UNIVERSITÉ DE LORRAINE

Ecole Doctorale IAEM

Thèse

Présentée et soutenue publiquement pour l'obtention du titre de

DOCTEUR DE L'UNIVERSITE DE LORRAINE

Mention : « Mathématiques »

par Ruotao Yang

TWISTED WHITTAKER CATEGORY ON AFFINE FLAGS AND CATEGORY OF REPRESENTATIONS OF MIXED QUANTUM GROUP

20 Juin 2020

Membres du jury :

Rapporteurs :

M.ROMAN Bezrukavnikov	Professeur, MIT	Boston
M.SAM Raskin	Professeur assistant, UT Austin	Austin

Examinateurs :

M.SERGEY Lysenko	Professeur, Université de Lorraine directeur de thèse	Nancy
M.DENNIS Gaitsgory	Professeur, Harvard University co-directeur de thèse	Boston
M.ALAIN Genestier	Professeur, Université de Lorraine	Nancy
M.VINCENT Lafforgue	Professeur, Université Grenoble Alpes	Grenoble
M.GERARD Laumon	Professeur, Université Paris Saclay	Orsay
M.OLIVIER Schiffmann	Professeur, Université Paris Saclay	Orsay
Mme.PENG Shan	Professeur, Tsinghua University	Beijing

Institut Élie Cartan de Lorraine (IECL)

TWISTED WHITTAKER CATEGORY ON AFFINE FLAGS AND CATEGORY OF REPRESENTATIONS OF MIXED QUANTUM GROUP

Ruotao Yang

Abstract

We prove the twisted Whittaker category on the affine flag variety and the category of representations of the mixed quantum group are equivalent.

Contents

1 Introduction

1.1 Notation

For simplicity, we assume G to be a semi-simple connected group which is also simply connected, B is its Borel subgroup, N its unipotent radical, $T \subset B$ is G's maximal torus, and $\check{G}$ is Langlands dual of G. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{t}$ and $\check{\mathfrak{g}}$ be their Lie algebras. We denote by Λ the coweight lattice of G, and denote by $\check{\Lambda}$ the weight lattice. By choosing a Borel, we define the semi-group of negative coweights and denote it by Λ^{neg}. Its inverse is denoted by Λ^{pos}. We denote by Λ^+ (resp. $\check{\Lambda}^+$) the semi-group of dominant coweights (resp. dominant weights). We denote by Δ the root system of $\check{G}$, and we denote $\alpha_1, \alpha_2, ..., \alpha_r$ the simple coroots. Let W denote the finite Weyl group and W^{ext} denote the extended affine Weyl group. Let w^0 be the longest element in W.

We will work with D-modules[1], so we need to assume that we are working over an algebraically closed field $\Bbbk$ of characteristic 0. Given a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on the corresponding de Rham prestack of $\mathcal{Y}$, we denote by $D_{\mathcal{G}}(\mathcal{Y})$ the category of $\mathcal{G}$-twisted D-modules. (for the precise definition, please see Appendix Section B.3)

Define $K = \Bbbk((t))$ and $O = \Bbbk[[t]]$. We will denote by $G(K)$ (resp. $N(K)$) the loop group of G (resp. N), and I the Iwahori subgroup of $G(O)$. Let χ denote a non-degenerate character of $N(K)$, normalized to have conductor 0 (see (3.1) for precise definition of χ). We denote by $Gr_G = G(K)/G(O)$ the affine Grassmannian and by $Fl_G = G(K)/I$ the affine flags.

Let X be a global curve (projective, connected and smooth) over $\Bbbk$. Given a scheme Y, and a closed subscheme Y', we denote by $D_{Y'}$ the formal completion of Y' in Y and denote by $D^*_{Y'}$ the complement of Y' in $D_{Y'}$.

1.2 Introduction to FLE

1.2.1 The statement of quantum local Langlands

Denote by $\kappa_{\mathfrak{g},c}$ the critical level of $\mathfrak{g}$, i.e, $-\frac{1}{2}$ times the Killing form. Given a Weyl group invariant symmetric bilinear form

$$\kappa : \Lambda \times \Lambda \longrightarrow \Bbbk \tag{1.1}$$

Because $\mathfrak{t} = \Lambda \underset{\mathbb{Z}}{\otimes} \Bbbk$, $\kappa - \kappa_{\mathfrak{g},c}$ gives rise to a symmetric bilinear form

$$\mathfrak{t} \times \mathfrak{t} \longrightarrow \Bbbk$$

If the above bilinear form is non-degenerated[2], we have an isomorphism $\mathfrak{t} \simeq \check{\mathfrak{t}}$, hence, we get a bilinear form

$$\check{\mathfrak{t}} \times \check{\mathfrak{t}} \longrightarrow \Bbbk$$

Regard $\check{\Lambda}$ as a lattice in $\check{\mathfrak{t}}$, we get a bilinear form. After adding $\check{\kappa}_{\check{\mathfrak{g}},c}$, we denote the resulted bilinear form by $\check{\kappa}$

$$\check{\kappa} : \check{\Lambda} \times \check{\Lambda} \longrightarrow \Bbbk$$

It is called the dual level of κ, denoted by $\check{\kappa}$.

If we denote by $G(K) - mod_\kappa$ the (2-)category of categories with an action of $G(K)$ at level κ (i.e, categories admitting an action of the monoidal category

[1] We could also work with ℓ-adic sheaves, the method is the same, but the gerbes used in this paper should be replaced by the gerbes in [GL2].

[1] A bilinear form κ is non-degenerate if for any $\lambda \in \Lambda$, $\kappa(\lambda, -) : \Lambda \longrightarrow \Bbbk$ is 0 implies $\lambda = 0$.

$D_\kappa(G(K))$ [3]) . Then, the quantum local Langlands conjecture in [GL1] and [Ga11] is

Conjecture 1.1.

$$G(K) - mod_\kappa \simeq \check{G}(K) - mod_{-\check{\kappa}+2\check{\kappa}_{\check{\mathfrak{g}},c}} \tag{1.2}$$

Notice that the groups of both sides are very large, this conjecture looks too difficult. The following conjecture about the *Whittaker* model and the *Kac-Moody* model provides us a way to approach Conjecture 1.1.

It is expected that the equivalence (1.2) satisfies the following property

Conjecture 1.2. *If one category $\mathcal{C}$ corresponds to a category $\check{\mathcal{C}}$ under the equivalence (1.2), then,*

1). the Whittaker model of $\mathcal{C}$ is equivalent to the Kac-Moody model of $\check{\mathcal{C}}$, i.e,

$$Whit(\mathcal{C}) \simeq KM(\check{\mathcal{C}}) \tag{1.3}$$

Here, $Whit(C) = \mathcal{C}^{N(K),\chi}$ and $KM(\check{\mathcal{C}}) = \check{\mathcal{C}}^{\check{G}(K),weak}$ is the $\check{G}(K)$-weak invariant[4] of $\check{\mathcal{C}}$.

2).

$$\mathcal{C}^{G(O)} \simeq \check{\mathcal{C}}^{\check{G}(O)} \tag{1.4}$$

3).

$$\mathcal{C}^{I} \simeq \check{\mathcal{C}}^{\check{I}} \tag{1.5}$$

1.2.2 Original FLE

We note that the assignment that sends a $G(K)$-module category at level κ to its (strong) $G(O)$-invariants is co-represented by $D_\kappa(Gr_G)$, and similarly for the Langlands dual counterpart. As a result, under the equivalence (1.2), $D_\kappa(Gr_G)$ corresponds to $D_{-\check{\kappa}+2\check{\kappa}_{\check{\mathfrak{g}},c}}(Gr_{\check{G}})$. Let us see what will happen if we apply (1.3) to the above fact.

If we denote by $\hat{\check{\mathfrak{g}}}_{\check{\kappa}}$ the central extension of $\check{\mathfrak{g}}((t))$ corresponding to $\check{\kappa}$, then we notice that the category of Kac-Moody modules is equivalent to the weak $\check{G}(K)$-invariant of the category of twisted D-modules on $\check{G}(K)$ with respect to the right action of $\check{G}(K)$ on $\check{G}(K)$, i.e,

$$\hat{\check{\mathfrak{g}}}_{\check{\kappa}} - mod \simeq D_{-\check{\kappa}+2\check{\kappa}_{\check{\mathfrak{g}},c}}(\check{G}(K))^{\check{G}(K),weak}$$

Take $\check{G}(O)$-invariants with respect to the left action of $\check{G}(O)$, we have

$$KL^{\check{\kappa}}(\check{G}) = \hat{\check{\mathfrak{g}}}_{\check{\kappa}} - mod^{\check{G}(O)} \simeq D_{-\check{\kappa}+2\check{\kappa}_{\check{\mathfrak{g}},c}}(Gr_{\check{G}})^{\check{G}(K),weak} = KM(D_{-\check{\kappa}+2\check{\kappa}_{\check{\mathfrak{g}},c}}(Gr_{\check{G}})) \tag{1.6}$$

[3]the quantum parameter κ determines a $\mathbb{G}_m$-gerbe on $G(K)$, we denote by $D_\kappa(G(K))$ the corresponding κ-twisted D-module category on $G(K)$. Please check [Yu] and [Zh3] for the precise definition of this category.

[4]We will recall the definition of weak invariant and strong invariant in Appendix A, readers could also check [Yu].

We define,

$$Whit_\kappa(Gr_G) = Whit(D_\kappa(Gr_G)) = D_\kappa(Gr_G)^{N(K),\chi}$$

Hence, apply (1.3), we could get a conjectural equivalence

Conjecture 1.3.

$$Whit_\kappa(Gr_G) \simeq KL^{\check{\kappa}}(\check{G}) \tag{1.7}$$

Conjecture 1.3 is called the fundamental local equivalence (FLE).

We could get Conjecture 1.3 another way. We note that the assignment:

$$\begin{aligned} G(K) - mod_\kappa &\longrightarrow DGCat_{cont} \\ \mathcal{C} &\rightsquigarrow KM(\mathcal{C}) \end{aligned} \tag{1.8}$$

is co-represented by $D_\kappa(\check{G}(K))^{\check{G}(K),weak}$, i.e,

$$KM(\mathcal{C}) \simeq Funct_{G(K)-mod_\kappa}(D_\kappa(\check{G}(K))^{\check{G}(K),weak}, \mathcal{C})$$

Here, $Funct_{G(K)-mod_\kappa}(D_\kappa(\check{G}(K))^{\check{G}(K),weak}, \mathcal{C})$ denotes the (∞)-category of functors from $\hat{\mathfrak{g}}_\kappa - mod$ to $\mathcal{C}$.

And the assignment that sends a $G(K)$-module category to its Whittaker model is co-represented by $Whit_\kappa(G(K))$. Hence,

$$Whit_\kappa(G(K)) \rightsquigarrow \hat{\check{\mathfrak{g}}}_{\check{\kappa}} - mod \tag{1.9}$$

under the equivalence (1.2). By applying (1.4), we could also get Conjecture 1.3.

By the main theorem of [KL],

$$Rep_q(\check{G}) \simeq KL^{\check{\kappa}}(\check{G}) \tag{1.10}$$

here, $Rep_q(\check{G})$ is the category of representations of the Lusztig quantum group $U_q(\check{G})$, and the quantum parameter

$$q : \Lambda \longrightarrow \Bbbk/\mathbb{Z}$$

is given by $q(\lambda) = \frac{1}{2} \cdot \kappa(\lambda, \lambda)(mod\mathbb{Z})$ for $\lambda \in \Lambda$.

Hence, Conjecture 1.3 is equivalent to the following equivalence

$$Whit_\kappa(Gr_G) \simeq Rep_q(\check{G}) \tag{1.11}$$

1.2.3 Iwahori FLE

By using the property (2) of Conjecture 1.2, we got the statement of FLE. Let us see what will happen if we apply the property (3) of Conjecture 1.2.

Apply the property (3) of Conjecture 1.2 to (1.9). Then, we obtain the following conjectural equivalence

Conjecture 1.4.

$$Whit_\kappa(Fl_G) \simeq \hat{\check{\mathfrak{g}}}_{\check{\kappa}} - \mathrm{mod}^{\check{I}} \tag{1.12}$$

This conjecture is called the *Iwahori FLE*.

The idea of proving Conjecture 1.4 is to find an analog of $Rep_q(\check{G})$ and then prove both sides of Conjecture 1.4 are equivalent to the analogous category of $Rep_q(\check{G})$.

The right replacement for $Rep_q(\check{G})$ is $Rep_q^{mix}(\check{G})$, the category of representations of the mixed quantum group. The category $Rep_q^{mix}(\check{G})$ is defined in [Ga1]. It could be regarded as the category of representations of a quantum group whose positive part is given by the Lusztig quantum group with locally nilpotent condition and the negative part is given by the Kac-De Concini quantum group. We will recall its definition in Section 3.2.

Theorem 1.1.
$$Whit_\kappa(Fl) \simeq Rep_q^{mix}(\check{G})$$

In this paper, we prove Theorem 1.1 with the method of [GL1] under some mild assumptions[5] on q.

1.2.4 Relationship

We will explain the relationship between the FLE and the Iwahori FLE. Roughly speaking, Theorem 1.1 could be regarded as an approximation of Conjecture 1.3.

In this section, let us denote by H the metaplectic Langlands dual group (for definition, please check 6.3.2 [GL2]) of G over $\Bbbk$ and we denote its distinguished defined Borel subgroup by B_H, its unipotent radical by N_H. Let us denote by $\mathbb{B}H$ the classifying stack of H, similarly for other groups.

Because of the geometric Satake equivalence ([Zhu]), the category of representations of H acts on both sides of Conjecture 1.3. And the conjectural equivalence is supposed to be compatible with the actions of the monoidal category $Rep(H) \simeq QCoh(\mathbb{B}H)$. In particular, we could regard the equivalence (1.7) as an equivalence over the classifying stack $\mathbb{B}H$.

Then, let us consider the following diagram:

$$\begin{tikzcd} \mathbb{B}T_H \arrow[r] & \mathbb{B}B_H \arrow[d] \\ & \mathbb{B}H \end{tikzcd} \tag{1.13}$$

We could approximate the equivalence (1.7) by its pullback to stack $\mathbb{B}B_H$, i.e, we hope to prove the following equivalence:

$$Whit_\kappa(Gr_G) \underset{QCoh(\mathbb{B}H)}{\otimes} QCoh(\mathbb{B}B_H) \simeq Rep_q(\check{G}) \underset{QCoh(\mathbb{B}H)}{\otimes} QCoh(\mathbb{B}B_H) \tag{1.14}$$

Consider its further pullback to $\mathbb{B}T_H$, we have:

$$Whit_\kappa(Gr_G) \underset{QCoh(\mathbb{B}H)}{\otimes} QCoh(\mathbb{B}T_H) \simeq Rep_q(\check{G}) \underset{QCoh(\mathbb{B}H)}{\otimes} QCoh(\mathbb{B}T_H) \tag{1.15}$$

[5] We require q avoids small torsion. For its definition, see Section 3.2.1.

(1.15) has already been proved in [GL1] Theorem 19.2.5.

When there is no twisting, it is proved in [AB] (combined with [Ras1]) that

$$Whit(Fl) \simeq QCoh(\check{\mathfrak{n}}/\check{B}) \tag{1.16}$$

When there is twisting, $Whit_q(Fl)$ and $QCoh(\mathfrak{n}_H/B_H)$ are no longer equivalent. But it could be seen that the category $QCoh(\mathfrak{n}_H/B_H)$ still acts on both sides of the equivalence in Theorem 1.1. It is expected that Theorem 1.1 is an equivalence over $\mathfrak{n}_H/B_H$.

The inclusion $\{0\} \in \mathfrak{n}$ gives rise to a map:

$$\mathbb{B}B_H \longrightarrow \mathfrak{n}_H/B_H \tag{1.17}$$

The pullback of both sides of Theorem 1.1 along with the above map (i.e, tensor with $QCoh(\mathfrak{n}_H/B_H)$ over $QCoh(\mathbb{B}B_H)$) is supposed to coincide with (1.14). Hence, from Theorem 1.1, we could derive Conjecture 1.3 to $\mathbb{B}B_H$.

1.3 Motivations of this work

In this section, we will explain why we expect Theorem 1.1 to be true and recall some former works which motivated this paper.

1.3.1 Casselman-Shalika theorem

The first work goes back to the geometric version Casselman-Shalika theorem in [FGV]. The original Casselman-Shalika theorem interprets the values of the spherical Whittaker function as characters of the irreducible representations of the Langlands dual group. The authors of *loc.cit* proved a generalization of the geometric version Casselman-Shalika formula, and hence, they proved the category of representations of the Langlands dual group could be interpreted by the Whittaker D-modules (equivalently, ℓ-adic sheaves). The latter statement could be stated as follows,

$$Whit((\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}) \simeq Rep(\check{G}) \tag{1.18}$$

Here, the algebraic stack $(\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}$ classifies the generalized N-bundles on a global curve X which are allowed to have a pole at a fixed point x. It will be defined in Definition 7.2.

Remark In [Ga2], the author considered the twisted version of the equivalence (1.18) and proved it for irrational q.

The geometric Casselman-Shalika formula also gives us a hint about how to construct a functor to relate the category of Whittaker sheaves and the category of representations. For example, if we want to compare the category of Whittaker sheaves on affine Grassmannian with the category of representations of $\check{G}$. Assume $\mathcal{F} \in Whit(Gr_G)$, then, the de Rham cohomology of $IC_\lambda \otimes \mathcal{F}$ could give information of the corresponding representation of $\mathcal{F}$. Here, IC_λ denotes the IC-extension of the constant sheaf on $G(O)t^\lambda G(O)/G(O)$, the $G(O)$-orbit passing through $t^\lambda \in Gr_G$.

The functor F^L (see Section 6.3 for definition) constructed in this paper (as well as [Ga2], [GL2], etc) also follows from this idea.

1.3.2 Work of [AB]

The study of Whittaker sheaves on affine flags goes back to [AB]. In [AB], the authors defined a category $Whit^{baby}(Fl)$ which is by definition the full subcategory of the D-module category on Fl_G with $(I^{0,-}, \phi)$-equivariant condition. Here, $I^{0,-}$ is the pro-p unipotent radical of the negative Iwahori subgroup I^- and ϕ denotes the character D-module on $I^{0,-}$ defined by the !-pullback of the exponential D-module[6] $exp := D_{\mathbb{A}^1} \cdot e^z$, $\partial e^z = e^z$ on $\mathbb{G}_a$ through the following morphism:

$$I^{0,-} \longrightarrow N^-(K) \overset{n^- \to w^0 n^- w^0}{\longrightarrow} N(K) \overset{Ad_{-\rho}}{\longrightarrow} N(K) \overset{\chi}{\longrightarrow} \mathbb{G}_a \tag{1.19}$$

In [AB], the authors constructed a functor:

$$Whit^{baby}(Fl) \longrightarrow QCoh(\check{\mathfrak{n}}/\check{B}) \tag{1.20}$$

and showed that it is an equivalence. It should be regarded as the degenerated case of Theorem 1.1 and we want to deform the above equivalence. [AB] gives us two hints about Theorem 4.1.

The first hint is how to characterize the equivalence given in Theorem 1.1. A crucial observation in [AB] is that the equivalence (1.20) is not only an equivalence of plain categories, but also an equivalence of highest weight categories. It means that we could define a collection of standard objects and costandard objects of both sides, and the functor of (1.20) preserves standards and costandards.

Hence, we may expect our main theorem Theorem 1.1 is also an equivalence of highest weight categories.

In order to clarify the highest weight categories structures of our theorem 1.1, first of all, we need to figure out what are the standards and costandards of (1.20). In (1.20), the standards of the right hand side are given by the pullback of the $\check{G}$-equivariant line bundle $O(\lambda)$ on $\check{G}/\check{B}$ along with the following map:

$$\check{\mathfrak{n}}/\check{B} \simeq \widetilde{\mathcal{N}}/\check{G} \simeq \mathcal{T}^*(\check{G}/\check{B}) \longrightarrow \check{G}/\check{B} \tag{1.21}$$

Here, $\widetilde{\mathcal{N}}$ denotes the Springer resolution of the nilpotent cone and $\mathcal{T}^*(\check{G}/\check{B})$ denotes the cotangent bundle of $\check{G}/\check{B}$.

The standards of the left hand side of (1.20) are given by applying the average functor to the BMW D-modules[7]. Let us be more precise. We denote by $D(Fl)^I$ the category of Iwahori-equivariant D-modules on Fl_G. There is a unique monoidal functor

$$\Lambda \longrightarrow D(Fl)^I \tag{1.22}$$

which sends $\lambda \in \Lambda^+$ to the !-extension of the perverse constant D-module (i.e, the constant D-module with a cohomological shift such that it concentrates in

[6] If we are working with ℓ-adic sheaves, we need to consider Artin-Schreier sheaf instead.

[7] In some papers, BMW D-modules are also called Wakimoto D-modules.

degree 0) on $I \cdot t^\lambda I/I \subset Fl$. For any $\lambda \in \Lambda$, the D-module corresponding to λ is denoted by $\mathfrak{J}_\lambda$. We call such D-modules the *BMW D*-modules (in some papers, they are also called the Wakimoto *D*-modules).

We denote the irreducible baby Whittaker D-module supporting on $I^{0,-} \cdot 1I/I \subset Fl$ by Δ_1^{baby}. Then, the convolution from right with Δ_1^{baby} defines a functor:

$$\begin{aligned} Av_\Phi : D^I(Fl) &\longrightarrow Whit^{baby}(Fl) \\ Av_\Phi : \mathcal{F} &\longrightarrow \Delta_1^{baby} \star \mathcal{F} \end{aligned} \tag{1.23}$$

The standard objects in $Whit^{baby}(Fl)$ are $\{Av_\Phi(\mathfrak{J}_\lambda), \lambda \in \Lambda\}$. It is shown in [AB] that under the equivalence

$$Whit^{baby}(Fl) \simeq QCoh(\check{\mathfrak{n}}/\check{B})$$

constructed by the authors, $Av_\Phi(\mathfrak{J}_\lambda)$ goes to $O(\lambda)$.

In [Ras1], the author proved that the baby Whittaker category is actually equivalent to the (adolescent) Whittaker category $Whit(Fl)$, and under this equivalence (with some non-essential modifications), we could see $Av_!^{N(K),\chi}(\mathfrak{J}_\lambda)$ corresponds to $Av_\Phi(\mathfrak{J}_\lambda)$. Here, the !-average functor $Av_!^{N(K),\chi}$ is the left adjoint functor of the forgetful functor

$$Whit(Fl) \longrightarrow D(Fl)$$

Hence, we expect that the standards in the left hand side of Theorem 1.1 could be given by applying the !-average functor to BMW D-modules and they will correspond to 'some distinguished objects' in $Rep_q^{mix}(\check{G})$.

The second hint given by [AB] is about what should $Rep_q^{mix}(\check{G})$ look like. Let us, at first, decode the definition of the category $QCoh(\check{\mathfrak{n}}/\check{B})$.

A quasi coherent sheaf on $\check{\mathfrak{n}}/\check{B}$ is given by a $\mathcal{O}(\check{\mathfrak{n}})$-module with a compatible action of $\check{B}$. It could be regarded as a vector space with compatible actions of $Sym(\check{\mathfrak{n}}^-)$, $\check{N}$ and $\check{T}$. An action of $\check{T}$ could give this vector space a Λ-graded vector space structure, an action of $\check{N}$ could be regarded as an action of $U(\check{\mathfrak{n}})$ with the locally nilpotent condition, and $Sym(\check{\mathfrak{n}}^-)$ could be regarded as the dual of $U(\check{\mathfrak{n}})$. It is to say that a quasi coherent sheaf on $\check{\mathfrak{n}}/\check{B}$ is a Λ-graded vector space with an action of $U(\check{\mathfrak{n}})$ and a compatible action of the graded dual of $U(\check{\mathfrak{n}})$.

Through the above description of $QCoh(\check{\mathfrak{n}}/\check{B})$, the standards $O(\lambda)$ could be described as the Verma module V_λ^{mix} (i.e, the induced module which is induced from the 1-dimensional module $\Bbbk^\lambda \in Rep(\check{B})$ of $\check{B}$ corresponding to the character λ) of $Rep^{mix}(\check{G})$.

To consider the deformed version of $QCoh(\check{\mathfrak{n}}/\check{B})$, we first consider the deformation of $U(\check{\mathfrak{n}})$. There is a candidate for the deformation of $U(\check{\mathfrak{n}})$, namely, the Lusztig quantum group $U_q(\check{N})$. Hence, the category of the mixed quantum group should be the category of Λ-graded vector spaces with an action of the positive part of the Lusztig quantum group $U_q(\check{N})$ (with the locally nilpotent condition) and a compatible action of the dual of the positive part of the Lusztig quantum group.

And we expect that there is an equivalence between $Whit_q(Fl)$ and $Rep_q^{mix}(\check{G})$ and this equivalence sends $Av_!^{N(K),\chi}(\mathfrak{J}_\lambda)$ to V_λ^{mix}.

Remark We may also expect that there is an analog of [AB]'s result about describing $Whit_q(Fl)$ as a quotient category of $D_q(Fl)^I$. But I do not know how to do it now. It is conjectured by D.Gaitsgory in a letter that:

Conjecture 1.5.

$$D_{\mathcal{G}^G}(Fl)^I \simeq IndCoh(St_H)$$

H denotes the metaplectic dual group of G and St denotes the Steinberg variety.

1.3.3 Small quantum groups

The representation category of the small quantum group has already been studied in [GL1]. In *loc.cit*, the authors proved that the Hecke eignsheaves of the twisted Whittaker categories on affine Grassmannian is equivalent to the same category of representations of small quantum groups. The small quantum group is very similar to the mixed quantum group, because the positive part and negative part of small quantum groups are also dual to each other. Hence, the method used in [GL1] indicates the way to prove our theorem 1.1 and offers us models for the construction of the functors and stacks used in our paper. For example, the key idea of the proof of our main theorem is to use local-global equivalence of Whittaker categories and then prove the theorem in the global setting, it comes from [GL1].

1.4 Some higher category

In this section, we will review some notions from the higher category. For a more detailed explanation, please check [Lu1], [Lu2] and Chapter 1 of [GR1]. *Categorical setting* The theory of infinite category is needed for this paper to give the local definition of Whittaker D-modules on affine flags. We denote by $1-Cat$ the ∞-category of $(\infty, 1)$-categories and by *Spc* the $(\infty, 1)$-category of spaces (i.e, ∞-groupoids).

Given two $(\infty, 1)$-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $Funct(\mathcal{C}, \mathcal{D})$ the $(\infty, 1)$-category of functors from $\mathcal{C}$ to $\mathcal{D}$.

Stable categories An $(\infty, 1)$-category $\mathcal{C}$ is stable, if it satisfies the following property:

1. It has pullbacks and push-outs (in particular, $\mathcal{C}$ has pullback and pushout for empty index set, hence, $\mathcal{C}$ has a final and and an initial objects)

2. The morphism from the initial object to the final object is an isomorphism. We denote it by 0.

3. The diagram:

$$\begin{tikzcd} c_0 \arrow[r] \arrow[d] & c_1 \arrow[d] \\ c_2 \arrow[r] & c_3 \end{tikzcd}$$

is a pullback diagram is equivalent to it is a push-out diagram.

For any such category $\mathcal{C}$, we could define shifts onside: $c[1] = 0 \underset{c}{\times} 0$ and $c[-1] = 0 \underset{c}{\sqcup} 0$

By 3). of the definition of a stable category, [1] and [−1] are inverse operators.

Limits and colimits In the context of infinite categories, limit and colimit are well-defined. It is one of the advantages of infinite category over triangulated category.

Given a functor $F : \mathcal{D} \longrightarrow \mathcal{C}$, the restriction along with F induces a functor for any category $\mathcal{E}$:

$$Funct(\mathcal{C}, \mathcal{E}) \longrightarrow Funct(\mathcal{D}, \mathcal{E})$$

The partially defined left (right) adjoint functor of the above functor is called the left (right) Kan extension.

In particular, if $\mathcal{C}$ is the ordinary category with one object and the morphism is id, then, we get the definition of colimit and limit. That is to say, we could regard a colimit (resp. limit) diagram in $\mathcal{E}$ as a functor F from an index category $\mathcal{D}$ to $\mathcal{E}$. Then, the colimit (resp. limit) of F is the resulted object in $\mathcal{E} = Funct(\mathcal{C}, \mathcal{E})$ given by the left (resp. right) Kan extension.

An $(\infty, 1)$-category is cocomplete, if it admits filtered colimits.

A continuous functor is a functor preserving filtered colimits.

The following lemma gives us a way to relate limit to colimit.

Given a functor

$$I \longrightarrow 1 - Cat$$

$$i \longrightarrow \mathcal{C}_i$$

Assume that each transition functor $\mathcal{C}_i \longrightarrow \mathcal{C}_j$ admits a continuous right adjoint functor. By passing to right adjoint, we get a functor:

$$I^{op} \longrightarrow 1 - Cat$$

Then, by Corollary 5.5.3.4 of [Lu2]

Lemma 1.1.

$$\underset{I}{colim} \mathcal{C}_i \simeq \underset{I^{op}}{lim} \mathcal{C}_i \tag{1.24}$$

Straightening/unstraightening In this part, we will use the notions in Vol 1 [GR1].

Given a category $\mathcal{C}$, we denote by $Cart/_{\mathcal{C}}$ the category of fibrations over $\mathcal{C}$ (see Chapter 1, 1.3 [GR1] vol 1). And we denote by $(Cart/_{\mathcal{C}})_{str}$ the 1-full subcategory of $Cart/_{\mathcal{C}}$ requiring the functors preserving fibration arrows.

Lemma 1.2. *There is a canonical equivalence*

$$(Cart/_{\mathcal{C}})_{str} \xrightarrow{\sim} Funct(\mathcal{C}, 1-Cat)$$

And under the above equivalence, the full subcategory of $(Cart/_{\mathcal{C}})_{str}$ consisting of fibrations over $\mathcal{C}$ in spaces (i.e, ∞-groupoid) corresponds to $Funct(\mathcal{C}, Spc)$

DG-categories We denote by $Vect$ the category of chain complexes over $\Bbbk$. Then, a differential graded category is a category enhanced over $Vect$. In this paper, when we talk about a category, we will mean a differential graded category. Without specific remarks, the DG-categories that we will consider are stable and cocomplete. It means filtered colimits exist. We denote by $DGCat$ the (∞-) category of such categories. And we denote by $DGCat_{cont}$ the 1-full subcategory of $DGCat$ consisting of continuous functors among cocomplete categories.

Compactly generated category We will use the notation $RHom_{\mathcal{C}}(c_1, c_2) \in Vect$ denote the morphisms from c_1 to c_2 in $\mathcal{C}$. And we denote by Hom the degree 0 of $RHom$.

As same as the ordinary category, we could define the notion of compact objects in a given DG-category. By definition, an object $c \in \mathcal{C}$ is compact, if $RHom_{\mathcal{C}}(c, -)$ is continuous. I.e,

$$RHom_{\mathcal{C}}(c, colimc_i) \simeq colimRHom_{\mathcal{C}}(c, c_i)$$

Given any DG-category, we denote by $\mathcal{C}^c$ the full category of compact objects in $\mathcal{C}$.

A category $\mathcal{C}$ is compactly generated if $\mathcal{C}^c$ generates $\mathcal{C}$, i.e, if an object $c_0 \in \mathcal{C}$ such that for $c' \in \mathcal{C}^c$, $RHom_{\mathcal{C}}(c', c_0) = 0$ could imply $c_0 = 0.82$

By Proposition 5.4.5 in Section 1 of [GR1], a compactly generated category is the minimal cocomplete stable DG-category containing its subcategory of compact objects.

Monoid Given an ∞ category $\mathcal{C}$, we define the simplicial object in $\mathcal{C}$ to be the functor from Δ^{op}, the simplex category, to $\mathcal{C}$.

If $\mathcal{C}$ admits all finite (including empty index set) product, we could define the Segal condition for the simplicial object. We say that a simplicial object

$$c. : \Delta^{op} \longrightarrow \mathcal{C}$$

satisfies the Segal condition if $c.$ preserves all limits.

we denote by $Monoid(\mathcal{C})$ the category of monoids in $\mathcal{C}$, i.e, the category of simplicial objects in $\mathcal{C}$ satisfying Segal condition.

And we denote by $ComMonoid(\mathcal{C})$ the category of commutative monoids in $\mathcal{C}$ if we replace the simplex category Δ by $fSet_*$, the category of pointed finite sets.

Prestack We denote by $PreStk$ the category of prestacks, i.e, the category of functors

$$(Sch^{aff})^{op} \longrightarrow Spc$$

Here, we denote by Sch^{aff} the category of affine schemes over $\Bbbk$.

Remark Many prestacks appear in this paper are $0-truncated$[8], such as the Ran space Ran_X, affine Grassmannian Gr_G, affine flags Fl_G and the loop group $G(K)$.

1.5 Acknowledgment

The author thanks Dennis Gaitsgory for suggesting this subject and also deeply thanks him for his guidance. Without his help, the author could not finish this subject alone. The author thanks Sergey Lysenko for helpful discussions and comments for the paper. The author also thanks to Lin Chen and Yuchen Fu for their careful reading and detailed suggestions about organizing this paper. The author also thanks Sam Raskin, Peng Zheng, Qiao Zhou, Zicheng Qian, Roman Travkin, Yifei Zhao, Jonathan Wise, Qile Chen, Lizao Ye and Simon Riche, Christoph Baerligea for their great help. The author also thanks Qiao Meng for her help of checking English grammar.

2 Geometric Preparation

We hope to introduce the precise statement of our main theorem as soon as possible, but before that, we need to introduce some important notions. In this section, we will define these basic geometric objects used in this paper, they are the bricks of the constructions in this paper.

We will, first of all, recall the definition of *Ran space* and *Configuration space* (Section 2.1 and Section 2.2), and then give the definition of affine flags and affine Grassmannian and their Ran-ify version in Section 2.3 and Section 2.4. In Section 2.5, we will explain the gerbes used in this paper which could give us the right deformation of (1.16).

2.1 Ran space

In this section, we define the Ran space.

Given a global curve X over $\Bbbk$, as [BD1], we can consider its Ran space Ran_X. This geometric object gives us a chance to relate local objects and global objects.

Definition 2.1. *We define the Ran space Ran_X to be the prestack classifying non-empty finite sets of $Maps(S, X)$, here, S is a finite type affine scheme.*

Let us explain why Ran_X is so important. By [BL] (or Section 2.4), the affine flag variety and the affine Grassmannian could be defined by choosing a point in a global curve X, A.Beilinson and V.Drinfeld told us that we could consider their Ran-ify objects, it means that we could choose several points and we could let the chosen points move and coincide (for further details, please check Section 2.4). This idea is very important because we can encode global

[8] That is to say, these prestacks take value in $Set \subset Spc$.

information inside. Another importance of Ran space is that we can talk about the factorization structure once we introduced it.

We denote by $fSet^{surj}$ the ordinary category of finite sets with surjective morphisms between them. Given any finite non-empty set $I \in fSet^{surj}$, we denote by X^I the $|I|$-th power of X. We have a natural map from X^I to Ran_X by mapping $\{x_i, i \in I\} \in X^I$ to $\{x_i, i \in I\} \in Ran_X$. And any surjection $f : I \twoheadrightarrow J$ induces a diagonal map:

$$\Delta_{I/J} : X^J \longrightarrow X^I \tag{2.1}$$

$$\{x_1, ...,\} \longrightarrow \{x_{f(1)}, ...\}$$

The image of this morphism is a subset of X^I consisting of points

$$\{x_1, x_2, ...,\} \in X^I$$

such that $x_{i_1} = x_{i_2}$ if $f(i_1) = f(i_2)$.

Note that we have

$$Ran_X \simeq \underset{I \in fSet^{surj}}{colim} X^I \tag{2.2}$$

One of the most important features of Ran_X is that Ran_X admits a (non-unital) monoid structure by taking union. Namely, we have a map

$$\bigcup : Ran_X \times Ran_X \to Ran_X \tag{2.3}$$

$$(I_1, I_2) \to I_1 \cup I_2$$

Definition 2.2. *Denote by $(Ran_X^J)_{disj}$ the open subset of Ran_X^J consisting of ordered $|I|$-points in Ran_X with disjoint supports, i.e, $(I_1, I_2, I_3, ..., I_{|J|}) \in Ran_X^J$ belongs to $(Ran_X^J)_{disj}$ if and only if $I_{j_1} \cap I_{j_2} = \emptyset$ for any $j_1 \neq j_2$.*

Remark We note that the map $\bigcup$ is finite and it is étale when restricting to $(Ran_X^J)_{disj}$.

We could define the Ran version affine Grassmannian $Gr_{T,Ran}$ (4.6.2 [GL1]):

Definition 2.3. *For any affine scheme of finite type $S \in Sch^{aff,ft}$, we ask $Maps(S, Gr_{T,Ran}) = \{(I, \mathcal{P}_T, \alpha) | \mathcal{P}_T$ a T bundle on $D_{S\times x}$, $\alpha : \mathcal{P}_T|_{D^*_{S\times x}} \cong \mathcal{P}^\circ_T|_{D^*_{S\times x}}\}$, here $\mathcal{P}^\circ_T$ denotes the trivial T-bundle.*

Inside $Gr_{T,Ran}$, we could define a closed substack denoted by $Gr^{neg}_{T,Ran}$.

Definition 2.4. *An S-point $(J, \mathcal{P}_T, \alpha)$ of $Gr_{T,Ran}$ is in $Gr^{neg}_{T,Ran}$ if it satisfies the following two conditions*

- *Regularity: for every dominant weight $\check{\lambda} \in \check{\Lambda}^+$, the meromorphic map of line bundles on $S \times X$*

$$\check{\lambda}(\mathcal{P}_T) \longrightarrow \check{\lambda}(\mathcal{P}^0_T) \tag{2.4}$$

induced by α, is regular.

- *Non-redundancy: for every point $s \in S$ and every element $j \in J$ there exists at least one $\check{\lambda} \in \check{\Lambda}^+$, for which the above map of line bundles has a zero at the point of X corresponding j to $s \longrightarrow S \longrightarrow X$.*

In addition to Ran_X, we may fix a point $x \in X$ and consider the corresponding fixed point version Ran space $Ran_{X,x}$. Sometimes, we will write it as Ran_x for simplicity if the curve X is clear.

Definition 2.5. *We define the prestack $Ran_{X,x}$ such that for any affine scheme of finite type S, $Maps(S, Ran_{X,x}) = \{I \subset Maps(S, X) | I$ is a non-empty set with a distinguished element $\tilde{x}\}$, here, $\tilde{x}$ denotes the constant map $\tilde{x} : S \to x \to X$.*

There is a forgetful functor:

$$Ran_x \longrightarrow Ran$$

Definition 2.6. *We define an open subset of $Ran_X^J \times Ran_{X,x}$, denoted by $(Ran^J \times Ran_x)_{disj}$: a point $(I_1, I_2, ...I_{|J|}, I)$ belongs to this open subset if and only if the supports of $(I_1, I_2, ...I_{|J|}, I)$ are pairwise disjoint, i.e,*

$$(Ran^J \times Ran_x)_{disj} := (Ran^J \times Ran)_{disj} \underset{Ran^J \times Ran}{\times} (Ran^J \times Ran_x)$$

In particular, if a point $(I_1, I_2, ...I_{|J|}, I)$ belongs to this open subset, then, $\tilde{x} \bigcap (I_1 \cup I_2 \cup ... \cup I_{|J|}) = \emptyset$.

Taking union defines a map

$$\bigcup_x : Ran_X^J \times Ran_{X,x} \to Ran_{X,x} \tag{2.5}$$

This map is finite and étale if we restrict it to $(Ran_X^J \times Ran_{X,x})_{disj}$. So, if we want to consider the pullback along with $\bigcup_x$, it does not matter whether we consider ! or $*$ pullback.

$\bigcup_x$ gives $Ran_{X,x}$ a module space structure of Ran_X which means that $Ran_{X,x}$ is a module of the non-unital algebra Ran_X in *Spc*.

2.2 Configuration spaces

In this section, we need to recall the definitions of the Configuration space $Conf(X, \Lambda^{neg})$ and its fixed point version $Conf(X, \Lambda^{neg})_{\infty \cdot x}$ ([GL1]), they can offer us factorization gadgets which will be useful for us in Section 4.

When we consider the categories of sheaves and gerbes on the configuration space, it is equivalent to consider the corresponding categories on $Gr_{T,Ran}$ (Lemma 2.2). And the former categories are easier to handle with than $Gr_{T,Ran}$.

Definition 2.7. *The scheme $Conf(X, \Lambda^{neg})$ is defined to be the prestack classifying the colored divisor of X with coefficient in Λ^{neg},*

i.e, it classifies:

$$D = \sum_k \lambda_k \cdot x_k, \lambda_k \in \Lambda^{neg} - 0 \tag{2.6}$$

Connected components of $Conf(X, \Lambda^{neg})$ are indexed by $\Lambda^{neg} - 0$,

$$Conf(X, \Lambda^{neg}) = \bigsqcup_{\Lambda^{neg}-0} Conf(X, \Lambda^{neg})^{\lambda}$$

Here, $Conf(X, \Lambda^{neg})^{\lambda}$ denotes the subscheme of $Conf(X, \Lambda^{neg})$ where we require the total degree of D (i.e, $\sum \lambda_k$) is λ.

Note that we assume that G is simply connected, so any coweight could be written as a sum of simple coroots with coefficients in $\mathbb{Z}$. Given a $\lambda \in \Lambda^{neg} - 0$, if we assume $\lambda = \sum_i n_i(-\alpha_i)$, then, it is easy to see that the connected component $Conf(X, \Lambda^{neg})^{\lambda}$ is isomorphic to

$$\prod_i X^{(n_i)}$$

Here, $X^{(n)}$ is the n-th unordered power of X.

For a non-empty finite set J, we define an open subscheme $Conf(X, \Lambda^{neg})^J_{disj} \subset Conf(X, \Lambda^{neg})^J$ to be the subscheme classifying the point $\{D_1, D_2, D_3, ...D_{|J|}\} \in Conf(X, \Lambda^{neg})^J$ such that for any $j_1 \neq j_2$, D_{j_1} and D_{j_2} have disjoint supports.

Remark given $D = \sum_{k \in J} \lambda_k \cdot x_k$, its support means its projection in Ran, i.e, the finite subset $\{x_1, x_2, ...x_{|J|}\} \subset X$.

Similar to Ran space, we have an operation that gives $Conf(X, \Lambda^{neg})$ a structure of non-unital commutative semi-group.

$$add : Conf(X, \Lambda^{neg})^J \longrightarrow Conf(X, \Lambda^{neg}) \tag{2.7}$$

$$D_1, D_2, D_3, ...D_{|J|} \longrightarrow D_1 + D_2 + ... + D_{|J|}$$

If we restrict this morphism to $Conf(X, \Lambda^{neg})^J_{disj}$, then, it is étale.

When G is semi-simple and simply connected, elements in $Conf(X, \Lambda^{neg})$ is bijective with the non-zero monoid morphisms from $\check{\Lambda}^+$ to the monoid of effective divisors on X:

$$Div(X) := \bigsqcup_{n \geq 0} X^{(n)} \tag{2.8}$$

Given any point in $Gr^{neg}_{T,Ran}$, we note that taking zeros of (2.4) gives rise to a monoidal functor:

$$\check{\Lambda}^+ \longrightarrow Div(X) \tag{2.9}$$

Hence, we obtain a map of prestacks:

$$Gr^{neg}_{T,Ran} \longrightarrow Conf(X, \Lambda^{neg}) \tag{2.10}$$

According to Lemma 8.14 in [Ga7] and Lemma 4.6.4 in [GL1], we have:

Lemma 2.1. *The morphism (2.10) induces an isomorphism of the sheafifications in the topology generated by finite surjective maps.*

In particular, (2.10) induces an equivalence between categories of gerbes on $Gr^{neg}_{T,Ran}$ and $Conf(X, \lambda^{neg})$.

Fix one point $x \in X$, we could consider a fixed point version Configuration space which allows the coefficient of x can be an arbitrary element in Λ instead of just in Λ^{neg}.

Definition 2.8. *We denote by $Conf(X, \Lambda^{neg})_{\infty \cdot x}$ the ind-scheme classifying the colored divisor on X with Λ-coefficient:*

$$D = \lambda_x \cdot x + \sum_k \lambda_k \cdot x_k \tag{2.11}$$

such that $\lambda_k \in \Lambda^{neg}$, $\lambda_x \in \Lambda$ and $x_k \neq x$.

To simplify the notation, we also denote this ind-scheme by $Conf_x$.

Definition 2.9. *We denote by $Conf_{\leq \mu \cdot x}$ the closed subscheme of $Conf_x$ where we require $\lambda_x \leq \mu$ in (2.11).*

We could give $Conf_x$ an ind-scheme structure by:

$$Conf_x \simeq \underset{\mu \in \Lambda}{colim}\, Conf_{\leq \mu \cdot x}$$

The transition morphism here is given by closed embedding:

$$Conf_{\leq \mu_1 \cdot x} \longrightarrow Conf_{\leq \mu_2 \cdot x}, \text{ if } \mu_2 - \mu_1 \in \Lambda^{pos} \tag{2.12}$$

The connected components decomposition of $Conf_{\leq \mu \cdot x}$ is given by

$$Conf_{\leq \mu \cdot x} = \bigsqcup_{\lambda \in \mu + \Lambda^{neg}} Conf^{\lambda}_{\leq \mu \cdot x} \tag{2.13}$$

where λ is the total degree of D (i.e, $\lambda = \lambda_x + \sum \lambda_k$).

If we assume that $\lambda - \mu = \sum_i m_i \cdot (-\alpha_i)$, then,

$$Conf^{\lambda}_{\leq \mu \cdot x} \simeq \prod_i X^{(m_i)}$$

Note that $Conf$ could act on $Conf_x$, i.e, for any finite set J with a distinguished element $*$, we have

$$add_x : Conf^{J-*} \times Conf_x \longrightarrow Conf_x \tag{2.14}$$

$$D_1, D_2, ..., D_{|J|-1}, D' \longrightarrow D_1 + D_2 + ... + D_{|J|-1} + D'$$

Regard $Conf$ as a (non-unital) algebra in $PreStk$, the above map gives $Conf_x$ a module structure of $Conf$.

Note that in $Conf^J \times Conf_x$, we could take out an open subscheme with the disjoint support condition. To be more precise, we denote by $(Conf^J \times Conf_x)_{disj}$ the subscheme of $Conf^J \times Conf_x$ consisting of points: $\{D_1, D_2, ..., D_{|J|}, D'\}$ where we ask the supports of $\{D_1, D_2, ..., D_{|J|}, D'\}$ to be pairwise disjoint.

Remark If we restrict add to $(Conf^J \times Conf_x)_{disj}$, then it is étale and finite. Hence, $add^! \simeq add^*$.

Definition 2.10. *For any affine scheme S of finite type, a S-point $(J, \mathcal{P}_T, \alpha)$ of Gr_{T,Ran_x} belongs to $(Gr^{neg}_{T,Ran})_{\infty\cdot x}$ if there exists a T -bundle $\mathcal{P}'_T$ on $S \times X$ and an isomorphism $\alpha' : \mathcal{P}'_T|_{S\times X-x} \simeq \mathcal{P}_T|_{S\times X-x}$, such that the resulted point $(J, \mathcal{P}'_T, \alpha' \circ \alpha)$ of Gr_{T,Ran_x} belongs to:*

$$Ran_x \underset{Ran}{\times} Gr^{neg}_{T,Ran}$$

Be similar to the definition of the map (2.10), evaluation on fundamental weights defines a map of prestacks:

$$(Gr^{neg}_{T,Ran})_{\infty\cdot x} \longrightarrow Conf_x \tag{2.15}$$

Lemma 2.2. *(4.6.7 [GL1]) The morphism (2.15) induces an isomorphism of the sheafifications in the topology generated by finite surjective maps.*

2.3 Factorization prestacks

In this section, we will recall the definitions of factorization prestacks and factorization module prestacks. They are the key structures of many prestacks that we will deal with in this work. The content of this section is a review of Section 1 and 3.1 of [GL1], the readers who are familiar with the definition of factorization prestacks could skip this section safely.

Given a prestack $\mathcal{Y}$ over Ran. Naively, the terminology *factorization* means the fiber of $\mathcal{Y}$ over the point $\{x_1, x_2, ...x_k\} \in Ran$ can be canonically identified with the product of the fiber over each $x_i \in \{x_1, x_2, ...x_k\}$. From this point of view, we can recover the fiber of a factorization object over an arbitrary point in Ran once we know its fiber over X.

It is easy to see that the scheme X^I admits a stratification given by X^β. Here, β is a pattern of $|I|$. We denote by $\mathcal{Y}_\beta$ the fiber product of $\mathcal{Y}$ and X^β over Ran_X:

$$\mathcal{Y}_\beta := \mathcal{Y} \underset{Ran_X}{\times} X^\beta$$

Remark If we know how to glue $\mathcal{Y}_\beta$ together in addition to the fiber of $\mathcal{Y}$ over X, we can recover $\mathcal{Y}$. It is the content of 1.10 [Ho] and [Ras1].

Definition 2.11. *A prestack $\mathcal{Y}$ over Ran_X is factorizable over Ran_X, if there are a collection of isomorphisms for any finite set J:*

$$\mathcal{Y} \underset{Ran_X}{\times} (Ran^J_X)_{disj} \cong \mathcal{Y}^J \underset{Ran^J_X}{\times} (Ran^J_X)_{disj}$$

and satisfy some higher associative properties.

Furthermore, given a factorization prestack $\mathcal{Y}_0$ over Ran_X, we can define the notion of factorization module prestack with respect to $\mathcal{Y}_0$ as follows:

Definition 2.12. *$\mathcal{Y}_0$ is a factorization prestack over Ran, then, a prestack $\mathcal{Y}_1$ over Ran_x is factorizable with respect to $\mathcal{Y}_0$, if for any J, we have a canonical isomorphism:*

$$\mathcal{Y}_1 \underset{Ran_{X,x}}{\times} (Ran_X^J \times Ran_{X,x})_{disj} \simeq \mathcal{Y}_0^J \times \mathcal{Y}_1 \underset{Ran_X^J \times Ran_{X,x}}{\times} (Ran_X^J \times Ran_{X,x})_{disj}$$

with additional higher associative properties.

In order to spell out the *higher associative properties* in the above definitions, we would apply the method used in [GL1] section 3.1.2.

The factorization property of Ran_X gives us a right-lax symmetric monoidal functor (for definition, see [GR1] 3.2.2 Chapter 1, Volume 1):

$$Ran_{fSet} : fSet \longrightarrow PreStk$$

which sends a finite set J to Ran^J_{disj}.

Given any $I, J \in fSet$, the right-lax symmetric monoidal functor structure means that we have the following map:

$$Ran^{I \sqcup J}_{disj} \longrightarrow Ran^I_{disj} \times Ran^J_{disj} \tag{2.16}$$

which is given by inclusion.

In particular, for any non-empty finite set I, we have:

$$Ran^I_{disj} \longrightarrow \underbrace{Ran^{\ominus}_{disj} \times Ran^{\ominus}_{disj} ... \times Ran^{\ominus}_{disj}}_{|I| \ many \ items} = Ran^I \tag{2.17}$$

Here, $\ominus$ means a set with one element.

Remark We use the notation $\ominus$ instead of $*$ here, because we will leave the later notation for the marked point in Definition 2.13.

Then, a factorization prestack $\mathcal{Y}$ over Ran could be regarded as a right-lax symmetric monoidal functor:

$$\mathcal{Y}_{fSet} : fSet \longrightarrow PreStk$$
$$J \longrightarrow \mathcal{Y}_J$$

with a natural transformation to Ran_{fSet} with the following conditions:

1. $\mathcal{Y}_J \longrightarrow Ran^J_{disj} \underset{Ran}{\times} \mathcal{Y}_{\ominus}$ induced by $\mathcal{Y}_J \longrightarrow \mathcal{Y}_{\ominus}$ and $\mathcal{Y}_J \longrightarrow Ran^J_{disj}$ is an isomorphism.
2. $\mathcal{Y}_J \longrightarrow Ran^J_{disj} \underset{Ran^J}{\times} (\mathcal{Y}_{\ominus})^J$ induced by $\mathcal{Y}_J \longrightarrow (\mathcal{Y}_{\ominus})^J$ and $\mathcal{Y}_J \longrightarrow Ran^J_{disj}$ is an isomorphism.

Consider the image of $\ominus$ under the functor $\mathcal{Y}_{fSet}$, we denote the resulted prestack by $\mathcal{Y}_0$. By definition, $\mathcal{Y}_0$ satisfies the factorization isomorphism in the definition 2.11.

Definition 2.13. *We denote by $fSet^{surj}_*$ the ordinary category consisting of pointed finite sets with surjective and point preserving morphism between them.*

Taking the disjoint union gives $fSet^{surj}$ a (non-unital) algebra structure. Furthermore, taking disjoint union gives $fSet_*^{surj}$ a module structure with respect to the algebra $fSet^{surj}$.

We note that the factorization property of Ran_x with respect to Ran gives rise to a module functor with respect to Ran_{fSet} (i.e, compatible with respect to the right lax symmetric monoidal structure of Ran_{fSet}):

$$\begin{gathered} fSet_*^{disj} \longrightarrow PreStk \\ Ran_{fSet_*} : (* \in J) \longrightarrow Ran_{disj}^J \underset{Ran}{\times} Ran_x \end{gathered} \tag{2.18}$$

here, the map $Ran_{disj}^J \longrightarrow Ran$ is given by the projection sending to the component corresponding to $*$.

The right-lax module functor structure means that for any $I \in fSet^{surj}$ and $(* \in J) \in fSet_*^{surj}$ we have a morphism (compatible with the right-lax structure of Ran_{fSet}) :

$$Ran_{disj}^{I \sqcup J} \underset{Ran}{\times} Ran_x \longrightarrow Ran_{disj}^I \times Ran_{disj}^J \underset{Ran}{\times} Ran_x \tag{2.19}$$

The above morphism is given by inclusion.

We define a factorization module prestack of $\mathcal{Y}_{fSet}$ to be a functor $\mathcal{Y}'_{fSet_*}$ which is compatible with respect to the monoidal structure of $\mathcal{Y}_{fSet}$:

$$\begin{gathered} \mathcal{Y}'_{fSet_*} : fSet_*^{surj} \longrightarrow PreStk \\ (* \in J) \longrightarrow \mathcal{Y}'_J \end{gathered} \tag{2.20}$$

with a natural transformation to Ran_{fSet_*} satisfying the following conditions:

1. $\mathcal{Y}'_J \longrightarrow Ran_{disj}^J \underset{Ran}{\times} Ran_x \underset{Ran_x}{\times} \mathcal{Y}'_*$ induced by $\mathcal{Y}'_J \longrightarrow Ran_{disj}^J \underset{Ran}{\times} Ran_x$ and $\mathcal{Y}'_J \longrightarrow \mathcal{Y}'_*$ is an isomorphism.
2. $\mathcal{Y}'_J \longrightarrow Ran_{disj}^J \underset{Ran}{\times} Ran_x \underset{Ran^{J-*} \times Ran_x}{\times} (\mathcal{Y}_*^{J-*} \times \mathcal{Y}'_*)$

Consider the image of $(* \in *)$ under the functor $\mathcal{Y}'_{fSet_*}$, we denote the resulted prestack by $\mathcal{Y}_1$. By definition, $\mathcal{Y}_1$ satisfies the factorization isomorphism in the definition 2.12.

We will call $\mathcal{Y}_0$ a factorization prestack over Ran and $\mathcal{Y}_1$ a factorization module prestack with respect to $\mathcal{Y}_0$ for simplicity.

By definition, if we denote by $\mathcal{Y}_{0,\bar{x}}$ (resp. $\mathcal{Y}_{1,\bar{x}'}$) the fiber of $\mathcal{Y}_0$(resp. $\mathcal{Y}_1$) over the point $\bar{x} \in Ran_X$(resp, $\bar{x}' \in Ran_{X,x}$). If $\bar{x} \cap \bar{x}' = \emptyset$, then,

$$\mathcal{Y}_{1,\bar{x}' \cup \bar{x}} \simeq \mathcal{Y}_{1,\bar{x}'} \times \mathcal{Y}_{0,\bar{x}}$$

2.4 Fl, Gr and their Ran-ify stacks

Let us recall the definitions of the most important geometric objects in this paper: affine flags Fl_G and affine Grassmannian Gr_G.

We start from classical (non- Ran-ify version) Gr_G and Fl_G. By the lemma of Beauville-Laszlo, we could define them as follows:

Definition 2.14. *Given a point $x \in X$, the prestack $Gr_{G,x}$ is defined by: $Maps(S, Gr_{G,x}) = \{ (\mathcal{P}_G, \alpha) | \ \mathcal{P}_G \in Bun_G(S),\ \alpha : \mathcal{P}_G|_{(X-x)\times S} \cong \mathcal{P}_G^\circ|_{(X-x)\times S}\}$, for any affine finite type scheme S. Here, $\mathcal{P}_G^\circ$ denotes the trivial G-bundle on $X \times S$.*

Definition 2.15. *We define the affine flag variety, $Fl_{G,x}$, to be the prestack classifying the following data: for any affine finite type scheme S, $Maps(S, Fl_{G,x}) := \{(\mathcal{P}_G, \alpha, \epsilon) | \mathcal{P}_G \in Bun_G(S), \alpha : \mathcal{P}_G|_{(X-x)\times S} \cong \mathcal{P}_G^\circ|_{(X-x)\times S}, \epsilon :$ a $B-$reduction of $\mathcal{P}_G$ at $x \times S\}$*

According to the Lemma of Drinfeld-Simpson (ref. [DS]), we could identify the affine Grassmannian $Gr_{G,x}$ with the *fpqc* stack quotient $G(K_x)/G(O_x)$, and identify $Fl_{G,x}$ with $G(K_x)/I_x$. Here, the Iwahori subgroup $I_x \in G(O_x)$ denotes the preimage of B under the evaluation map (at x) $G(O_x) \longrightarrow G$.

It is known that:

Lemma 2.3. *$Gr_{G,x}$ and $Fl_{G,x}$ are formally smooth ind-schemes.*

Next, we review their Ran-ify prestacks, $Gr_{G,Ran}$ and Fl_{G,Ran_x}.

[BD1] gives us a way to consider the Ran version of Gr_G and Fl_G. Namely, we can consider the prestack Gr_{G,Ran_X} (resp, $Fl_{Ran_{X,x}}$) which is defined over Ran_X (resp, $Ran_{X,x}$) such that, its fiber over the point x is $Gr_{G,x}$ (resp, $Fl_{G,x}$) and the fiber over a point $y \in X - x$ is $Gr_{G,y}$.

Remark It is known that affine flags is not factorizable, i.e, we cannot define a factorization version affine flags over Ran_X such that given $I_1, I_2 \in Ran_X$, if $I_1 \cap I_2 = \emptyset$, then the fiber of this prestack over $I_1 \cup I_2$ is isomorphic to the product of its fibers over I_1 and I_2. Actually, it is even impossible to define such a prestack over X^2. For more details, see [Ras1] page 3 footnote 7. However, it is indeed possible to define a factorization module prestack over $Ran_{X,x}$ with respect to the Ran version affine Grassmannian such that its fiber over x is given by $Fl_{G,x}$. To be more precise, it means given $I_1 \in Ran_X, (x \in I_2) \in Ran_{X,x}$, if $I_1 \cap I_2 = \emptyset$, then the fiber of this prestack over $I_1 \cup I_2$ is isomorphic to the product of its fiber over I_2 and the fiber of the factorization affine Grassmannian over I_1.

Definition 2.16. *Ran affine Grassmannian is defined to be the prestack assigns every finite type affine scheme $S \in Sch^{aff,ft}$ to the set consisting of $(I, \mathcal{P}_G, \alpha)$, here, $I \in Ran_X(S)$ a finite subset, $\mathcal{P}_G \in Bun_G(S)$ and $\alpha : \mathcal{P}_G|_{X\times S-\Gamma_I} \cong \mathcal{P}_G^\circ|_{X\times S-\Gamma_I}$.*

Here, we use Γ_I to represent the image of the corresponding graph of $I \in Maps(S, X)$.

An important feature of $Gr_{G,Ran}$ is that it is factorizable over Ran. That is to say, for any $J \in fSet$ we have a canonical isomorphism:

$$Gr_{G,Ran} \underset{Ran}{\times} Ran^J_{disj} \simeq Gr_{G,Ran} \boxtimes Gr_{G,Ran} \boxtimes ... \boxtimes Gr_{G,Ran}|_{Ran^J_{disj}} \tag{2.21}$$

and (2.21) satisfies higher associative conditions.

We could also define the Ran version affine flags.

Definition 2.17. *We define Ran version affine flag $Fl_{G,x}$ to be the functor assigns to every finite type affine scheme S to the set of data $(I, \mathcal{P}_G, \alpha, \epsilon)$, here $(I, \mathcal{P}_G, \alpha) \in Gr_{G,Ran_X}(S)$, ϵ is a B-reduction of $\mathcal{P}_G$ at $x \times S$.*

Fl_{G,Ran_x} is a factorization module prestack of $Gr_{G,Ran}$, i.e, for any $(* \in J) \in fSet_*$ we have:

$$\begin{gathered} Fl_{G,Ran_x} \underset{Ran_x}{\times} (Ran^{J-*} \times Ran_x)_{disj} \\ \simeq \\ Gr_{G,Ran} \boxtimes ... \boxtimes Fl_{G,Ran_x}|_{(Ran^{J-*} \times Ran_x)_{disj}} \end{gathered} \tag{2.22}$$

with higher associative conditions.

2.5 Gerbes used in this paper

The deformation of the right hand side of (1.16) is given by the category of the representations of the mixed quantum group and the deformation parameter is given by q. What is the corresponding deformation parameter of the left hand side, or, how to deform the left hand side of (1.16)?

The corresponding twisting gadget is given by $\mathbb{G}_m$-gerbes (its definition and how to twist a sheaf of category using a gerbe are explained in Appendix Section B). In general, given a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on a prestack, we could consider the corresponding twisted sheaf category.

In this section, we explain the gerbes used in this paper. For further explanation, we also refer to Section B.

We start with a gerbe $\mathcal{G}^G$ in $FactGrb^{reg}(Gr_{G,Ran})$, the category of factorization tame gerbes on Ran version affine Grassmannian (see Definition B.6). Consider the following diagram of prestacks:

$$\begin{tikzcd} & Gr_{B,Ran} \arrow[dl] \arrow[dr] & \\ Gr_{G,Ran} & & Gr_{T,Ran} \end{tikzcd} \tag{2.23}$$

The pullback of $\mathcal{G}^G$ along with the left morphism gives a tame factorization gerbe on $Gr_{B,Ran}$. By Lemma B.2 b), this regular factorization gerbe could descend to a regular factorization gerbe on $Gr_{T,Ran}$. We denote the resulted gerbe by $\mathcal{G}^T$.

By definition and Beauville-Laszlo Lemma, the Hecke prestack

$$Hecke_G = G(O)\backslash G(K)/G(O)$$

classifies the data: $(\mathcal{P}_{G,1}, \mathcal{P}_{G,2}, \alpha)$, here, $\mathcal{P}_{G,1}$ and $\mathcal{P}_{G,2}$ are G-bundles on X and $\alpha : \mathcal{P}_{G,1}|_{X-x} \simeq \mathcal{P}_{G,2}|_{X-x}$. Then, by Lemma B.3 a), $\mathcal{G}^G$ gives rise to a gerbe on the Hecke prestack. We still denote the descent gerbe on $Hecke_G$ by $\mathcal{G}^G$.

Let us fix a square root of the canonical line bundle ω on X and denote it by $\omega^{\otimes\frac{1}{2}}$. We define ω^ρ to be the T-bundle induced from $\omega^{\otimes\frac{1}{2}}$ by the morphism of group schemes

$$2\rho : \mathbb{G}_m \longrightarrow T \tag{2.24}$$

Here, ρ is defined to be the sum of all fundamental coweights.

Note that in the definition of $Gr_{G,Ran}$ (Definition 2.16), we could replace the trivial principal G-bundle $\mathcal{P}_G^0$ by any G-bundle. If replace $\mathcal{P}_G^0$ by the G-bundle $\mathcal{P}'_G := \omega^\rho \overset{T}{\times} G$ in the definition of $Gr_{G,Ran}$, then, the resulted prestack will be denoted by $Gr^{\omega^\rho}_{G,Ran}$. Similarly, we could also define $Fl^{\omega^\rho}_{G,Ran_x}$, $Gr^{\omega^\rho}_{G,x}$, $Fl^{\omega^\rho}_{G,x}$, $G(K)^{\omega^\rho}$, $Gr^{\omega^\rho}_{T,Ran}$, $(Gr^{\omega^\rho}_{T,Ran})^{neg}$, $(Gr^{\omega^\rho}_{T,Ran})^{neg}_{\infty\cdot x}$, etc.

By definition, taking $\mathcal{P}_{G,1}$ to be $\mathcal{P}'_G$ defines a morphism:

$$Gr^{\omega^\rho}_{G,Ran} \longrightarrow Hecke_G$$

Pullback $\mathcal{G}^G$ along with the above map, we get a factorization gerbe on $Gr^{\omega^\rho}_{G,Ran}$. With some abuse of notation, we also denote it by $\mathcal{G}^G$.

Do the same for $\mathcal{G}^T$, we could get a factorization gerbe $\mathcal{G}^T$ on $Gr^{\omega^\rho}_{T,Ran}$.

Similar to (2.10) and (2.15), by taking zeros, we have maps:

$$(Gr^{\omega^\rho}_{T,Ran})^{neg} \longrightarrow Conf \tag{2.25}$$

$$(Gr^{\omega^\rho}_{T,Ran})^{neg}_{\infty\cdot x} \longrightarrow Conf_x \tag{2.26}$$

By Lemma 2.1 and 2.2, (2.25) and (2.26) induce equivalences of factorization gerbes. Hence, we could descend a factorization gerbe $\mathcal{G}^T$ on $(Gr^{\omega^\rho}_{T,Ran})^{neg}$ (resp. $(Gr^{\omega^\rho}_{T,Ran})^{neg}_{\infty\cdot x}$) to a factorization gerbe on $Conf$ (resp. $Conf_x$). We denote the resulted gerbe by $\mathcal{G}^\Lambda$.

From now on, when we use the notations $\mathcal{G}^G$, $\mathcal{G}^T$ and $\mathcal{G}^\Lambda$, we will mean the gerbes obtained by the above procedures.

It is shown in [GL2] that any factorization gerbe on $Gr_{G,Ran}$ could descend to a gerbe on Bun_G, such that the pullback of the resulted gerbe along with the projection:

$$Gr_{G,Ran} \longrightarrow Bun_G \tag{2.27}$$

is $\mathcal{G}^G$. We will denote the resulted gerbe on Bun_G by the same notation.

3 Statement of the main theorem

In order to make our motivation be more clear, we introduce statement of the main theorem as quickly as possible. To achieve this goal, we should introduce two sides of the Iwahori FLE:

1. the category of Whittaker D-modules on affine flags
2. the category of representations of the mixed quantum group.

The introduction to the Whittaker category in Section 3.1 is the most economic one satisfying our requirement in Section 3. But it is not enough for the proof of the main theorem. Hence, we will give a more detailed exposition about the Whittaker category on Fl_G in Section 5.

Then, in Section 3.2, we will introduce the notion of the mixed quantum group and the category of its representations.

After the preparation given in Section 3.1 and 3.2, we could give an explicit statement of the main theorem (Theorem 3.1) of this paper in Section 3.3.

At last, in Section 3.4, we will explain the strategy to the proof of Theorem 3.1.

3.1 Definition of Whittaker category (through invariant)

Given a tame factorization gerbe $\mathcal{G}^G$ on affine flags. (For the definition of a tame factorization gerbe, please see Definition B.6 in Appendix B.) In this section, we will give a quick definition of the $\mathcal{G}^G$-twisted invariant Whittaker category (i.e, the category of $(N(K), \chi)$-invariant D-modules) with the D-module theory on infinite-dimensional schemes introduced in Section A. The references are [Ber], [Ga3] and [Ga4].

Consider a non-degenerated character χ on $N(K)$ which is defined as follows:

$$\chi : N(K) \overset{projection}{\longrightarrow} N(K)/[N(K), N(K)] \overset{\sim}{\longrightarrow} \mathbb{G}_a^r(K) \overset{add}{\longrightarrow} \mathbb{G}_a(K) \longrightarrow \mathbb{G}_a \quad (3.1)$$

Here, the last map $\mathbb{G}_a(K) \longrightarrow \mathbb{G}_a$ is given by sending $f(t) = \sum_{i \geq m_0} a_i t^i$ to a_{-1} and r is the rank of G.

Remark The definition of χ is not canonical, i.e, we need to pick a local parameter of K. We can fix this problem by replacing $N(K)$ by the ω^ρ-twisted ind-pro-scheme $N(K)^{\omega^\rho}$. We will introduce such ω^ρ-twisted groups later (Definition 6.1).

Note that according to Lemma A.1, $N(K)$ is an ind-pro group scheme. We can write $N(K)$ as:

$$N(K) \simeq \bigcup_{k \geq 0} Ad_{-k\rho} N(O) \quad (3.2)$$

With some abuse of notation, we also denote by χ the restriction of χ to $Ad_{-k\rho}N(O)$. Then, by (A.6) in Section A, we have:

$$Whit_q(Fl) = D_{\mathcal{G}^G}(Fl)^{N(K),\chi} = \lim_{oblv} D_{\mathcal{G}^G}(Fl)^{Ad_{-k\rho}N(O),\chi} \quad (3.3)$$

Here, q is the quadratic form attached to $\mathcal{G}^G$. For further explanation about how to get q from $\mathcal{G}^G$, please check Section B.4 and Section B.5.

Now, let us fix a nature number $k \geq 0$. By (3.3), we only need to define $D_{\mathcal{G}^G}(Fl)^{Ad_{-k\rho}N(O),\chi}$.

Note that by Lemma 2.3, Fl_G is an ind-scheme of ind-finite type, hence we could write Fl_G as a colimit of finite-dimensional scheme Y_i. What's more, we could assume that each Y_i is $Ad_{-k\rho}N(O)$ invariant. Then, by

$$D_{\mathcal{G}^G}(Fl_G) = \lim_{i} D_{\mathcal{G}^G}(Y_i)$$

we have:

$$D_{\mathcal{G}^G}(Fl)^{Ad_{-k\rho}N(O),\chi} = \lim_{i} D_{\mathcal{G}^G}(Y_i)^{Ad_{-k\rho}N(O),\chi} \tag{3.4}$$

The transition functor is given by !-pullback functor.

Next, we only need to define $D_{\mathcal{G}^G}(Y_i)^{Ad_{-k\rho}N(O),\chi}$. We note that $Ad_{-k\rho}N(O)$ is a pro-scheme of finite type, so we can write it as:

$$Ad_{-k\rho}N(O) = \lim_{l} N_k^l$$

such that each N_k^l is a finite-dimensional unipotent group scheme and the action of $Ad_{-k\rho}N(O)$ on Y_i factors through N_k^l. Then, we define

$$D_{\mathcal{G}^G}(Y_i)^{Ad_{-k\rho}N(O),\chi} := D_{\mathcal{G}^G}(Y_i)^{N_k^l,\chi} \tag{3.5}$$

The above definition is independent of the choice of N_k^l because for any $l' \geq l$, the kernel of $N_k^{l'} \longrightarrow N_k^l$ is unipotent.

Now, we have already reduced the definition of $Whit_q(Fl)$ to the theory of D-modules of finite dimensional schemes which is well-known.

By the way, through the construction above, we know that $Whit_q(Fl)$ is a full subcategory of $D_{\mathcal{G}^G}(Fl)$. We denote by $oblv_{N(K),\chi}$ the fully faithful forgetful functor:

$$oblv_{N(K),\chi} : Whit_q(Fl) \longrightarrow D_{\mathcal{G}^G}(Fl) \tag{3.6}$$

Because we are working with cocomplete DG-categories and $oblv_{N(K),\chi}$ preserves colimits, the functor $oblv_{N(K),\chi}$ admits a right adjoint functor by general categorical reason (adjoint functor theorem). It is denoted by $Av_*^{N(K),\chi}$.

$$Av_*^{N(K),\chi} : D_{\mathcal{G}^G}(Fl) \longrightarrow Whit_q(Fl) \tag{3.7}$$

What's more, $oblv_{N(K),\chi}$ admits a (partially defined) left adjoint functor which is denoted by $Av_!^{N(K),\chi}$.

Because $Av_!^{N(K),\chi}$ is a (partially defined) left adjoint functor, it commutes with filtered colimits. On the contrary, $Av_*^{N(K),\chi}$ is discontinuous.

With the ind-pro group scheme structure of $N(K)$, we may describe $Av_!^{N(K),\chi}$ and $Av_*^{N(K),\chi}$ as follows:

$$Av_*^{N(K),\chi} = \lim_k Av_*^{Ad_{-k\rho}N(O),\chi} \tag{3.8}$$

$$Av_!^{N(K),\chi} = \operatorname{colim}_k Av_!^{Ad_{-k\rho}N(O),\chi} \tag{3.9}$$

If $Av_!^{Ad_{-k\rho}N(O),\chi}(\mathcal{F})$ can be defined for any k, then, $Av_!^{N(K),\chi}(\mathcal{F})$ can be defined.

Remark In particular, $Av_!^{N(K),\chi}$ can be defined for (twisted) ind-holonomic D-modules.

3.2 Mixed quantum groups

In Section 3.2.1, we will introduce several different quantum groups, such as free quantum group, cofree quantum group, Lusztig quantum group, and Kac-DeConcini quantum group, etc. They will be used to give the definition of the category of representations of the mixed quantum group in Section 3.2.3. The readers who have already known the mixed quantum group in [Ga6] could skip this section safely.

