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Titre:
Réalisation de métriques $CAT(k)$ sur des surfaces fermées dans des espaces lorentziens de courbure constante

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Chapter 1

Introduction

1.1 Motivation : les métriques à courbure minorée sur les surfaces

Le problème des immersions isométriques est un problème de langue date. L'une des questions les plus célèbres qui ont été posées dans ce sujet était celle que H. Weyl a posé en 1916: Peut-on réaliser un plongement isométrique d'une métrique à courbure positive sur la sphère dans $\mathbb{R}^3$?

Ce problème a été résolu indépendamment par Nirenberg et Pogorelov dans les années 50 [Nir53].

A.D. Alexandrov a généralisé la question de Weyl, il a commencé à étudier pour la première fois les espace métriques de courbure positive (métrique non-régulière) et cela est venu en essayant de décrire la métrique induite sur le bord des corps convexes dans $\mathbb{R}^3$. Il a prouvé ce premier théorème célèbre.

Theorem 1.1.1 (Alexandrov)**.** *Soit d une métrique de courbure positive sur une sphère de dimension 2. Alors d est isométrique au bord d'un corps convexe dans $\mathbb{R}^3$.*

Donnons quelques détails. Appelons surface convexe le bord d'un convexe compact de l'espace euclidien. Alexandrov a caractérisé la distance intrinsèque induite par le fait d'être de courbure ≥ 0. Il a ensuite généralisé la notion de courbure ≥ 0 par le fait de comparer les triangles géodesiques de l'espace concerné avec des triangles dans le plan euclidien appelés " triangle de comparaison" (i.e. un triangle avec les mêmes longueurs de côtés). Une définition plus précise est que, pour tout point, il existe un voisinage tel que pour tout triangle géodésique dans ce voisinage, le triangle de comparaison dans le plan euclidien a des angles plus petits [Ale06]. Il a en fait montré que ces métriques caractérisent exactement les métriques sur les surfaces convexes. Plus précisément, pour toute métrique à courbure ≥ 0 sur la sphère, il existe une surface convexe isométrique dans l'espace euclidien. La preuve consiste à d'abord résoudre le problème pour les polyèdres, puis procéder par approximation.

Soit S une surface compacte. On définit facilement une métrique à courbure $\geq k$ sur S comme une distance intrinsèque géodésique, qui a la même propriété que au-dessus, mais où le triangle de comparaison est dans un plan de courbure k. Par une homothétie, il suffit de se ramener aux cas $k = -1, 0, 1$.

Une question naturelle est donc de savoir si ces métriques sont isométriques à la métrique induite sur des surfaces convexes dans des espaces de dimension 3 de courbure constante. Effectivement, on a le résultat suivant:

Theorem 1.1.2. *Toute métrique à courbure $\geq k$ sur une surface compacte est isométrique à la métrique intrinsèque induite sur une surface convexe dans un espace riemannien de courbure k.*

Cet énoncé est fait de la juxtapositions de différents cas. Voir l'introduction de [FIV16]. Nous n'entrons pas dans les détails car nous allons considérer des situations similaires.

1.2 Premier résultat de la thèse

La définition de métrique à courbure $\leq k$ sur une surface est similaire à celle de courbure $\geq k$, mais où les angles du triangles de comparaison sont cette fois plus grands. Les métriques à courbure $\leq k$, en particulier ≤ 0, sont devenus des objets fondamentaux en géométrie (voir par exemple [BH99]). Ce sont en fait des objets très généraux, ici nous supposons que la topologie de la métrique est celle de la surface.

Dans tous ce qui suit on va noter $M_k^{+,3}$ l'espace modèle riemannien de dimension 3 et de courbure constante $k \in \{-1, 0, 1\}$: respectivement l'espace hyperbolique, l'espace euclidien, et la sphère de dimension 3.

Introduisons aussi les analogues lorentziens : l'espace de Minkowski $\mathbb{R}^{3,1}$ (de courbure 0), l'espace de Sitter dS (de courbure 1) et l'espace Anti-de sitter AdS^3 (de courbure -1). On note ces espaces $M_k^{-,3}$ selon la courbure $k \in \{-1, 0, 1\}$.

Dans un espace riemannien, la courbure extrinsèque d'une surface (lisse) convexe est positive. En utilisant la formule de Gauss il est facile de voir qu'une surface strictement convexe dans $M_k^{+,3}$ est de courbure sectionnelle $\geq k$. Par ailleurs la même formule dans un espace lorentzien $M_k^{-,3}$ indique bien que la courbure sectionnelle d'une surface convexe de type espace est $\leq k$. L'idée générale est donc d'essayer de plonger isométriquement des surfaces compactes de courbure minorée dans un espace lorentzien de courbure sectionnelle constante.

Les deux premiers résultats dans ce sens ont concerné le cas de la sphère, pour le cas polyédral et pour le cas lisse respectivement.

Theorem 1.2.1 ([Riv86] et [HR93]). *Soit d une métrique sphérique à singularités coniques sur la sphère $\mathbb{S}^2$ tel que:*

1. Tous les angles coniques de $(\mathbb{S}^2, d)$ sont plus grand que 2π.

2. Toutes les géodésiques fermés contractiles dans $(\mathbb{S}^2, d)$ sont de longueur supérieur à 2π

Alors il existe une surface convexe polyédrale de type espace (unique modulo les isométries) dans l'espace de Sitter telle que la métrique induite est isométriques à $(\mathbb{S}^2, d)$.

Theorem 1.2.2 ([Sch96]). *Soit g une métrique Riemannienne de courbure < 1 sur la sphère. Supposons qu'il n'existe pas de géodésique contractile de longueur $\leq 2\pi$. Alors il existe une surface lisse convexe de type espace dans l'espace de Sitter isométrique à g.*

Notons que, hormis son intérêt intrinsèque en physique théorique, il y a une dualité entre l'espace hyperbolique et l'espace de Sitter. Elle peut être interprétée en utilisant le modèle de l'hyperboloïde de l'espace hyperbolique comme hypersphère dans $\mathbb{R}^{4,1}$. Cette dualité associe à chaque point de $\mathbb{H}^3$ un plan totalement géodésique de type espace dans l'espace de Sitter. Elle transforme une surface convexe de type espace de l'espace de Sitter en une surface convexe dans $\mathbb{H}^3$. Une isométrie de l'espace de Sitter qui fixe un point correspond à une isométrie de $\mathbb{H}^3$ qui fixe un plan.

