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Contributions en théorie du contrôle échantillonné optimal avec contraintes d'état et données non lisses

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To my grandfather,
To his love of life, plants and people

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Contents

Chapter 0

General introduction

Mathematical context. In mathematics a *dynamical system* describes the evolution of a point (usually called the *state* of the system) in an appropriate set following an evolution rule (known as the *dynamics* of the system). Dynamical systems are of many different natures and they can be categorized in different classes such as: continuous systems versus discrete systems, deterministic systems versus stochastic systems, etc. A *continuous system* is a dynamical system in which the state evolves in a continuous way in time (for instance, ordinary differential equations, evolution partial differential equations, etc.), while a *discrete system* is a dynamical system in which the state evolves in a discrete way in time (for instance, difference equations, quantum differential equations, etc.). A *control system* is a dynamical system in which a *control parameter* influences the evolution of the state. Optimal control theory is concerned with *optimal control problems* which consist of guiding a control system from some initial state to a desired final state while minimizing a given cost and being subject to some constraints. In the literature note that the control parameter is often taken to be *permanent* in the sense that its value can be modified at any instant in time. As an example, a continuous optimal permanent control problem in which the dynamics is described by a general nonlinear ordinary differential equation is given by

$$\left.\begin{array}{ll} \text{minimize} & g(x(T)), \\ \text{subject to} & x:[0,T]\to\mathbb{R}^n,\ u:[0,T]\to\mathbb{R}^m, \\ & \dot{x}(t)=f(x(t),u(t),t), \quad \text{a.e. } t\in[0,T], \\ & x(0)=x_0, \\ & u(t)\in \mathrm{U}, \quad \text{a.e. } t\in[0,T], \end{array}\right\} \tag{OCP}$$

where the cost function $g:\mathbb{R}^n\to\mathbb{R}$ and the dynamics $f:\mathbb{R}^n\times\mathbb{R}^m\times[0,T]\to\mathbb{R}^n$ are of sufficient regularity and where $x_0\in\mathbb{R}^n$, $T>0$, m, $n\in\mathbb{N}^*$ and U is a nonempty subset of $\mathbb{R}^m$. In Problem (OCP), x is the *state* function (also called *trajectory*) and u is the *control* function (also called *control*). In this general introduction we have chosen to consider a basic framework for Problem (OCP), but later in this manuscript we will consider more general problems with terminal state constraints, total cost in Bolza form, free final time problems, etc. It is usual to define the Hamiltonian $H:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times[0,T]\to\mathbb{R}$ associated to Problem (OCP) by the formula $H(x,u,p,t):=\langle p,f(x,u,t)\rangle_{\mathbb{R}^n}$ for all $(x,u,p,t)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times[0,T]$.

Established in [Pontryagin *et al.* 1962] by Pontryagin et al. at the end of the 1950's, the *Pontryagin maximum principle* (in short, PMP) is the milestone of optimal control theory. It provides first-order necessary optimality conditions. Roughly speaking, given a solution (x,u)

to Problem (OCP), the PMP states that there exists an *adjoint vector* $p : [0,T] \to \mathbb{R}^n$ such that the following conditions hold:

(i) **Adjoint equation:** p satisfies

$$-\dot{p}(t) = \nabla_1 H(x(t), u(t), p(t), t), \tag{1}$$

for almost every $t \in [0,T]$;

(ii) **Transversality condition on the adjoint vector:** p satisfies

$$p(T) = -\nabla g(x(T)); \tag{2}$$

(iii) **Hamiltonian maximization condition:** the condition

$$u(t) \in \operatorname*{arg\,max}_{\omega \in \mathrm{U}} H(x(t), \omega, p(t), t), \tag{3}$$

holds true for almost every $t \in [0,T]$.

The proof of the PMP follows from considering needle-like perturbations (which are perturbations of constant value over a small interval of time) of the optimal control. Then, by considering the corresponding perturbation on the cost function, one is able to define in a backward way in time the adjoint vector from the transversality condition and satisfying the adjoint equation, and finally obtain the Hamiltonian maximization condition. We mention here that, within the above framework of the PMP, the *Hamiltonian function* $\mathcal{H} : [0,T] \to \mathbb{R}$ is defined by $\mathcal{H}(t) := H(x(t), u(t), p(t), t)$ for almost all $t \in [0,T]$. In that context, the continuity of the Hamiltonian function is a very well-known fact (see, e.g., [Fattorini 1999, Theorem 2.6.3 p.73]). As a well known application of the PMP, if the Hamiltonian maximization condition allows to express the optimal control as a function of the augmented state-costate vector, then the PMP induces the so-called *indirect numerical method* which consists in numerically solving the boundary value problem satisfied by the augmented state-costate vector via a shooting method. Indirect numerical methods are opposed to *direct numerical methods* which consist in a full discretization of Problem (OCP) resulting in a constrained finite-dimensional optimization problem that can be numerically solved from various standard optimization algorithms and techniques. Soon afterwards and even nowadays, the PMP has been adapted to many situations, for control systems of different natures, with various constraints, etc. Several versions of the PMP were derived for discrete optimal permanent control problems concerning discrete control systems in which the dynamics is described by a difference equation (see, e.g., [Boltyanskii 1978, Halkin 1966, Holtzman & Halkin 1966]). In these discrete versions of the PMP, the historical Hamiltonian maximization condition does not hold in general (see a counterexample in [Boltyanskii 1978, Examples 10.1-10.4 p.59-62]) since one can no longer consider needle-like perturbations in the discrete setting. In that context, only basic convex perturbations of the optimal control can be considered and therefore one can only obtain a weaker condition known as a *nonpositive Hamiltonian gradient condition* (see, e.g., [Boltyanskii 1978, Theorem 42.1 p.330]). Note that some appropriate convexity conditions on the dynamics have been considered in order to recover the Hamiltonian maximization condition in the discrete case (see, e.g., [Holtzman & Halkin 1966]).

In this manuscript we are interested in *sampled-data control systems* in which the state evolves continuously in time while the control evolves discretely in time. More precisely the value of the control is authorized to be modified only at a finite number $N \in \mathbb{N}^*$ of times $t_i \in [0,T]$ called *sampling times*, and remains frozen elsewhere. Therefore *sampled-data controls* are modelled by piecewise constant functions respecting the partition of the interval $[0,T]$ defined by the sampling times. Sampled-data control systems, which have the peculiarity of presenting a mixed continuous/discrete structure, are often considered in scientific and engineering applications since in practice any control algorithm has to be implemented in discrete time while the state evolves continuously in time. They have been considered as models mostly in Engineering implemented by *digital controllers* which have a finite precision (see, e.g., [Santina & Stubberud 2005, Volz & Kazda 1966]). Sampled-data control systems are also used in Automation, notably in model predictive control algorithms in which the control value at each sampling time is chosen as the first value of a finite sequence of control values optimizing the given cost on a fixed finite horizon (see, e.g., [Grüne & Pannek 2017]). Numerous texts and articles have developed control theory for sampled-data control systems (see, e.g., [Ackermann 1985, Aström 1963, Aström & Wittenmark 1997, Fadali & Visioli 2013, Landau & Zito 2006] and references therein). For instance, global controllability for sampled-data control systems has been investigated in [Grasse & Sussmann 1990]. Optimal sampled-data control problems have been investigated in the literature with different approaches. One approach has been to apply $\mathcal{H}_2$-$\mathcal{H}_\infty$ optimization theory (see [Biryukov 2016, Chen & Francis 1996]) where the closed-loop transfer matrix under the $\mathcal{H}_2$- and $\mathcal{H}_\infty$- norms is taken as the criterion. Another approach involves the Karush-Kuhn-Tucker necessary conditions and dynamic programming (see [Bini & Buttazzo 2014]). However one should note that the aforementioned results are not formulated in terms of a PMP. Recently, Bourdin and Trélat have obtained in [Bourdin & Trélat 2016] a version of the PMP for general nonlinear optimal sampled-data control problems. In that sampled-data control framework, as in the purely discrete case addressed in the previous paragraph, the usual Hamiltonian maximization condition does not hold in general (for the same reason), and has to be replaced by a weaker condition known as a *nonpositive averaged Hamiltonian gradient condition* (see [Bourdin & Trélat 2016, Theorem 2.6 p.62]) given by

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t), u_i, p(t), t)\, dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0, \tag{4}$$

for all $\omega \in \mathrm{U}$ and all $i = 0, \ldots, N-1$, where u_i stands for the value of the optimal sampled-data control frozen on the *sampling interval* $[t_i, t_{i+1})$. Note that the PMP stated in [Bourdin & Trélat 2016, Theorem 2.6 p.62] considers the more general framework of *time scale calculus* and a version which does not take into account such a generality, and therefore closer to the considerations of this manuscript, can be found in [Bourdin & Trélat 2015, Theorem 1 p.81] or [Bourdin & Trélat 2016, Theorem 1.1 p.55]. We emphasize that this PMP is only concerned with optimal sampled-data control problems in which the sampling times t_i are fixed, without running inequality state constraints and with a smooth cost function g. As a continuation of the previously mentioned works, the main objective of this PhD thesis is to derive first-order necessary optimality conditions in the form of a PMP for optimal sampled-data problems which answer the following questions:

(i) What additional optimality conditions, if any, are necessary if one can freely choose the sampling times t_i?

(ii) What form does the PMP take in the presence of running inequality state constraints?

(iii) What form does the PMP take when the cost function g is nonsmooth?

The rest of this general introduction is devoted to the contributions of this manuscript which will answer these three main questions.

Contributions of Chapter 2. The first contribution of this manuscript, given in Chapter 2, is to answer Queston (i) and to derive a PMP for optimal sampled-data control problems with free sampling times. We mention that *optimal sampling times problems* have already been investigated in the literature but never from a PMP point of view until the paper [Bourdin & Dhar 2019] presented in Chapter 2 of this manuscript. For example many authors considered the related problem of finding the optimal fixed sampling period (or uniform time step) such as in [Levis & Schlueter 1971, Melzer & Kuo 1971]. Nonuniform sampling partitions have also been studied but in specific cases such as for the linear-quadratic integrator in [Schlueter 1973]. In [Schlueter & Levis 1973] the optimal sampled-data control problem is transformed into a purely discrete one by integrating the state over the sampling intervals and then is treated as an usual optimization problem. In Chapter 2 of this manuscript we present a PMP for general nonlinear optimal sampled-data control problems with free sampling times. Similarly to the PMP derived in [Bourdin & Trélat 2015, Theorem 1 p.81] or [Bourdin & Trélat 2016, Theorem 1.1 p.55] for fixed sampling times, by considering convex perturbations of the optimal sampled-data control, we obtain a first-order necessary optimality condition described by a nonpositive averaged Hamiltonian gradient condition (see Inequality (4)). Since the optimal sampled-data control is constant over the semi-open intervals $[t_i, t_{i+1})$, one can see that the Hamiltonian function $\mathcal{H}$ is continuous over these intervals. However, when considering fixed sampling times, the Hamiltonian function is not continuous over $[0, T]$ in general since it may admit discontinuities at the sampling times t_i (see Section 2.3 for an example). On the other hand, when considering free sampling times, we get in Chapter 2 a new and additional necessary optimality condition in the PMP called the *Hamiltonian continuity condition* given by

$$H(x(t_i), u_{i-1}, p(t_i), t_i) = H(x(t_i), u_i, p(t_i), t_i), \tag{5}$$

for all $i = 1, \ldots, N-1$. It follows that the continuity of the Hamiltonian function is recovered in the case of optimal sampled-data controls with optimal sampling times. The Hamiltonian continuity condition is obtained by considering special needle-like perturbations at the sampling times of the optimal sampled-data control (see Section 2.4.1.4 in Chapter 2 for details). We emphasize that the optimal sampled-data control problems with free sampling times which we consider in Chapter 2 have general *terminal state constraints* of the form $h(x(0), x(T)) \in \mathrm{S}$. Therefore, the strategy adopted in order to obtain a PMP is to penalize the distance to the state constraints in a corresponding cost functional and then to apply the Ekeland variational principle [Ekeland 1974, Theorem 1.1 p.324]. This leads us to consider a sequence of sampled-data controls converging to the optimal one. A first difficulty emerges in the fact that the associated sampling times do not necessarily converge to the optimal sampling times. Indeed a degenerate situation can occur if the optimal sampled-data control is constant over two consecutive sampling intervals. As a consequence, another obstacle is the possible phenomenon of accumulation of sampling times. These two difficulties are overcome by introducing a technical control set which guarantees that the sampling times produced by the Ekeland variational principle, firstly, remain unchanged for the ones corresponding to the consecutive sampling intervals on which the optimal sampled-data control is constant (avoiding thus the first difficulty) and, secondly, are contained in disjoint intervals for the others (avoiding thus the second difficulty). For more details on this technical control set we refer to Section 2.4.2 in Chapter 2. A final obstacle lies in the non-convexity of the set of piecewise constant functions with a fixed number N of sampling times. Therefore the standard procedure of considering convex perturbations of the control (as in [Bourdin & Trélat 2016, Lemma 4.17 p.84]) has to be adapted by considering convex perturbations respecting the same N-partition. We refer to the proof of Lemma 2.4.7 for details. Based on the Hamiltonian continuity condition (see Equality (5)), we are able to construct a shooting method which we use to solve two linear-quadratic optimal sampled-data control problems in Section 2.3 and to determine the corresponding optimal sampling times.

Contributions of Chapter 3. The next contribution of this manuscript, presented in Chapter 3, is a PMP for optimal sampled-data control problems in the presence of running inequality state constraints as given in the work [Bourdin & Dhar 2020] which provides an answer to Question (ii). Recall that an important part of optimal control theory is concerned with *state constrained optimal control problems* in which the state is restricted to a certain region of the state space. Indeed it is often undesirable and even inadmissible in scientific and engineering applications that the state crosses certain limits imposed in the state space for safety or practical reasons. Many examples can be found in mechanics and aerospace engineering (e.g., an engine may overheat or overload). State constrained optimal control problems are also encountered in management and economics (e.g., an inventory level may be limited in a production model). We refer to [Bonnard *et al.* 2003, Cots 2017, Kim *et al.* 2011, van Keulen *et al.* 2014, Van Reeven *et al.* 2016] and references therein for other examples. A first version of the PMP for continuous optimal permanent control problems with running state constraints was obtained by Gamkrelidze [Gamkrelidze 1960] (see also [Pontryagin *et al.* 1962, Theorem 25 p.311]) under some special assumptions on the structure of the optimal process. Later, these assumptions were somewhat excluded by Dubovitskii and Milyutin in the seminal work [Dubovitskii & Milyutin 1965, Section 7 p.37]. The contributions of Dubovitskii and Milyutin include, notably, general Lagrange multiplier rules for abstract optimization problems and the so-called method of *v-change of time variable* in view of generating needle-like variations by passing to a smooth control system (see more details in [Dmitruk 2009, Section 4]). Other methods have been developed in the literature in order to establish versions of the PMP for state constrained optimal control problems, such as the smoothly-convex structure of the controlled system in [Ioffe & Tihomirov 1979], the application of the Ekeland variational principle in [Vinter 2010], etc. A comprehensive survey [Hartl *et al.* 1995] of this field of research has been given in 1995 by Hartl, Sethi and Vickson.

Before being considered in the work [Bourdin & Dhar 2020] presented in Chapter 3 of this manuscript, to the best of our knowledge, optimal sampled-data control problems had never been investigated in the presence of running state constraints. In this work, the first objective was to bridge this gap in the literature by establishing a PMP for general nonlinear optimal sampled-data control problems in the presence of running inequality state constraints. Precisely, the running inequality state constraints are given in the form $h_j(x(t),t) \leq 0$ for all $t \in [0,T]$ where the function $h = (h_j)_{j=1,\dots,q} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^q$ is of sufficient regularity and $q \in \mathbb{N}^*$. In contrast to optimal sampled-data control problems with terminal state constraints (studied for example in Chapter 2 and which can be seen as finite-dimensional optimization problems), note that such problems can be seen as *semi-infinite*-dimensional optimization problems since the presence of running inequality state constraints imposes an infinite number of constraints (one at each instant of time). Similarly to Chapter 2, our strategy for obtaining the PMP in Chapter 3 is to penalize the distance to the state constraints in a corresponding cost functional and then to apply the Ekeland variational principle. To this aim, we invoke results on renorming Banach spaces in order to ensure the regularity of distance functions in the infinite-dimensional context (see Section 1.3.2 in Chapter 1 for details). Moreover, since the running inequality state constraints are described as constraints in the space of continuous functions, we obtain Lagrange multipliers which belong exactly to the dual space of continuous functions. Then, thanks to the Riesz representation theorem, these Lagrange multipliers are characterized by Borel measures associated to functions of bounded variation. Therefore, we obtain that, in the PMP for general nonlinear optimal sampled-data control problems in the presence of running inequality state constraints, the adjoint equation and transversality condition on the adjoint vector have to be replaced by a Cauchy-Stieltjes problem given by

$$\begin{cases} -dp = \nabla_1 H(x,u,p,\cdot) - \sum_{j=1}^q \nabla_1 h_j(x,\cdot)\, d\eta_j & \text{over } [0,T], \\ p(T) = -\nabla g(x(T)), \end{cases} \tag{6}$$

where the Lagrange multipliers $(d\eta_j)_{j=1\dots q}$ are finite nonnegative Borel measures associated to monotonically increasing functions of bounded variation $(\eta_j)_{j=1\dots q}$ satisfying the complementary slackness condition given by

$$\int_0^T h_j(x(t),t)\, d\eta_j(t) = 0, \tag{7}$$

for each $j = 1,\dots,q$. Since we found that the adjoint vector is in general (only) of bounded variation, one would expect to encounter some difficulties when implementing an indirect numerical method due to the possible jumps and singular part of the adjoint vector lying on parts of the optimal trajectory in contact with the boundary of the restricted state space. However, in our context of sampled-data controls and in contrary to the permanent control case, we found that the optimal trajectories have a common behavior which allows us to overcome these difficulties. Precisely, when we began studying optimal sampled-data control problems in the presence of running inequality state constraints in the work [Bourdin & Dhar 2020], we first numerically solved some simple problems using direct methods. Notably we observed that, in each problem, the optimal trajectory "bounces" against the boundary of the restricted state space, touching the state constraints at most at the sampling times. This behavior was the second major focus of the work [Bourdin & Dhar 2020] presented in Chapter 3 and is referred to as the *bouncing trajectory phenomenon*. Precisely, we proved that, under certain general hypotheses, any admissible trajectory (associated to a sampled-data control) necessarily bounces on the running inequality state constraints and, moreover, the rebounds occur at most at the sampling times (and thus are in a finite number and at precise instants). Taking advantage of this bouncing trajectory phenomenon we are able to use the PMP derived in the work [Bourdin & Dhar 2020] in order to implement an indirect numerical method which we use to numerically solve some simple examples of optimal sampled-data control problems with running inequality state constraints (see Section 3.4.1).

Contributions of Chapter 4. Chapter 4 is devoted to obtaining a PMP for optimal sampled-data control problemswith nonsmooth Mayer cost functions. Precisely, contrary to Chapters 2 and 3, the Mayer cost function $g : \mathbb{R}^n \to \mathbb{R}$ is taken to be (only) *locally Lipschitz* that is, at each point $x \in \mathbb{R}^n$, there exists a neighborhood of x on which g is Lipschitz continuous. Let us first mention that a vast literature is already dedicated to nonsmooth optimal control theory (nonsmooth cost functions arising in several natural contexts such as the minimization of a norm associated to a trajectory). We recall that, in the PMP for optimal permanent control problems with nonsmooth Mayer cost functions, the transversality condition on the adjoint vector is usually replaced by

$$-p(T) \in \partial g(x(T)). \tag{8}$$

where $\partial g(x(T))$ stands for the subdifferential (to be specified later) of g at $x(T)$ (see, e.g, [Vinter 2010, Theorem 6.2.3]). Several methods have been explored in order to establish PMPs for nonsmooth optimal permanent control problems. We can cite for example the method of quadratic inf-convolution in [Clarke 2008, Section 2.1 page 4] or the application of a nonsmooth Lagrange multiplier rule in [Vinter 2010, Theorem 5.6.2]. Most of the proofs found in the literature involve regularization methods. On the contrary, in the work [Adly *et al.* 2020] presented in Chapter 4, we were interested in developing a proof which directly follows from the tools of nonsmooth analysis (as presented in Section 1.4 of Chapter 1). Our investigation led us to consider the existence of a universal selection in the subdifferential of g at $x(T)$ which describes the transversality condition on the adjoint vector. Then, in the work [Adly *et al.* 2020], we determined the existence of such a universal selection by establishing a more general result asserting the existence of a universal separating vector for a given compact convex set. From the application of this result, which is called *universal separating vector theorem* (see Theorem 4.2.1 in Chapter 4 for details), we were able to derive a PMP for optimal permanent control problems with nonsmooth Mayer cost functions by a novel approach. Finally we applied again the universal separating vector theorem to obtain a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions which was the initial motivation of our work by answering Question (iii). We obtained that the necessary optimality conditions are the nonpositive averaged Hamiltonian gradient condition (see Inequality (4)), the adjoint equation (see Equation (1)) and the transversality condition on the adjoint vector described by an inclusion in the subdifferential of g at $x(T)$ (see Equation (8)). We emphasize that our novel approach in [Adly *et al.* 2020] for obtaining this PMP was also based on the combination of implicit spike variations and packages of needle-like perturbations of the optimal control.

Organization of the manuscript. This manuscript is composed of 4 chapters. Chapter 1 is devoted to the preliminary notions required throughout this manuscript in order to describe optimal sampled-data control problems. Precisely, we define the functional spaces to be encountered in optimal sampled-data control problems which include absolutely continuous functions, piecewise constant functions and functions of bounded variation. We also recall some results on the regularity of distance functions in order to handle the state constraints in the optimal sampled-data control problems investigated in Chapters 2 and 3. Finally, we give recalls on nonsmooth analysis which will be used when considering optimal sampled-data control problems with nonsmooth Mayer cost functions in Chapter 4.

Chapter 2 is devoted to a PMP for optimal sampled-data control problems with free sampling times (see Theorem 2.2.1). We also give a discussion on the continuity of the Hamiltonian function in Section 2.2.3 . In Section 2.3 we numerically solve two simple linear-quadratic optimal sampled-data control problems and we compare the two following situations: fixed sampling times versus free sampling times. Finally Section 2.4 is devoted to the detailed proof of Theorem 2.2.1.

In Chapter 3, we obtain a PMP for general nonlinear optimal sampled-data control problems in the presence of running inequality state constraints (see Theorem 3.2.1). In Section 3.3 we give heuristic descriptions and a sufficient condition for observing the bouncing trajectory phenomenon. In Section 3.4 we propose an indirect method for numerically solving optimal sampled-data control problems with running inequality state constraints based on Theorem 3.2.1 and with the aid of the bouncing trajectory phenomenon. Then we illustrate this method and highlight the bouncing trajectory phenomenon by numerically solving three simple examples. Finally Section 3.5 is devoted to the proof of Theorem 3.2.1.

Chapter 4 is devoted to obtaining a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions. We first present the problematic of universal separating vector which gives the context of the universal separating vector theorem (see Theorem 4.2.1). Section 4.2 is dedicated to the universal separating vector theorem obtained in [Adly *et al.* 2020] along with its proof. In Section 4.3, we show that it provides an alternative proof of a PMP for nonsmooth optimal permanent control problems with nonsmooth Mayer cost functions which makes direct use of the tools of nonsmooth analysis presented in Chapter 1. Finally Section 4.4 is devoted to a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions which was not presented in [Adly *et al.* 2020] whose proof again applies the universal separating vector theorem.

Finally, in the general conclusion of this manuscript, we review the outcome of the investigations undertaken during this PhD thesis. I also provide several possible perspectives including further personal research projects to be undertaken in the field of optimal sampled-data control theory.

Chapter 1

Preliminaries and notations

This chapter is devoted to the preliminary notions required throughout this manuscript in order to describe optimal sampled-data control problems. In Section 1.1 we define the functional spaces which describe the state functions and control functions to be encountered in optimal sampled-data control problems in Chapters 2, 3 and 4. Thus Section 1.1 will be devoted to basic results on absolutely continuous functions and piecewise constant functions. In Chapter 3 we consider optimal sampled-data control problems in the presence of running inequality state constraints which entails that the corresponding adjoint vectors are not absolutely continuous in general but (only) of bounded variation. Thus, in Section 1.2, we give recalls on functions of bounded variation. Afterwards, in Section 1.3, we recall some results on the regularity of distance functions in order to handle the state constraints in the optimal sampled-data control problems investigated in Chapters 2 and 3. Finally Section 1.4 is devoted to recalls on nonsmooth analysis which will be used when considering optimal sampled-data control problems with nonsmooth Mayer cost functions in Chapter 4.

1.1 Basic functional framework

In this section we give recalls on the basic functional spaces to be used throughout this manuscript. For optimal sampled-data control problems the state function (or trajectory) is absolutely continuous whereas the control function is only piecewise constant. Therefore in Section 1.1.1 we give some recalls on absolutely continuous functions and in Section 1.1.2 we give some recalls on piecewise constant functions. We now begin with the definitions of some other common functional spaces.

Let $n \in \mathbb{N}^*$ be a fixed positive integer and let $T > 0$ be fixed throughout this chapter. We denote by:

- $\mathrm{L}^1([0,T],\mathbb{R}^n)$ the Lebesgue space of integrable functions defined on $[0,T]$ with values in $\mathbb{R}^n$, endowed with its usual norm $\|\cdot\|_{\mathrm{L}^1}$;

- $\mathrm{L}^\infty([0,T],\mathbb{R}^n)$ the Lebesgue space of essentially bounded functions defined on $[0,T]$ with values in $\mathbb{R}^n$, endowed with its usual norm $\|\cdot\|_{\mathrm{L}^\infty}$;

- $\mathrm{C}([0,T],\mathbb{R}^n)$ the space of continuous functions defined on $[0,T]$ with values in $\mathbb{R}^n$, endowed with the standard uniform norm $\|\cdot\|_\infty$;

- $\mathrm{BF}([0,T],\mathbb{R}^n)$ the space of bounded functions defined on $[0,T]$ with values in $\mathbb{R}^n$, endowed with the standard uniform norm $\|\cdot\|_\infty$.

1.1.1 Absolutely continuous functions

In optimal control theory, it is usual to consider that the state function is absolutely continuous. We recall that a function $x : [0,T] \to \mathbb{R}^n$ is said to be absolutely continuous on $[0,T]$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\sum_{j\in J} \|x(b_j) - x(a_j)\|_{\mathbb{R}^n} < \varepsilon$ for any finite collection of disjoint subintervals $\{(a_j, b_j)\}_{j\in J}$ of $[0,T]$ satisfying $\sum_{j\in J} b_j - a_j < \delta$. We denote by $\mathrm{AC}([0,T],\mathbb{R}^n)$ the space of absolutely continuous functions. Moreover we have the following proposition which gives a characterisation of absolutely continuous functions.

Proposition 1.1.1. *Let $t_0 \in [0,T]$ and let $x : [0,T] \to \mathbb{R}^n$. Then $x \in \mathrm{AC}([0,T],\mathbb{R}^n)$ if and only if both of the following conditions are satisfied:*

(i) *x is differentiable almost everywhere on $[0,T]$ and $\dot{x} \in \mathrm{L}^1([0,T],\mathbb{R}^n)$;*

(ii) *It holds that*

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s)\, ds,$$

for all $t \in [0,T]$.

From Proposition 1.1.1 and the fundamental theorem of calculus we deduce the following result on the absolute continuity of primitives.

Proposition 1.1.2. *Let $t_0 \in [0,T]$ and let $y \in \mathrm{L}^1([0,T],\mathbb{R}^n)$. Let x be the function defined on $[0,T]$ by*

$$x(t) = \int_{t_0}^{t} y(s)\, ds,$$

for all $t \in [0,T]$. Then $x \in \mathrm{AC}([0,T],\mathbb{R}^n)$ and $\dot{x} = y$ almost everywhere on $[0,T]$.

1.1.2 Piecewise constant functions

In this manuscript we are interested in sampled-data control systems in which the value of the control is authorized to be modified only a finite number of times and remains frozen elsewhere. For this reason, the control function is described as a piecewise constant function. Thus, this section is devoted to some recalls on piecewise constant functions which will be necessary throughout this manuscript.

For all $N \in \mathbb{N}^*$, the set of all *N-partitions* of the interval $[0,T]$ is defined by

$$\mathcal{P}_N^T := \{\mathbb{T} = \{t_i\}_{i=0,\dots,N} \mid 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\}.$$

Then, for all $N \in \mathbb{N}^*$ and all $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}_N^T$, the set of all *piecewise constant functions over $[0,T]$ respecting the N-partition* $\mathbb{T}$ is defined by

$$\mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m) := \{u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m) \mid \forall i = 0,\dots,N-1, \\ \exists u_i \in \mathbb{R}^m,\ u(t) = u_i \text{ a.e. } t \in [t_i, t_{i+1}]\}.$$

In this paper, as usual in the Lebesgue space $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$, two functions in $\mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ which are equal almost everywhere on $[0,T]$ will be identified. Precisely, if $u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$, then u is identified to the function

$$u(t) = \begin{cases} u_i & \text{if} \quad t \in [t_i, t_{i+1}), \qquad i \in \{0,\dots,N-2\}, \\ u_{N-1} & \text{if} \quad t \in [t_{N-1}, t_N], \end{cases}$$

for all $t \in [0,T]$. Note that $\mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ is a linear subspace of $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$.

Remark 1.1.1. *Note that the inclusion* $\mathrm{PC}^{\mathbb{T}'}([0,T],\mathbb{R}^m) \subset \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ *holds true for all* M*,* $N \in \mathbb{N}^*$ *and all* $\mathbb{T}' \in \mathcal{P}^T_M$*,* $\mathbb{T} \in \mathcal{P}^T_N$ *such that* $\mathbb{T}' \subset \mathbb{T}$ *and* $M \leq N$*.*

Finally, for all $N \in \mathbb{N}^*$, the set of all *piecewise constant functions over* $[0,T]$ *respecting at least one* N*-partition* is defined by

$$\mathrm{PC}_N([0,T],\mathbb{R}^m) := \bigcup_{\mathbb{T} \in \mathcal{P}^T_N} \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m).$$

Note that $\mathrm{PC}_N([0,T],\mathbb{R}^m)$ is included in $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$, but it is not a linear subspace, neither a convex subset. This will present a challenge in order to obtain optimality conditions in Chapter 2 when we consider convex perturbations of the optimal sampled-data control.

Remark 1.1.2. *Similarly to Remark 1.1.1, note that* $\mathrm{PC}_M([0,T],\mathbb{R}^m) \subset \mathrm{PC}_N([0,T],\mathbb{R}^m)$ *for all* M*,* $N \in \mathbb{N}^*$ *such that* $M \leq N$*.*

1.2 Functions of bounded variation and Cauchy-Stieltjes problems

This section is devoted to recalls on functions of bounded variation and Cauchy-Stieltjes problems needed in Chapter 3. In Chapter 3, the optimal sampled-data control problems are subject to running inequality state constraints which are described by constraints in the space of continuous functions. Therefore we obtain Lagrange multipliers which belong exactly to the dual space of continuous functions. Then, thanks to the Riesz representation theorem, the Lagrange multipliers are characterized by measures associated to functions of bounded variation. Finally adjoint vectors are defined as solutions to Cauchy-Stieltjes problems associated to these measures. We begin this section with recalls on functions of bounded variation in Section 1.2.1. We then recall Riemann-Stieltjes integrals in Section 1.2.2 which will be instrumental to describe Cauchy-Stieltjes problems in Section 1.2.3.

1.2.1 Functions of bounded variation

In this section we give recalls on functions of bounded variation which will be encountered in Chapter 3. We refer to standard references and books such as [Bachman & Narici 2000, Burk 2007, Carothers 2000, Faraut 2012, Wheeden & Zygmund 2015] for some more details. Let $q \in \mathbb{N}^*$ be a fixed positive integer. We recall that a function $\eta : [0,T] \to \mathbb{R}^q$ is said to be of *bounded variation* on $[0,T]$ if the total variation given by the formula

$$V(\eta) := \sup_{\{t_i\}_i} \left\{ \sum_i \|\eta(t_{i+1}) - \eta(t_i)\|_{\mathbb{R}^q} \right\},$$

with the supremum taken over all finite partitions $\{t_i\}_i$ of the interval $[0, T]$, is finite. In what follows we denote by $\mathrm{BV}([0, T], \mathbb{R}^q)$ the space of all functions of bounded variation on $[0, T]$ with values in $\mathbb{R}^q$. A function $\eta \in \mathrm{BV}([0, T], \mathbb{R}^q)$ is said to be *normalized* if $\eta(0) = 0_{\mathbb{R}^q}$ and η is right-continuous on $(0, T)$. We denote the subspace of $\mathrm{BV}([0, T], \mathbb{R}^q)$ of all normalized functions of bounded variation by $\mathrm{NBV}([0, T], \mathbb{R}^q)$. We are now in position to state an important theorem on the decomposition of functions of bounded variation which will be essential to the content of Chapter 3.

Theorem 1.2.1. *Let $\eta \in \mathrm{BV}([0, T], \mathbb{R}^q)$. Then η admits a unique decomposition $\eta = \eta_{ac} + \eta_{sc} + \eta_{sa}$ where $\eta_{ac} \in \mathrm{AC}([0, T], \mathbb{R}^q)$ is absolutely continuous, $\eta_{sc} \in \mathrm{C}([0, T], \mathbb{R}^q)$ is singularly continuous (i.e, $\dot{\eta_{sc}} \equiv 0$ a.e. on $[0, T]$) and where η_{sa} defined by $\eta_{sa}(t) = \sum_{t \leq s} \eta(s+) - \eta(s-)$ is a saltus function.*

Proof. We refer to [Carothers 2000, Corollary 20.17 p. 373] for a proof in the case $q = 1$. □

1.2.2 Riemann-Stieltjes integrals

In this section our aim is to recall some notions on integration with respect to measures associated to functions of bounded variation. Recall that the *Riemann-Stieltjes integral* defined by

$$\int_0^T z(t)\, d\eta(t) := \lim \sum_i (\eta(t_{i+1}) - \eta(t_i)) z(t_i),$$

exists in $\mathbb{R}^n$ for all $z \in \mathrm{C}([0, T], \mathbb{R}^n)$ and all $\eta \in \mathrm{BV}([0, T], \mathbb{R})$, where the limit is taken over all finite partitions $\{t_i\}_i$ of the interval $[0, T]$ whose length tends to zero. We recall that the dual space $\mathrm{C}([0, T], \mathbb{R}^n)^*$ of $\mathrm{C}([0, T], \mathbb{R}^n)$ is the space of continuous linear functionals from $\mathrm{C}([0, T], \mathbb{R}^n)$ to $\mathbb{R}$. In the sequel we use the bracket notation $\langle \varphi, z \rangle_{\mathrm{C}^* \times \mathrm{C}} := \varphi(z) \in \mathbb{R}$ for all $\varphi \in \mathrm{C}([0, T], \mathbb{R}^n)^*$ and all $z \in \mathrm{C}([0, T], \mathbb{R}^n)$. We can now state the classical Riesz representation theorem which gives a description of the dual space of $\mathrm{C}([0, T], \mathbb{R})$ in terms of Riemann-Stieltjes integrals.

Theorem 1.2.2. (Riesz representation theorem). *Let $\varphi \in \mathrm{C}([0, T], \mathbb{R})^*$. There exists a unique $\eta \in \mathrm{NBV}([0, T], \mathbb{R})$ such that*

$$\langle \varphi, z \rangle_{\mathrm{C}^* \times \mathrm{C}} = \int_0^T z(t)\, d\eta(t),$$

for all $z \in \mathrm{C}([0, T], \mathbb{R})$. Moreover, $\langle \varphi, z \rangle_{\mathrm{C}^* \times \mathrm{C}} \geq 0$ for all $z \in \mathrm{C}([0, T], \mathbb{R}_+)$ if and only if $\eta \in \mathrm{NBV}([0, T], \mathbb{R})$ is monotonically increasing on $[0, T]$.

We refer to [Limaye 1996, Theorem 14.5 p.245-246] for a complete proof of Theorem 1.2.2. In Chapter 3 we will make use of the following weaker result for which we provide a complete proof.

Proposition 1.2.1. *Let $\varphi \in \mathrm{C}([0, T], \mathbb{R})^*$ such that $\langle \varphi, z \rangle_{\mathrm{C}^* \times \mathrm{C}} \geq 0$ for all $z \in \mathrm{C}([0, T], \mathbb{R}_+)$. Then there exists $\eta \in \mathrm{NBV}([0, T], \mathbb{R})$ such that η is monotonically increasing on $[0, T]$ and*

$$\langle \varphi, z \rangle_{\mathrm{C}^* \times \mathrm{C}} = \int_0^T z(t)\, d\eta(t),$$

for all $z \in \mathrm{C}([0, T], \mathbb{R})$. Moreover $\varphi = 0_{\mathrm{C}([0,T],\mathbb{R})^*}$ if and only if $\eta = 0_{\mathrm{NBV}([0,T],\mathbb{R})}$.

Proof. For the ease of notations, in this proof, we denote by $\mathrm{C} := \mathrm{C}([0,T],\mathbb{R})$, $\mathrm{C}^* := \mathrm{C}([0,T],\mathbb{R})^*$, $\mathrm{BF} := \mathrm{BF}([0,T],\mathbb{R})$ and $\mathrm{BF}^* := \mathrm{BF}([0,T],\mathbb{R})^*$. Moreover, we denote the dual bracket between the spaces $\mathrm{BF}([0,T],\mathbb{R})^*$ and $\mathrm{BF}([0,T],\mathbb{R})$ by $\langle \varphi, z\rangle_{\mathrm{BF}^*\times\mathrm{BF}} := \varphi(z) \in \mathbb{R}$ for all $\varphi \in \mathrm{BF}([0,T],\mathbb{R})^*$ and all $z \in \mathrm{BF}([0,T],\mathbb{R})$.

If $\varphi = 0_{\mathrm{C}^*}$ it suffices to consider $\eta = 0_{\mathrm{NBV}([0,T],\mathbb{R})}$. From now on let $\varphi \in \mathrm{C}^*$ be different from 0_{C^*} and let $\mathbf{1} \in \mathrm{C}$ denote the constant function equal to 1 on $[0,T]$. Since $\langle \varphi, z\rangle_{\mathrm{C}^*\times\mathrm{C}} \geq 0$ for all $z \in \mathrm{C}([0,T],\mathbb{R}_+)$ it holds that $\langle \varphi, \xi\rangle_{\mathrm{C}^*\times\mathrm{C}} \leq \langle \varphi, \mathbf{1}\rangle_{\mathrm{C}^*\times\mathrm{C}}$ for all $\xi \in \mathrm{C}$ such that $\|\xi\|_\infty \leq 1$. Thus $\|\varphi\|_{\mathrm{C}^*} \leq \langle \varphi, \mathbf{1}\rangle_{\mathrm{C}^*\times\mathrm{C}} \leq \|\varphi\|_{\mathrm{C}^*}$ and we obtain that $\|\varphi\|_{\mathrm{C}^*} = \langle \varphi, \mathbf{1}\rangle_{\mathrm{C}^*\times\mathrm{C}}$. From the Hahn-Banach theorem (see [Brezis 2011, Corollary 1.2 p.3]) there exists $\tilde{\varphi} \in \mathrm{BF}^*$ such that $\tilde{\varphi}$ extends φ up to BF and $\|\tilde{\varphi}\|_{\mathrm{BF}^*} = \|\varphi\|_{\mathrm{C}^*}$. Moreover $\|\tilde{\varphi}\|_{\mathrm{BF}^*} = \langle \tilde{\varphi}, \mathbf{1}\rangle_{\mathrm{BF}^*\times\mathrm{BF}}$. We wish to prove that $\langle \tilde{\varphi}, z\rangle_{\mathrm{BF}^*\times\mathrm{BF}} \geq 0$ for every $z \in \mathrm{BF}([0,T],\mathbb{R}_+)$. Let $z \in \mathrm{BF}([0,T],\mathbb{R}_+)$ such that $z \neq 0_{\mathrm{BF}}$. We define $\xi := \frac{2}{\|z\|_\infty} z - 1$. Then $\xi \in \mathrm{BF}$ and $\|\xi\|_\infty \leq 1$. Thus we obtain the inequality, $-\langle \tilde{\varphi}, \xi\rangle_{\mathrm{BF}^*\times\mathrm{BF}} \leq |\langle \tilde{\varphi}, \xi\rangle_{\mathrm{BF}^*\times\mathrm{BF}}| \leq \|\tilde{\varphi}\|_{\mathrm{BF}^*}\|\xi\|_\infty \leq \langle \tilde{\varphi}, \mathbf{1}\rangle_{\mathrm{BF}^*\times\mathrm{BF}}$. We conclude that $\langle \tilde{\varphi}, z\rangle_{\mathrm{BF}^*\times\mathrm{BF}} = \frac{\|z\|_\infty}{2}(\langle \tilde{\varphi}, \xi\rangle_{\mathrm{BF}^*\times\mathrm{BF}} + \langle \varphi, \mathbf{1}\rangle_{\mathrm{BF}^*\times\mathrm{BF}}) \geq 0$.

Let us define $\eta(t) := \langle \tilde{\varphi}, \mathbf{1}_{(0,t]}\rangle_{\mathrm{BF}^*\times\mathrm{BF}}$ for all $t \in [0,T]$. Clearly $\eta(0) = 0$. Moreover since $\mathbf{1}_{(0,t]} - \mathbf{1}_{(0,s]} \in \mathrm{BF}([0,T],\mathbb{R}_+)$ for all $0 \leq s \leq t \leq T$ and since $\langle \tilde{\varphi}, z\rangle_{\mathrm{BF}^*\times\mathrm{BF}} \geq 0$ for every $z \in \mathrm{BF}([0,T],\mathbb{R}_+)$ it follows that η is monotonically increasing on $[0,T]$ and different from $0_{\mathrm{NBV}([0,T],\mathbb{R})}$. Furthermore $\eta \in \mathrm{BV}([0,T],\mathbb{R})$ with $V(\eta) = \eta(T) - \eta(0)$. We wish to prove that $\langle \varphi, z\rangle_{\mathrm{C}^*\times\mathrm{C}} = \int_0^T z(t)\,d\eta(t)$ for all $z \in \mathrm{C}$. Now let $z \in \mathrm{C}$ and $\varepsilon > 0$. Since z is uniformly continuous on $[0,T]$ there exists $\delta > 0$ such that $\|\varphi\|_\infty |z(t) - z(s)| \leq \frac{\varepsilon}{2}$ for all $(t,s) \in [0,T]^2$ such that $|t - s| \leq \delta$. Let $\{t_i\}_i$ be a partition of $[0,T]$ such that $t_{i+1} - t_i \leq \delta$ and such that

$$\left| \int_0^T z(t)\,d\eta(t) - \sum_i z(t_i)(\eta(t_{i+1}) - \eta(t_i)) \right| \leq \frac{\varepsilon}{2}.$$

We define $u := \sum_k z(t_i)\mathbf{1}_{(t_i,t_{i+1}]} \in \mathrm{BF}$. By construction $\langle \tilde{\varphi}, u\rangle_{\mathrm{BF}^*\times\mathrm{BF}} = \sum_k z(t_i)(\eta(t_{i+1}) - \eta(t_i))$. Thus it holds that

$$\left| \int_0^T z(t)\,d\eta(t) - \langle \varphi, z\rangle_{\mathrm{C}^*\times\mathrm{C}} \right| \\ \leq \left| \int_0^T z(t)\,d\eta(t) - \langle \tilde{\varphi}, u\rangle_{\mathrm{BF}^*\times\mathrm{BF}} \right| + |\langle \tilde{\varphi}, u\rangle_{\mathrm{BF}^*\times\mathrm{BF}} - \langle \varphi, z\rangle_{\mathrm{C}^*\times\mathrm{C}}| \leq \varepsilon.$$

As $\varepsilon > 0$ can be taken to be as small as desired, it holds that $\int_0^T z(t)\,d\eta(t) = \langle \varphi, z\rangle_{\mathrm{C}^*\times\mathrm{C}}$ for all $z \in \mathrm{C}$. Moreover $\eta(T) = \eta(T) - \eta(0) = \int_0^T d\eta(t) = \langle \varphi, 1\rangle_{\mathrm{C}^*\times\mathrm{C}} = \|\varphi\|_\infty \neq 0$. Thus $\eta \neq 0_{\mathrm{BV}([0,T],\mathbb{R})}$.

Finally we prove that η can be taken to be right-continuous on $(0,T)$. Since η is monotonically increasing on $[0,T]$, the right-sided $\eta(t^+)$ exists and $\eta(t^+) = \inf\{\eta(\gamma) \mid t < \gamma < T\}$ for all $0 < t < T$. Furthermore it holds that $0 \leq \eta(s) \leq \eta(s^+) \leq \eta(t) \leq \eta(t^+) \leq \eta(T)$, for all $0 \leq s < t \leq T$. Let us define

$$\nu(t) := \begin{cases} 0 & \text{if } t = 0, \\ \eta(t^+) & \text{if } 0 < t < T, \\ \eta(T) & \text{if } t = T. \end{cases}$$

Then it follows that ν is monotonically increasing. We wish to prove that ν is right-continuous on $(0,T)$, that is, $\lim_{s\to t^+}\nu(s)=\nu(t)$ for all $t\in(0,T)$. By construction it holds that

$$\lim_{s\to t^+}\nu(s)=\lim_{s\to t^+}\eta(s^+)=\lim_{s\to t^+}\inf\{\eta(\gamma)\mid s<\gamma<T\}=\inf\{\eta(\gamma)\mid t<\gamma<T\}=\eta(t^+)=\nu(t),$$

for all $t\in(0,T)$. Thus $\nu\in\mathrm{NBV}([0,T],\mathbb{R})$. Let us check that $\int_0^T z(t)\,d\eta(t)=\int_0^T z(t)\,d\nu(t)$ for all $z\in\mathrm{C}$. Since η and ν are monotonically increasing they can differ only at discontinuity points of η (which are at most countable), we consider a sequence of partitions $(\{t_k^\ell\}_k)_\ell$ of $[0,T]$ such that no partition$\{t_k^\ell\}_k$ contains a discontinuity point of η and the length of the partitions tends to zero as ℓ tends to $+\infty$. Therefore

$$\int_0^T z(t)\,d\eta(t)=\lim_{\ell\to\infty}\sum_k z(t_k^\ell)(\eta(t_{k+1}^\ell)-\eta(t_k^\ell))=\lim_{\ell\to\infty}\sum_k z(t_k^\ell)(\nu(t_{k+1}^\ell)-\nu(t_k^\ell))=\int_0^T z(t)\,d\nu(t),$$

as was to be shown. □

We now give some recalls about the Lebesgue-Stieltjes integral. If $\eta\in\mathrm{NBV}([0,T],\mathbb{R})$ is monotonically increasing on $[0,T]$, η induces a finite nonnegative measure defined by $d\eta((a,b]):=\eta(b)-\eta(a)$ on the intervals $(a,b]\subset[0,T]$ for all $0\le a\le b\le T$. Using the Carathéodory extension theorem this measure is extended over the Borel algebra of $[0,T]$ and denoted by $d\eta$. Furthermore for all $z\in\mathrm{C}([0,T],\mathbb{R})$ the Riemann-Stieltjes integral of z with respect to $d\eta$ coincides with the Lebesgue-Stieltjes integral of z with respect to $d\eta$. Finally, the following Fubini-type formula holds:

$$\int_0^T\int_0^t z(t,s)\,ds\,d\eta(t)=\int_0^T\int_s^T z(t,s)\,d\eta(t)\,ds,$$

for all $z\in\mathrm{L}^\infty([0,T]^2,\mathbb{R})$ such that z is continuous in its first variable.

We conclude this section by defining some notations for integrals with respect to bounded variations which will be needed in Chapter 3. Let $q\in\mathbb{N}^*$. For all $(\eta_j)_{j=1,\dots,q}\in\mathrm{NBV}([0,T],\mathbb{R}^q)$ such that η_j is monotonically increasing for all $j=1,\dots,q$ and for all $z=(z_j)_{i=j,\dots,q}\in\mathrm{C}_q$, we denote by

$$\int_0^T\langle z(t),d\eta(t)\rangle_{\mathrm{C}^*\times\mathrm{C}}:=\sum_{j=1}^q\int_0^T z_j(t)\,d\eta_j(t)\in\mathbb{R}.$$

Let $r\in\mathbb{N}$. We denote by

$$\int_0^T A(t)\times d\eta(t):=\left(\sum_{j=1}^q\int_0^T a_{kj}(t)\,d\eta_j(t)\right)_{k=1,\dots,r}\in\mathbb{R}^r,$$

and

$$\int_0^T\langle y(t),A(t)\times d\eta(t)\rangle_{\mathrm{C}^*\times\mathrm{C}}:=\int_0^T\langle A(t)^\top\times y(t),d\eta(t)\rangle_{\mathrm{C}^*\times\mathrm{C}}\in\mathbb{R},$$

for all continuous matrices $A(\cdot)=(a_{kj}(\cdot))_{kj}:[0,T]\to\mathbb{R}^{r\times q}$ and all $y\in\mathrm{C}([0,T],\mathbb{R}^r)$.

1.2.3 About Cauchy-Stieltjes problems

We now give a short recap on linear Cauchy-Stieltjes problems which will be used to describe adjoint vectors in Chapter 3. Let $A \in \mathrm{L}^\infty([0,T],\mathbb{R}^{n\times n})$, $B \in \mathrm{L}^\infty([0,T],\mathbb{R}^{n\times n})$ and let $C_j \in \mathrm{C}([0,T],\mathbb{R}^n)$ and $\eta_j \in \mathrm{BV}([0,T],\mathbb{R})$ for every $j = 1,\dots,q$. We say that $x \in \mathrm{L}^\infty([0,T],\mathbb{R}^n)$ is a *solution* to the *forward linear Cauchy-Stieltjes problem* (FCSP) given by

$$\begin{cases} dx = (A \times x + B)\, dt + \sum_{j=1}^q C_j \, d\eta_j & \text{over } [0,T], \\ x(0) = x_0, \end{cases} \tag{FCSP}$$

where $x_0 \in \mathbb{R}^n$ is fixed, if x satisfies the integral representation

$$x(t) = x_0 + \int_0^t \Big(A(t) \times x(t) + B(t) \Big)\, dt + \sum_{j=1}^q \int_0^t C_j(t)\, d\eta_j(t),$$

for a.e. $t \in [0,T]$. Similarly we say that $p \in \mathrm{L}^\infty([0,T],\mathbb{R}^n)$ is a *solution* to the *backward linear Cauchy-Stieltjes problem* (BCSP) given by

$$\begin{cases} -dp = (A \times p + B)\, dt + \sum_{j=1}^q C_j \, d\eta_j & \text{over } [0,T], \\ p(T) = p_T, \end{cases} \tag{BCSP}$$

where $p_T \in \mathbb{R}^n$ is fixed, if p satisfies the integral representation

$$p(t) = p_T + \int_t^T \Big(A(t) \times p(t) + B(t) \Big)\, dt + \sum_{j=1}^q \int_t^T C_j(t)\, d\eta_j(t),$$

for a.e. $t \in [0,T]$. From usual contraction mapping techniques, one can easily prove that Problems (FCSP) and (BCSP) both admit a unique solution. Moreover, from standard identifications in $\mathrm{L}^\infty([0,T],\mathbb{R}^n)$, these solutions both belong to $\mathrm{BV}([0,T],\mathbb{R}^n)$ and the above integral representations are both satisfied for all $t \in [0,T]$. We refer to [Bourdin 2016, Appendices C and D] and references therein for details.