3.2.1 Five quantum groups

In Section B, from a tame factorization gerbe $\mathcal{G}^G$, we could get a symmetric bilinear form

$$b : \Lambda \times \Lambda \longrightarrow \Bbbk/\mathbb{Z} \tag{3.10}$$

Denote by b' a linear form, such that $b'(\lambda_1, \lambda_2) + b'(\lambda_2, \lambda_1) = b(\lambda_1, \lambda_2)$. We take the quadratic form q associated to b',

$$q : \Lambda \longrightarrow \Bbbk/\mathbb{Z} \tag{3.11}$$

In addition, we require q satisfies the following conditions:

a). q is W-invariant, i.e, $q(\lambda) = q(w(\lambda))$;

b). $b(\alpha, \lambda) = \langle \lambda, \check{\alpha} \rangle \cdot q(\alpha), \forall \alpha \in \Delta, \lambda \in \Lambda$.

c). q *avoids small torsion*[Ga6]: if for every simple factor in our root system and (any) long coroot α^l in it, we have $ord(q(\alpha^l)) \geq d + 1$, where d = 1, 2, 3 is the lacing number of that simple factor.

We let $Vect_q^\Lambda$ denote $Rep_q(\check{T})$, the category of representations of the quantum torus $\check{T}$. It could be regarded as the category of twisted Λ-graded vector space. If we do not have twisting here, the category $Vect^\Lambda$ admits a symmetric monoidal category structure given by the usual tensor product. $Vect_q^\Lambda$ has the same underlying category as $Vect^\Lambda$, the difference is that the braiding of $Vect_q^\Lambda$ is different from the symmetric one.

[8] By [GL1] 29.1, the objects defined in this section only depend on q. A choice of b' amounts to give an identification of $Vect_q^\Lambda$ and $Vect^\Lambda$.

If we denote by $\Bbbk_\lambda \in Vect_q^\Lambda$ the 1-dimensional vector space placed in the degree λ, then, the braiding of $Vect_q^\Lambda$ is generated by the braidings[9]:

$$\Bbbk_{\lambda_1+\lambda_2} = \Bbbk_{\lambda_1} \otimes \Bbbk_{\lambda_2} \overset{\times e^{2\pi i b'(\lambda_1,\lambda_2)}}{\longrightarrow} \Bbbk_{\lambda_2} \otimes \Bbbk_{\lambda_1} = \Bbbk_{\lambda_2+\lambda_1}, \text{ here } \lambda_1, \lambda_2 \in \Lambda \quad (3.12)$$

Given the fact that $Vect_q^\Lambda$ is a braided monoidal category, we could consider Hopf algebras inside. The first one we will consider is the free Hopf algebra $U_q^{fr}(\check{N})$.

We denote by $\Bbbk_i$ the 1-dimensional vector space placed in the degree α_i. $\oplus\Bbbk_i$ is the object in $Vect_q^\Lambda$ which is $\Bbbk$ in the degrees $\{\alpha_1, ..., \alpha_r\}$ and 0 elsewhere. Then, we define $U_q^{fr}(\check{N})$ to be the associative algebra generated by $\oplus\Bbbk_i$. In other words, $U_q^{fr}(\check{N})$ is the object satisfying the following universal property:

- *For any associative algebra A' in $Vect_q^\Lambda$, we have the canonical isomorphism:*

$$RHom_{Vect_q^\Lambda}(\oplus\Bbbk_i, A') \simeq RHom_{AAlg(Vect_q^\Lambda)}(U_q^{fr}(\check{N}^+), A') \quad (3.13)$$

Here, $AAlg(Vect_q^\Lambda)$ denotes the category of associative algebras in $Vect_q^\Lambda$.

Dually to the definition of $U_q^{fr}(\check{N})$, we may define a coassociative coalgebra cogenerated by $\Bbbk_i$ in $Vect_q^\Lambda$, and it is denoted by $U_q^{cofr}(\check{N})$. To be more precise, it is the object satisfying the universal property:

- For any coassociative coalgebra B in $Vect_q^\Lambda$

$$Hom_{Vect_q^\Lambda}(B, \oplus\Bbbk_i) = Hom_{CCAlg(Vect_q^\Lambda)}(B, U_q^{cofr}(\check{N})) \quad (3.14)$$

Here, $CCAlg(Vect_q^\Lambda)$ denotes the category of coassociative coalgebras in $Vect_q^\Lambda$.

Then, we define Lusztig quantum group U_q^L and Kac-DeConcini quantum group U_q^{KD}.

We choose v_i, such that,

$$v_i^2 = exp(2\pi i \cdot q(\alpha_i)) \quad (3.15)$$

The quantum numbers are defined to be

$$[n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}} = v_i^{n-1} + v_i^{n-3} + ... + v_i^{-n} \quad (3.16)$$

And the quantum factorial is defined to be:

$$[n]_i^! = \prod_{s=1}^{n} [s]_i \quad (3.17)$$

the quantum binomial coefficient is:

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i^!}{[m]_i^! [n-m]_i^!} \quad (3.18)$$

Definition 3.1. *The quantum Serre relation is defined to be:*

$$\sum_{p+p'=1-\langle\alpha_i,\check{\alpha}_j\rangle}(-1)^{p'}\begin{bmatrix}1-\langle\alpha_i,\check{\alpha}_j\rangle\\ p\end{bmatrix}_i \Bbbk_i^p\Bbbk_j\Bbbk_i^{p'}=0,\ \forall i,j \tag{3.19}$$

Definition 3.2. *The Kac-DeConcini quantum group $U_q^{KD}(\check{N}^+)$ is defined to be the quotient of $U_q^{fr}(\check{N}^+)$ by the quantum Serre relations.*

It can be shown (ref. [Ri]) that $U_q^{KD}(\check{N})$ also acquires a Hopf algebra structure: the product and coproduct of $U_q^{fr}(\check{N}^+)$ factor through the product and coproduct morphism of $U_q^{KD}(\check{N}^+)$.

By sending $\Bbbk_i$ to $\Bbbk_i$, we get a morphism from $U_q^{fr}(\check{N})$ to $U_q^{cofr}(\check{N})$. It can be shown ([Ri]) that it factors through $U_q^{KD}(\check{N})$, i.e,

$$U_q^{fr}(\check{N})\longrightarrow U_q^{KD}(\check{N})\longrightarrow U_q^{cofr}(\check{N}) \tag{3.20}$$

Definition 3.3. *We define the Lusztig quantum group $U_q^L(\check{N})$ to be the graded dual of $U_q^{KD}(\check{N})$.*

$U_q^L(\check{N}^+)$ could be regarded as a Hopf subalgebra in $U_q^{cofr}(\check{N}^+)$, it consists of linear functions on $U_q^{fr}(\check{N}^+)$ which is zero on the elements:

$$\sum_{p+p'=1-\langle\alpha_i,\check{\alpha}_j\rangle}(-1)^{p'}\begin{bmatrix}1-\langle\alpha_i,\check{\alpha}_j\rangle\\ p\end{bmatrix}_i \Bbbk_i^p\Bbbk_j\Bbbk_i^{p'},\forall i,j$$

Here, $\Bbbk_i$ and $\Bbbk_j$ are regarded as objects in $U_q^{fr}(\check{N})$. By the above description, the morphism $U_q^{fr}(\check{N})\longrightarrow U_q^{KD}(\check{N}^+)$ factors through $U_q^L(\check{N}^+)$.

We have the following sequence by [GL1]:

$$U_q^{fr}(\check{N})\twoheadrightarrow U_q^{KD}(\check{N})\twoheadrightarrow u_q(\check{N})\hookrightarrow U_q^L(\check{N})\hookrightarrow U_q^{cofr}(\check{N})$$

Remark In existed papers, the quantum group $u_q(\check{N})$ is called the small quantum group. For the theory of its representation category, please see [GL1] (q is rational) and [AG], [ABBGM] (q=1).

Remark The above definitions of U_q^L and U_q^{KD} are different from the ones in [Ga6] for general q. But when q *avoids small torsion*, the definitions are the same.

3.2.2 Relative Drinfeld center

In order to define the category of representations of the mixed quantum group, we will need the notion of the *relative Drinfeld center.*

Definition 3.4. *Given a monoidal category $\mathcal{O}$ and a braided monoidal category $\mathcal{C}$ which acts on $\mathcal{O}$ from right. Then, the relative Drinfeld center of $\mathcal{O}$ with respect to $\mathcal{C}$ is the universal braided monoidal category acting on $\mathcal{O}$ on left and commutes with the right action of $\mathcal{C}$. It is denoted by $Z_{Dr,\mathcal{C}}(\mathcal{O})$.*

It can be described as follows: the objects of $Z_{Dr,\mathcal{C}}(\mathcal{O})$ are $M \in \mathcal{O}$ with a collection of isomorphisms for $\forall c \in \mathcal{C}$:

$$\alpha_{M,c} : M \otimes c \xrightarrow{\sim} c \otimes M \tag{3.21}$$

such that they are compatible with the monoidal structure of $\mathcal{C}$, i.e,

$$\begin{tikzcd}
M \otimes (c_1 \otimes c_2) \arrow[r, "\alpha_{M,c_1 \otimes c_2}"] \arrow[d, "\sim"] & (c_1 \otimes c_2) \otimes M \\
(M \otimes c_1) \otimes c_2 \arrow[d, "\alpha_{M,c_1} \otimes id_{c_2}"] & c_1 \otimes (c_2 \otimes M) \arrow[u, "\sim"'] \\
(c_1 \otimes M) \otimes c_2 \arrow[r, "\sim"] & c_1 \otimes (M \otimes c_2) \arrow[u, "id_{c_2} \otimes \alpha_{M,c_2}"']
\end{tikzcd} \tag{3.22}$$

commutes.

And $\alpha_{M,c}$ is compatible with the right action of $\mathcal{O}$, i.e, for $\forall a \in \mathcal{O}, \forall c \in \mathcal{C}$

$$\begin{tikzcd}
M \otimes (c \cdot a) \arrow[r, "\alpha_{M,c \cdot a}"] \arrow[d, "\sim"] & (c \cdot a) \otimes M \arrow[d, "\sim"] \\
(M \otimes c) \cdot a \arrow[r, "\alpha_{M,c}"] & (c \otimes M) \cdot a
\end{tikzcd}$$

commutes.

3.2.3 Mixed representation category

With the preparations before, we could define $Rep_q^{mix}(\check{G})$, the category of representations of the mixed quantum group.

We denote by $U_q^L(\check{N}) - mod^{fin}$ the derived category of finite-dimensional $U_q^L(\check{N})$ modules in $Vect_q^{\Lambda}$. And we denote by $U_q^L(\check{N}) - mod^{loc.nil}$ the ind-completion of $U_q^L(\check{N}) - mod^{fin}$.

Remark An object in the category $U_q^L(\check{N}) - mod^{loc.nil}$ can be regarded as the union of its finite-dimensional submodules. Hence, the category $U_q^L(\check{N}) - mod^{loc.nil}$ can be regarded as the category of $U_q^L(\check{N})$-modules, and the augmentation ideal action is locally nilpotent. This description is the reason why we put "loc.nil" in the upper index.

Definition 3.5. *We define* $Rep_q^{mix}(\check{G}) := Z_{Dr,Vect_q^{\Lambda}}(U_q^L(\check{N}) - mod^{loc.nil})$

Let us explain why it is called the category of representations of the mixed quantum group. Recall that the relative Drinfeld center $Z_{Dr,Vect_q^{\Lambda}}(U_q^L(\check{N}) - mod^{loc.nil})$ admits a naturally defined forgetful monoidal functor to $U_q^L(\check{N}) - mod^{loc.nil}$

$$oblv_{U_q^L(\check{N})} : Z_{Dr,Vect_q^{\Lambda}}(U_q^L(\check{N}) - mod^{loc.nil}) \longrightarrow U_q^L(\check{N}) - mod^{loc.nil} \tag{3.23}$$

In addition, there is also a forgetful functor to $U_q^{KD}(\check{N}^-)$-mod,

$$oblv_{U_q^{KD}(\check{N}^-)}: Z_{Dr,Vect_q^\Lambda}(U_q^L(\check{N}) - mod^{loc.nil}) \longrightarrow U_q^{KD}(\check{N}^-) - mod \tag{3.24}$$

(3.24) is given by

$$\begin{aligned} Z_{Dr,Vect_q^\Lambda}(U_q^L(\check{N}) - mod^{loc.nil}) &\xrightarrow{\sim} Z_{Dr,Vect_q^\Lambda}(U_q^{KD}(\check{N}^-) - comod) \\ &\xrightarrow{R} U_q^{KD}(\check{N}^-) - mod \end{aligned} \tag{3.25}$$

The functor 'R' sends an object $(M \in U_q^{KD}(\check{N}^-) - comod, \alpha'_{M,c})$ to M with a $U_q^{KD}(\check{N}^-)$ action given by:

$$U_q^{KD}(\check{N}^-) \otimes M \overset{\alpha'_{M,U_q^{KD}(\check{N}^-)}}{\simeq} M \otimes U_q^{KD}(\check{N}^-) \overset{counit}{\longrightarrow} M \tag{3.26}$$

Hence, we could understand the relative Drinfeld center $Z_{Dr,Vect_q^\Lambda}(U_q^L(\check{N}) - mod^{loc.nil})$ as the category of $U_q^L(\check{N})$-module in $Vect_q^\Lambda$ such that the augmentation ideal action is locally nilpotent and also admits a compatible $U_q^{KD}(\check{N}^-)$-action.

We denote by $ind_{L\to Dr}$ (resp. $coind_{L\to Dr}$) the left (resp. right) adjoint functor of (3.23). Similarly, we denote by $ind_{KD\to Dr}$ (resp. $coind_{KD\to Dr}$) the left (resp. right) adjoint functor of (3.24). Then, we could define a collection of standard objects in $Rep_q^{mix}(\check{G})$ by:

$$V_\lambda^{mix} := ind_{L\to Dr}(\Bbbk^\lambda) \tag{3.27}$$

$$V_\lambda^{mix,\vee} := coind_{KD\to Dr}(\Bbbk^\lambda) \tag{3.28}$$

It is known that ([Ga1]):

$$RHom_{Rep_q^{mix}(\check{G})}(V_\lambda^{mix}, V_\mu^{mix,\vee}) = \Bbbk, if\ \lambda = \mu$$

and

$$RHom_{Rep_q^{mix}(\check{G})}(V_\lambda^{mix}, V_\mu^{mix,\vee}) = 0, if\ \lambda \neq \mu$$

3.3 Statement of the main theorem

Now, we could give the statement of the main theorem.

In Section 5.5, we will define the metaplectic BMW D-modules $\mathfrak{J}_\lambda$. And then, we will define a collection of standard objects Δ_λ in $Whit_q(Fl)$ indexed by Λ by:

$$\Delta_\lambda := Av_!^{N(K),\chi}(\mathfrak{J}_\lambda) \tag{3.29}$$

We define a t-structure of $Whit_q(Fl)$ such that, $\mathcal{F} \in Whit_q(Fl)^{\geq 0}$ if and only if

$$Hom_{Whit_q(Fl)}(\Delta_\lambda[k], \mathcal{F}) = 0, \forall \lambda \in \Lambda\ and\ k > 0 \tag{3.30}$$

Recall that in (3.27) in Section 3.2.3, we have already defined a collection of standard objects V_λ^{mix} in $Rep_q^{mix}(\check{G})$. Similarly, we could give a t-structure of $Rep_q^{mix}(\check{G})$ by $\mathcal{V} \in Rep_q^{mix}(\check{G})^{\geq 0}$ if and only if

$$Hom_{Rep_q^{mix}(\check{G})}(V_\lambda^{mix}, \mathcal{V}) = 0, \forall \lambda \in \Lambda \text{ and } k > 0 \tag{3.31}$$

Our main theorem says,

Theorem 3.1. *When q avoids small torsion, there exists a t-exact equivalence:*

$$Whit_q(Fl_G) \simeq Rep_q^{mix}(\check{G}) \tag{3.32}$$

which preserves standard objects, i.e, Δ_λ sends to V_λ^{mix}.

Let us explain here, why we define the t-structures of both sides of (3.1) by (3.30) and (3.31).

First of all, because any $N(K)$-orbit of Fl_G is of infinite dimension, the original t-structure of the category of D-modules behaves badly for Whittaker sheaves. For example, if we consider the Whittaker D-module $Av_!^{N(K),\chi}(\delta_0)$, then, by the same analysis as in Remark 6.3.7 [GL1], we could prove that this Whittaker D-module is infinitely connective. It means for any $n \in \mathbb{Z}$, we have:

$$oblv_{N(K),\chi} \circ Av_!^{N(K),\chi}(\delta_0) \in (D_{\mathcal{G}^G}(Fl))^{\leq n} \tag{3.33}$$

If we expect the equivalence of Theorem 3.1 to be t-exact, we need to find a t-structure on the left hand side which is friendly with the category of Whittaker sheaves.

Luckily, the naturally defined t-structure on the category of representations of the mixed quantum group could be described by some distinguished objects (by (3.31)) in the heart, such as V_λ^{mix} and $V_\lambda^{mix,\vee}$. Hence, we only need to look for the corresponding objects of V_λ^{mix} and $V_\lambda^{mix,\vee}$ on the left hand side of (3.32).

By the analysis in Section 1.3.2, we have already seen that the equivalence of our main theorem is supposed to send Δ_λ to V_λ^{mix}. Hence, we define the t-structure of $Whit_q(Fl_G)$ in this way.

3.4 Road map of proof

Strategy Let us explain how to prove Theorem 3.1.

1. The first step (Section 4) is to restate Theorem 3.1. We will replace $Rep_q^{mix}(\check{G})$ by a category of D-modules on the configuration space (to be more precise, it is the category of factorization modules on $Conf_x$ with respect to the factorization algebra Ω_q^L, we will explain these notions in the next section). The advantage of this new expression is that both sides of Theorem 4.1 are geometric objects and it is similar to the case of the Jacquet functor.

2. The second step (Section 6, in particular, Section 6.5) is to give a proof of Theorem 4.1 modulo Theorem 9.2 and Proposition 6.11. The proof of Theorem 9.2 (in Section 9.2) will be obtained during the proof of Proposition 6.11, hence, we only focus on the proof of the latter.

3. The third step (Section 7) is to use a global construction to decompose Proposition 6.11 as two parts: Theorem 7.1 and Proposition 7.4. The rest of this paper will be devoted to the proof of Theorem 7.1 and Proposition 7.4. In addition, the proof of Theorem 7.1 b). could be used to prove Theorem 9.2 without essential difference.

4. The fourth step (Section 8 and Section 9) will be devoted to the proof of Theorem 7.1.

5. And the fifth step (Section 10) will be devoted to the proof of Proposition 7.4

$$\begin{tikzcd}
 & & & Prop.7.4 \arrow[dl] \\
 & & Prop.6.11 \arrow[dl] & Thm.7.1a). \arrow[l] \\
Thm.3.1 \arrow[r, equal] & Thm.4.1 & & \\
 & & Thm.9.2 \arrow[ul] & Thm.7.1b). \arrow[l, Rightarrow] \arrow[uul]
\end{tikzcd}$$

Here, $A \longrightarrow B$ means that the A participates the proof of B, $A \Longrightarrow B$ means A essentially implies B.

Guide for readers

Let us explain the organization of the paper.

The subject of the first part is the introduction. It occupies Section 1- Section 5.

In Section 1, we introduce the notations, categorical settings in this paper. We also explain the motivation and the history of studying Whittaker sheaves and why we expect such a result.

Section 2 is devoted to the introduction of some geometric objects that we will use in this paper, such as factorization gerbes, affine flags, affine Grassmannians, Ran spaces, configuration spaces, etc.

Section 3 is a very short course on the categories that we deal with in this paper. Namely, $Whit_q(Fl)$, the Whittaker category on affine flags and $Rep_q^{mix}(\check{G})$, the category of representations of mixed quantum groups.

Section 4 is about the equivalence between the category of representations of mixed quantum groups and the category of factorization modules of factorization algebras. In this section, we will find a geometric replacement $(\Omega_q^L - FactMod)$

of $Rep_q^{mix}(\check{G})$ and once we established such an equivalence, we will only focus on the proof of the equivalence between $Whit_q(Fl)$ and the lately defined category (Theorem 4.1).

Section 5 serves as a complement of Section 3.1 which offers more details of $Whit_q(Fl)$. The principal goal of this section is to give the definition of BMW D-modules which will be used to construct standards Δ_λ in $Whit_q(Fl)$ and match the Verma module in $Rep_q^{mix}(\check{G})$.

The second part is Section 6.

In Section 6, we construct a functor $F^L : Whit_q(Fl) \longrightarrow \Omega_q^L - FactMod$. In Section 6.4, we will prove that the !-fiber of F^L is co-represented by Δ_λ based on Theorem 9.2. Then, in Section 6.5, we will make our first attempt to the proof of Theorem 4.1. Actually, we will prove that we only need to prove standards go to standards. It is the content of Proposition 6.11.

The third part of this paper is about proof of the key proposition, Proposition 6.11. The proof of Proposition 6.11 will be like peeling an onion. We follow the idea of [GL1], i.e, we want to construct a globally defined Whittaker category using Drinfeld compactification and then construct the corresponding functor. At last, prove this global functor could send the standard objects in the global Whittaker category to the standards in $\Omega_q^L - FactMod$.

In Section 7, we will give the definition of the global Whittaker category $Whit_q((\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x})$ and construct a functor F_{glob}^L maps the global Whittaker category to $\Omega_q^L - FactMod$.

In Section 8, we will prove the local Whittaker category on Fl_G and global category are equivalent.

In Section 9, we will compare the semi-infinite (I.e, $N(K)$-equivaraint+ $T(O)$-equivariant with respect to a character) D-module on Fl_G and the ! extension of the constant D-module on the Drinfeld compactification. As an application of this comparison, we could prove that the local functor and global functor are isomorphic.

In Section 10, we will prove that the global functor could send the global standard objects to standards of $\Omega_q^L - FactMod$.

Then, for the convenience of readers, we supply some appendix related to this paper.

4 Geometric Replacement of Rep_q^{mix}

The goal of this section is to replace Theorem 3.1 by a stronger theorem (Theorem 4.1) which is a totally geometric statement. In order to have a totally geometric expression of our main theorem (Theorem 3.1), we need to find a category of sheaves on some scheme which could replace the category $Rep_q^{mix}(\check{G})$. The solution is to replace $Rep_q^{mix}(\check{G})$ by the category of factorization modules with respect to a factorization algebra.

The origin of factorization algebras goes back to [BFS]. Then, it is known (such as [Lu1]) that the category of factorization modules of a certain factorization algebra could be used to describe the category of modules of an $\mathbb{E}_2$-algebra. The latter could be regarded as the category of representations of a Hopf algebra by the Koszul duality.

Let us explain the organization and content of this section:

In Section 4.1, we will explain the definition of factorization algebras and factorization modules (Section 4.1.1). Then, we will study the structure of $\mathcal{A} - FactMod$, the category of factorization modules of a certain factorization algebra $\mathcal{A}$ (Section 4.1.2). Also in this section, we could give an intrinsic definition of t-structure of $\mathcal{A} - FactMod$ (Proposition 4.1).

In Section 4.2, we will explain the relationship between the category of Hopf algebras and the category of factorization algebras, as well as their module categories. The factorization algebra which is important for us in this paper is Ω_q^L, we will explain this factorization algebra explicitly in Section 4.4 by a totally geometric way given in [Ga6].

In Section 4.5, as an application of the equivalence given in Section 4.2, we could give a new expression of Theorem 3.1.

4.1 Factorization modules

4.1.1 Factorization algebras and factorization modules

The object on the left hand side of our main theorem is in the geometric natural while the object on the right hand side is a representation category. The tool that makes the comparison easier is *factorization algebra*.

In this section, we will recall the notion *factorization algebra* introduced in [BFS]. And then, we will also study its module category. In particular, we will focus on the factorization algebra Ω_q^L in the next section. It is Ω_q^L-FactMod which is equivalent to the category of representations of mixed quantum groups and play a role as a replacement for the latter.

The advantage of the category of factorization modules over the category of the representations of the mixed quantum group is that the former one is also geometric and the twisting could be related to the twisting used in Whittaker category easily. In our expected equivalence, $Whit_q(Fl) \simeq \Omega_q^L - FactMod$, the left category is a collection of twisted D-modules defined on Fl_G and the right category is on the configuration space which is essentially the same as Gr^{neg}_{T,Ran_x} by Lemma 2.2. Hence, it falls into the case of a Jacquet functor.

The notion of factorization algebras have already been well-known for experts and there are many good studies about them and their module category, [BFS] etc. Naively speaking, factorization algebras are some D-modules (or, constructible sheaves) on the Configuration space $Conf(X, \Lambda^{neg})$ and compatible with the factorization property of $Conf(X, \Lambda^{neg}) = Conf$. If $\mathcal{F}$ is a factorization algebra on $Conf$, then, over a point

$$D = \sum_i \lambda_i \cdot x_i \tag{4.1}$$

, the fiber of $\mathcal{F}$ will be canonically isomorphic to

$$\prod_i \mathcal{F}_{\lambda_i \cdot x_i}$$

here, $\mathcal{F}_{\lambda_i \cdot x_i}$ denotes the fiber of $\mathcal{F}$ over $\lambda_i \cdot x_i$.

To spell out the precise definition, we restrict the addition map *add* in (2.7) to $Conf^J_{disj}$:

$$add|_{disj} : Conf^J_{disj} \longrightarrow Conf$$

Given a (twisted) D-module on $Conf(X, \Lambda^{neg})$, we could !-pull it back to $Conf^J_{disj}$, then, according to the factorizable property that $\mathcal{F}$ should satisfy, we have:

$$add^!|_{disj}(\mathcal{F}) \simeq \mathcal{F} \boxtimes \mathcal{F} \boxtimes ... \boxtimes \boxtimes \mathcal{F}|_{Conf^J_{disj}} \tag{4.2}$$

But we need to notice that there are some extra higher associative properties needed. To formulate the definition of these homotopy coherence, we still adopt the method used in [GL1].

Given a factorization tame gerbe $\mathcal{G}^\Lambda$ (see Section 2.5) on $Conf$, let us consider a right lax monoidal functor:

$$fSet^{surj} \longrightarrow DGCat \tag{4.3}$$

$$J \longrightarrow D_{(\mathcal{G}^\Lambda)^J}(Conf^J_{disj})$$

By Grothendieck straightening theorem (See Lemma 1.2), this right lax monoidal functor can be explained as a Cartesian fibration of symmetric monoidal categories:

$$D_{\mathcal{G}^\Lambda}(Conf^{fSet}_{disj}) \longrightarrow fSet^{surj} \tag{4.4}$$

It could be described as follows: the fiber over $J \in fSet^{surj}$ is given by $D_{(\mathcal{G}^\Lambda)^J}(Conf^J_{disj})$, and if $J \twoheadrightarrow I$ and $\mathcal{F}_I \in D_{(\mathcal{G}^\Lambda)^I}(Conf^I_{disj})$, then the object in $D_{(\mathcal{G}^\Lambda)^J}(Conf^J_{disj})$ is given by the !-pullback of $\mathcal{F}_I$ along with the map $Conf^J_{disj} \longrightarrow Conf^I_{disj}$ induced by $J \twoheadrightarrow I$.

We could define

Definition 4.1. *A factorization algebra is defined to be a symmetric monoidal Cartesian section of the Cartesian fibration (4.4).*

From this definition, we could see that a factorization algebra consists of an object $\mathcal{A}$ in $D_{\mathcal{G}^\Lambda}(Conf)$, a compatible collection of factorization isomorphisms (4.2) and homotopy-coherent conditions.

We could define a full subcategory $D_{\mathcal{G}^\Lambda}(Conf)^c$ of $D_{\mathcal{G}^\Lambda}(Conf)$, such that $\mathcal{F} \in D_{\mathcal{G}^\Lambda}(Conf)^c$ if it is compact when restricting to each connected component $Conf^\lambda \subset Conf$. We have $(D_{\mathcal{G}^\Lambda}(Conf))^\vee := Funct(D_{\mathcal{G}^\Lambda}(Conf), Vect)$

And the *Verdier* duality defines a coinvariant equivalence between compact objects of $D_{\mathcal{G}^\Lambda}(Conf)$ and $D_{(\mathcal{G}^\Lambda)^{-1}}(Conf)^\vee$.

$$\mathbb{D} : D_{\mathcal{G}^\Lambda}(Conf)^{c,op} \longrightarrow D_{(\mathcal{G}^\Lambda)^{-1}}(Conf)^c \tag{4.5}$$

An important feature of $\mathbb{D}$ is that it preserves the factorization property. That is to say that the Verdier dual of a factorization algebra is also a factorization algebra.

Given a factorization algebra $\mathcal{A}$ on $Conf$, we could define its module category on $Conf_x$. Naively speaking, if we consider the restriction of the map add_x in (2.14) to $(Conf^{J-*} \times Conf_x)_{disj}$

$$add_x|_{disj} : (Conf^{J-*} \times Conf_x)_{disj} \longrightarrow Conf_x$$

, a factorization module of $\mathcal{A}$ is a (suitable twisted) D-module $\mathcal{M}$ on $Conf_x$ such that:

$$add^!|_{disj}(\mathcal{M}) \simeq \mathcal{A} \boxtimes \mathcal{A} ... \boxtimes \mathcal{M} \tag{4.6}$$

with higher homotopy-coherence.

To be more precise, given the factorization tame gerbe $\mathcal{G}^\Lambda$ on $Conf$ and the gerbe $\mathcal{G}^\Lambda$ on $Conf_x$ which is factorizable with respect to $\mathcal{G}$. We consider a functor:

$$(fSet_*^{surj}) \longrightarrow DGCat \tag{4.7}$$

$$* \in J \longrightarrow D_{(\mathcal{G}^\Lambda)^{J-*} \boxtimes \mathcal{G}^\Lambda}((Conf^{J-*} \times Conf_x)_{disj})$$

By straightening (Lemma 1.2), it gives rise to a Cartesian fibration:

$$D_{\mathcal{G}^\Lambda}((Conf^{fSet_*^{surj}} \times Conf_x)_{disj}) \longrightarrow fSet_*^{surj} \tag{4.8}$$

The fiber of $D_{\mathcal{G}^\Lambda}((Conf^{fSet_*^{surj}} \times Conf_x)_{disj})$ over $(* \in J) \in fSet_*^{surj}$ is given by $D_{(\mathcal{G}^\Lambda)^{J-*} \boxtimes \mathcal{G}^\Lambda}((Conf^{J-*} \times Conf_x)_{disj})$.

This Cartesian fibration has an action of the symmetric monoidal category $D_{\mathcal{G}^\Lambda}(Conf^{fSet^{surj}})$ (see (4.4)) and this action is compatible with the action of $fSet^{surj}$ on $fSet_*^{surj}$ given by taking disjoint union. Recall that we defined a factorization algebra $\mathcal{A}$ to be a symmetric monoidal section of $D_{\mathcal{G}^\Lambda}(Conf^{fSet^{surj}}) \longrightarrow fSet^{surj}$. Hence, we could define:

Definition 4.2. *A factorization module of $\mathcal{A}$ is a Cartesian section of*

$$D_{\mathcal{G}^\Lambda}((Conf^{fSet_*^{surj}} \times Conf_x)_{disj}) \longrightarrow fSet_*^{surj} \tag{4.9}$$

which is compatible with the action of $\mathcal{A}$.

We denote by $\mathcal{A} - FactMod$ the category of factorization modules of $\mathcal{A}$.

For our future use, we recall the Verdier duality functor for ind-scheme (ind-stacks).

Lemma 4.1. *If $\mathcal{Y}$ is an ind-scheme of ind-finite type, then $D(\mathcal{Y})$ is compactly generated.*

Proof. It follows from the fact if $\mathcal{Y}$ can be written as a colimit of Z_i such that Z_i are of finite type and $Z_i \longrightarrow \mathcal{Y}$ is closed embedding for any i, then, we could take compact generators of each $D(Z_i)$ and they form a set of compact generators of $D(\mathcal{Y})$. □

Assume $\mathcal{Y}$ to be an ind-scheme of ind-finite type, and $\mathcal{G}$ is a gerbe on $\mathcal{Y}$. The same proof of Lemma 4.1 applies to twisted D-module category, one can show that $D_{\mathcal{G}}(\mathcal{Y})$ is compactly generated. In particular, it is dualizable.

Remark When there is no twisting, it is shown in [Ras3] and [Ber] that if $D(\mathcal{Y})$ is dualizable then its dual is exactly $D(\mathcal{Y})$ and the $*$-direct image functor becomes !-pullback functor under the dual functor. In the case of ind-scheme of ind-finite type (more generally, ind-placid scheme in [Ras3]), a dimension theory gives an identification of the category $D(\mathcal{Y})$ and its dual category.

When there is a twisting, we have a canonical identification

$$D_{\mathcal{G}}(\mathcal{Y})^{\vee} = Funct(D_{\mathcal{G}}(\mathcal{Y}), Vect) \simeq D_{\mathcal{G}^{-1}}(\mathcal{Y}) \tag{4.10}$$

The duality functor introduces an equivalence:

$$\mathbb{D} : D_{\mathcal{G}}(\mathcal{Y})^{c,op} \longrightarrow D_{(\mathcal{Y})^{-1}}(\mathcal{Y})^{c} \tag{4.11}$$

Inside $D_{\mathcal{G}}(\mathcal{Y})$, we could define a full subcategory $D_{\mathcal{G}}(\mathcal{Y})^{loc.c}$ of $D_{\mathcal{G}}(\mathcal{Y})$. It consists of the twisted D-modules $\mathcal{F}$ such that:

1. The support of $\mathcal{F}$ is a scheme.
2. For any quasi-compact open subscheme U in $\mathcal{Y}$, the restriction $\mathcal{F}|_U \in D_{\mathcal{G}}(U)^c$, i.e, $\mathcal{F}|_U \in D_{\mathcal{G}}(U)^{coh}$.

The functor (4.11) introduces the following Verdier duality:

$$\mathbb{D} : D_{\mathcal{G}}(\mathcal{Y})^{loc.c,op} \longrightarrow D_{\mathcal{G}^{-1}}(\mathcal{Y})^{loc.c} \tag{4.12}$$

In particular, we have a well-defined Verdier duality functor on $Conf_x$.

An important feature of this duality functor is that the dual of a factorization module of a factorization algebra $\mathcal{A}$ is a factorization module of the factorization algebra $\mathbb{D}(\mathcal{A})$. The Verider duality defines a coinvariant equivalence of categories of such factorization modules.

4.1.2 Structure of $\mathcal{A} - FactMod$

Given the factorization gerbe $\mathcal{G}^{\Lambda}$ on $Conf$ and a factorization algebra $\mathcal{A} \in D_{\mathcal{G}^{\Lambda}}(Conf)^c$. In addition, we assume that $\mathcal{A}$ is holonomic as a twisted D-module. In this section, we will list some general properties of the category $\mathcal{A} - FactMod$ which is defined in the last section.

Given $\mu \in \Lambda$, we denote by $Conf_{=\mu\cdot x}$ the locally closed subscheme of $Conf_x$ consisting of points: $\{D = \mu \cdot x + \sum \lambda_i x_i |\ \lambda_i \in \Lambda^{neg},\ x_i \neq x,\ \forall i\}$. It is an open subset of $Conf_{\leq\mu\cdot x}$ (for definition, please see Section 2.2). Between these prestacks, we have the following functors:

Definition 4.3.

$$j_{\lambda,Conf_x} : Conf_{=\lambda\cdot x} \longrightarrow Conf_{\leq\lambda\cdot x}$$
$$\bar{i}_{\lambda,Conf_X} : Conf_{\leq\lambda\cdot x} \longrightarrow Conf_x$$
$$i_{\lambda,Conf_x} := \bar{i}_{\lambda,Conf_X} \circ j_{\lambda,Conf_x}$$

If we restrict ourselves to (twisted) ind-holonomic D-modules, then, all six functors between (twisted) D-module categories can be defined.

The following easy lemma is useful for describing our t-structure on $\mathcal{A}-FactMod$,

Lemma 4.2. *$j_{\lambda,Conf_x}$ is affine for any $\lambda \in \Lambda$.*

Proof. We could restrict $j_{\lambda,Conf_x}$ to the connected component $Conf^{\mu}_{\leq \lambda\cdot x}$. Then, this open embedding is of the form: $(X-x)^{(n)} \longrightarrow X^{(n)}$. It is affine. □

Note that the restriction of the action (2.14) to $Conf^{J-*} \times Conf_{=\mu\cdot x}$ gives $Conf_{=\mu\cdot x}$ a factorization module prestack structure with respect to the factorization prestack $Conf$, hence, we could also define factorization module on $Conf_{=\mu\cdot x}$ with respect to a factorization algebra $\mathcal{A}$ on $Conf$. We denote by $\mathcal{A}-FactMod_{=\mu}$ the category of factorization modules on $Conf_{=\mu\cdot x}$ with respect to $\mathcal{A}$.

Denote by $\mathcal{G}|_{\lambda\cdot x}$ the restriction of $\mathcal{G}$ at $\lambda\cdot x$. Then, we have:

Lemma 4.3. *Taking !-fiber at $\lambda\cdot x$ defines an equivalence between $Vect_{\mathcal{G}|_{\lambda\cdot x}}$ and $\mathcal{A}-FactMod_{=\mu}$. Here, $Vect_{\mathcal{G}|_{\lambda\cdot x}}$ denotes the category of $\mathcal{G}|_{\lambda\cdot x}$-twisted vector spaces.*

Proof. First of all, we note that a factorization algebra $\mathcal{A}$ on $Conf$ could extend to a sheaf $\tilde{\mathcal{A}}$ on $Conf' = Conf \sqcup \{0\cdot x\}$ by letting the sheaf $\tilde{\mathcal{A}}$ on the component $\{0\cdot x\}$ be $\Bbbk$. Notice that for any $J \in fSet^{surj}$, we have the following morphism:

$$add' : (Conf')^J \longrightarrow Conf' \tag{4.13}$$

and we define $((Conf')^J)_{disj}$ to be the open subscheme of $(Conf')^J$ such that the supports are pairwise disjoint. Here, we regard the support of $\{0\cdot x\} \in Conf'$ as $\emptyset$.

We could see that the resulted sheaf $\tilde{\mathcal{A}}$ also requires a factorization algebra structure.

$$\tilde{\mathcal{A}} \boxtimes ...\tilde{\mathcal{A}}|_{((Conf')^J)_{disj}} \simeq add^{',*}(\tilde{\mathcal{A}})|_{((Conf')^J)_{disj}} \tag{4.14}$$

And a factorization module of $\mathcal{A}$ is also a factorization module of $\tilde{\mathcal{A}}$ with respect to the following factorization map:

$$\tilde{add} : ((Conf \sqcup 0\cdot x) \times Conf_x)_{disj} = (Conf \times Conf_x)_{disj} \bigsqcup Conf_x \longrightarrow Conf_x \tag{4.15}$$

Here, the map $Conf_x \longrightarrow Conf_x$ is given by identity.

Consider the restriction of the factorization morphism to $((Conf \sqcup \{0\cdot x\}) \times \{\lambda\cdot x\})_{disj}$:

$$((Conf \sqcup \{0\cdot x\}) \times \{\lambda\cdot x\})_{disj} \longrightarrow Conf_{=\lambda\cdot x}$$

It is an isomorphism of schemes. So, a factorization module with respect to $\mathcal{A}$ is determined by its ! fiber at $\lambda\cdot x$. The other direction is just by taking !-pullback. □

Under this equivalence, we could define our standard objects and costandard objects in the category of factorization modules now. We notice that if we assume $\mathcal{A}$ is holonomic, then, the object of $\mathcal{A} - FactMod_{=\mu}$ is holonomic as a twisted D-module on $Conf_{=\mu\cdot x}$. In particular, all six functors could be defined.

Definition 4.4. *In the category $\mathcal{A} - FactMod_{=\mu}$, we take out the generator of this category and consider its $*$ (resp, $!$)-direct image in $\mathcal{A} - FactMod$, we denote the resulted D-module by $\nabla_{\lambda,\mathcal{A}}$ (resp, $\Delta_{\lambda,\mathcal{A}}$).*

$\nabla_{\lambda,\mathcal{A}}$ is called costandard object of $\mathcal{A} - FactMod$ and $\Delta_{\lambda,\mathcal{A}}$ is called standard object.

By the definition of $\Delta_{\lambda,\mathcal{A}}$ and $\nabla_{\lambda,\mathcal{A}}$, we have the following property which explains why we call them standard objects and costandard objects:

Proposition 4.1. *Give $\lambda, \mu \in \Lambda$, we have:*

$$Hom_{\mathcal{A}-FactMod}(\Delta_{\lambda,\mathcal{A}}[k], \nabla_{\mu,\mathcal{A}}) = \Bbbk,\ if\ \lambda = \mu, k = 0$$

and

$$Hom_{\mathcal{A}-FactMod}(\Delta_{\lambda,\mathcal{A}}[k], \nabla_{\mu,\mathcal{A}}) = 0, otherwise.$$

From now on to the end of this chapter, we assume that $\mathcal{A}$ is in the heart of the t-structure of the category of D-modules when it is regarded as a (twisted) D- module.

Forgetting $\mathcal{A}$-module structure, we get a functor from $\mathcal{A} - FactMod$ to $D_{\mathcal{G}^\Lambda}(Conf_x)$, the next proposition gives us an intrinsic way to define the t-structure of $\mathcal{A} - FactMod$, i.e, we could describe the t-structure of $\mathcal{A} - FactMod$ by the objects inside.

Proposition 4.2. *An $\mathcal{A}$-factorization module $\mathcal{M}$ on $Conf_x$ is coconnective as a (twisted) D-module if and only for any $\lambda \in \Lambda$:*

$$Hom_{\mathcal{A}-FactMod}(\Delta_{\lambda,\mathcal{A}}[k], \mathcal{M}) = 0,\ if\ k > 0 \tag{4.16}$$

Proof. Consider the t-structure of $\mathcal{A} - FactMod$ given by (4.16). We should prove that the forgetful functor is t-exact.

Because we assume that $\mathcal{A}$ is in the heart, the forgetful functor is left t-exact.

For right t-exactness, we note that $\mathcal{A} - FactMod^{\leq 0}$ is generated under colimits by $\Delta_{\lambda,\mathcal{A}}$, so it suffices to prove that any $\Delta_{\lambda,\mathcal{A}}$ regarded as a plein D-module, it is a connective object. It is because the morphism $j_{\lambda,Conf}$ is an open affine embedding, it is t-exact. □

Note that both of $\Delta_{\lambda,\mathcal{A}}$ and $\nabla_{\lambda,\mathcal{A}}$ are in $D_{\mathcal{G}^\Lambda}(Conf_x)^{loc.c}$. If we apply Verdier duality functor to them, we have:

Proposition 4.3.

$$\mathbb{D}(\Delta_{\lambda,\mathcal{A}}) \simeq \nabla_{\lambda,\mathbb{D}(\mathcal{A})},\ \forall \lambda \in \Lambda$$

Here, $\Delta_{\lambda,\mathcal{A}} \in D_{\mathcal{G}^\Lambda}(Conf_x)$ and $\nabla_{\lambda,\mathbb{D}(\mathcal{A})} \in D_{(\mathcal{G}^\Lambda)^{-1}}(Conf_x)$

4.2 Factorization algebras and Hopf algebras

In this section, we will recall the equivalence between the category of factorization algebras and the category of Hopf algebras, as well as the category of their modules. In particular, at the end of this section, we will apply this equivalence to the mixed quantum group and obtain a factorization algebra denoted by Ω_q^L. Then, we will reach the equivalence:

$$Rep_q^{mix}(\check{G}) \cong \Omega_q^L - FactMod \tag{4.17}$$

From Hopf algebra (such as $U_q^L(\check{N})$) to factorization algebra, there are two steps. The first one is from Hopf algebra to $\mathbb{E}_2$-algebra, the second one is from $\mathbb{E}_2$-algebra to factorization algebra. We will review some notions in order. The main references of this section for us are [Ro] and [Lu1].

4.2.1 $\mathbb{E}_n$-algebras

In this section, we will recall the definition of $\mathbb{E}_n$-algebras. For more details, see Section 5 [Lu1].

$\mathbb{E}_2$-algebra is an important bridge for us to relate Hopf algebra and factorization algebra. Furthermore, because of the close relationship between $\mathbb{E}_n$-algebras and factorization objects, $\mathbb{E}_n$-algebras could offer us another way to study the factorization objects used in this paper, such as factorization gerbes, factorization spaces(stacks), factorization algebras, etc.

$\mathbb{E}_n$-algebra is a generalization of the associative algebra. In fact, it is the case when $n = 1$. There are many (equivalent) ways to introduce $\mathbb{E}_n$-algebra in the existed literature. The classical definition of an $\mathbb{E}_n$-algebra is given by induction: given a monoidal category $\mathcal{C}$, we define $\mathbb{E}_1 - alg(\mathcal{C})$, the category of $\mathbb{E}_1$-algebras, to be the category of the associative algebras inside. If we have already defined $\mathbb{E}_{n-1} - alg(\mathcal{C})$ for an $\mathbb{E}_n$-category $\mathcal{C}$, then, $\mathbb{E}_{n-1} - alg(\mathcal{C})$ is a monoidal category and we define the category of $\mathbb{E}_n$-algebras in $\mathcal{C}$ to be $\mathbb{E}_1(\mathbb{E}_{n-1} - alg(\mathcal{C}))$. In particular, a symmetric monoidal category is an $\mathbb{E}_n$-category for any $n \in \mathbb{N}$, we could define $\mathbb{E}_n$-algebras inside.

In this section we adopt the definition of $\mathbb{E}_n$-algebra given in [Ro]. The advantage of this version of the definition is that it arises naturally from the factorization object. As a result, it is more convenient for us to get a $\mathbb{E}_2$-algebra from a factorization algebra.

Remark For simplicity, we only consider the notion of $\mathbb{E}_n$-algebra in a symmetric monoidal category. The definition of $\mathbb{E}_n$-algebra in an $\mathbb{E}_n$-category should be similar.

The notion of $\mathbb{E}_n$-algebra is essentially higher categorical.

In ordinary category, an $\mathbb{E}_n$-algebra in $Vect$ is a commutative algebra when $n \geq 2$. An $\mathbb{E}_1$-algebra in the category of ordinary category is a monoidal category, and an $\mathbb{E}_n$-algebra is a braided monoidal category when $n \geq 2$ and symmetric if $n \geq 3$. But when we work with ∞-category, there will be some higher associative structures (homotopy coherent).

In the definition of Ran_X, we note that if we replace X by any smooth manifold M, we will get some topology space Ran_M. For any symmetric monoidal category $\mathcal{C}$, we could consider the constructible sheaves on Ran_M with value in $\mathcal{C}$. In this category, we could define factorization algebras by the same definition in Section 2.3. We denote by $FactAlg(M, \mathcal{C})$ the resulted category. It is called the category of factorization algebras with value in $\mathcal{C}$.

Taking M to be $\mathbb{R}^n$, then, we arrive the definition of $\mathbb{E}_n$-algebras,

Definition 4.5.

$$\mathbb{E}_n - alg(\mathcal{C}) := FactAlg(\mathbb{R}^n, \mathcal{C})$$

Remark $FactAlg(M, \mathcal{C})$ equips with a symmetric monoidal category given by tensoring on each power of $\mathcal{M}$.

The following lemma is claimed in [Ro],

Lemma 4.4. *Given two manifolds M and N, then,*

$$FactAlg(M \times N, \mathcal{C}) \simeq FactAlg(M, FactAlg(N, \mathcal{C}))$$

From this lemma, we recover the classical definition of $\mathbb{E}_n$-algebra. It is true because we observe that if $n = 1$, then $\mathbb{E}_1 - alg(\mathcal{C}) = Alg(\mathcal{C})$; when $n \geq 2$, $\mathbb{E}_n - alg(\mathcal{C}) = FactAlg(\mathbb{A}^1, FactAlg(\mathbb{A}^{n-1}, \mathcal{C})) = Alg(E_{n-1} - Alg(\mathcal{C}))$.

$\mathbb{R}^n$ admits an $O(n)$-action, and this action induces an action of $O(n)$ on $\mathbb{E}_n - alg(\mathcal{C})$. We denote by $\mathbb{E}_n - alg(\mathcal{C})_{O(n)}$ the resulted quotient category.

It is known that factorization algebra can be described by its infinisimal restriction to the tangent space of each point.

Lemma 4.5. *[Ro] The category of factorization algebras on a n-dimensional manifold M is equivalent to the category of locally constant family of factorization algebras on the tangent spaces for all points of M.*

i.e, it is the category of lift

$$\begin{tikzcd} & \mathbb{E}_n - alg(\mathcal{C}) \arrow[d] \\ M \arrow[ur, dotted] \arrow[r] & \mathbb{B}O(n) \end{tikzcd}$$

The map from $\mathbb{E}_n - alg(\mathcal{C})$ to $\mathbb{B}O(n)$ is given by $O(n)$ action on $\mathbb{E}_n - alg(\mathcal{C})$.

In this paper, we will concentrate on the case when $n = 2$.

If M is the underlying topology space of a global curve, then, we have a functor:

$$\mathbb{E}_2 - alg(\mathcal{C})_{SO(2)} \longrightarrow FactAlg(M, \mathcal{C})$$

We study the case when $\mathcal{C} = DGCat$. In this case, the $SO(2)$-equivariant structure can be expressed by the notion of ribbon twist.

Definition 4.6. *Let $(\mathcal{C}, \bigotimes, b')$ be a braided monoidal category. A ribbon twist on $\mathcal{C}$ is a natural transformation of braided monoidal functors between T and the identity functor. Explicitly, it consists of an isomorphism:*

$$\theta_X : X \longrightarrow X$$

for each $X \in \mathcal{C}$ such that for every $X, Y \in \mathcal{C}$ the diagram

$$\begin{tikzcd} X \bigotimes Y \arrow[r, "b(X,Y)"] \arrow[d, "\theta_{X \bigotimes Y}"] & X \bigotimes Y \arrow[d, "\theta_X \bigotimes \theta_Y"] \\ X \bigotimes Y \arrow[r, "id"] & X \bigotimes Y \end{tikzcd}$$

commutes.

In particular, $Vect_q^\Lambda$ is a braided monoidal category with a ribbon twist. It is given by

$$\theta_{\Bbbk_i} : \Bbbk_i \xrightarrow{\times q(\alpha_i)} \Bbbk_i$$

It comes from the fact:

$$b(x, y)q(x)q(y) = q(x + y)$$

4.2.2 From Hopf algebra to $\mathbb{E}_2$-algebra

Given a Hopf algebra A in $Vect_q^\Lambda$, we could attach to it a factorization algebra on $Conf$. In this section, we will explain the procedure of getting a factorization algebra from a Hopf algebra.

In this section, we will further require that $A \in Vect_q^{\Lambda^{pos}}$, $\dim(A^0)$=1 and $\dim(A^\lambda) < \infty$, for all λ.

We have a naturally defined functor $triv_A$ which sends a vector space to $A - mod^{loc.nil}$ with the same underlying vector space and with the trivial A action:

$$triv_A : Vect_q^\Lambda \longrightarrow A - mod^{loc.nil} \tag{4.18}$$

By general categorical argument, $triv_A$ admits a right adjoint functor. By definition of $A - mod^{loc.nil}$, finite-dimensional representations are compact. As a result, the functor $triv_A$ sends compact objects to compact objects.

Note that we have such a lemma from the category theory.

Lemma 4.6. *Given a pair of adjoint functors $(F \dashv G)$ between compact generated categories. If F preserves compactness, then, G is continuous.*

Hence, we have that the right adjoint functor of $triv_A$ is continuous,

$$inv_A : A - mod^{loc.nil} \longrightarrow Vect_q^\Lambda \tag{4.19}$$

Under our assumption for A ($A^0 = \Bbbk$ and $dimA^\lambda < \infty$), finite-dimensional vector spaces generate finite-dimensional A module category under the functor $triv_A$. Hence, as a corollary of the following lemma, inv_A is conservative.

Lemma 4.7. *If the image of a left adjoint functor F generates its target category under colimits, then, the right adjoint functor of F is conservative.*

The following lemma from Theorem 3.4.5 [Lu3] is a higher categorical analog of the classical result of [BW] and [Bec],

Lemma 4.8. *(Barr-Beck-Lurie) A right adjoint functor is mondaic if:*
· *it is conservative.*
· *preserves colimits.*

Note that if a right adjoint functor is continuous, then, it satisfies the last condition.

In particular, inv_A is monadic.

According to the definition of monadicity, the monad $\mathcal{T} = inv_A \circ triv_A$ induces an equivalence of categories, i.e, the functor :

$$inv_A : A - mod^{loc.nil} \longrightarrow Vect_q^{\Lambda}$$

factors as the decomposition:

$$A - mod^{loc.nil} \xrightarrow{\sim} \mathcal{T} - mod(Vect_q^{\Lambda}) \longrightarrow Vect_q^{\Lambda}$$

and the first functor is an equivalence.

Note that the monad $\mathcal{T}$ commutes with right action of $Vect_q^{\Lambda}$ on itself, so it is given by an associative algebra, we denote by B the resulted associative algebra. A is a Hopf algebra, the coalgebra structure of A gives $A - mod$ a monoidal category structure. The above equivalence gives $B - mod$ a monoidal category structure.

And we note that $B = \mathcal{T}(\Bbbk_0)$, it is the unital object in $\mathcal{T} - mod$. In particular, B acquires an $\mathbb{E}_2$-algebra structure. In fact, B is $Kosz(A^{\vee})$, the Koszul dual of $A^{\vee}$.

If we regard $\Bbbk_0$ and $\Bbbk_\lambda$ (for definition, we refer to Section 3.2.1) as A-modules with the trivial action, then the λ-component of B is given by

$$Hom_{A-mod}(\Bbbk_0, \Bbbk_\lambda)$$

The following lemma is claimed in [GL1], and will be proved in L.Chen and C.Fu's upcoming paper.

Lemma 4.9. *(A generation of 4.36 in [Fra]) Given an $\mathbb{E}_2$-category $\mathcal{O}$ and an $\mathbb{E}_2$-algebra B inside, then, we have:*

$$Z_{Dr,\mathcal{O}}(B - mod) \simeq B - mod_{\mathbb{E}_2}(\mathcal{O}) \tag{4.20}$$

Apply the above lemma in our case, we get the following equivalence:

$$\begin{aligned}
Z_{Dr,Vect_q^{\Lambda}}(U_q^L(\check{N}) - mod^{loc.nil}) \simeq & Z_{Dr,Vect_q^{\Lambda}}(U_q^{KD}(\check{N}^-) - comod) \\
\simeq & Z_{Dr,Vect_q^{\Lambda}}(Kosz(U_q^{KD}(\check{N}^-)) - mod) \\
\simeq & Kosz(U_q^{KD}(\check{N}^-)) - mod_{\mathbb{E}_2}
\end{aligned} \tag{4.21}$$

4.2.3 From $\mathbb{E}_2$-algebra to factorization algebra

In this section, we will apply the content of Chapter 5 of [Lu1] to obtain a factorization algebra from an $\mathbb{E}_2$-algebra.

Recall that in Section 4.2.1, given an $O(2)$-equivariant $\mathbb{E}_2$-category (for example, a ribbon twist category), we could get a factorization category on any global curve X, i.e, we have a functor:

$$\begin{aligned} Fact : \mathbb{E}_2 - alg(DGCat)_{O(2)} &\longrightarrow FactAlg(X, DGCat) \\ \mathcal{O} &\longrightarrow Fact(\mathcal{O}) \end{aligned} \tag{4.22}$$

The following lemma is from [Ro]:

Lemma 4.10. *The above functor (4.22) sends $Vect_q^{\Lambda}$ to $D_{\mathcal{G}^T}(Gr_{T,Ran})^{r.h}$. Here, $D^{r.h}$ means the category of regular holonomic D-modules.*

Furthermore, it is claimed in [GL1] that any $\mathbb{E}_2$-algebra object B in $\mathcal{O}$ corresponds to a factorization algebra Ω_B in $Fact(\mathcal{O})$. What's more, there is an equivalence between the category of $\mathbb{E}_2$-modules of B and the category of the factorization modules of Ω_B.

In our case, $\mathcal{O} = Vect_q^{\Lambda}$ and $Fact(\mathcal{O}) = D_{\mathcal{G}^{\Lambda}}(Gr_{T,Ran})^{r.h}$. In Lemma 4.10, if we ask the $\mathbb{E}_2$-algebra B to be in $Vect_q^{\Lambda^{neg}-0}$, then we could regard Ω_B as a regular holonomic D-module supporting on the closed subscheme $Gr_{T,Ran}^{neg}$ ([GL1] 29.4.2). According to Lemma 2.1, we could regard Ω_B as a factorization algebra in the category $D_{\mathcal{G}^{\Lambda}}(Conf(X, \Lambda^{neg}))$.

Hence, we have the following equivalence:

$$B - mod_{\mathbb{E}_2} \simeq \Omega_B - FactMod \tag{4.23}$$

The relationship between B and Ω_B is that the !-fiber of Ω_B at $\lambda \cdot x$ is given by λ-component of B. The relationship between an $\mathbb{E}_2$ module M of B and the corresponding factorization modules of Ω_B is that the !-fiber of the factorization module at $\lambda \cdot x$ is given by λ-component of M.

From now on, when we talk about the factorization algebra $\Omega(A)$ corresponding to a Hopf algebra A, it means the factorization algebra corresponds to the corresponding $\mathbb{E}_2$-algebra of that Hopf algebra, i.e, $\Omega(A) := \Omega_B$ where $B = Kosz(A^{\vee})$.

Take $A = U_q^L(\check{N})$, then,

Definition 4.7. *we denote by Ω_q^L the factorization algebra $\Omega(U_q^L(\check{N}))$. And we denote by Ω_q^{KD} the factorization algebra $\Omega(U_q^{KD}(\check{N}))$.*

Combined (4.23) with (4.21), we have:

Proposition 4.4.

$$Rep_q^{mix}(\check{G}) = Z_{Dr,Vect_q^{\Lambda}}(U_q^L(\check{N}) - mod^{loc.nil}) \simeq \Omega_q^L - FactMod \tag{4.24}$$

4.3 Description of $\Omega(A)$ and $\Omega(A,M)$

In this section, we will study the corresponding Ω_q^L-modules of standards V_λ^{mix} and costandards $V_\lambda^{mix,\vee}$ of $Rep_q^{mix}(\check{G})$ under the equivalence in Proposition 4.4.

Given a Hopf algebra A and an object M in the relative Drinfeld center $Z_{Dr,Vect_q^\Lambda}(A-mod^{loc.nil})$. In this section, we will give an explicit formula for the factorization algebra $\Omega(A)$ and the corresponding factorization module $\Omega(A,M)$.

By the description below (4.23), given an A-module M, the !-fiber at $\mu\cdot x$ of the corresponding factorization module is given by:

$$Z_{Dr,Vect_q^\Lambda}(A-mod^{loc.nil}) \xrightarrow{oblv_A} A-mod^{loc.nil} \xrightarrow{inv_A} Vect_q^\Lambda \longrightarrow Vect_q^\mu \simeq Vect$$

The third functor is given by projection to the λ-component and the last equivalence is given by taking a (non-canonical) trivialization of $\mathcal{G}^\lambda$.

And we apply it in the case when $A=U_q^L(\check{N}^+)$

$$\begin{aligned} Z_{Dr,Vect_q^\Lambda}(U_q^L(\check{N})-mod^{loc.nil}) \xrightarrow{oblv_{U_q^L(\check{N})}} U_q^L(\check{N})-mod^{loc.nil} \xrightarrow{inv_{U_q^L(\check{N})}} \\ \xrightarrow{inv_{U_q^L(\check{N})}} Vect_q^\Lambda \longrightarrow Vect_q^\mu \simeq Vect \end{aligned}$$

In other words, given an object M in $Z_{Dr,Vect_q^\Lambda}(U_q^L-mod^{loc.nil})$, then, the !-fiber of $\Omega(U_q^L(\check{N}),M)$ at $\lambda\cdot x$ is given by

$$Hom_{U_q^L(\check{N})}(\Bbbk_0,M)^\lambda \tag{4.25}$$

By taking duality, we could get the $*$-fiber of $\Omega(U_q^L(\check{N}),M)$ at $\lambda\cdot x$ is given by:

$$(M\underset{U_q^L(\check{N})}{\otimes}\Bbbk)^\lambda \tag{4.26}$$

Remark Note that because all (twisted) D-modules above are holonomic, we have a well-defined $*$-fiber functor.

Remark In the rest of the paper, we denote by $\Delta_{\lambda,L}^{fact}$ the D-module $\Delta_{\lambda,U_q^L(\check{N})}$ defined in Definition 4.4 and by $\Delta_{\lambda,DK}^{fact}$ the D-module $\Delta_{\lambda,U_q^{KD}(\check{N})}$. Similarly for $\nabla_{\lambda,L}^{fact}$.

By description (4.26) and the fact that the underlying $U_q^{KD}(\check{N}^-)$-module structure of V_λ^{mix} is the Verma module corresponding to λ, we get that V_λ^{mix} corresponds to $\Delta_{\lambda,L}^{fact}$ under the equivalence in Proposition 4.4.

Similarly, $V_\lambda^{mix,\vee}$ corresponds to $\nabla_{\lambda,L}^{fact}$.

4.4 An explicit description of Ω_q^L

In this section, we will recall a geometric construction of the factorization algebras Ω_q^L and Ω_q^{KD} given in [Ga6].

We note that $Conf(X, \Lambda^{neg})$ has an open subset $\overset{\circ}{Conf}$ removing all diagonals. That is to say if we write

$$Conf = \bigsqcup_{\lambda \in \Lambda^{neg}-0} Conf^{\lambda} \simeq \bigsqcup_{\lambda \in \Lambda^{neg}-0, \lambda = \sum n_i(-\alpha_i)} \prod_i X^{(n_i)}$$

Then $\overset{\circ}{Conf}$ is the open subset defined by

$$\overset{\circ}{Conf} = \bigsqcup_{\lambda \in \Lambda^{neg}-0} \overset{\circ}{Conf^{\lambda}} \simeq \bigsqcup_{\lambda \in \Lambda^{neg}-0, \lambda = \sum n_i(-\alpha_i)} \Pi_i \overset{\circ}{X^{(n_i)}}$$

Here, $\overset{\circ}{X^{(n_i)}}$ denotes the open subset of $X^{(n_i)}$ erasing all diagonals.

Using the expression given in (4.1), the point $D \in \overset{\circ}{Conf}$ is of the form $D = \sum -\alpha_{i_k} x_k$ such that $x_i \neq x_j$ if $i \neq j$.

Under our requirement for the quadratic form q in Section 3.2.1, the following lemma comes from [GL2]

Lemma 4.11. *$\mathcal{G}^{\Lambda}|_{\overset{\circ}{Conf}}$ is canonically trivialized.*

Under the above trivialization, we have:

$$D_{\mathcal{G}^{\Lambda}}(\overset{\circ}{Conf}) \simeq D(\overset{\circ}{Conf})$$

Given $n \in \mathbb{N}$, we could define the sign local system on $\overset{\circ}{X^{(n)}}$. To be more precise, we have a S_n-étale cover of $\overset{\circ}{X^{(n)}}$ given by $\overset{\circ}{X^n}$. S_n has an order 2 normal subgroup A_n. So, we could get a 2-cover of $\overset{\circ}{X^{(n)}}$. Consider the constant perverse D-module on this cover and $*$-push-out it to $\overset{\circ}{X^{(n)}}$. Then, it can be decomposed as the sum of constant D- module and another D-module. The latter is called the sign D- module.

Under the equivalence between $D_{\mathcal{G}^{\Lambda}}(\overset{\circ}{Conf})$ and $D(\overset{\circ}{Conf})$, the sign D-module gives rise to a twisted D-module on $\overset{\circ}{Conf}$, we denote it by $\overset{\circ}{\Omega}_{\mathcal{G}^{\Lambda}}$. It is known that this D-module only depends on the quadratic form q, hence, we could also denote $\overset{\circ}{\Omega}_{\mathcal{G}^{\Lambda}}$ by $\overset{\circ}{\Omega}_q$.

We can construct a factorization algebra Ω_q^L from $\overset{\circ}{\Omega}_q$ using the factorization property of Ω_q^L. Given $\lambda \in \Lambda^{neg} - 0$, assume that we have already known the restriction of Ω_q^L on $Conf^{\lambda'}$ for any $\lambda' > \lambda$. On $Conf^{\lambda}$, by the factorization property of Ω_q^L, we have already known the restriction of Ω_q^L on the complement of the main diagonal:

$$X \hookrightarrow Conf^{\lambda}$$

We denote the open embedding from the complement $Conf^{\lambda}\backslash X$ to $Conf^{\lambda}$ by $j_{\lambda,big}$

$$j_{\lambda,big} : Conf^{\lambda}\backslash X \longrightarrow Conf^{\lambda} \tag{4.27}$$

We could describe Ω_q^L inductively as follows (2.3 [Ga6]):

Proposition 4.5. *i). If $\lambda = w(\rho) - \rho$ and $l(w) = 2$, then,*

$$j_{\lambda,big,!} \circ j^!_{\lambda,big}(\Omega_q^L) \xrightarrow{\sim} \Omega_q^L$$

ii). If $\lambda = w(\rho) - \rho$ and $l(w) \leq 3$,

$$H^0(j_{\lambda,big,!} \circ j^!_{\lambda,big}(\Omega_q^L)) \simeq \Omega_q^L$$

iii). If λ is not of the form $w(\rho) - \rho$, then,

$$\Omega_q^L \simeq j_{\lambda,big,!*} \circ j^!_{\lambda,big}(\Omega_q^L))$$

4.5 Restatement of Theorem 3.1

This is a restatement of the main theorem of our paper,

Theorem 4.1. *When q avoids small torsion, there is a functor F^L which establishes a t-exact equivalence:*

$$F^L : Whit_q(Fl_x) \xrightarrow{\sim} \Omega_q^L - FactMod$$

and preserves standard objects and costandard objects.

From now on, we will prove this theorem instead of the original expression.

5 Whittaker category: t-structure and duality

In this section, we will give a more detailed description of the local Whittaker category on Fl_G. The principal goal of this section is to give the definitions of standard objects in $Whit_q(Fl_G)$ and the duality functor. The dual of the standard object will play an important role in the proof of our main theorem.

The content of this section:

In Section 5.1, we will study the $N(K)$-orbits of Fl_G and determine which orbits could admit a non-zero Whittaker D-modules supporting on it.

In Section 5.2, we will give the definition of metaplectic BMW D-modules $\mathfrak{J}_\lambda$ for dominant coweights. To achieve the definition of metaplectic BMW D-modules $\mathfrak{J}_\lambda$, we need to input another twisting on Fl_G from right, hence, the notations in this section will be a little bit complicated (especially in Section 5.2.2).

In Section 5.3, we will define the convolution product for metaplectic D-modules and prove that our metaplectic BMW D-modules behave similarly as non-twisting BMW D-modules.

In Section 5.4, we will extend the definition of $\mathfrak{J}_\lambda$ to any coweight λ using the convolution product given in the previous section.

In Section 5.5, finally, we could give the definition of the standards $\Delta_\lambda \in Whit_q(Fl_G)$ using the definition of $\mathfrak{J}_\lambda$. Then, we could also define costandards.

In Section 5.6, we will prove that the standard objects compactly generate $Whit_q(Fl)$.

In Section 5.7, we will study the t-structure of $Whit_q(Fl_G)$ given by Δ_λ and prove that our standards and costandards belong to the heart.

In Section 5.8, we will study the Whittaker category defined by coinvariant. Through this definition, we could define the Verdier duality for compact objects in $Whit_q(Fl)$.

In Section 5.9, we will use the duality functor to prove that costandards are compact when q is irrational which are not compact when q is rational and degenerated.

5.1 Relevant orbits

Note that $N(K)$ is a unipotent group, hence, the category of Whittaker D-modules on each orbit is equivalent to $Vect$ or 0. To study $Whit_q(Fl_G)$, we should at first study when a $N(K)$-orbit admits non-zero Whittaker D-modules supporting onside.

It is known that the $N(K)$-orbits of Fl_G are indexed by the extended Weyl group W^{ext}. Here, $W^{ext} \simeq W \ltimes \Lambda$. But not all orbits admit non-zero Whittaker objects supporting on them. W^{ext} admits a map to $G(K)$, we will denote the corresponding element in $G(K)$ by $\widetilde{w}$ as well.

Definition 5.1. *Given any $\widetilde{w} \in W^{ext}$, we denote by $S^{\widetilde{w}}_{Fl}$ the $N(K)$-orbit passing through $\widetilde{w} \cdot I/I$.*

And we denote by $\bar{S}^{\widetilde{w}}_{Fl}$ the closure of $S^{\widetilde{w}}_{Fl}$ in Fl_G.

Definition 5.2. *Given two elements $\widetilde{w},\ \widetilde{w}' \in W^{ext}$, we denote by*

$$\widetilde{w} \leq_N \widetilde{w}' \tag{5.1}$$

the condition: $S^{\widetilde{w}}_{Fl} \subset \bar{S}^{\widetilde{w}'}_{Fl}$

Remark By [Zhou], [La] and [FFKM], the partially defined order $\leq_N$ is given by *semi-infinite Bruhat order.*

Definition 5.3. *We call such orbits admitting non-zero Whittaker D-modules supporting on them to be relevant orbits. And if $N(K)\widetilde{w} \cdot I/I$ is a relevant $N(K)$-orbit, then we say $\widetilde{w} \in W^{ext}$ is relevant.*

It is easy to see that the necessary and sufficient condition for a $N(K)$-orbit $N(K)\widetilde{w} \cdot I/I \subset Fl$ to be relevant is:

$$Stab_{N(K)}(\widetilde{w}I/I) \subset Ker(\chi) \tag{5.2}$$

Definition 5.4. *Given an element* $\widetilde{w} \in W^{ext}$*, we denote by* $l(\widetilde{w})$ *the dimension of its Iwahori orbit in the affine flag. (or the length of* $\widetilde{w}$*)*

If $\tilde{w} = t^{\lambda}w$ *then, we have*

$$l(\tilde{w}) = \sum_{\check{\alpha}\in\check{\Delta}^+, w^{-1}(\check{\alpha})>0} |<\check{\alpha},\lambda>| + \sum_{\check{\alpha}\in\check{\Delta}^+, w^{-1}(\check{\alpha})<0} |<\check{\alpha},\lambda>+1| \tag{5.3}$$

Proposition 5.1. *A* $N(K)$*-orbit* $N(K)\widetilde{w}\cdot I/I \subset Fl$ *is revelent if and only if* $\widetilde{w}$ *can be written as the form:* $t^{-\rho}\ {}^{l}(t^{\mu})$ *for some* $t^{\mu} \in W^{ext}$*, here,* ${}^{l}(\widetilde{w}')$ *means the unique maximal length element in* $W\widetilde{w}' \subset W^{ext}$.

Proof. Denote by $\widetilde{\check{\Delta}}^{+}$ the sets of the roots corresponding to the Iwahori subgroup I. $\check{\Pi}^{+}$ denotes the set of positive simple roots of G. $\check{\Pi}^{-}$ denotes the set of negative simple roots G. r_i denotes the simple reflection in W given by the simple root $\check{\alpha}_i$ of G and δ is the positive imaginary root generator.

$$\begin{aligned}
&Stab_{N(K)}(\widetilde{w}\cdot I/I) = \widetilde{w}I\widetilde{w}^{-1}\cap N(K) \subset Ker(\chi)\\
&\Leftrightarrow \widetilde{w}(\widetilde{\check{\Delta}}^{+}) \cap \{\check{\Pi}^{+} - \delta\} = \emptyset\\
&\Leftrightarrow \widetilde{w}(\widetilde{\check{\Delta}}^{+}) \cap t^{-\rho}\{\check{\Pi}^{+}\} = \emptyset\\
&\Leftrightarrow t^{\rho}\widetilde{w}(\widetilde{\check{\Delta}}^{+}) \cap \{\check{\Pi}^{+}\} = \emptyset\\
&\Leftrightarrow \{\check{\Pi}^{-}\} \subset t^{\rho}\widetilde{w}(\widetilde{\check{\Delta}}^{+})\\
&\Leftrightarrow \widetilde{w}^{-1}t^{-\rho}(\check{\Delta}^{+}) \leq 0\\
&\Leftrightarrow l(\widetilde{w}^{-1}t^{-\rho}r_i) \leq l(\widetilde{w}^{-1}t^{-\rho}), \forall i \in \mathcal{J}\\
&\Leftrightarrow l(r_i t^{\rho}\widetilde{w}) \leq l(t^{\rho}\widetilde{w}), \forall i \in \mathcal{J}
\end{aligned} \tag{5.4}$$

So, by [Kac] we have $\widetilde{w} = t^{-\rho}\ {}^{l}(\widetilde{w}')$ for some element $\widetilde{w}'$. □

Using the length function (5.3), we could decode the relevant condition in a more explicit way. Through the proof of the proposition above, we could see if we write $\widetilde{w}$ as $t^{\lambda}w$, $t^{\lambda} \in \Lambda, w \in W$, then we have the following description of the relevant condition:

Corollary 5.1. *(i). If* $w^{-1}(\check{\alpha}_i) \in \check{\Delta}^{+}$*, then,* $(\lambda, \check{\alpha}_i) \geq 0$

(ii). If $w^{-1}(\check{\alpha}_i) \in \check{\Delta}^{-}$*, then,* $(\lambda, \check{\alpha}_i) \geq -1$

Here, $(-,-) : \Lambda \times \check{\Lambda} \longrightarrow \Bbbk$ *is the natural pairing.*

Note that the above corollary has an equivalent expression: given $\check{\alpha} \in \check{\Delta}^{+}$, then,

$$\begin{aligned}
(i).w^{-1}(\check{\alpha}) \in \check{\Delta}^{+} \Rightarrow (\lambda+\rho, \check{\alpha}) \geq 1\\
(ii).w^{-1}(\check{\alpha}) \in \check{\Delta}^{-} \Rightarrow (\lambda+\rho, \check{\alpha}) \geq 0
\end{aligned} \tag{5.5}$$

Remark In the above description, we allow the test object $\check{\alpha}$ to be in $\check{\Delta}^{+}$ instead of just in $\check{\prod}^{+}$. In some cases, the second description will be more convenient for us.

