{aligned}$$

and for almost all $0\le s_0\le s\le T$ it holds

$$\begin{aligned}\frac{1}{2}\big(\|\overline{\mathbf{v}}^\epsilon(s)\|^2+\|\sqrt{d}\,\overline{\boldsymbol{\omega}}^\epsilon(s)\|^2\big)+\int_{s_0}^s\Big[\nu\|\nabla\overline{\mathbf{v}}^\epsilon(t)\|^2+\int_\Omega\Big(\epsilon\chi_1:\mathbf{D}\overline{\mathbf{v}}^\epsilon+d^{1/3}\chi_2\cdot\overline{\boldsymbol{\omega}}^\epsilon\Big)\,\mathrm{d}\mathbf{x}\Big]\mathrm{d}t\\ =\frac{1}{2}\big(\|\overline{\mathbf{v}}^\epsilon(s_0)\|^2+\|\sqrt{d}\,\overline{\boldsymbol{\omega}}^\epsilon(s_0)\|^2\big).\end{aligned}$$

Hence, to show that $\overline{\mathbf{v}}^\epsilon$ is a solution to (4.4.43) one needs to prove that

$$\chi_1=|\mathbf{D}\overline{\mathbf{v}}^\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon \qquad\text{and}\qquad \chi_2=d^{2/3}\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon, \tag{4.4.45}$$

at least almost everywhere in $(0,T)\times\Omega$.

This can be proved by using the standard monotonicity argument (Minty-Browder trick) as developed in the time evolution problem in [Lad69, Lio69]. The only thing to be verified is that the function

$$\overline{\mathbf{v}}^{\epsilon,m}-\overline{\mathbf{v}}^\epsilon,$$

is a legitimate test function. This follows by the regularity of the time derivative we proved. Hence, the classical argument proceeds as in (cf. [Lio69, page 207]) showing that the approximate solution $\overline{\mathbf{v}}^\epsilon$ satisfies

$$\begin{aligned}&\epsilon\int_0^s\int_\Omega(\chi_1-|\mathbf{D}\boldsymbol{\phi}|\mathbf{D}\boldsymbol{\phi}):(\mathbf{D}\overline{\mathbf{v}}^\epsilon-\mathbf{D}\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\mathrm{d}t\ge 0,\\ &\int_0^s\int_\Omega(\chi_2-d^{2/3}\,|\text{curl }\boldsymbol{\phi}|\text{curl }\boldsymbol{\phi})\cdot(d^{1/3}\overline{\boldsymbol{\omega}}^\epsilon-d^{1/3}\text{curl }\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\qquad\qquad\qquad\qquad\|\\ &\int_0^s\int_\Omega(d\chi_2\,|\text{curl }\boldsymbol{\phi}|\text{curl }\boldsymbol{\phi})\cdot(\overline{\boldsymbol{\omega}}^\epsilon-\text{curl }\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\mathrm{d}t\ge 0,\end{aligned}$$

for a.e. $s \in [0,T]$ and for arbitrary $\boldsymbol{\phi} \in L^3(0,T;W^{1,3}_{0,\sigma}(\Omega))$, since they are both coming from monotone terms. This is enough to imply by monotonicity of the functions

$$\mathbf{B} \mapsto |\mathbf{B}|\mathbf{B} \qquad \text{and} \qquad \mathbf{b} \mapsto d^\alpha |\mathbf{b}|\mathbf{b},$$

(which is valid for all matrices $\mathbf{B}$, vectors $\mathbf{b}$, $\alpha \in \mathbb{R}^+$, and smooth functions d such that $d > 0$ for all $\mathbf{x} \in \Omega$, cf. [BB20, Lemma 3.2]) that the equalities in (4.4.45) hold true. We finally proved that there exists $\overline{\mathbf{v}}^\epsilon$ such that

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[\overline{\mathbf{v}}^\epsilon_t \cdot \boldsymbol{\phi} + d\, \overline{\boldsymbol{\omega}}^\epsilon_t \cdot \operatorname{curl} \boldsymbol{\phi} + (\overline{\boldsymbol{\omega}}^\epsilon \times \overline{\mathbf{v}}^\epsilon) \cdot \boldsymbol{\phi} \Big] \, \mathrm{d}\mathbf{x}\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \Big[\nu \, \nabla \overline{\mathbf{v}}^\epsilon : \nabla \boldsymbol{\phi} + \epsilon \, |\mathbf{D}\overline{\mathbf{v}}^\epsilon| \mathbf{D}\overline{\mathbf{v}}^\epsilon : \mathbf{D}\boldsymbol{\phi} + d \, |\overline{\boldsymbol{\omega}}^\epsilon| \, \overline{\boldsymbol{\omega}}^\epsilon \cdot \operatorname{curl} \boldsymbol{\phi} \Big] \, \mathrm{d}\mathbf{x}\mathrm{d}t = 0,
\end{aligned} \tag{4.4.46}$$

at least for all $\boldsymbol{\phi} \in L^3(0,T;W^{1,3}_{0,\sigma}(\Omega))$. Well-known estimates can be also applied to show that the solution $\overline{\mathbf{v}}^\epsilon$ is unique. □

