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par

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Backward stochastic differential equations driven by Gaussian Volterra processes

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"De la même façon que personne n'est capable d'expliquer pourquoi les étoiles sont belles, c'est difficile d'exprimer la beauté des mathématiques."

Yoko Ogawa

Chapter 1

General introduction

Introduction, French version

Cette thèse porte sur les équations différentielles stochastiques rétrogrades (EDSR) dirigées par un processus de Volterra.

L'objectif principal de cette thèse est l'étude de l'existence, de l'unicité et de la régularité de la solution des EDSR dirigées par un processus de Volterra et la résolution explicite dans le cas linéaire.

1.1 Structure de la thèse

Le titre de cette thèse porte sur les équations différentielles stochastiques rétrogrades. Ce type d'équations est utilisé en mathématiques financières et en contrôle stochastique (voir par exemple les travaux de S. Peng (1993) [29] et N. El Karoui, S. Peng et M.-C. Quenez (1997) [1]). C'est un thème qui s'est beaucoup développé ces dernières années et qui trouve des applications dans de nombreux domaines comme la biologie ou pour l'étude d'équations aux dérivées partielles. La théorie liée à ces équations a été développée par E. Pardoux et S. Peng dès 1990 [28]. La principale question sur la théorie des EDSR est l'existence et l'unicité des solutions. Ces équations sont apparues en finance à partir de la formule de Black et Scholes par exemple dans le cas des EDSR linéaires.

Cette thèse est organisée de la façon suivante:

1- Dans la section 1.2, nous présentons les équations différentielles stochastiques rétrogrades dirigées par un processus de Volterra gaussien. Ensuite, nous résumons nos principaux résultats développés dans cette thèse.

2- Dans le deuxième chapitre, nous définissons la classe des processus de Volterra étudiés dans

cette thèse. Cette classe de processus contient des mouvements browniens multi-fractionnaires et des processus d'Ornstein-Uhlenbeck multi-fractionnaires. Contrairement au mouvement brownien fractionnaire où le paramètre de Hurst H est constant, il devient pour le mouvement brownien multi-fractionnaire une fonction h supposée ici différentiable et à valeurs dans $(1/2, 1)$.

3- Dans le troisième chapitre, nous définissons l'opérateur de dérivation ou dérivée de Malliavin. Nous utiliserons régulièrement ces outils. Le calcul de Malliavin est un calcul différentiel sur l'espace de Wiener introduit par P. Malliavin en 1978 dans [2]. Nous citons en particulier les livres de D. Nualart [26] et de I. Nourdin et G. Peccati [3].

4- Dans le quatrième chapitre, nous traitons la formule d'Itô multi-dimensionnelle pour un processus de Volterra.

5- Dans le cinquième chapitre, nous développons la solution explicite des EDSR linéaires dirigées par une classe de processus de Volterra introduite au chapitre 2 et notés par X. L'objectif est d'obtenir la solution de l'EDSR linéaire avec l'EDP linéaire associée, dont la solution classique est explicitement donnée. Ceci généralise un résultat dans [7] obtenu pour le mouvement brownien fractionnaire. Nous définissons l'intégrale stochastique par rapport à X comme une intégrale de divergence et généralisons une formule d'Itô dans [5] au cas multi-dimensionnel. La formule d'Itô est ensuite appliquée à la solution de l'EDP associée afin d'obtenir une solution de l'EDSR. Une attention particulière est accordée au fait que la variance des processus de Volterra n'est pas nécessairement une fonction croissante du temps, mais en général uniquement a variation bornée. La solution explicite de l'EDP associée contient cette variance, et elle est donnée par une formule de type Feynman-Kac. L'application de cette formule aux EDSR est donc limitée aux intervalles de temps où cette variance est croissante.

6- Dans le sixième chapitre, nous établissons le lien entre les EDP et les EDSR dans le cas non linéaire. Dans un second temps, nous présentons les outils nécessaires pour prouver l'existence et l'unicité de la solution. En fait, ces outils sont: les développements en chaos, l'espérance quasi-conditionnelle, la formule de Clark-Ocone, et pour finir, nous introduisons une formule très pratique qui est la formule d'Itô dans le cas stochastique.

7- Dans le septième chapitre, nous appliquons les résultats que nous avons vu dans le chapitre précédent pour prouver l'existence et l'unicité de la solution des EDSR non linéaires.

1.2 Équations différentielles stochastiques rétrogrades dirigées par un processus de Volterra

Une équation différentielle stochastique rétrograde (EDSR) avec un générateur $f : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, à valeur terminale ξ_T et dirigée par un processus stochastique $X = (X^1, ..., X^n)$ est donnée par l'équation

$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T Z_s dX_s, \ 0 \leqslant t_0 \leqslant t \leqslant T. \tag{1.1}$$

Une solution est une paire de processus de carrés intégrables Y et $Z = (Z^1, ..., Z^n)$ qui sont adaptés à la filtration engendrée par X. Ces équations apparaissent particulièrement dans le contexte de la tarification des actifs et de la théorie de la couverture en finance et dans le contexte des problèmes de contrôle stochastique [33], [1], [24]. Les EDSR peuvent être considérées comme une alternative aux équations aux dérivées partielles connues d'avantage, car leurs solutions sont liées aux solutions classiques ou aux solutions de viscosité des EDP associées. En conséquence, les EDSR peuvent être utilisées pour la solution numérique des EDP non linéaires.

Suite au premier résultat d'existence et d'unicité démontré par E. Pardoux et S. Peng [28], les EDSR non-linéaires ont été étudiées de façon approfondie. Pour une synthèse de ces travaux de recherche, nous pouvons nous reporter aux manuels récents [11], [13], [27], [30], [33], [36]. Plus récemment, des EDSR dirigées par des mouvements browniens fractionnaires ont été étudiées (voir, par exemple, [7], [19], [18], [20], [25], [32], [34], [35]). Comme les mouvements browniens fractionnaires (avec l'indice de Hurst $H \in (0,1/2) \cup (1/2,1)$) ne sont ni des martingales ni des processus de Markov, de nouvelles méthodes ont été développées pour montrer l'existence et l'unicité de la solution de (1.1) dans certains espaces fonctionnels. L'intégrale stochastique $\int_t^T Z_s dX_s$ a souvent été définie comme une intégrale de divergence dans le cadre du calcul de Malliavin et la notion d'espérance quasi-conditionnelle a été introduite, étant donné que la notion classique d'espérance conditionnelle ne semble pas convenir à la preuve de l'existence et de l'unicité des solutions aux EDSR dont le processus n'est pas une martingale ([26], Lemme 1.2.5). Très peu d'articles traitent des EDSR pour des processus gaussiens plus généraux ([8], [9], [14]). Dans [8] l'intégrale stochastique $\int_t^T Z_s dX_s$ a été définie dans le sens Wick-Itô, et l'existence et l'unicité de la solution (1.1) a été prouvée pour une classe de processus gaussiens incluant le mouvement brownien fractionnaire. La preuve est basée sur un théorème de transfert qui vise à réduire la question de l'existence et de l'unicité de la solution à l'EDSR dirigée par le mouvement brownien. Dans [9], les auteurs ont montré qu'on peut supposer la condition terminale de carré intégrable si X est une martingale.

La principale motivation est l'étude de l'existence, de l'unicité et de la régularité de la solution des equations différentielles stochastiques rétrogrades dirigées par un processus de Volterra.

Dans cette thèse, nous avons généralisé le résultat de C. Bender [7] où l'auteur a donné la forme explicite de la solution de l'EDSR linéaire dirigée par un mouvement brownien fractionnaire dans le cas uni-dimensionnel. Dans notre travail, nous avons généralisé ce résultat au cas multi-dimensionnel et aux processus de Volterra. Puis, nous avons généralisé le résultat de L. Maticiuc et T. Nie [25] sur l'existence et l'unicité de la solution de l'EDSR non linéaire unidimensionnelle aux processus de Volterra.

Pour mettre en lumière les points importants, nous présentons dans cette introduction des énoncés simplifiés de nos résultats. Des énoncés plus complets sont exposés dans les différents chapitres présentés ci-après.

1.3 Présentation des résultats obtenus

Nous résumons les résultats obtenus et les notions que nous avons utilisé dans cette thèse.

1.3.1 Compléments sur le calcul de Malliavin pour un processus de Volterra

Soit $W = (W^1, ..., W^n)$ un mouvement brownien standard dans $\mathbb{R}^n$, défini sur l'espace de probabilité $(\Omega, \mathcal{F}, P)$ et soit $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, t \in [0, T]\}$ la filtration engendrée par W. On considère le processus de Volterra $X = (X^1,..., X^n)$ donné par

$$X_t^j = \int_0^t K^j(t, s)dW_s^j, \, j = 1,..., n, \tag{1.2}$$

où $K^j : [0,T]^2 \to \mathbb{R}$ est de carré intégrable et nous supposons K^j est du type de Volterra c'est-à-dire $K^j(t, s) = 0$, pour tout $t, s \in [0, T]$ tel que $s > t$.
K^j satisfait aux conditions suivantes:

(H1) $K^j(t, s)$ est continue pour tous $0 < s \leqslant t < T$ et continuement dérivable par rapport à t, pour tous $0 < s < t < T$,

(H2) Ils existent des constantes $0 < \alpha_j, \beta_j < \frac{1}{2}$ et $c_j > 0$ tel que

$$\left|\frac{\partial K^j}{\partial t}(t, s)\right| \leqslant c(t - s)^{\alpha_j - 1}\left(\frac{t}{s}\right)^{\beta_j}, \text{pour tous } 0 < s < t < T.$$

Le noyau K donné par (1.2) définit un opérateur linéaire dans $L^2([0,T])$ donné par $(K\sigma)_t = \int_0^t K(t, s)\sigma_s ds$, $\sigma \in L^2([0,T])$. Soit $\mathcal{E}$ l'ensemble des fonctions étagées sur $[0,T]$, et soit $K_T^* : \mathcal{E} \to L^2([0,T])$ défini par

$$(K_T^*\sigma)_u := \int_u^T \sigma_s \frac{\partial K}{\partial s}(s, u)ds. \tag{1.3}$$

L'opérateur K_T^* est l'adjoint de K ([5], Lemma 1).

Soit $\mathcal{H}$ la fermeture des fonctions indicatrices $1_{[0,t]}$, $t \in [0,T]$ par rapport au semi-produit scalaire

$$< 1_{[0,t]}, 1_{[0,s]} >_{\mathcal{H}} := < K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]} >_{L^2([0,T])} .$$

L'opérateur K_T^* est une isométrie entre $\mathcal{H}$ et un sous-espace de $L^2([0,T])$, et $\| \cdot \|_{\mathcal{H}}$ est une semi-norme sur $\mathcal{H}$. Soient

$$\phi(r,s) := \int_0^{inf(r,s)} \frac{\partial K}{\partial r}(r,t)\frac{\partial K}{\partial s}(s,t)dt,\ r \neq s,$$

$$\widetilde{\phi}(r,s) := \int_0^{inf(r,s)} \Big|\frac{\partial K}{\partial r}(r,t)\Big|\Big|\frac{\partial K}{\partial s}(s,t)\Big|dt,\ r \neq s.$$

Soit $|\ \mathcal{H}\ |$ la fermeture des fonctions indicatrices par rapport à la semi-norme

$$\| \varphi \|^2_{|\mathcal{H}|} = \int_0^T \left(\int_t^T |\ \varphi_r\ | \Big|\frac{\partial K}{\partial r}(r,t)\Big| dr \right)^2 dt = 2\int_0^T dr \int_0^r ds \widetilde{\phi}(r,s)\ |\ \varphi_r\ ||\ \varphi_s\ | .$$

Soit $\mathcal{S}$ l'ensemble des variables aléatoires de la forme $F = f(X(\varphi_1),....,X(\varphi_n))$, où $n \geqslant 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f et ses dérivées sont bornées) et $\varphi_1,...,\varphi_n \in \mathcal{H}$. La dérivée au sens de Malliavin de F est donnée par

$$D^X F := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(X(\varphi_1),....,X(\varphi_n))\varphi_j,$$

$D^X : L^p(\Omega) \longrightarrow L^p(\Omega;\mathcal{H})$ pour tout $p \geqslant 1$. Nous notons $\mathbb{D}^X_{1,p}$ la fermeture de $\mathcal{S}$ par rapport à la norme

$$\|F\|^p_{1,p} = \mathbb{E}|F|^p + \mathbb{E}\|D^X F\|^p_{\mathcal{H}}. \tag{1.4}$$

Nous appelons $Dom(\delta^X)$ l'ensemble des $u \in L^2(\Omega,\mathcal{H})$ pour lesquels il existe une constante $c > 0$ tel que

$$\Big|\mathbb{E}\Big[< D^X F, u >_{\mathcal{H}} \Big]\Big| \leqslant c\sqrt{\mathbb{E}[F^2]}, \text{pour tous } F \in \mathbb{D}^X_{1,2}.$$

Soit $\mathcal{S}^{\mathcal{H}}$ la classe de variables aléatoires tel que $u = \sum_{j=1}^n F_j h_j$ ($F_j \in \mathcal{S}$, $h_j \in \mathcal{H}$).

$\mathbb{D}^X_{1,p}(|\ \mathcal{H}\ |)$ est la complétion de $\mathcal{S}^{|\mathcal{H}|}$ sous la semi-norme

$$\| u \|^p_{1,p,|\mathcal{H}|} := \mathbb{E}\ \| u \|^p_{|\mathcal{H}|} + \mathbb{E}\ \| D^X u \|^p_{|\mathcal{H}|\otimes|\mathcal{H}|} ,$$

où

$$\| D^X u \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \int_{[0,T]^4} |\ D^X_s u_t\ ||\ D^X_{t'} u_{s'}\ |\ \widetilde{\phi}(s,s')\widetilde{\phi}(t,t')dsdtds'dt'. \tag{1.5}$$

L'espace $\mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$ est inclus dans le domaine de δ^X, et on a pour tout $u \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$,

$$\mathbb{E}\Big(\delta^X(u)^2\Big) \leqslant \mathbb{E}\ \| u \|^2_{|\mathcal{H}|} + \mathbb{E}\ \| D^X u \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} .$$

Pour $F \in \mathcal{S}$, soit

$$\mathbb{D}^X_s F := \int_0^T \phi(s,t) D^X_t F dt, \quad s \in [0,T]. \tag{1.6}$$

1.3.2 Formule d'Itô

Nous généralisons la formule d'Itô de [5] au cas multi-dimensionnel. Afin de présenter la formule d'Itô, on a besoin des définitions suivantes:

Pour $j = 1,...,n$, soit σ^j une fonction bornée sur $[0,T]$ et $b^j \in \mathcal{C}^1((0,T),\mathbb{R}) \cap \mathcal{C}([0,T],\mathbb{R})$. Le processus $N := (N^1,...,N^n)$ est défini par

$$N_t^j = b_t^j + \int_0^t \sigma_s^j \delta X_s^j,\ t \in [0,T], j = 1,...,n. \tag{1.7}$$

Soit $F \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ et supposons que

$$\max\left\{\left|F(t,x)\right|, \left|\frac{\partial F}{\partial t}(t,x)\right|, \left|\frac{\partial F}{\partial x_j}(t,x)\right|, \left|\frac{\partial^2 F}{\partial x_j^2}(t,x)\right|, j = 1,...,n\right\} \leqslant ce^{\lambda|x|^2} \tag{1.8}$$

pour tout $t \in [0,T]$ et $x \in \mathbb{R}^n$, où c, λ sont des constantes positives telles que $\lambda < \frac{1}{4}\min_i(\sup_{t\in[0,T]} Var(N_t^i))^{-1}$. Cela implique

$$\begin{aligned}\mathbb{E}\left|F(t,N_t)\right|^2 &\leqslant c^2\mathbb{E}exp(2\lambda|N_t|^2) \\ &= \frac{c^2}{(2\pi)^{n/2}}\prod_{j=1}^n \frac{1}{(Var(N_t^j))^{1/2}}\int exp\left(2\lambda x_j^2 - \frac{1}{2}\frac{(x_j - b_t^j)^2}{Var(N_t^j)}\right)dx_j < \infty,\end{aligned}$$

et la même propriété est valable pour $\partial/\partial t F(t,x)$, $\partial/\partial x_j F(t,x)$ et $\partial^2/\partial x_j^2 F(t,x)$, $j = 1,...,n$.

Maintenant, nous énonçons la formule d'Itô:

Théorème 4.2.1 *Soit N donné par (1.7). Supposons que, pour $j = 1,...,n$, le noyau K^j de X^j satisfait (H1) et (H2), que $b^j \in \mathcal{C}^1((0,T),\mathbb{R}) \cap \mathcal{C}([0,T],\mathbb{R})$, et que $\sigma = \{\sigma_t^j, t \in [0,T]\}$ est bornée. Si $F \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ satisfait (1.8), alors $\partial/\partial x_j F(\cdot, N_\cdot) \in \mathbb{D}_{1,2}^{X^j}(|\ \mathcal{H}^j|)$, $j = 1,...,n$ et, pour tout $t \in [0,T]$,*

$$\begin{aligned}F(t,N_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial s}(s,N_s)ds + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial x_j}(s,N_s)\left(\frac{d}{ds}b_s^j ds + \sigma_s^j \delta X_s^j\right) \\ &+ \frac{1}{2}\sum_{j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_j^2}(s,N_s)\frac{d}{ds}Var(N_s^j)ds.\end{aligned} \tag{1.9}$$

Dans [5], les auteurs ont étudié la formule d'Itô dans le cas uni-dimensionnel. Notre motivation dans cette partie est de donner une généralisation de la formule d'Itô au cas multi-dimensionnel pour une fonction F qui satisfait (1.8).

1.3.3 Résolution de l'EDSR linéaire

Nous montrons l'existence de la solution de l'EDSR linéaire en la ramenant à l'EDP associée, puis nous présentons la forme explicite de la solution de l'EDSR linéaire dirigée par le processus de Volterra donné par (1.2). Ce résultat généralise au cas multi-dimensionnel et aux processus de Volterra le résultat de C. Bender [7].

Pour $t_0 \geqslant 0$ notons par $\mathbb{L}^2(\mathbb{F}, \mathbb{R}^n)$ l'ensemble des processus Z $\mathbb{F}$-adaptés et à valeurs dans $\mathbb{R}^n$ tel que $\mathbb{E}(\int_{t_0}^T \mid Z_t \mid^2 dt) < \infty$. Nous considérons l'EDSR linéaire pour les processus $Y = (Y_t, t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R})$ et $Z = ((Z_t^1,..., Z_t^n), t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^n)$ donnée par

$$Y_t = g(N_T) - \int_t^T [\tilde{f}(s) + A_1(s)Y_s - A_2(s)Z_s]ds + \int_t^T Z_s \delta X_s, \ t \in [t_0,T], \tag{1.10}$$

où les fonctions g, $\tilde{f}$, A_1 et $A_2 := (A_2^1,..., A_2^n)$ sont supposées connues et l'intégrale $\int_t^T Z_s \delta X_s$ est définie comme une intégrale de divergence. (1.10) est associé à l'EDP linéaire du second ordre suivante avec condition terminale

$$\frac{\partial u}{\partial t}(t,x) = -\frac{1}{2}\sum_{j=1}^n \frac{d}{dt}Var(N_t^j)\frac{\partial^2 u}{\partial x_j^2}(t,x) + \sum_{j=1}^n [\sigma_t^j A_2^j(t) - \frac{d}{dt}b_t^j]\frac{\partial u}{\partial x_j}(t,x) + A_1(t)u(t,x) + \tilde{f}(t), \tag{1.11}$$

$$u(T,x) = g(x), \ (t,x) \in [t_0, T) \times \mathbb{R}^n.$$

En appliquant la formule d'Itô (Théorème 4.2.1), nous montrons que

$$Y_t := u(t, N_t) \text{ and } Z_t^j := -\sigma_t^j \frac{\partial u}{\partial x_j}(t, N_t), \ j = 1,..., n \tag{1.12}$$

est une solution de (1.10).

Avant d'énoncer le résultat qui nous donne la forme explicite de la solution de l'EDP, nous faisons les hypothèses suivantes:

(H3) Il existe des constantes $c, C > 0$ tels que $c < \sigma^j < C$, $j = 1,..., n$, and A_1,$\tilde{f}$, A_2 sont bornées.

(H4) g est continue, et il existe des constantes positives c' et $\lambda' < \min_{j=1,...,n}(16 \sup_{t\in[0,T]} Var(N_t^j))^{-1}$ tel que $\mid g(x) \mid \leqslant c' e^{\lambda' |x|^2}$ pour tout $x \in \mathbb{R}^n$.

(H5) $d/dtVar(N_t^j) > 0$ pour tout $t \in [t_0, T]$ et $\int_{t_0}^T (Var(N_T^j) - Var(N_t^j))^{-1/2} dt < \infty$, $j = 1,..., n$.

L'hypothèse (H5) est discutée à la fin du Chapitre 5.

Maintenant, nous énonçons le théorème qui donne la forme explicite de la solution de l'EDP:

Théorème 5.2.1 *Nous supposons (H1)-(H5), et soit* $v(t,z) = (2\pi t)^{-1/2}\exp(-z^2/2t)$. *Alors la solution de (1.10)* $(Y,(Z^1,...,Z^n))$ *est donnée par* (1.12) *, avec*

$$u(t,x) = -\int_t^T \exp\Big(\int_s^t A_1(r)dr\Big)\tilde{f}(s)ds + \exp\Big(-\int_t^T A_1(s)ds\Big) \times \int_{\mathbb{R}^n} g(y)\prod_{j=1}^n v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T (\sigma_s^j A_2^j(s) - \frac{d}{ds}b_s^j)ds - y_j\Big)dy. \tag{1.13}$$

Ce théorème se démontre en appliquant le Lemme 5.2.2 qui assure que l'hypothèse (1.8) du Théorème 4.2.1 est satisfaite pour F remplacé par u. La preuve du Théoreme 5.2.1 s'appuie sur un raisonnement similaire à celui de C. Bender [7], pour montrer la solution explicite de l'EDP (1.11) associée à l'EDSR (1.10).
L'unicité de la soluion de l'EDSR (1.10) suit de l'unicité de la solution de l'EDP (1.11) associée.

Lemme 5.2.2 *Soit* u *donné par* (1.13). *Alors, ils existent des constantes positives* M *et* $\lambda < \min_{j=1,...n}(4\sup_{t\in[0,T]} Var(N_t^j))^{-1}$ *tel que*

1) $\mid u(t,x) \mid \leqslant Me^{\lambda|x|^2}$, $(t,x) \in [t_0,T]\times\mathbb{R}^n$,

2) $\left|\frac{\partial u}{\partial x_j}(t,x)\right| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1/2}e^{\lambda|x|^2}$, $(t,x)\in[t_0,T)\times\mathbb{R}^n$,

3) $\left|\frac{\partial^2 u}{\partial x_j^2}(t,x)\right| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1}e^{\lambda|x|^2}$, $(t,x)\in[t_0,T)\times\mathbb{R}^n$.

La solution u de l'EDP satisfait donc la condition de croissance exponentielle. Elle satisfait la même hypothèse (H4) que la condition terminale de l'EDSR. Nous pouvons démontrer le lemme en appliquant la condition sur g.

1.3.4 EDSR non-linéaire au cas uni-dimensionnel

D'abord nous revisitons le lien entre les EDSR et l'EDP associée dans le cas où f n'est pas nécessairement linéaire. Dans un second temps, nous présentons les outils probabilistes pour prouver l'existence et l'unicité de la solution de l'EDSR non linéaire.
Nous considérons le processus de Volterra défini sur l'espace de probabilité $(\Omega,\mathcal{F},\mathbb{P})$, et donné par $X_t = \int_0^t K(t,s)dW_s$, $t\in[0,T]$. Soit $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, t\in[0,T]\}$ la filtration engendrée par X sous $\mathbb{P}$. Le processus N est donné par

$$N_t = \int_0^t \sigma_s \delta X_s,\ t\in[0,T]. \tag{1.14}$$

(1.14) est un cas particulier de (1.7) (pour $n = 1$ et $b = 0$).
Nous supposons que la fonction σ est bornée sur $[0, T]$ et que $\frac{d}{dt}Var(N_t)$ est strictement positive. Nous considérons l'EDSR non linéaire pour les processus $Y = (Y_t,\ t \in [t_0, T])$ et $Z = (Z_t,\ t \in [t_0, T])$ donnée par

$$Y_t = g(N_T) + \int_t^T f(s, N_s, Y_s, Z_s)ds + \int_t^T Z_s \delta X_s,\ \ t \in [t_0, T]. \tag{1.15}$$

(1.15) est associé à l'EDP du second ordre avec la condition terminale:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2 u}{\partial x^2}(t,x) - f(t,x,u(t,x), -\sigma_t \frac{\partial u}{\partial x}(t,x)), \\ u(T,x) = g(x),\ (t,x) \in [t_0, T] \times \mathbb{R}. \end{cases} \tag{1.16}$$

Supposons que (1.16) admet une solution classique u qui satisfait (1.8). En appliquant la formule d'Itô (Théorème 4.2.1) pour $u(t, N_t)$, nous obtenons

$$du(t, N_t) \ = \ \Big[\frac{\partial u}{\partial t}(t, N_t)dt + \frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2 u}{\partial x^2}(t, N_t)\Big]dt + \sigma_t \frac{\partial u}{\partial x}(t, N_t)\delta X_t.$$

Puis, nous remplaçons $\frac{\partial u}{\partial t}(t, N_t)$ par l'expression de l'EDP (1.16). En posant $Y_t = u(t, N_t)$ et $Z_t = -\sigma_t \frac{\partial u}{\partial x}(t, N_t)$, nous obtenons que (Y, Z) est une solution de l'EDSR

$$du(t, N_t) = -f(t, N_t, u(t, N_t), -\sigma_t \frac{\partial u}{\partial x}(t, N_t))dt + \sigma_t \frac{\partial u}{\partial x}(t, N_t)\delta X_t.$$

Ceci démontre l'existence de la solution de l'EDSR (1.15) en l'associant à l'EDP (1.16).

La notion de l'espérance quasi-conditionnelle est introduite dans la Section 6.3.2. Ce type d'espérance s'exprime en fonction de la décomposition en chaos. La notion de l'espérance conditionnelle classique ne nous permet pas de prouver l'existence et l'unicité de la solution de l'EDSR (1.15) si X n'est pas une martingale. Maintenant, le lemme suivant donne une formule très pratique en utilisant la notion de l'espérance quasi-conditionnelle et sera appliqué dans la démonstration de l'existence de la solution de l'EDSR (1.15).

Lemme 6.4.7 *Soit $s, t \in [0, T]$ tel que $0 \leqslant t < s \leqslant T$. Soit k une fonction réelle continue à croissance polynomiale et nous supposons que $Var(N_\cdot)$ est croissante. Alors $F := k(N_s) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ et*

$$\widehat{\mathbb{E}}[F|\mathcal{F}_t] = P_{Var(N_s)-Var(N_t)}k(N_t), \quad 0 \leqslant t < s \leqslant T. \tag{1.17}$$

où $P_t k(x) = \int_{\mathbb{R}} p_t(x-y)k(y)dy$ avec $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t},\ t > 0$.

La preuve de ce lemme se base sur la propriété de semi-groupe de P. Ce lemme joue un rôle important dans la preuve de la Proposition 7.1.1.

Nous donnons une formule de type de Clark-Ocone qui nous permet de construire la solution (Y_t, Z_t) de l'EDSR (1.18).

Proposition 6.4.13 *Soit* $F \in \mathbb{D}^X_{1,2}$ *et nous supposons que* $\widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t\Big] \in \mathbb{D}^X_{1,2}$, *pour tout* $t \in [0,T]$. *Alors* $F = \mathbb{E}[F] + \int_0^t \widehat{\mathbb{E}}\Big[D^X_s F \mid \mathcal{F}_s\Big]\delta X_s$.

Cette proposition généralise aux processus de Volterra la formule de Clark-Ocone pour le brownien.

Soit $\mathcal{C}^{k,l}_{pol}([0,T]\times\mathbb{R}^m)$ l'espace des fonctions $h \in \mathcal{C}^{k,l}$ sur $[0,T]\times\mathbb{R}^m$ tel que h et ses dérivées sont à croissance polynomiale.
Nous introduisons l'espace suivant:

$$\mathcal{V}_T = \{Y_\cdot = w(.,N_\cdot); w \in \mathcal{C}^{1,3}_{pol}([0,T]\times\mathbb{R}) \text{ avec } \frac{\partial w}{\partial t} \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R})\}.$$

Nous donnons la formule d'Itô la plus importante dans notre contexte.

Théorème 6.5.1 *Soit* $u \in \mathcal{V}_T$ *et* $f \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R})$. *Nous notons* $f_s = f(s,N_s)$, $s \in [0,T]$. *Soit*

$$C_t = C_0 + \int_0^t f_s ds + \int_0^t u_s \delta X_s, t \in [0,T].$$

Alors
i) $uC1_{[0,s]} \in Dom(\delta^X)$,
ii) $C_s \in \mathbb{D}^X_{1,2}$, $s \in [0,T]$,
iii) $(\mathbb{D}^X_s C_s)_{s\in[0,T]} \in L^2([0,T]\times\Omega)$,
et

$$C_t^2 = C_0^2 + 2\int_0^t C_s f_s ds + 2\int_0^t C_s u_s \delta X_s + 2\int_0^t u_s \mathbb{D}^X_s C_s ds, \text{ a.s. } t \in [0,T].$$

Cette formule d'Itô s'applique au carré de processus qui sont la somme d'une intégrale de divergence et d'une intégrale de Lebesgue. Ce résultat est très utile dans la preuve de l'existence et l'unicité de la solution de l'EDSR (1.15). La partie difficile de la démonstration est la formule suivante pour le carré de l'intégrale $M_t = \int_0^t u_s \delta X_s$.

$$M_t^2 = 2\int_0^t M_s u_s \delta X_s + 2\int_0^t u_s \mathbb{D}^X_s M_s ds, t \in [0,T].$$

Sa démonstration est basée essentiellement sur les propriétés du calcul de Malliavin.

Les hypothèses qui suivent servent à démontrer l'existence et l'unicité de la solution de (1.15) sans faire référence à l'EDP associée. Nous formulons maintenant les hypothèses sur σ, f, g dans (1.15)

(H6) $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ est une fonction continue déterministe telle que $\sigma_t \neq 0$, pour tout $t \in [0, T]$. Soit

$$\hat{\sigma}_t = \int_0^t \phi(t, r)\sigma_r dr, t \in [0, T].$$

De plus, nous supposons qu'il existe $\epsilon : [0, T] \longrightarrow \mathbb{R}_+$ continue tel que $m\epsilon(s) \leqslant \frac{\hat{\sigma}_s}{\sigma_s} \leqslant M\epsilon(s), s \in [0, T]$ pour certaines constantes $m, M > 0$ et tel que $\frac{1}{\epsilon}$ est intégrable sur $[t, T], t > 0$.

(H7) La fonction $f : [0, T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ appartient à l'espace $\mathcal{C}^{0,1}_{pol}([0, T] \times \mathbb{R}^3)$. Nous supposons en plus qu'il existe une constante L tel que, pour tout $t \in [0, T], x, y_1, y_2, z_1, z_2 \in \mathbb{R}$,

$$\mid f(t, x, y_1, z_1) - f(t, x, y_2, z_2) \mid \leqslant L(\mid y_1 - y_2 \mid + \mid z_1 - z_2 \mid),$$

et

$$\mathbb{E}\Big[\int_0^T \mid f(t, N_t, 0, 0) \mid^2 dt\Big] < \infty.$$

(H8) $\xi = g(N_T)$, où $g \in \mathcal{C}^1_{pol}(\mathbb{R})$.

L'hypothèse (H6) est discutée dans la Remarque 6.3.1.

Soit $\tilde{\mathcal{V}}_{T,\epsilon}$ la complétion de $\mathcal{V}_T$ sous la norme suivante

$$\| Y \|^2_\epsilon = \int_0^T \epsilon(t)\mathbb{E} \mid Y_t \mid^2 dt.$$

Pour $\epsilon = 1$ on a la norme de $L^2([0, T] \times \Omega)$.

1.3.5 Existence et unicité de la solution de l'EDSR non lineaire

Nous supposons que la variance de N_t soit croissante. Les résultats qui suivent généralisent à des processus de Volterra les résultats de L. Maticiuc et T. Nie [25].

Existence

Nous commençons par considérer l'équation suivante:

$$Y_t = \xi + \int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds + \int_t^T Z_s \delta X_s, 0 \leqslant t_0 \leqslant t \leqslant T. \tag{1.18}$$

où $\chi, \psi \in \mathcal{V}_T$.

Nous démontrons l'existence et l'unicité de la solution de (1.18) par la proposition suivante:

Proposition 7.1.1 *Sous les hypothèses (H1), (H2), (H6), (H7), (H8),* (1.18) *a une unique solution* $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$. *De plus, si* K_T^* *est injective, la solution a la forme suivante*

$$(Y_t, Z_t) = (u(t, N_t), -\sigma_t \frac{\partial}{\partial x} u(t, x), \textit{où } u \in \mathcal{V}_T.$$

Cette proposition se démontre comme suit:
Dans un premier temps, nous avons construit la solution de l'EDSR (Y, Z) à l'aide de la formule de Clark-Ocone, puis nous avons prouvé que $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$.
Dans un deuxième temps, sous l'hypothèse de l'injectivité de K_T^* la partie Z de la solution est donnée par $Z_t = -\sigma_t \frac{\partial}{\partial x} u(t, N_t)$ où $Y_t = u(t, N_t)$.

Avant d'énoncer le résultat principal, nous donnons une estimation qui sera utile dans la suite. Elle permettra de construire une suite de Cauchy et de démontrer l'existence de la solution de (1.15).

Proposition 7.1.2 *Supposons que* K_T^* *est injective et que les hypothèses (H1), (H2), (H6), (H7), (H8) sont satisfaites. Pour tous* $(U, V) \in \mathcal{V}_T \times \mathcal{V}_T$, *soit* $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$, *l'unique solution de l'EDSR*

$$Y_t = \int_t^T f(s, N_s, U_s, V_s) ds + \int_t^T Z_s \delta X_s,\ 0 \leqslant t \leqslant T.$$

Alors, pour tout $\beta > 0$, *il existe* $C(\beta) \in \mathbb{R}_+$ *(dépendant de* L, T *et* ϵ *de (H6)) tel que*

$$\begin{aligned} &\sup_{t\in[0,T]} e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 + \int_0^T e^{2\beta s}\mathbb{E} \mid Y_s \mid^2 ds + \int_0^T e^{2\beta s}\epsilon(s)\mathbb{E}Z_s^2 ds \\ &\leqslant C(\beta)\Big(\int_0^T e^{2\beta s}\mathbb{E} \mid U_s \mid^2 ds + \int_0^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid V_s \mid^2 ds + \int_0^T e^{2\beta s}\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2 ds \Big). \end{aligned} \tag{1.19}$$

De plus, la constante $C(\beta)$ *pourra être choisie de telle sorte que* $\lim_{\beta\to\infty} C(\beta) = 0$.

Cette estimation se démontre en appliquant le Théorème 6.5.1 à $\mathbb{E}e^{2\beta t} \mid Y_t \mid^2$ et en utilisant la condition de Lipschitz sur f et (H6).

Maintenant, nous énonçons le théorème qui assure l'existence de la solution de l'EDSR (1.15).

Théorème 7.1.3 *Supposons que K_T^* est injective. Sous les hypothèses (H1), (H2), (H6), (H7), (H8), l'EDSR*

$$Y_t = \xi + \int_t^T f(s, N_s, Y_s, Z_s)ds + \int_t^T Z_s \delta X_s,\ 0 < t_0 \leqslant t \leqslant T. \tag{1.20}$$

a une solution $(Y, Z) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$.

La preuve du théorème est basée sur l'estimation (1.19) en appliquant le principe du point fixe.

Unicité

Avant d'énoncer le résultat principal de l'unicité, nous introduisons les espaces suivants:

$\mathcal{M} = \{K/K_t = K_T + \int_t^T v_s ds + \int_t^T u_s \delta X_s, t \in [0,T]$ avec $u \in \mathcal{V}_T, v_s = v(s, N_s)$, où $v \in \mathcal{C}_{pol}^{0,1}([0,T] \times \mathbb{R}\}$.
Soit $\mathcal{S}_f$ l'ensemble des $(Y, Z) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$ tel que, pour tout $t \in (0,T]$,

$$(i)\ \mathbb{E} \mid Y_t \mid^2 + 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s} \mathbb{E} \mid Z_s \mid^2 ds = \mathbb{E} \mid \xi \mid^2 + 2\int_t^T \mathbb{E}\Big[Y_s f(s, N_s, Y_s, Z_s)\Big] ds.$$

$$(ii)\ \mathbb{E}[Y_t K_t] - \mathbb{E}\int_t^T \Big[-\frac{\hat{\sigma}_s}{\sigma_s} u_s + \mathbb{D}_s^X K_s\Big] Z_s ds = \mathbb{E}[\xi . K_T] + \mathbb{E}\int_t^T \Big[K_s f(s, N_s, Y_s, Z_s) + Y(s) v_s\Big] ds, K \in \mathcal{M}.$$

Si Y est une solution de (1.15) et que nous appliquons la formule d'Itô (Théorème 6.5.1) à Y^2, nous obtenons (i). De plus, si K est la solution de l'équation du même type que (1.15), une application de la formule d'Itô (Théorème 6.5.1) à K nous donne (ii). $\mathcal{S}_f$ semble donc un choix naturel pour l'espace des solutions de (1.15).

Théoreme 7.2.2 *Supposons que K_T^* est injective. Sous les hypothèses (H1), (H2), (H6), (H7), (H8), l'EDSR (1.20) admet une unique solution $(Y, Z) \in \mathcal{S}_f$.*

La preuve du théorème s'appuie sur la formule d'Itô (Théorème 6.5.1). Pour démontrer l'unicité nous supposons qu'il existe deux solutions (Y, Z) et $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}_f$, et nous appliquons la formule d'Itô (Théorème 6.5.1) à $Y - \tilde{Y}$. Ensuite, nous appliquons l'inégalité de Gronwall pour conclure.

1.4 Prépublications

Le contenu de cette thèse a fait l'objet des prépublications suivantes:

H. KNANI, M. DOZZI ([22]) Linear Backward Stochastic Differential Equations with Gaussian Volterra processes. Soumis pour publication.

H. KNANI ([21]) Partial differential equations associated to non-linear backward stochastic differential equations with Gaussian Volterra processes. Soumis pour publication.

Un autre papier sur l'EDSR non linéaire est en cours de préparation:

H. KNANI (2020) Non-linear backward stochastic differential equations with Gaussian Volterra processes. (En préparation)

1.5 Conclusions

Dans le *cas linéaire*, l'existence d'une solution classique de l'EDP (1.11) associée à l'EDSR (1.10) est démontrée en donnant la formule explicite de la solution sous les hypothèses (H1)-(H5). La formule d'Itô donne le lien entre les solutions de l'EDSR et de l'EDP associée, sous la condition de croissance exponentielle de la solution de l'EDP. Cette condition est satisfaite grâce à l'hypothèse que la condition terminale satisfait la même hypothèse de croissance.

La question de l'existence et de l'unicité de la solution est plus complexe dans le *cas non-linéaire*. Nous étudions deux approches: (i) avec l'EDP associée, (ii) par des méthodes probabilistes sans faire référence à l'EDP associée. Dans les deux cas on ne peut pas (pour le moment) se passer de l'injectivité de K_T^* (Lemme 6.1.6).

(i) On suppose qu'il existe une solution classique u de l'EDP (1.16) associée, et on en déduit une solution de l'EDSR (1.15). Sous l'hypothèse de l'injectivité de K_T^* la partie Z de la solution est donnée par $Z_t = -\sigma_t \frac{\partial}{\partial x} u(t, N_t)$, où $Y_t = u(t, N_t)$. La question de l'unicité n'est pas traitée en détail, elle peut être traitée pour l'EDP associée. L'hypothèse de l'injectivité a été discutée dans la Remarque 6.1.7. Le théorème de comparaison donné en Section 7.3 s'inscrit dans ce contexte.

(ii) La preuve se fait en 2 étapes: en un premier temps dans $\mathcal{V}_T$ pour l'équation préliminaire (7.1). Cette solution a la même structure que la solution obtenue par l'EDP associée et nécessite l'injectivité de K_T^* seulement pour la preuve que $Z_t = -\sigma_t \frac{\partial}{\partial x} u(t, N_t)$, $u \in \mathcal{V}_T$.
La seconde étape est l'existence et l'unicité de la solution $(Y, Z) \in \mathcal{V}_{T,1} \times \mathcal{V}_{T,\epsilon}$ de l'EDSR (1.15) sous l'hypothèse d'injectivité de K_T^*, et l'unicité est démontrée dans la classe de processus don-

née par $\mathcal{S}_f$ (Section 7.2). On connaît donc l'intégrabilité, mais pas la régularité de Y et Z. Cela fait penser à une solution non classique (éventuellement de viscosité) de l'EDP associée.

Pour les deux parties (i) et (ii) on suppose que le générateur fait partie de $C^{0,1}_{pol}([0,T]\times\mathbb{R}^3)$ et qu'il est Lipschitz en les 2 dernières variables (uniformément dans les deux premières). Nous ne pensons donc pas qu'une des deux méthodes (i) et (ii) est plus performante que l'autre.
Jusqu'à présent les EDSR ont été étudiées principalement pour le brownien et le brownien fractionnaire. L'objectif principal de ce travail est d'étendre cette étude à une classe de processus gaussiens de Volterra.

1.6 Perspectives

Concernant les perspectives apparaissant de ce travail, nous pouvons envisager les questions suivantes:
1) Remplacer $N_t = \int_0^t \sigma_s \delta X_s + \int_0^t b_s ds$ par une l'EDS suivante:

$$N_t = N_0 + \int_0^t \sigma(N_s)\delta X_s + \int_0^t b(N_s)ds,$$

avec σ, b linéaires. Résoudre l'EDS explicitement et résoudre l'EDSR appelée forward-backward en anglais.
Les EDS non linéaires n'ont pas encore été traitées pour les processus de Volterra.

2) Étude de l'application des EDSR dirigées par des processus de Volterra dans le contrôle stochastique (optimisation stochastique). Étude des applications plus récentes en biologie et en actuariat, traitées jusqu'à présent pour les EDSR dirigées par un brownien seulement.

3) Étude des EDSR "mixtes", c'est-à-dire les EDSR dirigées par un brownien et un processus fractionnaire simultanément. Ces deux processus modélisent des aléas différents dans l'évolution du système décrit par l'EDSR.

4) La solution numérique des EDSR dans le cas non linéaire, en un premier temps avec la solution classique de l'EDP associée.

5) Affaiblir les conditions sur le générateur dans (ii). L'existence d'une solution classique de l'EDP associée n'est alors plus garantie. Remplacer par exemple la condition de Lipschitz par une condition de monotonie. Ceci a été fait pour les EDSR dirigées par un brownien seulement.

Chapter 2

Gaussian Volterra processes

In this Chapter, we introduce a class of Gaussian Volterra processes and give some examples, including multifractional Brownian motion and multifractional Ornstein-Uhlenbeck processes.

2.1 Definition

Let $X = \{X_t, 0 \leqslant t \leqslant T\}$ be a zero mean continuous Gaussian process given by

$$X_t = \int_0^T K(t,s)dW_s \tag{2.1}$$

where $W = \{W_t, 0 \leqslant t \leqslant T\}$ is a standard Brownian motion and $K : [0,T]^2 \to \mathbb{R}$ is a square integrable kernel, i.e. $\int_{[0,T]^2} K(t,s)^2 dtds < +\infty$. We assume that K is of Volterra type, i.e, $K(t,s) = 0$ whenever $t < s$. Usually, the representation (2.1) is called a Volterra representation of X. Gaussian Volterra processes and their stochastic analysis have been studied e.g. in [5], [12] and [31]. In [5] K is called *regular* if it satisfies

(H) For all $s \in (0,T]$, $\int_0^T \mid K \mid ((s,T],s)^2 ds < \infty$, where $\mid K \mid ((s,T],s)$ denotes the total variation of $K(.,s)$ on $(s,T]$.

We assume the following condition on K which is more restrictive than *(H)* ([5], [12]):

(H1) $K(t,s)$ is continuous for all $0 < s \leqslant t < T$ and continuously differentiable in the variable t in $0 < s < t < T$,

(H2) For some $0 < \alpha, \beta < \frac{1}{2}$, there is a finite constant $c > 0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leqslant c(t-s)^{\alpha-1}\left(\frac{t}{s}\right)^{\beta}, \text{for all } 0 < s < t < T.$$

The covariance function of X is given by

$$R(t,s) := \mathbb{E}X_t X_s = \int_0^{inf(t,s)} K(t,u)K(s,u)du. \tag{2.2}$$

2.2 Examples

We discuss some examples of Gaussian Volterra processes that satisfy *(H1) and (H2)*.