1.3 Regularity of distance functions

Regularity results for distance functions are required in Chapters 2 and 3 in order to handle state constraints. Indeed our strategy is to penalize the distance to the state constraints in a corresponding cost functional and then to apply the Ekeland variational principle. In particular, in the case of terminal state constraints, we are only concerned with the regularity of distance functions in finite dimensions. Thus Section 1.3.1 is devoted to recalls on convex analysis in order to ensure the regularity of the distance function in the finite-dimensional case. In Chapter 3 we also penalize the running inequality state constraints. However, in this case, the running inequality state constraints are described by constraints in the space of continuous functions. Therefore, Section 1.3.2 is devoted to renorming Banach spaces and techniques required in order to ensure the regularity of distance functions in the infinite-dimensional context.

1.3.1 The finite-dimensional case

This section is devoted to recalls on the regularity of distance functions in the finite-dimensional case which will be used in order to handle terminal state constraints in Chapter 2. Let $\mathrm{S} \subset \mathbb{R}^n$ be a nonempty closed convex subset. We denote by $\mathrm{d}_\mathrm{S} : \mathbb{R}^n \to \mathbb{R}_+$ the standard distance function to S defined as $\mathrm{d}_\mathrm{S}(z) := \inf_{z' \in \mathrm{S}} \|z - z'\|_{\mathbb{R}^n}$ for all $z \in \mathbb{R}^n$. We recall that, for all $z \in \mathbb{R}^n$, there exists a unique element $\mathrm{P}_\mathrm{S}(z) \in \mathrm{S}$ (called the *projection* of z onto S) such that $\mathrm{d}_\mathcal{S}(z) = \|z - \mathrm{P}_\mathrm{S}(z)\|_{\mathbb{R}^n}$. It can easily be shown that the map $\mathrm{P}_\mathrm{S} : \mathbb{R}^n \to \mathrm{S}$ is 1-Lipschitz continuous. Moreover it holds that $\langle z - \mathrm{P}_\mathrm{S}(z), z' - \mathrm{P}_\mathrm{S}(z) \rangle_{\mathbb{R}^n} \leq 0$ for all $z \in \mathbb{R}^n$ and all $z' \in \mathrm{S}$. The *normal cone* to S at a given $z \in \mathrm{S}$ is defined by

$$\mathrm{N}_\mathrm{S}[z] := \{ z' \in \mathbb{R}^n \mid \forall z'' \in \mathrm{S}, \langle z', z'' - z \rangle_{\mathbb{R}^n} \leq 0 \}.$$

In particular, it is a closed convex cone containing $0_{\mathbb{R}^n}$. We end this subsection by recalling the three following useful lemmas.

Lemma 1.3.1. *It holds that* $z - \mathrm{P}_\mathrm{S}(z) \in \mathrm{N}_\mathrm{S}[\mathrm{P}_\mathrm{S}(z)]$ *for all* $z \in \mathbb{R}^n$.

Lemma 1.3.2. *Let* $(z_k)_{k \in \mathbb{N}}$ *be a sequence in* $\mathbb{R}^n$ *converging to some point* $z \in \mathrm{S}$ *and let* $(\zeta_k)_{k \in \mathbb{N}}$ *be a sequence in* $\mathbb{R}_+$. *If* $\zeta_k(z_k - \mathrm{P}_\mathrm{S}(z_k))$ *converges to some* $\overline{z} \in \mathbb{R}^n$, *then* $\overline{z} \in \mathrm{N}_\mathrm{S}[z]$.

In the next lemma we give a result on the regularity of the distance function which will be used in Section 2.4.3 of Chapter 2.

Lemma 1.3.3. *The map*

$$\begin{array}{rccl} \mathrm{d}^2_\mathrm{S} : & \mathbb{R}^n & \longrightarrow & \mathbb{R}_+ \\ & z & \longmapsto & \mathrm{d}^2_\mathrm{S}(z) := \mathrm{d}_\mathrm{S}(z)^2, \end{array}$$

is differentiable on $\mathbb{R}^n$, *and its differential* $D\mathrm{d}^2_\mathrm{S}(z)$ *at every* $z \in \mathbb{R}^n$ *can be expressed as*

$$D\mathrm{d}^2_\mathrm{S}(z)(z') = 2\langle z - \mathrm{P}_\mathrm{S}(z), z' \rangle_{\mathbb{R}^n},$$

for all $z' \in \mathbb{R}^n$.

1.3.2 The infinite-dimensional case

This section is devoted to recalls on the regularity of distance functions in the infinite-dimensional context which will be used in order to handle running inequality state constraints in Chapter 3. In the infinite-dimensional setting, renorming Banach spaces is required to obtain results on the regularity of distance functions. Let $(Z, \|\cdot\|)$ be a normed space. We recall that the *dual space* of $(Z, \|\cdot\|)$, which we denote by $Z^* := \mathcal{L}((Z, \|\cdot\|), \mathbb{R})$, is the space of linear continuous forms on $(Z, \|\cdot\|)$. We recall that Z^* can be endowed with the *dual norm* $\|\cdot\|^*$ defined by

$$\begin{array}{rccl} \|\cdot\|^* : & Z^* & \longrightarrow & \mathbb{R}_+ \\ & z^* & \longmapsto & \|z^*\|^* := \sup\limits_{\substack{z \in Z \\ \|z\| \leq 1}} |\langle z^*, z \rangle_{Z^* \times Z}|. \end{array}$$

In this situation we denote by $(Z^*, \|\cdot\|^*) := \mathrm{dual}(Z, \|\cdot\|)$. We recall the following proposition on renorming separable Banach spaces.

Proposition 1.3.1. *Let* $(Z, \|\cdot\|)$ *be a separable Banach space and let* $(Z^*, \|\cdot\|^*) = \mathrm{dual}(Z, \|\cdot\|)$*. Then there exists a norm* $\mathcal{N}$ *on* Z *equivalent to* $\|\cdot\|$ *such that:*

(i) $\mathcal{N}^*$ *is equivalent to* $\|\cdot\|^*$*;*

(ii) $\mathcal{N}^*$ *is strictly convex;*

where $(Z^*, \mathcal{N}^*) = \mathrm{dual}(Z, \mathcal{N})$*.*

Proof. We refer to [Li & Yong 1995, Theorem 2.18 p.42] or to [Bourdin 2016, Proposition 4 p.16] for a complete proof. □

Let $F : Z \to \mathbb{R}$ be a convex function. Throughout this manuscript we denote the Moreau *subdifferential* of F at a point $z \in Z$ by the set

$$\partial F(z) := \{z^* \in Z^* \mid \langle z^*, z' - z\rangle_{Z^* \times Z} \leq F(z') - F(z) \text{ for all } z' \in Z\}.$$

We recall that a function $F : Z \to \mathbb{R}$ is said to be *strictly Hadamard-differentiable* at a point $z \in Z$ with the strict Hadamard derivative $DF(z) \in Z^*$ if

$$\lim_{\substack{z' \to z \\ t \searrow 0}} \left[\sup_{z'' \in \mathrm{K}} \left| \frac{F(z' + tz'') - F(z')}{t} - \langle DF(z), z''\rangle_{Z^* \times Z} \right| \right] = 0,$$

for every compact set $\mathrm{K} \subset Z$. We refer to [Mordukhovich 2006b, p.312-313] for more details on the Hadamard derivative. Finally we denote by $\mathrm{d}_{\mathcal{S}} : Z \to \mathbb{R}$ the *distance function* to a nonempty subset $\mathcal{S} \subset Y$ defined by $\mathrm{d}_{\mathcal{S}}(z) := \inf_{z' \in \mathcal{S}} \|z - z'\|$ for all $z \in Z$, and by $\mathrm{d}^2_{\mathcal{S}} : Z \to \mathbb{R}$ the *squared distance function* defined by $\mathrm{d}^2_{\mathcal{S}}(z) := \mathrm{d}_{\mathcal{S}}(z)^2$ for all $z \in Z$. We conclude this section by recalling the following proposition on the regularity of distance functions which will be used in Section 3.5.1 of Chapter 3.

Proposition 1.3.2. *Let* $(Z, \|\cdot\|)$ *be a normed space. Let* $\mathcal{S} \subset Z$ *be a nonempty closed convex subset and let us assume that* $\|\cdot\|^*$ *is strictly convex, where* $(Z^*, \|\cdot\|^*) := \mathrm{dual}(Z, \|\cdot\|)$*. Then it holds that:*

(i) $\mathrm{d}_{\mathcal{S}}$ *is convex and* 1*-Lipschitz continuous;*

(ii) $\mathrm{d}_{\mathcal{S}}$ *is strictly Hadamard-differentiable on* $Z \setminus \mathcal{S}$ *with* $\|D\mathrm{d}_{\mathcal{S}}(z)\|^* = 1$ *and* $\partial \mathrm{d}_{\mathcal{S}}(z) = \{D\mathrm{d}_{\mathcal{S}}(z)\}$ *for all* $z \in Z \setminus \mathcal{S}$*;*

(iii) $\mathrm{d}^2_{\mathcal{S}}$ *is strictly Hadamard-differentiable on* $Z \setminus \mathcal{S}$ *with* $D\mathrm{d}^2_{\mathcal{S}}(z) = 2\mathrm{d}_{\mathcal{S}}(z) D\mathrm{d}_{\mathcal{S}}(z)$ *for all* $z \in Z \setminus \mathcal{S}$*;*

(iv) $\mathrm{d}^2_{\mathcal{S}}$ *is Fréchet-differentiable on* $\mathcal{S}$ *with* $D\mathrm{d}^2_{\mathcal{S}}(z) = 0_{Z^*}$ *for all* $z \in \mathcal{S}$*.*

Proof. The proof of (i) is a standard result. We refer to [Mordukhovich 2006b, Theorem 3.54 p.313] and [Bourdin 2016, Appendix B.2] for the proof of (ii). The proofs of (iii) and (iv) are straightforward. □

1.4 Some basics of nonsmooth analysis

Our aim here is to recall some basic notions and results from nonsmooth analysis. These notions will be essential in Chapter 4 where we will consider optimal sampled-data control problems with nonsmooth cost functions. We refer to [Clarke *et al.* 1998, Section 1 in Chapter 2] for more details on the definitions and propositions presented in this subsection. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz* if, at each point $x \in \mathbb{R}^n$, there exists a neighborhood of x on which φ is Lipschitz continuous. In that context the *Clarke generalized directional derivative* of φ at a given point $x \in \mathbb{R}^n$ in a given direction $v \in \mathbb{R}^n$ is defined by

$$\varphi^\circ(x;v) := \limsup_{\substack{y\to x\\ \eta\downarrow 0}} \frac{\varphi(y+\eta v)-\varphi(y)}{\eta}.$$

Then the *Clarke subdifferential* of φ at a point $x \in \mathbb{R}^n$ is defined as the set

$$\partial\varphi(x) := \{y \in \mathbb{R}^n \mid \forall v \in \mathbb{R}^n,\ \langle y,v\rangle_{\mathbb{R}^n} \leq \varphi^\circ(x;v)\}.$$

Here we have kept the same notation for the Clarke subdifferential as for the Moreau subdifferential since for convex functions the two notions coincide. Note that there exist several different notions of subdifferential in the literature such as the proximal subdifferential, the Dini subdifferential, the limiting subdifferential, etc. (see [Clarke 2001] for more details). However in this manuscript we will only make use of the Clarke subdifferential and thus avoid any possible conflict of notation. The next propositions are direct consequences of [Clarke *et al.* 1998, Propositions 1.1 and 1.5 in Chapter 2].

Proposition 1.4.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. It holds that $\varphi^\circ(x;v)$ is finite for all x, $v \in \mathbb{R}^n$. Furthermore, for every $x \in \mathbb{R}^n$, the function $\varphi^\circ(x;\cdot)$ is positively homogeneous and Lipschitz continuous.*

Proposition 1.4.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and $x \in \mathbb{R}^n$. It holds that $\partial\varphi(x)$ is a nonempty compact convex set of $\mathbb{R}^n$. Furthermore, for every $v \in \mathbb{R}^n$, there exists $\zeta_v \in \partial\varphi(x)$ such that*

$$\varphi^\circ(x;v) = \langle \zeta_v, v\rangle_{\mathbb{R}^n} = \max_{\zeta\in\partial\varphi(x)} \langle \zeta, v\rangle_{\mathbb{R}^n}.$$

CHAPTER 2

Optimal sampled-data control problems with free sampling times

This chapter is based on the article "Continuity/constancy of the Hamiltonian function in a Pontryagin maximum principle for optimal sampled-data control problems with free sampling times" by L. Bourdin and G. Dhar (see [Bourdin & Dhar 2019]). An additional numerical example is presented in this chapter which was not found in the paper [Bourdin & Dhar 2019] (see Section 2.3). Several proofs that had been omitted in the paper [Bourdin & Dhar 2019] are also provided in this chapter (see Section 2.4).

2.1 Introduction

This chapter is concerned with necessary optimality conditions for optimal sampled-data control problems where one is allowed to optimize the sampling times of the optimal control as well as the control values. Before giving a formal definition of *optimal sampled-data control problems* we give some recalls on terminology used to describe usual optimal control problems. To begin with, a *control system* is a dynamical system in which a control parameter influences the evolution of the state. An *optimal control problem* consists of determining a control which allows to steer the state of a control system from a specified configuration to some desired target while minimizing a given criterion. Established in [Pontryagin *et al.* 1962] by Pontryagin et al. at the end of the 1950's, the *Pontryagin maximum principle* (in short, PMP) is the milestone of optimal control theory. It provides first-order necessary optimality conditions for optimal control problems in which the dynamics is described by a general nonlinear ordinary differential equation. Roughly speaking, the classical PMP ensures the existence of an adjoint vector such that the optimal control satisfies the so-called *Hamiltonian maximization condition*. Soon afterwards and even nowadays, the PMP has been adapted to many situations, for control systems of different natures, with various constraints, etc. It is not the aim of the present chapter to give a state of the art. Nevertheless we precise that several versions of the PMP were derived for discrete optimal control problems in which the dynamics is described by a difference equation (see, e.g., [Boltyanskii 1978, Halkin 1966, Holtzman & Halkin 1966]). In these discrete versions of the PMP, the historical Hamiltonian maximization condition does not hold in general (see a counterexample in [Boltyanskii 1978, Examples 10.1-10.4 p.59-62]) and has to be replaced by a weaker condition known as a *nonpositive Hamiltonian gradient condition* (see, e.g., [Boltyanskii 1978, Theorem 42.1 p.330]). Note that some appropriate convexity conditions on the dynamics have been considered in order to recover the Hamiltonian maximization condition in the discrete case (see, e.g., [Holtzman & Halkin 1966]).

In the classical optimal control theory the control is often taken to be *permanent* in the sense that its value can be modified at any instant in time. In this chapter we are interested in *sampled-data control systems* in which the state evolves continuously in time while the control evolves discretely in time. More precisely the value of the control is authorized to be modified only a finite number of times. The times in which the control can be modified are usually called the *sampling times*. Note that sampled-data control systems have the peculiarity of presenting a mixed continuous/discrete structure. They have been considered as models mostly in Engineering implemented by *digital controllers* which have a finite precision (see, e.g., [Santina & Stubberud 2005, Volz & Kazda 1966]). Numerous texts and articles have developed control theory for sampled-data control systems (see, e.g., [Ackermann 1985, Aström 1963, Aström & Wittenmark 1997, Fadali & Visioli 2013, Landau & Zito 2006] and references therein). For instance, global controllability for sampled-data control systems has been investigated in [Grasse & Sussmann 1990]. Sampled-data control systems are used in Automation, notably in model predictive control algorithms in which the control value at each sampling time is chosen as the first value of a finite sequence of control values optimizing the given cost on a fixed finite horizon (see, e.g., [Grüne & Pannek 2017]). Optimal sampled-data control problems have been investigated in the literature with different approaches. One approach has been to apply $\mathcal{H}_2$-$\mathcal{H}_\infty$ optimization theory (see [Biryukov 2016, Chen & Francis 1996]) where the closed-loop transfer matrix under the $\mathcal{H}_2$- and $\mathcal{H}_\infty$- norms is taken as the criterion. Another approach involves the Karush-Kuhn-Tucker necessary conditions and dynamic programming (see [Bini & Buttazzo 2014]). However one should note that the aforementioned results are not formulated in terms of a PMP. Recently Bourdin and Trélat have obtained in [Bourdin & Trélat 2016] a version of the PMP for general nonlinear optimal sampled-data control problems. In that sampled-data control framework, as in the purely discrete case addressed in the previous paragraph, the usual Hamiltonian maximization condition does not hold in general and has to be replaced by a weaker condition known as a *nonpositive averaged Hamiltonian gradient condition* (see [Bourdin & Trélat 2016, Theorem 2.6 p.62]). Note that the PMP enunciated in [Bourdin & Trélat 2016, Theorem 2.6 p.62] is actually stated in the more general framework of *time scale calculus* and a version which does not take into account such a generality, and therefore closer to the considerations of the present chapter, can be found in [Bourdin & Trélat 2015, Theorem 1 p.81] or [Bourdin & Trélat 2016, Theorem 1.1 p.55]. Unfortunately this PMP is only concerned with fixed sampling times, and thus it does not take into account the possibility of free sampling times that can be chosen from a given interval. The main objective of the paper [Bourdin & Dhar 2019] in collaboration with Bourdin, whose content is presented in this chapter, was to fill this gap in the literature by deriving a PMP for general nonlinear optimal sampled-data control problems with free sampling times. We mention that *optimal sampling times problems* have already been investigated in the literature but, to the best of our knowledge, never from a PMP point of view. For example many authors consider the related problem of finding the optimal fixed sampling interval (or time step) such as in [Levis & Schlueter 1971, Melzer & Kuo 1971]. Nonuniform sampling partitions have also been studied but in specific cases such as for the linear-quadratic integrator in [Schlueter 1973]. In [Schlueter & Levis 1973] the optimal sampled-data control problem is transformed into a purely discrete one by integrating the state over the sampling intervals and then is treated as an usual optimization problem.

The main theoretical result of the paper [Bourdin & Dhar 2019] (presented in Theorem 2.2.1 in Section 2.2.2 of the present chapter) is a PMP for nonlinear optimal sampled-data control problems with free sampling times. Similarly to the PMP derived in [Bourdin & Trélat 2015, Theorem 1 p.81] or [Bourdin & Trélat 2016, Theorem 1.1 p.55] for fixed sampling times, we obtain a first-order necessary optimality condition described by a nonpositive averaged Hamiltonian gradient condition (see Inequality (2.2)). Furthermore, from the freedom of choosing sampling times, we get a new and additional necessary optimality condition (see Equality (2.3)) which happens to coincide with the continuity of the Hamiltonian function. In an autonomous context, even the constancy of the Hamiltonian function can be derived. We refer to Section 2.2.3 for a detailed discussion on the continuity/constancy of the Hamiltonian function. Moreover, in case of additional constraints on the size of sampling intervals (in practice one can expect a minimum size for instance), the continuity of the Hamiltonian function is replaced by a weaker inequality (see Remarks 2.2.14 and 2.2.15 for details).

We must remark that in the classical case of optimal permanent control problems, the (absolute) continuity of the Hamiltonian function is a very well-known fact (see, e.g., [Fattorini 1999, Theorem 2.6.3 p.73]). With the help of two simple linear-quadratic examples, we show in Section 2.3 that this classical property does not hold in general for optimal sampled-data control problems with fixed sampling times (see Figures 2.1 and 2.3). On the other hand, the present work proves that this continuity property is recovered when considering optimal sampling times, which is illustrated with the same aforementioned linear-quadratic examples (see Figures 2.2 and 2.4). In the second linear-quadratic example, which is autonomous, even the constancy of the Hamiltonian function is obtained when considering optimal sampling times (see Figure 2.4). Furthermore the linear-quadratic examples developed in Section 2.3 allow us to prove the interest of our main result since it is numerically solved by using, on one hand, the Riccati theory developed in [Bourdin & Trélat 2017, Theorem 2 and Corollary 1 p.276] and, on the other hand, a shooting method based on the Hamiltonian continuity condition derived in Theorem 2.2.1. We conclude this paragraph by mentioning that, in the context of hybrid optimal control problems, a similar Hamiltonian continuity condition at crossing times (resp. at switching times) can be found in [Haberkorn & Trélat 2011, Remark 1.3] by Haberkorn and Trélat (resp. in [Sussmann 1999, Definition 13] by Sussmann under the name of *Hamiltonian value condition*). Nevertheless, due to the nature of the sampling times and of the sampled-data controls considered in the present chapter, our main result (in particular the nonpositive averaged Hamiltonian gradient condition) cannot, to the best of our knowledge, be seen as a direct consequence of the works [Haberkorn & Trélat 2011, Sussmann 1999].

In this paragraph our aim is to give some details about the strategy adopted in the paper [Bourdin & Dhar 2019] and the major difficulties encountered. The proof of our main result is detailed in Section 2.4 and, similarly to [Bourdin & Trélat 2016, Theorem 2.6 p.62], it is based on the classical Ekeland variational principle [Ekeland 1974, Theorem 1.1 p.324]. This leads us to consider a sequence of sampled-data controls converging in L^1-norm to the optimal one. A first difficulty emerges in the fact that the associated sampling times do not necessarily converge to the optimal sampling times. Indeed a degenerate situation can occur if the optimal control is constant over two consecutive sampling intervals. Moreover another obstacle is the possible phenomenon of accumulation of sampling times. These two difficulties are overcome by introducing a technical control set (see Section 2.4.2) which guarantees that the sampling times produced by the Ekeland variational principle, firstly, remain unchanged for the ones corresponding to the consecutive sampling intervals on which the optimal control is constant (avoiding thus the first difficulty) and, secondly, are contained in disjoint intervals for the others (avoiding thus the second difficulty). We refer to Proposition 2.4.7 for details. A final obstacle lies in the non-convexity of the set of N-piecewise constant functions (where $N \in \mathbb{N}^*$ is fixed). Therefore the standard procedure of considering convex L^∞-perturbations of the control (as in [Bourdin & Trélat 2016, Lemma 4.17 p.84]) has to be adapted by considering convex L^∞-perturbations respecting the same N-partition.

This chapter is organized as follows. Section 2.2 is dedicated to the main result of the paper [Bourdin & Dhar 2019] (see Theorem 2.2.1). The optimal sampled-data control problem considered is presented in detail in Section 2.2.1 (see Problem (OSCP)). The corresponding Pontryagin maximum principle (Theorem 2.2.1) is stated in Section 2.2.2 and a list of general comments is given. Finally Section 2.2.3 is devoted to a discussion on the continuity/constancy of the Hamiltonian function. In Section 2.3 we numerically solve a simple linear-quadratic optimal sampled-data control problem and we compare the two following situations: fixed sampling times versus free sampling times. As expected from our main result, the Hamiltonian function admits discontinuities in the first case (see Figure 2.1), while it does not in the second case (see Figure 2.2). Finally Section 2.4 is devoted to the detailed proof of Theorem 2.2.1.

2.2 Main result and comments

This section is devoted to the main result of the paper [Bourdin & Dhar 2019]. In Section 2.2.1 we present the optimal sampled-data control problem. In Section 2.2.2, the corresponding Pontryagin maximum principle, which constitutes our main result, is stated. A list of general comments is in order. Section 2.2.3 is devoted to a discussion on the continuity/constancy of the Hamiltonian function.

2.2.1 The optimal sampled-data control problem: terminology and assumptions

Let m, n, j, $N \in \mathbb{N}^*$ be four positive integers fixed in the whole chapter. In the present chapter we focus on the general optimal sampled-data control problem (OSCP) given by

$$\left.\begin{array}{ll} \text{minimize} & g(x(0),x(T),T) + \displaystyle\int_0^T L(x(t),u(t),t)\,dt, \\ \text{subject to} & T>0 \text{ fixed or free}, \\ & \mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^T_N \text{ fixed or free}, \\ & x \in \mathrm{AC}([0,T],\mathbb{R}^n),\ u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m), \\ & \dot{x}(t) = f(x(t),u(t),t), \quad \text{a.e. } t \in [0,T], \\ & h(x(0),x(T),T) \in \mathrm{S}, \\ & u_i \in \mathrm{U}, \quad \text{for all } i = 0,\dots,N-1. \end{array}\right\} \qquad \text{(OSCP)}$$

A solution to Problem (OSCP) is a quadruple $(T,\mathbb{T},x,u)$ which satisfies all above constraints and which minimizes the cost among all quadruples satisfying these constraints. Our aim in this section is to fix the terminology and the assumptions associated to Problem (OSCP).

In Problem (OSCP), x is the *state* function (also called *trajectory*) and u is the *control* function. In the classical literature about the Pontryagin maximum principle (see, e.g., [Bressan & Piccoli 2007, Cesari 1983a, Hiriart-Urruty 2008, Pontryagin *et al.* 1962, Sethi & Thompson 2000, Trélat 2005, Vinter 2010] and references therein), the control u usually can be any function in $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$, satisfying the constraint $u(t) \in \mathrm{U}$ for almost every $t \in [0,T]$. In that case we say that the control is *permanent* in the sense that its value can be modified at any time $t \in [0,T]$. In the present chapter, the control u is constrained to be a piecewise constant function respecting at least one N-partition, where $N \in \mathbb{N}^*$ is fixed. In other words, the value of the control is authorized to be modified at most $N-1$ times. In that situation we say that the control is *nonpermanent*. The standard terminology adopted in the literature is to say that the control u in Problem (OSCP) is a *sampled-data control* (see, e.g., [Bini & Buttazzo 2014, Bourdin & Trélat 2015, Bourdin & Trélat 2016, Bourdin & Trélat 2017] and references therein).

In Problem (OSCP), the *final time* $T > 0$ can be fixed or not. In the case where the final time is free, it becomes a parameter to optimize. Similarly the N-partition $\mathbb{T} = \{t_i\}_{i=0,\dots,N}$ can be fixed or not in Problem (OSCP). For $i = 1,\dots,N-1$, the elements t_i of $\mathbb{T}$ are called the *sampling times* because they correspond to the times in which the value of the sampled-data control u can be modified. We distinguish two situations:

(i) If the N-partition is fixed in Problem (OSCP), we say that the sampling times t_i are fixed and Problem (OSCP) is an *optimal sampled-data control problem with fixed sampling times*;

(ii) If the N-partition is free in Problem (OSCP), we say that the sampling times t_i are free and they become $N-1$ parameters to optimize. In that case, Problem (OSCP) is said to be an *optimal sampled-data control problem with free sampling times.*

In this chapter we consider the following regularity and topology assumptions:

- the functions $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ and $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$, that describe respectively the *Mayer cost* $g(x(0),x(T),T)$ and the *Lagrange cost* $\int_0^T L(x(t),u(t),t)\,dt$, are of class C^1;

- the set $\mathrm{S} \subset \mathbb{R}^j$ is a nonempty closed convex subset of $\mathbb{R}^j$ and the function $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^j$, that describes the *terminal state constraint* $h(x(0),x(T),T) \in \mathrm{S}$, is of class C^1;

- the set $\mathrm{U} \subset \mathbb{R}^m$, that describes the *control constraint* $u_i \in \mathrm{U}$, is a nonempty closed convex subset of $\mathbb{R}^m$;

- the *dynamics* $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$, that drives the *state equation* $\dot{x}(t) = f(x(t),u(t),t)$, is of class C^1. In particular, for every compact subset $\mathrm{K} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$, there exists a nonnegative constant $C_\mathrm{K} \geq 0$ such that $\|\nabla_1 f(x,u,t)\|_{\mathbb{R}^{n\times n}} \leq C_\mathrm{K}$, $\|\nabla_2 f(x,u,t)\|_{\mathbb{R}^{n\times m}} \leq C_\mathrm{K}$ for all $(x,u,t) \in \mathrm{K}$, and such that
$$\|f(x_2,u_2,t) - f(x_1,u_1,t)\|_{\mathbb{R}^n} \leq C_\mathrm{K}(\|x_2 - x_1\|_{\mathbb{R}^n} + \|u_2 - u_1\|_{\mathbb{R}^m}), \tag{2.1}$$
for all (x_1,u_1,t), $(x_2,u_2,t) \in \mathrm{K}$. Throughout this chapter, we use the notation ∇ to denote the derivative of a mapping with respect to a variable.

Since the total cost $g(x(0),x(T),T) + \int_0^T L(x(t),u(t),t)\,dt$ considered in Problem (OSCP) is written as the sum of a Mayer cost and a Lagrange cost, it is said to be of *Bolza form*.

Remark 2.2.1. *In Problem (OSCP), since the N-partition depends on the final time, note that the problem has no sense if the final time is free while the N-partition is fixed. This case will not be treated. In order to deal with it, one should introduce a different framework that can handle partitions that are independent of the final time. It was not the aim of the paper [Bourdin & Dhar 2019] in which the objective was to focus on the free sampling times case.*

Remark 2.2.2. *In the free sampling times case, Problem (OSCP) can be rewritten by removing the third line, and by replacing "$u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$" in the fourth line by "$u \in \mathrm{PC}_N([0,T],\mathbb{R}^m)$".*

Remark 2.2.3. *Let $(T,\mathbb{T},x,u)$ be a solution to Problem (OSCP). One can easily deduce, respectively from Remarks 1.1.1 and 1.1.2, that:*

- *if the sampling times are fixed in Problem (OSCP) and $u \in \mathrm{PC}^{\mathbb{T}'}([0,T],\mathbb{R}^m)$ for some $\mathbb{T}' \in \mathcal{P}^T_M$ with $M \leq N$ such that $\mathbb{T}' \subset \mathbb{T}$, then the quadruple $(T,\mathbb{T}',x,u)$ is a solution to the same problem as Problem (OSCP) replacing N by M and $\mathbb{T}$ by $\mathbb{T}'$. A similar remark was already done in [Bourdin & Trélat 2016, Remark 3 p.60];*

- *if the sampling times are free in Problem (OSCP) and $u \in \mathrm{PC}^{\mathbb{T}'}([0,T],\mathbb{R}^m)$ for some $\mathbb{T}' \in \mathcal{P}^T_M$ with $M \leq N$, then the quadruple $(T,\mathbb{T}',x,u)$ is a solution to the same problem as Problem (OSCP) replacing N by M.*

Remark 2.2.4. *If the final time is fixed in Problem (OSCP), one can directly consider that the two functions g and h are both independent of T, and we directly write the Mayer cost as $g(x(0),x(T))$ and the terminal state constraint as $h(x(0),x(T)) \in \mathrm{S}$.*

Remark 2.2.5. *Let $(T,\mathbb{T},x,u)$ be a quadruple satisfying all constraints of Problem (OSCP). From the state equation and since u is a piecewise constant function, it is clear that x is not only absolutely continuous but also piecewise smooth of class C^1 over the interval $[0,T]$, in the sense that x is of class C^1 over each interval $[t_i,t_{i+1}]$.*

Remark 2.2.6. *A Filippov-type theorem for the existence of a solution to Problem (OSCP) in case of fixed sampling times was derived in [Bourdin & Trélat 2016, Theorem 2.1 p.61]. The paper [Bourdin & Dhar 2019] focuses only on necessary optimality conditions and thus it was not our aim to discuss the extension of the previously mentioned result to the case of free sampling times. Nevertheless we precise that, in the context of free sampling times, one would likely be faced with the same difficulty encountered in the proof of Theorem 2.2.1 developed in Section 2.4. Precisely, considering a minimizing sequence of sampled-data controls would lead to a sequence of partitions and thus to the possibility of accumulation of sampling times. As a consequence, a cautious and rigorous mathematical treatment would be required in order to give a meaning to the limit of the sequence of sampled-data controls when accumulations of sampling times appear. Moreover, note that the standard Filippov's theorem is usually established in case of permanent controls, that is, with controls that belong to the infinite dimensional space $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$, while the sampled-data control framework considered here (with fixed or free sampling times) can be seen as a finite dimensional optimization problem. This fundamental difference could potentially lead to existence results in case of sampled-data controls without the convexity assumption made on the augmented velocity set in the case of permanent controls.*

2.2.2 Pontryagin maximum principle and general comments

We recall that the main objective of the paper [Bourdin & Dhar 2019] was to state a Pontryagin maximum principle for Problem (OSCP). As mentioned in the previous section, one of the novelties of Problem (OSCP) with respect to the classical literature is to consider sampled-data controls. Note that this framework was already considered by Bourdin and Trélat in [Bourdin & Trélat 2015, Bourdin & Trélat 2016] in which a Pontryagin maximum principle was already established. However, in contrary to the framework considered in [Bourdin & Trélat 2015, Bourdin & Trélat 2016], the sampling times t_i in Problem (OSCP) are not necessarily fixed and can be free. Hence the major contribution of the paper [Bourdin & Dhar 2019] was to state a Pontryagin maximum principle that can handle, not only sampled-data controls, but also free sampling times. In that particular case, a new necessary optimality condition is derived (see Equality (2.3) in Theorem 2.2.1 below). This additional necessary optimality condition happens to coincide with the continuity of the Hamiltonian function. A discussion devoted to this phenomenon is provided in Section 2.2.3. Before recalling our main result, we first need to recall the following definitions.

Definition 2.2.1 (Hamiltonian). *The Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *associated to Problem* (OSCP) *is defined by* $H(x,u,p,p^0,t) := \langle p, f(x,u,t)\rangle_{\mathbb{R}^n} + p^0 L(x,u,t)$, *for all* $(x,u,p,p^0,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$.

Definition 2.2.2 (Submersiveness). *We say that the function* h *is submersive at a point* $(x_1,x_2,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ *if its differential at this point, that is, if*

$$Dh(x_1,x_2,t) = \begin{pmatrix} \nabla_1 h(x_1,x_2,t) & \nabla_2 h(x_1,x_2,t) & \nabla_3 h(x_1,x_2,t) \end{pmatrix} \in \mathbb{R}^{j\times(2n+1)},$$

is surjective.

We are now in a position to recall the main result established in the paper [Bourdin & Dhar 2019].

Theorem 2.2.1 (Pontryagin maximum principle). *Let* $(T,\mathbb{T},x,u)$ *be a solution to Problem* (OSCP). *If* h *is submersive at* $(x(0),x(T),T)$, *then there exists a nontrivial couple* $(p,p^0) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathbb{R}_-$ *such that:*

(i) ***Adjoint equation:*** *p (which is called the adjoint vector or the costate) satisfies*

$$-\dot{p}(t) = \nabla_1 H(x(t),u(t),p(t),p^0,t),$$

for almost every $t \in [0,T]$;

(ii) ***Transversality conditions on the adjoint vector:*** *p satisfies*

$$\begin{aligned} -p(0) &= p^0 \nabla_1 g(x(0),x(T),T) + \nabla_1 h(x(0),x(T),T)^\top \times \Psi, \\ p(T) &= p^0 \nabla_2 g(x(0),x(T),T) + \nabla_2 h(x(0),x(T),T)^\top \times \Psi, \end{aligned}$$

where $\Psi \in \mathbb{R}^j$ *is such that* $-\Psi \in \mathrm{N}_{\mathrm{S}}[h(x(0),x(T),T)]$ *(where* $\mathrm{N}_{\mathrm{S}}[h(x(0),x(T),T)]$ *stands for the normal cone of* S *at* $h(x(0),x(T),T)$ *defined in Section 1.3);*

(iii) ***Nonpositive averaged Hamiltonian gradient condition:*** *the condition*

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t), u_i, p(t), p^0, t)\, dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0, \tag{2.2}$$

is satisfied for all $\omega \in \mathrm{U}$ *and all* $i = 0, \dots, N-1$*;*

(iv) ***If moreover the sampling times are free in Problem*** (OSCP)***:*** *the optimal sampling times* t_i *satisfy the Hamiltonian continuity condition*

$$H(x(t_i), u_{i-1}, p(t_i), p^0, t_i) = H(x(t_i), u_i, p(t_i), p^0, t_i), \tag{2.3}$$

for all $i = 1, \dots, N-1$*;*

(v) ***If moreover the final time is free in Problem*** (OSCP)***:*** *the optimal final time* T *satisfies the transversality condition*

$$-H(x(T), u_{N-1}, p(T), p^0, T) = p^0 \nabla_3 g(x(0), x(T), T) + \nabla_3 h(x(0), x(T), T)^\top \times \Psi,$$

where $\Psi \in \mathbb{R}^j$ *is introduced in the transversality conditions on the adjoint vector.*

Section 2.4 is dedicated to the proof of Theorem 2.2.1. A list of comments is in order. We just point out, as detailed in Remark 2.2.12 below, that the submersion property considered in Theorem 2.2.1 is not restrictive. The reader who is interested in the continuity/constancy of the Hamiltonian function may jump directly to the specific Section 2.2.3.

Remark 2.2.7. *The nontrivial couple* (p, p^0) *in Theorem 2.2.1, which is a Lagrange multiplier, is defined up to a positive multiplicative scalar. In the normal case* $p^0 \neq 0$*, it is usual to normalize the Lagrange multiplier so that* $p^0 = -1$*.*

Remark 2.2.8. *Let us consider the framework of Theorem 2.2.1. One can easily see that the couple* (x, p) *satisfies the Hamiltonian system*

$$\dot{x}(t) = \nabla_3 H(x(t), u(t), p(t), p^0, t), \qquad -\dot{p}(t) = \nabla_1 H(x(t), u(t), p(t), p^0, t),$$

for almost every $t \in [0, T]$*.*

Remark 2.2.9. *Our strategy in Section 2.4 in order to prove Theorem 2.2.1 is based on the Ekeland variational principle [Ekeland 1974, Theorem 1.1 p.324]. It requires the closedness of* U *in order to define the corresponding penalized functional on a complete metric set (see details in Section 2.4.3). The closure of* U *is thus a crucial assumption in our strategy. On the other hand, the convexity of* U *is also an essential hypothesis for our strategy in order to consider convex* L^∞*-perturbation of the control (see the proof of Lemma 2.4.7).*

Remark 2.2.10. *The nonpositive averaged Hamiltonian gradient condition in Theorem 2.2.1 (see Inequality* (2.2)*) can be rewritten as*

$$\int_{t_i}^{t_{i+1}} \nabla_2 H(x(t), u_i, p(t), p^0, t)\, dt \in \mathrm{N_U}[u_i],$$

for all $i = 0, \ldots, N-1$. We deduce that

$$u_i = \mathrm{proj}_{\mathrm{U}}\left(u_i + \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t), u_i, p(t), p^0, t)\, dt \right),$$

for all $i = 0, \ldots, N-1$, where $\mathrm{proj}_{\mathrm{U}}$ stands for the classical projection operator onto U. In particular, if $\mathrm{U} = \mathbb{R}^m$ (that is, if there is no control constraint in Problem (OSCP)), then the nonpositive averaged Hamiltonian gradient condition can be rewritten as

$$\int_{t_i}^{t_{i+1}} \nabla_2 H(x(t), u_i, p(t), p^0, t)\, dt = 0_{\mathbb{R}^m},$$

for all $i = 0, \ldots, N-1$.

Remark 2.2.11. *In this remark, for simplicity, we suppose that the final time is fixed in Problem (OSCP). Our aim here is to describe some typical terminal state constraint $h(x(0), x(T)) \in \mathrm{S}$ and the corresponding transversality conditions on the adjoint vector derived in Theorem 2.2.1:*

- *If the terminal points are fixed in Problem (OSCP) (that is, $x(0) = x_0$ and $x(T) = x_f$), one may consider $j = 2n$, h as the identity function and $\mathrm{S} = \{x_0\} \times \{x_f\}$. In that case, the normal cone to S is the entire space, and thus the transversality conditions on the adjoint vector in Theorem 2.2.1 do not provide any additional information.*

- *If the initial point is fixed (that is, $x(0) = x_0$) and the final point $x(T)$ is free in Problem (OSCP), one may consider $j = 2n$, h as the identity function and $\mathrm{S} = \{x_0\} \times \mathbb{R}^n$. In that case, the nontriviality of the couple (p, p^0) and the second transversality condition on the adjoint vector in Theorem 2.2.1 imply that $p^0 \neq 0$ (which we normalize to $p^0 = -1$, see Remark 2.2.7) and $p(T) = -\nabla_2 h(x(0), x(T))$.*

- *If the initial point is fixed (that is, $x(0) = x_0$) and the final point $x(T)$ is subject to inequality constraints $h_\ell(x(T)) \geq 0$ for $\ell = 1, \ldots, n_h$, for some $n_h \in \mathbb{N}^*$, one may consider $j = n + n_h$, $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n+n_h}$ given by $h(x_1, x_2) := (x_1, h'(x_2))$ where $h' = (h_1, \ldots, h_{n_h}) : \mathbb{R}^n \to \mathbb{R}^{n_h}$ and $\mathrm{S} = \{x_0\} \times (\mathbb{R}_+)^{n_h}$. If h' is of class C^1 and the differential $Dh'(x_2) \in \mathbb{R}^{n_h \times n}$ is surjective at any point $x_2 \in h'^{-1}((\mathbb{R}_+)^{n_h})$, then the second transversality condition on the adjoint vector in Theorem 2.2.1 can be written as $p(T) = p^0 \nabla_2 g(x(0), x(T)) + \sum_{\ell=1}^{n_h} \lambda_\ell \nabla h_\ell(x(T))$, for some $\lambda_\ell \geq 0$ satisfying moreover the slackness condition $\lambda_\ell h_\ell(x(T)) = 0$, for all $\ell = 1, \ldots, n_h$.*

- *If there is no Mayer cost (that is, $g = 0$) and the periodic condition $x(0) = x(T)$ is considered in Problem (OSCP), one may consider $j = n$, $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $h(x_1, x_2) := x_2 - x_1$ and $\mathrm{S} = \{0_{\mathbb{R}^n}\}$. In that case the transversality conditions on the adjoint vector in Theorem 2.2.1 yield that $p(0) = p(T)$.*

We point out that, in all examples above, the submersiveness condition is satisfied.

Remark 2.2.12. *Let $(T, \mathbb{T}, x, u)$ be a solution to Problem (OSCP). If the submersion property is not satisfied, one can easily go back to the submersive case by noting that $(T, \mathbb{T}, x, u)$ is also a solution to the same problem as Problem (OSCP) replacing j by $\tilde{j} := 2n+1$, h by the identity function $\tilde{h}$ and S by the singleton $\tilde{\mathrm{S}} := \{x(0)\} \times \{x(T)\} \times \{T\}$. With this new problem the* *submersion property is obviously satisfied and Theorem 2.2.1 can be applied. However, with this new problem, the normal cone to* $\tilde{\mathrm{S}}$ *is the entire space, and thus the transversality conditions on the adjoint vector and on the final time do not provide any information. In other words, if the submersion property is not satisfied, then Theorem 2.2.1 is still valid by removing the two items* (ii) *and* (v)*.*

Remark 2.2.13. *Following the proof of Theorem 2.2.1 in Section 2.4, one can easily see that the theorem is still valid for a quadruple* $(T, \mathbb{T}, x, u)$ *which is solution to Problem* (OSCP) *in (only) a local sense to be precised.*

Remark 2.2.14. *In the case of free sampling times in Problem* (OSCP)*, one may be interested by the additional constraint* $t_{i+1} - t_i \geq \delta_{\min}$ *for all* $i = 0, \ldots, N-1$*, for some* $\delta_{\min} > 0$ *fixed. Following the proof of Theorem 2.2.1 in Section 2.4, one can easily see that Equality* (2.3) *is preserved for all* $i \in \{1, \ldots, N-1\}$ *such that* $\min(t_i - t_{i-1}, t_{i+1} - t_i) > \delta_{\min}$*, but has to be replaced by the weaker condition*

$$H(x(t_i), u_{i-1}, p(t_i), p^0, t_i) \leq H(x(t_i), u_i, p(t_i), p^0, t_i),$$

for all $i \in \{1, \ldots, N-1\}$ *such that* $t_i - t_{i-1} = \delta_{\min}$ *and* $t_{i+1} - t_i > \delta_{\min}$*, and by the weaker condition*

$$H(x(t_i), u_{i-1}, p(t_i), p^0, t_i) \geq H(x(t_i), u_i, p(t_i), p^0, t_i),$$

for all $i \in \{1, \ldots, N-1\}$ *such that* $t_i - t_{i-1} > \delta_{\min}$ *and* $t_{i+1} - t_i = \delta_{\min}$*. However, if* $t_i - t_{i-1} = t_{i+1} - t_i = \delta_{\min}$*, then no necessary optimality condition on* t_i *can be derived from our strategy in Section 2.4.*

Remark 2.2.15. *Remark 2.2.14 can be easily adapted to the case of the additional constraint* $t_{i+1} - t_i \leq \delta_{\max}$ *for all* $i = 0, \ldots, N-1$*, for some* $\delta_{\max} > 0$ *fixed. One can also obtain a similar remark as in Remark 2.2.14 by considering the additional constraint* $\delta_{\min} \leq t_{i+1} - t_i \leq \delta_{\max}$ *for all* $i = 0, \ldots, N-1$*, for some* $0 < \delta_{\min} < \delta_{\max}$ *fixed.*

Remark 2.2.16. *This comment highlights several research perspectives.*

(i) *In the context of linear-quadratic problems, the authors of [Bourdin & Trélat 2017] prove that the optimal sampled-data controls (with fixed sampling times) converge pointwisely to the optimal permanent control when the lengths of sampling intervals tend uniformly to zero. The convergence of the corresponding costs and the uniform convergence of the corresponding states and costates are also derived. An interesting research perspective would be to get similar convergence results in the context of the present work. Several directions can be investigated: nonlinear dynamics, terminal state constraints, free sampling times (whereas sampling times are fixed in [Bourdin & Trélat 2017]). In context of free sampling times, a wonderful challenge would be to study the asymptotic behavior when letting* N *tend to* $+\infty$ *(which is a weaker condition than the uniform convergence to zero of the lengths of sampling intervals).*

(ii) *In view of initializations of numerical algorithms, it would be relevant to get theoretical results about the distribution of optimal sampling times with respect to* N *and/or with respect to the data (cost, dynamics, constraints) of the considered problem.*

(iii) *Last (but not least) a relevant research perspective would concern the extension of the present chapter to the more general framework in which the values of the free sampling times t_i intervene explicitly in the cost to minimize and/or in the dynamics. Let us take this occasion to mention the paper [Bakir et al. 2020] in which the authors derive Pontryagin-type conditions for a specific problem from medicine that can be written as an optimal sampled-data control problem in which the sampling times t_i are free and intervene explicitly in the expression of the dynamics. We precise that, even in this very particular context, giving an expression of the necessary optimality conditions in an Hamiltonian form still remains an open mathematical question.*

Remark 2.2.17. *In this chapter, as explained in the Introduction, the proof of Theorem 2.2.1 is based on the Ekeland variational principle [Ekeland 1974]. Let us note that an alternative proof of Theorem 2.2.1 can be obtained by adapting a remarkable technique exposed in the paper [Dmitruk & Kaganovich 2011] by Dmitruk and Kaganovich that consists of mapping each sampling interval $[t_i, t_{i+1}]$ to the interval $[0,1]$. In that situation, the free sampling times t_i play the role of free terminal states which lead, through the application of the classical PMP, to transversality conditions which exactly coincide with the Hamiltonian continuity condition* (2.3)*, while the values u_i of the sampled-data control play the role of parameters which lead, through the application of a "PMP with parameters" (see, e.g., [Bourdin & Trélat 2013, Remark 5 p.3790]), to a necessary optimality condition written in integral form which exactly coincides with the nonpositive averaged Hamiltonian gradient condition* (2.2).

2.2.3 Continuity/constancy of the Hamiltonian function

This section is devoted to a discussion on the continuity/constancy of the Hamiltonian function. We first recall the definition of the Hamiltonian function within the framework of Theorem 2.2.1.

Definition 2.2.3 (Hamiltonian function). *With the framework of Theorem 2.2.1, the corresponding Hamiltonian function $\mathcal{H} : [0,T] \to \mathbb{R}$ is defined by*

$$\mathcal{H}(t) := H(x(t), u(t), p(t), p^0, t),$$

for all $t \in [0,T]$.

Remark 2.2.18. *If the final time is free in Problem* (OSCP)*, note that the transversality condition on the optimal final time T in Theorem 2.2.1 can be rewritten as*

$$-\mathcal{H}(T) = p^0 \nabla_3 g(x(0), x(T), T) + \nabla_3 h(x(0), x(T), T)^\top \times \Psi,$$

since $u(T) = u_{N-1}$.

Remark 2.2.19. *In the classical case of optimal permanent control problems, we recall that the (absolute) continuity of the Hamiltonian function $\mathcal{H}$ is a very well-known fact (see, e.g., [Fattorini 1999, Theorem 2.6.3 p.73]). Moreover it holds that*

$$\dot{\mathcal{H}}(t) = \nabla_5 H(x(t), u(t), p(t), p^0, t),$$

for a.e. $t \in [0,T]$.

Let us consider the framework of Theorem 2.2.1. Similarly to the trajectory x (see Remark 2.2.5), it can easily be seen from the adjoint equation that the adjoint vector p is not only absolutely continuous but also piecewise smooth of class C^1 over the interval $[0,T]$, in the sense that p is of class C^1 over each interval $[t_i,t_{i+1}]$. Since moreover u is piecewise constant, it is clear that the Hamiltonian function $\mathcal{H}$ is piecewise smooth of class C^1 over $[0,T]$, in the sense that $\mathcal{H}$ is of class C^1 over each semi-open interval $[t_i,t_{i+1})$ for $i=0,\ldots,N-2$ and over the closed interval $[t_{N-1},t_N]$. Moreover, since the couple (x,p) satisfies the Hamiltonian system (see Remark 2.2.8), it clearly holds that

$$\dot{\mathcal{H}}(t) = \nabla_5 H(x(t),u(t),p(t),p^0,t),$$

over each semi-open interval $[t_i,t_{i+1})$ for $i=0,\ldots,N-2$ and over the closed interval $[t_{N-1},t_N]$.

However, in contrary to the couple (x,p), the Hamiltonian function $\mathcal{H}$ is not continuous over $[0,T]$ in general. It may admit a discontinuity at each sampling times t_i. We provide an example of this phenomenon in Section 2.3 (see Figure 2.1 in which the sampling times are fixed). Nevertheless, if the sampling times are free in Problem (OSCP), Equality (2.3) in Theorem 2.2.1 implies that the optimal sampling times t_i satisfy

$$\lim_{\substack{t\to t_i\\ t<t_i}} \mathcal{H}(t) = H(x(t_i),u_{i-1},p(t_i),p^0,t_i) = H(x(t_i),u_i,p(t_i),p^0,t_i) = \mathcal{H}(t_i),$$

for all $i=1,\ldots,N-1$, which correspond exactly to the continuity of $\mathcal{H}$ at each optimal sampling time t_i. In that situation we conclude that the Hamiltonian function $\mathcal{H}$ is continuous over the whole interval $[0,T]$. The following result summarizes the previous remarks.

Proposition 2.2.1. *Let us consider the framework of Theorem 2.2.1. Then, the Hamiltonian function $\mathcal{H}$ is piecewise smooth of class C^1 over the interval $[0,T]$, in the sense that $\mathcal{H}$ is of class C^1 with the derivative*

$$\dot{\mathcal{H}}(t) = \nabla_5 H(x(t),u(t),p(t),p^0,t), \tag{2.4}$$

over each semi-open interval $[t_i,t_{i+1})$ for $i=0,\ldots,N-2$ and over the closed interval $[t_{N-1},t_N]$. Moreover:

(i) *If the sampling times are fixed in Problem (OSCP), then $\mathcal{H}$ may admit a discontinuity at each sampling time t_i.*

(ii) *If the sampling times are free in Problem (OSCP), then the Hamiltonian function $\mathcal{H}$ is continuous at each optimal sampling time t_i. In that case, $\mathcal{H}$ is continuous over the whole interval $[0,T]$.*

Proposition 2.2.1 is illustrated with a simple linear-quadratic example numerically solved in the next Section 2.3 (see Figures 2.1 and 2.2).