Remark 4.4.1. *Due to the regularity of the solution of the approximated system we can use the function $\overline{\mathbf{v}}^{\epsilon,m} - \overline{\mathbf{v}}^\epsilon$ as test function. In the case of the non-regularized system (4.4.42) we will see that localization in the space variable is needed and this is not compatible with the finite dimensional Galerkin approximation.*

4.4.4 Proof of Theorem 4.1.1

We now consider the original problem (without the ϵ-regularization) and give the proof of the main result of the paper.

Proof of Theorem 4.1.1. The proof is divided into two steps. Let us start with the existence part.

Step 1: Existence part. To construct weak solutions to (4.4.42) we consider the limit $\epsilon \to 0$ of solutions to (4.4.43). By the estimate coming from the energy inequality we also have by using Lemma 4.4.1 the following inequality, for all $s \in (0,T)$

$$\begin{aligned}
\frac{1}{2}\|\overline{\mathbf{v}}^\epsilon(s)\|^2 + \min\left\{1, \frac{1}{2C(\ell)}\right\} \|\sqrt{d}\, \nabla \overline{\mathbf{v}}^\epsilon(s)\|^2 + 2\nu \int_0^s \|\nabla \overline{\mathbf{v}}^\epsilon\|^2 \, \mathrm{d}t + 2\epsilon \int_0^s \|\mathbf{D}\overline{\mathbf{v}}^\epsilon\|_3^3 \, \mathrm{d}t \\
+ 2 \int_0^s \int_\Omega d \, |\overline{\boldsymbol{\omega}}^\epsilon|^3 \, \mathrm{d}\mathbf{x}\mathrm{d}t \leq \|\overline{\mathbf{v}}_0\|^2 + \|\sqrt{d}\, \overline{\boldsymbol{\omega}}_0\|^2,
\end{aligned}$$

which shows that

$$\overline{\mathbf{v}}^\epsilon \in L^\infty(0,T;L^2_\sigma(\Omega)^3) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega)) \qquad \text{and} \qquad d^{1/3}\overline{\boldsymbol{\omega}}^\epsilon \in L^3((0,T)\times\Omega),$$

with estimates independent of $\epsilon > 0$. We now extract further information from the other bound which is independent of ϵ, namely

$$\sqrt{d}\,\nabla\overline{\mathbf{v}}^\epsilon \in L^2(0,T;L^2(\Omega)^9),$$

coming from the other term on the left-hand side. We use now the inequality

$$\|\mathbf{v}\|_{H^{1/2}(\Omega)^3} \leq C\|\sqrt{d}\,\nabla\mathbf{v}\| \qquad \forall\,\mathbf{v}\in W_0^{1,2}(\Omega)^3,$$

which is a simplification of that proved in [ABLN20, Theorem 3.1] and $H^{1/2}(\Omega)$ denotes the famous critical fractional Sobolev space.

Here, we have the full gradient instead of the deformation tensor on the right-hand side and since we are working with the Galerkin approximations we need to verify it at least for functions in $W_0^{1,2}(\Omega)$, instead that for general distributions: This is why the estimate is less technical than that in [ABLN20]. By using the Sobolev embedding $H^{1/2}(\Omega)\subset L^3(\Omega)$, valid in three space dimensions, we finally have the following version of a classical Lions and Magenes result

$$\|\mathbf{v}\|_3 \leq C\|\mathbf{v}\|_{1/2,2} \leq C\|\sqrt{d}\,\nabla\mathbf{v}\| \qquad \forall\,\mathbf{v}\in W_0^{1,2}(\Omega)^3. \tag{4.4.47}$$