Example 1. The multi-fractional brownian motion (mbm) $(B_t^{h(t)}, 0 \leqslant t \leqslant T)$ with Hurst function $h : [0,T] \to [a,b] \subset (\frac{1}{2}, 1)$. Its kernel is given by [10]

$$K(t,s) = s^{1/2-h(t)} \int_s^t (y-s)^{h(t)-3/2} y^{h(t)-1/2} dy \quad \text{for all } s < t, \tag{2.3}$$

where h is assumed to be continously differentiable with bounded derivative.
Otherwise $K(t,s) = 0$.
Furthermore

$$\frac{\partial K}{\partial t}(t,s) = h'(t) s^{\frac{1}{2}-h(t)} \int_s^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} ln((\frac{y}{s}-1)y) dy + s^{\frac{1}{2}-h(t)} (t-s)^{h(t)-\frac{3}{2}} t^{h(t)-\frac{1}{2}}.$$

A straightforward calculation shows that (H2) is satisfied with $\alpha = a - \frac{1}{2}$, $\beta = b + \epsilon - \frac{1}{2}$ with ϵ small enough and c depends on a, b, T and ϵ. Let us verify (H2). Indeed,
Notice that $(\frac{y}{s}-1)y = 1$ iff $y^* = \frac{s}{2}\Big(1 + \sqrt{1+\frac{4}{s}}\Big)$.
Case 1. $(\frac{y}{s}-1)y > 1$. For $\epsilon > 0$ and small enough we get

$$\begin{aligned}
\left|\frac{\partial K}{\partial t}(t,s)\right| &\leqslant \Big|h'(t)\Big| s^{\frac{1}{2}-h(t)} \int_{y^*}^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} ln((\frac{y}{s}-1)y) dy + s^{\frac{1}{2}-h(t)} (t-s)^{h(t)-\frac{3}{2}} t^{h(t)-\frac{1}{2}} \\
&\leqslant \Big|h'(t)\Big| s^{\frac{1}{2}-h(t)} \int_{y^*}^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} ln((\frac{y}{s}-1)y) dy + s^{\frac{1}{2}-h(t)} (t-s)^{h(t)-\frac{3}{2}} t^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon \Big|h'(t)\Big| s^{\frac{1}{2}-h(t)} \int_{y^*}^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} (y-s)^{\epsilon} \Big(\frac{y}{s}\Big)^{\epsilon} dy + s^{\frac{1}{2}-h(t)} (t-s)^{h(t)-\frac{3}{2}} t^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon \Big|h'(t)\Big| \int_{y^*}^t (y-s)^{h(t)-\frac{3}{2}+\epsilon} \Big(\frac{y}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}} dy + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} (t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant C_\epsilon \Big|h'(t)\Big| \Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}} \frac{(t-s)^{h(t)-\frac{3}{2}+\epsilon}}{h(t)-\frac{1}{2}+\epsilon} (t-s) + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} (t-s)^{h(t)-\frac{3}{2}}
\end{aligned}$$

• If $t - s < 1$, we get

$$\begin{aligned}
\left|\frac{\partial K}{\partial t}(t,s)\right| &\leqslant C_\epsilon \Big|h'(t)\Big| \Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}} \frac{(t-s)^{h(t)-\frac{3}{2}}}{a-\frac{1}{2}+\epsilon} + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} (t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant C_\epsilon \Big|h'(t)\Big| \Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}} \frac{(t-s)^{h(t)-\frac{3}{2}}}{a-\frac{1}{2}+\epsilon} + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}+\epsilon} (t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant c_{\epsilon,T} \Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}} (t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant c_{\epsilon,T} \Big(\frac{t}{s}\Big)^{b+\epsilon-\frac{1}{2}} (t-s)^{a-\frac{3}{2}}.
\end{aligned}$$

Consequently, for some $0 < \alpha, \beta < \frac{1}{2}$ such that $\alpha = a - \frac{1}{2}$, $\beta = b + \epsilon - \frac{1}{2}$ and $\epsilon > 0$, there is a finite constant $c_{\epsilon,T} > 0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leqslant c_{\epsilon,T}(t-s)^{\alpha-1}\Big(\frac{t}{s}\Big)^{\beta}, \quad t \in (s,T].$$

• If $t - s > 1$, we obtain

$$\begin{aligned}
\left|\frac{\partial K}{\partial t}(t,s)\right| &\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}}\frac{(t-s)^{h(t)-\frac{3}{2}}}{a-\frac{1}{2}}(t-s)^{\epsilon+1} + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}}(t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{t}{s}\Big)^{\epsilon+h(t)-\frac{1}{2}}\frac{(t-s)^{h(t)-\frac{3}{2}}}{a-\frac{1}{2}}T^{\epsilon+1} + \Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}+\epsilon}(t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant c'_{\epsilon,T}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}+\epsilon}(t-s)^{h(t)-\frac{3}{2}} \\
&\leqslant c'_{\epsilon,T}\Big(\frac{t}{s}\Big)^{b-\frac{1}{2}+\epsilon}(t-s)^{b-\frac{3}{2}}.
\end{aligned}$$

Consequently, for some $0 < \alpha, \beta < \frac{1}{2}$ such that $\alpha = b - \frac{1}{2}$, $\beta = b + \epsilon - \frac{1}{2}$ and $\epsilon > 0$, there is a finite constant $c'_{\epsilon,T} > 0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leqslant c'_{\epsilon,T}(t-s)^{\alpha-1}\Big(\frac{t}{s}\Big)^{\beta}, \quad t \in (s,T].$$

Case 2. $0 < (\frac{y}{s} - 1)y < 1$. For $\epsilon > 0$ and small enough we get

$$\begin{aligned}
&\left|\frac{\partial K}{\partial t}(t,s)\right| \leqslant \Big|h'(t)\Big|s^{\frac{1}{2}-h(t)}\int_s^{y^*}(y-s)^{h(t)-\frac{3}{2}}y^{h(t)-\frac{1}{2}}\Big|ln((\frac{y}{s}-1)y)\Big|dy + s^{\frac{1}{2}-h(t)}(t-s)^{h(t)-\frac{3}{2}}t^{h(t)-\frac{1}{2}} \\
&\leqslant \Big|h'(t)\Big|s^{\frac{1}{2}-h(t)}\int_s^{y^*}(y-s)^{h(t)-\frac{3}{2}}y^{h(t)-\frac{1}{2}}C_\epsilon\Big[(\frac{y-s}{s})y\Big]^{-\epsilon}dy + s^{\frac{1}{2}-h(t)}(t-s)^{h(t)-\frac{3}{2}}t^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\int_s^{y^*}(y-s)^{h(t)-\frac{3}{2}-\epsilon}\Big(\frac{y}{s}\Big)^{h(t)-\frac{1}{2}-\epsilon}dy + (t-s)^{h(t)-\frac{3}{2}}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{y^*}{s}\Big)^{h(t)-\frac{1}{2}-\epsilon}\int_s^{y^*}(y-s)^{h(t)-\frac{3}{2}-\epsilon}dy + (t-s)^{h(t)-\frac{3}{2}}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}-\epsilon}\frac{(y^*-s)^{h(t)-\frac{1}{2}-\epsilon}}{h(t)-\frac{1}{2}-\epsilon} + (t-s)^{h(t)-\frac{3}{2}}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}-\epsilon}\frac{(t-s)^{h(t)-\frac{1}{2}-\epsilon}}{h(t)-\frac{1}{2}-\epsilon} + (t-s)^{h(t)-\frac{3}{2}}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}} \\
&\leqslant C_\epsilon\Big|h'(t)\Big|\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}-\epsilon}\frac{(t-s)^{h(t)-\frac{3}{2}-\epsilon}(t-s)}{a-\frac{1}{2}-\epsilon} + (t-s)^{h(t)-\frac{3}{2}}\Big(\frac{t}{s}\Big)^{h(t)-\frac{1}{2}}.
\end{aligned}$$

• If $t-s<1$, we get

$$\begin{aligned}\left|\frac{\partial K}{\partial t}(t,s)\right| &\leqslant C_\epsilon\left|h'(t)\right|\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}}\frac{(t-s)^{h(t)-\frac{3}{2}}(t-s)^{1-\epsilon}}{a-\frac{1}{2}-\epsilon}+(t-s)^{h(t)-\frac{3}{2}}\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}}\\ &\leqslant C_\epsilon\left|h'(t)\right|\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}+\epsilon}\frac{(t-s)^{h(t)-\frac{3}{2}}}{a-\frac{1}{2}-\epsilon}+(t-s)^{h(t)-\frac{3}{2}}\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}}\\ &\leqslant C_{\epsilon,T}\left(\frac{t}{s}\right)^{\epsilon+h(t)-\frac{1}{2}}(t-s)^{h(t)-\frac{3}{2}}\\ &\leqslant C_{\epsilon,T}\left(\frac{t}{s}\right)^{b+\epsilon-\frac{1}{2}}(t-s)^{a-\frac{3}{2}}.\end{aligned}$$

Consequently, for some $0<\alpha,\beta<\frac{1}{2}$ such that $\alpha=a-\frac{1}{2}$, $\beta=b+\epsilon-\frac{1}{2}$ and $\epsilon>0$, there is a finite constant $C_{\epsilon,T}>0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right|\leqslant C_{\epsilon,T}(t-s)^{\alpha-1}\left(\frac{t}{s}\right)^{\beta},\quad t\in(s,T].$$

• If $t-s>1$, we get

$$\begin{aligned}\left|\frac{\partial K}{\partial t}(t,s)\right| &\leqslant C_\epsilon\left|h'(t)\right|\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}}\frac{(t-s)^{h(t)-\frac{3}{2}}(t-s)}{a-\frac{1}{2}-\epsilon}+(t-s)^{h(t)-\frac{3}{2}}\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}}\\ &\leqslant C'_{\epsilon,T}\left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}+\epsilon}(t-s)^{h(t)-\frac{3}{2}}\\ &\leqslant C'_{\epsilon,T}\left(\frac{t}{s}\right)^{b-\frac{1}{2}+\epsilon}(t-s)^{b-\frac{3}{2}}.\end{aligned}$$

Consequently, for some $0<\alpha,\beta<\frac{1}{2}$ such that $\alpha=b-\frac{1}{2}$, $\beta=b+\epsilon-\frac{1}{2}$ and $\epsilon>0$, there is a finite constant $C'_{\epsilon,T}>0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right|\leqslant C'_{\epsilon,T}(t-s)^{\alpha-1}\left(\frac{t}{s}\right)^{\beta},\quad t\in(s,T].$$

The mbm defined by (2.3) generalizes fractional brownian motion (fbm) with Hurst index $H>1/2$. Mbm is a more flexible model than fbm since the Hölder continuity of its trajectories varies with h. The trajectories of mbm $B^{h(\cdot)}_\cdot$ are in fact locally Hölder continous of order $h(t)$ at t.

Example 2. The multi-fractional Ornstein-Uhlenbeck process $U=\{U_t,0\leqslant t\leqslant T\}$ given by $U_t=\int_0^t e^{-\theta(t-s)}dB_s^{h(s)}$, where $\theta>0$ is a parameter and B^h is the mbm of Example 1. The kernel of U is given by

$$\mathcal{K}(t,r)=\int_r^t e^{-\theta(t-s)}\frac{\partial K}{\partial s}(s,r)ds=K(t,r)-\theta\int_r^t e^{-\theta(t-s)}K(s,r)ds\quad\text{for all } r<t,$$

otherwise $\mathcal{K}(t,r) = 0$.
In fact we have

$$\begin{aligned}\int_0^t \mathcal{K}(t,r)dW_r &= \int_0^t K(t,r)dW_r - \theta \int_0^t \left(\int_r^t e^{-\theta(t-s)}K(s,r)ds\right) dW_r \\ &= B_t^{h(t)} - \theta \int_0^t e^{-\theta(t-s)} B_s^{h(s)} ds.\end{aligned}$$

An integration by parts gives the representation of U. We notice that in the framework of the divergence integral (Chapter 3) the integral with respect to mbm can be reduced to an integral with respect to brownian motion. (H2) is satisfied with the same values of α and β as in Example 1.
In fact, we have

$$\frac{\partial}{\partial t}\mathcal{K}(t,r) = \frac{\partial}{\partial t}K(t,r) + \int_r^t \theta e^{-\theta(t-s)} \frac{\partial}{\partial s}K(s,r)ds.$$

Therefore

$$\Big|\frac{\partial \mathcal{K}}{\partial t}(t,r)\Big| \leqslant \Big|\frac{\partial K}{\partial t}(t,r)\Big| + \theta \int_r^t \Big|\frac{\partial K}{\partial s}(s,r)\Big| ds.$$

But

$$\Big|\frac{\partial K}{\partial t}(t,r)\Big| \leqslant c(t-r)^{\alpha-1}\Big(\frac{t}{r}\Big)^{\alpha}$$

and

$$\begin{aligned}\int_r^t \Big|\frac{\partial K}{\partial s}(s,r)\Big| ds &\leqslant cr^{-\alpha}\int_r^t (s-r)^{\alpha-1}s^{\alpha}ds \\ &= c\Big(\frac{t}{r}\Big)^{\alpha}(t-r)^{\alpha}.\end{aligned}$$

Therefore,

$$\Big|\frac{\partial \mathcal{K}}{\partial s}(s,r)\Big| \leqslant c(t-r)^{\alpha-1}\Big(\frac{t}{r}\Big)^{\alpha}\Big(1 + c(t-r)\Big)$$

and therefore $\mathcal{K}$ satisfies (H2).

Example 3. The Liouville multi-fractional Brownian motion (LmBm) $(B_t^{L,h(t)}, t \in [0,T])$ with Hurst function h as in Example 1. Its kernel is given by $\tilde{\mathcal{K}}(t,r) = (t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r)$. We have

$$\begin{aligned}\frac{\partial \tilde{\mathcal{K}}}{\partial t}(t,r) &= \frac{\partial}{\partial t}\Big[\Big(h(t) - \frac{1}{2}\Big)ln(t-r)\Big](t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r) \\ &= \Big[h'(t)ln(t-r) + \Big(h(t) - \frac{1}{2}\Big)\frac{1}{t-r}\Big](t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r).\end{aligned}$$

If $t-r>1$, we obtain

$$\begin{aligned}
\Big|\frac{\partial\tilde{\mathcal{K}}}{\partial t}(t,r)\Big| &\leqslant \Big|h'(t)ln(t-r)+\Big(h(t)-\frac{1}{2}\Big)\frac{1}{t-r}\Big|(t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r)\\
&\leqslant \Big(\Big|h'(t)\Big|(t-r)^{\epsilon}+(b-\frac{1}{2})\frac{1}{t-r}\Big)(t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r)\\
&= \Big(\Big|h'(t)\Big|(t-r)^{\epsilon+1}+(b-\frac{1}{2})\Big)(t-r)^{h(t)-\frac{3}{2}}1_{(0,t]}(r)\\
&\leqslant \Big(\Big|h'(t)\Big|(t-r)^{h(t)+\epsilon+\frac{1}{2}}+(b-\frac{1}{2})\Big)(t-r)^{h(t)-\frac{3}{2}}\\
&\leqslant C_T(t-r)^{h(t)-\frac{3}{2}}\Big(\frac{t}{r}\Big)^{h(t)-\frac{1}{2}}\\
&\leqslant C_T(t-r)^{b-\frac{3}{2}}\Big(\frac{t}{r}\Big)^{b-\frac{1}{2}}.
\end{aligned}$$

Consequently, for some $0<\alpha,\beta<\frac{1}{2}$ such that $\alpha=\beta=b-\frac{1}{2}$ and $\epsilon>0$, there is a finite constant $C_T>0$ such that

$$\Big|\frac{\partial K}{\partial t}(t,s)\Big|\leqslant C_T(t-s)^{\alpha-1}\Big(\frac{t}{s}\Big)^{\beta},\quad t\in(s,T].$$

If $t-r<1$, we have

$$\begin{aligned}
\Big|\frac{\partial\tilde{\mathcal{K}}}{\partial t}(t,r)\Big| &\leqslant \Big|h'(t)ln(t-r)+\Big(h(t)-\frac{1}{2}\Big)\frac{1}{t-r}\Big|(t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r)\\
&\leqslant \Big(\Big|h'(t)\Big|(t-r)^{-\epsilon}+(b-\frac{1}{2})\frac{1}{t-r}\Big)(t-r)^{h(t)-\frac{1}{2}}1_{(0,t]}(r)\\
&= \Big(\Big|h'(t)\Big|(t-r)^{-\epsilon+1}+(b-\frac{1}{2})\Big)(t-r)^{h(t)-\frac{3}{2}}1_{(0,t]}(r)\\
&\leqslant c_T(t-r)^{h(t)-\frac{3}{2}}\Big(\frac{t}{r}\Big)^{h(t)-\frac{1}{2}}\\
&\leqslant c_T(t-r)^{a-\frac{3}{2}}\Big(\frac{t}{r}\Big)^{b-\frac{1}{2}}.
\end{aligned}$$

Consequently, for some $0<\alpha,\beta<\frac{1}{2}$ such that $\alpha=a-\frac{1}{2}$, $\beta=b-\frac{1}{2}$ and $\epsilon>0$, there is a finite constant $c_T>0$ such that

$$\Big|\frac{\partial K}{\partial t}(t,s)\Big|\leqslant c_T(t-s)^{\alpha-1}\Big(\frac{t}{s}\Big)^{\beta},\quad t\in(s,T].$$

For a proof of (H2) under a different assumption on h' we refer to [12].

Chapter 3

Complements on Malliavin calculus for Volterra processes

In this Chapter, we give some complements on the Malliavin calculus for Gaussian Volterra processes. The definitions of the Malliavin derivative and the divergence integral are classical, but we introduce several functional spaces that will be used in the following chapters, in particular for the proof of Itô's formula. The space $\mathcal{H}$ (Definition 3.0.2) introduced in this Chapter may be understood as a representation of the classical space known as reproducing kernel Hilbert space.

The kernel K in (2.1) defines a linear operator in $L^2([0,T])$ given by $(K\sigma)_t = \int_0^t K(t,s)\sigma_s ds$, $\sigma \in L^2([0,T])$. Let $\mathcal{E}$ be the set of step functions of $[0,T]$, and let $K_T^* : \mathcal{E} \to L^2([0,T])$ be defined by

$$(K_T^*\sigma)_u := \int_u^T \sigma_s \frac{\partial K}{\partial s}(s,u)ds. \tag{3.1}$$

The operator K_T^* is the adjoint of K ([5], Lemma 1).

Remarks 3.0.1. *a) For $s > t$, we have $(K_T^*\sigma 1_{[0,t]})_s = 0$, and we will denote $(K_T^*\sigma 1_{[0,t]})_s$ by $(K_t^*\sigma)_s$ where K_t^* is the adjoint of the operator K in the interval $[0,t]$.*

b) If $K(u,u) = 0$ for all $u \in [0,T]$, $(K_T^* 1_{[0,r]})_u = K(r,u)$ for $u < r$. Indeed, if $u \leqslant r$, we have

$$(K_T^* 1_{[0,r]})_u = \int_u^T 1_{[0,r]}(s)\frac{\partial K}{\partial s}(s,u)ds = \int_u^r \frac{\partial K}{\partial s}(s,u)ds.$$

Therefore

$$R(t,s) = \mathbb{E}\Big[X_t X_s\Big] = \int_0^{inf(t,s)} (K_T^* 1_{[0,t]})_u (K_T^* 1_{[0,s]})_u du$$
$$=< K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]} >_{L^2([0,T])} .$$

For $\sigma, \widetilde{\sigma} \in \mathcal{E}$ this may be extended to

$$X(\sigma) := \int_0^T (K_T^* \sigma)_s dW_s \text{ and } E\Big[X(\sigma)X(\widetilde{\sigma})\Big] =< K_T^* \sigma, K_T^* \widetilde{\sigma} >_{L^2([0,T])} .$$

Definition 3.0.2. *Let $\mathcal{H}$ be the closure of the linear span of the indicator functions $1_{[0,t]}$, $t \in [0,T]$ with respect to the semi-inner product*

$$< 1_{[0,t]}, 1_{[0,s]} >_{\mathcal{H}} :=< K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]} >_{L^2([0,T])} .$$

The operator K_T^* is an isometry between $\mathcal{H}$ and a closed subspace of $L^2([0,T])$, and $\| \cdot \|_{\mathcal{H}}$ is a semi-norm on $\mathcal{H}$. Furthermore, for $\varphi, \psi \in \mathcal{H}$,

$$\begin{aligned} < K_T^* \varphi, K_T^* \psi >_{L^2([0,T])} &= \int_0^T (K_T^* \varphi)_t (K_T^* \psi)_t dt \\ &= \int_0^T \int_t^T \varphi_r \frac{\partial K}{\partial r}(r,t) dr \int_t^T \psi_s \frac{\partial K}{\partial s}(s,t) ds dt \\ &= \int_0^T \int_0^T \Big(\int_0^{inf(r,s)} \frac{\partial K}{\partial r}(r,t) \frac{\partial K}{\partial s}(s,t) dt \Big) \varphi_r \psi_s ds dr. \end{aligned}$$

For further use let

$$\phi(r,s) := \int_0^{inf(r,s)} \frac{\partial K}{\partial r}(r,t) \frac{\partial K}{\partial s}(s,t) dt, \, r \neq s, \tag{3.2}$$

$$\widetilde{\phi}(r,s) := \int_0^{inf(r,s)} \Big| \frac{\partial K}{\partial r}(r,t) \Big| \Big| \frac{\partial K}{\partial s}(s,t) \Big| dt, \, r \neq s. \tag{3.3}$$

Note that $\phi(r,s) = \partial^2 / \partial s \partial r R(r,s)$ $(r \neq s)$ (ϕ may be infinite on the diagonal $r = s$). Let $| \mathcal{H} |$ be the closure of the linear span of indicator functions with respect to the semi-norm given by

$$\| \varphi \|_{|\mathcal{H}|}^2 = \int_0^T \Big(\int_t^T | \varphi_r | \Big| \frac{\partial K}{\partial r}(r,t) \Big| dr \Big)^2 dt = 2 \int_0^T dr \int_0^r ds \widetilde{\phi}(r,s) | \varphi_r | | \varphi_s | . \tag{3.4}$$

We briefly recall some basic elements of the stochastic calculus of variations with respect to X. We refer to [3] and [26] for a more complete presentation. Let $\mathcal{S}$ be the set of random variables

of the form $F = f(X(\varphi_1),....,X(\varphi_n))$, where $n \geqslant 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and its derivatives are bounded) and $\varphi_1,...,\varphi_n \in \mathcal{H}$. The derivative of F

$$D^X F := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X(\varphi_1),....,X(\varphi_n))\varphi_j,$$

is an $\mathcal{H}$-valued random variable, and D^X is a closable operator from $L^p(\Omega)$ to $L^p(\Omega;\mathcal{H})$ for all $p \geqslant 1$. We denote by $\mathbb{D}_{1,p}^X$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|D^X F\|_{\mathcal{H}}^p. \tag{3.5}$$

We denote by $Dom(\delta^X)$ the subset of $L^2(\Omega,\mathcal{H})$ composed of those elements u for which there exists a positive constant c such that

$$\left|\mathbb{E}\Big[< D^X F, u >_{\mathcal{H}} \Big]\right| \leqslant c\sqrt{\mathbb{E}[F^2]}, \text{for all } F \in \mathbb{D}_{1,2}^X. \tag{3.6}$$

For $u \in L^2(\Omega;\mathcal{H})$ in $Dom(\delta^X)$, $\delta^X(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$\mathbb{E}\Big[F\delta^X(u)\Big] = \mathbb{E}\Big[< D_\cdot^X F, u_\cdot >_{\mathcal{H}} \Big], F \in \mathbb{D}_{1,2}^X. \tag{3.7}$$

We also use the notation $\int_0^T u_t \delta X_t$ for $\delta^X(u)$. A class of processes that belong to the domain of δ^X is given as follows: let $\mathcal{S}^{\mathcal{H}}$ be the class of $\mathcal{H}$-valued random variables $u = \sum_{j=1}^n F_j h_j$ ($F_j \in \mathcal{S}, h_j \in \mathcal{H}$).
In the same way $\mathbb{D}_{1,p}^X(|\ \mathcal{H}\ |)$ is defined as the completion of $\mathcal{S}^{|\mathcal{H}|}$ under the semi-norm

$$\|\ u\ \|_{1,p,|\mathcal{H}|}^p := \mathbb{E}\ \|\ u\ \|_{|\mathcal{H}|}^p + \mathbb{E}\ \|\ D^X u\ \|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p\ ,$$

where

$$\|\ D^X u\ \|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 = \int_{[0,T]^4} |\ D_s^X u_t\ ||\ D_{t'}^X u_{s'}\ |\ \widetilde{\phi}(s,s')\widetilde{\phi}(t,t')dsdtds'dt'. \tag{3.8}$$

The space $\mathbb{D}_{1,2}^X(|\ \mathcal{H}\ |)$ is included in the domain of δ^X, and we have, for $u \in \mathbb{D}_{1,2}^X(|\ \mathcal{H}\ |)$,

$$\mathbb{E}\Big(\delta^X(u)^2\Big) \leqslant \mathbb{E}\ \|\ u\ \|_{|\mathcal{H}|}^2 + \mathbb{E}\ \|\ D^X u\ \|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2\ .$$

For $F \in \mathcal{S}$, let

$$\mathbb{D}_s^X F := \int_0^T \phi(s,t) D_t^X F dt, s \in [0,T]. \tag{3.9}$$

Then (3.9) implies

$$\int_{[0,T]^2} \mathbb{D}_s^X u_t \mathbb{D}_t^X v_s dsdt = \int_{[0,T]^4} D_s^X u_t D_{t'}^X v_{s'} \phi(s,s')\phi(t,t')dsdtds'dt'. \tag{3.10}$$

Proposition 3.0.3. *Let $f,\, g \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$. Then the integrals $\delta^X(f)$ and $\delta^X(g)$ exist in $L^2(\Omega)$ and*

$$\mathbb{E}\Big[\delta^X(f)\delta^X(g)\Big] = \mathbb{E} < f, g >_{\mathcal{H}} + \int_0^T ds \int_0^T dt \mathbb{E}\Big[\mathbb{D}^X_t f_s \mathbb{D}^X_s g_t\Big]. \tag{3.11}$$

Remark. With the choice $f = g$, Proposition 3.0.3 implies $\mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |) \subset dom(\delta^X)$. In fact, for $f \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$, by Proposition 3.0.3, we get

$$\begin{aligned}
\mathbb{E}\Big(\delta^X(f)^2\Big) &= \mathbb{E} \parallel f \parallel^2_{\mathcal{H}} + \int_0^T ds \int_0^T dt \mathbb{E}[\mathbb{D}^X_t f_s \mathbb{D}^X_s f_t] \\
&\leqslant \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \mathbb{E} \int_0^T ds \int_0^T dt \int_0^T dt' \int_0^T ds' \phi(t,t') D_{t'} f_s \phi(s,s') D_{s'} f_t \\
&\leqslant \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \mathbb{E} \int_0^T ds \int_0^T dt \int_0^T dt' \int_0^T ds' \tilde{\phi}(t,t') \mid D_{t'} f_s \mid \tilde{\phi}(s,s') \mid D_{s'} f_t \mid \\
&\leqslant \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \mathbb{E} \int_0^T ds \int_0^T dt \int_0^T dt' \int_0^T ds' \tilde{\phi}(t',t) \tilde{\phi}(s,s') \mid D_{t'} f_s \mid\mid D_{s'} f_t \mid \\
&= \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \mathbb{E} \parallel D^X f \parallel^2_{|\mathcal{H}|\otimes|\mathcal{H}|}.
\end{aligned}$$

We denote by $\mathbb{L}^X_{1,2}(|\ \mathcal{H}\ |)$ the space $|\ \mathcal{H}\ |$- valued random variables such that

$$\parallel u \parallel^2_{\mathbb{L}^X_{1,2}(|\mathcal{H}|)} := \mathbb{E} \parallel u \parallel^2_{|\mathcal{H}|} + \mathbb{E} \int_0^T \int_0^T \mid \mathbb{D}^X_s u_t \mid^2 dsdt < \infty.$$

Remark. $\mathbb{L}^X_{1,2}(|\ \mathcal{H}\ |) \subset dom(\delta^X)$. In fact,

$$\begin{aligned}
\mathbb{E}\Big(\delta^X(f)^2\Big) &= \mathbb{E} \parallel f \parallel^2_{\mathcal{H}} + \int_0^T ds \int_0^T dt \mathbb{E}[\mathbb{D}^X_t f_s \mathbb{D}^X_s f_t] \\
&\leqslant \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \frac{1}{2} \int_0^T ds \int_0^T dt \mathbb{E} \mid \mathbb{D}^X_t f_s \mid^2 + \frac{1}{2} \int_0^T ds \int_0^T dt \mathbb{E} \mid \mathbb{D}^X_s f_t \mid^2 \\
&= \mathbb{E} \parallel f \parallel^2_{|\mathcal{H}|} + \mathbb{E} \int_0^T ds \int_0^T dt \mathbb{E} \mid \mathbb{D}^X_t f_s \mid^2.
\end{aligned}$$

For the proof of Proposition 3.0.3 we need the following lemma.

Lemma 3.0.4. *Let $G \in \mathbb{D}^X_{1,2}$ and $f \in dom(\delta^X)$. Then*

$$\mathbb{E}\Big[\delta^X(f)G\Big] = \int_0^T \mathbb{E}\Big[f_t \mathbb{D}^X_t G\Big] dt.$$

Proof of the Lemma.

$$\begin{aligned}
\mathbb{E}\Big[\delta^X(f)G\Big] &= \mathbb{E}\Big[< D^X_\cdot G, f_\cdot >_{\mathcal{H}} \Big] = \mathbb{E}\Big[< (K^*_T D^X G)_\cdot, (K^* f)_\cdot >_{L^2([0,T])} \Big] \\
&= \mathbb{E}\Big[\int_0^T dt \int_t^T du_1 f_{u_1} \frac{\partial K}{\partial u_1}(u_1, t) \int_t^T du_2 D^X_{u_2} G \frac{\partial K}{\partial u_2}(u_2, t) \Big] \\
&= \mathbb{E}\Big[\int_0^T du_1 f_{u_1} \int_0^T du_2 D^X_{u_2} G \phi(u_1, u_2) \Big] = \mathbb{E}\Big[\int_0^T f_{u_1} \mathbb{D}^X_{u_1} G du_1 \Big]. \quad \blacksquare
\end{aligned}$$

Proof of Proposition 3.0.3. Let first $f, g \in \mathcal{S}^{|\mathcal{H}|}$. If $f = \sum_{j=1}^{n} F_j h_j$, $F_j \in \mathcal{S}$, $h_j \in| \mathcal{H} |$, lemma 2.2 in [26] shows that

$$\delta^X(f) = \sum_{j=1}^{n} \left(F_j X(h_j) - \langle D^X F_j, h_j \rangle_{\mathcal{H}} \right) \in \mathcal{S}.$$

A straightforward calculation shows $D^X_{\cdot} \delta^X(f) \in \mathcal{S}^{|\mathcal{H}|}$ and $\delta^X(D^X_{\cdot} f) \in \mathcal{S}^{|\mathcal{H}|}$. Moreover, by means of the notion of directional derivatives ([26], p.27), one shows

$$\left\langle f_{\cdot}, D^X_{\cdot}(\delta^X(g)) \right\rangle_{\mathcal{H}} = \left\langle f_{\cdot}, \delta^X(D^X_{\cdot} g) + g_{\cdot} \right\rangle_{\mathcal{H}}.$$

Therefore

$$\mathbb{E}\Big[\delta^X(f)\delta^X(g)\Big] = \mathbb{E}\left\langle f_{\cdot}, D^X_{\cdot}(\delta^X(g)) \right\rangle_{\mathcal{H}} = \mathbb{E} < f, g >_{\mathcal{H}} + \mathbb{E}\left\langle f_{\cdot}, \delta^X(D^X_{\cdot} g) \right\rangle_{\mathcal{H}},$$

Moreover

$$\begin{aligned}
\mathbb{E}\left\langle f_{\cdot}, \delta^X(D^X_{\cdot} g) \right\rangle_{\mathcal{H}} &= \mathbb{E}\left[\int_0^T (K_T^* f)_u \left(K_T^* \delta^X(D^X_{\cdot} g) \right)_u du \right] \\
&= \mathbb{E}\left[\int_0^T du \int_u^T ds f_s \frac{\partial K}{\partial s}(s,u) \int_u^T dr \delta^X(D^X_r g) \frac{\partial K}{\partial r}(r,u) \right] \\
&= \mathbb{E}\left[\int_0^T ds \int_0^T dr f_s \delta^X(D^X_r g) \left(\int_0^{r \wedge s} \frac{\partial K}{\partial s}(s,u) \frac{\partial K}{\partial r}(r,u) du \right) \right] \\
&= \mathbb{E}\left[\int_0^T ds \int_0^T dr f_s \delta^X(D^X_r g) \phi(s,r) \right] = \int_0^T ds \int_0^T dr \mathbb{E}[f_s \delta^X(D^X_r g)] \phi(s,r) \\
&= \int_0^T ds \int_0^T dr \mathbb{E}\Big[< D^X_r g_{\cdot}, D^X_{\cdot} f_s >_{\mathcal{H}} \Big] \phi(s,r) \\
&= \int_0^T ds \int_0^T dr \mathbb{E}\Big[< K_T^* D^X_r g_{\cdot}, K_T^* D^X_{\cdot} f_s >_{L^2([0,T])} \Big] \phi(s,r) \\
&= \int_0^T ds \mathbb{E}\Big[< K_T^* \int_0^T D^X_r g_{\cdot} \phi(s,r) dr, K_T^* D^X_{\cdot} f_s >_{L^2([0,T])} \Big] \\
&= \int_0^T ds \mathbb{E} < \int_0^T D^X_r g_{\cdot} \phi(s,r) dr, D^X_{\cdot} f_s >_{\mathcal{H}} = \int_0^T ds \mathbb{E}\left[f_s \delta^X \left(\int_0^T D^X_r g_{\cdot} \phi(s,r) dr \right) \right] \\
&= \int_0^T ds \mathbb{E}\left[f_s \delta^X \left(\mathbb{D}^X_s g_{\cdot} \right) \right] = \int_0^T ds \int_0^T dt \mathbb{E}\left[\mathbb{D}^X_s g_t \mathbb{D}^X_t f_s \right],
\end{aligned}$$

where we have used the preceding lemma in the last equality. This proves (3.11) for $f, g \in \mathcal{S}^{|\mathcal{H}|}$.

Let now $f \in \mathbb{D}^X_{1,2}(| \mathcal{H} |)$. There exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{S}^{|\mathcal{H}|}$ such that, as $n \to \infty$, $f_n \to f$ in $L^2(\Omega, \mathcal{H})$ and $D^X f_n \to D^X f$ in $L^2(\Omega, \mathcal{H} \otimes \mathcal{H})$. Therefore $\delta^X(f_n)$ converges in $L^2(\Omega)$. By proposition 1.3.6 in [26] the limit is $\delta^X(f)$. ■

Appendix

The aim of this annex is to show that the space $\mathcal{H}$ in Definition 3.0.2 is in fact a representation of the more classical *reproducing kernel Hilbert space* (RKHS) (see e.g. [4]) generated by the Gaussian Volterra process X. Let us recall that the RKHS is the closure with respect to the norm $\|\cdot\|_H$ defined below of the class $\widetilde{\mathcal{E}}$ of functions given by

$$\widetilde{\mathcal{E}} = \{f : [0,T] \to \mathbb{R},\, f(\cdot) = \sum_{i=1}^{n} a_i R(s_i,\cdot),\, a_i \in \mathbb{R},\, s_i \in [0,T],\, n \geqslant 1\}.$$

Consider the inner product $\langle\cdot,\cdot\rangle_H$ on $\widetilde{\mathcal{E}}$ given by

$$\langle f,g\rangle_H = \left\langle \sum_{i=1}^{n} a_i R(s_i,\cdot), \sum_{j=1}^{m} b_j R(t_j,\cdot) \right\rangle_H := \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j R(s_i,t_j).$$

The fact that R is non-negative definite implies that $\langle f,f\rangle_H \geqslant 0$ for all $f \in \widetilde{\mathcal{E}}$. Furthermore,

$$f(t) = \sum_{i=1}^{n} a_i R(s_i,t) = \left\langle \sum_{i=1}^{n} a_i R(s_i,t), R(t,\cdot) \right\rangle_H = \langle f, R(t,\cdot)\rangle_H,$$

and

$$|\, f(t)\,|^2 = |\, \langle f, R(t,\cdot)\rangle_H\,|^2 \leqslant \langle f,f\rangle_H \langle R(t,\cdot), R(t,\cdot)\rangle_H,$$

where the inequality is the Cauchy-Schwartz inequality for semi-inner products. Thus, if $\langle f,f\rangle_H = 0$, then $f = 0$ for all $t \in [0,T]$. Consequently $\langle f,g\rangle_H$ defines an (proper) inner product and a norm on $\widetilde{\mathcal{E}}$. The closure of $\widetilde{\mathcal{E}}$ under this norm is called the RKHS of X or R and denoted by H. In fact, for $(f_n)_{n\geqq 1} \subset \widetilde{\mathcal{E}}$,

$$|\, f_n(t) - f_m(t)\,|^2 = |\, \langle f_n - f_m, R(t,\cdot)\rangle_H\,|^2 \leqslant \|\, f_n - f_m\,\|_H^2 \|\, R(t,\cdot)\,\|_H^2 = \|\, f_n - f_m\,\|_H^2\, R(t,t).$$

Therefore, if $(f_n)_{n\geqq 1}$ is a Cauchy sequence with respect to $\|\cdot\|_H$, it is also pointwise Cauchy.

Lemma 3.0.5. *With f,g as above,*

$$\langle f,g\rangle_H = \left\langle \sum_{i=1}^{n} a_i 1_{[0,s_i]}, \sum_{j=1}^{m} b_j 1_{[0,t_j]} \right\rangle_{\mathcal{H}}.$$

Therefore $\mathcal{H}$ may be understood as a representation of H.

Proof.

$$\begin{aligned} f(t) &= \sum_{i=1}^{n} a_i R(s_i,t) = \sum_{i=1}^{n} a_i E[X_{s_i} X_t] = \sum_{i=1}^{n} a_i \int_0^{s_i\wedge t} (K_T^* 1_{[0,s_i]})_u (K_T^* 1_{[0,t]})_u du \\ &= \sum_{i=1}^{n} \int_0^{s_i\wedge t} \int_u^T \frac{\partial}{\partial v} K(v,u) a_i 1_{[0,s_i]}(v) dv (K_T^* 1_{[0,t]})_u du \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \sum_{i=1}^n a_i 1_{[0,s_i]}(v) \int_0^{v\wedge t} \frac{\partial}{\partial v} K(v,u)(K_T^* 1_{[0,t]})_u du dv \\
&= \int_0^T \sum_{i=1}^n a_i 1_{[0,s_i]}(v) \int_0^{v\wedge t} \frac{\partial}{\partial v} K(v,u) \int_u^T \frac{\partial}{\partial w} K(w,u) 1_{[0,t]}(w) dw du dv \\
&= \int_0^T \sum_{i=1}^n a_i 1_{[0,s_i]}(v) \int_0^T 1_{[0,t]}(w) \int_0^{w\wedge v} \frac{\partial}{\partial v} K(v,u) \frac{\partial}{\partial w} K(w,u) du dw dv \\
&= \int_0^T \sum_{i=1}^n a_i 1_{[0,s_i]}(v) \int_0^T 1_{[0,t]}(w) \phi(w,v) dw dv \\
&= \left\langle K_T^*(\sum_{i=1}^n a_i 1_{[0,s_i]}), K_T^* 1_{[0,t]}) \right\rangle_{L^2([0,T])} \\
&= \left\langle \sum_{i=1}^n a_i 1_{[0,s_i]}, 1_{[0,t]}) \right\rangle_{\mathcal{H}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle f, g\rangle_H &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j R(s_i,t_j) = \sum_{j=1}^m b_j f(t_j) \\
&= \sum_{j=1}^m b_j \left\langle \sum_{i=1}^n a_i 1_{[0,s_i]}, 1_{[0,t_j]}) \right\rangle_{\mathcal{H}} = \left\langle \sum_{i=1}^n a_i 1_{[0,s_i]}, \sum_{j=1}^m b_j 1_{[0,t_j]}) \right\rangle_{\mathcal{H}}. \quad \blacksquare
\end{aligned}
$$

Consider now $h \in \mathcal{H}$ with $\| h \|_{\mathcal{H}} = 0$, and let $(h_n)_{n\geqq 1} \subset \mathcal{E}$ such that $\| h_n \|_{\mathcal{H}} \to 0$ as $n \to +\infty$. Let $(f_n)_{n\geqq 1} \subset \widetilde{\mathcal{E}}$ be the sequence associated to $(h_n)_{n\geqq 1}$. Then

$$
0 = \lim_{n\to\infty} \| f_n \|_H = \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n a_i^n a_j^n R(s_i^n, s_j^n) = \lim_{n\to\infty} E(\sum_{i=1}^n a_i^n X_{s_i^n})^2 = \lim_{n\to\infty} EX(f_n)^2 = EX(f)^2.
$$

Therefore $X(f) = 0$ $P-a.s.$

Chapter 4

Itô formula

In this Chapter, we prove an Itô formula for multidimensional gaussian Volterra processes. It will be used in Chapter 5 to solve explicitely the partial differential equation associated to a linear BSDE and to derive the solution of the BSDE. Another Itô formulated that will be needed for non linear BSDEs appears in Chapter 6.

4.1 Introduction

Let $W = (W^1,...,W^n)$ a standard brownian motion in $\mathbb{R}^n$, defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F},\, t \in [0,T]\}$ be the filtration generated by W and augmented by the P-null sets. We consider the $\mathbb{R}^n$-valued Volterra processes $X = (X^1,...,X^n)$ given by

$$X_t^j = \int_0^t K^j(t,s)dW_s^j,\; j = 1,...,n, \tag{4.1}$$

where $K^j : [0,T]^2 \to \mathbb{R}$ satisfies the conditions *(H1)* and *(H2)*. Let σ^j, $j = 1,...,n$ be bounded functions on $[0,T]$, and let $b^j \in \mathcal{C}^1((0,T),\mathbb{R}) \cap \mathcal{C}([0,T],\mathbb{R})$, $j = 1,...,n$. The process $N := (N^1,...,N^n)$ is defined by

$$N_t^j = b_t^j + \int_0^t \sigma_s^j \delta X_s^j,\; t \in [0,T], j = 1,...,n. \tag{4.2}$$

We are now in a position to prove the Itô Formula that will be used in the next Chapter.

4.2 A multi-dimensional Itô formula

Let $F \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ and suppose that

$$\max\left(\left|F(t,x)\right|, \left|\frac{\partial F}{\partial t}(t,x)\right|, \left|\frac{\partial F}{\partial x_j}(t,x)\right|, \left|\frac{\partial^2 F}{\partial x_j^2}(t,x)\right|, j = 1,...,n\right) \leqslant ce^{\lambda|x|^2} \tag{4.3}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n$, where c, λ are positive constants such that $\lambda < \frac{1}{4}\min_j(\sup_{t\in[0,T]} Var(N_t^j))^{-1}$. This implies

$$\begin{aligned}\mathbb{E}\Big|F(t,N_t)\Big|^2 &\leqslant c^2\mathbb{E}exp(2\lambda|N_t|^2)\\ &= \frac{c^2}{(2\pi)^{n/2}}\prod_{j=1}^n \frac{1}{(Var(N_t^j))^{1/2}}\int exp\left(2\lambda x_j^2 - \frac{1}{2}\frac{(x_j-b_t^j)^2}{Var(N_t^j)}\right)dx_j < \infty,\end{aligned}$$

and the same property holds for $\partial/\partial t F(t,x)$, $\partial/\partial x_j F(t,x)$ and $\partial^2/\partial x_j^2 F(t,x)$, $j = 1,...,n$.

Theorem 4.2.1. *Let N be given by (4.2), and suppose that, for $j = 1,...,n$, the kernels K^j of X^j satisfy (H1) and (H2), $b^j \in \mathcal{C}^1((0,T),\mathbb{R}) \cap \mathcal{C}([0,T],\mathbb{R})$, and $\sigma = \{\sigma_t^j,\, t \in [0,T],\, j = 1,...,n\}$ is bounded. If $F \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$ satisfies (4.3), $\partial/\partial x_j F(\cdot, N_\cdot) \in \mathbb{D}_{1,2}^{X^j}(|\mathcal{H}^j|)$, $j = 1,...,n$ and, for all $t \in [0,T]$,*

$$\begin{aligned}F(t,N_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial s}(s,N_s)ds + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial x_j}(s,N_s)\Big(\frac{d}{ds}b_s^j ds + \sigma_s^j \delta X_s^j\Big)\\ &+ \frac{1}{2}\sum_{j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_j^2}(s,N_s)\frac{d}{ds}Var(N_s^j)ds. \qquad (4.4)\end{aligned}$$

Remarks. a) A more general model for N than in Section 4.1 is

$$\widetilde{N}_t^i = b_t^i + \sum_{j=1}^n \int_0^t \widetilde{\sigma}_s^{i,j}\delta X_s^j,\ i = 1,...,n, \qquad (4.5)$$

where $\widetilde{\sigma} = (\widetilde{\sigma}_t^{i,j},\ i,j = 1,...,n)$ is a matrix of bounded functions $\widetilde{\sigma}^{i,j}$ defined on $[0,T]$. Let $\widetilde{N} = (\widetilde{N}^1,...,\widetilde{N}^n)$. The components of $\widetilde{N}$ are dependent since $\widetilde{N}^i$ depends not only on a single random perturbation X^i, but on the others X^j $(j \neq i)$ as well. The model of Section 4.1 may be recovered by choosing the matrix $\widetilde{\sigma}$ diagonal with functions $\sigma^j := \widetilde{\sigma}^{j,j}$ in the diagonal. An Itô formula can be shown for $F(t,\widetilde{N_t})$ too, but, instead of the variances of N, the covariances of $\widetilde{N}$ appear now in the second order term. It reads

$$\begin{aligned}F(t,\widetilde{N}_t) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial s}(s,\widetilde{N}_s)ds + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(s,\widetilde{N}_s)\Big(\frac{d}{ds}b_s^i ds + \sum_{j=1}^n \widetilde{\sigma}_s^{i,j}\delta X_s^j\Big)\\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s,\widetilde{N}_s)\frac{d}{ds}Cov(\widetilde{N}_s^i,\widetilde{N}_s^j)ds. \qquad (4.6)\end{aligned}$$

Two problems arise with this model. The first is to find a positive constant λ in the growth condition (4.3) that implies $\mathbb{E}\Big|F(t,\widetilde{N_t})\Big|^2 < \infty$. It seems therefore that (4.6) can be shown for $F \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^n)$ only, i.e. if F and its derivatives are bounded, a hypothesis that is in general too restrictive for an application to the solution u of the PDE (5.2). The second problem

is that, in order to get the solution of the BSDE (5.1) with $\widetilde{N}$, the derivatives of the variances of N in the second order term of (5.2) should be replaced by the derivatives of the covariances of $\widetilde{N}$, and it would be necessary to assume that the matrix $(\frac{d}{dt}Cov(\widetilde{N}_t^i, \widetilde{N}_t^j),\ i,j = 1,...,n)$ is positive definite, a hypothesis which seems to be restrictive (see also the remark below). For this reason we prefer to consider BSDEs with N given by (4.2). We notice that

$$Var(N_t^j) = \mathbb{E}\left(\int_0^t \left(K_t^{*,j}\sigma^j\right)_s \delta W_s^j\right)^2 = \int_0^t \left(K_t^{*,j}\sigma^j\right)_s^2 ds, \text{ and } \frac{d}{dt}Var(N_t^j) = 2\sigma_t^j \mathbb{D}_t^j N_t^j,$$

b) In the model where $\widetilde{\sigma}$ is diagonal, X^i and X^j are defined with independent brownian motions W^i and W^j if $i \neq j$. This differs from the model where all the Volterra processes $\overline{X}^j$ are defined with the same brownian motion $\overline{W}$ as follows:

$$\overline{X}_t^j = \int_0^t K^j(t,s)\delta\overline{W}_s,\ \overline{N}_t^j = b_t^j + \int_0^t \sigma_s^j \delta\overline{X}_s^j,\ t \in [0,T],\ j = 1,...,n.$$

In this case the processes $\overline{N}^j$ are correlated, and the matrix $(\frac{d}{dt}Cov(\overline{N}_t^i, \overline{N}_t^j),\ i,j = 1,...,n)$ is not diagonal. For $n = 2$ a direct verification shows that this matrix is negative (semi-)definite.

Proof of Theorem 4.2.1

1. Let us first show that $\partial/\partial x_j F(\cdot, N_\cdot) \in \delta^{X^j}$ for all $j = 1,...,n$. For this we show that $\partial/\partial x_j F(\cdot, N_\cdot) \in \mathbb{D}_{1,2}^{X^j}(|\ \mathcal{H}^j\ |)$, where $\mathcal{H}^j$ is the space defined in Chapter 3 with X replaced by X^j. The terms ϕ^j and $\widetilde{\phi}^j$ refer now to the kernel K^j of X^j. The constants in the inequalities below may vary from line to line. By (3.4)

$$\begin{aligned}\mathbb{E}\left\|\frac{\partial F}{\partial x_j}(.,N_\cdot)\right\|_{\mathcal{H}^j}^2 &= \mathbb{E}\int_0^T \left(\int_s^T \left|\frac{\partial F}{\partial x_j}(t,N_t)\right| \left|\frac{\partial K^j}{\partial t}(t,s)\right| dt\right)^2 ds \\ &\leqslant c_j \mathbb{E}\int_0^T \left(\int_s^T \exp(\lambda \mid N_t \mid^2)(t-s)^{\alpha_j - 1}\left(\frac{t}{s}\right)^{\beta_j} dt\right)^2 ds\end{aligned}$$

Applying Hölder's inequality to $1 < p_j < \frac{1}{1-\alpha_j} < 2$ and q_j its conjugate, we get

$$\left(\int_s^T \exp(\lambda \mid N_t \mid^2)(t-s)^{\alpha_j-1}\left(\frac{t}{s}\right)^{\beta_j} dt\right)^2 \leqslant \left(\int_s^T \exp(q_j\lambda \mid N_t \mid^2) dt\right)^{\frac{2}{q_j}} \left(\int_s^T (t-s)^{p_j(\alpha_j-1)}\left(\frac{t}{s}\right)^{p_j\beta_j} dt\right)^{\frac{2}{p_j}}.$$

Then

$$\begin{aligned}\mathbb{E}\left(\int_0^T \exp(q_j\lambda \mid N_t \mid^2) dt\right)^{\frac{2}{q_j}} &\leqslant \mathbb{E}\left(\int_0^T \exp(q_j\lambda \sup_{t\in[0,T]} \mid N_t \mid^2) dt\right)^{\frac{2}{q_j}} \\ &\leqslant T^{2/q_j} \prod_{i=1}^n \mathbb{E}\exp(2\lambda \sup_{t\in[0,T]} (N_t^i)^2) dt < \infty \quad \text{(see [15])},\end{aligned}$$

for $\lambda < \frac{1}{4}\min_i(\sup_{t\in[0,T]} Var(N_t^i))^{-1}$.

Moreover, we will to prove $\int_0^T \left(\int_s^T (t-s)^{p_j(\alpha_j-1)}\left(\frac{t}{s}\right)^{p_j\beta_j} dt\right)^{\frac{2}{p_j}} ds < \infty$. In fact,

$$\int_s^T (t-s)^{p_j(\alpha_j-1)}\left(\frac{t}{s}\right)^{p\beta_j} dt = s^{-p_j\beta_j}\int_s^T (t-s)^{p_j(\alpha_j-1)} t^{p_j\beta_j} dt$$

$$\leqslant s^{-p_j\beta_j}\int_s^T (t-s)^{p_j(\alpha_j-1)}T^{p_j\beta_j}dt$$
$$= s^{-p_j\beta}T^{p_j\beta_j}\frac{(T-s)^{p_j(\alpha_j-1)+1}}{p_j(\alpha_j-1)+1}.$$

Therefore

$$\int_0^T\Big(\int_s^T(t-s)^{p_j(\alpha_j-1)}\Big(\frac{t}{s}\Big)^{p_j\beta_j}dt\Big)^{\frac{2}{p_j}}ds = \frac{T^{2\beta_j}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\int_0^T\Big(s^{-p_j\beta_j}(T-s)^{p_j(\alpha_j-1)+1}\Big)^{\frac{2}{p_j}}ds$$
$$= \frac{T^{2\beta_j}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\int_0^T s^{-2\beta_j}T^{2(\alpha_j-1)+\frac{2}{p_j}}\Big(1-\frac{s}{T}\Big)^{2(\alpha_j-1)+\frac{2}{p_j}}ds$$
$$= \frac{T^{2\beta_j}T^{2(\alpha_j-1)+\frac{2}{p_j}}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\int_0^1 u^{-2\beta_j}T^{-2\beta_j}(1-u)^{2(\alpha_j-1)+\frac{2}{p_j}}Tdu$$
$$= \frac{T^{2(\alpha_j-1)+\frac{2}{p_j}+1}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\mathcal{B}(1-2\beta_j,2(\alpha_j-1)+\frac{2}{p_j}+1).$$

It remains to show that $\mathbb{E}\Big\|D^{X^j}\frac{\partial F}{\partial x_j}(.,N_.)\Big\|_{|\mathcal{H}^j|\otimes|\mathcal{H}^j|}<\infty$. By (3.8)

$$\mathbb{E}\Big\|D^{X^j}\frac{\partial F}{\partial x_j}(.,N_.)\Big\|^2_{|\mathcal{H}^j|\otimes|\mathcal{H}^j|}$$
$$= \mathbb{E}\int_{[0,T]^4}\Big|D^{X^j}_s\frac{\partial F}{\partial x_j}(t,N_t)\Big|\Big|D^{X^j}_{t'}\frac{\partial F}{\partial x_j}(s',N_{s'})\Big|\widetilde{\phi^j}(s,s')\widetilde{\phi^j}(t,t')dsdtds'dt'$$
$$= \mathbb{E}\int_{[0,T]^4}\Big|\frac{\partial^2F}{\partial x_j^2}(t,N_t)\sigma^j_s\Big|\Big|\frac{\partial^2F}{\partial x_j^2}(s',N_{s'})\sigma^j_{t'}\Big|\widetilde{\phi^j}(s,s')\widetilde{\phi^j}(t,t')dsdtds'dt'$$
$$\leqslant c\int_0^T dt\int_0^T ds'\left[\mathbb{E}\left(\frac{\partial^2F}{\partial x_j^2}(t,N_t)\right)^2+\mathbb{E}\left(\frac{\partial^2F}{\partial x_j^2}(s',N_{s'})\right)^2\right]\int_0^T dt'\widetilde{\phi^j}(t,t')\int_0^T ds\widetilde{\phi^j}(s',s).$$

By (4.3),

$$\mathbb{E}\left(\frac{\partial^2F}{\partial x_j^2}(t,N_t)\right)^2\leqslant c\mathbb{E}\exp(2\lambda\mid N_t\mid^2)\leqslant c'\prod_{i=1}^n(1-4\lambda Var(N^i_t))^{-1/2},$$

and by the choice of λ the right side stays bounded in $t\in[0,T]$. The finiteness of the remaining integrals follows from *(H1), (H2)* applied to K^j.