Definition 5.5. *Let $Whit_q(S^{\widetilde{w}}_{Fl})$ denote the category of $Whittaker$ D-modules on $S^{\widetilde{w}}_{Fl}$.*

Notice $N(K)$ is ind-pro unipotent, we have:

Proposition 5.2.
$$Whit_q(S^{\tilde{w}}_{Fl}) \simeq Vect$$
if $\tilde{w}$ is relevant and
$$Whit_q(S^{\tilde{w}}_{Fl}) \simeq 0$$
otherwise.

Given any relevant $N(K)$-orbit $S^{\widetilde{w}}_{Fl} \subset Fl$, we define:

$$\bar{j}_{\widetilde{w},Fl} : S^{\widetilde{w}}_{Fl} \longrightarrow \bar{S}^{\widetilde{w}}_{Fl} \tag{5.6}$$

to be the corresponding open embedding.

Then, we could define direct image functors and pullback functors along with this morphism:

$$\bar{j}_{\widetilde{w},Fl,!} : Whit_q(S^{\widetilde{w}}_{Fl}) \rightleftharpoons Whit_q(\bar{S}^{\widetilde{w}}_{Fl}) : \bar{j}^{!}_{\widetilde{w},Fl} \tag{5.7}$$

$$\bar{j}^{*}_{\widetilde{w},Fl} : Whit_q(\bar{S}^{\widetilde{w}}_{Fl}) \rightleftharpoons Whit_q(S^{\widetilde{w}}_{Fl}) : \bar{j}_{\widetilde{w},Fl,*} \tag{5.8}$$

$\bar{j}_{\widetilde{w},Fl,!}$ is well- defined because $Whittaker$ D-modules are ind-holonomic. By the same reason, $*$-pullback of Whittaker D-module is also well-defined.

Let us denote
$$j_{\widetilde{w},Fl} : S^{\widetilde{w}}_{Fl} \longrightarrow Fl \tag{5.9}$$

$$j_{\widetilde{w},Fl,!} : Whit_q(S^{\widetilde{w}}_{Fl}) \rightleftharpoons Whit_q(Fl) : j^{!}_{\widetilde{w},Fl,} \tag{5.10}$$

$$j^{*}_{\widetilde{w},Fl} : Whit_q(Fl) \rightleftharpoons Whit_q(S^{\widetilde{w}}_{Fl}) : j_{\widetilde{w},Fl,*} \tag{5.11}$$

And
$$i_{\widetilde{w},Fl} : \bar{S}^{\widetilde{w}}_{Fl} \longrightarrow Fl \tag{5.12}$$

$$i_{\widetilde{w},Fl,!} : Whit_q(\bar{S}^{\widetilde{w}}_{Fl}) \rightleftharpoons Whit_q(Fl) : i^{!}_{\widetilde{w},Fl} \tag{5.13}$$

$$i^{*}_{\widetilde{w},Fl} : Whit_q(Fl) \rightleftharpoons Whit_q(\bar{S}^{\widetilde{w}}_{Fl}) : i_{\widetilde{w},Fl,*} \tag{5.14}$$

We could give a highest weight structure to $Whit_q(Fl)$ with the !-extension D-modules and $*$-extension D-modules from relevant orbits.

Definition 5.6. *Assume that $\widetilde{w}$ is relevant, then, we define:*

$$\Delta^{ver}_{\widetilde{w}} := Av_!^{N(K),\chi}(\delta_{\widetilde{w}})[-l(w_0 t^{\rho}\widetilde{w}) + l(t^{-\rho}w_0)]$$

$$\nabla^{ver}_{\widetilde{w}} := \bar{j}_{\widetilde{w},Fl,*} \circ \bar{j}^{!}_{\widetilde{w},Fl}(\Delta^{ver}_{\widetilde{w}})$$

Remark The cohomological shift here is to let $\Delta^{ver}_{\widetilde{w}}$ and $\nabla^{ver}_{\widetilde{w}}$ match with their global corresponding objects defined in Section 7.2.

By definition, we have the following property of $\Delta^{ver}_{\widetilde{w}}$ and $\nabla^{ver}_{\widetilde{w}}$:

Proposition 5.3. *i). $\Delta^{ver}_{\widetilde{w}}$ (resp. $\nabla^{ver}_{\widetilde{w}}$) compactly generated $Whit_q(Fl)$*

ii).

$$Hom_{Whit_q}(\Delta^{ver}_{\widetilde{w}}, \nabla^{ver}_{\widetilde{w}'}[k]) = 0$$

unless $\widetilde{w} = \widetilde{w}'$ and $k = 0$. And in this case,

$$Hom_{Whit_q}(\Delta^{ver}_{\widetilde{w}}, \nabla^{ver}_{\widetilde{w}}) = \Bbbk$$

Hence, we could describe the compact objects in $Whit_q(Fl)$ very explicitly.

Proposition 5.4. *An object $\mathcal{F} \in Whit_q(Fl)$ is compact if and only if $\mathcal{F}$ is a compact object in $Whit_q(\bar{S}^{\widetilde{w}}_{Fl})$ for some relevant orbit $N(K)\widetilde{w} \cdot I/I \subset Fl$.*

Proof. We note that the direct image along with a closed embedding preserves compact objects. Hence, we only need to prove any compact object $\mathcal{F}$ supports on $\bar{S}^{\widetilde{w}}_{Fl}$ for some $\widetilde{w} \in W^{ext}$.

Because $\{\Delta^{ver}_{\widetilde{w}}, \forall \widetilde{w}\ relevant\}$ are compact generators of $Whit_q(Fl)$ and in a compact generated category, compact objects are finite extensions of compact generators. As a result, any compact object supports on some $\bar{S}^{\widetilde{w}}_{Fl}$. □

Definition 5.7. *We define a t-structure of $Whit_q(Fl^{\omega^\rho}_x)$ as follows:*

- *A twisted Whittaker D-module $\mathcal{F}$ on $Fl^{\omega^\rho}_x$ is coconective if for any relevant λw, we have*

$$Hom_{Whit_q(Fl^{\omega^\rho}_x)}(\Delta^{ver}_{\lambda w}[k], \mathcal{F}) = 0, \forall k > 0$$

We denote the heart of this t-structure by $Whit_q(Fl)^{\heartsuit'}$ (we use $'$ here because we will define another t-structure which is more important for us and we leave our heart $\heartsuit$ for this one.)

Given a t-structure of $Whit_q(Fl)$ as above, then, we could prove :

Proposition 5.5. *i).$\Delta^{ver}_{\widetilde{w}}, \nabla^{ver}_{\widetilde{w}} \in Whit_q(Fl)^{\heartsuit'}$*

ii). Irreducible objects of $Whit_q(Fl)^{\heartsuit'}$ are given by

$$L^{ver}_{\widetilde{w}} := j_{\widetilde{w},Fl,!*} \circ j^!_{\widetilde{w},Fl}(\Delta^{ver}_{\widetilde{w}})$$

iii). $L^{ver}_{\widetilde{w}}$ is compact.

Proof. The first claim will be proved in Section 8.2 after introducing the global definition of the Whittaker category.

ii). Assume that $\mathcal{F} \in Whit_q(Fl)^{\heartsuit'}$ is irreducible, then, there exists a relevant extended Weyl group element, λw, such that there is a non-zero morphism $\Delta^{ver}_{\lambda w} \longrightarrow \mathcal{F}$, hence it is surjective by the irreducibility of $\mathcal{F}$. Similar, we could find a relevant extended Weyl group element $t^{\lambda'} w'$, such that there is a non-zero morphism $\mathcal{F} \longrightarrow \nabla^{ver}_{\lambda' w'}$, and it is injective also by the irreducibility of $\mathcal{F}$. Hence, the composition

$$\Delta^{ver}_{\lambda w} \longrightarrow \mathcal{F} \longrightarrow \nabla^{ver}_{\lambda' w'} \tag{5.15}$$

is non-zero. So, by Proposition 5.3 ii). $\lambda = \lambda'$ and $w = w'$. And $j_{t^\lambda w, Fl, !*} \circ j^!_{t^\lambda w, Fl}(\Delta^{ver}_{t^\lambda w})$ is the unique D-module satisfying the requirement for $\mathcal{F}$ in (5.15).

$L^{ver}_{\widetilde{w}}$ is irreducible: if not, it admits an irreducible submodule of the form $L^{ver}_{\widetilde{w}'}$. Hence, there is a non-zero map:

$$\Delta^{ver}_{\widetilde{w}'} \longrightarrow L^{ver}_{\widetilde{w}}$$

it implies $\widetilde{w} = \widetilde{w}'$.

iii). Because $L^{ver}_{\widetilde{w}}$ could be obtained by finite extension of $\Delta^{ver}_{\lambda w}$. □

Regarding $\Delta^{ver}_{\widetilde{w}}$ as standard objects and $\nabla^{ver}_{\widetilde{w}}$ as costandard objects, we can prove that it gives $Whit_q(Fl)$ a highest weight category structure. But the problem is that it is difficult to define a functor from $Whit_q(Fl)$ to $\Omega^L_q - FactMod$, such that, the standard objects, and costandard objects of both sides match under the given functor.

We need to define another highest weight category structure of $\Omega^L_q - FactMod$, such that standard objects(resp. costandard objects) go to the standard objects (resp. costandard objects) in $\Omega^L_q - FactMod$. The work of [AB] tells us that we should use the Whittaker object constructed from BMW D-modules.

5.2 Right monodromic D-modules

[AB] indicates us that we need to define the metaplectic BMW D-module $\mathfrak{J}_\lambda$ and use them to define our standard objects $\Delta_\lambda = Av_!^{N(K),\chi}(\mathfrak{J}_\lambda)$. So, how to define $\mathfrak{J}_\lambda$?

The naive attempt fails, the objects in $D_{\mathcal{G}^G}(Fl)^I$ are too few. That is to say, if we still consider the category $D_{\mathcal{G}^G}(Fl)^I$, and define our standard objects in $Whit_q(Fl)$ by applying the !-average functor to some distinguished objects in $D_{\mathcal{G}^G}(Fl)^I$, then, we cannot get many objects. The reason is that the trivialization of the twisting $\mathcal{G}^G$ on each Iwarohi orbit is not necessarily Iwahori equivariant! Instead, it is equivariant with respect to certain character D-module on I. For example, when q is irrational, on the I-orbit $I \cdot t^\lambda w \cdot I/I \subset Fl$, there is no I-equivariant D-module on this orbit unless $\lambda = 0$. Because of the twisting, we will have 'fewer objects' in $D_{\mathcal{G}^G}(Fl)^I$. The solution to fix this problem is to consider pro-p-Iwahori, I^0, equivariant D-modules, instead of considering Iwahori equivariant D-modules.

We could consider metaplectic Iwahori equivariant D-modules as I^0-equivalent D-modules on $G(K)/I^0$ which are monodromic with respect to left T-action and a certain character and also with respect to right T-action and some character.

Sections 5.2-5.4 are devoted to constructing the metaplectic BMW D-modules.

Let us, first of all, recall the classical BMW D-modules. They are a collection of Iwahori equivariant D-modules $\{\mathfrak{J}_\lambda\}$ indexed by Λ with the following property:

1. the convolution product gives a monoidal structure:
$$\mathfrak{J}_\eta \star \mathfrak{J}_\lambda \simeq \mathfrak{J}_{\eta+\lambda} \tag{5.16}$$
2. when λ is dominant, $\mathfrak{J}_\lambda$ should be isomorphic to the !-extension of its restriction to the Iwahori orbit $I \cdot \lambda I/I$ and when $-\lambda$ is dominant, $\mathfrak{J}_\lambda$ should be isomorphic to the $*$-extension of its restriction to the Iwahori orbit $I \cdot \lambda \cdot I/I$

Let us explain the organization of Section 5.2-5.4. First of all, in Section 5.2, we will introduce the equivariant D-modules that we will use. This section contains two parts:

1. $\mathcal{G}^G$-twisted I^0-equivariant D-module on Fl_G,
2. $\mathcal{G}^G \otimes \mathcal{G}^\alpha$-twisted I^0-equivariant and right monodromic D-module on $\widetilde{Fl} := G(K)/I^0$. Here, $\mathcal{G}^\alpha$ denotes the restriction of $\mathcal{G}^G$ at t^α and I^0 is the pro-p unipotent radical of I.

The second part is very important for us to form a meaningful convolution product.

In this section, we could also give the definition of the metaplectic BMW D-module $\mathfrak{J}_\lambda$ (resp. their variations ${}_\alpha(\mathfrak{J}_\lambda)_\mu$) for λ dominant.

Then, in Section 5.3, we will define a convolution product functor for the equivariant D-modules defined in Section 5.2.

Finally, in Section 5.4, we will construct the metaplectic BMW D-modules.

5.2.1 $\mathcal{G}^G$-twisted D-module on Fl

Assume $\tilde{w} = t^\lambda w, \lambda \in \Lambda, w \in W$. Denote by $Fl^{\widetilde{w}}$ the Iwahori orbit of Fl which contains $\widetilde{w}$. It is an affine space and hence contractible. Because I^0 is pro-unipotent, there is a unique (up to a non-canonical isomorphism) I^0-equivariant trivialization of the gerbe $\mathcal{G}^G$ on $Fl^{\widetilde{w}} \subset Fl$.

Hence, under this I^0-equivariant trivialization of the gerbe $\mathcal{G}^G$ on $Fl^{\widetilde{w}}$, we could regard the category $D_{\mathcal{G}^G}(Fl^{\widetilde{w}})^{I^0}$ as $D(Fl^{\widetilde{w}})^{I^0}$. By taking !-fiber at $\widetilde{w} \in Fl$, this category is (non-canonically) equivalent to $Vect$. We may take the constant local system on $Fl_G^{\widetilde{w}}$ and we denote it by $c_{\widetilde{w}}$. It is a generator of $D_{\mathcal{G}^G}(Fl^{\widetilde{w}})^{I^0} \simeq D(Fl^{\widetilde{w}})^{I^0}$.

Let
$$\mathfrak{J}_{\tilde{w}!} : D_{\mathcal{G}^G}(Fl^{\tilde{w}}) \longrightarrow D_{\mathcal{G}^G}(Fl)$$
denote the !-direct image functor of twisted D-module categories.

Definition 5.8. *We define $\mathfrak{J}_{\tilde{w}!}$ to be $\mathfrak{J}_{\tilde{w}!} = \mathfrak{J}_{\tilde{w}!}(c_{\tilde{w}})[l(\widetilde{w})]$.*

Remark Because $c_{\widetilde{w}}$ is an (ind-) holonomic D-module, !-direct image functor can be defined on such an object.

Similarly,

Definition 5.9. *We define* $\mathfrak{J}_{\tilde{w}*} = \mathfrak{J}_{\tilde{w}*}(c_{\tilde{w}})[l(\widetilde{w})]$.

Here, $\mathfrak{J}_{\tilde{w}*} : D_{\mathcal{G}^G}(Fl^{\tilde{w}}) \to D_{\mathcal{G}^G}(Fl)$ is the $*$- direct image functor of the twisted D-module category.

Remark The objects $\mathfrak{J}_{\tilde{w}*}$ and $\mathfrak{J}_{\tilde{w}!}$ that we constructed above are I^0-equivariant D-modules, but they are not necessary I-equivariant. In fact, $c_{\tilde{w}} \in D_{\mathcal{G}^G}(Fl^{\tilde{w}})^I$ if and only if $b(\lambda, \mu) = 0$ for any $\mu \in \Lambda$.

By our philosophy of BMW D-module, we expect that there will be a convolution product for $\mathfrak{J}_{\lambda,!}$ or $\mathfrak{J}_{\lambda,*}$ like (5.16).

But now, we meet a problem: there is no interesting monoidal category structure for $D_{\mathcal{G}^G}(Fl)^{I^0}$, i.e, there is no convolution product of a right I-equivariant D-module and a left I^0-equivariant D-module if the later is not left T-equivariant!

5.2.2 $\mathcal{G}^G \otimes \mathcal{G}^\alpha$-twisted right monodromic D-module on $G(K)/I^0$

In order to define a good convolution product, we need to modify a little here by adding some twisting on the right. The author thanks D.Gaitsgory again for sharing generously this idea.

As we said before, the D-modules that we constructed above, $c_{\tilde{w}}, \mathfrak{J}_{\tilde{w},!}, \mathfrak{J}_{\tilde{w},*}$ (here, $\widetilde{w} = t^\lambda w$), they are I^0-equivariant but not T-equivariant. But we notice that the I^0-equivariant trivialization of the gerbe $\mathcal{G}^G$ on $Fl^{\widetilde{w}}$ is T-monodromic with respect to the character $b_\lambda := b(\lambda, -) : \Lambda \to \Bbbk/\mathbb{Z}$, so all of the objects listed above are T-equivariant with respect to the Kummer D-module b_λ.

The exact sequence:

$$1 \to I^0 \to I \to T \to 1$$

is split, hence we may consider the right action of T on $\widetilde{Fl} := G(K)/I^0$. In particular, we can define the notion of D-module on $\widetilde{Fl}$ which is right monodromic with respect to a character of T.

In the function case, given two I^0-equivariant functions f_1, f_2 on $G(K)/I^0$, if f_1 is right (T, b)-equivariant and f_2 is left (T, b)-equivariant, then, we could define a function $f_1 \star f_2$ on $G(K)/I^0$ by

$$(f_1 \star f_2)(g) = \int_{x \in G(K)/I} f_1(x) f_2(x^{-1}g) dx$$

Definition 5.10. *Let* $D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha}(\widetilde{Fl})^{I,\lambda}_{\mu}$ *be the category of* $\mathcal{G}^G \bigotimes \mathcal{G}^\alpha$*-twisted left* (I, b_λ)*-equivariant and right* (T, b_μ)*-equivariant D-modules on* $\widetilde{Fl}$.

If $\widetilde{w} = t^\lambda w \in W^{ext}$, we note that there is a right T-equivariant projection

$$f_{\widetilde{w}} : \widetilde{Fl}^{\widetilde{w}} := I\widetilde{w}I/I_0 \longrightarrow T \tag{5.17}$$

$i_1 t^\lambda w i_2 \longrightarrow w^{-1} t_1 w t_2$, *here* $i_1 = i_{1,0}t_1, i_2 = i_{2,0}t_2, i_{1,0}, i_{2,0} \in I_0, t_1, t_2 \in T$

It is easy to check that the definition of $f_{\widetilde{w}}$ is independent of choices of i_1 and i_2.

Given the Kummer D-module b_μ on T, we consider its !-pullback along with the map $f_{\widetilde{w}}$. It is a right (T, b_μ)-equivariantand and left $(T, b_{w(\mu)})$-equivariant D-module on $\widetilde{Fl}^{\widetilde{w}}$. We denote it by $(c_{\widetilde{w}})_\mu \in D(\widetilde{Fl}^{\widetilde{w}})$.

We consider the I^0-equivariant trivialization of $\mathcal{G}^G$ on $Fl^{\tilde{w}} \subset Fl$ and pullback it to a trivialization of $\mathcal{G}^G$ on $\widetilde{Fl}^{\widetilde{w}}$, it is of course also I^0-equivariant. What's more, this trivialization is left (T, b_μ)-equivariant and right T-equivariant (because it is a pullback from Fl).

Under this trivialization, we could identify $D_{\mathcal{G}^G}(\widetilde{Fl}^{\widetilde{w}})$ with $D(\widetilde{Fl}^{\widetilde{w}})$. In particular, we could regard $(c_{\widetilde{w}})_\mu$ as an object in $D_{\mathcal{G}^G}(\widetilde{Fl}^{\widetilde{w}})$. And after taking into account the T-monodromy of the above I^0-equivariant trivialization of $\mathcal{G}^G$, $(c_{\tilde{w}})_\mu$ is a right (T, b_μ)-equivariant and left $(T, b_{\lambda+w(\mu)})$-equivariant D-module as a twisted D-module in $D_{\mathcal{G}^G}(\widetilde{Fl})$.

Remark Note that by definition, $(c_{\widetilde{w}})_\mu$ actually belongs to $D_{\mathcal{G}^G}(\widetilde{Fl})_\mu$.

Definition 5.11. *Let $(\mathfrak{J}_{\tilde{w},!})_\mu (resp, (\mathfrak{J}_{\tilde{w},*})_\mu)$ be the ! (resp, *)-extension of the perverse D-module $(c_{\tilde{w}})_\mu[l(\widetilde{w})]$ with respect to the morphism:*

$$\mathfrak{J}_{\widetilde{w}} : \widetilde{Fl}^{\tilde{w}} \longrightarrow \widetilde{Fl}$$

Similarly, we could define ${}_\alpha(\mathfrak{J}_{\widetilde{w},!})_\mu$ and ${}_\alpha(\mathfrak{J}_{\widetilde{w},*})_\mu \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha}(\widetilde{Fl})_\mu$.

From now on, if the symbol ${}_\alpha(?)_\mu$ appears in this paper, we should keep in mind that it indicates that this object is a $\mathcal{G}^G \bigotimes \mathcal{G}^\alpha$-twisted right (T, b_μ)-equivariant D-module on $\widetilde{Fl}$.

We need to note here that if we fix some trivialization of $\mathcal{G}^\alpha$, then, we will have an equivalence between $\mathcal{G}^G \bigotimes \mathcal{G}^\alpha$-twisted D-module category and $\mathcal{G}^G$-twisted D-module category. In particular, we could regard ${}_\alpha(?)_\mu$ as some $\mathcal{G}^G$-twisted D-module. And we could define the convolution product of these twisted BMW D-modules (with $\mathcal{G}^\alpha$ twisting) when take trivializations and regard them as $\mathcal{G}^G$-twisted D-modules. (no $\mathcal{G}^\alpha$ twisting)

5.3 Convolution product

The convolution product is given similar to the non-twisted case. Consider the following diagram:

$$\begin{tikzcd}
 & G(K) \times G(K)/I^0 \arrow[dl, "\pi_1"'] \arrow[d, "\pi"] \arrow[dr, "\pi_2"] & \\
G(K)/I^0 & G(K) \overset{I}{\times} G(K)/I^0 \arrow[d, "m"] & G(K)/I^0 \\
 & G(K)/I^0 &
\end{tikzcd} \tag{5.18}$$

Denote by $\tilde{\pi}$ the natural projection from $G(K)$ to $\widetilde{Fl}$,

$$\tilde{\pi} : G(K) \to G(K)/I_0 \tag{5.19}$$

Given $\mathcal{F}_1 \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha}(\widetilde{Fl})_\lambda$, we have that $\tilde{\pi}^!(\mathcal{F}_1)$ is a right (I, b_λ)-equivariant $\mathcal{G}^G \bigotimes \mathcal{G}^\alpha$-twisted D-module on $G(K)$. And we take $\mathcal{F}_2 \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\beta}(\widetilde{Fl})^{I,\lambda}_\eta$. Then, we consider the following diagram,

$$\ldots \rightrightarrows G(K) \times I \times G(K)/I_0 \rightrightarrows G(K) \times G(K)/I_0 \xrightarrow{m} G(K) \overset{I}{\times} G(K)/I_0 \tag{5.20}$$

The equivariant condition of $\mathcal{F}_1$ and $\mathcal{F}_2$ translates to the descent condition. So the external product $\tilde{\pi}^!(\mathcal{F}_1) \boxtimes \mathcal{F}_2$ can descend to a twisted D module $q^!(\mathcal{F}_1) \widetilde{\boxtimes} \mathcal{F}_2 \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha \times \mathcal{G}^G \bigotimes \mathcal{G}^\beta}(G(K) \overset{I}{\times} G(K)/I)$, s.t, $\pi^!(\tilde{\pi}^!(\mathcal{F}_1) \widetilde{\boxtimes} \mathcal{F}_2) \simeq \tilde{\pi}^!(\mathcal{F}_1) \boxtimes \mathcal{F}_2$.

Definition 5.12. *We define*

$$\mathcal{F}_1 \underset{I,\lambda}{\star} \mathcal{F}_2 := m_*((\tilde{\pi}^!(\mathcal{F}_1) \widetilde{\boxtimes} \mathcal{F}_2)) \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^{\alpha+\beta}}(\widetilde{Fl})_\eta \tag{5.21}$$

Here, the gerbes on $G(K) \times G(K)$ and $G(K)$ match because of the multiplicative property of $\mathcal{G}^G$.

Sometimes we will just write $\star$ instead of $\underset{I,\lambda}{\star}$ for simplicity.

Remark

$$m_!, m_* : D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha \times \mathcal{G}^G \bigotimes \mathcal{G}^\beta}(G(K) \overset{I}{\times} G(K)/I_0) \to D_{\mathcal{G}^G \bigotimes \mathcal{G}^{\alpha+\beta}}(G(K)/I_0)$$

It does not matter whether we consider !- or $*$-direct image functor, because m is (ind-) proper.

For $\widetilde{w} \in W^{ext}$, if we can write it as $\widetilde{w} = t^\lambda w$, then, we use symbol $\overline{\widetilde{w}}$ denote λ.

Remark ${}_\alpha(\mathfrak{J}_{\widetilde{w},?})_\mu$ ($?=$! or *) is left $(T, b_{\overline{\widetilde{w}\mu}})$-equivariant and right (T, b_μ)-equivariant.

In order to extend the definition of *metaplectic BMW D*-module to any coweight $\lambda \in \Lambda$, we need the following lemma which is an analog of lemma 8 in [AB].

Lemma 5.1. *$\tilde{w}, \tilde{w}' \in W^{ext}, l(\tilde{w}\tilde{w}') = l(\tilde{w}) + l(\tilde{w}')$, and $\mu \in \Lambda$, then,*

i),

$$(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}' t^\mu}} \star (\mathfrak{J}_{\widetilde{w}',!})_\mu \cong (\mathfrak{J}_{\widetilde{w}\widetilde{w}',!})_\mu$$

$$(\mathfrak{J}_{\widetilde{w},*})_{\overline{\widetilde{w}' t^\mu}} \star (\mathfrak{J}_{\widetilde{w}',*})_\mu \cong (\mathfrak{J}_{\widetilde{w}\widetilde{w}',*})_\mu$$

ii), $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu \cong (\delta_0)_\mu \cong (\mathfrak{J}_{\widetilde{w},*})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},!})_\mu$

Proof. the proof is similar to the non-twisted case. Here, we sketch the proof.

i) we only prove the first claim, the second one follows from the same argument.

$$(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}'t^\mu}} \star (\mathfrak{J}_{\widetilde{w}',!})_\mu \cong (\mathfrak{J}_{\widetilde{w}\widetilde{w}',!})_\mu$$

Consider the convolution diagram (5.18), by Cartan decomposition of $G(K)$ by Iwahori subgroup I, the map $m : G(K) \overset{I}{\times} G(K)/I^0 \to G(K)/I^0$ is an isomorphism after restricting to $I\widetilde{w}I \overset{I}{\times} I\widetilde{w}'I/I^0 \to I\widetilde{w}\widetilde{w}'I/I^0$. And we notice that the trivializations of gerbes on both sides are compatible.

Both $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}'t^\mu}} \star (\mathfrak{J}_{\widetilde{w}',!})_\mu$ and $(\mathfrak{J}_{\widetilde{w}\widetilde{w}',!})_\mu$ have zero $*$-fiber outside $I\widetilde{w}\widetilde{w}'I/I \subset Fl$.

Consider the $*$-fiber of $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}'t^\mu}} \star (\mathfrak{J}_{\widetilde{w}',!})_\mu$ over the point $\widetilde{w}\widetilde{w}' \in Fl$. It can be identified with the $*$-fiber of $(\mathfrak{J}_{\widetilde{w}\widetilde{w}',!})_{\widetilde{w}''}$ at the same point. This identification is I-equivariant.(both of them are T-equivariant with respect to a same character D-module and I^0-equivariant).

It can imply the first assertion.

ii).From the first assertion of this lemma, we may reduce the question to the case when $\tilde{w}$ is a simple reflection. In this case, $\overline{I\tilde{w}I/I} \subset Fl$ is isomorphic to $\mathbb{P}^1$.

Then, we consider $\overline{I\tilde{w}I \cdot I\tilde{w}^{-1}I/I^0}$, according to the Bruhat order, it is contained in $I\tilde{w}I/I \cup I/I^0$(note $\tilde{w}^2 = 1$). It is a closed subscheme in $\widetilde{Fl}$. We should prove that the !-fiber of $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu$ is zero at $\tilde{w}$ and canonically isomorphic with !-fiber of $(\delta_0)_\mu$ at 1.

The direct image along with the map: $G(K) \to G(K) : g \to g_0 g^{-1}$ induces a functor

$$\mathfrak{s}_{g_0} : D^{I,b_\lambda}_{\mathcal{G}^G}(\widetilde{Fl})_\mu \longrightarrow D_{(\mathcal{G}^G)^{-1} \bigotimes \mathcal{G}_{g_0}}(\widetilde{Fl})_{-\lambda} \tag{5.22}$$

Given any point $g_0 \in G(K)/I_0$, its preimage in $G(K) \underset{I}{\times} G(K)/I_0$ under m is identified with $G(K)/I$ by the composition of projection $p_1 : G(K) \underset{I}{\times} G(K)/I_0 \to G(K)/I_0$ and $G(K)/I^0 \longrightarrow G(K)/I$. We can regard $m^{-1}(g_0) \subset G(K) \underset{I}{\times} G(K)/I^0$ as Fl, and under this identification the !-restriction of $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu$ to $m^{-1}(g_0)$ is identified with

$$(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \overset{!}{\otimes} \mathfrak{s}_{g^0}((\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu) \in D_{\mathcal{G}^G_{g_0}}(Fl)$$

By base change, the !-fiber of $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu$ at the point g_0 is given by

$$H(Fl, (\mathfrak{J}_{\widetilde{w}}, !)_{\overline{\widetilde{w}^{-1}\mu}} \otimes \mathfrak{s}_{g^0}((\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu)) \in Vect_{\mathcal{G}^G_{g_0}}$$

here $Vect_{\mathcal{G}^G_{g_0}}$ means the twisted vector space category.

The !-fiber of $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \star (\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu$ at $\widetilde{w}$ is identified with

$$H(Fl, (\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \overset{!}{\otimes} \mathfrak{s}_{\widetilde{w}}((\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu))$$

Because $\widetilde{w}$ is a simple reflection, we have $\overline{I\widetilde{w}I} \cong \mathbb{P}^1$. Under this identification, the D-module $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \overset{!}{\bigotimes} \mathfrak{s}_{\widetilde{w}}((\mathfrak{J}_{\widetilde{w}^{-1},*})_\mu)$ can be identified with $*$-pushout from some $\mathbb{G}_m$-monodromic (with respect to certain Kummer D module) D-module on $\mathbb{G}_m$ to $\mathbb{A}^1$, and then !-pushout to $\mathbb{P}^1$. By Braden theorem, its cohomology is zero. (See [DG1]). And over the point $1 \in Fl$, the !-fiber can be calculated as

$$H(Fl, (\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \overset{!}{\bigotimes} \mathfrak{s}_1((\mathfrak{J}_{\widetilde{w}^{-1},*})_{\mu}))$$

And we notice that this tensor product is I-equivariant. Its restriction to $\mathbb{A}^1$ is isomorphic to the constant object.

By projection formula, $(\mathfrak{J}_{\widetilde{w},!})_{\overline{\widetilde{w}^{-1}\mu}} \overset{!}{\otimes} \mathfrak{s}_{\widetilde{w}}((\mathfrak{J}_{\widetilde{w}^{-1},*})_{\mu})$ is $*$-pushout from $\mathbb{A}^1$, and it is $\Bbbk$. $\square$

5.4 Metaplectic BMW D-modules

With the preparation given in the last several sections, finally, we are able to construct BMW D-modules in twisted cases.

Definition 5.13. *Given $\lambda \in \Lambda$, s.t, $\lambda = \lambda_1 - \lambda_2, \lambda_1, \lambda_2 \in \Lambda^+$. We define the metaplectic BMW D-module $(\mathfrak{J}_\lambda)_\mu \in D_{\mathcal{G}^G}(\widetilde{Fl})_\mu$ to be:*

$$(\mathfrak{J}_\lambda)_\mu = (\mathfrak{J}_{\lambda_1,!})_{-\lambda_2+\mu} \star (\mathfrak{J}_{-\lambda_2,*})_\mu \tag{5.23}$$

Similarly, we define the dual BMW D module:

$$(\mathfrak{J}^{\mathbb{D}}_\lambda)_\mu := (\mathfrak{J}_{\lambda_1,*})_{-\lambda_2+\mu} \star (\mathfrak{J}_{-\lambda_2,!})_\mu \tag{5.24}$$

The definitions are independent of choices of λ_1, λ_2 according to Lemma 5.1.

This definition can be generalized to the definition of ${}_\alpha(\mathfrak{J}_\lambda)_\mu \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha}(\widetilde{Fl})_\mu$ and ${}_\alpha(\mathfrak{J}^{\mathbb{D}}_\lambda)_\mu \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^\alpha}(\widetilde{Fl})_\mu$.

By Lemma 5.1, metaplectic BMW D-modules admit a convolution product: i.e,

$$(\mathfrak{J}_\eta)_{\lambda+\mu} \underset{I,\lambda+\mu}{\star} (\mathfrak{J}_\lambda)_\mu \simeq (\mathfrak{J}_{\eta+\lambda})_\mu \in D_{\mathcal{G}^G}(\widetilde{Fl})^{I,b_{\lambda+\mu+\eta}}_\mu \tag{5.25}$$

If we also add $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$ twistings:

$${}_\alpha(\mathfrak{J}_\eta)_{\lambda+\mu} \underset{I,\lambda+\mu}{\star} {}_\beta(\mathfrak{J}_\lambda)_\mu \simeq {}_{\alpha+\beta}(\mathfrak{J}_{\eta+\lambda})_\mu \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^{\alpha+\beta}}(\widetilde{Fl})^{I,b_{\lambda+\mu+\eta}}_\mu \tag{5.26}$$

5.5 Standards and Costandards

In this section, we will define the standard objects and costandard objects of $Whit_q(Fl)$ which will match the standard objects and costandard objects of $\Omega^L_q - FactMod$ constructed in Section 4.1.2.

From $D_{\mathcal{G}^G}(Fl)$ to $Whit_q(Fl)$ we have a functor denoted by $Av_*^{N(K),\chi}$ which is right adjoint to the forgetful functor. But this functor is not continuous, i.e, it does not commute with *colimits*. So, it is not practical for our categories (our categories are *cocomplete* DG-categories, so it would be convenient for us to ask functors among them commute with colimits).

Let $l_{k,k'}$ denote the $*$-fiber of $N_{k'}/N_k$ at 1. If we ignore the Galois action, then, $l_{k,k'}$ is just a shift by relative dimension. Given $k \leq k'$, there is a functor: $Av_*^{N_k,\chi} \longrightarrow l_{k,k'} \bigotimes Av_*^{N_{k'},\chi}$ given by the Lemma 4.2 in [Ber]. Now we could define another functor $Av_*^{N(K),\chi}$:

$$Av_*^{ren} := \underset{k}{colim}\, Av_*^{N_k,\chi} \bigotimes l_{0,k} \tag{5.27}$$

It is easy to see that $Av_*^{ren}(\mathcal{F}) \in Whit_q(Fl)$. What's more, it is defined by *colimit*, so it is continuous. We will see this functor can be used to describe many important Whittaker D-modules.

In the category $Whit_q(Fl)$, we define

Definition 5.14.

$$\Delta_\lambda = Av_!^{N(K),\chi}(\mathfrak{J}_\lambda) \tag{5.28}$$

$$\nabla_\lambda \simeq Av_*^{ren}(\underset{\alpha\in\Lambda^+}{colim}(\mathfrak{J}_{\alpha,*})_{\lambda-\alpha} \underset{I,\lambda-\alpha}{\star} \mathfrak{J}_{\lambda-\alpha,*}) \tag{5.29}$$

We call Δ_λ standard objects of $Whit_q(Fl)$ and ∇_λ costandard objects.

A basic property of these objects (also where the names 'standard' and 'costandard' come from) is the following proposition:

Proposition 5.6.

$$Hom_{Whit_q(Fl)}(\Delta_\lambda, \nabla_\mu[i]) = \Bbbk,\ if\ \lambda = \mu, i = 0$$

and

$$Hom_{Whit_q(Fl)}(\Delta_\lambda, \nabla_\mu[i]) = 0\ otherwise.$$

Proof.

$$\begin{aligned}
&Hom_{Whit_q(Fl)}(\Delta_\lambda, \nabla_\mu[i]) \\
=&Hom_{Whit_q(Fl)}(Av_!^{N(K),\chi}(\mathfrak{J}_\lambda), Av_*^{ren} \underset{\alpha\in\Lambda^+}{colim}((\mathfrak{J}_{\alpha,*})_{\mu-\alpha} \underset{I,\mu-\alpha}{\star} \mathfrak{J}_{\mu-\alpha,*}[i])) \\
=&\underset{\alpha\in\Lambda^+}{colim}\, Hom_{Whit_q(Fl)}(Av_!^{N(K),\chi}(\mathfrak{J}_\lambda), Av_*^{ren}((\mathfrak{J}_{\alpha,*})_{\mu-\alpha} \underset{I,\mu-\alpha}{\star} \mathfrak{J}_{\mu-\alpha,*}[i]) \\
=&\underset{\alpha\in\Lambda^+}{colim}\, Hom_{Whit_q(\widetilde{Fl})_{\mu-\alpha}}(Av_!^{N(K),\chi}(\mathfrak{J}_\lambda \star (\mathfrak{J}_{\alpha-\mu,!})_{\mu-\alpha}), Av_*^{ren}((\mathfrak{J}_{\alpha,*})_{\mu-\alpha} \\
&\underset{I,\mu-\alpha}{\star} \mathfrak{J}_{\mu-\alpha,*}[i] \star (\mathfrak{J}_{\alpha-\mu,!})_{\mu-\alpha}) \\
=&\underset{\alpha\in\Lambda^+}{colim}\, Hom_{Whit_q(\widetilde{Fl})_{\mu-\alpha}}(Av_!^{N(K),\chi}(\delta_{\lambda+\alpha-\mu})_{\mu-\alpha}[-2|\lambda+\alpha-\mu|], \\
&Av_*^{ren}(\delta_\alpha)_{\mu-\alpha}[-2|\alpha|+i])
\end{aligned}$$

By the following lemma 5.2, $Av_!^{N(K),\chi}(\delta_{\lambda+\alpha-\mu})_{\mu-\alpha}[-2|\lambda+\alpha-\mu|]$ is !-extension from its restriction to $S_{\widetilde{Fl}}^{\lambda+\alpha-\mu} \subset \widetilde{Fl}$ and $Av_*^{ren}(\delta_\alpha)_{\mu-\alpha}[-2|\alpha|+i]$ is the $*$-extension of its restriction to $S_{\widetilde{Fl}}^{\alpha} \subset \widetilde{Fl}$.

Hence, we have

$$Hom_{Whit_q(\widetilde{Fl})_{\mu-\alpha}}(Av_!^{N(K),\chi}(\delta_{\lambda+\alpha-\mu})_{\mu-\alpha}[-2|\lambda+\alpha-\mu|], Av_*^{ren}(\delta_\alpha)_{\mu-\alpha}[-2|\alpha|+i])$$

$= \Bbbk$ if and only if $\lambda = \mu$ and $i = 0$ and

$$Hom_{Whit_q(\widetilde{Fl})_{\mu-\alpha}}(Av_!^{N(K),\chi}(\delta_{\lambda+\alpha-\mu})_{\mu-\alpha}[-2|\lambda+\alpha-\mu|], Av_*^{ren}(\delta_\alpha)_{\mu-\alpha}[-2|\alpha|+i])$$

$= 0$ otherwise.

□

Lemma 5.2. *Given $\lambda \in \Lambda^+$, then, we have:*
i).

$$Av_*^{ren}(\delta_\lambda)[-2|\lambda|] \simeq j_{S^\lambda_{Fl},*} \circ j^!_{S^\lambda_{Fl}}(\Delta_\lambda) \simeq Av_*^{ren}(\mathfrak{J}_{\lambda,*})$$

ii).

$$\Delta_\lambda \cong \Delta_\lambda^{ver} := Av_!^{N(K),\chi}(\delta_\lambda)[-2|\lambda|]$$

Proof. i). If λ is dominant, then $N(K)t^\lambda I = N(K)It^\lambda I$. Hence, all these three D-modules are $*$-extension from their restrictions to $S^\lambda_{Fl} \subset Fl$. As they are $(N(K), \chi)$-equivariant, we only need to prove that their !-fiber at $t^\lambda \in Fl$ coincide.

Then, the isomorphism follows from

$$\begin{aligned} i^!_\lambda(Av_*^{N_k,\chi} \bigotimes l_{0,k}(\mathfrak{J}_{\lambda,!})) &\simeq i^!_\lambda(Av_*^{N_k,\chi} \bigotimes l_{0,k}(\mathfrak{J}_{\lambda,*})) \\ &\simeq i^!_\lambda(Av_*^{N_k,\chi} \bigotimes l_{0,k}(\delta_\lambda[-2|\lambda|])) \end{aligned}$$

The proof of the second claim is similar. □

5.6 Compact generation property

In this section, we prove that $Whit_q(Fl)$ is compactly generated by Δ_λ.

Recall Proposition 5.3, $\Delta^{ver}_{\widetilde{w}}$ compactly generate $Whit_q(Fl)$, so we want to study the relationship between Δ_λ and $\Delta^{ver}_{\widetilde{w}}$. If $\Delta^{ver}_{\widetilde{w}}$ can be generated by finite extensions by Δ_λ, then Δ_λ, $\lambda \in \Lambda$ compactly generate $Whit_q(Fl)$.

Let us denote $w_0 \in W$ the longest element in the finite Weyl element, we consider the relevant orbit $N(K)t^{-\rho}w_0 \subset Fl$. It is the minimal relevant $N(K)$-orbit, *i.e*, its closure contains only one relevant orbit (itself).

From this description, we have:

Lemma 5.3. *Any Whittaker D-module on $S^{-\rho w_0}_{Fl}$ is clean.*

On this orbit, we define a twisted D-module,

Definition 5.15. *the Steinberg Whittaker D-module St is defined to be*

$$Av_!^{N(K),\chi}(\delta_{t^{-\rho}w_0}[l(t^{-\rho}w_0)]) \tag{5.30}$$

According to [AB] lemma 4 c), we have the following property of St.

Lemma 5.4. *For any $w \in W$, we have $St \cong St \star \mathfrak{J}_{w,!} \cong St \star \mathfrak{J}_{w,*}$.*

With the cleanness property of St in Lemma 5.3, we claim the following property:

Proposition 5.7. $St \cong \Delta_{-\rho}$

Proof. We note $\Delta_{-\rho} \star (\mathfrak{J}_\rho)_{-\rho} \simeq (\Delta_0)_{-\rho} := Av_!^{N(K),\chi}((\delta_0)_{-\rho})$ and $(\mathfrak{J}_\rho)_{-\rho}$ is invertible. So, we need to prove that $St \star (\mathfrak{J}_\rho)_{-\rho} \cong (\Delta_0)_{-\rho}$. According to the lemma above, we have $St \star (\mathfrak{J}_\rho)_{-\rho} \cong St \star \mathfrak{J}_{w_0,*} \star (\mathfrak{J}_\rho)_{-\rho}$. Because $l(t^\rho) = l(w_0) + l(t^{-\rho}w_0)$, we have $\mathfrak{J}_{w_0,*} \star (\mathfrak{J}_\rho)_{-\rho} \cong (\mathfrak{J}_{w_0 t^\rho,!})_{-\rho}$.

And according to Lemma 5.5, when $t^{-\rho}w_0t^\lambda w$ is relevant, we have:

$$N(K)t^{-\rho}w_0It^\lambda wI = N(K)t^{-\rho}w_0t^\lambda wI$$

From this equality, we know $St \star (\mathfrak{J}_\rho)_{-\rho}$ is !-extension from some object on $N(K)I/I \subset Fl$. Only need to prove the ! fiber of $St \star (\mathfrak{J}_{w_0t^\rho,!})_{-\rho}$ at $1 \in Fl$ is canonically isomorphic to $\Bbbk$. It follows from the fact that the convolution product m in Diagram (5.18) is an isomorphism after restricting to:

$$N(K)t^{-\rho}w_0I \overset{I}{\times} Iw_0t^\rho I/I^0 \longrightarrow N(K)I/I^0$$

□

Lemma 5.5. *For $\lambda \in \Lambda, w \in W$, the following two conditions are equivalent:*
(i). $t^{-\rho}w_0t^\lambda w$ is relevant.
(ii).

$$N(K)t^{-\rho}w_0It^\lambda wI = N(K)t^{-\rho}w_0t^\lambda wI$$

Proof.

$$\begin{aligned} &N(K)t^{-\rho}w_0It^\lambda wI = N(K)t^{-\rho}w_0t^\lambda wI \\ \Leftrightarrow &Iwt^\lambda \subset N^-(K)wt^\lambda I \\ \Leftrightarrow &t^{-w(\lambda)}It^{w(\lambda)} \subset N^-(K)wIw^{-1} \end{aligned} \tag{5.31}$$

And the last condition is equivalent to the following conditions for λ and w:

- If $\alpha > 0, w^{-1}w_0(\alpha) > 0, \langle \alpha, w_0w(\lambda)\rangle \geq 1$
- If $\alpha > 0, w^{-1}w_0(\alpha) > 0, \langle \alpha, w_0w(\lambda)\rangle \geq 0.$

By (5.5), it is equal to say $t^{w_0w(\lambda)-\rho}w_0w$ is relevant. □

From now on until the end of this section,, in order to simplify the notations, we will only consider the case when there is no twisting. The proof of general cases is the same after modifying Iwahori equivariant property by adding a character and change notations by adding twistings.

The following lemma is well-known,

Lemma 5.6. *If $\mathcal{F}$ is I-equivariant, then, $Av_!^{N(K),\chi}(\mathcal{F}) \simeq \Delta_0 \star \mathcal{F}$.*

With this lemma, we could calculate the image of $\mathfrak{J}_{t^\lambda w,!}$ (resp. $\mathfrak{J}_{t^\lambda w,*}$) under the functor $St \star - : D(Fl) \longrightarrow Whit(Fl)$.

Lemma 5.7. *If $t^{-\rho}w_0t^\lambda w$ is relevant, then,*

$$Av_!^{N(K),\chi}(\delta_{t^{-\rho}w_0t^\lambda w})[-l(t^\lambda w) + l(t^{-\rho}w_0)] = St \star \mathfrak{J}_{t^\lambda w,!}$$

Proof. Because of the lemma 5.5 and 5.6, we know that both sides are !-extension from their restrictions to $S_{Fl}^{t^{-\rho}w_0t^\lambda w}$. Then, we only need to compare their !-fibers at $t^{-\rho}w_0t^\lambda w$. The isomorphism of their !-fibers comes followed the fact that the convolution diagram: $N(K)t^{-\rho}w_0I \overset{I}{\times} It^\lambda wI \longrightarrow N(K)t^{-\rho}w_0t^\lambda wI$ is an isomorphism if $t^{-\rho}w_0t^\lambda w$ is relevant. □

Given $\tilde{w} \in W^{ext}$, recall that we denote by ${}^l\tilde{w}$ the longest element in the left coset of finite Weyl group W containing $\widetilde{w}$ ($W\tilde{w} \subset W^{ext}$).

Definition 5.16. *Then, we define $\phi(\tilde{w}) = t^{-\rho}\, {}^l\tilde{w}$.*

Given an I-equivariant D-module supporting on the closure of $I\tilde{w}I/I \subset Fl$, we claim that we can calculate the support of $St \star \mathcal{F}$.

Proposition 5.8. *If $\mathcal{F} \in D^I(Fl)$ and $supp(\mathcal{F}) = \overline{I\tilde{w}I/I}$, then,*

$$supp(St \star \mathcal{F}) = \overline{N(K)\phi(\tilde{w})I/I} \tag{5.32}$$

Proof. We can prove it by inducting on the length of $\tilde{w}$.

If $l(\tilde{w}) = 0$, we have ${}^l(\tilde{w}) = w_0\tilde{w}$, so $\phi(\tilde{w}) = t^{-\rho}w_0\tilde{w}$. And because $\overline{I\widetilde{w}I}/I$ is a point, we have $supp(St \star \mathcal{F}) = \overline{N(K)t^{-\rho}w_0\widetilde{w}I/I}$. The claim is true.

We assume that for $l(\tilde{w}') < n$, we have already proved the claim. Given $\mathcal{F}$ supported on $\overline{I\widetilde{w}I}/I$, $l(\tilde{w}) = n$. Note that we can regard $\mathcal{F}$ as an extension of $\mathfrak{J}_{\widetilde{w},!}$ and some D-module $\mathcal{F}'$ whose support belongs to $\overline{N(K)\widetilde{w}I}\backslash N(K)\tilde{w}I/I \subset Fl$.

So, we only need to prove two following lemmas:

i). $supp(St \star \mathfrak{J}_{\tilde{w},!}) = \overline{N(K)\phi(\tilde{w})I/I}$.

ii). $supp(St \star \mathcal{F}') \subset \overline{N(K)\phi(\tilde{w})I/I}\backslash N(K)\phi(\tilde{w})I/I$.

For the first claim, we assume ${}^l(\tilde{w}) = w\tilde{w}$, then, we have $St \star \mathfrak{J}_{\tilde{w},!} = St \star \mathfrak{J}_{w_0w,*} \star \mathfrak{J}_{\tilde{w},!} = St \star \mathfrak{J}_{w_0w\tilde{w},!}$ by Lemma 5.1 and Lemma 5.4. Hence, we have: $\overline{N(K)\phi(\tilde{w})I/I} \subset supp(St \star \mathfrak{J}_{w_0w\tilde{w},!}) \subset \overline{N(K)t^{-\rho}w_0Iw_0w\tilde{w}I/I} = \overline{N(K)\phi(\tilde{w})I/I}$.

To prove the second claim, we need to prove if $\tilde{w}' \leq_I \tilde{w}$, then, $\phi(\tilde{w}') \leq_N \phi(\tilde{w})$. Here, $\leq_I$ means the Bruhat order, I.e, $\widetilde{w}' \leq_I \widetilde{w}$ if and only if $I\widetilde{w}'I \subset \overline{I\widetilde{w}I}$. And $\leq_N$ denotes the semi-infinite Bruhat order, i.e, $\widetilde{w}' \leq_N \widetilde{w}$ if and only if $N(K)\widetilde{w}'I \subset \overline{N(K)\widetilde{w}I}$.

It is true. By definition, $\tilde{w}' \leq_I \tilde{w}$ means $\overline{I\widetilde{w}'I} \subset \overline{I\widetilde{w}I}$. So, $\overline{G(O)\widetilde{w}'I} \subset \overline{G(O)\widetilde{w}I}$. Note that we have $\overline{G(O)\widetilde{w}I} = \overline{I^l(\widetilde{w})I}$, so $\overline{I^l(\widetilde{w}')I} \subset \overline{I^l(\widetilde{w})I}$.

We claim: when λ dominant, then, $N(K)It^\lambda wI = N(K)t^\lambda wI$. It is true. Because λ dominant $\Rightarrow t^{-\lambda}N^-(tO)t^\lambda \subset N^-(tO) \subset wIw^{-1} \Rightarrow N^-(tO) \subset t^\lambda wIw^{-1}t^{-\lambda} \Rightarrow I \subset N(K)t^\lambda wIw^{-1}t^{-\lambda} \Rightarrow N(K)t^\lambda wI = N(K)It^\lambda wI$.

If we write ${}^{l}(\tilde{w}) = t^{\lambda}w$, then, λ dominant. Hence, we have

$$\overline{N(K)^{l}(\tilde{w})I} = \overline{N(K)I^{l}(\tilde{w})I}$$

$$\overline{N(K)^{l}(\tilde{w}')I} = \overline{N(K)I^{l}(\tilde{w}')I}$$

As a result, we have:

$$\overline{N(K)^{l}(\tilde{w}')I} = \overline{N(K)I^{l}(\tilde{w}')I} \subset \overline{N(K)I^{l}(\tilde{w})I} = \overline{N(K)^{l}(\tilde{w})I}$$

□

As a direct corollary of this proposition, we can calculate the support of Δ_λ.

Corollary 5.2. *supp*$(\Delta_\lambda) = \overline{N(K)\phi(\rho+\lambda)I}/I$

Proof. It follows from $\Delta_\lambda = \Delta_0 \star \mathfrak{J}_\lambda = St \star \mathfrak{J}_{\rho,!} \star \mathfrak{J}_\lambda = St \star \mathfrak{J}_{\rho+\lambda}$. And the support of *supp*$(\mathfrak{J}_{\lambda+\rho}) = \overline{It^{\lambda+\rho}I}/I$. (by Lemma 5.8) □

The following lemma is from [Be2] (Lemma 11).

Lemma 5.8. *For any $\lambda \in \Lambda$, we have: supp$(\mathfrak{J}_\lambda) = \overline{It^{\lambda}I}/I$, and we have*

$$\mathfrak{J}_\lambda|_{\overline{G(O)t^\lambda I}}/I = \mathfrak{J}_{\lambda,!}|_{\overline{G(O)t^\lambda I}}/I$$

For general λ, Δ_λ is different from Δ_λ^{ver}. But we have the following property:

Corollary 5.3. $\Delta_\lambda|_{N(K)\phi(\lambda+\rho)I/I} \simeq \Delta^{ver}_{\phi(\lambda+\rho)}|_{N(K)\phi(\lambda+\rho)I/I}$.

Proof. According to Lemma 5.8, we know the support of Δ_λ is the closure of $\overline{N(K)\phi(\lambda+\rho)I/I}$. And according to Lemma 5.8 and the proof of Lemma 5.8, when restricting to $N(K)\phi(\lambda+\rho)I/I$, we have: $\Delta_\lambda|_{N(K)\phi(\lambda+\rho)I/I} \simeq St \star \mathfrak{J}_{\lambda,!}|_{N(K)\phi(\lambda+\rho)I/I} = \Delta^{ver}_{\phi(\lambda+\rho)}$. □

With the above preparation, we could get the most important proposition in this section:

Proposition 5.9. *$\{\Delta_\lambda, \lambda \in \Lambda\}$ is a collection of compact generators of $Whit_q(Fl)$.*

Proof. By definition of relevant orbits and ϕ, we know $\phi(\lambda), \lambda \in \Lambda$ is one-to-one corresponds to relevant orbits. Now the proposition directly follows from *supp*$(\Delta_\lambda) = \overline{N(K)\phi(t^{\rho+\lambda})I}/I$, and $rank(\Delta_\lambda|_{N(K)\phi(\rho+\lambda)I}) = 1$ (I.e, $\Delta_\lambda|_{N(K)\phi(\rho+\lambda)I}$ is a generator of $Whit_q(S_{Fl}^{\phi(\rho+\lambda)})$). □

The following Proposition is independent of the goal of this section, we write it just for completeness. We could describe Δ_λ explicitly if λ is anti-dominant.

Proposition 5.10. *If λ is strict dominant, then $\Delta_{w_0(\lambda)-\rho} = \nabla^{ver}_{t^\lambda w_0}$*

Proof. $\Delta_{w_0(\lambda)-\rho} = St \star \mathfrak{J}_{t^{w_0(\lambda)},*} = St \star \mathfrak{J}_{w_0,*} \star \mathfrak{J}_{t^{w_0(\lambda)},*} \star \mathfrak{J}_{w_0,!} \star \mathfrak{J}_{w_0,*} \simeq St \star \mathfrak{J}_{t^{w_0(w_0(\lambda))},*} \star \mathfrak{J}_{w_0,*} \simeq St \star \mathfrak{J}_{t^\lambda w_0,*} \simeq \nabla^{ver}_{t^\lambda w_0}$. □

5.7 t-structure given by Δ_λ

In this section, we will define the important t-structure on $Whit_q(Fl)$ with metaplectic BMW D-modules which have already been proved to be a collection of compact generators of $Whit_q(Fl)$ in the last section. We will denote this t-structure by t_2 and its heart is denoted by $Whit_q(Fl)^\heartsuit$.

This new $t-structure$ is important for us, because it is this t-structure which could make our main theorem 4.1 to be t-exact.

Recall the following lemma in [BR],

Lemma 5.9. *If we assume $\mathcal{C}$ to be a compactly generated category with compact generators $\{c_i\}$.*

Then, there is a t-structure given by

$$\mathcal{C}^{\geq 0} := \{c | Hom_{\mathcal{C}}(c_i[k], c) = 0, \forall i, \forall k > 0\}$$

and

$$\mathcal{C}^{<0} := \{c | Hom_{\mathcal{C}}(c, c') = 0, \forall c' \in \mathcal{C}^{\geq 0}\}$$

Definition 5.17. *In the category $Whit_q(Fl)$, we define a new t-structure by the compact generators Δ_λ, i.e,*

- *$\mathcal{F} \in Whit_q(Fl)^{\geq 0}$ if and only if*

$$Hom_{Whit_q(Fl)}(\Delta_\lambda[k], \mathcal{F}) = 0, \ \forall \lambda \in \Lambda \text{ and } k > 0$$

The first thing that we need to do is to see whether Δ_λ is in the heart of this t-structure.

Proposition 5.11. *Δ_λ and ∇_λ are in the heart of this t-structure.*

Proof. We prove this proposition as a corollary of our main theorem. Assume that we have already proved Theorem 4.1, then, because $\Delta^{fact}_{\lambda,L}$ and $\nabla^{fact}_{\lambda,L}$ are in the heart of the t-structure of $\Omega_q^L - FactMod$, we have Δ_λ and ∇_λ are in the heart (F^L preserves standards and costandards).

□

From the definition of the t-structure of $Whit_q(Fl)$, we know:

the t-structure is compatible with filtered colimits, i.e, $\mathcal{F}_i \in Whit_q(Fl)^{\geq 0}$ (resp, $Whit_q(Fl)^{\leq 0}$), then, $\underset{i\in I}{colim}\, \mathcal{F}_i \in \mathcal{F}^{\geq 0}$(resp, $Whit_q(Fl)^{\leq 0}$).

Indeed, because Δ_λ is compact,

$$RHom_{Whit_q(Fl)}(\Delta_\lambda, colim \mathcal{F}_i) = colim\ RHom_{Whit_q(Fl)}(\Delta_\lambda, \mathcal{F}_i) \tag{5.33}$$

And there is a natural isomorphism for any $c \in Whit_q(Fl)$:

$$RHom_{Whit_q(Fl)}(colim \mathcal{F}_i, c) = lim\ RHom_{Whit_q(Fl)}(\mathcal{F}_i, c) \tag{5.34}$$

Definition 5.18. *we denote L_λ the image of Δ_λ in ∇_λ in $Whit_q(Fl)^\heartsuit$.*

After we proved our main theorem, we will see that ∇_λ and Δ_λ are of finite length for all $\lambda \in \Lambda$ if and only if q is irrational. When q is rational, costandard objects and irreducible objects are not compact, but standard objects are still compact.

Given a relevant orbit $N(K)\tilde{w}I/I \subset Fl$, we denote by $\chi_{\tilde{w},!*}$ the Goresky-Macpherson extension (i.e, image of !-extension to *-extension) of the restriction of the D-module $Av_!^{N(K),\chi}(\delta_{\tilde{w}})[-l(w_0 t^\rho \widetilde{w}) + l(t^{-\rho} w_0)]$ to $N(K)\tilde{w}I/I \subset Fl$. We note by definition, it can be regarded as $St \star \mathfrak{J}_{t^{-\rho} w_0 \tilde{w},!*}$. Because every I-orbit in Fl is affine, we have $\mathfrak{J}_{t^{-\rho} w_0 \tilde{w},!*}$ is a D-module concentrated in degree 0.

Then, we propose a conjecture about how to describe L_λ.

Conjecture 5.1. *L_λ is isomorphic to $\underset{\lambda_1 \in \Lambda^+}{colim}\, \chi_{\lambda_1,!*} \star \mathfrak{J}_{\lambda-\lambda_1,*}$.*

5.8 Coinvariant

In Section A, we see that there are two different definitions of the Whittaker category: one is constructed as invariant and another one is constructed as coinvariant. In Section 3.1, we studied the definition given by invariants. Now, we will consider another definition given by coinvariants. Through considering this category, we could get the definition of Verdier duality functor for $Whit_q(Fl)$.

In [Ga5], D.Gaitsgory constructed a functor from the coinvariant Whittaker category to the invariant Whittaker category and the author in *loc.cit* proved that it is actually an equivalence of categories.

In this section, we will review some parts from [Ga5], for the readers who are familiar with this paper and the equivalence between coinvariant Whittaker category and invariant Whittaker category can safely skip this section.

To start with, for any DG-category $\mathcal{C}$ and its subcategory $\mathcal{C}'$, we could define its quotient DG category $\mathcal{C}_{co}$ by the universal property: for any DG-category $\mathcal{D}$, we have

$$Funct(\mathcal{C}_{co}, \mathcal{D}) \simeq Funct(\mathcal{C}, \mathcal{D}) \underset{Funct(\mathcal{C}', \mathcal{D})}{\times} Funct(\mathcal{C}', \mathcal{D}^{grp})$$

Here, $\mathcal{D}^{grp}$ denotes $1-full$ subcategory of $\mathcal{D}$ which contains only isomorphisms in $\mathcal{D}$.

Definition 5.19. *We define $Whit_q(Fl)_{co}$ to be the quotient DG-category of $D_{\mathcal{G}^G}(Fl)$ by the full subcategory generated by:*

$$Fib(Av_*^{N_k,\chi}(\mathcal{F}) \longrightarrow \mathcal{F}) \tag{5.35}$$

here $\mathcal{F} \in D_{\mathcal{G}^G}(Fl), N_k = Ad_{t^{-k\rho}}(N(O)), k \in \mathbb{Z}$.

Remark It is equivalent to the definition of coinvariant Whittaker category given in Section A.

We could write $Whit_q(Fl)_{co}$ as

$$\underset{k}{colim}\, D_{\mathcal{G}^G}(Fl)_{N_k,\chi}$$

Here, the transition functor is given by projection functor

$$D_{\mathcal{G}^G}(Fl)_{N_k,\chi} \longrightarrow D_{\mathcal{G}^G}(Fl)_{N_{k'},\chi}, k' \geq k$$

For the definition of $D_{\mathcal{G}^G}(Fl)_{N_k,\chi}$ please check Definition A.8.
And as N_k is pro-unipotent, the natural projection

$$D_{\mathcal{G}^G}(Fl)_{N_k,\chi} \longrightarrow D_{\mathcal{G}^G}(Fl)^{N_k,\chi}$$

induced by $Av_*^{N_k,\chi} : D_{\mathcal{G}^G}(Fl) \longrightarrow D_{\mathcal{G}^G}(Fl)^{N_k,\chi}$ is an equivalence. Hence, we could identify $Whit_q(Fl)_{co}$ as

$$\underset{k}{colim}\, D_{\mathcal{G}^G}(Fl)^{N_k,\chi} \tag{5.36}$$

The transition functor for this *colimit* is given by

$$D_{\mathcal{G}^G}(Fl)^{N_k,\chi} \to D_{\mathcal{G}^G}(Fl)^{N_{k'},\chi} : \mathcal{F} \longrightarrow Av_*^{N_{k'},\chi}(\mathcal{F}), k \leq k' \tag{5.37}$$

Recall that we have $Whit_q(Fl) \simeq \underset{k}{lim}\, D_{\mathcal{G}^G}(Fl)^{N_k,\chi}$, the transition functor is given by forgetful functor.

In (5.27), we have already defined a functor $Av_*^{ren} : D_{\mathcal{G}^G}(Fl) \longrightarrow Whit_q(Fl)$

$$Av_*^{ren}(\mathcal{F}) := \underset{k}{colim}\, l_{0,k} \bigotimes Av_*^{N_k,\chi}(\mathcal{F})$$

Remark. The name of the functor Av_*^{ren} used in [Ga5] is 'Ps-Id'

Av_*^{ren} maps all morphisms of the form (5.35) to isomorphisms. So, it induces a functor from $Whit_q(Fl)_{co}$ to $Whit_q(Fl)$. With some abuse of notations, we also denote Av_*^{ren} this functor. The main theorem in [Ga5] is that:

Lemma 5.10.

$$Av_*^{ren} : Whit_q(Fl)_{co} \longrightarrow Whit_q(Fl) \tag{5.38}$$

is an equivalence of categories.

Let us denote $\nabla_{\lambda,co}$ the image of

$$\underset{\alpha\in\Lambda^+}{colim}(\mathfrak{J}_{\alpha,*})_{-\alpha+\lambda} \underset{I,-\alpha+\lambda}{\star} (\mathfrak{J}_{-\alpha+\lambda,*}) \in D_{\mathcal{G}^G}(Fl) \tag{5.39}$$

in $Whit_q(Fl)_{co}$ under the projection $D_{\mathcal{G}^G}(Fl) \to Whit_q(Fl)_{co}$.

According to the definition of ∇_λ and $\nabla_{\lambda,co}$, we get the following proposition:

Proposition 5.12.

$$Av_*^{ren}(\nabla_{\lambda,co}) \simeq \nabla_\lambda$$

for any $\lambda \in \Lambda$.

As a corollary of Lemma 5.10, we have,

Corollary 5.4. $Whit_q(Fl)_{co}$ *is dualizable.*

Proof. By Lemma 5.10, $Whit_q(Fl)_{co}$ is equivalent to $Whit_q(Fl)$. $\square$

Let us denote $Whit_q(Fl)_{co}^{\vee}$ the dual DG-category of $Whit_q(Fl)_{co}$. By definition, it is the functor category given by

$$Funct(Whit_q(Fl)_{co}, Vect)$$

Proposition 5.13. *The duality functor defines an equivalence:*

$$\mathbb{D} : Whit_q(Fl)_{co}^{\vee} \simeq Whit_{q^{-1}}(Fl) \tag{5.40}$$

Proof. By universal property,

$$Funct(Whit_q(Fl)_{co}, Vect) \subset Funct(D_{\mathcal{G}^G}(Fl), Vect)$$

is the full subcategory generated by the functor from $D_{\mathcal{G}^G}(Fl)$ to $Vect$ such that all counit morphims of the form $Av_*^{N_k,\chi}(\mathcal{F}) \to \mathcal{F}$ maps to isomorphisms in $Vect$. And the right hand side of (5.40) is a full subcategory generated by $(N(K), \chi)$-equivariant D-modules, i.e, by all $\mathcal{F} \in D_{\mathcal{G}^G}(Fl)$, s.t, the counit map $Av_*^{N_k,\chi}(\mathcal{F}) \to \mathcal{F}$ is an isomorphism in $D_{\mathcal{G}^G}(Fl)$, for any k.

Note that Verdier duality functor gives us an equivalence

$$D_{\mathcal{G}^G}(Fl)^{\vee} := Funct(D_{\mathcal{G}^G}(Fl), Vect) \simeq D_{(\mathcal{G}^G)^{-1}}(Fl) \tag{5.41}$$

We only need to prove that the corresponding subcategories of both sides of (5.41) coincide. That is to say, given a functor

$$D_{\mathcal{G}^G}(Fl) \longrightarrow Vect$$

, it belongs to $Whit_q(Fl)_{co}^{\vee}$ if and only if it corresponds to $Whit_{q^{-1}}(Fl)$ under the equivalence (5.41).

First, let us consider the 'only if' direction. Indeed, given such a functor belongs to $Whit_q(Fl)_{co}^{\vee}$. Then, it is of the form:

$$D_{\mathcal{G}^G}(Fl) \longrightarrow Vect$$

$$\mathcal{F} \longrightarrow \langle \mathcal{F}, \mathcal{F}_0 \rangle \tag{5.42}$$

for some $\mathcal{F}_0 \in D_{(\mathcal{G}^G)^{-1}}(Fl)$.

By definition, we have: for any $\mathcal{F} \in D_{\mathcal{G}^G}(Fl)$ and k,

$$\langle Av_*^{N_k,\chi}(\mathcal{F}), \mathcal{F}^0 \rangle \xrightarrow{\sim} \langle \mathcal{F}, \mathcal{F}^0 \rangle$$

$\Rightarrow$

$$\langle \mathcal{F}, Av_*^{N_k,\chi}(\mathcal{F}^0) \rangle \xrightarrow{\sim} \langle \mathcal{F}, \mathcal{F}^0 \rangle$$

From here, we get $Av_*^{N_k,\chi}(\mathcal{F}^0) \simeq \mathcal{F}^0$ for any k. Hence, $\mathcal{F}^0$ is an object in $Whit_q(Fl)$.

The other direction is similar: if $\mathcal{F}^0 \in Whit_{q^{-1}}(Fl)$, then the functor corresponding to $\mathcal{F}^0$ belongs to $Whit_q(Fl)_{co}^{\vee}$. $\square$

Definition 5.20. *We define the Verdier duality functor:*

$$\mathbb{D} : (Whit_q(Fl)^c)^{op} \longrightarrow Whit_{q^{-1}}(Fl)^c$$

to be the composition of functors:

$$(Whit_q(Fl)^c)^{op} \longrightarrow Whit_q(Fl)^c_{co} \xrightarrow[(5.40)]{\sim} Whit_{q^{-1}}(Fl)^c \tag{5.43}$$

We claim that the dual of standard object Δ_λ could be described by the dual BMW sheaf:

Proposition 5.14. *For any $\lambda \in \Lambda$, $\Delta_\lambda \in Whit_q(Fl)$, we have*

$$\mathbb{D}(\Delta_\lambda) \simeq Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \tag{5.44}$$

Proof. Av_*^{ren} maps $\mathfrak{J}_\lambda^{\mathbb{D}}$ to $Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}})$, we only need to prove under the equivalence: $Whit_q(Fl)^\vee \simeq Whit_{q^{-1}}(Fl)_{co}$, Δ_λ goes to $\mathfrak{J}_\lambda^{\mathbb{D}}$.

We only need to prove for any $\mathcal{F} \in Whit_q(Fl)$, we have

$$RHom_{Whit_q(Fl)}(\Delta_\lambda, \mathcal{F}) \simeq \langle \mathcal{F}, \mathfrak{J}_\lambda^{\mathbb{D}} \rangle$$

It follows from:

$$\begin{aligned} RHom_{Whit_q(Fl)}(\Delta_\lambda, \mathcal{F}) \simeq & RHom_{D_{\mathcal{G}^G}(Fl)}(\mathfrak{J}_\lambda, \mathcal{F}) \\ \simeq & \langle \mathcal{F}, \mathfrak{J}_\lambda^{\mathbb{D}} \rangle \end{aligned}$$

The first isomorphism is because of adjoint property, and the second comes from the fact $\mathfrak{J}_\lambda$ goes to $\mathfrak{J}_\lambda^{\mathbb{D}}$ under the equivalence

$$(D_{\mathcal{G}^G}(Fl))^\vee \simeq D_{(\mathcal{G}^G)^{-1}}(Fl)$$

and the definition of the pairing between $Whit_q(Fl)$ and $Whit_{q^-1}(Fl)_{co}$. □

5.9 An application of duality functor

This section does not participate in the proof of the main theorem, it just studies some special phenomenon when q is irrational.

In general, the support of ∇_λ contains infinite many $N(K)$-orbits in Fl, but when q is irrational, its support is contained in the union of finite many $N(K)$-orbits.

Proposition 5.15. *When q is irrational, we have $\nabla_\lambda \star \mathfrak{J}_{w_0,!} \simeq \mathbb{D}(\Delta_\lambda) \star \mathfrak{J}_{w_0,*}$. Here, ∇_λ is in $Whit_{q^{-1}}(Fl)$, $\Delta_\lambda \in Whit_q(Fl)$.*

In particular, ∇_λ supports on finite many $N(K)$-orbits.

Proof. □

And in the irrational case, we could relate Δ_λ with $\mathbb{D}(\Delta_\lambda)$.

Proposition 5.16. *If q is irrational and $\lambda \in \Lambda^+$, then, for $\Delta_{\lambda,q} \in Whit_q(Fl)$ and $\Delta_{\lambda,q^{-1}} \in Whit_{q^{-1}}(Fl)$, we have:*

$$\Delta_{q,\lambda} \simeq \mathbb{D}(\Delta_{q^{-1},\lambda})$$

Proof. It follows from $\Delta_{q,\lambda} \simeq St \underset{I,\rho+\lambda}{\star} (\mathfrak{J}_{\rho+\lambda,!}) \simeq St \underset{I,\rho+\lambda}{\star} (\mathfrak{J}_{\rho+\lambda,*}) \simeq \mathbb{D}(\Delta_{q^{-1},\lambda})$. □

Corollary 5.5. *When q is irrational, and $\lambda \in \Lambda^+$, $\Delta_\lambda \star \mathfrak{J}_{w_0,*} \star \mathfrak{J}_{w_0,*} \simeq \nabla_\lambda \simeq \nabla^{ver}_{\lambda w_0} \star \mathfrak{J}_{w_0,*}$.*

Conjecture 5.2. *If q is irrational, then:*
$\{\Delta_\lambda \star j_{w_0,!}, \lambda \in \Lambda\}$ and $\{\Delta^{ver}_{\widetilde{w}}, \widetilde{w}$ relevant$\}$ are the same.

If G is of type A^1, then, it is true.

6 Proof of the main theorem

The organization of this section is as follows:

In Section 6.1, we want to introduce a closed substack $(\bar{S}^0_{Fl,Ran_x})_{\infty\cdot x}$ of Fl_{G,Ran_x}. Theorem 6.1 ensures that we could regard a Whittaker D-module on Fl_G as a Whittaker D-module on $(\bar{S}^0_{Fl,Ran_x})_{\infty\cdot x}$.

In Section 6.2, we introduce configuration version affine flags and affine Grassmannian. There is a pull back functor from Fl_{G,Ran_x} to $Fl_{G,Conf_x}$, hence, we may regard a Whittaker D-module on Fl_G as a Whittaker D-module defined on configuration version affine flags.

In Section 6.3, we will give the definition of local functors used in this paper. In particular, the functor F^L appeared in Theorem 4.1 will be defined in this section.

In Section 6.4, we will give a description of the functors defined in the previous section by calculating the !-fibers of F^L and F^{KD} at $\lambda \cdot x \in Conf_x$.

In Section 6.5, we will prove the main theorem modulo Proposition 6.11.

In Section 6.6, we will supply proof of Theorem 6.1 used in Section 6.1.