Pour le genre > 1, on a le premier résultat suivant, qui introduit les *surfaces convexes fuchsiennes*: ce sont des surfaces convexes de type espace invariantes sous l'action d'un groupe d'isométrie qui fixent un point et agissent de manière cocompacte sur une surface à distance constante de ce point. Des détails sont donnés dans le corps du texte.

Theorem 1.2.3 ([LS00]).

1. *Soit g une métrique riemannienne de courbure < 1 sur une surface S de genre > 1. Supposons qu'il n'existe pas de géodésique contractile de longueur $\leq 2\pi$. Alors il existe une surface convexe fuchsienne dans l'espace de Sitter dont le quotient est isométrique à (S, g).*

2. *Soit g une métrique riemannienne de courbure < 0 sur une surface compacte de genre > 1. Alors il existe une surface fuchsienne convexe dans l'espace de Minkowski dont le quotient est isométrique à (S, g).*

3. *Soit g une métrique riemannienne de courbure* < -1 *sur une surface compacte de genre* > 1. *Alors il existe une surface fuchsienne convexe l'espace Anti-de Sitter dont le quotient est isométrique à* (S, d).

L'analogue polyédral est énoncé dans [Fil11] mais nous n'en auront pas besoin. Dans le cas polyédral plat, une preuve variationnelle a été donnée dans [Bru17]. À partir du cas lisse plat, par approximation, la généralisation du théorème d'Alexandrov au cas lorentzien a été obtenue :

Theorem 1.2.4 ([FS18]). *Soit* (S, d) *une métrique à courbure* ≤ 0 *sur une surface compacte de genre* > 1.

Alors il existe une surface fuchsienne convexe dans l'espace de Minkowski telle que la métrique induite sur le quotient est isométrique à (S, d).

On se pose alors la question naturelle suivante :

Conjecture 1.2.5. *Soit* (S, d) *une surface compacte avec une métrique à courbure* $\leq k$*, qui n'ait pas de géodésiques contractiles de longueur* $\leq 2\pi$. *Alors il existe un plongement isométrique convexe de* (S, d) *dans un espace lorentzien de dimension* 3 *de courbure* k.

Notons que par le théorème de Hadamard, l'hypothèse sur les géodésiques contractiles ne concerne que le cas $k = 1$.

Pour le cas $k = 0$, le théorème 1.2.4 répond à la question (pour le cas du tore plat, il se plonge trivialement dans $\mathbb{T}^2 \times \mathbb{R}$ munit de la métrique lorentzienne produit). En effet, le groupe fuchsien du théorème 1.2.4 agit sur le cône futur de l'origine, qui contient la surface fuchsienne. La variété en question dans la conjecture ci-dessus est le quotient du cône par le groupe.

Le résultat principal de cette thèse est le résultat suivant, qui correspond au cas $k = -1$ de la conjecture ci-dessus.

Theorem 1.2.6. *Soit* (S, d) *une métrique à courbure* ≤ -1 *sur une surface compacte de genre* > 1. *Alors il existe une surface fuchsienne convexe dans l'espace de Anti-de Sitter telle que la métrique induite sur le quotient est isométrique à* (S, d).

On expliquera dans la suite pourquoi le cas $k = 1$ est plus difficile à obtenir.

1.3 Plan de la thèse et méthode employée

Le premier chapitre est dédié à une introduction rapide à la géométrie semi-Riemannienne (Lorentzienne) dans lequel on introduit les espaces qu'on a besoin dans cette thèse, l'espace anti-de Sitter, hyperbolique et de Sitter en essayant de relever les propriétés demandées dans la résolution des problèmes considérés dans ce manuscrit.

Dans le deuxieme chapitre on montre qu'une métrique à courbure $\leq k$ est limite uniforme de métriques polyedrales. En ayant cette approximation, on cosidère les cas suivants:

- $k = -1$: La preuve du théorème 1.2.6 a une stratégie similaire à celle du théorème 1.2.4, pour cela:

  1. L'approximation sera par les métriques lisses, donc on montre de plus que ces métriques poléyèdrales obtenues admettent une approximation par des métriques lisses (section 3.4) et donc, on obtient qu'une métrique de courbure ≤ -1 est limite uniforme de métriques riemanniennes à courbure < -1.
  2. On utilise le théorème 1.2.3 pour obtenir une suite de surface convexes fuchsiennes lisses.

     Grâce aux propriétés des surfaces convexes, et comme les métriques induites convergent, on montre que, quitte à extraire une sous-suite, ces surfaces convergent, ainsi que la suite de groupes fuchsiens associés.

     On conclut en montrant que la suite des métriques induites correspondantes converge.

  La preuve du théoreème 1.2.6 occupe donc la section 4.1 de la présente thèse.

- $k = 1$: Pour avoir une preuve de la conjecture 1.2.5, il reste à considérer le cas des métriques à courbure ≤ 1. Ces métriques existent pour tout type topologique de surfaces compactes. Pour le tore, on voudrait réaliser les métriques par des surfaces convexes *paraboliques*, c'est-à-dire invariantes par un groupe qui fixe une horosphère dans l'espace de de Sitter. Pour le genre supérieur à 1, la réalisation sera comme d'habitudes par des surfaces convexes fuchsiennes.

  Une différence avec les cas $k = -1, 0$ pour $k = 1$, c'est que l'argument du 1 ci-dessus n'est pas évident. C'est peut-être un résultat folklorique, mais nous sommes incapables de trouver une référence. La difficulté réside exactement dans la possibilité d'avoir une triangulation convenable, ou les sommets sont exactement les singularités de la métrique. Les surfaces ayant cette propriété s'appelle "les surface de courbure intégrale bornée (BIC)". Cette étape est evidente à prouver pour le cas $k = -1$ mais pas pour $k = 1$. Ceci est discuté en détail dans la section 3.2.