This is still not enough for our purposes, but we pass at the estimate obtained testing with $\overline{\mathbf{v}}_t^\epsilon$. We can also write the following estimate, which follows as in [ABLN20, Section 4]

$$\begin{aligned}\Big|\int_\Omega(\overline{\boldsymbol{\omega}}^\epsilon\times\overline{\mathbf{v}}^\epsilon)\cdot\overline{\mathbf{v}}_t^\epsilon\,d\mathbf{x}\Big| &\leq \|\overline{\mathbf{v}}_t^\epsilon\|_3\|\overline{\mathbf{v}}^\epsilon\|_6\|\overline{\boldsymbol{\omega}}^\epsilon\| \leq C\|\overline{\mathbf{v}}_t\|_3\|\nabla\overline{\mathbf{v}}^\epsilon\|^2\\ &\leq \frac{1}{2}\min\left\{1,\frac{1}{2C(\ell)}\right\}\|\sqrt{d}\,\nabla\overline{\mathbf{v}}_t^\epsilon\|^2 + C_1(\ell)\|\nabla\overline{\mathbf{v}}^\epsilon\|^4,\end{aligned} \tag{4.4.48}$$

valid for smooth functions for some $C_1(\ell)$. At the level of the Galerkin approximation we can use the above estimate and then the bound is inherited by the limit in $m\to+\infty$. Hence, by testing by $\overline{\mathbf{v}}_t^{\epsilon,m}$ the Galerkin system and by using Lemma 4.4.1, with the estimation on the convective term (4.4.48), we get (after passing to the limit $m\to+\infty$) the following differential inequality

$$\begin{aligned}\frac{1}{2}\|\overline{\mathbf{v}}_t^\epsilon\|^2 + \frac{1}{2}\min\left\{1,\frac{1}{2C(\ell)}\right\}\|\sqrt{d}\,\nabla\overline{\mathbf{v}}_t^\epsilon\|^2 + \frac{d}{dt}\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}^\epsilon\|^2 + \frac{d}{dt}\frac{\epsilon}{3}\|\mathbf{D}\overline{\mathbf{v}}^\epsilon\|_3^3&\\ + \frac{d}{3dt}\int_\Omega d\,|\overline{\boldsymbol{\omega}}^\epsilon|^3\,d\mathbf{x} \leq C_1(\ell)\|\nabla\overline{\mathbf{v}}^\epsilon\|^4&.\end{aligned}$$

In particular, for all $s\in(0,T)$ it holds

$$\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}^\epsilon(s)\|^2 \leq \frac{\nu}{2}\|\nabla\overline{\mathbf{v}}_0\|^2 + \frac{\epsilon}{3}\|\mathbf{D}\overline{\mathbf{v}}_0\|_3^3 + \frac{1}{3}\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,d\mathbf{x} + C_1(\ell)\int_0^s\|\nabla\overline{\mathbf{v}}^\epsilon\|^4\,dt,$$

which shows that, by using the Gronwall's lemma (see for example [ABLN20, Lemma 4.1])

$$
\begin{aligned}
\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}^\epsilon(s)\|^2 &\leq \left(\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}_0\|^2+\frac{\epsilon}{3}\|\mathbf{D}\overline{\mathbf{v}}_0\|_3^3+\frac{1}{3}\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x}\right)\exp\left\{C_1(\ell)\int_0^s\|\nabla\overline{\mathbf{v}}^\epsilon\|^2\,\mathrm{d}t\right\}\\
&\leq \left(\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}_0\|^2+\frac{1}{3}\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x}\right)\exp\left\{\frac{C_1(\ell)}{2\nu}\left(\|\overline{\mathbf{v}}_0\|^2+\|\sqrt{d}\,\overline{\boldsymbol{\omega}}_0\|^2\right)\right\}\\
&=: F(\ell,\overline{\mathbf{v}}_0),
\end{aligned}
$$

where we have used the uniform estimate for $\overline{\mathbf{v}}^\epsilon$ in $L^2(0,T;W_0^{1,2}(\Omega)^3$, previously proved. Therefore, from the above differential inequality we get, for all $s\in(0,T)$

$$
\begin{aligned}
&\frac{1}{2}\int_0^s\left(\|\overline{\mathbf{v}}_t^\epsilon\|^2+\min\left\{1,\frac{1}{2C(\ell)}\right\}\|\sqrt{d}\,\nabla\overline{\mathbf{v}}_t^\epsilon\|^2\right)\,\mathrm{d}t+\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}^\epsilon(s)\|^2+\frac{1}{3}\int_\Omega d\,|\overline{\boldsymbol{\omega}}^\epsilon(s)|^3\,\mathrm{d}\mathbf{x}\\
&+\frac{\epsilon}{3}\|\mathbf{D}\overline{\mathbf{v}}^\epsilon(s)\|_3^3\leq\frac{\nu}{2}\|\nabla\overline{\mathbf{v}}_0\|^2+\frac{1}{3}\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x}+C_1(\ell)F(\ell,\overline{\mathbf{v}}_0)\left(\|\overline{\mathbf{v}}_0\|^2+\|\sqrt{d}\,\overline{\boldsymbol{\omega}}_0\|^2\right),
\end{aligned}
$$

for all $\epsilon>0$. The latter implies in particular that

$$
\overline{\mathbf{v}}_t^\epsilon\in L^2(0,T;L^3(\Omega)^3\cap H^{1/2}(\Omega)^3)\qquad\text{and}\qquad\overline{\mathbf{v}}^\epsilon\in L^\infty(0,T;W_{0,\sigma}^{1,2}(\Omega)),
$$

with bounds uniform in $\epsilon>0$. (The validity of the estimates can be justified working again with the Galerkin approximation showing estimates not depending on m in a standard way.) We can now use this information to pass to the limit as $\epsilon\to0$.