6.1 Whittaker category on Fl_{G,Ran_x}

The construction of the functor F^L uses a lot of factorization prestacks. Recall that we have already defined Fl_{G,Ran_x} and $Gr_{G,Ran}$ in Section 2.4, in this section, we will study the corresponding Whittaker categories onside. We will focus on the Whittaker category defined on a closed substack of Fl_{G,Ran_x}, and then compare it with the Whittaker category on $Ran_x \times Fl_x$. And the most important stuff of this section is Theorem 6.1 which claims that these two categories are equivalent. But because the proof of this theorem is quite technical and is independent with other content of this paper, hence, we postpone the proof of Theorem 6.1 in Section 6.6. Another important proposition introduced in this section is Corollary 6.1, from this Corollary, we could get a factorization structure of the functor F^L constructed in Section 6.3.

In this section, our schemes and groups will be schemes and groups over the *Ran* space (or fixed point *Ran* space).

First of all, let us explain the definition of Whittaker D-module on Fl_{G,Ran_x} and related geometric objects.

By replacing $\mathcal{P}^0_G$ by $\mathcal{P}'_G := \omega^\rho \overset{T}{\times} G$, we could get a ω^ρ-twisted version affine flags $Fl^{\omega^\rho}_G$ and affine Grassmannian $Gr^{\omega^\rho}_G$, similar for $N(K)^{\omega^\rho}$, $G(K)^{\omega^\rho}$, etc. On $N(K)^{\omega^\rho}$, we have a canonically defined non-degenerated character:

$$\chi : N(K)^{\omega^\rho}_x \overset{projection}{\longrightarrow} N(K)^{\omega^\rho}_x/[N(K)^{\omega^\rho}_x, N(K)^{\omega^\rho}_x] \overset{\sim}{\longrightarrow} \omega^r|_{D^*_x} \overset{add}{\longrightarrow} \omega|_{D^*_x} \overset{res}{\longrightarrow} \mathbb{G}_a \tag{6.1}$$

Let us define some stacks over *Ran* and Ran_x:

Definition 6.1. *Let $G(K)^{\omega^\rho}_{Ran}$ denote the group prestack classifying the data: $(\mathcal{P}_G, \alpha, I, \gamma)$, here, $(\mathcal{P}_G, \alpha, I)$ is an element in $Gr^{\omega^\rho}_{G,Ran}$ and γ is an isomorphism between $\mathcal{P}_G$ and $\mathcal{P}'_G$ on D_I.*

$G(K)^{\omega^\rho}_{Ran_x}$ is the fiber product of $G(K)^{\omega^\rho}_{Ran}$ with $Ran_{X,x}$ over Ran_X

From definitions, $G(K)^{\omega^\rho}_{Ran}$ is a factorization space over Ran_X, and $G(K)^{\omega^\rho}_{Ran_x}$ is a factorization module space with respect to $G(K)^{\omega^\rho}_{Ran}$. The fiber of $G(K)^{\omega^\rho}_{Ran}$ over one point y is given by:

$$G(K)^{\omega^\rho}_y$$

We could define a factorization ind-pro group subscheme of $G(K)^{\omega^\rho}_{Ran}$, such that the fiber over the point $y \in X$ is given by $N(K)^{\omega^\rho}_y$. We define $N(K)^{\omega^\rho}_{Ran}$ to be the fiber product of $N(K)^{\omega^\rho}_{Ran_x}$ and Ran_x over *Ran*.

We have a natural action of $N(K)^{\omega^\rho}_{Ran}$ on $Gr^{\omega^\rho}_{Ran}$, and an action of $N(K)^{\omega^\rho}_{Ran_x}$ on $Gr^{\omega^\rho}_{Ran_x}$ and $Fl^{\omega^\rho}_{Ran_x}$.

Now, we define a closed $N(K)^{\omega^\rho}_{Ran_x}$-invariant subspace $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \subset Fl^{\omega^\rho}_{G,Ran_x}$, it is defined as:

Definition 6.2. *A point $(I, \mathcal{P}_G, \alpha, \epsilon) \in Fl^{\omega^\rho}_{Ran_{X,x}}$ belongs to $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$ if and only if for any dominant weight $\check{\lambda} \in \check{\Lambda}^+$, the composite map:*

$$\kappa^{\check{\lambda}} : \check{\lambda}(\omega^\rho) \overset{\sim}{\longrightarrow} (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \tag{6.2}$$

is regular on $X - x$. Here, the second map is the map mapping to the highest weight vector in $\mathcal{V}^{\check{\lambda}}_G$. $\mathcal{V}^{\check{\lambda}}_G$ is the Weyl module of G corresponding to the highest weight $\check{\lambda}$ and $\mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$ is defined to be $\mathcal{P}_G$'s associated vector bundle. And $\mathcal{P}'_G := \omega^\rho \overset{T}{\times} G$.

If we require (6.2) to be regular on the whole curve X, then, we denote the resulted stack by $\bar{S}^{w_0}_{Ran_x,Fl}$.

Remark We introduce $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$ is because it is very important for us to do a global and local comparison in Section 8 and Section 9. We will see later that we have a naturally defined map:

$$\pi_{Fl,Ran_x} : (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \longrightarrow (\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x} \tag{6.3}$$

What's more, the map $\pi_{Fl,x} : Fl_x^{\omega^\rho} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ factorizes π_{Fl,Ran_x}. We will use this observation in the proof of Theorem 7.1.

We could also define the stack $\bar{S}^0_{Ran,Gr} \subset Gr^{\omega^\rho}_{G,Ran}$ as well as $(\bar{S}^0_{Ran_x,Gr})_{\infty\cdot x} \subset Gr^{\omega^\rho}_{Ran_x}$ (we use the same notations as in [Ga1]):

Definition 6.3. *A point $(I, \mathcal{P}_G, \alpha) \in Gr^{\omega^\rho}_{Ran}$ belongs to $\bar{S}^\circ_{Ran,Gr}$, if we have: for any dominant weight $\check{\lambda} \in \check{\Lambda}^+$, the composite map:*

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \tag{6.4}$$

is regular on X.

If we replace the condition 'regular on X' by 'regular on X-x' (after we fixed a point x, so we also need to replace Ran_X by $Ran_{X,x}$), then, we denote this substack in $Gr^{\omega^\rho}_{Ran_{X,x}}$ by $(\bar{S}^0_{Gr,Ran_x})_{\infty\cdot x}$.

The most important feature of these stacks are their factorization property.

Proposition 6.1. *a). We have $\bar{S}^0_{Ran,Gr}$ is a factorization space, i.e,*

$$\bar{S}^0_{Gr,Ran} \underset{Ran_x}{\times} (Ran \times Ran)_{disj} \simeq \bar{S}^0_{Ran,Gr} \times \bar{S}^0_{Gr,Ran} \underset{Ran\times Ran}{\times} (Ran \times Ran)_{disj} \tag{6.5}$$

with homotopy coherence.

b). $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$ and $(\bar{S}^0_{Gr,Ran_x})_{\infty\cdot x}$ factorize w.r.t $\bar{S}^0_{Ran,Gr}$, i.e,

$$\begin{gathered}(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj} \\ \simeq \\ \bar{S}^0_{Ran,Gr} \times (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \underset{Ran\times Ran_x}{\times} (Ran \times Ran_x)_{disj}\end{gathered} \tag{6.6}$$

$$\begin{gathered}(\bar{S}^0_{Gr,Ran_x})_{\infty\cdot x} \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj} \\ \simeq \\ \bar{S}^0_{Ran,Gr} \times (\bar{S}^0_{Gr,Ran_x})_{\infty\cdot x} \underset{Ran\times Ran_x}{\times} (Ran_X \times Ran_x)_{disj}\end{gathered} \tag{6.7}$$

with homotopy coherence.

Proof. We only prove the first claim of b). The proofs for other claims are similar (just forgetting the Iwahori structure).

Given a point $s_1 = (\mathcal{P}_{G,1}, (x \in I_1), \alpha_1, \epsilon)$ in $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$ and a point $s_2 = (\mathcal{P}_{G,2}, I_2, \alpha_2)$ in $\bar{S}^0_{Ran,Gr}$. Because $I_1 \cap I_2 = \emptyset$, $X\backslash I_1 \cup X\backslash I_2 = X$. We define a new point $s_3 = (\mathcal{P}_{G,3}, (x \in I_1 \sqcup I_2), \alpha_3, \epsilon')$ in $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$ by the following assignment: We let $\mathcal{P}_{G,3}$ be $\mathcal{P}_{G,1}$ on $X - I_2$ and be $\mathcal{P}_2$ on $X - I_1$. On $X - I_1 \cup I_2$, the transition map is given by $\alpha_1\alpha_2^{-1}$. And α_3 is α_1 on $X - I_2$ and is α_2 on $X - I_1$. Because $\mathcal{P}_{G,3}$ and $\mathcal{P}_{G,1}$ are the same on $X - I_2$, in particular, we could identify their fiber over x. We let ϵ' be ϵ under the identification of $\mathcal{P}_{G,3}$ and $\mathcal{P}_{G,1}$ over x.

The inverse map is easy. Given a point $(\mathcal{P}_{G,3}, (x \in I_1 \sqcup I_2), \alpha_3, \epsilon')$ in $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x}$, we could recover $s_1 = (\mathcal{P}_{G,1}, (x \in I_1), \alpha_1, \epsilon)$ in $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x}$ and a point $s_2 = (\mathcal{P}_{G,2}, I_2, \alpha_2)$ in $\bar{S}^0_{Ran,Gr}$ by the following assignments:

$\mathcal{P}_{G,1} = \mathcal{P}_{G,3}$ on $X - I_2$, and equals $\mathcal{P}'_G$ on $X - I_1$. The transition map is given by α_3 on $X - I_2$ and id on $X - I_1$. ϵ is given by ϵ' with the identification of $\mathcal{P}_{G,3}$ and $\mathcal{P}_{G,1}$ over x.

s_2 is given similarly. □

Remark In order to simplify the notations, from now on, we will use the following simplified notations:

$$((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x})_{disj} := (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x} \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj}$$

$$(\bar{S}^0_{Ran_X,Gr} \times (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x})_{disj} := (\bar{S}^0_{Ran,Gr} \times (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x}) \underset{Ran \times Ran_x}{\times} (Ran \times Ran_x)_{disj}$$

$$((\bar{S}^0_{Gr,Ran_x})_{\infty \cdot x})_{disj} := (\bar{S}^0_{Gr,Ran_x})_{\infty \cdot x} \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj}$$

and

$$(\bar{S}^0_{Ran_X,Gr} \times (\bar{S}^0_{Gr,Ran_x})_{\infty \cdot x})_{disj} := (\bar{S}^0_{Ran,Gr} \times (\bar{S}^0_{Gr,Ran_x})_{\infty \cdot x}) \underset{Ran \times Ran_x}{\times} (Ran_X \times Ran_x)_{disj}$$

The stack $\bar{S}^0_{Ran_X,Gr}$ admits an open dense $N(K)^{\omega^\rho}_{Ran_X}$-orbit $S^0_{Ran,Gr}$.

Definition 6.4. *We define $S^0_{Ran,Gr} \subset \bar{S}^0_{Ran,Gr}$ (or $S_{Ran_x,Gr} \subset (\bar{S}^0_{Gr,Ran_x})_{\infty \cdot x}$ or $S^0_{Ran_x,Fl} \subset (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty \cdot x}$) to be the substack consisting of the points such that maps $\kappa^{\check{\lambda}}$ are injective on X (resp, $X - x$) for any $\check{\lambda} \in \check{\Lambda}^+$.*

Recall in Section 5.1, we denoted by $S^{\widetilde{w}}_{Fl}$ the $N(K)^{\omega^\rho}$-orbit passing through $\widetilde{w} \in Fl^{\omega^\rho}$. Similarly, we could also define the $N(K)^{\omega^\rho}$-orbit passing through $t^\lambda \in Gr$ and denote it by S^λ_{Gr}. For example, the subscheme $S^0_{Gr,y} \subset Gr^{\omega^\rho}_{G,y}$ classifies the data $(\mathcal{P}_G, \alpha)$ such that $\forall \check{\lambda} \in \check{\Lambda}^+$, the map

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$$

is injective on X. Its closure is denoted by $\bar{S}^0_{Gr,y} \subset Gr^{\omega^\rho}_{G,y}$. The difference between the definition of $\bar{S}^0_{Gr,y}$ and $S^0_{Gr,y}$ is that we only ask $\kappa^{\check{\lambda}}$ to be regular, i.e, it may has torsion.

We note that the fiber of $(\bar{S}^0_{Fl,Ran_x})_{\infty \cdot x}$ over the point $I = \{x, x_1, x_2, ..., x_k\} \in Ran_{X,x}$ is isomorphic to the product $Fl_x \times \prod_{1 \le i \le k} \bar{S}^0_{Gr,x_i}$. And the fiber of $\bar{S}^0_{Gr,Ran_X}$ over the point $I = \{x_1, x_2, ..., x_k\} \in Ran_X$ is isomorphic to $\prod_{1\leq i\leq k} \bar{S}^0_{Gr,x_i}$.

Note that the gerbe $\mathcal{G}^G$ is multiplicative, and the gerbe $\mathcal{G}^G$ on $Fl^{\omega^\rho}_{G,Ran_x}$ is equivalent w.r.t $\mathcal{G}^G$ on $G(K)^{\omega^\rho}_{Ran_x}$ under the action of $G(K)^{\omega^\rho}_{Ran_x}$ on $Fl^{\omega^\rho}_{G,Ran_x}$. In particular, the gerbe $\mathcal{G}^G$ on $Fl^{\omega^\rho}_{G,Ran_x}$ is equivalent w.r.t $\mathcal{G}^G$ on $N(K)^{\omega^\rho}_{Ran_x}$ under the action of $N(K)^{\omega^\rho}_{Ran_x}$ on $Fl^{\omega^\rho}_{G,Ran_x}$. Because N is unipotent, we have $N(K)^{\omega^\rho}_{Ran_x}$ is an affine space. Hence, there is a canonical trivialization of the restriction of $\mathcal{G}^G$ on $N(K)^{\omega^\rho}_{Ran_x}$. So, the gerbe $\mathcal{G}^G$ on $Fl^{\omega^\rho}_{G,Ran_x}$ is equivalent with respect to the action of $N(K)^{\omega^\rho}_{Ran_x}$. It means that we can talk about $N(K)^{\omega^\rho}_{Ran_x}$-equivariant trivialization of the gerbe on each $N(K)^{\omega^\rho}_{Ran_x}$ orbit (up to a non-canonical isomorphism which involves the choice of the trivialization at a point).

In particular, given any character D-module χ on $N(K)^{\omega^\rho}_{Ran_x}$, we may consider the category of $(N(K)^{\omega^\rho}_{Ran_x}, \chi_{Ran})$-equivariant D-modules.

If we denote by χ_{Ran} the composition of maps as follows:

$$\chi_{Ran} : N(K)^{\omega^\rho}_{Ran_x} \longrightarrow N^{\omega^\rho}_{Ran_x}(K)/[N(K)^{\omega^\rho}_{Ran_x}, N(K)^{\omega^\rho}_{Ran_X}] \longrightarrow \omega^r_{Ran_x}(K) \longrightarrow \xrightarrow{\text{take residue}} \mathbb{G}^r_{a,Ran_x} \longrightarrow \mathbb{G}_{a,Ran_x} \tag{6.8}$$

then, we could define the (twisted) Whittaker category on $Fl^{\omega^\rho}_{G,Ran_x}$,

Definition 6.5.

$$Whit_q(Fl^{\omega^\rho}_{G,Ran_x}) := D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})^{N(K)_{Ran_x},\chi_{Ran}}$$

Remark The twisting ω^ρ does not change the category essentially. This twisting only let us have a canonical choice of χ_{Ran}.

We note that $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$ is invariant under the $N(K)^{\omega^\rho}_{Ran_x}$-action. We denote by $Whit_q(Fl^{\omega^\rho}_{G,Ran_x})^{\leq 0}$ the full subcategory of $Whit_q(Fl^{\omega^\rho}_{G,Ran_x})$ consisting of objects supporting on $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}$.

The aim of this section to compare $Whit_q(Fl^{\omega^\rho}_{G,Ran_x})^{\leq 0}$ with the Whittaker category on $Fl^{\omega^\rho}_x \times Ran_x$. Let us consider the product space $Ran_x \times Fl^{\omega^\rho}_x$. The $N(K)^{\omega^\rho}_x$ action on $Fl^{\omega^\rho}_{G,x}$ gives a $N(K)^{\omega^\rho}_x$-action on $Ran_x \times Fl^{\omega^\rho}_{G,x}$. The pullback of the gerbe $\mathcal{G}^G$ on $Fl^{\omega^\rho}_{G,x}$ to $Ran_x \times Fl^{\omega^\rho}_x$ is still denoted by $\mathcal{G}^G$. This gerbe is $N(K)^{\omega^\rho}_x$-equivariant, we define the *Whittaker category* on this product space to be:

$$Whit_q(Ran_x \times Fl^{\omega^\rho}_x) := D_{\mathcal{G}^G}(Ran_x \times Fl^{\omega^\rho}_{G,Ran_x})^{N(K)^{\omega^\rho}_x,\chi} \tag{6.9}$$

We have a canonical morphism:

$$unit : Fl^{\omega^\rho}_x \times Ran_x \to Fl^{\omega^\rho}_{Ran_x} \tag{6.10}$$

Definition 6.6. *unit* : $Fl^{\omega^\rho}_x \times Ran_x \to Fl^{\omega^\rho}_{Ran_x}$ *sends* $(\mathcal{P}_G, \alpha, \epsilon), x \in I$ *to* $(x \in I, \mathcal{P}_G, \alpha, \epsilon)$.

similarly, we have:

$$unit_{Gr} : \bar{S}^{0}_{Gr} \times Ran \to \bar{S}^{0}_{Gr,Ran} \tag{6.11}$$

A useful observation is that the functor *unit* factors through the closed subspace $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \hookrightarrow Fl^{\omega^\rho}_{G,Ran_x}$. It is true because α is an isomorphism on $X - x$, so for any $\check{\lambda} \in \check{\Lambda}^+, \kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G}$ is a regular morphism (in fact, it is even injective) of vector bundles outside x by definition. Hence, we could also denote by *unit* the map:

$$unit : Fl^{\omega^\rho}_x \times Ran_x \to (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \tag{6.12}$$

Note that $! - pullback$ along with the projection

$$pr_{Ran_x} : Ran_x \times Fl^{\omega^\rho}_x \to Fl^{\omega^\rho}_x$$

gives rise to a functor

$$pr^!_{Ran_x} : D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x) \to D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x) \tag{6.13}$$

This map commutes with $N(K)^{\omega^\rho}_x$-action, so we have $pr^!_{Ran_x}$ induces a functor between corresponding Whittaker categories:

$$Whit_q(Fl^{\omega^\rho}_x) \to Whit_q(Fl^{\omega^\rho}_x \times Ran_x) \tag{6.14}$$

Now, let us consider $unit^! : D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x)$.

We claim that this map could induce a functor between the corresponding Whittaker categories:

$$unit^! : Whit_q((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}) \to Whit_q(Ran_x \times Fl^{\omega^\rho}_x)$$

Let us explain why the functor $unit^!$ can induce a functor between the Whittaker categories.

Remark The Whittaker conditions of two sides of the functor $unit^!$ are different: the Whittaker condition in the definition of $D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})$ is $(N(K)^{\omega^\rho}_{Ran_x}, \chi_{Ran})$-equivariant and the other side is $(N(K)^{\omega^\rho}_x, \chi)$-equivariant.

Despite of the difference of the equivariant conditions, $unit^!$ sends Whittaker objects to Whittaker objects. Consider the closed subgroup N' in $N(K)^{\omega^\rho}_{Ran_x}$ whose fiber over a point $\{x, x_1, x_2, ..., x_k\} \in Ran_x$ is given by $N(K)^{\omega^\rho}_x \times \prod_i N(O)^{\omega^\rho}_{x_i}$. Restrict to x gives a projection:

$$N' \longrightarrow N(K)^{\omega^\rho}_x \tag{6.15}$$

We note that the map *unit* is compatible with N'-action, where the action of N' on $Fl^{\omega^\rho}_x \times Ran_x$ is given by the projection above and the action of $N(K)^{\omega^\rho}_x$ on $Fl^{\omega^\rho}_x$. Then, we notice that the kernel of the projection (6.15) is a product of $N(O)_{x_i}$ which is pro-unipotent, so we have an equivalence:

$$oblv_{N'\to N(K)^{\omega^\rho}_x} : D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x)^{N',\chi} \xrightarrow{\sim} D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x)^{N(K)^{\omega^\rho}_x,\chi}$$

So, $unit^!$ induces a functor:

$$D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_{Ran_x},\chi_{Ran_x}} \longrightarrow D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x)^{N',\chi} \simeq D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times Ran_x)^{N(K)^{\omega^\rho}_x,\chi} \tag{6.16}$$

i.e,

$$unit^! : \ Whit_q((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}) \longrightarrow Whit_q(Fl^{\omega^\rho}_x \times Ran_x) \tag{6.17}$$

Similarly, if we consider the !-pullback of $unit_{Gr}$. Then, by the same analysis as above, we could get a functor:

$$unit^!_{Gr} : Whit_q(\bar{S}^0_{Ran,Gr}) \longrightarrow Whit_q(\bar{S}^0_{Gr} \times Ran) \tag{6.18}$$

In Section 6.6, we will apply the same method used in [Ga5] to prove:

Theorem 6.1. *Functors (6.17) and (6.18) are equivalences.*

Note that the projection:

$$\bar{S}^0_{Ran,Gr} \longrightarrow Ran \tag{6.19}$$

admits a canonical section:

$$s_{Ran} : Ran \longrightarrow \bar{S}^0_{Ran,Gr} \tag{6.20}$$

Let Vac denote the $Whittaker$ D-module on $\bar{S}^0_{Ran,Gr}$ uniquely characterized by the property that its pullback to Ran is the dualizing D-module on Ran, i.e, $s^!_{Ran}(Vac) \simeq \omega_{Ran}$.

As a direct corollary from Theorem 6.1, we have the following factorization property:

Corollary 6.1. *a). Vac is a factorization algebra on $\bar{S}^0_{Gr,Ran}$, i.e,*

$$Vac \boxtimes Vac|_{(\overline{S}^0_{Gr,Ran} \times \overline{S}^0_{Gr,Ran})_{disj}} \cong Vac|_{(\overline{S}^0_{Gr,Ran})_{disj}}$$

b). Any $\mathcal{F} \in Whit_q((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})$ is factorization module of Vac, i.e:

$$Vac \boxtimes \mathcal{F}|_{(\overline{S}^0_{Gr,Ran} \times (\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})_{disj}} \cong \mathcal{F}|_{(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})_{disj}}$$

Proof. We only show b). a) follows from the same proof.

For any $x \in \bar{S}^0_{Ran,Gr} - S^0_{Ran,Gr}$, we have $stab_{N(K)^{\omega^\rho}_{Ran}}(x) \nsubseteq Ker(\chi_{Ran})$.(given a point $x \in Fl^{\omega^\rho}_{Ran}$, the condition $stab_{N(K)^{\omega^\rho}_{Ran}}(x) \subseteq Ker(\chi_{Ran})$ is equivalent to x lies in $S^\lambda_{Ran,Gr}$, $\lambda \in \Lambda^+$). It implies $Whit_q(\bar{S}^0_{Ran,Gr} - S^0_{Ran,Gr}) = 0$. So, we have, $Whit_q(\bar{S}^0_{Ran,Gr}) \simeq Whit_q(S^0_{Ran,Gr})$

Taking !-fiber at the canonical section of Ran in $S^0_{Ran,Gr}$ gives us an equivalence: $Whit_q(S^0_{Ran,Gr}) \simeq D(Ran)$. Indeed, it is because $N(K)^{\omega^\rho}_{Ran}$ acts transversely on $S^0_{Ran,Gr}$ over Ran and $N(K)^{\omega^\rho}_{Ran}$ is ind-pro-unipotent.

We consider the following diagram:

$$\begin{tikzcd}
Ran_x \times Fl_x^{\omega^\rho} \arrow[r, "unit"] & (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \\
(Ran \times Ran_x \times Fl_x^{\omega^\rho})_{disj} \arrow[u, "add\times id"] \arrow[r, "s_{Ran}\times unit"] & (\bar{S}^0_{Gr,Ran} \times (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x})_{disj} \arrow[u]
\end{tikzcd}$$

It is commutative. Both of the images of a point $(I_0, I_1, \mathcal{P}_G, \alpha, \epsilon) \in (Ran \times Ran_x \times Fl_x^{\omega^\rho})_{disj}$ through the clockwise composition and counter-clockwise composition are $(I_0 \cup I_1, \mathcal{P}_G, \alpha|_{X-I_0\cup I_1}) \in (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$.

Given a $Whittaker$ D-module $\mathcal{F}$ on $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$, we need to prove

$$(add \times id)^! \circ unit^!(\mathcal{F}) \simeq \omega_{Ran} \boxtimes unit^!(\mathcal{F})|_{disj}$$

And it follows from the fact any (twisted) D-module on Ran_x factorizes with respect to ω_{Ran}, i.e, for any (twisted) D-module $\mathcal{M} \in D(Ran_x)$, we have $add^!(\mathcal{M})|_{disj} \simeq \omega \boxtimes \mathcal{M}|_{disj}$. □

6.2 Configuration version of Gr and Fl

Recall that in Theorem 4.1, we need to construct a functor sending to $Conf_x$. Hence, it is convenient to consider the stack over the Configuration space. In this section, we will explain the configuration version of Gr and Fl.

Definition 6.7. *Let $Gr^{\omega^\rho}_{G,Conf}$ be the prestack over $Conf$ that classifies the data: $(D, \mathcal{P}_G, \alpha)$, here,*

$$D = \sum_{1\le i\le k} \lambda_i x_i \in Conf_x \tag{6.21}$$

$\mathcal{P}_G \in Bun_G, \alpha : \mathcal{P}'_G|_{X-I} \cong \mathcal{P}_G|_{X-I}$, $I = \{x_1, x_2, ..., x_k\}$,.

Definition 6.8. *Let $Fl^{\omega^\rho}_{G,Conf_x}$ be the prestack over $Conf_x$ that classifies the data: $(D, \mathcal{P}_G, \alpha, \epsilon)$, here,*

$$D = \lambda \cdot x + \sum_{1\le i\le k} \lambda_i \cdot x_i \in Conf_x \tag{6.22}$$

$\mathcal{P}_G \in Bun_G$, $\alpha : \mathcal{P}'_G|_{X-I} \cong \mathcal{P}_G|_{X-I}$, $I = \{x, x_1, x_2, ..., x_k\}$, and ϵ is a B-reduction of $\mathcal{P}_G$ at x.

Given a point in $Conf_x$ (resp, $Conf$) of the form (6.22) (resp. (6.21)). Then, it is easy to see that the fiber of $Fl^{\omega^\rho}_{G,Conf_x}$ (resp. $Gr^{\omega^\rho}_{G,Conf}$) over this point is canonically isomorphic to:

$$Fl^{\omega^\rho}_x \times \prod_{1\le i\le k} Gr^{\omega^\rho}_{x_i} \tag{6.23}$$

(resp.

$$\prod_{1\leq i\leq k} Gr^{\omega^\rho}_{x_i} \tag{6.24}$$

)

Now, we define some substacks in the stacks defined above.

Definition 6.9. *We denote by* $\bar{S}^0_{Gr,Conf}$ *the closed sub prestack of* $Gr^{\omega^\rho}_{G,Conf}$ *such that,*

$$\bar{S}^0_{Ran,Gr} \underset{Ran}{\times} Gr^{neg}_{T,Ran} \simeq \bar{S}^0_{Conf,Gr} \underset{Conf}{\times} Gr^{neg}_{T,Ran} \tag{6.25}$$

as substacks of the following isomorphism:

$$Gr^{\omega^\rho}_{G,Ran} \underset{Ran}{\times} Gr^{neg}_{T,Ran} \simeq Gr^{\omega^\rho}_{G,Conf} \underset{Conf}{\times} Gr^{neg}_{T,Ran} \tag{6.26}$$

Definition 6.10. *We denote by* $\bar{S}^{w_0}_{Fl,Conf_x}$ *the closed sub prestack of* $Fl^{\omega^\rho}_{G,Conf_x}$ *consisting of the points* $(D, \mathcal{P}_G, \alpha, \epsilon)$ *s.t,*

$$\bar{S}^{w_0}_{Fl,Ran_x} \underset{Ran_x}{\times} Gr^{neg}_{T,Ran_x} \simeq \bar{S}^{w_0}_{Conf_x,Fl} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x} \tag{6.27}$$

here, we regard both sides as substacks of the following isomorphism:

$$Fl^{\omega^\rho}_{G,Ran_x} \underset{Ran_x}{\times} Gr^{neg}_{T,Ran_x} \simeq Fl^{\omega^\rho}_{G,Conf_x} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x} \tag{6.28}$$

And we denote by $(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x}$ *the closed sub prestack of* $Fl^{\omega^\rho}_{G,Conf_x}$ *consisting of the points* $(D, \mathcal{P}_G, \alpha, \epsilon)$ *s.t,*

$$(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \underset{Ran_x}{\times} Gr^{neg}_{T,Ran_x} \simeq (\bar{S}^{w_0}_{Conf_x,Fl})_{\infty\cdot x} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x} \tag{6.29}$$

By Lemma 2.2, $Gr^{neg}_{T,Ran_x} \to Conf_x$ induced an isomorphism on the topology generated by finite surjective morphisms. Hence, the pullback functors induce equivalences of the category of gerbes and the corresponding twisted categories:

$$D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Conf_x} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x}) \simeq D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Conf_x}) \tag{6.30}$$

$$D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Conf_x,Fl} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x}) \simeq D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Conf_x,Fl}) \tag{6.31}$$

There is a functor from $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})$ to $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Conf_x})$ given by first pullback to $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x} \underset{Ran_x}{\times} Gr^{neg}_{T,Ran_x})$ and then use the equivalence with $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Conf_x} \underset{Conf_x}{\times} Gr^{neg}_{T,Ran_x})$ given by (6.28), and then descend to a D-module in $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Conf_x})$ by (6.30). To simplify, we will also call this functor to be pullback from $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{Ran_x})$ to $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{Conf_x})$.

According to the same proof as Proposition 6.1, we have:

$$\begin{array}{c} \bar{S}^0_{Gr,Conf} \underset{Conf_x}{\times} (Conf \times Conf)_{disj} \\ \simeq \\ \bar{S}^0_{Gr,Conf} \times \bar{S}^0_{Gr,Conf} \underset{Conf \times Conf}{\times} (Conf \times Conf)_{disj} \end{array} \tag{6.32}$$

$$\begin{array}{c} (\bar{S}^{w_0}_{Conf_x,Fl})_{\infty \cdot x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \\ \simeq \\ \bar{S}^0_{Gr,Conf} \times (\bar{S}^{w_0}_{Conf_x,Fl})_{\infty \cdot x} \underset{Conf \times Conf_x}{\times} (Conf \times Conf_x)_{disj} \end{array} \tag{6.33}$$

Definition 6.11. *a). We let $S^{Conf}_{Conf,Gr}$ be the prestack classifying the data $(D, \mathcal{P}_G, \alpha)$, such that for any $\check{\lambda}$ dominant, the induced maps:*

$$\kappa^{\check{\lambda}}: \ (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle}(-\langle \check{\lambda}, D \rangle) \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$$

which are a priori defined on $X - I$ extend to injective maps on the whole curve X and satisfy Plücker relations.

b). We let $S^{Conf_x}_{Conf_x,Fl}$ be the prestack classifying a data from $S^{Conf}_{Conf,Gr}$ and a B-reduction of $\mathcal{P}_G$ at x.

But the ones that we will use in the future are their negative counterparts.

Definition 6.12. *We let $S^{-,Conf}_{Conf,Gr}$ be the prestack classifying the data $(D, \mathcal{P}_G, \alpha)$, such that for any $\check{\lambda}$ dominant, the induced maps:*

$$\kappa^{-,\check{\lambda}}: \ '\mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to' \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \to (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle}(\langle \check{\lambda}, D \rangle)$$

which are a priori defined on $X - I$ extend to surjective maps on the whole curve X and satisfy Plücker relations. Here, $'\mathcal{V}^{\check{\lambda}}$ is the dual Weyl module of G corresponding to dominant weight $\check{\lambda}$.

$\bar{S}^{-,Conf}_{Conf,Gr}$ is defined to be the closure of $S^{-,Conf}_{Conf,Gr}$ in $Gr^{\omega^\rho}_{G,Conf}$. It classifies the data $(D, \mathcal{P}_G, \alpha)$, such that the induced maps:

$$\kappa^{-,\check{\lambda}} :' \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to' \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \to (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle}(\langle \check{\lambda}, D \rangle)$$

which are a priori defined on $X - I$ extends to genuine maps on the whole curve X and satisfy Plücker relations.

We let $S^{-,Conf_x}_{Conf_x,Fl}$ be the prestack classifying a data from $S^{-,Conf}_{Conf,Gr}$ and a B-reduction of $\mathcal{P}_G$ at x.

$\bar{S}^{-,Conf_x}_{Conf_x,Fl}$ is defined to be the closure of $S^{-,Conf_x}_{Conf_x,Fl}$ in $Fl^{\omega^\rho}_{G,Conf_x}$. It classifies the data $(D, \mathcal{P}_G, \alpha, \epsilon)$, s.t the induced maps $\kappa^{-,\check{\lambda}}$ which are a priori defined on $X - I$ extends to genuine maps on the whole curve X and satisfy Plücker relations.

By [GL1], $S^{-,Conf}_{Conf,Gr}$ is a factorization space.

$$\begin{gathered} S^{-,Conf}_{Conf,Gr} \underset{Conf}{\times} (Conf \times Conf)_{disj} \\ \simeq \\ S^{-,Conf}_{Conf,Gr} \times S^{-,Conf}_{Conf,Gr} \underset{Conf \times Conf}{\times} (Conf \times Conf)_{disj} \end{gathered} \tag{6.34}$$

And by the same proof of Proposition 6.1, we could see that $S^{-,Conf_x}_{Conf_x,Fl}$ is a factorization module space with respect to $S^{-,Conf}_{Conf,Gr}$.

The prestack $S^{-,Conf_x}_{Conf_x,Fl}$ admits a stratification given by the 'relative position' of B reduction given by ϵ and B^--reduction at x given by $\kappa^{-,\check{\lambda}}$. To be more precise, the morphisms $\kappa^{-,\check{\lambda}}$ are surjective, so they induce a B^--reduction at x, i.e, we have a map:

$$S^{-,Conf_x}_{Fl,Conf_x} \to pt/B^- \tag{6.35}$$

given by sending a point of $S^{-,Conf_x}_{Fl,Conf_x}$ to its reduced B^--bundle at x. In addition, ϵ gives a map:

$$S^{-,Conf_x}_{Fl,Conf_x} \to pt/B \tag{6.36}$$

We note that their compositions with inductions to G-bundle coincide (both of them are $\mathcal{P}_G$ at x), so we have a map of relative position:

$$rp : S^{-,Conf_x}_{Fl,Conf_x} \to pt/B \underset{pt/G}{\times} pt/B^- \simeq B\backslash G/B^- \tag{6.37}$$

The Bruhat decomposition gives double coset decomposition of (B, B^-) in G:

$$G = \bigsqcup_{w\in W} BwB^-$$

And it induces a decomposition of $B\backslash G/B^-$, we denote by Br^w the cell corresponding to BwB^-.

$$B\backslash G/B^- = \bigsqcup_{w\in W} Br^w \tag{6.38}$$

Definition 6.13. *We denote by $S^{-,w,Conf_x}_{Fl,Conf_x}$ the preimage of the w-cell Br^w in $S^{-,Conf_x}_{Fl,Conf_x}$, $w \in W$.*

$$\begin{tikzcd} S^{-,w,Conf_x}_{Fl,Conf_x} \arrow[r] \arrow[d] & S^{-,Conf_x}_{Fl,Conf_x} \arrow[d] \\ Br^w \arrow[r] & B\backslash G/B^- \end{tikzcd} \tag{6.39}$$

In particular, Br^1 is the unique open cell of $B\backslash G/B^-$, the corresponding subspace $S^{-,1,Conf_x}_{Fl,Conf_x}$ is dense in $S^{-,Conf_x}_{Fl,Conf_x}$.

Follows from the same proof as Proposition 6.1, we could get $S^{-,w,Conf_x}_{Fl,Conf_x}$ and $S^{-,Conf_x}_{Fl,Conf_x}$ (and their closure in $Fl^{\omega^\rho}_{Conf_x}$) factorize with respect to $S^{-,Conf}_{Gr,Conf}$. I.e,

Proposition 6.2.

$$\begin{gathered} S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \\ \cong \\ S^{-,Conf}_{Gr,Conf} \times S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf \times Conf_x}{\times} (Conf \times Conf_x)_{disj} \end{gathered} \tag{6.40}$$

$$\begin{gathered} \bar{S}^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \\ \cong \\ \bar{S}^{-,Conf}_{Gr,Conf} \times \bar{S}^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf \times Conf_x}{\times} (Conf \times Conf_x)_{disj} \end{gathered} \tag{6.41}$$

It is easy to see that the fiber of $S^{-,w,Conf_x}_{Fl,Conf_x}$ over the point D of the form (6.22) is canonically isomorphic to:

$$S^{-,t^\lambda w}_{Fl,x} \times \prod_{1\leq i\leq k} S^{-,\lambda_i}_{Gr,x_i} \tag{6.42}$$

Here, $S^{-,t^\lambda w}_{Fl,x} \subset Fl^{\omega^\rho}_x$ denotes the $N^-(K)^{\omega^\rho}$-orbit passing $t^\lambda w \in Fl^{\omega^\rho}_x$, and $S^{-,\lambda_i}_{Gr,x_i} \subset Gr^{\omega^\rho}_{G,x}$ denotes the $N^-(K)^{\omega^\rho}$-orbit passing $t^{\lambda_i} \in Gr^{\omega^\rho}_{G,x_i}$.

Remark The above identification of the fiber is compatible with the one given in (6.23).

We denote by $\bar{j}^-_{w,Fl,Conf_x}$ the open immersion

$$\bar{j}^-_{w,Fl,Conf_x} : S^{-,w,Conf_x}_{Fl,Conf_x} \longrightarrow \bar{S}^{-,w,Conf_x}_{Fl,Conf_x} \tag{6.43}$$

$\overline{S}^{-,Conf_x}_{Fl,Conf_x}$ is a substack in $Fl^{\omega^\rho}_{G,Conf_x}$, we still denote the restriction of $\mathcal{G}^G$ to this substack by $\mathcal{G}^G$. And $\overline{S}^{-,Conf_x}_{Fl,Conf_x}$ admits a map to $Conf_x$, we could consider the gerbe $\mathcal{G}^\Lambda$ on it by pullback $\mathcal{G}^\Lambda$ from $Conf_x$. (definitions please see Section 2.5)

Definition 6.14. *We define a gerbe on* $\overline{S}^{-,Conf_x}_{Fl,Conf_x}$ *by*

$$\mathcal{G}^{G,T,ratio} := (\mathcal{G}^\Lambda)^{-1} \bigotimes (\mathcal{G}^G) \tag{6.44}$$

The restriction of $\mathcal{G}^{G,T,\mathrm{ratio}}$ to $S^{-,1,Conf_x}_{Fl,Conf_x} \subset \overline{S}^{-,1,Conf_x}_{Fl,Conf_x}$ admits a canonical $N^-(K)^{\omega^\rho}_{Conf_x}$-equivariant trivialization. We have:

$$D_{\mathcal{G}^{G,T,\mathrm{ratio}}}(S^{-,1,Conf_x}_{Fl,Conf_x})^{N^-(K)^{\omega^\rho}_{Conf_x}} \simeq D(S^{-,1,Conf_x}_{Fl,Conf_x})^{N^-(K)^{\omega^\rho}_{Conf_x}}$$

Definition 6.15. *Let us denote by* $\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}}$ *the* $\mathcal{G}^{G,T,ratio}$*-twisted D-module corresponding to the dualizing D-module on* $S^{-,1,Conf_x}_{Fl,Conf_x}$.

In Section 9, we will study the existence of !-extension D-module for a *semi-infinite* (i.e, $N(K)$-equivariant+$T(O)$-equivaraint with respect to a character) D-module on a $N(K)$ (sometimes, a $N(K)_{Ran}$)-orbit. In particular, by the same proof of Corollary 9.2, we have the following existence property:

Corollary 6.2. *The left adjoint functor*

$$\bar{j}^{-}_{w,Fl,Conf_x,!}: D_{\mathcal{G}^{G,T,ratio}}(S^{-,w,Conf_x}_{Fl,Conf_x}) \longrightarrow D_{\mathcal{G}^{G,T,ratio}}(\bar{S}^{-,w,Conf_x}_{Fl,Conf_x})$$

of

$$\bar{j}^{-,!}_{w,1,Fl,Conf_x}: D_{\mathcal{G}^{G,T,ratio}}(\bar{S}^{-,w,Conf_x}_{Fl,Conf_x}) \longrightarrow D_{\mathcal{G}^{G,T,ratio}}(S^{-,w,Conf_x}_{Fl,Conf_x})$$

can be defined on $N^-(K)^{\omega^\rho}_{Conf_x}$*-equivariant D-modules.*

Similarly for the semi-infinite D-module on $Gr^{\omega^\rho}_{Conf}$.

Warning The proof of this corollary is quite far from the subject of this section, hence, readers could pass its proof in the first reading and check it when necessary.

In particular, we could define the !-extension of $\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}}$ and $\omega_{S^{-,Conf}_{Conf,Gr}}$. In order to shorten the notation, let us denote by $j_!(\omega_{S^{-,Conf}_{Conf,Gr}})$ (resp. $j_!(\omega_{S^{-,w,Conf_x}_{Fl,Conf_x}})$) the $\bar{j}^{-}_{w,Gr,Conf,!}(\omega_{S^{-,Conf}_{Conf,Gr}})$ (resp. $\bar{j}^{-}_{w,Fl,Conf_x,!}(\omega_{S^{-,w,Conf_x}_{Fl,Conf_x}})$).

These twisted semi-infinite D-modules satisfy factorization properties. Recall that by [GL1], $j_!(S^{-,Conf}_{Conf,Gr})$ is a factorization algebra on $\bar{S}^{-,Conf}_{Conf,Gr}$, i.e,

$$\begin{aligned} & j_!(\omega_{S^{-,Conf}_{Conf,Gr}})|_{S^{-,Conf}_{Conf,Gr} \underset{Conf}{\times} (Conf\times Conf)_{disj}} \\ & \cong \\ & j_!(\omega_{S^{-,Conf}_{Conf,Gr}}) \boxtimes j_!(\omega_{S^{-,Conf}_{Conf,Gr}})|_{S^{-,Conf}_{Gr,Conf}\times S^{-,Conf}_{Conf,Gr} \underset{Conf\times Conf}{\times}(Conf\times Conf)_{disj}} \end{aligned} \tag{6.45}$$

By Proposition 6.2, we have that $\omega_{S^{-,Conf_x}_{Fl,Conf_x}}$ factorizes w.r.t $\omega_{S^{-,Conf}_{Gr,Conf}}$. i.e,

$$\begin{aligned} & \omega_{S^{-,w,Conf_x}_{Fl,Conf_x}}|_{S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf_x}{\times} (Conf\times Conf_x)_{disj}} \\ & \cong \\ & \omega_{S^{-,Conf}_{Gr,Conf}} \boxtimes \omega_{S^{-,w,Conf_x}_{Fl,Conf_x}}|_{S^{-,Conf}_{Gr,Conf}\times S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf\times Conf_x}{\times}(Conf\times Conf_x)_{disj}} \end{aligned} \tag{6.46}$$

As a corollary, we have:

$$\begin{aligned} & j_!(\omega_{S^{-,w,Conf_x}_{Fl,Conf_x}})|_{S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf_x}{\times} (Conf\times Conf_x)_{disj}} \\ & \cong \\ & j_!(\omega_{S^{-,Conf}_{Gr,Conf}}) \boxtimes j_!(\omega_{S^{-,w,Conf_x}_{Fl,Conf_x}})|_{S^{-,Conf}_{Gr,Conf}\times S^{-,w,Conf_x}_{Fl,Conf_x} \underset{Conf\times Conf_x}{\times}(Conf\times Conf_x)_{disj}} \end{aligned} \tag{6.47}$$

6.3 Constructions of functors

In this section, we will define the functor F^L:

$$F^L : Whit_q(Fl_G^{\omega^\rho}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x) \tag{6.48}$$

To start with, let us summarize the stacks defined in previous sections of this paper in the following diagram:

$$\begin{tikzcd}
 & Fl^{\omega^\rho}_{G,Conf_x} & & \\
 & (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \arrow[u] \arrow[d] & & \overline{S^{-,Conf_x}_{Conf_x,Fl}} \arrow[ull] \\
Fl^{\omega^\rho}_{G,Ran_x} \times Ran_x \arrow[r, "unit"] \arrow[d, "pr_{Ran_x \times Fl \to Fl}"] & (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} & & (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S^{-,Conf_x}_{Conf_x,Fl}} \arrow[u] \arrow[ull] \arrow[dd, "v_{Conf_x}"] \\
Fl^{\omega^\rho}_{G,x} & & & \\
 & & & Conf_x
\end{tikzcd} \tag{6.49}$$

The morphism $unit : Fl_x^{\omega^\rho} \times Ran_x \longrightarrow (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$ is given by (6.12).

Construction The construction of the local functor F^L can be constructed through the following steps:

1. Given a twisted Whittaker D-module $\mathcal{F} \in Whit_q(Fl_x^{\omega^\rho})$, first of all, we !-pullback it to $Fl^{\omega^\rho}_{G,Ran_x} \times Ran_x$. We get a twisted Whittaker D-module on $Fl_x^{\omega^\rho} \times Ran_x$.

2. By Theorem 6.1, the !-pullback along with the morphism *unit* defines an equivalence between the category of twisted Whittaker D-module on $Fl_x^{\omega^\rho} \times Ran_x$ and the Whittaker D-module on $Fl^{\omega^\rho}_{G,Ran_x}$ supporting on $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$. So the twisted D-module obtained in the first step will correspond to a twisted D-module on $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$.

3. Under the isomorphism (9.41), we have the following identification:

$$(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \underset{Ran_x}{\times} (Gr_{T,Ran_x})^{neg}_{\infty\cdot x} \simeq (\bar{S}^{w_0}_{Conf_x,Fl})_{\infty\cdot x} \underset{Conf_x}{\times} (Gr_{T,Ran_x})^{neg}_{\infty\cdot x}$$

we could !-pullback the resulted twisted D-module in the step 2) to $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \underset{Ran_x}{\times} (Gr_{T,Ran_x})^{neg}_{\infty\cdot x}$ and then, by Lemma 2.15, it could descend to a twisted D-module on $(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x}$. We denote this twisted D-module by $sprd_{Fl}(\mathcal{F})$.

$$sprd_{Fl} : Whit_q(Fl_G) \longrightarrow D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x}) \tag{6.50}$$

4. Taking the restriction of $sprd_{Fl}(\mathcal{F})$ to $(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S}^{-,Conf_x}_{Conf_x,Fl}$, and we take its !-tensor product with the !-restriction of the semi-infinite !-extension D-module $j_!(\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}})$ on $\overline{S}^{-,Conf_x}_{Conf_x}$ defined in Section 6.2.

5. Then, take ! (or equivalently, take $*$)-direct image along with the projection v_{Conf_x} with cohomology shift $\langle\lambda, 2\check{\rho}\rangle$ on the connected component $Conf^{\lambda}_x$ of $Conf_x$. We denote this shift by $[deg]$.

The resulted D-module $F^L(\mathcal{F})$ belongs to $D_{\mathcal{G}^\Lambda}(Conf_x)$. Indeed, $sprd_{Fl}(\mathcal{F}) \in D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x})$ and $j_!(\omega_{S^{-,Conf_x}_{Conf_x,Fl}}) \in D_{(\mathcal{G}^{G,T,\text{ratio}})^{-1}}(\bar{S}^{-,Conf_x}_{Conf_x,Fl})$. By the definition of $\mathcal{G}^{G,T,\text{ratio}}$, it is the quotient of $\mathcal{G}^G$ by $\mathcal{G}^\Lambda$. Hence, the resulted D-module is $\mathcal{G}^\Lambda$-twisted.

To summarize,

$$F^L : Whit_q(Fl^{\omega^\rho}_{G,x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

$$\mathcal{F} \longrightarrow v_{Conf_x,*}(sprd_{Fl}(\mathcal{F})|_{(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}} \overset{!}{\otimes} j_!(\omega_{S^{-,Conf_x}_{Conf_x,Fl}}))[deg] \tag{6.51}$$

Similarly, we consider the following diagram,

$$\begin{tikzcd}
 & Gr^{\omega^\rho}_{G,Conf} & \\
 & \bar{S}^0_{Gr,Conf} \arrow[u] \arrow[d] & \overline{S^{-,Conf}_{Conf,Gr}} \arrow[ul] \\
\bar{S}^0_{Gr} \times Ran \arrow[r, "unit_{Gr}"] \arrow[d, "pr_{Ran\times Gr\to Gr}"] & (\bar{S}^0_{Gr,Ran}) & (\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Conf,Gr}} \arrow[u] \arrow[ul] \arrow[dd, "v_{Conf}"] \\
\bar{S}^0_{Gr} & & \\
 & & Conf
\end{tikzcd} \tag{6.52}$$

By applying the same steps 1), 2) and 3) as the above construction (with tiny modifications), we could get a functor:

$$sprd_{Gr} : Whit_q(\bar{S}^0_{Gr}) \longrightarrow D_{\mathcal{G}^G}((\bar{S}^0_{Gr,Conf})) \tag{6.53}$$

Then, we define a functor:

$$F^L_{Gr} : Whit_q(\bar{S}^0_{Gr}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf)$$

given by:

$$\mathcal{F} \longrightarrow v_{Conf,*}(sprd_{Gr}(\mathcal{F})|_{(\bar{S}^0_{Gr,Conf})} \overset{!}{\otimes} j_!(\omega_{S^{-,Conf}_{Conf,Gr}}))[deg] \tag{6.54}$$

It is easy to see that $Whit_q(\bar{S}^0_{Gr}) \simeq Whit_q(S^0_{Gr}) \simeq Vect$, hence, there exists a unique irreducible Whittaker D-module on $\bar{S}^0_{Gr}$.

Definition 6.16. *We denote by $\Omega^{L'}_{q,fact} \in D_{\mathcal{G}^\Lambda}(Conf)$ the image of the unique irreducible Whittaker D-module on $\bar{S}^0_{Gr}$ under the functor (6.54).*

The most important feature of $\Omega^{L'}_{q,fact}$ is its factorization property.

Proposition 6.3. *$\Omega^{L'}_{q,fact}$ is a factorization algebra on $Conf$.*

Proof. Because v_{Conf} is compatible with the factorization, hence, it suffices to show that

$$sprd_{Gr}(\mathcal{F})|_{(\bar{S}^0_{Gr,Conf})} \overset{!}{\otimes} j_!(\omega_{S^{-,Conf}_{Conf,Gr}})[deg]$$

is factorizable.

By construction, the image of the unique irreducible Whittaker D-module on $\bar{S}^0_{Gr}$ under $sprd_{Gr}$ is given by the pullback of Vac to $\bar{S}^0_{Gr,Conf}$. According to Corollary 6.1, Vac is factorizable. And the pullback from $\bar{S}^0_{Gr,Ran}$ to $\bar{S}^0_{Gr,Conf}$ is compatible with the factorization structure, hence, the image of the irreducible object in $Whit_q(\bar{S}^0_{Gr})$ under the functor $sprd_{Gr}$ is still factorizable.

What's more, by (6.45), we have $j_!(\omega_{S^{-,Conf}_{Conf,Gr}})$ is factorizable.

Now, Proposition 6.3 comes from the fact that the tensor product of factorization algebras is still factorizable. ∎

Given the factorization algebra $\Omega^{L'}_{q,fact}$, we could define the category of $\Omega^{L'}_{q,fact}$ factorization modules on $Conf_x$.

$$\Omega^{L,'}_{q,fact} - FactMod \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

According to the above constructions of F^L and $\Omega^{L,'}_{q,fact}$, we claim that the functor F^L factors through $\Omega^{L'}_{q,fact} - FactMod$.

Proposition 6.4. *Given any $\mathcal{F} \in Whit_q(Fl)$, its image $F^L(\mathcal{F})$ has a naturally defined $\Omega^{L'}_{q,fact}$ factorization module structure.*

Proof. Because v_{Conf_x} is compatible with the factorization of v_{Conf}, hence, it suffices to show that

$$sprd_{Fl}(\mathcal{F})|_{((\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x})} \overset{!}{\otimes} j_!(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}})[deg]$$

factorizes with respect to $sprd_{Gr}(\mathcal{F})|_{(\bar{S}^0_{Gr,Conf})} \overset{!}{\otimes} j_!(\omega_{S^{-,Conf}_{Conf,Gr}})[deg]$.

According to Corollary 6.1, we have that any Whittaker D-module on $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$ factorizes with respect to Vac. And the pullback from $(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}$ to $(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x}$ is compatible with the factorization structure, hence, the image of any Whittaker object on $Fl_{G,x}$ under the functor $sprd_{Fl}$ is a factorization module with respect to the pullback of Vac.

What's more, by (6.47), $j_!(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}})$ factorizes with respect to $j_!(\omega_{S^{-,Conf}_{Conf,Gr}})$.

Now, Proposition 6.3 comes from the fact that the tensor product of factorization modules is still a factorization module with respect to the tensor product of corresponding factorization algebras. □

Hence, we could write the functor F^L as:

$$F^L : Whit_q(Fl) \longrightarrow \Omega^{L,'}_{q,fact} - FactMod \tag{6.55}$$

Similar to the constructions of F^L and F^L_{Gr}, if we replace the semi-infinite !-extension D-module by semi-infinite $*$-extension D-module in the construction of F^L_{Gr}, we could also get functors:

$$F^{KD} : Whit_q(Fl^{\omega^\rho}_{G,x}) \longrightarrow D_{\mathcal{G}^\wedge}(Conf_x)$$

$$\mathcal{F} \longrightarrow v_{Conf_x,*}(sprd_{Fl}(\mathcal{F})|_{(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x}} \overset{!}{\otimes} j_*(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}}))[deg] \tag{6.56}$$

$$F^{KD}_{Gr} : Whit_q(\bar{S}^0_{Gr}) \longrightarrow D_{\mathcal{G}^\wedge}(Conf)$$

$$\mathcal{F} \longrightarrow v_{Conf,*}(sprd_{Gr}(\mathcal{F})|_{(\bar{S}^0_{Gr,Conf})} \overset{!}{\otimes} j_*(\omega_{S^{-,Conf}_{Conf,Gr}}))[deg] \tag{6.57}$$

Definition 6.17. *We denote by $\Omega^{KD,'}_{q,fact} \in D_{\mathcal{G}^\wedge}(Conf)$ the image of the unique irreducible Whittaker D-module on $\bar{S}^0_{Gr}$ under the functor (6.57). And we denote by $\Omega^{KD,'}_{q,fact} - FactMod$ the category of factorization modules of $\Omega^{KD,'}_{q,fact}$.*

And we have the analog of Proposition 6.3.

Proposition 6.5. *F^{KD} factors through $\Omega^{KD,'}_{q,fact} - FactMod$, i.e, it gives rise to a functor*

$$F^{KD} : Whit_q(Fl^{\omega^\rho}_{G,x}) \longrightarrow \Omega^{KD,'}_{q,fact} - FactMod \tag{6.58}$$

Remark The functor F^{KD} is not the one that we will use to construct equivalence, but it will be used to prove the factorization structure of the global functor F^{KD}_{glob} constructed in Definition 7.17 and the latter will be important for us in the proof of Theorem 4.1.

6.4 Calculation of the !-fiber of F^L and F^{KD}

Note that because of Section 4.1.2, in the category of $\Omega_{\mathcal{A}} - FactMod$, we know that the standard object $\Delta_{\lambda,\mathcal{A}}$ could be uniquely characterized by its $*$-fiber at $\mu \cdot x, \mu \in \Lambda$ and the costandard object $\nabla_{\lambda,\mathcal{A}}$ could be uniquely characterized by its !-fiber at $\mu \cdot x, \mu \in \Lambda$. Hence, in order to prove that F^L sends standard objects to standard objects, costandard objects to costandard objects, we only need to find an explicit expression of the !-fibers and $*$-fibers of the image of F^L.

The theory of D-modules is friendly with !-fiber. There are two reasons: the first one is because the !-pullback functor is always well-defined, the second one is because we have a base change theorem (Lemma A.5), hence, the calculation will be much easier than the calculation of $*$-fiber.

In this section, we will give an explicit formula for the !-fiber of F^L at $\lambda \cdot x$.

The most important observation of this section is Proposition 6.6, i.e, the twisted BMW standard object Δ_λ corepresents the functor $H(Fl, - \otimes j_!(\omega_{S^{-,\lambda}_{Fl,x}}))$. Here, $\omega_{S^{-,\lambda}_{Fl,x}}$ denotes the dualizing sheaf (with a cohomological shift $[2|\lambda|]$) on $S^{-,\lambda}$. And we have a similar result for F^{KD} (Proposition 6.7).

By definition, given a point $D = \lambda \cdot x + \sum_i \lambda_i \cdot x_i \in Conf_x$ such that $x, x_1, ..., x_k$ are pairwise disjoint. Under the identification (6.23), the fiber of $(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty \cdot x}$ over D is canonically isomorphic to:

$$Fl_x^{\omega^\rho} \times \prod_i \bar{S}^0_{Gr,x_i} \tag{6.59}$$

And the fiber of $\bar{S}^{-,Conf_x}_{Conf_x,Fl}$ over D is canonically isomorphic to:

$$\bar{S}^{-,\lambda}_{Fl,x} \times \prod_i \bar{S}^{-,\lambda_i}_{Gr,x_i} \tag{6.60}$$

Hence, the fiber of $\bar{S}^{-,Conf_x}_{Conf_x,Fl} \cap (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty \cdot x}$ over $\lambda \cdot x$ is given by $\bar{S}^{-,\lambda}_{Fl,x}$. Considering the following Cartesian diagram:

$$\begin{tikzcd} \bar{S}^{-,\lambda}_{Fl,x} \arrow[r] \arrow[d] & \bar{S}^{-,Conf_x}_{Conf_x,Fl} \cap (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty \cdot x} \arrow[d, "v_{Conf_x}"] \\ \lambda \cdot x \arrow[r, "i_\lambda"] & Conf_x \end{tikzcd} \tag{6.61}$$

We denote the embedding of $\bar{S}^{-,\lambda}_{Fl,x}$ into $\bar{S}^{-,Conf_x}_{Conf_x,Fl}$ by $i^{-,Conf_x}_{\lambda,Conf_x}$,

$$i^{-,Conf_x}_{\lambda,Conf_x} : \bar{S}^{-,\lambda}_{Fl,x} \longrightarrow \bar{S}^{-,Conf_x}_{Conf_x,Fl} \tag{6.62}$$

And we denote by $i^{w_0}_{\lambda,Conf_x}$:

$$i^{w_0}_{\lambda,Conf_x} : Fl_x^{\omega^\rho} \longrightarrow (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty \cdot x} \tag{6.63}$$

Then, after taking a trivialization of $\mathcal{G}^\lambda$, by base change theorem, we have

$$\begin{aligned} i^!_\lambda(F^L(\mathcal{F})) \simeq & i^!_\lambda \circ v_{Conf_x,*}(sprd_{Fl}(\mathcal{F})|_{((\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x})} \overset{!}{\otimes} \\ & j_!(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}}))[2|\lambda|] \\ \simeq & H(Fl^{\omega^\rho}_x, i^{w_0,!}_{\lambda,Conf_x}(sprd_{Fl}(\mathcal{F})) \overset{!}{\otimes} \\ & i^{-,Conf_x,!}_{\lambda,Conf_x}(j_!(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}}))[2|\lambda|]) \end{aligned} \tag{6.64}$$

By the construction (6.50), we could see:

$$i^{w_0,!}_{\lambda,Conf_x}(sprd_{Fl}(\mathcal{F})) \simeq \mathcal{F},\ \forall \lambda \in \Lambda, \mathcal{F} \in Whit_q(Fl^{\omega^\rho}_x)$$

Later, in Section 9, we will prove the following statement (Theorem 9.2 and (9.63) in Corollary 9.7):

$$i^{-,Conf_x,!}_{\lambda,Conf_x}(j_!(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}}))[2|\lambda|] \simeq j_!(\omega_{S^{-,\lambda}_{Fl,x}}) := t^\lambda j_!(\omega_{S^{-,0}_{Fl,x}})[2|\lambda|]$$

As a result,

Proposition 6.6. *By taking a trivialization of $\mathcal{G}^\lambda$,*

$$i^!_\lambda(F^L(\mathcal{F})) \simeq H(Fl^{\omega^\rho}_x, \mathcal{F} \overset{!}{\otimes} j_!(\omega_{S^{-,\lambda}_{Fl,x}})) \tag{6.65}$$

Similarly, we have:

Proposition 6.7. *By taking a trivialization of $\mathcal{G}^\lambda$,*

$$i^!_\lambda(F^{DK}(\mathcal{F})) \simeq H(Fl^{\omega^\rho}_x, \mathcal{F} \overset{!}{\otimes} j_*(\omega_{S^{-,\lambda}_{Fl,x}})) \tag{6.66}$$

The following corollary related our functor F^L with the standard objects that we constructed in Definition 5.14.

Corollary 6.3. *Given $\lambda \in \Lambda$ and fix a trivialization of $\mathcal{G}^\lambda$, then, we have:*

$$i^!_\lambda(F^L(\mathcal{F})) \simeq RHom_{Whit_q}(\Delta_\lambda, \mathcal{F}) \tag{6.67}$$

Proof. In order to simplify the notations, let us just omit the twisting notations.

According to Proposition 6.6, we have to prove:

$$H(Fl^{\omega^\rho}_x, \mathcal{F} \overset{!}{\otimes} j_!(\omega_{S^{-,\lambda}_{Fl,x}})) \simeq RHom_{Whit_q}(\Delta_\lambda, \mathcal{F})$$

According to the definition of the Whittaker category, $\mathcal{F}$ is $(N(K)^{\omega^\rho}_x, \chi)$-equivariant. In particular, it is $(N(O)^{\omega^\rho}_x, \chi)$-equivariant, and we notice that $\chi|_{N(O)^{\omega^\rho}_x}$ is trivial.

So, we have:

$$
\begin{aligned}
H(Fl_x^{\omega^\rho}, \mathcal{F} \overset{!}{\otimes} j_!(\omega_{S^{-,\lambda}_{Fl,x}})) =& H(Fl_x^{\omega^\rho}, \mathcal{F} \overset{!}{\otimes} Av_*^{N(O)}(j_!(\omega_{S^{-,\lambda}_{Fl,x}})))\\
\underset{Prop.6.8}{=}& H(Fl_x^{\omega^\rho}, \mathcal{F} \overset{!}{\otimes} \mathfrak{J}^{\mathbb{D}}_\lambda)\\
=& RHom_{Whit_q}(Av_!^{N(K),\chi}(\delta_0), \mathcal{F} \star \mathfrak{J}_{-\lambda})\\
=& RHom_{Whit_q}(Av_!^{N(K),\chi}(\mathfrak{J}_0), \mathcal{F} \star \mathfrak{J}_{-\lambda})\\
=& RHom_{Whit_q}(Av_!^{N(K),\chi}(\mathfrak{J}_0) \star \mathfrak{J}_\lambda, \mathcal{F})\\
=& RHom_{Whit_q}(\Delta_\lambda, \mathcal{F})
\end{aligned}
$$

□

Remark The isomorphism of (6.67) is actually canonical, because both sides are defined rely on a choice of the twisting $\mathcal{G}^\lambda$, and they are compatible.

Proposition 6.8. *Given $\mu \in \Lambda$,*

$$
Av_*^{N(O)}(j_{\mu,!}(\omega_{S^{-,\mu}_{Fl,x}})) \simeq \mathfrak{J}^{\mathbb{D}}_\mu \tag{6.68}
$$

Proof. First, we write $j_{\mu,!}(\omega_{S^{-,\mu}_{Fl,x}})$ as $\underset{\alpha-\mu\in\Lambda^+}{colim}\ t^\alpha{}_\alpha(\mathfrak{J}_{-\alpha+\mu,!})[\langle\alpha, 2\check{\rho}\rangle]$. And we notice that ${}_\alpha(\mathfrak{J}_{-\alpha+\mu,!})$ is $(I, b_{\mu-\alpha})$-equivariant. And for an $(I, b_{\mu-\alpha})$-equivariant D-module $\mathcal{F}$, we have $Av_*^{N(O)}(t^\alpha \cdot \mathcal{F})[\langle\alpha, 2\check{\rho}\rangle] \simeq Av_*^{I,\mu-\alpha/T(O),\mu-\alpha}(t^\alpha \cdot \mathcal{F})[\langle\alpha, 2\check{\rho}\rangle] \simeq {}_{-\alpha}(\mathfrak{J}_{\alpha,*})_{\mu-\alpha} \underset{I,\mu-\alpha}{\star} \mathcal{F}$. So, we have:

$$
\begin{aligned}
Av_*^{N(O)}(\underset{\alpha,\alpha-\mu\in\Lambda^+}{colim}\ t^\alpha{}_\alpha(\mathfrak{J}_{-\alpha+\mu,!})[\langle\alpha, 2\check{\rho}\rangle]) \simeq& \underset{\alpha,\alpha-\mu\in\Lambda^+}{colim}(\mathfrak{J}_{\alpha,*})_{\mu-\alpha} \star (\mathfrak{J}_{-\alpha+\mu,!})\\
\simeq& \mathfrak{J}^{\mathbb{D}}_\mu
\end{aligned} \tag{6.69}
$$

□

Similarly, we have:

Proposition 6.9. *Given $\mu \in \Lambda$, then, we have*

$$
Av_*^{N(O)}(j_{\mu,*}(\omega_{S^{-,\mu}_{Fl,x}})) = \underset{\alpha\in\Lambda^+}{colim}(\mathfrak{J}_{\alpha,*})_{\eta-\alpha} \star (\mathfrak{J}_{-\alpha+\mu,*})
$$

6.5 Proposition 6.11⇒ main theorem

This section we will be devoted to the proof of Theorem 4.1. But let us replace the factorization algebra Ω_q^L by $\Omega_{q,fact}^{L,'}$. In fact, we will see in Corollary 9.8 and Lemma 9.11, when q avoids small torsion, Ω_q^L and $\Omega_{q,fact}^{L,'}$ are isomorphic.

Assume $\Omega_q^L \simeq \Omega_q^{L,'}$ when q avoids small torsion (it is the statement of Corollary 9.10 which will be proved with a lemma from [Ga6]), then Theorem

4.1 could be implied by the following more general theorem:
Theorem 4.1'. For any q,

$$F^L : Whit_q(Fl_x^{\omega^\rho}) \longrightarrow \Omega^{L,'}_{q,fact} - FactMod$$

is a t-exact equivalence.

First of all, let us check the compatibility of costandard objects under F^L.

Proposition 6.10.

$$F^L(\nabla_\lambda) \simeq \nabla_{\lambda,\Omega^{L,'}_{q,fact}} \tag{6.70}$$

Proof. Followed from Proposition 5.6 and the isomorphism (6.67) $\square$

Then, we prove Theorem 4.1' with the following proposition:

Proposition 6.11. *Given $\lambda \in \Lambda$, then, the functor*

$$F^L : Whit_q(Fl_x^{\omega^\rho}) \longrightarrow \Omega^{L,'}_{q,fact} - FactMod$$

sends standards to standards, i.e,

$$F^L(\Delta_\lambda) = \Delta_{q,\Omega^{L,'}_{q,fact}}$$

The proof of Proposition 6.11 will occupy the rest of the paper and finally be given in Section 10.5. Now, we will give the proof of Theorem 4.1'.

Proof. (of Theorem 4.1').

Because standards Δ_λ generates the category $Whit_q(Fl_x^{\omega^\rho})$ and standards $\Delta_{\lambda,\Omega^{L,'}_q}$ generates the category $\Omega^{L,'}_{q,fact} - FactMod$, so we could conclude that the functor F^L is essentially surjective by Proposition 6.11.

The fully faithfullness comes from the isomorphism (6.67).

We need to prove that the following map is an isomorphism,

$$RHom_{Whit_q(Fl_x^{\omega^\rho})}(\mathcal{F}_1, \mathcal{F}_2) \simeq RHom_{\Omega^{L,'}_{q,fact} - FactMod}(F^L(\mathcal{F}_1), F^L(\mathcal{F}_2))$$

for any $\mathcal{F}_1, \mathcal{F}_2 \in Whit_q(Fl_x^{\omega^\rho})$.

Because standards $\Delta_\lambda[k]$ generate the category $Whit_q(Fl_x^{\omega^\rho})$ by colimits, so it is equivalent to prove:

$$RHom_{Whit_q(Fl_x^{\omega^\rho})}(\Delta_\lambda, \mathcal{F}_2) \simeq RHom_{\Omega^{L,'}_{q,fact} - FactMod}(F^L(\Delta_\lambda), F^L(\mathcal{F}_2))$$

for any $\mathcal{F}_2 \in Whit_q(Fl_x^{\omega^\rho})$.

It is true by the following reason:

$$\begin{aligned} RHom_{Whit_q(Fl_x^{\omega^\rho})}(\Delta_\lambda, \mathcal{F}_2) & \underset{F^L, isomorphism\ 6.67}{\xrightarrow{\sim}} i^!_\lambda(F^L(\mathcal{F}_2)) \\ & \underset{prop\ 6.11}{=} RHom_{\Omega^{L,'}_{q,fact} - FactMod}(F^L(\Delta_\lambda), F^L(\mathcal{F}_2)) \end{aligned}$$

And by definition of $t-structures$ on both sides, both sides are defined by the 'Hom' with standard objects, and we have already seen from above (Proposition 6.10 and 6.11) that the functor F^L preserves standards and costandards. Hence, we get F^L is $t-exact$. $\square$

6.6 Proof of Theorem 6.1

This section supplies the proof of Theorem 6.1. It is different from the theme of this paper, hence, readers could skip this part and it won't influence the understanding of other parts.

Now, we essentially copy the proof used in [Ga5] to prove Theorem 6.1. The proof of (6.17) and (6.18) are the same, so we only prove the second one.

The strategy is as follows:

1. We first reduce the question to X^I from Ran_x.
2. Then, we prove Theorem 6.1 on $X^\beta \subset X^I$ which forms a stratification of X^I. From here, we know that the functor $unit^!$ is conservative.
3. The last step is to prove the left adjoint functor of $unit^!$ is fully faithful.

We recall that Ran_x can be written as $\underset{\{*\in I\}\in fSet_*^{surj}}{colim} X^I \underset{X}{\times}\{x\}$. Similarly, we have:

$$(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \simeq \underset{\{*\in I\}}{colim}(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \underset{Ran_x}{\times} (X^I \underset{X}{\times}\{x\}) \tag{6.71}$$

We denote by $(\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x}$ the fiber product $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} \underset{Ran_x}{\times} (X^I \underset{X}{\times}\{x\})$, we have $(\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x} = \underset{\{*\in I\}}{colim}(\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x}$. From this identification, we have:

$$D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x}) \simeq \underset{\{*\in I\}}{lim}\, D_{\mathcal{G}^G}((\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x})$$

here, given any surjective map $\phi : \{* \in I\} \longrightarrow \{* \in J\}$, s.t $\phi(*) = *$, the transition functor is given by $\Delta^!_\phi$: here, the induced inclusion map $\Delta_\phi : X^J \underset{X}{\times}\{x\} \longrightarrow X^I \underset{X}{\times}\{x\}$ is given by: $(x = x_0, x_1, ..., x_J) \to (x = x'_0, x'_1 = x_{\phi(1)}, ..., x'_I = x_{\phi(I)})$.

After adding equivariant conditions, we have the following description of $D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_{Ran_x},\chi_{Ran_x}}$:

$$D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Ran_x,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_{Ran_x},\chi_{Ran_x}} \simeq \lim_{\{*\in I\}} D_{\mathcal{G}^G}((\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_I,\chi_I}$$

It suffices to prove:

Proposition 6.12.

$$D_{\mathcal{G}^G}((\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_I,\chi_I} \xrightarrow{\sim} D_{\mathcal{G}}(Fl^{\omega^\rho}_x \times X^I_x)^{N(K)^{\omega^\rho}_x,\chi} \tag{6.72}$$

And Theorem 6.1 is the limit of the claim above.

Given a point in $X^I_x := X^I \underset{X}{\times} \{x\}$, it means $|I|$ many points in X(some of points can be coincident). We can classify them according to the pattern of collision of points. We can give X^I_x a stratification as follows:

$$X^I_x = \bigsqcup_{\beta \text{ a pattern of } |I|} X^\beta_x$$

Denote

$$(\bar{S}^{w_0}_{\beta,Fl})_{\infty\cdot x} := (\bar{S}^{w_0}_{I,Fl})_{\infty\cdot x} \underset{X^I_x}{\times} X^\beta_x$$

We claim:

Proposition 6.13. $D_{\mathcal{G}^G}((\bar{S}^{w_0}_{\beta,Fl})_{\infty\cdot x})^{N(K)^{\omega^\rho}_\beta,\chi_\beta} \xrightarrow{\sim} D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times X^\beta_x)^{N(K)^{\omega^\rho_\beta},\chi_\beta}$

Proof. If we restrict our claim to a pattern β, all geometric objects considered in this section become much easier to describe.

For any pattern β, we could find a number $n \in \mathbb{N}$, s.t, $X^\beta_x \simeq (X-x)^n - Diag$. Here, $Diag$ denotes the diagonal divisor of $(X-x)^n$. (actually, n equals the number of different points in the pattern β -1).