Ensuite, les suites de surfaces convexes dans de Sitter peuvent présenter un type original de dégénérescence, qui peut se traduire de manière intrinsèque par la condition sur la longueur des géodésique. Enfin, nous discuton rapidement la réalisation de métrique dans le cas $k = 1$, la preuve n'a pas encore été complétée, on va parler de tous ça dans la section 4.2.

1.4 Références

Comme les preuves de ces théorèmes se font par approximation lisse ou polyédrale, indiquons des références dans les tableaux suivants.

Métriques			Polyèdre		Références
genre	courbure	Singularité conique	Type	Espace ambiant	
0	0	positive	compact	Euclidien	Alexandrov
0	1	positive	compact	Sphère	Alexandrov
0	-1	positive	compact	Hyperbolique	Alexandrov
0	1	négative	compact	de Sitter	[HR93]
1	0	positive	parabolique	Euclidien	Cas trivial
1	0	négative	parabolique	Minkowski	Cas trivial
1	-1	positive	parabolique	Hyperbolique	[FI09]
1	1	négative	parabolique	de Sitter	[FI11]
> 1	-1	positive	fuchsienne	Hyperbolique	[Fil07]
> 1	1	negative	fuchsienne	de Sitter	[Sch]
> 1	0	negative	fuchsienne	Minkowski	[Fil11]
> 1	-1	negative	fuchsienne	Anti-de Sitter	[Fil11]

Table 1.1: Le cas polyédral

Métriques		Lisse		Références
genre	courbure sectionnelle	Type	Espace ambiant	
0	> 0	compact	Euclidien	Nirenberg-Pogorelov
0	> 1	compact	Sphère	Alexandrov
0	> -1	compact	Hyperbolique	Alexandrov
0	< 1	compact	de Sitter	[Sch96]
1	> 0	parabolique	Euclidien	Cas trivial
1	< 0	parabolique	Minkowski	Cas trivial
1	> -1	parabolique	Hyperbolique	[Sch06]
1	< 1	parabolique	de Sitter	question ouverte
> 1	> -1	fuchsienne	Hyperbolique	Conséquence de [Sch06]
> 1	< 0	fuchsienne	Minkowski	[LS00]
> 1	< -1	fuchsienne	Anti-de Sitter	[LS00]
> 1	< 1	fuchsienne	de Sitter	[LS00]

Table 1.2: Le cas lisse

Métriques		Quelconque		Références
genre	courbure	Type	Espace ambiant	
0	≥ 0	compact	Euclidien	Alexandrov
0	≥ -1	compact	Hyperbolique	Alexandrov
0	≥ 1	compact	Sphère	Alexandrov
0	≤ 1	compact	de Sitter	en cours de résolution (H. LABENI)
1	≥ 0	parabolique	Euclidien	Cas trivial
1	≤ 0	parabolique	Minkowski	Cas trivial
1	≥ -1	parabolique	Hyperbolique	[FIV16]
1	≤ 1	parabolique	de Sitter	en cours de résolution (H. LABENI)
> 1	≤ 0	fuchsienne	Minkowski	[FS18]
> 1	≤ -1	fuchsienne	Anti-de Sitter	[LAB20](H. LABENI)
> 1	≥ -1	fuchsienne	Hyperbolique	Conséquence de [Slu18]
> 1	≤ 1	fuchsienne	de Sitter	en cours de résolution (H. LABENI)

Table 1.3: Le cas général

1.5 D'autres questions ouvertes

Dans le cas d'une surface lisse S de genre > 1 avec une métrique riemannienne g de courbure > -1, on peut montrer, dans l'esprit du théorème 1.1.2, qu'il existe une surface convexe fuchsienne lisse dont le quotient est isométrique à (S, g).

Une généralisation est le résultat suivant prouvé par Labourie 1992 [Lab92] (pour l'existence) et Schlenker [Sch06] (pour l'existence et l'unicité) :

Theorem 1.5.1. *Soient g, h deux métriques riemanniennes de courbure sectionnelle > -1 sur une surface compacte de genre > 1. Alors il existe une unique variété à bord quasi-fuchsienne hyperbolique et tel que le bord est convexe et isométrique à g et h.*

On précise que le théorème 1.5.1 a été démontré dans le cas ou le bord est lisse et il en sera de même pour le théorème 1.5.3 ci-dessous.

Avec un argument d'approximation, Slutskiy a démontré le théorème suivant :

Theorem 1.5.2 ([Slu18]). *Soient g, h deux métriques à courbure ≥ -1 sur une surface compacte de genre > 1. Alors il existe une variété à bord quasi-fuchsienne hyperbolique et tel que le bord est convexe et isométrique à g et h.*

Un résultat récent de A. Tamburelli [Tam18] qui est l'analogue du théorème 1.5.1 dit que :

Theorem 1.5.3. *Soient g, h deux métriques riemanniennes de courbure sectionnelle < -1 sur une surface compact de genre > 1. Alors il existe une variété à bord globalement hyperbolique Anti-de Sitter et telle que le bord est convexe et isométrique à g et h.*

Maintenant passant au cas général (quand le bord n'est pas lisse) la question qui se pose naturellement est la suivante :

Question 1.5.4. *Soient g, h deux métriques de courbure ≤ -1 sur une surface compacte S de genre > 1. Existe-il une variété à bord globalement hyperbolique Anti-de Sitter tel que le bord est convexe, de type espace et isométrique à g et h ?*

Notons que le théorème 1.2.6 est un cas particulier de la question dessus dans le cas ou $g = h$, on va expliquer cela dans la section 4.1.6.

Chapter 2

Preliminaries

Semi-Riemannian geometry [1] [O'N83] is the study of manifolds endowed with metric tensors of arbitrary signature. The most important special cases of the semi-Riemannian geometry are the Riemannian geometry [dC16] where the metric is positive definite and the Lorentzian geometry [BEE96] which has a very important place in physic and especially in the theory of relativity [SW77, Syn56] where the space-time is mathematically studied as the Minkowski space. The latter plays also an important role in Einstein's theory of relativity where the gravitation is no longer treated as a force, but as a deformation of the space-time.