In particular, by the a priori estimates, and since $d>0$ for all $\mathbf{x}\in\Omega$ observe that we can infer

$$
d_K\int_0^T\int_K|\overline{\boldsymbol{\omega}}^\epsilon|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t\leq\int_0^T\int_\Omega d\,|\overline{\boldsymbol{\omega}}^\epsilon|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t,
$$

with $0<d_K:=\min_{\mathbf{x}\in K}d(\mathbf{x})$. Next, being the right-hand side bounded independently of $\epsilon>0$ this shows that we have (up to a sub-sequence) L^3-weak convergence in $(0,T)\times K$. Considering a family of closed balls $\overline{B}_{q,r_q}\subset\Omega$ with rational center $q\in\mathbb{Q}^3$ and rational radius $r_q\in\mathbb{Q}^+$ which form a covering of Ω, and using a diagonal argument we can show that we can find a sub-sequence $\{\overline{\boldsymbol{\omega}}^\epsilon\}$ converging in L^3 in any compact set of $(0,T)\times\Omega$. Moreover, one has also the weak-$*$ convergence in $L^\infty(0,T;L^3(K))$.

By collecting all information coming from the above a priori estimates, we can infer that there exists

$$
\overline{\mathbf{v}}\in W^{1,2}(0,T;L^3_\sigma(\Omega)^3\cap H^{1/2}(\Omega)^3)\cap L^\infty(0,T;W_{0,\sigma}^{1,2}(\Omega)),
$$

with

$$
\overline{\boldsymbol{\omega}}\in L^\infty(0,T;L^3_{loc}(\Omega)^3),
$$

such that

$$\overline{\mathbf{v}}^\epsilon \overset{*}{\rightharpoonup} \overline{\mathbf{v}} \qquad \text{in } L^\infty(0,T;W^{1,2}_{0,\sigma}(\Omega)), \tag{4.4.49}$$

$$\sqrt{d}\,\overline{\boldsymbol{\omega}}^\epsilon \overset{*}{\rightharpoonup} \sqrt{d}\,\overline{\boldsymbol{\omega}} \qquad \text{in } L^\infty(0,T;L^2(\Omega)^3), \tag{4.4.50}$$

$$\overline{\mathbf{v}}^\epsilon_t \rightharpoonup \overline{\mathbf{v}}_t \qquad \text{in } L^2(0,T;H^{1/2}(\Omega)^3 \cap L^3_\sigma(\Omega)^3), \tag{4.4.51}$$

$$\sqrt{d}\,\overline{\boldsymbol{\omega}}^\epsilon_t \rightharpoonup \sqrt{d}\,\overline{\boldsymbol{\omega}}_t \qquad \text{in } L^2(0,T;L^2(\Omega)^3), \tag{4.4.52}$$

$$\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon \rightharpoonup \mathbf{0} \qquad \text{in } L^{3/2}(0,T;L^{3/2}(\Omega)^9), \tag{4.4.53}$$

$$d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \rightharpoonup \chi \qquad \text{in } L^{3/2}(0,T;L^{3/2}(\Omega)^3), \tag{4.4.54}$$

$$d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \overset{*}{\rightharpoonup} \chi \qquad \text{in } L^\infty(0,T;L^{3/2}(\Omega)^3), \tag{4.4.55}$$

$$\overline{\boldsymbol{\omega}}^\epsilon \rightharpoonup \overline{\boldsymbol{\omega}} \qquad \text{in } L^3(0,T;L^3(K)^3),\ \forall\, K \subset\subset \Omega, \tag{4.4.56}$$

$$\overline{\boldsymbol{\omega}}^\epsilon \overset{*}{\rightharpoonup} \overline{\boldsymbol{\omega}} \qquad \text{in } L^\infty(0,T;L^3(K)^3),\ \forall\, K \subset\subset \Omega, \tag{4.4.57}$$

and by Aubin-Lions lemma

$$\overline{\mathbf{v}}^\epsilon \to \overline{\mathbf{v}} \qquad \text{in } L^2(0,T;W^{3/4,2}_{0,\sigma}(\Omega)^3) \subset L^2(0,T;L^4(\Omega)^3), \tag{4.4.58}$$