2. We proceed to the proof of the Itô formula. Let

$$N^{j,\varepsilon}_t = b^j_t+\int_0^t\Big(\int_s^t\sigma^j_r\frac{\partial K^j}{\partial r}(r+\varepsilon,s)dr\Big)\delta W^j_s = b^j_t+\int_0^t\Big(\int_0^r\sigma^j_r\frac{\partial K^j}{\partial r}(r+\varepsilon,s)\delta W^j_s\Big)dr,\ \forall t\leqslant T-\varepsilon.$$

Then, $F(t, N_t^{1,\varepsilon},...,N_t^{n,\varepsilon})$ has locally bounded variation, and we can write

$dF(t, N_t^{1,\varepsilon},..., N_t^{n,\varepsilon}) = \frac{\partial F}{\partial t}(t, N_t^{1,\varepsilon},..., N_t^{n,\varepsilon})dt + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(t, N_t^{1,\varepsilon},..., N_t^{n,\varepsilon})dN_t^{j,\varepsilon}.$

Moreover, $dN_t^{j,\varepsilon} = \sigma_t^j \int_0^t \frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j dt + \frac{d}{dt}b_t^j dt$. Therefore

$$dF(t, N_t^{1,\varepsilon},..., N_t^{n,\varepsilon}) = \Big(\frac{\partial F}{\partial t}(t, N_t^{\varepsilon}) + \frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\frac{d}{dt}b_t^j\Big)dt + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\sigma_t^j \int_0^t \frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j dt,$$

with the notation $N^{\varepsilon} = (N^{1,\varepsilon},..., N^{n,\varepsilon})$. Furthermore,

$$\begin{aligned}\frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\sigma_t^j \int_0^t \frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j &= \sigma_t^j\Big[\int_0^t \frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j \\ &\quad + \int_0^t D_s^{W^j}\Big(\frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\Big)\frac{\partial K^j}{\partial t}(t+\varepsilon, s)ds\Big].\end{aligned}$$

We have

$$\begin{aligned}D_s^{W^j}\Big(\frac{\partial F}{\partial x_j}(t, N_t^{\varepsilon})\Big) = \frac{\partial^2 F}{\partial x_j^2}(t, N_t^{\varepsilon})D_s^{W^j}N_t^{j,\varepsilon} &= \frac{\partial^2 F}{\partial x_j^2}(t, N_t^{\varepsilon})D_s^{W^j}\Big(\int_0^t \Big[\int_u^t \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, u)dr\Big]\delta W_u^j\Big) \\ &= \frac{\partial^2 F}{\partial x_j^2}(t, N_t^{\varepsilon})\underbrace{\int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, s)dr}_{:=\Big(K_t^{*,\varepsilon,j}\sigma^j\Big)_s} \\ &= \frac{\partial^2 F}{\partial x_j^2}(t, N_t^{\varepsilon})\Big(K_t^{*,\varepsilon,j}\sigma^j\Big)_s.\end{aligned}$$

Then,

$$\begin{aligned}F(t, N_t^{\varepsilon}) = F(0,0) &+ \int_0^t \frac{\partial F}{\partial s}(s, N_s^{\varepsilon})ds + \sum_{j=1}^{n}\int_0^t \Big(\int_s^t \sigma_r^j \frac{\partial F}{\partial x_i}(r, N_r^{\varepsilon})\frac{\partial K^j}{\partial r}(r+\varepsilon, s)dr\Big)\delta W_s^j \\ &+ \frac{1}{2}\sum_{j=1}^{n}\int_0^t \frac{\partial^2 F}{\partial x_i^2}(r, N_r^{\varepsilon})\frac{\partial}{\partial r}\Big(\int_0^r \Big(K_r^{*,\varepsilon,j}(\sigma^j)\Big)_s^2 ds\Big)dr + \sum_{j=1}^{n}\int_0^t \frac{\partial F}{\partial x_j}(s, N_s^{\varepsilon})\frac{d}{ds}b_s^j ds.\end{aligned}$$

And we can write

$$F(t, N_t^{\varepsilon}) = F(0,0) + S_{1,\varepsilon} + S_{2,\varepsilon} + S_{3,\varepsilon} + S_{4,\varepsilon}.$$

where

$$\begin{aligned}S_{1,\varepsilon} &= \int_0^t \frac{\partial F}{\partial s}(s, N_s^{\varepsilon})ds \\ S_{2,\varepsilon} &= \sum_{j=1}^{n}\int_0^t \Big(\int_s^t \sigma_r^j \frac{\partial F}{\partial x_j}(r, N_r^{\varepsilon})\frac{\partial K^j}{\partial r}(r+\varepsilon, s)dr\Big)\delta W_s^j = \sum_{j=1}^{n} A_{2,\varepsilon}^j \\ S_{3,\varepsilon} &= \frac{1}{2}\sum_{j=1}^{n}\int_0^t \frac{\partial^2 F}{\partial x_j^2}(r, N_r^{\varepsilon})\frac{\partial}{\partial r}\Big(\int_0^r \Big(K_r^{*,\varepsilon,j}(\sigma^j)\Big)_s^2 ds\Big)dr = \sum_{j=1}^{n} A_{3,\varepsilon}^j\end{aligned}$$

$$S_{4,\varepsilon} = \sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial x_j}(s, N_s^\varepsilon) \frac{d}{ds} b_s^j ds = \sum_{j=1}^{n} A_{4,\varepsilon}^j.$$

Now, the proof breaks down into 4 steps.

Step 1. We will prove

$$\lim_{\varepsilon \to 0} \mathbb{E} \Big| S_{1,\varepsilon} - \int_0^t \frac{\partial F}{\partial s}(s, N_s) ds \Big|^2 = 0. \tag{4.7}$$

We have

$$\mathbb{E} \Big| \int_0^t \frac{\partial F}{\partial s}(s, N_s^\varepsilon) ds - \int_0^t \frac{\partial F}{\partial s}(s, N_s) ds \Big|^2 \leqslant T \int_0^t \mathbb{E} \Big| \frac{\partial F}{\partial s}(s, N_s^\varepsilon) - \frac{\partial F}{\partial s}(s, N_s) \Big|^2 ds.$$

Moreover, $\frac{\partial F}{\partial s}(s, .)$ is continuous, $\lim_{\varepsilon \to 0} \frac{\partial F}{\partial s}(s, N_s^\varepsilon) = \frac{\partial F}{\partial s}(s, N_s)$ and we have
$\Big| \frac{\partial F}{\partial s}(s, N_s^\varepsilon) \Big|^2 \leqslant exp(2\lambda \mid N_s^\epsilon \mid^2)$.
Furthermore, $\mathbb{E} \Big| \frac{\partial F}{\partial s}(s, N_s^\varepsilon) \Big|^2 \leqslant c' \prod_{i=1}^n (1 - 4\lambda Var(N_s^{i,\epsilon}))^{-\frac{1}{2}} \leqslant c' \prod_{i=1}^n (1 - 4\lambda \sup_{s \in [0,T]} Var(N_s^{i,\epsilon}))^{-\frac{1}{2}}$.
Moreover,

$$\begin{aligned} Var(N_s^{i,\epsilon}) &= \mathbb{E} \Big[\int_0^s \Big(\int_t^s \sigma_r^i \frac{\partial K^i}{\partial r}(r + \epsilon, t) dr \Big) dW_t \Big]^2 \\ &= \int_0^s \Big(\int_t^s \sigma_r^i \frac{\partial K^i}{\partial r}(r + \epsilon, t) dr \Big)^2 dt \\ &\leqslant \int_0^s \Big(\int_t^s \mid \sigma_r^i \mid \Big| \frac{\partial K^i}{\partial r}(r + \epsilon, t) \Big| dr \Big)^2 dt \\ &\leqslant M_1 \int_0^s \Big(\int_t^s \Big| \frac{\partial K^i}{\partial r}(r + \epsilon, t) \Big| dr \Big)^2 dt. \end{aligned} \tag{4.8}$$

We prove now that the right side of (4.8) is uniformly bounded. In fact, we have

$$\begin{aligned} \int_t^s \Big| \frac{\partial K^i}{\partial r}(r + \epsilon, t) \Big| dr &\leqslant c_i \int_t^s (r + \epsilon - t)^{\alpha_i - 1} \Big(\frac{r + \epsilon}{t} \Big)^{\beta_i} dr \\ &\leqslant c_i t^{-\beta_i} \int_t^s (r - t)^{\alpha_i - 1} (r + \epsilon)^{\beta_i} dr \\ &\leqslant c_i t^{-\beta_i} \frac{(s + \epsilon)^{\beta_i} (s - t)^{\alpha_i}}{\alpha_i}. \end{aligned}$$

Then

$$\Big(\int_t^s \Big| \frac{\partial K^i}{\partial r}(r + \epsilon, t) \Big| dr \Big)^2 \leqslant c_i' t^{-2\beta_i} (s + \epsilon)^{2\beta_i} (s - t)^{2\alpha_i}.$$

Therefore

$$\int_0^s \Big(\int_t^s \Big| \frac{\partial K^i}{\partial r}(r + \epsilon, t) \Big| dr \Big)^2 dt \leqslant c_i' (T + \epsilon)^{2\beta_i} \int_0^s t^{-2\beta_i} s^{2\alpha_i} \Big(1 - \frac{t}{s} \Big)^{2\alpha_i} dt$$

$$\leqslant c_i'' \int_0^1 u^{-2\beta_i} s^{-2\beta_i} s^{2\alpha_i} (1-u)^{2\alpha_i} s du$$
$$\leqslant c_i'' s^{2\alpha_i - 2\beta_i + 1} \mathcal{B}(-2\beta_i + 1, 2\alpha_i + 1) \leqslant K,$$

since $2\alpha_i - 2\beta_i + 1 > 0$.

By applying the dominated convergence theorem, $\int_0^t \mathbb{E}\Big|\frac{\partial F}{\partial s}(s, N_s^\varepsilon) - \frac{\partial F}{\partial s}(s, N_s)\Big|^2 ds \underset{\epsilon \to 0}{\longrightarrow} 0$. (4.7) follows.

Step 2. Let

$$(\alpha_{2,\varepsilon}^j)^2 = \mathbb{E}\int_0^t \Big| K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s^\varepsilon)\sigma_s^j\Big) - K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s)\sigma_s^j\Big)\Big|^2 ds.$$

Let us show $\lim\limits_{\varepsilon \to 0} \alpha_{2,\varepsilon}^j = 0$.

In fact,

$$(\alpha_{2,\varepsilon}^j)^2 \leqslant \mathbb{E}\int_0^t \Big| K_t^{*,\varepsilon,j}\Big[\frac{\partial F}{\partial x_j}(s, N_s^\varepsilon)\sigma_s^j - \frac{\partial F}{\partial x_j}(s, N_s)\sigma_s^j\Big]\Big|^2 ds$$
$$+ \mathbb{E}\int_0^t \Big| K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s)\sigma_s^j\Big) - K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s)\sigma_s^j\Big)\Big|^2 ds. \tag{4.9}$$

For the first term we get

$$\mathbb{E}\int_0^t \Big| K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s^\varepsilon)\sigma_s^j\Big) - K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s, N_s)\sigma_s^j\Big)\Big|^2 ds$$
$$\leqslant c_j \mathbb{E}\int_0^t \Big| \int_s^t \Big[\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big]\sigma_r^j (r-s)^{\alpha_j - 1}\Big(\frac{r}{s}\Big)^{\beta_j} dr\Big|^2 ds$$
$$= c_j \int_0^t \mathbb{E}\Big(\int_s^t \Big[\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big]\sigma_r^j (r-s)^{\alpha_j - 1}\Big(\frac{r}{s}\Big)^{\beta_j} dr\Big)^2 ds$$
$$\leqslant c_j \int_0^t \int_s^t \mathbb{E}\Big|\Big[\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big]\sigma_r^j\Big|^2 (r-s)^{\alpha_j - 1}\Big(\frac{r}{s}\Big)^{\beta_j} dr ds$$
$$\leqslant M_j^2 c_j \int_0^t \int_s^t \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 (r-s)^{\alpha_j - 1}\Big(\frac{r}{s}\Big)^{\beta_j} dr ds$$
$$\leqslant M_j^2 c_j \int_0^t r^{\beta_j} \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 \Big(\int_0^r (r-s)^{\alpha_j - 1} s^{-\beta_j} ds\Big) dr$$
$$= M_j^2 c_j \int_0^t r^{\beta_j} \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 \Big(\int_0^r r^{\alpha_j - 1}\Big(1 - \frac{s}{r}\Big)^{\alpha_j - 1} s^{-\beta_j} ds\Big) dr$$
$$= M_j^2 c_j \int_0^t r^{\beta_j} \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 \Big(\int_0^1 r^{\alpha_j - 1}(1-u)^{\alpha_j - 1} u^{-\beta_j} r^{-\beta_j} r du\Big) dr$$
$$= M_j^2 c_j \mathcal{B}(\alpha_j, 1-\beta_j) \int_0^t r^{\alpha_j} \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 dr$$
$$\leqslant M_j^2 c_j \mathcal{B}(\alpha_j, 1-\beta_j) T^{\alpha_j} \int_0^t \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r, N_r^\varepsilon) - \frac{\partial F}{\partial x_j}(r, N_r)\Big|^2 dr.$$

Since, $\frac{\partial F}{\partial x_j}(r,.)$ is continuous, $\lim_{\varepsilon\to 0}\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)=\frac{\partial F}{\partial x_j}(r,N_r)$ and $\left|\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)\right|^2\leqslant exp(2\lambda\mid N_r^\epsilon\mid^2)$.
Furthermore, $\mathbb{E}\left|\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)\right|^2\leqslant c'\prod_{j=1}^n(1-4\lambda Var(N_r^{j,\epsilon}))^{-\frac{1}{2}}$ and using (4.8). Thus, $\mathbb{E}\left|\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)\right|^2\leqslant K_1$.
By applying the dominated convergence theorem, $\int_0^t\mathbb{E}\left|\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)-\frac{\partial F}{\partial x_j}(r,N_r)\right|^2dr\underset{\epsilon\to 0}{\longrightarrow}0$. Thus,

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^t\left|K_t^{*,\varepsilon,j}\Big[\Big(\frac{\partial F}{\partial x_j}(s,N_s^\varepsilon)-\frac{\partial F}{\partial x_j}(s,N_s)\Big)\sigma_s^j\Big]\right|^2ds=0.$$

We show now that the second term in (4.9) converges to zero where $\epsilon\to 0$.
Indeed, we have

$$K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s)\sigma_s^j\Big)-K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s)\sigma_s^j\Big)=\int_s^t\sigma_r^j\frac{\partial F}{\partial x_j}(r,N_r)\Big[\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big]dr$$

Then, we have

$$\begin{aligned}
&\mathbb{E}\int_0^t\left|K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s)\sigma_s^j\Big)-K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s)\sigma_s^j\Big)\right|^2ds\\
&\leqslant\mathbb{E}\int_0^t\Big|\int_s^t\sigma_r^j\frac{\partial F}{\partial x_j}(r,N_r)\Big[\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big]dr\Big|^2ds\\
&\leqslant\int_0^t\mathbb{E}\Big(\int_s^t\Big|\sigma_r^j\frac{\partial F}{\partial x_j}(r,N_r)\Big|\Big|\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big|dr\Big)^2ds\\
&\leqslant M_j\int_0^t\int_s^t\mathbb{E}\Big|\frac{\partial F}{\partial x_j}(r,N_r)\Big|^2\Big|\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big|drds\\
&\leqslant M_j'\int_0^t\int_s^t\Big|\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big|drds.
\end{aligned}$$

Furthermore, $\lim_{\varepsilon\to 0}\frac{\partial K^j}{\partial r}(r+\varepsilon,s)=\frac{\partial K^j}{\partial r}(r,s)$ and we have

$$\begin{aligned}
\Big|\frac{\partial K^j}{\partial r}(r+\varepsilon,s)-\frac{\partial K^j}{\partial r}(r,s)\Big|&\leqslant c_j\Big[(r+\varepsilon-s)^{\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{\beta_j}+(r-s)^{\alpha_j-1}\Big(\frac{r}{s}\Big)^{\beta_j}\Big]\\
&\leqslant c_j\Big[(r-s)^{\alpha_j-1}\Big(\frac{r+1}{s}\Big)^{\beta_j}+(r-s)^{\alpha_j-1}\Big(\frac{r}{s}\Big)^{\beta_j}\Big].
\end{aligned}$$

Moreover, $\int_0^t\Big(\int_s^t(r-s)^{\alpha_j-1}\Big(\frac{r}{s}\Big)^{\beta_j}dr\Big)ds<+\infty$. Applying the dominated convergence theorem we get

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^t\left|K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s^\varepsilon)\sigma_s^j\Big)-K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(s,N_s)\sigma_s^j\Big)\right|^2ds=0.$$

The proof of step 2 is complete.

Step 3.

Let $\alpha_{3,\varepsilon}^j=\mathbb{E}\Big|A_{3,\varepsilon}^j-\frac{1}{2}\int_0^t\frac{\partial^2F}{\partial x_j^2}(r,N_r)\frac{\partial}{\partial r}\Big(\int_0^r\Big(K_r^{*,j}(\sigma^j)\Big)_s^2ds\Big)dr\Big|$.

Let us prove $\lim_{\varepsilon\to 0}\alpha^j_{3,\varepsilon}=0$. In fact, we have

$$\begin{aligned}\alpha^j_{3,\varepsilon}&=\mathbb{E}\Big|\frac{1}{2}\int_0^t\frac{\partial^2F}{\partial x_j^2}(r,N^\varepsilon_r)\frac{\partial}{\partial r}\Big(\int_0^r\Big(K^{*,\epsilon,j}_r(\sigma^j)\Big)^2_s ds\Big)dr-\frac{1}{2}\int_0^t\frac{\partial^2F}{\partial x_j^2}(r,N_r)\frac{\partial}{\partial r}\Big(\int_0^r\Big(K^{*,j}_r(\sigma^j)\Big)^2_s ds\Big)dr\Big|\\
&\leqslant\underbrace{\frac{1}{2}\mathbb{E}\Big|\int_0^t\frac{\partial^2F}{\partial x_j^2}(r,N_r)\Big(\frac{\partial}{\partial r}\int_0^r\Big[\Big(K^{*,\epsilon,j}_r(\sigma^j)\Big)^2_s-\Big(K^{*,j}_r(\sigma^j)\Big)^2_s\Big]ds\Big)dr\Big|}_{:=\alpha^j_{31,\varepsilon}}\\
&+\underbrace{\frac{1}{2}\mathbb{E}\Big|\int_0^t\Big[\frac{\partial^2F}{\partial x_j^2}(r,N^\varepsilon_r)-\frac{\partial^2F}{\partial x_j^2}(r,N_r)\Big]\frac{\partial}{\partial r}\Big(\int_0^r\Big(K^{*,\epsilon,j}_r(\sigma^j)\Big)^2_s ds\Big)dr\Big|}_{:=\alpha^j_{32,\varepsilon}}.\end{aligned}$$

By applying Cauchy-Schwarz inequality, we get

$$2\alpha^j_{32,\varepsilon}\leqslant c'\Big(\mathbb{E}\int_0^t\Big[\frac{\partial^2F}{\partial x_j^2}(r,N^\varepsilon_r)-\frac{\partial^2F}{\partial x_j^2}(r,N_r)\Big]^2dr\Big)^{\frac{1}{2}}\Big(\int_0^T\Big|\frac{\partial}{\partial r}\int_0^r\Big(K^{*,\varepsilon,j}_r(\sigma^j)\Big)^2_s ds\Big|^2dr\Big)^{\frac{1}{2}}.$$

Furthermore,

$$\begin{aligned}\frac{\partial}{\partial r}\int_0^r\Big(K^{*,\epsilon,j}_r(\sigma^j)\Big)^2_s ds&=\frac{\partial}{\partial r}\int_0^r\Big(\int_s^r\sigma^j_\theta\frac{\partial K^j}{\partial\theta}(\theta+\varepsilon,s)d\theta\Big)^2ds=\int_0^r\frac{\partial}{\partial r}\Big(\int_s^r\sigma^j_\theta\frac{\partial K^j}{\partial\theta}(\theta+\varepsilon,s)d\theta\Big)^2ds\\
&=\int_0^r 2\frac{\partial}{\partial r}\Big(\int_s^r\sigma^j_\theta\frac{\partial K^j}{\partial\theta}(\theta+\varepsilon,s)d\theta\Big)\int_s^r\sigma^j_\theta\frac{\partial K^j}{\partial\theta}(\theta+\varepsilon,s)d\theta ds\\
&\leqslant 2\int_0^r\Big|\sigma^j_r\frac{\partial K^j}{\partial r}(r+\varepsilon,s)\Big|\Big(\int_s^r\Big|\sigma^j_\theta\frac{\partial K^j}{\partial\theta}(\theta+\varepsilon,s)\Big|d\theta\Big)ds\\
&\leqslant 2c_j^2\int_0^r|\sigma^j_r|(r+\varepsilon-s)^{\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{\beta_j}\Big(\int_s^r|\sigma^j_\theta|(\theta+\varepsilon-s)^{\alpha_j-1}\Big(\frac{\theta+\varepsilon}{s}\Big)^{\beta_j}d\theta\Big)ds\\
&\leqslant 2c_j^2M_j^2\int_0^r(r+\varepsilon-s)^{\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{\beta_j}\Big(\int_s^r(\theta+\varepsilon-s)^{\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{\beta_j}d\theta\Big)ds.\end{aligned}$$

So,

$$\begin{aligned}\Big|\frac{\partial}{\partial r}\int_0^r\Big(K^{*,\epsilon,j}_r(\sigma^j)\Big)^2_s ds\Big|^2&\leqslant\frac{4c_j^4M_j^4}{\alpha_j^2}\Big(\int_0^r(r+\varepsilon-s)^{\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{2\beta_j}(r+\varepsilon-s)^{\alpha_j}ds\Big)^2\\
&\leqslant\frac{4c_j^4M_j^4}{\alpha_j^2}\Big(\int_0^r(r+\varepsilon-s)^{2\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{2\beta_j}ds\Big)^2\\
&=\frac{4c_j^4M_j^4}{\alpha_j^2}\Big(\int_0^r(r+\varepsilon)^{2\alpha_j-1}\Big(1-\frac{s}{r+\varepsilon}\Big)^{2\alpha_j-1}\Big(\frac{r+\varepsilon}{s}\Big)^{2\beta_j}ds\Big)^2\\
&=\frac{4c_j^4M_j^4}{\alpha_j^2}\Big((r+\varepsilon)^{2\alpha_j-1}\int_0^{r/r+\varepsilon}(1-v)^{2\alpha_j-1}v^{-2\beta_j}(r+\varepsilon)dr\Big)^2\\
&\leqslant\frac{4c_j^4M_j^4}{\alpha_j^2}(r+\varepsilon)^{4\alpha_j}\Big(\int_0^{r/r+\varepsilon}(1-v)^{2\alpha_j-1}v^{-2\beta_j}dv\Big)^2\\
&\leqslant\frac{4c_j^4M_j^4}{\alpha_j^2}(r+\varepsilon)^{4\alpha_j}\mathcal{B}(\frac{r}{r+\varepsilon};2\alpha_j,-2\beta_j+1)^2\end{aligned}$$

$$\leqslant \frac{4c_j^4 M_j^4}{\alpha_j^2}(r+1)^{4\alpha_j}\mathcal{B}(2\alpha_j, -2\beta_j+1)^2$$

Then,

$$\int_0^T \Big|\frac{\partial}{\partial r}\int_0^r \Big(K_r^{*,\epsilon,j}(\sigma^j)\Big)_s^2 ds\Big|^2 dr \leqslant \frac{4c_j^4 M_j^4}{\alpha_j^2}\mathcal{B}(2\alpha_j, -2\beta_j+1)^2 \int_0^T (r+1)^{4\alpha_j} dr$$
$$\leqslant \frac{4c_j^4 M_j^4}{\alpha_j^2}\mathcal{B}(2\alpha_j, -2\beta_j+1)^2 \frac{(T+1)^{4\alpha_j+1}}{4\alpha_j+1} < \infty.$$

Now we prove

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^t \Big(\frac{\partial^2 F}{\partial x_j^2}(r, N_r^\varepsilon) - \frac{\partial^2 F}{\partial x_j^2}(r, N_r)\Big)^2 dr = 0. \tag{4.10}$$

Indeed, $\frac{\partial^2 F}{\partial x_j^2}(r,.)$ is continuous and $\lim_{\varepsilon\to 0}\frac{\partial^2 F}{\partial x_j^2}(r, N_r^\varepsilon) = \frac{\partial^2 F}{\partial x_j^2}(r, N_r)$.

Moreover, $\mathbb{E}\Big|\frac{\partial^2 F}{\partial x_j^2}(r, N_r^\varepsilon)\Big|^2 \leqslant c'\prod_{j=1}^n (1-4\lambda Var(N_r^{j,\epsilon}))^{-\frac{1}{2}} \leqslant c'\prod_{j=1}^n (1-4\lambda \sup_{r\in[0,T]} Var(N_r^{j,\epsilon}))^{-\frac{1}{2}}$

and using (4.8). Thus, $\mathbb{E}\Big|\frac{\partial^2 F}{\partial x_j^2}(r, N_r^\varepsilon)\Big|^2 \leqslant K_1$.

Applying the dominated convergence theorem, we get (4.10). Thus, $\lim_{\varepsilon\to 0}\alpha_{32,\varepsilon}^j = 0$ since the second member is bounded.

Now we prove that $\lim_{\varepsilon\to 0}\alpha_{31,\varepsilon}^j = 0$. By applying the Cauchy-Schwarz inequality

$$2\alpha_{31,\varepsilon}^j \leqslant \Big(\mathbb{E}\int_0^t \Big(\frac{\partial^2 F}{\partial x_j^2}(r, N_r)\Big)^2 dr\Big)^{\frac{1}{2}} \Big(\int_0^T \Big|\frac{\partial}{\partial r}\int_0^r \Big[(K_r^{*,\varepsilon,j}(\sigma^j))_s^2 - (K_r^{*,j}(\sigma^j))_s^2\Big] ds\Big|^2 dr\Big)^{\frac{1}{2}}$$
$$\leqslant kT^{\frac{1}{2}}\Big(\int_0^T \Big|\frac{\partial}{\partial r}\int_0^r \Big[(K_r^{*,\varepsilon,j}(\sigma^j))_s^2 - (K_r^{*,j}(\sigma^j))_s^2\Big] ds\Big|^2 dr\Big)^{\frac{1}{2}}.$$

It remains to be verified that $\lim_{\varepsilon\to 0}\int_0^T \Big|\frac{\partial}{\partial r}\int_0^r \Big[(K_r^{*,\varepsilon,j}(\sigma^j))_s^2 - (K_r^{*,j}(\sigma^j))_s^2\Big] ds\Big|^2 dr = 0.$

On the one hand,

$$\frac{\partial}{\partial r}\int_0^r \Big[\Big(K_r^{*,\varepsilon,j}(\sigma^j)\Big)_s^2 - \Big(K_r^{*,j}(\sigma^j)\Big)_s^2\Big] ds = 2\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta+\varepsilon, s) d\theta\Big) ds$$
$$- 2\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r, s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta, s) d\theta\Big) ds.$$

Therefore

$$\Big|\frac{\partial}{\partial r}\int_0^r \Big[\Big(K_r^{*,\varepsilon,j}(\sigma^j)\Big)_s^2 - \Big(K_r^{*,j}(\sigma^j)\Big)_s^2\Big] ds\Big|^2 \leqslant 4\Big|\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r, s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta, s) d\theta\Big) ds\Big|^2$$
$$+ 4\Big|\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta+\varepsilon, s) d\theta\Big) ds\Big|^2. \tag{4.11}$$

Moreover

$$\begin{aligned}
\Big|\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta,s)d\theta\Big)ds\Big| &\leqslant M_j \int_0^r (r-s)^{\alpha_j-1}\Big(\frac{r}{s}\Big)^{\beta_j}\Big(\int_s^r (\theta-s)^{\alpha_j-1}\Big(\frac{\theta}{s}\Big)^{\beta_j}d\theta\Big)ds \\
&\leqslant M_j T^{\beta_j}\int_0^r (r-s)^{\alpha_j-1}s^{-2\beta_j}\Big(\int_s^r (\theta-s)^{\alpha_j-1}\theta^{\beta_j}d\theta\Big)ds \\
&\leqslant M_j \frac{T^{\beta_j}}{\alpha_j}\int_0^r (r-s)^{\alpha_j-1}s^{-2\beta_j}r^{\beta_j}(r-s)^{\alpha_j}ds \\
&\leqslant C_j'\int_0^r r^{\beta_j}(r-s)^{2\alpha_j-1}s^{-2\beta_j}ds \\
&\leqslant C_j'\int_0^r r^{\beta_j}r^{2\alpha_j-1}\Big(1-\frac{s}{r}\Big)^{2\alpha_j-1}s^{-2\beta_j}ds \\
&= C_j'\int_0^1 r^{\beta_j}r^{2\alpha_j-1}(1-u)^{2\alpha_j-1}u^{-2\beta_j}r^{-2\beta_j}r du \\
&\leqslant C_j' r^{2\alpha_j-\beta_j}\mathcal{B}(2\alpha_j,1-2\beta_j) \\
&\leqslant C_j'' r^{2\alpha_j-\beta_j}.
\end{aligned}$$

Therefore

$$\Big|\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\Big(\int_s^r \sigma_\theta^j \frac{\partial K^j}{\partial \theta}(\theta,s)d\theta\Big)ds\Big|^2 \leqslant C_j''' r^{4\alpha_j-2\beta_j}.$$

A similar caluclation for the second term of (4.11). Thus,

$$\int_0^T \Big|\frac{\partial}{\partial r}\int_0^r \Big[(K_r^{*,\varepsilon,j}(\sigma^j))_s^2 - (K_r^{*,j}(\sigma^j))_s^2\Big]ds\Big|^2 dr \leqslant C_j'''\int_0^T r^{4\alpha_j-2\beta_j}dr < \infty.$$

Thus, $\lim_{\varepsilon\to 0}\alpha_{31,\varepsilon}^j = 0$. Consequently, $\lim_{\varepsilon\to 0}\alpha_{3,\varepsilon}^j = 0$.

Step 4.

We pose $\alpha_{4,\varepsilon}^j = \Big(\mathbb{E}\Big|\int_0^t \frac{\partial F}{\partial x_j}(s,N_s^\varepsilon)\frac{d}{ds}b_s^j ds - \int_0^t \frac{\partial F}{\partial x_j}(s,N_s)\frac{d}{ds}b_s^j ds\Big|^2\Big)^{\frac{1}{2}}$.

We have $(\alpha_{4,\varepsilon}^j)^2 \leqslant m_j\int_0^t \mathbb{E}\Big|\frac{\partial F}{\partial x_j}(s,N_s^\varepsilon) - \frac{\partial F}{\partial x_j}(s,N_s)\Big|^2 ds$.

Using again the dominated convergence theorem, we obtain $\lim_{\varepsilon\to 0}\alpha_{4,\varepsilon}^j = 0$.

Finally, steps 3 and 4 imply that the square integrable random variable G^0 given by

$$G^0 = -\frac{1}{2}\sum_{j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_j^2}(r,N_r)\frac{\partial}{\partial r}\Big(\int_0^r \Big(K_s^{*,j}(\sigma^j)\Big)_r^2 ds\Big)dr - \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial x_j}(s,N_s)\frac{d}{ds}b_s^j ds,$$

It satisfies

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[H.\sum_{j=1}^n \int_0^t K_t^{*,\varepsilon,j}\Big(\frac{\partial F}{\partial x_j}(.,N_.)\sigma_.^j\Big)_s \delta W_s^j\Big] = \mathbb{E}\Big[H.G^0\Big], \text{ for any smooth random variable } H.$$

By step 2, we get

$$G^0 = \sum_{j=1}^n \int_0^t K_t^{*,j}\Big(\frac{\partial F}{\partial x_j}(.,N_.)\sigma_.^j\Big)_s \delta W_s^j. \ \blacksquare$$

Chapter 5

Solvability of linear Backward Stochastic Differential Equation (BSDE)

In this Chapter, we present new results related to the explicit form of the solution of linear BSDE's driven by Volterra processes in terms of the associated PDEs. These results generalize those given by Bender [7].

5.1 Connection between partial differential equations and BSDEs

Let $t_0 \geqslant 0$ be fixed, and denote by $\mathbb{L}^2(\mathbb{F},\mathbb{R}^n)$ the set of $\mathbb{F}$-adapted $\mathbb{R}^n$-valued processes Z such that $\mathbb{E}(\int_{t_0}^T \mid Z_t \mid^2 dt) < \infty$. We consider the linear BDSE for the processes $Y = (Y_t,\, t \in [t_0,T]) \in \mathbb{L}^2(\mathbb{F},\mathbb{R})$ and $Z = ((Z_t^1,...,Z_t^n),\, t \in [t_0,T]) \in \mathbb{L}^2(\mathbb{F},\mathbb{R}^n)$ given by

$$Y_t = g(N_T) - \int_t^T [f(s) + A_1(s)Y_s - A_2(s)Z_s]ds + \int_t^T Z_s \delta X_s,\ \ t \in [t_0,T], \tag{5.1}$$

where the real-valued functions g, f, A_1 and the $\mathbb{R}^n-$valued function A_2 are supposed to be known and the integral $\int_t^T Z_s \delta X_s$ is defined as a divergence integral as in Chapter 3. (5.1) is shown to be associated to the following second order linear PDE with terminal condition

$$\frac{\partial u}{\partial t}(t,x) = -\frac{1}{2}\sum_{j=1}^n \frac{d}{dt}Var(N_t^j)\frac{\partial^2 u}{\partial x_j^2}(t,x) + \sum_{j=1}^n [\sigma_t^j A_2^j(t) - \frac{d}{dt}b_t^j]\frac{\partial u}{\partial x_j}(t,x) + A_1(t)u(t,x) + f(t), \tag{5.2}$$

$$u(T,x) = g(x),\ (t,x) \in [t_0,T) \times \mathbb{R}^n.$$

By means of the Itô formula in Chapter 4 we show that

$$Y_t := u(t, N_t) \text{ and } Z_t^j := -\sigma_t^j \frac{\partial u}{\partial x_j}(t, N_t),\ j = 1,...,n \tag{5.3}$$

is a solution of (5.1). (5.2) is solved explicitely in this Chapter.
We will show later in this Chapter that u in fact satisfies the growth condition (4.3) under a suitable growth condition on the final condition in (5.1).
The Itô formula (4.4), applied to u reads

$$\begin{aligned} u(t, N_t) = u(T, N_T) &- \int_t^T \frac{\partial u}{\partial s}(s, N_s)ds - \sum_{j=1}^n \int_t^T \frac{\partial u}{\partial x_j}(s, N_s)\Big(\frac{d}{ds}b_s^j ds + \sigma_s^j \delta X_s^j\Big) \\ &- \frac{1}{2}\sum_{j=1}^n \int_t^T \frac{\partial^2 u}{\partial x_j^2}(s, N_s)\frac{d}{ds}Var(N_s^j)ds. \end{aligned} \tag{5.4}$$

An application of (5.2) to the second term on the right hand side of (5.4) yields

$$\begin{aligned} u(t, N_t) \quad = \quad & u(T, N_T) - \int_t^T \Big(f(s) + A_1(s)u(s, N_s) + \sum_{j=1}^n A_2^j(s)\sigma_s^j \frac{\partial u}{\partial x_j}(s, N_s)\Big)ds \\ & - \sum_{j=1}^n \int_t^T \sigma_s^j \frac{\partial u}{\partial x_j}(s, N_s)\delta X_s^j. \end{aligned} \tag{5.5}$$

Evidently, $(Y, (Z^1,..., Z^n))$ given by (5.3) solve (5.1) and are adapted to $\mathbb{F}$.

5.2 Explicit solution

In order to solve (5.2) explicitely, we have to assume, in addition to *(H1)* and *(H2)* for the kernels K^j, some regularity and integrability conditions. *(H5)* will be discussed later.

(H3) There exist constants $c, C > 0$ such that $c < \sigma^j < C$, $j = 1,...,n$, and A_1, f, $A_2 := (A_2^1,..., A_2^n)$ are bounded.

(H4) g is continous, and there exist positive constants c' and $\lambda' < \min\limits_{j=1,...,n}(16 \sup\limits_{t\in[0,T]} Var(N_t^j))^{-1}$ such that $\mid g(x) \mid \leqslant c' e^{\lambda' |x|^2}$ for all $x \in \mathbb{R}^n$.

(H5) $d/dtVar(N_t^j) > 0$ for all $t \in [t_0,T]$ and $\int_{t_0}^T (Var(N_T^j) - Var(N_t^j))^{-1/2} dt < \infty$, $j = 1,...,n$.

Theorem 5.2.1. *Assume that (H1)-(H5) hold, and let* $v(t,z) = (2\pi t)^{-1/2}\exp(-z^2/2t)$. *Then* $(Y,(Z^1,...,Z^n))$ *given by* (5.3) *solves* (5.1), *where the classical solution of* (5.2) *is given by*

$$u(t,x) = -\int_t^T \exp\Big(\int_s^t A_1(r)dr\Big)f(s)ds + \exp\Big(-\int_t^T A_1(s)ds\Big) \times \int_{\mathbb{R}^n} g(y)\prod_{j=1}^n v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T (\sigma_s^j A_2^j(s) - \frac{d}{ds}b_s^j)ds - y_j\Big)dy. \quad (5.6)$$

We show now that u verifies the growth condition (4.3).

Lemma 5.2.2. *Let* u *be given by* (5.6). *Then there are positive constants* M *and* $\lambda < \min_{j=1,...n}(4\sup_{t\in[0,T]} Var(N_t^j))^{-1}$ *such that*

1) $|\ u(t,x)\ | \leqslant Me^{\lambda|x|^2}$, $(t,x)\in[t_0,T]\times\mathbb{R}^n$,

2) $\Big|\frac{\partial u}{\partial x_j}(t,x)\Big| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1/2}e^{\lambda|x|^2}$, $(t,x)\in[t_0,T)\times\mathbb{R}^n$,

3) $\Big|\frac{\partial^2 u}{\partial x_j^2}(t,x)\Big| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1}e^{\lambda|x|^2}$, $(t,x)\in[t_0,T)\times\mathbb{R}^n$.

Proof. 1) We write r_s^i for $\sigma_s^i A_2^i(s) - \frac{d}{ds}b_s^i$ and D_t^i for $Var(N_T^i) - Var(N_t^i)$ in (5.6). Then, we have, for some constant $C > 0$,

$$\begin{aligned}
&\Big|u(t,x)\Big| \\
&= \Big| -\int_t^T e^{\int_s^t A_1(u)du} f(s)ds + e^{-\int_t^T A_1(s)ds}\int_{\mathbb{R}^n} g(y)\prod_{i=1}^n v(D_t^i, x_i - \int_t^T r_s^i ds - y_i)dy\Big| \\
&\leqslant C\int_{\mathbb{R}^n} exp\Big(\lambda'\Big\|x - \int_t^T r_s ds - z\Big\|^2\Big)\prod_{i=1}^n \frac{exp\Big(-\frac{z_i^2}{2D_t^i}\Big)}{\sqrt{2\pi D_t^i}}dz
\end{aligned}$$

Let us first prove that

$$\begin{aligned}
&exp\Big(\lambda'\Big\|x - \int_t^T r_s ds - z\Big\|^2\Big)\prod_{i=1}^n \frac{exp\Big(-\frac{z_i^2}{2D_t^i}\Big)}{\sqrt{2\pi D_t^i}} \\
&\leqslant exp\Big(2\lambda'\Big\|x - \int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n \frac{exp\Big[-\frac{z_i^2}{D_t^i}\Big(\frac{1}{2} - 2\lambda' Var N_T^i\Big)\Big]}{\sqrt{2\pi D_t^i}}
\end{aligned} \quad (5.7)$$

Let's check

$$exp\Big(\lambda'\Big\|x-\int_t^T r_s ds - z\Big\|^2\Big) \leqslant exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n exp\Big(2\lambda'\frac{VarN_T^i}{D_t^i}z_i^2\Big)$$

In fact, we have

$$\begin{aligned} exp\Big(\lambda'\Big\|x-\int_t^T r_s ds - z\Big\|^2\Big) &\leqslant exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)exp\Big(2\lambda'\|z\|^2\Big) \\ &\leqslant exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)exp\Big(2\lambda'\sum_{i=1}^n \frac{VarN_T^i}{D_t^i}z_i^2\Big) \\ &= exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n exp\Big(2\lambda'\frac{VarN_T^i}{D_t^i}z_i^2\Big) \end{aligned}$$

(5.7) follows now easily.
Consequently, using (5.7) we obtain

$$\begin{aligned} \Big|u(t,x)\Big| &\leqslant Cexp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^n}\prod_{i=1}^n exp\Big(\frac{\frac{1}{2}-2\lambda' VarN_T^i}{D_t^i}z_i^2\Big)\frac{1}{\sqrt{2\pi D_t^i}}dz \\ &= Cexp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n \frac{1}{\sqrt{2\pi D_t^i}}\int_{\mathbb{R}} exp\Big(\frac{1-4\lambda' VarN_T^i}{2D_t^i}z_i^2\Big)dz_i \\ &= Cexp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n \frac{1}{\sqrt{2\pi D_t^i}}\sqrt{\frac{2\pi D_t^i}{1-4\lambda' VarN_T^i}} \\ &= Cexp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{i=1}^n \frac{1}{\sqrt{1-4\lambda' VarN_T^i}} \\ &\leqslant C\frac{1}{\Big[\sqrt{1-4\lambda'\max\limits_i VarN_T^i}\Big]^n}exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big) \\ &= C_1 exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big) \\ &= C_1 exp\Big(4\lambda'\parallel x\parallel^2\Big)exp\Big(4\lambda'\Big\|\int_t^T r_s ds\Big\|^2\Big) = Mexp\Big(\lambda\parallel x\parallel^2\Big). \end{aligned}$$

Then, 1) follows by choosing $M = C_1 exp\Big(4\lambda'\Big\|\int_t^T r_s ds\Big\|^2\Big)$ and $\lambda = 4\lambda'$.
2) We prove that $\Big|\frac{\partial}{\partial x_i}u(t,x)\Big| \leqslant M\frac{1}{\sqrt{D_t^i}}e^{\lambda\|x\|^2}$. In fact,

$$\begin{aligned} &\frac{\partial}{\partial x_i}u(t,x) \\ &= e^{-\int_t^T A_1(s)ds}\int_{\mathbb{R}^n} g(y)\frac{\partial}{\partial \xi_i}v(D_t^i,\xi_i)\Big|_{\xi_i = x_i - \int_t^T r_s^i ds - y_i}\prod_{j\neq i}^n v(D_t^j, x_j - \int_t^T r_s^j ds - y_j)dy \end{aligned}$$

$$= e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{1}{\sqrt{2\pi D_t^i}} \Big(- \frac{x_i - \int_t^T r_s^i ds - y_i}{D_t^i} \Big) exp\Big(- \frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i} \Big)$$
$$\times \prod_{j \neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j} \Big) dy.$$

Thus

$$\Big| \frac{\partial}{\partial x_i} u(t,x) \Big| \leqslant C \int_{\mathbb{R}^n} e^{\lambda' \|y\|^2} \frac{1}{\sqrt{2\pi D_t^i}} \Big| - \frac{x_i - \int_t^T r_s^i ds - y_i}{D_t^i} \Big| exp\Big(- \frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i} \Big)$$
$$\times \prod_{j \neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j} \Big) dy.$$

Then, by the change of variables $z_i = x_i - \int_t^T r_s^i ds - y_i$, we get

$$\Big| \frac{\partial}{\partial x_i} u(t,x) \Big| \leqslant C \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi D_t^i}} \Big| \frac{z_i}{D_t^i} \Big| exp\Big(- \frac{z_i^2}{2D_t^i} \Big) e^{\lambda' \Big\| x - \int_t^T r_s ds - z \Big\|^2} \prod_{j \neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{z_j^2}{2D_t^j} \Big) dz.$$

Prove that

$$exp\Big(\lambda' \Big\| x - \int_t^T r_s ds - z \Big\|^2 \Big) \prod_{j \neq i} \frac{exp\Big(- \frac{z_j^2}{2D_t^j} \Big)}{\sqrt{2\pi D_t^j}} \leqslant exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds \Big\|^2 \Big) exp(2\lambda' z_i^2)$$
$$\times \prod_{j \neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j} \Big(\frac{1}{2} - 2\lambda' Var N_T^j \Big) \Big]}{\sqrt{2\pi D_t^j}}$$

In fact, similarly to (5.7), we have

$$exp\Big(\lambda' \Big\| x - \int_t^T r_s ds - z \Big\|^2 \Big) \leqslant exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds \Big\|^2 \Big) exp\Big(2\lambda' \|z\|^2 \Big)$$
$$= exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds \Big\|^2 \Big) exp(2\lambda' z_i^2) \prod_{j \neq i} exp(2\lambda' z_j^2)$$
$$\leqslant exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds \Big\|^2 \Big) exp(2\lambda' z_i^2) \prod_{j \neq i} exp\Big(2\lambda' \frac{Var N_T^j}{D_t^j} z_j^2 \Big).$$

Consequently,

$$\Big| \frac{\partial}{\partial x_i} u(t,x) \Big|$$

$$\leqslant C\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\Big|\frac{z_i}{D_t^i}\Big|exp\Big(-\frac{z_i^2}{2D_t^i}\Big)exp(2\lambda' z_i^2)exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz$$
$$=Cexp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\frac{1}{\sqrt{D_t^i}}\frac{\Big|z_i\Big|}{\sqrt{D_t^i}}exp(2\lambda' z_i^2)$$
$$\times exp\Big(-\frac{z_i^2}{2D_t^i}\Big)\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz.$$

Moreover, for any $\epsilon>0$, there is a constant K_ϵ such that $\frac{\Big|z_i\Big|}{\sqrt{D_t^i}}\leqslant K_\epsilon exp\Big(\frac{\epsilon}{D_t^i}z_i^2\Big)$.

$$\Big|\frac{\partial}{\partial x_i}u(t,x)\Big|$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\frac{1}{\sqrt{D_t^i}}exp\Big(\frac{\epsilon}{D_t^i}z_i^2\Big)$$
$$\times exp(2\lambda' z_i^2)exp\Big(-\frac{z_i^2}{2D_t^i}\Big)\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\frac{1}{\sqrt{D_t^i}}exp\Big(\frac{(\epsilon-\frac{1}{2}+2\lambda' VarN_T^i)}{D_t^i}z_i^2\Big)$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}D_t^i}exp\Big(\frac{-(\frac{1}{2}-\epsilon-2\lambda' VarN_T^i)}{D_t^i}z_i^2\Big)dz_i\Big]$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' S^j(T)\Big)\Big]}{\sqrt{2\pi D_t^j}}dz',\quad z'=(z_1,...,z_{i-1},z_{i+1},...,z_n)$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\int_{\mathbb{R}^{n-1}}\Big[\frac{1}{\sqrt{2\pi}D_t^i}\sqrt{\frac{2\pi D_t^i}{1-2\epsilon-4\lambda' VarN_T^i}}\Big]$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_t^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz'$$
$$=CK_\epsilon\frac{1}{\sqrt{1-2\epsilon-4\lambda' VarN_T^i}}\frac{1}{\sqrt{D_t^i}}exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\prod_{j\neq i}\int_{\mathbb{R}}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_t^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz_j$$

$$= CK_\epsilon \frac{1}{\sqrt{1-2\epsilon-4\lambda' VarN_T^i}} \frac{1}{\sqrt{D_t^i}} exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds \Big\|^2\Big) \prod_{j\neq i} \frac{1}{\sqrt{2\pi D_t^j}} \sqrt{\frac{2\pi D_t^j}{1-4\lambda' VarN_T^j}}$$

$$\leqslant CK_\epsilon \frac{1}{\sqrt{1-2\epsilon-4\lambda' VarN_T^i}} \frac{1}{\sqrt{D_t^i}} exp(4\lambda' \parallel x \parallel^2) exp\Big(4\lambda' \Big\| \int_t^T r_s ds \Big\|^2\Big) \frac{1}{\Big[\sqrt{1-4\lambda' \max\limits_j VarN_T^j}\Big]^{n-1}}$$

$$\leqslant \frac{M}{\sqrt{D_t^i}} exp\Big(\lambda \parallel x \parallel^2\Big) \text{ with a suitable constant } M \text{ and } \lambda = 4\lambda'.$$

3) Now, we prove that $\Big|\frac{\partial^2}{\partial x_i^2} u(t,x)\Big| \leqslant M \frac{1}{\Big(D_t^i\Big)^{\frac{3}{2}}} e^{\lambda \|x\|^2}$.