In this chapter, we introduce the basic geometric concepts. To start with, semi-Riemmannian metric tensors, semi-Riemannian manifolds, connexion, curvature.. etc. Then we pass to the geometry of constant sectional curvature introducing hyper-quadrics and the model spaces we are using in this work (Minkowski, hyperbolic, de Sitter and anti-de Sitter space (section 2.3)) trying to highlight the most important properties and features that we will need in this thesis.

2.1 Semi-Riemannian geometry

Definition 2.1.1. *A Riemannian metric g on M is a $(0,2)$ tensor field of M, which is symmetric, non degenerate and positive definite .*

1. In general, the name pseudo-Riemannian geometry is also used instead of semi-Riemannian geometry.

A Riemannian manifold is a couple (M, g) where M is a differentiable manifold and g is a Riemannian metric.

Semi-Riemannian manifolds are generalization of Riemannian manifolds, where the metric is not necessarily positive definite. So, a semi-Riemannian metric g on M is a $(0, 2)$ tensor field of M, which is symmetric and non degenerate.

Definition 2.1.2. *A semi-Riemannian manifold is a couple (M, g) where M is a manifold and g is a semi-Riemannian metric.*

If the metric g is of constant signature $(-, +, .., +)$, then g is called a Lorentzian metric, and (M, g) is called a Lorentzian manifold.

A tangent vector v to M is said to be space-like, time-like or light-like following its pseudo-norm $g(v, v)$.

$$\text{It is called:} \begin{cases} \text{space-like} & if\ g(v,v) > 0. \\ \text{time-like} & if\ g(v,v) < 0. \\ \text{light-like} & if\ g(v,v) = 0. \end{cases}$$

A differentiable curve $\gamma : [0, 1] \to M$ is space-like, (resp. time-like, light-like) if its tangent vector is space-like (resp. time-like, light-like) at every point. The length of a differentiable curve γ will be

$$\text{length}(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))} \qquad \text{if } \gamma \text{ is a space-like curve.}$$

$$\text{length}(\gamma) = \int_0^1 \sqrt{-g(\gamma'(t), \gamma'(t))} \qquad \text{if } \gamma \text{ is a time-like curve.}$$

Orientation

Let p be a point on the manifold M, and T_pM the tangent space at p to M. The set of light-like vectors together with null vectors divides T_pM in two regions:

The first region is constituted by two convex open cones formed by time-like tangent vectors, one opposite to the other. The second region is constituted by space-like tangent vectors (Figure 2.1). It is then easy to remark that the set of time-like vectors in the tangent bundle TM is either connected or formed by two connected components.

We say that M is time-orientable if the set of time-like vectors is formed by two components. In this case, a time orientation is simply the choice of one of these components. Now, let's choose one component and denote it C^+: A non-space-like tangent vector is future-directed (resp. past directed) if it is not the null vector and it lies (resp. it does not lie) in the chosen component C^+.

Let $p \in M$, the future of p (resp. the past) is the subset denoted by $I^+(p)$ (resp. $I^-(p)$) of M formed by the points of M which are connected to p by a future directed time-like (or light-like) curve.

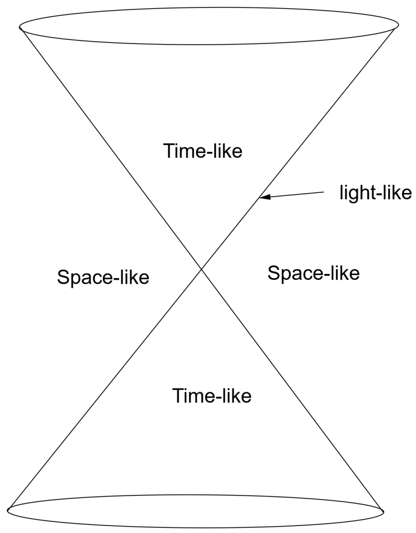

Figure 2.1: The time cone

2.1.1 Connection

There are no changes required in extending the notion of connexion and curvature to semi-Riemannian manifolds because the proofs uses only the non-degeneracy of the metric. Let's start by the following definitions.

A linear connection on a semi-Riemannian manifold M is a map :

2. $\Gamma(TM)$: space of vector fields.
3. $C^\infty(M)$: set of differentiable functions on M.

$$\nabla \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_X Y.$$

such that for all $X, Y, Z \in \Gamma(TM)$.[2] and $f \in C^\infty(M)$.[3]., we have:

- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,
- $\nabla_{X+fY} Z = \nabla_X Z + f\nabla_Y Z$.

As in the Riemannian case, on a semi-Riemannian manifold M there is a unique linear connection which is symmetric and compatible with the metric g, it is the Levi-Civita connection. We define it to be the unique connection on TM verifying, for all $X, Y, Z \in \Gamma(TM)$:

- $\nabla_X Y = \nabla_Y X - [X, Y]$, ($\nabla$ is said to be without torsion).
- $X(g(Y, X)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, ($\nabla$ is said to be compatible with the metric g).

The Levi-Civita connection determines the Riemann curvature tensor R. It is defined as follows

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We give without demonstration some properties of the Riemann tensor. If $X, Y, Z, W \in \Gamma(TM)$ then

- $R(X, Y)Z = -R(Y, X)Z$,
- $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

2.1.2 Curvature

From the Riemann curvature tensor R, we can construct different notions of curvature. Here we introduce the notion of "sectional curvature", then we introduce quickly other notions of curvature which are less important in our thesis.

The sectional curvature of a plane π in TM is defined by $sec(\pi) = R(X, Y, X, Y)$ where X, Y form an orthonormal basis of π. A manifold is of sectional curvature $\geq k$ (resp. $< k$) if for every plane $sec(\pi) \geq k$ (resp. $< k$). Let

$$Q(X, Y) = g(X, X)g(Y, Y) - g(X, Y)^2$$

A tangent plane π is non-degenerate if and only if $Q(X, Y) \neq 0$

Lemma 2.1.3. *Let π be a non-degenerate tangent plane to M at p. The number*

$$K(X, Y) = \frac{g(R(X, Y)Y, X)}{Q(X, Y)}$$

is independent of the choice of basis X, Y for π, and is called the "sectional curvature" $K(\pi)$ of π.