All terms in the equation with the weak formulation (4.4.44) for $\overline{\mathbf{v}}^\epsilon$ pass to the limit, except the nonlinear one concerning the Baldwin & Lomax stress tensor. We obtain then

$$\int_0^T \int_\Omega \overline{\mathbf{v}}_t \cdot \boldsymbol{\phi} + d\,\overline{\boldsymbol{\omega}}_t \cdot \operatorname{curl} \boldsymbol{\phi} + (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}}) \cdot \boldsymbol{\phi} + \nu \nabla \overline{\mathbf{v}} : \nabla \boldsymbol{\phi} + \chi \cdot \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}\mathbf{x}\mathrm{d}t = 0, \tag{4.4.59}$$

for all smooth test functions $\boldsymbol{\phi}$ with compact support in $(0,T) \times \Omega$.

The last step is to show that the limit $\overline{\mathbf{v}}$ (and it curl $\overline{\boldsymbol{\omega}}$) satisfies the system (4.4.42) in a weak sense. To this end it would be classical to take the difference between the equation satisfied by $\overline{\mathbf{v}}^\epsilon$ and that satisfied by $\overline{\mathbf{v}}$, test by the difference and show that the limit vanishes. This is needed to show that

$$d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \to d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}},$$

at least a.e. in $(0,T) \times \Omega$. All the other terms work fine, the only problem is then to make sure that the integral below is well-defined

$$\int_0^T \int_\Omega (d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) \cdot (\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}}) \, \mathrm{d}\mathbf{x}\mathrm{d}t \to 0, \tag{4.4.60}$$

and to show that it vanishes. The a priori estimates we have on the solution are not enough for this results: the integral in (4.4.60) can be well-defined if taken over a compact subset of $K \subset \Omega$, being $\overline{\boldsymbol{\omega}} \in L^3_{loc}(\Omega)^3$ for a.e. $t \in [0,T]$, but not over the whole domain Ω.

In order to overcome this problem we have to localize. So let us fix an open ball $B := B(\mathbf{x}, R) \subset \Omega$ and take a cut-off function $0 \leq \eta \in C_0^\infty(\Omega))$ such that

$$\begin{cases} \eta(x) = 1 & \text{if } x \in \overline{B/2} := \overline{B(\mathbf{x}, R/2)}, \\ \eta(x) = 0 & \text{if } x \in \Omega \backslash B. \end{cases}$$

In this way, since for a.e. $t \in (0, T)$ it follows that $\overline{\mathbf{v}}(t) \in L^3(\Omega)^3$, and $\overline{\boldsymbol{\omega}}(t) \in L^3(B)^3$ we have that

$$\begin{aligned} \eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})_{|\partial B} &= 0, \\ \operatorname{div}(\eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})) &= \nabla\eta \cdot (\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) \in L^3(B), \\ \operatorname{curl}(\eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})) &= \nabla\eta \times (\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) + \eta\,(\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}}) \in L^3(B)^3, \end{aligned}$$

it follows then by (4.2.11) that $\eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) \in L^3(0, T; W_0^{1,3}(B)^3)$. Concerning the regularity, for all $\epsilon > 0$ the vector $\eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})$ will be suitable as test function, but it still not allowed since $\eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})$ is not divergence-free. So in order to be able to use it we need to subtract its divergence. This can be done by means of the Bogovskiĭ operator $\mathrm{Bog}_B(\,.\,)$ associated to the ball B. Note that we are using it for all fixed $t \in [0, T]$ and this does not create problems since the functions are smooth enough to consider the time as a parameter. Hence, a legitimate test function is the following one

$$\Phi^\epsilon := \begin{cases} \eta\,(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) - \mathrm{Bog}_B(\nabla\eta \cdot (\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})) & \text{in } B, \\ \mathbf{0} & \text{in } \Omega \backslash B. \end{cases}$$

From the continuity of the Bogovskiĭ operator as in Proposition 4.2.1 we can infer that $\operatorname{supp} \Phi^\epsilon \subset B$ for all $t \in [0, T]$ and

$$\Phi^\epsilon \in L^\infty(0, T; W_{0,\sigma}^{1,2}(\Omega)) \cap L^3(0, T; W_{0,\sigma}^{1,3}(\Omega)).$$

Moreover, from the convergence of the approximated sequence we also have, by interpolation, that $\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}} \to 0$ in $L^3(0, T; L^3(\Omega)^3)$, hence

$$\Phi^\epsilon \to \mathbf{0} \quad \text{in } L^3(0, T; L^3(\Omega)^3), \tag{4.4.61}$$

$$\Phi^\epsilon \rightharpoonup \mathbf{0} \quad \text{in } L^3(0, T; W_0^{1,3}(B)^3), \tag{4.4.62}$$

$$\mathrm{Bog}_B(\nabla\eta \cdot (\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})) \to \mathbf{0} \quad \text{in } L^3(0, T; W_0^{1,3}(B)^3). \tag{4.4.63}$$