We conclude this section by discussing the case where Problem (OSCP) is *autonomous*, in the sense that the dynamics f and the Lagrange cost function L are independent of the variable t. In that case, the Hamiltonian H is also independent of the variable t. In that situation, from Equality (2.4), we deduce that the Hamiltonian function $\mathcal{H}$ is constant over each semi-open interval $[t_i, t_{i+1})$ for $i = 0, \ldots, N-2$ and over the closed interval $[t_{N-1}, t_N]$. If moreover the sampling times are free in Problem (OSCP), we deduce from Proposition 2.2.1 that the Hamiltonian function $\mathcal{H}$ is constant over the whole interval $[0, T]$. In the next section, a simple autonomous linear-quadratic example is solved which illustrates the constancy of the Hamiltonian function in the case of free sampling times (see Figure 2.4).

2.3 Numerical illustrations with two simple linear-quadratic examples

Our aim in this section is to illustrate our discussion in Section 2.2.3 about the continuity/constancy of the Hamiltonian function. Bourdin and Trélat have recently extended in [Bourdin & Trélat 2017, Section 3 p.275] the classical Riccati theory to the sampled-data control framework. In particular they provide in [Bourdin & Trélat 2017, Theorem 2 and Corollary 1 p.276] a numerical way to compute the optimal sampled-data control for linear-quadratic problems in the case of fixed sampling times. We adapt this method along with the Hamiltonian continuity condition given by Equation (2.3) in order to numerically solve two simple linear-quadratic examples with free sampling times.

2.3.1 Example illustrating the continuity of the Hamiltonian function

We focus in this section on the following unidimensional linear-quadratic optimal sampled-data control problem ($\mathrm{OSCP_{lq}}$) given by

$$\left.\begin{aligned} &\text{minimize} && x(1)^2 + \int_0^1 3x(t)^2 + u(t)^2 \, dt, \\ &\text{subject to} && \mathbb{T} = \{t_i\}_{i=0,\ldots,N} \in \mathcal{P}_N^1 \text{ free}, \\ &&& x \in \mathrm{AC}([0,1], \mathbb{R}), \ u \in \mathrm{PC}^{\mathbb{T}}([0,1], \mathbb{R}), \\ &&& \dot{x}(t) = x(t) - u(t) + t, \quad \text{a.e. } t \in [0,1], \\ &&& x(0) = -4, \end{aligned}\right\} \qquad (\mathrm{OSCP_{lq}})$$

with different values of $N \in \mathbb{N}^*$. Note that Problem ($\mathrm{OSCP_{lq}}$) satisfies all the assumptions of Section 2.2.1 with the final time $T = 1$ being fixed.

For the needs of this section, for all N-partitions $\mathbb{T} \in \mathcal{P}_N^1$, we will denote by ($\mathrm{OSCP_{lq}}$-$\mathbb{T}$) the same problem as Problem ($\mathrm{OSCP_{lq}}$) replacing "free" by "fixed", that is, Problem ($\mathrm{OSCP_{lq}}$-$\mathbb{T}$) corresponds to Problem ($\mathrm{OSCP_{lq}}$) but with the fixed partition $\mathbb{T}$. As recalled in the beginning of the section, [Bourdin & Trélat 2017, Theorem 2 and Corollary 1 p.276] allows us to numerically compute, for all N-partitions $\mathbb{T} \in \mathcal{P}_N^1$, the optimal cost (denoted by $C_{\mathbb{T}}$) and the Hamiltonian function (denoted by $\mathcal{H}_{\mathbb{T}}$) corresponding to Problem ($\mathrm{OSCP_{lq}}$-$\mathbb{T}$). Hence, in order to numerically solve Problem ($\mathrm{OSCP_{lq}}$) (with free sampling times), we can follow two different methods:

(i) Firstly we directly minimize the optimal cost mapping $\mathbb{T} \longmapsto C_{\mathbb{T}}$ (using the MATLAB function *fmincon*).

(ii) Secondly, following the Hamiltonian continuity conditions (2.3) in Theorem 2.2.1, we apply a shooting method (based on the MATLAB function *fsolve*) on the Hamiltonian discontinuities mapping given by

$$\mathbb{T} = \{t_i\}_{i=0,\dots,N} \longmapsto \Big(\mathcal{H}_{\mathbb{T}}(t_i) - \lim_{\substack{t \to t_i \\ t < t_i}} \mathcal{H}_{\mathbb{T}}(t) \Big)_{i=1,\dots,N-1}.$$

Both methods yield the same optimal sampling times. Hence Problem ($\mathrm{OSCP_{lq}}$) is numerically solved and we present hereafter some numerical simulations for different values of N. In particular we compare the results with the fixed uniform partition case (see Table 2.1). As expected, one can clearly observe that the optimal cost $C_{\mathbb{T}}$ is lower for the optimal sampling times than for the fixed uniform partition.

N	Fixed uniform partition	$C_{\mathbb{T}}$	Optimal sampling times	$C_{\mathbb{T}}$
2	$\mathbb{T} = \{0, 0.5, 1\}$	46.6828	$\mathbb{T} = \{0, 0.3592, 1\}$	46.0285
4	$\mathbb{T} = \{0, 0.25, 0.5, 0.75, 1\}$	44.5131	$\mathbb{T} = \{0, 0.1574, 0.3544, 0.6163, 1\}$	44.3159
8	$\mathbb{T} = \{0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1\}$	43.9704	$\mathbb{T} = \{0, 0.0744, 0.1567, 0.2487, 0.3529, 0.4729, 0.6140, 0.7847, 1\}$	43.9191

Table 2.1: Comparison of optimal costs $C_{\mathbb{T}}$ (fixed uniform partition versus optimal sampling times).

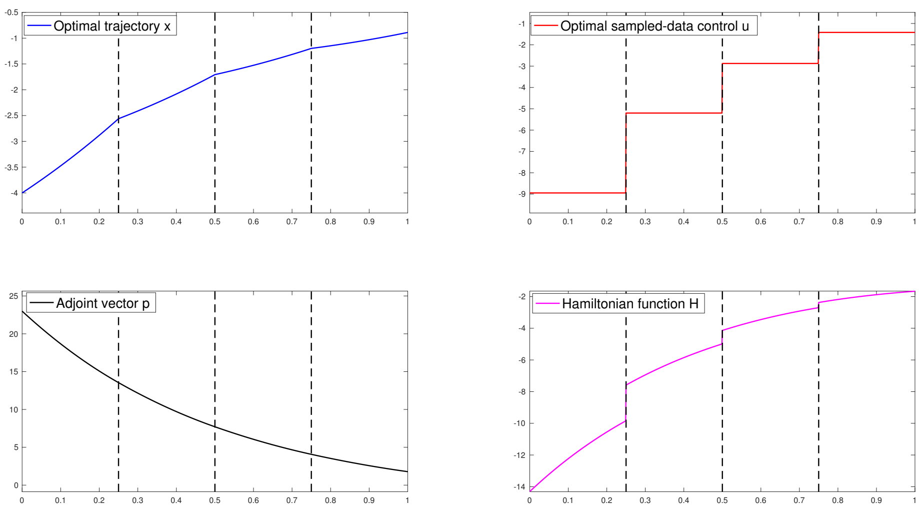

Figure 2.1: The case $N = 4$ of Problem ($\mathrm{OSCP_{lq}}$) with fixed uniform partition.

In Figure 2.1 (with $N = 4$ and fixed uniform partition), as expected from Section 2.2.3, we observe that the Hamiltonian function $\mathcal{H}$ is continuous over each semi-open interval $[t_i, t_{i+1})$ for $i = 0, 1, 2$ and over the closed interval $[t_3, t_4]$. However, since the uniform partition is not optimal in that situation, the Hamiltonian function $\mathcal{H}$ has discontinuities at each t_i. On the contrary, in Figure 2.2 (with $N = 4$ and optimal sampling times), we observe that the Hamiltonian function

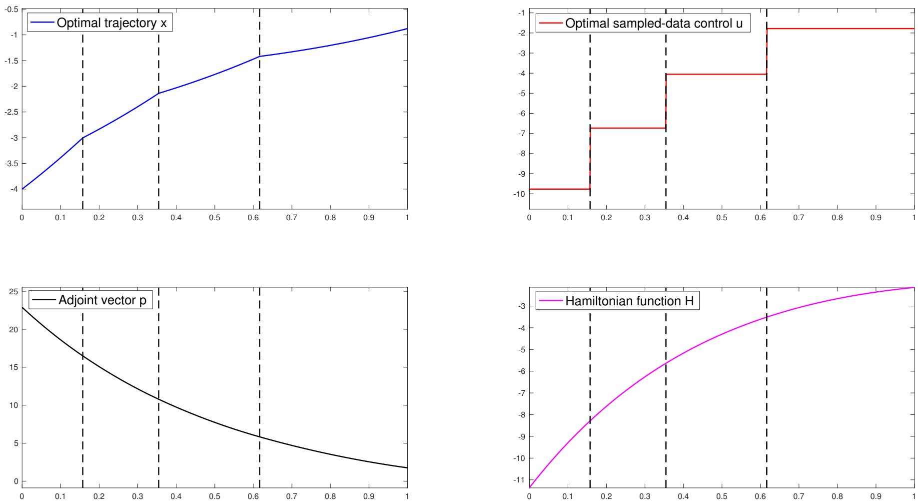

Figure 2.2: The case $N = 4$ of Problem (OSCP_{lq}) with optimal sampling times.

$\mathcal{H}$ is continuous over the whole interval $[0, 1]$. These numerical results are coherent with the discussion addressed in Section 2.2.3.

2.3.2 Autonomous example illustrating the constancy of the Hamiltonian function

We will focus in this section on the following unidimensional autonomous linear-quadratic optimal sampled-data control problem (OSCP_{alq}) given by

$$\left.\begin{array}{ll} \text{minimize} & \dfrac{5}{2}x(1)^2 + \displaystyle\int_0^1 x(t)^2 + \dfrac{5}{2}u(t)^2\,dt, \\ \text{subject to} & \mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}_N^1 \text{ free}, \\ & x \in \text{AC}([0,1],\mathbb{R}),\ u \in \text{PC}^{\mathbb{T}}([0,1],\mathbb{R}), \\ & \dot{x}(t) = x(t) - 10u(t), \quad \text{a.e. } t \in [0,1], \\ & x(0) = -4, \end{array}\right\} \qquad (\text{OSCP}_{\text{alq}})$$

with different values of $N \in \mathbb{N}^*$. Note that Problem (OSCP_{alq}) satisfies all the assumptions of Section 2.2.1 and of Theorem 2.2.1, with the final time $T = 1$ being fixed.

Problem (OSCP_{alq}) is numerically solved and we present hereafter some numerical simulations for different values of N. In particular we compare the results with the fixed uniform partition case (see Table 2.2). As expected, one can clearly observe that the optimal cost $C_{\mathbb{T}}$ is lower for the optimal sampling times than for the fixed uniform partition.

In Figure 2.3 (with $N = 4$ and fixed uniform partition), as expected from Section 2.2.3 since the problem is autonomous, we observe that the Hamiltonian function $\mathcal{H}$ is constant over each semi-open interval $[t_i, t_{i+1})$ for $i = 0, \dots, 2$ and over the closed interval $[t_3, t_4]$. However, since the uniform partition is not optimal in that situation, the Hamiltonian function $\mathcal{H}$ has

N	Fixed uniform partition	$C_{\mathbb{T}}$	Optimal sampling times	$C_{\mathbb{T}}$
2	$\mathbb{T} = \{0, 0.5, 1\}$	4.2607	$\mathbb{T} = \{0, 0.2013, 1\}$	3.3128
4	$\mathbb{T} = \{0, 0.25, 0.5, 0.75, 1\}$	3.3139	$\mathbb{T} = \{0, 0.072, 0.176, 0.3641, 1\}$	3.0284
8	$\mathbb{T} = \{0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1\}$	3.0518	$\mathbb{T} = \{0, 0.0318, 0.0685, 0.1121, 0.1657, 0.2352, 0.3341, 0.5055, 1\}$	2.9765

Table 2.2: Comparison of optimal costs $C_{\mathbb{T}}$ (fixed uniform partition versus optimal sampling times).

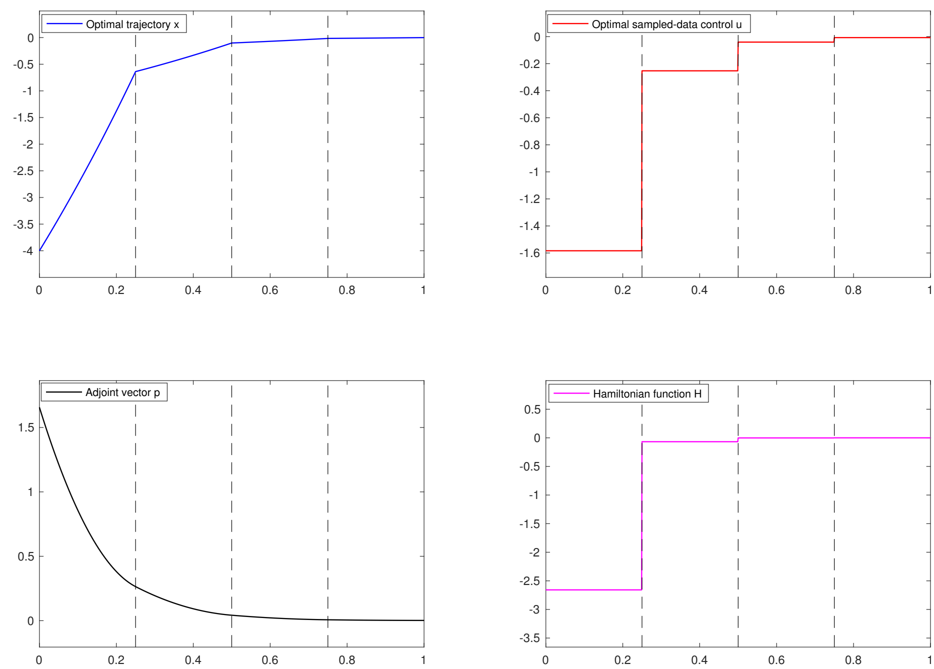

Figure 2.3: The case $N = 4$ of Problem (OSCP$_{\text{alq}}$) with fixed uniform partition.

discontinuities at each t_i. On the contrary, in Figure 2.4 (with $N = 4$ and optimal sampling times), we observe that the Hamiltonian function $\mathcal{H}$ is constant over the whole interval $[0, 1]$. These numerical results are coherent with the discussion addressed in Section 2.2.3.

To conclude this section, note that the above numerical results emphasize the effectiveness of our two methods in order to compute the optimal sampling times of a simple linear-quadratic example. Numerous perspectives can be investigated by using other methods than the Riccati theory from [Bourdin & Trélat 2017, Section 3 p.275] and by considering more sophisticated problems such as nonlinear multidimensional problems, handling terminal state and/or control constraints, with or without free final time, etc.

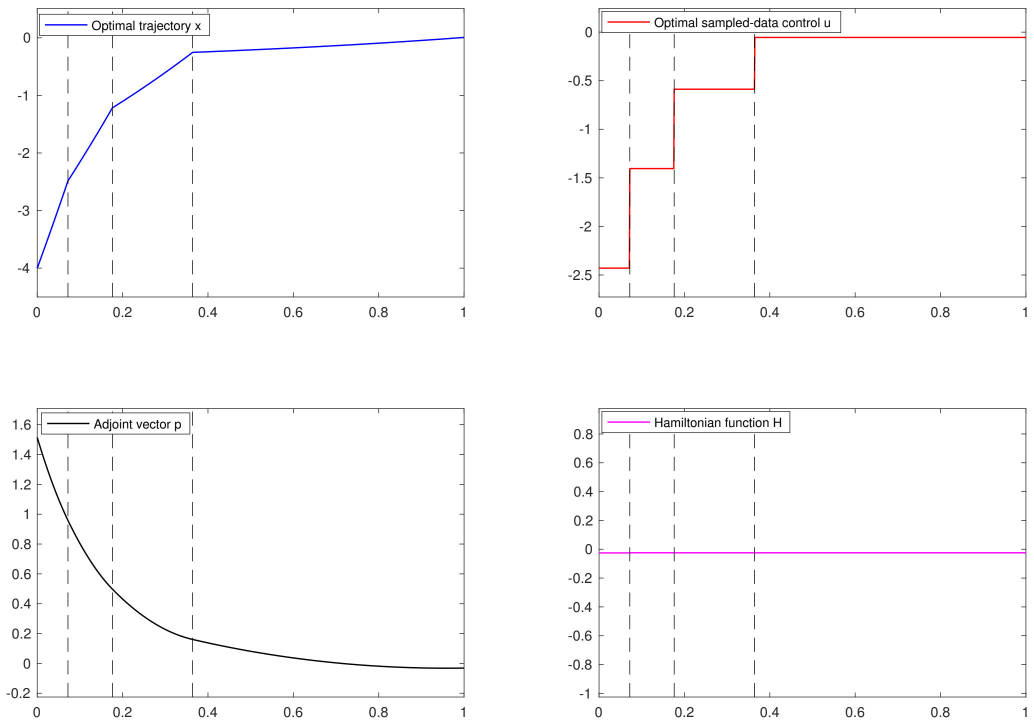

Figure 2.4: The case $N = 4$ of Problem ($\mathrm{OSCP_{alq}}$) with optimal sampling times.

2.4 Proof of the Theorem 2.2.1

This section is devoted to the detailed proof of Theorem 2.2.1. Our proof is based on the sensitivity analysis of the state equation in Section 2.4.1 and on the application of the Ekeland variational principle in Section 2.4.3 in the case $L = 0$ (without Lagrange cost). In Section 2.4.2 we introduce a technical control set which guarantees that we can extract a subsequence from the sequence of sampled-data controls produced by the Ekeland variational principle which converges almost everywhere to the optimal sampled-data control with, moreover, the sampling times converging to the optimal sampling times (see Proposition 2.4.7 for details). The case $L \neq 0$ (with the Lagrange cost) is treated afterwards in Section 2.4.4.

2.4.1 Sensitivity analysis of the state equation

In this section we focus on the Cauchy problem given by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t), & \text{a.e. } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{CP}$$

for any $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$. We first recall some definitions and results from the classical Cauchy-Lipschitz (or Picard-Lindelöf) theory.

Definition 2.4.1. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$. A (local) solution to the Cauchy problem (CP) is a couple (x, I) such that:*

(i) *I is an interval such that $\{0\} \subsetneq I \subset \mathbb{R}_+$;*

(ii) *$x \in \mathrm{AC}([0,t],\mathbb{R}^n)$, with $\dot{x}(t) = f(x(t),u(t),t)$ for almost every $t \in [0,t]$, for all $t \in I$;*

(iii) *$x(0) = x_0$.*

Let (x_1, I_1) and (x_2, I_2) be two (local) solutions to the Cauchy problem (CP). We say that (x_2, I_2) is an extension (resp. strict extension) to (x_1, I_1) if $I_1 \subset I_2$ (resp. $I_1 \subsetneq I_2$) and $x_2(t) = x_1(t)$ for all $t \in I_1$. A maximal solution to the Cauchy problem (CP) is a (local) solution that does not admit any strict extension. Finally a global solution to the Cauchy problem (CP) is a solution (x, I) such that $I = \mathbb{R}_+$. In particular a global solution is necessarily a maximal solution.

Lemma 2.4.1. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^n$. Any (local) solution to the Cauchy problem (CP) can be extended into a maximal solution.*

Lemma 2.4.2. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m) \times \mathbb{R}^n$. A couple (x, I) is a (local) solution to the Cauchy problem (CP) if and only if:*

(i) *I is an interval such that $\{0\} \subsetneq I \subset \mathbb{R}_+$;*

(ii) *$x \in \mathrm{C}(I,\mathbb{R}^n)$;*

(iii) *x satisfies the integral representation $x(t) = x_0 + \int_0^t f(x(s),u(s),s)\,ds$ for all $t \in I$.*

Proposition 2.4.1. *For all $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m) \times \mathbb{R}^n$, the Cauchy problem (CP) admits a unique maximal solution denoted by $(x(\cdot,u,x_0), I(u,x_0))$. Moreover the maximal interval $I(u,x_0)$ is semi-open and we write $I(u,x_0) = [0,t(u,x_0))$ where $t(u,x_0) \in (0,+\infty]$. Furthermore, if $t(u,x_0) < +\infty$, that is, if the maximal solution $(x(\cdot,u,x_0), I(u,x_0))$ is not global, then $x(\cdot,u,x_0)$ is not bounded over $I(u,x_0) = [0,t(u,x_0))$.*

Remark 2.4.1. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m) \times \mathbb{R}^n$. The maximal solution $(x(\cdot,u,x_0), I(u,x_0))$ to the Cauchy problem (CP) coincides with the maximal extension (see Lemma 2.4.1) of any other local solution.*

Our aim in the next subsections is to study the behaviour of $x(\cdot,u,x_0)$ with respect to perturbations on the control u and on the initial condition x_0 in order to later quantify the change of the cost function with respect to these perturbations.

2.4.1.1 A general continuity result

Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^n$. In the sequel, for the ease of notations, we denote by $\|\cdot\|_{\mathrm{L}^\infty} := \|\cdot\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m)}$ and we introduce two sets:

(i) For all $R \geq \|u\|_{\mathrm{L}^\infty}$ and all $0 < t < t(u,x_0)$, we denote by

$$\mathrm{K}((u,x_0),(R,t)) := \{(y,v,t) \in \mathbb{R}^n \times \overline{\mathrm{B}}_{\mathbb{R}^m}(0_{\mathbb{R}^m},R) \times [0,t] \mid \|y - x(t,u,x_0)\|_{\mathbb{R}^n} \leq 1\}.$$

Firstly note that $\mathrm{K}((u,x_0),(R,t))$ is convex with respect to its first two variables. Secondly, since $x(\cdot,u,x_0)$ is continuous over $[0,t]$, then $\mathrm{K}((u,x_0),(R,t))$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$. Thus we denote by $C_{\mathrm{K}}((u,x_0),(R,t)) \geq 0$ the Lipschitz constant of f over the compact subset $\mathrm{K}((u,x_0),(R,t))$ (see Inequality (2.1) in Section 2.2.1).

(ii) For all $R \geq \|u\|_{\mathrm{L}^\infty}$ and all $0 < t < t(u, x_0)$, we denote by

$$\mathcal{V}((u,x_0),(R,t),\varepsilon)$$
$$:= \left\{ (u', x_0') \in \left(\overline{\mathrm{B}}_{\mathrm{L}^1}(u,\varepsilon) \cap \overline{\mathrm{B}}_{\mathrm{L}^\infty}(0_{\mathrm{L}^\infty}, R) \right) \times \overline{\mathrm{B}}_{\mathbb{R}^n}(x_0,\varepsilon) \;\middle|\; u' = u \text{ over } [t,+\infty) \right\},$$

for all $\varepsilon > 0$, which can be seen as a neighborhood of the couple (u, x_0) in the $\mathrm{L}^1([0,t],\mathbb{R}^m) \times \mathbb{R}^n$-space. The second part of the above definition, imposing that $u' = u$ over $[t, +\infty)$, allows us in the sequel to endow the above set with the $\mathrm{L}^1([0,t],\mathbb{R}^m) \times \mathbb{R}^n$-distance.

In the next proposition we state a continuous dependence result for the trajectory $x(\cdot, u, x_0)$ with respect to the couple (u, x_0). Note that the proof has been omitted in the paper [Bourdin & Dhar 2019] but it is provided in this manuscript.

Proposition 2.4.2. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$. For all $R \geq \|u\|_{\mathrm{L}^\infty}$ and all $0 < t < t(u, x_0)$, there exists $\varepsilon > 0$ such that*

$$\forall (u', x_0') \in \mathcal{V}((u,x_0),(R,t),\varepsilon), \quad t(u', x_0') > t.$$

Moreover, considering the $\mathrm{L}^1([0,t],\mathbb{R}^m) \times \mathbb{R}^n$-distance over the set $\mathcal{V}((u,x_0),(R,t),\varepsilon)$, the map

$$(u', x_0') \in \mathcal{V}((u,x_0),(R,t),\varepsilon) \longmapsto x(\cdot, u', x_0') \in \mathrm{C}([0,t],\mathbb{R}^n),$$

is Lipschitz continuous and

$$(x(t, u', x_0'), u'(t), t) \in \mathrm{K}((u,x_0),(R,t)),$$

for a.e. $t \in [0,t]$ and for all $(u', x_0') \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$.

Proof. Let $R \geq \|u\|_{\mathrm{L}^\infty}$ and $0 < t < t(u, x_0)$. In this proof, for the ease of notations, we denote by $\mathrm{K} := \mathrm{K}((u,x_0),(R,t))$ and by $C_\mathrm{K} := C_\mathrm{K}((u,x_0),(R,t))$. We fix a constant $0 < \varepsilon < 1$ such that $\varepsilon(1 + C_\mathrm{K})e^{C_\mathrm{K} t} < 1$. Let $(u', x_0') \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$ and let us introduce the set

$$A := \{ t \in [0, t(u', x_0')) \cap [0,t] \mid \|x(t,u',x_0') - x(t,u,x_0)\|_{\mathbb{R}^n} > 1 \}.$$

By contradiction let us assume that $A \neq \emptyset$ and let $\bar{t} := \inf A$. From the definition of $\bar{t}$, there exists a monotonically decreasing sequence $(t_k)_{k \in \mathbb{N}}$ in A which converges to $\bar{t}$. Since $x(\cdot, u, x_0)$ and $x(\cdot, u', x_0')$ are continuous, it holds that $\|x(\bar{t}, u', x_0') - x(\bar{t}, u, x_0)\|_{\mathbb{R}^n} \geq 1$. Moreover $\bar{t} > 0$ since $\|x(0,u',x_0') - x(0,u,x_0)\|_{\mathbb{R}^n} = \|x_0' - x_0\|_{\mathbb{R}^n} \leq \varepsilon < 1$. Therefore $\|x(t,u',x_0') - x(t,u,x_0)\|_{\mathbb{R}^n} \leq 1$ for all $t \in [0, \bar{t})$. It follows that $(x(t,u',x_0'), u'(t), t) \in \mathrm{K}$ for a.e. $t \in [0,\bar{t})$. From the integral representations of $x(\cdot, u, x_0)$ and $x(\cdot, u', x_0')$, it holds that

$$x(t,u',x_0') - x(t,u,x_0) = x_0' - x_0 + \int_0^t f(x(s,u',x_0'), u'(s), s) - f(x(s,u,x_0), u(s), s)\, ds,$$

for all $t \in [0, \bar{t}]$. Therefore we get from the Lipschitz continuity of f over K that

$$\|x(t,u',x_0') - x(t,u,x_0)\|_{\mathbb{R}^n}$$
$$\leq \|x_0' - x_0\|_{\mathbb{R}^n} + C_\mathrm{K} \int_0^t \|x(s,u',x_0') - x(s,u,x_0)\|_{\mathbb{R}^n} + \|u'(s) - u(s)\|_{\mathbb{R}^m}\, ds$$
$$\leq \varepsilon(1 + C_\mathrm{K}) + C_\mathrm{K} \int_0^t \|x(s,u',x_0') - x(s,u,x_0)\|_{\mathbb{R}^n} ds,$$

for all $t \in [0, \bar{t}]$. Thus $\|x(t, u', x'_0) - x(t, u, x_0)\|_{\mathbb{R}^n} \leq \varepsilon(1 + C_{\mathrm{K}})e^{C_{\mathrm{K}}t} < 1$ for all $t \in [0, \bar{t}]$ from the Gronwall lemma. This gives us a contradiction at $t = \bar{t}$ and so $A = \emptyset$. We conclude that $x(\cdot, u', x'_0)$ is bounded over $[0, t(u', x'_0)) \cap [0, t]$ which implies from Proposition 2.4.1 that $\tau(u', x'_0) > t$. Moreover, since $\tau(u', x'_0) > t$ and $A = \emptyset$, we deduce that $(x(t, u', x'_0), u'(t), t) \in \mathrm{K}$ for a.e. $t \in [0, t]$. Thus we get from the Lipschitz continuity of f over K that

$$\|x(t, u', x'_0) - x(t, u, x_0)\|_{\mathbb{R}^n} \\ \leq \|x'_0 - x_0\|_{\mathbb{R}^n} + C_{\mathrm{K}}\|u' - u\|_{\mathrm{L}^1} + C_{\mathrm{K}} \int_0^t \|x(s, u', x'_0) - x(s, u, x_0)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [0, t]$. The Gronwall lemma leads to

$$\|x(t, u', x'_0) - x(t, u, x_0)\|_{\mathbb{R}^n} \leq (\|x'_0 - x_0\|_{\mathbb{R}^n} + C_{\mathrm{K}}\|u' - u\|_{\mathrm{L}^1})e^{C_{\mathrm{K}}t},$$

for all $t \in [0, t]$, which completes the proof. □

Remark 2.4.2. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$. Let $R \geq \|u\|_{\mathrm{L}^\infty}$ and $0 < t < t(u, x_0)$. Let $\varepsilon > 0$ given in Proposition 2.4.2. Let $(u_k, x_{0,k})_{k\in\mathbb{N}}$ be a sequence in $\mathcal{V}((u, x_0), (R, t), \varepsilon)$ and let $(u', x'_0) \in \mathcal{V}((u, x_0), (R, t), \varepsilon)$. From Proposition 2.4.2, if $(u_k, x_{0,k})$ converges to (u', x'_0) in $\mathrm{L}^1([0, t], \mathbb{R}^m) \times \mathbb{R}^n$, then the sequence $(x(\cdot, u_k, x_{0,k}))_{k\in\mathbb{N}}$ uniformly converges to $x(\cdot, u', x'_0)$ over $[0, t]$.*

2.4.1.2 Perturbation of the control

In the next proposition we state a differentiability result for the trajectory $x(\cdot, u, x_0)$ with respect to a convex L^∞-perturbation of the control u. Note that the proof had been omitted in the paper [Bourdin & Dhar 2019] but is provided in this manuscript.

Proposition 2.4.3. *Let $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$ and $0 < t < t(u, x_0)$. Let $v \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ be fixed. We consider the convex L^∞-perturbation given by*

$$u_v(\cdot, \alpha) := \begin{cases} u + \alpha(v - u) & \text{over} \quad [0, t), \\ u & \text{over} \quad [t, +\infty), \end{cases}$$

for all $0 \leq \alpha \leq 1$. Then:

(i) *there exists $0 < \alpha_0 \leq 1$ such that $t(u_v(\cdot, \alpha), x_0) > t$ for all $0 \leq \alpha \leq \alpha_0$;*

(ii) *the map*

$$\alpha \in [0, \alpha_0] \longmapsto x(\cdot, u_v(\cdot, \alpha), x_0) \in \mathrm{C}([0, t], \mathbb{R}^n),$$

is differentiable at $\alpha = 0$ and its derivative is equal to w_v being the unique solution (that is global) to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t, u, x_0), u(t), t) \times w(t) \\ \qquad\qquad + \nabla_2 f(x(t, u, x_0), u(t), t) \times (v(t) - u(t)), \qquad a.e.\ t \in [0, t], \\ w(0) = 0_{\mathbb{R}^n}. \end{cases}$$

Proof. First of all note that the variation vector w_v is global since the corresponding Cauchy problem is linear. Let $R := \max\{\|u\|_{\mathrm{L}^\infty}, \|v\|_{\mathrm{L}^\infty}\}$. For the ease of notations we denote by $\mathrm{K} := \mathrm{K}((u,x_0),(R,t))$ and by $C_{\mathrm{K}} := C_{\mathrm{K}}((u,x_0),(R,t))$ (see the beginning of Section 2.4.1.1 for these two notations). From Proposition 2.4.2, there exists $\varepsilon > 0$ such that $t(u',x'_0) > t$ for all $(u',x'_0) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$. Let us take $0 < \alpha_0 \leq 1$ small enough such that $\alpha_0\|v-u\|_{\mathrm{L}^1} \leq \varepsilon$. Then it holds that $\|u_v(\cdot,\alpha) - u\|_{\mathrm{L}^1} \leq \alpha\|v-u\|_{\mathrm{L}^1} \leq \varepsilon$ and $\|u_v(\cdot,\alpha)\|_{\mathrm{L}^\infty} \leq R$ for all $0 \leq \alpha \leq \alpha_0$. It follows that $(u_v(\cdot,\alpha),x_0) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$ and $t(u_v(\cdot,\alpha),x_0) > t$ for all $0 \leq \alpha \leq \alpha_0$. The first item of Proposition 2.4.3 is proved. Since $(u_v(\cdot,\alpha),x_0) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$ for all $0 \leq \alpha \leq \alpha_0$ and $u_v(\cdot,\alpha)$ converges to u in $\mathrm{L}^1([0,t],\mathbb{R}^m)$ as α tends to zero, we know from Proposition 2.4.2 that $x(\cdot,u_v(\cdot,\alpha),x_0)$ converges uniformly to $x(\cdot,u,x_0)$ over $[0,t]$ as α tends to zero and that $(x(t,u_v(\cdot,\alpha),x_0),u_v(t,\alpha),t) \in \mathrm{K}$ for a.e. $t \in [0,t]$ and for all $0 \leq \alpha \leq \alpha_0$. Now let us define the function

$$\chi(t,\alpha) := \frac{x(t,u_v(\cdot,\alpha),x_0) - x(t,u,x_0)}{\alpha} - w_v(t),$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. We will prove that $\chi(\cdot,\alpha)$ uniformly converges to the zero function on $[0,t]$ as α tends to 0. From the integral representation of $\chi(\cdot,\alpha)$, it holds that

$$\begin{aligned}\chi(t,\alpha) = \int_0^t & \frac{f(x(s,u_v(\cdot,\alpha),x_0),u_v(s,\alpha),s) - f(x(s,u,x_0),u(s),x_0)}{\alpha} \\ & - \nabla_1 f(x(s,u,x_0),u(s),s) \times w_v(s) - \nabla_2 f(x(s,u,x_0),u(s),s) \times (v(s)-u(s))\, ds,\end{aligned}$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. Expanding this expression using Taylor's theorem with integral remainder, we obtain

$$\begin{aligned}\chi(t,\alpha) = & \int_0^t \left(\int_0^1 \nabla_1 f(\star_{\alpha\theta s})\, d\theta \right) \times \chi(s,\alpha)\, ds \\ & + \int_0^t \left(\int_0^1 \nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\, d\theta \right) \times w_v(s)\, ds \\ & + \int_0^t \left(\int_0^1 \nabla_2 f(\star_{\alpha\theta s}) - \nabla_2 f(x(s,u,x_0),u(s),s)\, d\theta \right) \times (v(s)-u(s))\, ds,\end{aligned}$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$, where

$$\star_{\alpha\theta s} := (x(s,u,x_0) + \theta(x(s,u_v(\cdot,\alpha),s) - x(s,u,x_0)), u(s) + \theta(u_v(s,\alpha) - u(s)), s) \in \mathrm{K},$$

since K is convex with respect to its first two variables. From the Triangle inequality, it holds that

$$\|\chi(t,\alpha)\|_{\mathbb{R}^n} \leq \Phi(\alpha) + C_{\mathrm{K}} \int_0^t \|\chi(s,\alpha)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$, where the term $\Phi(\alpha)$ is defined to be:

$$\begin{aligned}\Phi(\alpha) := & \int_0^t \int_0^1 \|\nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}}\, d\theta \|w_v(s)\|_{\mathbb{R}^n}\, ds \\ & + \int_0^t \int_0^1 \|\nabla_2 f(\star_{\alpha\theta s}) - \nabla_2 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times m}}\, d\theta \|v(s)-u(s)\|_{\mathbb{R}^m}\, ds,\end{aligned}$$

for all $0 \leq \alpha \leq \alpha_0$. From the Gronwall lemma, it holds that

$$\|\chi(t,\alpha)\| \leq \Phi(\alpha)e^{C_{\mathrm{K}}t},$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. Since the estimate on the right-hand side is independent of t, we only need to prove that $\Phi(\alpha)$ tends to 0 as α tends to 0. Since $\star_{\alpha\theta s} \in \mathrm{K}$, it holds that $\|\nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}} \leq 2C_{\mathrm{K}}$ and $\|\nabla_2 f(\star_{\alpha\theta s}) - \nabla_2 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}} \leq 2C_{\mathrm{K}}$ for a.e. $s \in [0,t]$ and for all $\theta \in [0,1]$ and all $\alpha \in (0,\alpha_0]$. Moreover, since $\nabla_1 f$ and $\nabla_2 f$ are continuous, since $x(\cdot,u_v(\cdot,\alpha),x_0)$ uniformly converges to $x(\cdot,u,x_0)$ over $[0,t]$ and since $u_v(s,\alpha)$ converges to $u(s)$ for a.e. $s \in [0,t]$ as α tends to 0, we get from the Lebesgue dominated convergence theorem that $\Phi(\alpha)$ tends to 0 as α tends to 0. The proof is complete. □

We conclude this section by a technical lemma on the convergence of the variation vectors. This result is needed in the proof of our main result (see Section 2.4.3.2). Note that the proof had been omitted in the paper [Bourdin & Dhar 2019] but is provided in this manuscript.

Lemma 2.4.3. *Let $(u,x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m) \times \mathbb{R}^n$. Let $R \geq \|u\|_{\mathrm{L}^\infty}$ and $0 < t < t(u,x_0)$. We take $\varepsilon > 0$ as in Proposition 2.4.2. Let $(u_k,x_{0,k})_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{V}((u,x_0),(R,t),\varepsilon)$ such that $x_{0,k}$ converges to x_0 and $u_k(t)$ converges to $u(t)$ for a.e. $t \in [0,t]$. Let $(v_k)_{k\in\mathbb{N}}$ be a sequence in $\mathrm{L}^\infty([0,t],\mathbb{R}^m)$ converging in $\mathrm{L}^1([0,t],\mathbb{R}^m)$ to some $v \in \mathrm{L}^\infty([0,t],\mathbb{R}^m)$. Finally let $w^k_{v_k}$ be the unique solution (that is global) to the linear Cauchy problem given by*

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t,u_k,x_{0,k}),u_k(t),t) \times w(t) \\ \qquad\qquad +\nabla_2 f(x(t,u_k,x_{0,k}),u_k(t),t) \times (v_k(t) - u_k(t)), \qquad a.e.\ t \in [0,t], \\ w(0) = 0_{\mathbb{R}^n}, \end{cases}$$

for all $k \in \mathbb{N}$. Then the sequence $(w^k_{v_k})_{k\in\mathbb{N}}$ uniformly converges to w_v over $[0,t]$ where w_v is defined as in Proposition 2.4.3.

Proof. First of all, for all $k \in \mathbb{N}$, note that the variation vector $w^k_{v_k}$ is global since the corresponding Cauchy problem is linear. For the ease of notations, we denote by $\mathrm{K} := \mathrm{K}((u,x_0),(R,t))$ and by $C_{\mathrm{K}} := C_{\mathrm{K}}((u,x_0),(R,t))$ (see the beginning of Section 2.4.1.1 for these two notations). From the integral representations of $w^k_{v_k}$ and w_v, it holds that

$$\begin{aligned} w^k_{v_k}(t) - w_v(t) &= \int_0^t \nabla_1 f(x(s,u_k,x_{0,k}),u_k(s),s) \times (w^k_{v_k}(s) - w_v(s))\,ds \\ &+ \int_0^t \big[\nabla_1 f(x(s,u_k,x_{0,k}),u_k(s),s) - \nabla_1 f(x(s,u,x_0),u(s),s)\big] \times w_v(s)\,ds \\ &+ \int_0^t \big[\nabla_2 f(x(s,u_k,x_{0,k}),u_k(s),s) - \nabla_2 f(x(s,u,x_0),u(s),s)\big] \times (v(s) - u(s))\,ds \\ &+ \int_0^t \nabla_2 f(x(s,u_k,x_{0,k}),u_k(s),s) \times (v_k(s) - v(s) + u(s) - u_k(s))\,ds, \end{aligned}$$

for all $t \in [0,t]$ and all $k \in \mathbb{N}$. From Proposition 2.4.2, it holds that $(x(s,u_k,x_{0,k}),u_k(s),s) \in \mathrm{K}$ and thus $\|\nabla_1 f(x(s,u_k,x_{0,k}),u_k(s),s)\|_{\mathbb{R}^{n\times n}} \leq C_{\mathrm{K}}$ and $\|\nabla_2 f(x(s,u_k,x_{0,k}),u_k(s),s)\|_{\mathbb{R}^{n\times m}} \leq C_{\mathrm{K}}$

for a.e. $s \in [0,t]$ and all $k \in \mathbb{N}$. From the Triangle inequality, it holds that

$$\|w^k_{v_k}(t) - w_v(t)\|_{\mathbb{R}^n} \leq \Gamma_k + C_{\mathrm{K}} \int_0^t \|w^k_{v_k}(s) - w_v(s)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [0,t]$ and all $k \in \mathbb{N}$, where the term Γ_k is defined to be

$$\begin{aligned}\Gamma_k := {} & C_{\mathrm{K}} \int_0^t \|v_k(s) - v(s)\|_{\mathbb{R}^m} + \|u(s) - u_k(s)\|_{\mathbb{R}^m}\, ds \\ & + \int_0^t \|\nabla_1 f(x(s,u_k,x_{0,k}),u_k(s),s) - \nabla_1 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}} \|w_v(s)\|_{\mathbb{R}^n}\, ds \\ & + \int_0^t \|\nabla_2 f(x(s,u_k,x_{0,k}),u_k(s),s) - \nabla_2 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times m}} \|v(s) - u(s)\|_{\mathbb{R}^m}\, ds.\end{aligned}$$

From the Gronwall Lemma, we obtain that

$$\|w^k_{v_k}(t) - w_v(t)\|_{\mathbb{R}^n} \leq \Gamma_k e^{C_{\mathrm{K}} t},$$

for all $t \in [0,t]$ and all $k \in \mathbb{N}$. Since the estimate on the right-hand side is independent of t, we only need to prove that Γ_k tends to 0 as k tends to $+\infty$. This can be done with the Lebesgue dominated convergence theorem, similarly to the end of the proof of Proposition 2.4.3. ☐

2.4.1.3 Perturbation of the initial condition

In the next proposition we prove a differentiability result for the trajectory $x(\cdot,u,x_0)$ with respect to a simple perturbation of the initial condition x_0. Note that the proof had been omitted in the paper [Bourdin & Dhar 2019] but is provided in this manuscript.

Proposition 2.4.4. *Let $(u,x_0) \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m) \times \mathbb{R}^n$ and $0 < t < t(u,x_0)$. Let $y \in \mathbb{R}^n$ be fixed. Then:*

(i) there exists $\alpha_0 > 0$ such that $t(u,x_0+\alpha y) > t$ for all $0 \leq \alpha \leq \alpha_0$;

(ii) the map

$$\alpha \in [0,\alpha_0] \longmapsto x(\cdot,u,x_0+\alpha y) \in \mathrm{C}([0,t],\mathbb{R}^n),$$

is differentiable at $\alpha = 0$ and its derivative is equal to w_y the unique solution (that is global) to the linear homogeneous Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t,u,x_0),u(t),t) \times w(t), & \text{a.e. } t \in [0,t], \\ w(0) = y. \end{cases}$$

Proof. First of all note that the variation vector w_y is global since the corresponding Cauchy problem is linear. Let $R := \|u\|_{\mathrm{L}^\infty}$. For the ease of notations we denote by $\mathrm{K} := \mathrm{K}((u,x_0),(R,t))$ and by $C_{\mathrm{K}} := C_{\mathrm{K}}((u,x_0),(R,t))$ (see the beginning of Section 2.4.1.1 for these two notations). From Proposition 2.4.2, there exists $\varepsilon > 0$ such that $t(u',x'_0) > t$ for all $(u',x'_0) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$. Let us take $\alpha_0 > 0$ small enough such that $\alpha_0\|y\|_{\mathbb{R}^n} \leq \varepsilon$. Then it holds that $\|x_0 + \alpha y - x_0\|_{\mathbb{R}^n} \leq \alpha\|y\|_{\mathbb{R}^n} \leq \varepsilon$ for all $0 \leq \alpha \leq \alpha_0$. It follows that $(u,x_0+\alpha y) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$ and $t(u,x_0+\alpha y) > t$ for all $0 \leq \alpha \leq \alpha_0$. The first item of Proposition 2.4.4 is proved. Since $(u, x_0+\alpha y) \in \mathcal{V}((u,x_0),(R,t),\varepsilon)$ for all $0 \leq \alpha \leq \alpha_0$ and since $x_0+\alpha y$ converges to x_0 as α tends to zero, we know from Proposition 2.4.2 that $x(\cdot, u, x_0+\alpha y)$ converges uniformly to $x(\cdot,u,x_0)$ over $[0,t]$ as α tends to zero and that $(x(t,u,x_0+\alpha y), u(t), t) \in \mathrm{K}$ for a.e. $t \in [0,t]$ and for all $0 \leq \alpha \leq \alpha_0$. Now let us consider the function given by

$$\chi(t,\alpha) := \frac{x(t,u,x_0+\alpha y) - x(t,u,x_0)}{\alpha} - w_y(t),$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. We will prove that $\chi(\cdot,\alpha)$ uniformly converges to the zero function on $[0,t]$ as α tends to 0. From the integral representation of $\chi(\cdot,\alpha)$, it holds that

$$\chi(t,\alpha) = \int_0^t \frac{f(x(s,u,x_0+\alpha y),u(s),s) - f(x(s,u,x_0),u(s),s)}{\alpha} \\ - \nabla_1 f(x(s,u,x_0),u(s),s) \times w_y(s)\, ds,$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. Expanding this expression using Taylor's theorem with integral remainder, we obtain

$$\chi(t,\alpha) = \int_0^t \left(\int_0^1 \nabla_1 f(\star_{\alpha\theta s})\, d\theta \right) \times \chi(s,\alpha)\, ds \\ + \int_0^t \left(\int_0^1 \nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\, d\theta \right) \times w_y(s)\, ds,$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$, where

$$\star_{\alpha\theta s} := (x(s,u,x_0) + \theta(x(s,u,x_0+\alpha y) - x(s,u,x_0)), u(s), s) \in \mathrm{K},$$

since K is convex with respect to its first two variables. From the Triangle inequality, it holds that

$$\|\chi(t,\alpha)\|_{\mathbb{R}^n} \leq \Phi(\alpha) + C_{\mathrm{K}} \int_0^t \|\chi(s,\alpha)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$, where the term $\Phi(\alpha)$ is defined to be:

$$\Phi(\alpha) := \int_0^t \int_0^1 \|\nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}}\, d\theta \|w_y(s)\|_{\mathbb{R}^n}\, ds.$$

From the Gronwall lemma, it holds that

$$\|\chi(t,\alpha)\|_{\mathbb{R}^n} \leq \Phi(\alpha) e^{C_{\mathrm{K}} t},$$

for all $t \in [0,t]$ and all $\alpha \in (0,\alpha_0]$. Since the estimate on the right-hand side is independent of t, we only need to prove that $\Phi(\alpha)$ tends to 0 as α tends to 0. This can be done with the Lebesgue dominated convergence theorem, similarly to the end of the proof of Proposition 2.4.3. □

We conclude this section by a technical lemma on the convergence of the variation vectors. This result is needed in the proof of our main result (see Section 2.4.3.2). Note that the proof of this result was omitted in the paper [Bourdin & Dhar 2019] but is included in this manuscript.

Lemma 2.4.4. *Let* $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$*. Let* $R \geq \|u\|_{\mathrm{L}^\infty}$ *and* $0 < t < t(u, x_0)$*. We take* $\varepsilon > 0$ *as in Proposition 2.4.2. Let* $(u_k, x_{0,k})_{k\in\mathbb{N}}$ *be a sequence of elements in* $\mathcal{V}((u, x_0), (R, t), \varepsilon)$ *such that* $x_{0,k}$ *converges to* x_0 *and* $u_k(t)$ *converges to* $u(t)$ *for a.e.* $t \in [0, t]$*. Let* $y \in \mathbb{R}^n$ *be fixed. Finally let* w^k_y *be the unique solution (that is global) to the linear homogeneous Cauchy problem given by*

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t, u_k, x_{0,k}), u_k(t), t) \times w(t), & \text{a.e. } t \in [0, t], \\ w(0) = y, \end{cases}$$

for all $k \in \mathbb{N}$*. Then the sequence* $(w^k_y)_{k\in\mathbb{N}}$ *uniformly converges to* w_y *over* $[0, t]$ *where* w_y *is defined as in Proposition 2.4.4.*

Proof. First of all, for all $k \in \mathbb{N}$, note that the variation vector w^k_y is global since the corresponding Cauchy problem is linear. For the ease of notations we denote by $\mathrm{K} := \mathrm{K}((u, x_0), (R, t))$ and by $C_\mathrm{K} := C_\mathrm{K}((u, x_0), (R, t))$ (see the beginning of Section 2.4.1.1 for these two notations). From the integral representations of w^k_y and w_y, it holds that

$$\begin{aligned} w^k_y(t) - w_y(t) = &\int_0^t \nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) \times (w^k_y(s) - w_y(s))\, ds \\ &+ \int_0^t (\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) - \nabla_1 f(x(s, u, x_0), u(s), s)) \times w_y(s)\, ds, \end{aligned}$$

for all $t \in [0, t]$ and all $k \in \mathbb{N}$. From Proposition 2.4.2, it holds that $(x(s, u_k, x_{0,k}), u_k(s), s) \in \mathrm{K}$ and thus $\|\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s)\|_{\mathbb{R}^{n\times n}} \leq C_\mathrm{K}$ for a.e. $s \in [0, t]$ and all $k \in \mathbb{N}$. From the Triangle inequality, it holds that

$$\|w^k_y(t) - w_y(t)\|_{\mathbb{R}^n} \leq \Gamma_k + C_\mathrm{K} \int_0^t \|w^k_y(s) - w_y(s)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [0, t]$ and all $k \in \mathbb{N}$, where the term Γ_k is defined to be:

$$\Gamma_k := \int_0^t \|\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) - \nabla_1 f(x(s, u, x_0), u(s), s)\|_{\mathbb{R}^{n\times n}} \|w_y(s)\|_{\mathbb{R}^n}\, ds.$$

From the Gronwall lemma, we obtain

$$\|w^k_y(t) - w_y(t)\|_{\mathbb{R}^n} \leq \Gamma_k e^{C_\mathrm{K} t},$$

for all $t \in [0, t]$ and all $k \in \mathbb{N}$. Since the estimate on the right-hand side is independent of t, we only need to prove that Γ_k tends to 0 as k tends to $+\infty$. This can be done with the Lebesgue dominated convergence theorem, similarly to the end of the proof of Proposition 2.4.3. □

2.4.1.4 Perturbation of a switching time

Let us introduce the following notion of *switching time* for a control $u \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

Definition 2.4.2 (Switching time)**.** *Let* $u \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$*. We say that* $r > 0$ *is a switching time of* u *if there exist* $0 < \eta_r \leq r$ *and* $u(r^-)$*,* $u(r^+) \in \mathbb{R}^m$ *such that:*

(i) $u = u(r^-)$ *almost everywhere over* $[r - \eta_r, r)$*;*

(ii) $u = u(r^+)$ *almost everywhere over* $[r, r+\eta_r)$.

This notion is particularly relevant when dealing with piecewise constant controls as in Problem (OSCP). Indeed, let $u \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that $u \in \mathrm{PC}^{\mathbb{T}}([0,t], \mathbb{R}^m)$ for some $t > 0$, $N \in \mathbb{N}^*$ and $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N$. Then t_i is a switching time of u with $u(t_i^-) = u_{i-1}$, $u(t_i^+) = u_i$ and $\eta_{t_i} = \min(t_i - t_{i-1}, t_{i+1} - t_i) > 0$ for every $i \in \{1, \dots, N-1\}$.

In the next proposition we prove a differentiability result for the trajectory $x(\cdot, u, x_0)$ with respect to a perturbation of a switching time of the control u.