Then, we have the following identifications:

$$(\bar{S}^{w_0}_{\beta,Fl})_{\infty\cdot x} \simeq (((X-x)^n - Diag) \underset{Ran}{\times} \bar{S}^0_{Ran,Gr}) \times Fl^{\omega^\rho}_x \tag{6.73}$$

and

$$N(K)^{\omega^\rho}_\beta := (((X-x)^n - Diag) \underset{Ran}{\times} N(K)^{\omega^\rho}_{Ran}) \times N(K)^{\omega^\rho}_x \tag{6.74}$$

Because there is no Whittaker D-module supported on $\bar{S}^0_{Ran,Gr} - S^0_{Ran,Gr}$,

$$D_{\mathcal{G}^G}((((X-x)^n - Diag) \underset{Ran}{\times} \bar{S}^0_{Ran,Gr}) \times Fl^{\omega^\rho}_x)^{N(K)^{\omega^\rho}_\beta,\chi}$$

is equivalent to

$$D_{\mathcal{G}^G}((((X-x)^n - Diag) \underset{Ran}{\times} S^0_{Ran,Gr}) \times Fl^{\omega^\rho}_x)^{N(K)^{\omega^\rho}_\beta,\chi}$$

And we note that under the identification (6.74), N'_β is isomorphic to

$$(((X-x)^n - Diag) \underset{Ran}{\times} N(O)^{\omega^\rho}_{Ran}) \times N(K)^{\omega^\rho}_x$$

, and $N(O)^{\omega^\rho}_{Ran}$ is a pro-unipotent group over Ran such that the restriction of χ_{Ran} onside is trivial. As a corollary, $D_{\mathcal{G}^G}((((X-x)^n - Diag) \underset{Ran}{\times} S^0_{Ran,Gr}) \times Fl^{\omega^\rho}_x)^{N(K)^{\omega^\rho}_\beta,\chi}$ is equivalent to $D_{\mathcal{G}^G}((((X-x)^n - Diag) \times Fl^{\omega^\rho}_x)^{N(K)^{\omega^\rho}_x,\chi} \simeq D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x \times X^\beta_x)^{N(K)^{\omega^\rho_\beta},\chi_\beta}$. ∎

The next step is to glue the equivalence of categories given by Proposition 6.13 to an equivalence over X_x^I.

By Proposition 6.13, $unit_I^!$ is conservative (if $unit_I^!(\mathcal{F}_1) \simeq unit_I^!(\mathcal{F}_2)$, then, $unit_I^!(\mathcal{F}_1)|_{X_x^\beta \times Fl_x^{\omega^\rho}} \simeq unit_I^!(\mathcal{F}_2)|_{X_x^\beta \times Fl_x^{\omega^\rho}}$ for any β. Hence, by proposition 6.13, $\mathcal{F}_1|_{Fl_\beta^{\omega^\rho}} \simeq \mathcal{F}_2|_{Fl_\beta^{\omega^\rho}}$. $Fl_I^{\omega^\rho}$ is the union of $Fl_\beta^{\omega^\rho}$, hence, $\mathcal{F}_1 \simeq \mathcal{F}_2$). Now, by lemma 6.1 we need to prove $unit^!$ admits a left adjoint functor $(unit_I^!)^L$ such that $unit_I^! \circ (unit^!)^L \simeq id$. In fact, we only need to prove the existence of the functor $(unit_I^!)^L$.

Lemma 6.1. *Given a pair of adjoint functors $F \dashv G$ between two categories:*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

i).if $id \cong G \circ F$, then, F, G induce an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if G is conservative.

ii). if G is conservative, then, $\mathcal{G}$ is an equivalence, if and only if $id \xrightarrow{\sim} G \circ F$.

Now, we prove Proposition 6.12 with the following Lemma:

Lemma 6.2. *$unit_I^! : Whit_q((\bar{S}_{I,Fl}^{w_0})_{\infty \cdot x}) \longrightarrow Whit_q(X_x^I \times Fl_x^{\omega^\rho})$ admits a left adjoint functor $(unit_I)^L : Whit_q(X^I \times Fl_x^{\omega^\rho}) \longrightarrow Whit_q((\bar{S}_{I,Fl}^{w_0})_{\infty \cdot x})$ and this left adjoint functor commutes with $D(X_x^I)$-action, i.e,*

The proof of Lemma 6.2 will be given in Appendix C.

Proof. (Proposition 6.12) We denote ι_β the locally closed embedding,

$$\iota_\beta : X_x^\beta \longrightarrow X_x^I$$

Because X_x^I is the union of X_x^β, in order to prove $id \xrightarrow{\sim} unit_I^! \circ (unit_I^!)^L$, it suffices to prove:

$$\iota_\beta^! \xrightarrow{\sim} \iota_\beta^! \circ unit_I^! \circ (unit_I^!)^!$$

Note that the above functor can be factorized as:

$$\iota_\beta^! \longrightarrow unit_\beta^! \circ (unit_\beta^!)^L \circ \iota_\beta^! \longrightarrow (\iota_\beta)^! \circ unit_I^! \circ (unit_I^!)^L$$

We have $id \xrightarrow{\sim} unit_\beta^! \circ (unit_\beta^!)^L$ by proposition 6.13. So we have to prove:

$$unit_\beta^! \circ (unit_\beta^!)^L \circ (\iota_\beta)^! \xrightarrow{\sim} (\iota_\beta)^! \circ unit_I^! \circ (unit_I^!)^L$$

We have $\iota_\beta^! \circ unit_I^! \simeq unit_\beta^! \circ \iota_\beta^!$. So we only need to prove:

$$(unit_\beta^!)^L \circ (\iota_\beta)^! \xrightarrow{\sim} (\iota_\beta)^! \circ (unit_I^!)^L$$

Given two twisted D-modules $\mathcal{F}_1 \in Whit_q(X_x^I \times Fl_x^{\omega^\rho})$, $\mathcal{F}_2 \in Whit_q(Fl_\beta^{\omega^\rho})$

$$
\begin{aligned}
& RHom_{Whit_q(Fl_\beta^{\omega^\rho})}((unit_\beta^!)^L \circ (\iota_\beta)^!(\mathcal{F}_1), \mathcal{F}_2) \\
\simeq & RHom_{Whit_q(X_x^\beta \times Fl_x^{\omega^\rho})}((\iota_\beta)^!(\mathcal{F}_1), unit_\beta^!(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(X_x^I \times Fl_x^{\omega^\rho})}((\iota_\beta)_* \circ (\iota_\beta)^!(\mathcal{F}_1), (\iota_\beta)_* \circ unit_\beta^!(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(X_x^I \times Fl_x^{\omega^\rho})}((\iota_\beta)_* \circ (\iota_\beta)^!(\mathcal{F}_1), (unit_I^!) \circ (\iota_\beta)_*(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(X_x^I \times Fl_x^{\omega^\rho})}((\iota_\beta)_* \circ (\omega_{X_x^\beta}) \overset{!}{\bigotimes}(\mathcal{F}_1), (unit_I^!) \circ (\iota_\beta)_*(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(Fl_I^{\omega^\rho})}((unit_I^!)^L((\iota_\beta)_* \circ (\omega_{X_x^\beta}) \overset{!}{\bigotimes}(\mathcal{F}_1)), (\iota_\beta)_*(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(Fl_I^{\omega^\rho})}((\iota_\beta)_* \circ (\omega_{X_x^\beta}) \overset{!}{\bigotimes}((unit_I^!)^L(\mathcal{F}_1)), (\iota_\beta)_*(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(Fl_I^{\omega^\rho})}((\iota_\beta)_* \circ (\iota_\beta)^! \circ ((unit_I^!)^L(\mathcal{F}_1)), (\iota_\beta)_*(\mathcal{F}_2)) \\
\simeq & RHom_{Whit_q(Fl_\beta^{\omega^\rho})}((\iota_\beta)^! \circ ((unit_I^!)^L(\mathcal{F}_1)), \mathcal{F}_2)
\end{aligned}
$$

It implies $(unit_\beta^!)^L \circ (\iota_\beta)^! \simeq (\iota_\beta)^! \circ ((unit_I^!)^L$.

As a direct corollary,

$$
unit^! : Whit_q((\bar{S}^{w_0}_{I,Fl})_{\infty \cdot x}) \longrightarrow Whit_q(Ran_x \times Fl_x^{\omega^\rho})
$$

is an equivalence. □

7 Whittaker category: global definition

In Section 6.5, we have already reduced the proof of the main theorem (i.e, Theorem 4.1) to the Proposition 6.11. From now on, we will focus on the proof of this proposition. As we noted before, it is hard to calculate $*$-fiber. But luckily, we could use duality functor to transfer the calculation of $*$-fiber to a calculation of !-fiber. To make the calculation possible, we introduce the global counterparts of $Whit_q(Fl)$ and F^L. In this section, our aim is to construct a globally defined Whittaker category on affine flags and a functor from such a category to $\Omega_{q,fact}^{L,'} - FactMod$ and show they are the same as locally defined ones.

The content of this section:

In Section 7.1, we will use Drinfeld compactification to define some algebraic stacks.

Then, in Section 7.2, we will define the global Whittaker category by inputting an equivariant condition on the category of D-modules on the Drinfeld compactification. And we will study this Whittaker category by giving Drinfeld compactification a stratification.

In Section 7.3, we will introduce the global counterpart of the !-extension semi-infinite D-module $j_*(\omega_{S^{-,1,Conf_x}_{Conf_x,Fl}})$, etc.

In Section 7.4, we will recall the definition of Zastava space and its affine flags variant.

In Section 7.5, we will define the functor F^L_{glob} which sends the global Whittaker category to the category of factorization modules.

In Section 7.6, we will decompose the proof of Proposition 6.11 into two parts: Theorem 7.1 and Proposition 7.4.

7.1 Drinfeld compactifications

Fix $x \in X$, in this section, we will use [Ga1], [FGV] 's method to define the Whittaker category by a global construction. The key tool in this section is Drinfel'd compactification. We will define $Whit_{q,Fl}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$, the (twisted) global Whittaker category on affine flag, as a full subcategory in $D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}}')_{\infty\cdot x})$ satisfying some equivariant property with respect to a certain groupoid.

Given a scheme S and a G bundle $\mathcal{P}_G$ on S, a data of N-reduction of $\mathcal{P}_G$ on S is equivalent to a collection of injective morphisms of vector bundles $\{\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}, \forall\check{\lambda} \in \Lambda^+\}$, such that $\kappa^{\check{\lambda}}$ satisfy Plücker relations. Drinfel'd compactification means, we only require $\kappa^{\check{\lambda}}$ to be regular instead of injective, i.e, the quotient is not necessary torsion free.

Now, we define several algebraic stacks using Drinfeld compactification. We will use them to construct the global twisted Whittaker category.

Definition 7.1. $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$

Let $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ be the stack classifying the following data: $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \forall\check{\lambda} \in \Lambda^+\}, \epsilon)$. Here, $\mathcal{P}_G \in Bun_G$, $\kappa^{\check{\lambda}}$ is a family of maps of coherent sheaves: $\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$, such that it is regular outside $x \in X$. ϵ is a B-reduction of $\mathcal{P}_G$ at x.

Ignore the Iwahori structure at x, we may construct the following stacks

Definition 7.2. *Let $(\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}$ be the stack classifying the following data: $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \forall\check{\lambda} \in \Lambda^+\},)$. Here, $\mathcal{P}_G \in Bun_G$, $\kappa^{\check{\lambda}}$ is a family of maps of coherent sheaves: $\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$, it is regular on $X - x$.*

If we require $\kappa^{\check{\lambda}}$ to be regular on the whole curve X, then the resulted stack is denoted by $\overline{Bun_N^{\omega^\rho}}$.

The stacks defined above are Artin stacks, we have well-defined category of D-modules on such stacks.

If we pull the gerbe $\mathcal{G}^G$ on Bun_G back to the stacks defined above through the obvious morphisms (forget $\kappa^{\check{\lambda}}, \epsilon$, etc):

$$(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \longrightarrow Bun_G$$

$$(\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x} \longrightarrow Bun_G$$

$$\overline{Bun_N^{\omega^\rho}} \longrightarrow Bun_G$$

we may get gerbes on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$, $(\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}$ and $\overline{Bun_N^{\omega^\rho}}$ and we could consider the corresponding twisted D-module categories.

In the construction of the global counterpart of F^L, we also need to consider the following algebraic stacks:

Definition 7.3. *We define Bun'_{B^-} to be the algebraic stack classifying a B^--bundle on X with a B-reduction at x of the induced G-bundle and Bun'_N to be the algebraic stack classifying a N-bundle on X with a B-reduction at x of the induced G-bundle.*

We denote by $\overline{Bun_{B^-}}'$ the Drinfeld compactification of Bun'_{B^-}.

By forgetting the Iwahori structure, we could define $\overline{Bun_{B^-}}$.

Bun'_{B^-} has a map to $B^-\backslash G/B$. We will denote the preimage of $Br^w, w \in W$ in Bun'_{B^-} by $Bun^w_{B^-}$. For convenience, we denote by Bun''_{B^-} the stack $Bun^1_{B^-}$. And through this section, if we mark a bar $^-$ over some stack, it means the Drinfeld compactification of the corresponding stack.

7.2 Global Whittaker category

In this section, we will define the global Whittaker category which is expected to be equivalent to the locally defined one.

Let us define the global Whittaker category

$$Whit_{q,Fl}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}) \subset D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$$

and

$$Whit_{q,Gr}((\overline{Bun_N^{\omega^\rho}})) \subset D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}}))$$

We perform the definition of $Whit_{q,Fl}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$, the definition of the latter is by the same way.

Given a point $\bar{y} = \{y_1, y_2, ..., y_n\}$ in Ran, which is disjoint with x, i.e, $x \neq y_i$ for any i. We can define an open substack $((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}}$ in $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$,

Definition 7.4. *$((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}} \subset (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ is the open substack where we require the maps $\kappa^{\check{\lambda}}$ to be injective on the fiber over any $y_i \in \bar{y}$.*

It means that $\kappa^{\check{\lambda}}$ are injective bundle maps (quotient are quotient free) on the neighborhood of $\bar{y}$. Because κ^{λ} are injective maps near $\bar{y}$, they give rise to a N-reduction of $\mathcal{P}_G$ near $\bar{y}$, it means:

$$\mathcal{P}_G|_{D_{\bar{y}}} \cong \mathcal{P}_B|_{D_{\bar{y}}} \overset{B}{\times} G$$

for some B bundle $\mathcal{P}_B$ on $D_{\bar{y}}$ and $\beta^T_{\bar{y}} : \mathcal{P}_B \overset{B}{\times} T \cong \omega^\rho$.

By [BD], we could construct a $N(O)^{\omega^\rho}_{\bar{y}}$-principal bundle over the stack $((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}}$. It is denoted by $_{\bar{y}}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}}$ which classifies the data $\{(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \forall \check{\lambda} \in \Lambda^+\}, \beta_{\bar{y}}, \epsilon) | \beta_{\bar{y}} : \omega^\rho|_{D_{\bar{y}}} \overset{T}{\times} B \cong \mathcal{P}_B, \beta_{\bar{y}} \overset{B}{\times} T = \beta^T_{\bar{y}}\}$.

By some standard glue procedure, we can extend the $N(O)^{\omega^\rho}_{\bar{y}}$-action on ${}_{\bar{y}}((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}}$ to an action of $N(K)^{\omega^\rho}_{\bar{y}}$. The proof used in Lemma 3.2.7 in [FGV] can be used to prove this claim without an essential difference: in the proof in [FGV], there is a chosen identification $\mathcal{P}_{1,G}|_{X-\bar{y}} = \mathcal{P}_{2,G}|_{X-\bar{y}}$, and because our x is not in $\bar{y}$, so the G fiber of $\mathcal{P}_{1,G}$ and $\mathcal{P}_{2,G}$ over x can be identified. And we may just ask the action of $N(K)^{\omega^\rho}_{\bar{y}}$ does not change ϵ.

Note $N(K)^{\omega^\rho}_{\bar{y}} \cong \prod_{1\le i\le k} N(K)^{\omega^\rho}_{y_i}$.

We define a character $\chi_{\bar{y}}$ on $N(K)^{\omega^\rho}_{\bar{y}}$ as follows:

$$\begin{aligned}\chi_{\bar{y}} : N(K)^{\omega^\rho}_{\bar{y}} \xrightarrow{pr} N(K)^{\omega^\rho}_{\bar{y}}/[N(K)^{\omega^\rho}_{\bar{y}}, N(K)^{\omega^\rho}_{\bar{y}}] \longrightarrow \\ \xrightarrow{\sim} \prod^{r} \omega|_{D^*_{y_i}} \xrightarrow{res} \prod^{r} \mathbb{G}_a \xrightarrow{add} \mathbb{G}_a\end{aligned} \tag{7.1}$$

here r is the rank of G.

With respect to this character of $N(K)^{\omega^\rho}_{\bar{y}}$, we could define $(N(K)^{\omega^\rho}_{\bar{y}}, -\chi_{\bar{y}})$-equivariant object on a scheme or an algebraic stack admitting a $N(K)^{\omega^\rho}_{\bar{y}}$ action. In particular, ${}_{\bar{y}}((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}}$.

Definition 7.5. *We denote by* $Whit_q(((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}})$ *the category*

$$D_{\mathcal{G}^G}({}_{\bar{y}}((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}})^{N(K)^{\omega^\rho}_{\bar{y}}, -\chi_{\bar{y}}}$$

Remark Every object in the category $Whit_q(((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}})$ is $(N(K)^{\omega^\rho}_{\bar{y}}, -\chi_{\bar{y}})$-equivariant, in particular, $N(O)^{\omega^\rho}_{\bar{y}}$-equivariant. We note that the group $N(O)^{\omega^\rho}_{\bar{y}}$ is a pro-unipotent group scheme. As a result, any object in $Whit_q(((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}})$ could descend to a twisted D-module on $((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}}$. Hence, we could also regard objects in the category $Whit_q(((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}})$ as (twisted) D-modules on $((\overline{Bun}^{\omega^\rho}_N)'_{\infty\cdot x})_{good\ at\ \bar{y}}$.

To study the basic property of the global Whittaker category, we should give $(\overline{Bun}^{\omega^\rho}_N)'_{\infty x, good\ at\ \bar{y}}$ a locally finite stratification, and study the Whittaker D-modules supported onside:

Given a point $\bar{x} = \{x, x_1, ..., x_m\} \in Ran_x$ which is disjoint with $\bar{y}$ and a series of coweights $\mu_x, \mu_1, ..., \mu_m$ such that $\mu_x \in \Lambda$ and $\mu_i \in \Lambda^{neg}$. If $w \in W$, let $(Bun^{\omega^\rho}_N)'^{,w}_{\infty x, good\ at\ \bar{y}, \bar{x}, \bar{\mu}}$ be the locally finite algebraic substack of $(Bun'^{,\omega^\rho}_N)_{\infty x, good\ at\ \bar{y}}$ with the condition:

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\lambda, \check{2}\rho\rangle} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\sum_k \langle\check{\lambda}, \mu_k\rangle x_k)$$

are injective bundle maps and the relative position map to

$$B\backslash G/B \simeq pt/B \underset{pt/G}{\times} pt/B \tag{7.2}$$

given by B-reductions of $\mathcal{P}_G$ at x is in $Br^{w,+} := B\backslash BwB/B$.

The inclusion map is a locally closed embedding

$$(Bun_N^{',\omega^\rho})^w_{\infty x, good\ at\ \bar{y},\bar{x},\bar{\mu}} \hookrightarrow (\overline{Bun_N^{',\omega^\rho}})_{\infty x, good\ at\ \bar{y}} \tag{7.3}$$

We consider the restriction of the $N(O)^{\omega^\rho}_{\bar{y}}$-bundle ${}_{\bar{y}}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}}$ on $(\overline{Bun_N^{',\omega^\rho}})_{\infty x, good\ at\ \bar{y}}$ to $(Bun_N^{',\omega^\rho})^w_{\infty x, good\ at\ \bar{y},\bar{x},\bar{\mu}}$. We denote it by $B^{\bar{x},\bar{\mu},w}_{\bar{y}}$. Note that $B^{\bar{x},\bar{\mu},w}_{\bar{y}}$ is $N(K)^{\omega^\rho}_{\bar{y}}$-invariant, hence, we could consider the Whittaker D-modules onside.

With the method given in [FGV], we have the following property:

Proposition 7.1. *If $t^{\mu_x}w$ satisfies the relevant condition in Section 5.1, and $\mu_i = 0$, then*

$$D_{\mathcal{G}^G}(B^{\bar{x},\bar{\mu},w}_{\bar{y}})^{N(K)^{\omega^\rho}_{\bar{y}},\chi} \cong Vect$$

Otherwise, it is equivalent to 0.

Proof. We only consider the case of $\bar{y} = y$, our cases are similar.

Changing ω^ρ by any T-bundle $\mathcal{P}_T$, we could define the group ind-scheme $N(K)^{\mathcal{P}_T}_y$. Similarly, we could define $N(O)^{\mathcal{P}_T}_y$ and $N^{\mathcal{P}_T}_{out,y} := Maps(X - y, N^{\mathcal{P}_T})$. If we fix an isomorphism $\mathcal{P}_T \simeq \omega^\rho$ in the neighborhood of y, then we could identify $N(K)^{\mathcal{P}_T}_y$ with $N(K)^{\omega^\rho}_y$. Let us denote by $D = \mu_x \cdot x + \sum_i \mu_i \cdot x_i$.

We denote by

$$N^{\omega^\rho(-D),w}_{out,y} \subset N^{\omega^\rho(-D)}_{out,y}$$

the subgroup which maps to $wN^{\omega^\rho(-D)}_x w^{-1} \cap N^{\omega^\rho(-D)}_x \subset N^{\omega^\rho(-D)}_x$ under the map:

$$N^{\omega^\rho(-D)}_{out,y} \to N^{\omega^\rho(-D)}_x$$

given by restricting to x. We have the identification:

$$B^{\bar{x},\bar{\mu},w}_y \cong N^{\omega^\rho(-D),w}_{out,y} \backslash N(K)^{\omega^\rho}_y \tag{7.4}$$

We notice that $N(K)^{\omega^\rho}_y$ is pro-ind unipotent and $N^{\omega^\rho(-D)}_{out,y}$ is connected, as a result, $Whit_q(B^{\bar{x},\bar{\mu},w}_y) \cong Vect$ if the restriction of χ to $N^{\omega^\rho(-D),w}_{out,y}$ is trivial and $Whit_q(B^{\bar{x},\bar{\mu},w}_y) \cong 0$ otherwise.

Consider the character map $\chi_y : N(K)^{\omega^\rho}_y \to \mathbb{G}_a$ defined in (7.1), it is equal to the following composition given in [FGV]:

$$\begin{aligned} \chi_y : N(K)^{\omega^\rho}_y &\longrightarrow \prod_i H^0(X - y, \omega(\langle -D, \check{\alpha}_i\rangle) \longrightarrow \\ \longrightarrow \prod_i H^0(D^*_y, \omega(\langle -D, \check{\alpha}_i\rangle)) &\longrightarrow \omega^r_{D^*_y} \xrightarrow{residue} \mathbb{G}^r_a \longrightarrow \mathbb{G}_a \end{aligned} \tag{7.5}$$

The image of $N^{\omega^\rho(-D),w}_{out,y}$ in $\prod_i H^0(X - y, \omega(\langle -D, \check{\alpha}_j\rangle))$ is

$$\prod_{w^{-1}(\check{\alpha_j})>0} H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle)) \times \prod_{w^{-1}(\check{\alpha_j})<0} H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle - x)) \tag{7.6}$$

If our $t^{\mu_x}w$ is not relevant, then, we could take one j, such that, $w^{-1}(\check{\alpha_j})<0$ and $\langle D,\check{\alpha_j}\rangle < -1$ or one j, such that $w^{-1}(\check{\alpha_j})>0$ and $\langle D,\check{\alpha_j}\rangle<0$.

Assume the first case, then, we may take an element

$$\gamma \in H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle - x))$$

s.t, its image under the composition:

$$H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle - x)) \to H^0(D_x^*,\omega(\langle -D,\check{\alpha_j}\rangle - x)) \to \omega_{D_x^*} \to \mathbb{G}_a$$

does not vanish. And its image under the map:

$$H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle)) \to H^0(D_{x_i}^*,\omega(\langle -D,\check{\alpha_j}\rangle)) \to \omega_{D_{x_i}^*} \to \mathbb{G}_a$$

vanishes for any x_i. Then, by residue formula, the image of $\gamma \in H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle)) \to \omega_{D_y^*} \to \mathbb{G}_a$ does not equal to 0. So, χ_y is not trivial on $N_{out,y}^{\omega^\rho(-D),w}$.

The analysis for the second case is similar.

Also, we could prove that if some $\mu_i \neq 0$, then we could take $\gamma \in H^0(X-y,\omega(\langle -D,\check{\alpha_j}\rangle))$ such that its image under the character map is non-zero. Hence, we could prove that the restriction of χ_y is non-trivial when restricted to $N_{out,y}^{\omega^\rho(-D),w}$.

Now, if we assume that μ_x and μ_i satisfy the hypothesis, the restriction of $\chi_{\bar{y}}$ to $N_{out,\bar{y}}^{\omega^\rho(-D),w}$ is indeed trivial. It is because in this case, it is easy to see that the image of (7.6) in $\omega^r_{D_y^*}$ is holomorphic, hence, it is the image under residue map is 0. □

$((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}}$ forms an open covering of $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$. The relationship between them is:

- if $\bar{y}_1 \cup \bar{y}_2 = \bar{y}$, then,

  $$((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}} = ((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}_1} \cap ((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})_{good\ at\ \bar{y}_2}$$

  On this intersection, there are three different versions of Whittaker categories. Namely,

1. $(N_{\bar{y}_1},-\chi_{\bar{y}_1})$-equivariant D-modules

2. $(N_{\bar{y}_2}, -\chi_{\bar{y}_2})$-equivariant D-modules

3. $(N_{\bar{y}}, -\chi_{\bar{y}})$-equivariant D-modules

By a similar proof of Lemma 4.14 in [Ga8], we could get three categories are equivalent. Hence, we could define:

Definition 7.6. *we define twisted Whittaker D-module on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ to be the twisted D-module on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ such that its restriction to $(\overline{Bun_N^{\omega^\rho}})'_{\infty x, good\ at\ \bar{y}}$ is a twisted Whittaker D-module for any $\bar{y}$ disjoint with x. And we denote the category of twisted Whittaker D-modules on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ by $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$.*

Definition 7.7. *Let us denote by $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}$ the substack of $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ classifying the data such that $\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\check{\rho}\rangle} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\sum_k \langle\check{\lambda},\mu_k\rangle x_k)$ is injective for any dominant weight $\check{\lambda}$, and the relative position of the B-reduction given by ϵ and B-reduction given by $\kappa^{\check{\lambda}}$ is of relative position w.*

Similarly, we define $(Bun_N^{\omega^\rho})_{\lambda\cdot x}$ the algebraic substack of $(\overline{Bun_N^{\omega^\rho}})$ classifying the data such that $\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\check{\rho}\rangle} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\sum_k \langle\check{\lambda},\mu_k\rangle x_k)$ is injective for any dominant weight $\check{\lambda}$.

We denote by $\bar{j}_{glob,t^\lambda w,Fl,Ran_x}$ the open embedding:

$$\bar{j}_{glob,t^\lambda w,Fl,Ran_x} : (Bun_N^{\omega^\rho})'_{\lambda\cdot x,w} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\lambda\cdot x,w} \tag{7.7}$$

Here, $(\overline{Bun_N^{\omega^\rho}})'_{\lambda\cdot x,w}$ is the closure of $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}$ in $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$.

As Definition 7.6, we could define the category of twisted Whittaker D-modules on $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}$. According to Proposition 7.1:

Corollary 7.1.

$$Whit_q((Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}) \simeq Vect$$

if $t^\lambda w$ is relevant, and

$$Whit_q((Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}) = 0$$

otherwise.

By the same proof, we have:

Corollary 7.2.

$$Whit_q((Bun_N^{\omega^\rho})_{\lambda\cdot x}) \simeq Vect$$

if $\lambda = 0$, and

$$Whit_q((Bun_N^{\omega^\rho})_{\lambda\cdot x}) = 0$$

otherwise.

In particular,

$$Whit_q(\overline{Bun_N^{\omega^\rho}}) \simeq Vect$$

Definition 7.8. *On each relevant stratum $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}$, there is a unique irreducible twisted D-module up to a scalar isomorphism, then, we denote its !-extension by $\Delta^{ver}_{glob,\lambda w}$ and its $*$-extension to be $\nabla^{ver}_{glob,\lambda w}$.*

Given two strata $(Bun_N^{\omega^\rho})'_{\lambda_1 x,w_1}$ and $(Bun_N^{\omega^\rho})'_{\lambda_2 x,w_2}$, then, the closure relation is also given by the semi-infinite Bruhat closure relation.

i.e, $(Bun_N^{\omega^\rho})'_{\lambda_1 x,w_1} \subset \overline{(Bun_N^{\omega^\rho})}'_{\lambda_2 x,w_2}$ if and only if

$$t^{\lambda_1} w_1 \leq_N t^{\lambda_2} w_2$$

We note that the relevant strata have a minimal one and the relevant strata are filtered. As a direct corollary,

Proposition 7.2. *$\{\Delta^{ver}_{glob,\lambda w}, t^\lambda w \text{ relevant}\}$ (resp. $\{\nabla^{ver}_{glob,\lambda w}, t^\lambda w \text{ relevant}\}$) compactly generates $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$*

7.3 Global semi-infinite !-extension D module

The semi-infinite D-modules $j_!(\omega_{S^{-,Conf}_{Gr,Conf}})$, $j_!(\omega_{S^{-,w,Conf_x}_{Fl,Conf_x}})$ defined in Section 6.2 have corresponding globally defined objects. In this section, we will introduce these D-modules.

The gerbe $\mathcal{G}^G$ is canonically trivial on $Bun_N^{\omega^\rho}$ and $(Bun_N^{\omega^\rho})'_{0,w_0}$, hence, we could consider the constant D-modules on them. We denote them by $c_{Bun_N^{\omega^\rho}}$ and $c_{(Bun_N^{\omega^\rho})'_{0,w_0}}$ respectively.

Denote $d_g := dim(Bun_N^{\omega^\rho})$

Definition 7.9. *The !-extension of $c_{Bun_N^{\omega^\rho}}[d_g]$ (respectively, $c_{(Bun_N^{\omega^\rho})'_{0,w_0}}[d_g + dimN]$) to $\overline{Bun_N^{\omega^\rho}}$ (respectively, !-extension to $\overline{Bun_N^{\omega^\rho,'}}$) is denoted by $j_{!,glob,Gr}$ (respectively, $j_{!,glob,Fl}$).*

Remark The function that we add shifts in the definition is to make the objects perverse, i.e, concentrated in degree 0.

We could also consider the twisted !-extension D-module on the closure of Bun_{B^-}. First of all, let us introduce a gerbe on $\overline{Bun'_{B^-}}$ which corresponds to the gerbe $\mathcal{G}^{G,T,ratio}_{glob}$ defined in Definition 6.14 (6.44).

Definition 7.10. *Under the diagrams below:*

$$\begin{tikzcd} & \overline{Bun_{B^-}} \arrow[dl, tail] \arrow[dr, tail] & \\ Bun_G & & Bun_T \end{tikzcd}$$

$$\begin{tikzcd} & \overline{Bun'_{B^-}} \arrow[dl, tail] \arrow[dr, tail] & \\ Bun'_G & & Bun_T \end{tikzcd}$$

We define gerbes $\mathcal{G}^{G,T,ratio}_{glob}$ on $\overline{Bun_{B^-}}$ and $\overline{Bun_{B^-}}'$ to be $(\mathcal{G}^G)\bigotimes(\mathcal{G}^T)^{-1}$, i.e, the ratio of $\mathcal{G}^G$ (pullback from Bun_G) and $\mathcal{G}^T$ (pullback from Bun_T).

By the definition of $\mathcal{G}^G$ and $\mathcal{G}^T$ in Section 2.5, we could see that the gerbe $\mathcal{G}^{G,T,ratio}_{glob}$ is canonically trivial on $Bun''_{B^-} \subset \overline{Bun_{B^-}}'$ and $Bun_{B^-} \subset \overline{Bun_{B^-}}$. Hence, we could regard the category of twisted D-modules on Bun''_{B^-} and Bun_{B^-} as the category of non-twisted D-modules. Similar to Definition 7.9, we may define the twisted !-extension D-modules on $\overline{Bun_{B^-}}$ and $\overline{Bun_{B^-}}'$.

Definition 7.11. *We denote by $j^-_{!,glob,Bun_T\times Ran_x,Fl}$ (resp. $j^-_{!,glob,Bun_T\times Ran,Gr}$) the !-extension of the* ***perverse*** *constant D-modules on Bun''_{B^-} (resp. Bun_{B^-}).*

Remark We give these !-extension D-modules such complicated names just in order to match their names after introducing relative stacks in Section 9.3. (Bun_B could be regarded as a relative version of Bun_N over Bun_T)

Let us define some substacks in $\overline{Bun_N^{\omega^\rho}}$ and $\overline{Bun_N^{\omega^\rho}}'$.

Definition 7.12. *Given $\eta \in \Lambda^{neg}$, then $Bun_N^{\omega^\rho,\eta} \subset \overline{Bun_N^{\omega^\rho}}$ denotes the algebraic substack such that the total defect of the generalized defect of B-bundle is exactly $-\eta$.*

Given $\widetilde{w} = t^\eta w$, $\eta \in \Lambda^{neg}$, then $Bun_N^{\omega^\rho,\widetilde{w},'} \subset \overline{Bun_N^{\omega^\rho}}'$ denotes the algebraic substack such that the total defect of the generalized defect of B-bundle is exactly $-\eta$ and the relative position is w.

7.4 Zastava spaces

In this section, we will recall the definition of Zastava spaces and their modifications. They will play an important role in our global construction of the equivalence functor and they will be a useful tool for us to compare local and global semi-infinite D-modules.

Recall that we denote by Bun'_G the algebraic stack classifying a G-bundle with a B-reduction and we define Bun'_{B^-} to be the algebraic stack classifying B^--bundle with a B-reduction at x of its induced G-bundle.

We denote by $\bar{\mathfrak{p}}' : (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \to Bun'_G$ and $\bar{\mathfrak{q}}' : \overline{Bun'_{B^-}} \to Bun'_G$ the natural maps. Similarly, we denote by $\bar{\mathfrak{p}} : (\overline{Bun_N^{\omega^\rho}}) \to Bun_G$ and $\bar{\mathfrak{q}} : \overline{Bun_{B^-}} \to Bun_G$ the natural maps.

Definition 7.13. *a). We define: $Bun^\eta_{B^-} \subset Bun_{B^-}$ classifies the data:*

$$(\mathcal{P}_T, \mathcal{P}_G, \{\kappa^{-,\check{\lambda}}, \forall\check{\lambda} \in \Lambda^+\})$$

where we ask the degree of $\mathcal{P}_T$ is of degree $-\eta + (2g-2)\rho$,

b). If we have $\widetilde{w} = t^\eta w$, we define: $Bun^{',\widetilde{w}}_{B^-} \subset Bun'_{B^-}$ classifies the data $(\mathcal{P}_T, \mathcal{P}_G, \{\kappa^{-,\check{\lambda}}, \forall\check{\lambda} \in \Lambda^+\}, \epsilon)$ where we ask the degree of $\mathcal{P}_T$ is of degree $-\eta + (2g-2)\rho$, and the relative position of this B^--reduction and the Iwahori-structure given by ϵ is w.

Definition 7.14. *We define the affine Grassmannian Zastava space and its modifications as follows:*

$(Z_{Gr})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}_{\infty\cdot x} \underset{Bun_G}{\times'} \overline{Bun_{B^-}}$

$(Z_{Gr})^{\circ}_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}_{\infty\cdot x} \underset{Bun_G}{\times'} Bun_{B^-}$

$Z_{Gr} := \overline{(Bun_N^{\omega^\rho})} \underset{Bun_G}{\times'} \overline{Bun_{B^-}}$

$Z^{\circ}_{Gr} := \overline{(Bun_N^{\omega^\rho})} \underset{Bun_G}{\times'} Bun_{B^-}$

$\tilde{Z}^{\circ}_{Gr} := (Bun_N^{\omega^\rho}) \underset{Bun_G}{\times'} Bun_{B^-}$

$\tilde{Z}_{Gr} := (Bun_N^{\omega^\rho}) \underset{Bun_G}{\times'} \overline{Bun_{B^-}}$

$Z^{\eta,\circ}_{Gr} := Z^{\circ}_{Gr} \underset{Conf}{\times} Conf^{\eta}$, *if* $\eta \in \Lambda^{neg}$

Here, $\times'$ means that we take an open subset of the product such that the composition of $\kappa^{\check{\lambda}}$ and $\kappa^{-,\check{\lambda}}$ is non-zero.

We denote by $\bar{\mathfrak{q}}_Z : Z_{Gr} \longrightarrow \overline{Bun_N^{\omega^\rho}}$ and by $\bar{\mathfrak{p}}_Z : Z_{Gr} \longrightarrow \overline{Bun_{B^-}}$

Then, we define the affine flags Zastava space and its variations as follows:

Definition 7.15.

$$Z_{Fl,x} := (\overline{Bun_N^{\omega^\rho}})' \underset{Bun'_G}{\times'} \overline{Bun''_{B^-}}$$

$$Z'_{Fl,x} := (\overline{Bun_N^{\omega^\rho}})' \underset{Bun'_G}{\times'} Bun'_{B^-}$$

$$Z^{\circ}_{Fl,x} := (\overline{Bun_N^{\omega^\rho}})' \underset{Bun'_G}{\times'} Bun''_{B^-}$$

$$Z^{\widetilde{w},\circ}_{Fl,x} := (\overline{Bun_N^{\omega^\rho}})' \underset{Bun'_G}{\times'} Bun'^{,\widetilde{w}}_{B^-}$$

$$Z^{\widetilde{w}}_{Fl,x} := (\overline{Bun_N^{\omega^\rho}})' \underset{Bun'_G}{\times'} \overline{Bun'^{,\widetilde{w}}_{B^-}}$$

$$\tilde{Z}^{\widetilde{w},\circ}_{Fl,x} := Bun_N^{\omega^\rho,w_0,'} \underset{Bun'_G}{\times'} Bun'^{,\widetilde{w}}_{B^-}$$

Definition 7.16.

$$(Z_{Fl,x})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}'_{\infty\cdot x} \underset{Bun'_G}{\times'} \overline{Bun''_{B^-}}$$

$$(Z'_{Fl,x})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}'_{\infty\cdot x} \underset{Bun'_G}{\times'} Bun'_{B^-}$$

$$(Z^{\circ}_{Fl,x})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}'_{\infty\cdot x} \underset{Bun'_G}{\times'} Bun''_{B^-}$$

$$(Z^{\widetilde{w},\circ}_{Fl,x})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}'_{\infty\cdot x} \underset{Bun'_G}{\times'} Bun'^{,\widetilde{w}}_{B^-}$$

$$(Z^{\widetilde{w}}_{Fl,x})_{\infty\cdot x} := \overline{(Bun_N^{\omega^\rho})}'_{\infty\cdot x} \underset{Bun'_G}{\times'} \overline{Bun'^{,\widetilde{w}}_{B^-}}$$

We denote by $\bar{\mathfrak{q}}'_Z : (Z_{Fl,x})_{\infty\cdot x} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ and by $\bar{\mathfrak{p}}'_Z : (Z_{Fl,x})_{\infty\cdot x} \longrightarrow \overline{Bun_{B^-}}'$. And we denote by $\mathfrak{q}'_Z : (Z_{Fl,x})^\circ_{\infty\cdot x} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ and by $\mathfrak{p}'_Z : (Z_{Fl,x})^\circ_{\infty\cdot x} \longrightarrow \overline{Bun_{B^-}}'$.

Taking poles of the composition of $\kappa^{\check{\lambda}}$ and $\kappa^{-,\check{\lambda}}$ gives us maps:

$$v_{Fl,x}:\ (Z_{Fl,x})_{\infty\cdot x} \to Conf_x \tag{7.8}$$

$$v_{Gr,x}:\ (Z_{Gr})_{\infty\cdot x} \to Conf_x \tag{7.9}$$

$$v_{Gr}:\ Z_{Gr} \to Conf \tag{7.10}$$

The map $v_{Fl,x}$ factors through $(Z_{Gr})_{\infty\cdot x}$, i.e, it decomposes as follows: $(Z_{Fl,x})_{\infty\cdot x} \cong (Z_{Gr})_{\infty\cdot x} \underset{Bun_G}{\times} Bun'_G \to (Z_{Gr})_{\infty\cdot x} \to Conf_x$.

According to [FGV], we could see that the Zastava space Z_{Gr} is a factorization space over $Conf$:

Lemma 7.1.

$$Z_{Gr} \underset{Conf}{\times} (Conf \times Conf)_{disj} \cong (Z_{Gr} \times Z_{Gr}) \underset{Conf\times Conf}{\times} (Conf \times Conf)_{disj}$$

Imitate the construction in [FGV], we could see that the affine flag Zastava space $(Z_{Fl,x})_{\infty\cdot x}$ is a factorization module space with respect to affine Grassmannian Zastava space Z_{Gr}:

Proposition 7.3.

$$\begin{gathered} (Z_{Fl,x})_{\infty\cdot x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \\ \cong \\ (Z_{Gr} \times (Z_{Fl,x})_{\infty\cdot x}) \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \end{gathered} \tag{7.11}$$

Proof. Assume that given a point $z_1 = (\mathcal{P}_T, \mathcal{P}_G, \kappa^\lambda, \kappa^{-,\lambda}) \in Z_{Gr}$ and a point $z_2 = (\mathcal{P}'_T, \mathcal{P}'_G, \kappa^{',\lambda}, \kappa^{',-,\lambda}, \epsilon) \in (Z_{Fl,x})_{\infty\cdot x}$, if the support of z_1 and the support of z_2(contains x) does not intersect, then, we need to construct the corresponding point $z_3 = (\mathcal{P}''_T, \mathcal{P}''_G, \kappa^{'',\lambda}, \kappa^{'',-,\lambda}, \epsilon'') \in (Z_{Fl,x})_{\infty\cdot x}$ in $(Z_{Fl,x})_{\infty\cdot x}$.

We define $\mathcal{P}''_G|_{X-supp(z_1)}$to be $\mathcal{P}'_G|_{X-supp(z_1)}$ and we define $\mathcal{P}''_G|_{X-supp(z_2)}$ to be $\mathcal{P}_G|_{X-supp(z_2)}$. Outside $supp(z_1)$, morphisms $\kappa^{',\lambda}$ determine an isomorphism $\mathcal{P}_G|_{X-supp(z_1)} \cong \mathcal{P}^1_G|_{X-supp(z_1)}$. And outside $supp(z_2)$, $\kappa^{',\lambda}$ determines an isomorphism $\mathcal{P}''_G|_{X-supp(z_2)} \cong \mathcal{P}^1_G|_{X-supp(z_1)}$. It determines the isomorphism of $\mathcal{P}_G$ and $\mathcal{P}'_G$ on the intersection of $X - supp(z_1)$ and $X - supp(z_2)$. ϵ'' is given by ϵ' because the fiber of $\mathcal{P}'_G$ and $\mathcal{P}''_G$ on x can be identified.

The opposite direction is given similarly.

∎

7.5 Construction of global functors

In order to prove Proposition 6.11, we need to use a global explanation of the functor F^L.To start with, let us consider the following diagram:

$$
\begin{tikzcd}
 & (Z_{Fl,x})_{\infty\cdot x} \arrow[dl, "\bar{\mathfrak{q}}'_Z"'] \arrow[d, "v"] \arrow[dr, "\bar{\mathfrak{p}}'_Z"] & \\
(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x} & Conf_x & \overline{Bun_{B^-}}'
\end{tikzcd}
\tag{7.12}
$$

We define global functors as follows:

Definition 7.17.

$$F^L_{glob} : Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^\wedge}(Conf_x)$$

$$
\begin{aligned}
F^L_{glob}(\mathcal{F}) := v_{Fl,x,!}(\bar{\mathfrak{q}}_Z^{',!}(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}_Z^{',!}(j^-_{!,glob,Bun_T\times Ran_x,Fl}[dimBun'_G])) \\
\in D_{\mathcal{G}^\wedge}(Conf_x)
\end{aligned}
\tag{7.13}
$$

$$F^{KD}_{glob} : Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^\wedge}(Conf_x)$$

$$
\begin{aligned}
F^{DK}_{glob}(\mathcal{F}) := v_{Fl,x,!}(\bar{\mathfrak{q}}_Z^{',!}(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}_Z^{',!}(j^-_{*,glob,Bun_T\times Ran_x,Fl}[dimBun'_G])) \\
\in D_{\mathcal{G}^\wedge}(Conf_x)
\end{aligned}
\tag{7.14}
$$

Similarly, we consider the diagram without Iwahori structures,

$$
\begin{tikzcd}
 & Z_{Gr} \arrow[dl, "\bar{\mathfrak{q}}_Z"'] \arrow[d, "v"] \arrow[dr, "\bar{\mathfrak{p}}_Z"] & \\
(\overline{Bun_N^{\omega^\rho}}) & Conf & \overline{Bun_{B^-}}
\end{tikzcd}
\tag{7.15}
$$

We define affine Grassmannian global functors as follows:

Definition 7.18.

$$F^L_{glob,Gr} : Whit_q((\overline{Bun_N^{\omega^\rho}})) \longrightarrow D_{\mathcal{G}^\wedge}(Conf)$$

$$F^L_{glob,Gr}(\mathcal{F}) := v_{Gr,!}(\bar{\mathfrak{q}}^!_Z(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}^!_Z(j^-_{!,glob,Bun_T\times Ran,Gr}[dimBun_G])) \in D_{\mathcal{G}^\Lambda}(Conf) \tag{7.16}$$

$$F^{KD}_{glob,Gr} : Whit_q((\overline{Bun^{\omega^\rho}_N})) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf)$$

$$F^{KD}_{glob,Gr}(\mathcal{F}) := v_{Gr,!}(\bar{\mathfrak{q}}^!_Z(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}^!_Z(j^-_{*,glob,Bun_T\times Ran,Gr}[dimBun_G])) \in D_{\mathcal{G}^\Lambda}(Conf) \tag{7.17}$$

By Corollary 7.2, we have $Whit_q((\overline{Bun^{\omega^\rho}_N})) \simeq Vect$, hence, we could take the unique irreducible object inside. We denote it by $\mathcal{F}_\emptyset$. We could define two twisted D-modules on $Conf$ by applying $F^L_{glob,Gr}$ and $F^{KD}_{glob,Gr}$ to $\mathcal{F}_\emptyset$.

Definition 7.19. *The twisted D-module $\Omega^{L'}_q \in D_{\mathcal{G}^\Lambda}(Conf)$ is defined to be*

$$\Omega^{L'}_q := v_{Gr,*}(\bar{\mathfrak{q}}^!_Z(\mathcal{F}_\emptyset) \overset{!}{\otimes} \bar{\mathfrak{p}}^!_Z(j^-_{!,glob,Bun_T\times Ran,Gr}[dimBun_G]))$$

And the twisted D-module $\Omega^{DK'}_q \in D_{\mathcal{G}^\Lambda}(Conf)$ is defined to be

$$\Omega^{KD'}_q := v_{Gr,*}(\bar{\mathfrak{q}}^!_Z(\mathcal{F}_\emptyset) \overset{!}{\otimes} \bar{\mathfrak{p}}^!_Z(j^-_{!,glob,Bun_T\times Ran,Gr}[dimBun_G]))$$

By using the same method of Proposition 6.3, we could prove that $\Omega^{L'}_q$ and $\Omega^{DK'}_q$ are factorization algebras on $Conf$. Furthermore, if we apply the same method in [Ga2] we could get the factorization property of F^L_{glob} and F^{KD}_{glob} with respect to $\Omega^{L'}_q$ and $\Omega^{KD'}_q$. But we do not perform the proof of this method here, it is because that the factorization property of F^L_{glob} and F^{KD}_{glob} follows directly from the isomorphism of F^L (resp. F^{KD}) and F^L_{glob} (resp. F^{KD}_{glob}) in Proposition 7.1 b)., hence, we postpone the proof of factorization property of F^L_{glob} and F^{KD}_{glob} in Section 9.6 after we proved Proposition 7.1 b)..

7.6 Theorem 7.1+Proposition 7.4$\Rightarrow$ Proposition 6.11

Note that there is a projection from $Fl^{\omega^\rho}_x$ to $(\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}$, namely,

$$\pi_{Fl,x} : Fl^{\omega^\rho}_x \longrightarrow (\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x} \tag{7.18}$$

This morphism sends $(\mathcal{P}_G, \alpha, \epsilon) \in Fl^{\omega^\rho}_x$ to $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \forall\check{\lambda} \in \check{\Lambda}^+\}, \epsilon)$ Here, $\kappa^{\check{\lambda}}$ is induced from α, i.e,

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\check{\rho}\rangle} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}^0_G} \xrightarrow{\alpha} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$$

Similarly, forgetting ϵ, we have a projection functor from $\bar{S}^0_{Gr}$ to $\overline{Bun^{\omega^\rho}_N}$:

$$\pi_{Gr,x} : \bar{S}^0_{Gr} \longrightarrow \overline{Bun^{\omega^\rho}_N} \tag{7.19}$$

It is easy to see that Proposition 6.11 could be divided into two steps:
The first step is to prove:

Theorem 7.1. *a).*

$$\pi^!_{Fl,x} : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow Whit_q(Fl^{\omega^\rho}_x)$$

is an equivalence.

$$\pi^!_{Gr,x} : Whit_q(\overline{Bun^{\omega^\rho}_N}) \longrightarrow Whit_q(\bar{S}^0_{Gr})$$

is an equivalence.

b). Under the equivalence:

$$\pi^!_{x,Fl}[d_g] : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \xrightarrow{\sim} Whit_q(Fl^{\omega^\rho}_{G,x})$$

we have:

$$F^L : Whit_q(Fl^{\omega^\rho}_{G,x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

and

$$F^L_{glob} : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

are isomorphic.

Under the equivalence:

$$\pi^!_{x,Gr}[d_g] : Whit_q(\overline{Bun^{\omega^\rho}_N}) \xrightarrow{\sim} Whit_q(\bar{S}^0_{Gr})$$

we have:

$$F^L_{Gr} : Whit_q(\bar{S}^0_{Gr}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf)$$

and

$$F^L_{glob,Gr} : Whit_q(\overline{Bun^{\omega^\rho}_N}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf)$$

are isomorphic.

The second step is to prove:

Proposition 7.4.

$$F^L_{glob}(\Delta^\lambda_{glob}) \simeq \Delta_{\lambda,\Omega^{L,\prime}_q} \tag{7.20}$$

8 Proof of Theorem 7.1 a). Local Global comparison

In this section, we will prove the local definition and global definition of the Whittaker category on affine flags are equivalent. It is the statement of Theorem 7.1.

8.1 Local Whittaker vs Global Whittaker

We want to study the relationship between the local Whittaker category $Whit_q(Fl_x^{\omega^\rho})$ and the global Whittaker category $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ (and the relationship between $Whit_q(\bar{S}^0_{Gr})$ and $Whit_q(\overline{Bun_N^{\omega^\rho}})$).

We consider the !-pullback functors along with the morphisms $\pi_{Fl,x}$ and $\pi_{Gr,x}$. They define functors:

$$\pi^!_{Fl,x} : D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^G}(Fl_x^{\omega^\rho}) \tag{8.1}$$

$$\pi^!_{Gr,x} : D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})) \longrightarrow D_{\mathcal{G}^G}(\bar{S}^0_{Gr}) \tag{8.2}$$

Lemma 8.1. *a). Functors (8.1) and (8.2) send the Whittaker subcategories $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}) \subset D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ (resp. $Whit_q((\overline{Bun_N^{\omega^\rho}}))$) to $Whit_q(Fl_x^{\omega^\rho}) \subset D_{\mathcal{G}^G}(Fl_x^{\omega^\rho})$ (resp. $Whit_q(\bar{S}^0_{Fl}) \subset D_{\mathcal{G}^G}(\bar{S}^0_{Fl})$).*

b).
$$\pi^!_{Fl,x} : Whit_q((Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}) \simeq Whit_q(S^{\lambda w}_{Fl})$$

and
$$\pi^!_{Gr,x} : Whit_q((Bun_N^{\omega^\rho})_{\lambda\cdot x}) \simeq Whit_q(S^{\lambda}_{Gr})$$

Proof. The proof of this lemma is an analog of Theorem 5.1.4 in [Ga5]. Here, we only prove the affine flags part of Lemma 8.1, the case of affine Grassmannian is similar to the case of affine flags.

It is easy to see that a twisted D-module $\mathcal{F}$ on $Fl_x^{\omega^\rho}$ belongs to the subcategory $Whit_q(Fl_x^{\omega^\rho})$ if and only if its !-restriction to each relevant $N(K)_x^{\omega^\rho}$-orbit satisfies the *Whittaker equivalent* condition. And we notice for any $t^\lambda w \in W^{ext}$, the morphism $\pi_{Fl,x}$ restricts to a morphism

$$S^{\lambda w}_{Fl,x} \longrightarrow (Bun_N^{\omega^\rho})'_{\lambda\cdot x,w} \tag{8.3}$$

So, we only need to prove for any relevant $t^\lambda w \in W^{ext}$, the pullback functor sends the corresponding twisted Whittaker category $Whit_q((Bun_N^{\omega^\rho})'_{\lambda\cdot x,w})$ to the twisted Whittaker subcategory $Whit_q(S^{\lambda w}_{Fl,x})$ in $D_{\mathcal{G}^G}(S^{\lambda w}_{Fl,x})$ and it is an equivalence.

Recall that any twisted *Whittaker* D-module $\mathcal{F}$ on $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w}$ could be regarded as a twisted D-module on $(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w,good\ at\ \bar{y}}$ descend from $B^{x,\lambda,w}_{\bar{y}}$.

We notice that the morphism (8.3) factors through

$$S^{\lambda w}_{Fl,x} \longrightarrow (Bun_N^{\omega^\rho})'_{\lambda w,good\ at\ \bar{y}} \tag{8.4}$$

and the latter lifts to a morphism:

$$N(K)^{\omega^\rho}_{\bar{y}} \times S^{\lambda w}_{Fl,x} \longrightarrow B^{x,\lambda,w}_{\bar{y}} \tag{8.5}$$

and the morphism (8.5) is $N(K)^{\omega^\rho}_{\bar{y}}$-equivariant where the action of $N(K)^{\omega^\rho}_{\bar{y}}$ on $N(K)^{\omega^\rho}_{\bar{y}} \times S^{\lambda w}_x$ is right multiplication on $N(K)^{\omega^\rho}_{\bar{y}}$ and the action on the

$B_{\bar{y}}^{x,\lambda,w}$ is given by the standard gluing procedure which extends the natural $N(O)_{\bar{y}}^{\omega^\rho}$ action.

By the equivariance property of the morphism, the !-pullback of $Whittaker$ D-module to $N(K)_{\bar{y}}^{\omega^\rho} \times S_{Fl,x}^{\lambda w}$ is $(N(K)_{\bar{y}}^{\omega^\rho}, -\chi_{\bar{y}})$-equivariant. What's more, the pullback of any twisted D-module from $B_{\bar{y}}^{x,\lambda,w}$ is $N_{out,x,\bar{y}}$-equivariant with respect to the diagonal action on $N(K)_{\bar{y}}^{\omega^\rho} \times S_{Fl,x}^{\lambda w}$. Here, $N_{out,x,\bar{y}}$ is the group scheme classifying the map from $X - x - \bar{y}$ to N. It could be regarded as a subgroup of $N(K)_x$ or $N(K)_{\bar{y}}$. It is easy to see that $\chi_x|_{N_{out,x,\bar{y}}} + \chi_{\bar{y}}|_{N_{out,x,\bar{y}}} = 0$.

Then, it is easy to see pullback along with:

$$\begin{aligned} S_{Fl,x}^{\lambda w} &\longrightarrow N(K)_{\bar{y}}^{\omega^\rho} \times S_{Fl,x}^{\lambda w} \\ s &\longrightarrow (1, s) \end{aligned} \tag{8.6}$$

defines an equivalence of category of $N_{out,x,\bar{y}}$-equivariant and $(N(K)_{\bar{y}}^{\omega^\rho}, -\chi_{\bar{y}})$-equivariant D-modules on $N(K)_{\bar{y}}^{\omega^\rho} \times S_{Fl,x}^{\lambda w}$ and the category of $(N_{out,x,\bar{y}}, \chi_x)$-equivariant D-modules on $S_{Fl,x}^{\lambda w}$.

Hence, we know that the pullback of Whittaker D-module along with the morphism (8.3) establishes an equivalence of $Whit_q((Bun_N^{\omega^\rho})'_{\lambda \cdot x})$ and the category of $(N_{out,x,\bar{y}}, \chi_x)$-equivariant D-modules on $S_{Fl,x}^{\lambda w}$. We only need to prove that the latter is equivalent to $Whit_q(S_{Fl,x}^{\lambda w})$.

It follows from the fact that the unipotent group $N_{out,x,\bar{y}}$-acts transversely on $S_{Fl,x}^{\lambda w}$ and the stabilizer is connected, so taking !-fiber at λw defines an equivalence: if $t^\lambda w$ is relevant,

$$D_{\mathcal{G}^G}(S_{Fl,x}^{\lambda w})^{N_{out,x,\bar{y}},\chi_x} \simeq Vect \tag{8.7}$$

and we know taking !-fiber at λw defines an equivalence(Proposition 5.2): if $t^\lambda w$ is relevant,

$$Whit_q(S_{Fl,x}^{\lambda w}) \simeq Vect \tag{8.8}$$

Hence,

$$D_{\mathcal{G}^G}(S_{Fl,x}^{\lambda w})^{N_{out,x,\bar{y}},\chi_x} \simeq Whit_q(S_{Fl,x}^{\lambda w}) \tag{8.9}$$

□

The main theorem of this section is to prove Theorem 7.1 a).

The proof can be divided into two parts. The first step is to prove $\pi_x^!$ is essentially surjective. The second step is to prove it is fully faithful.

The surjectivity is not very difficult. It follows from Proposition 8.1. We could easily give an explicit description of the image of Verma costandard objects.

Corollary 8.1. $\pi_{Fl,x}^!(\nabla_{glob,\lambda w}^{ver})[d_g] = \nabla_{\lambda w}^{ver}$

Proof. It follows from the base change of $*$-extension and !-pullback and Proposition 8.1. □

By orthogonal property of Verma standard objects and Verma costandard objects, we could prove that $\Delta^{ver}_{\lambda w}$ and $\Delta^{ver}_{glob,\lambda w}$ match under the functor $\pi^!_{Fl,x}$(after a cohomological shift).

Corollary 8.2. $\pi^!_{Fl,x}(\Delta^{ver}_{glob,\lambda w})[d_g] = \Delta^{ver}_{\lambda w}$

To prove the functor $\pi^!_{Fl,x} : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow Whit_q(Fl^{\omega^\rho}_x)$ is fully faithful, it suffices to prove

$$\pi^!_{Fl,x} : D_{\mathcal{G}^G}((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^G}(Fl^{\omega^\rho}_x)$$

is a fully faithful functor. Indeed, it is because both two subcategories are fully faithful subcategories.

To prove the latter, we need to introduce a notation of *universally homological contractible* introduced in D.Gaitsgory's paper [Ga7] Section 3.

Definition 8.1. *Assume X and Y are two prestacks. $f : X \longrightarrow Y$ is called a universally homological contractible map, if for any prestack Z and any morphism $Z \longrightarrow Y$, the resulted functor:*

$$D(Z) \longrightarrow D(Z \underset{Y}{\times} X)$$

is fully faithful.

Given an algebraic group H, use the same notation in [Ga7], we can define the prestack Bun^{gen}_H classifying the following data:

- an open subset U in $X \times S$ such that each fiber over $x \in X$ is dense in U.
- a H bundle $\mathcal{P}_H$ on U,

i.e, Bun^{gen}_H classifies a generic H bundle on $X \times S$.

Take H to be N, then, according to [Ga5] Theorem A 1.10, we have:

Lemma 8.2. $Gr_{G,Ran} \longrightarrow Bun^{gen}_H \underset{Bun^{gen}_G}{\times} Bun_G$ *is universally homological contractible.*

From Lemma 8.2, we get:

Corollary 8.3. $D_{\mathcal{G}^G}((\bar{S}^{w_0}_{Fl,Ran_x})_{\infty x}) \longrightarrow D_{\mathcal{G}^G}((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x})$ *is fully faithful*

Proof. It follows from the Cartesian diagram:

$$\begin{tikzcd}
(\bar{S}^{w_0}_{Fl,Ran_x})_{\infty x} \arrow[r] \arrow[d] & Gr_{G,Ran} \arrow[d] \\
(\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x} \arrow[r] & Bun^{gen}_H \underset{Bun^{gen}_G}{\times} Bun_G
\end{tikzcd}$$

∎

Proof. (of Theorem 7.1 a).)

Surjectivity of $\pi^!_{Fl,x}$ has already been proved in Corollary 8.1 and Corollary 8.2, we only need to prove that $\pi^!_{Fl,x}$ is fully faithful.

We notice that the morphism:

$$\pi_{Fl,x} : Fl_x^{\omega^\rho} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$$

can be decomposed as the composition:

$$Fl_x^{\omega^\rho} \longrightarrow Ran_x \times Fl_x^{\omega^\rho} \xrightarrow{unit} (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty x} \xrightarrow{\pi_{Ran_x}} (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$$

The !-pullback along with π_{Ran_x} is fully faithful by Corollary 8.3, the !-pullback along with *unit* is an equivalence if we restrict to *Whittaker* subcategories according to Theorem 6.1.

So we have the !-pullback functor along with the morphism:

$$Ran_x \times Fl_x^{\omega^\rho} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$$

is fully faithful if we restrict to *Whittaker* subcategories.

We note that this morphism can be decomposed as:

$$Ran_x \times Fl_x^{\omega^\rho} \longrightarrow Fl_x^{\omega^\rho} \xrightarrow{\pi_{Fl,x}} (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$$

So, $\pi^!_{Fl,x}$ is fully faithful. □

By Corollary 8.1, 8.2 and 7.2, we have:

Corollary 8.4. *The equivalence in Theorem 7.1 is compatible with duality functor, i.e,*

$$\mathbb{D} \circ (\pi^!_x[d_g]) \simeq (\pi^!_x[d_g]) \circ \mathbb{D} \tag{8.10}$$

8.2 Compatibility of t-structures

In this section, we define two t-structures t_1' and t_2' on $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ and compare with the t-structures defined in Definition 5.7 and 5.17 on $Whit_q(Fl_x^{\omega^\rho})$ under the functor $\pi^!_{Fl,x}[d_g]$.

The fist t-structure t_1' on the algebraic stack $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ is the classically defined t-structure of D-module category. We denote by $\heartsuit'$ the heart of the category of twisted D-modules with respect to this t-structure. Let us study its relationship with t_1.

By an analog of Proposition 3.3.1 in [FGV], we have:

Lemma 8.3. *The open embedding:*

$$(Bun_N^{\omega^\rho})'_{\lambda\cdot x,w} \longrightarrow (\overline{Bun_N^{\omega^\rho}})'_{\lambda\cdot x,w}$$

is affine.

Recall that the Verma standard object $\Delta^{ver}_{glob,\lambda w}$ is the !-extension of the irreducible object in $Whit_q((Bun_N^{\omega^\rho})'_{\lambda w})^{\heartsuit'}$. Hence, we have:

$\Delta^{ver}_{glob,\lambda w} \in Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})^{\heartsuit'}$.

And the t-structure t'_1 on $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ can be described intrinsically as follows:

Definition 8.2. *A twisted Whittaker D-module $\mathcal{F}$ on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ is coconective if for any relevant λw, we have*

$$Hom_{Whit_{q,Fl}(G)}(\Delta^{ver}_{glob,\lambda w}[k], \mathcal{F}) = 0, \textit{ if } k>0$$

By Definition 5.7, the t-structure t_1 of $Whit_q(Fl_x^{\omega^\rho})$ is given by:

· *A twisted Whittaker D-module $\mathcal{F}$ on $Fl_x^{\omega^\rho}$ is coconective if for any relevant λw, we have*

$$Hom_{Whit_q(Fl_x^{\omega^\rho})}(\Delta^{ver}_{\lambda w}[k], \mathcal{F}) = 0, \textit{ if } k>0$$

We notice that the functor $\pi_x^![d_g]$ sends $\Delta^{ver}_{glob,\lambda w}$ to $\Delta^{ver}_{\lambda w}$ and $\nabla^{ver}_{glob,\lambda w}$ to $\nabla^{ver}_{\lambda w}$, so the functor $\pi_x^![d_g]$ is $t-exact$ with respect to these two t-structures. In particular, we proved the Proposition 5.5 i)..

Proof. (of Proposition 5.5 i).) Because $\pi_x^![d_g]$ is t-exact and Corollary 8.1 and 8.2. ∎

Now, we need to define a new t-structure of $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ which is compatible with the naturally defined $t-structure$ on $\Omega_q^L - FactMod$ defined in Proposition 4.2, i.e, the t-structure of local Whittaker category defined in Definition 5.17.

Although we do not really need to explain the 'right' $t-structure$ on $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ (we could define it directly by the equivalence between $Whit_q(Fl_x^{\omega^\rho})$ and $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$ given by $\pi_x^![d_g]$), we still present it in an explicit way in order to be complete.

We hope we could define this $t-structure$ by choosing a collection of compact generators and then define it like before. So, our aim is to describe the twisted D-module on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ corresponding to Δ_λ under the equivalence given by $\pi^!_{Fl,x}[d_g]$.

If $\lambda \in \Lambda^+$, then, according to Corollary 8.2, the corresponding twisted D-module is just $\Delta^{ver}_{glob,\lambda w}$.

In order to find the twisted D-module corresponding to Δ_λ for general relevant λ, we need to introduce a right convolution action of twisted BMW D-modules on $Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})$.

Consider the algebraic stack $(\widetilde{\overline{Bun_N^{\omega^\rho}}})'_{\infty\cdot x}$ which classifies the following data:

- A point $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \epsilon)$ of $(\widetilde{\overline{Bun_N^{\omega^\rho}}})'_{\infty\cdot x}$
- An isomorphism of $\mathcal{P}_G$ and $\mathcal{P}'_G$ on D_x, the formal disc of x, and ϵ corresponds to the B-reduction of $\mathcal{P}'_G$ given by $\omega^\rho \overset{T}{\times} B$.

It is an I-bundle over $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$.

By a standard gluing procedure, we can extend this action to an action of $G(K)_x^{\omega^\rho}$.

In order to simplify the notations, in this part, we only consider non-twisted case. For twisted case, it is the same except we also need to put right twist.

Consider the following diagram:

$$
\begin{tikzcd}
& (\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \times G(K)/I \arrow[dl, "\pi_1^{glob}"'] \arrow[d, "\pi^{glob}"] \arrow[dr, "\pi_2^{glob}"] & \\
(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x} & (\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \overset{I}{\times} G(K)/I \arrow[d, "m^{glob}"] & G(K)/I \\
& (\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x}/I &
\end{tikzcd}
\tag{8.11}
$$

So, given a $Whittaker$ D-module $\mathcal{F}_1$ on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$, and a BMW D-module $\mathcal{F}_2$, we may define their convolution product:

$$
\mathcal{F}_1 \star \mathcal{F}_2 := m_*^{glob}(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2) \tag{8.12}
$$

Here, $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$ is the module descend from $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ on $(\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \times G(K)/I$.

It is easy to see that the convolution product is compatible with the equivalence given in Theorem 7.1, i.e:

Proposition 8.1. $\pi_{Fl,x}^!(\mathcal{F}_1 \star \mathcal{F}_2) \simeq \pi_{Fl,x}^!(\mathcal{F}_1) \star \mathcal{F}_2$

As an application of Proposition 8.1, we could define the corresponding D-modules on $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ corresponding to Δ_λ:

Definition 8.3. *Given $\lambda \in \Lambda$, and $\lambda = \lambda_1 - \lambda_2, \lambda_1, \lambda_2 \in \Lambda^+$,*

We define the standard objects $\Delta_{glob,\lambda}$ to be

$$
\Delta_{glob,\lambda_1}^{ver} \star J_{-\lambda_2}
$$

By definition, we have:

Corollary 8.5. $\pi_{Fl,x}^!(\Delta_{glob,\lambda})[d_g] \simeq \Delta_\lambda$

And by orthogonal property and compact generation property, we have:

Corollary 8.6. $\pi_{Fl,x}^!(\nabla_{glob,\lambda})[d_g] \simeq \nabla_\lambda$

9 Proof of Theorem 7.1 b). Semi-infinite equivariant D-modules

In this section, we will prove Theorem 7.1 b). Because we have already established an equivalence between local and global Whittaker categories, hence, we should pay attention to the comparison of local semi-infinite D-module and the global ones defined in Section 7.3. To be more precise, we will consider the !-extension and $*$-extension D-module on $Fl^{\omega^\rho}_{Conf}$, $Fl^{\omega^\rho}_{G,Conf_x}$ and Drinfeld compactifications.

The content of this section:

In Section 9.1, we will study the existence of !-extension of semi-infinite D-modules.

In Section 9.2, we will compare the local semi-infinite D-module and global semi-infinite D-module.(Theorem 9.1)

But in order to satisfy our requirements to prove Theorem 7.1 b)., we need to consider a relative version of Theorem 9.1. Hence, we study the relative stacks in Section 9.3.

In Section 9.4, we will introduce the relative version of Theorem 9.1. The proof of the relative version is the same as non-relative one, so we omit its proof.

In Section 9.5, we will give the Zastava spaces another description using factorization stacks. Hence, we could compare F^L and F^L_{glob}.

In Section 9.6, we will prove Theorem 7.1 b)., i.e, we prove $F^L \simeq F^L_{glob}$.

9.1 Local !-extension semi-infinite D-module

In this section , we will study the existence of !-extension of semi-infinite D-module. The most important property of this section is Corollary 9.2 which said !-direct image functor could be defined for these D-modules.

We mimic the method in the first section of [Ga4]. In *loc.cit*, the author proved the existence of !-extension of semi-infinite D-module on affine Grassmannian when there is no twisting. The proof in *loc.cit* could be used to prove the existence of !-extension of semi-infinite twisted D-module on affine flags in the same way.

We define a substack S^w_{Fl,Ran_x} of Fl_{Ran_x}, it classifies $(\mathcal{P}_G, \alpha, \epsilon) \in Fl_{Ran_x}$ such that the induced map $\check{\lambda}(\omega^\rho) \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$ is injective on X and the relative position of the N-reduction and ϵ is w. And we denote by $\bar{S}^w_{Fl,Ran_x}$ the closure of S^w_{Fl,Ran_x} in Fl_{Ran_x}. We denote by $\bar{j}_{w,Fl,Ran_x}$ the open immersion

$$\bar{j}_{w,Fl,Ran_x} : S^w_{Fl,Ran_x} \longrightarrow \bar{S}^w_{Fl,Ran_x} \tag{9.1}$$

we want to prove that $\bar{j}_{w,Fl,Ran_x,!}$ can be defined on the subcategory of $N(K)^{\omega^\rho}_{Ran_x}$-equivariant twisted D-modules on S^w_{Fl,Ran_x}.

Definition 9.1. *Given $(* \in I)$, we let $Fl^{\omega^\rho}_I$ denote the fiber products:*

$$Fl^{\omega^\rho}_I := Fl^{\omega^\rho}_{Ran_x} \underset{Ran_x}{\times} X^I_x$$

Here, $X^I_x := X^I \underset{X}{\times} x$

Definition 9.2. *Inside* $Conf \times Ran_x$*, we define a substack* $(Conf \times Ran_x)^{\subset}$ *to be the substack consisting of the points* (D, I) *such that the support of* D *is contained in the support of* I*.*

Definition 9.3. *Given* $\widetilde{w} = t^\eta w \in W^{ext}$ *such that* $\eta \in \Lambda^{neg}$*, we define a locally closed substack* $S^{\widetilde{w}}_{Fl,Ran_x}$ *in* $(Conf^\eta \times Ran_x)^{\subset} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{Ran_x}$*, as follows:*

- *A point* $(D, I, \alpha, \epsilon) \in (Conf^\eta \times Ran_x)^{\subset} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{Ran_x}$ *belongs to* $S^{\widetilde{w}}_{Fl,Ran_x}$ *if the induced morphisms*

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle}(-\langle\check{\lambda}, D\rangle) \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \longrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G},$$

which are a priori defined on $X - I$ *extends to injective maps on the whole curve* X *and the induced* B^-*-reduction and the* B*-reduction given by* ϵ *is of relative position* w*.*

And we denote by $\bar{S}^{\widetilde{w}}_{Fl,Ran_x}$ *the closure of* $S^{\widetilde{w}}_{Fl,Ran_x}$ *in*

$$(Conf^\eta \times Ran_x)^{\subset} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{Ran_x}$$

Their fiber product with X^I_x over Ran_x is denoted by $S^{\widetilde{w}}_{I,Fl}$ and $\bar{S}^{\widetilde{w}}_{I,Fl}$.