We then obtain from the weak formulation of the regularized problem (4.4.46) the following equality

$$
\begin{aligned}
I &:= \int_0^T \int_\Omega \eta \big(d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}\big) \cdot \big(\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}}\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= -\int_0^T \int_\Omega \big(d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}\big) \cdot \nabla\eta \times \big(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&- \int_0^T \int_\Omega \big(d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}\big) \cdot \operatorname{curl}\Big[\mathrm{Bog}_B(\nabla\eta\cdot(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}))\Big]\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&- \nu \int_0^T \int_\Omega \mathbf{D}(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) : \mathbf{D}\Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega \big(\overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} - \overline{\boldsymbol{\omega}}^\epsilon \times \overline{\mathbf{v}}^\epsilon\big) \cdot \Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&+ \int_0^T \int_\Omega \big(\chi - d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}\big) \cdot \operatorname{curl}\,\Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t - \epsilon \int_0^T \int_\Omega |\mathbf{D}\overline{\mathbf{v}}^\epsilon|\mathbf{D}\overline{\mathbf{v}}^\epsilon : \mathbf{D}\Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&- \int_0^T \int_\Omega (\overline{\mathbf{v}}_t^\epsilon - \overline{\mathbf{v}}_t) \cdot \Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^T \int_\Omega d\,(\overline{\boldsymbol{\omega}}_t^\epsilon - \overline{\boldsymbol{\omega}}_t) \cdot \operatorname{curl}\,\Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&=: (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII).
\end{aligned}
$$

The strong $L^3(0,T;L^3(\Omega)^3)$ convergence of $\overline{\mathbf{v}}^\epsilon$ and the continuity of the Bogovskiĭ operator, with (4.4.63) imply that (I) and (II) vanish as $\epsilon \to 0$ (we also used that the function d is uniformly bounded). We write then the following equality

$$
\begin{aligned}
(III) = &-\nu \int_0^T \int_\Omega \eta |\mathbf{D}(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t - \nu \int_0^T \int_\Omega \mathbf{D}(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) : \nabla\eta \otimes (\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}})\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&+ \nu \int_0^T \int_\Omega \mathbf{D}(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}) : \mathbf{D}\big[\mathrm{Bog}_B(\nabla\eta\cdot(\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}))\big]\,\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}
$$

where the first term is non-positive and the second and third one vanish on account of (4.4.58) and (4.4.63). The convergence of the term (IV) follows from uniform bounds in $L^2(0,T;W^{1,2}(\Omega)^3)$ and (4.4.61). The term $(V) \to 0$ due to (4.4.62) and the bound in $L^{3/2}((0,T)\times B)$ of χ and $|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}$. Next, $(VI) \to 0$, due the $L^3(0,T;W^{1,3}(B)^3)$ bound of $\overline{\mathbf{v}}^\epsilon - \overline{\mathbf{v}}$ and (4.4.53).

Concerning the terms involving the time derivative, which are the new ones with respect to the steady problem treated in [BB20], it follows that they both vanish as $\epsilon \to 0$. In fact, in (VII) the term $\overline{\mathbf{v}}_t^\epsilon - \overline{\mathbf{v}}_t$ is bounded in $L^2(0,T;L^2(\Omega)^3)$, by (4.4.59), while Φ^ϵ vanishes strongly in $L^2(0,T;L^2(\Omega)^3)$. Moreover, regarding $(VIII)$, we rewrite it as

$$
\int_0^T \int_\Omega (\sqrt{d}\,\overline{\boldsymbol{\omega}}_t^\epsilon - \sqrt{d}\,\overline{\boldsymbol{\omega}}_t) \cdot \sqrt{d}\,\Phi^\epsilon\,\mathrm{d}\mathbf{x}\mathrm{d}t,
$$

and observe that the quantity $\sqrt{d}\,\overline{\boldsymbol{\omega}}_t^\epsilon - \sqrt{d}\,\overline{\boldsymbol{\omega}}_t$ is bounded in $L^2(0,T;L^2(\Omega)^3)$ by (4.4.52), while $\sqrt{d}\,\Phi^\epsilon$ converges strongly to zero in $L^2(0,T;L^2(\Omega)^3)$ by (4.4.58).