Proposition 2.4.5. *Let* $(u, x_0) \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}^n$ *and* $0 < t < t(u, x_0)$. *Let* $0 < r < t$ *be a switching time of* u *and let* $\mu \in \{-1, 1\}$. *We consider the perturbation*

$$u^\mu_r(\cdot, \alpha) := \begin{cases} u(r^-) & \text{over } [r - \eta_r, r + \mu\alpha), \\ u(r^+) & \text{over } [r + \mu\alpha, r + \eta_r), \\ u & \text{otherwise,} \end{cases}$$

for all $0 \leq \alpha \leq \frac{\eta_r}{2}$. *Then:*

(i) *there exists* $0 < \alpha_0 \leq \frac{\eta_r}{2}$ *such that* $t(u^\mu_r(\cdot, \alpha), x_0) > t$ *for all* $0 \leq \alpha \leq \alpha_0$;

(ii) *for any* $0 < \lambda \leq t - r$ *fixed, the map*

$$\alpha \in [0, \alpha_0] \longmapsto x(\cdot, u^\mu_r(\cdot, \alpha), x_0) \in \mathrm{C}([r + \lambda, t], \mathbb{R}^n),$$

is differentiable at $\alpha = 0$ *and its derivative is equal to* w^μ_r *being the unique solution (that is global) to the linear homogeneous Cauchy problem given by*

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t, u, x_0), u(t), t) \times w(t), & \text{a.e. } t \in [r, t], \\ w(r) = \mu\Big(f(x(r, u, x_0), u(r^-), r) - f(x(r, u, x_0), u(r^+), r)\Big). \end{cases}$$

Proof. We only prove the case $\mu = 1$ (the proof for the case $\mu = -1$ is similar). First of all note that the variation vector w^μ_r is global (in the sense that it is defined over the whole interval $[r, t]$) since the corresponding Cauchy problem is linear. Let $R := \|u\|_{\mathrm{L}^\infty}$. For the ease of notations we denote by $\mathrm{K} := \mathrm{K}((u, x_0), (R, t))$ and by $C_\mathrm{K} := C_\mathrm{K}((u, x_0), (R, t))$ (see the beginning of Section 2.4.1.1 for these two notations). From Proposition 2.4.2, there exists $\varepsilon > 0$ such that $t(u', x'_0) > t$ for all $(u', x'_0) \in \mathcal{V}((u, x_0), (R, t), \varepsilon)$. Let us take $0 < \alpha_0 \leq \frac{\eta_r}{2}$ small enough such that $r + \alpha_0 < t$ and $2R\alpha_0 \leq \varepsilon$. Then it holds that $u^\mu_r(\cdot, \alpha) = u$ over $[t, +\infty)$, $\|u^\mu_r(\cdot, \alpha) - u\|_{\mathrm{L}^1} \leq 2R\alpha \leq \varepsilon$ and $\|u^\mu_r(\cdot, \alpha)\|_{\mathrm{L}^\infty} \leq R$ for all $0 \leq \alpha \leq \alpha_0$. It follows that $(u^\mu_r(\cdot, \alpha), x_0) \in \mathcal{V}((u, x_0), (R, t), \varepsilon)$ and $t(u^\mu_r(\cdot, \alpha), x_0) > t$ for all $0 \leq \alpha \leq \alpha_0$. The first item of Proposition 2.4.4 is proved. Since $(u^\mu_r(\cdot, \alpha), x_0) \in \mathcal{V}((u, x_0), (R, t), \varepsilon)$ for all $0 \leq \alpha \leq \alpha_0$ and $u^\mu_r(\cdot, \alpha)$ converges to u in $\mathrm{L}^1([0, t], \mathbb{R}^m)$ as α tends to zero, we know from Proposition 2.4.2 that $x(\cdot, u^\mu_r(\cdot, \alpha), x_0)$ converges uniformly to $x(\cdot, u, x_0)$ over $[0, t]$ as α tends to zero and that $(x(t, u^\mu_r(\cdot, \alpha), x_0), u^\mu_r(t, \alpha), t) \in \mathrm{K}$ for a.e. $t \in [0, t]$ and for all $0 \leq \alpha \leq \alpha_0$. Now let us define the function

$$\chi(t, \alpha) := \frac{x(t, u^\mu_r(\cdot, \alpha), x_0) - x(t, u, x_0)}{\alpha} - w^\mu_r(t),$$

for all $t \in [r, t]$ and all $\alpha \in (0, \alpha_0]$. Let $0 < \lambda \leq t - r$ be fixed. We will prove that $\chi(\cdot, \alpha)$ uniformly converges to the zero function on $[r + \lambda, t]$ as α tends to 0. From the integral representation of $\chi(\cdot, \alpha)$, it holds that

$$\chi(t,\alpha) = \chi(r+\alpha,\alpha) + \int_{r+\alpha}^{t} \left[\frac{f(x(s,u_r^\mu(\cdot,\alpha),x_0),u(s),s) - f(x(s,u,x_0),u(s),s)}{\alpha} - \nabla_1 f(x(s,u,x_0),u(s),s) \times w_r^\mu(s) \right] ds,$$

for all $t \in [r+\alpha, t]$ and all $\alpha \in (0, \alpha_0]$. Expanding this expression using Taylor's theorem with integral remainder, we obtain

$$\begin{aligned} \chi(t,\alpha) = \chi(r+\alpha,\alpha) &+ \int_{r+\alpha}^{t} \int_0^1 \nabla_1 f(\star_{\alpha\theta s})\, d\theta \times \chi(s,\alpha)\, ds \\ &+ \int_{r+\alpha}^{t} \left(\int_0^1 \nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\, d\theta \right) \times w_r^\mu(s)\, ds, \end{aligned}$$

for all $t \in [r+\alpha, t]$ and all $\alpha \in (0, \alpha_0]$, where

$$\star_{\alpha\theta s} := (x(s,u,x_0) + \theta(x(s,u_r^\mu(\cdot,\alpha),x_0) - x(s,u,x_0)), u(s), s) \in \mathrm{K},$$

since K is convex with respect to its first two variables. From the Triangle inequality it holds that

$$\|\chi(t,\alpha)\|_{\mathbb{R}^n} \leq \|\chi(r+\alpha,\alpha)\|_{\mathbb{R}^n} + \Phi(\alpha) + C_{\mathrm{K}} \int_{r+\alpha}^{t} \|\chi(s,\alpha)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [r+\alpha, t]$ and all $\alpha \in (0, \alpha_0]$, where the term $\Phi(\alpha)$ is defined to be:

$$\Phi(\alpha) := \int_r^t \int_0^1 \|\nabla_1 f(\star_{\alpha\theta s}) - \nabla_1 f(x(s,u,x_0),u(s),s)\|_{\mathbb{R}^{n\times n}}\, d\theta \|w_r^\mu(s)\|_{\mathbb{R}^n}\, ds.$$

From the Gronwall lemma, it holds that

$$\|\chi(t,\alpha)\|_{\mathbb{R}^n} \leq (\|\chi(r+\alpha,\alpha)\|_{\mathbb{R}^n} + \Phi(\alpha)) e^{C_{\mathrm{K}} t},$$

for all $t \in [r+\alpha, t]$ and all $\alpha \in (0, \alpha_0]$. Since we only want to prove the uniform convergence of $\chi(\cdot, \alpha)$ to the zero function on $[r + \lambda, t]$ as α tends to 0 and since the estimate on the right-hand side is independent of t, we only need to prove that $\chi(r + \alpha, \alpha)$ tends to $0_{\mathbb{R}^n}$ and $\Phi(\alpha)$ tends to 0 as α tends to zero. The convergence of $\Phi(\alpha)$ can be obtained with the Lebesgue dominated convergence theorem. Now let us prove that $\chi(r + \alpha, \alpha)$ tends to $0_{\mathbb{R}^n}$ as α tends to zero. Since $x(r, u_r^\mu(\cdot, \alpha), x_0) = x(r, u, x_0)$ and from the integral representations of $x(\cdot, u_r^\mu(\cdot, \alpha), x_0)$ and $x(\cdot, u, x_0)$, it holds that

$$\begin{aligned} \chi(r+\alpha,\alpha) &= \int_r^{r+\alpha} \frac{f(x(s,u_r^\mu(\cdot,\alpha),x_0),u_r^\mu(s,\alpha),s) - f(x(s,u,x_0),u(s),s)}{\alpha}\, ds - w_r^\mu(r+\alpha) \\ &= \int_r^{r+\alpha} \frac{f(x(s,u_r^\mu(\cdot,\alpha),x_0),u(r^-),s) - f(x(s,u,x_0),u(r^+),s)}{\alpha}\, ds - w_r^\mu(r+\alpha) \\ &= \int_r^{r+\alpha} \frac{f(x(s,u,x_0),u(r^-),s) - f(x(s,u,x_0),u(r^+),s)}{\alpha}\, ds - w_r^\mu(r+\alpha) \\ &+ \int_r^{r+\alpha} \frac{f(x(s,u_r^\mu(\cdot,\alpha),x_0),u(r^-),s) - f(x(s,u,x_0),u(r^-),s)}{\alpha}\, ds, \end{aligned}$$

for all $\alpha \in (0,\alpha_0]$. Let us deal with the three terms above. Since the first above integrand is continuous, it is clear that r is a Lebesgue point and we get that

$$\lim_{\alpha\to 0}\int_r^{r+\alpha}\frac{f(x(s,u,x_0),u(r^-),s)-f(x(s,u,x_0),u(r^+),s)}{\alpha}\,ds$$
$$=f(x(r,u,x_0),u(r^-),r)-f(x(r,u,x_0),u(r^+),r)=w_r^\mu(r).$$

Secondly, from the continuity of w_r^μ, we know that $w_r^\mu(r+\alpha)$ tends to $w_r^\mu(r)$ as α tends to 0. Finally, using the Lipschitz continuity of f over K, we get that

$$\left\|\int_r^{r+\alpha}\frac{f(x(s,u_r^\mu(\cdot,\alpha),x_0),u(r^-),s)-f(x(s,u,x_0),u(r^-),s)}{\alpha}\,ds\right\|_{\mathbb{R}^n}$$
$$\leq\frac{C_{\mathrm{K}}}{\alpha}\int_r^{r+\alpha}\|x(s,u_r^\mu(\cdot,\alpha),x_0)-x(s,u,x_0)\|_{\mathbb{R}^n}\,ds\leq C_{\mathrm{K}}\|x(\cdot,u_r^\mu(\cdot,\alpha),x_0)-x(\cdot,u,x_0)\|_{\mathrm{C}([0,t],\mathbb{R}^n)}.$$

Since $x(\cdot,u_r^\mu(\cdot,\alpha),x_0)$ converges uniformly to $x(\cdot,u,x_0)$ over $[0,t]$ as α tends to 0, the proof is complete. □

We conclude this section by a technical lemma on the convergence of the variation vectors. This result is needed in the proof of our main result (see Section 2.4.3.2).

Lemma 2.4.5. *Let $(u,x_0)\in\mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^n$. Let $R\geq\|u\|_{\mathrm{L}^\infty}$ and let $0<t<t(u,x_0)$. We take $\varepsilon>0$ as in Proposition 2.4.2. Let $(u_k,x_{0,k})_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{V}((u,x_0),(R,t),\varepsilon)$ such that $x_{0,k}$ converges to x_0 and $u_k(t)$ converges to $u(t)$ for a.e. $t\in[0,t]$. Let $0<r<t$ be a switching time of u and r_k be a switching time of u_k for all $k\in\mathbb{N}$. Let us assume that r_k converges to r and that $u_k(r_k^-)$ and $u_k(r_k^+)$ converge respectively to $u(r^-)$ and $u(r^+)$. Finally let $\mu\in\{-1,1\}$ and let $w_{r_k}^{\mu,k}$ be the unique solution (that is global) to the linear homogeneous Cauchy problem given by*

$$\begin{cases}\dot{w}(t)=\nabla_1 f(x(t,u_k,x_{0,k}),u_k(t),t)\times w(t), & a.e.\ t\in[r_k,t],\\ w(r_k)=\mu\Big(f(x(r_k,u_k,x_{0,k}),u_k(r_k^-),r_k)-f(x(r_k,u_k,x_{0,k}),u_k(r_k^+),r_k)\Big),\end{cases}$$

for all $k\in\mathbb{N}$. Then, for any $0<\lambda\leq t-r$ fixed, the sequence $(w_{r_k}^{\mu,k})_{k\in\mathbb{N}}$ uniformly converges to w_r^μ over $[r+\lambda,t]$, where the variation vector w_r^μ is defined as in Proposition 2.4.5.

Proof. First of all, for all $k\in\mathbb{N}$, note that the variation vector $w_{r_k}^{\mu,k}$ is global (in the sense that it is defined over the whole interval $[r_k,t]$) since the corresponding Cauchy problem is linear. In this proof we denote by $\mathrm{K}:=\mathrm{K}((u,x_0),(R,t))$ and by $C_{\mathrm{K}}:=C_{\mathrm{K}}((u,x_0),(R,t))$ (see the beginning of Section 2.4.1.1 for these two notations). From Proposition 2.4.2, it is clear that $\|\nabla_1 f(x(t,u_k,x_{0,k}),u_k(t),t)\|_{\mathbb{R}^{n\times n}}\leq C_{\mathrm{K}}$ for a.e. $t\in[0,t]$ and all $k\in\mathbb{N}$. From the integral representation of $w_{r_k}^{\mu,k}$, it holds that

$$w_{r_k}^{\mu,k}(t)=w_{r_k}^{\mu,k}(r_k)+\int_{r_k}^t\nabla_1 f(x(s,u_k,x_{0,k}),u_k(s),s)\times w_{r_k}^{\mu,k}(s)\,ds,$$

for all $t\in[r_k,t]$ and all $k\in\mathbb{N}$. We deduce that

$$\|w_{r_k}^{\mu,k}(t)\|_{\mathbb{R}^n}\leq\|w_{r_k}^{\mu,k}(r_k)\|_{\mathbb{R}^n}+C_{\mathrm{K}}\int_{r_k}^t\|w_{r_k}^{\mu,k}(s)\|_{\mathbb{R}^n}\,ds,$$

and, from the Gronwall lemma, that $\|w^{\mu,k}_{r_k}(t)\|_{\mathbb{R}^n} \leq \|w^{\mu,k}_{r_k}(r_k)\|_{\mathbb{R}^n} e^{C_{\mathrm{K}} t}$ for all $t \in [r_k, t]$ and all $k \in \mathbb{N}$. From Proposition 2.4.2, we know that $x(r_k, u_k, x_{0,k})$ converges to $x(r, u, x_0)$ and, from the continuity of f and the hypotheses, it is clear that $w^{\mu,k}_{r_k}(r_k)$ tends to $w^{\mu}_r(r)$. We deduce that there exists a constant $C \geq 0$ such that $\|w^{\mu,k}_{r_k}(t)\|_{\mathbb{R}^n} \leq C$ for all $t \in [r_k, t]$ and all $k \in \mathbb{N}$. Now we define $\overline{r}_k := \max(r_k, r)$ for all $k \in \mathbb{N}$. Note that $\overline{r}_k$ tends to r. From the integral representations of $w^{\mu,k}_{r_k}$ and w^{μ}_r, it holds that

$$w^{\mu,k}_{r_k}(t) - w^{\mu}_r(t) = w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k) + \int_{\overline{r}_k}^{t} \nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) \times (w^{\mu,k}_{r_k}(s) - w^{\mu}_r(s))\, ds$$
$$+ \int_{\overline{r}_k}^{t} (\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) - \nabla_1 f(x(s, u, x_0), u(s), s)) \times w^{\mu}_r(s)\, ds,$$

for all $t \in [\overline{r}_k, t]$ and all $k \in \mathbb{N}$. From the Triangle inequality, it holds that

$$\|w^{\mu,k}_{r_k}(t) - w^{\mu}_r(t)\|_{\mathbb{R}^n} \leq \|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n} + \Gamma_k + C_{\mathrm{K}} \int_{\overline{r}_k}^{t} \|w^{\mu,k}_{r_k}(s) - w^{\mu}_r(s)\|_{\mathbb{R}^n}\, ds,$$

for all $t \in [\overline{r}_k, t]$ and all $k \in \mathbb{N}$, where the term Γ_k is defined to be:

$$\Gamma_k := \int_{r}^{t} \|\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s) - \nabla_1 f(x(s, u, x_0), u(s), s)\|_{\mathbb{R}^{n\times n}} \|w^{\mu}_r(s)\|_{\mathbb{R}^n}\, ds.$$

From the Gronwall lemma, we obtain

$$\|w^{\mu,k}_{r_k}(t) - w^{\mu}_r(t)\|_{\mathbb{R}^n} \leq (\|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n} + \Gamma_k) e^{C_{\mathrm{K}} t},$$

for all $t \in [\overline{r}_k, t]$ and all $k \in \mathbb{N}$. Since we only want to prove the uniform convergence of $w^{\mu,k}_{r_k}$ to w^{μ}_r on $[r + \lambda, t]$ (and since $\overline{r}_k$ converges to r) and since the estimate on the right-hand side is independent of t, we only need to prove that $\|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n}$ and Γ_k converge to 0 as k tends to $+\infty$. The convergence of Γ_k can be obtained with the Lebesgue dominated convergence theorem. Now let us prove that $\|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n}$ tends to 0 as k tends to $+\infty$. It holds that

$$\|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n}$$
$$\leq \|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu,k}_{r_k}(r_k)\|_{\mathbb{R}^n} + \|w^{\mu,k}_{r_k}(r_k) - w^{\mu}_r(r)\|_{\mathbb{R}^n} + \|w^{\mu}_r(r) - w^{\mu}_r(\overline{r}_k)\|_{\mathbb{R}^n},$$

for all $k \in \mathbb{N}$. Let us deal with the three terms above. Firstly, from the integral representation of $w^{\mu,k}_{r_k}$, it holds that

$$\|w^{\mu,k}_{r_k}(\overline{r}_k) - w^{\mu,k}_{r_k}(r_k)\|_{\mathbb{R}^n} \leq \int_{r_k}^{\overline{r}_k} \|\nabla_1 f(x(s, u_k, x_{0,k}), u_k(s), s)\|_{\mathbb{R}^{n\times n}} \|w^{\mu,k}_{r_k}(s)\|_{\mathbb{R}^n}\, ds$$
$$\leq C_{\mathrm{K}} C (\overline{r}_k - r_k),$$

for all $k \in \mathbb{N}$. Secondly we have already mentioned that $w^{\mu,k}_{r_k}(r_k)$ tends to $w^{\mu}_r(r)$ as k tends to $+\infty$. Thirdly we use the continuity of w^{μ}_r to conclude the proof. □

2.4.2 A technical control set

In this section our aim is to introduce a technical control set which guarantees that the sampling times produced by the Ekeland variational principle in the next Section 2.4.3, firstly, remain unchanged for the ones corresponding to the consecutive sampling intervals on which the optimal control is constant and, secondly, are contained in disjoint intervals for the others. In particular, Proposition 2.4.7 guarantees that from any sequence of sampled-data controls which belong to the aforementioned technical control set and which converges in the L^1-norm, one can extract a subsequence which converges almost everywhere with, moreover, the sampling times converging as well. We begin by introducing the following notions.

Let $t > 0$ and $N \in \mathbb{N}^*$ be fixed. For all $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N$ and $u \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$, we denote by

$$\|\mathbb{T}\| := \min\{t_{i+1} - t_i \mid i = 0,\dots,N-1\} > 0,$$

and we define the set

$$\mathcal{P}^t_{N,(u,\mathbb{T})} := \left\{ \mathbb{T}' = \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N \mid \forall i = 1,\dots,N-1,\ |t'_i - t_i| \leq \delta_{\{u_{i-1}\neq u_i\}} \frac{\|\mathbb{T}\|}{4} \right\},$$

where $\delta_{\{u_{i-1}\neq u_i\}} = 1$ if $u_{i-1} \neq u_i$, and $\delta_{\{u_{i-1}\neq u_i\}} = 0$ otherwise. In particular, if $\mathbb{T}' = \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$, it holds that

$$0 = t'_0 < t_1 - \frac{\|\mathbb{T}\|}{4} \leq t'_1 \leq t_1 + \frac{\|\mathbb{T}\|}{4} < t_2 - \frac{\|\mathbb{T}\|}{4} \leq t'_2 \leq t_2 + \frac{\|\mathbb{T}\|}{4} < \dots$$
$$\dots < t_{N-2} - \frac{\|\mathbb{T}\|}{4} \leq t'_{N-2} \leq t_{N-2} + \frac{\|\mathbb{T}\|}{4} < t_{N-1} - \frac{\|\mathbb{T}\|}{4} \leq t'_{N-1} \leq t_{N-1} + \frac{\|\mathbb{T}\|}{4} < t'_N = t,$$

with $t'_i = t_i$ for all $i \in \{1,\dots,N-1\}$ such that $u_{i-1} = u_i$. Hence, for all $\mathbb{T}' = \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$, the elements $t'_i = t_i$ remain unchanged when u is constant over two consecutive sampling intervals $[t_{i-1}, t_i)$ and $[t_i, t_{i+1})$ and all the elements t'_i live in intervals which are (strictly) disjoint. Finally we introduce the following technical control set

$$\mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m) := \bigcup_{\mathbb{T}' \in \mathcal{P}^t_{N,(u,\mathbb{T})}} \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m). \tag{2.5}$$

Of course note that $\mathbb{T} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ and thus $u \in \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$. Also note that the inclusion $\mathcal{P}^t_{N,(u,\mathbb{T})} \subset \mathcal{P}^t_N$ holds and thus $\mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ is included in $\mathrm{PC}_N([0,t],\mathbb{R}^m) \subset \mathrm{L}^\infty([0,t],\mathbb{R}^m)$, but is not a linear subspace, neither a convex subset.

Lemma 2.4.6. *Let $t > 0$ and $N \in \mathbb{N}^*$. Let $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N$ and $u \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$. Then $\mathbb{T}$ is the unique element $\mathbb{T}' \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ such that $u \in \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m)$.*

Proof. Let $\mathbb{T}' = \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ be such that $u \in \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m)$. Let us assume by contradiction that $\mathbb{T}' \neq \mathbb{T}$. Let $i \in \{1,\dots,N-1\}$ such that $t_i \notin \mathbb{T}'$. Necessarily it holds that $u_{i-1} \neq u_i$ and there exists $j \in \{i-1, i\}$ such that $t'_j < t_i < t'_{j+1}$. Since $u \in \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m)$, there exists $c \in \mathbb{R}^m$ such that $u(t) = c$ for a.e. $t \in [t'_j, t'_{j+1}]$. Since $u(t) = u_{i-1}$ for a.e. $t \in [t_{i-1}, t_i]$ and $u(t) = u_i$ for a.e. $t \in [t_i, t_{i+1}]$, we deduce that $c = u_{i-1}$ and $c = u_i$ which raises a contradiction since $u_{i-1} \neq u_i$. The proof is complete. □

Proposition 2.4.6. *Let $t > 0$ and $N \in \mathbb{N}^*$. Let $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N$ and $u \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$. The set $\mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ is a closed subset of $\mathrm{L}^1([0,t],\mathbb{R}^m)$.*

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ converging in $\mathrm{L}^1([0,t],\mathbb{R}^m)$ to some $u' \in \mathrm{L}^1([0,t],\mathbb{R}^m)$. Our aim is to prove that $u' \in \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$. The proof is divided in three steps.

First step: Let $\mathbb{T}_k = \{t_{i,k}\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ be a partition associated to u_k for all $k \in \mathbb{N}$. It holds for all $k \in \mathbb{N}$ that

$$0 = t_{0,k} < t_1 - \frac{\|\mathbb{T}\|}{4} \leq t_{1,k} \leq t_1 + \frac{\|\mathbb{T}\|}{4} < t_2 - \frac{\|\mathbb{T}\|}{4} \leq t_{2,k} \leq t_2 + \frac{\|\mathbb{T}\|}{4} < \dots$$
$$\dots < t_{N-2} - \frac{\|\mathbb{T}\|}{4} \leq t_{N-2,k} \leq t_{N-2} + \frac{\|\mathbb{T}\|}{4} < t_{N-1} - \frac{\|\mathbb{T}\|}{4} \leq t_{N-1,k} \leq t_{N-1} + \frac{\|\mathbb{T}\|}{4} < t_{N,k} = t,$$

and $t_{i,k} = t_i$ for all $i \in \{1,\dots,N-1\}$ such that $u_{i-1} = u_i$. Extracting a finite number of subsequences (that we do not relabel), we know that, for all $i \in \{0,\dots,N\}$, $t_{i,k}$ converges to some t'_i satisfying

$$0 = t'_0 < t_1 - \frac{\|\mathbb{T}\|}{4} \leq t'_1 \leq t_1 + \frac{\|\mathbb{T}\|}{4} < t_2 - \frac{\|\mathbb{T}\|}{4} \leq t'_2 \leq t_2 + \frac{\|\mathbb{T}\|}{4} < \dots$$
$$\dots < t_{N-2} - \frac{\|\mathbb{T}\|}{4} \leq t'_{N-2} \leq t_{N-2} + \frac{\|\mathbb{T}\|}{4} < t_{N-1} - \frac{\|\mathbb{T}\|}{4} \leq t'_{N-1} \leq t_{N-1} + \frac{\|\mathbb{T}\|}{4} < t'_N = t,$$

and $t'_i = t_i$ for all $i \in \{1,\dots,N-1\}$ such that $u_{i-1} = u_i$. Hence we have obtained a partition $\mathbb{T}' := \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$.

Second step: Extracting a subsequence (that we do not relabel) from the partial converse of the Lebesgue dominated convergence theorem, we know that $u_k(t)$ converges to $u'(t)$ for a.e. $t \in [0,t]$. We introduce the subset A of $[0,t]$ of full measure defined by

$$A := \{t \in [0,t] \mid u_k(t) \text{ converges to } u'(t)\},$$

and the subset B of $[0,t]$ of full measure defined by $B := \cap_{k\in\mathbb{N}} B_k$ where

$$B_k := \bigcup_{i=0}^{N-1} \{t \in [t_{i,k}, t_{i+1,k}) \mid u_k(t) = u_{i,k}\},$$

for all $k \in \mathbb{N}$.

Third step: Let $i \in \{0,\dots,N-1\}$ and let $t \in (t'_i, t'_{i+1}) \cap (A \cap B)$. For $k \in \mathbb{N}$ sufficiently large, it holds that $t \in (t_{i,k}, t_{i+1,k})$. Since $t \in A \cap B$, we know that $u_k(t) = u_{i,k}$ which converges to $u'(t)$. Since the convergence of $u_{i,k}$ to $u'(t)$ is independent of the choice of $t \in (t'_i, t'_{i+1}) \cap (A \cap B)$, we deduce that u is equal almost everywhere over $[t'_i, t'_{i+1}]$ to a constant. Since the last sentence is true for every $i \in \{0,\dots,N-1\}$, we conclude that $u' \in \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m) \subset \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$. The proof is complete. □

Proposition 2.4.7. *Let $t > 0$ and $N \in \mathbb{N}^*$. Let $\mathbb{T} = \{t_i\}_{i=0,\dots,N} \in \mathcal{P}^t_N$ and $u \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ converging in $\mathrm{L}^1([0,t],\mathbb{R}^m)$ to u. Let $\mathbb{T}_k = \{t_{i,k}\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ be a partition associated to u_k for all $k \in \mathbb{N}$. Then there exists a subsequence of $(u_k)_{k\in\mathbb{N}}$ (that we do not relabel) such that:*

(i) $u_k(t)$ *converges to* $u(t)$ *for a.e.* $t \in [0,t]$*;*

(ii) $t_{i,k}$ *converges to* t_i *for all* $i = 0,\dots,N$*;*

(iii) $u_{i,k}$ *converges to* u_i *for all* $i = 0,\dots,N-1$*.*

Proof. Following exactly the same steps as in the proof of Proposition 2.4.6 (replacing u' by u), we construct a partition $\mathbb{T}' = \{t'_i\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ such that $u \in \mathrm{PC}^{\mathbb{T}'}([0,t],\mathbb{R}^m)$. From Lemma 2.4.6, it implies that $\mathbb{T}' = \mathbb{T}$. From the construction of $\mathbb{T}'$, we conclude that, up to subsequences (which we do not relabel), $t_{i,k}$ converges to t_i for all $i = 0,\dots,N$. Let us prove the last statement. Let us consider the sets A and B defined in the proof of Proposition 2.4.6 and let us introduce the subset B' of $[0,t]$ of full measure defined by

$$B' := \bigcup_{i=0}^{N-1} \{t \in [t_i,t_{i+1}) \mid u(t) = u_i\}.$$

Let $i = 0,\dots,N-1$ and let $t \in (t_i,t_{i+1}) \cap (A \cap B \cap B')$. For $k \in \mathbb{N}$ sufficiently large, it holds that $t \in (t_{i,k},t_{i+1,k})$. Moreover, since $t \in A \cap B \cap B'$, we know that $u_k(t) = u_{i,k}$ converges to $u(t) = u_i$. Since the last statement is true for all $i = 0,\dots,N-1$, the proof is complete. □

2.4.3 Application of the Ekeland variational principle in the case $L = 0$

We are now in a position to give a proof of Theorem 2.2.1 based on the following simplified version of the Ekeland variational principle (see [Ekeland 1974, Theorem 1.1 p.324]).

Proposition 2.4.8 (Ekeland variational principle)**.** *Let* $(\mathrm{E},\mathrm{d_E})$ *be a complete metric set. Let* $\mathcal{J} : \mathrm{E} \to \mathbb{R}^+$ *be a continuous nonnegative map. Let* $\varepsilon > 0$ *and* $\lambda^* \in \mathrm{E}$ *such that* $\mathcal{J}(\lambda^*) \leq \varepsilon$*. Then there exists* $\lambda_\varepsilon \in \mathrm{E}$ *such that* $\mathrm{d_E}(\lambda_\varepsilon,\lambda^*) \leq \sqrt{\varepsilon}$*, and* $-\sqrt{\varepsilon}\,\mathrm{d_E}(\lambda,\lambda_\varepsilon) \leq \mathcal{J}(\lambda) - \mathcal{J}(\lambda_\varepsilon)$ *for all* $\lambda \in \mathrm{E}$*.*

We first apply the Ekeland variational principle in the case where $L = 0$ in Problem (OSCP) (with no Lagrange cost). The case $L \neq 0$ (with Lagrange cost) is treated in the next Section 2.4.4. In this section we will also assume that the final time and the N-partition are free in Problem (OSCP) (the two simpler cases where only the final time is fixed, and where both of them are fixed can both be treated in very similar ways).

Let $(T,\mathbb{T},x,u)$ be a solution to Problem (OSCP). In the sequel we will consider that $u \in \mathrm{L}^\infty(\mathbb{R}_+,\mathbb{R}^m)$ by considering the extension

$$\begin{cases} u & \text{over } [0,T), \\ u_{N-1} & \text{over } [T,+\infty). \end{cases}$$

In particular, using the notations of Section 2.4.1, note that $x = x(\cdot,u,x(0))$ and that $\tau(u,x(0)) > T$. In the rest of the proof we fix some t_0, t such that

$$t_0 := T - \frac{T-t_{N-1}}{3} \qquad \text{and} \qquad T < t < \min\left(T + \frac{T-t_{N-1}}{3}, t(u,x(0))\right).$$

In particular it holds that $t_{N-1} < t_0 < T < t < t(u,x(0))$. Replacing $t_N = T$ by $t_N = t$, it holds that $\mathbb{T} \in \mathcal{P}^t_N$ and, with the above extension of u, it holds that $u \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$. We conclude by noting that, with the new value of $\|\mathbb{T}\|$, it holds that $t_{N-1} + \frac{\|\mathbb{T}\|}{4} < t_0$.

2.4.3.1 Fix $R \in \mathbb{N}$ such that $R \geq \|u\|_{L^\infty}$

In this section we fix $R \in \mathbb{N}$ such that $R \geq \|u\|_{L^\infty}$ and we denote by

$$\mathcal{V}_\varepsilon^R := \big\{(u', x_0') \in \mathcal{V}((u, x(0)), (R, t), \varepsilon) \mid u' \in \mathrm{PC}_{N,(u,\mathbb{T})}([0, t], \mathbb{R}^m) \text{ with } u'(t) \in \mathrm{U} \text{ for a.e. } t \in [0, t]\big\},$$

where $\varepsilon > 0$ is given in Proposition 2.4.2. We endow the set $\mathcal{V}_\varepsilon^R \times [t_0, t]$ with the $\mathrm{L}^1([0, t], \mathbb{R}^m) \times \mathbb{R}^n \times \mathbb{R}$-distance. Endowed with this distance, it can be seen from Proposition 2.4.6, from the closedness assumption on U and from the partial converse of the Lebesgue dominated convergence theorem that $\mathcal{V}_\varepsilon^R \times [t_0, t]$ is a complete metric set.

Let us consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to zero such that $0 < \sqrt{\varepsilon_k} < \varepsilon$ for all $k \in \mathbb{N}$. Then we define the penalized functional $\mathcal{J}_k^R : \mathcal{V}_\varepsilon^R \times [t_0, t] \to \mathbb{R}_+$ by

$$\mathcal{J}_k^R(u', x_0', T') := \sqrt{\Big(g(x_0', x(T', u', x_0'), T') - g(x(0), x(T), T) + \varepsilon_k\Big)^{+2} + \mathrm{d}_\mathrm{S}^2\Big(h(x_0', x(T', u', x_0'), T')\Big)},$$

for all $(u', x_0', T') \in \mathcal{V}_\varepsilon^R \times [t_0, t]$ and all $k \in \mathbb{N}$.

Since g, h and d_S^2 are continuous and from Proposition 2.4.2, it follows that $\mathcal{J}_k^R$ is a continuous nonnegative map over $\mathcal{V}_\varepsilon^R \times [t_0, t]$ for all $k \in \mathbb{N}$. Furthermore it is clear that $\mathcal{J}_k^R(u, x(0), T) = \varepsilon_k$ for all $k \in \mathbb{N}$. Therefore, from the Ekeland variational principle (see Proposition 2.4.8), we conclude that there exists a sequence $(u_k, x_{0,k}, T_k)_{k\in\mathbb{N}} \subset \mathcal{V}_\varepsilon^R \times [t_0, t]$ such that

$$\mathrm{d}_{\mathrm{L}^1([0,t],\mathbb{R}^m)\times\mathbb{R}^n\times\mathbb{R}}((u_k, x_{0,k}, T_k), (u, x(0), T)) \leq \sqrt{\varepsilon_k}, \tag{2.6}$$

and

$$-\sqrt{\varepsilon_k}\, \mathrm{d}_{\mathrm{L}^1([0,t],\mathbb{R}^m)\times\mathbb{R}^n\times\mathbb{R}}((u', x_0', T'), (u_k, x_{0,k}, T_k)) \leq \mathcal{J}_k^R(u', x_0', T') - \mathcal{J}_k^R(u_k, x_{0,k}, T_k), \tag{2.7}$$

for all $(u', x_0', T') \in \mathcal{V}_\varepsilon^R \times [t_0, t]$ and all $k \in \mathbb{N}$.

By contradiction let us assume that there exists $(u', x_0', T') \in \mathcal{V}_\varepsilon^R \times [t_0, t]$ such that $\mathcal{J}_k^R(u', x_0', T') = 0$. In particular we have $0 < T' \leq t$. Let us denote by $x' = x(\cdot, u', x_0') \in \mathrm{AC}([0, T'], \mathbb{R}^n)$. Since $u' \in \mathrm{PC}_{N,(u,\mathbb{T})}([0, t], \mathbb{R}^m)$, there exists $\mathbb{T}' = \{t_i'\}_{i=0,\ldots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ such that $u' \in \mathrm{PC}^{\mathbb{T}'}([0, t], \mathbb{R}^m)$. Since $\mathbb{T}' = \{t_i'\}_{i=0,\ldots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$, we know that $t_{N-1}' \leq t_{N-1} + \frac{\|\mathbb{T}\|}{4} < t_0 \leq T' \leq t$. Then, replacing $t_N' = t$ by $t_N' = T'$, we get that $\mathbb{T}' \in \mathcal{P}_N^{T'}$ and $u' \in \mathrm{PC}^{\mathbb{T}'}([0, T'], \mathbb{R}^m)$. Moreover it holds that $\dot{x}'(t) = f(x'(t), u'(t), t)$ and $u'(t) \in \mathrm{U}$ for almost every $t \in [0, T']$. Since $\mathcal{J}_k^R(u', x_0', T') = 0$, we deduce moreover that $h(x'(0), x'(T'), T') \in \mathrm{S}$. Thus the quadruple $(T', \mathbb{T}', x', u')$ satisfies all constraints of Problem (OSCP) and thus $g(x'(0), x'(T'), T') \geq g(x(0), x(T), T)$ from optimality of the quadruple $(T, \mathbb{T}, x, u)$. This raises a contradiction with the equality $\mathcal{J}_k^R(u', x_0', T') = 0$. We conclude that $\mathcal{J}_k^R(u', x_0', T') > 0$ for all $(u', x_0', T') \in \mathcal{V}_\varepsilon^R \times [t_0, t]$.

From the above paragraph we can correctly define the couple $(\psi_k^{0R}, \psi_k^R) \in \mathbb{R} \times \mathbb{R}^j$ as

$$\psi_k^{0R} := \frac{-1}{\mathcal{J}_k^R(u_k, x_{0,k}, T_k)} \Big(g(x_{0,k}, x(T_k, u_k, x_{0,k}), T_k) - g(x(0), x(T), T) + \varepsilon_k\Big)^+$$

and

$$\psi_k^R := \frac{-1}{\mathcal{J}_k^R(u_k,x_{0,k},T_k)}\Big(h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k) - \mathrm{P_S}(h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k))\Big),$$

for all $k \in \mathbb{N}$. Note that $\psi_k^{0R} \in \mathbb{R}_-$ and $-\psi_k^R \in \mathrm{N_S}[\mathrm{P_S}(h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k))]$ from Lemma 1.3.1 for all $k \in \mathbb{N}$.

Since $(u_k,x_{0,k},T_k) \in \mathcal{V}_\varepsilon^R \times [t_0,t]$, we know that $u_k \in \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ for all $k \in \mathbb{N}$. Let us denote by $\mathbb{T}_k = \{t_{i,k}\}_{i=0,\dots,N} \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ a partition associated to u_k for all $k \in \mathbb{N}$. Moreover, from Inequality (2.6), the sequence $(u_k)_{k\in\mathbb{N}}$ converges to u in $\mathrm{L}^1([0,t],\mathbb{R}^m)$. Thus we can extract from Proposition 2.4.7 a subsequence (which we do not relabel) such that $u_k(t)$ converges to $u(t)$ for almost every $t \in [0,t]$, $t_{i,k}$ converges to t_i for all $i = 0,\dots,N$ and $u_{i,k}$ converges to u_i for all $i = 0,\dots,N-1$. From Inequality (2.6), we know that $x_{0,k}$ and T_k converge respectively to $x(0)$ and T. From Proposition 2.4.2, we deduce that $x(T_k,u_k,x_{0,k})$ converges to $x(T)$ and thus $h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)$ converges to $h(x(0),x(T),T) \in \mathrm{S}$. Finally, from the definition of $\mathcal{J}_k^R$, it is clear that $|\psi_k^{0R}|^2 + \|\psi_k^R\|^2_{\mathbb{R}^j} = 1$ for all $k \in \mathbb{N}$. By a compactness argument, we can extract subsequences (which we do not relabel) such that ψ_k^{0R} converges to some $\psi^{0R} \in \mathbb{R}_-$ and ψ_k^R converges to some $\psi^R \in \mathbb{R}^j$ which satisfies $-\psi^R \in \mathrm{N_S}[h(x(0),x(T),T)]$ from Lemma 1.3.2. Note that $|\psi^{0R}|^2 + \|\psi^R\|^2_{\mathbb{R}^j} = 1$.

2.4.3.2 Crucial inequalities depending on R fixed in the previous section

In this section we will use Inequality (2.7) along with the perturbations defined in Section 2.4.1 to obtain four crucial inequalities (depending on R fixed in the previous section). The perturbations will be considered on u_k, $x_{0,k}$, $t_{i,k}$, but also on T_k.

Lemma 2.4.7. *Let $v \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$ taking values in $\mathrm{U} \cap \overline{\mathrm{B}}_{\mathbb{R}^m}(0_{\mathbb{R}^m},R)$. Then the inequality*

$$\Big\langle \psi^{0R}\nabla_2 g(x(0),x(T),T) + \nabla_2 h(x(0),x(T),T)^\top \times \psi^R, w_v(T)\Big\rangle_{\mathbb{R}^n} \leq 0, \tag{2.8}$$

where w_v is defined in Proposition 2.4.3, holds true.

Proof. The proof is divided in three steps.

First step: For all $k \in \mathbb{N}$, let us define

$$v_k(t) := v_i \quad \text{if } t \in [t_{i,k},t_{i+1,k}) \text{ for some } i \in \{0,\dots,N-1\},$$

for all $t \in [0,t)$. Then $v_k \in \mathrm{PC}^{\mathbb{T}_k}([0,t],\mathbb{R}^m)$ for all $k \in \mathbb{N}$ and, since $t_{i,k}$ converges to t_i for all $i = 0,\dots,N$, it is clear that the sequence $(v_k)_{k\in\mathbb{N}}$ converges to v in $\mathrm{L}^1([0,t],\mathbb{R}^m)$. It is also true that v_k takes its values in $\mathrm{U} \cap \overline{\mathrm{B}}_{\mathbb{R}^m}(0_{\mathbb{R}^m},R)$ for all $k \in \mathbb{N}$.

Second step: Let us fix $k \in \mathbb{N}$. We define as in Proposition 2.4.3 the convex perturbation

$$u_{k,v_k}(\cdot,\alpha) := \begin{cases} u_k + \alpha(v_k - u_k) & \text{over } [0,t), \\ u_k = u & \text{over } [t,+\infty), \end{cases}$$

for all $0 \leq \alpha \leq 1$. First of all, note that $u_{k,v_k}(\cdot,\alpha) \in \mathrm{PC}^{\mathbb{T}_k}([0,t],\mathbb{R}^m) \subset \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ and, since U is convex, that $u_{k,v_k}(\cdot,\alpha)$ takes its values in U for all $0 \leq \alpha \leq 1$. Moreover, it holds that $\|u_{k,v_k}(\cdot,\alpha)\|_{\mathrm{L}^\infty} \leq R$ and

$$\|u_{k,v_k}(\cdot,\alpha) - u\|_{\mathrm{L}^1} \leq \|u_{k,v_k}(\cdot,\alpha) - u_k\|_{\mathrm{L}^1} + \|u_k - u\|_{\mathrm{L}^1} \leq \alpha\|v_k - u_k\|_{\mathrm{L}^1} + \sqrt{\varepsilon_k}.$$

Since $\sqrt{\varepsilon_k} < \varepsilon$, it follows that there exists $0 < \alpha_0 \leq 1$ small enough such that $(u_{k,v_k}(\cdot,\alpha), x_{0,k}) \in \mathcal{V}^R_\varepsilon$ for all $\alpha \in [0,\alpha_0]$. From Inequality (2.7) we obtain

$$-\sqrt{\varepsilon_k}\|u_{k,v_k}(\cdot,\alpha) - u_k\|_{\mathrm{L}^1} \leq \mathcal{J}^R_k(u_{k,v_k}(\cdot,\alpha), x_{0,k}, T_k) - \mathcal{J}^R_k(u_k, x_{0,k}, T_k),$$

and thus

$$\begin{aligned} & -\sqrt{\varepsilon_k}\|v_k - u_k\|_{\mathrm{L}^1} \\ \leq & \frac{1}{\mathcal{J}^R_k(u_{k,v_k}(\cdot,\alpha), x_{0,k}, T_k) + \mathcal{J}^R_k(u_k, x_{0,k}, T_k)} \times \frac{\mathcal{J}^R_k(u_{k,v_k}(\cdot,\alpha), x_{0,k}, T_k)^2 - \mathcal{J}^R_k(u_k, x_{0,k}, T_k)^2}{\alpha}, \end{aligned}$$

for all $\alpha \in (0,\alpha_0]$. Taking the limit as α tends to 0 and using the definitions of ψ^{0R}_k and ψ^R_k, we obtain from Proposition 2.4.3 that

$$\begin{aligned} \Big\langle \psi^{0R}_k \nabla_2 g(x_{0,k}, x(T_k, u_k, x_{0,k}), T_k) + \nabla_2 h(x_{0,k}, x(T_k, u_k, x_{0,k}), T_k)^\top \times \psi^R_k, w^k_{v_k}(T_k) \Big\rangle_{\mathbb{R}^n} \\ \leq \sqrt{\varepsilon_k}\|v_k - u_k\|_{\mathrm{L}^1}. \end{aligned}$$

where $w^k_{v_k}$ is defined in Lemma 2.4.3.

Third step: We take the limit of the above inequality as k tends to $+\infty$. Since g and h are both of class C^1 and from the uniform convergence of $(w^k_{v_k})_{k\in\mathbb{N}}$ to w_v over $[0,t]$ (see Lemma 2.4.3), it holds that

$$\Big\langle \psi^{0R} \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi^R, w_v(T) \Big\rangle_{\mathbb{R}^n} \leq 0.$$

The proof is complete. □

Lemma 2.4.8. *Let $y \in \mathbb{R}^n$ be fixed. Then the inequality*

$$\begin{aligned} & \Big\langle \psi^{0R} \nabla_1 g(x(0), x(T), T) + \nabla_1 h(x(0), x(T), T)^\top \times \psi^R, y \Big\rangle_{\mathbb{R}^n} \\ & + \Big\langle \psi^{0R} \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi^R, w_y(T) \Big\rangle_{\mathbb{R}^n} \leq 0, \end{aligned} \tag{2.9}$$

where w_y is defined in Proposition 2.4.4, holds true.

Proof. The proof is divided in two steps.

First step: Let us fix $k \in \mathbb{N}$. It holds that

$$\|x_{0,k} + \alpha y - x(0)\|_{\mathbb{R}^n} \leq \alpha\|y\|_{\mathbb{R}^n} + \|x_{0,k} - x(0)\|_{\mathbb{R}^n} \leq \alpha\|y\|_{\mathbb{R}^n} + \sqrt{\varepsilon_k},$$

for all $\alpha \geq 0$. Since $\sqrt{\varepsilon_k} < \varepsilon$, it follows that there exists $0 < \alpha_0 \leq 1$ small enough such that $(u_k, x_{0,k} + \alpha y) \in \mathcal{V}^R_\varepsilon$ for all $0 \leq \alpha \leq \alpha_0$. From Inequality (2.7) we obtain

$$-\sqrt{\varepsilon_k}\|x_{0,k} + \alpha y - x_{0,k}\|_{\mathbb{R}^n} \leq \mathcal{J}^R_k(u_k, x_{0,k} + \alpha y, T_k) - \mathcal{J}^R_k(u_k, x_{0,k}, T_k),$$

and thus

$$-\sqrt{\varepsilon_k}\|y\|_{\mathbb{R}^n} \leq \frac{1}{\mathcal{J}_k^R(u_k,x_{0,k}+\alpha y,T_k)+\mathcal{J}_k^R(u_k,x_{0,k},T_k)} \times \frac{\mathcal{J}_k^R(u_k,x_{0,k}+\alpha y,T_k)^2-\mathcal{J}_k^R(u_k,x_{0,k},T_k)^2}{\alpha},$$

for all $\alpha \in (0,\alpha_0]$. Taking the limit as α tends to 0 and using the definitions of ψ_k^{0R} and ψ_k^R, we obtain from Proposition 2.4.4 that

$$\left\langle \psi_k^{0R}\nabla_1 g(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)+\nabla_1 h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)^\top \times \psi_k^R, y \right\rangle_{\mathbb{R}^n} + \left\langle \psi_k^{0R}\nabla_2 g(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)+\nabla_2 h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)^\top \times \psi_k^R, w_y^k(T_k) \right\rangle_{\mathbb{R}^n} \leq \sqrt{\varepsilon_k}\|y\|_{\mathbb{R}^n},$$

where w_y^k is defined in Lemma 2.4.4.

Second step: We take the limit of the above inequality as k tends to $+\infty$. Since g and h are both of class C^1 and from the uniform convergence of $(w_k^y)_{k\in\mathbb{N}}$ to w_y over $[0,t]$ (see Lemma 2.4.4), it holds that

$$\left\langle \psi^{0R}\nabla_1 g(x(0),x(T),T)+\nabla_1 h(x(0),x(T),T)^\top \times \psi^R, y \right\rangle_{\mathbb{R}^n} + \left\langle \psi^{0R}\nabla_2 g(x(0),x(T),T)+\nabla_2 h(x(0),x(T,u,x(0)),T)^\top \times \psi^R, w_y(T) \right\rangle_{\mathbb{R}^n} \leq 0.$$

The proof is complete. □

Lemma 2.4.9. *Let $i \in \{1,\dots,N-1\}$ such that $u_{i-1} \neq u_i$ and let $\mu \in \{-1,1\}$. Then the inequality*

$$\left\langle \psi^{0R}\nabla_2 g(x(0),x(T),T)+\nabla_2 h(x(0),x(T),T)^\top \times \psi^R, w_{t_i}^\mu(T) \right\rangle_{\mathbb{R}^n} \leq 0, \tag{2.10}$$

where $w_{t_i}^\mu$ is defined in Proposition 2.4.5, holds true.

Proof. The proof is divided in two steps.

First step: Since $t_{i,k}$ converges to t_i and since $t_i - \frac{\|\mathbb{T}\|}{4} \leq t_{i,k} \leq t_i + \frac{\|\mathbb{T}\|}{4}$, we fix $k \in \mathbb{N}$ sufficiently large in order to guarantee that $t_i - \frac{\|\mathbb{T}\|}{8} \leq t_{i,k} \leq t_i + \frac{\|\mathbb{T}\|}{8}$. Since $u_k \in \mathrm{PC}^{\mathbb{T}_k}([0,t],\mathbb{R}^m)$, the point $t_{i,k}$ is a switching time of u_k with $\eta_{t_{i,k}} = \min(t_{i,k}-t_{i-1,k}, t_{i+1,k}-t_{i,k}) > 0$. We define the perturbation $u_{k,t_{i,k}}^\mu(\cdot,\alpha)$ as

$$u_{k,t_{i,k}}^\mu(\cdot,\alpha) := \begin{cases} u_k(t_{i,k}^-) = u_{i-1,k} & \text{over } [t_{i,k}-\eta_{t_{i,k}}, t_{i,k}+\mu\alpha), \\ u_k(t_{i,k}^+) = u_{i,k} & \text{over } [t_{i,k}+\mu\alpha, t_{i,k}+\eta_{t_{i,k}}), \\ u_k & \text{otherwise,} \end{cases}$$

for all $0 \leq \alpha \leq \frac{\eta_{t_{i,k}}}{2}$. Considering $\mathbb{T}_k^{i,\alpha}$ the N-partition given by

$$0 = t_{0,k} < t_{1,k} < \ldots < t_{i-1,k} < t_{i,k}+\mu\alpha < t_{i+1,k} < \ldots < t_{N-1,k} < t_{N,k} = t,$$

it is clear that $u^\mu_{k,t_{i,k}}(\cdot,\alpha) \in \mathrm{PC}^{\mathbb{T}^{i,\alpha}_k}([0,t],\mathbb{R}^m)$ for all $0 \leq \alpha \leq \frac{\eta_{t_{i,k}}}{2}$. Since $t_i - \frac{\|\mathbb{T}\|}{8} \leq t_{i,k} \leq t_i + \frac{\|\mathbb{T}\|}{8}$, then $t_i - \frac{\|\mathbb{T}\|}{4} \leq t_{i,k} + \mu\alpha \leq t_i + \frac{\|\mathbb{T}\|}{4}$ and thus $\mathbb{T}^{i,\alpha}_k \in \mathcal{P}^t_{N,(u,\mathbb{T})}$ and $u^\mu_{k,t_{i,k}}(\cdot,\alpha) \in \mathrm{PC}_{N,(u,\mathbb{T})}([0,t],\mathbb{R}^m)$ for small enough $0 \leq \alpha \leq \frac{\eta_{t_{i,k}}}{2}$. Note that $u^\mu_{k,t_{i,k}}(\cdot,\alpha)$ takes its values in U for all $0 \leq \alpha \leq \frac{\eta_{t_{i,k}}}{2}$. It holds that $\|u^\mu_{k,t_i}(\cdot,\alpha)\|_{\mathrm{L}^\infty} \leq R$ and

$$\|u^\mu_{k,t_i}(\cdot,\alpha) - u\|_{\mathrm{L}^1} \leq \|u^\mu_{k,t_i}(\cdot,\alpha) - u_k\|_{\mathrm{L}^1} + \|u_k - u\|_{\mathrm{L}^1} \leq 2R\alpha + \sqrt{\varepsilon_k},$$

for all $0 \leq \alpha \leq \frac{\eta_{t_{i,k}}}{2}$. Since $\sqrt{\varepsilon_k} < \varepsilon$, we conclude that there exists $0 < \alpha_0 \leq \frac{\eta_{t_{i,k}}}{2}$ small enough such that $(u^\mu_{k,t_i}(\cdot,\alpha), x_{0,k}) \in \mathcal{V}^R_\varepsilon$ for all $0 \leq \alpha \leq \alpha_0$. From Inequality (2.7) we obtain

$$-\sqrt{\varepsilon_k}\|u^\mu_{k,t_i}(\cdot,\alpha) - u_k\|_{\mathrm{L}^1} \leq \mathcal{J}^R_k(u^\mu_{k,t_i}(\cdot,\alpha), x_{0,k}, T_k) - \mathcal{J}^R_k(u_k, x_{0,k}, T_k),$$

and thus

$$\begin{aligned} & -2R\sqrt{\varepsilon_k} \\ & \leq \frac{1}{\mathcal{J}^R_k(u^\mu_{k,t_i}(\cdot,\alpha), x_{0,k}, T_k) + J^R_k(u_k, x_{0,k}, T_k)} \times \frac{\mathcal{J}^R_k(u^\mu_{k,t_i}(\cdot,\alpha), x_{0,k}, T_k)^2 - J^R_k(u_k, x_{0,k}, T_k)^2}{\alpha}, \end{aligned}$$

for all $\alpha \in (0,\alpha_0]$. Taking the limit as α tends to 0 and using the definitions of ψ^{0R}_k and ψ^R_k, we obtain from Proposition 2.4.5 that

$$\left\langle \psi^{0R}_k \nabla_2 g(x_{0,k}, x(T_k,u_k,x_{0,k}), T_k) + \nabla_2 h(x_{0,k}, x(T_k,u_k,x_{0,k}), T_k)^\top \times \psi^R_k, w^{\mu,k}_{t_{i,k}}(T_k) \right\rangle_{\mathbb{R}^n} \leq 2R\sqrt{\varepsilon_k},$$

where $w^{\mu,k}_{t_{i,k}}$ is defined in Lemma 2.4.5.