Now, let S be an embedded surface in a semi-Riemannian manifold M and $x \in S$. The Gauss map is defined to be the map that associates to each point p of S the point $N(p) \in \mathbb{S}^2$, where N is the unit normal vector to the surface S. The operator of Weingarten (also called the shape operator) denoted by B is the map defined as follows

$$\begin{aligned} B \colon T_x S &\to T_x S \\ X &\mapsto -\nabla_X N. \end{aligned}$$

Principal, mean and Gauss curvature: the principal curvatures of the surface S are the eigenvalues of B. The mean curvature is the half of the trace of B and the Gauss curvature K_{Gauss} is the determinant of B (i.e. the product of principal curvatures).

Gauss formula: Let N be a hypersurface in a manifold M there is a well known formula ([O'N83], Corollary 20, page 107) relating the sectional curvature of the manifold M at a point x (denoted by K^M_{sec}) and the Gaussian curvature of the hypersurface N (denoted by K^N_{Gauss}). It is given as follows:

If the normal at x is space-like then:

$$K_{sec}^{N} = K_{sec}^{M} + K_{Gauss}^{N}$$

If the normal at x is time-like then:

$$K_{sec}^{N} = K_{sec}^{M} - K_{Gauss}^{N}$$

2.1.3 Geodesics

Geodesics are a generalization of the Euclidean notion of "straight lines". They become an essential ingredient of semi-Reimannian geometry, as they are for Riemannian one. Although there are usually a lot of similitude of notions between the two geometries, but it is not the case for geodesics. In semi-Riemannian case there are other difficulties and complications, we will see some of them in the sequel.

Parallel transport

Let ∇ be a linear connection on M, $\gamma : [0,1] \longrightarrow M$ a curve and Y a vector field along γ (i.e. a smooth mapping $Y : [0,1] \longrightarrow TM$ such that $Y(t) \in T_{\gamma(t)}M$ for each $t \in [0,1]$). For $t_0 \in [0,1]$ we may locally extend Y to a smooth vector field defined on a neighborhood of $\gamma(t_0)$. Then we may consider the vector field $\nabla_{\gamma'}Y(t)$ along γ. The preceding arguments show that this vector field along γ is independent of the local extension, and consequently $\nabla_{\gamma'}Y(= Y')$ is well defined. A vector field Y along γ is said to move by parallel translation along γ if it satisfies: $\nabla_{\gamma'}Y(t) = 0$ for all $t \in [0,1]$.

Definition 2.1.4. *A geodesic $\gamma : [0,1] \longrightarrow M$ is a smooth curve of M such that the tangent vector γ' moves by parallel translation along γ. In other words, γ is a geodesic if*

$$\nabla_{\gamma'}\gamma' = 0.$$

As we saw before the square of the norm of a vector can be negative then the length of a curve can be imaginary. A remarkable difference in semi-Riemannian geometry, is that the tensor metric does not induce a distance on M (because it is not positive definite) as in Riemannian setting, in this case there is no meaning to talk about the notion of “minimization of distances" for the geodesics (a geodesic can even be maximizing). In this case we will call "distance between two points" joined by a time-like geodesic the module of the length of the geodesic.

Exponential function

As in the Riemannian manifolds, the exponential map $exp_p : T_pM \longrightarrow M$ is defined as follows: given $v \in T_pM$, let $\gamma_v(t)$ denote the unique geodesic in M with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Then the exponential map $exp_p(v)$ of v is given by $exp_p(v) = \gamma_v(1)$ provided $\gamma_v(1)$ is defined.

Definition 2.1.5. *We say that M is geodesically complete if every geodesic is defined for all times, (in other words) the exponential map is defined everywhere.*

2.1.4 Isometries

Let M_1 and M_2 be two semi-Riemannian manifolds with metric tensors g_1 and g_2. An isometry between M_1 and M_2 is a diffeomorphism $\phi : M_1 \longrightarrow M_2$ that preserves metric tensors i.e.

$$g_1(X, Y) = g_2(df(X), df(Y)), \quad \forall X, Y \in \Gamma(TM)$$

this means that g_1 is the pull back of g_2 by ϕ and we write

$$\phi^*(g_2) = g_1$$

Example 2.1.6. *The identity map $Id : (M_1, g) \longrightarrow (M_1, g)$ is an isometry (Id is a diffeomorphism and $Id^* = Id$).*

Isometric immersions: An immersion of M_1 into M_2 is just a smooth mapping $\phi : M_1 \longrightarrow M_2$ such that $d\phi_x$ is injective for all $x \in M_1$. An isometric immersion of M_1 into M_2 is a smooth immersion such that

$$\phi^*(g_2) = g_1$$

An isometric embedding is an injective isometric immersion.

2.2 Hyperquadrics

In this section the concept of hyperquadrics is introduced, roughly speaking they are a family to which the sphere and the hyperbolic space belong. We give also some of their useful properties, which will be used further. In the following we denote by $\mathbb{R}^n_p$ the space $\mathbb{R}^n$ endowed with the quadratic form

$$\langle x, x\rangle^n_p = -x_1^2 - ... - x_p^2 + x_{p+1}^2 + ... + x_n^2$$

Let's first introduce the familiar special case of the hyperquadrics, the sphere.

Sphere

The round sphere $\mathbb{S}^2$ can be seen as the set of unit vectors of the Euclidean space. The following well known points about the sphere can be easily checked [O'N83] .