In this way we proved that

$$\begin{aligned}
\min_{\mathbf{x}\in\overline{B/2}} d(\mathbf{x}) &\int_0^T \int_{B/2} (|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) \cdot (\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}})\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= \min_{\mathbf{x}\in\overline{B/2}} d(\mathbf{x}) \int_0^T \int_{B/2} \eta\,(|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) \cdot (\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}})\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&\leq \int_0^T \int_{B/2} d(\mathbf{x})\,\eta\,(|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) \cdot (\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}})\,\mathrm{d}\mathbf{x} \\
&\leq \int_0^T \int_{B} d(\mathbf{x})\,\eta\,(|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon - |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}) \cdot (\overline{\boldsymbol{\omega}}^\epsilon - \overline{\boldsymbol{\omega}})\,\mathrm{d}\mathbf{x}\mathrm{d}t \to 0,
\end{aligned}$$

which is enough to prove that $|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \to |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}$ a.e. in $(0,T)\times B/2$. The arbitrariness of the ball $B \subset \Omega$ implies that

$$|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \to |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}} \qquad \text{a.e. in } (0,T)\times\Omega.$$

This proves, by the identification of weak and almost everywhere limits, the validity of the limit $d\,|\overline{\boldsymbol{\omega}}^\epsilon|\overline{\boldsymbol{\omega}}^\epsilon \rightharpoonup d\,|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}$, at least in $L^{3/2}(0,T;L^{3/2}(\Omega)^3)$ ending the proof of the existence part, since $\overline{\mathbf{v}}$ satisfies

$$\begin{aligned}
&\int_0^T \int_\Omega [\overline{\mathbf{v}}_t \cdot \boldsymbol{\phi} + d\,\overline{\boldsymbol{\omega}}_t \cdot \operatorname{curl}\,\boldsymbol{\phi} + (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}}) \cdot \boldsymbol{\phi}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega [\nu\,\nabla\overline{\mathbf{v}} : \nabla\boldsymbol{\phi} + d\,|\overline{\boldsymbol{\omega}}|\,\overline{\boldsymbol{\omega}} \cdot \operatorname{curl}\,\boldsymbol{\phi}]\,\mathrm{d}\mathbf{x}\mathrm{d}t = 0,
\end{aligned}$$

for all $\boldsymbol{\phi} \in C^\infty_{0,\sigma}((0,T)\times\Omega)^3$.

Observe that the hypotheses on the initial datum $\overline{\mathbf{v}}_0 \in W^{1,3}_{0,\sigma}(\Omega)$ are enough to make the integrals well-defined. In the limit only the weighted estimate $\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x} < \infty$ is needed. So at the price of further technical questions related to approximation by smooth functions in weighted space as in Kufner [Kuf85], one can relax the hypotheses on the initial datum as follows:

$$\overline{\mathbf{v}}_0 \in W^{1,2}_{0,\sigma}(\Omega) \qquad \text{with} \qquad \int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x} < \infty,$$

such that there exists a sequence $\overline{\mathbf{v}}_0^\epsilon \in W^{1,3}_{0,\sigma}(\Omega)$ satisfying

$$\overline{\mathbf{v}}_0^\epsilon \to \overline{\mathbf{v}}_0 \ \text{ in } W^{1,2}_{0,\sigma}(\Omega) \qquad \text{and} \qquad \int_\Omega d\,|\overline{\boldsymbol{\omega}}_0^\epsilon|^3\,\mathrm{d}\mathbf{x} \leq 2\int_\Omega d\,|\overline{\boldsymbol{\omega}}_0|^3\,\mathrm{d}\mathbf{x}.$$

We continue now with the uniqueness part.

Step 2: Uniqueness part. Since we proved existence of rather regular weak solutions, we can now prove their uniqueness. As usual we suppose that there exists two solutions $\overline{\mathbf{v}}_1$, $\overline{\mathbf{v}}_2$ corresponding to the same initial datum. We take the difference and it follows that all estimates satisfied by the velocity are inherited by the difference and hence $\delta\,\overline{\mathbf{v}} := \overline{\mathbf{v}}_1 - \overline{\mathbf{v}}_2$ and $\delta\,\overline{\boldsymbol{\omega}} := \overline{\boldsymbol{\omega}}_1 - \overline{\boldsymbol{\omega}}_2$ satisfy in particular the following

$$\begin{aligned}
\delta\,\overline{\mathbf{v}} &\in L^\infty(0,T;W^{1,2}_{0,\sigma}(\Omega)),\\
\sqrt{d}\,(\delta\,\overline{\boldsymbol{\omega}}) &\in L^\infty(0,T;L^2(\Omega)^3),\\
\delta\,\overline{\mathbf{v}}_t &\in L^2(0,T;H^{1/2}(\Omega)^3\cap L^3_\sigma(\Omega)),\\
\sqrt{d}\,(\delta\,\overline{\boldsymbol{\omega}}_t) &\in L^2(0,T;L^2(\Omega)^3),\\
\delta\,\overline{\boldsymbol{\omega}} &\in L^\infty(0,T;L^3(K)^3), \quad \forall\, K \subset\subset \Omega.
\end{aligned}$$