Second step: We take the limit of the above inequality as k tends to $+\infty$. Since g and h are of class C^1 and, since $t_i < t_0$, from the uniform convergence of $(w^{\mu,k}_{t_{i,k}})_{k\in\mathbb{N}}$ to $w^\mu_{t_i}$ over $[t_0,t]$ (see Lemma 2.4.5), it holds that

$$\left\langle \psi^{0R} \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi^R, w^\mu_{t_i}(T) \right\rangle_{\mathbb{R}^n} \leq 0.$$

The proof is complete. □

Lemma 2.4.10. *The equality*

$$\left\langle \psi^{0R} \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi^R, f(x(T), u_{N-1}, T) \right\rangle_{\mathbb{R}^n} + \psi^{0R} \nabla_3 g(x(0), x(T), T) + \nabla_3 h(x(0), x(T), T)^\top \times \psi^R = 0, \quad (2.11)$$

holds.

Proof. The proof is divided in two steps.

First step: Let $\mu \in \{-1,1\}$. Since $(T_k)_{k\in\mathbb{N}}$ converges to $T \in (t_0,t)$, then $T_k \in (t_0,t)$ for $k \in \mathbb{N}$ sufficiently large. Let us fix such an integer $k \in \mathbb{N}$. Thus there exists $\alpha_0 > 0$ small enough such that $(x_{0,k}, u_k, T_k + \mu\alpha) \in \mathcal{V}^R_\varepsilon \times [t_0,t]$ for all $0 \leq \alpha \leq \alpha_0$. From Inequality (2.7) we obtain

$$-\sqrt{\varepsilon_k}\ |T_k + \mu\alpha - T_k| \leq \mathcal{J}^R_k(u_k, x_{0,k}, T_k + \mu\alpha) - \mathcal{J}^R_k(u_k, x_{0,k}, T_k),$$

and thus

$$-\sqrt{\varepsilon_k} \leq \frac{1}{\mathcal{J}_k^R(u_k,x_{0,k},T_k+\mu\alpha)+\mathcal{J}_k^R(u_k,x_{0,k},T_k)} \times \frac{\mathcal{J}_k^R(u_k,x_{0,k},T_k+\mu\alpha)^2-\mathcal{J}_k^R(u_k,x_{0,k},T_k)^2}{\alpha},$$

for all $\alpha \in (0,\alpha_0]$. Taking the limit as α tends to 0 and using the definitions of ψ_k^{0R} and ψ_k^R, we obtain from the differentiability of $x(\cdot,u_k,x_{0,k})$ at T_k (since u_k is constant over the interval $[t_0,t] \subset [t_{N-1,k},t_{N,k}]$ and since $T_k \in (t_0,t)$) that

$$\mu\Big\langle \psi_k^{0R}\nabla_2 g(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)+\nabla_2 h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)^\top \times \psi_k^R \\ ,f(x(T_k,u_k,x_{0,k}),u_k(T_k),T_k)\Big\rangle_{\mathbb{R}^n} \\ +\mu\psi_k^{0R}\nabla_3 g(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)+\mu\nabla_3 h(x_{0,k},x(T_k,u_k,x_{0,k}),T_k)^\top \times \psi_k^R \leq \sqrt{\varepsilon_k},$$

where $u_k(T_k)=u_{N-1,k}$.

Second step: We take the limit of the above inequality as k tends to $+\infty$. Let us recall that $u_{N-1,k}$ converges to u_{N-1}. Furthermore, since f is continuous, since g and g are of class C^1, and since $u_k(T_k)$ converges to $u(T)$ from Proposition 2.4.7, it holds that

$$\mu\Big\langle \psi^{0R}\nabla_2 g(x(0),x(T),T)+\nabla_2 h(x(0),x(T),T)^\top \times \psi^R, f(x(T),u_{N-1},T)\Big\rangle_{\mathbb{R}^n} \\ +\mu\psi^{0R}\nabla_3 g(x(0),x(T),T)+\mu\nabla_3 h(x(0),x(T),T)^\top \times \psi^R \leq 0.$$

Since μ can be chosen arbitrarily in $\{-1,1\}$, the proof is complete. □

2.4.3.3 Crucial inequalities letting R go to $+\infty$

In the previous section we have obtained Inequalities (2.8), (2.9) and (2.10) and Equality (2.11) which are valid for $R \in \mathbb{N}$ being fixed such that $R \geq \|u\|_{\mathrm{L}^\infty}$. In particular Inequality (2.8) is satisfied only for $v \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$ taking values in $\mathrm{U} \cap \overline{\mathrm{B}}_{\mathbb{R}^m}(0_{\mathbb{R}^m},R)$. Our goal in this section is to get rid of the dependence in R. From the equality $|\psi^{0R}|^2+\|\psi^R\|^2_{\mathbb{R}^j}=1$ (see the end of Section 2.4.3.1), we can extract subsequences (that we do not relabel) such that $(\psi^{0R})_{R\in\mathbb{N}}$ converges to some ψ^0 in $\mathbb{R}$ and $(\psi^R)_{R\in\mathbb{N}}$ converges to some ψ in $\mathbb{R}^j$ when $R \to \infty$. It clearly holds that $|\psi^0|^2+\|\psi\|^2_{\mathbb{R}^j}=1$ and, since $\mathbb{R}_-$ and $\mathrm{N_S}[h(x(0),x(T),T)]$ are closed, that $\psi^0 \in \mathbb{R}_-$ and $-\psi \in \mathrm{N_S}[h(x(0),x(T),T)]$.

Now let us fix $v \in \mathrm{PC}^{\mathbb{T}}([0,t],\mathbb{R}^m)$ taking values in U. Considering $R \in \mathbb{N}$ large enough in order to get that $R \geq \|u\|_{\mathrm{L}^\infty}$ and $R \geq \|v\|_{\mathrm{L}^\infty}$, we know from Lemma 2.4.7 that Inequality (2.8) is satisfied. Taking the limit as R tends to $+\infty$ we conclude that

$$\langle \psi^0\nabla_2 g(x(0),x(T),T)+\nabla_2 h(x(0),x(T),T)^\top \times \psi, w_v(T)\rangle_{\mathbb{R}^n} \leq 0. \tag{2.12}$$

Similarly, letting R go to $+\infty$ in Inequalities (2.9) and (2.10) and in Equality (2.11), we get that

$$\langle \psi^0\nabla_1 g(x(0),x(T),T)+\nabla_1 h(x(0),x(T),T)^\top \times \psi, y\rangle_{\mathbb{R}^n} \\ +\langle \psi^0\nabla_2 g(x(0),x(T),T)+\nabla_2 h(x(0),x(T),T)^\top \times \psi, w_y(T)\rangle_{\mathbb{R}^n} \leq 0, \tag{2.13}$$

for any $y \in \mathbb{R}^n$, that

$$\langle \psi^0 \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi, w^{\mu}_{t_i}(T) \rangle_{\mathbb{R}^n} \leq 0, \tag{2.14}$$

for any $i \in \{1, \dots, N-1\}$ such that $u_{i-1} \neq u_i$ and any $\mu \in \{-1, 1\}$, and that

$$\begin{aligned} &\langle \psi^0 \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \psi, f(x(T), u_{N-1}, T) \rangle_{\mathbb{R}^n} \\ &\qquad + \psi^0 \nabla_3 g(x(0), x(T), T) + \nabla_3 h(x(0), x(T), T)^\top \times \psi = 0. \end{aligned} \tag{2.15}$$

2.4.3.4 End of the proof

Now we can end the proof of Theorem 2.2.1 (in the case $L = 0$) with the introduction of the adjoint vector p. Before coming to this point, let us first define $p^0 := \psi^0$ and $\Psi := \psi$. In particular note that $p^0 \in \mathbb{R}_-$, that $\Psi \in \mathbb{R}^j$ is such that $-\Psi \in \mathrm{N_S}[h(x(0), x(T), T)]$ and that $|p^0|^2 + \|\Psi\|^2_{\mathbb{R}^j} = 1$.

We define the adjoint vector $p \in \mathrm{AC}([0, T], \mathbb{R}^n)$ as the unique solution (that is global) to the backward linear Cauchy problem given by

$$\begin{cases} \dot{p}(t) = -\nabla_1 f(x(t), u(t), t)^\top \times p(t), \quad \text{a.e. } t \in [0, T], \\ p(T) = p^0 \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \Psi. \end{cases}$$

From the Duhamel formula, recall that

$$p(t) = \Phi(T, t)^\top \times \Big(p^0 \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \Psi \Big),$$

for all $t \in [0, T]$, where $\Phi(\cdot, \cdot) : [0, T]^2 \to \mathbb{R}^{n \times n}$ stands for the state transition matrix associated to the matrix function $t \mapsto \nabla_1 f(x(t), u(t), t)$. We refer to [Sontag 1998, Appendix C.4] for more details on state-transition matrices.

Adjoint equation and transversality condition on the adjoint vector. From the above definition of the adjoint vector p, it is clear that the adjoint equation in Theorem 2.2.1 and the transversality condition $p(T) = p^0 \nabla_2 g(x(0), x(T), T) + \nabla_2 h(x(0), x(T), T)^\top \times \Psi$ are satisfied. Moreover, from the Duhamel formula, it holds that $w_y(T) = \Phi(T, 0) \times y$ and thus Inequality (2.13) can be rewritten as

$$\langle p^0 \nabla_1 g(x(0), x(T), T) + \nabla_1 h(x(0), x(T), T)^\top \times \Psi + p(0), y \rangle_{\mathbb{R}^n} \leq 0,$$

for all $y \in \mathbb{R}^n$. Thus we conclude that the transversality condition $-p(0) = p^0 \nabla_1 g(x(0), x(T), T) + \nabla_1 h(x(0), x(T), T)^\top \times \Psi$ holds.

Nonpositive averaged Hamiltonian gradient condition. Let us fix $\omega \in \mathrm{U}$ and $i \in \{0, \dots, N-1\}$. Let us consider $v \in \mathrm{PC}^{\mathbb{T}}([0, T], \mathbb{R}^m)$ defined by

$$v(t) := \begin{cases} \omega & \text{if } t \in [t_i, t_{i+1}), \\ u(t) & \text{otherwise,} \end{cases}$$

for all $t \in [0,T]$. From the Duhamel formula given by

$$w_v(T) = \int_0^T \Phi(T,t) \times \nabla_2 f(x(t),u(t),t) \times (v(t)-u(t))\, dt,$$

Inequality (2.12) can be rewritten as

$$\int_0^T \left\langle \nabla_2 f(x(t),u(t),t)^\top \times p(t), v(t)-u(t) \right\rangle_{\mathbb{R}^m} dt \leq 0,$$

that is

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t),u_i,p(t),p^0,t)\, dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0.$$

Transversality conditions on the optimal sampling times. Let us fix some $i \in \{1,\dots,N-1\}$ and $\mu \in \{-1,1\}$. If $u_{i-1} = u_i$, then the transversality condition (2.3) in Theorem 2.2.1 is obviously satisfied. Now let us assume that $u_{i-1} \neq u_i$. From the Duhamel formula given by

$$w^\mu_{t_i}(T) = \mu\Phi(T,t_i) \times \Big(f(x(t_i),u_{i-1},t_i) - f(x(t_i),u_i,t_i) \Big),$$

Inequality (2.14) can be rewritten as

$$\mu \langle p(t_i), f(x(t_i),u_{i-1},t_i) - f(x(t_i),u_i,t_i) \rangle_{\mathbb{R}^n} \leq 0.$$

Since μ can be arbitrarily chosen in $\{-1,1\}$ and from the definition of the Hamiltonian H, we get that

$$H(x(t_i),u_{i-1},p(t_i),p^0,t_i) = H(x(t_i),u_i,p(t_i),p^0,t_i).$$

Transversality condition on the optimal final time. Equality (2.15) can be directly rewritten as

$$-H(x(T),u_{N-1},p(T),p^0,T) = p^0 \nabla_3 g(x(0),x(T),T) + \nabla_3 h(x(0),x(T),T)^\top \times \Psi.$$

Nontriviality of the couple (p,p^0). Let us assume by contradiction that the couple (p,p^0) is trivial. Then $p(0) = p(T) = 0_{\mathbb{R}^n}$ and $p^0 = 0$. We get from the transversality conditions on the adjoint vector and on the optimal final time that $Dh(x(0),x(T),T)^\top \times \Psi = 0_{\mathbb{R}^{2n+1}}$. From the submersion property, we deduce that $\Psi = 0_{\mathbb{R}^j}$ which raises a contradiction with the equality $|p^0|^2 + \|\Psi\|^2_{\mathbb{R}^j} = 1$.

2.4.4 The case $L \neq 0$

In the previous section we have proved Theorem 2.2.1 in the case $L = 0$ (without Lagrange cost). This section is dedicated to the case $L \neq 0$. Let $(T,\mathbb{T},x,u)$ be a solution to Problem (OSCP) (with $L \neq 0$) and let us assume that h is submersive at $(x(0),x(T),T)$. Let us define

$$X(t) := \int_0^t L(x(s),u(s),s)\, ds,$$

for all $t \in [0,T]$. One can easily see that the augmented quadruple $(T, \mathbb{T}, (x,X), u)$ is a solution to the augmented optimal sampled-data control problem of Mayer form given by

$$\left.\begin{array}{ll} \text{minimize} & \tilde{g}((x,X)(0),(x,X)(T),T), \\[2mm] \text{subject to} & T>0 \text{ fixed or free}, \\[2mm] & \mathbb{T}=\{t_i\}_{i=0,\ldots,N} \in \mathcal{P}^T_N \text{ fixed or free}, \\[2mm] & (x,X) \in \mathrm{AC}([0,T],\mathbb{R}^{n+1}),\ u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m), \\[2mm] & \begin{pmatrix} \dot{x} \\ \dot{X} \end{pmatrix}(t) = \begin{pmatrix} f(x(t),u(t),t) \\ L(x(t),u(t),t) \end{pmatrix}, \quad \text{a.e. } t \in [0,T], \\[2mm] & \tilde{h}((x,X)(0),(x,X)(T),T) \in \tilde{\mathrm{S}}, \\[2mm] & u_i \in \mathrm{U}, \quad \text{for all } i=0,\ldots,N-1, \end{array}\right\} \qquad (\mathrm{OSCP_{aug}})$$

where $\tilde{g} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\tilde{g}((x_1,X_1),(x_2,X_2),t) := g(x_1,x_2,t) + X_2,$$

for all $((x_1,X_1),(x_2,X_2),t) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}_+$, where $\tilde{h} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}^{j+2}$ is defined by

$$\tilde{h}((x_1,X_1),(x_2,X_2),t) := (h(x_1,x_2,t),X_1,X_2),$$

for all $((x_1,X_1),(x_2,X_2),t) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}_+$ and where

$$\tilde{\mathrm{S}} := \mathrm{S} \times \{0\} \times \mathbb{R}.$$

Since h is submersive at $(x(0),x(T),T)$, note that $\tilde{h}$ is submersive at $((x,X)(0),(x,X)(T),T)$ and thus Theorem 2.2.1 can be applied to Problem ($\mathrm{OSCP_{aug}}$) (which has no Lagrange cost). We deduce the existence of a nontrivial augmented couple $((p,q),p^0) \in \mathrm{AC}([0,T],\mathbb{R}^{n+1}) \times \mathbb{R}_-$ satisfying all necessary conditions listed in Theorem 4.3.1 adapted to the augmented Problem ($\mathrm{OSCP_{aug}}$). In particular we get that $\dot{q}=0$, $q(T)=p^0$ and thus $q(t)=p^0$ for all $t \in [0,T]$. The rest of the proof is straightforward from all necessary conditions provided in Theorem 2.2.1 (in the case of a Mayer problem).

Chapter 3

Optimal sampled-data controls with running inequality state constraints

This chapter is based on the article "Optimal sampled-data controls with running inequality state constraints: Pontryagin maximum principle and bouncing trajectory phenomenon" by L. Bourdin and G. Dhar (see [Bourdin & Dhar 2020]).

3.1 Introduction

In this chapter we derive a Pontryagin maximum principle for general nonlinear optimal sampled-data control problems in the presence of running inequality state constraints as given in the work [Bourdin & Dhar 2020]. Recall that the *Pontryagin maximum principle* (in short, PMP) which was established in [Pontryagin *et al.* 1962] by Pontryagin et al. at the end of the 1950's, is the milestone of optimal control theory. As a well known application, if the *Hamiltonian maximization condition* allows to express the optimal control as a function of the augmented state-costate vector, then the PMP induces the so-called *indirect numerical method* which consists in numerically solving the boundary value problem satisfied by the augmented state-costate vector via a shooting method. Indirect numerical methods are opposed to *direct numerical methods* which consist in a full discretization of the optimal control problem resulting into a constrained finite-dimensional optimization problem that can be numerically solved from various standard optimization algorithms and techniques.

An important generalization of the PMP concerns *state constrained optimal control problems* in which the state is restricted to a certain region of the state space. Indeed it is often undesirable and even inadmissible in scientific and engineering applications that the state crosses certain limits imposed in the state space for safety or practical reasons. Many examples can be found in mechanics and aerospace engineering (e.g., an engine may overheat or overload). We refer to [Bonnard *et al.* 2003, Cots 2017, Chertovskih *et al.* 2018, Chertovskih *et al.* 2020, Kim *et al.* 2011, van Keulen *et al.* 2014, Van Reeven *et al.* 2016] and references therein for other examples. State constrained optimal control problems are also encountered in management and economics (e.g., an inventory level may be limited in a production model). We refer to [Cho *et al.* 1993, Maurer *et al.* 2005, Puchkova *et al.* 2014, Sethi & Thompson 2000] and references therein for other examples. A first version of the PMP for optimal control problems with running inequality state constraints was obtained by Gamkrelidze [Gamkrelidze 1960] (see also [Pontryagin *et al.* 1962, Theorem 25 p.311]) under some special assumptions on the structure of the optimal process. Later, these assumptions were somewhat excluded by Dubovitskii and Milyutin in the seminal work [Dubovitskii & Milyutin 1965, Section 7 p.37]. The contributions of Dubovitskii and Milyutin in the development of the PMP for optimal control problems with various constraints, in the 1960's and later years, have been the subject of a survey written by Dmitruk [Dmitruk 2009] in 2009. These contributions include, notably, general Lagrange multiplier rules for abstract optimization problems and the so-called method of *v-change of time variable* in view of generating needle-like variations by passing to a smooth control system (see more details in [Dmitruk 2009, Section 4]). These approaches have been revisited in a book of lectures on extremum problems written by Girsanov [Girsanov 1972] in 1972, and extended recently in a series of papers by Dmitruk and Osmolovskii [Dmitruk & Osmolovskii 2014, Dmitruk & Osmolovskii 2017, Dmitruk & Osmolovskii 2018, Dmitruk & Osmolovskii 2019] with applications to various optimal control problems such as with integral equations, state constraints, mixed state-control constraints, etc. Other methods have been developed in the literature in order to establish versions of the PMP for state constrained optimal control problems, such as the smoothly-convex structure of the controlled system in [Ioffe & Tihomirov 1979], the application of the Ekeland variational principle in [Vinter 2010], etc. A comprehensive survey [Hartl *et al.* 1995] of this field of research has been given in 1995 by Hartl, Sethi and Vickson. Note that the PMP for optimal control problems is more intricate in the presence of running inequality state constraints because the adjoint vector is not absolutely continuous in general (while it is in the state unconstrained case), but (only) of bounded variation (see Section 1.2 for recalls on functions of bounded variation). Therefore theoretical and numerical difficulties may arise due to the possible pathological behavior of the adjoint vector which consists in jumps and singular part lying on parts of the optimal trajectory in contact with the boundary of the restricted state space. As a consequence a wide portion of the literature is devoted to the analysis of the costate's behavior and some constraint qualification conditions have been established in order to ensure that the adjoint vector has no singular part (see, e.g., [Bettiol & Frankowska 2008, Bonnard *et al.* 2003, Dmitruk 2009, Hartl *et al.* 1995, Jacobson *et al.* 1971, Maurer 1977]). We briefly conclude this paragraph by mentioning that the related theme of state constrained discrete optimal control problems has also been investigated in the literature (see, e.g., [Cots *et al.* 2018, Proposition 2 p.13]).

Before being presented in the work [Bourdin & Dhar 2020], to the best of our knowledge, optimal sampled-data control problems had never been investigated in the presence of state constraints. In the work [Bourdin & Dhar 2020] presented in this chapter, the first objective was to bridge this gap in the literature by establishing a PMP for general nonlinear optimal sampled-data control problems in the presence of running inequality state constraints (see Theorem 3.2.1 in Section 3.2). In contrast to the previous chapter in which we considered optimal sampled-data control problems with terminal state constraints (which can be seen as finite-dimensional optimization problems), note that such problems can be seen as *semi-infinite*-dimensional optimization problems since the presence of running inequality state constraints imposes an infinite number of constraints (one at each instant of time). In the work [Bourdin & Dhar 2020], in the same spirit as Bourdin and Trélat in [Bourdin & Trélat 2016] and the paper [Bourdin & Dhar 2020] presented in Chapter 2, our proof is based on the Ekeland variational principle. Similarly to the PMP derived in [Bourdin & Trélat 2016, Theorem 2.6 p.62] for state unconstrained optimal sampled-data control problems, we obtained a first-order necessary optimality condition described by a nonpositive averaged Hamiltonian gradient condition. Moreover, as in the case of optimal permanent control problems with running inequality state constraints, we found that the adjoint vector is in general (only) of bounded variation. Therefore one would expect to encounter the same difficulties as in the permanent control case when implementing an indirect numerical method due to the possible jumps and singular part of the adjoint vector. However, in our context of sampled-data controls, we found that the optimal trajectories have a common behavior which allows us to overcome these difficulties. Precisely, when we began studying optimal sampled-data control problems in the presence of running inequality state constraints in the work [Bourdin & Dhar 2020], we first numerically solved some simple problems using direct methods. Notably we observed that, in each problem, the optimal trajectory "bounces" against the boundary of the restricted state space, touching the state constraint at most at the sampling times. This behavior was the second major focus of the work [Bourdin & Dhar 2020] and is referred to as the *bouncing trajectory phenomenon*. We proved that, under certain general hypotheses, any admissible trajectory necessarily bounces on the running inequality state constraints and, moreover, the rebounds occur at most at the sampling times (and thus are in a finite number and at precise instants). We refer to Section 3.3 for details. Inherent to this behavior, the singular part of the adjoint vector derived in our PMP vanishes and its discontinuities are reduced to a finite number of jumps which occur exactly at the sampling times. Taking advantage of these informations, we are able in Section 3.4 to implement an indirect numerical method which we use to numerically solve three simple examples of optimal sampled-data control problems with running inequality state constraints. We take this occasion to mention that a similar trajectory phenomenon has already been observed in the literature on state constrained optimal permanent control problems. Precisely, Milyutin provides an example in his doctoral dissertation [Milyutin 1966] in 1966 (see also [Dmitruk 2009, p.940]) in which the optimal trajectory touches the state constraint a countably infinite number of times before landing on it. As a consequence, in that example, the corresponding adjoint vector admits a countably infinite number of jumps. This example was also given independently by Robbins [Robbins 1980] in 1980.

This chapter is organized as follows. In Section 3.2 we first present the optimal sampled-data control problem with running inequality state constraints considered in this work (see Problem ($\mathrm{OSCP_{sc}}$)) accompanied by some background on sampled-data controls and a list of comments. The corresponding Pontryagin maximum principle is stated thereafter (see Theorem 3.2.1) and a list of general comments follows. In Section 3.3 we give heuristic descriptions and a sufficient condition for observing the bouncing trajectory phenomenon. In Section 3.4 we propose an indirect method for numerically solving optimal sampled-data control problems with running inequality state constraints based on our main result and with the aid of the bouncing trajectory phenomenon. Then we illustrate this method and highlight the bouncing trajectory phenomenon by numerically solving three simple examples. Finally Section 3.5 is devoted to the proof of the main result (Theorem 3.2.1).

3.2 Main result and comments

This section is dedicated to the statement of the main result of the work [Bourdin & Dhar 2020]. In Section 3.2.1 below, we introduce the general optimal sampled-data control problem with running inequality state constraints considered in [Bourdin & Dhar 2020], and we fix the terminology and assumptions used all along this chapter. In Section 3.2.2 we state the corresponding

Pontryagin maximum principle (see Theorem 3.2.1) and a list of comments follows.

3.2.1 A general optimal sampled-data control problem with running inequality state constraints

Let n, m, q, $N \in \mathbb{N}^*$ be four fixed positive integers. Let us fix a positive real number $T > 0$, as well as an N-partition $\mathbb{T} = \{t_i\}_{i=0,\ldots,N}$ of the interval $[0, T]$. In the present chapter we focus on the general optimal sampled-data control problem with running inequality state constraints given by

$$\left.\begin{array}{ll} \text{minimize} & g(x(T)) + \displaystyle\int_0^T L(x(t), u(t), t)\, dt, \\[2ex] \text{subject to} & x \in \mathrm{AC}([0,T], \mathbb{R}^n),\ u \in \mathrm{PC}^{\mathbb{T}}([0,T], \mathbb{R}^m), \\[1ex] & \dot{x}(t) = f(x(t), u(t), t), \quad \text{a.e. } t \in [0,T], \\[1ex] & x(0) = x_0, \\[1ex] & h_j(x(t), t) \leq 0, \quad \text{for all } t \in [0,T] \text{ and all } j = 1, \ldots, q, \\[1ex] & u_i \in \mathrm{U}, \quad \text{for all } i = 0, \ldots, N-1. \end{array}\right\} \qquad (\mathrm{OSCP}_{\mathrm{sc}})$$

A couple (x, u) is said to be *admissible* for Problem ($\mathrm{OSCP}_{\mathrm{sc}}$) if it satisfies all its constraints. A *solution* to Problem ($\mathrm{OSCP}_{\mathrm{sc}}$) is an admissible couple (x, u) which minimizes the *Bolza cost* given by $g(x(T)) + \int_0^T L(x(t), u(t), t)\, dt$ among all admissible couples.

Throughout the chapter we will make use of the following regularity and topology assumptions:

- the functions $g : \mathbb{R}^n \to \mathbb{R}$ and $L : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$, that describe respectively the *Mayer cost* $g(x(T))$ and the *Lagrange cost* $\int_0^T L(x(t), u(t), t)\, dt$, are of class C^1;
- the *dynamics* $f : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$, that drives the *state equation* $\dot{x}(t) = f(x(t), u(t), t)$, is continuous and of class C^1;
- the function $h = (h_j)_{j=1,\ldots,q} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^q$, that describes the *running inequality state constraints* $h_j(x(t), t) \leq 0$, is continuous and of class C^1 in its first variable;
- the set $\mathrm{U} \subset \mathbb{R}^m$, that describes the *control constraint* $u(t) \in \mathrm{U}$, is a nonempty closed convex subset of $\mathbb{R}^m$;
- the initial condition $x_0 \in \mathbb{R}^n$ is fixed.

Note that we make use of the same regularity and topology assumptions for the dynamics f and for the control constraint set U as those introduced in Section 2.2.1 of Chapter 2. This will allow us to use results from the sensitivity analysis given in Section 2.4.1 from Chapter 2 later when we discuss the proof of the main result of this chapter (see Section 3.5 for details).

Optimal sampled-data control problems have been investigated in the literature (see, e.g., [Aström 1963, Bini & Buttazzo 2014, Bourdin & Dhar 2019, Bourdin & Trélat 2015, Bourdin & Trélat 2016, Bourdin & Trélat 2017]) with general terminal constraints on $x(0)$ and $x(T)$, free final time, free sampling times, etc. Before being presented in the work [Bourdin & Dhar 2020], to the best of our knowledge, running inequality state constraints had never been investigated with sampled-data controls. The aim of [Bourdin & Dhar 2020] was to fill this gap in the literature, and thus we focused on the running inequality state constraints $h_j(x(t), t) \leq 0$ in Problem ($\mathrm{OSCP_{sc}}$). As a consequence, for the sake of simplicity, we took the decision not to consider general terminal constraints in Problem ($\mathrm{OSCP_{sc}}$). Indeed we only considered the basic case in which the initial condition $x(0) = x_0$ is fixed and the final condition $x(T)$ is free. Similarly we also chose to consider that the final time $T > 0$ and the partition $\mathbb{T} = \{t_i\}_{i=0,\dots,N}$ are fixed. If the reader is interested in techniques allowing to handle general terminal constraints, free final time, free sampling times, etc., we refer to the references mentioned above.

Remark 3.2.1. *This comment highlights some research perspectives.*

(i) *Existence theorems for optimal permanent control problems with running inequality state constraints can be found in a text by Clarke (see [Clarke 1990, Theorem 5.4.4 p.222]). Furthermore, existence theorems for related problems such as optimal permanent control problems with state constraints where the constraint is given as an inclusion into a general subset of the state space can be found in works of Cesari (see [Cesari 1983a, Theorem 9.2.i p.311]) and Rockafellar (see [Rockafellar 1972, Theorem 2 p.696]). A Filippov-type theorem for the existence of a solution to Problem* ($\mathrm{OSCP_{sc}}$) *without running inequality state constraints was derived in [Bourdin & Trélat 2016, Theorem 2.1 p.61]. The work [Bourdin & Dhar 2020] presented in this chapter only focuses on necessary optimality conditions and thus was not concerned with the extension of the previously mentioned result to the case with running inequality state constraints. Existence theorems for optimal sampled-data control problems with running inequality state constraints will not be presented in this chapter, however they constitute an interesting perspective for further research.*

(ii) *In the paper [Bourdin & Dhar 2019] presented in Chapter 2 we consider optimal sampled-data control problems with free sampling times and obtain a corresponding necessary optimality condition which happens to coincide with the continuity of the Hamiltonian function. It would be relevant to extend the scope of Problem* ($\mathrm{OSCP_{sc}}$) *to study optimal sampling times in the presence of running inequality state constraints.*

(iii) *Several papers in the literature consider optimal permanent control problems with constraints of different natures, for instance with state constraints where the constraint is given as an inclusion into a general subset of the state space (see, e.g., [Cesari 1983a, Rockafellar 1972]) or with mixed state-control constraints of the form $h(x(t), u(t), t) \leq 0$ (see, e.g., [Dmitruk & Osmolovskii 2014, Hartl et al. 1995]). A possible challenge would be to extend Problem* ($\mathrm{OSCP_{sc}}$) *to the previous mentioned contexts.*

(iv) *As addressed in Chapter 2 (see Remark 2.2.16), a last (but not least) a relevant research perspective would concern the extension of Problem* ($\mathrm{OSCP_{sc}}$) *to the more general framework in which the values of the sampling times t_i intervene explicitly in the cost to minimize and/or in the dynamics.*

3.2.2 Pontryagin maximum principle

The main objective of the work [Bourdin & Dhar 2020] was to derive a Pontryagin maximum principle for Problem (OSCP_{sc}). Let us recall here that establishing a consensual version of the Pontryagin maximum principle for optimal permanent control problems in the presence of running inequality state constraints still constitutes a wonderful mathematical challenge. We refer to the Introduction for a brief bibliographic recap and we refer to [Aronna *et al.* 2016, Arutyunov *et al.* 2011, Bonnans & de la Vega 2010, Bonnans *et al.* 2013, Bonnans & Hermant 2009, Bonnans & Hermant 2009, Dmitruk & Osmolovskii 2014, Dmitruk & Osmolovskii 2017, Dmitruk & Osmolovskii 2019, Vinter 2010] for recent contributions with various generalizations.

The novelty of the work [Bourdin & Dhar 2020] presented in this chapter was to deal with nonpermanent controls, precisely, with sampled-data controls. As given in Definition 2.2.1 we recall that the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times [0,T] \to \mathbb{R}$ associated with Problem (OSCP_{sc}) is defined by $H(x,u,p,p^0,t) := \langle p, f(x,u,t)\rangle_{\mathbb{R}^n} + p^0 L(x,u,t)$, for all $(x,u,p,p^0,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times [0,T]$. We also refer to Section 1.2.2 for recalls on functions of bounded variations (in particular, the spaces $\text{BV}([0,T],\mathbb{R}^q)$ and $\text{NBV}([0,T],\mathbb{R}^q)$) and on Cauchy-Stieltjes problems. We now state the main result of the work [Bourdin & Dhar 2020] which was given by the following theorem.

Theorem 3.2.1 (Pontryagin maximum principle). *Let (x,u) be a solution to Problem (OSCP_{sc}). Then there exists a nontrivial couple (p^0,η), where $p^0 \leq 0$ and $\eta = (\eta_j)_{j=1,\dots,q} \in \text{NBV}([0,T],\mathbb{R}^q)$, such that the nonpositive averaged Hamiltonian gradient condition*

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t),u_i,p(t),p^0,t)\,dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0, \tag{3.1}$$

holds for all $\omega \in \mathrm{U}$ and all $i = 0,\dots,N-1$, where the adjoint vector $p \in \text{BV}([0,T],\mathbb{R}^n)$ (also called costate) is the unique solution to the backward linear Cauchy-Stieltjes problem given by

$$\begin{cases} -dp = \Big(\nabla_1 f(x,u,\cdot)^\top \times p + p^0 \nabla_1 L(x,u,\cdot)\Big)\,dt - \sum_{j=1}^q \nabla_1 h_j(x,\cdot)\,d\eta_j & \text{over } [0,T], \\ p(T) = p^0 \nabla g(x(T)). \end{cases} \tag{3.2}$$

Moreover the complementary slackness condition:

$$\eta_j \text{ is monotonically increasing on } [0,T] \text{ and } \int_0^T h_j(x(t),t)\,d\eta_j(t) = 0, \tag{3.3}$$

is satisfied for each $j = 1,\dots,q$.

Section 3.5 is dedicated to the detailed proof of Theorem 3.2.1. A list of comments is presented hereafter.

Remark 3.2.2. *The nontrivial couple (p^0,η) provided in Theorem 3.2.1, which corresponds to a Lagrange multiplier, is defined up to a positive multiplicative scalar. In the normal case $p^0 \neq 0$ it is usual to normalize the Lagrange multiplier so that $p^0 = -1$. The case $p^0 = 0$ is* *usually called the abnormal case. We observe that, if the running inequality state constraints are never activated (that is, if $h_j(x(t),t) < 0$ for all $t \in [0,T]$ and all $j = 1,\dots,q$), then the case is normal. Outside of this trivial situation, sufficient conditions ensuring normality is a difficult topic which has been widely developed in the literature on constrained optimal permanent control problems (see, e.g., [Bettiol & Frankowska 2007, Clarke 1976, Malanowski 2003, Rampazzo & Vinter 1999] and references therein). To the best of our knowledge, the extension of such results to the present sampled-data control setting is an open challenge in the literature, and thus perspectives for further research works are possible in that direction. Since it is not our aim in this chapter to discuss this point in more depth, we will opt in practice (as in Examples 1, 2 and 3 in Section 3.4) for proofs by contradiction in order to show that the case is normal.*

Remark 3.2.3. *If there is no running inequality state constraint in Problem* ($\mathrm{OSCP_{sc}}$)*, that is, considering $h_j = -1$ for all $j = 1,\dots,q$ for example, then Theorem 3.2.1 recovers the standard Pontryagin maximum principle for optimal sampled-data control problems obtained for example in [Bourdin & Trélat 2015, Bourdin & Trélat 2016] or as in the paper [Bourdin & Dhar 2019] in the case of fixed sampling times (see Theorem 2.2.1 in Chapter 2).*

Remark 3.2.4. *This comment echoes the Remark 2.2.13 given in Chapter 2 regarding Problem* (OSCP)*. Following the proof in Section 3.5, one can easily see that Theorem 3.2.1 is still valid for a couple (x,u) which is a solution to Problem* ($\mathrm{OSCP_{sc}}$) *in (only) a local sense to be precised. For the ease of statement, we took the decision to establish Theorem 3.2.1 for a couple (x,u) which is a solution to Problem* ($\mathrm{OSCP_{sc}}$) *in a global sense.*

Remark 3.2.5. *In the context of Theorem 3.2.1 and using the definition of the Hamiltonian, note that the state equation can be written as*

$$\dot{x}(t) = \nabla_3 H(x(t), u(t), p(t), p^0, t),$$

for a.e. $t \in [0,T]$, and that the adjoint equation can be written as

$$-dp = \nabla_1 H(x,u,p,p^0,\cdot)\, dt - \sum_{j=1}^{q} \nabla_1 h_j(x,\cdot)\, d\eta_j,$$

over $[0,T]$.

Remark 3.2.6. *It is frequent in the literature (see, e.g., [Vinter 2010, Theorem 9.3.1]) to find the adjoint vector $p \in \mathrm{BV}([0,T],\mathbb{R}^n)$ written as the sum $p = p_1 + p_2$ where $p_1 \in \mathrm{AC}([0,T],\mathbb{R}^n)$ is the unique solution to the backward linear Cauchy problem*

$$\begin{cases} \dot{p_1}(t) = -\nabla_1 H(x(t), u(t), p(t), p^0, t), & \text{a.e. } t \in [0,T], \\ p_1(T) = p^0 \nabla g(x(T)), \end{cases}$$

and where $p_2 \in \mathrm{BV}([0,T],\mathbb{R}^n)$ is defined by

$$p_2(t) := -\sum_{j=1}^{q} \int_t^T \nabla_1 h_j(x(s),s)\, d\eta_j(s),$$

for all $t \in [0,T]$. This decomposition easily follows from the integral representation of the solutions to backward linear Cauchy-Stieltjes problem recalled in Section 1.2.

Remark 3.2.7. *In the context of Theorem 3.2.1, note that the complementary slackness condition implies that, for all $j = 1, \ldots, q$, the function η_j remains constant on any open subinterval $(t_1, t_2) \subset \{t \in [0, T] \mid h_j(x(t), t) < 0\}$ with $t_1 < t_2$. Denoting by $d\eta_j$ the finite nonnegative Borel measure associated with η_j (see Section 1.2 for more details), we deduce that*

$$\mathrm{supp}(d\eta_j) \subset \{t \in [0, T] \mid h_j(x(t), t) = 0\},$$

for all $j = 1, \ldots, q$, where $\mathrm{supp}(d\eta_j)$ stands for the classical notion of support of the measure $d\eta_j$.

Remark 3.2.8. *Note that the necessary optimality conditions of Theorem 3.2.1 are not of interest when the running inequality state constraints are degenerate. For example this can occur in the case $q = 2$ if the optimal trajectory x activates the two running inequality state constraints at some time $t = \bar{t}$ with moreover $\nabla_1 h_1(x(\bar{t}), \bar{t}) = -\nabla_1 h_2(x(\bar{t}), \bar{t})$. In that context, taking $(p, p^0) = (0_{\mathrm{BV}}, 0)$ and the measures $d\eta_1$ and $d\eta_2$ being both the Dirac measure concentrated at $t = \bar{t}$, we obtain that the triplet $(p, p^0, d\eta)$ satisfies all conditions of Theorem 3.2.1 which thus provides no additional information. We refer to [Vinter 2010, Remark (b) p.330] for a similar remark in the classical case of permanent controls.*

Remark 3.2.9. *In the classical case of state unconstrained optimal permanent control problems, the Pontryagin maximum principle induces an indirect numerical method based on the resolution by a shooting method of the boundary value problem satisfied by the augmented state-costate vector (see, e.g., [Trélat 2005, p.170-171] for details). Recall that:*

(i) *In the presence of state constraints, the indirect numerical method can be adapted. However, some theoretical and numerical difficulties may appear due to the possible pathological behavior of the adjoint vector (see Section 3.4 for more details).*

(ii) *The indirect numerical method has also been adapted to the case of (state unconstrained) optimal sampled-data control problems in [Bourdin & Trélat 2015, Bourdin & Trélat 2016], and also in case of free sampling times in [Bourdin & Dhar 2019] as presented in Chapter 2.*

Before being presented in the work [Bourdin & Dhar 2020], to the best of our knowledge, the indirect numerical method had never been adapted to optimal sampled-data control problems in the presence of running inequality state constraints. This gap was filled in the literature by using the Pontryagin maximum principle derived in Theorem 3.2.1. Of course, in the context of Theorem 3.2.1, it might be possible that the adjoint vector $p \in \mathrm{BV}([0, T], \mathbb{R}^n)$ is pathological and/or admits an infinite number of discontinuities, but it is shown in Sections 3.3 and 3.4 that, under certain (quite unrestrictive) hypotheses, the implementation of the indirect numerical method is simplified due to the particular behavior of the optimal trajectory (called the bouncing trajectory phenomenon).

Remark 3.2.10. *This comment echoes Remark 2.2.17 given in Chapter 2 regarding an alternative technique of proof for a PMP for optimal sampled-data control problems. In this chapter, as explained in the Introduction, the proof of Theorem 3.2.1 is based on the Ekeland variational principle [Ekeland 1974]. Let us note that an alternative proof of Theorem 3.2.1 can be obtained by adapting a remarkable technique exposed in the paper [Dmitruk & Kaganovich 2011]* *by Dmitruk and Kaganovich that consists of mapping each sampling interval* $[t_i, t_{i+1}]$ *to the interval* $[0, 1]$ *and by taking the values* u_i *of the sampled-data control to be additional parameters. Then, through the application of the Pontryagin maximum principle for optimal permanent control problems with running inequality state constraints (see, e.g., [Bourdin 2016, Theorem 1] and [Vinter 2010, Theorem 9.5.1 p. 339-340]), one obtains the adjoint equation (3.2) and complementary slackness condition (3.3) given in Theorem 3.2.1. Moreover the application of a "Pontryagin maximum principle with parameters" (see, e.g, [Bourdin & Trélat 2013, Remark 5 p.3790]) leads to a necessary optimality condition written in integral form which coincides with the nonpositive averaged Hamiltonian gradient condition (3.1).*

3.3 Bouncing trajectory phenomenon

When we undertook to study optimal sampled-data control problems in the presence of running inequality state constraints in the work [Bourdin & Dhar 2020], one of our first actions was to numerically solve some simple problems using a direct method (see Section 3.4 for some details on direct methods in optimal control theory). On this occasion we observed that the optimal trajectories returned by the algorithm had a common behavior with respect to the running inequality state contraints. Precisely the optimal trajectories were "bouncing" on them. We refer to Figure 3.3 and Section 3.4 for some examples illustrating this observation which we refer to as the *bouncing trajectory phenomenon*. Actually, when dealing with sampled-data controls and running inequality state constraints, the bouncing trajectory phenomenon concerns, not only the optimal trajectories, but all admissible trajectories.

In this section our aim is to give a detailed description of this new observation which was; to the best of our knowledge, first expounded in the work [Bourdin & Dhar 2020] (which does not appear in general in the classical theory, that is, with permanent controls). Precisely in [Bourdin & Dhar 2020] we showed that, under certain hypotheses, an admissible trajectory of Problem ($\mathrm{OSCP_{sc}}$) necessarily bounces on the running inequality state constraints and, moreover, the activating times occur at most at the sampling times t_i (and thus in a finite number and at precise instants). As detailed later in Section 3.4, this feature presents some benefits from a numerical point of view.

In Section 3.3.1 below we initiate an heuristic discussion allowing to understand why, usually, the admissible trajectories of Problem ($\mathrm{OSCP_{sc}}$) bounce on the running inequality state constraints and, moreover, at most at the sampling times t_i. Then we provide in Section 3.3.2 a mathematical framework and rigorous justifications which allow us to specify a sufficient condition ensuring this behavior (see Proposition 3.3.1).

Throughout this section, for simplicity, we will assume that $q = 1$, that is, there is only one running inequality state constraint in Problem ($\mathrm{OSCP_{sc}}$) denoted by $h(x(t), t) \leq 0$. Nevertheless the results and comments of this section can be extended to multiple running inequality state constraints, that is, for $q \geq 2$. Furthermore we will assume that the dynamics f and the running inequality state constraint function h are of class C^∞ in all variables. In particular note that any admissible trajectory of Problem ($\mathrm{OSCP_{sc}}$) is thus piecewise smooth of class C^∞, in the sense that it is of class C^∞ over each sampling interval $[t_i, t_{i+1}]$.

3.3.1 Expected behavior of an admissible trajectory

We start this section by recalling some standard terminology from [Hartl *et al.* 1995, p.183] or [Sethi & Thompson 2000, p.105]. Let x be an admissible trajectory of Problem ($\mathrm{OSCP_{sc}}$). An element $t \in [0,T]$ is called an *activating time* if it satisfies $h(x(t),t) = 0$. An interval $[t_1,t_2] \subset [0,T]$, with $t_1 < t_2$, is called a *boundary interval* if $h(x(t),t) = 0$ for all $t \in [t_1,t_2]$. Note that any point of a boundary interval is an activating time, while the reverse is not true in general. In what follows, we say that the trajectory x exhibits the *bouncing trajectory phenomenon* if the set of activating times contains no boundary interval.

Our aim in this section is to give some heuristic descriptions (and illustrative figures) of the main reason why a bouncing trajectory phenomenon is common when dealing with sampled-data controls in the presence of running inequality state constraints (see (i) below) and why, moreover, the activating times occur at most at the sampling times t_i only (see (ii) below). The mathematical framework and rigorous justifications will be provided in Section 3.3.2.

(i) In the classical theory (that is, with permanent controls), a boundary interval may correspond to a *feedback control*, that is, to an expression of the control as a function of the state. Such an expression usually leads to a nonconstant control. More generally, a running inequality state constraint usually cannot be activated by a trajectory on an interval $[t_1,t_2]$, with $t_1 < t_2$, on which the associated (permanent) control is constant. We refer to Figure 3.1 for an illustration. Therefore, since we deal with piecewise constant controls in Problem ($\mathrm{OSCP_{sc}}$), one should expect that an admissible trajectory of Problem ($\mathrm{OSCP_{sc}}$) does not contain any boundary interval and thus exhibits a bouncing trajectory phenomenon. In order to guarantee the validity of this remark, it is sufficient to make an assumption on f and h which prevents the existence of an admissible trajectory x of Problem ($\mathrm{OSCP_{sc}}$) and an interval $[t_1,t_2] \subset [0,T]$, with $t_1 < t_2$, for which $\varphi^{(\ell)}(t) = 0$ for all $\ell \in \mathbb{N}$ and all $t \in [t_1,t_2]$, where φ is defined by $\varphi(t) := h(x(t),t)$ for all $t \in [t_1,t_2]$. This will be done in Section 3.3.2 (see Hypothesis (H1)).

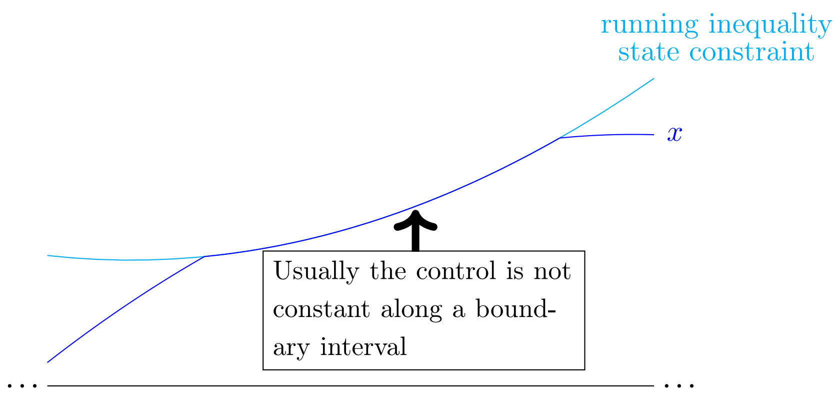

Figure 3.1: In the classical theory (that is, with permanent controls), a boundary interval is usually associated with a nonconstant control

(ii) Let $t \in [0,T]$ be a left isolated (resp. right isolated) activating time of an admissible trajectory x of Problem ($\mathrm{OSCP_{sc}}$). In what follows we denote by u the corresponding control. Let us assume that t is not a sampling time, that is, $t \in (t_i, t_{i+1})$ for some $i \in \{0, \dots, N-1\}$. Usually the trajectory x "hits" (resp. "exits") the running inequality state constraint transversally at t. Since the control value $u(t) = u_i$ is fixed all along the sampling interval $[t_i, t_{i+1}]$, the trajectory x then "crosses" the running inequality state constraint immediately after t (resp. immediately before t), which contradicts the admissibility of x. We refer to Figure 3.2 for an illustration. Hence, in order to preserve the admissibility of x, we understand that the control value must change at t, that is, since u is a sampled-data control, that t must be one of the sampling times t_i. From this simple heuristic discussion, one should expect that an admissible trajectory of Problem (OSCP_{sc}) has no left or right isolated activating time outside of the sampling times t_i. In order to guarantee the validity of this remark, it is sufficient to make an assumption on f and h which prevents the existence of an admissible trajectory of Problem (OSCP_{sc}) which "hits" or "exits" the running inequality state constraint tangentially. This will be done in Section 3.3.2 (see Hypothesis (H2)). Actually our Hypothesis (H2) will even guarantee that an admissible trajectory of Problem (OSCP_{sc}) has no activating time outside of the sampling times t_i.