Indeed, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial x_i^2} u(t,x) \\
&= e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{1}{\sqrt{2\pi D_t^i}} \Big(\frac{x_i - \int_t^T r_s^i ds - y_i}{D_t^i}\Big)^2 exp\Big(-\frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i}\Big) \\
&\times \prod_{j\neq i}^n \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(-\frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j}\Big) dy \\
&+ e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{1}{\sqrt{2\pi D_t^i}} (-1) \frac{1}{D_t^i} exp\Big(-\frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i}\Big) \\
&\times \prod_{j\neq i}^n \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(-\frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j}\Big) dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big|\frac{\partial^2}{\partial x_i^2} u(t,x)\Big| \\
&\leqslant C e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} e^{\lambda' \|y\|^2} \frac{1}{\sqrt{2\pi D_t^i}} \Big(\frac{(x_i - \int_t^T r_s^i ds - y_i)}{D_t^i}\Big)^2 exp\Big(-\frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i}\Big) \\
&\times \prod_{j\neq i}^n \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(-\frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j}\Big) dy \\
&+ e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} e^{\lambda' \|y\|^2} \frac{1}{\sqrt{2\pi D_t^i}} \frac{1}{D_t^i} exp\Big(-\frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i}\Big)
\end{aligned}$$

$$\times \prod_{j\neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j}\Big) dy.$$

We pose again $z_i = x_i - \int_t^T r_s^i ds - y_i$ and obtain

$$\Big|\frac{\partial^2}{\partial x_i^2} u(t,x)\Big| \leqslant C \int_{\mathbb{R}^n} e^{\lambda' \Big\| x - \int_t^T r_s ds - z\Big\|^2} \frac{2\pi}{[2\pi D_t^i]^{\frac{3}{2}}} \frac{z_i^2}{D_t^i} exp\Big(- \frac{z_i^2}{2D_t^i}\Big) \prod_{j\neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{z_j^2}{2D_t^j}\Big) dz$$
$$+ C \int_{\mathbb{R}^n} e^{\lambda' \Big\| x - \int_t^T r_s ds - z\Big\|^2} \frac{1}{[2\pi D_t^i]^{\frac{3}{2}}} exp\Big(- \frac{z_i^2}{2D_t^i}\Big) \prod_{j\neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{z_j^2}{2D_t^j}\Big) dz.$$

Similarly as before we see that

$$exp\Big(\lambda' \Big\| x - \int_t^T r_s ds - z\Big\|^2\Big) \prod_{j\neq i} \frac{exp\Big(- \frac{z_i^2}{2D_t^j}\Big)}{\sqrt{2\pi D_t^j}} \leqslant exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds\Big\|^2\Big) exp(2\lambda z_i^2)$$
$$\times \prod_{j\neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j}\Big(\frac{1}{2} - 2\lambda' Var N_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}.$$

Consequently,

$$\Big|\frac{\partial^2}{\partial x_i^2} u(t,x)\Big|$$
$$\leqslant C \int_{\mathbb{R}^n} \frac{2\pi}{[2\pi D_t^i]^{\frac{3}{2}}} \frac{z_i^2}{D_t^i} exp\Big(- \frac{z_i^2}{2D_t^i}\Big) exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds\Big\|^2\Big) exp(2\lambda' z_i^2) \prod_{j\neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j}\Big(\frac{1}{2} - 2\lambda' Var N_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz$$
$$+ C \int_{\mathbb{R}^n} \frac{1}{[2\pi D_t^i]^{\frac{3}{2}}} exp\Big(- \frac{z_i^2}{2D_t^i}\Big) exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds\Big\|^2\Big) exp(2\lambda' z_i^2) \prod_{j\neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j}\Big(\frac{1}{2} - 2\lambda' Var N_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz$$
$$\leqslant C \frac{2\pi exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}} D_t^i} \int_{\mathbb{R}^{n-1}} \Big[\int_{\mathbb{R}} z_i^2 exp\Big(- \frac{1 - 4\lambda' Var N_T^i}{2D_t^i} z_i^2\Big) dz_i\Big]$$
$$\times \prod_{j\neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j}\Big(\frac{1}{2} - 2\lambda' Var N_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz'$$
$$+ C \frac{exp\Big(2\lambda' \Big\| x - \int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}}} \int_{\mathbb{R}^{n-1}} \Big[\int_{\mathbb{R}} exp\Big(- \frac{1 - 4\lambda' Var N_T^i}{2D_t^i} z_i^2\Big) dz_i\Big]$$

$$\times \prod_{j\neq i} \frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz', \quad z' = (z_1,...,z_{i-1},z_{i+1},...,z_n)$$

$$\leqslant C\frac{2\pi exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}} D_t^i}\int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}} u_i^2 \frac{D_t^i}{1-4\lambda' VarN_T^i} exp\Big(-\frac{1}{2}u_i^2\Big)\frac{\sqrt{D_t^i}}{\sqrt{1-4\lambda' VarN_T^i}}du_i\Big]$$

$$\times \prod_{j\neq i} \frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz'$$

$$+ C\frac{exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}}}\int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}} exp\Big(-\frac{1}{2}v_i^2\Big)\frac{\sqrt{D_t^i}}{\sqrt{1-4\lambda' VarN_T^i}}dv_i\Big]$$

$$\times \prod_{j\neq i} \frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}} dz'$$

$$\leqslant C\frac{(2\pi)^{\frac{3}{2}} exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\sqrt{D_t^i}}{[2\pi D_t^i]^{\frac{3}{2}}(1-4\lambda' VarN_T^i)^{\frac{3}{2}}}\prod_{j\neq i}\frac{1}{\sqrt{2\pi D_t^j}}\int_{\mathbb{R}} exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]dz_j$$

$$+ C\frac{exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}}}\frac{\sqrt{2\pi}\sqrt{D_t^i}}{\sqrt{1-4\lambda' VarN_T^i}}\prod_{j\neq i}\frac{1}{\sqrt{2\pi D_t^j}}\int_{\mathbb{R}} exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]dz_j$$

$$= C\frac{(2\pi)^{\frac{3}{2}} exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\sqrt{D_t^i}}{[2\pi D_t^i]^{\frac{3}{2}}(1-4\lambda' VarN_T^i)^{\frac{3}{2}}}\prod_{j\neq i}\frac{1}{\sqrt{2\pi D_t^j}}\sqrt{\frac{2\pi D_t^j}{1-4\lambda' VarN_T^j}}$$

$$+ C\frac{exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}}}\frac{\sqrt{2\pi}\sqrt{D_t^i}}{\sqrt{1-4\lambda' VarN_T^i}}\prod_{j\neq i}\frac{1}{\sqrt{2\pi D_t^j}}\sqrt{\frac{2\pi D_t^j}{1-4\lambda' VarN_T^j}}$$

$$\leqslant C\frac{exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)\sqrt{D_t^i}}{(D_t^i)^{\frac{3}{2}}(1-4\lambda'(VarN_T^i)^{\frac{3}{2}}}\prod_{j\neq i}\frac{1}{\sqrt{1-4\lambda'\max\limits_j VarN_T^j)}}$$

$$+ C\frac{exp\Big(2\lambda'\Big\|x-\int_t^T r_s ds\Big\|^2\Big)}{[2\pi D_t^i]^{\frac{3}{2}}}\frac{\sqrt{2\pi}\sqrt{D_t^i}}{\sqrt{1-4\lambda' VarN_T^i}}\prod_{j\neq i}\frac{1}{\sqrt{1-4\lambda'\max\limits_j VarN_T^j)}}$$

$\leqslant M\frac{1}{D_t^i} exp(\lambda \parallel x \parallel^2)$ with a suitable constant M and $\lambda = 4\lambda'$. ■

Remark. Since u solves the PDE (5.2) we obtain, by the assumed regularity of the coefficients and the preceding lemma, that $\mid \frac{\partial u}{\partial t}(t,x) \mid$ satisfies the same type of inequality as in Lemma 5.2.2.

We are now in a position to prove the main result of this chapter:

Proof of Theorem 5.2.1 An explicit calculation of $\partial/\partial t u(t,x)$ shows that u is in fact the (classical) solution of (5.2). Indeed, we have

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= f_t - \int_t^T e^{\int_s^t A_1(u)du} f_s ds A_1(t) \\
&+ e^{-\int_t^T A_1(s)ds} A_1(t) \int_{\mathbb{R}^n} g(y) \prod_{i=1}^n v\Big(Var(N_T^i) - Var(N_t^i), x_i - \int_t^T r_s^i ds - y_i\Big) dy \\
&+ e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \sum_{i=1}^n \frac{\partial}{\partial \tau} v\Big(\tau, x_i - \int_t^T r_s^i ds - y_i\Big) \frac{\partial}{\partial t}\Big(-Var(N_t^i)\Big) \\
&\times \prod_{j\neq i} v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T r_s^j ds - y_j\Big) dy \\
&+ e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \sum_{i=1}^n \frac{\partial}{\partial \xi_i} v\Big(Var(N_T^i) - Var(N_t^i), \xi_i\Big) r_t^i \\
&\times \prod_{j\neq i} v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T r_s^j ds - y_j\Big) dy
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}u(t,x) &= e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{\partial}{\partial \xi_i} v\Big(Var(N_T^i) - Var(N_t^i), \xi_i\Big)\Big|_{\xi_i = x_i - \int_t^T r_s^i ds - y_i} \\
&\times \prod_{j\neq i} v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T r_s^j ds - y_j\Big) dy
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}u(t,x) &= e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{\partial^2}{\partial \xi_i^2} v\Big(Var(N_T^i) - Var(N_t^i), \xi_i\Big)\Big|_{\xi_i = x_i - \int_t^T r_s^i ds - y_i} \\
&\times \prod_{j\neq i} v\Big(Var(N_T^j) - Var(N_t^j), x_j - \int_t^T r_s^j ds - y_j\Big) dy
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial t}u(t,x) = f_t + A_1(t)u(t,x) - \frac{1}{2}\sum_{i=1}^n \frac{\partial}{\partial t} Var(N_t^i) \frac{\partial^2}{\partial x_i^2} u(t,x) + \sum_{i=1}^n r_t^i \frac{\partial}{\partial x_i} u(t,x).$$

By the preceding lemma u satisfies (5.5) with T replaced by $T-\varepsilon$. Since $\mathbb{E}\exp(2\lambda \mid N_t \mid^2)$ is bounded on $[0,T]$ for $\lambda < \min\limits_{j=1,\dots n} (4 \sup\limits_{t\in[0,T]} Var(N_t^j))^{-1}$, we obtain

$$\mathbb{E}\int_t^T \mid f(s) + A_1(s)u(s,N_s) + \sum_{j=1}^n A_2^j(s)\sigma_s^j \frac{\partial u}{\partial x_j}(s,N_s) \mid ds \leqslant C \sum_{j=1}^n \int_t^T (Var(N_T^j) - Var(N_t^j))^{-1/2} dt < \infty$$

for all $t < T$ by *(H5)*. By continuity of u and N, the terms in the first line of (5.5) with T replaced by $T - \varepsilon$ converge to the terms in (5.5) as $\varepsilon \to 0$. The divergence integrals converge also in the sense

$$\mathbb{E}\left[\int_t^{T-\varepsilon} \sigma_s^j \frac{\partial u}{\partial x_j}(s,N_s)\delta X_s^j F\right] \underset{\varepsilon\to 0}{\to} \mathbb{E}\left[\int_t^T \sigma_s^j \frac{\partial u}{\partial x_j}(s,N_s)\delta X_s^j F\right]$$

for all $j = 1,...,n$ and $F \in \mathcal{S}$. ■

Remark. We discuss now the **hypothesis *(H5)*.** The positivity of $\frac{d}{dt}Var(N_t^j)$ means that $Var(N_t^j)$ is (strictly) increasing on $[t_0,T]$. We note that

$$\frac{d}{dt}Var(N_t^j) = \frac{d}{dt}\int_0^t (K_t^{*,j}\sigma^j)_s^2 ds = 2\int_0^t \sigma_t^j \sigma_u^j \int_0^u \frac{\partial K^j}{\partial t}(t,s)\frac{\partial K^j}{\partial u}(u,s)dsdu = 2\sigma_t^j \int_0^t \sigma_u^j \phi^j(t,u)du.$$

Since $\sigma^j > 0$ by *(H3)*, a sufficient condition for $\frac{d}{dt}Var(N_t^j) > 0$ for all $t \in [t_0,T]$ is $\phi^j > 0$ on $[0,T]^2 \setminus [0,t_0]^2$. This is the case if $\frac{\partial K^j}{\partial u}(u,s) > 0$ for all $(u,s) \in [0,T]^2, u > s$. For mbm (Example 1) such cases can easily be found for Hurst functions h that are increasing or decreasing on $[0,T]$, but other examples of mbm show that $Var(N_t^j)$ increasing on $[t_0,T]$ is possible for suitable values of t_0 even if h is only of bounded variation.
For $h(t) = \frac{1}{5}sin(t) + \frac{3}{4}$ (of bounded variation) $Var(N_t)$ is increasing on certain intervals and decreasing on others. For $h(t) = \frac{1}{4}(1 - e^{-t}) + \frac{3}{5}$ (increasing) $Var(N_t)$ is increasing, and for $h(t) = \frac{1}{4}e^{-t} + \frac{3}{5}$ (decreasing) $Var(N_t)$ is increasing.

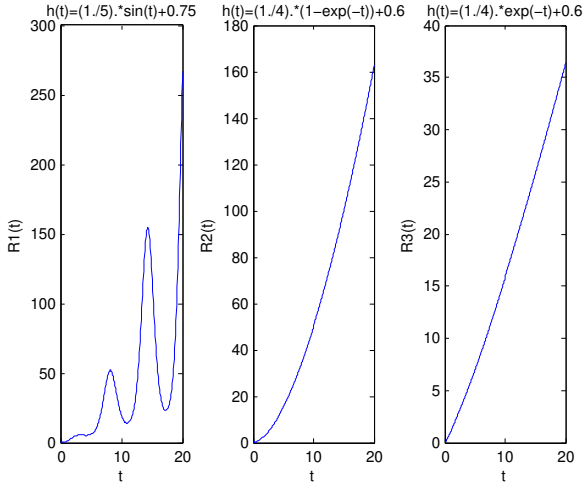


FIG. 5.1 : Graph $R(t) := Var(N_t)$ for $h(t) = \frac{1}{5}sin(t) + \frac{3}{4}$, $h(t) = \frac{1}{4}(1 - e^{-t}) + \frac{3}{5}$, $h(t) = \frac{1}{4}e^{-t} + \frac{3}{5}$

We note that for fractional brownian motion B^H with Hurst index $H > 1/2$, $\phi^H(t,u) = \frac{\partial^2}{\partial t \partial u}\mathbb{E}B_t^H B_u^H = C_H \mid t-u \mid^{2H-2} > 0$, where $C_H > 0$ is a constant depending on H.
We notice that the function ϕ in the case of the mbm, it is not necessarily positive.
For $h(t) = \frac{1}{5}sin(t) + \frac{3}{4}$, ϕ is positive or negative.
If $h(t) = \frac{1}{4}e^{-t} + \frac{3}{5}$, ϕ is positive.

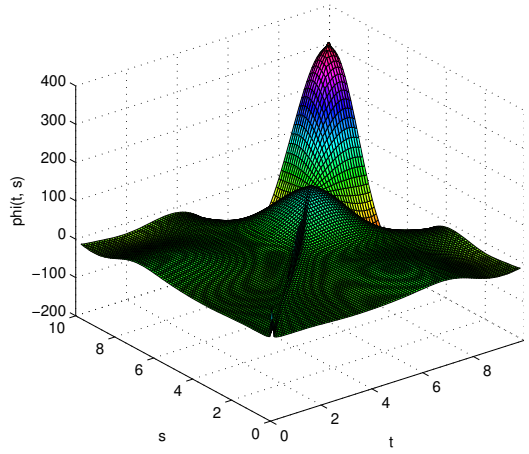


FIG. 5.2 : Graph ϕ for $h(t) = \frac{1}{5}sin(t) + \frac{3}{4}$

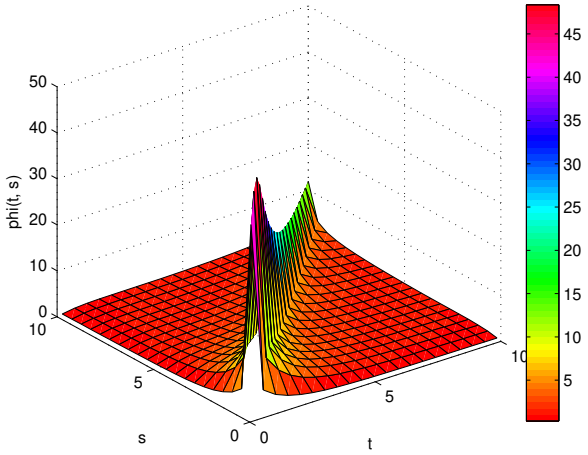


FIG. 5.3 : Graph ϕ for $h(t) = \frac{1}{4}e^{-t} + \frac{3}{5}$

Nota I thank Pr. Jean Rodolphe ROCHE who helped me in this numerical methods.

Let us comment now on the hypothesis of integrability of $(D_t^j)^{-1/2} := (Var(N_T^j) - Var(N_t^j))^{-1/2}$ near T. We have

$$\begin{aligned} D_t^j &= \mathbb{E}\Big[\Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)^2 - \Big(\int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)^2\Big] \\ &= \mathbb{E}\Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr + \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big) \\ &\times \Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr - \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)\Big]. \end{aligned}$$

We pose

$$a_1 = \int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr,$$

and

$$b_{1,t} = \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr.$$

Then

$$\begin{aligned} a_1 &= \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr + \int_0^t \delta W_s^j \int_t^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr + \int_t^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr \\ &:= b_{1,t} + b_{2,t} + b_{3,t}, \end{aligned}$$

with $a_1 - b_{1,t} = b_{2,t} + b_{3,t}$ we get

$$\begin{aligned} D_t^j &= \mathbb{E}\Big[(2b_{1,t} + b_{2,t} + b_{3,t})(b_{2,t} + b_{3,t})\Big] \\ &= \mathbb{E}\Big[2b_{1,t}b_{2,t} + 2b_{1,t}b_{3,t} + b_{2,t}^2 + 2b_{2,t}b_{3,t} + b_{3,t}^2\Big] = \mathbb{E}\Big[2b_{1,t}b_{2,t} + b_{2,t}^2 + b_{3,t}^2\Big]. \end{aligned}$$

Furthermore

$$\begin{aligned}\mathbb{E}\Big[2b_{1,t}b_{2,t}\Big] &= \int_0^t ds\Big[\int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr \int_t^T \sigma_{r'}^j \frac{\partial K^j}{\partial r'}(r',s)dr'\Big] \\ &= \int_0^T dr \int_0^T dr' \sigma_r^j \sigma_{r'}^j 1_{[0,t]\times[t,T]}(r,r') \int_0^{inf(r,r')} \frac{\partial K^j}{\partial r}(r,s)\frac{\partial K^j}{\partial r'}(r',s)ds \\ &= \int_0^t dr \int_t^T dr' \sigma_r^j \sigma_{r'}^j \phi(r,r').\end{aligned}$$

Moreover

$$\mathbb{E}[b_{2,t}^2] = \int_0^t ds \left(\int_t^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2,$$

and

$$\mathbb{E}[b_{3,t}^2] = \int_t^T ds \left(\int_s^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2.$$

Consequently

$$\begin{aligned}D_t^j &= \int_0^t dr \int_t^T dr' \sigma_r^j \sigma_{r'}^j \phi^j(r,r') + \int_0^t ds \left(\int_t^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2 + \int_t^T ds \left(\int_s^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2 \\ &=: A_t^1 + A_t^2 + A_t^3.\end{aligned}$$

Under the hypothesis *(H3)* for σ^j and if $\phi^j > 0$, a sufficient condition for $\int_{t_0}^T (D_t^j)^{-1/2} dt < \infty$ is

$$A_t^3 = \int_t^T (K_T^{*,j}\sigma^j)_s^2 ds \geqslant c(T-t)^a$$

for some constant $c > 0$ and $a \in (0,2)$ as $t \nearrow T$. For fractional brownian motion B^H with $H > 1/2$ this condition is satisfied with $a = H + 1/2$. For the Volterra processes in Examples 1-3 this condition is satisfied with $a = 2h(T)$ if h is such that $\frac{\partial K}{\partial u}(u,s) > 0$, for $(u,s) \in (t_0,T)^2, u > s$.

5.3 Application

A possible application of the result of this Chapter is to formulate and to solve, in the context of Volterra processes, the well known problem of self-financing trading strategies against a positive contingence claim ξ at a time T. Volterra processes are in fact more flexible than brownian motion to modelize the different kinds of uncertainties that occur on financial markets.

The model

We begin with the typical setup for continuous-time asset pricing: the basic securities consist of $n+1$ assets. One of them is a locally riskless (the money market instrument or bond) with price per unit P^0 governed by the equation

$$dP_t^0 = P_t^0 r_t dt,$$

wher r_t is the short rate. In addition to the bond, n risky securities (the stocks) are continuously traded. The price process P^j for one share of jth is modeled by the linear stochastic differential equation

$$dP_t^j = P_t^j(b_t^j dt + \sigma_t^j \delta X_t^j),$$

where b_t^j is the appreciation rate and σ_t^j is the volatility. Then $\theta_t^j := (b_t^j - r_t)/\sigma_t^j$ is called the risk premium for the risky asset j.
The problem is to find the part π_t^j of the wealth V_t to invest at time t in the asset j, $j = 1,...,n$. Of course this decision can only be based on the current information $\mathcal{F}_t$, i.e. the processes π_t^j and $\pi_t^0 := V_t - \sum_{j=1}^n \pi_t^j$ must be $\mathcal{F}_t$-adapted. In the model of Harrison and Pliska ([16]) V_t satisfies the equation $dV_t = r_t V_t dt + \sum_{j=1}^n (b_t^j - r_t)\pi_t^j dt + \sum_{j=1}^n \pi_t^j \sigma_t^j \delta X_t^j$ with final condition $V_T = \xi$. If ξ is of the form $g(N_T)$, $(V_t,\pi_t^1,...,\pi_t^n)$, $t \in [t_0,T]$, is the solution of (5.1) with $A_1(s) = r_s$, and $A_2^j(s) = \theta_s^j \sigma_s^j$, $j = 1,...,n$.

Chapter 6

Non linear BSDE in the one-dimensional case

In this Chapter, we establish the relation between BSDE (6.1) and PDE (6.2). We present the tools that interest us to prove the existence and the uniqueness of solutions of non linear BSDE without making reference to the associated PDE (6.2). For the construction of the solution we need the notion of quasi-conditional expectation, a Clark-Ocone type formula and another Itô formula for Volterra processes.

For the rest of the work, $(X_t, t \in [0,T])$ is a Volterra process that satisfies $(H1)$ and $(H2)$ and $\mathcal{S}$ be the set of random variables of the form $F = f(X(\varphi_1),....,X(\varphi_n))$, where $n \geqslant 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^\infty$ -function such that f and its partial derivatives have at most **polynomial growth**, and $\varphi_1,...,\varphi_n \in \mathcal{H}$.

6.1 Connection between partial differential equations and BSDEs revisited

We consider X defined on the probability space $(\Omega, \mathcal{F}, P)$ and given by (2.1). Let $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, t \in [0,T]\}$ the filtration generated by X and augmented by the P-null sets. Let N be given by (1.14), where σ is a bounded function on $[0,T]$ and suppose that $\frac{d}{dt}Var(N_t) > 0$ for all $t \in (0,T)$.

Let $t_0 \geqslant 0$ be fixed, and denote by $\mathbb{L}^2(\mathbb{F},\mathbb{R})$ the set of $\mathbb{F}$-adapted $\mathbb{R}$-valued processes Z such that $\mathbb{E}(\int_{t_0}^T \mid Z_t \mid^2 dt) < \infty$. We consider the non linear BDSE for the processes $Y = (Y_t, t \in [t_0,T]) \in \mathbb{L}^2(\mathbb{F},\mathbb{R})$ and $Z = (Z_t, t \in [t_0,T]) \in \mathbb{L}^2(\mathbb{F},\mathbb{R})$ given by

$$Y_t = g(N_T) + \int_t^T f(s, N_s, Y_s, Z_s)ds + \int_t^T Z_s \delta X_s, \ t \in [t_0,T], \tag{6.1}$$

We show that (6.1) is associated to the following second order PDE with terminal condition

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2 u}{\partial x^2}(t,x) - f(t,x,u(t,x), -\sigma_t\frac{\partial u}{\partial x}(t,x)), \\ u(T,x) = g(x),\ (t,x) \in [t_0,T]\times\mathbb{R} \end{cases} \tag{6.2}$$

The main result is as follows.

Theorem 6.1.1. *If (6.2) has a classical solution $u(t,x)$ that satisfies (4.3) with F replaced by u, then $(Y_t,Z_t) := (u(t,N_t), -\sigma_t\frac{\partial u}{\partial x}(t,N_t))$, $t\in(t_0,T)$, satisfies (6.1).*

Proof. Applying the Itô formula (Theorem 4.4) for $u(t,N_t)$, we get

$$du(t,N_t) = \Big(\frac{\partial u}{\partial t}(t,N_t) + \frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2 u}{\partial x^2}(t,N_t)\Big)dt + \sigma_t\frac{\partial u}{\partial x}(t,N_t)\delta X_t.$$

We set $Y_t = u(t,N_t)$ and $Z_t = -\sigma_t\frac{\partial u}{\partial x}(t,N_t)$. Then the pair (Y,Z) is a solution of BSDE

$$du(t,N_t) = -f(t,N_t,u(t,N_t), -\sigma_t\frac{\partial u}{\partial x}(t,N_t))dt + \sigma_t\frac{\partial u}{\partial x}(t,N_t)\delta X_t.$$

From (4.3) we conclude $\mathbb{E}\int_{t_0}^T Z_s^2 ds < \infty$ and $\mathbb{E}\int_{t_0}^T Y_s^2 ds < \infty$.
This proves the theorem. ■

Remark 6.1.2. *Theorem 6.1.1 suggests interesting interactions between the BSDE (6.1) and the PDE (6.2). Existence and uniqueness theorems for (6.2), not only for classical solutions, could be applied to show existence and uniqueness of solutions of (6.1) and vice versa. Let us summarize such results given in [6], [23] and [27] for BSDE driven by brownian motion and in [18] for BSDE driven by fractional brownian motion.*

If the PDE associated to the BSDE has a classical solution, a solution of the BSDE is usually deduced by a theorem analogous to Theorem 6.1.1. The uniqueness of the solution of the BSDE within the class of solutions expressed in terms of classical solutions of the associated PDE follows then from the uniqueness of (classical) solutions of the PDE.

Viscosity solutions u of the PDE: the existence of u is usually derived from the solution of the BSDE and uniqueness of u is proved in [27] under Lipschitz conditions on the generator f within the class of solutions u with polynomial growth at infinity. We refer to [23] for existence and uniqueness results under different assumptions on f.

Sobolev (or weak) solutions of the PDE: the existence and uniqueness of such solutions to semilinear PDEs has been shown independently of BSDEs in [6] and [23] for differential operators whose coefficients depend on the space variable only. The solution of the BSDE is then deduced by a theorem analogous to Theorem 6.1.1. Since in our PDEs the coefficients depend on the time variable, these results are not directly applicable to our case.

Mild (or evolution) solutions of the PDE: the mild form of (6.2) is given by

$$u(t,x) = \int G(T,t,x-y)g(y)dy - \int_t^T \int G(s,t,x-y)f(s,y,u(s,y),-\sigma_s\frac{\partial}{\partial x}u(s,y))dyds, \tag{6.3}$$

where

$$G(s,t,x) = (2\pi)^{-1/2}(Var(N_s)-Var(N_t))^{-1/2}\exp\left(-\frac{x^2}{2(Var(N_s)-Var(N_t))}\right).$$

In fact, Theorem 5.2.1 applied to $n=1, b=A_1=A_2=f=0$ gives $u(t,x) = \int G(T,t,x-y)g(y)dy$ in the linear case and (6.3) follows by classical arguments for the mild form of a nonlinear PDE. In the case where X is fractional brownian motion with Hurst index $H > 1/2$ it is shown in [18], under regularity conditions similar to (H6)-(H8) (see below, evaluated for fractional brownian motion) that the solution of (6.3) is unique and twice differentiable in x. The authors apply this result to show uniqueness of the solution of the BSDE in their case.

A second result on the relation between (6.1) and (6.2) will be given in Proposition 6.1.8. The proof of this proposition needs some auxiliary results.
Let

$$p_{|t|}(x) = \frac{1}{\sqrt{2\pi \mid t \mid}}e^{-x^2/2|t|},\ t \in \mathbb{R}^*. \tag{6.4}$$

Let k be a continuous function, such that the following is well defined:

$$P_{|t|}k(x) = \int_{\mathbb{R}} p_{|t|}(x-y)k(y)dy. \tag{6.5}$$

Remark 6.1.3. *a)*

$$\frac{\partial}{\partial t}P_{|t|}k(x) = \frac{1}{2}sig(t)\frac{\partial^2}{\partial x^2}P_{|t|}k(x). \tag{6.6}$$

In fact,
If $t>0$, we have $\frac{\partial}{\partial t}p_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p_t(x)$, then $\frac{\partial}{\partial t}P_tk(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}P_tk(x)$.
If $t<0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}P_{-t}k(x) &= \tfrac{\partial}{\partial t}\int_{\mathbb{R}} p_{-t}(x-y)k(y)dy \\ &= \int_{\mathbb{R}}\frac{\partial}{\partial t}p_{-t}(x-y)k(y)dy.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{d}{dt}\Big(\frac{1}{\sqrt{2\pi(-t)}}e^{x^2/2t}\Big) &= \frac{d}{dt}\Big((2\pi(-t))^{-\frac{1}{2}}e^{x^2/2t}\Big) \\ &= \frac{d}{dt}\Big((2\pi(-t))^{-\frac{1}{2}}\Big)e^{x^2/2t} + (2\pi(-t))^{-\frac{1}{2}}\frac{d}{dt}e^{x^2/2t}\end{aligned}$$

$$= \Big(\frac{\pi}{(-2\pi t)^{\frac{3}{2}}} - \frac{x^2}{2t^2\sqrt{-2\pi t}}\Big)e^{x^2/2t}.$$

Therefore

$$\frac{d}{dt}p_{-t}(x) = \frac{e^{x^2/2t}}{\sqrt{2\pi(-t)}}\Big[-\frac{1}{2t} - \frac{x^2}{2t^2}\Big].$$

Moreover

$$\frac{d}{dx}\Big(\frac{1}{\sqrt{2\pi(-t)}}e^{x^2/2t}\Big) = \frac{1}{\sqrt{2\pi(-t)}}\frac{x}{t}e^{x^2/2t},$$

$$\frac{d^2}{dx^2}\Big(\frac{1}{\sqrt{2\pi(-t)}}\frac{x}{t}e^{x^2/2t}\Big) = \frac{e^{x^2/t^2}}{\sqrt{2\pi(-t)}}\Big[\frac{1}{t} + \frac{x^2}{t^2}\Big].$$

Therefore

$$\frac{\partial}{\partial t}p_{-t}(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}p_{-t}(x).$$

Consequenly

$$\frac{\partial}{\partial t}P_{-t}k(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}P_{-t}k(x).$$

The following theorem will be needed in the proof of Lemma 6.1.6 below.

Theorem 6.1.4. *Let $Var(N_t)$ be increasing. Assume that h is a continuous function, and suppose that there exist positive constants c and $\lambda' < (8Var(N_T))^{-1}$ such that $\mid h(x) \mid \leqslant ce^{\lambda' x^2}$ for all $x \in \mathbb{R}$. Then*

$$h(N_s) = P_{Var(N_s)}h(0) + \int_0^s \frac{\partial}{\partial x}\Big[P_{Var(N_s)-Var(N_t)}h\Big](N_t)\sigma_t\delta X_t,\ 0 \leqslant s < T.$$

Proof. We want to apply the Itô formula (Theorem 4.4) to $F(t,N_t) = \Big[P_{Var(N_s)-Var(N_t)}h\Big](N_t)$, $t < s$.
We begin by verifying the hypotheses of the Itô formula:

- $$\mid F(t,x) \mid = \mid P_{VarN_s - VarN_t}h(x) \mid = \int_{\mathbb{R}} p_{VarN_T - VarN_t}(x-y) \mid h(y) \mid dy$$
$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(VarN_s - VarN_t)}}e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} \mid h(y) \mid dy$$
$$\leqslant c\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(VarN_s - VarN_t)}}e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}}e^{\lambda' y^2}dy$$
$$= \frac{c}{\sqrt{2\pi(VarN_s - VarN_t)}}\int_{\mathbb{R}} e^{-\frac{z^2}{2(VarN_s - VarN_t)}}e^{\lambda'(x+z)^2}dz$$

$$
\begin{aligned}
&= \frac{ce^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{2(VarN_s - VarN_t)}} e^{2\lambda' z^2} dz \\
&\leqslant \frac{ce^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{2(VarN_s - VarN_t)}} e^{2\lambda' \frac{VarN_s}{(VarN_s - VarN_t)} z^2} dz \\
&= \frac{ce^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{(VarN_s - VarN_t)}(\frac{1}{2} - 2\lambda' VarN_s)} dz \\
&= \frac{ce^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \sqrt{\frac{2\pi(VarN_s - VarN_t)}{1 - 4\lambda' VarN_s}} = \frac{ce^{2\lambda' x^2}}{\sqrt{1 - 4\lambda' VarN_s}} \\
&\leqslant Me^{\lambda x^2},
\end{aligned}
$$

where the last inequality follows by choosing $M = \frac{c}{\sqrt{1-4\lambda VarN_T}}$ and $\lambda = 2\lambda'$.

$$
\begin{aligned}
\bullet \Big| \frac{\partial}{\partial x} F(t,x) \Big| &\leqslant \Big| \frac{\partial}{\partial x} P_{(VarN_s - VarN_t)} h(x) \Big| \\
&\leqslant \int_{\mathbb{R}} \frac{2\pi}{(2\pi(VarN_s - VarN_t))^{\frac{3}{2}}} \mid x - y \mid e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} \mid h(y) \mid dy \\
&\leqslant \frac{2\pi c}{(2\pi(VarN_T - VarN_t))^{\frac{3}{2}}} \int_{\mathbb{R}} \mid x - y \mid e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} e^{\lambda' y^2} dy \\
&= \frac{2\pi c}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} \int_{\mathbb{R}} \mid z' \mid e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\lambda'(x-z')^2} dz' \\
&\leqslant \frac{2\pi c}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} \int_{\mathbb{R}} \mid z' \mid e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{2\lambda' x^2} e^{2\lambda'(z')^2} dz' \\
&\leqslant \frac{ce^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} \frac{\mid z' \mid}{\sqrt{VarN_s - VarN_t}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\frac{2\lambda' VarN_s}{(VarN_s - VarN_t)}(z')^2} dz', \quad (t < s).
\end{aligned}
$$

Moreover, for any $\epsilon > 0$, there is a constant K_ϵ such that $\frac{|z'|}{\sqrt{VarN_s - VarN_t}} \leqslant K_\epsilon e^{\frac{\epsilon}{VarN_s - VarN_t}(z')^2}$. Then, we get

$$
\begin{aligned}
\Big| \frac{\partial}{\partial x} F(t,x) \Big| &\leqslant cK_\epsilon \frac{e^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} e^{\frac{\epsilon}{VarN_s - VarN_t}(z')^2} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\frac{2\lambda' VarN_s}{VarN_s - VarN_t}(z')^2} dz' \\
&= cK_\epsilon \frac{e^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} e^{-\frac{(z')^2}{VarN_s - VarN_t}(\frac{1}{2} - \epsilon - 2\lambda' VarN_s)} dz' \\
&= cK_\epsilon \frac{e^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}(1 - 2\epsilon - 4\lambda' VarN_s)} dz' \\
&= cK_\epsilon \frac{e^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \frac{1}{\sqrt{1 - 2\epsilon - 4\lambda' VarN_s}} \leqslant Me^{\lambda x^2}, \quad \text{(for fixed } t, t < s)
\end{aligned}
$$

with a suitable constant M and $2\lambda' = \lambda$.

$$
\begin{aligned}
&\bullet \Big| \frac{\partial^2}{\partial x^2} F(t,x) \Big| \\
&= \Big| \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{2\pi}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} \mid x - y \mid e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} h(y) dy \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_{\mathbb{R}} \frac{2\pi}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} e^{-\frac{(x-y)^2}{2|VarN_T - VarN_t|}} \mid h(y) \mid dy \\
&+ \int_{\mathbb{R}} \frac{2\pi \mid x-y \mid^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} \mid h(y) \mid dy \\
&\leqslant 2Me^{\lambda x^2} + 2\pi c \int_{\mathbb{R}} \frac{(z')^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)|}} e^{\lambda'(x-z')^2} dz' \\
&\leqslant 2Me^{\lambda x^2} + 2\pi c \int_{\mathbb{R}} \frac{(z')^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{2\lambda' x^2} e^{2\lambda'(z')^2} dz' \\
&\leqslant 2Me^{\lambda x^2} + 2\pi c e^{2\lambda' x^2} \int_{\mathbb{R}} \frac{(z')^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\frac{2\lambda' VarN_s}{VarN_s - VarN_t}(z')^2} dz' \\
&\leqslant 2Me^{\lambda x^2} + 2\pi c \frac{e^{2\lambda' x^2}}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} \int_{\mathbb{R}} (z')^2 e^{-\frac{(z')^2}{VarN_s - VarN_t}(\frac{1-4\lambda' VarN_s}{2})} dz' \\
&\leqslant 2Me^{\lambda x^2} + 2\pi c \frac{e^{2\lambda' x^2}}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} \int_{\mathbb{R}} u^2 \frac{VarN_s - VarN_t}{1 - 4\lambda' VarN_s} e^{-\frac{1}{2}u^2} \sqrt{\frac{VarN_s - VarN_t}{1 - 4\lambda' VarN_s}} du \\
&= 2Me^{\lambda x^2} + 2\pi c \frac{e^{2\lambda' x^2}}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} \frac{(VarN_s - VarN_t)^{\frac{3}{2}}}{(1 - 4\lambda' VarN_s)^{\frac{3}{2}}} \int_{\mathbb{R}} u^2 e^{-\frac{1}{2}u^2} du \\
&= 2Me^{\lambda x^2} + 2\frac{ce^{2\lambda' x^2}}{(2\pi)^{\frac{3}{2}}(VarN_s - VarN_t)(1 - 4\lambda' VarN_s)^{\frac{3}{2}}} \leqslant M' e^{\lambda x^2}, \quad \text{(for fixed } t,\ t < s)
\end{aligned}
$$

with a suitable constant M' and $2\lambda' = \lambda$.
Moreover, for fixed t, $t < s$ we have $\frac{\partial}{\partial t} P_{VarN_s - VarN_t} h(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} P_{VarN_s - VarN_t|} h(x)$, and we conclude $\mid \frac{\partial}{\partial t} F(t,x) \mid \leqslant M" e^{\lambda x^2}$ with a suitable constant $M"$.
Applying the Itô formula (Theorem 4.4) to $F(t,N_t) = \Big[P_{Var(N_s) - Var(N_t)} h\Big](N_t)$ on $[0, s-\epsilon]$, we obtain

$$
\begin{aligned}
F(s-\epsilon, N_{s-\epsilon}) = F(0,0) \quad &+ \quad \int_0^{s-\epsilon} \frac{\partial}{\partial x} \Big[P_{Var(N_s) - Var(N_t)} h\Big](N_t) \sigma_t \delta X_t + \int_0^{s-\epsilon} \frac{\partial}{\partial t} \Big[P_{Var(N_s) - Var(N_t)} h\Big](N_t) dt \\
&+ \quad \frac{1}{2} \int_0^{s-\epsilon} \frac{\partial^2}{\partial x^2} \Big[P_{Var(N_s) - Var(N_t)} h\Big](N_t) \frac{d}{dt} Var(N_t) dt.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} P_{Var(N_s) - Var(N_t)} h(x) &= \frac{\partial}{\partial \tau} P_\tau h(x) \mid_{\tau = VarN_s - VarN_t} \frac{\partial \tau}{\partial t} \mid_{\tau = VarN_s - VarN_t} \\
&= \frac{1}{2} \frac{\partial^2}{\partial x^2} P_\tau h(x) \mid_{\tau = VarN_s - VarN_t} (-\frac{d}{dt} Var(N_t)) \\
&= -\frac{1}{2} \frac{d}{dt} Var(N_t) \frac{\partial^2}{\partial x^2} P_{VarN_s - VarN_t} h(x). \qquad (6.7)
\end{aligned}
$$

Using (6.7), we get

$$
P_{VarN_s - VarN_{s-\epsilon}} h(N_{s-\epsilon}) - P_{Var(N_s)} h(0) \quad = \quad \int_0^{s-\epsilon} \frac{\partial}{\partial x} \big[P_{Var(N_s) - Var(N_t)} h\big](N_t) \sigma_t \delta X_t.
$$

We tend ϵ to 0 and we obtain

$$h(N_s) = P_{Var(N_s)}h(0) + \int_0^s \frac{\partial}{\partial x}\big[P_{Var(N_s)-Var(N_t)}h\big](N_t)\sigma_t \delta X_t. \tag{6.8}$$

■

Remarks 6.1.5. *a) If $VarN_t$ is decreasing, we have for all $0 \leqslant t < s < T$*

$$\begin{aligned}
\frac{\partial}{\partial t}P_{|Var(N_s)-Var(N_t)|}h(x) &= \frac{\partial}{\partial \tau}P_\tau h(x)\mid_{\tau=|Var(N_s)-Var(N_t)|}\frac{\partial \tau}{\partial t}\mid_{\tau=VarN_t-VarN_s}\\
&= -\frac{1}{2}\frac{\partial^2}{\partial x^2}P_\tau h(x)\mid_{\tau=VarN_t-VarN_s}\frac{d}{dt}Var(N_t)\\
&= -\frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2}{\partial x^2}P_{VarN_t-VarN_s}h(x)\\
&= -\frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2}{\partial x^2}P_{|Var(N_s)-Var(N_t)|}h(x).
\end{aligned}$$

b) If $VarN_t$ is decreasing, we must add the following hypothesis in the previous theorem $\frac{Var(N_s)}{|Var(N_s)-Var(N_t)|} \geqslant 1$, for all $0 \leqslant t < s < T$.

Lemma 6.1.6. *Suppose that K_T^* is injective. Let $b(s,x)$ and $a(s,x)$, $s \in (t_0,T)$, $x \in \mathbb{R}$ be continuous functions, b continuously differentiable with respect to x and of a polynomial growth. Let a be in the domain of δ^X. Let N given by (1.14). If*

$$\int_{t_0}^t b(s,N_s)ds + \int_{t_0}^t a(s,N_s)\delta X_s = 0, \textit{ for all } t \in (t_0,T), \tag{6.9}$$

then

$$b(s,x) = a(s,x) = 0 \quad \textit{for all } s \in (t_0,T),\ x \in \mathbb{R}. \tag{6.10}$$

Proof. Using (6.4), (6.5) and applying Theorem 6.1.4, we want to show

$$b(s,N_s) = \mathbb{E}[b(s,N_s)] + \int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dy\Big)\sigma_u \delta X_u. \tag{6.11}$$

In fact, on the one hand, for all $u \in (0,s)$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dy &= \frac{\partial}{\partial x}\int_{\mathbb{R}} p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dy\\
&= \frac{\partial}{\partial x}P_{Var(N_s)-Var(N_u)}b(s,N_u).
\end{aligned}$$

Moreover, we have

$$\mathbb{E}\Big[b(s,N_s)\Big] = \int_{\mathbb{R}} b(s,x)\frac{e^{-\frac{x^2}{2Var(N_s)}}}{\sqrt{2\pi Var(N_s)}}dx, \quad since \quad N_s \sim \mathcal{N}(0,Var(N_s)).$$

Therefore,

$$\mathbb{E}\Big[b(s,N_s)\Big] = \int_{\mathbb{R}} b(s,x)p_{Var(N_s)}(x)dx = \int_{\mathbb{R}} b(s,x)p_{Var(N_s)}(-x)dx = P_{Var(N_s)}b(s,0).$$

On the other hand, we apply Theorem 6.1.4 to $h(x) = b(s,x)$ (for fixed s), we obtain

$$b(s,N_s) \;=\; P_{Var(N_s)}b(s,0) + \int_0^s \frac{\partial}{\partial x}\Big[P_{Var(N_s)-Var(N_u)}b(s,.)\Big](N_u)\sigma_u\delta X_u.$$

Therefore,

$$\begin{aligned} b(s,N_s) &= P_{Var(N_s)}b(s,0) + \int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u \\ &= \mathbb{E}[b(s,N_s)] + \int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u. \quad (6.12) \end{aligned}$$

Now, we show that

$$\begin{aligned} &\int_{t_0}^t \Big[\int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u\Big]ds \\ &= \int_{t_0}^t \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dyds\delta X_u. \quad (6.13) \end{aligned}$$

In fact, let $F \in L^2(\Omega,\mathcal{H})$

$$\begin{aligned} &\mathbb{E}\Big[\int_{t_0}^t \Big[\int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u\Big]dsF\Big] \\ &= \int_{t_0}^t \mathbb{E}\Big[\int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u F\Big]ds \\ &= \int_{t_0}^t \mathbb{E}\Big[< \int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_.)}(N_.-y)b(s,y)dy\sigma_., D_.^X F >_{\mathcal{H}}\Big]ds \\ &= \mathbb{E}\Big[< \int_.^t \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_.)}(N_.-y)b(s,y)dy\Big)ds\sigma_., D_.^X F >_{\mathcal{H}}\Big] \\ &= \mathbb{E}\Big[\int_{t_0}^t \sigma_u\Big(\int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dyds\Big)\delta X_u F\Big]. \end{aligned}$$

From (6.12) and (6.13) we get

$$\begin{aligned} \int_{t_0}^t b(s,N_s)ds &= \int_{t_0}^t \mathbb{E}[b(s,N_s)]ds + \int_{t_0}^t \Big[\int_0^s \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dy\Big)\sigma_u\delta X_u\Big]ds. \\ &= \int_{t_0}^t \mathbb{E}[b(s,N_s)]ds + \int_{t_0}^t \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x}p_{Var(N_s)-Var(N_u)}(N_u-y)b(s,y)dyds\delta X_u. \end{aligned}$$

Then from (6.9) we get

$$0 = \int_{t_0}^t b(s,N_s)ds + \int_{t_0}^t a(s,N_s)\delta X_s$$

$$= \int_{t_0}^{t} \mathbb{E}[b(s,N_s)]ds + \int_{t_0}^{t} a(s,N_s)\delta X_s + \int_{t_0}^{t} \sigma_u \Big[\int_u^t \Big(\int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dy \Big) ds \Big] \delta X_u$$
$$= \int_{t_0}^{t} \Big[a(u,N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dyds \Big] \delta X_u + \int_{t_0}^{t} \mathbb{E}[b(s,N_s)]ds.$$

Thus

$$\int_{t_0}^{t} \mathbb{E}[b(s,N_s]ds \;=\; 0, \tag{6.14}$$

And

$$\int_{t_0}^{t} \Big[a(u,N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dyds \Big] \delta X_u \;=\; 0. \tag{6.15}$$

Let

$$Z(u,N_u) \;=:\; a(u,N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s)-Var(N_u)}(N_u - y)b(s,y)dyds$$

(6.15) implies that $Var(\int_{t_0}^{t} Z(u,N_u)\delta X_u) = 0$. Therefore $K_t^* Z(.,N_.)_u = 0$ $u-$a.e. in $[t_0,t]$.
By injectivity of K_t^*, we get

$$Z(u,N_u) = 0, \text{for all } u \in [t_0,t].$$

Therefore

$$a(u,z) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s)-Var(N_u)}(z - y)b(s,y)dyds = 0,$$

for all $z \in \mathbb{R}$ ([19]). Now by differentiating with respect to t, we get

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_t)-Var(N_u)}(z - y)b(t,y)dy = 0.$$

for all $t > u$ and $z \in \mathbb{R}$.
An integration by parts formula yields

$$\int_{\mathbb{R}} p_{Var(N_t)-Var(N_u)}(z - y)\frac{\partial}{\partial y} b(t,y)dy = 0.$$

Let $u \to t$ then, we see that $\frac{\partial}{\partial y} b(t,y) = 0$ for all $t \in (t_0, T-\epsilon)$ and $y \in \mathbb{R}$. This means that there is a $b_1(t)$ such that $b(t,y) = b_1(t)$. Now from (6.14) we have

$$\int_{t_0}^{t} b_1(s)ds = 0, \; t_0 \leqslant t \leqslant T - \epsilon.$$

This implies that $b_1(t) = 0$ for all $t \in (t_0,T)$ and accordingly $b(t,y) = 0$ for all $t \in (t_0,T)$ and $y \in \mathbb{R}$. Thus, $a(t,y) = 0$. ■

Remarks 6.1.7. *In the proof of Lemma 6.1.6 we have*

$$Var\int_{t_0}^{t} Z(u,N_u)\delta X_u = Var\int_{t_0}^{t} K_t^* Z(\cdot,N.)_u \delta W_u = 0$$

which implies that $K_t^* Z(\cdot,N.)_u = \int_u^t \frac{\partial}{\partial v}K(v,u)Z(v,N_v)dv = 0 \ \ (u,\omega)-a.e.$ *on* $[t_0,t]\times\Omega$, *for all* $t\in(t_0,T]$.