As Lemma 1.3.9 in [Ga4], we have:

Lemma 9.1. *The composition:*

$$S^{\widetilde{w}}_{Fl,Ran_x} \longrightarrow (Conf^\eta \times Ran_x)^{\subset} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{Ran_x} \longrightarrow Fl^{\omega^\rho}_{Ran_x} \tag{9.2}$$

is a locally closed embedding. And on the geometric point level, every point in $\bar{S}^{w_0}_{Fl,Ran_x}$ *is in the image of a unique* $S^{\widetilde{w}}_{Fl,Ran_x}$ *such that* $\eta \in \Lambda^{neg}$*.*

Chosen a regular dominant coweight, and consider the resulted $\mathbb{G}_m$-action on the fibers of $\bar{S}^{w_0}_{Fl,I}$ over X^I_x, then, as same as Lemma 1.7.2 in [Ga4],

Lemma 9.2. *The attracting locus of this* $\mathbb{G}_m$ *action is*

$$\bigsqcup_{\widetilde{w}=t^\eta w\in W^{ext},\ \eta\in\Lambda^{neg}} S^{\widetilde{w}}_{Fl,I}$$

The repelling locus is

$$\bigsqcup_{\widetilde{w}=t^\eta w\in W^{ext},\ \eta\in\Lambda^{neg}} S^{-,\widetilde{w}}_{Fl,I}$$

Definition 9.4. *We denote functors among stacks as follows:*

$$\bar{j}^{-}_{Fl,\widetilde{w},Ran_x} : S^{-,\widetilde{w}}_{Fl,Ran_x} \longrightarrow \bar{S}^{-,\widetilde{w}}_{Fl,Ran_x}$$

$$i^{-}_{Fl,\widetilde{w},Ran_x} : \bar{S}^{-,\widetilde{w}}_{Fl,Ran_x} \longrightarrow Fl^{\omega^\rho}_{G,Ran_x}$$

$$j^{-}_{Fl,\widetilde{w},Ran_x} := i^{-}_{Fl,\widetilde{w},Ran_X} \circ \bar{j}^{-}_{Fl,\widetilde{w},Ran_x}$$

Similar definition applies to $\bar{j}^{-}_{Fl,\widetilde{w},I}, i^{-}_{Fl,\widetilde{w},I}$, *and* $j^{-}_{Fl,\widetilde{w},I}$.
Then, we define its negative counterparts:

$$\bar{j}_{Fl,\widetilde{w},Ran_x} : S^{\widetilde{w}}_{Fl,Ran_x} \longrightarrow \bar{S}^{\widetilde{w}}_{Fl,Ran_x}$$

$$i_{Fl,\widetilde{w},Ran_X} : \bar{S}^{\widetilde{w}}_{Fl,Ran_x} \longrightarrow Fl^{\omega^\rho}_{G,Ran_x}$$

$$j_{Fl,\widetilde{w},Ran_x} := i_{Fl,\widetilde{w},Ran_X} \circ \bar{j}_{Fl,\widetilde{w},Ran_x}$$

Similar definition applies to $\bar{j}_{Fl,\widetilde{w},I}, i_{Fl,\widetilde{w},I}$, *and* $j_{Fl,\widetilde{w},I}$.
There is a natural projection:

$$p^{\widetilde{w}} : S^{\widetilde{w}}_{Fl,Ran_x} \longrightarrow (Conf^{\eta} \times Ran_x)^{\subset}$$

as well as $p^{-,\widetilde{w}}$, $p^{\widetilde{w}}_I$ *and* $p^{-,\widetilde{w}}_I$.
$p^{\widetilde{w}}$ *and* $p^{-,\widetilde{w}}$ *admit canonical sections, we denote them by*

$$s_{\widetilde{w}} : (Conf^{\eta} \times Ran_x)^{\subset} \longrightarrow S^{\widetilde{w}}_{Fl,Ran_x}$$

and

$$s_{-,\widetilde{w}} : (Conf^{\eta} \times Ran_x)^{\subset} \longrightarrow S^{-,\widetilde{w}}_{Fl,Ran_x}$$

The images of $s_{\widetilde{w}}$ and $s_{-,\widetilde{w}}$ coincide in $Fl^{\omega^\rho}_{G,Ran_x}$.

The semi-infinite D-modules on $\bar{S}^{w_0}_{Fl,I}$ could be studied by studying the semi-infinite D-modules on the stratification of $\bar{S}^{w_0}_{Fl,I}$ given by $\bar{S}^{\widetilde{w}}_{Fl,I}$:

Lemma 9.3. $p^{\widetilde{w},!}_I : D((Conf^{\eta} \times X^I_x)^{\subset}) \longrightarrow D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I})^{N(K)^{\omega^\rho}_I}$ *induces an equivalence of DG-categories.*

Proof. It is because $N(K)^{\omega^\rho}_I$ acts transitively and the stabilizer of $s_{\widetilde{w},I}((Conf^{\eta} \times X^I)^{\subset})$ is a pro-unipotent group scheme over X^I_x. □

Proposition 9.1. *The partially left adjoint functor*

$$j^{*}_{Fl,\widetilde{w},I} : D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Fl,I}) \longrightarrow D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I})$$

of

$$j_{Fl,\widetilde{w},I,*} : D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I}) \longrightarrow D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Fl,I})$$

can be defined on $N(K)^{\omega^\rho}_I$*-equivariant objects.*

Proof. Given any $N(K)_I^{\omega^\rho}$-equivariant D-module $\mathcal{F}$ in $D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Fl,I})$, according to the lemma above, we only need to prove the D-module $s^!_{\widetilde{w}} \circ j^*_{Fl,\widetilde{w},I}(\mathcal{F}) \in ProD((Conf^\eta \times X_x^I)^\subset)$ actually lies in $D((Conf^\eta \times X_x^I)^\subset)$.

Indeed, according to the Braden theorem, we have

$$s^!_{\widetilde{w}} \circ j^*_{Fl,\widetilde{w},I}(\mathcal{F}) \simeq s^*_{-,\widetilde{w}} \circ j^{-,!}_{Fl,\widetilde{w},I}(\mathcal{F}) \simeq p_{-,\widetilde{w},*} \circ j^{-,!}_{Fl,\widetilde{w},I}(\mathcal{F}) \in D((Conf^\eta \times X_x^I)^\subset)$$

□

As a corollary of the lemma above, we have the existence of !-extended functor for $N(K)$-equivariant D-module.

Corollary 9.1. *The left adjoint functor $j_{Fl,\widetilde{w},I,!}$ of $j^!_{Fl,\widetilde{w},I} : D_{\mathcal{G}^G}(Fl_I)^{\omega^\rho} \longrightarrow D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I})$ can be defined on $N(K)_I^{\omega^\rho}$-equivariant D-modules.*

Proof. Follows from the duality functor on D-module category and under duality functor $f_!$ goes to f^*. □

Lemma 9.4. *Assume $(* \in I_2) \twoheadrightarrow (* \in I_1)$, and denote by $\Delta : X_x^{I_1} \longrightarrow X_x^{I_1}$ the corresponding map. We consider the following commutative diagram:*

$$\begin{tikzcd} S^{\widetilde{w}}_{Fl,I_1} \arrow[r, "\Delta"] \arrow[d, "j_{Fl,\widetilde{w},I_1}"] & S^{\widetilde{w}}_{Fl,I_2} \arrow[d, "j_{Fl,\widetilde{w},I_2}"] \\ \bar{S}^{w_0}_{Fl,I_1} \arrow[r, "\Delta"] & \bar{S}^{w_0}_{Fl,I_2} \end{tikzcd}$$

We have:

$$\Delta^! \circ j^*_{Fl,\widetilde{w},I_2} \simeq j^*_{Fl,\widetilde{w},I_1} \circ \Delta^!$$

Proof. Consider the diagram:

$$
\begin{tikzcd}
 & X_x^{I_1} \arrow[rr, tail, "\Delta"] \arrow[dl, "s_{\widetilde{w}}"'] \arrow[dd, shift right, "s_{-,\widetilde{w}}"'] & & X_x^{I_2} \arrow[dl, "s_{\widetilde{w}}"'] \arrow[dd, shift right, "s_{-,\widetilde{w}}"'] \\
S^{\widetilde{w}}_{Fl,I_1} \arrow[rr, tail, "\Delta"] \arrow[dd, "j_{Fl,\widetilde{w},I_1}"'] & & S^{\widetilde{w}}_{Fl,I_2} \arrow[dd, "j_{Fl,\widetilde{w},I_2}"'] & \\
 & S^{-,\widetilde{w}}_{Fl,I_1} \arrow[rr, tail, "\Delta"] \arrow[dl, "j^{-}_{Fl,\widetilde{w},I_1}"] \arrow[uu, shift right, "p^{-}_{\widetilde{w}}"'] & & S^{-,\widetilde{w}}_{Fl,I_2} \arrow[dl, tail, "j^{-}_{Fl,\widetilde{w},I_2}"] \arrow[uu, shift right, "p_{-,\widetilde{w}}"'] \\
Fl^{\omega^\rho}_{I_1} \arrow[rr, tail, "\Delta"] & & Fl^{\omega^\rho}_{I_2} &
\end{tikzcd}
$$

It is suffice to prove $\Delta^! \circ s^!_{\widetilde{w}} \circ j^*_{Fl,\widetilde{w},I_2} \simeq s^!_{\widetilde{w}} \circ j^*_{Fl,\widetilde{w},I_1} \circ \Delta^!$. According to the Braden theorem in [DG1], it is equivalent to the isomorphism:

$$\Delta^! \circ p_{-,\widetilde{w},*} \circ j^{-,!}_{Fl,\widetilde{w},I_2} \simeq p_{-,\widetilde{w},*} \circ j^{-,!}_{Fl,\widetilde{w},I_1} \circ \Delta^!$$

And the above isomorphism follows from base change.

□

Lemma 9.5. *j^*_{Fl,Ran_x} can be defined on $N(K)^{\omega^\rho}_{Ran_x}$-equivariant objects.*

Proof. According to lemmas above, given $\mathcal{F} \in D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{Ran_x})$ satisfies the $N(K)^{\omega^\rho}_I$ equivariant condition , we can define a collection of $N(K)^{\omega^\rho}_I$-equivariant D-modules $\{j^*_{Fl,\widetilde{w},I} \circ \Delta^!(\mathcal{F}) \in D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I})\}$, and they give rise a $N(K)^{\omega^\rho}_{Ran_x}$-equivariant (also $T(O)^{\omega^\rho}_{Ran_x}$ monodromic) D-module. And the adjoint property can be shown:

$$
\begin{aligned}
&Hom_{D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,Ran_x})}(j^*_{Fl,\widetilde{w},Ran_x}(\mathcal{F}),\mathcal{G})\\
\simeq &Hom_{D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,Ran_x})}(colim\Delta_! \circ j^*_{Fl,\widetilde{w},I}\circ \Delta^!(\mathcal{F}),\mathcal{G})\\
\simeq &limHom_{D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,Ran_x})}(\Delta_! \circ j^*_{Fl,\widetilde{w},I}\circ \Delta^!(\mathcal{F}),\mathcal{G})\\
\simeq &limHom_{D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,I})}(j^*_{Fl,\widetilde{w},I}\circ \Delta^!(\mathcal{F}),\Delta^!\circ(\mathcal{G}))\\
\simeq &limHom_{D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,I})}(\Delta^!(\mathcal{F}),j_{Fl,\widetilde{w},I,*}\circ \Delta^!\circ(\mathcal{G}))\\
\simeq &limHom_{D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,I})}(\Delta^!(\mathcal{F}),\Delta^!\circ j_{Fl,\widetilde{w},Ran_x,*}(\mathcal{G}))\\
\simeq &limHom_{D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})}(\Delta_!\circ\Delta^!(\mathcal{F}),j_{Fl,\widetilde{w},Ran_x,*}(\mathcal{G}))\\
\simeq &Hom_{D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})}(colim\Delta_!\circ\Delta^!(\mathcal{F}),j_{Fl,\widetilde{w},Ran_x,*}(\mathcal{G}))\\
\simeq &Hom_{D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})}(\mathcal{F},j_{Fl,\widetilde{w},Ran_x,*}(\mathcal{G}))
\end{aligned}
$$

□

By duality, we could get

Corollary 9.2. *The left adjoint functor*

$$j_{Fl,\widetilde{w},Ran_x,!}: D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,Ran_x}) \longrightarrow D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x})$$

of

$$j^!_{Fl,\widetilde{w},Ran_x}: D_{\mathcal{G}^G}(Fl^{\omega^\rho}_{G,Ran_x}) \longrightarrow D_{\mathcal{G}^G}(S^{\widetilde{w}}_{Fl,Ran_x})$$

can be defined on $N(K)^{\omega^\rho}_{Ran_x}$-equivariant D-modules.

In particular, we could define the !-extension of the dualizing D-module on $S^{w_0}_{Fl,Ran_x}$.

Definition 9.5. *We denote by $j_{!,Ran_x,Fl}$ the !-extension of the dualizing D-module on $S^{w_0}_{Ran_x,Fl}$.*

Applying the same method used in this section, we could get:

Corollary 9.3. *The !-direct image functor from $S^0_{Ran,Gr}$ to $Gr^{\omega^\rho}_{G,Ran}$ is well-defined for the dualizing D-module on $S^0_{Ran,Gr}$.*

We denote it by $j_{!,Ran,Gr}$.

9.2 Local semi-infinite vs Global semi-infinite

In this section, we will show the following theorem:

Theorem 9.1. *Denote*

$$\pi_{Ran_x,Gr}: \bar{S}^0_{Ran,Gr} \longrightarrow \overline{Bun^{\omega^\rho}_N}$$

$$\pi_{Ran_x,Fl}:\bar{S}^{w_0}_{Ran_x,Fl}\longrightarrow \overline{Bun_N^{\omega^\rho,'}}$$

Then, we have:

$$\pi^!_{Ran,Gr}(j_{!,glob,Gr})[d_g]\simeq j_{!,Ran,Gr}$$

$$\pi^!_{Ran_x,Fl}(j_{!,glob,Fl})[d_g+dimN]\simeq j_{!,Ran_x,Fl}\in D_{\mathcal{G}}(\bar{S}^{w_0}_{Fl,Ran_x})$$

Proof. Both of claims could be proved as a tiny variation of theorem 3.3.3 in [Ga4], here, we only perform how to prove the second claim, the first claim follows from the same way. We consider the Cartesian diagram:

$$\begin{tikzcd} \bar{S}^{w_0}_{Ran_x,Fl} \arrow[r, "\pi_{Ran_x,Fl}"] & \overline{Bun_N^{\omega^\rho,'}} \\ S^{w_0}_{Ran_x,Fl} \arrow[u, "\bar{j}_{w_0,Fl,Ran_x}"] \arrow[r, "\pi_{Ran_x,Fl}"] & (Bun_N^{\omega^\rho})'_{0,w_0} \arrow[u, "\bar{j}_{glob,w_0,Fl,Ran_x}"'] \end{tikzcd}$$

The theorem above is equivalent to prove a base change theorem(for the constant object on $(Bun_N^{\omega^\rho})'_{0,w_0}$):

$$\pi^!_{Ran_x,Fl}\circ\bar{j}_{glob,w_0,Fl,Ran_x,!}\simeq\bar{j}_{w_0,Fl,Ran_x,!}\circ\pi^!_{Ran_x,Fl} \tag{9.3}$$

By the proof of Proposition 8.1, we know both sides of (9.3) are in the category $D_{\mathcal{G}^G}(\bar{S}^{w_0}_{Ran_x,Fl})^{N(K)_{Ran_x}}$, and (9.3) is an isomorphism is equivalent to the claim that the !-pullback of both sides of (9.3) to $S^{\widetilde{w}}_{Ran_x,Fl}$ is an isomorphism for any $\widetilde{w}\leq_N w_0$. (That is to say, $\widetilde{w}=t^\eta w$, $\eta\in\Lambda^{neg}$)

Hence, the theorem 9.1 comes from the following lemma. □

Lemma 9.6. *Given any $\widetilde{w}<_N w_0$, we have:*

$$j^*_{\widetilde{w},Fl,Ran_x}\circ\pi^!_{Ran_x,Fl}(j_{!,glob,Fl})=0 \tag{9.4}$$

$$\begin{tikzcd} \bar{S}^{w_0}_{Ran_x,Fl} \arrow[r, "\pi_{Ran_x,Fl}"] & \overline{Bun_N^{\omega^\rho,'}} \\ S^{\widetilde{w}}_{Ran_x,Fl} \arrow[u, "j_{\widetilde{w},Fl,Ran_x}"] \arrow[r, "\pi_{Ran_x,Fl}"] & Bun_N^{\omega^\rho,\widetilde{w},'} \arrow[u, "j_{glob,\widetilde{w},Fl}"'] \end{tikzcd}$$

Let us only prove the claim (9.4) over X^I_x, the claim over Ran_x follows directly from the claim over X^I_x by taking the limit.

Before we start our proof of Theorem 9.1, we summarize the notations used in this section in the following diagram. If readers forget the notations, please could go back and check the notations here.

$$
\begin{tikzcd}
& \overline{Bun_N^{\omega^\rho}}' & Z^{\widetilde{w}}_{Fl,x} \arrow[l, "\mathfrak{q}'_Z"'] \arrow[dl, "s"] & (Conf^\eta \times X^I_x)^{\subset} \underset{Conf^\eta}{\times} Z^{\widetilde{w}}_{Fl,x} \arrow[l, "pr^{\widetilde{w}}_I \times id_{Z^{\widetilde{w}}_{Fl,x}}"'] \arrow[dl, "id\times v"] \arrow[dd, equal, "\wr"] \\
Bun_N^{\omega^\rho,\widetilde{w},'} \arrow[ur, "j_{glob,\widetilde{w},Fl}"] & Conf^\eta \arrow[l, "q^{\widetilde{w}}"] \arrow[ur, "p"] & (Conf^\eta \times X^I_x)^{\subset} \arrow[l, "pr^I_{\widetilde{w}}"] \arrow[dd, equal] & \\
& \bar{S}^{w_0}_{Fl,I} \arrow[uu, "\pi_I"] \arrow[dl, "j_{\widetilde{w},Fl,I}"] \arrow[dd, "j_{\widetilde{w},Fl,I}"] & & \bar{S}^{w_0}_{Fl,I} \cap S^{-,\widetilde{w}}_{Fl,I} \arrow[ll, "j_{\widetilde{w},Fl,I}"] \arrow[dl, "p^{-,\widetilde{w}}"] \arrow[dd, "i^{w_0}"] \\
S^{\widetilde{w}}_{Fl,I} \arrow[uu, "\pi_{I,\widetilde{w},Fl}"] \arrow[rr, shift left, "p^{\widetilde{w}}"] & & (Conf^\eta \times X^I_x)^{\subset} \arrow[ll, shift left, "s_{I,\widetilde{w}}"] \arrow[ur, "s^-_{I,\widetilde{w}}"] & \\
& Fl_I & & S^{-,\widetilde{w}}_{Fl,I} \arrow[ll, "j^-_{\widetilde{w},Fl,I}"]
\end{tikzcd}
$$

Proof. (of Lemma 9.6)

According to Lemma 9.3, proving Lemma 9.6 suffices to prove: for any $\widetilde{w} <_N w_0$,

$$s^!_{I,\widetilde{w}} \circ j^*_{\widetilde{w},Fl,I} \circ \pi^!_{I,Fl}(j_{!,glob,Fl}) = 0 \tag{9.5}$$

By the Braden theorem, for any semi-infinite D-module $\mathcal{F}$ on $Fl_{G,I}$ supporting on $\bar{S}^{w_0}_{Fl,I}$, we have:

$$p^{-,\widetilde{w}}_* \circ j^{-,!}_{\widetilde{w},Fl,I} \simeq s^!_{I,\widetilde{w}} \circ j^*_{\widetilde{w},Fl,I} \tag{9.6}$$

$$p_*^{-,\widetilde{w}} \simeq s_{I,\widetilde{w}}^{-,*} \tag{9.7}$$

Hence, we have:

$$\begin{aligned} & s^!_{I,\widetilde{w}} \circ j^*_{\widetilde{w},Fl,I} \circ \pi^!_{I,Fl}(j_{!,glob,Fl}) \\ \underset{(9.6)}{\simeq} & s^{-,*}_{I,\widetilde{w}} \circ j^{-,!}_{\widetilde{w},Fl,I} \circ \pi^!_{I,Fl}(j_{!,glob,Fl}) \\ \underset{(9.7)}{\simeq} & p_*^{-,\widetilde{w}} \circ j^{-,!}_{\widetilde{w},Fl,I} \circ \pi^!_{I,Fl}(j_{!,glob,Fl}) \end{aligned} \tag{9.8}$$

Hence, Lemma follows from the following lemma. □

Lemma 9.7.

$$p_*^{-,\widetilde{w}} \circ j^{-,!}_{\widetilde{w},Fl,I} \circ \pi^!_{I,Fl}(j_{!,glob,Fl}) = 0$$

In order to prove above is 0, we need to use the notion of (affine flags) Zastava space introduced in Section 7.4.

$\widetilde{w} = t^\eta w$ with $\eta \in \Lambda^{neg}$. Note that under the map

$$v : Z^{\widetilde{w}}_{Fl,x} \longrightarrow Conf_x$$

$Z^{\widetilde{w},\circ}_{Fl,x}$ maps to $Conf^\eta$. Here, we define $Conf^0 := pt = \{0 \cdot x\}$.

Easily from constructions, we have the following identifications:

Lemma 9.8. *There is a canonical isomorphism:*

$$\bar{S}^{w_0}_{Fl,I} \cap S^{-,\widetilde{w}}_{Fl,I} \simeq (Conf^\eta \times X^I_x)^\subset \underset{Conf^\eta}{\times} Z^{\widetilde{w},\circ}_{Fl,x} \tag{9.9}$$

And under the above isomorphism, the morphism

$$\bar{S}^{w_0}_{Fl,I} \cap S^{-,\widetilde{w}}_{Fl,I} \overset{j^-_{\widetilde{w},Fl,I}}{\longrightarrow} \bar{S}^{w_0}_{Fl,I} \overset{\pi_{I,Fl}}{\longrightarrow} \overline{Bun_N^{\omega^\rho,\prime}} \tag{9.10}$$

coincides with

$$(Conf^\eta \times X^I_x)^\subset \underset{Conf^\eta}{\times} Z^{\widetilde{w},\circ}_{Fl,x} \overset{pr^{\widetilde{w}}_I \times id_{Z^{\widetilde{w}}_{Fl,x}}}{\longrightarrow} Z^{\widetilde{w},\circ}_{Fl,x} \overset{\mathfrak{q}'_Z}{\longrightarrow} \overline{Bun_N^{\omega^\rho,\prime}} \tag{9.11}$$

And

$$\bar{S}^{w_0}_{Fl,I} \cap S^{-,\widetilde{w}}_{Fl,I} \overset{i_{w_0,Fl,I}}{\longrightarrow} S^{-,\widetilde{w}}_{Fl,I} \overset{p^{-,\widetilde{w}}}{\longrightarrow} (Conf^\eta \times X^I_x)^\subset \tag{9.12}$$

coincides with

$$(Conf^\eta \times X^I_x)^\subset \underset{Conf^\eta}{\times} Z^{\widetilde{w},\circ}_{Fl,x} \overset{id \times v}{\longrightarrow} (Conf^\eta \times X^I_x)^\subset \tag{9.13}$$

So, by above identifications, we could identify Lemma 9.7 as

$$p_*^{-,\widetilde{w}} \circ j_{\widetilde{w},Fl,I}^{-,!} \circ \pi_{I,Fl}^!(j_{!,glob,Fl}) \tag{9.14}$$

$$\underset{Lemma 9.8}{\simeq}(id \times v)_* \circ (pr_I^{\widetilde{w}} \times id_{Z^{\widetilde{w}}})^! \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \tag{9.15}$$

Consider the following diagram:

$$\begin{tikzcd}
(Conf^\eta \times X_x^I)^{\subset} \underset{Conf^\eta}{\times} Z_{Fl,x}^{\widetilde{w},\circ} \arrow[r, "pr_I^{\widetilde{w}} \times id_{Z_{Fl,x}^{\widetilde{w}}}"] \arrow[d, "id\times v"] & Z_{Fl,x}^{\widetilde{w},\circ} \arrow[d, "v"] \\
(Conf^\eta \times X_x^I)^{\subset} \arrow[r, "pr_I^{\widetilde{w}}"] & Conf^\eta
\end{tikzcd} \tag{9.16}$$

It is easy to see that it is Cartesian. So, by base change theorem, we have:

$$\begin{aligned} &(id \times v)_* \circ (pr_I^{\widetilde{w}} \times id_{Z^{\widetilde{w}}})^! \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \\ \simeq &(pr_I^{\widetilde{w}})^! \circ v_* \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \end{aligned} \tag{9.17}$$

To prove Lemma 9.7, it suffices to prove that:

Lemma 9.9. $v_* \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) = 0$

Proof. Let us study the morphism $v : Z_{Fl,x}^{\widetilde{w},\circ} \longrightarrow Conf^\eta$, we claim it admits a canonical section.

Indeed, note that we have a distinguished point in $Bun_N^{\omega^\rho,w,'}$ given by the trivial N-bundle and a B reduction of relative position w, hence, we have a canonical map:

$$q^{\widetilde{w}} : Conf^\eta \longrightarrow Bun_N^{\widetilde{w},'} \tag{9.18}$$

given by

$$Conf^\eta \longrightarrow Conf^\eta \times Bun_N^{\omega^\rho,w,'} \tag{9.19}$$

induced by $pt \longrightarrow Bun_N^{\omega^\rho,w,'}$.

Similarly, we have the map:

$$q^{-,\widetilde{w}} : Conf^\eta \longrightarrow Bun_{B^-}^{\widetilde{w},'} \tag{9.20}$$

By (9.18) and (9.20), we got a map:

$$s_{\widetilde{w}} : Conf^\eta \longrightarrow Z_{Fl,x}^{\widetilde{w},\circ} \tag{9.21}$$

which is a section of v.

We take a dominant coweight, the action of $\mathbb{G}_m$ on N induces an action on $\overline{Bun_N^{\omega^\rho}}'$, and $j_{!,glob,Fl}$ is $\mathbb{G}_m$-equivariant with respect to a character. Hence, $\mathfrak{q}_Z^{',!}(j_{!,glob,Fl})$ also shares the same equivariant property. Because $\mathbb{G}_m$ acts on the fiber of v and by the choice of coweight, this action contracts to the image of $s_{\widetilde{w}}$. By Proposition 3.2.2 of [DG], we have:

$$v_* \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \simeq s_{\widetilde{w}}^* \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \tag{9.22}$$

Hence, Lemma 9.9 follows from the Lemma 9.10. □

Lemma 9.10. *$\forall \widetilde{w} <_N w_0$, we have:*

$$s_{\widetilde{w}}^* \mathfrak{q}_Z^{',!}(j_{glob,Fl}) = 0$$

Proof. Consider the Cartesian diagram:

$$\begin{tikzcd} Bun_N^{\widetilde{w},'} \arrow[r, "j_{glob,\widetilde{w},Fl}"] & \overline{Bun_N^{\omega^\rho,'}} \\ Bun_N^{\widetilde{w},'} \underset{\overline{Bun_N^{\omega^\rho,'}}}{\times} Z_{Fl,x}^{\widetilde{w}} \arrow[u, "q_Z^{\widetilde{w}}"] \arrow[r, "j_{Z,\widetilde{w},Fl}"] & Z_{Fl,x}^{\widetilde{w}} \arrow[u, "\mathfrak{q}'_Z"'] \end{tikzcd} \tag{9.23}$$

We only need to prove:

$$j_{Z,\widetilde{w},Fl}^* \circ \mathfrak{q}_Z^{',!}(j_{!,glob,Fl}) \simeq q_Z^{\widetilde{w},!} \circ j_{glob,\widetilde{w},Fl}^*(j_{!,glob,Fl}) \tag{9.24}$$

Indeed, if (9.24) is true, then, we consider the composition:

$$s_{\widetilde{w}} : Conf^\eta \longrightarrow Bun_N^{\omega^\rho,\widetilde{w},'} \underset{\overline{Bun_N^{\omega^\rho,'}}}{\times} Z_{Fl,x}^{\widetilde{w}} \longrightarrow Z_{Fl,x}^{\widetilde{w}}$$

Then, it is easy to see:

$$s_{\widetilde{w}}^* \mathfrak{q}_Z^{',!}(j_{glob,Fl}) = 0$$

Now, we focus on (9.24). The proof is an analog of the proof in [Ga3] 3.6.5 b).

The proof of 9.24 can be divided into two parts: the first step is to prove the lemma for η very anti-dominant, and the second step is to reduce the problem to the case above by factorization property.

I). When η is very anti-dominant, we have the map: $Bun_{B^-}^{\widetilde{w},'} \to Bun'_G$ is smooth, and by definition $Z_{Fl,x}^{\widetilde{w},\circ}$ is an open subset of $\overline{Bun_N^{\omega^\rho,'}} \underset{Bun'_G}{\times} Bun_{B^-}^{\widetilde{w}}$, we have $\mathfrak{q}'_Z : Z_{Fl,x}^{\widetilde{w},\circ} \longrightarrow \overline{Bun_N^{\omega^\rho,'}}$ is smooth. By smoothness property, !-pullback and $*$-pullback differ by a shift, so we have the claim holds in this case.

II). For general η, we could always find a very anti-dominant coweight $\mu \in \Lambda^{neg}$, s.t, $\eta + \mu$ is very anti-dominant. We take $\widetilde{w}' := t^{\eta+\mu} w$.

In this case, we consider the relative version of $\overline{Bun_N^{\omega^\rho,'}}$ over $Conf^\mu$. We denote it by $(Conf^\mu \times \overline{Bun_N^{\omega^\rho,'}})^{good}$. By definition, it is an open subset of

$(Conf^{\mu}\times \overline{Bun_N^{w_0}})$ where we ask all $\kappa^{\check{\lambda}}$ are injective on the neighborhoods of the support of the point from $Conf^{\mu}$.

It has a $N(O)^{\omega^{\rho}}_{Conf^{\mu}}$-torsor by adding an isomorphism data of the corresponding N-bundle on the neighborhood of the support of the point from $Conf^{\mu}$, we denote this $N(O)^{\omega^{\rho}}_{Conf^{\mu}}$-torsor by ${}_{\mu}(\overline{Bun_N^{\omega^{\rho}}})')_{good\ at\ \mu}$. This $N(O)^{\omega^{\rho}}_{Conf^{\mu}}$ action extends to an action of $N(K)^{\omega^{\rho}}_{Conf^{\mu}}$([BD1]). $N(K)^{\omega^{\rho}}_{Conf^{\mu}}$ admits an ind-pro group scheme structure over $Conf^{\mu}$, it is the union of its pro-scheme over $Conf^{\mu}$. We could take a big enough sub pro-group N' of $N(K)^{\omega^{\rho}}_{Conf^{\mu}}$, which contains $N(O)^{\omega^{\rho}}_{Conf^{\mu}}$ and makes the following diagram commute:

$$\begin{tikzcd}
(Z^{\mu,\circ}_{Gr}\times Z^{\widetilde{w},\circ}_{Fl})_{disj} \arrow[r] \arrow[d, "v\times id"] & (Z^{\widetilde{w}',\circ}_{Fl}) \underset{Conf^{\mu+\eta}}{\times} (Conf^{\mu}\times Conf^{\eta})'_{disj} \arrow[d, "Z"', "id_{Conf^{\eta}}\times \mathfrak{q}"] \\
(Conf^{\mu}\times Z^{\widetilde{w},\circ}_{Fl})'_{disj} \arrow[d, "Z"', "id_{Conf^{\eta}}\times \mathfrak{q}"] & (Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good} \arrow[d] \\
(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good} \arrow[r] \arrow[d] & {}_{\mu}(\overline{Bun_N^{\omega^{\rho}}})')_{good\ at\ \mu}/N' \\
(\overline{Bun_N^{\omega^{\rho},'}}) &
\end{tikzcd} \tag{9.25}$$

The morphism in the first column is given by factorization. The lower left morphism, lower middle morphism and lower right morphism are given by projections. Given an object in $(Z^{\mu,\circ}_{Gr}\times Z^{\widetilde{w},\circ}_{Fl})_{disj}$, its images in left lower $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good}$ and in right middle $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good}$ have the same generalized N-reductions on the complement of the support of the point in $Conf^{\mu}$ and same Iwahori structure at x. And quotient by $N(K)^{\omega^{\rho}}_{Conf^{\mu}}$ could ignore the generalized N-reduction near its support. Hence, we could take a big enough N' such that the diagram commute. Further more, we could require the middle lower and right lower morphisms are smooth.

We need to prove that the !-pullback of $j_{!,glob,Fl}$ along with the composition of the left three maps is isomorphic to the !-extension D-module from $(Z^{\circ\mu}_{Gr}\times \tilde{Z}^{\widetilde{w},\circ}_{Fl,x})_{disj}$.

Because $Conf^{\eta}$ is smooth and $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good}$ is an open subset of $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})$, hence, the lower left morphism is smooth. We should prove the !-pullback along with $v\times id\circ(id_{Conf^{\eta}}\times \mathfrak{q}'_Z)$ of the !-extension D-module on $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good}$ extended from $(Conf^{\mu}\times Bun_N^{\omega^{\rho},w_0,'})^{good}$ is the !-extension D-module on $(Z^{\mu,\circ}_{Gr}\times Z^{\widetilde{w},\circ}_{Fl})_{disj}$.

The map $(Conf^{\mu}\times \overline{Bun_N^{\omega^{\rho},'}})^{good}\longrightarrow (Conf^{\mu}\times {}_{\mu}(\overline{Bun_N^{\omega^{\rho}}})')_{good\ at\ \mu}/N'$ is smooth, hence, we should prove the pullback of !-extension D-module on

${}_{\mu}(\overline{Bun_N^{\omega^\rho}})')_{good\ at\ \mu}/N'$ along with the counter-clock composition is the !-extension D-module on $(Z_{Gr}^{\mu,\circ} \times Z_{Fl}^{\widetilde{w},\circ})_{disj}$ which corresponds to the !-extension D-module on $(Z_{Fl}^{\widetilde{w}',\circ})_{disj}$ extended from $(\tilde{Z}_{Fl}^{\widetilde{w}',\circ})_{disj}$ by the factorization isomorphism.

Hence, it suffices to prove: the !-pullback along with $id_{Conf^\eta} \times \mathfrak{q}'_Z$ of the !-extension D-module on $(Conf^\mu \times \overline{Bun_N^{\omega^\rho,'}})^{good}$ is the !-extension D-module on $(Z_{Fl}^{\widetilde{w}',\circ}) \underset{Conf^{\mu+\eta}}{\times} (Conf^\mu \times Conf^\eta)_{disj}$.

Note that: $(Conf^\mu \times \overline{Bun_N^{\omega^\rho,'}})^{good} \longrightarrow \overline{Bun_N^{\omega^\rho,'}}$ is smooth, hence, the !-extension D-module on $(Conf^\mu \times \overline{Bun_N^{\omega^\rho,'}})^{good}$ is the !-pullback of !-extension D-module on $\overline{Bun_N^{\omega^\rho,'}}$.

Hence, we should prove the !-pullback of the !-extension D-module on $\overline{Bun_N^{\omega^\rho,'}}$ along with:

$$(Z_{Fl}^{\widetilde{w}',\circ}) \underset{Conf^{\mu+\eta}}{\times} (Conf^\mu \times Conf^\eta)_{disj} \longrightarrow \overline{Bun_N^{\omega^\rho,'}}$$

is the !-extension D-module.

But according to our assumption on $\widetilde{w}'$, it has already been proved in the first part of the proof.

□

Through the proof of Theorem 9.1, we could notice that the Ran space does not matter the proof: that is to say, we could replace Fl_{Ran_x} by Fl_G and then a tiny modification of the proof could be used to prove the following theorem:

Theorem 9.2.

$$\pi^!_{Fl,x}(j_{!,glob,Fl})[d_g + dimN] \simeq j_{!,Ran_x,Fl} \tag{9.26}$$

What's more, we could replace Ran_x by any other prestack equipped with a map to $T(O)_{Ran_x}\backslash Ran_x$, and by the same proof, we could prove similar statements for relative stacks. We will achieve this in Section 9.4. Before that, we need to introduce the notion of relative stacks.

9.3 Relative stacks

In this section, we will offer a different method to study the Configuration affine Grassmannian and affine flags. The tool is the relative stacks.

Recall that the stack $\bar{S}_{Fl}^{w_0} \subset Fl_x^{\omega^\rho}$ classifies the data $(\mathcal{P}_G, \alpha, \epsilon)$, such that the induced map $\kappa^{\check{\lambda}}$ is regular for every $\check{\lambda} \in \check{\lambda}^+$

In order to satisfy our further requirement, we need to consider its relative version. It means we need to consider a relative version affine flag variety parameterized by a base stack. The idea also comes from [GL1].

In *loc.cit*, athours considered algebraic stacks over the bases $(Gr_{T,Ran}^{\omega^\rho})^{neg}$ and $(Gr_{T,Ran}^{\omega^\rho})^{neg}_{\infty\cdot x}$. We will use same bases.

Let us start with some notations:

Definition 9.6. *Given a prestack $\mathcal{Y}$ equipped with a map to $T(O)_{Ran}\backslash Ran$, we define affine Grassmannian over $\mathcal{Y}$ to be:*

$$_{\mathcal{Y}}Gr := \mathcal{Y} \underset{T(O)_{Ran}\backslash Ran}{\times} T(O)_{Ran}\backslash Gr_{G,Ran} \tag{9.27}$$

Given a presheaf $\mathcal{Y}$ equips with a map to $T(O)_{Ran_x}\backslash Ran_x$, we define affine flags over $\mathcal{Y}$ to be:

$$_{\mathcal{Y}}Fl := \mathcal{Y} \underset{T(O)_{Ran_x}\backslash Ran_x}{\times} T(O)_{Ran_x}\backslash Fl_{G,Ran_x} \tag{9.28}$$

To be more precise, $_{\mathcal{Y}}Gr$ parameterizes the data $(\mathcal{P}_G, I, y, \mathcal{P}_T, \alpha)$, $\mathcal{P}_G$ is a G bundle on X, $I \in Ran$, $\mathcal{P}_T$ is a T bundle on X determined by y, α is an isomorphism of G-bundles between $\mathcal{P}_G$ and $\mathcal{P}_T \underset{T}{\times} G$.

$_{\mathcal{Y}}Fl$ parameterizes the data $(\mathcal{P}_G, I, y, \mathcal{P}_T, \alpha, \epsilon)$, $\mathcal{P}_G$ is a G bundle on X, $I \in Ran_x$, $\mathcal{P}_T$ is a T bundle on X determined by y, α is an isomorphism of G-bundles between $\mathcal{P}_G$ and $\mathcal{P}_T \underset{T}{\times} G$. ϵ is a B reduction of $\mathcal{P}_G$ at x.

That is to say, the difference between $_{\mathcal{Y}}Fl$ and Fl^{ω^ρ} is that we replace ω^ρ by a family of T-bundles on D_I parameterized by $\mathcal{Y}$.

According to the construction and the same proof as the proof of Proposition 6.1, we can see the following factorization property of $_{\mathcal{Y}_0}Gr$ and $_{\mathcal{Y}_1}Fl$.

Proposition 9.2. *If $\mathcal{Y}_1$ is factorizable with respect to $\mathcal{Y}_0$, then, we have: $_{\mathcal{Y}_1}Fl$ is factorizable with respect to $_{\mathcal{Y}_0}Gr$, i.e, there is a canonical isomorphism:*

$$_{\mathcal{Y}_1}Fl \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj} \simeq {}_{\mathcal{Y}_0}Gr \times {}_{\mathcal{Y}_1}Fl \underset{Ran \times Ran_x}{\times} (Ran \times Ran_x)_{disj} \tag{9.29}$$

which is compatible with higher associative properties.

And the gerbes on the corresponding stacks are compatible with the factorization.

Given an element $\widetilde{w}$ in extended Weyl group W^{ext}, like Definition 9.3 we may also consider the corresponding substack $_{\mathcal{Y}}S^{\widetilde{w}}_{Fl}$ in $_{\mathcal{Y}}Fl$ as well as $_{\mathcal{Y}}\bar{S}^{\widetilde{w}}_{Fl}$.

For example, for the longest element w_0 in finite Weyl group, $_{\mathcal{Y}}\bar{S}^{w_0}_{Fl} \subset {}_{\mathcal{Y}}Fl$ is the closed substack in $_{\mathcal{Y}}Fl$ where we ask the induced map $\breve{\kappa}^{\check{\lambda}}$ to be regular for every $\check{\lambda} \in \Lambda^+$.

Definition 9.7. *We denote by $\bar{j}_{\mathcal{Y},\widetilde{w},Fl}$ the open embedding:*

$$\bar{j}_{\mathcal{Y},\widetilde{w},Fl} : {}_{\mathcal{Y}}S^{\widetilde{w}}_{Fl} \hookrightarrow {}_{\mathcal{Y}}\bar{S}^{\widetilde{w}}_{Fl} \tag{9.30}$$

by $i_{\mathcal{Y},\widetilde{w},Fl}$ the proper morphism:

$$\bar{i}_{\mathcal{Y},\widetilde{w},Fl} : {}_{\mathcal{Y}}\bar{S}^{\widetilde{w}}_{Fl} \longrightarrow {}_{\mathcal{Y}}Fl \tag{9.31}$$

and denote by $j_{\mathcal{Y},\widetilde{w},Fl}$ the locally closed embedding:

$$j_{\mathcal{Y},\widetilde{w},Fl} := i_{\mathcal{Y},\widetilde{w},Fl} \circ \bar{j}_{\mathcal{Y},\widetilde{w},Fl} \tag{9.32}$$

Similarly, given an element $\lambda \in \Lambda$, we could define the corresponding substack ${}_{\mathcal{Y}}S^{\lambda}_{Gr} \subset {}_{\mathcal{Y}}Gr$ as well as ${}_{\mathcal{Y}}\bar{S}^{\lambda}_{Gr}$. The corresponding maps are denoted by:

$$\bar{j}_{\mathcal{Y},\widetilde{w},Gr} : {}_{\mathcal{Y}}S^{\widetilde{w}}_{Gr} \hookrightarrow {}_{\mathcal{Y}}\bar{S}^{\widetilde{w}}_{Gr} \tag{9.33}$$

$$\bar{i}_{\mathcal{Y},\widetilde{w},Gr} : {}_{\mathcal{Y}}\bar{S}^{\widetilde{w}}_{Gr} \longrightarrow {}_{\mathcal{Y}}Gr \tag{9.34}$$

$$j_{\mathcal{Y},\widetilde{w},Gr} := i_{\mathcal{Y},\widetilde{w},Gr} \circ \bar{j}_{\mathcal{Y},\widetilde{w},Gr} \tag{9.35}$$

In ${}_{\mathcal{Y}}Fl$, we could define another substack denoted by $({}_{\mathcal{Y}}\bar{S}^{w_0}_{Fl})_{\infty \cdot x}$ which is a relative version of $(\bar{S}^{w_0}_{Fl})_{\infty\cdot x}$. By definition, it classifies the data $(\mathcal{P}_G, I, y, \mathcal{P}_T, \alpha, \epsilon)$ as the data in ${}_{\mathcal{Y}}Fl$, but here, we ask maps $\kappa^{\check{\lambda}}$ induced by α are regular on $X - x$ for all $\check{\lambda} \in \Lambda^+$.

In [GL2], the authors proved $\mathcal{G}^G$ on $Gr_{G,Ran}$ is $G(O)_{Ran}$-equivariant. In particular, it is $T(O)_{Ran}$-equivariant. It descends to a gerbe on $T(O)_{Ran}\backslash Gr_{G,Ran}$. Its pullback to ${}_{\mathcal{Y}}Gr$ is denoted by ${}_{\mathcal{Y}}\mathcal{G}^G$. The gerbe $\mathcal{G}^G$ on Fl_{G,Ran_x} is given by the pullback of $\mathcal{G}^G$ on $Gr_{G,Ran}$. Through the construction, we see this gerbe is also $T(O)_{Ran_x}$-equivariant. As a result, it descends to a gerbe on $T(O)_{Ran_x}\backslash Fl_{G,Ran_x}$. And then, we pullback this gerbe to ${}_{\mathcal{Y}}Fl$ under the projection: ${}_{\mathcal{Y}}Fl \longrightarrow T(O)_{Ran_x}\backslash Fl$, we denote it by ${}_{\mathcal{Y}}\mathcal{G}^G$.

According to the same proof in Corollary 9.2, we could get the following propositions:

Corollary 9.4. *a). The* !*-extension functor*

$$j_{\mathcal{Y},Gr,0,!} : D_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}S^0_{Gr}) \longrightarrow D_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}Gr^{\omega^\rho}) \tag{9.36}$$

can be defined on $N(K)_{\mathcal{Y}\times Ran}$-equivariant objects.

Similarly,

b). The !*-extension functor*

$$j_{\mathcal{Y},Fl,w_0,!} : D_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}S^{w_0}_{Fl}) \longrightarrow D_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}Fl^{\omega^\rho}) \tag{9.37}$$

can be defined on $N(K)_{\mathcal{Y}\times Ran_x}$-equivariant objects.

Note the gerbe ${}_{\mathcal{Y}}\mathcal{G}^G$ on ${}_{\mathcal{Y}}S^0_{Gr}$ is canonically trivialized, so, the dualizing D-module on it can be defined. We denote it by $\omega_{{}_{\mathcal{Y}}S^0_{Gr}}$.

Definition 9.8. *Apply* !*-extension functor to $\omega_{{}_{\mathcal{Y}}S^0_{Gr}} \in D_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}S^0_{Gr})$, we obtain a twisted D-module on ${}_{\mathcal{Y}}Gr$. We denote it by $j_{!,\mathcal{Y},Gr}$.*

Similarly, we can define the !*-extension of the dualizing D-module $\omega_{{}_{\mathcal{Y}}S^{w_0}_{Fl}}$. We denote it by $j_{!,\mathcal{Y},Fl}$.*

$*$-extension functor is always well-defined in the context of D-modules, so, we could also define $*$-extension of the dualizing D-module $\omega_{{}_{\mathcal{Y}}S^0_{Gr}}$ on $\mathcal{Y}S^0_{Gr}$ and $\omega_{{}_{\mathcal{Y}}S^{w_0}_{Fl}}$ on ${}_{\mathcal{Y}}S^{w_0}_{Fl}$, we denote the resulted twisted D-modules by $j_{*,\mathcal{Y},Gr}$ and $j_{*,\mathcal{Y},Fl}$ respectively.

In this paper, we will concentrate on the cases:

1). ${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr$ and ${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl$ 2). ${}_{Bun_T\times Ran}Gr$ and ${}_{Bun_T\times Ran_x}Fl$.

9.3.1 ${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr$ and ${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl$

${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr$ (or, ${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl$) maps to $T(O)_{Ran}\backslash Ran$ (or $T(O)_{Ran_x}\backslash Ran_x$) by sending $(I, \mathcal{P}_T, \alpha_T)$ to $(I, \mathcal{P}_T)$.

We can describe ${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl$ as the following data: $(\mathcal{P}_T, \mathcal{P}_G, I, \alpha_G, \alpha_T, \epsilon)$. Here, $\mathcal{P}_G \in Bun_G$, ($\mathcal{P}_T \in Bun_T$, $I \in Ran_x$, α_T) is an element in $(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}$, α_G is a G-bundle isomorphism between $\mathcal{P}_G$ and $\mathcal{P}_T \overset{T}{\times} G$ on $X\backslash I$. The description of ${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr$ is similar, but without ϵ and I is an element in Ran instead of in Ran_x.

According to the above descriptions, there are canonical isomorphisms of stacks:

$$ {}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr \simeq (Gr^{\omega^\rho}_{T,Ran})^{neg} \underset{Ran}{\times} Gr^{\omega^\rho}_{G,Ran} \tag{9.38} $$

$$ {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl \simeq (Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{G,Ran_x} \tag{9.39} $$

Under the above isomorphisms, we have:

$$ {}_{\mathcal{Y}}\mathcal{G}^G \simeq (\mathcal{G}^T)^{-1} \boxtimes \mathcal{G}^G $$

It follows from the multiplicativity property of $\mathcal{G}^G$.

By definition, we have the following isomorphisms of stacks:

$$ (Gr^{\omega^\rho}_{T,Ran})^{neg} \underset{Ran}{\times} Gr^{\omega^\rho}_{G,Ran} \simeq (Gr^{\omega^\rho}_{T,Ran})^{neg} \underset{Conf}{\times} Gr^{\omega^\rho}_{G,Conf} \tag{9.40} $$

$$ (Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x} \underset{Ran_x}{\times} Fl^{\omega^\rho}_{G,Ran_x} \simeq (Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x} \underset{Conf_x}{\times} Fl^{\omega^\rho}_{G,Conf_x} \tag{9.41} $$

Under the isomorphism (9.40) and (9.41), the substacks $S^{Conf}_{Gr,Conf}$ and $S^{w,Conf_x}_{Fl,Conf_x}$ that we defined in Section 6.2 could be regarded as substacks of ${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr$ and ${}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}_{\infty\cdot x}}Fl$, i.e,

$$ (Gr^{\omega^\rho}_{T,Ran})^{neg} \underset{Conf}{\times} S^{Conf}_{Gr,Conf} \simeq {}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}S^0_{Gr} \tag{9.42} $$

$$ (Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x} \underset{Conf_x}{\times} S^{w,Conf_x}_{Fl,Conf_x} \simeq {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}S^w_{Fl} \tag{9.43} $$

Similar for their closure and the negative counterparts($S^{-,w,Conf_x}_{Fl,Conf_x}$, $S^{-,Conf}_{Gr,Conf}$, etc and their closure).

From the construction, we have the following important factorization property:

Proposition 9.3. *$j_{!,(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x},Fl}$ factorizes with respect to $j_{!,(Gr^{\omega^\rho}_{T,Ran})^{neg},Gr}$, i.e,*

$$j_{!,(Gr^{\omega^\rho}_{T,Ran})^{neg},Gr} \boxtimes j_{!,(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x},Fl}|_A$$
$$\simeq$$
$$j_{!,(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x},Fl}|_B$$

Under the isomorphism:

$$B := {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj}$$
$$\simeq$$
$$A := {}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}Gr \times {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}Fl \underset{Ran \times Ran_x}{\times} (Ran \times Ran_x)_{disj}$$

By replacing N by its negative counterpart N^- in (9.42) and (9.43), we have the equivalences:

$$D_{\mathcal{G}^{G,T,\mathrm{ratio}}}(S^{-,Conf}_{Gr,Conf}) \simeq D_{\mathcal{Y}(\mathcal{G}^G)^{-1}}({}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}S^{-,0}_{Gr}) \tag{9.44}$$

and

$$D_{\mathcal{G}^{G,T,\mathrm{ratio}}}(S^{-,w,Conf_x}_{Fl,Conf_x}) \simeq D_{\mathcal{Y}(\mathcal{G}^G)^{-1}}({}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}S^{-,w}_{Fl}) \tag{9.45}$$

which are compatible with

$$D_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\bar{S}^{-,Conf}_{Gr,Conf}) \simeq D_{\mathcal{Y}(\mathcal{G}^G)^{-1}}({}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}\bar{S}^{-,0}_{Gr}) \tag{9.46}$$

$$D_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\bar{S}^{-,w,Conf_x}_{Fl,Conf_x}) \simeq D_{\mathcal{Y}(\mathcal{G}^G)^{-1}}({}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}\bar{S}^{-,w}_{Fl}) \tag{9.47}$$

Then, under the above equivalences, Proposition 9.3(replacing N by N^-) essentially recovers Proposition 6.47.

9.3.2 ${}_{Bun_T \times Ran}Gr$ and ${}_{Bun_T \times Ran_x}Fl$

${}_{Bun_T \times Ran_x}Fl$ classifies the data: $(\mathcal{P}_T, \mathcal{P}_G, \alpha_G, \epsilon)$ as in the case of ${}_{Gr_{T,Ran_x}}Fl$. (but the data used in ${}_{Bun_T \times Ran_x}Fl$ does not contain α_T, so, we do not have a canonical map from ${}_{Bun_T \times Ran_x}Fl$ to Fl^{ω^ρ}) ${}_{Bun_T \times Ran_x}Fl$ is similar.

The factorization property of $j_{!,Bun_T \times Ran_x,Fl}$ says:

Proposition 9.4. *$j_{!,Bun_T \times Ran_x,Fl}$ factorizes with respect to $j_{!,Bun_T \times Ran,Gr}$, i.e,*

$$j_{!,Bun_T \times Ran,Gr} \boxtimes j_{!,Bun_T \times Ran_x,Fl}|_A$$
$$\simeq$$
$$j_{!,Bun_T \times Ran_x,Fl}|_B$$

Under the isomorphism:

$$B := {}_{Bun_T \times Ran_x}Fl \underset{Ran_x}{\times} (Ran \times Ran_x)_{disj}$$
$$\simeq$$
$$A := {}_{Bun_T \times Ran}Gr \times {}_{Bun_T \times Ran_x}Fl \underset{Ran \times Ran_x}{\times} (Ran \times Ran_x)_{disj}$$

9.4 Compare of !-extension D-module on relative stacks

We will compare D-modules $j^-_{!,glob,Bun_T\times Ran,Gr}$ (resp. $j^-_{!,glob,Bun_T\times Ran_x,Fl}$) with the semi-infinite D-modules $j_!(\omega_{S^{-,Conf}_{Gr,Conf}})$ (resp. $j_!(\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}})$). It is the content of Corollary 9.7. $j^-_{!,glob,Bun_T\times Ran,Gr}$ and $j^-_{!,glob,Bun_T\times Ran_x,Fl}$ were used in the construction of $F^L_{glob,Gr}$ and F^L_{glob}, $j_!(\omega_{S^{-,Conf}_{Gr,Conf}})$ and $j_!(\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}})$ were used in the construction of F^L_{Gr} and F^L. The Corollary 9.7 is important for us in the comparison of locally defined functor and globally defined functor.

If we forget the isomorphism of G-bundles on the open subset of X, we could relate ${}_{\mathcal{Y}}Fl$ and ${}_{\mathcal{Y}}Gr$ with the corresponding global objects.

Similar to Section 9.2, in relative setting, we have the following canonical maps:

$$\pi_{Bun_T\times Ran_x,Fl} : {}_{Bun_T\times Ran_x}\bar{S}^{w_0}_{Fl} \longrightarrow {}_{Bun_T}\overline{Bun^{\omega^\rho}_N}' = \overline{Bun_B}' \tag{9.48}$$

Similarly,

$$\pi_{Bun_T\times Ran,Gr} : {}_{Bun_T\times Ran}\bar{S}^{0}_{Gr} \longrightarrow {}_{Bun_T}\overline{Bun^{\omega^\rho}_N} = \overline{Bun_B} \tag{9.49}$$

it sends $(\mathcal{P}_T, I, \mathcal{P}_G, \alpha_G)$ to $(\mathcal{P}_T, \mathcal{P}_G, \kappa^{\check{\lambda}})$, here $\kappa^{\check{\lambda}}$ is the map induced by α.

We are going to introduce the gerbes on $\overline{Bun_B}'$ and $\overline{Bun_B}'$ corresponding to ${}_{Bun_T\times Ran_x}\mathcal{G}^G$ and ${}_{Bun_T\times Ran}\mathcal{G}^G$ under (9.48) and (9.49). We also denote them by $\mathcal{G}^{G,T,ratio}_{glob}$.

Definition 9.9. *Under the diagrams below:*

$$\begin{tikzcd} & \overline{Bun_B} \arrow[dl] \arrow[dr] & \\ Bun_G & & Bun_T \end{tikzcd}$$

$$\begin{tikzcd} & \overline{Bun'_B} \arrow[dl] \arrow[dr] & \\ Bun'_G & & Bun_T \end{tikzcd}$$

We define gerbes $\mathcal{G}^{G,T,ratio}_{glob}$ on $\overline{Bun_B}$ and $\overline{Bun_B}'$ to be $(\mathcal{G}^G)\bigotimes(\mathcal{G}^T)^{-1}$.

By construction, it is easy to see that the gerbe $\mathcal{G}^{G,T,\text{ratio}}$ corresponds to ${}_{Bun_T\times Ran_x}\mathcal{G}^G$ and ${}_{Bun_T\times Ran}\mathcal{G}^G$ under (9.46) and (9.47).

Similar to the case of $\overline{Bun_{B^-}}$ and $\overline{Bun'_{B^-}}$, we could define the relative global !-extension D-modules on $\overline{Bun'_B}$ and $\overline{Bun_B}$:

Definition 9.10. *We denote by $j_{!,glob,Bun_T\times Ran_x,Fl}$ (resp. $j_{!,glob,Bun_T\times Ran,Gr}$) the !-extension of the perverse constant D-module on Bun''_B (resp. the perverse constant D-module on Bun_B).*

By the same proof of Theorem 9.1 (also, Theorem 14.4.8 in [GL1]), we have the following theorem of comparison between $j_{!,Bun_T\times Ran_x,glob,Fl}$ (resp. $j_{!,Bun_T\times Ran,glob,Gr}$) and $j_{!,Bun_T\times Ran_x,Fl}$ (resp. $j_{!,Bun_T\times Ran,Gr}$).

Theorem 9.3.

$$\begin{gathered}\pi^!_{Bun_T\times Ran}(j_{!,Bun_T\times Ran,glob,Gr})[d_g+dim(Bun_T)+deg]\\ \simeq \\ j_{!,Bun_T\times Ran,Gr}\in D_{\mathcal{G}^{G,T,ratio}}({}_{Bun_T\times Ran}\bar{S}^0_{Gr})\end{gathered} \tag{9.50}$$

$$\begin{gathered}\pi^!_{Bun_T\times Ran_x}(j_{!,Bun_T\times Ran_x,glob,Fl})[d_g+dim(Bun_T)+deg+dimN]\\ \simeq \\ j_{!,Bun_T\times Ran_x,Fl}\in D_{\mathcal{G}^{G,T,ratio}}({}_{Bun_T\times Ran_x}\bar{S}^{w_0}_{Fl})\end{gathered} \tag{9.51}$$

Here, $deg=\langle\lambda,2\check{\rho}\rangle$ *on* $Bun_T^{\lambda}\underset{Bun_T}{\times}Bun_B$.

Consider the projection:

$$(Gr_{T,Ran_x})^{neg}_{\infty\cdot x}\longrightarrow Bun_T\times Ran_x \tag{9.52}$$

It induces a map:

$${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}(\bar{S}^{w_0}_{Fl})\longrightarrow {}_{Bun_T\times Ran_x}(\bar{S}^{w_0}_{Fl}) \tag{9.53}$$

By the definition of ${}_{\mathcal{Y}}\mathcal{G}^G$ indicated in Section 9.3, the pullback of the gerbe ${}_{Bun_T\times Ran_x}\mathcal{G}^G$ along with (9.53) is canonically isomorphic to ${}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}\mathcal{G}^G$.

And we have the following Cartesian diagram:

$$\begin{tikzcd} {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}(S^{w_0}_{Fl}) \arrow[r] \arrow[d] & {}_{Bun_T\times Ran_x}(S^{w_0}_{Fl}) \arrow[d] \\ {}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}(\bar{S}^{w_0}_{Fl}) \arrow[r] \arrow[d] & {}_{Bun_T\times Ran_x}(\bar{S}^{w_0}_{Fl}) \arrow[d] \\ (Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x} \arrow[r] & Bun_T\times Ran_x \end{tikzcd} \tag{9.54}$$

Hence, the !-pullback of $j_{!,Bun_T\times Ran_x,Fl}$ is isomorphic to $j_{!,(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x},Fl}$. We compose the map (9.53) with $\pi_{Bun_T\times Ran_x,Fl}$, then, we have the map:

$$\pi_{Fl,+}:{}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x}}(\bar{S}^{w_0}_{Fl})\longrightarrow\overline{Bun'_B} \tag{9.55}$$

By Theorem 9.3,

Corollary 9.5.

$$\pi^!_{Fl,+}(j_{!,glob,Bun_T\times Ran_x,Fl})[d_g+dim(Bun_T)+deg+dimN]\simeq j_{!,(Gr^{\omega^\rho}_{T,Ran_x})^{neg}_{\infty\cdot x},Fl} \tag{9.56}$$

Similarly,

$$\begin{tikzcd}
{}_{(Gr^{\omega^\rho}_{T,Ran_x})^{neg}}(S^0_{Gr}) \arrow[r] \arrow[d] & {}_{Bun_T\times Ran}(S^0_{Gr}) \arrow[d] \\
{}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}(\bar{S}^0_{Gr}) \arrow[r] \arrow[d] & {}_{Bun_T\times Ran}(\bar{S}^0_{Gr}) \arrow[d] \\
(Gr^{\omega^\rho}_{T,Ran})^{neg} \arrow[r] & Bun_T\times Ran
\end{tikzcd} \tag{9.57}$$

we have the map:

$$\pi_{Gr,+}:{}_{(Gr^{\omega^\rho}_{T,Ran})^{neg}}(\bar{S}^0_{Gr})\longrightarrow \overline{Bun_B} \tag{9.58}$$

And

Corollary 9.6.

$$\pi^!_{Gr,+}(j_{!,glob,Bun_T\times Ran,Gr})[d_g+dim(Bun_T)+deg]\simeq j_{!,(Gr^{\omega^\rho}_{T,Ran})^{neg},Gr} \tag{9.59}$$

Because N orbits and N^- orbits are essentially same(just differ by a w_0 twist), we could get similar results if we replace N by N^-.

Then, by (9.44), (9.45), (9.46) and (9.47), we could relate global twisted constant D-modules with semi-infinite twisted D-modules on Configuration affine Grassmannian and affine flags. Namely, if we denote by

$$\pi_{Gr,-}:\bar{S}^{-,Conf}_{Conf,Gr}\longrightarrow \overline{Bun_{B^-}} \tag{9.60}$$

and

$$\pi_{Fl,-}:\bar{S}^{-,Conf_x}_{Conf_x,Fl}\longrightarrow \overline{Bun'_{B^-}} \tag{9.61}$$

the natural maps, then, we have:

Corollary 9.7.

$$\pi^!_{Gr,-}(j^-_{!,glob,Bun_T\times Ran,Gr})[dim.Bun_{B^-}]\simeq j_!(\omega_{S^{-,Conf}_{Gr,Conf}}) \tag{9.62}$$

and

$$\pi^!_{Fl,-}(j^-_{!,glob,Bun_T\times Ran_x,Fl})[dim.Bun_{B^-}]\simeq j_!(\omega_{S^{-,1,Conf_x}_{Fl,Conf_x}}) \tag{9.63}$$

Here, the shift $[dim.Bun_{B^-}]=dim.Bun^{\lambda}_{B^-}$ on the connected component $dim.Bun^{\lambda}_{B^-}$. And the shift $[dim.Bun_{B^-}]=dim.Bun^{\lambda,'}_{B^-}$ on the connected component $dim.Bun^{\lambda,'}_{B^-}$.

9.5 Local description of Zastava spaces

In order to compare the locally defined functor and the globally defined functor, we also need to identify affine flags Zastava space $(Z_{Fl,x})_{\infty\cdot x}$ with $\bar{S}^{-,Conf_x}_{Conf_x,Fl} \cap (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x}$ and identify $Z_{Gr,x}$ with $\bar{S}^{-,Conf}_{Conf,Gr} \cap (\bar{S}^{0}_{Gr,Conf})$ (and also identify their substacks under this identification). After these identifications, Theorem 7.1 b). will follow directly.

Proposition 9.5. *a).*

$$(\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \xrightarrow{\sim} Z_{Gr,x} \tag{9.64}$$

And under the above identification, we have:

$$(\bar{S}^0_{Gr,Conf}) \cap S^{-,Conf}_{Gr,Conf} \xrightarrow{\sim} Z^{\circ}_{Gr,x} \tag{9.65}$$

b).

$$(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S}^{-,Conf_x}_{Fl,Conf_x} \xrightarrow{\sim} (Z_{Fl,x})_{\infty\cdot x} \tag{9.66}$$

And under the above isomorphism, the following isomorphisms hold:

$$(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap S^{-,Conf_x}_{Fl,Conf_x} \xrightarrow{\sim} Z'_{Fl,x} \tag{9.67}$$

$$(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap S^{-,1,Conf_x}_{Fl,Conf_x} \xrightarrow{\sim} Z^{\circ}_{Fl,x} \tag{9.68}$$

$$(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S^{-,w,Conf_x}_{Fl,Conf_x}} \xrightarrow{\sim} (Z^{w}_{Fl,x})_{\infty\cdot x} \tag{9.69}$$

$$(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap S^{-,w,Conf_x}_{Fl,Conf_x} \xrightarrow{\sim} (Z^{w,\circ}_{Fl,x})_{\infty\cdot x} \tag{9.70}$$

And the above isomorphisms are compatible with the projections to $Conf_x$.

Proof. The proof of Proposition 9.5 is given in [GL1], Proposition 20.2.2. In order to be complete, we copy the proof here.

a). We want to construct two maps between $(\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}}$ and $Z_{Gr,x}$ which are inverse to each other. The map $(\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \longrightarrow Z_{Gr,x}$ is easy. Given a point $(D, \mathcal{P}_G, \alpha)$ in Gr_{Conf}, α induces a collection of morphisms $\{\kappa^{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}$. By the definition of $\bar{S}^0_{Gr,Conf}$, $\kappa^{\check{\lambda}}$ is regular on $X - x$. Similarly, $\kappa^{-,\check{\lambda}}$ is genuine on X. These two collections of morphisms are induced by an isomorphism α on $X - supp(D)$, hence, the resulted morphisms

$$(\omega^{\frac{1}{2}})^{\langle \lambda, \check{2}\rho\rangle} \xrightarrow{\kappa^{\check{\lambda}}} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \longrightarrow' \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \xrightarrow{\kappa^{-,\check{\lambda}}} \check{\lambda}(\omega^{\rho}(D))$$

is transversal on $X - supp(D)$. So, a point in $(\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \longrightarrow Z_{Gr,x}$ determines a point in $Z_{Gr,x}$.

Now, we construct an inverse map $Z_{Gr,x} \longrightarrow (\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}}$. Given a point $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}, \{\kappa^{-,\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}, \mathcal{P}_T)$ in the Zastava space $Z_{Gr,x}$, by the transversity of $\kappa^{\check{\lambda}}$ and $\kappa^{-,\check{\lambda}}$, the composition:

$$(\omega^{\frac{1}{2}})^{\langle\lambda,\check{2}\rho\rangle} \xrightarrow{\kappa^{\check{\lambda}}} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \longrightarrow' \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \xrightarrow{\kappa^{-,\check{\lambda}}} \check{\lambda}(\mathcal{P}_T) \tag{9.71}$$

determines a point $D \in Conf$. And (9.71) is an isomorphism on the complement of $supp(D)$. Because $N\backslash NB^-/B^- \simeq pt$, (9.71) determines an isomorphism of $\mathcal{P}_G$ on the complement of $supp(D)$. Hence, it determines a point in $(\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}}$.

To prove b). We only need to notice that both sides of (9.64) tensor with pt/B over pt/G gives an identification:

$$(\bar{S}^{w_0}_{Fl,Conf_x}) \cap \overline{S}^{-,Conf_x}_{Fl,Conf_x} \xrightarrow{\sim} (Z_{Fl,x}) \tag{9.72}$$

And if we allow $\kappa^{\check{\lambda}}$ have a pole at x, then, apply the same proof, we could get (9.66).

Both sides of (9.67) are given by (9.66) with the condition that the Iwahori structure is transversal with the B^--reduction of $\mathcal{P}_G$ at x.

All other identifications follow similarly. □

By the construction of the identifications given above, we have the following commutativity:

Proposition 9.6. *The following diagrams commute:*

$$\begin{tikzcd} (\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \arrow[r] \arrow[d] & Z_{Gr,x} \arrow[d] \\ (\bar{S}^0_{Gr,Conf}) \arrow[r] & \overline{Bun_N^{\omega^\rho}} \end{tikzcd} \tag{9.73}$$

$$\begin{tikzcd} (\bar{S}^0_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \arrow[r] \arrow[d] & Z_{Gr,x} \arrow[d] \\ \overline{S^{-,Conf}_{Gr,Conf}} \arrow[r] & \overline{Bun_{B^-}} \end{tikzcd} \tag{9.74}$$

$$\begin{tikzcd} (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S}^{-,Conf_x}_{Fl,Conf_x} \arrow[r] \arrow[d] & (Z_{Fl,x})_{\infty\cdot x} \arrow[d] \\ (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \arrow[r] & (\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x} \end{tikzcd} \tag{9.75}$$

$$\begin{tikzcd}
(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S}^{-,Conf_x}_{Fl,Conf_x} \arrow[r] \arrow[d] & (Z_{Fl,x})_{\infty\cdot x} \arrow[d] \\
\overline{S}^{-,Conf_x}_{Fl,Conf_x} \arrow[r] & (\overline{Bun_{B^-}})'
\end{tikzcd} \tag{9.76}$$

$$\begin{tikzcd}
(\bar{S}^{0}_{Gr,Conf}) \cap \overline{S^{-,Conf}_{Gr,Conf}} \arrow[rr] \arrow[dr] & & Z_{Gr,x} \arrow[dl] \\
& Conf &
\end{tikzcd} \tag{9.77}$$

$$\begin{tikzcd}
(\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \cap \overline{S}^{-,Conf_x}_{Fl,Conf_x} \arrow[rr] \arrow[dr] & & (Z_{Fl,x})_{\infty\cdot x} \arrow[dl] \\
& Conf_x &
\end{tikzcd} \tag{9.78}$$

By the above identifications, we could get the fiber of $(Z_{Fl,x})_{\infty\cdot x} \longrightarrow Conf_x$ over $D = \lambda \cdot x + \sum_i \lambda_i \cdot x_i$ is given by:

$$Fl_x^{\omega^\rho} \times \prod_i \bar{S}^0_{Gr,x_i} \cap \bar{S}^{-,\lambda_i}_{Gr,x_i}$$

9.6 Compare between local functor and global functor

In this section, we will prove that the functor F^L, F^{KD}, F^L_{Gr} and F^{KD}_{Gr} are essentially the same as their corresponding globally defined functors $F^L_{glob}, F^{KD}_{glob}, F^L_{glob,Gr}$ and $F^{KD}_{glob,Gr}$.

First of all, let us combine the diagrams (6.52) and (7.15) together.

$$\begin{tikzcd}
 & & & Gr^{\omega^\rho}_{G,Conf} & \\
 & & \bar{S}^0_{Gr,Conf} \arrow[ur] \arrow[d] & & \overline{S^{-,Conf}_{Conf,Gr}} \arrow[ul] \arrow[dd, "\pi_{Gr,-}"] \\
 & \bar{S}^0_{Ran} \times Ran \arrow[r, "unit"] \arrow[dl, "pr_{Ran\times Gr\to Gr}"] & (\bar{S}^0_{Gr,Ran}) \arrow[d] & Z_{Gr} \arrow[ul] \arrow[ur] \arrow[dl, "\bar{\mathsf{q}}_Z"] \arrow[dr, "\bar{\mathsf{p}}_Z"] \arrow[dd, "v"] & \\
\bar{S}^0_x & & (\overline{Bun^{\omega^\rho}_N}) & & \overline{Bun_{B^-}} \\
 & & & Conf &
\end{tikzcd}$$

as well as diagrams (6.49) and (7.12):

$$\begin{tikzcd}
 & & Fl^{\omega^\rho}_{G,Conf_x} & \\
 & (\bar{S}^{w_0}_{Fl,Conf_x})_{\infty\cdot x} \arrow[ur] \arrow[d] & & \overline{S^{-,Conf_x}_{Conf_x}, Fl} \arrow[ul] \arrow[dd, "\pi_{Fl,-}"] \\
Fl^{\omega^\rho}_{G,Ran_x} \times Ran_x \arrow[r, "unit"] \arrow[d, "pr_{Ran_x\times Fl\to Fl}"] & (\bar{S}^{w_0}_{Fl,Ran_x})_{\infty\cdot x} \arrow[d] & (Z_{Fl,x})_{\infty\cdot x} \arrow[ul] \arrow[ur] \arrow[dl, "\bar{\mathsf{q}}'_Z"] \arrow[dr, "\bar{\mathsf{p}}'_Z"] \arrow[dd, "v"] & \\
Fl^{\omega^\rho}_x & (\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x} & & \overline{Bun_{B^-}}' \\
 & & Conf_x &
\end{tikzcd}$$

Proof. (of Theorem 7.1 b).) We prove the second part of the claim, i.e, $F^L_{Gr} \simeq F^L_{glob,Gr}$. The other part is the same.

By construction:

$$F^L_{Gr}(\pi^!_{Gr,x}(\mathcal{F})) := v_{Conf,*}(sprd_{Gr}(\pi^!_{Gr,x}(\mathcal{F}))|_{(\bar{S}^0_{Gr,Conf})} \overset{!}{\otimes} j_!(\omega_{\overline{S^{-,Conf}_{Conf,Gr}}}))[deg]$$

and

$$F^{L}_{glob,Gr}(\mathcal{F}) := v_{Gr,!}(\bar{\mathfrak{q}}^{!}_{Z}(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}^{!}_{Z}(j^{-}_{!,glob,Bun_T\times Ran,Gr}))[dimBun_G]$$

We need to prove that they are isomorphic.