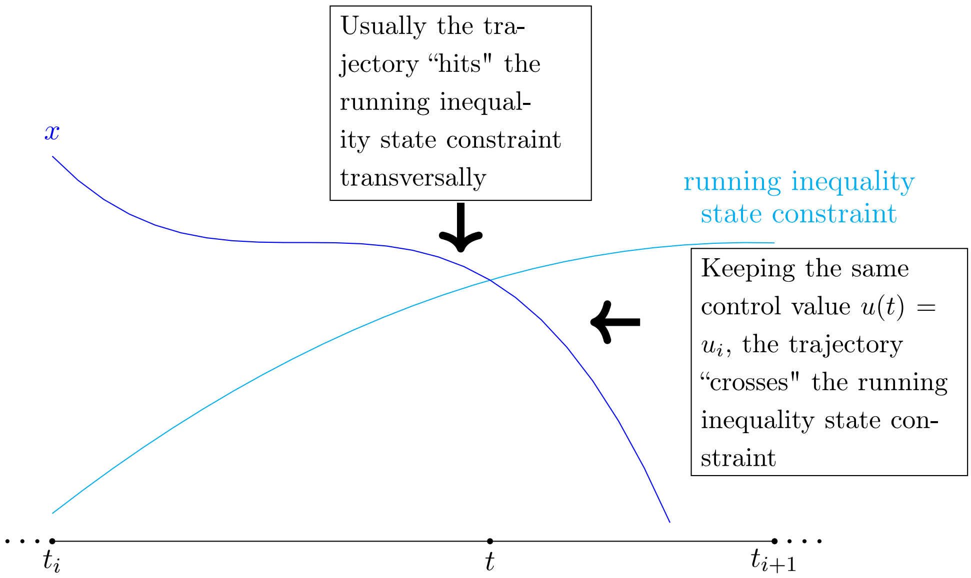

Figure 3.2: Illustration of a trajectory x hitting transversally the running inequality state constraint at some left isolated activating time t which belongs to the interior (t_i, t_{i+1}) of a sampling interval.

We conclude from (i) and (ii) that one should expect the admissible trajectories of Problem (OSCP_{sc}) to exhibit the bouncing trajectory phenomenon and, moreover, so that the activating times occur at most at the sampling times t_i (and thus in a finite number and at precise instants). We refer to Figure 3.3 for an illustration of this feature. Note that, even if activating times are sampling times, the reverse is not true in general.

We conclude this section by mentioning that the above descriptions are only heuristic and, of course, one can easily find counterexamples in which the behavior of Figure 3.3 is not observed. Nonetheless we emphasize that the bouncing trajectory phenomenon is quite ordinary when

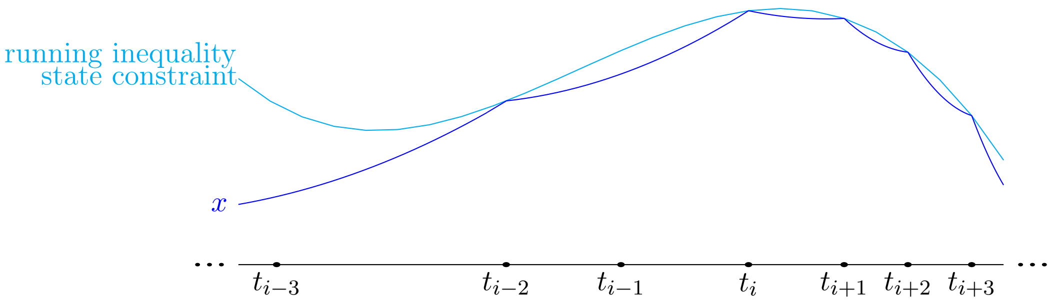

Figure 3.3: Illustration of the expected behavior of an admissible trajectory x of Problem ($\mathrm{OSCP_{sc}}$).

dealing with sampled-data controls and running inequality state constraints, as guaranteed by the mathematical justifications provided in Section 3.3.2 below and as illustrated by the examples numerically solved in Section 3.4.

3.3.2 A sufficient condition for the bouncing trajectory phenomenon

Our aim in this section is to provide a rigorous mathematical framework describing the heuristic discussion provided in the previous Section 3.3.1. In particular we will formulate a sufficient condition (see Proposition 3.3.1 below) ensuring the bouncing trajectory phenomenon and that the rebounds occur at most at the sampling times t_i.

To this aim, and similarly to [Hartl *et al.* 1995, p.183], we introduce the functions $h^{[\ell]} : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ defined by the induction

$$\begin{cases} h^{[0]}(y,\omega,t) := h(y,t), \\ \forall \ell \in \mathbb{N}, \quad h^{[\ell+1]}(y,\omega,t) := \langle \nabla_1 h^{[\ell]}(y,\omega,t), f(y,\omega,t) \rangle_{\mathbb{R}^n} + \nabla_3 h^{[\ell]}(y,\omega,t), \end{cases}$$

for all $(y,\omega,t) \in \mathbb{R}^n \times \mathbb{R}^m \times [0,T]$. We introduce the subset

$$\mathcal{M} := \{(y,t) \in \mathbb{R}^n \times ([0,T] \setminus \mathbb{T}) \mid h(y,t) = 0\},$$

and we denote by

$$\ell'(y,\omega,t) := \min\{\ell \in \mathbb{N} \mid h^{[\ell]}(y,\omega,t) \neq 0\} \in \mathbb{N}^* \cup \{+\infty\},$$

for all $(y,t) \in \mathcal{M}$ and all $\omega \in \mathrm{U}$. Finally we introduce the set

$$\mathrm{U}_{(y,t)} := \{\omega \in \mathrm{U} \mid \ell'(y,\omega,t) \text{ is finite and even}\},$$

for all $(y,t) \in \mathcal{M}$. We now state the main result of this section.

Proposition 3.3.1. *Assume that $q = 1$ and that f and h are of class C^∞ in all their variables. If the hypotheses*

$$\forall (y,t) \in \mathcal{M}, \quad \forall \omega \in \mathrm{U}, \quad \ell'(y,\omega,t) < +\infty, \tag{H1}$$

and

$$\forall (y,t) \in \mathcal{M}, \quad \forall \omega \in \mathrm{U}_{(y,t)}, \quad h^{[\ell'(y,\omega,t)]}(y,\omega,t) > 0, \tag{H2}$$

are both satisfied, then the activating times of an admissible trajectory x of Problem (OSCP$_{\text{sc}}$) *are sampling times. In particular x exhibits the bouncing trajectory phenomenon and the rebounds occur at most at the sampling times t_i (and thus in a finite number and at precise instants).*

Proof. Let (x,u) be an admissible couple of Problem (OSCP$_{\text{sc}}$). Let $t \in [0,T]$ be an activating time and assume by contradiction that $t \in (t_i, t_{i+1})$ for some $i = 0, \ldots, N-1$. In particular we have $(x(t), t) \in \mathcal{M}$. Since $u_i \in \mathrm{U}$, from Hypothesis (H1), we know that $\ell' := \ell'(x(t), u_i, t) < +\infty$ and it holds that $h^{[\ell']}(x(t), u_i, t) \neq 0$. From Taylor's theorem it holds that

$$h(x(t+\varepsilon), t+\varepsilon) = \varepsilon^{\ell'} \left(\frac{h^{[\ell']}(x(t), u_i, t)}{\ell'!} + R(\varepsilon) \right),$$

for all $\varepsilon \in \mathbb{R}$ such that $t + \varepsilon \in (t_i, t_{i+1})$, where the remainder term R satisfies $\lim_{\varepsilon \to 0} R(\varepsilon) = 0$. Thus there exists $\bar{\varepsilon} > 0$ such that $(t - \bar{\varepsilon}, t + \bar{\varepsilon}) \subset (t_i, t_{i+1})$ and $h(x(t'), t')$ has the same sign than $(t'-t)^{\ell'} h^{[\ell']}(x(t), u_i, t)$ for all $t' \in (t - \bar{\varepsilon}, t + \bar{\varepsilon})$ with $t' \neq t$. We now distinguish two cases: ℓ' odd and ℓ' even. If ℓ' is odd, then there clearly exists $t' \in (t - \bar{\varepsilon}, t + \bar{\varepsilon})$ with $t' \neq t$ such that $h(x(t'), t') > 0$ which raises a contradiction with the admissibility of (x,u). If ℓ' is even, then $u_i \in \mathrm{U}_{(x(t),t)}$ and, from Hypothesis (H2), it holds that $h^{[\ell']}(x(t), u_i, t) > 0$. We easily deduce that there exists $t' \in (t - \bar{\varepsilon}, t + \bar{\varepsilon})$ with $t' \neq t$ such that $h(x(t'), t') > 0$ which raises the same contradiction. The proof is complete. □

Remark 3.3.1. *We emphasize that Hypotheses* (H1) *and* (H2) *are assumptions which guarantee the validity of the arguments presented heuristically in the items* (i) *and* (ii) *of Section 3.3.1.*

In the context of Proposition 3.3.1, it is ensured that an admissible trajectory of Problem (OSCP$_{\text{sc}}$) activates the running inequality state constraint at most at the sampling times t_i (and thus in a finite number and at precise instants). We will see in Section 3.4 below that this bouncing trajectory phenomenon (with localized rebounds) presents some benefits from a numerical point of view. Taking this advantage we will numerically solve some simple examples in which Hypotheses (H1) and (H2) are both satisfied and we will observe optimal trajectories bouncing on the running inequality state constraint considered.

3.4 Numerical experiments

Two predominant kinds of numerical methods are known in classical optimal control theory (that is, with permanent controls) without running inequality state constraints. The first kind is usually called *direct numerical methods* and they consist in making a full discretization of the optimal control problem which results in a constrained finite-dimensional optimization problem that can be numerically solved from various standard optimization algorithms and techniques. The second strategy is called *indirect numerical methods* because they are based on the Pontryagin maximum principle. Precisely, if the Hamiltonian maximization condition allows to express the optimal control u as a function of the state x and of the (absolutely continuous) adjoint vector p, then the indirect numerical methods consist in the numerical resolution by a shooting method of the boundary value problem satisfied by the augmented state-costate vector (x,p). We emphasize that neither direct nor indirect methods are fundamentally better than the other. We refer for instance to [Trélat 2005, p.170-171] for details and discussions on the advantages and drawbacks of each kind of methods.

In the presence of running inequality state constraints, direct numerical methods can be adapted easily. On the contrary, solving optimal permanent control problems with running inequality state constraints might be more intricate when using indirect numerical methods. Indeed, in that situation, the adjoint vector p is not absolutely continuous in general, but (only) of bounded variation. From the Lebesgue decomposition (see Theorem 1.2.1 in Chapter 1), we can write

$$p = p_{ac} + p_{sc} + p_s,$$

where p_{ac} is the *absolutely continuous part*, p_{sc} is the *singularly continuous part* and p_s is the *saltus* (or *pure jump part*) of p. From the complementary slackness condition, it is well known that the adjoint vector p is absolutely continuous on intervals with no activating time of the optimal trajectory x. On the other hand, on boundary intervals, the adjoint vector p may have an infinite number of unlocalized jumps or a pathological behavior due to its singular part. As a consequence, an important part of the literature is devoted to the analysis of the costate's behavior and some constraint qualification conditions have been established. We refer for instance to [Bettiol & Frankowska 2008, Bonnard *et al.* 2003, Hartl *et al.* 1995, Jacobson *et al.* 1971, Maurer 1977].

In this chapter we have presented a Pontryagin maximum principle (Theorem 3.2.1) given in the work [Bourdin & Dhar 2020] and our aim in this section is to propose an indirect method for numerically solving optimal sampled-data control problems with running inequality state constraints as seen in the work [Bourdin & Dhar 2020]. As in the classical theory (with permanent controls), it appears that the adjoint vector obtained in Theorem 3.2.1 is (only) a function of bounded variation and we will *a priori* encounter the same difficulties outlined above. Nevertheless, as detailed in Section 3.3, we have proved in Proposition 3.3.1 that, under (quite unrestrictive) Hypotheses (H1) and (H2), the optimal trajectory x of Problem ($\mathrm{OSCP}_{\mathrm{sc}}$) activates the running inequality state constraint at most at the sampling times t_i. As detailed in Section 3.4.1 below, it follows that the corresponding adjoint vector p has no singular part and admits a finite number of jumps which are localized at most at the sampling times t_i. Taking advantage of these informations, we will propose in Section 3.4.1 a simple indirect method in order to numerically solve optimal sampled-data control problems with running inequality state constraints of the form of Problem ($\mathrm{OSCP}_{\mathrm{sc}}$) under Hypotheses (H1) and (H2).

In Sections 3.4.2, 3.4.3 and 3.4.4, this indirect method is implemented in order to numerically solve three simple examples. We precise that the parameters of these examples have been chosen in order to obtain figures which illustrate and highlight the bouncing trajectory phenomenon and the jumps of the adjoint vector. Furthermore note that the numerical results returned by the indirect method suggests the convergence of the optimal sampled-data controls to the optimal permanent control as N tends to $+\infty$. This provides a very interesting perspective to investigate in future works. We mention that such a result has already been established in [Bourdin & Trélat 2017] in the case of unconstrained linear-quadratic problems.

We conclude this paragraph by noting that the indirect numerical method proposed in Section 3.4.1 (and its implementation in Sections 3.4.2, 3.4.3 and 3.4.4) is based on the assumption that there exists a solution to Problem ($\mathrm{OSCP}_{\mathrm{sc}}$). This question of existence has not been addressed in the present chapter and constitutes an open question for future works (see Remark 2.2.16 for more details).

3.4.1 A shooting function for an indirect method

In this section our aim is to provide an indirect method, based on the Pontryagin maximum principle given in Theorem 3.2.1, which allows to numerically solve some optimal sampled-data control problems with running inequality state constraints. This numerical method can be implemented in the normal case as well as in the abnormal case (in the sense of Remark 3.2.2).

Let (x,u) be a solution to Problem ($\mathrm{OSCP_{sc}}$). We denote by p^0, η, p the elements provided by the Pontryagin maximum principle given in Theorem 3.2.1. As explained at the beginning of Section 3.4, the adjoint vector p may have a pathological behavior which would imply some theoretical and/or numerical difficulties. Our aim in the sequel is to take advantage of Proposition 3.3.1 established in Section 3.3. To this aim, we assume in the sequel that $q=1$, that f and h are of class C^∞ in all variables and that Hypotheses (H1) and (H2) are satisfied. As a consequence, it follows from Proposition 3.3.1 that x activates the running inequality state constraint at most at the sampling times t_i. From the complementary slackness condition in Theorem 3.2.1, we deduce that η admits exactly $(N+1)$ nonnegative jumps localized exactly at the sampling times t_i, and that η remains constant over (t_0,t_1) and over all $[t_i,t_{i+1})$ with $i=1,\dots,N-1$. In what follows we denote the nonnegative jumps of η as follows:

$$\eta^{[0]} := \eta(t_0^+) - \eta(t_0) = \eta(t_0^+), \quad \eta^{[1]} := \eta(t_1) - \eta(t_0^+) \quad \text{and} \quad \forall i = 2,\dots,N, \quad \eta^{[i]} := \eta(t_i) - \eta(t_{i-1}).$$

From the adjoint equation in Theorem 3.2.1, it follows that the adjoint vector p has no singular part, that it admits $(N+1)$ jumps localized exactly at the sampling times t_i, and that p remains absolutely continuous over (t_0,t_1) and over all $[t_i,t_{i+1})$ with $i=1,\dots,N-1$. Moreover, from the integral representation of p, the jumps of the adjoint vector are given by

$$p^{[0]} := p(t_0^+) - p(t_0) = \eta^{[0]} \nabla_1 h(x(t_0),t_0) \quad \text{and} \quad \forall i = 1,\dots,N, \quad p^{[i]} := p(t_i) - p(t_i^-) = \eta^{[i]} \nabla_1 h(x(t_i),t_i).$$

The general indirect numerical method proposed in this chapter is based on the shooting map

$$\left(x_T, (\eta^{[i]})_{i=0,\dots,N} \right) \longmapsto \left(x(0) - x_0, \left(\eta^{[i]} h(x(t_i),t_i) \right)_{i=0,\dots,N} \right),$$

where:

(i) we provide a guess of the final value $x(T)=x_T$ and of the nonnegative jumps $\eta^{[i]}$ for all $i=0,\dots,N$;

(ii) we compute $p(T) = p^0 \nabla g(x(T))$;

(iii) we numerically solve the state and adjoint equations in a backward way (from $t=T$ to $t=0$), by using the nonpositive averaged Hamiltonian gradient condition in order to compute the control values u_i for all $i=0,\dots,N-1$;

(iv) we finally compute $x(0)-x_0$ and $\eta^{[i]} h(x(t_i),t_i)$ for all $i=0,\dots,N$.

As illustrations of the above indirect numerical method, we solve three simple examples in Sections 3.4.2, 3.4.3 and 3.4.4 below. We precise that we used the MATLAB function *fsolve* in order to find the zeros of the above shooting function. We also mention that we used the basic forward Euler method in order to numerically solve the state and adjoint equations (but numerous other approaches can be considered, such as Runge-Kutta methods for example). Finally we emphasize that the numerical results obtained and presented hereafter have all been confirmed by direct numerical approaches (using a basic forward Euler discretization of the whole problem resulting into a constrained finite-dimensional optimization problem solved numerically by the MATLAB function *fmincon*).

3.4.2 Example 1: a problem with a parabolic running inequality state constraint

We first consider the following optimal sampled-data control problem with running inequality state constraint given by

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\int_0^4 x(t) + \frac{1}{4}u(t)^2\, dt \\[2ex] \text{subject to} & x \in \mathrm{AC}([0,4],\mathbb{R}),\ u \in \mathrm{PC}^{\mathbb{T}}([0,4],\mathbb{R}), \\[1ex] & \dot{x}(t) = u(t), \quad \text{a.e. } t \in [0,4], \\[1ex] & x(0) = 6, \\[1ex] & \frac{1}{2}(t-2)^2 + 2 - x(t) \leq 0, \quad \text{for all } t \in [0,4], \\[1ex] & u_i \in [-3,+\infty), \quad \text{for all } i = 0,\ldots,N-1, \end{array}\right\} \tag{E1}$$

for fixed uniform N-partitions $\mathbb{T}$ of the interval $[0,4]$ with different values of $N \in \mathbb{N}^*$. This simple problem coincides with a calculus of variations problem (with running inequality state constraints on the trajectory and its derivative, and also constraining the derivative to be piecewice constant).

Let us check that Problem (E1) satisfies Hypotheses (H1) and (H2). To this aim we follow the notations introduced in Section 3.3.2. For all $(y,t) \in \mathcal{M}$ and all $\omega \in [-3,+\infty)$ it holds that $h^{[2]}(y,\omega,t) = 1$ and so Hypothesis (H1) is satisfied. We deduce that, for all $(y,t) \in \mathcal{M}$ and all $\omega \in \mathrm{U}_{(y,t)}$, we have $\ell'(y,\omega,t) = 2$ and $h^{[\ell'(y,\omega,t)]}(y,\omega,t) = 1 > 0$, so Hypothesis (H2) is satisfied as well. We conclude from Proposition 3.3.1 that all admissible trajectories activate the running inequality state constraint at most at the sampling times t_i.

In what follows we assume that there exists an optimal couple (x,u) for Problem (E1) and we denote by p^0, η, p the elements provided by the Pontryagin maximum principle given in Theorem 3.2.1. Let us check that the case is normal (in the sense of Remark 3.2.2). Assume by contradiction that $p^0 = 0$. We have the adjoint equation $-dp = d\eta$ over $[0,4]$ with $p(4) = 0$. Therefore $p(t) = \int_t^4 d\eta(s) = \eta(4) - \eta(t)$ for all $t \in [0,4]$. Then, from the nontriviality of the couple (p^0,η), it follows that $\eta \neq 0_{\mathrm{NBV}([0,T],\mathbb{R})}$ and thus, from the complementary slackness condition, we deduce that x necessarily activates the running inequality state constraint. Let $\bar{t} \in [0,4]$ denote the first activating time. From Proposition 3.3.1, we know that $\bar{t} = t_{\widehat{i}}$ for some $\widehat{i} \in$

$\{0, \ldots, N\}$. Since $x(0) = 6$, we have $\widehat{i} \geq 1$. It follows that $p(t) > 0$ for all $t \in [0, t_1)$. Finally, from the nonpositive averaged Hamiltonian gradient condition at $i = 0$, it follows that $u_0 \geq \omega$ for all $\omega \in [-3, +\infty)$ which is absurd.

From the previous paragraph, we normalize $p^0 = -1$ (see Remark 3.2.2). Since we are in the context of Proposition 3.3.1, we can now apply the shooting method detailed in Section 3.4.1. As expected, we observe in Figure 3.4 (with $N = 5$) that the optimal trajectory returned by the algorithm activates the running inequality state constraint at most at the sampling times t_i (represented by dashed lines). As also expected, the jumps of the adjoint vector occur at the same activating times. Figures 4.3 and 3.6 continue to illustrate this bouncing trajectory phenomenon for larger values of N (respectively with $N = 10$ and $N = 40$). Furthermore, in Figures 3.4, 4.3 and 3.6, we observe that the adjoint vector has no jump at sampling times which are not activating times.

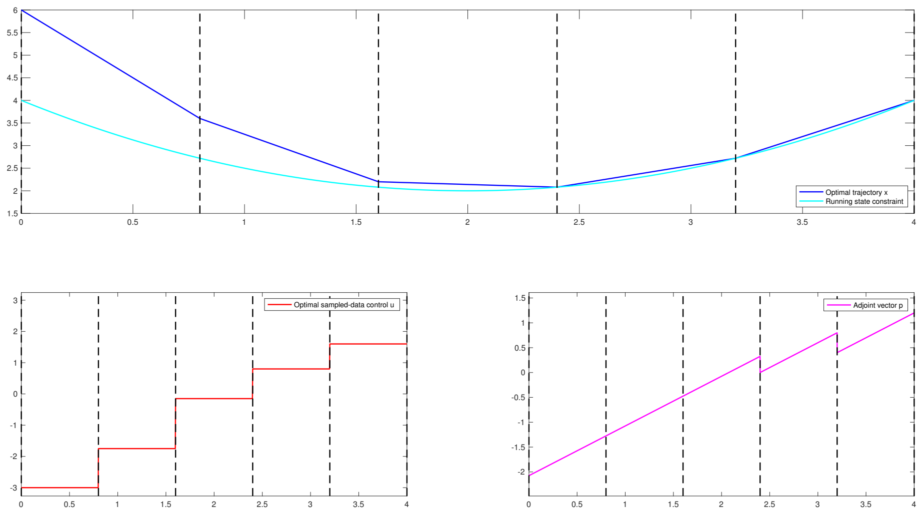

Figure 3.4: Example 1 with $N = 5$.

Remark 3.4.1. *Actually, in that simple Example 1, the state and adjoint equations are very simple and can be solved explicitly. As a consequence, the shooting map can even be expressed in the closed form given by*

$$\Big(x_T, (\eta^{[i]})_{i=0,\ldots,N}\Big) \longmapsto$$
$$\left(x_T - x_0 - \sum_{i=0}^{N-1} u_i(t_{i+1} - t_i), \left(\eta^{[i]} \left[\frac{1}{2}(t_i - 2)^2 + 2 - x_T + \sum_{k=i}^{N-1} u_k(t_{k+1} - t_k) \right] \right)_{i=0,\ldots,N-1} \right),$$

where

$$u_i = \max \left\{ 2 \left(\frac{t_i + t_{i+1}}{2} - T + \sum_{k=i+1}^{N} \eta^{[k]} \right), -3 \right\},$$

for all $i = 0, \ldots, N-1$.

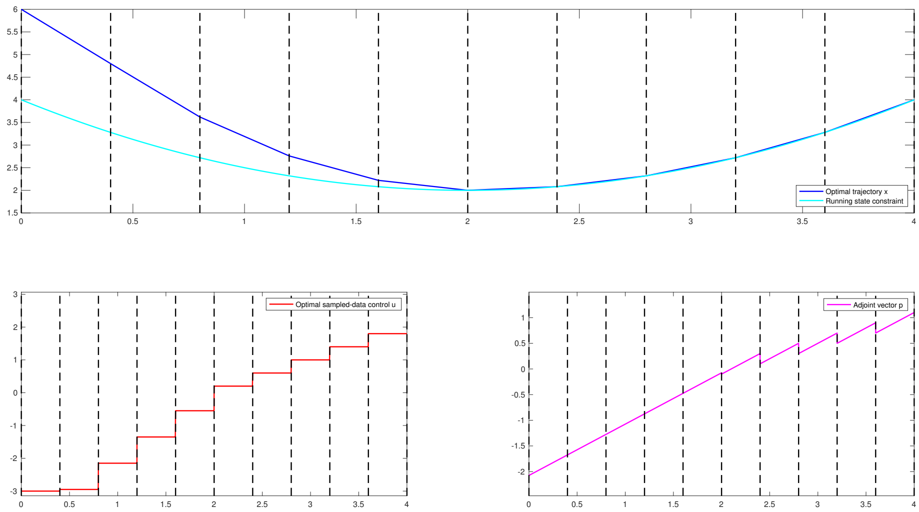

Figure 3.5: Example 1 with $N = 10$.

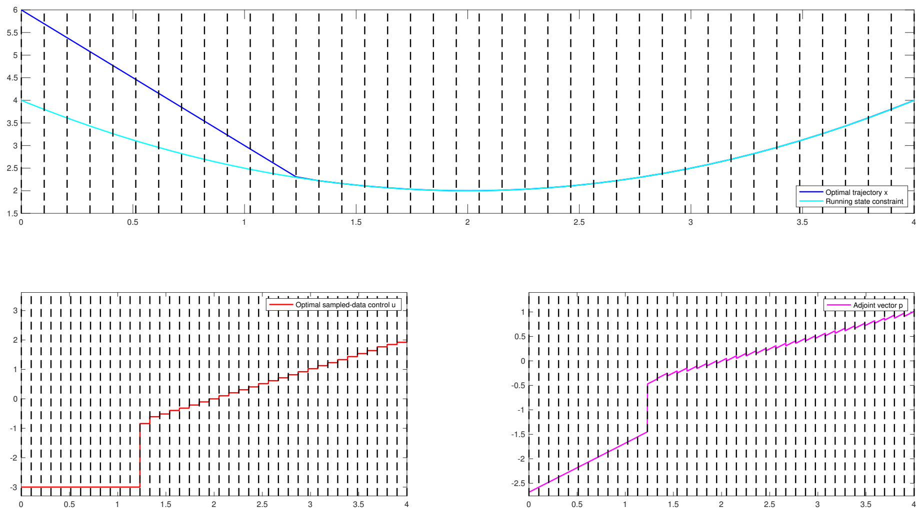

Figure 3.6: Example 1 with $N = 40$.

3.4.3 Example 2: an optimal consumption problem with an affine running inequality state constraint

The second example is the optimal sampled-data control problem with running inequality state constraint given by

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\int_0^{12} (u(t)-1)x(t)\,dt \\ \text{subject to} & x \in \mathrm{AC}([0,12],\mathbb{R}),\ u \in \mathrm{PC}^{\mathbb{T}}([0,12],\mathbb{R}), \\ & \dot{x}(t) = u(t)x(t), \quad \text{a.e. } t \in [0,12], \\ & x(0) = 1, \\ & x(t) - 10t - 2 \leq 0, \qquad \text{for all } t \in [0,12], \\ & u_i \in [0,1], \quad \text{for all } i = 0,\dots,N-1, \end{array}\right\} \tag{E2}$$

for fixed uniform N-partitions $\mathbb{T}$ of the interval $[0,12]$ with different values of $N \in \mathbb{N}^*$. This problem corresponds to a classical optimal consumption problem (see, e.g., [Evans 2013, p.5]) revisited with sampled-data controls and running inequality state constraint.

Let us check that Problem (E2) satisfies Hypotheses (H1) and (H2). To this aim we follow the notations introduced in Section 3.3.2. Let us assume by contradiction that there exist $(y,t) \in \mathcal{M}$ and $\omega \in [0,1]$ such that $\ell'(y,\omega,t) = +\infty$. Then it follows that $h^{[0]}(y,\omega,t) = h^{[1]}(y,\omega,t) = h^{[2]}(y,\omega,t) = 0$. From $h^{[0]}(y,\omega,t) = 0$, it holds that $y > 0$ and, from $h^{[1]}(y,\omega,t) = 0$, it holds that $\omega > 0$. Therefore $h^{[2]}(y,\omega,t) = \omega^2 y > 0$ which raises a contradiction. Thus Hypothesis (H1) is satisfied. From a similar reasoning, we prove that Hypothesis (H2) is also satisfied. We conclude from Proposition 3.3.1 that all admissible trajectories activate the running inequality state constraint at most at the sampling times t_i.

In what follows we assume that there exists an optimal couple (x,u) for Problem (E2) and we denote by p^0, η, p the elements provided by the Pontryagin maximum principle given in Theorem 3.2.1. Note that necessarily $x(t) > 0$ for all $t \in [0,12]$. Let us check that the case is normal (in the sense of Remark 3.2.2). Assume by contradiction that $p^0 = 0$. We have the adjoint equation $dp = d\eta$ over $[0,12]$ with $p(12) = 0$. Therefore $p(t) = -\int_t^{12} d\eta(s) = \eta(t) - \eta(12)$ for all $t \in [0,12]$. Then, from the nontriviality of the couple (p^0,η), it follows that $\eta \neq 0_{\mathrm{NBV}([0,T],\mathbb{R})}$ and thus, from the complementary slackness condition, we deduce that x necessarily activates the running inequality state constraint. Let $\bar{t} \in [0,12]$ denote the first activating time. From Proposition 3.3.1, we know that $\bar{t} = t_{\widehat{i}}$ for some $\widehat{i} \in \{0,\dots,N\}$. Since $x(0) = 1$, we know that $\widehat{i} \geq 1$. It follows that $p(t) < 0$ for all $t \in [0,t_{\widehat{i}})$. Finally, since $x(t) > 0$ for all $t \in [0,12]$ and from the nonpositive averaged Hamiltonian gradient condition at $i = 0,\dots,\widehat{i}-1$, we get that $u_0 = \ldots = u_{\widehat{i}-1} = 0$, which gives $x(t) = 1$ for all $t \in [0,\bar{t}]$. This raises a contradiction since $x(\bar{t}) = 10\bar{t} + 2 > 1$.

From the previous paragraph, we normalize $p^0 = -1$ (see Remark 3.2.2). Since we are in the context of Proposition 3.3.1, we can now apply the shooting method detailed in Section 3.4.1. In Figure 3.7 (with $N = 2$) we observe that the optimal trajectory returned by the algorithm activates the running inequality state constraint at most at the sampling times t_i (represented by dashed lines). Figures 3.8 and 3.9 continue to illustrate this bouncing trajectory phenomenon for larger values of N (respectively with $N = 4$ and $N = 6$). Furthermore, in Figures 3.8 and 3.9, we observe that the adjoint vector has no jump at sampling times which are not activating times.

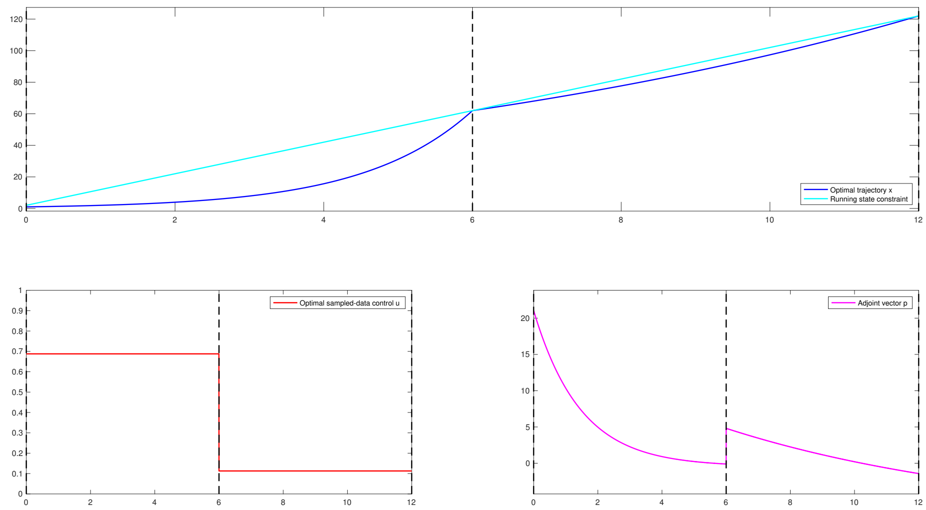

Figure 3.7: Example 2 with $N = 2$.

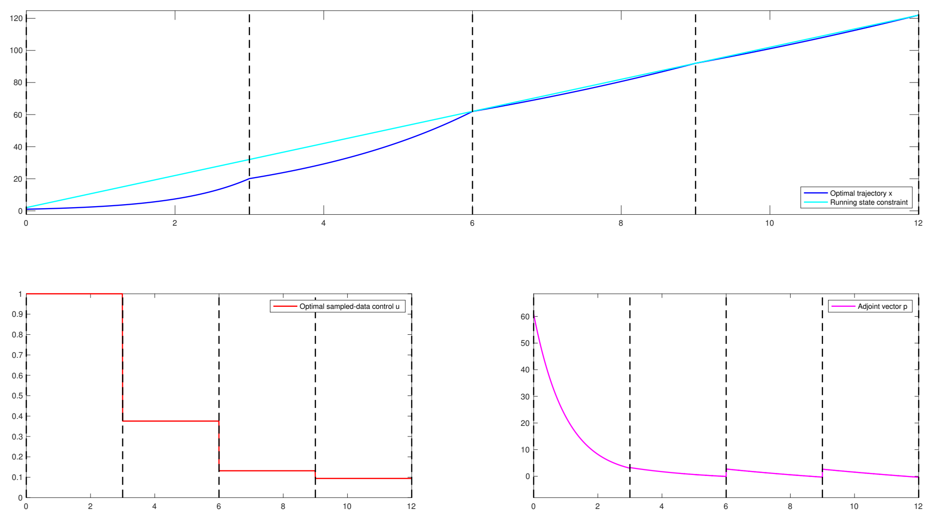

Figure 3.8: Example 2 with $N = 4$.

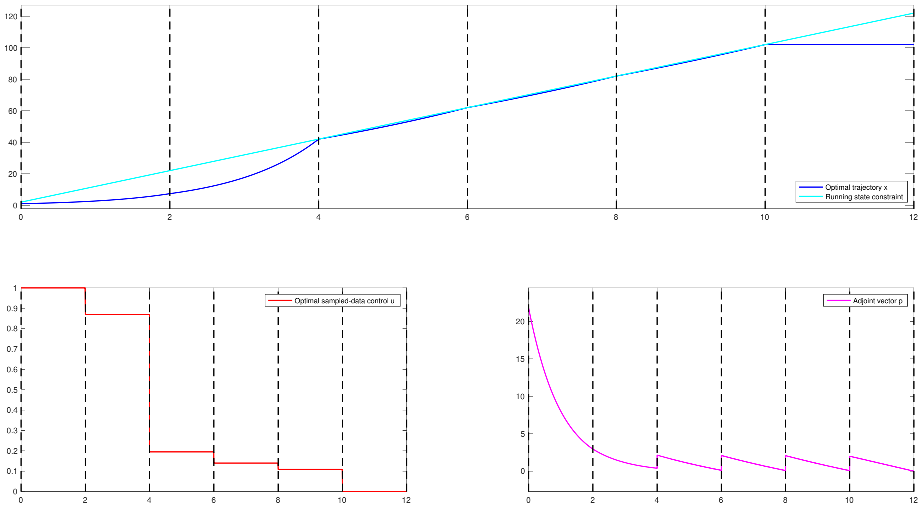

Figure 3.9: Example 2 with $N = 6$.

3.4.4 Example 3: a two-dimensional problem with a linear running inequality state constraint

As a third and last example we consider the optimal sampled-data control problem with running inequality state constraint given by

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\int_0^2 x_2(t) + \frac{1}{4}u(t)^2\,dt \\[2ex] \text{subject to} & x \in \mathrm{AC}([0,2],\mathbb{R}^2),\ u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}), \\[1ex] & \begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} x_2(t) - u(t) \\ x_1(t) + x_2(t) + u(t) \end{pmatrix}, \quad \text{a.e. } t \in [0,2], \\[1ex] & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 0.05 \\ -0.1 \end{pmatrix}, \\[1ex] & x_1(t) - 16x_2(t) - 2 \leq 0, \qquad \text{for all } t \in [0,2], \\[1ex] & u_i \in [-0.1,+\infty), \quad \text{for all } i = 0,\dots,N-1, \end{array}\right\} \tag{E3}$$

for fixed uniform N-partitions $\mathbb{T}$ of the interval $[0,2]$ with different values of $N \in \mathbb{N}^*$. This problem constitutes a two-dimensional problem with a linear running inequality state constraint.

Let us check that Problem (E3) satisfies Hypotheses (H1) and (H2). We denote by $y := (y_1, y_2) \in \mathbb{R}^2$ and we follow the notations introduced in Section 3.3.2. Let us assume by contradiction that there exist $(y,t) \in \mathcal{M}$ and $\omega \in [-0.1,+\infty)$ such that $\ell'(y,\omega,t) = +\infty$. Then it follows the system of linear equalities given by

$$\begin{aligned} h^{[0]}(y,\omega,t) &= y_1 - 16y_2 - 2 = 0, \\ h^{[1]}(y,\omega,t) &= -16y_1 - 15y_2 - 17\omega = 0, \\ h^{[2]}(y,\omega,t) &= -15y_1 - 31y_2 + \omega = 0, \end{aligned}$$

which has a unique solution given by $(y_1, y_2, \omega) = \frac{1}{9}(2, -1, -1)$ which raises a contradiction since $\omega \in [-0.1, +\infty)$. Thus Hypothesis (H1) is satisfied. If $h^{[0]}(y, \omega, t) = h^{[1]}(y, \omega, t) = 0$ for some $(y, t) \in \mathcal{M}$ and some $\omega \in [-0.1, +\infty)$, it follows that $-15y_1 - 31y_2 = 2 + 17\omega$. Therefore $h^{[2]}(y, \omega, t) = -15y_1 - 31y_2 + \omega = 2 + 18\omega \geq 0.2 > 0$. Thus Hypothesis (H2) is also satisfied. We conclude from Proposition 3.3.1 that all admissible trajectories activate the running inequality state constraint at most at the sampling times t_i.

In what follows we assume that there exists an optimal couple (x, u) for Problem (E3) and we denote by p^0, η, p the elements provided by the Pontryagin maximum principle given in Theorem 3.2.1. Let us check that the case is normal (in the sense of Remark 3.2.2). Assume by contradiction that $p^0 = 0$. From the previous paragraph, we know that x activates the running inequality state constraint at most at the sampling times t_i. From the complementary slackness condition in Theorem 3.2.1, we deduce that η admits exactly $(N+1)$ nonnegative jumps localized exactly at the sampling times t_i, and that η remains constant over (t_0, t_1) and over all $[t_i, t_{i+1})$ with $i = 1, \ldots, N-1$. Similarly to Section 3.4.1, we denote by $\eta^{[i]}$, for all $i = 0, \ldots, N$, the $N+1$ jumps of η. From the nontriviality of the couple (p^0, η), we know that the jumps $\eta^{[i]}$ are not all zero. Since the initial condition $x(0)$ does not activate the running inequality state constraint, we know that $\eta^{[0]} = 0$. Now take $\widehat{i} \in \{1, \ldots, N\}$ such that $\eta^{[\widehat{i}]} > 0$ is the last nonzero jump of η. On the other hand, since $p(T) = 0_{\mathbb{R}^2}$ and from the adjoint equation considered over the time interval $[t_{\widehat{i}}, T]$, we obtain that $p(t_{\widehat{i}}) = 0_{\mathbb{R}^2}$. From the adjoint equation considered over the time interval $[t_{\widehat{i}-1}, t_{\widehat{i}}]$, we obtain that

$$p(t_{\widehat{i}}^-) = p(t_{\widehat{i}}) - \eta^{[\widehat{i}]} \begin{pmatrix} 1 \\ -16 \end{pmatrix} = \eta^{[\widehat{i}]} \begin{pmatrix} -1 \\ 16 \end{pmatrix},$$

and $p(t) = e^{(t - t_{\widehat{i}})A} \times p(t_{\widehat{i}}^-)$ for all $t \in [t_{\widehat{i}-1}, t_{\widehat{i}})$ where

$$A := \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}.$$

We get that

$$\begin{aligned} \int_{t_{\widehat{i}-1}}^{t_{\widehat{i}}} \nabla_2 H(x(s), u_{\widehat{i}-1}, p(s), p^0, s)\, ds &= \int_{t_{\widehat{i}-1}}^{t_{\widehat{i}}} p_2(s) - p_1(s)\, ds = \int_{t_{\widehat{i}-1}}^{t_{\widehat{i}}} \left\langle p(s), \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}^2} ds \\ &= \eta^{[\widehat{i}]} \left\langle \int_{t_{\widehat{i}-1}}^{t_{\widehat{i}}} e^{(s - t_{\widehat{i}})A}\, ds \times \begin{pmatrix} -1 \\ 16 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}^2} \\ &= \eta^{[\widehat{i}]} \left\langle A^{-1} \times \left(\mathrm{Id}_2 - e^{-(t_{\widehat{i}} - t_{\widehat{i}-1})A} \right) \times \begin{pmatrix} -1 \\ 16 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}^2}. \end{aligned}$$

The sign of this term is independent of the value of $\eta^{[\widehat{i}]} > 0$. We compute numerically the above term with different positive values of $t_{\widehat{i}} - t_{\widehat{i}-1}$ belonging to $(0, 2]$ (in particular $\frac{2}{4}$, $\frac{2}{5}$ and $\frac{2}{8}$ which are the values used in the next paragraph) and we always obtain a positive value which raises a contradiction with the nonpositive averaged Hamiltonian gradient condition provided in Theorem 3.2.1.

From the previous paragraph, we normalize $p^0 = -1$ (see Remark 3.2.2). Since we are in the context of Proposition 3.3.1, we can now apply the shooting method detailed in Section 3.4.1.

In Figure 3.10 (with $N = 4$) we observe as expected a bouncing trajectory phenomenon. Figures 3.11 and 3.12 give illustrations for larger values of N (respectively with $N = 5$ and $N = 8$).

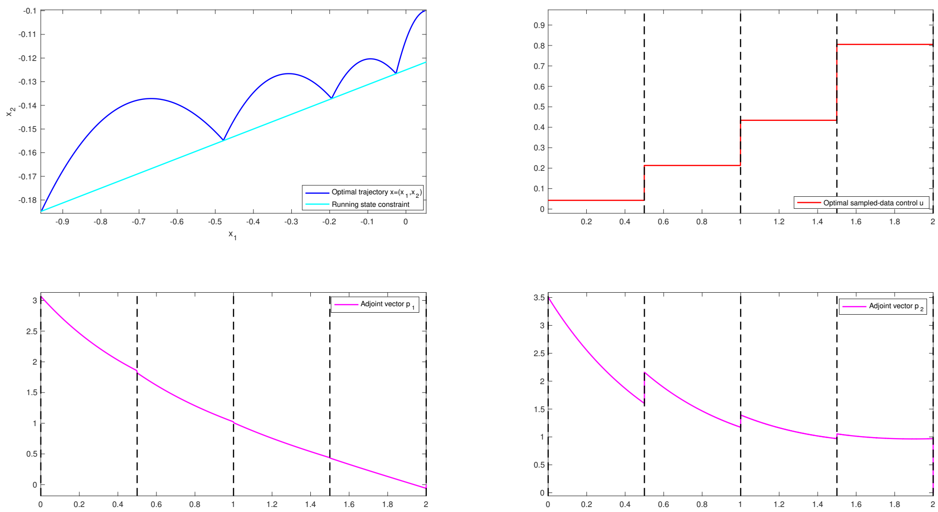

Figure 3.10: Example 3 with $N = 4$.

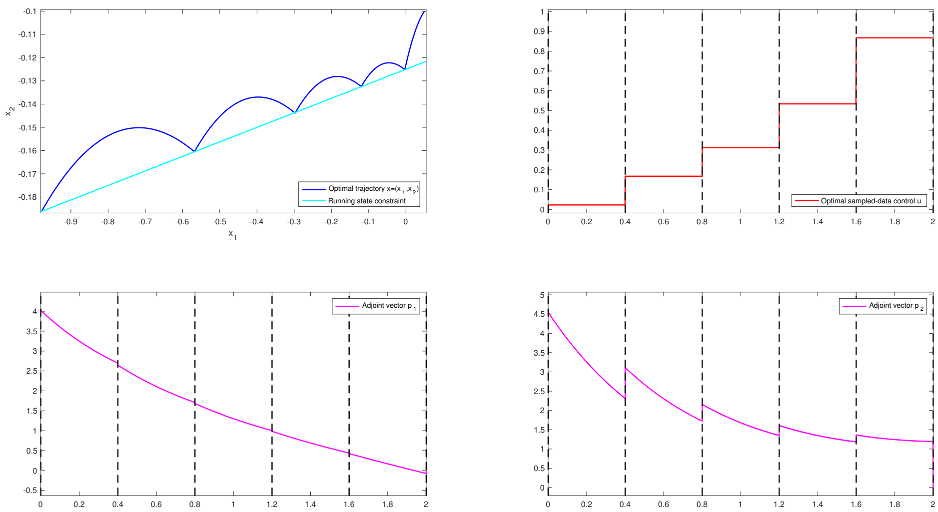

Figure 3.11: Example 3 with $N = 5$.

3.5 Proof of Theorem 3.2.1

This section is dedicated to the detailed proof of Theorem 3.2.1. In the whole section we will assume that $L = 0$ in Problem ($\mathrm{OSCP}_{\mathrm{sc}}$) (the case $L \neq 0$ can be treated similarly as in Section 2.4.4 in Chapter 2). Our method of proof is similar to the one followed in Section 2.4 of Chapter 2 and we will use results from the sensitivity analysis given in Section 2.4.1. Precisely, the Ekeland variational principle is applied in Section 3.5.1 on an appropriate penalized functional in order to derive a crucial inequality (see Inequality (3.8)) and we conclude the proof of

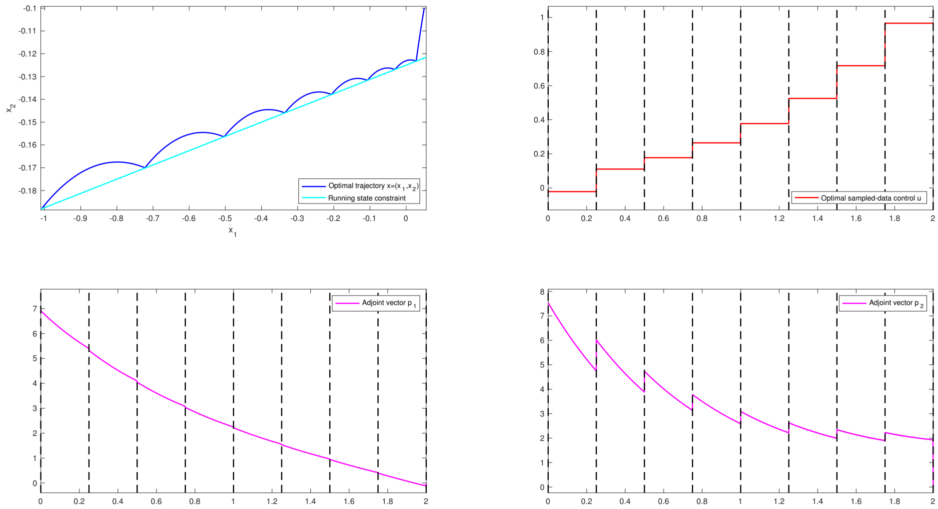

Figure 3.12: Example 3 with $N = 8$.

Theorem 3.2.1 in Section 3.5.2 by introducing an appropriate adjoint vector p.

We first remark that the running inequality state constraints in Problem ($\mathrm{OSCP_{sc}}$) can be written as $\mathfrak{h}(x) \in \mathcal{S}$ where:

- $\mathfrak{h} : \mathrm{C}([0,T],\mathbb{R}^n) \to \mathrm{C}([0,T],\mathbb{R}^q)$ is defined as $\mathfrak{h}(x) := h(x,\cdot)$ for all $x \in \mathrm{C}([0,T],\mathbb{R}^n)$. Note that $\mathfrak{h}$ is of class C^1 with $D\mathfrak{h}(x)(x') = \nabla_1 h(x,\cdot) \times x'$ for all x, $x' \in \mathrm{C}([0,T],\mathbb{R}^n)$;
- $\mathcal{S} := \mathrm{C}([0,T],(\mathbb{R}_-)^q)$ where $\mathbb{R}_- := (-\infty, 0]$. We emphasize that $\mathcal{S} \subset \mathrm{C}([0,T],\mathbb{R}^q)$ is a nonempty closed convex cone of $\mathrm{C}([0,T],\mathbb{R}^q)$ with a nonempty interior.

Recall that $(\mathrm{C}([0,T],\mathbb{R}^q), \|\cdot\|_\infty)$ is a separable Banach space. Applying Proposition 1.3.1 in Chapter 1, we endow $\mathrm{C}([0,T],\mathbb{R}^q)$ with an equivalent norm $\|\cdot\|_{\mathrm{C}_q}$ such that the associated dual norm $\|\cdot\|_{\mathrm{C}_q^*}$ is strictly convex. We denote by $\mathrm{d}_\mathcal{S} : \mathrm{C}([0,T],\mathbb{R}^q) \to \mathbb{R}$ the 1-Lipschitz continuous distance function to $\mathcal{S}$ (see Section 1.3 for details). Then, from Proposition 1.3.2, we know that $\mathrm{d}_\mathcal{S}$ and $\mathrm{d}_\mathcal{S}^2$ are strictly Hadamard-differentiable on $\mathrm{C}([0,T],\mathbb{R}^q) \setminus \mathcal{S}$ with $D\mathrm{d}_\mathcal{S}^2(x) = 2\mathrm{d}_\mathcal{S}(x) D\mathrm{d}_\mathcal{S}(x)$ and $\|D\mathrm{d}_\mathcal{S}(x)\|_{\mathrm{C}_q^*} = 1$ for all $x \in \mathrm{C}([0,T],\mathbb{R}^q) \setminus \mathcal{S}$, and that $\mathrm{d}_\mathcal{S}^2$ is Fréchet-differentiable on $\mathcal{S}$ with $D\mathrm{d}_\mathcal{S}^2(x) = 0_{\mathrm{C}([0,T],\mathbb{R}^q)^*}$ for all $x \in \mathcal{S}$.