- $\mathbb{S}^2$ is a smooth surface of sectional curvature equal to 1 at every point.
- *Geodesics* of $\mathbb{S}^2$ are given by intersecting $\mathbb{S}^2$ with linear planes, hence they are great circles. Now, given a tangent vector v to the sphere in a point x, the curve defined by
$$t \mapsto (\sin t)v + (\cos t)x$$
for $t \in [0, 2\pi[$ is a geodesic, and by the way, all geodesics of the sphere can be written in this form.
- *Isometries* of $\mathbb{S}^2$ are isometries of the Euclidean space which preserve the Euclidean scalar product. One can verify that this is the orthogonal group $O(3)$ (we recall that $O(3) = \{M \in Mat_3(\mathbb{R})|^t MM = 1\}$ where Mat_3 is the set of (3×3)-dimensional matrices))
- The distance between two points x, y on the sphere is the length of the (smallest) geodesic between x and y and by a direct computation, one can easily check that this is equal to the angle between the vectors x and y in $\mathbb{R}^3$, also we have the following relation between distance on the sphere and the Euclidean scalar product
$$\cos d(x, y) = \langle x, y\rangle$$

Pseudo-spheres and pseudo-hyperbolics

We introduce the pseudo-sphere of radius 1 in $\mathbb{R}^{n+1}_p$ as the hyperquadric defined as follows

$$\mathbb{S}^n_p = \{x \in \mathbb{R}^{n+1}_p, \langle x, x\rangle^{n+1}_p = 1\}$$

The pseudo-hyperbolic space of radius 1 in $\mathbb{R}^{n+1}_{v+1}$ is defined as follows

$$\mathbb{H}^n_p = \{x \in \mathbb{R}^{n+1}_{p+1}, \langle x, x\rangle^{n+1}_{p+1} = -1\}$$

We give one of the important basic properties in the following Lemma

Lemma 2.2.1. *The pseudo-sphere $\mathbb{S}^n_p$ is diffeomorphic to $\mathbb{S}^{n-p} \times \mathbb{R}^p$; the pseudo-hyperbolic space $\mathbb{H}^n_p$ is diffeomorphic to $\mathbb{S}^p \times \mathbb{R}^{n-p}$.*

Proof. The map ϕ_S from $\mathbb{S}^{n-p} \times \mathbb{R}^p$ to $\mathbb{R}^{n+1-p} \times \mathbb{R}^p_p \simeq \mathbb{R}^{n+1}_p$ defined by

$$\phi_S(x, y) \longmapsto (x\sqrt{1 + ||y||^2}, y)$$

is a diffeomorphism its inverse is given by

$$\phi_S^{-1}(x, y) \longmapsto (x\frac{1}{\sqrt{1 + ||y||^2}}, y)$$

Where $||.||$ is the Euclidean norm. The same proof holds for $\mathbb{H}^n_p$.

□

The following points are worth mentioning, one can see ([O'N83], Chapter 4) for more details and proofs.

For pseudo-spheres: Let's now give some properties of the pseudo-spheres which in some sense are the analogous of the preceding properties for the sphere. Note that, in general the proof of these properties is done following the same way as for the spheres (see [O'N83] for more details and proofs).

- $\mathbb{S}^n_p$ is a smooth hypersurface of constant sectional curvature equal to 1.
- Geodesics of $\mathbb{S}^n_p$ are given by intersecting $\mathbb{S}^n_p$ with planes in $\mathbb{R}^{n+1}_p$ which crosses the origin.
- totally geodesic sub-manifolds are the intersections of $\mathbb{S}^n_p$ with vector subspaces.
- *Isometries:* The isometries of the pseudo-sphere $\mathbb{S}^n_p$ are the restrictions of isometries of the ambient space which preserve the bilinear form $\langle x, x\rangle^{n+1}_p$, this is the group $O(p, n + 1 - p)$.[4]

For pseudo-hyperbolics: Similarly we give the following analogous properties for Pseudo-hyperbolic, we have:

4. Recall that $O(p, q)$ is the subgroup of $GL_{p+q}(\mathbb{R})$ of isometries preserving the quadratic form $\langle ., .\rangle^q_p$ so that $O(p, q) = \{M \in GL_{p+q}(\mathbb{R}), \mathbb{R}|MI_{p,q}M^t = I_{p,q}\}$, where I_p is the $(p \times p)$-identity matrix for $p, q \geq 1$ and $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$.

- $\mathbb{H}^n_p$ is a smooth hypersurface of constant sectional curvature equal to -1.
- Geodesics on $\mathbb{H}^n_p$ are given by intersecting $\mathbb{H}^n_p$ with planes in $\mathbb{R}^{n+1}_{p+1}$ which crosse the origin.
- Totally geodesic sub-manifolds are the intersections $\mathbb{H}^n_p$ of with vector sub-spaces.
- *Isometries:* The isometries of the pseudo-sphere are the restrictions of isometries of the ambient space which preserve the bilinear form $\langle x, x\rangle^{n+1}_{p+1}$, this is the group $O(p+1, n-p)$.

2.3 Model spaces $M^{\epsilon,n}_k$

In this thesis we are most interested in Anti-de Sitter and de Sitter space. In this section we give a geometric description to these two spaces, we will try to focus especially on the projective models (Klein models), trying also to raise the most important features and properties that we need in the resolution of the main problems considered in this thesis. Also, naturally we pass quickly by some basic definitions and properties of the Minkowski and hyperbolic space.

The notation $M^{\epsilon,n}_K$, where $K \in \{-1, 0, 1\}$ designates the curvature of the space, ϵ takes the sign $+$ (resp. $-$) to designate that the space is Riemannian (resp. semi-Riemannian) and $n \in \mathbb{N}$ is the dimension of the space, so that we have

$M^{-,3}_1 =$ de Sitter space	$M^{-,2+1}_0 = \mathrm{Mink}(2,1)$	$M^{+,2}_{-1} = \mathbb{H}^2$
$M^{-,3}_{-1} =$ Anti-de Sitter space	$M^{-,3+1}_0 = \mathrm{Mink}(3,1)$	$M^{+,3}_{-1} = \mathbb{H}^3$

2.3.1 Minkowski space and hyperbolic plane

$(2+1)$-**Minkowski space:** The $(2+1)$-dimensional Minkowski space denoted by $\mathbb{R}^{2,1}$ is the vector space $\mathbb{R}^3$ endowed with the bilinear quadratic form

$$\langle x, y\rangle^1_2 = -x_1y_1 + x_2y_2 + x_3y_3$$

It is a simply connected complete Lorentzian $(2+1)$-manifold of curvature 0. It is an orientable and time-orientable Lorentz manifold. Let's mention quickly some of its main properties:

- *Geodesics* in Minkowski space are straight lines. As usual there are three types of geodesics (up to isometry) classified according to the restriction of the bilinear form $\langle ., . \rangle_2^1$ on them. If it is positive (resp. negative, null) then it is called space-like (resp. time-like, light-like).