We would like to conclude that $Z(v,N_v)=0 \ \ (v,\omega)-a.e.$ *on* $[t_0,t]\times\Omega$.

a) An evident hypothesis is the injectivity of $K_t^* Z(\cdot,N.)_u$ *as a function of* $u\in[t_0,t]$.

Let us look for a sufficient condition for injectivity:

$$0=\int_{t_0}^{t} K_t^* Z(\cdot,N.)_u du = \int_{t_0}^{t}\int_{u}^{t}\frac{\partial}{\partial s}K(s,u)Z(s,N_s,\omega)dsdu = \int_{t_0}^{t} Z(s,N_s,\omega)\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)duds.$$

Suppose that $K_t^* Z(\cdot,N.)_u = 0 \ \ u-a.e.$ *on* $[t_0,t]$ *for all* $t\in(t_0,T]$ *and* $Z>0$ *on* $(a,b)\subset[t_0,T]$. *Then*

$$\begin{aligned}
0 &= \int_{t_0}^{b} K_b^* Z(\cdot,N.)_u du - \int_{t_0}^{a} K_a^* Z(\cdot,N.)_u du \\
&= \int_{t_0}^{b} Z(s,N_s,\omega)\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)duds - \int_{t_0}^{a} Z(s,N_s,\omega)\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)duds \\
&= \int_{a}^{b} Z(s,N_s,\omega)\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)duds
\end{aligned}$$

Let $\widetilde{K}_{t_0}(s)=\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)du$. *If we suppose that* $\widetilde{K}_{t_0}(s)\neq 0$ *on* $[t_0,T]$ *(and therefore does not change sign on* $[t_0,T]$*), we obtain a contradiction. Therefore, a sufficient condition for* K_T^* *to be injective is* $\widetilde{K}_{t_0}(s)\neq 0$ *on* $[t_0,T]$. *This last hypothesis is satisfied in particular if* $\frac{\partial}{\partial s}K(s,u)>0$ *(or* <0*) for* $u\in(t_0,s)$ *for all* $s\in(t_0,T)$. *By Remark 6.3.1 the positivity of* $\frac{\partial}{\partial s}K(s,u)$ *implies that* $VarN_t$ *is increasing in* $t\in[t_0,T]$.
One could also try to find a sufficient condition for the injectivity from the equality

$$0=\int_{t_0}^{t}(K_t^*Z)_u^2 du = \int_{t_0}^{t}\int_{t_0}^{t} Z_r Z_s \phi_{t_0}(r,s)dsdr, \text{ where } \phi_{t_0}(r,s)=\int_{t_0}^{r\wedge s}\frac{\partial}{\partial s}K(s,v)\frac{\partial}{\partial r}K(r,v)dv.$$

However, it seems that the hypothesis of the positivity of ϕ_{t_0} *does not imply* $Z=0$.

b) For all $h\in L^2([t_0,t])$

$$\begin{aligned}
\int_{t_0}^{t} K_t^* Z(\cdot,N.,\omega)_u h(u)du &= \int_{t_0}^{t}\int_{u}^{t}\frac{\partial}{\partial s}K(s,u)Z(s,N_s,\omega)dsh(u)du \\
&= \int_{t_0}^{t} Z(s,N_s,\omega)\int_{t_0}^{s}\frac{\partial}{\partial s}K(s,u)h(u)duds.
\end{aligned}$$

Suppose that the right side of the equality above is zero for all $h \in L^2([t_0,t])$. Under which condition on K does this imply $Z(s,N_s,\omega)=0 \;\; s-a.e$ on $[t_0,t]$?

c) We could also try to argue with the chaos composition of $\int_{t_0}^t Z(u,N_u)\delta X_u$ which is unique. Is Z unique? If so, $Z=0$. The question is to find an analogy to theorem 12.1 in [17].

We are now ready to formulate our second result on the relation between BSDE and PDE.

Proposition 6.1.8. *Suppose that K_T^* is injective. Let (6.1) have a solution of the form $(Y_t = u(t,N_t), Z_t = v(t,N_t))$, where u is as in Theorem 6.1.1 and $v \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ satisfies (4.3). Suppose that $f \in \mathcal{C}^{0,1}([0,T]\times\mathbb{R}^3)$ and is of polynomial growth. Then $v(t,x) = -\sigma_t \frac{\partial}{\partial x}u(t,x)$.*

Proof. We apply the Itô formula (Theorem 4.2.1) to u and get

$$du(t,N_t) \;\; = \;\; \Big[\frac{\partial}{\partial t}u(t,N_t) + \frac{1}{2}\frac{d}{dt}Var(N_t)\frac{\partial^2}{\partial x^2}u(t,N_t)\Big]dt + \sigma_t\frac{\partial}{\partial x}u(t,N_t)\delta X_t.$$

Moreover, we can write

$$u(t,N_t) - g(N_T) = -\int_t^T \Big[\frac{\partial}{\partial s}u(s,N_s) + \frac{1}{2}\frac{d}{ds}Var(N_s)\frac{\partial^2}{\partial x^2}u(s,N_s)\Big]ds - \int_t^T \sigma_s\frac{\partial}{\partial x}u(s,N_s)\delta X_s.$$

Using (6.1), we get

$$\begin{aligned}&\int_t^T f(s,N_s,u(s,N_s),v(s,N_s))ds + \int_t^T v(s,N_s)\delta X_s\\ &= -\int_t^T \Big[\frac{\partial}{\partial s}u(s,N_s) + \frac{1}{2}\frac{d}{ds}Var(N_s)\frac{\partial^2}{\partial x^2}u(s,N_s)\Big]ds - \int_t^T \sigma_s\frac{\partial}{\partial x}u(s,N_s)\delta X_s.\end{aligned}$$

For $t=t_0$, we obtain

$$\begin{aligned}&\int_{t_0}^T f(s,N_s,u(s,N_s),v(s,N_s))ds + \int_{t_0}^T v(s,N_s)\delta X_s\\ &= -\int_{t_0}^T \Big[\frac{\partial}{\partial s}u(s,N_s) + \frac{1}{2}\frac{d}{ds}Var(N_s)\frac{\partial^2}{\partial x^2}u(s,N_s)\Big]ds - \int_{t_0}^T \sigma_s\frac{\partial}{\partial x}u(s,N_s)\delta X_s.\end{aligned}$$

Subtracting the two equations we get for all $t_0 \leqslant t \leqslant T-\epsilon$

$$\begin{aligned}&\int_{t_0}^t f(s,N_s,u(s,N_s),v(s,N_s))ds + \int_{t_0}^t v(s,N_s)\delta X_s\\ &= -\int_{t_0}^t \Big[\frac{\partial}{\partial s}u(s,N_s) + \frac{1}{2}\frac{d}{ds}Var(N_s)\frac{\partial^2}{\partial x^2}u(s,N_s)\Big]ds - \int_{t_0}^t \sigma_s\frac{\partial}{\partial x}u(s,N_s)\delta X_s.\end{aligned}$$

Using (6.2), we obtain

$$\begin{aligned}&\int_{t_0}^t \Big[f(s,N_s,u(s,N_s),v(s,N_s)) - f(s,N_s,u(s,N_s),-\sigma_s\frac{\partial}{\partial x}u(s,N_s))\Big]ds\\ &+ \int_{t_0}^t \Big[v(s,N_s) + \sigma_s\frac{\partial}{\partial x}u(s,N_s)\Big]\delta X_s = 0.\end{aligned}$$

Since u, v satisfy (4.3) and $f \in \mathcal{C}^{0,1}([0,T] \times \mathbb{R}^3)$ and is of polynomial growth, we apply now Lemma 6.1.6 and obtain

$$v(t,x) = -\sigma_t \frac{\partial}{\partial x} u(t,x),\ \forall t \in (t_0,T), x \in \mathbb{R}. \blacksquare$$

Remark 6.1.9. *If the PDE (6.2) has a unique classical solution (see in particular the last part of Remark 6.1.2), then under the hypotheses of Proposition 6.1.8 the BSDE (6.1) has a unique solution.*

In fact, let us suppose that (6.1) has two solutions (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) : $Y_t^1 = u^1(t, N_t)$, $Z_t^1 = -\sigma_t \frac{\partial}{\partial x} u^1(t, N_t)$ and $Y_t^2 = u^2(t, N_t)$, $Z_t^2 = -\sigma_t \frac{\partial}{\partial x} u^2(t, N_t)$. By uniqueness of solutions of (6.2), $u^1(t,.) = u^2(t,.)$ and therefore, for $t \in [t_0, T]$ $Y_t^1 = Y_t^2$, $Z_t^1 = Z_t^2$.

6.2 Transfer formula

Theorem 6.2.1. *(Transfer formula) Let X be given by (2.1). Let D^W be the Malliavin derivative with respect to the Brownian motion. Then, $K_T^* D_\cdot^X = D_\cdot^W$ on $\mathbb{D}_{1,2}^X$.*

Proof. We have $F = f(X(\varphi_1), ..., X(\varphi_n)) = f(W(K_T^*\varphi_1), ..., W(K_T^*\varphi_n))$, $n \geqslant 1$ and $F \in \mathcal{S}$.

Recall that $X(\varphi_i) = \int_0^T \varphi_i(s)\delta X_s = \int_0^T (K_T^*\varphi_i)_s \delta W_s = W(K_T^*\varphi_i), i = 1,..., n, \varphi_1,..., \varphi_n \in \mathcal{H}$.

Furthermore $D_t^X F := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), ..., X(\varphi_n))\varphi_i(t) \in \mathcal{H}$, and

$$\begin{aligned}
(K_T^* D_\cdot^X F)_t &= \int_t^T D_u^X F \frac{\partial K}{\partial u}(u,t)du \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), ..., X(\varphi_n)) \int_t^T \varphi_i(u) \frac{\partial K}{\partial u}(u,t)du \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), ..., X(\varphi_n))(K_T^*\varphi_i)_t = D_t^W F, t \in [0,T].
\end{aligned}$$

We extend the equality $(K_T^* D_\cdot^X F)_t = D_t^W F$ to the closure of the linear combinations of $\mathcal{S}$ by means of the norm

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\|D^X F\|_{\mathcal{H}}^2.$$

a) $F_1, F_2 \in \mathcal{S}, a_1, a_2 \in \mathbb{R}$: $(K_T^* D_\cdot^X (a_1F_1 + a_2F_2))_t = D_t^W (a_1F_1 + a_2F_2)$ because K^*, D^X and D^W are linear operators.

b) Let $(F_n)_{n\in\mathbb{N}} \subset span(\mathcal{S})$ such that $(F_n)_{n\in\mathbb{N}}$ converges in norm $\|.\|_{1,2}$ to F. Then, $F \in \mathbb{D}_{1,2}^X$. Thus, $\|D^X F - D^X F_n\|_{\mathcal{H}} \to 0$, with $n \longmapsto \infty$. Thus, $\|K_T^* D^X F - K_T^* D^X F_n\|_{L^2(\Omega, L^2(0,T))} \to 0$, with $n \longmapsto \infty$ by isometry.

Since $K_T^* D^X F_n = D^W F_n$, $(D^W F_n)_{n\in\mathbb{N}}$ is a convergent sequence in $L^2(\Omega, L^2(0,T))$. By proposition 1.2.1 ([26]) the limit in $L^2(\Omega, L^2(0,T))$ is $D^W F$. Therefore, $K_T^* D_\cdot^X = D_\cdot^W$. $\blacksquare$

6.3 Assumptions and solution spaces

In this section we formulate the assumptions we need for the proofs, carried out in Chapter 7, of the existence and uniqueness of the solution of (6.1) without using the associated PDE. We define the spaces where these solutions live.

We formulate now the assumptions on σ, f, g in (6.1):

(H6) $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is a deterministic continuous function such that $\sigma_t \neq 0$, for all $t \in [0,T]$ and σ is bounded. Let

$$\hat{\sigma}_t = \int_0^t \phi(t,r)\sigma_r dr, t \in [0,T].$$

Moreover, we assume that there exists $\epsilon : [0,T] \longrightarrow \mathbb{R}_+$ is continuous function such that $m\epsilon(s) \leqslant \frac{\hat{\sigma}_s}{\sigma_s} \leqslant M\epsilon(s), s \in [0,T]$ for some constants $m, M > 0$ and such that $\frac{1}{\epsilon}$ is integrable on $[t,T]$ for all $t > 0$.

(H7) The function $f : [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ belongs to the space $\mathcal{C}^{0,1}_{pol}([0,T] \times \mathbb{R}^3)$, where $\mathcal{C}^{k,l}_{pol}([0,T] \times \mathbb{R}^m)$ is the space of all $\mathcal{C}^{k,l}$-functions over $[0,T] \times \mathbb{R}^m$ which, together with their derivatives, are of polynomial growth. Moreover we assume that there exists a constant $L > 0$ such that, for all $t \in [0,T]$, $x, y_1, y_2, z_1, z_2 \in \mathbb{R}$,

$$\mid f(t,x,y_1,z_1) - f(t,x,y_2,z_2) \mid \leqslant L(\mid y_1 - y_2 \mid + \mid z_1 - z_2 \mid).$$

And

$$\mathbb{E}\Big[\int_0^T \mid f(t,N_t,0,0) \mid^2 dt\Big] < \infty.$$

(H8) $\xi = g(N_T)$, where $g \in \mathcal{C}^1_{pol}(\mathbb{R})$.

Remarks 6.3.1. *We discuss now the hypothesis (H6).*
a) • We have $\frac{d}{dt}VarN_t = 2\sigma_t\hat{\sigma}_t$. *Indeed,*

$$\begin{aligned}
\frac{d}{dt}VarN_t &= \frac{d}{dt}\int_0^t (K_t^*\sigma)_s^2 ds \\
&= \frac{d}{dt}\int_0^t \Big(\int_s^t \frac{\partial}{\partial u}K(u,s)\sigma_u du\Big)^2 ds \\
&= 2\int_0^t \Big(\int_s^t \frac{\partial}{\partial u}K(u,s)\sigma_u du\Big)\frac{\partial}{\partial t}K(t,s)\sigma_t ds \\
&= 2\int_0^t du\sigma_t\sigma_u \int_0^{inf(t,u)} \frac{\partial}{\partial t}K(t,s)\frac{\partial}{\partial u}K(u,s)ds \\
&= 2\sigma_t\int_0^t \sigma_u\phi(t,u)du := 2\sigma_t\hat{\sigma}_t.
\end{aligned}$$

• $\frac{\hat{\sigma}_t}{\sigma_t} = \frac{\frac{d}{dt}VarN_t}{2\sigma_t^2} > 0$ *if* $\frac{d}{dt}VarN_t > 0$.
• *If* $\sigma = 1$, *we have* $\int_0^t \phi(t,r)dr = \frac{1}{2}\frac{d}{dt}VarX_t$.
b) If $\phi(t,u) > 0, 0 \leqslant u < t \leqslant T$ *we can choose* $\epsilon(t) = \int_0^t \phi(t,u)du = \frac{1}{2}\frac{d}{dt}VarX_t > 0$ *in (H6). Indeed, if there exists* $c, C > 0$ *such that* $c < \sigma < C$, *we have* $\frac{c}{C}\epsilon(t) \leqslant \frac{\hat{\sigma}_t}{\sigma_t} \leqslant \frac{C}{c}\epsilon(t)$.
c) If we suppose that $\frac{d}{dt}VarX_t$ *is continuous on* $[s,T]$, *and we have* $\frac{d}{dt}VarX_t \geqslant \delta > 0$, *then* $\frac{1}{\epsilon}$ *is integrable on* $[s,T]$ *for all* $s > 0$.

We introduce the following space:

Definition 6.3.2. *Let* $\mathcal{V}_T$ *denote the set of processes* $(Y_t, t \in [0,T])$ *of the form*

$$\mathcal{V}_T = \{Y_\cdot = w(.,N_\cdot); w \in \mathcal{C}^{1,3}_{pol}([0,T]\times\mathbb{R}) \text{ with } \frac{\partial w}{\partial t} \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R})\}.$$

Remark 6.3.3. *It may be surprising that a third order derivative with respect to* x *of* w *and a derivative with respect to* x *of* $\frac{\partial}{\partial t}w$ *is required in Definition 6.3.2. They are in fact needed for an application of Lemma 6.1.6 in part e) of the proof of Proposition 7.1.1. The definition of* V_T *corresponds to the definition in [25].*

Let $\tilde{\mathcal{V}}_{T,\epsilon}$ be the completion of $\mathcal{V}_T$ under the following norm

$$\| Z \|^2_\epsilon = \int_0^T \epsilon(t)\mathbb{E} \mid Z_t \mid^2 dt.$$

And let

$$\| Y \|^2_2 = \int_0^T \mathbb{E} \mid Y_t \mid^2 dt.$$

We make the link between the 3 spaces: $\mathcal{V}_T, \mathbb{D}^X_{1,2}(\mid \mathcal{H} \mid), Dom(\delta^X)$.

Lemma 6.3.4. *We have* $\mathcal{V}_T \subset \mathbb{D}^X_{1,2}(\mid \mathcal{H} \mid) \subset Dom(\delta^X)$.

Proof. Let $u \in \mathcal{V}_T$ and we prove $u \in \mathbb{D}^X_{1,2}(\mid \mathcal{H} \mid)$.
a) We first prove that $\mathbb{E} \| u \|^2_{\mathcal{H}} < \infty$.
We have

$$\begin{aligned}\| u \|^2_{\mathcal{H}} &= \int_0^T \Big(\int_t^T u(r, N_r)\frac{\partial}{\partial r}K(r,t)dr \Big)^2 dt \\ &\leqslant \int_0^T \Big(\int_t^T \mid u(r,N_r) \mid \Big|\frac{\partial}{\partial r}K(r,t)\Big| dr \Big)^2 dt.\end{aligned}$$

Applying Hölder's inequality to $p < \frac{1}{1-\alpha} < 2$ (α is a constant in (H2)) and q its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$), we get

$$\Big(\int_t^T \mid u(r,N_r) \mid \Big|\frac{\partial}{\partial r}K(r,t)\Big| dr \Big)^2$$

$$
\begin{aligned}
&\leqslant C\Big(\int_t^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}\Big(\int_t^T (r-t)^{p(\alpha-1)}\Big(\frac{r}{t}\Big)^{p\beta}dr\Big)^{\frac{2}{p}}\\
&= Ct^{-2\beta}T^{2\beta}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}\Big(\int_t^T (r-t)^{p(\alpha-1)}dr\Big)^{\frac{2}{p}}\\
&= Ct^{-2\beta}T^{2\beta}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}\Big[\frac{(T-t)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big]^{\frac{2}{p}}, (p(\alpha-1)+1>0)\\
&= Ct^{-2\beta}T^{2\beta}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}\frac{(T-t)^{2(\alpha-1)+\frac{2}{p}}}{(p(\alpha-1)+1)^{\frac{2}{p}}}\\
&= C't^{-2\beta}T^{2\beta}(T-t)^{2(\alpha-1)+\frac{2}{p}}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}. \qquad (6.16)
\end{aligned}
$$

We want to prove that $\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} < \infty$. Indeed, we have $\frac{2}{q} < 1$, then Jensen's inequality can be applied of a concave function to get

$$\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} \leqslant \Big(\mathbb{E}\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}.$$

It must be demonstrated that $\int_0^T \mathbb{E}\mid u(r,N_r)\mid^q dr < \infty$.
Since $u \in \mathcal{V}_T$, we have, for some $C>0$, $k \geqslant 1$,

$$\mid u(r,N_r)\mid \leqslant C\Big(1+\mid N_r\mid +...+\mid N_r\mid^k\Big), r\in[0,T].$$

Then we have

$$\mid u(r,N_r)\mid^q = \Big(C\sum_{i=0}^k \mid N_r\mid^i\Big)^q \leqslant C'\Big(\sum_{i=0}^k \mid N_r\mid^i \vee 1\Big)^{2m'} = C'\sum_{i=0}^{2km'} \mid N_r\mid^i \vee 1$$

where $[q] < 2m'$.
Using (H6) we have $\mid \sigma \mid \leqslant M'$, then one can check that for all $r\in[0,T]$

$$
\begin{aligned}
\mathbb{E}(N_r^2) = \int_0^T K_T^*(\sigma 1_{[0,r]})_s^2 ds &= \int_0^T \Big[\int_s^T \sigma_u 1_{[0,r]}(u)\frac{\partial}{\partial u}K(u,s)du \int_s^T \sigma_v 1_{[0,r]}(v)\frac{\partial}{\partial v}K(v,s)dv\Big]ds\\
&\leqslant M'\int_0^T du\int_0^T dv\tilde{\phi}(u,v) \leqslant 2M'\int_0^T du\int_0^u dv\tilde{\phi}(u,v) < \infty.
\end{aligned}
$$

For all $r\in[0,T]$ and $p\geqslant 1$, we have $\mathbb{E}\mid N_r\mid^{2p} \leqslant C_p \parallel \sigma \parallel_{\mathcal{H}}^{2p} < \infty$. Thus, we get

$$\int_0^T \mathbb{E}\mid u(r,N_r)\mid^q dr < \infty. \qquad (6.17)$$

Using (6.16) and (6.17), we get

$$\mathbb{E}\parallel u \parallel_{\mathcal{H}}^2 \leqslant \mathbb{E}\int_0^T \Big(\int_t^T \mid u(r,N_r)\mid \Big|\frac{\partial}{\partial r}K(r,t)\Big|dr\Big)^2 dt$$

$$\begin{aligned}
&\leqslant C^{"}\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} \int_0^T (T-t)^{2(\alpha-1)+\frac{2}{p}} t^{-2\beta} dt\\
&\leqslant C^{"}\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} \int_0^1 (sT)^{-2\beta}(1-s)^{2(\alpha-1)+\frac{2}{p}} T^{2(\alpha-1)+\frac{2}{p}} T ds\\
&\leqslant C^{"}\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} T^{-2\beta+2\alpha+\frac{2}{p}-1} \int_0^1 s^{-2\beta}(1-s)^{2(\alpha-1)+\frac{2}{p}} ds\\
&= C^{"}\mathbb{E}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}} T^{-2\beta+2\alpha+\frac{2}{p}-1} \mathcal{B}(1-2\beta, 2(\alpha-1)+\frac{2}{p}+1) < \infty.
\end{aligned}$$

b) Now, we will prove that $\mathbb{E} \parallel D^X u \parallel^2_{\mid\mathcal{H}\mid\otimes\mid\mathcal{H}\mid} < \infty$. Indeed, applying (3.8), we obtain

$$\begin{aligned}
&\mathbb{E} \parallel D^X u \parallel^2_{\mid\mathcal{H}\mid\otimes\mid\mathcal{H}\mid} = \mathbb{E}\int_{[0,T]^4} \mid D^X_s u(t,N_t) \mid\mid D^X_{t'} u(s',N_{s'}) \mid \tilde{\phi}(t,t')\tilde{\phi}(s,s') ds dt ds' dt'\\
&= \mathbb{E}\int_{[0,T]^4} \Big|\frac{\partial}{\partial x} u(t,N_t)\Big| \mid \sigma_s \mid \Big|\frac{\partial}{\partial x} u(s',N_{s'})\Big| \mid \sigma_{t'} \mid \tilde{\phi}(t,t')\tilde{\phi}(s,s') ds dt ds' dt'\\
&\leqslant C\int_{[0,T]^4} \mathbb{E}\Big[\Big|\frac{\partial}{\partial x} u(t,N_t)\Big|\Big|\frac{\partial}{\partial x} u(s',N_{s'})\Big|\Big]\tilde{\phi}(t,t')\tilde{\phi}(s,s') ds dt ds' dt'\\
&\leqslant \frac{C}{2}\int_{[0,T]^4} \Big(\mathbb{E}\Big[\Big|\frac{\partial}{\partial x} u(t,N_t)\Big|^2\Big] + \mathbb{E}\Big[\Big|\frac{\partial}{\partial x} u(s',N_{s'})\Big|^2\Big]\Big)\tilde{\phi}(t,t')\tilde{\phi}(s,s') ds dt ds' dt'\\
&\leqslant \frac{C}{2}\int_{[0,T]^4} \Big[\mathbb{E}\Big(1+\mid N_t \mid + ... + \mid N_t \mid^k\Big) + \mathbb{E}\Big(1+\mid N_{s'} \mid + ... + \mid N_{s'} \mid^{k'}\Big)\Big]\tilde{\phi}(t,t')\tilde{\phi}(s,s') ds dt ds' dt'.
\end{aligned}$$

We can use the same calculations as in part a). Consequently, $\mathbb{E} \parallel D^X u \parallel^2_{\mid\mathcal{H}\mid\otimes\mid\mathcal{H}\mid} < \infty$.
Thus, $u \in \mathbb{D}^X_{1,2}(\mid \mathcal{H} \mid)$.
Finally, we have $\mathbb{D}^X_{1,2}(\mid \mathcal{H} \mid) \subset Dom(\delta^X)$. ■

6.4 Probabilistic methods for non linear BSDEs

We start by the presentation of some tools we need for the proofs.

6.4.1 Quasi-conditional expectation

For any $n \geqslant 1$, we introduce the set $\mathcal{H}^{\otimes n}$ of all real symmetric Borel functions f_n of n variables such that

$$\parallel f_n \parallel^2_{\mathcal{H}^{\otimes n}} := \int_{[0,T]^n\times[0,T]^n} \prod_{i=1}^n \phi(s_i,r_i) f_n(s_1,...,s_n) f_n(r_1,...,r_n) ds_1...ds_n dr_1...dr_n < \infty. \quad (6.18)$$

Then, one can define the iterated integral

$$I^X_n(f_n) = n! \int_{0\leqslant t_1<...<t_n} f_n(t_1,...,t_n) dX_{t_1}...dX_{t_n}$$

in the sense of Itô Skorohod. For $n = 0$, $f = f_0$ a constant, we set $I_0^X(f_0) = f_0$ and $\| f_0 \|^2_{\mathcal{H}^{\otimes 0}} = f_0^2$.
Recall from corollary 2.7.8 ([3]) that any $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be expanded as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^X(f_n).$$

for some unique collection of kernels $f_n \in \mathcal{H}^{\otimes n}$, $n \geqslant 1$.
The following isometry property of integrals holds (proposition 2.7.5, [3]):
Fix integers $1 \leqslant q \leqslant p$, as well as $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$. We have

$$\mathbb{E}[I_p^X(f) I_q^X(g)] = \begin{cases} p! < f, g >_{\mathcal{H}^{\otimes p}} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathbb{E}[I_p^X(f)^2] = p! \| f \|^2_{\mathcal{H}^{\otimes p}}$.

Remark 6.4.1. *For $n = 2$, (6.18) does not coincide with $\| \cdot \|_{|\mathcal{H}|\otimes|\mathcal{H}|}$ in Chapter 3.*

The notion of quasi-conditional expectation has been introduced, since the classical notion of conditional expectation seems not to be convenient for a proof of the existence and uniqueness of solutions to BSDEs whose driving process is not a martingale.

Let $\{\mathcal{F}_t\}_{t \geqslant 0}$ be the filtration generated by X. The following remark is introduced by lemma 1.2.5 in [26].

Remark 6.4.2. *Let $F = I_n^X(f_n)$ where $f_n = 1_{B_1 \times ... \times B_n}$ and $B_1, ..., B_n$ are mutually disjoint Borel sets with finite Lebesgue measure.*
Then

$$\begin{aligned} \mathbb{E}[F \mid \mathcal{F}_t] = \mathbb{E}[X(B_1)...X(B_n) \mid \mathcal{F}_t] &= \mathbb{E}\Big[\prod_{i=1}^{n} (X(B_i \cap [0,t]) + X(B_i \cap [0,t]^c)) \mid \mathcal{F}_t \Big] \\ &= I_n^X(1_{(B_1 \cap [0,t]) \times ... \times (B_n \cap [0,t])}) + \mathbb{E}\Big(\prod_{i=1}^{n} X(B_i \cap [0,t]^c) \mid \mathcal{F}_t \Big) \\ &= I_n^X(f_n 1_{[0,t]}^{\otimes n}) + \mathbb{E}\Big(\prod_{i=1}^{n} X(B_i \cap [0,t]^c) \mid \mathcal{F}_t \Big). \end{aligned}$$

We have $\mathbb{E}\Big(\prod_{i=1}^{n} X(B_i \cap [0,t]^c) \mid \mathcal{F}_t \Big) \neq 0$ if X has not independent increments. In general, it is difficult to treat this term, so the notion of quasi-conditional expectation has been defined as follows (definition 12, [25]).

Definition 6.4.3. *For $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ let $f_n \in \mathcal{H}^{\otimes n}$, $n \geqslant 0$ such that $F = \sum_{n=0}^{\infty} I_n^X(f_n)$. The quasi-conditional expectation is defined as*

$$\widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t \Big] = \sum_{n=0}^{\infty} I_n^X(f_n 1_{[0,t]}^{\otimes n}), \tag{6.19}$$

if the series on the right side converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. *Here*

$$1_{[0,t]}^{\otimes n}(t_1,...,t_n) = 1_{[0,t]}(t_1)...1_{[0,t]}(t_n).$$

Remark 6.4.4. $\widehat{\mathbb{E}}\Big[\widehat{\mathbb{E}}[F \mid \mathcal{F}_t] \mid \mathcal{F}_s\Big] = \widehat{\mathbb{E}}[F \mid \mathcal{F}_s]$, *for* $0 \leqslant s \leqslant t \leqslant T$.

Remark 6.4.5. ***(Semi-group property)*** *Let* g *be a continuous function and let* $Var(N_.)$ *increasing or decreasing. Let us show that* $P_{|Var(N_T)-Var(N_t)|} = P_{|Var(N_T)-Var(N_s)|}P_{|Var(N_s)-Var(N_t)|}g(x), (t,s,x) \in [t_0,T]^2 \times \mathbb{R}$ *and* $t < s < T$. *Here* P *is the heat kernel (6.5) evaluated in the scale of* $Var(N_.)$.

- If $Var(N_.)$ is increasing then $Var(N_t) \leqslant Var(N_s) \leqslant Var(N_T)$. Therefore

$$\begin{aligned} P_{|Var(N_T)-Var(N_t)|}g(x) &= P_{Var(N_T)-Var(N_t)}g(x). \\ &= P_{Var(N_T)-Var(N_s)}P_{Var(N_s)-Var(N_t)}g(x). \\ &= P_{|Var(N_T)-Var(N_s)|}P_{|Var(N_s)-Var(N_t)|}g(x). \end{aligned}$$

- If $Var(N_.)$ is decreasing, $-Var(N_.)$ is increasing, we get

$$\begin{aligned} P_{|Var(N_T)-Var(N_t)|}g(x) &= P_{Var(N_t)-Var(N_T)}g(x) = P_{-Var(N_T)-(-Var(N_t))}g(x) \\ &= P_{-Var(N_T)+Var(N_s)}P_{-Var(N_s)+Var(N_t)}g(x). \\ &= P_{|Var(N_s)-Var(N_T)|}P_{|Var(N_t)-Var(N_s)|}g(x). \\ &= P_{|Var(N_T)-Var(N_s)|}P_{|Var(N_s)-Var(N_t)|}g(x). \end{aligned}$$

The following lemma is motivated by lemma 3.3 in [19].

Lemma 6.4.6. *Let* $g \in Dom(\delta^X)$ *and* $t \in [0,T]$, *then* $\widehat{\mathbb{E}}\Big[\int_t^T g_u \delta X_u | \mathcal{F}_t\Big] = 0$, ***P-a.s.***

Proof. Let $g_u = \sum_{n=0}^{\infty} I_n^X(f_n(u))$ be the chaos expansion of g_u, where $f_n(u)$ is a function of n variables: $f_n(s_1,...,s_n,u)$. By proposition 1.3.7 in [26], we get

$$\int_t^T g_u \delta X_u = \sum_{n=0}^{\infty} \int_t^T I_n^X(f_n(u))\delta X_u = \sum_{n=0}^{\infty} I_{n+1}^X(\tilde{f}_{n+1}).$$

where $\tilde{f}_{n+1}$ is the symmetrization of a function of $n+1$ variables, given by

$$\tilde{f}_{n+1}(s_1,...,s_{n+1},s) = \frac{1}{n+2}\Big[f_{n+1}(s_1,...,s_{n+1},s) + \sum_{i=1}^{n+1} f_{n+1}(s_1,...,s_{i-1},s,s_{i+1},...,s_{n+1},s_i)\Big].$$

Now, applying the definition of quasi-conditional expectation we obtain

$$\widehat{\mathbb{E}}\Big[\int_t^T g_u \delta X_u | \mathcal{F}_t\Big] = \widehat{\mathbb{E}}\Big[\sum_{n=1}^{\infty} I_{n+1}^X(\tilde{f}_{n+1}) | \mathcal{F}_t\Big]$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} I_{n+1}^{X}(\tilde{f}_{n+1} 1_{[0,t]}^{\otimes n+1}) \\
&= \sum_{n=0}^{\infty} \int_t^T I_n^X(f_n(.,u) 1_{[0,t]}^{\otimes n}) \delta X_u \\
&= \sum_{n=0}^{\infty} \int_0^T I_n^X(f_n(.,u) 1_{[0,t]}^{\otimes n} 1_{[t,T]}(u)) \delta X_u = 0. \ \blacksquare
\end{aligned}
$$

Lemma 6.4.7. *Let $F = k(N_s)$, where $k : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function of polynomial growth and we suppose that $Var(N_.)$ is increasing. Then $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and*

$$
\widehat{\mathbb{E}}[F|\mathcal{F}_t] = P_{Var(N_s)-Var(N_t)} k(N_t), \quad 0 \leqslant t < s \leqslant T. \tag{6.20}
$$

where P is the heat kernel (6.5) evaluated in the scale of $Var(N_.)$.

Proof. a) We prove $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$:
In fact, we have

$$
\mathbb{E} \mid N_s \mid^p \leqslant C_p \Big(\parallel \sigma \parallel_{\mathcal{H}}^2 \Big)^{\frac{p}{2}} < \infty.
$$

Since k is a measurable real valued function of polynomial growth we have, for some $C > 0$, $p \geqslant 1$,

$$
\mathbb{E} \mid k(N_s) \mid^2 \leqslant C\mathbb{E}\Big(1 + \mid N_s \mid + \ldots + \mid N_s \mid^p \Big).
$$

b) We apply now Itô's formula (Theorem 4.2.1) to $G(t,x) = P_{Var(N_s)-Var(N_t)} k(x)$ to show (6.20). In fact, $G(t,x) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ and satisfies (4.3) (we can do a similar calculation as in proof of Theorem 6.1.4). Moreover, $\partial/\partial x G(\cdot, N_.) \in \mathbb{D}_{1,2}^X(\mid \mathcal{H}|)$ (a similar calculation as in proof of Theorem 4.2.1).
We obtain

$$
G(t,N_t) = G(0,0) \quad + \quad \int_0^t \frac{\partial}{\partial x} \Big[P_{Var(N_s)-Var(N_v)} k \Big] (N_v) \sigma_v \delta X_v. \tag{6.21}
$$

Using Theorem 6.1.4, we get

$$
k(N_s) = P_{VarN_s} k(0) + \int_0^s \frac{\partial}{\partial x} \Big[P_{Var(N_s)-Var(N_v)} k \Big] (N_v) \sigma_v \delta X_v. \tag{6.22}
$$

Then, substracting the two equations (6.22) and (6.21) we get

$$
k(N_s) \quad = \quad P_{Var(N_s)-Var(N_t)} k(N_t) + \int_t^s \frac{\partial}{\partial x} \Big[P_{Var(N_s)-Var(N_v)} \Big] k(N_v) \sigma_v \delta X_v. \tag{6.23}
$$

and hence,

$$
\mathbb{E} k(N_s) = \mathbb{E} P_{Var(N_s)-Var(N_t)} k(N_t). \tag{6.24}
$$

For $t = 0$, we have

$$k(N_s) = P_{Var(N_s)}k(0) + \int_0^s \frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(N_v)\sigma_v \delta X_v. \tag{6.25}$$

Applying quasi-conditional expectation, we get

$$\begin{aligned}\widehat{\mathbb{E}}\Big[F|\mathcal{F}_t\Big] &= P_{Var(N_s)}k(0) + \int_0^t \frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(N_v)\sigma_v \delta X_v \\ &+ \widehat{\mathbb{E}}\Big[\int_t^s \frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(N_v)\sigma_v \delta X_v|\mathcal{F}_t\Big].\end{aligned}$$

Moreover, we have $\partial/\partial x G(\cdot, N_\cdot) \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}|) \subset Dom(\delta^X)$, then we apply Lemma 6.4.6 and obtain

$$\widehat{\mathbb{E}}\Big[F|\mathcal{F}_t\Big] = P_{Var(N_s)}k(0) + \int_0^t \frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(N_v)\sigma_v \delta X_v.$$

We take $s = t$ in (6.25), we obtain

$$k(N_t) = P_{Var(N_t)}k(0) + \int_0^t \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)}k(N_v)\sigma_v \delta X_v. \tag{6.26}$$

From the semi-group property, we have

$$P_{Var(N_s)-Var(N_v)}k(x) = P_{Var(N_s)-Var(N_t)}P_{Var(N_t)-Var(N_v)}k(x).$$

Let us show

$$\frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(x) = P_{Var(N_s)-Var(N_t)}\frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)}k(x), v < t < s$$

Indeed, we have

$$\begin{aligned}\frac{\partial}{\partial x} P_{Var(N_s)-Var(N_v)}k(x) &= \frac{\partial}{\partial x} P_{Var(N_s)-Var(N_t)}(P_{Var(N_t)-Var(N_v)})k(x) \\ &= \frac{\partial}{\partial x}(P_{Var(N_t)-Var(N_v)}k * p_{Var(N_s)-Var(N_t)})(x) \ (* \text{ is convolution product}) \\ &= \frac{\partial}{\partial x}(p_{Var(N_s)-Var(N_t)} * P_{Var(N_t)-Var(N_v)}k)(x) \\ &= p_{Var(N_s)-Var(N_t)} * \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)}k(x) \\ &= P_{Var(N_s)-Var(N_t)}\Big(\frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)}k(x)\Big).\end{aligned} \tag{6.27}$$

Now, we show that

$$\int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y)\Big(\int_0^t \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)}k(N_v)\sigma_v \delta X_v\Big)dy$$

$$= \int_0^t \Big(\int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v dy \Big) \delta X_v. \tag{6.28}$$

Indeed, let $F \in L^2(\Omega, \mathcal{H})$

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \Big(\int_0^t \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v \Big) dy \Big) F \Big] \\
&= \int_{\mathbb{R}} dy p_{Var(N_s)-Var(N_t)}(x-y) \mathbb{E}\Big[\Big(\int_0^t \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} \sigma_v \delta X_v \Big) F \Big] \\
&= \int_{\mathbb{R}} dy p_{Var(N_s)-Var(N_t)}(x-y) \mathbb{E}\Big[< \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_\cdot)} k(N_\cdot)\sigma_\cdot, D^X_\cdot F >_{\mathcal{H}} \Big] \\
&= \mathbb{E}\Big[< \int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_\cdot)} k(N_\cdot)\sigma_\cdot dy, D^X_\cdot F >_{\mathcal{H}} \Big] \\
&= \mathbb{E}\Big[\int_0^t \Big(\int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v dy \Big) \delta X_v . F \Big].
\end{aligned}$$

By (6.27) we have

$$\int_0^t \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v \tag{6.29}$$

$$\begin{aligned}
&= \int_0^t P_{Var(N_s)-Var(N_t)} \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v \quad \text{(Using (6.27))} \\
&= \int_0^t \Big(\int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v dy \Big) \delta X_v \\
&= \int_{\mathbb{R}} p_{Var(N_s)-Var(N_t)}(x-y) \Big(\int_0^t \frac{\partial}{\partial y} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v \Big) dy \quad \text{(Using (6.28))}
\end{aligned}$$

$$= P_{Var(N_s)-Var(N_t)} \int_0^t \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v. \tag{6.30}$$

Using (6.30) and (6.26), we get

$$\begin{aligned}
\widehat{\mathbb{E}}\Big[k(N_s) | \mathcal{F}_t \Big] &= P_{Var(N_s)-Var(N_t)} P_{Var(N_t)} k(0) + P_{Var(N_s)-Var(N_t)} \int_0^t \frac{\partial}{\partial x} P_{Var(N_t)-Var(N_v)} k(N_v)\sigma_v \delta X_v \\
&= P_{Var(N_s)-Var(N_t)} k(N_t). \ \blacksquare
\end{aligned}$$

Remark 6.4.8. *If $VarN_t$ is decreasing, we must add the following hypothesis in the previous lemma: $\frac{Var(N_s)}{|Var(N_s)-Var(N_t)|} \geqslant 1$, for all $0 \leqslant t < s < T$. So that we can apply the Itô formula.*

Lemma 6.4.9. *Let $VarN_\cdot$ be increasing on $[0,T]$. Assume that $h \in \mathcal{C}^{0,1}_{pol}([0,T] \times \mathbb{R})$ and put $h_s = h(s, N_s)$, $s \in [0,T]$. Then,*

$$\hat{\mathbb{E}}\Big[\int_t^T h_s ds \mid \mathcal{F}_t \Big] = \int_t^T \hat{\mathbb{E}}\Big[h_s \mid \mathcal{F}_t \Big] ds.$$

Proof. Since $h \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R})$, then we have $h_s \in L^2(\Omega,\mathcal{F},\mathbb{P})$, for all $s\in[0,T]$.
By corollary 2.7.8 ([3]), there exist $h_{n,s} \in \mathcal{H}^{\otimes n}, n \geqslant 0$ such that $h_s = \sum_{n=0}^{\infty} I_n^X(h_{n,s})$, with $I_0^X(h_{0,s}) = \mathbb{E}(h_s)$.
Moreover, $\sum_{n=0}^{\infty} I_n^X(h_{n,s})$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$.
a) For $G = \int_t^T h_s ds \in L^2(\Omega,\mathcal{F},\mathbb{P})$ ($h_s \in L^2(\Omega,\mathcal{F},\mathbb{P})$), there exist $G_n \in \mathcal{H}^{\otimes n}, n\geqslant 0$ such that $G = \sum_{n=0}^{\infty} I_n^X(G_n)$ with $I_0^X(G_0) = \mathbb{E}(G)$. Let us show that we can choose $G_n = \int_t^T h_{n,s} ds$, $n \geqslant 0$.

$$\begin{aligned}
\mathbb{E}\int_t^T \Big|\sum_{n=0}^{N} I_n^X(h_{n,s})\Big|^2 ds &= \int_t^T \mathbb{E}\Big|\sum_{n=0}^{N} I_n^X(h_{n,s})\Big|^2 ds \\
&= \int_t^T \mathbb{E}\Big|\sum_{n=0}^{N} I_n^X(h_{n,s}) \sum_{m=0}^{N} I_m^X(h_{m,s})\Big| ds \\
&\leqslant \int_t^T \sum_{n,m=0}^{N} \mathbb{E} \mid I_n^X(h_{n,s}) I_m^X(h_{m,s}) \mid ds \\
&= \int_t^T \sum_{n=0}^{N} \mathbb{E} \mid I_n^X(h_{n,s}) \mid^2 ds \\
&= \mathbb{E}\int_t^T \sum_{n=0}^{N} \mid I_n^X(h_{n,s}) \mid^2 ds \\
&\leqslant \mathbb{E}\int_t^T \sum_{n=0}^{\infty} \mid I_n^X(h_{n,s}) \mid^2 ds \\
&= \mathbb{E}\int_t^T \Big|\sum_{n=0}^{\infty} I_n^X(h_{n,s})\Big|^2 ds = \mathbb{E}\int_t^T \mid h_s \mid^2 ds.
\end{aligned}$$

Then, $I_n^X(h_{n,s}) \in L^2(\Omega\times[t,T])$.
b) For arbitrary $F \in L^2(\Omega,\mathcal{F},\mathbb{P})$ with the chaos expansion $F = \sum_{n=0}^{\infty} I_n^X(l_n)$, we have

$$\begin{aligned}
\mathbb{E}[F I_n^X(G_n)] = \mathbb{E}\Big[\sum_{m=0}^{\infty} I_m^X(l_m) I_n^X(G_n)\Big] &= \mathbb{E}\Big[I_n^X(l_n) \sum_{m=0}^{\infty} I_m^X(G_m)\Big] \\
&= \mathbb{E}[I_n^X(l_n) G] \\
&= \mathbb{E}\Big[I_n^X(l_n) \int_t^T h_s ds\Big].
\end{aligned}$$

Moreover, we have

$$\mathbb{E}\Big[I_n^X(l_n)\int_t^T h_s ds\Big] = \mathbb{E}\Big[\int_t^T h_s I_n^X(l_n) ds\Big] = \int_t^T \mathbb{E}[h_s I_n^X(l_n)] ds. \tag{6.31}$$

Using (6.31), we obtain

$$\mathbb{E}[F I_n^X(G_n)] = \mathbb{E}\Big[\int_t^T h_s I_n^X(l_n) ds\Big] = \int_t^T \mathbb{E}[h_s I_n^X(l_n)] ds$$

$$
\begin{aligned}
&= \int_t^T \mathbb{E}\Big[\sum_{m=0}^{\infty} I_m^X(h_{m,s}) I_n^X(l_n)\Big] ds \\
&= \int_t^T \mathbb{E}[I_n^X(h_{n,s}) I_n^X(l_n)] ds \\
&= \int_t^T \mathbb{E}\Big[\sum_{m=0}^{\infty} I_m^X(l_m) I_n^X(h_{n,s})\Big] ds \\
&= \int_t^T \mathbb{E}[F I_n^X(h_{n,s})] ds \\
&= \mathbb{E}\Big[F \int_t^T I_n^X(h_{n,s}) ds\Big].
\end{aligned}
$$

We deduce

$$
I_n^X(G_n) = \int_t^T I_n^X(h_{n,s}) ds, n \geqslant 0. \tag{6.32}
$$

c) Now, we want to prove

$$
I_n^X\Big(\int_t^T h_{n,s} ds\Big) = \int_t^T I_n^X(h_{n,s}) ds. \tag{6.33}
$$

Let $f \in L^2(\Omega, \mathcal{H}^{\otimes n})$. By definition 2.5.2 of [3],

$$
\begin{aligned}
\mathbb{E}\Big[I_n^X\Big(\int_t^T h_{n,s} ds\Big) f\Big] &= \mathbb{E}\Big[< \int_t^T h_{n,s} ds, D_\cdot^X f >_{\mathcal{H}^{\otimes n}} \Big] \\
&= \mathbb{E} \int_t^T < h_{n,s}, D_\cdot^X f >_{\mathcal{H}^{\otimes n}} ds \\
&= \int_t^T \mathbb{E} < h_{n,s}, D_\cdot^X f >_{\mathcal{H}^{\otimes n}} ds \\
&= \int_t^T \mathbb{E}[I_n^X(h_{n,s}) f] ds \\
&= \mathbb{E}\Big[\int_t^T I_n^X(h_{n,s}) ds f\Big].
\end{aligned}
$$

By (6.32) we get

$$
I_n^X(G_n) = I_n^X\Big(\int_t^T h_{n,s} ds\Big).
$$

Consequently, $G_n = \int_t^T h_{n,s} ds, n \geqslant 0$, and $1_{[0,t]}^{\otimes n} G_n = 1_{[0,t]}^{\otimes n} \int_t^T h_{n,s} ds, n \geqslant 0$.
d) We show that

$$
\hat{\mathbb{E}}\Big[\int_t^T h_s ds \mid \mathcal{F}_t\Big] = \int_t^T \hat{\mathbb{E}}\Big[h_s \mid \mathcal{F}_t\Big] ds.
$$

d1)

$$
\lim_{N \to +\infty} \sum_{n=0}^{N} I_n^X\Big(\int_t^T h_{n,s} ds\Big) = \lim_{N \to +\infty} \sum_{n=0}^{N} I_n^X(G_n) = \sum_{n=0}^{\infty} I_n^X(G_n) = G = \int_t^T h_s ds.
$$

Since $\int_t^T h_s ds \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, we conclude $\sum_{n=0}^{\infty} I_n^X\Big(\int_t^T h_{n,s} ds\Big) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.
d2)

$$\begin{aligned}
\hat{\mathbb{E}}\Big[\int_t^T h_s ds \mid \mathcal{F}_t\Big] &= \hat{\mathbb{E}}\Big[\sum_{n=0}^{\infty} I_n^X(G_n) \mid \mathcal{F}_t\Big] \\
&= \sum_{n=0}^{\infty} I_n^X(1_{[0,t]}^{\otimes n} G_n) \\
&= \sum_{n=0}^{\infty} I_n^X(1_{[0,t]}^{\otimes n} \int_t^T h_{n,s} ds), \quad (G_n = \int_t^T h_{n,s} ds) \\
&= \sum_{n=0}^{\infty} \int_t^T I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds
\end{aligned} \tag{6.34}$$

by (6.33). By Lemma 6.4.7, $\hat{\mathbb{E}}[h_s \mid \mathcal{F}_t]$ exists and $\hat{\mathbb{E}}[h_s \mid \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}))$.
d3) Now, we prove that the last term in (6.34) is equal to $\int_t^T \hat{\mathbb{E}}[h_s \mid \mathcal{F}_t] ds$. We have

$$\Big|\sum_{n=0}^{N} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n})\Big| \leqslant \Big|\sum_{n=0}^{\infty} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n})\Big| = \mid \hat{\mathbb{E}}[h_s \mid \mathcal{F}_t] \mid .$$

Applying the dominate convergence theorem, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_t^T I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_t^T I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds \\
&= \lim_{N\to\infty} \int_t^T \sum_{n=0}^{N} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds \\
&= \int_t^T \lim_{N\to\infty} \sum_{n=0}^{N} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds \\
&= \int_t^T \sum_{n=0}^{\infty} I_n^X(h_{n,s} 1_{[0,t]}^{\otimes n}) ds = \int_t^T \hat{\mathbb{E}}[h_s \mid \mathcal{F}_t] ds. \ \blacksquare
\end{aligned}$$

6.4.2 Clark-Ocone Formula

In this part, we prove the Clark-Ocone formula. It will be used for the construction of Y_t and Z_t in the proof of the existence of a solution of BSDE (6.1).