3.5.1 Application of the Ekeland variational principle

Let $(x,u) \in \mathrm{AC}([0,T],\mathbb{R}^n) \times \mathrm{PC}^\mathbb{T}([0,T],\mathbb{R}^m)$ be a solution to Problem ($\mathrm{OSCP_{sc}}$). Recall that in the sensitivity analysis given in Section 2.4.1 in Chapter 2, the Cauchy problem (CP) is considered over $\mathbb{R}_+$ and a global solution is defined over all of $\mathbb{R}_+$. In contrast, the state equation given in Section 3.2.1 is defined over the compact interval $[0,T]$. It can be shown that the results given in Section 2.4.1 hold for the state equation given in Section 3.2.1 by the same techniques. We also remark that, in contrast to Chapter 2, the optimal sampled-data control problem with running inequality state constraints considered in Section 3.2.1 has a fixed initial condition $x(0) = x_0$. For this reason in the sequel, we simply denote by $x(\cdot,u') := x(\cdot,u',x_0)$ for all $u' \in \mathrm{PC}^\mathbb{T}([0,T],\mathbb{R}^m)$. In what follows we consider the positive real number $\varepsilon > 0$ given by Proposition 2.4.2 and we introduce the set

$$\mathcal{V}_\varepsilon := \{u' \in \overline{\mathrm{B}}_{\mathrm{L}^\infty}(u,\varepsilon) \mid u' \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m) \text{ and } u'(t) \in \mathrm{U} \text{ for all } t \in [0,T]\}.$$

From the closedness assumption on U, one can easily prove that $(\mathcal{V}_\varepsilon, \|\cdot\|_{\mathrm{L}^\infty})$ is a complete metric set. Let us choose a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $0 < \sqrt{\varepsilon_k} < \varepsilon$ for all $k \in \mathbb{N}$ and satisfying $\lim_{k\to\infty} \varepsilon_k = 0$. We introduce the penalized functional

$$\begin{array}{rccl} \mathcal{J}_k : & \mathcal{V}_\varepsilon & \longrightarrow & \mathbb{R}_+ \\ & u' & \longmapsto & \mathcal{J}_k(u') := \sqrt{\left(\left(g(x(T,u')) - g(x(T)) + \varepsilon_k\right)^+\right)^2 + \mathrm{d}^2_{\mathcal{S}}\Big(\mathfrak{h}(x(\cdot,u'))\Big)}, \end{array}$$

for all $k \in \mathbb{N}$. From Proposition 2.4.2 in Chapter 2, note that $\mathcal{J}_k$ is correctly defined for all $k \in \mathbb{N}$. Also, from Proposition 2.4.2 and from the continuities of g, $\mathfrak{h}$ and $\mathrm{d}^2_{\mathcal{S}}$ (see Proposition 1.3.2 in Chapter 1), it follows that $\mathcal{J}_k$ is continuous as well for all $k \in \mathbb{N}$. Note that $\mathcal{J}_k$ is nonnegative and, since the constraint $\mathfrak{h}(x) \in \mathcal{S}$ is satisfied, it holds that $\mathcal{J}_k(u) = \varepsilon_k$ for all $k \in \mathbb{N}$. Therefore, from the Ekeland variational principle (see Proposition 2.4.8 in Chapter 2), we conclude that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{V}_\varepsilon$ such that

$$\|u_k - u\|_{\mathrm{L}^\infty} \leq \sqrt{\varepsilon_k}, \tag{3.4}$$

and

$$-\sqrt{\varepsilon_k}\, \|u' - u_k\|_{\mathrm{L}^\infty} \leq \mathcal{J}_k(u') - \mathcal{J}_k(u_k), \tag{3.5}$$

for all $u' \in \mathcal{V}_\varepsilon$ and all $k \in \mathbb{N}$. In particular, from Inequality (3.4), note that the sequence $(u_k)_{k\in\mathbb{N}}$ converges to u in $\mathrm{L}^\infty([0,T],\mathbb{R}^m)$. From optimality of the couple (x,u), note that $\mathcal{J}_k(u') > 0$ for all $u' \in \mathcal{V}_\varepsilon$ and all $k \in \mathbb{N}$. We thus define correctly the couple $(\lambda_k, \psi^*_k) \in \mathbb{R} \times \mathrm{C}^*([0,T],\mathbb{R}^q)$ as

$$\lambda_k := \frac{1}{\mathcal{J}_k(u_k)} \Big(g(x(T,u_k)) - g(x(T)) + \varepsilon_k\Big)^+ \geq 0,$$

and

$$\psi^*_k := \begin{cases} \dfrac{1}{\mathcal{J}_k(u_k)} \mathrm{d}_{\mathcal{S}}\Big(\mathfrak{h}(x(\cdot,u_k))\Big) D\mathrm{d}_{\mathcal{S}}\Big(\mathfrak{h}(x(\cdot,u_k))\Big) & \text{if } \ \mathfrak{h}(x(\cdot,u_k)) \notin \mathcal{S}, \\ 0_{\mathrm{C}^*_q} & \text{if } \ \mathfrak{h}(x(\cdot,u_k)) \in \mathcal{S}, \end{cases}$$

for all $k \in \mathbb{N}$. From Proposition 1.3.2 it holds that $|\lambda_k|^2 + \|\psi^*_k\|^2_{\mathrm{C}^*_q} = 1$ for all $k \in \mathbb{N}$. As a consequence, we can extract subsequences (which we do not relabel) such that $(\lambda_k)_{k\in\mathbb{N}}$ converges to some $\lambda \geq 0$ and $(\psi^*_k)_{k\in\mathbb{N}}$ weakly* converges to some $\psi^* \in \mathrm{C}^*([0,T],\mathbb{R}^q)$. In particular it holds that $|\lambda|^2 + \|\psi^*\|^2_{\mathrm{C}^*_q} \leq 1$. At this step note that we cannot ensure that the couple (λ,ψ^*) is not trivial. The nontriviality is guaranteed by the next proposition.

Proposition 3.5.1. *The couple $(\lambda,\psi^*) \in \mathbb{R} \times \mathrm{C}^*([0,T],\mathbb{R}^q)$ is nontrivial and it holds that*

$$\langle \psi^*, \psi - \mathfrak{h}(x) \rangle_{\mathrm{C}^*_q \times \mathrm{C}_q} \leq 0, \tag{3.6}$$

for all $\psi \in \mathcal{S}$.

Proof. Let $k \in \mathbb{N}$ be fixed. From Proposition 1.3.2, if $\mathfrak{h}(x(\cdot,u_k)) \notin \mathcal{S}$, then $D\mathrm{d}_{\mathcal{S}}(\mathfrak{h}(x(\cdot,u_k))) \in \partial \mathrm{d}_{\mathcal{S}}(\mathfrak{h}(x(\cdot,u_k)))$. Hence, if $\mathfrak{h}(x(\cdot,u_k)) \notin \mathcal{S}$, it holds that

$$\Big\langle D\mathrm{d}_{\mathcal{S}}\Big(\mathfrak{h}(x(\cdot,u_k))\Big), \psi - \mathfrak{h}(x(\cdot,u_k))\Big\rangle_{\mathrm{C}_q^* \times \mathrm{C}_q} \leq \mathrm{d}_{\mathcal{S}}(\psi) - \mathrm{d}_{\mathcal{S}}\Big(\mathfrak{h}(x(\cdot,u_k)\Big) \leq 0,$$

for all $\psi \in \mathcal{S}$. As a consequence, in both cases $\mathfrak{h}(x(\cdot,u_k)) \in \mathcal{S}$ and $\mathfrak{h}(x(\cdot,u_k)) \notin \mathcal{S}$, it holds that

$$\langle \psi_k^*, \psi - \mathfrak{h}(x(\cdot,u_k))\rangle_{\mathrm{C}_q^* \times \mathrm{C}_q} \leq 0, \tag{3.7}$$

for all $\psi \in \mathcal{S}$. Using Proposition 2.4.2 and taking the limit as k tends to $+\infty$, we get Inequality (3.6). Now let us prove that the couple $(\lambda, \psi^*) \in \mathbb{R} \times \mathrm{C}^*([0,T],\mathbb{R}^q)$ is nontrivial. Since $\mathcal{S}$ has a nonempty interior, there exists $\xi \in \mathcal{S}$ and $\delta > 0$ such that $\xi + \delta\psi \in \mathcal{S}$ for all $\psi \in \overline{\mathrm{B}}_{\mathrm{C}_q}(0_{\mathrm{C}_q},1)$. Hence we obtain from Inequality (3.7) that

$$\delta \langle \psi_k^*, \psi \rangle_{\mathrm{C}_q^* \times \mathrm{C}_q} \leq \langle \psi_k^*, \mathfrak{h}(x(\cdot,u_k)) - \xi \rangle_{\mathrm{C}_q^* \times \mathrm{C}_q},$$

for all $\psi \in \overline{\mathrm{B}}_{\mathrm{C}_q}(0_{\mathrm{C}_q},1)$ and all $k \in \mathbb{N}$. We deduce that

$$\delta \|\psi_k^*\|_{\mathrm{C}_q^*} = \delta\sqrt{1-|\lambda_k|^2} \leq \langle \psi_k^*, \mathfrak{h}(x(\cdot,u_k)) - \xi \rangle_{\mathrm{C}_q^* \times \mathrm{C}_q},$$

for all $k \in \mathbb{N}$. Using Proposition 2.4.2 and taking the limit as k tends to $+\infty$, we obtain that

$$\delta\sqrt{1-|\lambda|^2} \leq \langle \psi^*, \mathfrak{h}(x) - \xi \rangle_{\mathrm{C}_q^* \times \mathrm{C}_q}.$$

Since $\delta > 0$, the last inequality implies that the couple (λ, ψ^*) is nontrivial which completes the proof. ☐

Finally, in the next result, we use Inequality (3.5) with convex L^∞-perturbations (see Section 2.4.1.2 in Chapter 2) of the control u_k in order to establish a crucial inequality.

Proposition 3.5.2. *The inequality*

$$\lambda\Big\langle \nabla g(x(T)), w_v(T)\Big\rangle_{\mathbb{R}^n} + \Big\langle \psi^*, \nabla_1 h(x,\cdot) \times w_v(\cdot)\Big\rangle_{\mathrm{C}_q^* \times \mathrm{C}_q} \geq 0, \tag{3.8}$$

holds for all $v \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U, *where $w_v(\cdot)$ is the variation vector defined in Proposition 2.4.3.*

Proof. Let $v \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U. We fix $k \in \mathbb{N}$. Since U is convex, it is clear that the convex L^∞-pertubation of the control u_k associated with v, defined by $u_{k,v}(t,\alpha) := u_k(t) + \alpha(v(t) - u_k(t))$ for all $t \in [0,T]$ and all $0 \leq \alpha \leq 1$, belongs to $\mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ and takes values in U. Furthermore it holds that $\|u_{k,v}(\cdot,\alpha) - u\|_{\mathrm{L}^\infty} \leq \alpha\|v-u_k\|_{\mathrm{L}^\infty} + \|u_k - u\|_{\mathrm{L}^\infty} \leq \alpha\|v-u_k\|_{\mathrm{L}^\infty} + \sqrt{\varepsilon_k}$. Since $\sqrt{\varepsilon_k} < \varepsilon$, we deduce that $u_{k,v}(\cdot,\alpha) \in \mathcal{V}_\varepsilon$ for small enough $\alpha > 0$. From Inequality (3.5) we get that

$$-\sqrt{\varepsilon_k}\,\|v-u_k\|_{\mathrm{L}^\infty} \leq \frac{1}{\mathcal{J}_k(u_{k,v}(\cdot,\alpha)) + \mathcal{J}_k(u_k)} \times \frac{\mathcal{J}_k(u_{k,v}(\cdot,\alpha))^2 - \mathcal{J}_k(u_k)^2}{\alpha},$$

for small enough $\alpha > 0$. From Proposition 2.4.3 and from strict Hadamard-differentiability of $\mathrm{d}^2_{\mathcal{S}}$ over $\mathrm{C}([0,T],\mathbb{R}^q)\backslash\mathcal{S}$ and Fréchet-differentiability of $\mathrm{d}^2_{\mathcal{S}}$ over $\mathcal{S}$ (see Proposition 1.3.2), taking the limit as α tends to 0, we get that

$$-\sqrt{\varepsilon_k}\,\|v-u_k\|_{\mathrm{L}^\infty} \leq \frac{1}{2\mathcal{J}_k(u_k)}\Bigg[2\Big(g(x(T,u_k))-g(x(T))+\varepsilon_k\Big)^+\Big\langle \nabla g(x(T,u_k)), w_{k,v}(T)\Big\rangle_{\mathbb{R}^n} + \Big\langle 2\mathrm{d}_{\mathcal{S}}(\mathfrak{h}(x(\cdot,u_k)))D\mathrm{d}_{\mathcal{S}}(\mathfrak{h}(x(\cdot,u_k))), \nabla_1 h(x(\cdot,u_k),\cdot)\times w_{k,v}(\cdot)\Big\rangle_{\mathrm{C}_q^*\times\mathrm{C}_q}\Bigg],$$

with the convention that the second term on the right-hand side is zero if $\mathfrak{h}(x(\cdot,u_k))\in\mathcal{S}$. Using the definition of λ_k and ψ_k^*, we deduce that

$$-\sqrt{\varepsilon_k}\,\|v-u_k\|_{\mathrm{L}^\infty} \leq \lambda_k\Big\langle \nabla g(x(T,u_k)), w_{k,v}(T)\Big\rangle_{\mathbb{R}^n} + \Big\langle \psi_k^*, \nabla_1 h(x(\cdot,u_k),\cdot)\times w_{k,v}(\cdot)\Big\rangle_{\mathrm{C}_q^*\times\mathrm{C}_q}.$$

We take the limit of this inequality as k tends to $+\infty$. From the smoothness of g and h and from Proposition 2.4.2 and Lemma 2.4.3, Inequality (3.8) is proved. □

3.5.2 Introduction of the adjoint vector

We can now conclude the proof of Theorem 3.2.1 (in the case $L=0$) by introducing the adjoint vector p. We refer to Section 1.2 in Chapter 1 for notations and background concerning Stieltjes integrations and linear Cauchy-Stieltjes problems.

Introduction of the nontrivial couple (p^0,η) and complementary slackness condition. We introduce $p^0 := -\lambda \leq 0$ and we write $\psi^* = (\psi_j^*)_{j=1,\dots,q}$ where $\psi_j^* \in \mathrm{C}([0,T],\mathbb{R})^*$ for every $j=1,\dots,q$. From the Riesz representation theorem (see Proposition 1.2.2), there exists a unique $\eta_j \in \mathrm{NBV}([0,T],\mathbb{R})$ such that

$$\langle \psi_j^*, \psi\rangle_{\mathrm{C}_1^*\times\mathrm{C}_1} = \int_0^T \psi(t)\,d\eta_j(t),$$

for all $\psi\in\mathrm{C}_1$ and all $j=1,\dots,q$. Furthermore $\psi_j^* = 0_{\mathrm{C}_1^*}$ if and only if $\eta_j = 0_{\mathrm{NBV}([0,T],\mathbb{R})}$. Thus it follows from Proposition 3.5.1 that the couple (p^0,η) is not trivial, where $\eta := (\eta_j)_{j=1,\dots,q} \in \mathrm{NBV}([0,T],\mathbb{R}^q)$. Moreover, from Inequality (3.6) (and the fact that $\mathcal{S}$ is a cone containing $\mathfrak{h}(x)$), one can easily deduce that $\langle \psi_j^*, \mathfrak{h}_j(x)\rangle_{\mathrm{C}_1^*\times\mathrm{C}_1} = 0$, that is,

$$\int_0^T h_j(x(t),t)\,d\eta_j(t) = 0,$$

for all $j=1,\dots,q$. Finally one can similarly deduce from Inequality (3.6) that $\langle \psi_j^*, \psi\rangle_{\mathrm{C}_1^*\times\mathrm{C}_1} \geq 0$ for all $\psi\in\mathrm{C}([0,T],\mathbb{R})$ and all $j=1,\dots,q$. From Proposition 1.2.2, it follows that η_j is monotonically increasing on $[0,T]$ for all $j=1,\dots,q$.

Adjoint equation. We define the adjoint vector $p\in\mathrm{BV}([0,T],\mathbb{R}^n)$ as the unique solution to the backward linear Cauchy-Stieltjes problem given by

$$\begin{cases} -dp = \Big(\nabla_1 f(x,u,\cdot)^\top \times p\Big)\,dt - \sum_{j=1}^q \nabla_1 h_j(x,\cdot)\,d\eta_j \quad \text{over } [0,T], \\ p(T) = p^0\nabla g(x(T)). \end{cases}$$

From the Duhamel formula for backward linear Cauchy-Stieltjes problems and using notations introduced in Section 1.2, it holds that

$$p(t) = \Phi(T,t)^\top \times \Big(p^0 \nabla g(x(T))\Big) - \int_t^T \Phi(s,t)^\top \times \nabla_1 h(x(s),s)^\top \times d\eta(s),$$

for all $t \in [0,T]$, where $\Phi(\cdot,\cdot) : [0,T]^2 \to \mathbb{R}^{n\times n}$ stands for the state-transition matrix associated with $\nabla_1 f(x,u,\cdot) \in \mathrm{L}^\infty([0,T],\mathbb{R}^{n\times n})$ (see [Sontag 1998, Appendix C.4] for more details on state-transition matrices).

Nonpositive averaged Hamiltonian gradient condition. From Inequality (3.8) and using notations introduced in Section 1.2, it holds that

$$\lambda \Big\langle \nabla g(x(T)), w_v(T,u) \Big\rangle_{\mathbb{R}^n} + \int_0^T \Big\langle \nabla_1 h(x(t),t) \times w_v(t,u), d\eta(t) \Big\rangle \geq 0,$$

for all $v \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U. From the definition of the variation vector $w_v(\cdot,u)$ and the classical Duhamel formula for standard forward linear Cauchy problems, it holds that

$$w_v(t,u) = \int_0^t \Phi(t,s) \times \nabla_2 f(x(s),u(s),s) \times (v(s)-u(s))\, ds,$$

for all $t \in [0,T]$. Substituting this expression into the previous inequality and using the last Fubini formula given in Section 1.2, it follows that

$$\begin{aligned} &\int_0^T \Big\langle \Phi(T,s)^\top \times \Big(p^0 \nabla g(x(T))\Big), \nabla_2 f(x(s),u(s),s) \times (v(s)-u(s)) \Big\rangle_{\mathbb{R}^n} ds \\ &\quad - \int_0^T \Big\langle \nabla_2 f(x(s),u(s),s) \times (v(s)-u(s)), \int_s^T \Phi(t,s)^\top \times \nabla_1 h(x(t),t)^\top \times d\eta(t) \Big\rangle_{\mathbb{R}^n} ds \leq 0, \end{aligned}$$

for all $v \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U. Finally, grouping like terms, we exactly obtain

$$\int_0^T \Big\langle p(t), \nabla_2 f(x(t),u(t),t) \times (v(t)-u(t)) \Big\rangle_{\mathbb{R}^n} dt \leq 0,$$

for all $v \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U. For all $i = 0,\dots,N-1$ and all $\omega \in \mathrm{U}$, let us consider $v_{i,\omega} \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m)$ with values in U as

$$v_{i,\omega}(t) := \begin{cases} \omega & \text{if} \quad t \in [t_i,t_{i+1}), \\ u(s) & \text{if} \quad t \notin [t_i,t_{i+1}), \end{cases}$$

for all $t \in [0,T]$. Substituting v by $v_{i,\omega}$ in the above inequality and from the definition of the Hamiltonian H, we exactly get that

$$\Big\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t),u_i,p(t),p^0,t)\, dt, \omega - u_i \Big\rangle_{\mathbb{R}^m} \leq 0,$$

for all $\omega \in \mathrm{U}$ and all $i = 0,\dots,N-1$. The proof of Theorem 3.2.1 is complete (in the case $L=0$). The proof in the case $L \neq 0$ follows from a similar argument as that given in Section 2.4.4 in Chapter 2.

Chapter 4

A universal separating vector theorem with applications to optimal control problems with nonsmooth Mayer cost functions

This chapter is based on the work "A universal separating vector theorem with application to nonsmooth optimal control problems" by S. Adly, L. Bourdin and G. Dhar which has recently been submitted for publication (see [Adly et al. 2020]). An additional application of the universal separating vector theorem is given in this chapter to obtain a Pontryagin maximum principle for optimal sampled-data control problems with nonsmooth Mayer cost functions which was not presented in [Adly et al. 2020].

4.1 Introduction

Content of the present chapter. Our main objective in the present chapter is to establish a Pontryagin maximum principle (in short, PMP) for optimal sampled-data control problems with nonsmooth Mayer cost functions. To this aim, we have studied the vast literature on nonsmooth optimal permanent control theory in which several methods have been explored in order to establish PMPs for optimal permanent control problems with nonsmooth data. We can cite for example the method of quadratic inf-convolution in [Clarke 2008, Section 2.1 page 4] or the application of a nonsmooth Lagrange multiplier rule in [Vinter 2010, Theorem 5.6.2]. Most of the proofs found in the literature involve regularization methods. On the contrary, in the work [Adly *et al.* 2020], we were interested in developing a proof which directly follows from the tools of nonsmooth analysis (as presented in Section 1.4 of Chapter 1). Our investigation led us to consider the existence of a universal selection in the subdiffeential of the nonsmooth Mayer cost function. In the work [Adly *et al.* 2020], we determined the existence of such a universal selection by establishing a more general result asserting the existence a universal separating vector for a given compact convex set not containing the origin. From the application of this result, which is called *universal separating vector theorem* (see Theorem 4.2.1 for details), we were able to derive a PMP for optimal permanent control problems with nonsmooth Mayer cost functions. Note that our novel approach in [Adly *et al.* 2020] was also based on the combination of implicit spike variations and packages of needle-like perturbations of the optimal control (see Proposition 4.3.1 and Lemma 4.3.1 respectively). Finally, in this chapter, we apply again the universal separating vector theorem to obtain a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions which was the initial motivation of the present work and which was not presented in the article [Adly *et al.* 2020].

Brief presentation of the problematic of universal separating vector. Let $n \in \mathbb{N}^*$ be a positive integer fixed throughout the chapter. The aim of this paragraph is to present the problematic of universal separating vector considered in the work [Adly *et al.* 2020]. For this purpose we first introduced the following basic notion of *separating vector*.

Definition 4.1.1 (Separating vector)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty convex set not containing the origin $0_{\mathbb{R}^n}$. A vector $v \in \mathbb{R}^n$ is said to be a separating vector of C if $\langle c, v\rangle_{\mathbb{R}^n} < 0$ for all $c \in C$.*

Let $C \subset \mathbb{R}^n$ be a nonempty convex set not containing the origin $0_{\mathbb{R}^n}$. Using the usual terminology found in the convex analysis literature (see, e.g., [Brezis 2011, Section 1.2]), every separating vector of C defines a hyperplane which separates C with the singleton $\{0_{\mathbb{R}^n}\}$. In the work [Adly *et al.* 2020] we adopted the notion of *separating vector* (instead of *separating hyperplane*) because our problematic led us to consider convex combinations of separating vectors, while the concept of convex combinations of separating hyperplanes would have been confusing. Finally, since the inequality in Definition 4.1.1 is strict and its left-hand term can be arbitrarily close to zero, note that the notion of *separating vector* of C is not exactly equivalent to the standard notion of *separating hyperplane* of C with the singleton $\{0_{\mathbb{R}^n}\}$ (neither in the large sense, nor in the strict one).

Remark 4.1.1. *Let $C \subset \mathbb{R}^n$ be a nonempty convex set not containing the origin $0_{\mathbb{R}^n}$. If C is not closed, then C may not admit any separating vector (a simple two-dimensional unclosed counterexample is given by $C := ([-1, 1] \times (0, 1]) \cup \{(-1, 0)\}$, see Figure 4.1). Otherwise, if C is closed, the next proposition is a direct consequence of the classical Hahn-Banach separation theorem (see, e.g., [Brezis 2011, Theorem 1.7]).*

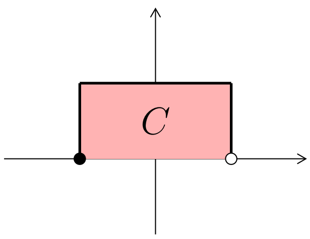

Figure 4.1: Two-dimensional unclosed counterexample from Remark 4.1.1.

Proposition 4.1.1. *If $C \subset \mathbb{R}^n$ is a nonempty closed convex set not containing the origin $0_{\mathbb{R}^n}$, then C admits at least one separating vector.*

Before presenting the problematic of universal separating vector, let us start with a preliminary question. Consider $\{C_i\}_{i\in\mathcal{I}}$ being a family of nonempty closed convex sets of $\mathbb{R}^n$ not containing the origin $0_{\mathbb{R}^n}$ such that their union $C := \cup_{i\in\mathcal{I}} C_i$ is also closed and convex. From Proposition 4.1.1, each C_i admits a separating vector v_i. Furthermore, again from Proposition 4.1.1, we know that C admits a separating vector. The preliminary question is the following: does there exist a separating vector of C which is a convex combination of the vectors v_i? This issue is illustrated in the two-dimensional Figure 4.2 in the case where $\mathcal{I} = \{1, 2\}$.

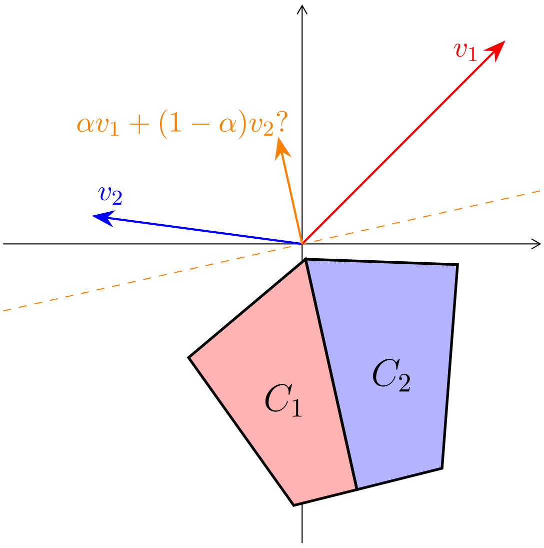

Figure 4.2: In this two-dimensional illustration, the vectors v_1 and v_2 are respectively separating vectors of the nonempty closed convex sets C_1 and C_2 not containing the origin $0_{\mathbb{R}^2}$. Does there exist a separating vector of the closed convex set $C := C_1 \cup C_2$ which is a convex combination of v_1 and v_2 (that is, which can be written as $\alpha v_1 + (1-\alpha)v_2$ for some $\alpha \in [0,1]$)?

The (more general) problematic of universal separating vector is expounded as follows. Let $C \subset \mathbb{R}^n$ be a nonempty convex set and $\mathcal{V} \subset \mathbb{R}^n$ be a nonempty set such that, for all $c \in C$, there exists a vector $v_c \in \mathcal{V}$ which is a separating vector of the singleton $\{c\}$. Our problematic then asks whether there exists a *universal* vector v in the convex envelope of $\mathcal{V}$ which is a separating vector of the whole set C.[1] The study of this issue was given in the work [Adly *et al.* 2020] and is presented in the next Section 4.2. Despite the simplicity of the statement, and that one might expect an easy and positive answer, it turned out that closing this mathematical challenge was not a trivial task. In particular, we showed in the main result of the work [Adly *et al.* 2020] (see Theorem 4.2.1) and with several counterexamples that the existence of such a universal vector v in the convex envelope of $\mathcal{V}$ requires the compactness of C in general.

Organization of the chapter. This chapter is organized as follows. Section 4.2 is dedicated to the universal separating vector theorem obtained in [Adly *et al.* 2020] along with its proof. In Section 4.3, we show that it provides an alternative proof of a PMP for optimal permanent control problems with nonsmooth Mayer cost functions which makes direct use of the tools of nonsmooth analysis presented in Chapter 1. Section 4.4 is devoted to a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions which was not presented in [Adly *et al.* 2020] whose proof again applies the universal separating vector theorem. Finally the proofs of two technical propositions are postponed to Section 4.5.

[1] Note that the previous preliminary question is a particular case of this general problematic, by considering in particular $\mathcal{V} := \{v_i\}_{i \in \mathcal{I}}$.

4.2 A universal separating vector theorem

This section is devoted to the universal separating vector theorem obtained in [Adly *et al.* 2020] and recalled in Section 4.2.1 (see Theorem 4.2.1). This theorem is based on a compactness assumption commented afterwards in a list of remarks with examples and counterexamples. Then Section 4.2.2 is devoted to the proof of Theorem 4.2.1.

4.2.1 Main result and comments

For the needs of this section we recall that the *convex envelope*, denoted by $\mathrm{Conv}(\mathcal{V})$, of a nonempty set $\mathcal{V} \subset \mathbb{R}^n$ is defined as the smallest convex set of $\mathbb{R}^n$ containing $\mathcal{V}$, and that we have

$$\mathrm{Conv}(\mathcal{V}) = \left\{ \sum_{i=1}^{N} \alpha_i v_i \in \mathbb{R}^n \mid N \in \mathbb{N}^*,\ \alpha = (\alpha_1, \dots, \alpha_N) \in \Delta_N \text{ and } \forall i = 1, \dots, N,\ v_i \in \mathcal{V} \right\},$$

where Δ_N stands for the *simplex* defined by

$$\Delta_N := \left\{ \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_+ \mid \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$

We refer for example to [Rockafellar 1970, page 12] for more details. We are now in a position to state the universal separating vector theorem obtained in [Adly *et al.* 2020]. Its proof is postponed to the next Section 4.2.2.

Theorem 4.2.1 (A universal separating vector theorem). *Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set and $\mathcal{V} \subset \mathbb{R}^n$ be a nonempty set. Suppose that*

$$\forall c \in C, \quad \exists v_c \in \mathrm{Conv} V, \quad \langle c, v_c \rangle_{\mathbb{R}^n} < 0,$$

then

$$\exists v \in \mathrm{Conv}(\mathcal{V}), \quad \forall c \in C, \quad \langle c, v \rangle_{\mathbb{R}^n} < 0.$$

The conclusion of Theorem 4.2.1 can be seen as a transposition of the quantifiers in the hypothesis. This justifies the terminology of *universal* separating vector used throughout this chapter. Note that Theorem 4.2.1 is based on a compactness assumption that cannot be removed in general, as discussed in the following list of remarks with several counterexamples.

Remark 4.2.1. *This first remark discusses two trivial cases. Obviously, in the one-dimensional setting $n = 1$, the compactness hypothesis on C in Theorem 4.2.1 is superfluous (since, by the other assumptions, C is included either in $\mathbb{R}^*_+$ or in $\mathbb{R}^*_-$, and $\mathcal{V}$ contains at least one element either in $\mathbb{R}^*_-$ or $\mathbb{R}^*_+$ respectively). Also trivially, if $\mathcal{V}$ contains exactly one nonzero vector, then the compactness hypothesis on C in Theorem 4.2.1 is superfluous.*

Remark 4.2.2. *In the higher-dimensional setting $n \geq 2$ and when $\mathcal{V}$ contains at least two different nonzero vectors, the compactness hypothesis on C in Theorem 4.2.1 cannot be removed in general:*

- *a two-dimensional bounded unclosed counterexample is given by $C := ([-1, 1] \times (0, 1]) \cup \{(-1, 0)\}$ with $\mathcal{V}$ containing $v_1 := (1, 0)$ and $v_2 := (0, -1)$. Indeed, in that situation, recall that C has no separating vector (see Remark 4.1.1 and Figure 4.1).*

- *a two-dimensional unbounded closed counterexample is given by* $C := \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ xy \geq 1\}$ *with* $\mathcal{V} := (0,1] \times \{-1\}$. *In that context, note that* v *is a separating vector of* C *if and only if* $v \in \mathbb{R}^2_- \backslash \{0_{\mathbb{R}^2}\}$. *However* $\mathrm{Conv}(\mathcal{V}) \cap (\mathbb{R}^2_- \backslash \{0_{\mathbb{R}^2}\}) = \mathcal{V} \cap (\mathbb{R}^2_- \backslash \{0_{\mathbb{R}^2}\}) = \emptyset$. *We refer to Figure 4.3 for an illustration.*

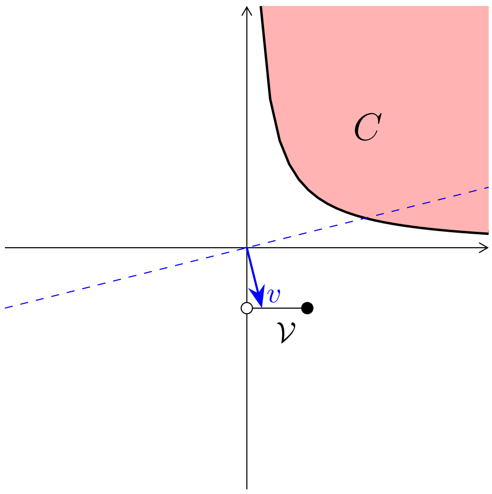

Figure 4.3: Two-dimensional unbounded closed counterexample from Remark 4.2.2.

Remark 4.2.3. *Note that the convexity (or not) of the set* $\mathcal{V}$ *does not play any role in Theorem 4.2.1 since its convex envelope is considered in the conclusion of the theorem. In particular this emphasizes that* C *and* $\mathcal{V}$ *do not play any symmetrical or similar roles. Nonetheless, in the higher-dimensional setting* $n \geq 2$, *one may wonder if the compactness hypothesis on* C *in Theorem 4.2.1 could be weakened if additional assumptions were made on the set* $\mathcal{V}$. *This remark is dedicated to a discussion on that issue. Firstly, from the first counterexample in Remark 4.2.2, the closedness of* C *appears clearly as a necessary assumption. As a consequence, in the rest of this remark, we will focus on the possibility of relaxing (only) the boundedness of* C *in Theorem 4.2.1. In the unbounded closed counterexample provided in Remark 4.2.2, the associated set* $\mathcal{V}$ *is bounded and unclosed. Considering* $\mathcal{V}' := \{(x, -\frac{1}{x}) \mid x \in (0,1]\}$ *(resp.* $\mathcal{V}'' := \{(x, -\sqrt{x}) \mid x \in [0,1]\}$*), we get the same counterexample but with the associated set* $\mathcal{V}'$ *unbounded and closed (resp.* $\mathcal{V}''$ *compact). This underlines that the compactness (or not) of* $\mathcal{V}$ *does not allow to weaken the boundedness assumption on* C *in Theorem 4.2.1. Finally one might believe that the finiteness (with at least two different nonzero vectors) of* $\mathcal{V}$ *does, but the three-dimensional unbounded closed counterexample given by*

$$C := \{(x,y,z) \in \mathbb{R}^3 \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1,\ z \geq 1,\ yz - (x+1)^2 \geq 0\},$$

with $\mathcal{V} := \{v_1, v_2\}$, *where* $v_1 := (1,0,0)$ *and* $v_2 := (0,-1,0)$, *proves that the boundedness of* C *cannot be relaxed in Theorem 4.2.1 even if* $\mathcal{V}$ *contains exactly two different nonzero vectors.*

Remark 4.2.4. *In the same spirit as Remark 4.2.3, one would be eager to look for other assumptions made on* $\mathcal{V}$ *which would allow to weaken the boundedness hypothesis made on* C *in Theorem 4.2.1. Note that in the work [Adly et al. 2020] our aim was not to pursue our study in that direction. Indeed, we emphasize that, in our practical applications of Theorem 4.2.1 (such* *as in Sections 4.3 and 4.4 for nonsmooth optimal control problems), it turns out that the set $\mathcal{V}$ is very general and does not satisfy any interesting property a priori, while the set C is indeed compact (see the proofs of Theorems 4.3.1 and 4.4.1 for details).*

We conclude this section with the contrapositive of Theorem 4.2.1 which will play a central role in Sections 4.3 and 4.4 where we will derive PMPs for nonsmooth optimal permanent and sampled-data control problems.

Proposition 4.2.1 (Contrapositive of Theorem 4.2.1)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set and $\mathcal{V} \subset \mathbb{R}^n$ be a nonempty set. Suppose that*

$$\forall v \in \mathrm{Conv}(\mathcal{V}), \quad \exists c_v \in C, \quad \langle c_v, v \rangle_{\mathbb{R}^n} \geq 0,$$

then

$$\exists c \in C, \quad \forall v \in \mathcal{V}, \quad \langle c, v \rangle_{\mathbb{R}^n} \geq 0.$$

4.2.2 Proof of Theorem 4.2.1

This section is dedicated to the proof of Theorem 4.2.1. We proceed step by step as follows:

(i) After a technical lemma (Lemma 4.2.1), we first prove Theorem 4.2.1 in the special case where $\mathcal{V}$ contains at most two vectors and $n \in \{1, 2\}$ (see Proposition 4.2.2);

(ii) Then we prove Theorem 4.2.1 in the special case where $\mathcal{V}$ contains at most two vectors and $n \in \mathbb{N}^*$ (see Proposition 4.2.3);

(iii) Then we prove Theorem 4.2.1 in the special case where $\mathcal{V}$ contains a finite number $N \in \mathbb{N}^*$ of vectors and $n \in \mathbb{N}^*$ (see Proposition 4.2.4).

(iv) Finally Theorem 4.2.1 is proved in its entirety at the end of the section.

Let us start with the following technical lemma.

Lemma 4.2.1. *Let $C \subset \mathbb{R}^2$ be a two-dimensional nonempty compact convex set not containing the origin $0_{\mathbb{R}^2}$. Then there exist c_1, $c_2 \in C$ such that $C \subset \{\lambda_1 c_1 + \lambda_2 c_2 \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \backslash \{0_{\mathbb{R}^2}\}\}$.*

Proof. Applying Proposition 4.1.1, C admits a separating vector $v \in \mathbb{R}^2$. Note that $v \neq 0_{\mathbb{R}^2}$ and let us fix some $c_0 \in C$. Since C is compact and v is a separating vector of C, we are able to define the constants

$$\varepsilon := \min_{c \in C} \frac{\langle c_0, v \rangle_{\mathbb{R}^2}}{\langle c, v \rangle_{\mathbb{R}^2}} \quad \text{and} \quad M := \max_{c \in C} \frac{\langle c_0, v \rangle_{\mathbb{R}^2}}{\langle c, v \rangle_{\mathbb{R}^2}},$$

which satisfy $\varepsilon \in (0, 1]$ and $M \in [1, +\infty)$. We now define the set

$$S := [\varepsilon, M]C = \{\mu c \mid \mu \in [\varepsilon, M], \ c \in C\}.$$

Note that $C \subset S$ and that S is a compact set. Let us prove that S is convex. Let s', $s'' \in S$, and $\alpha \in [0, 1]$. Then it holds that $s' = \mu' c'$ and $s'' = \mu'' c''$ for some μ', $\mu'' \in [\varepsilon, M]$ and c',

$c'' \in C$. Note that $\alpha\mu' + (1-\alpha)\mu'' \in [\varepsilon, M]$ and, since C is convex and $\frac{\alpha\mu'}{\alpha\mu'+(1-\alpha)\mu''} \in [0,1]$, it holds that

$$(\alpha\mu' + (1-\alpha)\mu'')\left(\frac{\alpha\mu'}{\alpha\mu' + (1-\alpha)\mu''}c' + \left(1 - \frac{\alpha\mu'}{\alpha\mu' + (1-\alpha)\mu''}\right)c''\right) \in S,$$

from which we conclude that $\alpha s' + (1-\alpha)s'' \in S$.

We now introduce the set $T := S \cap (c_0 + L)$ where $L := \{w \in \mathbb{R}^2 \mid \langle w, v\rangle_{\mathbb{R}^2} = 0\}$ is a line in $\mathbb{R}^2$ (since $v \neq 0_{\mathbb{R}^2}$) which contains the origin $0_{\mathbb{R}^2}$. Note that T is compact and convex and, since $c_0 \in C \subset S$, that T is also nonempty. Then, since T is a nonempty compact convex set included in the line $(c_0 + L)$, we deduce that T is a compact segment and, since $T \subset S$, that $T = [s_1, s_2]$ with s_1, $s_2 \in S$. In particular there exist μ_1, $\mu_2 \in [\varepsilon, M]$ and c_1, $c_2 \in C$ such that $s_1 = \mu_1 c_1$ and $s_2 = \mu_2 c_2$.

We are now in a position to conclude the proof. Let $c \in C$. Consider $\mu := \frac{\langle c_0, v\rangle_{\mathbb{R}^2}}{\langle c, v\rangle_{\mathbb{R}^2}} \in [\varepsilon, M]$. It holds that $\mu c \in S$ and, since $\langle \mu c - c_0, v\rangle_{\mathbb{R}^2} = 0$, that $\mu c \in (c_0 + L)$. It follows that $\mu c \in T = [s_1, s_2]$. So there exists $\alpha \in [0,1]$ such that $\mu c = \alpha s_1 + (1-\alpha)s_2$ and thus

$$c = \frac{\alpha\mu_1}{\mu}c_1 + \frac{(1-\alpha)\mu_2}{\mu}c_2 \in \{\lambda_1 c_1 + \lambda_2 c_2 \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \backslash \{0_{\mathbb{R}^2}\}\},$$

which completes the proof. □

Remark 4.2.5. *Two comments on Lemma 4.2.1:*

(i) *Lemma 4.2.1 is not valid without the compactness hypothesis on C. A bounded unclosed counterexample is given by $C := \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + (y-2)^2 < 1\}$ and an unbounded closed counterexample is given by $C := \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ xy \geq 1\}$.*

(ii) *Lemma 4.2.1 is intrinsic to the two-dimensional setting $n = 2$ since there is no direct analogue in the higher-dimensional setting $n \geq 3$. For example, a three-dimensional counterexample is given by*

$$C := \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z^2,\ 1 \leq z \leq 2\},$$

for which it does not exist any c_1, c_2, $c_3 \in C$ such that $C \subset \{\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 \mid (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+ \backslash \{0_{\mathbb{R}^3}\}\}$.

Proposition 4.2.2. *Let $n \in \{1,2\}$ and $C \subset \mathbb{R}^n$ be a (one- or two-dimensional) nonempty compact convex set and let v_1, $v_2 \in \mathbb{R}^n$. Suppose that, for all $c \in C$, there exists $i \in \{1,2\}$ such that $\langle c, v_i\rangle_{\mathbb{R}^n} < 0$. Then there exists $\alpha \in [0,1]$ such that $\langle c, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0$ for all $c \in C$.*

Proof. Proposition 4.2.2 is trivial for the one-dimensional case $n = 1$ (since then C is included in $\mathbb{R}^*_+$ or in $\mathbb{R}^*_-$ and at least one of the v_i belongs to $\mathbb{R}^*_-$ or $\mathbb{R}^*_+$ respectively). In what follows we will focus only on the two-dimensional case $n = 2$. It is clear from the hypothesis that C does not contain the origin $0_{\mathbb{R}^2}$. From Lemma 4.2.1, there exist c_1, $c_2 \in C$ such that $C \subset \{\lambda_1 c_1 + \lambda_2 c_2 \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \backslash \{0_{\mathbb{R}^2}\}\}$. As a consequence, in order to conclude this proof, our aim is to prove that there exists $\alpha \in [0,1]$ such that $\langle c_1, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0$ and $\langle c_2, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0$. Up to relabelling v_1 and v_2, we assume in what follows that $\langle c_1, v_1\rangle_{\mathbb{R}^2} < 0$. The worst case is then given by $\langle c_2, v_1\rangle_{\mathbb{R}^2} \geq 0$, $\langle c_2, v_2\rangle_{\mathbb{R}^2} < 0$ and $\langle c_1, v_2\rangle_{\mathbb{R}^2} \geq 0$ (otherwise the proof is easily concluded by taking $\alpha = 0$ or $\alpha = 1$). In the worst case let us define the function $\Phi : [0,1] \to \mathbb{R}$ by $\Phi(\gamma) := \langle c_2, \gamma v_1 + (1-\gamma)v_2\rangle_{\mathbb{R}^2}$ for all $\gamma \in [0,1]$. Note that Φ is affine with $\Phi(0) < 0$, $\Phi(1) \geq 0$ and $\Phi(\beta) = 0$ where

$$\beta := \frac{\langle c_2, v_2\rangle_{\mathbb{R}^2}}{\langle c_2, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_1\rangle_{\mathbb{R}^2}} \in (0,1].$$

In particular Φ is negative over the interval $[0,\beta)$. Now let us prove that $\langle c_1, \beta v_1 + (1-\beta)v_2\rangle_{\mathbb{R}^2} < 0$. Since

$$\left(1 - \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}}\right) c_1 + \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}} c_2 \in C,$$

and

$$\left\langle \left(1 - \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}}\right) c_1 + \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}} c_2, v_2 \right\rangle_{\mathbb{R}^2} = 0,$$

we deduce from the hypothesis that

$$\left\langle \left(1 - \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}}\right) c_1 + \frac{\langle c_1, v_2\rangle_{\mathbb{R}^2}}{\langle c_1, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_2\rangle_{\mathbb{R}^2}} c_2, v_1 \right\rangle_{\mathbb{R}^2} < 0,$$

which exactly corresponds to $\langle c_1, \beta v_1 + (1-\beta)v_2\rangle_{\mathbb{R}^2} < 0$ by substituting the value of β and using the fact that $\langle c_2, v_2\rangle_{\mathbb{R}^2} - \langle c_2, v_1\rangle_{\mathbb{R}^2} < 0$. Consequently, there exists $\alpha \in [0,\beta)$ (sufficiently close to β) such that $\langle c_1, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^2} < 0$ and $\Phi(\alpha) = \langle c_2, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^2} < 0$, which completes the proof. □

Proposition 4.2.3. *Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set and let v_1, $v_2 \in \mathbb{R}^n$. Suppose that, for all $c \in C$, there exists $i \in \{1,2\}$ such that $\langle c, v_i\rangle_{\mathbb{R}^n} < 0$. Then there exists $\alpha \in [0,1]$ such that $\langle c, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0$ for all $c \in C$.*

Proof. Note that v_1 and v_2 cannot be both the null vector. We define the (one- or two-dimensional) space $V := \mathrm{span}\{v_1, v_2\}$ and the nonempty compact convex set $\widetilde{C} := \mathrm{proj}_V(C)$ of V, where $\mathrm{proj}_V : \mathbb{R}^n \to V$ stands for the usual linear projection operator onto V. Note that, for all $\widetilde{c} \in \widetilde{C}$, there exists $c \in C$ such that $\widetilde{c} = \mathrm{proj}_V(c)$ and it holds that $\langle \widetilde{c} - c, v_i\rangle_{\mathbb{R}^n} = 0$ for all $i \in \{1,2\}$. Then, it follows from the hypothesis that, for all $\widetilde{c} \in \widetilde{C}$, there exists $i \in \{1,2\}$ such that $\langle \widetilde{c}, v_i\rangle_{\mathbb{R}^n} = \langle \widetilde{c} - c, v_i\rangle_{\mathbb{R}^n} + \langle c, v_i\rangle_{\mathbb{R}^n} < 0$. From Proposition 4.2.2, there exists $\alpha \in [0,1]$ such that $\langle \widetilde{c}, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0$ for all $\widetilde{c} \in \widetilde{C}$. Now let $c \in C$. It holds that

$$\langle c, \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} = \langle c - \mathrm{proj}_V(c), \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} + \langle \mathrm{proj}_V(c), \alpha v_1 + (1-\alpha)v_2\rangle_{\mathbb{R}^n} < 0,$$

since $\mathrm{proj}_V(c) \in \widetilde{C}$. The proof is thereby completed. □

Proposition 4.2.4. *Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set and let $v_1, \ldots, v_N \in \mathbb{R}^n$ with $N \in \mathbb{N}^*$. Suppose that, for all $c \in C$, there exists $i \in \{1, \ldots, N\}$ such that $\langle c, v_i\rangle_{\mathbb{R}^n} < 0$. Then there exists $\alpha = (\alpha_1, \ldots, \alpha_N) \in \Delta_N$ such that $\langle c, \sum_{i=1}^N \alpha_i v_i\rangle_{\mathbb{R}^n} < 0$ for all $c \in C$.*

Proof. Proposition 4.2.4 is trivial for $N = 1$ and is true for $N = 2$ from Proposition 4.2.3. Now let us proceed by induction. Assume that the result is true for $N = q-1$ with $q \geq 3$. We define the set $Q := \{c \in C \mid \langle c, v_q\rangle_{\mathbb{R}^n} \geq 0\}$. If $Q = \emptyset$, then we take $\alpha_1 = \ldots = \alpha_{q-1} = 0$ and $\alpha_q = 1$ to conclude. Suppose now that $Q \neq \emptyset$. Then Q is a nonempty compact convex set of $\mathbb{R}^n$ and, from the hypothesis, there exists $i \in \{1, \ldots, q-1\}$ such that $\langle c, v_i\rangle_{\mathbb{R}^n} < 0$ for all $c \in Q$. Using the induction hypothesis, there exists $\beta = (\beta_1, \ldots, \beta_{q-1}) \in \Delta_{q-1}$ such that $\langle c, \sum_{i=1}^{q-1} \beta_i v_i \rangle_{\mathbb{R}^n} < 0$ for all $c \in Q$. We define $w_1 := \sum_{i=1}^{q-1} \beta_i v_i$ and $w_2 := v_q$. Let $c \in C$. If $c \in Q$, then $\langle c, w_1 \rangle_{\mathbb{R}^n} < 0$, otherwise $c \notin Q$ and $\langle c, w_2 \rangle_{\mathbb{R}^n} < 0$. Thus, applying Proposition 4.2.3, we obtain that there exists $\mu \in [0,1]$ such that $\langle c, \mu w_1 + (1-\mu) w_2 \rangle_{\mathbb{R}^n} < 0$ for all $c \in C$. Taking $\alpha_i := \mu \beta_i$ for all $i = 1, \ldots, q-1$ and $\alpha_q := 1 - \mu$, it holds that $\langle c, \sum_{i=1}^{q} \alpha_i v_i \rangle_{\mathbb{R}^n} < 0$ for all $c \in C$ and the induction is complete. □

We are now in a position to provide the proof of Theorem 4.2.1. From hypothesis of Theorem 4.2.1, the covering of C by open sets given by

$$C \subset \bigcup_{v \in \mathcal{V}} \{ w \in \mathbb{R}^n \mid \langle w, v \rangle_{\mathbb{R}^n} < 0 \},$$

holds true. Using the compactness of C, we can extract a finite set $\{v_1, \ldots, v_N\} \subset \mathcal{V}$ with $N \in \mathbb{N}^*$ such that

$$C \subset \bigcup_{i=1}^{N} \{ w \in \mathbb{R}^n \mid \langle w, v_i \rangle_{\mathbb{R}^n} < 0 \}.$$

From Proposition 4.2.4, there exists $v \in \mathrm{Conv}(\{v_1, \ldots, v_N\}) \subset \mathrm{Conv}(\mathcal{V})$ such that $\langle c, v \rangle_{\mathbb{R}^n} < 0$ for all $c \in C$. The proof of Theorem 4.2.1 is complete.

Remark 4.2.6. *Note that the compactness hypothesis on C has been used at several occasions in the proof of Theorem 4.2.1 (for example in Lemma 4.2.1, which is crucial for Proposition 4.2.2, but also in Proposition 4.2.3 in order to ensure the compactness of $\widetilde{C} := \mathrm{proj}_V(C)$ and, finally, at the end of this subsection for extracting a finite open covering of C).*

Remark 4.2.7. *In this chapter, as given in the work [Adly et al. 2020], we state the universal separating vector theorem (Theorem 4.2.1) in the finite-dimensional setting (which is sufficient for our applications in Sections 4.3 and 4.4). A natural perspective for further research works concerns the extension of Theorem 4.2.1 to the infinite-dimensional Hilbert or Banach setting with a weak compactness assumption. For example, let us mention that Proposition 4.2.4 is still valid in the Hilbert setting by assuming that C is a nonempty weakly compact convex set. The extension of Theorem 4.2.1 in its entirety may require a new approach.*

4.3 Application in optimal permanent control theory

This section is dedicated to an application of the universal separating vector theorem (Theorem 4.2.1) in nonsmooth optimal permanent control theory. In Section 4.3.1, an optimal permanent control problem with a nonsmooth Mayer cost function is introduced. Then a discussion is provided in Section 4.3.2 in order to motivate the use of the universal separating vector theorem with the aid of the tools of nonsmooth analysis presented in Section 1.4 of Chapter 1. Finally a PMP for optimal permanent control problems with nonsmooth Mayer cost functions is recovered in Section 4.3.3 by applying the contrapositive Proposition 4.2.1 as in [Adly *et al.* 2020].

4.3.1 A basic optimal permanent control problem with a nonsmooth Mayer cost function

In what follows we fix two positive integers m, $n \in \mathbb{N}^*$ and a positive real number $T > 0$. In this section we focus on the basic optimal permanent control problem with nonsmooth Mayer cost function (OPCP$_{\rm ns}$) given by

$$\begin{cases} \text{minimize} & g(x(T)), \\ \text{subject to} & x \in \mathrm{AC}([0,T],\mathbb{R}^n), \quad u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m), \\ & \dot{x}(t) = f(x(t),u(t),t), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x_0, \\ & u(t) \in \mathrm{U}, \quad \text{a.e. } t \in [0,T]. \end{cases} \tag{OPCP$_{\rm ns}$}$$

In this section we will make use of the following regularity and topology assumptions:

- the function $g : \mathbb{R}^n \to \mathbb{R}$ that describes the *Mayer cost* $g(x(T))$ is locally Lipschitz;
- the *dynamics* $f : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$, that drives the *state equation* $\dot{x}(t) = f(x(t),u(t),t)$, is continuous and of class C^1;
- the set $\mathrm{U} \subset \mathbb{R}^m$, that describes the *control constraint* $u(t) \in \mathrm{U}$, is a nonempty subset of $\mathbb{R}^m$;
- the initial condition $x_0 \in \mathbb{R}^n$ is fixed.