  The totally geodesic planes are the affine planes in Minkowski space. Again, they are classified (up to isometry) by the restriction of the bilinear form $\langle ., . \rangle_2^1$ on them. Hence we say that a plane P is space-like (resp. time-like, light-like) if $\langle ., . \rangle_2^1|_P$ is positive (resp. negative, null).

- *Isometries:* The group of all isometries of the Minkowski space is isomorphic to

$$\mathbb{R}^{2,1} \rtimes O(2,1)$$

  We mention also that the group of orientation-preserving and time-preserving isometries denoted by $Isom_0(\mathbb{R}^{2,1})$ is isomorphic to

$$\mathbb{R}^{2,1} \rtimes SO_0(2,1).^5$$

Hyperbolic plane $\mathbb{H}^2$: The hyperbolic plane has several models. The well known are the hyperboloid and the projective model (Klein). The most one we are interested in and using it in this thesis is the projective model, but first we start by defining the hyperboloid model,

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} | \langle x, x \rangle_2^1 = -1, x_0 > 0\}$$

The hyperboloid has two sheets, distinguished by the sign of the first coordinate. We define the hyperbolic space to be the sheet with $x_0 > 0$. Let's mention briefly the following properties:

- *Geodesic:* In the hyperboloid model the geodesics are given by intersecting $\mathbb{H}^2$ with planes of $\mathbb{R}^{2,1}$ passing through the origin, hence they are straight lines.

5. Recall that the group $SO(p,q)$ is defined by $SO(p,q) = \{A \in O(p,q) | det(A) = 1\}$. It is the group of linear isometries which preserve orientation of the Minkowski space. The connected component of the identity $SO_0(p,q)$ is the group of linear transformations which preserve orientation and time-orientation.

- *Isometries* of $\mathbb{H}^2$ are given by restricting the linear isometries of the ambient space $\mathbb{R}^{2,1}$. The group of orientation-preserving isometries of $\mathbb{H}^2$ is $SO_0(2,1)$.

Klein model of $\mathbb{H}^2$: We define the Klein model of the hyperbolic plane as a domain of the projective space $\mathbb{RP}^2$. It is given as follows

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} | \langle x, x\rangle < 0\}/ \sim$$

where $x \sim x'$ if and only if there exists λ such that $x = \lambda x'$. The boundary of $\mathbb{H}^2$ is given as follows

$$\partial_\infty \mathbb{H}^2 = \{[x] \in \mathbb{RP}^2 | \langle x, x\rangle = 0\}/ \sim$$

Image in an affine chart: Let $\varphi_0 : \mathbb{RP}^2 \setminus \{x_0 = 0\} \to \mathbb{R}^2$ be an affine chart of $\mathbb{RP}^2$ defined by:

$$\varphi_0([x_0, x_1, x_2]) = (\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (\bar{x}_1, \bar{x}_2) \tag{2.1}$$

Then $\varphi_0(\mathbb{H}^2 \setminus \{x_0 = 0\})$ gives,
$-x_0^2 + x_1^2 + x_2^2 < 0 \Rightarrow -(\frac{x_0}{x_0})^2 + (\frac{x_1}{x_0})^2 + (\frac{x_2}{x_0})^2 < 0$, so in this affine chart $\mathbb{H}^2$ is the domain

$$\{(\bar{x}_1, \bar{x}_2) : \bar{x}_1 + \bar{x}_2 < 1\}$$

and this is the interior of a disc. Now, analogously to the Minkowski space, we will give the following properties of the hyperbolic plane (one can see [Rat94], for more proofs and details):

- *Geodesics* in the projective model are given by straight lines.

- *Isometries:* The group of isometries in this model satisfies the following isomorphism

$$Isom(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$$

2.3.2 Hyperbolic space and de Sitter space

(3+1)**-Minkowski space:** Following the same way as in the lower dimensional case, we define the (3+1)-dimensional Minkowski space, to be the vector space $\mathbb{R}^4$ endowed with the bilinear form

$$\langle x, y\rangle_3^1 = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

It is a simply connected complete Lorentzian (3 + 1)-manifold of curvature 0, denoted by $\mathbb{R}^{3,1}$. It is also an orientable and time-orientable Lorentzian manifold.

Hyperbolic space $\mathbb{H}^3$: As in the 2-dimensional case, we define quickly the hyperboloid model of $\mathbb{H}^3$ as follows

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{3,1} | \langle x, x\rangle_3^1 = -1, x_0 > 0\}$$

It is easy to check that $\mathbb{H}^3$ is a simply connected complete Riemannian manifold of constant curvature -1. Again, the following properties can be easily checked [reference for more details]

- *Geodesics* of $\mathbb{H}^3$ are given by intersecting $\mathbb{H}^3$ with 2-planes of $\mathbb{R}^{3,1}$ which pass through the origin.

  If a unit speed geodesic starting at a point p with initial vector v, then it has the following parametrization

  $$\gamma(t) = \cosh(t)p + \sinh(t)v$$

  Totally geodesic planes of $\mathbb{H}^3$ are given by intersecting $\mathbb{H}^3$ with a time-like plane which passes through the origin.