Proposition 6.4.10. *([3], p. 40) Let $F \in L^2(\Omega)$ be such that $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^X(f_n)$ with $f_n \in \mathcal{H}^{\otimes n}$. Then $F \in \mathbb{D}_{1,2}^X$ if and only if $\sum_{n=1}^{\infty} nn! \parallel f_n \parallel_{\mathcal{H}^{\otimes n}}^2 < \infty$ and in this case*

$$D^X F = \sum_{n=1}^{\infty} n I_{n-1}^X(f_n). \tag{6.35}$$

Proposition 6.4.11. *Suppose that $F \in \mathbb{D}^X_{1,2}$ be a square integrable random variable and suppose that $\widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t\Big] \in \mathbb{D}^X_{1,2}$. Then we have*

$$D^X_s \widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t\Big] = \widehat{\mathbb{E}}\Big[D^X_s F \mid \mathcal{F}_t\Big] 1_{[0,t]}(s). \tag{6.36}$$

Proof. Using Proposition 6.4.10 and by definition of quasi-conditional expectation we obtain

$$\begin{aligned}
\widehat{\mathbb{E}}\Big[D^X_s F \mid \mathcal{F}_t\Big] 1_{[0,t]}(s) &= \widehat{\mathbb{E}}\Big[\sum_{n=1}^{\infty} n I^X_{n-1}\Big(f_n(.,s)\Big) \mid \mathcal{F}_t\Big] 1_{[0,t]}(s) \\
&= \sum_{n=1}^{\infty} n I^X_{n-1}\Big(f_n(.,s) 1^{\otimes n-1}_{[0,t]}\Big) 1_{[0,t]}(s) \\
&= D^X_s\Big(\sum_{n=0}^{\infty} I^X_n(f_n 1^{\otimes n}_{[0,t]})\Big) \\
&= D^X_s \widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t\Big]. \blacksquare
\end{aligned}$$

Lemma 6.4.12. *Let, for $t > 0$, $u_t \in \mathbb{D}^X_{1,2}$ and let the chaos expansion of u_t be given by*

$$u_t = \sum_{n=0}^{\infty} I^X_n(f_n(.,t)), \tag{6.37}$$

where for each $n \geqslant 1$ $f_n(.,t) \in \mathcal{H}^{\otimes n}$. Then, u belongs to $Dom(\delta^X)$ if and only if the series $\sum_{n=0}^{\infty} I^X_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$ and $\delta^X(u) = \sum_{n=0}^{\infty} I^X_{n+1}(\tilde{f}_n)$, where $\tilde{f}_n$ is the symmetrization of f_n in all its variables given by

$$\tilde{f}_n(t_1,...,t_n,t) = \frac{1}{n+1}\Big[f_n(t_1,...,t_n,t) + \sum_{i=1}^{n} f_n(t_1,...,t_{i-1},t,t_{i+1},...,t_n,t_i)\Big].$$

Proof. Let $g \in \mathcal{H}^{\otimes n}$ $(n \geqslant 1)$ and $G := I^X_n(g)$. Then

$$\begin{aligned}
\mathbb{E}\Big[< D^X G, u >_{\mathcal{H}} \Big] &= \mathbb{E}\Big[\int_0^T \int_0^T (D^X_t G) u_r \phi(t,r) dt dr\Big] \\
&= \mathbb{E}\Big[\int_{[0,T]^2} n I^X_{n-1}(g(.,t)) \sum_{m=0}^{\infty} I^X_m(f_m(.,r)) \phi(t,r) dt dr\Big] \\
&= \sum_{m=0}^{\infty} \int_{[0,T]^2} n \mathbb{E}\Big[I^X_{n-1}(g(.,t)) I^X_m(f_m(.,r))\Big] \phi(t,r) dt dr \\
&= \int_{[0,T]^2} n(n-1)! < f_{n-1}(.,r), g(.,t) >_{\mathcal{H}^{\otimes n-1}} \phi(t,r) dt dr \\
&= n! < f_{n-1}, g >_{\mathcal{H}^{\otimes n}} \\
&= n! < \tilde{f}_{n-1}, g >_{\mathcal{H}^{\otimes n}}
\end{aligned}$$

$$= \mathbb{E}[I_n^X(\tilde{f}_{n-1})I_n^X(g)] = \mathbb{E}[I_n^X(\tilde{f}_{n-1})G]. \tag{6.38}$$

After this preliminary computation, first assume that $u \in Dom(\delta^X)$. Then the above yields

$$\mathbb{E}\Big[\delta^X(u)G\Big] = \mathbb{E}\Big[< D^X G, u >_{\mathcal{H}} \Big] = \mathbb{E}[I_n^X(\tilde{f}_{n-1})G], \quad \text{for } G = I_n^X(g).$$

Suppose that $\delta^X(u) = \mathbb{E}(\delta^X(u)) + \sum_{m=1}^{\infty} I_m^X(g_m)$.
Then,

$$\mathbb{E}[\delta^X(u)G] = \mathbb{E}\Big[\sum_{m=1}^{\infty} I_m^X(g_m)G\Big] = \mathbb{E}[I_n^X(g_n)G].$$

Moreover, we have $\mathbb{E}[\delta^X(u)G] = \mathbb{E}[I_n^X(\tilde{f}_{n-1})G]$ for all $n \geqslant 1$. Thus, $I_n^X(g_n) = I_n^X(\tilde{f}_{n-1})$ for all $n \geqslant 1$. Consequently, $\delta^X(u) = \sum_{n=1}^{\infty} I_n^X(\tilde{f}_{n-1}) = \sum_{n=0}^{\infty} I_{n+1}^X(\tilde{f}_n)$. By orthogonality, $\mathbb{E}[\delta^X(u)^2] = \sum_{n=0}^{\infty}(n+1)! \parallel \tilde{f}_n \parallel^2_{\mathcal{H}^{\otimes n+1}} < \infty$ ([3]).
Conversely, suppose $V := \sum_{n=0}^{\infty} I_{n+1}^X(\tilde{f}_n)$ exists in $L^2(\Omega)$ and we prove that $u \in Dom(\delta^X)$.
The preceding computation yields

$$\mathbb{E}\int_{[0,T]^2} u_r D_t^X G\phi(t,r)dtdr = \mathbb{E}[VG]$$

for all $G = I_n^X(g)$, hence, by linearity,

$$\mathbb{E}\int_{[0,T]^2} u_r D_t^X \Big(\sum_{n=0}^{N} I_n^X(g)\Big)\phi(t,r)dtdr = \mathbb{E}\Big[V\sum_{n=0}^{N} I_n^X(g)\Big] \text{for all } N \geqslant 0$$

So,

$$\Big|\mathbb{E}\int_{[0,T]^2} u_r D_t^X F_N \phi(t,r)dtdr\Big| \leqslant \Big\|V\Big\|_2^2 \Big\|F_N\Big\|_2^2, \tag{6.39}$$

for any random variable $F_N = \sum_{n=0}^{N} I_n^X(g)$ where the series converges in $L^2(\Omega)$. By density, (6.39) holds for any random variable $F := \sum_{n=0}^{\infty} I_n^X(g)$ in $\mathbb{D}_{1,2}^X$. Using (3.6), we conclude that $u \in Dom(\delta^X)$ and $\delta^X(u) = V$. ■

Now, we prove a formula of the type of Clark-Ocone:

Proposition 6.4.13. *(Clark-Ocone Formula) Let $F \in \mathbb{D}_{1,2}^X$ and suppose that $\widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_t\Big] \in \mathbb{D}_{1,2}^X$, for all $t \in [0,T]$. Then $F = \mathbb{E}[F] + \int_0^t \widehat{\mathbb{E}}\Big[D_s^X F \mid \mathcal{F}_s\Big]\delta X_s$, P-a.s.*

Proof. Let $F = \sum_{n=0}^{\infty} I_n^X(f_n)$. Using (6.35) and (6.19) we deduce

$$\begin{aligned} D_s^X \widehat{\mathbb{E}}\Big[F \mid \mathcal{F}_s\Big] &= D_s^X \sum_{n=0}^{\infty} I_n^X(f_n 1_{[0,s]}^{\otimes n}) \\ &= \sum_{n=1}^{\infty} n I_{n-1}^X\Big(f_n(.,s) 1_{[0,s]}^{\otimes n-1}\Big) \\ &= \sum_{n=1}^{\infty} n I_{n-1}^X\Big(f_n(s_1,...,s_{n-1},s) 1_{[0,s]}^{\otimes n-1}(s_1,...,s_{n-1})\Big). \end{aligned}$$

Applying (6.36), we get $\widehat{\mathbb{E}}\Big[D_s^X F \mid \mathcal{F}_s\Big] = \sum_{n=1}^{\infty} n I_{n-1}^X\Big(f_n(s_1,...,s_{n-1},s) 1_{[0,t]}^{\otimes n-1}(s_1,...,s_{n-1})\Big)$.
Set $\phi_s = \widehat{\mathbb{E}}\Big[D_s^X F \mid \mathcal{F}_s\Big]$. Then $\phi_s = \sum_{n=1}^{\infty} I_{n-1}^X\Big(\tilde{f}_n(s_1,...,s_{n-1},s) 1_{[0,t]}^{\otimes n-1}(s_1,...,s_{n-1})\Big)$.
Now, applying Lemma 6.4.12, we obtain

$$\begin{aligned} \delta^X(\phi) &= \sum_{n=1}^{\infty} I_n^X\Big(\tilde{f}_n(s_1,...,s_{n-1},s) 1_{[0,t]}^{\otimes n-1}(s_1,...,s_{n-1})\Big) \\ &= \sum_{n=1}^{\infty} I_n^X(f_n) \text{ converges in } L^2(\Omega). \end{aligned}$$

Consequently, $\phi \in Dom(\delta^X)$ and we have $\delta^X(\phi) = F - \mathbb{E}[F]$. This proves the Proposition. ■

6.5 Itô formula for stochastic integrals

Theorem 6.5.1. *Let $u \in \mathcal{V}_T$ and $f \in \mathcal{C}_{pol}^{0,1}([0,T] \times \mathbb{R})$. We write f_s for $f(s, N_s)$, $s \in [0,T]$. Then for the Itô type process*

$$C_t = C_0 + \int_0^t f_s ds + \int_0^t u_s \delta X_s, t \in [0,T],$$

we have
i) $uC1_{[0,t]} \in Dom(\delta^X)$,
ii) $C_s \in \mathbb{D}_{1,2}^X, s \in [0,T]$,
iii) $(\mathbb{D}_s^X C_s)_{s\in[0,T]} \in L^2([0,T] \times \Omega)$,
and

$$C_t^2 = C_0^2 + 2\int_0^t C_s f_s ds + 2\int_0^t C_s u_s \delta X_s + 2\int_0^t u_s \mathbb{D}_s^X C_s ds, \text{ P-a.s. } t \in [0,T].$$

For the proof we need the following lemma

Lemma 6.5.2. *Let $u \in \mathcal{V}_T$ and $M_t = \int_0^t u_s \delta X_s, t \in [0,T]$. Then $uM \in dom(\delta^X)$, $(\mathbb{D}_s^X M_s)_{s\in[0,T]} \in L^2([0,T]\times\Omega)$, and*

$$M_t^2 = 2\int_0^t M_s u_s \delta X_s + 2\int_0^t u_s \mathbb{D}_s^X M_s ds, \ \textit{P-a.s.}\ t \in [0,T].$$

Proof. a) We prove that

$$\mathbb{E}[F\delta^X(u)] = \mathbb{E}\int_0^T u_t \mathbb{D}_t^X F dt, \text{for all } F \in \mathcal{S}. \tag{6.40}$$

In fact,

$$\begin{aligned}
\mathbb{E}[F\delta^X(u)] &= \mathbb{E}\int_0^T\int_0^T D_s^X F u_t \phi(t,s) dt ds \quad \text{(Using (3.7))} \\
&= \mathbb{E}\int_0^T u_t\Big(\int_0^T D_s^X F \phi(t,s) ds\Big) dt \\
&= \mathbb{E}\int_0^T u_t \mathbb{D}_t^X F dt.
\end{aligned}$$

b) Now, we calculate the expectation of $M_t^2 F$:

$$\begin{aligned}
\mathbb{E}[M_t^2 F] = \mathbb{E}\Big[M_t(M_t F)\Big] &= \mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s)\delta X_s (M_t F)\Big] \\
&= \mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X (M_t F) ds\Big] \quad \text{(Using (6.40))} \\
&= \mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s)[(\mathbb{D}_s^X M_t)F + M_t \mathbb{D}_s^X F] ds\Big] \quad \text{(By linearity of } D^X\text{)} \\
&= \mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s)(\mathbb{D}_s^X M_t) ds F\Big] + \mathbb{E}\Big[\Big(\int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X F ds\Big) M_t\Big].
\end{aligned} \tag{6.41}$$

For the first term in (6.41), we get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \mathbb{D}_s^X M_t (u_s F) ds\Big] \\
&= \mathbb{E}\Big[\int_0^t\int_0^t D_\theta^X M_t (u_s F)\phi(s,\theta) ds d\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t\int_0^t \Big(u_\theta + \int_\theta^t D_\theta^X u_r \delta X_r\Big)(u_s F)\phi(s,\theta) ds d\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t\int_0^t u_\theta \phi(s,\theta)(u_s F) ds d\theta\Big] + \mathbb{E}\Big[\int_0^t\int_0^t \Big(\int_0^t D_\theta^X u_r \delta X_r\Big)(u_s F)\phi(s,\theta) ds d\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t\int_0^t u_\theta \phi(s,\theta)(u_s F) ds d\theta\Big] + \mathbb{E}\Big[\int_0^t\int_0^t \Big(\int_0^t D_\theta^X u_r \phi(s,\theta) d\theta\Big)\delta X_r (u_s F) ds,
\end{aligned}$$

in the above equality, we use a similar calculation as for (6.13).Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \mathbb{D}_s^X M_t(u_sF)ds\Big] \\
&= \mathbb{E}\Big[\int_0^t\int_0^t u_\theta\phi(s,\theta)d\theta(u_sF)ds\Big] + \mathbb{E}\Big[\int_0^t\Big(\int_0^t \mathbb{D}_s^X u_r\delta X_r\Big)(u_sF)ds\Big] \\
&= \mathbb{E}\Big[\Big(\int_0^t u_s\Big(\int_0^t \phi(s,\theta)u_\theta d\theta\Big)ds\Big)F\Big] + \mathbb{E}\Big[\Big(\int_0^t u_s\Big(\int_0^t \mathbb{D}_s^X u_r\delta X_r\Big)ds\Big)F\Big] \\
&= \mathbb{E}\Big[\Big(\int_0^t u_s\Big(\int_0^t \phi(s,\theta)u_\theta d\theta\Big)ds\Big)F\Big] + \int_0^t \mathbb{E}\Big[\int_0^t \mathbb{D}_s^X u_\theta \mathbb{D}_\theta^X(u_sF)d\theta\Big]ds \quad \text{(Using (6.40).}
\end{aligned} \tag{6.42}$$

For the second term in (6.41), we get

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X F ds\Big)M_t\Big] &= \mathbb{E}\Big[\Big(\int_0^t u_s\mathbb{D}_s^X F ds\Big)\Big(\int_0^t u_\theta\delta X_\theta\Big)\Big] \\
&= \mathbb{E}\Big[\int_0^\theta u_\theta\mathbb{D}_\theta^X\Big(\int_0^t u_s\mathbb{D}_s^X F ds\Big)d\theta\Big] \quad \text{(Using (6.40))} \\
&= \mathbb{E}\Big[\int_0^t u_\theta\int_0^t \phi(\theta,r)D_r^X\Big(\int_0^t u_s\mathbb{D}_s^X F ds\Big)drd\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t u_\theta\Big[\int_0^t \phi(\theta,r)\Big(\int_0^t D_r^X(u_s\mathbb{D}_s^X F)ds\Big)dr\Big]d\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t u_\theta\Big[\int_0^t\int_0^t \phi(\theta,r)D_r^X(u_s\mathbb{D}_s^X F)drds\Big]d\theta\Big] \\
&= \mathbb{E}\Big[\int_0^t u_\theta\int_0^t \mathbb{D}_\theta(u_s\mathbb{D}_s^X F)dsd\theta\Big]
\end{aligned} \tag{6.43}$$

Plugging (6.42) and (6.43) into (6.41), we get

$$\begin{aligned}
\mathbb{E}[M_t^2F] &= \mathbb{E}\Big[\Big(\int_0^t u_s\mathbb{D}_s^X M_t ds\Big)F\Big] + \mathbb{E}\Big[\Big(\int_0^t u_s\mathbb{D}_s^X F ds\Big)M_t\Big] \\
&= \mathbb{E}\Big[\Big(\int_0^t\int_0^t u_su_\theta\phi(s,\theta)dsd\theta\Big)F\Big] + \mathbb{E}\int_0^t\int_0^t \mathbb{D}_s^X u_\theta\mathbb{D}_\theta^X(u_sF)d\theta ds \\
&+ \mathbb{E}\Big[\int_0^t\int_0^t \mathbb{D}_\theta^X(u_s\mathbb{D}_s^X F)u_\theta d\theta ds\Big].
\end{aligned}$$

The above equality is symmetric in (s,θ). Indeed, we can write

$$\mathbb{E}[M_t^2F] - \mathbb{E}\Big[\Big(\int_0^t ds\int_0^t d\theta u_su_\theta\phi(s,\theta)\Big)F\Big] = \mathbb{E}\int_0^t ds\int_0^t d\theta\Big[\mathbb{D}_s^X u_\theta\mathbb{D}_\theta^X(u_sF) + \mathbb{D}_\theta^X(u_s\mathbb{D}_s^X F)u_\theta\Big].$$

On the one hand, we have

$$\begin{aligned}
\int_0^t ds\int_0^s d\theta u_su_\theta\phi(s,\theta) + \int_0^t ds\int_s^t d\theta u_su_\theta\phi(s,\theta) &= \int_0^t ds\int_0^s d\theta u_su_\theta\phi(s,\theta) + \int_0^t d\theta\int_0^\theta ds u_su_\theta\phi(s,\theta) \\
&= 2\int_0^t u_s\Big(\int_0^s d\theta u_\theta\phi(s,\theta)\Big)ds.
\end{aligned}$$

On the other hand, we pose $I(s,\theta) = \int_0^t ds \int_0^t d\theta \Big[\mathbb{D}_s^X u_\theta \mathbb{D}_\theta^X (u_s F) + \mathbb{D}_\theta^X (u_s \mathbb{D}_s^X F) u_\theta\Big]$.
Then, we have

$$
\begin{aligned}
I(s,\theta) &= \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) D_w^X (u_s F) dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(\theta,w) D_w^X (u_s \mathbb{D}_s^X F) u_\theta dw \\
&= \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) [(D_w^X u_s) F + u_s D_w^X F] dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(\theta,w) [(D_w^X u_s) \mathbb{D}_s^X F + u_s D_w^X (\mathbb{D}_s^X F)] u_\theta dw \\
&= \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) (D_w^X u_s) F dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) u_s D_w^X F dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(\theta,w) \Big[(D_w^X u_s) \int_0^T \phi(s,v) D_v^X F dv + u_s \int_0^T \phi(s,v) D_w^X D_v^X F dv\Big] u_\theta dw \\
&= \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) (D_w^X u_s) F dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(s,v) D_v^X u_\theta dv \int_0^T \phi(\theta,w) u_s D_w^X F dw \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(\theta,w) (D_w^X u_s) dw \int_0^T \phi(s,v) u_\theta D_v^X F dv \\
&+ \int_0^t ds \int_0^t d\theta \int_0^T \phi(\theta,w) u_s \Big(\int_0^T \phi(s,v) u_\theta D_w^X D_v^X F dv \Big) dw.
\end{aligned}
$$

So we can see that each term on the right side of the equality above is symmetric also in (s,θ).
Then, we obtain

$$
\begin{aligned}
\mathbb{E}[M_t^2 F] &= 2\mathbb{E}\Big[\int_0^t \int_0^s \phi(s,\theta) u_s u_\theta ds d\theta F\Big] + 2\mathbb{E} \int_0^t \int_0^s \mathbb{D}_s^X u_\theta \mathbb{D}_\theta^X (u_s F) d\theta ds \\
&\quad + 2\mathbb{E}\Big[\int_0^t \int_0^s \mathbb{D}_\theta^X (u_s \mathbb{D}_s^X F) u_\theta d\theta ds\Big] \\
&:= 2I_1 + 2I_2 + 2I_3.
\end{aligned}
$$

We calculate I_3, we obtain

$$
\begin{aligned}
I_3 &= \mathbb{E}\Big[\int_0^t \int_0^s \mathbb{D}_\theta^X (u_s \mathbb{D}_s^X F) u_\theta d\theta ds\Big] = \int_0^t \mathbb{E}\Big[\int_0^T u_\theta 1_{[0,s]}(\theta) \mathbb{D}_\theta^X (u_s \mathbb{D}_s^X F) d\theta\Big] ds \\
&= \int_0^t \mathbb{E}\Big[\int_0^T u_\theta 1_{[0,s]}(\theta) \delta X_\theta u_s \mathbb{D}_s^X F\Big] ds = \int_0^t \mathbb{E}[M_s u_s \mathbb{D}_s^X F] ds \\
&= \mathbb{E}\Big[\int_0^t M_s u_s \mathbb{D}_s^X F ds\Big].
\end{aligned}
$$

For I_2, we get

$$I_2 = \mathbb{E}\Big[\int_0^t\int_0^s \mathbb{D}_s^X u_\theta \mathbb{D}_\theta^X(u_s F)d\theta ds\Big] = \int_0^t \mathbb{E}\Big[\int_0^s \mathbb{D}_s^X u_\theta \mathbb{D}_\theta^X(u_s F)d\theta\Big]ds$$
$$= \int_0^t \mathbb{E}\Big[u_s F\int_0^s \mathbb{D}_s^X u_\theta \delta X_\theta\Big]ds = \mathbb{E}\int_0^t\Big(\int_0^s \mathbb{D}_s^X u_\theta \delta X_\theta\Big)u_s F ds.$$

Furthermore, we have

$$I_1 + I_2 = \mathbb{E}\Big[\int_0^t\int_0^s \phi(s,\theta)u_s u_\theta ds d\theta F\Big] + \mathbb{E}\Big[\int_0^t\Big(\int_0^s \mathbb{D}_s^X u_\theta \delta X_\theta\Big)u_s F ds\Big]$$
$$= \mathbb{E}\Big[\int_0^t u_s\Big(\int_0^s \phi(s,\theta)u_\theta d\theta\Big)ds F\Big] + \mathbb{E}\Big[\int_0^t\Big(\int_0^s \mathbb{D}_s^X u_\theta \delta X_\theta\Big)u_s ds F\Big]$$
$$= \mathbb{E}\Big[\int_0^t u_s\Big(\int_0^s \phi(s,\theta)u_\theta d\theta + \int_0^s \mathbb{D}_s^X u_\theta \delta X_\theta\Big)ds F\Big] = \mathbb{E}\Big[F\int_0^t u_s \mathbb{D}_s^X M_s ds\Big]. \quad (6.44)$$

Therefore, we have

$$\mathbb{E}[M_t^2 F] = 2\mathbb{E}[I_1 + I_2 + I_3] = 2\mathbb{E}\Big[\Big(\int_0^t u_s M_s \mathbb{D}_s^X F ds + F\int_0^t u_s \mathbb{D}_s^X M_s ds\Big)\Big].$$

Thus, for all $F \in \mathcal{S}$, we have

$$\mathbb{E}\Big[\Big(M_t^2 - 2\int_0^t u_s \mathbb{D}_s^X M_s ds\Big)F\Big] = 2\mathbb{E}\Big[\int_0^s u_s M_s \mathbb{D}_s^X F ds\Big] = 2\mathbb{E}\Big[F\int_0^t u_s M_s \delta X_s\Big]. \quad (6.45)$$

c) We show $M^2 \in L^2(\Omega)$.
By theorem 2.5.5 ([3]), we have

$$\mathbb{E} \mid \delta^X(u) \mid^4 \leqslant c_4 \parallel u \parallel^4_{\mathbb{D}^X_{1,4}(\mathcal{H})}$$
$$= c_4\Big[\mathbb{E} \parallel u \parallel^4_{\mathcal{H}} + \mathbb{E} \parallel D^X u \parallel^4_{\mathcal{H}\otimes\mathcal{H}}\Big]. \quad (6.46)$$

c_1) On the one hand, since $u \in \mathcal{V}_T$, we have

$$\parallel u \parallel^4_{\mathcal{H}} = \Big[\int_0^T\Big(\int_t^T u(r,N_r)\frac{\partial}{\partial r}K(r,t)dr\Big)^2 dt\Big]^2 \leqslant \Big[\int_0^T\Big(\int_t^T \mid u(r,N_r)\mid \Big|\frac{\partial}{\partial r}K(r,t)\Big|dr\Big)^2 dt\Big]^2$$

Applying Hölder's inequality again to $p < \frac{1}{1-\alpha} < 2$, and q its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$), we get as in (6.16)

$$\Big(\int_t^T \mid u(r,N_r)\mid \Big|\frac{\partial}{\partial r}K(r,t)\Big|dr\Big)^2 \leqslant ct^{-2\beta}T^{2\beta}(T-t)^{2(\alpha-1)+\frac{2}{p}}\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{2}{q}}.$$

Consequently,

$$\parallel u \parallel^4_{\mathcal{H}} \leqslant C'\Big(\int_0^T t^{-2\beta}(T-t)^{2(\alpha-1)+\frac{2}{p}}dt\Big)^2\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{4}{q}}$$
$$\leqslant C'T^{-4\beta+4\alpha+\frac{4}{p}-2}\mathcal{B}(1-2\beta,2\alpha+\frac{2}{p}+1)^2\Big(\int_0^T \mid u(r,N_r)\mid^q dr\Big)^{\frac{4}{q}}.$$

Thus,

$$\mathbb{E} \parallel u \parallel^4_{\mathcal{H}} \leqslant C^{"}\mathbb{E}\Big[\int_0^T \mid u(r,N_r) \mid^q dr\Big]^{\frac{4}{q}}, \quad \text{where } C^{"} \text{ depends on } \alpha, \beta, T, p.$$

Now, we want to prove

$$\mathbb{E}\Big[\int_0^T \mid u(r,N_r) \mid^q dr\Big]^{\frac{4}{q}} < \infty. \tag{6.47}$$

In fact,

• If $1 < \frac{4}{q} < 2$, we have

$$\mathbb{E}\Big[\int_0^T \mid u(r,N_r) \mid^q dr\Big]^{\frac{4}{q}} \leqslant T^{\frac{4}{q}}\mathbb{E}\Big(\int_0^T \mid u(r,N_r) \mid^4 dr\Big) < \infty \quad \text{(Using (6.17))}.$$

• If $\frac{4}{q} < 1$, then Jensen's inequality can be applied of a concave function to get

$$\mathbb{E}\Big[\int_0^T \mid u(r,N_r) \mid^q dr\Big]^{\frac{4}{q}} \leqslant \Big(\mathbb{E}\int_0^T \mid u(r,N_r) \mid^q dr\Big)^{\frac{4}{q}} < \infty \quad \text{(Using (6.17))}.$$

c_2) We prove that

$$\mathbb{E} \parallel D^X u \parallel^4_{\mathcal{H}^{\otimes 2}} = \mathbb{E}\Big(\int_{[0,T]^4} \phi(s_1,r_1)\phi(s_2,r_2)D^X_{s_1}u_{s_2}D^X_{r_1}u_{r_2}ds_1ds_2dr_1dr_2\Big)^2 < \infty.$$

In fact, we have

$$\begin{aligned}
&\Big(\int_{[0,T]^4} \phi(s_1,r_1)\phi(s_2,r_2)D^X_{s_1}u_{s_2}D^X_{r_1}u_{r_2}ds_1ds_2dr_1dr_2\Big)^2\\
&= \Big(\int_{[0,T]^4} \phi(s_1,r_1)\phi(s_2,r_2)D^X_{s_1}u(s_2,N_{s_2})D^X_{r_1}u(r_2,N_{r_2})ds_1ds_2dr_1dr_2\Big)^2\\
&= \Big(\int_{[0,T]^4} \phi(s_1,r_1)\phi(s_2,r_2)\sigma_{s_1}\frac{\partial u}{\partial x}(s_2,N_{s_2})\sigma_{r_1}\frac{\partial u}{\partial x}(r_2,N_{r_2})ds_1ds_2dr_1dr_2\Big)^2\\
&\leqslant M'\Big(\int_{[0,T]^4} \mid \phi(s_1,r_1) \mid\mid \phi(s_2,r_2) \mid (1+\mid N_{s_2} \mid +...+ \mid N_{s_2} \mid^k)(1+\mid N_{r_2} \mid +...+ \mid N_{r_2} \mid^{k'})ds_1ds_2dr_1dr_2\Big)^2\\
&= M'\Big(\int_{[0,T]^2} \mid \phi(s_1,r_1) \mid ds_1dr_1\Big)^2\\
&\times \Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big(1+\mid N_{s_2} \mid +...+ \mid N_{s_2} \mid^k\Big)\Big(1+\mid N_{r_2} \mid +...+ \mid N_{r_2} \mid^{k'}\Big)ds_2dr_2\Big)^2\\
&= M'\Big[\int_0^T \Big(\int_0^T \mid \phi(s_1,r_1) \mid dr_1\Big)ds_1\Big]^2\\
&\times \Big[\int_0^T \Big(1+\mid N_{r_2} \mid +...+ \mid N_{r_2} \mid^{k'}\Big)\Big(\int_0^T \mid \phi(s_2,r_2) \mid \Big(1+\mid N_{s_2} \mid +...+ \mid N_{s_2} \mid^k\Big)ds_2\Big)dr_2\Big]^2.
\end{aligned}$$

c_{21}) Now, we prove that

$$\Big[\int_0^T \Big(\int_0^T \mid \phi(s_1,r_1) \mid dr_1\Big)ds_1\Big]^2 = 2\int_0^T\int_0^{s_1} \mid \phi(s_1,r_1) \mid dr_1ds_1 < \infty.$$

In fact,

$$\begin{aligned}\int_0^{s_1} \mid \phi(s_1,r_1) \mid dr_1 &\leqslant \int_0^{s_1}\int_0^{inf(s_1,r_1)} \Big|\frac{\partial K}{\partial s_1}(s_1,v)\Big|\Big|\frac{\partial K}{\partial r_1}(r_1,v)\Big| dv dr_1 \\ &\leqslant \int_0^{s_1}\int_0^{r_1} (s_1-v)^{\alpha-1}\Big(\frac{s_1}{v}\Big)^{\beta}(r_1-v)^{\alpha-1}\Big(\frac{r_1}{v}\Big)^{\beta} dv dr_1 \\ &= s_1^{\beta}\int_0^{s_1} r_1^{\beta}\Big(\int_0^{r_1}(s_1-v)^{\alpha-1}(r_1-v)^{\alpha-1}v^{-2\beta}dv\Big)dr_1.\end{aligned}$$

Now, we will study this integral $\int_0^{r_1}(s_1-v)^{\alpha-1}(r_1-v)^{\alpha-1}v^{-2\beta}dv$:
We have

$$\begin{aligned}\int_0^{r_1}(s_1-v)^{\alpha-1}(r_1-v)^{\alpha-1}v^{-2\beta}dv &\leqslant (s_1-r_1)^{\alpha-1}\int_0^{r_1}(r_1-v)^{\alpha-1}v^{-2\beta}dv \\ &= (s_1-r_1)^{\alpha-1}r_1^{\alpha-2\beta}\mathcal{B}(1-2\beta,\alpha).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{s_1} r_1^{\beta}(s_1-r_1)^{\alpha-1}r_1^{\alpha-2\beta}dr_1 &= \int_0^{s_1} r_1^{\alpha-\beta}(s_1-r_1)^{\alpha-1}dr_1 \\ &= \int_0^1 u^{\alpha-\beta}s_1^{\alpha-\beta}s_1^{\alpha-1}(1-u)^{\alpha-1}s_1 du \\ &= s_1^{2\alpha-\beta}\mathcal{B}(\alpha-\beta+1,\alpha).\end{aligned}$$

Consequently,

$$\int_0^{s_1} \mid \phi(s_1,r_1) \mid dr_1 \leqslant s_1^{2\alpha}\mathcal{B}(\alpha-\beta+1,\alpha). \tag{6.48}$$

Thus,

$$\Big[\int_0^T\Big(\int_0^T \mid \phi(s_1,r_1) \mid dr_1\Big)ds_1\Big]^2 \leqslant 2\mathcal{B}(\alpha-\beta+1,\alpha)\int_0^T s_1^{2\alpha}ds_1 < \infty.$$

c_{22}) Now, we prove that

$$\mathbb{E}\Big[\int_0^T\Big(1+\mid N_{r_2}\mid +...+\mid N_{r_2}\mid^{k'}\Big)\Big(\int_0^T \mid \phi(s_2,r_2)\mid\Big(1+\mid N_{s_2}\mid +...+\mid N_{s_2}\mid^{k}\Big)ds_2\Big)dr_2\Big]^2 < \infty.$$

Indeed,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T\Big(1+\mid N_{r_2}\mid +...+\mid N_{r_2}\mid^{k'}\Big)\Big(\int_0^T \mid \phi(s_2,r_2)\mid\Big(1+\mid N_{s_2}\mid +...+\mid N_{s_2}\mid^{k}\Big)ds_2\Big)dr_2\Big]^2 \\ &\leqslant \int_0^T \mathbb{E}\Big(1+\mid N_{r_2}\mid +...+\mid N_{r_2}\mid^{k'}\Big)^2 dr_2 \int_0^T \mathbb{E}\Big(\int_0^T \mid \phi(s_2,r_2)\mid\Big(1+\mid N_{s_2}\mid +...+\mid N_{s_2}\mid^{k}\Big)ds_2\Big)^2 dr_2.\end{aligned}$$

Applying Hölder's inequality to $p < \frac{1}{1-\alpha} < 2$, and q its conjugate ($\frac{1}{p}+\frac{1}{q}=1$),

$$\mathbb{E}\Big(\int_0^T \mid \phi(s_2,r_2)\mid\Big(1+\mid N_{s_2}\mid +...+\mid N_{s_2}\mid^{k}\Big)ds_2\Big)^2$$

$$\leqslant \mathbb{E}\Big[\Big(\int_0^T \mid \phi(s_2,r_2)\mid^p ds_2\Big)^{\frac{2}{p}}\Big(\int_0^T \Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q ds_2\Big)^{\frac{2}{q}}\Big]$$
$$= \Big(\int_0^T \mid \phi(s_2,r_2)\mid^p ds_2\Big)^{\frac{2}{p}}\mathbb{E}\Big[\Big(\int_0^T \Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q ds_2\Big)^{\frac{2}{q}}\Big].$$

i) We prove that

$$\mathbb{E}\Big[\Big(\int_0^T \Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q ds_2\Big)^{\frac{2}{q}}\Big] < \infty.$$

In fact, we have $\frac{2}{q} < 1$, then Jensen's inequality can be applied for a concave function to get

$$\mathbb{E}\Big[\Big(\int_0^T \Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q ds_2\Big)^{\frac{2}{q}}\Big] \leqslant \Big[\mathbb{E}\int_0^T \Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q ds_2\Big]^{\frac{2}{q}}$$
$$< \infty, \quad \text{(using (6.17))}.$$

ii) We prove that

$$\Big(\int_0^T \mid \phi(s_2,r_2)\mid^p ds_2\Big)^{\frac{2}{p}} < \infty.$$

In fact,

$$\int_0^T \mid \phi(s_2,r_2)\mid^p ds_2 \leqslant \int_0^T \Big(\int_0^{inf(s_2,r_2)} \Big|\frac{\partial K}{\partial s_2}(s_2,u)\Big|\Big|\frac{\partial K}{\partial r_2}(r_2,u)\Big|du\Big)^p ds_2$$
$$\leqslant \int_0^T \Big(\int_0^{inf(s_2,r_2)} (s_2-u)^{\alpha-1}\Big(\frac{s_2}{u}\Big)^\beta (r_2-u)^{\alpha-1}\Big(\frac{r_2}{u}\Big)^\beta du\Big)^p ds_2$$
$$= \int_0^T \Big(\int_0^{inf(s_2,r_2)} (s_2-u)^{\alpha-1} s_2^\beta r_2 \beta (r_2-u)^{\alpha-1} u^{-2\beta} du\Big)^p ds_2$$
$$= r_2^{p\beta}\int_0^T s_2^{p\beta}\Big(\int_0^{inf(s_2,r_2)} (s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$$
$$= r_2^{p\beta}\int_0^{r_2} s_2^{p\beta}\Big(\int_0^{s_2} (s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$$
$$+ r_2^{p\beta}\int_{r_2}^T s_2^{p\beta}\Big(\int_0^{r_2} (s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$$

Now, we will to study this integral $r_2^{p\beta}\int_0^{r_2} s_2^{p\beta}\Big(\int_0^{s_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$:
we begin to study this integral $\int_0^{s_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du$.
We have

$$\int_0^{s_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du \leqslant (r_2-s_2)^{\alpha-1}\int_0^{s_2}(s_2-u)^{\alpha-1}u^{-2\beta}du$$
$$= (r_2-s_2)^{\alpha-1}s_2^{\alpha-2\beta}\mathcal{B}(1-2\beta,\alpha).$$

Therefore,

$$r_2^{p\beta}\int_0^{r_2} s_2^{p\beta}\Big(\int_0^{s_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$$

$$
\begin{aligned}
&\leqslant r_2^{p\beta}\mathcal{B}^p(1-2\beta,\alpha)\int_0^{r_2} s_2^{p\beta}(r_2-s_2)^{p(\alpha-1)}s_2^{p(\alpha-2\beta)}ds_2\\
&= r_2^{p\beta}\mathcal{B}^p(1-2\beta,\alpha)\int_0^{r_2} s_2^{p(\alpha-\beta)}(r_2-s_2)^{p(\alpha-1)}ds_2\\
&= r_2^{p\beta}\mathcal{B}^p(1-2\beta,\alpha)\int_0^{1} v^{p(\alpha-\beta)}r_2^{p(\alpha-\beta)}r_2^{p(\alpha-1)}(1-v)^{p(\alpha-1)}r_2dv\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(2\alpha-1)+1}\int_0^{1} v^{p(\alpha-\beta)}(1-v)^{p(\alpha-1)}dv\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(2\alpha-1)+1}\mathcal{B}(p(\alpha-\beta)+1,p(\alpha-1)+1). \qquad (6.49)
\end{aligned}
$$

Now, we will to study this integral $r_2^{p\beta}\int_{r_2}^T s_2^{p\beta}\Big(\int_0^{r_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2$:
we begin to study this integral $\int_0^{r_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du$.
We have

$$
\begin{aligned}
\int_0^{r_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du &\leqslant (s_2-r_2)^{\alpha-1}\int_0^{r_2}(r_2-u)^{\alpha-1}u^{-2\beta}du\\
&= (s_2-r_2)^{\alpha-1}\int_0^{1} r_2^{\alpha-1}(1-v)^{\alpha-1}v^{-2\beta}r_2^{-2\beta}r_2dv\\
&= (s_2-r_2)^{\alpha-1}r_2^{\alpha-2\beta}\mathcal{B}(1-2\beta,\alpha).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&r_2^{p\beta}\int_{r_2}^T s_2^{p\beta}\Big(\int_0^{r_2}(s_2-u)^{\alpha-1}(r_2-u)^{\alpha-1}u^{-2\beta}du\Big)^p ds_2\\
&\leqslant r_2^{p\beta}\mathcal{B}^p(1-2\beta,\alpha)\int_{r_2}^T s_2^{p\beta}(s_2-r_2)^{p(\alpha-1)}r_2^{p(\alpha-2\beta)}ds_2\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(\alpha-\beta)}\int_{r_2}^T s_2^{p\beta}(s_2-r_2)^{p(\alpha-1)}ds_2\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(\alpha-\beta)}T^{p\beta}\int_{r_2}^T (s_2-r_2)^{p(\alpha-1)}ds_2\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(\alpha-\beta)}T^{p\beta}\frac{(T-r_2)^{p(\alpha-1)+1}}{p(\alpha-1)+1}, \quad (p(\alpha-1)>-1)\\
&\leqslant \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(\alpha-\beta)}T^{p\beta}\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\\
&= \mathcal{B}^p(1-2\beta,\alpha)r_2^{p(\alpha-\beta)}\frac{T^{p(\alpha+\beta-1)+1}}{p(\alpha-1)+1}. \qquad (6.50)
\end{aligned}
$$

Using (6.49) and (6.50), we obtain

$$
\begin{aligned}
\int_0^T \mid\phi(s_2,r_2)\mid^p ds_2 &\leqslant \mathcal{B}^p(1-2\beta,\alpha)\mathcal{B}(p(\alpha-\beta)+1,p(\alpha-1)+1)r_2^{p(2\alpha-1)+1}\\
&\quad+\mathcal{B}^p(1-2\beta,\alpha)\frac{T^{p(\alpha+\beta-1)+1}}{p(\alpha-1)+1}r_2^{p(\alpha-\beta)}.
\end{aligned}
$$

Therefore

$$\Big(\int_0^T \mid \phi(s_2,r_2)\mid^p ds_2\Big)^{\frac{2}{p}} \leqslant C_p\Big[\mathcal{B}^2(1-2\beta,\alpha)\mathcal{B}^{\frac{2}{p}}(p(\alpha-\beta)+1,p(\alpha-1)+1)r_2^{2(2\alpha-1)+\frac{2}{p}} + \mathcal{B}^2(1-2\beta,\alpha)\frac{T^{2(\alpha+\beta-1)+\frac{2}{p}}}{[p(\alpha-1)+1]^{\frac{2}{p}}}r_2^{2(\alpha-\beta)}\Big], \quad \Big(\frac{2}{p}>1\Big).$$

Moreover, $\int_0^T r_2^{2(2\alpha-1)+\frac{2}{p}}dr_2 < \infty$, ($2(2\alpha-1)+\frac{2}{p} > \frac{2}{p}-2 > -1$) and $\int_0^T r_2^{2(\alpha-\beta)}dr_2 < \infty$, $(2(\alpha-\beta) > -1)$.

A similar calculation as (6.17), we obtain $\int_0^T \mathbb{E}\Big(1+\mid N_{r_2}\mid+...+\mid N_{r_2}\mid^{k'}\Big)^2 dr_2 < \infty$.

d) We show now that $\int_0^t u_s \mathbb{D}_s^X M_s ds \in L^2(\Omega)$. For this we show that both terms in (6.44) are in $L^4(\Omega\times[0,T])$.

Now, we prove $\int_0^s u_r\phi(s,r)dr \in L^4(\Omega\times[0,T])$. In fact,

$$\begin{aligned}
\mathbb{E}\Big[\int_0^s u_r\phi(s,r)dr\Big]^4 &= \mathbb{E}\Big[\int_0^s u_r\Big(\int_0^{\inf(s,r)}\frac{\partial}{\partial s}K(s,u)\frac{\partial}{\partial r}K(r,u)du\Big)dr\Big]^4\\
&\leqslant \mathbb{E}\Big[\int_0^s \mid u_r\mid\Big(\int_0^{inf(s,r)}(s-u)^{\alpha-1}\Big(\frac{s}{u}\Big)^{\beta}(r-u)^{\alpha-1}\Big(\frac{r}{u}\Big)^{\beta}du\Big)dr\Big]^4\\
&= \mathbb{E}\Big[\int_0^s \mid u_r\mid s^{\beta}r^{\beta}\Big(\int_0^r(s-u)^{\alpha-1}(r-u)^{\alpha-1}u^{-2\beta}du\Big)dr\Big]^4\\
&\leqslant \mathbb{E}\Big[\int_0^s \mid u_r\mid s^{\beta}r^{\beta}(s-r)^{\alpha-1}r^{\alpha-2\beta}\mathcal{B}(1-2\beta,\alpha)dr\Big]^4\\
&= s^{4\beta}\mathcal{B}(1-2\beta,\alpha)^4\mathbb{E}\Big[\int_0^s \mid u_r\mid r^{\alpha-\beta}(s-r)^{\alpha-1}dr\Big]^4\\
&\leqslant s^{4\beta}\mathcal{B}(1-2\beta,\alpha)^4\mathbb{E}\Big[\int_0^s \mid u(r,N_r)\mid^q dr\Big]^{\frac{4}{q}}\Big(\int_0^s r^{p(\alpha-\beta)}(s-r)^{(\alpha-1)p}dr\Big)^{\frac{4}{p}}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\int_0^s r^{p(\alpha-\beta)}(s-r)^{(\alpha-1)p}dr &= \int_0^1 \theta^{p(\alpha-\beta)}s^{p(\alpha-\beta)}s^{(\alpha-1)p}(1-\theta)^{(\alpha-1)p}sd\theta\\
&= s^{p(\alpha-\beta)+(\alpha-1)p+1}\int_0^1\theta^{p(\alpha-\beta)}(1-\theta)^{(\alpha-1)p}d\theta\\
&= s^{p(2\alpha-\beta-1)+1}\mathcal{B}((\alpha-\beta)p+1,(\alpha-1)p+1) < \infty.
\end{aligned}$$

Using (6.47) and $s^{4\beta}s^{[p(2\alpha-\beta-1)+1]\frac{4}{p}}$ is integrable on $[0,T]$ (because $8\alpha-4+\frac{4}{p} > 4\alpha > 0$), we get $\int_0^s u_r\phi(s,r)dr \in L^4(\Omega\times[0,T])$.

e) Now, we want to prove that $\int_0^T \mathbb{D}_s^X u_r 1_{]0,s]}(r)\delta X_r \in L^4(\Omega\times[0,T])$.