Remark 4.3.1. *Note that it is not our aim in this section to consider a very general nonsmooth optimal permanent control problem. For instance we do not consider general terminal state constraints, neither a free final time problem and, above all, the nonsmoothness of Problem* (OPCP$_{\rm ns}$) *lies only in the Mayer cost function (and not in the dynamics). For complete studies of more general nonsmooth optimal permanent control problems, we refer to standard references such as [Clarke & Vinter 1989, Clarke 1990, Clarke et al. 1998, Mordukhovich 2006a, Mordukhovich 2006c, Mordukhovich & Shvartsman 2013, Vinter 2010, Warga 1975] and references therein.*

As usual in optimal control theory, we recall that the (scalar) Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ associated to Problem (OPCP$_{\rm ns}$) is defined by $H(x,u,p,t) := \langle p, f(x,u,t) \rangle_{\mathbb{R}^n}$ for all $(x,u,p,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times [0,T]$. For the needs of this chapter, as given in [Adly *et al.* 2020], we introduce the following unusual notion of *vector Hamiltonian function* associated to Problem (OPCP$_{\rm ns}$) given by $\overline{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times n} \times [0,T] \to \mathbb{R}^n$ defined by $\overline{H}(x,u,P,t) := P^\top \times f(x,u,t)$ for all $(x,u,P,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times n} \times [0,T]$. Here $\mathbb{R}^{n\times n}$ stands for the set of $n \times n$ matrices and $\top$ denotes the classical transposition operation. We note that

$$\overline{H}(x,u,P,t) = \begin{pmatrix} H(x,u,p_1,t) \\ H(x,u,p_2,t) \\ \vdots \\ H(x,u,p_n,t) \end{pmatrix} = \begin{pmatrix} \langle p_1, f(x,u,t) \rangle_{\mathbb{R}^n} \\ \langle p_2, f(x,u,t) \rangle_{\mathbb{R}^n} \\ \vdots \\ \langle p_n, f(x,u,t) \rangle_{\mathbb{R}^n} \end{pmatrix},$$

for all $(x,u,P,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times n} \times [0,T]$, where the $p_i \in \mathbb{R}^n$ stand for the columns of the matrix $P = (p_1\ p_2\ \dots\ p_n) \in \mathbb{R}^{n\times n}$.

4.3.2 Motivation for a universal selection

For the needs of this section we refer to Section 1.4 in Chapter 1 for recalls on nonsmooth analysis (notably when considering the Clarke generalized derivative g° and the Clarke subdifferential ∂g of the cost function g). Here our aim is to motivate the use of the universal separating vector theorem (Theorem 4.2.1) in order to derive a PMP for Problem ($\mathrm{OPCP_{ns}}$). For this purpose we first need to establish two preliminary propositions which are based on standard techniques of perturbation of the optimal control. The proof of the next proposition (which is quite technical and thus postponed to Section 4.5) is based on the classical tool of *implicit spike variation* of the optimal control (see, e.g., [Bergounioux & Zidani 1999, Bourdin 2016, Casas 1997, Fattorini 1993, Li & Yao 1985, Li & Yong 1991, Li & Yong 1995, Yu 2013]).

Proposition 4.3.1. *Let (x, u) be a solution to Problem ($\mathrm{OPCP_{ns}}$). Then there exists an adjoint matrix $P \in \mathrm{AC}([0, T], \mathbb{R}^{n\times n})$ such that:*

(i) ***Adjoint equation:*** *P satisfies*

$$-\dot{P}(t) = \nabla_1 \overline{H}(x(t), u(t), P(t), t),$$

for a.e. $t \in [0, T]$;

(ii) ***Transversality condition:*** *P satisfies*

$$-P(T) = \mathrm{Id}_n,$$

where $\mathrm{Id}_n \in \mathbb{R}^{n\times n}$ stands for the $n \times n$ identity matrix;

(iii) ***Integral pseudo-maximization condition:*** *the inequality*

$$g^\circ \left(x(T); \int_0^T \overline{H}(x(t), u(t), P(t), t) - \overline{H}(x(t), u'(t), P(t), t)\, dt \right) \geq 0,$$

is satisfied for all $u' \in \mathrm{L}^\infty([0, T], \mathrm{U})$.

Remark 4.3.2. *As in classical optimal control theory where the terminology of adjoint vector is commonly used in order to designate (actually) a vector function, we decided to use the terminology of adjoint matrix in order to designate (actually) a matrix function.*

Remark 4.3.3. *Note that the adjoint matrix provided in Proposition 4.3.1 is given by $P(t) = -\Phi(T, t)^\top$ for all $t \in [0, T]$, where $\Phi(\cdot, \cdot) : [0, T]^2 \to \mathbb{R}^{n\times n}$ stands for the state-transition matrix (or fundamental matrix solution) associated to the matrix function $\nabla_1 f(x, u, \cdot)$. We refer to [Sontag 1998, Appendix C.4] for more details on state-transition matrices.*

The next proposition follows from Proposition 4.3.1 and the application of the usual technique of *needle-like perturbation* of the optimal control (see, e.g., [Bourdin & Trélat 2013, Bourdin & Trélat 2016, Cesari 1983b, Fattorini 1999, Liberzon 2012, Pontryagin *et al.* 1962]).

Proposition 4.3.2. *Let (x, u) be a solution to Problem ($\mathrm{OPCP_{ns}}$). Then the adjoint matrix P provided in Proposition 4.3.1 is such that:*

(iii') ***Pseudo-maximization condition:*** *the inequality*

$$g^\circ\Big(x(T); \overline{H}(x(s),u(s),P(s),s) - \overline{H}(x(s),\omega,P(s),s)\Big) \geq 0,$$

is satisfied for a.e. $s \in [0,T]$ *and all* $\omega \in \mathrm{U}$.

Proof. Let $s \in [0,T)$ be a Lebesgue point of the essentially bounded vector function $\overline{H}(x,u,P,\cdot)$ and let $\omega \in \mathrm{U}$. We introduce the needle-like perturbation $u_\eta \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ of u defined by

$$u_\eta(t) := \begin{cases} \omega & \text{over the interval } [s,s+\eta), \\ u(t) & \text{otherwise}, \end{cases}$$

for a.e. $t \in [0,T]$ and for all $\eta > 0$ sufficiently small. Note that u_η is with values in U. From the integral pseudo-maximization condition in Proposition 4.3.1, it holds that

$$g^\circ\left(x(T); \int_0^T \overline{H}(x(t),u(t),P(t),t) - \overline{H}(x(t),u_\eta(t),P(t),t)\,dt\right) \geq 0,$$

which gives

$$g^\circ\left(x(T); \int_s^{s+\eta} \overline{H}(x(t),u(t),P(t),t) - \overline{H}(x(t),\omega,P(t),t)\,dt\right) \geq 0.$$

Since $\eta > 0$ and $g^\circ(x(T);\cdot)$ is positively homogeneous (see Proposition 1.4.1 in Chapter 1), we get that

$$g^\circ\left(x(T); \frac{1}{\eta}\int_s^{s+\eta} \overline{H}(x(t),u(t),P(t),t) - \overline{H}(x(t),\omega,P(t),t)\,dt\right) \geq 0.$$

Since s is a Lebesgue point of the function $\overline{H}(x,u,P,\cdot)$ and since $g^\circ(x(T);\cdot)$ is Lipschitz continuous (see Proposition 1.4.1), taking the limit $\eta \to 0^+$ leads to

$$g^\circ\Big(x(T); \overline{H}(x(s),u(s),P(s),s) - \overline{H}(x(s),\omega,P(s),s)\Big) \geq 0,$$

which completes the proof. □

We are now in a position to motivate the use of the universal separating vector theorem (Theorem 4.2.1) in order to derive a PMP for Problem ($\mathrm{OPCP}_{\mathrm{ns}}$). Consider the framework of Proposition 4.3.2.

- In the smooth case where g is continuously differentiable at $x(T)$, then $\partial g(x(T)) = \{\nabla g(x(T))\}$ is reduced to a singleton (see [Clarke *et al.* 1998, Proposition 3.1 in Chapter 2]). In that context we obtain from Proposition 1.4.2 and the pseudo-maximization condition in Proposition 4.3.2 that

$$g^\circ\Big(x(T); \overline{H}(x(s),u(s),P(s),s) - \overline{H}(x(s),\omega,P(s),s)\Big) \\ = \Big\langle P(s)\times\nabla g(x(T)), f(x(s),u(s),s) - f(x(s),\omega,s)\Big\rangle_{\mathbb{R}^n} \geq 0,$$

  for a.e. $s \in [0,T]$ and all $\omega \in \mathrm{U}$. Defining the *adjoint vector* $p \in \mathrm{AC}([0,T],\mathbb{R}^n)$ by $p(t) := P(t)\times\nabla g(x(T))$ for all $t \in [0,T]$, we recover the usual necessary optimality conditions of the classical PMP (see, e.g., [Pontryagin *et al.* 1962, Theorem 3 page 50]) given by:

(i) **Adjoint equation:** p satisfies

$$-\dot{p}(t) = \nabla_1 H(x(t), u(t), p(t), t),$$

for a.e. $t \in [0, T]$;

(ii) **Transversality condition:** p satisfies

$$-p(T) = \nabla g(x(T));$$

(iii) **Maximization condition:** the inequality

$$H(x(s), u(s), p(s), s) - H(x(s), \omega, p(s), s) \geq 0,$$

is satisfied for a.e. $s \in [0, T]$ and all $\omega \in \mathrm{U}$.

- In the nonsmooth case, in which $\partial g(x(T))$ is not reduced to a singleton a priori, we obtain from Proposition 1.4.2 and the pseudo-maximization condition in Proposition 4.3.2 that

$$g^\circ\Big(x(T); \overline{H}(x(s), u(s), P(s), s) - \overline{H}(x(s), \omega, P(s), s)\Big)$$
$$= \Big\langle P(s) \times \xi_{(s,v)}, f(x(s), u(s), s) - f(x(s), \omega, s)\Big\rangle_{\mathbb{R}^n} \geq 0,$$

for a.e. $s \in [0, T]$ and all $\omega \in \mathrm{U}$, where the selection $\xi_{(s,\omega)} \in \partial g(x(T))$ depends strongly on the couple (s, ω). In order to follow the same strategy as above in the smooth case, we would need the existence of a *universal* selection $\xi \in \partial g(x(T))$ (that is, independent of the couple (s, ω)) such that

$$\Big\langle P(s) \times \xi, f(x(s), u(s), s) - f(x(s), \omega, s)\Big\rangle_{\mathbb{R}^n} \geq 0,$$

for a.e. $s \in [0, T]$ and all $\omega \in \mathrm{U}$. In such a context we would define the *adjoint vector* $p \in \mathrm{AC}([0, T], \mathbb{R}^n)$ by $p(t) := P(t) \times \xi$ for all $t \in [0, T]$ and we would recover the necessary optimality conditions of the PMP for nonsmooth Mayer cost functions given, for example, in [Vinter 2010, Theorem 6.2.1]. The existence of such a universal selection $\xi \in \partial g(x(T))$ is exactly the topic of the next subsection and will be obtained by applying the universal separating vector theorem (Theorem 4.2.1). Let us precise that in [Vinter 2010, Theorem 6.2.1] the PMP is stated in terms of the limiting subdifferential rather than the Clarke subdifferential considered in this chapter.

4.3.3 Application of the universal separating vector theorem

In this section we will apply the universal separating vector theorem (Theorem 4.2.1) in order to prove the existence of a universal selection $\xi \in \partial g(x(T))$ as explained at the end of the previous subsection. We first start with the following technical lemma based on the tool of *package of needle-like perturbations* of the optimal control (see, e.g., [Bohner *et al.* 2017, Bourdin & Trélat 2017, Dmitruk & Osmolovskii 2014, Korytowski 2014]).

Lemma 4.3.1. *Let* (x, u) *be a solution to Problem* ($\mathrm{OPCP_{ns}}$) *and consider the framework of Proposition 4.3.1. Let* $\pi := (\overline{s}, \overline{\omega}, \overline{\alpha}) \in \mathcal{L}^q \times \mathrm{U}^r \times \Delta_r$*, with* q*,* $r \in \mathbb{N}^*$*, be a package which consists of:*

- *A q-tuple* $\overline{s} := \{s_i\}_{i=1,\dots,q} \in \mathcal{L}^q$ *such that* $0 \leq s_1 < s_2 < \ldots < s_q < T$*, where* $\mathcal{L} \subset [0,T)$ *stands for the set of Lebesgue points of the essentially bounded vector function* $\overline{H}(x,u,P,\cdot)$*;*
- *A r-tuple* $\overline{\omega} := \{\omega_i^j\}_{\substack{i=1,\dots,q,\\ j=1,\dots,r_i}} \in \mathrm{U}^r$*, with* $r_i \in \mathbb{N}^*$ *for all* $i \in \{1,\dots,q\}$*, where* $r := \sum_{i=1}^q r_i$*;*
- *A r-tuple* $\overline{\alpha} := (\alpha_i^j)_{\substack{i=1,\dots,q,\\ j=1,\dots,r_i}} \in \Delta_r$*.*

Then there exists $\xi_\pi \in \partial g(x(T))$ *such that*

$$\left\langle \xi_\pi, \sum_{i=1}^q \sum_{j=1}^{r_i} \alpha_i^j \Big(\overline{H}(x(s_i),u(s_i),P(s_i),s_i) - \overline{H}(x(s_i),\omega_i^j,P(s_i),s_i) \Big) \right\rangle_{\mathbb{R}^n} \geq 0.$$

Proof. We consider the package of needle-like perturbations $u_\eta \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ of u defined by

$$u_\eta(t) := \begin{cases} \omega_i^j & \text{over the interval } [s_i+(j-1)\alpha_i^j\eta, s_i+j\alpha_i^j\eta),\ \forall j \in \{1,\dots,r_i\},\ \forall i \in \{1,\dots,q\}, \\ u(t) & \text{otherwise,} \end{cases}$$

for a.e. $t \in [0,T]$ and for all $\eta > 0$ sufficiently small. Note that u_η is with values in U. From the integral pseudo-maximization condition in Proposition 4.3.1, it holds that

$$g^\circ \left(x(T); \int_0^T \overline{H}(x(t),u(t),P(t),t) - \overline{H}(x(t),u_\eta(t),P(t),t)\, dt \right) \geq 0,$$

which gives

$$g^\circ \left(x(T); \sum_{i=1}^q \sum_{j=1}^{r_i} \int_{s_i+(j-1)\alpha_i^j\eta}^{s_i+j\alpha_i^j\eta} \overline{H}(x(t),u(t),P(t),t) - \overline{H}(x(t),\omega_i^j,P(t),t)\, dt \right) \geq 0.$$

With the same arguments as in the proof of Proposition 4.3.2, we obtain

$$g^\circ \left(x(T); \sum_{i=1}^q \sum_{j=1}^{r_i} \alpha_i^j \Big(\overline{H}(x(s_i),u(s_i),P(s_i),s_i) - \overline{H}(x(s_i),\omega_i^j,P(s_i),s_i) \Big) \right) \geq 0.$$

Finally, from Proposition 1.4.2 in Chapter 1, there exists an element $\xi_\pi \in \partial g(x(T))$ such that

$$\begin{aligned} g^\circ & \left(x(T); \sum_{i=1}^q \sum_{j=1}^{r_i} \alpha_i^j \Big(\overline{H}(x(s_i),u(s_i),P(s_i),s_i) - \overline{H}(x(s_i),\omega_i^j,P(s_i),s_i) \Big) \right) \\ & = \left\langle \xi_\pi, \sum_{i=1}^q \sum_{j=1}^{r_i} \alpha_i^j \Big(\overline{H}(x(s_i),u(s_i),P(s_i),s_i) - \overline{H}(x(s_i),\omega_i^j,P(s_i),s_i) \Big) \right\rangle_{\mathbb{R}^n} \geq 0, \end{aligned}$$

which concludes the proof. □

We are now in a position to apply the universal separating vector theorem (Theorem 4.2.1) in order to recover the following PMP for Problem ($\mathrm{OPCP_{ns}}$) using only the tools of nonsmooth analysis as presented in [Adly *et al.* 2020].

Theorem 4.3.1. *Let (x, u) be a solution to Problem* ($\mathrm{OPCP}_{\mathrm{ns}}$). *Then there exists an adjoint vector $p \in \mathrm{AC}([0, T], \mathbb{R}^n)$ such that:*

(i) ***Adjoint equation:*** *p satisfies*

$$-\dot{p}(t) = \nabla_1 H(x(t), u(t), p(t), t),$$

for a.e. $t \in [0, T]$;

(ii) ***Transversality condition:*** *p satisfies*

$$-p(T) \in \partial g(x(T));$$

(iii) ***Maximization condition:*** *the inequality*

$$H(x(s), u(s), p(s), s) - H(x(s), \omega, p(s), s) \geq 0,$$

is satisfied for a.e. $s \in [0, T]$ and all $\omega \in \mathrm{U}$.

Proof. Consider the framework of Lemma 4.3.1 and let us define the set $\mathcal{V} \subset \mathbb{R}^n$ by

$$\mathcal{V} := \Big\{ \overline{H}(x(s), u(s), P(s), s) - \overline{H}(x(s), \omega, P(s), s) \mid (s, \omega) \in \mathcal{L} \times \mathrm{U} \Big\}.$$

From Lemma 4.3.1, for every $v \in \mathrm{Conv}(\mathcal{V})$, there exists $\xi_v \in \partial g(x(T))$ such that $\langle \xi_v, v \rangle_{\mathbb{R}^n} \geq 0$. Since $\partial g(x(T))$ is a nonempty compact convex set (see Proposition 1.4.2 in Chapter 1) and from the contrapositive of the universal separating vector theorem (Proposition 4.2.1), there exists a universal vector $\xi \in \partial g(x(T))$ such that $\langle \xi, v \rangle_{\mathbb{R}^n} \geq 0$ for all $v \in \mathcal{V}$, which gives exactly

$$\langle \xi, \overline{H}(x(s), u(s), P(s), s) - \overline{H}(x(s), \omega, P(s), s) \rangle_{\mathbb{R}^n} \geq 0,$$

for a.e. $s \in [0, T]$ and all $\omega \in \mathrm{U}$. Then the end of the proof is similar to the discussion provided at the end of Section 4.3.2 in the smooth case (where g is continuously differentiable at $x(T)$), that is, by introducing the adjoint vector $p \in \mathrm{AC}([0, T], \mathbb{R}^n)$ defined by $p(t) := P(t) \times \xi$ for all $t \in [0, T]$. □

Remark 4.3.4. *As mentioned in Remark 4.3.1, numerous texts in the literature are already dedicated to nonsmooth optimal permanent control theory, and to more general nonsmooth optimal permanent control problems. In these references, several methods have been explored in order to establish nonsmooth versions of the PMP. We can cite for example the method of quadratic inf-convolution in [Clarke 2008, Section 2.1 page 4] or the application of a nonsmooth Lagrange multiplier rule in [Vinter 2010, Theorem 5.6.2]. Our novel approach is based on the combination of implicit spike variations and packages of needle-like perturbations of the optimal control (in Proposition 4.3.1 and Lemma 4.3.1 respectively) and, finally, on the application of the universal separating vector theorem (Theorem 4.2.1), precisely of its contrapositive (Proposition 4.2.1).*

4.4 Application in optimal sampled-data control theory

The aim of this section is to apply the universal separating vector theorem (Theorem 4.2.1) in order to establish a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions. We emphasize that such a result has not been presented in the work [Adly *et al.* 2020] and, to the best of our knowledge, has not been considered elsewhere in the literature. In Section 4.4.1, an optimal sampled-data control problem with a nonsmooth Mayer cost function (see Problem ($\mathrm{OSCP_{ns}}$)) is introduced. Then a discussion is provided in Section 4.4.2 in order to motivate the use of the universal separating vector theorem (Theorem 4.2.1). Finally a PMP for Problem ($\mathrm{OSCP_{ns}}$) is obtained in Section 4.4.3 by applying the universal separating vector theorem.

4.4.1 An optimal sampled-data control problem with a nonsmooth Mayer cost function

Let m, n, $N \in \mathbb{N}^*$ be three fixed positive integers. Let us fix a positive real number $T > 0$, as well an N-partition $\mathbb{T} = \{t_i\}_{i=0,\dots,N}$ of the interval $[0,T]$. In this section we focus on the basic nonsmooth optimal sampled-data control problem with nonsmooth Mayer cost function ($\mathrm{OSCP_{ns}}$) given by

$$\left\{\begin{array}{ll} \text{minimize} & g(x(T)), \\ \text{subject to} & x \in \mathrm{AC}([0,T],\mathbb{R}^n), \quad u \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathbb{R}^m), \\ & \dot{x}(t) = f(x(t),u(t),t), \quad \text{a.e. } t \in [0,T], \\ & x(0) = x_0, \\ & u_i \in \mathrm{U}, \quad \text{for all } i = 0,\dots,N-1. \end{array}\right. \tag{$\mathrm{OSCP_{ns}}$}$$

In this section we will make use of the following regularity and topology assumptions:

- the function $g : \mathbb{R}^n \to \mathbb{R}$ that describes the *Mayer cost* $g(x(T))$ is locally Lipschitz;
- the *dynamics* $f : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$, that drives the *state equation* $\dot{x}(t) = f(x(t),u(t),t)$, is continuous and locally Lipschitz;
- the set $\mathrm{U} \subset \mathbb{R}^m$, that describes the *control constraint* $u_i \in \mathrm{U}$, is a nonempty convex subset of $\mathbb{R}^m$;
- the initial condition $x_0 \in \mathbb{R}^n$ is fixed.

We remark, in contrast to Problem ($\mathrm{OPCP_{ns}}$), the control constraint set U in Problem ($\mathrm{OSCP_{ns}}$) is assumed to be convex in order to apply convex L^∞-perturbations of the control in the next section.

4.4.2 Motivation for a universal selection

In this section our aim is to motivate the use of the universal separating vector theorem (Theorem 4.2.1) in order to derive a PMP for Probem ($\mathrm{OSCP_{ns}}$). We begin with the following preliminary proposition on optimality conditions for Problem ($\mathrm{OSCP_{ns}}$) (whose proof is postponed to Section 4.5.3).

Proposition 4.4.1. *Let* (x,u) *be a solution to Problem* ($\mathrm{OSCP_{ns}}$). *Then there exists an adjoint matrix* $P \in \mathrm{AC}([0,T],\mathbb{R}^{n\times n})$ *such that:*

(i) ***Adjoint equation:*** P *satisfies*

$$-\dot{P}(t) = \nabla_1 \overline{H}(x(t),u(t),P(t),t),$$

for a.e. $t \in [0,T]$*;*

(ii) ***Transversality condition:*** P *satisfies*

$$-P(T) = \mathrm{Id}_n,$$

where $\mathrm{Id}_n \in \mathbb{R}^{n\times n}$ *stands for the* $n \times n$ *identity matrix;*

(iii) ***Integral pseudo-nonpositive averaged Hamiltonian gradient condition:*** *the inequality*

$$g^\circ \left(x(T); -\int_0^T \nabla_2 \overline{H}(x(t),u(t),P(t),t) \times (u'(t)-u(t))\, dt \right) \geq 0,$$

for all $u' \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathrm{U})$.

We are now in a position to motivate the use of the universal separating vector theorem (Theorem 4.2.1) in order to obtain a PMP for Problem ($\mathrm{OSCP_{ns}}$). Consider the framework of Proposition 4.4.1.

- In the smooth case where g is continuously differentiable at $x(T)$, then $\partial g(x(T)) = \{\nabla g(x(T))\}$ is reduced to a singleton (see [Clarke *et al.* 1998, Proposition 3.1 in Chapter 2]). Let $\omega \in \mathrm{U}$ and let $i \in \{0,\dots,N-1\}$. Let $u' \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathrm{U})$ be defined by

$$u'(t) := \begin{cases} \omega & \text{if } t \in [t_i,t_{i+1}), \\ u(t) & \text{otherwise,} \end{cases}$$

for all $t \in [0,T]$. Then we obtain from Proposition 1.4.2 in Chapter 1 and the integral pseudo-nonpositive averaged Hamiltonian gradient condition in Proposition 4.4.1 that

$$g^\circ \Big(x(T); -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\, dt \times (\omega - u_i) \Big)$$
$$= \Big\langle -\int_{t_i}^{t_{i+1}} \nabla_2 f(x(t),u(t),t)^\top \times P(t) \times \nabla g(x(T))\, dt, \omega - u_i \Big\rangle_{\mathbb{R}^m} \geq 0,$$

for all $\omega \in \mathrm{U}$ and $i = 0,\dots,N-1$. Defining the *adjoint vector* $p \in \mathrm{AC}([0,T],\mathbb{R}^n)$ by $p(t) := P(t) \times \nabla g(x(T))$ for all $t \in [0,T]$, we recover the usual nonpositive averaged Hamiltonian gradient condition given by

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t),u_i,p(t),t)\, dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0,$$

for all $\omega \in \mathrm{U}$ and $i = 0,\dots,N-1$.

- In the nonsmooth case, $\partial g(x(T))$ is not reduced to a singleton a priori. Let $\omega \in \mathrm{U}$ and let $i \in \{0,\dots,N-1\}$. Let $u' \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathrm{U})$ be defined by

$$u'(t) := \begin{cases} \omega & \text{if } t \in [t_i, t_{i+1}), \\ u(t) & \text{otherwise,} \end{cases}$$

for all $t \in [0,T]$. We obtain from Proposition 1.4.2 in Chapter 1 and the integral pseudo-nonpositive averaged Hamiltonian gradient condition in Proposition 4.4.1 that

$$g^\circ\left(x(T); -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega - u_i) \right) = \left\langle \xi_{(\omega,i)}, -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega - u_i) \right\rangle_{\mathbb{R}^m} \geq 0,$$

for all $\omega \in \mathrm{U}$ and $i = 0,\dots,N-1$, where the selection $\xi_{(\omega,i)} \in \partial g(x(T))$ depends strongly on the couple (ω,i). In order to follow the same strategy as above in the smooth case, we would need the existence of a *universal* selection $\xi \in \partial g(x(T))$ (that is, independent of the couple (ω,i)) such that

$$\left\langle \xi, -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega - u_i) \right\rangle_{\mathbb{R}^n} \geq 0,$$

for all $\omega \in \mathrm{U}$ and $i = 0,\dots,N-1$. In such a context we would define the *adjoint vector* $p \in \mathrm{AC}([0,T],\mathbb{R}^n)$ by $p(t) := P(t) \times \xi$ for all $t \in [0,T]$ and recover the usual nonpositive averaged Hamiltonian gradient condition of the PMP for optimal sampled-data control problems (see [Bourdin & Trélat 2016, Theorem 2.6 p.62]). The existence of such a universal selection $\xi \in \partial g(x(T))$ will be obtained by applying the universal separating vector theorem (Theorem 4.2.1) in the next section.

4.4.3 Application of the universal separating vector theorem

In this section we will apply the universal separating vector theorem (Theorem 4.2.1) in order to prove the existence of a universal selection $\xi \in \partial g(x(T))$ as explained at the end of the previous subsection. We first start with the following technical lemma.

Lemma 4.4.1. *Let (x,u) be a solution to Problem (OSCP$_{\mathrm{ns}}$) and consider the framework of Proposition 4.4.1. Let $\pi := (\overline{t},\overline{\omega},\overline{\alpha}) \in \mathbb{T}^q \times \mathrm{U}^r \times \Delta_r$, with q, $r \in \mathbb{N}^*$, be a package which consists of:*

- *A q-tuple $\overline{t} := \{t_i\}_{i=1,\dots,q} \in \mathbb{T}^q$ such that $0 \leq t_1 < t_2 < \ldots < t_q < T$;*
- *A r-tuple $\overline{\omega} := \{\omega_i^j\}_{\substack{i=1,\dots,q,\\ j=1,\dots,r_i}} \in \mathrm{U}^r$, with $r_i \in \mathbb{N}^*$ for all $i \in \{1,\dots,q\}$, where $r := \sum_{i=1}^q r_i$;*
- *A r-tuple $\overline{\alpha} := (\alpha_i^j)_{\substack{i=1,\dots,q,\\ j=1,\dots,r_i}} \in \Delta_r$.*

Then there exists $\xi_\pi \in \partial g(x(T))$ such that

$$\left\langle \xi_\pi, \sum_{i=1}^q \sum_{j=1}^{r_i} \alpha_i^j \left(-\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega_i^j - u_i) \right) \right\rangle_{\mathbb{R}^n} \geq 0.$$

Proof. We consider the perturbation $u' \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ of u defined by

$$u'(t) := \begin{cases} u_i + \sum_{j=1}^{r_i} \alpha_i^j (\omega_i^j - u_i) & \text{over the interval } [t_i, t_{i+1}) \text{ for } i = 1,\dots,q, \\ u(t) & \text{otherwise,} \end{cases}$$

for all $t \in [0,T]$. Note that u' is with values in U. From the integral pseudo-nonpositive averaged Hamiltonian gradient condition in Proposition 4.4.1, it holds that

$$g^\circ \left(x(T); - \int_0^T \nabla_2 \overline{H}(x(t),u(t),P(t),t) \times (u'(t) - u(t))\, dt \right) \geq 0,$$

which gives

$$g^\circ \left(x(T); \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_i^j \left(- \int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\, dt \times (\omega_i^j - u_i) \right) \right) \geq 0.$$

Finally from Proposition 1.4.2 in Chapter 1 there exists an element $\xi_\pi \in \partial g(x(T))$ such that

$$\begin{aligned} g^\circ & \left(x(T); \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_i^j \left(- \int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\, dt \times (\omega_i^j - u_i) \right) \right) \\ &= \left\langle \xi_\pi, \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_i^j \left(- \int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\, dt \times (\omega_i^j - u_i) \right) \right\rangle_{\mathbb{R}^n} \geq 0, \end{aligned}$$

which concludes the proof. □

We are now in a position to apply the universal separating vector theorem (Theorem 4.2.1) in order to obtain the following PMP for Problem ($\mathrm{OSCP_{ns}}$).

Theorem 4.4.1. *Let (x,u) be a solution to Problem ($\mathrm{OSCP_{ns}}$). Then there exists an adjoint vector $p \in \mathrm{AC}([0,T],\mathbb{R}^n)$ such that:*

(i) ***Adjoint equation:*** *p satisfies*

$$-\dot{p}(t) = \nabla_1 H(x(t),u(t),p(t),t),$$

for a.e. $t \in [0,T]$;

(ii) ***Transversality condition:*** *p satisfies*

$$-p(T) \in \partial g(x(T));$$

(iii) ***Nonpositive averaged Hamiltonian gradient condition:*** *the inequality*

$$\left\langle \int_{t_i}^{t_{i+1}} \nabla_2 H(x(t),u_i,p(t),t)\, dt, \omega - u_i \right\rangle_{\mathbb{R}^m} \leq 0,$$

is satisfied for all $\omega \in \mathrm{U}$ and all $i = 0,\dots,N-1$.

Proof. Consider the framework of Lemma 4.4.1 and let us define the set $\mathcal{V} \subset \mathbb{R}^n$ by

$$\mathcal{V} := \Big\{ -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega - u_i) \mid (\omega,i) \in \mathrm{U} \times \{0,\dots,N-1\} \Big\}.$$

From Lemma 4.4.1, for every $v \in \mathrm{Conv}(\mathcal{V})$, there exists $\xi_v \in \partial g(x(T))$ such that $\langle \xi_v, v\rangle_{\mathbb{R}^n} \geq 0$. Since $\partial g(x(T))$ is a nonempty compact convex set (see Proposition 1.4.2 in Chapter 1) and from the contrapositive of the universal separating vector theorem (Proposition 4.2.1), there exists a universal vector $\xi \in \partial g(x(T))$ such that $\langle \xi, v\rangle_{\mathbb{R}^n} \geq 0$ for all $v \in \mathcal{V}$, which gives exactly

$$\Big\langle \xi, -\int_{t_i}^{t_{i+1}} \nabla_2 \overline{H}(x(t),u(t),P(t),t)\,dt \times (\omega - u_i) \Big\rangle_{\mathbb{R}^n} \geq 0,$$

for all $\omega \in \mathrm{U}$ and $i = 0,\dots,N-1$. Then the end of the proof is similar to the discussion provided at the end of Section 4.4.2 in the smooth case (where g is continuously differentiable at $x(T)$), that is, by introducing the adjoint vector $p \in \mathrm{AC}([0,T],\mathbb{R}^n)$ defined by $p(t) := P(t) \times \xi$ for all $t \in [0,T]$. □

4.5 Proofs of Propositions 4.3.1 and 4.4.1

This section is dedicated to the proof of Propositions 4.3.1 and 4.4.1. In Section 4.5.1 we first present the sensitivity analysis of the state equation of Problem ($\mathrm{OPCP}_{\mathrm{ns}}$) under implicit spike variation of the control. The proof of Proposition 4.3.1 is provided in Section 4.5.2. Finally the very similar proof of Proposition 4.4.1 is given afterwards in Section 4.5.3.

4.5.1 Sensitivity analysis under implicit spike variation

In this section we focus on the Cauchy problem given by

$$\begin{cases} \dot{x}(t) = f(x(t),u(t),t), & \text{a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{CP}$$

for any $u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$. We refer to Section 2.4.1 in Chapter 2 for some notions and recalls on the sensitivity analysis of Problem (CP). We now introduce the following notion of controls admissible for globality.

Definition 4.5.1. *A control $u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ is said to be admissible for globality if the corresponding maximal solution $(x(\cdot,u),I(u))$ is global, that is, if $I(u) = [0,T]$. In what follows we denote by $\mathcal{AG} \subset \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ the set of all controls admissible for globality.*

We now give some recalls on implicit spike variation of a control $u \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ starting with the following lemma (see [Li & Yong 1995, Paragraph 3.2 page 143]).

Lemma 4.5.1. *Let $h \in \mathrm{L}^\infty([0,T],\mathbb{R}^n)$. Then, for all $0 < \eta \leq 1$, there exists a measurable set $\mathrm{Q}_\eta \subset [0,T]$, with a Lebesgue measure equal to ηT, such that*

$$\sup_{t\in[0,T]} \left\| \int_0^t \left(1 - \frac{1}{\eta}\mathbb{1}_{\mathrm{Q}_\eta}(s)\right) h(s)\,ds \right\|_{\mathbb{R}^n} \leq \eta,$$

where $\mathbb{1}_{\mathrm{Q}_\eta}$ denotes the indicator function of Q_η.

Let $u \in \mathcal{AG}$ and let $u' \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$. We introduce the implicit spike variation $u_\eta \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$ of u defined by

$$u_\eta(t) := \begin{cases} u'(t) & \text{over } \mathrm{Q}_\eta, \\ u(t) & \text{otherwise,} \end{cases}$$

for a.e. $t \in [0,T]$ and all $0 \leq \eta \leq 1$, where $\mathrm{Q}_0 := \emptyset$ and Q_η is given in Lemma 4.5.1 by considering $h(t) := f(x(t,u),u'(t),t) - f(x(t,u),u(t),t)$ for a.e. $t \in [0,T]$ when $0 < \eta \leq 1$. The next proposition can be found in [Li & Yong 1995, Lemma 4.3 page 152].

Proposition 4.5.1. *Let $u \in \mathcal{AG}$ and let $u' \in \mathrm{L}^\infty([0,T],\mathbb{R}^m)$. Then:*

(i) *there exists $0 < \eta_0 \leq 1$ such that $u_\eta \in \mathcal{AG}$ for all $0 \leq \eta \leq \eta_0$;*

(ii) *the map $\eta \in [0,\eta_0] \mapsto x(T,u_\eta) \in \mathbb{R}^n$ is differentiable at $\eta = 0$ and its derivative is equal to $w(T)$, where w is the the unique maximal solution (which is global) to the linear Cauchy problem given by*

$$\begin{cases} \dot{w}(t) = \nabla_1 f(x(t,u),u(t),t) \times w(t) + f(x(t,u),u'(t),t) - f(x(t,u),u(t),t), & \text{a.e. } t \in [0,T], \\ w(0) = 0_{\mathbb{R}^n}. \end{cases}$$

4.5.2 Proof of Proposition 4.3.1

We are now in a position to give the proof of Proposition 4.3.1. Let (x,u) be a solution to Problem ($\mathrm{OPCP}_{\mathrm{ns}}$). Let us define $P(t) := -\Phi(T,t)^\top$ for all $t \in [0,T]$, where $\Phi(\cdot,\cdot) : [0,T]^2 \to \mathbb{R}^{n\times n}$ stands for the state-transition matrix (or fundamental matrix solution) associated to the matrix function $\nabla_1 f(x,u,\cdot)$ (see [Sontag 1998, Appendix C.4] for more details on state-transition matrices). In particular the adjoint equation and the tranversality condition are satisfied. Let us prove the integral pseudo-maximization condition. For this purpose let $u' \in \mathrm{L}^\infty([0,T],\mathrm{U})$. From Proposition 4.5.1, there exists $\eta_0 > 0$ such that $x(T,u_\eta) = x(T) + \eta w(T) + \eta R(\eta)$ for all $0 \leq \eta \leq \eta_0$, where $R : [0,\eta_0] \to \mathbb{R}^n$ is a remainder term such that $\|R(\eta)\|_{\mathbb{R}^n}$ tends to 0 as η tends to 0. Then it holds that

$$\begin{aligned} g^\circ(x(T);w(T)) &= \limsup_{\substack{y\to x(T)\\ \eta\downarrow 0}} \frac{g(y+\eta w(T)) - g(y)}{\eta} \\ &\geq \limsup_{\eta\downarrow 0} \frac{g(x(T)+\eta R(\eta)+\eta w(T)) - g(x(T)+\eta R(\eta))}{\eta}, \\ &= \limsup_{\eta\downarrow 0} \left(\frac{g(x(T,u_\eta)) - g(x(T))}{\eta} - \frac{g(x(T)+\eta R(\eta)) - g(x(T))}{\eta} \right). \end{aligned}$$

Since g is locally Lipschitz, the second quotient tends to zero as η tends to 0. Moreover, since u_η is with values in U and from optimality of u, it is clear that the first quotient is nonnegative. We finally obtain that $g^\circ(x(T);w(T)) \geq 0$. Finally recalling the Duhamel formula given by

$$w(T) = \int_0^T \Phi(T,t) \times \Big(f(x(t),u'(t),t) - f(x(t),u(t),t) \Big)\, dt,$$

and since $P(t) = -\Phi(T,t)^\top$ for all $t \in [0,T]$, the integral pseudo-maximization condition is obtained. The proof is complete.

4.5.3 Proof of Proposition 4.4.1

We now give the proof of Proposition 4.4.1 which follows very similarly to the proof of Proposition 4.3.1 in Section 4.5.2. Let (x,u) be a solution to Problem ($\mathrm{OSCP}_{\mathrm{ns}}$). We define $P(t) := -\Phi(T,t)^\top$ for all $t \in [0,T]$, where $\Phi(\cdot,\cdot) : [0,T]^2 \to \mathbb{R}^{n\times n}$ stands for the state-transition matrix associated to the matrix function $\nabla_1 f(x,u,\cdot)$. It follows that the adjoint equation and the tranversality condition are satisfied. Let us prove the integral pseudo-nonpositive averaged Hamiltonian gradient condition. For this purpose, let us fix $u' \in \mathrm{PC}^{\mathbb{T}}([0,T],\mathrm{U})$. We consider the convex L^∞-perturbation $u_{u'}(\cdot,\alpha)$ for all $0 \leq \alpha \leq 1$ (see Proposition 2.4.3 in Chapter 2 for recalls). From Proposition 2.4.3, there exists $\alpha_0 > 0$ such that $x(T,u_{u'}(\cdot,\alpha)) = x(T,u)+\alpha w_{u'}(T)+\alpha R(\alpha)$ for all $0 \leq \alpha \leq \alpha_0$, where $R : [0,\alpha_0] \to \mathbb{R}^n$ is a remainder term such that $\|R(\alpha)\|_{\mathbb{R}^n}$ tends to 0 as α tends to 0. Then following the same arguments as in Section 4.5.2 we obtain that $g^\circ(x(T);w_{u'}(T)) \geq 0$. Finally recalling the Duhamel formula given by

$$w_{u'}(T) = \int_0^T \Phi(T,t) \times \nabla_2 f(x(t),u(t),t) \times (u'(t)-u(t))\,dt,$$

and since $P(t) = -\Phi(T,t)^\top$ for all $t \in [0,T]$, the integral pseudo-nonpositive averaged Hamiltonian gradient condition is obtained. The proof is complete.

CHAPTER 5

General conclusion

In this general conclusion, we begin by reviewing the outcome of the investigations conducted in this PhD thesis. Then we give several perspectives based on the results obtained, including some personal further research projects to be undertaken in the field of optimal sampled-data control theory.

Outcome of our investigations. The work presented in this dissertation has provided first-order necessary optimality conditions for optimal sampled-data control problems in the form of a Pontryagin maximum principle (in short, PMP) in diverse contexts. In some sense, our work is a continuation of the paper [Bourdin & Trélat 2016] in which the usual Hamiltonian maximization condition for optimal permanent control problems has to be replaced, when considering optimal sampled-data control problems, by a weaker condition known as a nonpositive averaged Hamiltonian gradient condition. In this PhD thesis, we have extended the PMP for optimal sampled-data control problems to several situations, including when:

- one can freely choose the sampling times;
- there are running inequality state constraints;
- the Mayer cost function is nonsmooth.

In Chapter 2, considering that the sampling times can be freely chosen, we obtained a new and additional necessary optimality condition in the PMP which happens to coincide with the continuity of the Hamiltonian function. Recall that the Hamiltonian function for optimal sampled-data control problems is not continuous in general when the sampling times are fixed. Our result asserts that the continuity of the Hamiltonian function is recovered in the case of optimal sampled-data controls with optimal sampling times. Finally we were able to implement a shooting method based on this new optimality condition in order to numerically determine the optimal sampling times in two linear-quadratic examples.

In Chapter 3, when considering running inequality state constraints, we obtained a PMP in which the adjoint vector is a solution to a Cauchy-Stieltjes problem defined by Borel measures associated to functions of bounded variation. We also found that, under certain general hypotheses, the admissible trajectories (associated to sampled-data controls) have a common behavior where they "bounce" against the boundary of the restricted state space, touching the state constraints at most at the sampling times. Taking advantage of this bouncing trajectory phenomenon, we were able to use the PMP derived in Chapter 3 to implement an indirect method to numerically solve some simple examples of optimal sampled-data control problems with running inequality state constraints.

In Chapter 4, we obtained a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions. Precisely, the Mayer cost function is taken to be (only) locally Lipschitz. We were interested in developing a proof which directly follows from the tools of nonsmooth analysis, and our investigation led us to consider the existence of a selection in the subdifferential of the nonsmooth Mayer cost function. In fact, we were able to assure the existence of this selection by establishing a more general result asserting the existence of a universal separating vector for a given compact convex set. From the application of this result, called universal separating vector theorem, we obtained a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions where the transversality conditon on the adjoint vector is given by an inclusion in the subdifferential of the nonsmooth Mayer cost function.

Perspectives. When considering optimal sampled-data control problems with free sampling times, the optimal cost can only remain the same or decrease as one increases the allowed number of sampling times. On the other hand, for practical applications, one is generally limited in the number of times one can change the value of the control and it can be cumbersome to administer a control which takes a large number of sampling times. This leads to a natural perspective: how to handle optimal sampled-data control problems when the cost includes a penalization on the number of sampling times? This question is challenging since it involves an integer variable to be optimized and as one does not a priori have an estimate on how the value of the optimal cost changes as one increases the allowed number of sampling times.

Another possible question one can ask for optimal sampled-data control problems with free sampling times is how to obtain necessary optimality conditions for more general dynamics, in particular which depend on the value of the sampling times t_i. As an example, this situation arises in medical applications such as the one considered in the recent work [Bakir *et al.* 2020] where one wants to find the optimal sampling times to send electrical impulses in order to stimulate a muscle and maximize the force generated.

In our numerical simulations, we observed that when, one reduces the sampling period, the values of the optimal sampled-data control seem to approach the values of the optimal permanent control. Thus a natural perspective to investigate is whether or not one can prove the convergence of optimal sampled-data controls to the optimal permanent control as the sampling period tends to zero. As a first step towards this direction, let us mention that the work [Bourdin & Trélat 2017] proves such convergence results in the context of unconstrained linear-quadratic problems.

To the best of our knowledge, optimality conditions have not been considered for optimal sampled-data control problems with nonsmooth dynamics described, for example, by differential inclusions or complementarity systems. Since nonsmooth dynamical systems are used to model several physical systems such as electrical circuits and mechanical systems with impacts (see [Acary & Brogliato 2008] and [Adly 2017] for some examples), one perspective is to study optimal sampled-data control problems with nonsmooth dynamics and determine whether such conditions can be given in the form of a PMP using the tools of nonsmooth analysis. Nonsmooth optimal permanent control problems have been considered extensively in the literature by Bettiol, Clarke, Frankowska, Vinter, etc. (see, e.g., [Bettiol & Frankowska 2007, Clarke & Vinter 1989, Clarke 2001, Frankowska & Mazzola 2011, Vinter 2010]) whose works provide a basis for future investigations in the case of sampled-data controls.

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Contributions in optimal sampled-data control theory with state constraints and nonsmooth data

Abstract: This dissertation is concerned with first-order necessary optimality conditions in the form of a Pontryagin maximum principle (in short, PMP) for optimal sampled-data control problems with free sampling times, running inequality state constraints and nonsmooth Mayer cost functions.

Chapter 1 is devoted to notations and basic framework needed to describe the optimal sampled-data control problems to be encountered in the manuscript.

In Chapter 2, considering that the sampling times can be freely chosen, we obtain an additional necessary optimality condition in the PMP called the *Hamiltonian continuity condition*. Recall that the Hamiltonian function, which describes the evolution of the Hamiltonian taking values of the optimal trajectory and of the optimal sampled-data control, is in general discontinuous when the sampling times are fixed. Our result proves that the continuity of the Hamiltonian function is recovered in the case of optimal sampled-data controls with optimal sampling times. Finally we implement a shooting method based on the Hamiltonian continuity condition in order to numerically determine the optimal sampling times in two linear-quadratic examples.

In Chapter 3, we obtain a PMP for optimal sampled-data control problems with running inequality state constraints. In particular we obtain that the adjoint vectors are solutions to Cauchy-Stieltjes problems defined by Borel measures associated to functions of bounded variation. Moreover, we find that, under certain general hypotheses, any admissible trajectory (associated to a sampled-data control) necessarily bounces on the running inequality state constraints. Taking advantage of this *bouncing trajectory phenomenon*, we are able to use the PMP to implement an indirect numerical method which we use to numerically solve some simple examples of optimal sampled-data control problems with running inequality state constraints.

In Chapter 4, we obtain a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions. Our proof directly follows from the tools of nonsmooth analysis and does not involve any regularization technique. We determine the existence of a selection in the subdifferential of the nonsmooth Mayer cost function by establishing a more general result asserting the existence a universal separating vector for a given compact convex set. From the application of this result, which is called *universal separating vector theorem*, we obtain a PMP for optimal sampled-data control problems with nonsmooth Mayer cost functions where the transversality conditon on the adjoint vector is given by an inclusion in the subdifferential of the nonsmooth Mayer cost function.

To obtain the optimality conditions in the form of a PMP, we use different techniques of perturbations of the optimal control. In order to handle the state constraints, we penalize the distance to them in a corresponding cost functional and then apply the Ekeland variational principle. In particular, we invoke some results on renorming Banach spaces in order to ensure the regularity of distance functions in the infinite-dimensional context. Finally we use standard notions in nonsmooth analysis such as the Clarke generalized directional derivative and the Clarke subdifferential to study optimal sampled-data control problems with nonsmooth Mayer cost functions.

Keywords: Optimal control, sampled-data control, Pontryagin maximum principle, Ekeland variational principle, optimal sampling times, Hamiltonian continuity, running state constraints, Cauchy-Stieltjes problems, nonsmooth analysis, indirect numerical methods, shooting methods.

Contributions en théorie du contrôle échantillonné optimal avec contraintes d'état et données non lisses

Résumé : L'objectif de cette thèse est d'obtenir des conditions nécessaires d'optimalité du premier ordre sous la forme d'un principe du maximum de Pontryagin (en abrégé PMP) pour des problèmes de contrôle échantillonné optimal avec temps d'échantillonnage libres, contraintes d'état et coûts de Mayer non lisses.

Le Chapitre 1 est consacré aux notations et espaces fonctionnels utiles pour décrire les problèmes de contrôle échantillonné optimal qui seront rencontrés dans le manuscrit.

Dans le Chapitre 2, nous obtenons une condition nécessaire d'optimalité lorsque les temps d'échantillonnage peuvent être choisis librement qui est appelée *condition de continuité de la fonction Hamiltonienne*. Rappelons que la fonction Hamiltonienne qui décrit l'évolution du Hamiltonien avec les valeurs de la trajectoire optimale et du contrôle échantillonné optimal est, en général, discontinue quand les temps d'échantillonnage sont fixés. Notre résultat démontre que la continuité de la fonction Hamiltonienne est retrouvée pour les contrôles échantillonnés optimaux avec temps d'échantillonnage optimaux. Pour terminer, nous implémentons une méthode de tir basée sur la condition de continuité de la fonction Hamiltonienne pour déterminer numériquement les temps d'échantillonnage optimaux dans deux exemples linéaires-quadratiques.

Dans le Chapitre 3, nous obtenons un PMP pour des problèmes de contrôle échantillonné optimal avec contraintes d'état. Nous obtenons que les vecteurs adjoints sont solutions de problèmes de Cauchy-Stieltjes définis par des mesures de Borel associées à des fonctions à variation bornée. De plus, nous trouvons que, sous quelques hypothèses assez générales, toute trajectoire admissible (associée à un contrôle échantillonné) rebondit nécessairement sur les contraintes d'état. Nous exploitons ce *phénomène de trajectoires rebondissantes* pour implémenter une méthode indirecte qu'on utilise pour résoudre numériquement quelques exemples simples de problèmes de contrôle échantillonné optimal avec contraintes d'état.

Dans le Chapitre 4, nous obtenons un PMP pour des problèmes de contrôle échantillonné optimal avec coûts de Mayer non lisses. Notre preuve est uniquement basée sur les outils de l'analyse non lisse et n'utilise aucune technique de régularisation. Nous déterminons l'existence d'une sélection dans le sous-différentiel de la fonction de coût de Mayer non lisse en établissant un résultat plus général sur l'existence d'un vecteur séparant universel pour les ensembles convexes compacts. En appliquant ce résultat, appelé *théorème de vecteur séparant universel*, nous obtenons un PMP pour des problèmes de contrôle échantillonné optimal avec coûts de Mayer non lisses où la condition de transversalité sur le vecteur adjoint est donnée par une inclusion dans le sous-différentiel de la fonction de coût de Mayer non lisse.

Pour obtenir les conditions d'optimalité sous la forme d'un PMP, nous utilisons différentes techniques de perturbation sur le contrôle optimal. Pour traiter les contraintes d'état, nous pénalisons la distance à ces contraintes dans une fonctionelle et nous appliquons le principe variationnel d'Ekeland. En particulier, nous invoquons des résultats sur la renormalisation des espaces de Banach pour assurer la régularité de la fonction distance dans les contextes de dimension infinie. Enfin nous utilisons des notions standards de l'analyse non lisse, telles que les dérivées directionnelles généralisées de Clarke et le sous-différentiel de Clarke, pour étudier les problèmes de contrôle échantillonné optimal avec coûts de Mayer non lisses.

Mots clés : Contrôle optimal, contrôle échantillonné, principe du maximum de Pontryagin, principe variationnel d'Ekeland, temps d'échantillonnage optimaux, continuité de la fonction Hamiltonienne, contraintes d'état, problèmes de Cauchy-Stieltjes, analyse non lisse, méthodes numériques indirectes, méthodes de tir.