By Proposition 9.5 and Proposition 9.6, we only need to prove that, under the identification (9.64), we have:

$$\begin{gathered} sprd_{Gr}(\pi^{!}_{Gr,x}(\mathcal{F}))|_{(\bar{S}^{0}_{Gr,Conf})} \overset{!}{\otimes} j_{!}(\omega_{S^{-,Conf}_{Conf,Gr}})[deg+d_g] \\ \simeq \\ \bar{\mathfrak{q}}^{!}_{Z}(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}^{!}_{Z}(j^{-}_{!,glob,Bun_T\times Ran,Gr})[dimBun_G] \end{gathered} \tag{9.79}$$

By (9.62) of Corollary 9.7, we only need to prove:

$$sprd_{Gr}(\pi^{!}_{Gr,x}(\mathcal{F}))|_{Z_{Gr,x}} \simeq \bar{\mathfrak{q}}^{!}_{Z}(\mathcal{F})$$

It suffices to prove that:

$$sprd_{Gr}(\pi^{!}_{Gr,x}(\mathcal{F})) \simeq \pi^{!}_{Gr,Conf}(\mathcal{F})$$

And it suffices to prove:

$$(unit^{!}_{Gr})^{L} pr^{!}_{Ran\times Gr\to Gr}(\pi^{!}_{Gr,x}(\mathcal{F})) \simeq \pi^{!}_{Gr,Ran}(\mathcal{F})$$

By Theorem 6.1, we only need to prove the above isomorphism after pullback along with:

$$unit_{Gr} : Ran \times \bar{S}^{0}_{Gr,x} \longrightarrow \bar{S}^{0}_{Ran}$$

And then, Theorem 7.1 b). comes from the following isomorphism:

$$\begin{aligned} unit^{!}_{Gr} \circ (unit^{!}_{Gr})^{L} \circ pr^{!}_{Ran\times Gr\to Gr} \circ \pi^{!}_{Gr,x} &\simeq pr^{!}_{Ran\times Gr\to Gr} \circ \pi^{!}_{Gr,x} \\ &\simeq unit^{!}_{Gr} \circ \pi^{!}_{Gr,Ran} \end{aligned} \tag{9.80}$$

□

Apply the same argument as Theorem 7.1 b)., we could get the following dual proposition:

Proposition 9.7. *a).Under the equivalence:*

$$\pi^{!}_{x,Fl}[d_g] : Whit_q((\overline{Bun^{\omega^\rho}_{N}})'_{\infty\cdot x}) \xrightarrow{\sim} Whit_q(Fl^{\omega^\rho}_{G,x})$$

$$F^{DK} : Whit_q(Fl^{\omega^\rho}_{G,x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

and

$$F^{DK}_{glob} : Whit_q((\overline{Bun^{\omega^\rho}_{N}})'_{\infty\cdot x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf_x)$$

are isomorphic.

b). Under the equivalence:

$$\pi^!_{x,Fl}[d_g] : Whit_q(\overline{Bun^{\omega^\rho}_N}) \xrightarrow{\sim} Whit_q(\bar{S}^0_{Gr})$$

$$F^{KD}_{Gr} : Whit_q(Gr^{\omega^\rho}_{G,x}) \longrightarrow D_{\mathcal{G}^\Lambda}(Conf)$$

Immediate from Theorem 7.1 b). and Proposition 9.7, we have:

Corollary 9.8.
$$\Omega^{L'}_{q,fact} \simeq \Omega^{L'}_q \tag{9.81}$$
and
$$\Omega^{KD'}_{q,fact} \simeq \Omega^{KD'}_q \tag{9.82}$$

Proof. Apply Theorem 7.1 b). to $\mathcal{F}_\emptyset$. □

Combined Proposition 7.1 b)., 6.4, 6.5 with Corollary 9.8, we have:

Corollary 9.9. *F^L_{glob} factors through $\Omega^{L,'}_q - FactMod$ and F^{DK}_{glob} factors through $\Omega^{DK,'}_q - FactMod$:*

$$F^L_{glob} : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow \Omega^{L,'}_q - FactMod$$

and

$$F^{DK}_{glob} : Whit_q((\overline{Bun^{\omega^\rho}_N})'_{\infty\cdot x}) \longrightarrow \Omega^{DK,'}_q - FactMod$$

In the paper [Ga6], the author proved

Lemma 9.11. *When q avoids small torsion, then,*

$$\Omega^L_q \simeq \Omega^{L,'}_q$$
$$\Omega^{DK}_q \simeq \Omega^{DK,'}_q$$

Hence, we could get the following statement used in Section 6.9 to get Theorem 4.1 from Theorem 4.1',

Corollary 9.10. *When q avoids small torsion,*

$$\Omega^L_q \simeq \Omega^{L,'}_{q,fact}$$

$$\Omega^{DK}_q \simeq \Omega^{DK,'}_{q,fact}$$

10 Proof of Proposition 7.4: Duality

In this section, we will study the relationship between F^L and F^{KD}, we want to prove that:

$$F^L_{glob}(\mathbb{D}(\mathcal{F})) \simeq \mathbb{D}F^{KD}_{glob}(\mathcal{F}) : Whit_q((\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x})^{loc.c} \longrightarrow \Omega_q^{L,'} - FactMod$$

The method is given by reducing the above isomorphism to some stack where we can apply the ULA property.

Before that, let us recall some notions.

10.1 Universal local acyclicity

The key to prove the main theorem is the Universal local acyclicity (ULA) property of $j_{!,Bun_T\times Ran_x,glob,Fl}$ with respect to $\overline{Bun_{B^-}}' \longrightarrow Bun'_G$.

In the [BG], the authors induced a notion of Universal local acyclicity which is similar to the definition given in [D]. In general, a sheaf on $\mathcal{Y}_1$ is ULA with respect to a morphism $f : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$ is to say its singular support over each fiber of f is the same. Another equivalent description is about finite generation property of the stalk which will be given in Lemma 10.3.

Given a map between two smooth algebraic stacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$, a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ and two holonomic D-modules $\mathcal{F}_1, \mathcal{F}_2 \in D(\mathcal{Y}_2)$, we have the natural transformation between $f^*[\mathcal{Y}_1 - dim\mathcal{Y}_2]$ and $f^![-dim\mathcal{Y}_1 + dim\mathcal{Y}_2]$, i.e:

$$can_f : f^*(\mathcal{F})[dim\mathcal{Y}_1 - dim\mathcal{Y}_2] \longrightarrow f^!(\mathcal{F})[-dim\mathcal{Y}_1 + dim\mathcal{Y}_2] \tag{10.1}$$

Given a morphism g between two smooth algebraic stacks X and Y. Denote by $\Gamma_g : X \to X \times Y$ the graph map.

In the (10.1), we consider the case when $\mathcal{Y}_1 = X$, and $\mathcal{Y}_2 = X \times Y$.

In this case, we get a morphism :

$$can_{\Gamma_g} : \Gamma_g^*(\mathcal{F})[-dimY] \longrightarrow \Gamma_g^!(\mathcal{F})[dimY]$$

If our $\mathcal{F}$ has the form $\mathcal{F}_X \boxtimes \mathcal{F}_Y \in D(X) \boxtimes D(Y) \simeq D(X \times Y)$, then, the morphism above can be rewritten as:

$$can_{\Gamma_g} : \mathcal{F}_X \overset{*}{\bigotimes} \mathcal{F}_Y[-dimY] \longrightarrow \mathcal{F}_X \overset{!}{\bigotimes} \mathcal{F}_Y[dimY] \tag{10.2}$$

Definition 10.1. *A D-module $\mathcal{F}_X \in D(X)$ is called locally acyclic with respect to*

$$g : X \longrightarrow Y$$

if for any holonomic D-module $\mathcal{F}_2 \in D(Y)$, the canonical map can_{Γ_g} in (10.2) is an isomorphism.

A D-module $\mathcal{F}_X \in D(X)$ is called universally locally acyclic with respect to $g : X \longrightarrow Y$ if for any smooth morphism $Y' \longrightarrow Y$, the pullback of $\mathcal{F}_X$ to $X \underset{Y}{\times} Y'$ is local acyclic with respect to $X \underset{Y}{\times} Y' \longrightarrow Y'$.

From now on, we will write 'ULA' for universally local acyclic.

There are some basic properties of ULA listed in [BG]

Lemma 10.1. *Given a D-module $\mathcal{F} \in D(X)$ and a morphism $f : X \longrightarrow Y$,*

a). ULA property is a smooth local property on the source. I.e, if there is a smooth (resp. smooth and surjective) morphism $s : X' \longrightarrow X$, then the pullback of $\mathcal{F}$ is ULA w.r.t $f \circ s$ if (resp. if and only if) $\mathcal{F}$ is ULA w.r.t f.

b). Closed embedding keeps ULA property. I.e, if we have a closed embedding $i : X \longrightarrow X'$. And the map $f : X \longrightarrow Y$ factors through $f' : X' \longrightarrow Y$. Then, we have $\mathcal{F}$ is ULA w.r.t f if and only if the direct image of $\mathcal{F}$ on X' is ULA w.r.t f'.

c). If $X \longrightarrow Y$ factors as $X \xrightarrow{p} X' \longrightarrow Y$, and $X \longrightarrow X'$ is proper, then, $\mathcal{F}$ is ULA with respect to f could imply $p_!(F)$ is ULA with respect to the map: $X' \longrightarrow Y$.

d). $\mathcal{F}$ is ULA if and only if $\mathbb{D}(\mathcal{F})$ is ULA.

By b) of Lemma 10.1, we could define the notion of ULA for any morphism.

Definition 10.2. *Given morphism $f : X \longrightarrow Y$ between algebraic stacks, and Y is smooth. Then, we say it is ULA if there is a closed embedding $X \longrightarrow X'$, and the direct image of $\mathcal{F}$ on X' is ULA w.r.t f'.*

A direct corollary of the definition of ULA is the following:

Lemma 10.2. *Consider the following Cartesian diagram of algebraic stacks:*

$$\begin{tikzcd} X \underset{W}{\times} Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "q_2"] \\ Y \arrow[r, "q_1"] & W \end{tikzcd}$$

If W is a smooth algebraic stack of dimension d, and $\mathcal{F}_1 \in D(X), \mathcal{F}_2 \in D(Y)$. If we assume that $\mathcal{F}_1$ is ULA w.r.t q_2. Then, the following canonical map:

$$p_1^*(\mathcal{F}_1) \overset{*}{\bigotimes} p_2^*(\mathcal{F}_2)[-d] \xrightarrow{\sim} p_1^!(\mathcal{F}_1) \overset{!}{\bigotimes} p_2^!(\mathcal{F}_2)[d]$$

is isomorphic

There is an equivalent description of ULA property. The following lemma establishes a link between the singular support of a holonomic D-module and the ULA property.

Lemma 10.3. *Given a smooth morphism $f : X \longrightarrow Y$, and a D-module $\mathcal{F}$ on X, then, $\mathcal{F}$ is ULA w.r.t f if and only if $df^{-1}(SS(\mathcal{F}))\backslash\{0\} = \emptyset$.*

10.2 Duality of F^L_{glob} and F^{KD}_{glob}

First of all, let us introduce a ULA theorem of $j^-_{!,Bun_T\times Ran_x,glob,Fl}$ whose proof will be given in Section 10.3.

We denote by $\overline{Bun_{B^-}}'^{,\lambda}$ the substack of $\overline{Bun_{B^-}}'$ such that the degree of the induced T-bundle is of degree $-\lambda+(2g-2)\rho$. Let us denote by $\overline{Bun_{B^-}}'^{,\lambda}_{\leq\mu}$ the open substack of $\overline{Bun_{B^-}}'^{,\lambda}$ such that the total order of degeneracy of the generalized B^--reductions of is no more than μ.

Theorem 10.1. *There exists an integer d depends only on the genius of X, such that, for any $\mu\in\Lambda^{pos}$, and $\lambda\in\Lambda$ satisfying the condition (X):*

$$(X):\langle-\lambda-\mu,\check{\alpha}_i\rangle>d$$

The restriction of $j^-_{!,Bun_T\times Ran_x,glob,Fl}$ to $\overline{Bun_{B^-}}'^{,\lambda}_{\leq\mu}$ is ULA with respect to the natural projection:

$$\overline{Bun_{B^-}}'\longrightarrow Bun'_G \tag{10.3}$$

In this section, we will prove the following theorem with Theorem 10.1:

Theorem 10.2. *F^L_{glob} and F^{DK}_{glob} intertwine with the Verdier duality functor $\mathbb{D}$, i.e,*

$$F^L_{glob}\mathbb{D}\simeq\mathbb{D}F^{DK}_{glob} \tag{10.4}$$

By Corollary 8.4, the duality functor intertwines with the isomorphism between local and global functors. Hence, Theorem 10.2 could imply the following corollary,

Corollary 10.1. *F^L and F^{DK} intertwine with the Verdier duality functor $\mathbb{D}$, i.e, F^L and F^{DK} intertwine with the Verdier duality functor $\mathbb{D}$, i.e,*

$$\mathbb{D}F^L\simeq F^{DK}\mathbb{D}$$

The rest of this section will be devoted to the proof of Theorem 10.2. The method is mimicing the proof in [GL1] Section 21.2.

Step I:

By (7.16) and (7.17), we have:

$$\begin{gathered}\mathbb{D}(v_!(\bar{\mathfrak{q}}^!_Z(\mathcal{F})\overset{!}{\otimes}\bar{\mathfrak{p}}^!_Z(j^-_{*,glob,Bun_T\times Ran,Fl}[dimBun'_G])))\\ \simeq\\ v_!(\bar{\mathfrak{q}}^!_Z(\mathbb{D}(\mathcal{F}))\overset{!}{\otimes}\bar{\mathfrak{p}}^!_Z(j^-_{!,glob,Bun_T\times Ran,Fl}[dimBun'_G]))\end{gathered}$$

By definition, v is $ind-proper$, hence, it suffices to prove:

$$\begin{gathered}\mathbb{D}(\bar{\mathfrak{q}}^!_Z(\mathcal{F})\overset{!}{\otimes}\bar{\mathfrak{p}}^!_Z(j^-_{*,glob,Bun_T\times Ran,Fl}[dimBun'_G])))\\ \simeq\\ \bar{\mathfrak{q}}^!_Z(\mathbb{D}(\mathcal{F}))\overset{!}{\otimes}\bar{\mathfrak{p}}^!_Z(j^-_{!,glob,Bun_T\times Ran,Fl}[dimBun'_G])\end{gathered} \tag{10.5}$$

Because we are working with $ind-holonomic$ D-modules, six functors are well-defined for such objects. Hence, we only need to prove:

$$\begin{gathered}\bar{\mathfrak{q}}_Z^*(\mathcal{F})\overset{*}{\otimes}\bar{\mathfrak{p}}_Z^*(\mathbb{D}(j^-_{*,glob,Bun_T\times Ran,Fl})))[-dimBun'_G]\\ \simeq \\ \bar{\mathfrak{q}}_Z^!(\mathcal{F})\overset{!}{\otimes}\bar{\mathfrak{p}}_Z^!(j^-_{!,glob,Bun_T\times Ran,Fl})[dimBun'_G]\end{gathered} \tag{10.6}$$

By the definition of $j_{*,glob,Bun_T\times Ran,Fl}$ and $j_{!,glob,Bun_T\times Ran,Fl}$, they are dual to each other. Hence, (10.6) is equal to:

$$\begin{gathered}\bar{\mathfrak{q}}_Z^*(\mathcal{F})\overset{*}{\otimes}\bar{\mathfrak{p}}_Z^*(j^-_{!,glob,Bun_T\times Ran,Fl}))[-dimBun'_G]\\ \simeq \\ \bar{\mathfrak{q}}_Z^!(\mathcal{F})\overset{!}{\otimes}\bar{\mathfrak{p}}_Z^!(j^-_{!,glob,Bun_T\times Ran,Fl})[dimBun'_G]\end{gathered} \tag{10.7}$$

We are in the same situation as Lemma 10.2.

If we could prove that $j^-_{!,glob,Bun_T\times Ran,Fl}$ is ULA with respect to the projection (10.3), then, (10.7) follows from Lemma 10.2. But in fact, we do not need such a strong property. It is because $\mathcal{F}$ is not an arbitrary D-module on $\overline{Bun_N^{\omega^\rho}}'_{\infty\cdot x}$. Instead, it is a global Whittaker D-module, hence, the expected isomorphism (10.7) satisfies a factorization property with respect to the factorization property of Zastava space. We could recover the isomorphism (10.7) from its restriction to an open subset.

Let us recall the following diagram:

$$\begin{tikzcd} (Z_{Fl,x})_{\infty\cdot x} \arrow[r, "\bar{\mathfrak{p}}'_Z"] \arrow[d, "\bar{\mathfrak{q}}'_Z"] & \overline{Bun_{B^-}}' \arrow[d, "\bar{\mathfrak{q}}'"] \\ \overline{Bun_N^{\omega^\rho}}'_{\infty\cdot x} \arrow[r, "\bar{\mathfrak{p}}'"] & Bun'_G \end{tikzcd}$$

It is not a Cartesian diagram, but we note that $(Z_{Fl,x})_{\infty\cdot x}$ is an open subset of the fiber product $\overline{Bun_{B^-}}'\underset{Bun'_G}{\times}\overline{Bun_N^{\omega^\rho}}'_{\infty\cdot x}$.

We may replace the algebraic stack $\overline{Bun_{B^-}}'$ by its open substack $\overline{Bun_{B^-}}'^{,\lambda}_{\leq\mu}$ such that λ,μ satisfy the condition (X).

With some abuse of notations, we still use the same notations for functors after restriction.

Definition 10.3. *Denote by* $(Z_{Fl,x})^{\lambda;\leq\mu}_{\infty\cdot x}$ *the preimage of* $\overline{Bun_{B^-}}'^{,\lambda}_{\leq\mu}$ *in* $(Z_{Fl,x})_{\infty\cdot x}$.

We will get the following diagram:

$$
\begin{tikzcd}
(Z_{Fl,x})^{\lambda,\leq\mu}_{\infty\cdot x} \arrow[r, "\bar{\mathfrak{p}}'^{-}_Z"] \arrow[d, "\bar{\mathfrak{q}}'_Z"] & \overline{Bun}_{B^-}{}'^{,\lambda}_{\leq\mu} \arrow[d, "\bar{\mathfrak{q}}'"] \\
\overline{Bun}^{\omega^\rho}_{N}{}'_{\infty\cdot x} \arrow[r, "\bar{\mathfrak{p}}'"] & Bun'_G
\end{tikzcd}
$$

Let us recall Theorem 10.1 given in this section, we have the restriction of $j_{!,Bun_T\times Ran_x,glob,Fl}$ on $\overline{Bun}_{B^-}{}'^{,\lambda}_{\leq\mu}$ is ULA with respect to the projection $\overline{Bun}_{B^-}{}'^{,\lambda}_{\leq\mu} \longrightarrow Bun'_G$. Combined with Lemma 10.2, we have:

Corollary 10.2. *For any twisted D-module $\mathcal{F} \in D_{\mathcal{G}^G}(\overline{Bun}^{\omega^\rho}_{N}{}'_{\infty\cdot x})$, (10.7) is an isomorphism on $(Z_{Fl,x})^{-\lambda,\leq\mu}_{\infty\cdot x}$.*

Denote

$$
(Z_{Fl,x})^{s}_{\infty\cdot x} := \bigcup_{\lambda\in\Lambda,\mu\in\Lambda^{pos},\ condition(X)} (Z_{Fl,x})^{\lambda,\leq\mu}_{\infty\cdot x} \tag{10.8}
$$

Then, by above corollary, (10.7) is an isomorphism on $(Z_{Fl,x})^{s}_{\infty\cdot x}$.

Step II

Now, we want to extend this isomorphism to the whole affine flag Zastava space $(Z_{Fl,x})_{\infty\cdot x}$. We now prove (10.7) with the factorization property of $(Z_{Fl,x})_{\infty\cdot x}$ with respect to $Z_{Gr,x}$.

Recall that we denote by Z^{λ}_{Gr} the fiber product:

$$
Z_{Gr} \underset{Conf}{\times} Conf^{\lambda}
$$

Similarly, we denote by $(Z_{Fl,x})^{\lambda}_{\infty\cdot x}$ the fiber product $(Z_{Fl,x})_{\infty\cdot x} \underset{Conf_x}{\times} Conf^{\lambda}_x$.

By Proposition 7.3, affine flag Zastava space $(Z_{Fl,x})_{\infty\cdot x}$ factorizes with respect to Z_{Gr}, i.e,

$$
\begin{gathered}
Z_{Gr} \times (Z_{Fl,x})_{\infty\cdot x} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \\
\simeq \\
(Z_{Fl,x})_{\infty\cdot x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj}
\end{gathered}
\tag{10.9}
$$

Note that this isomorphism is compatible with degree by definition, i.e,

$$
\begin{gathered}
Z^{\lambda_1}_{Gr} \times (Z_{Fl,x})^{\lambda_2}_{\infty\cdot x} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \\
\simeq \\
(Z_{Fl,x})^{\lambda_2+\lambda_1}_{\infty\cdot x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj}
\end{gathered}
\tag{10.10}
$$

Restrict to the following map:

$$\begin{gathered} Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \\ \longrightarrow \\ (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2+\lambda_1} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \end{gathered} \tag{10.11}$$

Here, $Z_{Gr}^{\circ,\lambda_1} = Z_{Gr}^{\circ} \cap Z_{Gr}^{\lambda_1}$.

We note that because the general B^--bundle given by the Z_{Gr}°,λ_1} is indeed a B^--bundle (it does not have defect). As a result, given arbitrary element in $z_2 \in (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2}$ and arbitrary element z_1 in Z_{Gr}°,λ_1}, the corresponding object in the right hand side of (10.11) in $(Z_{Fl,x})_{\infty\cdot x}^{\lambda_2+\lambda_1}$ has the same order of degeneracy of generalized B^--bundle as z_2.

$Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj}$ admits two morphisms to $(Z_{Fl,x})_{\infty\cdot x}$:

one is given by the second projection:

$$r_1 : Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \longrightarrow (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2} \tag{10.12}$$

another one is given by the factorization map (10.11) composed with the second projection:

$$\begin{aligned} r_2 : Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2} \underset{Conf\times Conf_x}{\times} (Conf \times Conf_x)_{disj} \longrightarrow \\ \xrightarrow{(10.11)} (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2+\lambda_1} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \longrightarrow \\ \longrightarrow (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2+\lambda_1} \end{aligned} \tag{10.13}$$

Remark The images of same point under r_1 and r_2 have the same degree of degeneracy.

Remark r_1 and r_2 are smooth.

The key observation of proof of Theorem 10.2 is:

- inside $Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2}$, given any $\mu \in \Lambda^{pos}$, we could take out an open subset $(Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2})_\mu$ whose image under r_2 in $(Z_{Fl,x})_{\infty\cdot x}$ lies in $(Z_{Fl,x})_{\infty\cdot x}^{s}$, and if we let λ_1 and μ vary, then, the collection of stacks $\{(Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2})_\mu, \mu \in \Lambda^{pos}, \lambda_1 \in \Lambda^{neg}\}$ can give a smooth cover of $(Z_{Fl,x})_{\infty\cdot x}^{\lambda_2}$ by the map r_1.

Now, let us explain the construction of $(Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2})_\mu$.

Given $\mu \in \Lambda^{pos}$, $\lambda_1 \in \Lambda^{neg}$ and $\lambda_2 \in \Lambda$, a point of $(Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2})_{disj}$ belongs to $(Z_{Gr}^{\circ,\lambda_1} \times (Z_{Fl,x})_{\infty\cdot x}^{\lambda_2})_\mu$ if and only if :

1. the order of degeneracy of the generalized B^--bundle is no more than μ (it is an open condition),

2. $\lambda_1+\lambda_2$ and μ satisfy the condition (X).

If we allow λ_1 and μ vary, we could see all the $(Z_{Gr}^{\circ,\lambda_1}\times(Z_{Fl,x})^{\lambda_2}_{\infty\cdot x})_\mu$ form a smooth cover of $(Z_{Fl,x})^{\lambda_2}_{\infty\cdot x}$ by r_1. The claim that we want to prove is a smooth local claim, so we only need to prove the !-pullback to $(Z_{Gr}^{\circ,\lambda_1}\times(Z_{Fl,x})^{\lambda}_{\infty\cdot x})_\mu$ of the morphism (10.7) is an isomorphism for any $\mu\in\Lambda^{pos}$ and $\lambda_1\in\Lambda^-$.

We note the following lemma from the argument of Section 3.9 in [Ga3]:

Lemma 10.4. *Given a D-module $\mathcal{F}\in Whit_q(Fl)$, there is a smooth local system $\mathcal{L}$ such that:*

$$r_2^!\mathcal{F}|_{(Z_{Fl,x})^{\lambda_1+\lambda_2}_{\infty\cdot x}}\simeq\mathcal{L}\otimes r_1^!\mathcal{F}|_{(Z_{Fl,x})^{\lambda_2}_{\infty\cdot x}}$$

And we note that r_1 and r_2 are smooth. As a result, we only need to prove that the morphism (10.7) is an isomorphism after pulling back from $(Z_{Fl,x})^{\lambda_1+\lambda_2}_{\infty\cdot x}$ along with r_2 and then restrict to $(Z_{Gr}^{\circ,\lambda_1}\times(Z_{Fl,x})^{\lambda}_{\infty\cdot x})_\mu$.

We take the open subset $(Z_{Fl,x})^{s,\lambda_1+\lambda_2}_{\infty\cdot x}:=(Z_{Fl,x})^{s}_{\infty\cdot x}\cap(Z_{Fl,x})^{\lambda_1+\lambda_2}_{\infty\cdot x}\subset(Z_{Fl,x})^{\lambda_1+\lambda_2}_{\infty\cdot x}$, by Corollary 10.2, we know our claim is true onside. Hence, the pullback of the morphism in (10.7) to the open subset $r_2^{-1}((Z_{Fl,x})^{s,\lambda_1+\lambda_2}_{\infty\cdot x})$ in $Z_{Gr}^{\circ,\lambda_1}\times(Z_{Fl,x})^{\lambda_2}_{\infty\cdot x}$ is still an isomorphism. Now, the claim follows from the fact that $(Z_{Gr}^{\circ,\lambda_1}\times(Z_{Fl,x})^{\lambda_2}_{\infty\cdot x})_\mu$ is contained in $r_2^{-1}((Z_{Fl,x})^{s,\lambda_1+\lambda_2}_{\infty\cdot x})$ by our choice of λ_1,λ_2 and μ.

So, we proved the claim in Theorem 10.2.

10.3 Kontsevich compactification

In this section, we will prove Theorem 10.1. Instead of proving such a theorem, we prove the equivalent theorem for its positive counterpart, i.e, we replace B^- by B. The condition (X) also changes:

Condition (X)': for any negative root $\check{\alpha}^-$, we have $\langle\check{\alpha}^-,\lambda+\mu\rangle>2g-2$

Let us denote by $\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}$ the substack of $\overline{Bun_B^{w_0}}$ consisting of the points such that the degree of the induced T-bundle is λ and the total degree of degeneracy is no more than μ.

Theorem 10.3. *If $\lambda\in\Lambda,\mu\in\Lambda^{pos}$ satisfy the condition (X)', then, the restriction of $j_{!,glob,Bun_T\times Ran_x,Fl}$ to $\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}$ is ULA w.r.t $\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}\longrightarrow Bun_G'$*

In this section, we will meet several different compactifications of Bun_B', if we continue to use the notation $j_{!,glob,Bun_T\times Ran_x,Fl}$ to represent the !-extension D-module of the constant D-module on $Bun_B^{w_0,'}$, it will make readers forget its base stack. Hence, in order to emphasize that it is defined on Drinfeld compactification, we use $j_{!,\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}}$ to denote the restriction of $j_{!,glob,Bun_T\times Ran_x,Fl}$ on $\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}$.

The idea of proving Theorem 10.3 is an analog of the proof of Theorem 4.2.1 in [Camp]. We will mimic [Camp] to get a proof of Theorem 10.3.

In order to prove the ULA property of $j_{!,\overline{Bun_B^{w_0}}'^{,\lambda}_{\leq\mu}}$ with respect to the proper morphism: $\mathfrak{q}' : \overline{Bun_B^{w_0}}'^{,\lambda}_{\leq\mu} \longrightarrow Bun'_G$, we need a resolution of singularity of $\overline{Bun_B^{w_0}}'^{,\lambda}_{\leq\mu}$ such that $Bun_B^{w_0}{}'^{,\lambda}_{\leq\mu}$ is an open substack of it and the complement of $Bun_B^{w_0}{}'^{,\lambda}_{\leq\mu}$ is a normal crossing divisor. The construction of this resolution of singularity will be given in Definition 10.7 and denoted by $\overline{Bun_B^{w_0}}^{D,'}$. Now, before reaching the definition of $\overline{Bun_B^{w_0}}^{D,'}$, let us do some preparation.

We define:

Definition 10.4. *M'_X denotes the moduli stack classifying the data:*

1. *A connected nodal curve C of genus g.*
2. *A proper morphism $p : C \longrightarrow X$*
3. *p is of degree 1.*
4. *$R^1 p_* O_C = 0$*
5. *s is a marked point in the smooth locus of C.*

The moduli stack classifies 1-4 is denoted by M_X in [Camp].

Let us analyze the curve C in the above definition. Denote by $\overset{\circ}{X}$ the smooth locus of X. For a geometric point of M'_X, because p is of degree 1 and it is proper, p admits a canonical section $X \longrightarrow C$. Hence, we could regard X as an irreducible component of C. Because C is of genus g, $C \backslash \overset{\circ}{X}$ is of genus 0 and simply connected. Hence, the dual graph of C is a tree and each irreducible component of $C \backslash \overset{\circ}{X}$ is $\mathbb{P}^1$.

Note that we have a natural map

$$ev_p : M'_X \to X$$

which maps the marked point to a point in X. We denote its fiber over a point x by $M'_{X\,x}$.

Remark It does not matter whether we consider the fiber in derived sense or classical sense, because M'_X has a deformation along X, it is smooth over X.

We note that $M'_{X,x}$ has a stratification given by the pattern of nodal points and the marked point.

Definition 10.5. *Given a point of $M'_{X,x}$, (C,p,s), we take the dual graph of C and we mark the subgraph of C_x(the fiber of C over x), we mark the vertex carrying the marked point s and we also mark the vertex of X. The resulted graph will be called the enhanced dual graph of (C,p,s). Points in $M'_{X,x}$ have the same enhanced dual graph that will be called have the same type.*

Points of the same type β form a locally closed substack of $M'_{X,x}$, we denote it by $M'_{X,x,\beta}$

Definition 10.6. *We define $B^{K,'}$ to be the stack classifying the data: $(C, p, \mathcal{P}_G, \mathcal{P}_B, s, \epsilon)$, where*

1. *(C, s, p) is a point in $M'_{X,x}$,*
2. *$\mathcal{P}_G$ is a G-bundle on X,*
3. *$\mathcal{P}_B$ is a B-reduction of $\mathcal{P}_G$ after pullback back to C,*
4. *ϵ is a B-reduction of $\mathcal{P}_G$ at x*
5. *the automorphism group of $\mathcal{C} \longrightarrow (G/B)_{\mathcal{P}_G}$ with a marked point s is finite.*

If we replace $M'_{X,x}$ by M_X and forgetting s in 5, we denote by $\overline{Bun_B}^K$ the resulted stack, it is called Kontsevich compactification of Bun_B in [Camp].

We denote by $B^{K,',\lambda}$ the substack of $B^{K,'}$ such that the degree of $\mathcal{P}_B$ is λ. By forgetting C, p, s, we get a morphism from $B^{K,'}$ to Bun'_G.

Proposition 10.1. *i). $B^{K,'}$ is an Artin stack locally of finite type.*
ii). $B^{K,',\lambda} \longrightarrow Bun'_G$ is proper.

Proof. i). Given a S point in Bun'_G. Then, consider its fiber in $B^{K,'}$. It is a closed substack of the Deligne-Mumford stack $\overline{\mathcal{M}_{g,1}}(G/B)_{\mathcal{P}_G}$ of stable maps with a marked point defined on arithmetic genus g curves where the composition with $(G/B)_{P_G} \longrightarrow X \times S$ has degree one and the image of s under $C \longrightarrow (G/B)_{P_G} \longrightarrow X \times S$ is $\{x\} \times S$. It is proved in [AO] that $\overline{\mathcal{M}_{g,1}}(G/B)_{\mathcal{P}_G}$ is a Deligne-Mumford stack locally of finite type, hence, the fiber of S in $B^{K,'}$ is also a Deligne-Mumford stack locally of finite type. And because Bun'_G is an Artin stack locally of finite type, $B^{K,'}$ is an Artin stack locally of finite type.

ii). It is because $\overline{\mathcal{M}_{g,1}}(G/B)_{\mathcal{P}_G}$ is proper after fixing the degree λ, hence the fiber of S in $B^{K,'}$ is also proper. □

If we denote by $M'_{X,x} \times pt/B$ the algebraic stack classifies a data of $M'_{X,x}$ and a B-bundle at x,(equivalent, because s is a section of x in C, we could regard the B-bundle at s canonically as B-bundle at s), then, we have the following morphism between algebraic stacks:

$$l : B^{K,'} \longrightarrow (M'_{X,x} \times pt/B) \underset{pt/G}{\times} Bun'_G \tag{10.14}$$

It maps a point $(C, p, \mathcal{P}_G, \mathcal{P}_B, s, \epsilon)$ to $(C, p, s, \mathcal{P}_B|_s, \mathcal{P}_G, \epsilon)$

We could also define the degree of degeneracy for the points of the algebraic stack $B^{K,'}$. If we consider the components of C except X, we have $C \backslash \mathring{X} = \bigcup_{i \leq n} C_i$. The degree of degeneracy a point of $B^{K,'}$ is given by

$$-\sum_{i \in n} deg(\mathcal{P}_B|_{C_i})$$

Remark C_i is different from X, hence, p is constant on C_i. We recall that $\mathcal{P}_B$ is a B-reduction of the pullback of $\mathcal{P}_G$. In particular, $\mathcal{P}_B$ is a B-reduction of the trivial G-bundle when restricted to C_i. As a result, the degree of $\mathcal{P}_B$ on C_i is negative and the defect of any point in $B^{K,'}$ is positive.

Given a positive coweight μ, we define $B^{K,',\lambda}_{\leq\mu}$ to be the open substack of $B^{K,',\lambda}$ such that the defect is no more than μ.

We claim:

Proposition 10.2. *If λ, μ satisfy the condition (X)', then, after restricting to $B^{K,',\lambda}_{\leq\mu}$, the morphism l is smooth.*

Proof. Given a point $(C, p, s, \mathcal{P}_B|_s, \mathcal{P}_G, \epsilon)$ in $(M'_{X,x} \times pt/B) \underset{pt/G}{\times} Bun'_G$, the fiber of l over this point is an open subset of

$$Maps_X(C, (G/B)_{\mathcal{P}_G}) \underset{Maps(s,(G/B)_{\mathcal{P}_G})}{\times} pt$$

with stable condition. Here, the map $pt \longrightarrow Maps(s, (G/B)_{\mathcal{P}_G})$ is given by $\mathcal{P}_B|_s$.

By Proposition 4.1.1 of [Camp], $Maps_X(C, (G/B)_{\mathcal{P}_G})$ and $Maps(s, (G/B)_{\mathcal{P}_G})$ are smooth, hence we only need to prove the tangent map induces a surjective map on H^0.

The tangent complex of $Maps_X(C, (G/B)_{\mathcal{P}_G})$ is given by

$$R\Gamma(C, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B})$$

So, we only need to prove $H^0(C, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}) \longrightarrow (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B|s}$ is surjective. So we should prove $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ is globally generated.

Denote by $\mathcal{P}_T$ the T bundle induced from $\mathcal{P}_B$.

If we denote λ_X the degree of $\mathcal{P}_B$ restricted to X. Then, we have:

$\lambda - \lambda_X \geq -\mu \Longrightarrow \lambda_X \leq \lambda + \mu$, so we have $\langle \lambda_X, \check{\alpha}^- \rangle > 2g - 2$

$(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ has a filtration such that the associated graded vector bundle is $\bigoplus_{\alpha\in\Delta^-} \alpha(\mathcal{P}_T)$. Because the degree of $\alpha(\mathcal{P}_T)|_X$ is bigger than $2g-2$, hence it is globally generated. In particular, we have $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}|_X$ is global generated.

Then, we induct on the number of components n of C except X. According to the proof above, we have the result when $n = 0$. Now, assume we have the result for $n - 1$, we choose a leaf C_0 of the tree C, i.e, C_0 is an irreducible component except X and contains a single nodal c. Denote $C' := C \backslash (C_0 \backslash \{c\})$

According to our inductive hypothesis, we have $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ is globally generated on C' and we note that $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ is also globally generated on C_0 because the degree of $\mathcal{P}_B|_{C_0}$ is positive. As a result, we have $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}$ is globally generated on the whole curve C. □

The algebraic stack $\overline{Bun^{w_0}_B}^{D,'}$ that we want to construct is a fiber product of $B^{K,'}$ with the Bott-Samelson(-Demazure-Hansen) resolution. The Bott-Samelson resolution defines a resolution of singularity of G/B (e.g, [BS], [K] and [Ra]). It admits a birational map to G/B and it is an isomorphism on the open B-orbit

Bw_0B/B. Outside Bw_0B/B, the complement closed substack in the Bott-Samelson resolution is a normal crossing divisor. Another good property of the Bott-Samelson resolution is that it admits a left B action and each irreducible component of the complement of Bw_0B/B is B-invariant. So, we could get a resolution of singularity of $B\backslash G/B \simeq pt/B \underset{pt/G}{\times} pt/B$. And on the open Bruhat cell $Br^{w_0,+}$, this resolution is an isomorphism. The complement of $Br^{w_0,+}$ is a normal crossing divisor.

We denote this resolution of singularity of $B\backslash G/B$ by BS:

$$BS \longrightarrow B\backslash G/B \tag{10.15}$$

$B^{K,'}$ admits two maps to pt/B: one is given by ϵ and another one is given by restricting $\mathcal{P}_B$ to s and then pullback to x by the section information s. It is easy to see that the compositions of these two morphisms with the induction: $pt/B \longrightarrow pt/G$ are isomorphic. Hence, it induces a morphism:

$$B^{K,'} \longrightarrow B\backslash G/B$$

Finally, we could give the definition of $\overline{Bun_B^{w_0}}^{D,'}$.

Definition 10.7. *We define* $\overline{Bun_B^{w_0}}^{D,'}$ *to be* $B^{K,'} \underset{B\backslash G/B}{\times} BS$

Similarly, we could define $\overline{Bun_B^{w_0}}^{D,',\lambda}$ *and* $\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu}$.

Because (10.15) is locally of finite type and $B^{K,'}$ is locally of finite type, hence, $\overline{Bun_B^{w_0}}^{D,'}$ is an (Artin) stack locally of finite type.

In Section 3 of [Camp], the author constructed a functor from $\overline{Bun_B}^K$ to $\overline{Bun_B}$

$$\overline{Bun_B}^K \longrightarrow \overline{Bun_B} \tag{10.16}$$

And it is proved in *loc.cit*, the functor (10.16) preserves degree and degree of defect.

It is easy to see that the functor (10.16) gives rise to a functor:

$$\overline{Bun_B^{w_0}}^{D,'} \longrightarrow \overline{Bun_B^{w_0}}' \tag{10.17}$$

And the functor (10.17) preserves degree and degree of defect. It is easy to see that (10.17) is an isomorphism on $Bun_B^{w_0,'}$. Now, we want to analyze the complement of $Bun_B^{w_0,'}$ in $\overline{Bun_B^{w_0}}^{D,'}$. We claim:

Proposition 10.3. *The complement of* $(Bun_B^{w_0,',\lambda})_{\leq\mu}$ *in* $\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu}$ *is a normal crossing divisor.*

Proof. From the morphism l in 10.14, we obtain the following morphism by base change:

$$\overline{Bun_B^{w_0}}^{D,'} \longrightarrow (M'_{X,x} \times BS) \underset{pt/G}{\times} Bun_G \tag{10.18}$$

What's more, if we restrict this morphism to $\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu}$, then, it is smooth by a base change of Proposition 10.2. In particular, the morphism:

$$\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu} \longrightarrow M'_{X,x} \times BS$$

is smooth. (Because $Bun_G \longrightarrow pt/G$ is smooth.)

$Bun_B^{w_0,'}$ is the preimage of the open locus $\{X, id : X \to X, x\} \times Br^{w_0,+}$ of $M'_{X,x} \times BS$. Now the claim follows from the fact that the pullback of a normal crossing divisor under a smooth morphism is still a normal crossing divisor and the following lemma. $\square$

Lemma 10.5. *i). $M'_{X,x} \times BS$ is smooth.*

ii). The complement of $\{X, id : X \to X, x\} \times Br^{w_0,+}$ in $M'_{X,x} \times BS$ is a normal crossing divisor.

Proof. i). We only need to prove $M'_{X,x}$ is smooth. Because $M'_X \longrightarrow X$ smooth, hence it suffices to prove that M'_X is smooth.

Taking $\mathcal{K} = [\mathcal{T}C(-s) \longrightarrow p^*\mathcal{T}X]$, tangent complex of M'_X is given by

$$R\Gamma(C, K)[1]$$

We should show $R^{\geq 2}\Gamma(C, K) = 0$. If $i \geq 3$, then, $R^i\Gamma(C, K) = 0$ is zero because $R^{\geq 2}\Gamma(C, \mathcal{T}C(-s)) = 0$ and $R^{\geq 2}\Gamma(C, p^*\mathcal{T}X) = 0$. If $i = 2$, then, we should prove

$$H^1(X, \mathcal{T}C(-s)) \longrightarrow H^1(X, p^*\mathcal{T}X)$$

is surjective

Using Reimann-Roch, we only need to prove:

$$H^0(C, \omega_C \bigotimes p^*\mathcal{T}^*X) \longrightarrow H^0(C, \omega_C \bigotimes \mathcal{T}^*C(s))$$

is injective.

It is true because it factors as:

$$H^0(C, \omega_C \bigotimes p^*\mathcal{T}^*X) \longrightarrow H^0(C, \omega_C \bigotimes \mathcal{T}^*C) \longrightarrow H^0(C, \omega_C \bigotimes \mathcal{T}^*C(s))$$

The second morphism is injective and the first morphism

$$H^0(C, \omega_C \bigotimes p^*\mathcal{T}^*X) \longrightarrow H^0(C, \omega_C \bigotimes \mathcal{T}^*C)$$

is injective by [Camp].

ii). The complement of $Br^{w_0,+}$ in BS is a normal crossing divisor, hence, we only need to prove that the complement of $\{X, id : X \to X, x\}$ in $M'_{X,x}$ is a normal crossing divisor. It could be proved by the calculation of dimension of deformations.

Fix a type β, we assume that $M'_{X,x,\beta}$ has q nodal points. That is to say, for any point $(C, p, s) \in M'_{X,x,\beta}$, C has q nodal points. Then, it is easy to see that

$X'_{X,x,\beta}$ is the intersection of q branches and each branch is given by smoothing a nodal point.

Now, the claim follows from the fact that the codimension of the strata $M'_{X,x,\beta}$ is exactly q. Indeed, having one more nodal point means we have one more $\mathbb{P}^1$ with a special point(the nodal point) for curves in $M'_{X,x,\beta}$, the automorphism dimension of $\mathbb{P}^1$ with a point is 2; meanwhile, the new nodal point could move along with the curve, hence, adding one more nodal point also has an effect of adding 1 more dimension freedom of moving the nodal point. To summarize these two effects, adding a nodal point means codimension adds 2-1=1.

So, these q branches intersects transversely at $M'_{X,x,\beta}$. $\square$

Let us analyze Theorem 10.3. It suffices to prove Theorem 10.3 for the twisting given by the line bundle corresponding to the divisor supported on $\overline{Bun_B^{w_0}}^{D,\lambda}_{\leq\mu} - (Bun_B^{w_0,',\lambda})_{\leq\mu}$. Then, because our gerbe is trivial on $(Bun_B^{w_0,',\lambda})_{\leq\mu}$, the twisting is given by $O(D')$, where $D' = \sum_i r'_i D'_i$, D'_i is an irreducible component of $\overline{Bun_B^{w_0}}^{D,\lambda}_{\leq\mu} - (Bun_B^{w_0,',\lambda})_{\leq\mu}$.

Using the relationship between singular support and ULA, we could get a convenient criteria for ULA for the !-extension D-module from an open subset whose complement is a normal crossing divisor. Given a normal crossing divisor $D = \sum r_i D_i$ in an algebraic stack X. Note that D could be written as the disjoint union of X_n, $n = 1, 2, ...$ Here, X_n is the locus where locally the point can be regarded as the transversal intersection of n hyperplanes. X_n gives a stratification of D. On the complement of D in X, the line bundle $O(\sum r_i D_i)$ is canonically trivialized. We could consider constant D-module on it under this trivialization of twisting, and then !-extend it to the twisted D-module category on X. We denote this D-module by $j_!(c_{X_0})$. The proof of Lemma 4.3.1 in [Camp] can be used to prove the following lemma.

Lemma 10.6. *If $X_n \longrightarrow Y$ is smooth for all n, then, we have $j_!(c_{X_0})$ is ULA w.r.t the map $f : X \longrightarrow Y$*

Proof. ULA property is a smooth local property, we could reduce the question to the case X and Y are schemes. Then, we use the criteria given by Lemma 10.3.

According to the lemma above, we only need to prove $(df)^{-1}(SS)\backslash\{0\} = \emptyset$.

And the normal crossing ensures that for a point x in X_n SS lies in the span of the n lines in T^*_x determined by the n hyperplanes determined by the divisor D. It is the kernel of $T^*_x X \longrightarrow T^*_x X_n$.

By the smoothness of f_n, the image $df_{f(x)}(Y)$ in $T^*_x X$ lies in $T^*_x X_n$. We got $(df)^{-1}(SS)\backslash\{0\} = \emptyset$. $\square$

Now the theorem follows from applying the above lemma to our case. Let us denote by $j_{!,\overline{(Bun_B^{w_0})}^{D,'}}$ the !-extension of constant D-module on $Bun_B^{w_0,'}$ to $\overline{(Bun_B^{w_0})}^{D,'}$ in the category of twisted D-modules. We denote its restriction to $\overline{(Bun_B^{w_0})}^{D,',\lambda}_{\leq\mu}$ by $j_{!,\overline{(Bun_B^{w_0})}^{D,',\lambda}_{\leq\mu}}$.

Proof. (of Theorem 10.3) By Proposition 10.3, the boundary of $(Bun_B^{w_0,'})^{\lambda}_{\leq\mu}$ in $\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu}$ is a normal crossing divisor and each strata of this normal crossing divisor is smooth over Bun'_G. So, by Lemma 10.6, we have $j_{!,(\overline{Bun_B^{w_0})}^{D,',\lambda}_{\leq\mu}}$ is ULA w.r.t $\overline{Bun_B^{w_0}}^{D,',\lambda}_{\leq\mu} \longrightarrow Bun'_G$.

We note that the restriction of $\overline{Bun_B^{w_0}}^{D,'} \longrightarrow \overline{Bun_B^{w_0}}'$ to $Bun_B^{w_0,'}$ is an isomorphism and the !-direct image of $j_{!,\overline{(Bun_B^{w_0})}^{D,',\lambda}_{\leq\mu}}$ is isomorphic to $j_{!,\overline{Bun_B^{w_0}}^{',\lambda}_{\leq\mu}}$. Now, the theorem follows from Lemma 10.1 and the following Proposition 10.4. □

Proposition 10.4. *$\overline{Bun_B^{w_0}}^{D,'} \longrightarrow \overline{Bun_B^{w_0}}'$ is proper.*

Proof. The above morphism decomposes as $\overline{Bun_B^{w_0}}^{D,'} \longrightarrow B^{K,'}$ and $B^{K,'} \longrightarrow \overline{Bun_B^{w_0}}'$.

$\overline{Bun_B^{w_0}}^{D,'} \longrightarrow B^{K,'}$ is proper because $BS \to B\backslash G/B$ is proper.

And by Proposition 10.1 and the fact that $\overline{Bun_B^{w_0}}'$ is proper over Bun'_G, $B^{K,'} \longrightarrow \overline{Bun_B^{w_0}}'$ is proper. □

10.4 Another proof of ULA property

In this section, we will give another proof of Theorem 10.3. The author thanks D.Gaitsgory for his observation which makes the proof of Theorem 10.3 much more easier and we do not need to use Bott-Samelson resolution.

Note that we could right convolve any twisted D-module on $Bun_B^{w_0,'}$ with (metaplectic) BMW D-modules like the diagram (8.11) and functor (8.12).

We could define an I-bundle of Bun'_G by adding one more data of isomorphism of $\mathcal{P}_G$ and $\mathcal{P}_G^0$ on D_x and ϵ is given by the trivial B-bundle at x under this isomorphism. We denote it by $\widetilde{Bun_G}'$. The algebraic stack $(\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ in Diagram 8.11 is the pullback of $\widetilde{Bun_G}'$ to $(\overline{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$. We denote the pullback of $\widetilde{Bun_G}'$ to $(Z_{Fl,x})_{\infty\cdot x}$ (resp. $\overline{Bun_{B^-}}'$) by $(\widetilde{Z_{Fl,x}})_{\infty\cdot x}$ (resp. $\widetilde{Bun_{B^-}}'$). Then, replacing $(\widetilde{Bun_N^{\omega^\rho}})'_{\infty\cdot x}$ in Diagram (8.11) by $(\widetilde{Z_{Fl,x}})_{\infty\cdot x}$ (resp. $\widetilde{Bun_{B^-}}'$), we could define the convolution product of metaplectic BMW D-module on twisted D-modules on $(Z_{Fl,x})_{\infty\cdot x}$ (resp. $\overline{Bun_{B^-}}'$).

Recall that we want to prove the following isomorphism for any $\mathcal{F} \in D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x})$,

$$\begin{gathered} \bar{\mathfrak{q}}_Z^*(\mathcal{F}) \overset{*}{\otimes} \bar{\mathfrak{p}}_Z^*(j_{!,\overline{Bun_{B^-}}'})[-dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}} \\ \simeq \\ \bar{\mathfrak{q}}_Z^!(\mathcal{F}) \overset{!}{\otimes} \bar{\mathfrak{p}}_Z^!(j_{!,\overline{Bun_{B^-}}'})[dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}} \end{gathered} \tag{10.19}$$

(for definition of $(Z_{Fl,x})^{\lambda,\leq\mu}_{\infty\cdot x}$, please see Definition 10.3 and $j_{!,\overline{Bun_{B^-}}'}$ is the !-extension of the constant D-module(concentrated in degree 0) on Bun''_{B^-})

(10.19) is equivalent to the following:

$$\begin{aligned}
&\bar{\mathfrak{q}}^*_Z(\mathcal{F}) \overset{*}{\otimes} (\bar{\mathfrak{p}}^*_Z(j_{!,\overline{Bun_{B^-}}^{w_0,'}}) \star j_{w_0,!})[-dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}} \\
&\simeq \\
&\bar{\mathfrak{q}}^!_Z(\mathcal{F}) \overset{!}{\otimes} (\bar{\mathfrak{p}}^!_Z(j_{!,\overline{Bun_{B^-}}^{w_0,'}}) \star j_{w_0,!})[dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}}
\end{aligned} \tag{10.20}$$

($j_{!,\overline{Bun_{B^-}}^{w_0,'}}$ is the !-extension of constant D-module(concentrated in degree 0) on $Bun^{w_0,'}_{B^-}$ to $\overline{Bun_{B^-}}'$)

(10.20) is equivalent to the following:

$$\begin{aligned}
&(\bar{\mathfrak{q}}^*_Z(\mathcal{F}) \star j_{w_0,*}) \overset{*}{\otimes} \bar{\mathfrak{p}}^*_Z(j_{!,\overline{Bun_{B^-}}^{w_0,'}})[-dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}} \\
&\simeq \\
&(\bar{\mathfrak{q}}^!_Z(\mathcal{F}) \star j_{w_0,*}) \overset{!}{\otimes} \bar{\mathfrak{p}}^!_Z(j_{!,\overline{Bun_{B^-}}^{w_0,'}})[dimBun'_G]|_{(Z_{Fl,x})^{\lambda,\leq\mu}}
\end{aligned} \tag{10.21}$$

And because the functor:

$$\begin{aligned}
D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}) &\longrightarrow D_{\mathcal{G}^G}((\overline{Bun_N^{\omega^\rho}})_{\infty\cdot x}) \\
\mathcal{F} &\longrightarrow \mathcal{F} \star j_{w_0,*}
\end{aligned} \tag{10.22}$$

is invertible, hence it induces an equivalence.

In particular, we only need to prove that $j_{!,\overline{Bun_{B^-}}^{w_0,'}}$ is ULA with respect to:

$$\overline{Bun_{B^-}}^{w_0,'} \longrightarrow Bun'_G$$

By replacing B^- by its positive counterpart, it is equivalent to prove the following theorem:

Theorem 10.4. *If $\lambda \in \Lambda, \mu \in \Lambda^{pos}$ satisfy the condition (X)', then, $Bun_B{}^{1,',\lambda}_{\leq\mu}$ is ULA with respect to $\overline{Bun_B}{}^{1,',\lambda}_{\leq\mu} \longrightarrow Bun'_G$.*

In this section, we will prove the above theorem.

First of all, let us construct a Kontsevich style compactification of $Bun_B{}^{1,',\lambda}_{\leq\mu}$.

Definition 10.8. *Denote by B^K the algebraic stack classifies the following data:*

· *A nodal projective curve C.*

· *A commutative diagram:*

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & pt/B \arrow[d] \\
X \arrow[r] & pt/G
\end{tikzcd}$$

i.e, a B-bundle $\mathcal{P}_B$ on C, a G-bundle $\mathcal{P}_G$ on X, and they are compatible under the pullback $C \longrightarrow X$.

$\cdot$ A B-reduction ϵ of $\mathcal{P}_G$ at x.

$\cdot$ A point s in the smooth locus of C over x.

$\cdot$ We require the map $C \to (G/B)_{\mathcal{P}_G} = X \underset{pt/G}{\times} pt/B$ (with a marked point s) to be stable.(i.e, the automorphism group in the moduli stack of maps of curves to X is finite)

$\cdot$ $C \longrightarrow X$ is of degree 1.

$\cdot$ The restriction of the map $C \to (G/B)_{\mathcal{P}_G}$ to s coincides with the one given by the pullback of the Iwahori structure at x.

We note that in B^K we could take out an open subset classifies the data $(X, id: X \longrightarrow X, x, \mathcal{P}_G, \mathcal{P}_B, \epsilon)$, i.e, we require p to be an isomorphism, $C \simeq X$, $s = x$. This substack is isomorphic to $Bun_B^{1,'}$.

The assignment sends $(C, p, s, \mathcal{P}_G, \mathcal{P}_B, \epsilon)$ to $(\mathcal{P}_G, \epsilon)$ gives rise to a map from B^K to Bun'_G.

Proposition 10.5. *B^K is an Artin stack locally of finite type and it is proper over Bun'_G.*

Proof. The fiber of a point $\mathcal{P}_G \in Bun_G$ in B^K is as same as the fiber of a point $(\mathcal{P}_G, \epsilon)$ in $B^{K,'}$. Hence, the first claim follows the same reason as Proposition 10.1.

For the second claim, we note that the proper morphism $B^K \longrightarrow Bun_G$ factors through $Bun'_G \longrightarrow Bun_G$. And the latter is proper. Hence, B^K is proper over Bun'_G. □

As a direct corollary,

Corollary 10.3.

$$B^K \longrightarrow \overline{Bun_B}' \tag{10.23}$$

is proper.

Proof. $B^K \longrightarrow \overline{Bun_B}' \longrightarrow Bun'_G$ is proper by the above proposition. Now the claim follows from the fact that the second morphism is also proper. □

We denote by $B^{K,\lambda}_{\leq\mu}$ the substack of B^K such that the degree of $\mathcal{P}_B$ is λ and $-\sum_i deg(\mathcal{P}_B|_{C_i}) \leq \mu$. Here, $C_1, C_2, ...$ denote the irreducible components of C except X. It is known(by the same method in [Camp]) that the map (10.23) preserves the degree of degeneracy and the degree. Let us denote by $j_{!,B^K}$ the !-extension of constant D-module on $Bun_B^{1,'}$ in the category of twisted D-modules. We denote its restriction to $B^{K,\lambda}_{\leq\mu}$ by $j_{!,B^{K,\lambda}_{\leq\mu}}$.

Note that the !-direct image of $j_{!,B^{K,\lambda}_{\leq\mu}}$ along with 10.23 is isomorphic to $j_{!,\overline{Bun^1_B}^{',\lambda}_{\leq\mu}}$. Hence, by Lemma 10.4 c), we only need to prove that $j_{!,B^{K,\lambda}_{\leq\mu}}$ is proper with respect to:

$$B^{K,\lambda}_{\leq\mu} \longrightarrow Bun'_G \tag{10.24}$$

By Lemma 10.6, we only need to prove that the complement of $Bun^{1\ ',\lambda}_{B\leq\mu}$ in $B^{K,\lambda}_{\leq\mu}$ is a normal crossing divisor. And the stratum X_n that is given by codimension n in $B^{K,\lambda}_{\leq\mu}$ is smooth over Bun'_G. And the claims follow from the following proposition:

Proposition 10.6.

$$B^{K,\lambda}_{\leq\mu} \longrightarrow M'_{X,x} \times Bun'_G \tag{10.25}$$

is smooth.

Proof. Given a point $(C, p, s, \mathcal{P}_G, \epsilon)$ in $M'_{X,x} \times Bun'_G$, its fiber in $B^{K,\lambda}_{\leq\mu}$ is given by the open locus of $Maps_X(C, (G/B)_{\mathcal{P}_G}) \underset{Maps(s,(G/B)_{\mathcal{P}_G})}{\times} pt$ given by stability condition.

By the proof of Proposition 10.2, $Maps_X(C, (G/B)_{\mathcal{P}_G})$ is smooth over $Maps(s, (G/B)_{\mathcal{P}_G})$. □

10.5 Proof of Proposition 7.4

This section we will devote to the proof of the Proposition 7.4.

And recall that in Section 6.5, we proved that Theorem 4.1 could be implied by Proposition 6.11 and which could be implied by Theorem 7.1+ Proposition 7.4, hence, it means that we will finish our proof of the main theorem in this section.

We define $\widetilde{\nabla}_{\lambda,glob}$ to be the Verdier dual of $\Delta_{\lambda,glob}$. Now, we claim:

Proposition 10.7.

$$F^{DK}_{glob}(\widetilde{\nabla}_{\lambda,glob}) \simeq \nabla_{\lambda,\Omega_q^{DK,'}}$$

Proof. In order to simplify the notations, we omit the twist notation here.

By the same proof of the first claim in Theorem 7.1 b), we have $F^{KD} \simeq F^{KD}_{glob} \circ \pi_{Fl,x}[d_g]$. And we note that the Verdier duality functor commutes with $\pi_{Fl,x}[d_g]$, hence, we only need to prove that the image of $\widetilde{\nabla}_\lambda := \mathbb{D}(\Delta_\lambda)$ under the functor F^{KD} is isomorphic to $\nabla_{\lambda,\Omega_q^{DK,'}}$.

Recall that in Proposition 5.14, we see that $\widetilde{\nabla}_\lambda \simeq \mathbb{D}(\Delta_\lambda)$ could be written as

$$Av^{ren}_*(\mathfrak{J}^{\mathbb{D}}_\lambda)$$

By Corollary 6.7 , in order to show the proposition, it suffices to show:

$$H(Fl^{\omega^\rho}_x, Av^{ren}_*(\mathfrak{J}^{\mathbb{D}}_\lambda) \overset{!}{\otimes} j_*(\omega_{S^{-,\mu}_x})) = 0$$

if $\lambda \neq \mu$ and

$$H(Fl^{\omega^\rho}_x, Av^{ren}_*(\mathfrak{J}^{\mathbb{D}}_\lambda) \overset{!}{\otimes} j_*(\omega_{S^{-,\mu}_x})) = \Bbbk$$

if $\lambda = \mu$.

Because $Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}})$ is $N(O)^{\omega^\rho}$-equivariant, so we could apply the functor $Av_*^{N(O)}$ to $j_*(\omega_{S_x^-,\mu})$. And according to Proposition 6.9, we have:

$$Av_*^{N(O)}(j_*(\omega_{S_x^-,\mu})) \simeq \underset{\eta\in\Lambda^+}{colim}\, \mathfrak{J}_{\eta,*} \star \mathfrak{J}_{-\eta+\mu,*}$$

By the proof of Lemma 5.1, we have:

$$\begin{aligned} H(Fl_x^{\omega^\rho}, Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \overset{!}{\otimes} j_*(\omega_{S_x^-,\mu})) = RHom_{D_{\mathcal{G}G}(Fl_x^{\omega^\rho})}(\delta_0, \\ Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \star \underset{\eta\in\Lambda^+}{colim}\, \mathfrak{J}_{\eta-\mu,*} \star \mathfrak{J}_{-\mu,*}) \end{aligned} \tag{10.26}$$

Use the analysis above, we have:

$$\begin{aligned}
&RHom_{D_{\mathcal{G}G}(Fl_x^{\omega^\rho})}(\delta_0, Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \star \underset{\eta\in\Lambda^+}{colim}\, \mathfrak{J}_{\eta-\mu,*} \star \mathfrak{J}_{-\eta,*})\\
=&RHom_{D_{\mathcal{G}G}(Fl_x^{\omega^\rho})}(\delta_0, \underset{\eta\in\Lambda^+}{colim}(Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \star \mathfrak{J}_{\eta-\mu,*} \star \mathfrak{J}_{-\eta,*}))\\
=&\underset{\eta\in\Lambda^+}{colim}\, RHom_{D_{\mathcal{G}G}(Fl_x^{\omega^\rho})}(\mathfrak{J}_{\eta,!}, Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \star \mathfrak{J}_{\eta-\mu,*})\\
=&\underset{\eta\in\Lambda^+}{colim}\, RHom_{Whit_q(Fl_x^{\omega^\rho})}(Av_!^{N(K),\chi}(\mathfrak{J}_{\eta,!}), Av_*^{ren}(\mathfrak{J}_\lambda^{\mathbb{D}}) \star \mathfrak{J}_{\eta-\mu,*})\\
=&\underset{\eta\in\Lambda^+}{colim}\, RHom_{Whit_q(Fl_x^{\omega^\rho})}(Av_!^{N(K),\chi}(\mathfrak{J}_{\eta,!}), Av_*^{ren}(\mathfrak{J}_{\lambda+\eta-\mu}^{\mathbb{D}}))\\
=&\underset{\eta\in\Lambda^+}{colim}\, RHom_{Whit_q(Fl_x^{\omega^\rho})}(Av_!^{N(K),\chi}(\delta_{\eta,!})[-2|\eta|], Av_*^{ren}(\delta_{\lambda+\eta-\mu})[-2|\lambda+\eta-\mu|])
\end{aligned}$$

And the last one is 0 if $\lambda \neq \mu$ and is $\Bbbk$ if $\lambda = \mu$.

□

Recall Proposition 4.3, we have:

$$\nabla_{\lambda,\Omega_{q^{-1}}^{DK,'}} \simeq \mathbb{D}(\Delta_{\lambda,\Omega_q^{L,'}})$$

Proof. (of Proposition 7.4)

According to Theorem 10.2,

$$\begin{aligned}
F_{glob}^L(\Delta_{\lambda,glob}) =&F_{glob}^L(\mathbb{D}(\widetilde{\nabla}_{\lambda,glob}))\\
=&\mathbb{D}F_{glob}^{DK}(\widetilde{\nabla}_{\lambda,glob})\\
=&\mathbb{D}(\nabla_{\lambda,\Omega_{q^{-1}}^{DK,'}})\\
=&\Delta_{q,\Omega_q^{L,'}}
\end{aligned}$$

□

Combined Proposition 7.4 and Theorem 7.1, we could prove Proposition 6.11:

Proof. (of Proposition 6.11)

$$\begin{aligned} F^L(\Delta_\lambda) =& F^L \circ \pi_{Fl,x}(\Delta_{\lambda,glob})[d_g] \\ =& F^L_{glob}(\Delta_{\lambda,glob}) \\ =& \Delta_{q,\Omega_q^{L,\prime}} \end{aligned}$$

∎

Hence, by Section 6.9, we proved Theorem 4.1.

Appendices

A D-module category on ind-pro-schemes

In this paper, we will meet lots of schemes(or stacks) not of finite type. But fortunately, many of they are equipped with ind-pro schemes structure. Namely, they could be regarded as filtered colimit of pro-finite schemes and the transition maps are given by closed embeddings. In existed literature, *e.x*, [Ber], there is a definition of the D-module category on such schemes.

A.1 D-modules on pro-schemes

The definition of D-modules used in this paper comes from applying Kan extensions to the classical definitions of D-module categories on schemes of finite type. Let us start with the usual definition of D-module category on a finite type scheme.

If $S \in Sch^{ft}$, we could define $D(S)$, the category of D-modules on S, *i.e*, we have a functor:

$$D^* : Sch^{ft} \longrightarrow DGCat \tag{A.1}$$

which sends S to $D(S)$ and $f : S_1 \longrightarrow S_2 \in Sch^{ft}$ to $f_* : D(S_1) \longrightarrow D(S_2)$. Similarly, we have:

$$D^! : (Sch^{ft})^{op} \longrightarrow DGCat \tag{A.2}$$

which sends S to $D(S)$ and $f : S_1 \longrightarrow S_2 \in Sch^{ft}$ to $f^! : D(S_2) \longrightarrow D(S_1)$.

Denote by Sch^{pro} the category of pro-finite schemes $Pro^{sm,aff,surj}Sch^{ft}$. It equals to the projective limit of Sch^{ft} with smooth affine and surjective morphisms. By definition, each object of Sch^{pro} could be regarded as an inverse limit of finite type schemes with smooth, affine and surjective transition morphisms.

Definition A.1. *We define the functor:*

$$D^* : Sch^{pro} \longrightarrow DGCat$$

by extending the functor D in (A.1) by the right Kan extension with respect to the inclusion $Sch^{ft} \hookrightarrow Sch^{pro}$

To be more precise, if we are given $Z \in Sch^{pro}$, and it can be written as:

$$Z = \lim_{\pi_{i,j}} Z_i, \ Z_i \in Sch^{ft} \tag{A.3}$$

the transition morphisms are affine smooth and surjective

$$\pi_{i,j} : Z_i \longrightarrow Z_j$$

Then, we have

$$D^*(Z) := \lim_{\pi_{i,j,*}} D(Z_i)$$

here, the transition functors are $*$-direct image functor:

$$\pi_{i,j,*} : D(Z_i) \longrightarrow D(Z_j)$$

Due to the lemma ..., we can replace $\pi_{i,j,*}$ by its left adjoint functor $\pi_{i,j}^*$, and then we can write $D^*(S)$ as $\underset{\pi_{i,j}^*}{colim}\, D(Z_i)$.

By taking duality, we have another definition of D-modules for a pro-scheme $Z \in Sch^{pro}$:

$$D^! : (Sch^{pro})^{op} \longrightarrow DGCat$$

By definition, given a Z of the form (A.3), we have:

$$D^!(Z) := \underset{\pi_{i,j}^!}{colim}\, D(Z_i) \simeq \lim_{\pi_{i,j,*}[-2d_{i,j}]} D(Z_i)$$

here, $d_{i,j} := dim(Z_i/Z_j)$

According to this identification, if we are given a trivialization of dimension, i.e, an assignment $Z_i \rightsquigarrow d(i)$, such that $d_{i,j} = d_i - d_j$, then, we have an equivalence between $D^!(Z)$ and $D^*(Z)$:

$$\eta_Z := \lim_i id_i[-2d_i] : D^!(Z) \xrightarrow{\sim} D^*(Z) \tag{A.4}$$

here, $id_i[-2d_i] : D(Z_i) \longrightarrow D(Z_i)$ denotes the identity functor shift ed by $-2d_i$.

If $f : X \to Y \in Sch^{pro}$, then, we denote by f_* the corresponding functor of $D^*(f)$. And similarly, we define $f^!$ to be $D^!(f)$. Their left adjoint functors f^* and $f_!$ are only partially defined. For example, f^* and $f_!$ are defined for holonomic D-modules.

A.2 D-module on ind-pro-schemes

The inductive limit of the category of Sch^{pro} with closed embedding morphisms will be called the category of ind-pro-schemes. To be more precise,

Definition A.2. *$IndSch^{pro} := Ind^{cl}(Sch^{pro})$, i.e, a presheaf of Sch^{aff} belongs to $IndSch^{pro}$ if and only if it can be written as a filtered colimit of pro-schemes with closed embedding as transition maps.*

From now until the end of this section, we use H to denote an ind-pro-scheme. Then, we may extend the functors D^* and $D^!$ to $IndSch^{pro}$.

Definition A.3. *We define $D^* : IndSch^{pro} \longrightarrow DGCat$ to be left Kan extension of $D^* : Sch^{pro} \longrightarrow DGCat$ along with the inclusion $Sch^{pro} \hookrightarrow IndSch^{pro}$.*

Similarly, we define $D^! : IndSch^{pro} \longrightarrow DGCat$ to be right Kan extension of $D^! : Sch^{pro} \longrightarrow DGCat$ along with the inclusion $Sch^{pro} \hookrightarrow IndSch^{pro}$.

Let us write the definitions more explicitly. Given $H \in IndCoh^{pro}$ and we assume that we can present H as $colim_i Z^i \simeq \underset{i}{colim}\, \underset{j}{lim}\, Z^i_j$. Here, $Z^i_j \in Sch^{ft}$. Morphisms between $Z^i_{j_1}$ and $Z^i_{j_2}$ are affine smooth surjective morphisms, and morphisms between $Z^{i_1} := \underset{j}{lim}\, Z^{i_1}_j$ and $Z^{i_2} := \underset{j}{lim}\, Z^{i_2}_j$ are closed embeddings. Then, we have:

$$D^*(Z) = \underset{i}{colim}\, \underset{j}{lim}\, D(Z^i_j)$$

$$D^!(Z) = \underset{i}{lim}\, \underset{j}{colim}\, D(Z^i_j)$$

Let f_* denote $D^*(f)$ and $f^!$ denote $D^!(f)$. And let f^* and $f_!$ denote their left adjoint functors respectively(if exist).

Lemma A.1. *$G(K)$ and $N(K) \in IndSch^{pro}$*

What's more, it is not difficult to see that $N(K)$ could be written as the union of its open compact subgroups. If we denote $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+_{\check{G}}} \alpha = \sum_i \omega_i$, then,

$$N(K) = \bigcup_{k \geq 0} Ad_{t^{-k\rho}}(N(O)) = \bigcup_{k \geq 0} \underset{r}{lim}\, Ad_{t^{-k\rho}}(N(O))/Ad_{t^{-k\rho}}(N^r)$$

$N^r := N(O) \cap K^r$ and K^r denotes the r-th congruence group of $G(K)$. So, $N(K)$ is an ind pro object in the category of group schemes.

We can describe $D^*(N(K))$ and $D^!(N(K))$ more detailed with the lemma in *loc.cit*.

Lemma A.2.

$$D^*(N(K)) \simeq \underset{r,\pi_*}{lim}\, D^*(N(K)/N^r) \simeq \underset{r,\pi^*}{colim}\, D^*(N(K)/N^r)$$

$$D^!(N(K)) \simeq \underset{r,\pi^!}{colim}\, D^!(N(K)/N^r)$$

According to Lemma A.2 and (A.4), we can relate $D^!(N(K))$ and $D^*(N(K))$ by the following lemma:

Lemma A.3. *A trivialization of dimensions of $\{N^r\}$ gives an equivalence:*

$$D^!(N(K)) \simeq D^*(N(K))$$

In this paper, we are interested in $N(K)$-action, Iwahori-action and $N^-(K)$-action. The action of $N(K)$ and $N^-(K)$ are essentially the same and because I-orbits in Fl_G are of finite-dimensional which are well-known, so we only need to specify the definition of $N(K)$-action and how to define its invariant, coinvariant.

A.3 Weak action and strong action

In the classical case, given a finite type group $\mathcal{G}$ and a vector space V, the functions on $\mathcal{G}$ is a coalgebra. The coalgebra structure is given by the pullback along with the multiplication map m. A structure of $\mathcal{G}$-representation on V is a coaction of the coalgebra $\Gamma(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ on V, *i.e*,

$$V \longrightarrow \Gamma(\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \bigotimes V$$

satisfies some coassociative conditions.

If we want to categorify this notation, we need to replace functions by sheaves. There are two kinds of sheaf theories: quasi-coherent sheaves and D-modules.

Given a finite type $\mathcal{G}$, $*$-pullback along with the multiplication map provides $QCoh(\mathcal{G})$ a comonoidal category structure,

Definition A.4. *For a DG category $\mathcal{C}$ with a weak $\mathcal{G}$-action, we mean $\mathcal{C}$ is a comodule category of $(QCoh(\mathcal{G}), m^*)$. By duality, it could be regarded as module category of $(QCoh(\mathcal{G}), m_!)$.*

We denote by $QCoh(\mathcal{G}) - mod$ the category of such DG-categories.

But when we define actions of pro-finite group scheme, such as $G(O)$, weak action definition is different. [Ras 5].

Definition A.5. *The category of categories with weak $G(O)$-action is defined to be:*

$$G(O) - mod_{weak} := \underset{n}{colim} QCoh(K^n) - mod \in DGCat_{cont} \tag{A.5}$$

Remark The functor $G(O)_m od_{weak} \longrightarrow QCoh(G(O)) - mod$ is not conservative.

Similar as Definition A.4, we could define a strong action for a group scheme(even for ind-pro-scheme).

Given an ind-pro group scheme $\mathcal{G}$. According to Definition A.3, we could define two DG-categories $D^*(\mathcal{G})$ and $D^!(\mathcal{G})$. The following lemma in [Ber] equips them with a monoidal structure and a comonoidal structure respectively.

Lemma A.4. *The $*$-direct image functor along with*

$$m: \ \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

defines $D^*(\mathcal{G})$ a monoidal category structure

$$m_* : D^*(\mathcal{G}) \bigotimes D(\mathcal{G})^* \xrightarrow{\sim} D^*(\mathcal{G} \times \mathcal{G})$$

Dually, $m^!$ provides $D^!$ a structure of comonoidal category.

Definition A.6. *For a DG category $\mathcal{C}$ with a strong $\mathcal{G}$-action(or infinitesimal trivialized action), we mean $\mathcal{C}$ is a module category of $(D^*(\mathcal{G}), m_*)$.*

Dually, it equivalents to $\mathcal{C} \in (D^!(\mathcal{G}), m^!) - comod(DGCat)$

A.4 $\mathcal{G}$-invariant and $\mathcal{G}$-coinvariant

After defining the $\mathcal{G}$-action on a DG category, now, we could define its invariant and coinvariant.

Note: In this paper, we mainly deal with the D-module category, so we only study strong $\mathcal{G}$-action. For weak ones, please check [Ras 5].

Recall in the classical case, for $V \in Rep(\mathcal{G})$, we define the invariant $V^{\mathcal{G}}$ and $V_{\mathcal{G}}$ as follows:

$$V^{\mathcal{G}} := Hom_{\mathcal{G}}(k, V)$$

$$V_{\mathcal{G}} := \Bbbk \bigotimes_{\mathcal{G}} V$$

Analogously, we can define the categorical invariant and coinvariant for a monoidal category $\mathcal{M}$ and its module category $\mathcal{C}$.

Definition A.7. *Given a monoidal category $\mathcal{M}$, and a left module category $\mathcal{C}$ of $\mathcal{M}$, we define:*

$$\mathcal{C}^{\mathcal{M}} := Funct_{\mathcal{M}}(Vect, \mathcal{C})$$

$$\mathcal{C}_{\mathcal{M}} := Vect \bigotimes_{\mathcal{M}} \mathcal{C}$$

Here, $Vect$ is the unit in the monoidal category $DGCat$ given by the tensor product and it is a naturally defined trivial $\mathcal{M}$-module category.

In particular, we may apply the above definition to the monoidal category $(D^*(\mathcal{G}), m_*)$ and a category with a strong $\mathcal{G}$-action.

In this case, the trivial action of $D^*(\mathcal{G})$ on $Vect$ is given by the functor:

$$M \bigotimes c \in D^*(G) \bigotimes Vect \longrightarrow R\Gamma(M) \bigotimes c \in Vect$$

, here, $R\Gamma$ denotes the de Rham cohomology functor, it can be defined to be the $*$-direct image functor along with the morphism: $\mathcal{G} \to pt$.