From (6.46), we have

$$\mathbb{E}\Big[\int_0^T \mathbb{D}_s^X u_r 1_{[0,s]}(r)\delta X_r\Big]^4 \leqslant c_4\Big[\mathbb{E}\parallel \mathbb{D}_s^X u 1_{[0,s]}\parallel_{\mathcal{H}}^4 + \mathbb{E}\parallel D^X(\mathbb{D}_s^X u 1_{[0,s]})\parallel_{\mathcal{H}^{\otimes 2}}^4\Big]. \tag{6.51}$$

e_1) For the first term in (6.51), we get

$$\| \mathbb{D}_s^X u 1_{[0,s]} \|_{\mathcal{H}}^4 = \Big[\int_0^T \Big(\int_t^T \mathbb{D}_s^X u(r, N_r) 1_{[0,s]}(r) \frac{\partial}{\partial r} K(r,t) dr \Big)^2 dt \Big]^2$$
$$\leqslant \Big[\int_0^T \Big(\int_t^s \mid \mathbb{D}_s^X u(r, N_r) \mid \Big| \frac{\partial}{\partial r} K(r,t) \Big| dr \Big)^2 dt \Big]^2.$$

Moreover, as in (6.16) we get

$$\Big(\int_t^s \mid \mathbb{D}_s^X u(r, N_r) \mid \Big| \frac{\partial}{\partial r} K(r,t) \Big| dr \Big)^2 \leqslant c t^{-2\beta} (T-t)^{2(\alpha-1)+\frac{2}{p}} T^{2\beta} \Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{2}{q}}.$$

Then,

$$\| \mathbb{D}_s^X u 1_{[0,s]} \|_{\mathcal{H}}^4 \leqslant c' T^{4\beta} \Big(\int_0^T t^{-2\beta} (T-t)^{2(\alpha-1)+\frac{2}{p}} dt \Big)^2 \Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{4}{q}}$$
$$= c' \Big(T^{-2\beta+2\alpha+\frac{2}{p}-1} \mathcal{B}(1-2\beta, 2(\alpha-1)+\frac{2}{p}+1) \Big)^2) \Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{4}{q}}$$
$$= c' T^{4\alpha+\frac{4}{p}-2} \mathcal{B}(1-2\beta, 2(\alpha-1)+\frac{2}{p}+1)^2 \Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{4}{q}}.$$

Now, we want to prove $\mathbb{E}\Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{4}{q}} < \infty$. In fact,

$$\mathbb{E}\Big(\int_0^s \mid \mathbb{D}_s^X u(r, N_r) \mid^q dr \Big)^{\frac{4}{q}} \leqslant c\mathbb{E}\Big[\int_0^s \Big(\int_0^r \mid \phi(s,v) \mid \Big| \frac{\partial}{\partial x} u(r, N_r) \Big| dv \Big)^q dr \Big]^{\frac{4}{q}}$$
$$= c\mathbb{E}\Big[\int_0^s \Big| \frac{\partial}{\partial x} u(r, N_r) \Big|^q \Big(\int_0^r \mid \phi(s,v) \mid dv \Big)^q dr \Big]^{\frac{4}{q}}.$$

Moreover, we have

$$\int_0^r \mid \phi(s,v) \mid dv \leqslant \int_0^r \Big(\int_0^{inf(s,v)} \Big| \frac{\partial}{\partial s} K(s,u) \Big| \Big| \frac{\partial}{\partial v} K(v,u) \Big| du \Big) dv$$
$$\leqslant c' \int_0^r \Big(\int_0^{inf(s,v)} (s-u)^{\alpha-1} \Big(\frac{s}{u} \Big)^\beta (v-u)^{\alpha-1} \Big(\frac{v}{u} \Big)^\beta du \Big) dv$$
$$= c' s^\beta \int_0^r v^\beta \Big(\int_0^{inf(s,v)} (s-u)^{\alpha-1} (v-u)^{\alpha-1} u^{-2\beta} du \Big) dv.$$

Now, we will to study this integral $\int_0^{inf(s,v)} (s-u)^{\alpha-1} (v-u)^{\alpha-1} u^{-2\beta} du$.
We have $s > r$, ($s > v$ for all $v \in (0,r)$)

$$\int_0^{inf(s,v)} (s-u)^{\alpha-1} (v-u)^{\alpha-1} u^{-2\beta} du \leqslant (s-v)^{\alpha-1} \int_0^v (v-u)^{\alpha-1} u^{-2\beta} du$$
$$\leqslant (s-v)^{\alpha-1} v^{\alpha-2\beta} \mathcal{B}(1-2\beta, \alpha).$$

Consequently,

$$\int_0^r v^{\alpha-\beta} (s-v)^{\alpha-1} dv \leqslant \int_0^s v^{\alpha-\beta} (s-v)^{\alpha-1} dv = \int_0^1 \theta^{\alpha-\beta} s^{\alpha-\beta} s^{\alpha-1} (1-\theta)^{\alpha-1} s d\theta$$

$$= s^{2\alpha-\beta}\mathcal{B}(\alpha-\beta+1,\alpha)$$

Then, we have

$$\Big(\int_0^r \phi(s,v)dv\Big)^q \leqslant c_2 s^{2q\alpha}.$$

Therefore

$$\begin{aligned}\mathbb{E}\Big[\int_0^s \mid \mathbb{D}_s^X u(r,N_r)\mid^q dr\Big]^{\frac{4}{q}} &\leqslant c_2\mathbb{E}\Big[\int_0^s \Big|\frac{\partial}{\partial x}u(r,N_r)\Big|^q s^{2q\alpha}dr\Big]^{\frac{4}{q}}\\ &= c_2 s^{8\alpha}\mathbb{E}\Big[\int_0^s \Big|\frac{\partial}{\partial x}u(r,N_r)\Big|^q dr\Big]^{\frac{4}{q}}.\end{aligned}$$

A similar calculation as for (6.47) allows us to conclude

$$\mathbb{E}\Big[\int_0^s \Big|\frac{\partial}{\partial x}u(r,N_r)\Big|^q dr\Big]^{\frac{4}{q}} < \infty.$$

e_2) For the second term in (6.51), we get

$$\begin{aligned}&\mathbb{E}\parallel D^X(\mathbb{D}_s^X u 1_{[0,s]})\parallel^4_{\mathcal{H}^{\otimes 2}}\\ &= \mathbb{E}\Big(\int_{[0,T]^4}\phi(s_1,r_1)\phi(s_2,r_2)D^X_{s_1}(\mathbb{D}_s^X u_{s_2}1_{[0,s]}(s_2))D^X_{r_1}(\mathbb{D}_s^X u_{r_2}1_{[0,s]}(r_2))ds_1ds_2dr_1dr_2\Big)^2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}D^X_{s_1}(\mathbb{D}_s^X u_{s_2}1_{[0,s]}(s_2)) &= D^X_{s_1}\Big(\int_0^T \phi(s,v)D_v^X u(s_2,N_{s_2})1_{[0,s]}(s_2)dv\Big)\\ &= D^X_{s_1}\Big(\int_0^{s_2} \phi(s,v)D_v^X u(s_2,N_{s_2})dv\Big)1_{[0,s]}(s_2)\\ &= \int_0^{s_2}\phi(s,v)D^X_{s_1}D_v^X u(s_2,N_{s_2})dv1_{[0,s]}(s_2)\\ &= \int_0^{s_2}\phi(s,v)\sigma_{s_1}\sigma_v\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})dv1_{[0,s]}(s_2).\end{aligned}$$

A similar calculation, we get

$$D^X_{r_1}(\mathbb{D}_s^X u_{r_2}1_{[0,s]}(r_2)) = \int_0^{r_2}\phi(s,v')\sigma_{r_1}\sigma_{v'}\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})dv'1_{[0,s]}(r_2).$$

Therefore,

$$\begin{aligned}&\mathbb{E}\parallel D^X(\mathbb{D}_s^X u 1_{[0,s]})\parallel^4_{\mathcal{H}^{\otimes 2}}\\ &\leqslant M'\mathbb{E}\Big[\int_{[0,T]^4}\mid\phi(s_1,r_1)\mid\mid\phi(s_2,r_2)\mid 1_{[0,s]}(s_2)1_{[0,s]}(r_2)\Big(\int_0^{s_2}\mid\phi(s,v)\mid\Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|dv\Big)\\ &\times\Big(\int_0^{r_2}\mid\phi(s,v')\mid\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|dv'\Big)ds_1ds_2dr_1dr_2\Big]^2\end{aligned}$$

$$\leqslant M'\Big[\Big(\int_{[0,T]^2} \mid \phi(s_1,r_1) \mid ds_1dr_1\Big)^2\mathbb{E}\Big(\int_{[0,s]^2} \mid \phi(s_2,r_2) \mid \Big(\int_0^s \mid \phi(s,v) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|dv\Big)$$
$$\times\Big(\int_0^s \mid \phi(s,v') \mid \Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|dv'\Big)ds_2dr_2\Big)^2\Big]$$
$$\leqslant M'\Big[\Big(\int_{[0,T]^2} \mid \phi(s_1,r_1) \mid ds_1dr_1\Big)^2\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big(\int_0^s \mid \phi(s,v) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|dv\Big)$$
$$\times\Big(\int_0^s \mid \phi(s,v') \mid \Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|dv'\Big)ds_2dr_2\Big)^2\Big].$$

On the one hand, we have $\int_{[0,T]^2} \mid \phi(s_1,r_1) \mid ds_1dr_1 < \infty$ (the finiteness of this integrals follows from (H1), (H2)).
On the other hand, we prove that

$$\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big(\int_0^s \mid \phi(s,v) \mid \frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})dv\Big)\Big(\int_0^s \phi(s,v')\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})dv'\Big)ds_2dr_2\Big)^2\Big] < \infty.$$

Indeed, we have

$$\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big(\int_0^s \mid \phi(s,v) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|dv\Big)\Big(\int_0^s \mid \phi(s,v') \mid \Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|dv'\Big)ds_2dr_2\Big)^2\Big]$$
$$= \mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|\Big(\int_0^s \mid \phi(s,v) \mid dv\Big)\Big(\int_0^s \mid \phi(s,v') \mid dv'\Big)ds_2dr_2\Big)^2$$
$$\leqslant s^{8\alpha}\mathcal{B}^4(\alpha-\beta+1,\alpha)\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|ds_2dr_2\Big)^2, \quad \text{(Using (6.48)).}$$

Now, we prove that

$$\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|ds_2dr_2\Big)^2 < \infty.$$

In fact, we have

$$\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|ds_2dr_2\Big)^2$$
$$\leqslant C\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big(1+ \mid N_{s_2} \mid +...+ \mid N_{s_2} \mid^k\Big)\Big(1+ \mid N_{r_2} \mid +...+ \mid N_{r_2} \mid^k\Big)ds_2dr_2\Big)^2.$$

Applying Hölder's inequality to $p < \frac{1}{1-\alpha} < 2$, and q its conjugate ($\frac{1}{p}+\frac{1}{q}=1$), we get

$$\mathbb{E}\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid \Big|\frac{\partial^2 u}{\partial x^2}(s_2,N_{s_2})\Big|\Big|\frac{\partial^2 u}{\partial x^2}(r_2,N_{r_2})\Big|ds_2dr_2\Big)^2$$
$$\leqslant C\Big(\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid^p ds_2dr_2\Big)^{\frac{2}{p}}$$
$$\times\,\mathbb{E}\Big(\int_{[0,T]^2}\Big(1+ \mid N_{s_2} \mid +...+ \mid N_{s_2} \mid^k\Big)^q\Big(1+ \mid N_{r_2} \mid +...+ \mid N_{r_2} \mid^k\Big)^q ds_2dr_2\Big)^{\frac{2}{q}}.$$

Using (6.49), we get $\int_{[0,T]^2} \mid \phi(s_2,r_2) \mid^p ds_2dr_2 \leqslant C\int_0^T r_2^{p(2\alpha-1)+1}dr_2 + C'\int_0^T r_2^{p(\alpha-\beta)}dr_2 < \infty$, $p(2\alpha-1)+1 > -1$ and $p(\alpha-\beta) > -1$.
Moreover, we have $\frac{2}{q} < 1$ (because $\frac{1}{q} < 1+(\alpha-1) < \frac{1}{2}$) , then Jensen's inequality can be applied for a concave function to get

$$\begin{aligned}
&\mathbb{E}\Big(\int_{[0,T]^2}\Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q\Big(1+\mid N_{r_2}\mid+...+\mid N_{r_2}\mid^k\Big)^q ds_2dr_2\Big)^{\frac{2}{q}}\\
&\leqslant\Big(\int_{[0,T]^2}\mathbb{E}\Big[\Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^q\Big(1+\mid N_{r_2}\mid+...+\mid N_{r_2}\mid^k\Big)^q\Big]ds_2dr_2\Big)^{\frac{2}{q}}\\
&\leqslant\Big(\frac{1}{2}\Big)^{\frac{2}{q}}\Big(\int_{[0,T]^2}\mathbb{E}\Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^{2q}+\mathbb{E}\Big(1+\mid N_{r_2}\mid+...+\mid N_{r_2}\mid^k\Big)^{2q}ds_2dr_2\Big)^{\frac{2}{q}}\\
&\leqslant c\Big[\Big(\int_0^T\mathbb{E}\Big(1+\mid N_{s_2}\mid+...+\mid N_{s_2}\mid^k\Big)^{2q}ds_2\Big)+\Big(\int_0^T\mathbb{E}\Big(1+\mid N_{r_2}\mid+...+\mid N_{r_2}\mid^k\Big)^{2q}dr_2\Big)\Big]^{\frac{2}{q}} < \infty.
\end{aligned}$$

Since $\int_0^T s^{8\alpha}ds < \infty$. Therefore $\int_0^s \mathbb{D}_s^X u_r\delta X_r \in L^4(\Omega\times[0,T])$.
f) We have now shown that $\int_0^t u_s\mathbb{D}_s^X M_s ds$, $M_t^2 \in L^2(\Omega,\mathcal{F},\mathbb{P})$. From (6.45), we obtain

$$\int_0^T u_s M_s 1_{[0,t]}(s)\delta X_s \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{H}).$$

From (3.6) and (6.45), we have $uM \in Dom(\delta^X)$ and we get

$$M_t^2 = 2\int_0^t M_s u_s\delta X_s + 2\int_0^t u_s\mathbb{D}_s^X M_s ds, t\in[0,T]. \blacksquare$$

Proof of Theorem 6.5.1:
Let $M_t := \int_0^t u_s\delta X_s$ and $B_t := C_0 + \int_0^t f_s ds, t\in[0,T]$.
From the previous Lemma 6.5.2, we have $uM1_{[0,t]} \in Dom(\delta^X)$, for all $t\in[0,T]$, and

$$M_t^2 = 2\int_0^t u_s M_s\delta X_s + 2\int_0^t u_s\mathbb{D}_s^X M_s ds, t\in[0,T].$$

Furthermore, we have

$$B_t^2 = C_0^2 + 2\int_0^t f_s B_s ds, t\in(0,T].$$

Moreover, $uB1_{[0,t]} \in Dom(\delta^X)$, for all $t\in[0,T]$, and

$$M_tB_t = \int_0^t u_s B_s\delta X_s + \int_0^t f_s M_s ds + \int_0^t u_s\mathbb{D}_s^X B_s ds, t\in[0,T].$$

In fact, on the one hand, since $B_tF \in \mathbb{D}_{1,2}^X$ and $F\in\mathcal{S}$, for all $s\in[0,t]$ we have

$$\mathbb{D}_s^X(B_tF) = \mathbb{D}_s^X(B_sF) + \mathbb{D}_s^X((B_t-B_s)F)$$

$$
\begin{aligned}
&= \mathbb{D}_s^X(B_sF) + \int_0^T \phi(s,v) D_v^X((B_t - B_s)F)dv \\
&= \mathbb{D}_s^X(B_sF) + \int_0^T \phi(s,v) D_v^X\Big(\int_s^t f_r dr F\Big)dv \\
&= \mathbb{D}_s^X(B_sF) + \int_0^T \phi(s,v)\Big[\int_s^t D_v^X f_r dr F + \int_s^t f_r dr D_v^X F\Big]dv \\
&= \mathbb{D}_s^X(B_sF) + \int_0^T \phi(s,v)\Big[\int_s^t dr[D_v^X f_r F + f_r D_v^X F]dv \\
&= \mathbb{D}_s^X(B_sF) + \int_0^T \phi(s,v)\int_s^t dr D_v^X(f_r F)dv \\
&= \mathbb{D}_s^X(B_sF) + \int_s^t dr \int_0^T dv\phi(s,v) D_v^X(f_r F) \\
&= \mathbb{D}_s^X(B_sF) + \int_s^t dr \mathbb{D}_s^X(f_r F). \qquad (6.52)
\end{aligned}
$$

On the other hand, using (6.40) and (6.52), we get

$$
\begin{aligned}
\mathbb{E}\Big[M_t B_t F\Big] &= \mathbb{E}\Big[\Big(\int_0^T u_s 1_{[0,t]}(s)\delta X_s\Big) B_t F\Big] \\
&= \mathbb{E}\Big[\int_0^t u_s \mathbb{D}_s^X(B_t F)ds\Big] \\
&= \mathbb{E}\Big[\int_0^t u_s\Big(\mathbb{D}_s^X(B_sF) + \int_s^t \mathbb{D}_s^X(f_r F)dr\Big)ds\Big] \quad \text{(Using (6.52))} \\
&= \mathbb{E}\Big[\int_0^s u_s \mathbb{D}_s^X(B_sF)ds + \int_0^t u_s\Big(\int_s^t \mathbb{D}_s^X(f_r F)dr\Big)ds\Big] \\
&= \mathbb{E}\Big[\int_0^t u_s[(\mathbb{D}_s^X B_s)F + \mathbb{D}_s^X F B_s]ds\Big] + \mathbb{E}\Big[\int_0^t \int_s^t u_s \mathbb{D}_s^X(f_r F)drds\Big] \\
&= \mathbb{E}\Big[\Big(\int_0^t u_s \mathbb{D}_s^X B_s ds\Big)F\Big] + \mathbb{E}\Big[\int_0^t u_s \mathbb{D}_s^X F B_s ds\Big] + \int_0^t \mathbb{E}\Big[\int_0^r u_s \mathbb{D}_s^X(f_r F)ds\Big]dr \\
&= \mathbb{E}\Big[\Big(\int_0^t u_s \mathbb{D}_s^X B_s ds\Big)F\Big] + \mathbb{E}\Big[\int_0^t u_s \mathbb{D}_s^X F B_s ds\Big] + \mathbb{E}\Big[\int_0^t M_r f_r F dr\Big].
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s) B_s \mathbb{D}_s^X F ds\Big] = \mathbb{E}[M_t B_t F] - \mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X B_s ds F\Big] - \mathbb{E}\Big[\int_0^T M_r f_r 1_{[0,t]}(r)dr F\Big].
$$

Consequently,

$$
\mathbb{E}\Big[\int_0^T u_s 1_{[0,t]}(s) B_s \delta X_s F\Big] = \mathbb{E}\Big[\Big(M_t B_t - \int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X B_s ds - \int_0^T M_r f_r 1_{[0,t]}(r)dr\Big)F\Big], F \in \mathcal{S}. \qquad (6.53)
$$

Since $uM1_{[0,t]} \in Dom(\delta^X)$, and as in the proof of Lemma 6.5.2, we can see that $M_t B_t - \int_0^T u_s 1_{[0,t]}(s)\mathbb{D}_s^X B_s ds - \int_0^T M_r f_r 1_{[0,t]}(r) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then $uB1_{[0,t]} \in Dom(\delta^X)$.

$$M_t B_t = \int_0^T u_s 1_{[0,t]}(s) B_s \delta X_s + \int_0^T u_s 1_{[0,t]}(s) \mathbb{D}_s^X B_s ds + \int_0^T M_r f_r 1_{[0,t]}(r) dr.$$

We pose $C_t = M_t + B_t$ and using the linearity of $Dom(\delta^X)$, then $uC1_{[0,t]} \in Dom(\delta^X)$ and from $C_t^2 = M_t^2 + B_t^2 + 2M_t B_t$, we get from Lemma 6.5.2

$$\begin{aligned} C_t^2 &= 2\int_0^t u_s M_s \delta X_s + 2\int_0^t u_s \mathbb{D}_s^X M_s ds + C_0^2 + 2\int_0^t f_s B_s ds + 2\int_0^t u_s B_s \delta X_s + 2\int_0^t f_s M_s ds \\ &\quad + 2\int_0^t u_s \mathbb{D}_s^X B_s ds \\ &= C_0^2 + 2\int_0^t u_s C_s \delta X_s + 2\int_0^t u_s \mathbb{D}_s^X C_s ds + 2\int_0^t f_s C_s ds. \blacksquare \end{aligned}$$

Chapter 7

Existence and uniqueness of solutions for a class of non linear BSDEs

In this Chapter, we present new results related to the existence and uniqueness of the solution of the BSDE (6.1). These results generalize those given by Maticiuc and Nie [25].
We suppose that $Var(N_t)$ is increasing on $[t_0, T]$.

7.1 Existence

We begin by considering the following equation:

$$Y_t = \xi + \int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds + \int_t^T Z_s \delta X_s,\ 0 \leqslant t_0 \leqslant t \leqslant T, \tag{7.1}$$

where $\chi, \psi \in \mathcal{V}_T$.

First, we show the existence and uniqueness of the solution of (7.1).

Proposition 7.1.1. *Under the assumptions (H1), (H2), (H6), (H7), (H8), (7.1) has a unique solution $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$. If K_T^* is injective this solution is given by $(Y_t, Z_t) = (u(t, N_t), -\sigma_t \frac{\partial}{\partial x} u(t, N_t))$, where $u \in \mathcal{V}_T$.*

Proof. We have

$$g(N_T) + \int_0^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds \in L^2(\Omega, \mathcal{F}, P).$$

Let

$$\begin{aligned} M_t &= \widehat{\mathbb{E}}\Big[g(N_T) + \int_0^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds \mid \mathcal{F}_t\Big] \\ &= \widehat{\mathbb{E}}\Big[g(N_T) \mid \mathcal{F}_t\Big] + \widehat{\mathbb{E}}\Big[\int_0^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds \mid \mathcal{F}_t\Big] \end{aligned}$$

$$
\begin{aligned}
&= P_{Var(N_T)-Var(N_t)}g(N_t) + \int_t^T P_{Var(N_u)-Var(N_t)}f(u,N_t,\chi(u,N_t),\psi(u,N_t))du \\
&+ \int_0^t f(u,N_u,\chi(u,N_u),\psi(u,N_u))du \qquad (7.2)
\end{aligned}
$$

by Lemma 6.4.9 and Lemma 6.4.7. Using proposition 2.3.7 ([3]), we obtain for $s < t$

$$
\begin{aligned}
D_s^X M_t &= \sigma_s P_{Var(N_T)-Var(N_t)}g'(N_t) + \sigma_s \int_t^T P_{Var(N_u)-Var(N_t)}\tilde{f}(u,N_t)du \\
&+ \sigma_s \int_s^t \tilde{f}(u,N_u)du. \qquad (7.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{f}(u,x) &= \frac{\partial}{\partial x}f(u,x,\chi(u,x),\psi(u,x)) + \frac{\partial}{\partial y}f(u,x,\chi(u,x),\psi(u,x))\chi_x(u,x) \\
&+ \frac{\partial}{\partial z}f(u,x,\chi(u,x),\psi(u,x))\psi_x(u,x). \qquad (7.4)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\widehat{\mathbb{E}}\Big[D_s^X M_t \mid \mathcal{F}_s\Big] &= \sigma_s \widehat{\mathbb{E}}\Big[P_{Var(N_T)-Var(N_t)}g'(N_t) \mid \mathcal{F}_s\Big] + \sigma_s \int_t^T \widehat{\mathbb{E}}\Big[P_{Var(N_u)-Var(N_t)}\tilde{f}(u,N_t) \mid \mathcal{F}_s\Big]du \\
&+ \sigma_s \int_s^t \widehat{\mathbb{E}}\Big[\tilde{f}(u,N_u) \mid \mathcal{F}_s\Big]du \quad \text{(By Lemma 6.4.9)} \\
&= \sigma_s P_{Var(N_T)-Var(N_t)}\widehat{\mathbb{E}}\Big[g'(N_t) \mid \mathcal{F}_s\Big] + \sigma_s \int_t^T P_{Var(N_u)-Var(N_t)}\widehat{\mathbb{E}}\Big[\tilde{f}(u,N_t) \mid \mathcal{F}_s\Big]du \\
&+ \sigma_s \int_s^t P_{Var(N_u)-Var(N_s)}\tilde{f}(u,N_s)du \\
&= \sigma_s P_{Var(N_T)-Var(N_t)}P_{Var(N_t)-Var(N_s)}g'(N_s) + \sigma_s \int_s^t P_{Var(N_u)-Var(N_s)}\tilde{f}(u,N_s)du \\
&+ \sigma_s \int_t^T P_{Var(N_u)-Var(N_t)}P_{Var(N_t)-Var(N_s)}\tilde{f}(u,N_s) \quad \text{(By Lemma 6.4.7)} \\
&= \sigma_s P_{Var(N_T)-Var(N_s)}g'(N_s) + \sigma_s \int_t^T P_{Var(N_u)-Var(N_s)}\tilde{f}(u,N_s)du \\
&+ \sigma_s \int_s^t P_{Var(N_u)-Var(N_s)}\tilde{f}(u,N_s)du \quad \text{(Using semi-group property)} \\
&= \sigma_s P_{Var(N_T)-Var(N_s)}g'(N_s) + \sigma_s \int_s^T P_{Var(N_u)-Var(N_s)}\tilde{f}(u,N_s)du. \qquad (7.5)
\end{aligned}
$$

a) We prove that $M_t \in \mathbb{D}_{1,2}^X$, for all $t \in [0,T]$. Indeed, we have

$$
\begin{aligned}
\mathbb{E} \mid M_t \mid^2 &\leqslant 3\mathbb{E}\Big|\widehat{\mathbb{E}}\big[g(N_T)|\mathcal{F}_t\big]\Big|^2 + 3\mathbb{E}\Big|\widehat{\mathbb{E}}\big[\int_0^t f(s,N_s,\chi(s,N_s),\psi(s,N_s))ds \mid \mathcal{F}_t\big]\Big|^2 \\
&+ 3\mathbb{E}\Big|\widehat{\mathbb{E}}\big[\int_t^T f(s,N_s,\chi(s,N_s),\psi(s,N_s))ds \mid \mathcal{F}_t\big]\Big|^2.
\end{aligned}
$$

Using (6.20) and inequality of Cauchy-Schwartz, we obtain

$$\begin{aligned}
&\widehat{\mathbb{E}}[g(N_T)|\mathcal{F}_t] \\
&= P_{Var(N_T)-Var(N_t)}g(N_t) \\
&= \int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(N_t-y)g(y)dy \\
&= \int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(N_t-y)^{\frac{1}{2}}g(y)p_{Var(N_T)-Var(N_t)}(N_t-y)^{\frac{1}{2}}dy \\
&\leqslant \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(N_t-y)g(y)^2dy\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(N_t-y)dy\Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(N_t-y)g(y)^2dy\Big)^{\frac{1}{2}} = \Big(\mathbb{E}\Big[\mid g(N_T)\mid^2 |\mathcal{F}_t\Big]\Big)^{\frac{1}{2}}. \qquad (7.6)
\end{aligned}$$

Thus

$$\mathbb{E}\Big|\widehat{\mathbb{E}}\big[g(N_T)\mid\mathcal{F}_t\big]\Big|^2 \leqslant \mathbb{E}\mid g(N_T)\mid^2. \qquad (7.7)$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}\Big|\widehat{\mathbb{E}}\Big[\int_t^T f(s,N_s,\chi(s,N_s),\psi(s,N_s))ds\mid\mathcal{F}_t\Big]\Big|^2 \\
&= \mathbb{E}\Big|\int_t^T \widehat{\mathbb{E}}\Big[f(s,N_s,\chi(s,N_s),\psi(s,N_s))\mid\mathcal{F}_t\Big]ds\Big|^2 \quad \text{(By Lemma 6.4.9)} \\
&\leqslant T\int_t^T \mathbb{E}\Big|\widehat{\mathbb{E}}\Big[f(s,N_s,\chi(s,N_s),\psi(s,N_s))\mid\mathcal{F}_t\Big]\Big|^2 ds \\
&\leqslant T\int_t^T \mathbb{E}\Big|f(s,N_s,\chi(s,N_s),\psi(s,N_s))\Big|^2 ds, \quad \text{(Using (7.7))}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\Big|\widehat{\mathbb{E}}\Big[\int_0^t f(s,N_s,\chi(s,N_s),\psi(s,N_s))ds\mid\mathcal{F}_t\Big]\Big|^2 \\
&= \mathbb{E}\Big|\int_0^t \widehat{\mathbb{E}}\Big[f(s,N_s,\chi(s,N_s),\psi(s,N_s))\mid\mathcal{F}_t\Big]ds\Big|^2 \quad \text{(By Lemma 6.4.9)} \\
&\leqslant T\mathbb{E}\int_0^T \mid f(s,N_s,\chi(s,N_s),\psi(s,N_s))\mid^2 ds.
\end{aligned}$$

Consequently, $\mathbb{E}\mid M_t\mid^2 < \infty$.
b) We prove now that $\mathbb{E}\parallel D_s^X M_t\parallel^2_{|\mathcal{H}|} < \infty, s\leqslant t$.

$$\begin{aligned}
\mathbb{E}\parallel D_s^X M_t\parallel^2_{|\mathcal{H}|} &= \mathbb{E}\int_0^t\int_0^t \tilde{\phi}(r,u)\mid D_r^X M_t\mid\mid D_u^X M_t\mid dudr \\
&= \int_0^t\int_0^t \tilde{\phi}(r,u)\mathbb{E}\Big[\mid D_r^X M_t\mid\mid D_u^X M_t\mid\Big]dudr \\
&\leqslant \frac{1}{2}\int_0^t\int_0^t \tilde{\phi}(r,u)\mathbb{E}\Big[(D_r^X M_t)^2+(D_u^X M_t)^2\Big]dudr
\end{aligned}$$

$$= \frac{1}{2}\int_0^t\int_0^t \tilde{\phi}(r,u)\mathbb{E}\Big[(D_r^X M_t)^2\Big]dudr + \frac{1}{2}\int_0^t\int_0^t \tilde{\phi}(r,u)\mathbb{E}\Big[(D_u^X M_t)^2\Big]dudr$$

Using (7.3) for $s = r$, we get

$$\mathbb{E} \mid D_r^X M_t \mid^2 \leqslant 3\sigma_r^2\mathbb{E} \mid P_{Var(N_T)-Var(N_t)}g'(N_t) \mid^2 + 3\sigma_r^2\mathbb{E}\Big(\int_r^t \tilde{f}(u,N_u)du\Big)^2$$
$$+ 3\sigma_r^2\mathbb{E}\Big(\int_t^T P_{Var(N_u)-Var(N_t)}\tilde{f}(u,N_t)du\Big)^2.$$

Here, we have

$$\mathbb{E} \mid P_{Var(N_T)-Var(N_t)}g'(N_t) \mid^2 \leqslant \mathbb{E} \mid g'(N_T) \mid^2 .$$

And

$$\mathbb{E}\Big(\int_r^t \tilde{f}(u,N_u)du\Big)^2 \leqslant c\int_0^T \mathbb{E}(\tilde{f}(u,N_u))^2du.$$

Moreover,

$$\begin{aligned}\mathbb{E}\Big(\int_t^T P_{Var(N_u)-Var(N_t)}\tilde{f}(u,N_t)du\Big)^2 &\leqslant T\mathbb{E}\int_t^T \Big(P_{Var(N_u)-Var(N_t)}\tilde{f}(u,N_t)\Big)^2du\\ &= T\int_t^T \mathbb{E}\Big(\mathbb{E}\Big[\tilde{f}(u,N_u) \mid \mathcal{F}_t\Big]\Big)^2du\\ &\leqslant T\int_t^T \mathbb{E}\tilde{f}^2(u,N_u)du.\end{aligned}$$

Consequently,

$$\mathbb{E} \mid D_r^X M_t \mid^2 \leqslant 3\Big[\sigma_r^2\mathbb{E} \mid g'(N_T) \mid^2\Big] + 3\Big[\sigma_r^2\int_0^T \mathbb{E}(\tilde{f}(u,N_u))^2du\Big]$$
$$+ 3T\Big[\sigma_r^2\int_0^T \mathbb{E}(\tilde{f}(u,N_u))^2du\Big] < \infty.$$

Using (H2), we conclude $\int_0^t\int_0^t \tilde{\phi}(r,u)drdu < \infty$. Then, $\mathbb{E} \parallel D_s^X M_t \parallel^2_{|\mathcal{H}|} < \infty$.
c) Therefore $M_t \in \mathbb{D}^X_{1,2}$ for all $t \in [0,T]$ and $\widehat{\mathbb{E}}\Big[M_t \mid \mathcal{F}_s\Big] = M_s$ (using Remark 6.4.4), then $\widehat{\mathbb{E}}\Big[M_t \mid \mathcal{F}_s\Big] \in \mathbb{D}^X_{1,2}$. Applying Clark- Ocone formula (Proposition 6.4.13), we get

$$M_t = \mathbb{E}M_t + \int_0^t \widehat{\mathbb{E}}\Big[D_s^X M_t \mid \mathcal{F}_s\Big]\delta X_s. \tag{7.8}$$

Now, we show that $\mathbb{E}\Big[M_t\Big] = \mathbb{E}\Big[M_T\Big]$ and $\widehat{\mathbb{E}}\Big[D_s^X M_t \mid \mathcal{F}_s\Big] = \widehat{\mathbb{E}}\Big[D_s^X M_T \mid \mathcal{F}_s\Big], P.a.s,\ s \in [0,t]$.
From (6.20), we obtain $\widehat{\mathbb{E}}[g(N_T)|\mathcal{F}_t] = P_{Var(N_T)-Var(N_t)}g(N_t)$.
Then $\mathbb{E}\Big[\widehat{\mathbb{E}}[g(N_T)|\mathcal{F}_t]\Big] = \mathbb{E}\Big[P_{Var(N_T)-Var(N_t)}g(N_t)\Big]$ and using (6.24), we obtain $\mathbb{E}\Big[\widehat{\mathbb{E}}[g(N_T)|\mathcal{F}_t]\Big] = \mathbb{E}\Big[g(N_T)\Big]$. Moreover, we have

$$\mathbb{E}\Big[\widehat{\mathbb{E}}\Big(\int_t^T f(s,N_s,\chi(s,N_s),\psi(s,N_s))ds \mid \mathcal{F}_t\Big)\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big(\int_t^T \widehat{\mathbb{E}}\Big[f(s, N_s, \chi(s, N_s), \psi(s, N_s)) \mid \mathcal{F}_t \Big] ds \Big) \quad \text{(By Lemma 6.4.9)} \\
&= \mathbb{E} \int_t^T P_{Var(N_s)-Var(N_t)} f(t, N_t, \chi(t, N_t), \psi(t, N_t)) ds \quad \text{(By Lemma 6.4.7)} \\
&= \int_t^T \mathbb{E} f(s, N_s, \chi(s, N_s), \psi(s, N_s)) ds \quad \text{(Using (6.24)).}
\end{aligned}
$$

Therefore, (7.2) becomes for $t = T$

$$
M_T = g(N_T) + \int_0^T f(s, N_s, \chi(s, N_s), \psi(s, N_s)) ds,
$$

and

$$
\begin{aligned}
\mathbb{E}\Big[M_t\Big] &= \mathbb{E}\Big(\widehat{\mathbb{E}}\Big[g(N_T) \mid \mathcal{F}_t\Big]\Big) + \mathbb{E} \int_0^t f(s, N_s, \chi(s, N_s), \psi(s, N_s)) ds \\
&+ \mathbb{E}\Big(\widehat{\mathbb{E}}\Big[\int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s)) ds \mid \mathcal{F}_t\Big]\Big) \\
&= \mathbb{E}\Big[g(N_T)\Big] + \mathbb{E} \int_0^T f(s, N_s, \chi(s, N_s), \psi(s, N_s)) ds = \mathbb{E}\Big[M_T\Big].
\end{aligned}
$$

Now, we want to prove that

$$
\widehat{\mathbb{E}}\Big[D_s^X M_t \mid \mathcal{F}_s\Big] = \widehat{\mathbb{E}}\Big[D_s^X M_T \mid \mathcal{F}_s\Big], P.a.s,\ s \in [0, t].
$$

Since (7.5) does not depend on t, it is a direct consequence of (7.5), we get $\widehat{\mathbb{E}}\Big[D_s^X M_t \mid \mathcal{F}_s\Big] = \widehat{\mathbb{E}}\Big[D_s^X M_T \mid \mathcal{F}_s\Big], P.a.s,\ s \in [0, t]$. From (7.8), we deduce that

$$
M_t = \mathbb{E} M_T + \int_0^t \widehat{\mathbb{E}}\Big[D_s^X M_T \mid \mathcal{F}_s\Big] \delta X_s.
$$

Let us now introduce the pocess $Z_s = -\widehat{\mathbb{E}}\Big[D_s^X M_T \mid \mathcal{F}_s\Big], s \in [0, T]$.

d) We show that $Z \in \mathcal{V}_T$ to get the integral $\int_0^t Z_s \delta X_s$ is well defined. We conclude by Lemma 6.3.4. Indeed, by (7.5) we have $-Z_s = \sigma_s P_{Var(N_T)-Var(N_s)} g'(N_s) + \sigma_s \int_s^T P_{Var(N_u)-Var(N_s)} \tilde{f}(u, N_s) du$. Furthermore, we have

$$
\begin{aligned}
P_{Var(N_T)-Var(N_t)} g'(x) &= \int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(x - y) g'(y) dy \\
&= \int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(x - y)^{\frac{1}{2}} g'(y) p_{Var(N_T)-Var(N_t)}(x - y)^{\frac{1}{2}} dy \\
&\leqslant \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(x - y) g'(y)^2 dy \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(x - y) dy \Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(x - y) g'(y)^2 dy \Big)^{\frac{1}{2}}
\end{aligned}
$$

$$= \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(z) g'(x-z)^2 dz \Big)^{\frac{1}{2}}. \tag{7.9}$$

Since g is of polynomial growth, $P_{Var(N_T)-Var(N_s)}g'$ is of polynomial growth. Furthermore, by hypothesis $\chi, \psi \in \mathcal{V}_T$, from (H7) and (7.4), $P_{Var(N_u)-Var(N_s)}\tilde{f}(u, N_s) \in \mathcal{V}_T$. Thus, $Z \in \mathcal{V}_T$.
Now, we define

$$Y_t = M_t - \int_0^t f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds, \;\; s \in [0, T].$$

Let us show that $Y \in \mathcal{V}_T$. We have

$$\begin{aligned} Y_T - Y_t &= -\int_0^T Z_r \delta X_r + \int_0^t Z_r \delta X_r - \int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds \\ &= -\int_t^T Z_r \delta X_r - \int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds. \end{aligned}$$

Moreover,

$$Y_T = M_T - \int_0^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds = g(N_T) = \xi.$$

So, we get

$$\begin{aligned} Y_t &= Y_T + \int_t^T Z_r \delta X_r + \int_t^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds \\ &= \xi + \int_t^T Z_r \delta X_r + \int_t^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds. \end{aligned} \tag{7.10}$$

From Lemma 6.4.9 and applying the quasi-conditional expectation, we get

$$\begin{aligned} &Y_t = \widehat{\mathbb{E}}\Big[Y_t \mid \mathcal{F}_t\Big] = \widehat{\mathbb{E}}\Big[g(N_T) + \int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds \mid \mathcal{F}_t\Big] \\ &= \widehat{\mathbb{E}}\Big[g(N_T) \mid \mathcal{F}_t\Big] + \widehat{\mathbb{E}}\Big[\int_t^T f(s, N_s, \chi(s, N_s), \psi(s, N_s))ds \mid \mathcal{F}_t\Big] \\ &= P_{Var(N_T)-Var(N_t)}g(N_t) + \int_t^T P_{Var(N_s)-Var(N_t)} f(s, N_t, \chi(s, N_t), \psi(s, N_t))ds \quad \text{(Lemma 6.4.7, Lemma 6.4.9)} \\ &= P_{Var(N_T)-Var(N_t)}g(N_t) + \int_t^T P_{Var(N_s)-Var(N_t)} f_1(s, N_t)ds. \end{aligned} \tag{7.11}$$

where $f_1(s, N_t) := f(s, N_t, \chi(s, N_t), \psi(s, N_t))$.
Using (7.9), we do a similar calculation by replacing g' by g, we obtain

$$P_{Var(N_T)-Var(N_t)}g(x) \leqslant \Big(\int_{\mathbb{R}} p_{Var(N_T)-Var(N_t)}(z) g(x-z)^2 dz \Big)^{\frac{1}{2}}. \tag{7.12}$$

Since g is of polynomial growth, $P_{Var(N_T)-Var(N_t)}g(N_t) \in \mathcal{V}_T$. Using (7.12) with g remplaced by f_1 we get $P_{Var(N_s)-Var(N_t)}f_1(s,N_t) \in \mathcal{V}_T$. Thus, $Y \in \mathcal{V}_T$.

Consequently, from (7.10) we have constructed a solution $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$ for BSDE (7.1).
e) In this part of the proof we assume that K_T^* is injective. Let $u, v \in \mathcal{V}_T$ such that $Y_t = u(t, N_t), Z_t = v(t, N_t), t \in [0, T]$. We show now $v(t, x) = -\sigma_t \frac{\partial u}{\partial t}(t, x)$.
Indeed, by appling Itô Formula (Theorem 4.2.1), we have

$$u(t, N_t) \;=\; g(N_T) - \int_t^T \Big[\frac{\partial}{\partial t}u(s, N_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2}{\partial x^2}u(s, N_s)\Big]ds - \int_t^T \sigma_s \frac{\partial}{\partial x}u(s, N_s)\delta X_s,$$

where $\tilde{\sigma}_t = \frac{d}{dt}Var(N_t)$.
From (7.1), we can conclude

$$\begin{aligned}&\int_t^T f(s,N_s,\chi(s, N_s),\psi(s, N_s))ds + \int_t^T v(s, N_s)\delta X_s \\ &= -\int_t^T \sigma_s \frac{\partial}{\partial x}u(s, N_s)\delta X_s - \int_t^T \Big[\frac{\partial}{\partial t}u(s, N_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2}{\partial x^2}u(s, N_s)\Big]ds. \qquad (7.13)\end{aligned}$$

By Lemma 6.1.6, we obtain

$$v(t, x) = -\sigma_t \frac{\partial}{\partial x}u(t, x), \text{for all } \; t \in [0, T], \; x \in \mathbb{R}.$$

f) Now, we prove the uniqueness. We suppose that there is another solution $Y^1, Z^1 \in \mathcal{V}_T$. Then, we apply Itô formula (Theorem 6.5.1), and get

$$\begin{aligned}(Y_t - Y_t^1)^2 &= 2\int_t^T (Y_s - Y_s^1)(Z_s - Z_s^1)\delta X_s - 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}(Z_s - Z_s^1)(Z_s - Z_s^1)ds \\ &= 2\int_t^T (Y_s - Y_s^1)(Z_s - Z_s^1)\delta X_s - 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}(Z_s - Z_s^1)^2 ds.\end{aligned}$$

Taking the expectation, we get

$$\mathbb{E}(Y_t - Y_t^1)^2 + 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E}(Z_s - Z_s^1)^2 ds = 2\mathbb{E}\int_t^T (Y_s - Y_s^1)(Z_s - Z_s^1)\delta X_s.$$

Then,

$$\mathbb{E} \mid Y_t - Y_t^1 \mid^2 + 2m\int_t^T \epsilon(s)\mathbb{E}(Z_s - Z_s^1)^2 ds \;\leqslant 2\mathbb{E}\int_t^T (Y_s - Y_s^1)(Z_s - Z_s^1)\delta X_s, \quad \text{(Using (H6))}.$$

Furthermore, we have $(Y - Y^1)(Z - Z^1) \in \mathcal{V}_T$, then

$$\mathbb{E}\int_t^T (Y_s - Y_s^1)(Z_s - Z_s^1)\delta X_s = 0. \qquad (7.14)$$

Using (7.14), we obtain

$$\mathbb{E} \mid Y_t - Y_t^1 \mid^2 + 2m\int_t^T \epsilon(s)\mathbb{E}(Z_s - Z_s^1)^2 ds = 0.$$

Consequenly, $\mathbb{E} \mid Y_t - Y_t^1 \mid^2 = 0$ and $\int_t^T \epsilon(s)\mathbb{E}(Z_s - Z_s^1)^2 ds = 0$.
We conclude $Y = Y^1$ P.a.s, $t \in [0, T]$ and $Z = Z^1$, P.a.s, $s \in [t, T]$. ■

Now, we establish an a priori estimate for BSDEs which will be useful later in the section.

Proposition 7.1.2. *Assume that K_T^* is injective and let the assumptions (H1), (H2), (H6), (H7), (H8) be satisfied. For $(U,V) \in \mathcal{V}_T \times \mathcal{V}_T$, let $(Y,Z) \in \mathcal{V}_T \times \mathcal{V}_T$ be the unique solution of the BSDE*

$$Y_t = \int_t^T f(s, N_s, U_s, V_s)ds + \int_t^T Z_s \delta X_s,\ 0 \leqslant t_0 \leqslant t \leqslant T.$$

Then, for all $\beta > 0$, there exists $C(\beta) \in \mathbb{R}_+$ (depending also on L, T and ϵ in (H6)) such that

$$\begin{aligned}&\sup_{t\in[0,T]} e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 + \int_0^T e^{2\beta s}\mathbb{E} \mid Y_s \mid^2 ds + \int_0^T e^{2\beta s}\epsilon(s)\mathbb{E}Z_s^2 ds \\ &\leqslant C(\beta)\Big(\int_0^T e^{2\beta s}\mathbb{E} \mid U_s \mid^2 ds + \int_0^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid V_s \mid^2 ds + \int_0^T e^{2\beta s}\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2 ds\Big).\end{aligned}$$

Moreover, $C(\beta)$ can be chosen such that $\lim_{\beta\to\infty} C(\beta) = 0$.