It follows that if we write the equation satisfied by the difference $\delta\,\overline{\mathbf{v}}$, we can rigorously test by the difference itself. All terms work directly, the only one that needs to be checked is the monotone one. In fact, if we write

$$\int_0^T\int_\Omega \big(d\,|\overline{\boldsymbol{\omega}}_1|\overline{\boldsymbol{\omega}}_1 - d\,|\overline{\boldsymbol{\omega}}_2|\overline{\boldsymbol{\omega}}_2\big)\cdot\big(\overline{\boldsymbol{\omega}}_1-\overline{\boldsymbol{\omega}}_2\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{4.4.64}$$

this would be surely finite if $\overline{\boldsymbol{\omega}}_i \in L^3(0,T;L^3(\Omega)^3)$, which we do not know. Nevertheless we can observe that, for all $i,j = 1,2$

$$\left|\int_0^T\int_\Omega d\,|\overline{\boldsymbol{\omega}}_i|\overline{\boldsymbol{\omega}}_i\cdot\overline{\boldsymbol{\omega}}_j\,\mathrm{d}\mathbf{x}\mathrm{d}t\right| \le \left(\int_0^T\int_\Omega d\,|\overline{\boldsymbol{\omega}}_i|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{2/3}\left(\int_0^T\int_\Omega d\,|\overline{\boldsymbol{\omega}}_j|^3\,\mathrm{d}\mathbf{x}\mathrm{d}t\right)^{1/3} < \infty,$$

hence the integral in (4.4.64) is well defined, and then by monotonicity it follows that

$$\int_0^T\int_\Omega \big(d\,|\overline{\boldsymbol{\omega}}_1|\overline{\boldsymbol{\omega}}_1 - d|\overline{\boldsymbol{\omega}}_2|\overline{\boldsymbol{\omega}}_2\big)\cdot\big(\overline{\boldsymbol{\omega}}_1-\overline{\boldsymbol{\omega}}_2\big)\,\mathrm{d}\mathbf{x}\mathrm{d}t \ge 0.$$

This proves that

$$\frac{1}{2}\|\delta\,\overline{\mathbf{v}}(s)\|^2 + \frac{1}{2}\|\sqrt{d}\,(\delta\,\overline{\boldsymbol{\omega}}(s))\|^2 + \frac{\nu}{2}\int_0^s \|\nabla(\delta\,\overline{\mathbf{v}})\|^2\,\mathrm{d}t \le \frac{C}{\nu}\int_0^s \|\nabla\overline{\mathbf{v}}_2\|^4\|\delta\,\overline{\mathbf{v}}\|^2\,\mathrm{d}t,$$

by using the standard inequalities for the nonlinear term (as in [ABLN20, Section 4]), since $\delta\,\overline{\mathbf{v}}(0) \equiv \mathbf{0}$. The bound $\nabla\overline{\mathbf{v}}_2 \in L^\infty(0,T;L^2(\Omega)^9)$ and the Gronwall lemma implies that $\|\delta\,\overline{\mathbf{v}}(s)\| \equiv 0$ for all $s \in [0,T]$, hence the uniqueness follows. □

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Titre : "Quelques Résultats sur les Modèles de Turbulence."

Mot clés : Équations de Navier-Stokes, Modèles de turbulents, Taux de convergence.

Résumé : L'objectif de la thèse est double : d'une part la thèse propose de nouveaux modèles turbulents et leur analyse également. Plus précisément, sur la base d'une modélisation de turbulence de base, de nouvelles formes d'hypothèse de Boussinesq - qui prennent en compte la rétrodiffusion d'énergie - sont obtenues. Ensuite, des outils d'analyse fonctionnelle sont appliqués pour prouver l'existence et l'unicité de solutions faibles aux modèles proposés. D'autre part, le manuscrit donne le taux de convergence des modèles de α-régularisation aux équations de Navier-Stokes. Plus précisément, l'erreur de modélisation est étudiée dans le cas d'un réglage périodique bidimensionnel de l'espace.

Title: "Some Results on Turbulent Models."

Keywords: Navier-Stokes equations, Turbulent models, Rate of convergence.

Abstract: The aim of the dissertation is twofold: On one hand the thesis provides new turbulent models and their analysis as well. More precisely, based on basic turbulence modeling new forms of Boussinesq assumption –which take into account of backscatter of energy– are obtained. Then functional analysis tools are applied to prove the existence and uniqueness of weak solutions to the proposed models. On the other hand the manuscript gives the rate of convergence of α-regularization models to the Navier-Stokes equations. More exactly, the modeling error is investigated in the case of two-dimensional space-periodic setting.