Remark For a finite type scheme, the $*$-direct image functor along with the projection map to pt coincides with the definition of taking the de Rham cohomology.

Let us write the definition given above more explicitly:

Consider the bar resolution of $Vect$ in right $D^*(\mathcal{G})$-module:

$$... \Rrightarrow D^*(\mathcal{G}) \bigotimes D^*(\mathcal{G}) \bigotimes D^*(\mathcal{G}) \Rrightarrow D^*(\mathcal{G}) \bigotimes D^*(\mathcal{G}) \rightrightarrows D^*(\mathcal{G})$$

By definition, $\mathcal{C}_{\mathcal{G}} = Vect \underset{D^*(\mathcal{G})}{\otimes} \mathcal{C}$ is given by the simplicial category:

$$... \Rrightarrow D^*(\mathcal{G}) \bigotimes D^*(\mathcal{G}) \bigotimes \mathcal{C} \Rrightarrow D^*(\mathcal{G}) \bigotimes \mathcal{C} \rightrightarrows \mathcal{C}$$

and $\mathcal{C}^{\mathcal{G}} = Funct_{D^!(\mathcal{G})}(Vect, \mathcal{C})$ is given by the totalization of the simplicial:

$$\bigotimes \mathcal{C} \rightrightarrows D^!(\mathcal{G}) \bigotimes \mathcal{C} \rightrightarrows D^!(\mathcal{G}) \bigotimes D^!(\mathcal{G}) \bigotimes \mathcal{C} \rightrightarrows ...$$

The character appeared in the definition of Whittaker D-module is not trivial, hence, we also need to define $(\mathcal{G}, \chi)$-equivariant D-module. But the definitions of $(\mathcal{G}, \chi)$-invariant and coinvariant are not very far from above.

Given a group $\mathcal{G}$ and one character $\chi : \mathcal{G} \to \mathbb{A}^1$, we could consider the !-pullback of the exponential D-module $exp := D_{\mathbb{A}^1}/(\partial - 1)D_{\mathbb{A}^1}$ to $\mathcal{G}$. It is a character D-module on $\mathcal{G}$. With some abuse of notation, we also denote it by $\chi \in D^!(\mathcal{G})$.

Then, we may define another action of $D^*(\mathcal{G})$ on $Vect$ with respect to χ:

$$M \otimes c \in D^*(\mathcal{G}) \otimes Vect \longrightarrow \langle M, \chi \rangle \otimes c \in Vect$$

$\langle , \rangle$ denotes the natural pairing between D^* and $D^!$

$$\langle , \rangle : D^* \otimes D^! \longrightarrow Vect$$

Let $Vect_\chi$ denote the resulted module category of $D^*(\mathcal{G})$.

Then, given a left module DG category of $D^*(\mathcal{G})$, we define its (G, χ)-invariant and coinvariant as follows:

Definition A.8.

$$\mathcal{C}^{\mathcal{G},\chi} := (\mathcal{C} \otimes Vect_\chi)^{\mathcal{G}}$$

$$\mathcal{C}_{\mathcal{G},\chi} := (\mathcal{C} \otimes Vect_\chi)_{\mathcal{G}}$$

The forgetful functor $oblv_{\mathcal{G},\chi}$ defines a functor from $\mathcal{C}^{\mathcal{G},\chi}$ to $\mathcal{C}$. We denote $Av_*^{\mathcal{G},\chi}$ the right adjoint functor of $oblv_{\mathcal{G},\chi}$. And we denote $Av_!^{\mathcal{G},\chi}$ the (partially defined) left adjoint functor of $oblv_{\mathcal{G},\chi}$.

By definition, if $\mathcal{G}$ is an ind-pro-scheme and can be written as $\mathcal{G} := colim \mathcal{G}_k$, such that $\mathcal{G}_k$ are pro-finite group schemes. Then, we have:

$$\mathcal{C}^{\mathcal{G},\chi} \simeq \lim_{oblv} \mathcal{C}^{\mathcal{G}_k,\chi} \tag{A.6}$$

and

$$\mathcal{C}_{\mathcal{G},\chi} \simeq \operatorname{colim}_{k} \mathcal{C}^{\mathcal{G}_k,\chi} \tag{A.7}$$

The transition functors of (A.7) are given by projections.

(lemma 4.2.5 important!)

A.5 Base change theorem

In the proof of the main theorem of this paper, the base change theorem plays an important role. Usually, the classical base change theorem is for finite type schemes. But the schemes that we will meet are no longer of finite type. In order to satisfy our purpose, we need to use the proper base change theorem given in Vol 2. [GR1].

Before that, we need to introduce some notations.

Definition A.9. *Given a morphism between two prestacks*

$$f : \mathcal{Y} \longrightarrow \mathcal{Y}'$$

, then it is $ind-schematic$ if for any affine scheme S over $\mathcal{Y}'$, the fiber product given by $\mathcal{Y} \underset{\mathcal{Y}'}{\times} S$ is an ind-scheme of finite type.

An ind-schematic morphism is $ind-proper$ if the fiber product given above can be written as a filtered colimit with transition maps are given by closed embedding

$$\mathcal{Y} \underset{\mathcal{Y}'}{\times} S \simeq colim\ S_k$$

and $\forall k$, $S_k \longrightarrow S$ is proper.

Then, we have the following proper base-change theorem in Vol. 2. [GR1]:

Lemma A.5. *Given a Cartesian diagram of prestacks locally of finite type:*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "g'"] \arrow[d, "f'"] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X}' \arrow[r, "g"] & \mathcal{Y}'
\end{tikzcd}$$

If $f : \mathcal{Y} \longrightarrow \mathcal{Y}'$ is an ind-proper ind-schematic morphism, then, $(f_*,\ f^!)$ and $(f'_*,\ (f')^!)$ are adjoint pairs. And the morphism induced by the adjunctions is an isomorphism, i.e,

$$f'_* \circ g^! \simeq (g')^! \circ f_*$$

Remark The fiber product considered in this lemma is the fiber product in the derived sense. In general, it is different from classical fiber product. But they will coincide when at least one of the morphisms f and g is flat.

It will be enough for our goal.

A.6 Other definitions

We could also define the category of D-modules as the category of quasi-coherent sheaves(or Ind-Coherent sheaves) on the de Rham stack as in [GR1]. In order to introduce this definition of D-modules, let us recall some notions in order.

Definition A.10. *Given a derived prestack $\mathcal{Y}$, we could define its de Rham prestack $\mathcal{Y}_{dR}$ as follows:*

$$Maps(S, \mathcal{Y}_{dR}) := Maps(S^{red,cl}, \mathcal{Y}), \forall S \in DGSch^{aff}_{/\mathcal{Y}}$$

Here, $DGSch^{aff}_{/\mathcal{Y}}$ denotes the category of derived affine schemes over $\mathcal{Y}$. $S^{red,cl}$ denoted the reduced scheme of the classical scheme part of S.

In [GR2], the authors defined left D-module category on a prestack $\mathcal{Y}$ to be the DG category of quasi-coherent sheaves on its de Rham prestack and right D-module category to the category of Ind-coherent sheaves on its de Rham prestack. The first notion always can be defined, but the second one behaves well only for the prestacks locally of finite type. The categories defined by these two definitions are equivalent for all prestacks(locally of finite type, in derived algebraic geometry, it should be replaced by the term ' almost locally of finite type').

Remark A DG prestack:

$$\mathcal{Y} : DGSch^{aff} \longrightarrow Spc$$

is said to be almost locally of finite type if:

· $\mathcal{Y}$ is convergent

· $\mathcal{Y}$ is the left Kan extension of its restriction to $DGSch^{aff}_{laft}$. $DGSch^{aff}_{laft}$ denoted the category of affine DG scheme almost locally of finite type.

Remark If readers are more familiar with S. Raskin's D-module theory on infinite dimensional varieties, then, the infinite type variety that we will meet are $ind-placid$ in his work [Ras3].

And we note that the definitions of the category of D-modules of [GR1] and [Sam Raskin D-modules on infinite dimensional varieties] and [Ber] will be equivalent if we only consider the case of locally of finite type classical prestacks. The difference is that [GR1] develops theorem of D-modules in derived algebraic geometry case.

Remark The affine flag variety Fl is an $ind-scheme\ of\ finite\ type$ and $ind-proper$(over the base), so both of theories can be applied to study the D-module category on it. And they are equivalent.

B Metaplectic parameters

B.1 Sheaves of categories

In order to give the definition of our twisting, we need to consider the sheaf of categories.

Definition B.1. *Given a prestack $\mathcal{Y}$ locally of finite type, a sheaf of categories on $\mathcal{Y}$ means a functor:*

$$\begin{aligned} (Sch^{aff,ft}_{/\mathcal{Y}})^{op} &\longrightarrow DGCat \\ (S \xrightarrow{y} \mathcal{Y}) &\longrightarrow \mathcal{C}(S, y) \in D(S)-mod \end{aligned} \tag{B.1}$$

This assignment satisfies the coherent property, i.e, for any morphism $f : S_1 \to S_2$ over $\mathcal{Y}$: The map :

$$\mathcal{C}(S_2, y_2) \to \mathcal{C}(S_1, y_1)$$

is $D(S_2)$-linear. And the functor in $D(S_1) - mod$:

$$\mathcal{C}(S_2, y_2) \bigotimes_{D(S_2)} D(S_1) \longrightarrow \mathcal{C}(S_1, y_1)$$

is an equivalence. These equivalences satisfy functorial properties.

It is easy to see that $(S \xrightarrow{y} \mathcal{Y}) \longrightarrow D(S, y)$ is a sheaf of categories.

B.2 Gerbes

We review the definition of gerbe here.

In order to consider the deformation of the result (1.16) proved in [AB], we need to consider the category of twisted D-modules. Notice that the category of (non-twisted) D-modules on $\mathcal{Y}$ could be regarded as the section of the constant sheaf of categories on $\mathcal{Y}$. We may hope to define the category of twisted D-modules on $\mathcal{Y}$ as the section of a twisted sheaf of categories. Gerbe offers us the tool to twist a sheaf of categories. In the context of D-modules, we need to consider the $\mathbb{G}_m$-gerbe on the corresponding de Rham prestack $\mathcal{Y}_{dR}$, i.e, a $\mathbb{B}\mathbb{G}_m$-bundle on $\mathcal{Y}_{dR}$, if we regard $\mathbb{B}\mathbb{G}_m$ as a commutative group stack. The references of this section are [GL2] and [Zh1].

We denote by $Ptd(\mathcal{C})$ the category of pointed objects, i.e, $(* \longrightarrow \rfloor \in \mathcal{C})$. Here, $*$ denotes the final object in $\mathcal{C}$. What's more, we denote by $Grp(Spc)$ the category of group objects in $\mathcal{C}$.

Given a category $\mathcal{C}$ admitting all finite products, we define the category of $\mathbb{E}_n$-algebras in $\mathcal{C}$ inductively.

Definition B.2. *If $n \geq 0$, we define the category of $\mathbb{E}_n$-algebras in $\mathcal{C}$ by*

$$\mathbb{E}_n(\mathcal{C}) = Monoid(\mathbb{E}_{n-1}(\mathcal{C}))$$

$$\mathbb{E}_0(\mathcal{C}) = Ptd(\mathcal{C})$$

Remark If $\mathcal{C}$ is $1 - Cat$ equipped with the product of categories, the resulted category $\mathbb{E}_n(1 - Cat)$ is called the category of $\mathbb{E}_n$-categories.

Let Ω denote the loop functor which sends a pointed object $(* \to S)$ to $* \times_S *$. The object $* \times_S *$ is a group object in $\mathcal{C}$. What's more, it is known that if S is an $\mathbb{E}_k$-algebra, then, $\Omega(S)$ is an $\mathbb{E}_{k+1}$-algebra.

Let $\mathbb{B}$ denote the left adjoint functor (if exists) of Ω, and is called the delooping functor

$$\begin{aligned} \mathbb{B} : Grp(\mathcal{C}) &\longrightarrow Ptd(\mathcal{C}) \\ G &\longrightarrow (* \in pt/G) \end{aligned}$$

Here, $*$ denotes the distinguished point in pt/G given by the trivial G-bundle.

The functor $\mathbb{B}$ exists if $\mathcal{C} = Spc$ or $PreStk$, hence, we could consider the following adjoint functor pair

$$\mathbb{B} : Grp(Prestk) \rightleftarrows Ptd(PreStk) : \Omega \tag{B.2}$$

If $\mathcal{A}$ is a commutative group object in $PreStk$, then, $\mathbb{B}(\mathcal{A})$ could be regarded as a commutative group object in $PreStk$ in a natural way. In particular, we could consider twofold of $\mathbb{B}$, namely, $\mathbb{B}^2(\mathcal{A})$.

We denote by $\mathbb{B}^2_{\acute{e}t}\mathcal{A}$ the étale sheafification of $\mathbb{B}^2\mathcal{A}$.

Definition B.3. *An étale $\mathcal{A}$-gerbe on a prestack $\mathcal{Y}$ is a map from $\mathcal{Y}$ to $\mathbb{B}^2_{\acute{e}t}\mathcal{A}$.*

Definition B.4. *Given a prestack $\mathcal{Y}$, we denote by $Grb(\mathcal{Y})$ the category*

$$Grb(\mathcal{Y}) := Maps(\mathcal{Y}_{dR}, \mathbb{B}^2_{\acute{e}t}\mathbb{G}_m) \tag{B.3}$$

B.3 Twisting construction

In this section, we will review how to twist a sheaf of categories

$$\{(S \xrightarrow{y} \mathcal{Y}) \longrightarrow \mathcal{C}(S, y)\} \tag{B.4}$$

by a gerbe. The references for us are Section 1.7 [GL2] and S.Lysenko's unpublished notes.

We note that there is a naturally defined monoidal functor of prestacks

$$\mathbb{B}_{\acute{e}t}\mathbb{G}_m(S_{dR}) \longrightarrow Funct(\mathcal{C}(S, y), \mathcal{C}(S, y)), \forall\ S \xrightarrow{y} \mathcal{Y} \tag{B.5}$$

Here, $\mathbb{B}_{\acute{e}t}\mathbb{G}_m(S_{dR})$ classifies the rank 1 local system on S, we could also denote it by $Loc^1(S)$. Loc^1 is a group object in $PreStk$ and $Grb = \mathbb{B}_{\acute{e}t}Loc^1$, i.e, Grb classifies *étale* Loc^1-torsors.

Given an affine scheme S of finite type, $D(S)$ admits an action of $Loc^1(S)$ given by the tensor product of D-modules. Hence, given an *étale* Loc^1-torsor, $\mathcal{G}$, we could define $D_{\mathcal{G}}(S)$ to be the associated category. To be more precise, consider the sheaf of categories in (B.4), we define a new sheaf of categories

$$\{(S \xrightarrow{y} \mathcal{Y}) \longrightarrow \mathcal{C}_{\mathcal{G}}(S, y)\} \tag{B.6}$$

as following. For any affine scheme $S \xrightarrow{y} \mathcal{Y}$ of finite type such that $\mathcal{G}$ is trivial on it, we let

$$C_{\mathcal{G}}(S, y) = (* \underset{Grb(S)}{\times} *) \bigotimes_{Loc^1(S)} \mathcal{C}(S, y) \tag{B.7}$$

Here, the action of $Loc^1(S)$ on $\mathcal{C}(S, y)$ is given by (B.5) and the two maps $* \longrightarrow Grb(S)$ are given by the trivial one and the composition:

$$S \xrightarrow{y} \mathcal{Y} \xrightarrow{\mathcal{G}} Grb \tag{B.8}$$

By the description of $C_{\mathcal{G}}(S)$ given in (B.7), we could understand the category $D_{\mathcal{G}}(\mathcal{Y})$ in this way:

For an affine finite type scheme S over $\mathcal{Y}$, each trivialization of $\mathcal{G}$ on S_{dR} gives rise to an identification:

$$D_{\mathcal{G}}(S) \simeq D(S)$$

The change of trivialization of $\mathcal{G}|_S$ given by $\mathcal{L}$, a rank 1 local system on S, changes the above equivalence by composing with the equivalence given by tensoring with the local system $\mathcal{L}$.

For a more precise description of $D_{\mathcal{G}}(\mathcal{Y})$, see [?] 3.5.1.

Remark Intuitively(topologically), an object $\mathcal{F}$ in $D_{\mathcal{G}}(\mathcal{Y})$ could be understood as a collection of twisted vector space $\Bbbk_{\mathcal{G}(x)} - mod$ for any $x \in \mathcal{Y}$ and for any 'path' $\gamma : x \longrightarrow y$ in $\mathcal{Y}_{dR}$, there is an isomorphism of $\gamma^*(\mathcal{F}_x)$ and $\mathcal{F}_y$ with higher coherence homotopy conditions.

B.3.1 Factorization gerbes

In this section, we define the notion of the factorization gerbe on the Configuration space $Conf$(for the definition of this space, please check Section 2.2). For other prestacks (such as $Gr_{G,Ran}$, $Gr^{neg}_{T,Ran}$, etc) equipped with a factorization property, we could define the factorization gerbe onside in the same way.

The definition of factorization gerbes are similar to the definition of factorization prestacks in Section 2.3.

A factorization gerbe on $Conf$ is a right lax symmetric monoidal functor:

$$(fSet^{surj})^{op} \longrightarrow DGCat_{cont}$$

$$J \longrightarrow Grb(Conf^J_{disj})$$

By Grothendieck straightening theorem, the above functor gives rise to a Cartesian fibration of symmetric monoidal categories:

$$Grb_{fSet} \longrightarrow fSet^{surj}$$

The fiber of $J \in fSet^{surj}$ is given by $Grb(Conf^J_{disj})$.

Then, a factorization gerbe is a symmetric monoidal Cartesian section of the above Cartesian fibration.

If we denote the fiber of the above Cartesian section over J by $\mathcal{G}^J \in Grb(Conf^J_{disj})$, then, by the definition of the symmetric monoidal Cartesian section, we have:

$\cdot$ $\mathcal{G}_J \xrightarrow{\sim} j^!_{disj}(\mathcal{G}^{\boxtimes J}_{\ominus})$ Here, $j_{disj} : (Conf^J)_{disj} \longrightarrow Conf^J$ is an open embedding.

$\cdot$ $\mathcal{G}_J \xrightarrow{\sim} add^!_{disj}(\mathcal{G}_{\ominus})$

To simplify the notion, we just say $\mathcal{G} := \mathcal{G}_{\ominus}$ is a factorizable gerbe.

Similarly, we could define the notion of gerbe on $Conf_x$ which are factorizable with respect to certain factorization gerbe on $Conf$.

B.4 Relate $\mathcal{G}$ with q in Constructible sheaf setting

Note that in our expected equivalence:

$$Whit_q(Fl) := D_{\mathcal{G}^G}(Fl)^{N(K),\chi} \simeq Rep_q^{mix}(\check{G})$$

We have two different parameters, $\mathcal{G}^G$ and q. In order to make sense of our expected equivalence, we should first find a way to link a gerbe to a quadratic form on Λ with value in $\Bbbk^{\times}$(or $\Bbbk/\mathbb{Z}$).

In this section, we will recall the method in [GL2] how the authors associated a gerbe(in Constructible sheaf setting) a quadratic form.

First of all, we consider the combination version of $Gr_{T,Ran}$,

$$Gr_{T,comb} := \underset{(I,\lambda^I)}{colim}\, X^I$$

The index category is taken to be the category consisting of the object ($I \in fSet$, $\lambda^I : I \longrightarrow \Lambda$). The morphism is given by

$$\phi : I \twoheadrightarrow J, \lambda^J(j) = \sum_{i \in f^{-1}(j)} \lambda^I(i)$$

Then, the transition map

$$X^J \longrightarrow X^I$$

is given by sending $(x_1, x_2, ..., x_{|J|})$ to $(x_{\phi(1)}, x_{\phi(2)}, ..., x_{\phi(|I|)})$.

Note that given an element in $(I, \lambda^I, (x_1, ..., x_{|I|}) \in X^I)$, we can get an element $(\mathcal{O}(\sum \lambda^I(i)x_i), \alpha)$ in $Gr_{T,I}$(hence, an element in $Gr_{T,Ran}$). Here, α is the natural trivialization of $\mathcal{O}(\sum \lambda^I(i)x_i)$ on $X - \{x_1, x_2, ..., x_{|J|}\}$

This assignment is compatible with the factorization property, and gives rise to a morphism between factorization prestacks:

$$Gr_{T,comb} \longrightarrow Gr_{T,Ran}$$

We denote by the same notation the pullback of a gerbe $\mathcal{G}$ on $Gr_{T,Ran}$ to $Gr_{T,comb}$. And its restriction to X^I corresponds to (I, λ^I) is denoted by $\mathcal{G}_{\lambda^I}$.

Note that $\mathcal{G}$ has the following factorization property:

- If $\phi : I \twoheadrightarrow J$, then, we have:

$$\mathcal{G}_{\lambda^I}|_{\phi,disj} \simeq (\boxtimes \mathcal{G}_{\lambda^{I_j}})|_{\phi,disj} \tag{B.9}$$

  Here, $X^I_{\phi,disj}$ is denoted by the open subset of X^I, such that $x_{i_1} \neq x_{i_2}$ if $\phi(i_1) = \phi(i_2)$.

In particular, we take $I = \{1, 2\}$, and λ^I is defined to be:

$$1 \to \lambda_1, 2 \to \lambda_2$$

Then, we denote by $\mathcal{G}_{\lambda_1,\lambda_2}$ the gerbe $\mathcal{G}_{\lambda^I}$.

Similarly, we denote by $\mathcal{G}_{\lambda_1}$ and $\mathcal{G}_{\lambda_2}$ the restriction of $\mathcal{G}$ on X corresponding to $1 \longrightarrow \lambda_1$ and $2 \longrightarrow \lambda_2$ respectively.

According to the factorization isomorphism (B.9), we have an isomorphism of gerbes:

$$\mathcal{G}_{\lambda_1,\lambda_2}|_{X^2\backslash X} \simeq (\mathcal{G}_{\lambda_1} \boxtimes \mathcal{G}_{\lambda_2})|_{X^2\backslash X} \tag{B.10}$$

In [GL2], the authors proved that (B.10) can give an element in Tate twist $\Bbbk^{tors,\times}(-1)$ which is denote by $b(\lambda_1, \lambda_2)$. In *loc.cit*, the authors also proved that the map:

$$b : \Lambda \times \Lambda \longrightarrow e^{tors,\times}(-1)$$

is actually a bilinear form.

B.5 Tame gerbe theory

The content of this section is a review of Y.Zhao's tame gerbe theory in [Zh1].

Let us mimic the construction in the last section. It seems everything is good until the step of getting a number $b(\lambda_1, \lambda_2)$ from the isomorphism (B.10).

Given a scheme X with a smooth effective Cartier divisor Y. Consider the following map:

$$\Bbbk/\mathbb{Z} \longrightarrow \text{gerbes on } X_{dR} \text{ with a trivialization on } (X\backslash Y)_{dR} \tag{B.11}$$

(B.11) is given as follows: locally, we could assume that Y is cut out by a function g on X. Then, this functor sends $c \in \Bbbk/\mathbb{Z}$ to a gerbe on X_{dR} with a trivialization outside Y_{dR} given by the Kummer local system Φ_c. A different trivialization η of $O(Y)$ will give rise to a transition local system η^c.

But the above assignment is not bijective!

Example If $X = \mathbb{A}^1$ with a parameter t and we take effective Cartesian divisor corresponding to $\{0\}$. We consider the D-module $exp(t^{-1})$ onside. Then, the trivial gerbe admits a trivialization on $\mathbb{A} - \{0\}$ given by $exp(t^{-1})$ is not in the image of $\Bbbk/\mathbb{Z}$ under (B.11).

Hence, we know that in the context of D-modules, the naive $\mathbb{G}_m$ gerbe is not the right twisting machine to work with, we could not use this procedure to get a bilinear form from arbitrary factorization gerbe $\mathcal{G} \in Grb(Gr_{T,comb})$. The reason is that it is not 'topological' enough.

In [Zh1], the author fixes this problem by taking a 1-full substack in Grb. Let us denote by $LocSys_1$ the prestack classifies the category of rank 1 local systems and by $LocSys_1^{reg}$ the sub-category consisting of the regular local systems.

Definition B.5. *We define the prestack*

$$Grb^{reg} := pt/LocSys_1^{reg}$$

Given a prestack $\mathcal{Y}$, $Grb^{reg}(\mathcal{Y})$ denotes the category of the étale sheafication of $LocSys_1^{reg}$-bundles(in étale topology) on $\mathcal{Y}$.

Then, in the upcoming work of [Zh1], the author proves the following lemma:

Lemma B.1. *Given a smooth scheme X and an effective Cardier s divisor Y inside. Then, the following functor is bijective.*

$$\Bbbk/\mathbb{Z} \longrightarrow \textit{tame gerbes on } X_{dR} \textit{ with a trivialization on } (X\backslash Y)_{dR} \tag{B.12}$$

What's more, in *loc.cit*, the author proves that:

Lemma B.2. *a).(descend along with finite surjective morphism) If $X \longrightarrow Y$ is a finite, surjective map between smooth schemes, then, the pullback defines an equivalence:*

$$Grb^{reg}(Y) \simeq Grb^{reg}(X)$$

b).($\mathbb{A}^1$-homotopy) There is a canonical equivalence:

$$Grb^{reg}(X) \simeq Grb^{reg}(X \times \mathbb{A}^1)$$

Definition B.6. *We denote by $FactGrb^{reg}(Gr_{G,Ran})$ the category of factorization object in $Grb^{reg}(Gr_{G,Ran})$. And we call the object inside the regular factorization gerbe on $Gr_{G,Ran}$.*

Then, any regular factorization gerbe on $Gr_{G,Ran}$ satisfies the following properties:

Lemma B.3. *a).($G(O)_{Ran}$-equivariant) Any regular factorization gerbe on $Gr_{G,Ran}$ is equivariant with respect to the $G(O)_{Ran}$-action on $Gr_{G,Ran}$.*

b). The pullback of regular factorization gerbes on $Gr_{G,Ran}$ along with the pojection:

$$G(K)_{Ran} \longrightarrow Gr_{Gr,Ran}$$

defines an equivalence between $FactGrb^{reg}(Gr_{G,Ran})$ and the category of multiplicative regular gerbes on $G(K)_{Ran}$.

By the same procedure as in Section B.4, we got a bilinear form:

$$b : \Lambda \times \Lambda \longrightarrow \Bbbk/\mathbb{Z}$$

and if we take a bilinear form b', such that $b'(\lambda_1, \lambda_2) + b'(\lambda_2, \lambda_1) = b(\lambda_1, \lambda_2)$, then the associated quadratic form of b' is denoted by q:

$$q : \Lambda \longrightarrow \Bbbk/\mathbb{Z}$$

C Proof of Lemma 6.2

In this section, we essentially copy the proof from [Ga5].

Before we start the proof, let us give some notations.

Notation: We denote by K^j the j-th congruence subgroup and by I^j the preimage of $N(O)/N^j$ in $G(O)/K^j$. i.e, we have the following Cartesian diagram:

$$\begin{tikzcd} I^j \arrow[r] \arrow[d] & G(O)^{\omega^\rho} \arrow[d] \\ N(O)^{\omega^\rho}/N(O)^{\omega^\rho} \cap K^j \arrow[r] & G(O)^{\omega^\rho}/K^j \end{tikzcd}$$

Let G'_{Ran_x} denote the closed subgroup of $G(K)^{\omega^\rho}_{Ran_x}$ such that the fiber over a point $\{x, y_1, y_2, ..., y_k\}$ is given by $\prod G(O)^{\omega^\rho}_{y_i} \times G(K)^{\omega^\rho}_x$. Given any subgroup $H \subset G(K)^{\omega^\rho}_x$, we denote by $\bar{H}$ the preimage of H under the projection $G'_I \longrightarrow G(K)_x$.

We note the map in (6.12) is a closed embedding, so we could define $unit_! : D_{\mathcal{G}^G}(X^I_x \times Fl^{\omega^\rho}_x) \longrightarrow D_{\mathcal{G}^G}((\bar{S}^{w_0}_{I,Fl})_{\infty \cdot x})$ on the whole category, but the image of the functor $unit_!$ is not $(N(K)^{\omega^\rho}_I, \chi)$-equivariant. According to adjointness of $(unit_!, unit^!)$ and $(Av_!^{N(K)^{\omega^\rho}_I, \chi}, oblv)$, $(Av_!^{N(K)^{\omega^\rho}_I, \chi} \circ unit_!, unit^!)$ is an adjoint pair. We need to prove that the functor $Av_!^{N(K)^{\omega^\rho}_I, \chi}$ can be defined on the image of $Whit_q(X^I \times Fl^{\omega^\rho}_x)$ under the functor $unit_!$ and it commutes with the action of $D(X^I_x)$.

It is known that the category $Whit_q(X^I_x \times Fl^{\omega^\rho}_x)$ is generated by the image of $D_{\mathcal{G}^G}(X^I_x \times Fl^{\omega^\rho}_x)^{I^j, \chi}$ under the functor $Av_!^{N(K), \chi} \circ oblv_{I^j, \chi}$. We consider the prestack $Fl^{\omega^\rho}_I$, note the image of $X^I \times Fl^{\omega^\rho}_x$ under $unit_!$ is G'_I-invariant and the action of G'_I on the image of $X^I \times Fl^{\omega^\rho}_x$ factors through the action of $G(K)$ on $Fl^{\omega^\rho}_I$. In particular, it is true for $N(\bar{K}) \subset G'_I$.

Hence, we have the following isomorphism of functors:

$$D_{\mathcal{G}^G}(X_x \times Fl^{\omega^\rho}_x)^{I^j, \chi} \longrightarrow Whit_q(Fl^{\omega^\rho}_I)$$

$$Av_!^{N(K)_I, \chi_I} \circ unit_! \simeq Av_I^{N(K)_I, \chi_I} \circ unit_! \circ Av_!^{N(K), \chi} \tag{C.1}$$

In particular, if we could prove that the functor $Av_!^{N(K)_I, \chi_I} \circ unit_!$ could be defined on $D_{\mathcal{G}^G}(X_x \times Fl^{\omega^\rho}_x)^{I^j, \chi}$ and commutes with the action of $D(X^I_x)$, then, the functor $Av_I^{N(K)_I, \chi_I} \circ unit_!$ could be defined on the category of Whittaker modules on $X^I_x \times Fl^{\omega^\rho}_x$ and commutes with the action of $D(X^I_x)$. And hence, Lemma 6.2 could be proved.

According to the above analysis, the image of $D_{\mathcal{G}^G}(X_x \times Fl^{\omega^\rho}_x)^{I^j, \chi}$ under $unit_!$ is in $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_I)^{\bar{I}^j, \chi}$. Hence, we have to prove:

Lemma C.1. *the functor $Av_!^{N(K)^{\omega^\rho}_I, \chi}$ can be defined on $D_{\mathcal{G}^G}(Fl^{\omega^\rho}_I)^{\bar{I}^j, \chi}$.*

We define

$$N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho}$$

to be the quotient scheme of $N(K)_I^{\omega^\rho} \times Fl_I^{\omega^\rho}$ by the diagonal action of $N(K)_I^{\omega^\rho} \cap \bar{I}^j$. Because the gerbe $\mathcal{G}^G$ is multiplicative, and the character D-module χ is right $N(K)_I^{\omega^\rho} \cap \bar{I}^j$-equivariant with respect to χ, hence, for any $(N(K)_I^{\omega^\rho} \cap \bar{I}^j, \chi)$-equivariant D-module $\mathcal{F} \in D_{\mathcal{G}^G}(Fl_I^{\omega^\rho})$, the D-module

$$\chi \boxtimes \mathcal{F} \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho} \times Fl_I^{\omega^\rho})$$

can descend to a D-module

$$\chi \widetilde{\boxtimes} \mathcal{F} \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho})$$

The left multiplication of $N(K)_I^{\omega^\rho}$ on $Fl_I^{\omega^\rho}$ gives a functor:

$$act : N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho} \longrightarrow Fl_I^{\omega^\rho} \tag{C.2}$$

It is known that the image of $\mathcal{F}$ under the functor:

$$D_{\mathcal{G}^G}(Fl_I^{\omega^\rho})^{\bar{I}^j,\chi} \overset{oblv}{\to} D_{\mathcal{G}^G}(Fl_I^{\omega^\rho}) \overset{Av_!^{N(K)_I^{\omega^\rho},\chi}}{\to} D_{\mathcal{G}^G}(Fl_I^{\omega^\rho})^{N(K)^{\omega^\rho},\chi}$$

is given by !-direct image of the desecent D-module $\chi \widetilde{\boxtimes} \mathcal{F}$ on

$$N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho}$$

along with the functor (C.2). Hence, we only need to prove that the !-direct image functor can be defined on $\chi \widetilde{\boxtimes} \mathcal{F}$ and commutes with $D(X_x^I)$ action.

Denote by G^j the subgroup of $G(K)$ given by $Ad_{-j\rho}(G(O))$.

Consider the prestack $N(K)_I^{\omega^\rho}\overline{G^j} \subset G(K)_I^{\omega^\rho}$, $\overline{G^j}$ acts on this scheme from right.

$$N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho}$$

it is isomorphic to the product:

$$N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho}$$

And because of the same reason as before, we have a well-defined D-module category $D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho})$.

We have:

$$D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho}) \simeq D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap \bar{I}^j}{\times} Fl_I^{\omega^\rho}) \tag{C.3}$$

We may regard $\chi \,\widetilde{\boxtimes}\, \mathcal{F}$ as a twisted D-module on $N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho}$.

Under the equivalence (C.3), the functor $act_!$ is isomorphic to the !-direct image functor along with the action map:

$$act : N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho} \longrightarrow Fl_I^{\omega^\rho} \tag{C.4}$$

So, we need to prove the !-extension can be defined for the above action map. Let $\overline{N(K)_I^{\omega^\rho}\overline{G^j}}$ denote the closure of $N(K)_I^{\omega^\rho}\overline{G^j}$ in $G(K)_I^{\omega^\rho}$.

Note the map above factors as the following composition:

$$N(K)_I^{\omega^\rho}\overline{G^j} \overset{N(K)_I^{\omega^\rho}\cap\overline{G^j}}{\times} Fl_I^{\omega^\rho} \xrightarrow{\mathfrak{j}} \overline{N(K)_I^{\omega^\rho}\overline{G^j}} \overset{N(K)_I^{\omega^\rho}\cap G(O)^j}{\times} Fl_I^{\omega^\rho} \xrightarrow{\bar{act}} Fl_I^{\omega^\rho} \tag{C.5}$$

The morphism

$$\overline{N(K)_I^{\omega^\rho}\overline{G^j}} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho} \xrightarrow{\bar{act}} Fl_I^{\omega^\rho}$$

is ind-proper. Hence, !-direct image is well-defined for $\bar{act}$.

Then, Lemma 6.2 comes from the following lemma ([locglob p rop A2.4]):

Lemma C.2. *For any (I^j, χ)-equivariant D-module $\mathcal{F}$ on $Fl_I^{\omega^\rho}$, $\chi^!(exp) \,\widetilde{\boxtimes}\, \mathcal{F} \in D_{\mathcal{G}^G \bigotimes \mathcal{G}^G}(N(K)_I^{\omega^\rho}\overline{G^j} \overset{\overline{G^j}}{\times} Fl_I^{\omega^\rho})$ is clean with respect to $\mathfrak{j}$.*

Proof. The proof of this lemma could be decomposed by two steps.

The first one is to prove that the pullback of $\chi^!(exp) \,\widetilde{\boxtimes}\, \mathcal{F}$ along with the projection:

$$N(K)_I^{\omega^\rho}\bar{G}^j \times Fl_I^{\omega^\rho} \longrightarrow N(K)_I^{\omega^\rho}\bar{G}^j \overset{\bar{G}^j}{\times} Fl_I^{\omega^\rho} \longrightarrow N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap\bar{I}^j}{\times} Fl_I^{\omega^\rho} \tag{C.6}$$

is $(N(K)_I^{\omega^\rho}, \chi)$-equivariant with respect to the left action of $N(K)_I^{\omega^\rho}$ from left on $N(K)_I^{\omega^\rho}\bar{G}^j$ and also $\bar{K}_j^j$-equivariant with respect the right action:

$$\bar{K}_j^j \times N(K)_I^{\omega^\rho}\bar{G}^j \overset{\bar{G}^j}{\times} Fl_I^{\omega^\rho} \longrightarrow N(K)_I^{\omega^\rho}\bar{G}^j \overset{\bar{G}^j}{\times} Fl_I^{\omega^\rho}$$

$$k, (ng, s) \rightsquigarrow (ngk^{-1}, s)$$

here $K_j^j := Ad_{-j\rho}(K^j)$.

To prove this claim, first of all, we notice that the morphism (C.6) is $N(K)_I^{\omega^\rho}$-equivariant, hence, the pullback is still $(N(K)_I^{\omega^\rho}, \chi)$-equivariant. Then, we could write $N(K)_I^{\omega^\rho}\bar{G}^j \overset{\bar{G}^j}{\times} Fl_I^{\omega^\rho}$ as $N(K)_I^{\omega^\rho} \overset{N(K)_I^{\omega^\rho}\cap\bar{G}^j}{\times} \bar{G}^j \times Fl_I^{\omega^\rho}$. Then the K_j^j-equivariant claim is equivalent to the K_j^j-equivariant claim for the action of K_j^j on $\bar{G}^j \times Fl_I^{\omega^\rho}$ given by:

$$K_j^j \times \bar{G}^j \times Fl_I^{\omega^\rho} \longrightarrow \bar{G}^j \times Fl_I^{\omega^\rho}$$

$$k, (g, s) \rightsquigarrow (gk^{-1}, s)$$

because $\bar{K}_j^j$ is a normal subgroup of $\bar{G}^j$, hence, the action map:

$$\bar{G}^j \times Fl_I^{\omega^\rho} \longrightarrow Fl_x^{\omega^\rho}$$

is K_j^j-equivariant. Now, the first claim follows from the fact that if $j > 1$, $K_j^j \subset I^j$ and $\chi|_{K_j^j}$ is trivial.

The second step is to prove that any twisted D-module on $N(K)_I^{\omega^\rho} \bar{G}^j \overset{\bar{G}^j}{\times} Fl_I^{\omega^\rho}$ descend from a twisted D-module satisfying the equivariant properties in the first step is clean with respect to the extension j in (C.5).

We only need to prove that the restriction of the $*$-extension of a twisted $(N(K)_I, \chi)$-equivariant D-module on $N(K)_I^{\omega^\rho} \bar{G}^j / K_j^j \times Fl_I^{\omega^\rho}$ is zero outside

$$N(K)_I^{\omega^\rho} \bar{G}^j / K_j^j \times Fl_I^{\omega^\rho}$$

It is enough to prove that the restriction of χ on the stabilizer of $N(K)_I^{\omega^\rho}$ at any point outside $N(K)_I^{\omega^\rho} \bar{G}^j / K_j^j \times Fl_I^{\omega^\rho}$ is not trivial.

Given such a point, assume that it is $nt^{\lambda_I} g / K_j^j \in N(K)_I^{\omega^\rho} t^{\lambda_I} \bar{G}^j / K_j^j \times Fl_I^{\omega^\rho}$, here $\lambda_I := \{\lambda_x = \lambda_1, \lambda_2, ..., \lambda_{|I|}\}$ such that $\lambda_x \notin \Lambda^+$. Then, its stabilizer is $nt^{\lambda_I} N(\bar{O}) t^{-\lambda_I} n^{-1}$. It is not in the kernel of χ_I.

□

References

[AB] S. Arkhipov and R.Bezrukavnikov, *Perverse sheaves on affine flags and langlands dual group*, Isr. J. Math. (2009) 170: 135. https://doi.org/10.1007/s11856-009-0024-y

[AG] S.Arkhipov, D.Gaitsgory *Another realization of the category of modules over the small quantum group*, arXiv:math/0010270 [math.QA]

[ABBGM] S. Arkhipov, R. Bezrukavnikov, A. Braverman, D. Gaitsgory, I. Mirković *Modules over the small quantum group and semi-infinite flag manifold*, arXiv:math/0505280 [math.AG]

[AO] D. Abramovich and F. Oort, *Stable maps and Hurwitz schemes in mixed characteristics*, in Advances in algebraic geometry motivated by physics, Contemp. Math. 276 (2001), 89-100

[BB1] A.Beilinson, J.Bernstein *A proof of Jantzen conjectures*,

[BB2] A. Beilinson, J. Bernstein, *A generalization of Casselman's submodule theorem*, Representation theory of reductive groups (Park City, Utah, 1982), 35–52, Progr. Math., 40, Birkhäuser Boston, Boston, MA, 1983.

[BD1] A. Beilinson and V. Drinfeld. *Quantization of Hitchin's integrable system and Hecke eigensheaves.*, (1991): 1297-1301.

[BD2] A. Beilinson and V. Drinfeld. *Chiral algebras*, Vol. 51. American Mathematical Soc., 2004.

[Be1] R. Bezrukavnikov, *Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group*, arXiv:math/0201256 [math.RT]

[Be2] R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves* , Inv. Math. 166 (2006), no 2, 327–357.

[Be3] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, arXiv:1209.0403 [math.RT]

[Bec] J.M.Beck, *Triples, algebras and cohomology*, arXiv:1603.05593 [math.AG]

[Bei] A.Beilinson, *Constructible sheaves are holonomic*, Sel. Math. New Ser. (2016) 22: 1797. https://doi.org/10.1007/s00029-016-0260-z

[BeL] R. Bezrukavnikov, A. Lachowska *The small quantum group and the Springer resolution* , arXiv:math/0609819 [math.RT]

[Ber] D. Beraldo *Loop Group Actions on Categories and Whittaker Invariants*, arXiv:1608.00284 [math.AG]

[BFGM] A. Braverman, M. Finkelberg, D. Gaitsgory, I. Mirković *Intersection cohomology of Drinfeld's compactifications*, Selecta Mathematica, 2002, vol. 8, no 3, p. 381-418.

[BFS] R. Bezrukavnikov, M. Finkelberg and V. Schechtman, *Factorization algebras and quantum groups*, Lecture Notes in Mathematics 1691 (1998).

[BG] A.Braverman, D.Gaitsgory *Geometric Eisenstein series*, Invent. math. (2002) 150: 287. https://doi.org/10.1007/s00222-002-0237-8

[BL] A. Beauville and Y.Laszlo *Un lemme de descente*, Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique 320.3 (1995): 335-340.

[BR] A.Beligiannis, I.Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883, viii+207 pp. Theorem III.2.3

[BS] R.Bott and H.Samelson, *Applications of the theory of Morse to symmetric spaces*, American Journal of Mathematics, Vol. 80, No. 4 (Oct., 1958), pp. 964-1029

[BW] M. Barr, C. Wells, *Toposes, Triples and Theories*, Grundlehren der math. Wissenschaften 278, Springer-Verlag 1983

[Camp] J.Campbell *A resolution of singularities for Drinfeld's compactification by stable maps*, arXiv:1606.01518 [math.AG]

[D] P. Deligne *SGA 4 1/2–Cohomologie étale*, Lecture Notes in Mathematics, 1977, vol. 569.

[DG1] V.Drinfeld, D.Gaitsgory *On a theorem of Braden*, arXiv:1308.3786 [math.AG]

[DG2] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, GAFA, 23 (2013), 149–294.

[DG3] V. Drinfeld and D. Gaitsgory *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv:1112.2402, Cambridge Math Journal, 3 (2015), 19–125.

[DS] V.Drinfeld and C.Simpson *B-structures on G-bundles and local triviality*, Mathematical Research Letters 2.6 (1995): 823-829

[DK] C. De Concini, V. Kac *Representations of quantum groups at roots of 1*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 471–506, Progr. Math. 92, Birkhäuser Boston, Boston, MA, 1990.

[FFKM] B. Feigin, M. Finkelberg, A. Kuznetsov, I. Mirkovic, *Semi-infinite flags. II. Local and global intersection cohomology of quasimaps' spaces,* , Differential topology, infinite- dimensional Lie algebras, and applications, 113–148, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999. arXiv:1603.05593 [math.AG]

[FG] E. Frenkel, D.Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, arXiv:0712.0788 [math.RT]

[FGV] E. Frenkel, D. Gaitsgory and K. Vilonen *Whittaker patterns in the geometry of moduli spaces of bundles on curves* , Annals of Math. 153 (2001), no. 3, 699–748. Vol. 51. American Mathematical Soc., 2004.

[FP] W. Fulton, R. Pandharipande *Notes on stable maps and quantum cohomology*, arXiv:alg-geom/9608011

[Fra] J. Francis, *The tangent complex and Hochschild cohomology of E_n-rings*, Compositio Math. 149 (2013), 430– 480.

[FrG] J.Francis, D.Gaitsgory *Chiral Koszul duality*, Vol. 51. American Mathematical Soc., 2004.

[Ga1] D. Gaitsgory *A conjectural extension of the Kazhdan-Lusztig equivalence*, arXiv:1810.09054 [math.RT]

[Ga2] D.Gaitsgory, *Twisted Whittaker model and factorization algebras*, Sel. math., New ser. (2008) 13: 617. https://doi.org/10.1007/s00029-008-0053-0

[Ga3] D. Gaitsgory *The semi-infinite intersection cohomology sheaf*, arXiv:1703.04199 [math.AG]

[Ga4] D. Gaitsgory *The semi-infinite intersection cohomology sheaf-II: the Ran space version*, arXiv:1708.07205 [math.AG]

[Ga5] D. Gaitsgory *The local and global versions of the Whittaker category*, arXiv:1811.02468 [math.AG]

[Ga6] D. Gaitsgory *On factorization algebras arising in the quantum geometric Langlands theory*, arXiv:1909.09775 [math.AG]

[Ga7] D. Gaitsgory *The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve*, arXiv: 1505.02331 [math.AG]

[Ga8] D. Gaitsgory *On a vanishing conjecture appearing in the geometric Langlands correspondence*, Ann. Math. 160 (2004), 617–682.

[Ga9] D.Gaitsgory, *Sheaves of categories and the notion of 1-affineness*, arXiv:1306.4304 [math.AG]

[Ga10] D. Gaitsgory, *Ind-coherent sheaves*, arXiv:1105.4857 [math.AG]

[Ga11] D. Gaitsgory, *Introduction to quantum local geometric Langlands*, http://www.iecl.univ-lorraine.fr/ Sergey.Lysenko/notes_talks_winter2018/GL-1%28Dennis%29.pdf

[GL1] D. Gaitsgory and S. Lysenko *Metaplectic Whittaker category and quantum groups : the "small" FLE*, arXiv:1903.02279 [math.AG]

[GL2] D. Gaitsgory and S. Lysenko *Parameters and duality for the metaplectic geometric Langlands theory*, arXiv:1608.00284 [math.AG]

[GR1] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry.*, Vol. 1. American Mathematical Soc., 2017.

[GR2] D. Gaitsgory and N. Rozenblyum. *Crystals and D-modules*, arXiv:1111.2087 [math.AG]

[Ho] Quoc P. Ho *Factorization algebras and categories*,

[Ja] J. C. Jantzen, *Lectures on quantum groups Graduate Studies in Mathematics 6, American Mathematical Society*, Graduate Studies in Mathematics 6, American Mathematical Society, Providence, RI, 1996.Lectures on quantum groups Vol. 51. American Mathematical Soc., 2004.

[K] G. R. Kempf, *Linear systems on homogeneous spaces*, (1976). Annals of Mathematics, 557-591

[Kac] V. G. Kac, *Infinite-dimensional Lie algebras*, (1990). Cambridge university press,

[KL] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, J. Amer. Math. Soc. 6 (1993), 905–1011 and 7 (1994), 335–453.

[La] M.Lanini *Semi-infinite combinatorics in representation theory*, arXiv:1505.01046 [math.RT]

[Lys1] S. Lysenko, *Twisted Whittaker models for metaplectic groups*, GAFA 27, 289–372.

[Lys2] S.Lysenko *Twisted geometric Langlands correspondence for a torus*, IMRN, 18, (2015), 8680-8723, arXiv:1312.4310,

[Lu1] J.Lurie *Higher Algebra*, www.math.harvard.edu/ lurie/papers

[Lu2] J.Lurie *Higher topos theory*, Princeton University Press, 2009.

[Lu3] J.Lurie *Derived Algebraic Geometry II: Noncommutative Algebra*, arXiv:math/0702299 [math.CT]

[Lus] G.Lusztig *Introduction to quantum groups*, Springer Science and Business Media, 2010.

[Ra] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splittings of diagonals*, (1987). Publications Mathématiques de l'IHÉS, 65, 61-90.

[Ras1] S.Raskin *Chiral Principal Series Categories*,

[Ras2] S.Raskin *W-algebras and Whittaker categories*, arXiv:1611.04937 [math.RT]

[Ras3] S.Raskin *D-modules on infinite dimensional varieties*, 2015,

[Ras4] S.Raskin *Chiral categories*,

[Ras5] S.Raskin *Weak actions of loop groups*,

[Rei] R.C. Reich *Twisted geometric Satake equivalence via gerbes on the factorizable grassmannian*, arXiv:1012.5782 [math.RT]

[Ri] S. Riche, *Tutorial on quantum groups.*,

[Ro] N. Rozenblyum *Tutorial on factorization vs braided monoidal categories*,

[Sai] T.Saito *The characteristic cycle and the singular support of a constructible sheaf*, arXiv:1510.03018 [math.AG

[So] C. Sorger *Lectures on moduli of principal G-bundles over algebraic curves*, Lectures on moduli of principal G-bundles over algebraic curves. No. INIS-XA–803. 2000.

[Yu] Y.Fu *Tutorial 1: Group Actions on Categories*,

[Zh1] Y.Zhao *Tame gerbes*,

[Zh2] Y.Zhao *Quantum parameters of the geometric Langlands theory*, arXiv:1708.05108 [math.AG]

[Zh3] Y.Zhao *Notes o quantum parameters (GL-2)*, http://www.iecl.univ-lorraine.fr/ Sergey.Lysenko/notes_talks_winter2018/GL-2%28Yifei%29.pdf

[Zhou] Q.Zhou, *Convex Polytopes for the Central Degeneration of the Affine Grassmannian*, arXiv:1604.08641 [math.AG]

[Zhu] X.Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, arXiv:1603.05593 [math.AG]

Cette thèse travaille sur l'équivalence locale fondamentale du programme de Langlands quantique. L'équivalence fondamentale consiste à établir une équivalence entre le modèle de Whittaker et le modèle de Kazhan-Lusztig.

Supposons que G soit un groupe réductif, B est son sous-groupe Borel, N son radical unipotent, T est le tore maximal de G et $\check{G}$ est Langlands dual de G. $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{t}$ et $\check{\mathfrak{g}}$ sont leurs algèbres de Lie.

Nous allons travailler sur les D-modules, donc nous devons supposer que nous travaillons sur un corps commutatif fermé algébrique $\Bbbk$ de caractéristique 0. En fait, nous pourrions travailler avec une autre théorie de la faisceau. Par exemple, la catégorie des faisceaux constructibles. Pour plus de simplicité, supposons que notre corps commutatif soit le corps complexe, G est un groupe reductif défini sur $\mathbb{C}$.

Nous notons Λ le réseau de copoids de G, et dénotons $\check{\Lambda}$ le réseau de poids. $K := \Bbbk((t))$ et $O = \Bbbk[[t]]$. On notera par $G(K)$ (resp. $N(K)$) le groupe de boucles de G (resp. N), et I le sous-groupe Iwahori de $G(O)$. Soit χ un caractère non dégénéré de $N(K)$. En choisissant un Borel, nous pourrions définir copoids négative Λ^{neg}. Soit X une courbe globale sur $\Bbbk$. L'espace de configuration $Conf$ est défini comme le schéma classifiant $D = \sum \lambda_i x_i$ de telle sorte que $\lambda_i \in \Lambda^{neg} \backslash 0$. Fixez $x \in X$, laissez $Conf_x$ classifier les données $D = \lambda x + \sum \lambda_i x_i$ de telle sorte que $\lambda_i \in \Lambda^{neg} \backslash 0$.

Nous dénotons:

$$Gr := G(K)/G(O)$$

$$Fl := G(K)/I$$

Il est bien connu sous le nom d'équivalence de Satake que le groupe de Grothendieck de représentations complexes de $\check{G}$ (groupe duel Langlands de G) est équivalent à l'anneau des fonctions $G(O)$-invariantes avec support compact sur la grassmanienne affine.

En tant que catégorisation de cette équivalence, nous avons l'équivalence Satake géométrique qui identifie $Sph(G)$, la catégorie des faisceaux pervers G(O) équivalentes sur grassmannin affine en tant que catégorie de représentation dimensionnelle finie du groupe duel de Langlands de G.

Maintenant, le côté droit, considéré comme une catégorie monoidale torsadé, admet une déformation de famille à un paramètre dans la catégorie $Rep(U_q(\check{G}))$, où, par $U_q(\check{G})$, nous notons le Lusztig groupe quantique.

Une question naturelle est: quelle est la déformation correspondante du côté droit?

La déformation correspondante est les D-modules torsadé. Généralement, étant donné un schéma X et un bundle de BG_m sur X, nous pourrions définir les D-modules torsadés correspondants sur X.

Étant donné une forme bilinéaire symétrique invariante non dégénérée du groupe de Weyl:

$$\kappa : \Lambda \times \Lambda \longrightarrow \Bbbk \tag{0.1}$$

il donne une forme bilinéaire symétrique:

$$\mathfrak{t} \times \mathfrak{t} \longrightarrow \Bbbk$$

Par (0.1), nous avons un isomophisme $\mathfrak{t} \simeq \check{\mathfrak{t}}$, donc nous obtenons une forme bilinéaire:

$$\check{\mathfrak{t}} \times \check{\mathfrak{t}} \longrightarrow \Bbbk$$

De là, nous obtenons une forme bilinéaire:

$$\check{\kappa} : \check{\Lambda} \times \check{\Lambda} \longrightarrow \Bbbk$$

Par le travail de S.Lysenko et D.Gaitsgory, nous savons qu'à partir d'une telle forme quadratique q, nous pouvons l'associer à un gerbe sur Gr (et Fl).

Il est naturel de considérer la déformation du côté $Sph(G)$. Nous pouvons considérer les D-modules monodromes sur G'/G(O). (G' est l'extension central de G(K)). Le passage de $\check{G}$ à sa déformation quantique devrait correspondre au remplacement des D-modules (ou faisceaux pervers) par les D-modules sur G' /G(O) qui sont monodromes le long de la fibre avec la monodromie. Mais cette tentative n'était pas correcte:

$$Rep_q(\check{G}) \not\simeq Sph_\kappa(G)$$

Par q, nous désignons la forme quadratique associée de κ, κ désigne le gerbe associé à la forme quadratique κ, c'est-à-dire que le paramètre quantique

$$q : \Lambda \longrightarrow \Bbbk^*$$

est donné par

$$q(\lambda) = exp(\pi i \kappa(\lambda, \lambda))$$

Une autre façon de considérer la déformation consiste à utiliser le modèle de Whittaker.

Notez qu'il existe une équivalence:

$$Whit(Gr) \simeq Sph(G)$$

Ici, Whit(Gr) est la catégorie de D-module $(N(K), \chi)$-equivalent sur Gr. Et Sph(Gr) est la catégorie de D-module G(O)-equivalent sur Gr.

Nous nous attendons donc à ce que nous puissions remplacer le côté gauche par la catégorie de Whittaker, puis prendre la déformation de l'équivalence. Parce que le retrait du gerbe κ à $G(K)$ est canoniquement trivial sur $N(K)$, nous définissons la catégorie Whittaker torsadée sur Fl_G (resp. Gr_G):

$$Whit_\kappa(Gr) := D_\kappa(Gr)^{N(K),\chi}$$

D.Gaitsgory a conjecturé que:

$$Whit_q(Gr) = Rep(U_q(\check{G}))$$

Nous notons par $KL^{\check{\kappa}}(\check{G})$ la catégorie des modules Kac-Moody $\check{G}(O)$ - intégrables. Ici, $\check{\mathfrak{g}}^{\check{\kappa}}$ désigne l'extension centrale de $\check{\mathfrak{g}}((t))$ par $\check{\kappa}$.

Et selon Kazhdan-Lusztig, nous avons que

$$KL^{\check{\kappa}}(\check{G}) = Rep(U_q(\check{G}))$$

Donc, l'équivalence locale fondamentale dit:

Conjecture 0.1.

$$Whit_\kappa(Gr_G) \simeq KL^{\check{\kappa}}(\check{G}) \tag{0.2}$$

L'équivalence locale fondamentale d'origine est très difficile, nous pourrions considérer quelques objets liés à $Whit_q(Gr)$. Par exemple, $Whit_q(Fl)$.

Récemment, D.Gaitsgory a proposé une version ramifiée de l'équivalence fondamentale locale. Dans ce cas, la partie de la catégorie des représentations n'est plus $Rep_q(\check{G})$, elle est supposée être $Rep_q^{mix}(\check{G})$, ici, 'mix' signifie la catégorie des représentations du groupe quantique mixte

Intuitivement, une représentation du groupe quantique mixte est un espace vectoriel Λ-dégradé avec une action de la partie positive du Lusztig groupe quantique $U_q(\check{N})$ (avec la condition localement nilpotente) et un action compatible du dual de la partie positive du Lusztig groupe quantique.

Pour donner la définition précise, nous devons utiliser le centre de Drinfeld relatif:

Definition 0.1.

$$Rep_q^{mix}(\check{G}) := Z_{Dr,Rep_q(\check{T})}(Rep_q(\check{N})^{loc.nil})$$

Ici, $Rep_q(\check{N})^{loc.nil}$ est l'ind-complétion de la catégorie des représentations dimensionnelles finies de la partie positive du groupe quantique de Lusztig.

Theorem 0.1.

$$Whit_\kappa(Fl) \simeq Rep_q^{mix}(\check{G})$$

Dans ma thèse de doctorat, j'ai démontré le théorème 0.1 avec la méthode de [6]. Cette thèse pourrait être considérée comme une version déformée de la théorie de S.Arkhipov-R.Bezrukavnikov dans [AB].

En effet, $QCoh(\check{\mathfrak{n}}/Ad\check{B})$ pourrait être considéré comme la catégorie des espaces vectoriels Λ-dégradés admettant une action localement nilpotente de $U(\check{\mathfrak{n}})$ et une action compatible de $sym(\check{\mathfrak{n}}^-)$. C'est-à-dire,

$$QCoh(\check{\mathfrak{n}}/Ad\check{B}) \simeq Rep^{mix}(\check{G})$$

Pour prouver le théorème 0.1, nous le reformulons. Selon le théorème de Lurie, nous pourrions réaliser $Rep_q^{mix}(\check{G})$ en tant que modules factorisables sur une fasceau factorisable.

Étant donné X une courbe projective lisse connectée,

Definition 0.2. *Le schéma $Conf(X, \Lambda^{neg})$ classe le diviseur coloré de X avec les coefficients dans Λ^{neg},*

c'est-à-dire qu'il classe:

$$D = \sum_k \lambda_k \cdot x_k, \lambda_k \in \Lambda^{neg} - 0 \tag{0.3}$$

Notation: nous désignons simplement $Conf(X, \Lambda^{neg})$ par $Conf$.

Nous définissons un sous-schéma ouvert $Conf^2_{disj} \subset Conf^2$ comme étant le sous-schéma classant le point $\{D_1, D_2\} \in Conf^2$ de telle sorte que D_1 et D_2 ont des supports disjoints.

Nous avons une opération qui donne à $Conf$ une structure de semi-groupe commutatif non unitaire.

$$add : Conf^2 \longrightarrow Conf \tag{0.4}$$

$$D_1, D_2 \longrightarrow D_1 + D_2$$

Si nous limitons ce morphisme à $Conf^2_{disj}$, c'est étale.

Étant donné Ω sur $Conf$, on l'appelle une algèbre de factorisation si:

$$add^!|_{Conf^2_{disj}}(\Omega) \simeq \Omega \boxtimes \Omega|_{Conf^2_{disj}} \tag{0.5}$$

Par l'esprit de [7] et [8], $Rep_q^{mix}(\check{G})$ est équivalent à $\Omega_q^L - FactMod$, la catégorie de modules factorisables sur $Conf_x$ par rapport à une algèbre factorisable Ω_q^L.

Avec cette équivalence, les deux côtés du théorème 0.1 deviennent des objets géométriques. J'ai construit un foncteur

$$F^L : Whit_\kappa(Fl) \longrightarrow \Omega_q^L - FactMod$$

L'équivalence que nous avons prouvée n'est pas seulement une équivalence de catégories, mais aussi une équivalence de «catégories de poids les plus hautes». Autrement dit, nous pourrions construire des objets standard $\Delta_{\lambda,q} \in Whit_q(Fl_G)$, $\Delta^{L,fact}_{\lambda,q} \in \Omega^L_q - FactMod$ et objets costandard $\nabla_{\lambda,q} \in Whit_q(Fl_G)$, $\nabla^{L,fact}_{\lambda,q} \in \Omega^L_q - FactMod$ et le foncteur les conserve .

Le guide pour nous de définir $\Delta_{\lambda,q}$ et $\nabla_{\lambda,q}$ sont [AB] et [R1].

Rappelons que sous l'équivalence suivante:

$$\begin{aligned} Whit(Fl_G) &\simeq QCoh(\check{\mathfrak{n}}/Ad\check{B}) \\ Av_!^{N(K),\chi}(\mathfrak{J}_\lambda) &\longrightarrow O(\lambda) \end{aligned} \tag{0.6}$$

Ici, $\mathfrak{J}_\lambda$ est la faiseau Bezrukavnikov-Mirkovic-Wakimoto (c'est-à-dire la faiseau Wakimoto). Ils sont définis comme étant l'image du foncteur monoïdal unique: $\Lambda \longrightarrow D(Fl)^I$ tel que pour λ dominant, $\mathfrak{J}_\lambda$ est le !-extension du D-module constant sur le I-orbite passant par $t^\lambda \in Fl_G$.

Si nous utilisons un langage de représentation, $O(\lambda)$ correspond au module Verma $V_\lambda^{mix} := Ind_{Rep(\check{B})}^{QCoh(\check{\mathfrak{n}}/Ad\check{B})}(\Bbbk^\lambda)$.

Et sous l'équivalence entre $QCoh(\check{\mathfrak{n}}/Ad\check{B})$ et la catégorie des modules de factorisation sur Ω^L, V_λ^{mix} est le !-extension du module de factorisation irréductible unique sur Ω^L sur $Conf_{=\lambda\cdot x}$.

Maintenant, nous donnons les définitions des standards et des costandards,

- Dans $\Omega_q^L - FactMod$, l'objet standard indexé par λ est le !-extension du module irréductible unique Ω_q^L sur $Conf_{=\lambda\cdot x} := \{D = \lambda \cdot x + \sum \lambda_i \cdot x_i | \lambda_i \in \Lambda^{neg}, x_i \neq x\}$ Nous le désignons par $\Delta_{\lambda,q}^{L,fait}$.
- Dans $Whit_q(Fl_G)$, l'objet standard indexé par λ est l'image de la BMW faiseau métaplectique $\mathfrak{J}_\lambda$ sous le foncteur de !- moyenne. Nous le désignons par $\Delta_{\lambda,q}$.
- Les objets costandard sont les objets qui sont orthogonaux à droite aux objets standard.
- L'objet irréductible indexé par λ est l'image du standard indexé par λ dans le costandard indexé par λ.

Nous définissons $Fl_{G,Conf_x}$ comme étant la version de configuration des drapeaux affines. C'est définie sur $Conf_x$ telle que sa fibre sur le point

$$D := \lambda \cdot x + \sum_k \lambda_k \cdot x_k \in Conf_x \tag{0.7}$$

est donné par

$$Fl_{G,x} \times \prod_k Gr_{G,x_k}$$

A partir de $\mathcal{F} \in Whit_q(Fl_G)$, nous définissons un D-module torsadé sur $Fl_{G,Conf_x}$, noté $sprd(\mathcal{F})$, de sorte que sa restriction à la fibre sur D est

$$\mathcal{F} \boxtimes \mathcal{F}_0 ... \boxtimes \mathcal{F}_0$$

Ici, $\mathcal{F}_0$ est le D-module Whittaker irréductible unique sur la fermeture du N (K)-orbite passant par $1 \in Gr_G$.

Dans $Fl_{G,Conf_x}$, nous définissons un module D torsadé $j_!(\Omega_{Conf_x})$ sur $Fl_{G,Conf_x}$ de telle sorte que sa! Restriction à la fibre supérieure à D est donné par l'extension! de:

$$\omega_{Fl,\lambda} \otimes \omega_{Gr,\lambda_1} \boxtimes ... \boxtimes \omega_{Gr,\lambda_k}$$

$\omega_{Fl,\lambda}$ est la gerbe dualisante sur l'orbite $N^-(K)$ - passant par $t^\lambda \in Fl_G$ et similaire pour les autres, si nous remplaçons Fl_G par Gr_G.

Il y a un foncteur:

$$v_{Fl} : Fl_{G,Conf_x} \longrightarrow Conf_x$$

F^L est défini comme:

$$F^L(\mathcal{F}) := v_{Fl,*}(sprd(\mathcal{F}) \overset{!}{\otimes} j_!(\Omega_{Conf_x}))[shift] \tag{0.8}$$

De même, nous définissons la version de configuration affine Grassmannienne $Gr_{G,Conf}$ telle que sa fibre sur

$$D = \sum_k \lambda_k \cdot x_k \in Conf$$

est donné par

$$\prod_k Gr_{G,x_k}$$

Nous définissons Vac comme un D-module torsadé sur $Gr_{G,Conf}$, sa restriction à la fibre au-dessus de D est donnée par le produit:

$$\mathcal{F}_0 \boxtimes \mathcal{F}_0 \boxtimes ... \boxtimes \mathcal{F}_0$$

Sur Gr_{Conf}, nous définissons un D-module torsadé $j_!(\Omega_{Conf})$ sur Gr_{Conf} de telle sorte que sa! Restriction à la fibre sur D soit donnée par le !-extension de:

$$\omega_{Gr,\lambda_1} \boxtimes ... \boxtimes \omega_{Gr,\lambda_k}$$

Définissez $v_{Gr,*}$ comme la projection de $Gr_{G,Conf}$ à $Conf$.

$$\Omega_q^{L,'} := v_{Gr,*}(Vac \overset{!}{\otimes} j_!(\Omega_{Conf}))[shift]$$

Notez que $Fl_{G,Conf_x}$ est factorisable par rapport à $Gr_{G,Conf}$:

$$Fl_{G,Conf_x} \underset{Conf_x}{\times} (Conf \times Conf_x)_{disj} \simeq (Gr_{Conf} \times Fl_{G,Conf_x}) \underset{Conf \times Conf_x}{\times} (Conf \times Conf_x)_{disj}$$

Pour tout $\mathcal{F} \in Whit_q(Fl_G)$, $sprd(\mathcal{F})$ est factorisable sur Vac. $j_!(\omega_{Conf_x})$ est factorisable par rapport à $j_!(\omega_{Conf})$. v_{Fl} et v_{Gr} sont compatibles avec la structure de factorisation.

Par conséquent, $F^L(\mathcal{F})$ est un module de factorisation par rapport à $\Omega_q^{L,'}$.

Nous prouvons que:

Proposition 0.1.

$$i_\lambda^!(F^L(\mathcal{F})) \simeq H^\bullet(Fl_G, \mathcal{F} \overset{!}{\otimes} j_!(\omega_{Fl,\lambda})) \tag{0.9}$$

Dans la formule ci-dessus, $\omega_{Fl,\lambda}$ est la gerbe de dualisation métaplectique sur le $N^-(K)$-orbite $N^-(K)t^\lambda I/I \subset Fl_G$.

Nous pouvons montrer que la !-fibre du foncteur F^L à $\lambda \cdot x \in Conf_x$ est représentée par $\Delta_{\lambda,q}$, c'est-à-dire,

$$i_\lambda^! \circ F^L(\mathcal{F}) = RHom_{Whit_q(Fl_G)}(\Delta_{\lambda,q}, \mathcal{F})$$

De cette description, nous savons: F^L préserve les objets costandards.

Il suffit de prouver que $\Delta_{\lambda,q}$ va à $\Delta_{\lambda,q}^{L,fact}$ sous le foncteur F^L.

En fait, prouver que F^L préserve les objets standard est le composant principal de la preuve. Parce que la théorie des D-modules se comporte bien avec !-pullback au lieu de *-fiber (par exemple, *-pullback n'est pas toujours bien défini et le théorème de changement de base est pour !-pullback), le calcul de la *-fiber semble très difficile.

Pour surmonter cette difficulté, nous adaptons la méthode utilisée dans [GL2], c'est-à-dire en utilisant la dualité de Verdier pour transférer le calcul de *-fiber de à un calcul de! -fibre.

Le foncteur de dualité sur $Whit_q(Fl_G)$ n'est pas tautologique. A savoir, $Whit_q(Fl_G)$ est défini comme des invariants (de N (K) contre un caractère), une dualité a priori prend des valeurs dans la catégorie des coinvariants. Cependant, en raison de [R2], nous savons que la catégorie de Whittaker invariante et la catégorie de coinvariant sont en fait équivalentes par une procédure non tautologique de * moyenne contre la faiseau de dualisation torsadée χ sur $N(K)$.

Par conséquent, nous avons un foncteur de dualité:

$$\mathbb{D} : Whit_q(Fl_G)^{c,op} \longrightarrow Whit_{q^{-1}}(Fl_G)^c$$

Nous construisons un autre foncteur F^{KD}.

$$F^{KD} : Whit_q(Fl_G) \longrightarrow \Omega_q^{KD} - FactMod \tag{0.10}$$

Ici, Ω_q^{KD} est le dual Verdier de $\Omega_{q^{-1}}^L$.

Semblable à $\Omega_q^L - FactMod$, $\Omega_q^{KD} - FactMod$ admet un ensemble d'objets standard et d'objets costandard. Nous les désignons respectivement par $\Delta_{\lambda,q}^{KD,fait}$ et $\nabla_{\lambda,q}^{KD,fait}$.

$\nabla_{\lambda,q}^{KD,fait}$ est le *-extension du module unique irréductible sur Ω_q^{KD} sur $Conf_{=\lambda\cdot x}$.

Par construction, $\mathbb{D}(\Delta_{\lambda,q}^{L,fait}) = \nabla_{\lambda,q^{-1}}^{KD,fait}$.

Il suffit de montrer les deux propositions suivantes.

Proposition 0.2.

$$F^{KD}(\mathbb{D}(\Delta_{\lambda,q^{-1}})) = \nabla_{\lambda,q}^{KD,fait} \tag{0.11}$$

Proposition 0.3.

$$\mathbb{D} \circ F^L \simeq F^{KD} \circ \mathbb{D} : Whit_q(Fl_G) \longrightarrow \Omega_{q^{-1}}^{KD} - FactMod \tag{0.12}$$

Nous résolvons le problème en résolvant les deux propositions ci-dessus.

Le foncteur F^{KD} construit dans l'article pourrait être décrit par sa !-fibre à $\lambda \cdot x \in Conf$.

Proposition 0.4.

$$i_\lambda^!(F^{KD}(\mathcal{F})) \simeq H^\bullet(Fl_G, \mathcal{F} \overset{!}{\otimes} j_*(\omega_{Fl,\lambda})) \tag{0.13}$$

La première proposition découle d'un calcul direct.

La deuxième proposition est loin d'être tautologique. Afin de prouver cette proposition de dualité, nous devons introduire une construction globale.

Definition 0.3. *Soit* $(\overline{Bun_N})'_{\infty\cdot x}$ *la champ classant les données suivantes:* $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \forall\check{\lambda} \in \Lambda^+\}, \epsilon)$. *Ici,* $\mathcal{P}_G \in Bun_G$, $\kappa^{\check{\lambda}}$ *est une famille des morphismes des faiseaux cohérentes:* $\kappa^{\check{\lambda}} : O \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}$, *de sorte qu'il soit régulier en dehors de* $x \in X$. ϵ *est une réduction de* B *de* $\mathcal{P}_G$ *à* x.

Le gerbe $\mathcal{G}^G$ sur Fl_G descend vers un gerbe sur $(\overline{Bun_N})'_{\infty\cdot x}$, nous définissons $Whit_{q,glob}(Fl_G)$ en tant que sous-catégorie de $D_{\mathcal{G}^G}((\overline{Bun_N})'_{\infty\cdot x})$ en imposant la condition d'équivariance par rapport à un certain groupoïde unipotent .

Il existe un morphisme naturel:

$$\pi_{Fl} : Fl_G \longrightarrow (\overline{Bun_N})'_{\infty\cdot x}$$

Ce morphisme induit un foncteur:

$$\pi^!_{Fl} : Whit_{q,glob}(Fl_G) \longrightarrow Whit_q(Fl_G)$$

En suivant la méthode de [G2], nous prouvons:

Theorem 0.2. $\pi^!_{Fl}$ *est une équivalence de catégories.*

Nous construisons les foncteurs globalement:

$$F^L_{glob} : Whit_{q,glob}(Fl_G) \longrightarrow \Omega^L_q - FactMod \tag{0.14}$$

$$F^{KD}_{glob} : Whit_{q,glob}(Fl_G) \longrightarrow \Omega^{KD}_q - FactMod \tag{0.15}$$

Les foncteurs définis localement et les foncteurs globalement sont isomorphes:

Proposition 0.5.

$$F^L \circ \pi^!_{Fl} \simeq F^L_{glob}$$

$$F^{KD} \circ \pi^!_{Fl} \simeq F^{KD}_{glob}$$

Par conséquent, F^L et F^{KD} s'entrelacent avec le foncteur de dualité de Verduer est équivalent à la proposition suivante:

Proposition 0.6.

$$\mathbb{D} \circ F^L_{glob} \simeq F^{KD}_{glob} \circ \mathbb{D} : Whit_{q,glob}(Fl_G) \longrightarrow \Omega^{KD}_{q^{-1}} - FactMod \tag{0.16}$$

Nous pourrions prouver la propriété de dualité avec la notion d'acyclicité universellement locale.