Proof. Applying Itô formula (Theorem 6.5.1) to $Y_t = Y_0 - \int_0^t f(s, N_s, U_s, V_s)ds - \int_0^t Z_s \delta X_s$, we get

$$Y_t^2 = 2\int_t^T Y_s f(s, N_s, U_s, V_s)ds + 2\int_t^T Y_s Z_s \delta X_s + 2\int_t^T Z_s \mathbb{D}_s^X Y_s ds. \tag{7.15}$$

Moreover, using Proposition 7.1.1, we have $Y_s = u(s, N_s)$, and

$$D_r^X Y_s = D_r^X u(s, N_s) = \frac{\partial}{\partial x}u(s, N_s)\sigma_r 1_{[0,s]}(r) = -\frac{\sigma_r}{\sigma_s} Z_s 1_{[0,s]}(r).$$

Then,

$$\mathbb{D}_s^X Y_s = \int_0^s D_r^X Y_s \phi(r,s)dr = \int_0^s -\frac{\sigma_r}{\sigma_s} Z_s \phi(r,s)dr = -\frac{\hat{\sigma}_s}{\sigma_s} Z_s.$$

Therefore

$$\mathbb{E}Y_t^2 = 2\int_t^T \mathbb{E}[Y_s f(s, N_s, U_s, V_s)]ds - 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E}Z_s^2 ds. \tag{7.16}$$

By (H6), we get

$$\begin{aligned}\mathbb{E}Y_t^2 + 2m\int_t^T \epsilon(s)\mathbb{E} \mid Z_s \mid^2 ds &\leqslant \mathbb{E}Y_t^2 + 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E} \mid Z_s \mid^2 ds \\ &\leqslant 2\int_t^T \mathbb{E}\Big[\mid Y_s \mid\mid f(s, N_s, U_s, V_s) \mid \Big]ds.\end{aligned}$$

By Lipschiz continuity of f, we have

$$yf(t, N_t, u(t), v(t)) = yf(t, N_t, u(t), v(t)) - yf(t, N_t, 0, 0) + yf(t, N_t, 0, 0).$$

Moreover,

$$\mid yf(t, N_t, u(t), v(t)) \mid \ \leqslant\ \mid y \mid L(\mid u(t) \mid + \mid v(t) \mid) + \mid y \mid\mid f(t, N_t, 0, 0) \mid .$$

Taking the expectation we get

$$\mathbb{E}[Y_t]^2 + 2m\int_t^T \epsilon(s)\mathbb{E} \mid Z_s \mid^2 ds \leqslant 2\int_t^T \mathbb{E}[\mid Y_s \mid (L \mid U_s \mid + L \mid V_s \mid + \mid f(s, N_s, 0, 0) \mid)]ds$$
$$\leqslant 2\int_t^T \Big(\mathbb{E}[Y_s]^2\Big)^{\frac{1}{2}} \Big(\mathbb{E}(L \mid U_s \mid + L \mid V_s \mid + \mid f(s, N_s, 0, 0) \mid)^2\Big)^{\frac{1}{2}} ds.$$

Let $x(t) = [\mathbb{E}(Y_s)^2]^{\frac{1}{2}}$, $t \in [0, T]$. Then

$$x(t)^2 \leqslant 2\sqrt{3}\int_t^T x(s)\Big[\mathbb{E}\Big(L^2 \mid U_s \mid^2 + L^2 \mid V_s \mid^2 + \mid f(s, N_s, 0, 0) \mid^2\Big)\Big]^{\frac{1}{2}} ds, t \in [0, T]. \quad (7.17)$$

To estimate $x(t)$, we will apply Corollary 6.64 [27] and (7.17), we obtain

$$x(t) \leqslant \sqrt{3}\int_t^T \Big[\mathbb{E}\Big(L^2 \mid U_s \mid^2 + L^2 \mid V_s \mid^2 + \mid f(s, N_s, 0, 0) \mid^2\Big)\Big]^{\frac{1}{2}} ds, t \in [0, T].$$

Hence for any $\beta > 0$,

$$\begin{aligned}
(\mathbb{E} \mid Y_s \mid^2)^{\frac{1}{2}} &\leqslant \sqrt{3}\int_t^T \Big[\mathbb{E}\Big(L^2 \mid U_s \mid^2 + L^2 \mid V_s \mid^2 + \mid f(s, N_s, 0, 0) \mid^2\Big)\Big]^{\frac{1}{2}} ds \\
&\leqslant \sqrt{3}\int_t^T \Big[L\Big(\mathbb{E} \mid U_s \mid^2\Big)^{\frac{1}{2}} + L\Big(\mathbb{E} \mid V_s \mid^2\Big)^{\frac{1}{2}} + \Big(\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2\Big)^{\frac{1}{2}}\Big] ds \\
&\leqslant \sqrt{3}L\int_t^T \Big(\mathbb{E} \mid U_s \mid^2\Big)^{\frac{1}{2}} + \Big(\mathbb{E} \mid V_s \mid^2\Big)^{\frac{1}{2}} ds + \sqrt{3}\int_t^T \Big(\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2\Big)^{\frac{1}{2}} ds \\
&\leqslant \sqrt{3}L\int_t^T e^{-\beta s}\Big[\mathbb{E}e^{2\beta s} \mid U_s \mid^2\Big]^{\frac{1}{2}} ds + \sqrt{3}\int_t^T e^{-\beta s}\Big[e^{2\beta s}\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2\Big]^{\frac{1}{2}} ds \\
&+ \sqrt{3}L\int_t^T \frac{1}{\sqrt{\epsilon(s)}} e^{-\beta s}\Big[\epsilon(s)e^{2\beta s}\mathbb{E} \mid V_s \mid^2\Big]^{\frac{1}{2}} ds.
\end{aligned}$$

Then,

$$\begin{aligned}
(\mathbb{E} \mid Y_s \mid^2)^{\frac{1}{2}} \leqslant{}& \sqrt{3}L\Big(\int_t^T e^{-2\beta s} ds\Big)^{\frac{1}{2}} \Big(\int_t^T e^{2\beta s}\mathbb{E} \mid U_s \mid^2 ds\Big)^{\frac{1}{2}} \\
&+ \sqrt{3}L\Big(\int_t^T \frac{e^{-2\beta s}}{\epsilon(s)} ds\Big)^{\frac{1}{2}} \Big(\int_t^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid V_s \mid^2) ds\Big)^{\frac{1}{2}} \\
&+ \sqrt{3}\Big(\int_t^T e^{-2\beta s} ds\Big)^{\frac{1}{2}} \Big(\int_t^T e^{2\beta s}\mathbb{E} \mid f(s, N_s, 0, 0) \mid^2) ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

We use the following notations:

$$A_t := \Big(\int_t^T e^{2\beta s}\mathbb{E} \mid U_s \mid^2 ds\Big)^{\frac{1}{2}},$$
$$B_t := \Big(\int_t^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid V_s \mid^2) ds\Big)^{\frac{1}{2}},$$

and

$$C_t := \Big(\int_t^T e^{2\beta s} \mathbb{E} \mid f(s, N_s, 0, 0) \mid^2 ds \Big)^{\frac{1}{2}}.$$

Since $\int_t^T e^{-2\beta s} ds = \frac{1}{2\beta}\Big(e^{-2\beta t} - e^{-2\beta T}\Big)$, then

$$\begin{aligned} e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 &\leqslant \frac{9L^2}{2\beta} A_t^2 + 9L^2 B_t^2 \int_t^T \frac{e^{2\beta(t-s)}}{\epsilon(s)} ds + \frac{9}{2\beta} C_t^2 \\ &\leqslant \Big[\frac{9L^2}{2\beta} + \frac{9}{2\beta} + 9L^2 \int_t^T \frac{e^{2\beta(t-s)}}{\epsilon(s)} ds\Big](A_t^2 + B_t^2 + C_t^2). \end{aligned}$$

Moreover, we have $\int_t^T \frac{e^{2\beta(t-s)}}{\epsilon(s)} ds \xrightarrow[\beta \to +\infty]{} 0$ if $\int_t^T \frac{ds}{\epsilon(s)} < +\infty \quad (H6)$.
Thus, there exists $C(\beta)$ with $\lim\limits_{\beta \to \infty} C(\beta) = 0$, such that

$$e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 \leqslant C(\beta)\Big(A_t^2 + B_t^2 + C_t^2\Big), t \in [0, T]. \tag{7.18}$$

Now, we apply the Itô formula to $e^{2\beta t}\mathbb{E} \mid Y_t \mid^2$. On the one hand, using (7.15), we get

$$\mid Y_t \mid^2 = 2\int_t^T Y_s f(s, N_s, U_s, V_s) ds + 2\int_t^T Y_s Z_s \delta X_s - 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s} \mid Z_s \mid^2 ds.$$

Then,

$$d\mathbb{E} \mid Y_t \mid^2 = -2\mathbb{E}[Y_t f(t, N_t, U_t, V_t)]dt + 2\frac{\hat{\sigma}_t}{\sigma_t}\mathbb{E} \mid Z_t \mid^2 dt. \tag{7.19}$$

On the other hand, using (7.19), we obtain

$$d(e^{2\beta t}\mathbb{E} \mid Y_t \mid^2) = 2\beta e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 dt - 2e^{2\beta t}\mathbb{E}[Y_t f(t, N_t, U_t, V_t)]dt + 2e^{2\beta t}\frac{\hat{\sigma}_t}{\sigma_t}\mathbb{E} \mid Z_t \mid^2 dt.$$

Consequently,

$$\begin{aligned} e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 + \int_t^T 2\beta e^{2\beta s}\mathbb{E} \mid Y_s \mid^2 ds + 2\int_t^T e^{2\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E} \mid Z_s \mid^2 ds &= 2\int_t^T e^{2\beta s}\mathbb{E}[Y_s f(s, N_s, U_s, V_s)]ds \\ &\leqslant 2\int_t^T e^{2\beta s}\mathbb{E}[\mid Y_s \mid\mid f(s, N_s, U_s, V_s) \mid]ds. \end{aligned}$$

Thus,

$$\begin{aligned} &e^{2\beta t}\mathbb{E} \mid Y_t \mid^2 + 2\beta \int_t^T e^{2\beta s}\mathbb{E} \mid Y_s \mid^2 ds + 2m\int_t^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid Z_s \mid^2 ds \\ &\leqslant 2\int_t^T e^{2\beta s}\mathbb{E}\Big[\mid Y_s \mid\mid f(s, N_s, U_s, V_s) \mid \Big]ds \\ &\leqslant 2\int_t^T e^{2\beta s}\mathbb{E}\Big[\mid Y_s \mid \Big(L \mid U_s \mid + L \mid V_s \mid) + \mid f(s, N_s, 0, 0) \mid \Big)\Big]ds \quad \text{(By lipschitz property of } f) \end{aligned}$$

$$
\begin{aligned}
&\leqslant 2L\int_t^T\Big[\mathbb{E}\Big(e^{2\beta s}\mid Y_s\mid^2\Big)\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(e^{2\beta s}\mid U_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2L\int_t^T\Big[\mathbb{E}\Big(\frac{e^{2\beta s}}{\epsilon(s)}\mid Y_s\mid^2\Big)\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(e^{2\beta s}\epsilon(s)\mid V_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2\int_t^T\Big[\mathbb{E}(e^{2\beta s}\mid Y_s\mid^2)\Big]^{\frac{1}{2}}\Big[\mathbb{E}(e^{2\beta s}\mid f(s,N_s,0,0)\mid^2)\Big]^{\frac{1}{2}}ds\\
&\leqslant 2L\int_t^T\Big[C(\beta)\Big(A_s^2+B_s^2+C_s^2\Big)\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(e^{2\beta s}\mid U_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2L\int_t^T\Big[\frac{C(\beta)}{\epsilon(s)}\Big(A_s^2+B_s^2+C_s^2\Big)\Big]^{\frac{1}{2}}\times\Big[\mathbb{E}\Big(e^{2\beta s}\epsilon(s)\mid V_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2\int_t^T\Big[C(\beta)\Big(A_s^2+B_s^2+C_s^2\Big)\Big]^{\frac{1}{2}}\Big[\mathbb{E}\Big(e^{2\beta s}\mid f(s,N_s,0,0)\mid^2\Big)\Big]^{\frac{1}{2}}ds\ \text{(Using (7.18))}\\
&\leqslant 2L\sqrt{C(\beta)}(A_t^2+B_t^2+C_t^2)^{\frac{1}{2}}\int_t^T\Big[\mathbb{E}\Big(e^{2\beta s}\mid U_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2L\sqrt{C(\beta)}(A_t^2+B_t^2+C_t^2)^{\frac{1}{2}}\int_t^T\frac{1}{\sqrt{\epsilon(s)}}\Big[\mathbb{E}\Big(e^{2\beta s}\epsilon(s)\mid V_s\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&+2\sqrt{C(\beta)}(A_t^2+B_t^2+C_t^2)^{\frac{1}{2}}\int_t^T\Big[\mathbb{E}\Big(e^{2\beta s}\mid f(s,N_s,0,0)\mid^2\Big)\Big]^{\frac{1}{2}}ds\\
&\leqslant 2L\sqrt{C(\beta)}(A_t+B_t+C_t)\sqrt{T-t}A_t+2L\sqrt{C(\beta)}(A_t+B_t+C_t)\Big(\int_t^T\frac{1}{\epsilon(s)}ds\Big)^{\frac{1}{2}}B_t\\
&+2\sqrt{C(\beta)}(A_t+B_t+C_t)\sqrt{T-t}C_t\\
&\leqslant C_2\sqrt{C(\beta)}(A_t^2+B_t^2+C_t^2).
\end{aligned}
\tag{7.20}
$$

Furthermore, using (7.18), we obtain $\sup_{t\in[0,T]}e^{2\beta t}\mathbb{E}\mid Y_t\mid^2\leqslant C(\beta)(A_0^2+B_0^2+C_0^2)$.
Moreover, by (7.20) for $t=0$, we get

$$
\begin{aligned}
&2\beta\int_0^T e^{2\beta s}\mathbb{E}\mid Y_s\mid^2 ds+2m\int_0^T e^{2\beta s}\epsilon(s)\mathbb{E}Z_s^2ds\\
&\leqslant C_2\sqrt{C(\beta)}(A_0^2+B_0^2+C_0^2).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sup_{t\in[0,T]}e^{2\beta t}\mathbb{E}\mid Y_t\mid^2+\int_0^T e^{2\beta s}\mathbb{E}\mid Y_s\mid^2 ds+\int_0^T e^{2\beta s}\epsilon(s)\mathbb{E}Z_s^2ds\\
&\leqslant\sup_{t\in[0,T]}e^{2\beta t}\mathbb{E}\mid Y_t\mid^2+2\beta\int_0^T e^{2\beta s}\mathbb{E}\mid Y_s\mid^2 ds+2m\int_0^T e^{2\beta s}\epsilon(s)\mathbb{E}Z_s^2ds\\
&\leqslant C(\beta)(A_0^2+B_0^2+C_0^2)+C_2\sqrt{C(\beta)}(A_0^2+B_0^2+C_0^2)\\
&\leqslant C'(\beta)(A_0^2+B_0^2+C_0^2).
\end{aligned}
$$

The proof is completed. ■

Theorem 7.1.3. *Assume that K_T^* is injective and let the assumptions (H1), (H2), (H6), (H7), (H8) be satisfied. Then the BSDE*

$$Y_t = \xi + \int_t^T f(s, N_s, Y_s, Z_s)ds + \int_t^T Z_s \delta X_s,\ 0 < t_0 \leqslant t \leqslant T.$$

has a solution $(Y, Z) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$.

Proof. The existence of the solution is obtained by the Banach fixed point theorem.
Let us consider the mapping $\Gamma : \mathcal{V}_T \times \mathcal{V}_T \to \mathcal{V}_T \times \mathcal{V}_T$ given by $(U, V) \longmapsto \Gamma(U, V) = (Y, Z)$, where (Y, Z) is a solution of the following generalized BSDE:

$$Y_t \ = \ \xi + \int_t^T f(s,N_s, U_s, V_s)ds + \int_t^T Z_s \delta X_s.$$

Let us show that Γ is a contraction for the norm $\| \ (u, v) \ \|_{2,\epsilon}:=\| \ u \ \|_2 + \| \ v \ \|_\epsilon$, for $(u, v) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$.
For $(U, V), (U', V') \in \mathcal{V}_T \times \mathcal{V}_T$ and $(Y, Z) = \Gamma(U, V)$, $(Y', Z') = \Gamma(U', V')$, we set $\triangle Y = Y - Y'$, $\triangle Z = Z - Z'$, $\triangle U = U - U'$ and $\triangle V = V - V'$. A similar calculation as the steps of the proofs of the previous Proposition 7.1.2, shows that there exists $C(\beta)$ which may depend on L, T and ϵ such that $\lim\limits_{\beta\to\infty} C(\beta) = 0$, and

$$\begin{aligned} &\sup_{t\in[0,T]} e^{2\beta t}\mathbb{E} \mid \triangle Y_t \mid^2 + \int_t^T e^{2\beta s}\mathbb{E} \mid \triangle Y_s \mid^2 ds + \int_t^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid \triangle Z_s \mid^2 ds \\ &\leqslant C(\beta)\Big(\int_t^T e^{2\beta s}\mathbb{E} \mid \triangle U_s \mid^2 ds + \int_t^T e^{2\beta s}\epsilon(s)\mathbb{E} \mid \triangle V_s \mid^2 ds\Big). \end{aligned} \tag{7.21}$$

Taking β large enough such that $C(\beta) \leqslant \frac{1}{2}$, Γ becomes a strict contraction in $\mathcal{V}_T \times \mathcal{V}_T$, for the norm $\| \ (., .) \ \|_{2,\epsilon}$.
Now, we define $\{(Y^k, Z^k)\}_{k\in\mathbb{N}}$ recursively by putting $Y_t^0 = \chi(t, N_t)$, $Z_t^0 = \psi(t, N_t)$ for $\chi, \psi \in \mathcal{V}_T$ through the BSDE

$$Y_t^{k+1} \ = \ \xi + \int_t^T f(s, N_s, Y_s^k, Z_s^k)ds + \int_t^T Z_s^{k+1}\delta X_s, \quad t \in [0, T], k \geqslant 0.$$

From Proposition 7.1.1, we know that for all $k \geqslant 0$, $Y_t^k = u_k(t, N_t)$, $Z_t^k = v_k(t, N_t)$, for suitable $u_k, v_k \in \mathcal{V}_T$ such that $Z_t^k = -\sigma_t \frac{\partial}{\partial x} u_k(t, N_t)), t \in [t_0, T]$.
Since Γ is a contraction, $\{(Y^k, Z^k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$, then there exists $(Y, Z) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$ such that (Y^k, Z^k) converges to (Y, Z) in $\tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$. We prove that (Y, Z) satisfies

$$Y_t \ = \ \xi + \int_t^T f(s, N_s, Y_s, Z_s)ds + \int_t^T Z_s \delta X_s, \quad t \in (0, T].$$

On the one hand, we have

$$\lim_{k\to\infty} \mathbb{E} \mid Y_t^k - Y_t \mid^2 = 0, t \in [0, T].$$

On the other hand, for arbitrary $t_0 > 0$ and for all $t \in [t_0, T]$,

$$\lim_{k\to\infty} \Big\{ \int_t^T \mathbb{E} \mid Y_s^k - Y_s \mid^2 ds + \int_t^T \epsilon(s)\mathbb{E} \mid Z_s^k - Z_s \mid^2 ds \Big\} = 0.$$

We show now, $\lim_{k\to\infty} \Big(-Y_t^{k+1} + \xi + \int_t^T f(s, N_s, Y_s^k, Z_s^k)ds\Big) = -Y_t + \xi + \int_t^T f(s, N_s, Y_s, Z_s)ds$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. In fact,

$$\begin{aligned}
&\mathbb{E}\Big|Y_t^k - Y_t + \int_t^T \Big[f(s, N_s, Y_s^k, Z_s^k) - f(s, N_s, Y_s, Z_s)\Big]ds\Big|^2 \\
&\leqslant 2\mathbb{E} \mid Y_t^k - Y_t \mid^2 + 2\mathbb{E}\Big| \int_t^T \Big[f(s, N_s, Y_s^k, Z_s^k) - f(s, N_s, Y_s, Z_s)\Big]ds\Big|^2 \\
&\leqslant 2\mathbb{E} \mid Y_t^k - Y_t \mid^2 + 2L^2T \int_t^T \mathbb{E} \mid Y_s^k - Y_s \mid^2 ds + 2\mathbb{E}\Big(\int_t^T \frac{1}{\epsilon(s)}ds\Big)\Big(\int_t^T \epsilon(s) \mid Z_s^k) - Z_s \mid^2 ds\Big) \\
&\leqslant 2\mathbb{E} \mid Y_t^k - Y_t \mid^2 + 2L^2T \int_t^T \mathbb{E} \mid Y_s^k - Y_s \mid^2 ds + 2L^2c \int_t^T \epsilon(s)\mathbb{E} \mid Z_s^k) - Z_s \mid^2 ds \underset{k\to+\infty}{\longrightarrow} 0
\end{aligned}$$

Next, we prove that $Z_s^k \mathbf{1}_{[t,T]} \to Z_s \mathbf{1}_{[t,T]}$ in $L^2(\Omega, \mathcal{F}, P, \mathcal{H})$. Indeed,

$$\begin{aligned}
&\mathbb{E} \parallel Z^k 1_{[t,T]} - Z 1_{[t,T]} \parallel^2_{L^2(\Omega,\mathcal{H})} \\
&= \mathbb{E} \parallel K_T^*(Z^k 1_{[t,T]} - Z 1_{[t,T]}) \parallel^2_{L^2([0,T])} \\
&= \mathbb{E} \int_0^T \Big(\int_s^T (Z^k - Z)_u 1_{[t,T]}(u) \frac{\partial}{\partial u} K(u, s)du\Big)^2 ds \\
&= \mathbb{E} \int_0^T ds\Big(\int_s^T (Z^k - Z)_u 1_{[t,T]}(u) \frac{\partial}{\partial u} K(u, s)du\Big)\Big(\int_s^T (Z^k - Z)_v 1_{[t,T]}(v) \frac{\partial}{\partial v} K(v, s)dv\Big) \\
&= \mathbb{E} \int_0^T (Z^k - Z)_u 1_{[t,T]}(u)du \int_0^T (Z^k - Z)_v 1_{[t,T]}(v)dv \int_0^{inf(u,v)} \frac{\partial}{\partial u} K(u, s) \frac{\partial}{\partial v} K(v, s)ds \\
&= \mathbb{E} \int_t^T du \int_t^T dv (Z^k - Z)_u (Z^k - Z)_v \phi(u, v) \\
&\leqslant \mathbb{E} \int_t^T du\mathbb{E}[(Z^k - Z)_u^2]\Big(\int_0^u \phi(u, v)dv\Big) + \mathbb{E} \int_t^T dv\mathbb{E}[(Z^k - Z)_v^2]\Big(\int_0^v \phi(u, v)du\Big) \\
&= 2 \int_t^T du\mathbb{E}[(Z^k - Z)_u^2] \int_0^u \phi(u, v)dv \\
&= 2 \int_t^T du\mathbb{E}[(Z^k - Z)_u^2] \int_0^u \int_0^{inf(u,v)} \frac{\partial}{\partial u} K(u, s) \frac{\partial}{\partial v} K(v, s)dsdv \\
&\leqslant 2 \int_t^T du\mathbb{E}[(Z^k - Z)_u^2] \int_0^u \int_0^{inf(u,v)} \Big|\frac{\partial}{\partial u} K(u, s)\Big| \Big|\frac{\partial}{\partial v} K(v, s)\Big| dsdv \\
&\leqslant c \int_t^T du\mathbb{E}[(Z^k - Z)_u^2] \int_0^u \int_0^{inf(u,v)} (u - s)^{\alpha-1} \Big(\frac{u}{s}\Big)^\beta (v - s)^{\alpha-1} \Big(\frac{v}{s}\Big)^\beta dsdv \\
&\leqslant c \int_t^T du\mathbb{E}[(Z^k - Z)_u^2] u^\beta \int_0^u v^\beta \Big(\int_0^{inf(u,v)} (u - s)^{\alpha-1}(v - s)^{\alpha-1} s^{-2\beta} ds\Big) dv.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_0^{inf(u,v)}(u-s)^{\alpha-1}(v-s)^{\alpha-1}s^{-2\beta}ds &\leqslant (u-v)^{\alpha-1}\int_0^v(v-s)^{\alpha-1}s^{-2\beta}ds\\ &= (u-v)^{\alpha-1}\int_0^v v^{\alpha-1}(1-\frac{s}{v})^{\alpha-1}s^{-2\beta}ds\\ &= (u-v)^{\alpha-1}v^{-2\beta}v^{\alpha-1}\int_0^1(1-\theta)^{\alpha-1}\theta^{-2\beta}vd\theta\\ &= v^{\alpha-2\beta}(u-v)^{\alpha-1}\mathcal{B}(\alpha,1-2\beta).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\int_t^T du\mathbb{E}[(Z^k-Z)_u^2]u^\beta\int_0^u v^\beta\Big(\int_0^{inf(u,v)}(u-s)^{\alpha-1}(v-s)^{\alpha-1}s^{-2\beta}ds\Big)dv\\ &\leqslant c\mathcal{B}(\alpha,1-2\beta)\int_t^T du\mathbb{E}[(Z^k-Z)_u^2]u^\beta\int_0^u v^{\alpha-\beta}(u-v)^{\alpha-1}dv\\ &\leqslant c\mathcal{B}(\alpha,1-2\beta)\int_t^T du\mathbb{E}[(Z^k-Z)_u^2]u^\beta\int_0^1(\theta u)^{\alpha-\beta}u^{\alpha-1}(1-\theta)^{\alpha-1}ud\theta\\ &\leqslant c\mathcal{B}(\alpha,1-2\beta)\int_t^T du\mathbb{E}[(Z^k-Z)_u^2]u^\beta u^{2\alpha-\beta}\int_0^1\theta^{\alpha-\beta}(1-\theta)^{\alpha-1}d\theta\\ &\leqslant c\mathcal{B}(\alpha,1-2\beta)\int_t^T du\mathbb{E}[(Z^k-Z)_u^2]u^{2\alpha}\mathcal{B}(\alpha-\beta+1,\alpha)\underset{k\to+\infty}{\longrightarrow}0.\end{aligned}$$

Consequently, $Z_s^k \underset{k\to+\infty}{\longrightarrow} Z_s$ in $L^2(\Omega,\mathcal{F},P;\mathcal{H})$.
Moreover, for all $F\in\mathcal{S}$, we have

$$\begin{aligned}\mathbb{E}\Big(<D_\cdot^X F, Z_\cdot \mathbf{1}_{[t,T]}>_{\mathcal{H}}\Big) &= \lim_{k\to\infty}\mathbb{E}\Big(<D_\cdot^X F, Z_\cdot^{k+1}\mathbf{1}_{[t,T]}>_{\mathcal{H}}\Big)\\ &= \lim_{k\to\infty}\mathbb{E}\Big(F.\int_t^T Z_s^{k+1}\delta X_s\Big)\quad \text{(By Proposition 7.1.1, we have } Z^{k+1}\in Dom(\delta^X)\text{)}\\ &= \mathbb{E}\Big[F\Big(Y_t-\xi-\int_t^T f(s,N_s,Y_s,Z_s)ds\Big)\Big].\end{aligned}$$

By proposition 1.3.6 in [26] we get $Z1_{[t,T]}\in Dom(\delta^X)$ and $Y_t-\xi-\int_t^T f(s,N_s,Y_s,Z_s)ds = \delta^X(Z1_{[t,T]})$.
Consequently, we have

$$Y_t=\xi+\int_t^T Z_s\delta X_s+\int_t^T f(s,N_s,Y_s,Z_s)ds.\quad \text{for all } t\in[t_0,T] \text{ with } t_0>0.$$

The proof is completed. ■

7.2 Uniqueness

In this section, we will follow the method of Maticiuc and Nie [25] but several authors have concluded the uniqueness by a fixed point theorem argument.

We introduce the following spaces:
$\mathcal{M} = \Big\{K/K_t = K_T + \int_t^T v_s ds + \int_t^T u_s \delta X_s, t \in [0,T] \ with \ u \in \mathcal{V}_T, v_s = v(s,N_s), where \ \ v \in \mathcal{C}^{0,1}_{pol}([0,T]\times\mathbb{R}\Big\}$.
Let $\mathcal{S}_f$, the set of $(Y,Z) \in \tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$ such that, for $t \in (0,T]$,

$$(i)\ \mathbb{E} \mid Y_t \mid^2 + 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E} \mid Z_s \mid^2 ds = \mathbb{E} \mid \xi \mid^2 + 2\int_t^T \mathbb{E}\Big[Y(s) f(s,N_s,Y_s,Z_s)\Big] ds.$$

$$(ii)\ \mathbb{E}[Y_t K_t] - \mathbb{E}\int_t^T \Big[-\frac{\hat{\sigma}_s}{\sigma_s} u_s + \mathbb{D}^X_s K_s\Big] Z_s ds = \mathbb{E}[\xi . K_T] + \mathbb{E}\int_t^T \Big[K_s f(s,N_s,Y_s,Z_s) + Y_s v_s\Big] ds, K \in \mathcal{M}.$$

Remark 7.2.1. *To prove i) and ii), assume that K_T^* is injective. If Y is a solution of (6.1), Itô's formula (Theorem 6.5.1) applied to Y^2 is in fact (i). Also, if K is a solution of an equation of the same type as (6.1), an application of Itô's formula (Theorem 6.5.1) to K gives (ii). In this sense $\mathcal{S}_f$ is a natural choice for a space of solutions.*

Theorem 7.2.2. *Assume that K_T^* is injective and let the assumptions (H1), (H2), (H6), (H7), (H8) be satisfied. Then the BSDE*

$$Y_t = \xi + \int_t^T f(s,N_s,Y_s,Z_s) ds + \int_t^T Z_s \delta X_s,\ 0 < t_0 \leqslant t \leqslant T.$$

has a unique solution $(Y,Z) \in \mathcal{S}_f$.

Proof. We show first that the solution (Y,Z) we constructed in the proof of Theorem 7.1.3 belongs to $\mathcal{S}_f$.
Indeed, the sequence $\{(Y^k,Z^k)\}_{k\in\mathbb{N}}$ introduced in the proof of Theorem 7.1.3 is in $\mathcal{V}_T \times \mathcal{V}_T$ and converges to (Y,Z) in $\tilde{\mathcal{V}}_{T,1} \times \tilde{\mathcal{V}}_{T,\varepsilon}$. Applying the Itô formula (Theorem 6.5.1) to $(Y^{k+1})^2$, using $\mathbb{D}^X_s Y_s = -\frac{\hat{\sigma}_s}{\sigma_s} Z_s$ and taking the expectation, we have

$$\mathbb{E} \mid Y_t^{k+1} \mid^2 + 2\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E} \mid Z_s^{k+1} \mid^2 ds = \mathbb{E} \mid \xi \mid^2 + 2\int_t^T \mathbb{E}\Big[Y_s^{k+1} f(s,N_s,Y_s^k,Z_s^k)\Big] ds. \quad (7.22)$$

Moreover, we know

$$\lim_{k\to\infty} \mathbb{E} \mid Y_t^k - Y_t \mid^2 = 0, t \in [0,T],$$

and

$$\lim_{k\to\infty}\Big(\int_t^T \mathbb{E}\mid Y_s^k - Y_s\mid^2 ds + \int_t^T \epsilon(s)\mathbb{E}\mid Z_s^k - Z_s\mid^2 ds\Big) = 0, t\in[0,T].$$

Letting $k\to\infty$ in (7.22), for $t_0>0$ we get

$$\mathbb{E}\mid Y_t\mid^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}_s}{\sigma_s}\mid Z_s\mid^2 ds = \mathbb{E}\mid\xi\mid^2 + 2\int_t^T \mathbb{E}\Big[Y_s f(s,N_s,Y_s,Z_s)\Big]ds, \quad t\in[t_0,T]. \tag{7.23}$$

On the other hand, for any $K\in\mathcal{M}$, we have

$$Y_t^{k+1}K_t = \frac{1}{4}\Big[(Y_t^{k+1}+K_t)^2 - (K_t - Y_t^{k+1})^2\Big],$$

and from Itô formula (Theorem 6.5.1), we get (after a long but straightforward computation)

$$\mathbb{E}[Y_t^{k+1}K_t] - \mathbb{E}\int_t^T\Big[-\frac{\hat{\sigma}_s}{\sigma_s}u_s + \mathbb{D}_s^X K(s)\Big]Z_s^{k+1}ds = \mathbb{E}[\xi.K_T] + \mathbb{E}\int_t^T\Big[K_s f(s,N_s,Y_s^k,Z_s^k) + Y_s^{k+1}v_s\Big]ds.$$

Letting $k\to\infty$, for $t_0>0$ we obtain

$$\mathbb{E}[Y_tK_t] - \mathbb{E}\int_t^T\Big[-\frac{\hat{\sigma}_s}{\sigma_s}u_s + \mathbb{D}_s^X K_s\Big]Z_s ds = \mathbb{E}[\xi.K_T] + \mathbb{E}\int_t^T\Big[K_s f(s,N_s,Y_s,Z_s) + Y_s v_s\Big]ds, \quad t\in[t_0,T].$$

Consequently, $(Y,Z)\in\mathcal{S}_f$.
Now, we show the uniqueness of the solution (Y,Z) in the class $\mathcal{S}_f$. We suppose that $(\tilde{Y},\tilde{Z})\in\mathcal{S}_f$ is another solution of BSDE (6.1). Then, for $t_0>0$

$$\mathbb{E}\mid\tilde{Y}_t\mid^2 + 2\int_t^T\frac{\hat{\sigma}_s}{\sigma_s}\mathbb{E}\mid\tilde{Z}_s\mid^2 ds = \mathbb{E}\mid\xi\mid^2 + 2\int_t^T\mathbb{E}\Big[\tilde{Y}_s f(s,N_s,\tilde{Y}_s,\tilde{Z}_s)\Big]ds, \quad t\in[t_0,T]. \tag{7.24}$$

Furthermore, for $t_0>0$

$$\begin{aligned}&\mathbb{E}[\tilde{Y}_tY_t^{k+1}] - \mathbb{E}\int_t^T\Big[-\frac{\hat{\sigma}_s}{\sigma_s}\tilde{Z}_s + \mathbb{D}_s^X\tilde{Y}_s\Big]Z_s^{k+1}ds = \mathbb{E}[\mid\xi\mid^2]\\ &+\mathbb{E}\int_t^T\Big[\tilde{Y}_s f(s,N_s,Y_s^k,Z_s^k) + Y_s^{k+1}f(s,N_s,\tilde{Y}_s,\tilde{Z}_s)\Big]ds, \quad \text{for } t\in[t_0,T].\end{aligned}$$

Thus, for $t_0>0$

$$\begin{aligned}&\mathbb{E}[\tilde{Y}_tY_t^{k+1}] + 2\mathbb{E}\int_t^T\frac{\hat{\sigma}_s}{\sigma_s}\tilde{Z}_sZ_s^{k+1}ds\\ &= \mathbb{E}[\mid\xi\mid^2] + \mathbb{E}\int_t^T\Big[\tilde{Y}_s f(s,N_s,Y_s^k,Z_s^k) + Y_s^{k+1}f(s,N_s,\tilde{Y}_s,\tilde{Z}_s)\Big]ds, \quad \text{for } t\in[t_0,T],\end{aligned}$$

and letting $k\to\infty$, we have

$$\mathbb{E}[\tilde{Y}_tY_t] + 2\mathbb{E}\int_t^T\frac{\hat{\sigma}_s}{\sigma_s}\tilde{Z}_sZ_s ds$$

$$= \mathbb{E}[\mid \xi \mid^2] + \mathbb{E}\int_t^T \Big[\tilde{Y}_s f(s, N_s, Y_s, Z_s) + Y_s f(s, N_s, \tilde{Y}_s, \tilde{Z}_s)\Big] ds, \quad \text{for } t \in [t_0, T] \text{ and } t_0 > 0. \tag{7.25}$$

From (7.23), (7.24) and (7.25), we obtain for $t \in [t_0, T]$ and $t_0 > 0$

$$\begin{aligned}
&\mathbb{E} \mid Y_t - \tilde{Y}_t \mid^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}_s}{\sigma_s} \mid Z_s - \tilde{Z}_s \mid^2 ds \\
&= 2\mathbb{E}\int_t^T \Big[Y_s f(s, N_s, Y_s, Z_s) - Y_s f(s, N_s, \tilde{Y}_s, \tilde{Z}_s) + \tilde{Y}_s f(s, N_s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{Y}_s f(s, N_s, Y_s, Z_s)\Big] ds \\
&= 2\mathbb{E}\int_t^T \Big[\Big(Y_s - \tilde{Y}_s\Big) f(s, N_s, Y_s, Z_s) - \Big(Y_s - \tilde{Y}_s\Big) f(s, N_s, \tilde{Y}_s, \tilde{Z}_s)\Big] ds \\
&= 2\mathbb{E}\int_t^T \Big[Y_s - \tilde{Y}_s\Big]\Big[f(s, N_s, Y_s, Z_s) - f(s, N_s, \tilde{Y}_s, \tilde{Z}_s)\Big] ds \\
&\leqslant 2L\mathbb{E}\int_t^T \mid Y_s - \tilde{Y}_s \mid \Big[\mid Y_s - \tilde{Y}_s \mid + \mid Z_s - \tilde{Z}_s \mid\Big] ds \\
&\leqslant 2L\mathbb{E}\int_t^T \mid Y_s - \tilde{Y}_s \mid^2 + \mid Y_s - \tilde{Y}_s \mid\mid Z_s - \tilde{Z}_s \mid ds \\
&= 2L\mathbb{E}\int_t^T \mid Y_s - \tilde{Y}_s \mid^2 ds + 2L\mathbb{E}\int_t^T \mid Y_s - \tilde{Y}_s \mid\mid Z_s - \tilde{Z}_s \mid ds \\
&\leqslant 2L\mathbb{E}\int_t^T \mid Y_s - \tilde{Y}_s \mid^2 ds + \mathbb{E}\int_t^T m\epsilon(s) \mid Z_s - \tilde{Z}_s \mid^2 ds + \frac{L^2}{m}\mathbb{E}\int_t^T \frac{1}{\epsilon(s)} \mid Y_s - \tilde{Y}_s \mid^2 ds.
\end{aligned}$$

Consequently,

$$\mathbb{E} \mid Y_t - \tilde{Y}_t \mid^2 + m\mathbb{E}\int_t^T \epsilon(s) \mid Z_s - \tilde{Z}_s \mid^2 ds \leqslant \int_t^T \Big(2L + \frac{L^2}{m\epsilon(s)}\Big)\mathbb{E} \mid Y_s - \tilde{Y}_s \mid^2 ds. \tag{7.26}$$

Thus,

$$\mathbb{E} \mid Y_t - \tilde{Y}_t \mid^2 \leqslant \int_t^T \Big(2L + \frac{L^2}{m\epsilon(s)}\Big)\mathbb{E} \mid Y_s - \tilde{Y}_s \mid^2 ds.$$

Applying Gronwall's inequality, we get

$$\mathbb{E} \mid Y_t - \tilde{Y}_t \mid^2 = 0, \quad \text{for } t \in [t_0, T] \text{ and } t_0 > 0. \tag{7.27}$$

From (7.26) and (7.27), we obtain $m\displaystyle\int_t^T \epsilon(s)\mathbb{E} \mid Z_s - \tilde{Z}_s \mid^2 ds = 0$, for $t \in [t_0, T]$ and $t_0 > 0$.
Then, $\mathbb{E} \mid Z_s - \tilde{Z}_s \mid^2 = 0$, $s \in [t, T]$ with $t > 0$.
Thus, we obtain $Y_t = \tilde{Y}_t$ P.a.s, $t \in [t_0, T]$ with $t_0 > 0$ and $Z_s = \tilde{Z}_s$, P.a.s, $s \in [t, T]$ with $t > 0$.
The proof is completed. ■

7.3 Comparison Theorem

Theorem 7.3.1. *For $i = 1, 2$ we consider the BSDE's*

$$Y_t^i = g^i(N_T^i) + \int_t^T f^i(s, N_s^i, Y_s^i, Z_s^i) ds + \int_t^T Z_s^i \delta X_s^i, \ 0 < t_0 \leqslant t \leqslant T, \tag{7.28}$$

where $N_t^i = \int_0^t \sigma_s^i \delta X_s^i$ *and* $c_0 < \sigma^i < C_0$ *for some constants* $c_0, C_0 > 0$. *Suppose that both BSDE's satisfy the hypotheses of Theorem 6.1.1 and Proposition 6.1.8. If* $f^1(\cdot,\cdot,y,z) \geqslant f^2(\cdot,\cdot,y,z)$ *for all* y, z, *and* $g^1 \geqslant g^2$, *then* $Y_t^1 \geqslant Y_t^2$ $P-a.s.$ *for all* $t \in [t_0, T]$.

Proof. Let $u^i(t,x)$ be the solution to (6.2) with f replaced by f^i, g replaced by g^i. Then the solution to the above equation (7.28) is given by $Y_t^i = u^i(t, N_t)$. It suffices to prove $u^1(t,x) \geqslant u^2(t,x)$.
Denote $\rho_t = \sqrt{\frac{d}{dt}Var(N_t)}$, and $\zeta_t = \int_0^t \rho_s \delta W_s$, where W is standard Brownian motion.
Applying Itô's formula with respect to W, we have

$$\begin{aligned} du^i(t,\zeta_t) &= \rho_t \frac{\partial u^i}{\partial x}(t,\zeta_t)dW_t + \frac{\partial u^i}{\partial t}(t,\zeta_t)dt + \frac{1}{2}\frac{\partial^2 u^i}{\partial x^2}(t,\zeta_t)\frac{d}{dt}Var(N_t)dt \\ &= -f^i(t,\zeta_t,u^i(t,\zeta_t),-\sigma_t\frac{\partial u^i}{\partial x}(t,\zeta_t))dt + \rho_t\frac{\partial u^i}{\partial x}(t,\zeta_t)dW_t. \end{aligned}$$

Thus, $(\tilde{Y}_t^i, \tilde{Z}_t^i) = (u^i(t,\zeta_t), \rho_t \frac{\partial u^i}{\partial x}(t,\zeta_t))$ is a solution to the following BSDE

$$\begin{cases} d\tilde{Y}_t^i = -f^i(t,\zeta_t,\tilde{Y}_t^i,-\sigma_t\rho_t^{-1}\tilde{Z}_t^i)dt + \tilde{Z}_t^i dW_t, \quad 0 < t_0 \leqslant t \leqslant T \\ \tilde{Y}_T^i = g^i(\zeta_T) \end{cases}$$

By the classical comparison theorem ([36]), $\tilde{Y}_t^1 \geqslant \tilde{Y}_t^2$ almost surely. Thus, $u^1(t,\zeta_t) \geqslant u^2(t,\zeta_t)$. Since ζ_t is a gaussian random variable with positive variance, from lemma 3.7 ([19]), we conclude $u^1(t,x) \geqslant u^2(t,x)$. ■

Bibliography

[1] N. EL KAROUI, S. PENG AND M. C QUENEZ. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1 – 71, 1997.

[2] P. MALLIAVIN. Stochastic calculus of variations and hypoelliptic operators. *In: Proc. Inter. Symp. on Stoch. Diff. Equations, Kyoto*, pages 195 – 263, 1978.

[3] I. NOURDIN AND GIOVANNI PECCATI. *Normal approximations with Malliavin calculus*. From Stein's method to universality. Cambridge Tracts in Mathematics 192, Cambridge University Press, 2012.

[4] R.J. ADLER. An introduction to continuity, extrema, and related topics for general gaussian processes. *Institute of Mathematical Statistics, Lecture Notes - Monograph Series*, 12, 1990.

[5] E. ALÒS, O. MAZET AND D. NUALART. Stochastic calculus with respect to Gausssian processes. *Ann. Probab.*, 29:766 – 801, 2001.

[6] K. BAHLALI, E. ESSAKY, M. HASSANI. Existence and uniqueness of multidimensional bsdes and of systems of degenerate pdes with superlinear generator. *SIAM J. Math. Anal.*, 47:4251–4288, 2015.

[7] C. BENDER. Explicit solutions of a class of linear fractional BSDEs. *Syst. Control Letters*, 54:671 – 680, 2005.

[8] C. BENDER. Backward SDE's driven by gaussian processes. *Stoch. Process. Appl.*, 124:2892 – 2916, 2014.

[9] C. BENDER, L. VIITASAARI. A general non-existence result for linear BSDEs driven by a Gaussian process. *Stochastic Process. Appl.*, 127:1204 – 1233, 2017.

[10] B. BOUFOUSSI, M. DOZZI, R. MARTY. Local time and Tanaka formula for a Volterra-type multifractional Gaussian process. *Bernoulli*, 16:1294 – 1311, 2010.

[11] R. CARMONA. *Lectures on BSDEs, stochastic control and stochastic differential games with financial applications*. SIAM, 2016.

[12] P. ČOUPEK, B. MASLOWSKI. Stochastic evolution equations with Volterra noise. *Stochastic Processes and their Applications*, 127:877 – 900, 2017.

[13] S. CRÉPEY. *Financial modeling, a backward stochastic differential equations perspective*. Springer Verlag, 2013.

[14] J. DIEHL, P. FRIZ. Backward stochastic differential equations with rough drivers. *Ann. Probab.*, 40:1715 – 1758, 2014.

[15] X. FERNIQUE . *Régularité des trajectoires des fonctions aléatoires gaussiennes*. Springer, 1975.

[16] M. HARRISON, S.R. PLISKA. Martingales and stochastic integrals in the theory of continous trading. *Stoch. Proc. Appl.*, 11:215–260, 1981.

[17] Y. HU. *Integral transformations and anticipative calculus for fractional Brownian motions*. Mem. Amer. Math. Soc., 2005.

[18] Y. HU, D. OCONE, J. SONG. Some results on backward stochastic differential equations driven by frcational brownian motions. *Stochastic Analysis and Applications to Finance, Interdiscip. Math. Sci*, 13:225 – 242, 2012.

[19] Y. HU, S. PENG. Backward stochastic differential equation driven by fractional brownian motion. *SIAM J. Control Optim.*, 48:1675 – 1700, 2009.

[20] K. JAŃCZAK-BORKOWSKA. Generalized BSDEs driven by fractional brownian motion. *Stat. Probab. Lett.*, 83:805 – 811, 2013.

[21] H. KNANI. Partial differential equations associated to the non-linear backward stochastic differential equations with Gaussian Volterra processes. submitted.

[22] H. KNANI, M. DOZZI. Linear Backward Stochastic Differential Equations with Gaussian Volterra processes. submitted.

[23] M. KOBYLANSKI. Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Prob.*, 28:558–602, 2000.

[24] T. MASTROLIA . Density analysis of non-markovian BSDEs and applications to biology and finance. *Stochastic Process*, 128:897–938, 2018.

[25] L. MATICIUC, T. NIE. Fractional backward stochastic differential equations and fractional backward variational inequalities. *J. Theor. Probab.*, 28:337 – 395, 2015.

[26] D. NUALART . *The Malliavin calculus and related topics*. 2nd edn. Springer, 2006.

[27] E. PARDOUX, A. RĂŞCANU. *Stochastic differential equations, backward SDEs, partial differential equations*. Springer Stochastic Modelling and Applied Probability, 2014.

[28] E. PARDOUX, S. PENG. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.*, 14:55 – 61, 1990.

[29] S. PENG . Backward stochastic differential equations and applications to optimal control. *Applied Mathematics and Optimization*, 27(2):125–144, 1993.

[30] H. PHAM. *Continous-time stochastic control and optimization with financial applications*. Springer Verlag, 2009.

[31] T. SOTTINEN, L. VIITASAARI. Stochastic analysis of gaussian processes via fredholm representation. *Intern. J.Stoch. Anal.*, 2016.

[32] A. SOW, B. DIOUF. Fractional anticipated BSDEs with stochastic lipschitz coefficients. *Random Oper. Stoch. Eq.*, 26:143 – 161, 2018.

[33] N. TOUZI. *Optimal stochastic control, stochastic target problems and backward SDE*. Springer Verlag, 2012.

[34] J. WEN, Y. SHI. Anticipative backward stochastic differential equations driven by fractional brownian motion. *Stat. Probab. Lett.*, 122:118 – 127, 2017.

[35] H. ZHANG. Properties of solutions of fractional backward stochastic differential equations. *Appl. Math. Comp.*, 228:446 – 453, 2014.

[36] J. ZHANG. *Backward Stochastic differential equations, probability theory and stochastic modelling*. Springer Verlag, 2017.

Abstract: This thesis treats of backward stochastic differential equations (BSDE) driven by a class of Gaussian Volterra processes that includes multifractional Brownian motion and multifractional Ornstein-Uhlenbeck processes. In the first part we study multidimensional BSDE with generators that are linear functions of the solution. By means of an Itô formula for Volterra processes, a linear second order partial differential equation (PDE) with terminal condition is associated to the BSDE. Under an integrability condition on a functional of the second moment of the Volterra process in a neighbourhood of the terminal time, we solve the associated PDE explicitely and deduce the solution of the linear BSDE. We discuss an application in the context of self-financing trading stategies. The second part of the thesis treats of non-linear BSDE driven by the same class of Gaussian Volterra processes. The main results are the existence and uniqueness of the solution in a space of regular functionals of the Volterra process, and a comparison theorem for the solutions of BSDE. We give two proofs for the existence and uniqueness of the solution, one is based on the associated PDE and a second one without making reference to this PDE, but with probabilistic and functional theoretic methods. Especially this second proof is technically quite complex, and, due to the absence of martingale properties in the context of Volterra processes, requires to work with different norms on the underlying Hilbert space that is defined by the kernel of the Volterra process. For the construction of the solution we need the notion of quasi-conditional expectation, a Clark-Ocone type formula and another Itô formula for Volterra processes. Contrary to the more classical cases of BSDE driven by Brownian or fractional Brownian motion, an assumption on the behaviour of the kernel of the driving Volterra process is in general necessary for the wellposedness of the BSDE. For multifractional Brownian motion this assumption is closely related to the behaviour of the Hurst function.

Key words. Backward stochastic differential equation, Gaussian Volterra process, partial differential equation with terminal condition, Feynman-Kac formula, divergence integral, Malliavin calculus, chaos expansion, quasi-conditional expectation, Clark-Ocone formula.

Résumé: Cette thèse porte sur les équations différentielles stochastiques rétrogrades (EDSR) dirigées par une classe de processus de Volterra qui contient le mouvement brownien multifractionnaire et le processus Ornstein-Uhlenbeck multifractionnaire. Dans la première partie, nous étudions la solution des EDSRs multidimensionnelles avec des générateurs linéaires. Par la formule d'Itô pour les processus de Volterra nous réduisons l'EDSR à une équation aux dérivées partielles (EDP) de second ordre linéaire avec la condition terminale. Sous une condition d'intégrabilité dans un voisinage du temps terminal de la variance du processus de Volterra, nous résolvons l'EDP associée explicitement et en déduisons la solution des EDSR linéaire. Puis, nous discutons une application dans le contexte des stratégies autofinancées. La seconde partie de la thèse traite des EDSRs non linéaires dirigées par la même classe de processus de Volterra. Les résultats principaux sont l'existence et l'unicité de la solution de l'EDSR dans un espace de fonctionnelles régulières du processus de Volterra et un théorème de comparaison qui porte sur les générateurs et les conditions terminales. Nous donnons deux preuves de l'existence et de l'unicité de la solution de l'EDSR, l'une basée sur l'EDP associée et l'autre sans référence à l'EDP, mais avec des méthodes probabilistes. Cette seconde preuve est techniquement difficile et, en raison de l'absence de propriétés de martingale dans le contexte des processus de Volterra, la preuve nécessite différentes normes sur l'espace de Hilbert sous-jacent défini par le noyau du processus de Volterra. Pour la construction de la solution, nous avons besoin de la notion de l'espérance quasi-conditionnelle, d'une formule de type Clark-Ocone et d'une autre formule d'Itô pour les processus de Volterra. Contrairement au cas classique des EDSR dirigées par le mouvement brownien ou brownien fractionnaire, une hypothèse sur le comportement du noyau est nécessaire pour l'existence et l'unicité de la solution de l'EDSR. Pour le mouvement brownien multifractionnaire, cette hypothèse est liée à la fonction de Hurst.

Mots clés. Équations différentielles stochastiques rétrogrades, processus de Volterra Gaussien, équation aux dérivées partielles avec condition terminale, formule de Feynman-Kac, intégrale de divergence, calcul de Malliavin, expression de chaos, espérance quasi-conditionnelle, formule de Clark-Ocone.