- *Isometries:* The group of orientation-preserving isometries of $\mathbb{H}^3$ is the group of linear isometries of $\mathbb{R}^{3,1}$ which preserve orientation and do not switch the two connected components of the quadric $\{\langle x, x\rangle_3^1 = -1\}$, it is given by

  $$Isom(\mathbb{H}^3) = SO_0(3; 1)$$

  ,

Klein model of $\mathbb{H}^3$: Again, and similarly to the lower dimensional case, we define the Klein model (projective model) of the three dimensional hyperbolic space as follows:

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{3,1} | \langle x, x\rangle_3^1 < 0\} / \sim$$

where $x \sim x'$ if and only if there exists λ such that $x = \lambda x'$. The boundary of $\mathbb{H}^3$ is given as follows

$$\partial_\infty \mathbb{H}^3 = \{[x] \in \mathbb{RP}^3 | \langle x, x\rangle_3^1 = 0\}/\sim$$

Image in an affine chart: By definition $\mathbb{H}^3$ is a subset of the projective space. In order to better visualize it, we look at its intersection with an affine chart and see its image in $\mathbb{R}^3$. Let $\varphi_0 : \mathbb{RP}^3 \setminus \{x_0 = 0\} \to \mathbb{R}^3$ be an affine chart of $\mathbb{RP}^3$ defined by:

$$\varphi_0([x_0, x_1, x_2, x_3]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{2.2}$$

Then $\varphi_0(\mathbb{H}^3 \setminus \{x_0 = 0\})$ gives,

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 < 0 \Rightarrow -1 + \left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 + \left(\frac{x_3}{x_0}\right)^2 < 0$$

so in this affine chart $\mathbb{H}^3$ fills the domain

$$\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 < 1,$$

which is the interior of a ball. In this chart, the boundary at infinity $\partial_\infty \mathbb{H}^3$ is the round sphere,

$$\Omega = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 | \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = 1\},$$

Again, we give the following corresponding properties for the Klein model of the hyperbolic 3-space.

- *Geodesic* lines (resp. hyperplanes) in this model are given by intersecting $\mathbb{H}^3$ with lines (resp. hyperplanes) of $\mathbb{RP}^3$. It follows directly that in the affine chart geodesics are straight lines and totally geodesic planes are the intersection of Ω with an affine plane.

- *Isometries:* The group of orientation-preserving isometries of $\mathbb{H}^3$ is $PSO(3, 1)$.

De-Sitter space: In this section we recall the basic theory of de Sitter space. Good refrence for this material is [O'N83].

Let consider the $(3 + 1)$-Minkowski space $\mathbb{R}^{3,1}$ with the bilinear form

$$\langle x, y\rangle_3^1 = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$$

We define the quadratic model $\widehat{dS}$ to be the the pseudo-sphere

$$\widehat{dS} = \{x \in \mathbb{R}^4 | \langle x, x \rangle_3^1 = -1\},$$

endowed with the induced metric. It is a Lorentzian manifold of constant sectional curvature 1, which is simply connected and diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. (Lemma 2.2.1). Although de Sitter space is orientable, it is not time orientable.

Klein model of dS: In the following, we will define the Klein model of de Sitter space, and see its image in an affine chart. Let $x, y \in \mathbb{R}^4$ we say that $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $x = \lambda y$.

Definition 2.3.1. *We define the de Sitter space of dimension* 3 *as follows:*

$$dS = \widehat{dS}/\sim$$

endowed with the quotient metric.

Image in an affine chart: by definition dS is a subset of the projective space $\mathbb{RP}^3$. In order to better visualize it, we look at its intersection with an affine chart and see its image in $\mathbb{R}^3$. Let $\varphi_0 : \mathbb{RP}^3 \setminus \{x_0 = 0\} \to \mathbb{R}^3$ be an affine chart of $\mathbb{RP}^3$ defined by:

$$\varphi_0([x_0, x_1, x_2, x_3]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{2.3}$$

Then $\varphi_0(dS \setminus \{x_0 = 0\})$ gives,

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 > 0 \Rightarrow -1 + \left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 + \left(\frac{x_3}{x_0}\right)^2 > 0$$

so in this affine chart dS fills the domain

$$\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 > 1,$$

which is the exterior of a ball. The interior of this ball is the hyperbolic 3-space already seen above.

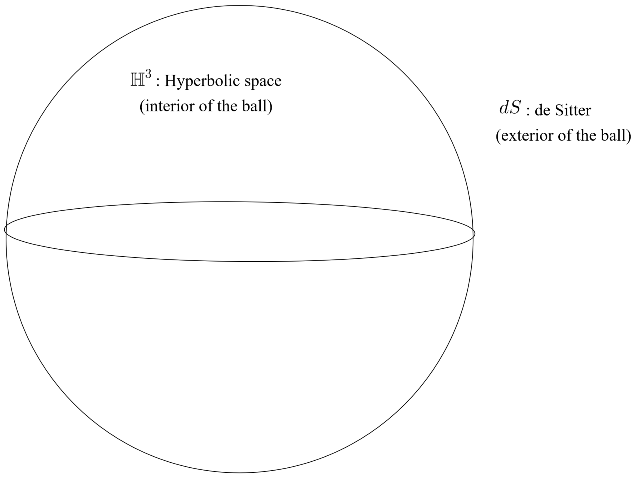

Figure 2.2: In an affine chart $\{x_0 \neq 0\}$, dS is the exterior of the ball.

Analogously to the preceding cases the following properties are worth mentioning:

- *Geodesics of dS* are given by intersecting dS with projective lines of $\mathbb{RP}^3$. Hence, it is clear from the construction that in the affine chart φ_0, geodesics (resp. totally geodesic planes) are given by the intersection between affine lines (resp. affine planes) in $\mathbb{R}^3$ with the exterior of the ball describes above. A plane P is space like if the restriction of the induced metric on P is positive-definite. A convex space-like surface in de Sitter space is a convex surface which has only space-like planes as support planes (see section 4.2.1). Recall also that the ideal boundary at infinity $\partial_\infty dS$ is given by

$$\partial_\infty dS = \{[x] \in \mathbb{RP}^3 | \langle x, x \rangle_3^1 = 0\} / \sim .$$

  We can distinguish the type of geodesics in de Sitter space as follows (Figure 2.3):

  - A geodesic in dS is time-like if it meets $\partial_\infty \mathbb{H}^3$ in two different points.
  - A geodesic in dS is light-like if it meets $\partial_\infty \mathbb{H}^3$ in only one point .
  - A geodesic in dS is space-like if it is not contained in $\partial_\infty \mathbb{H}^3$.

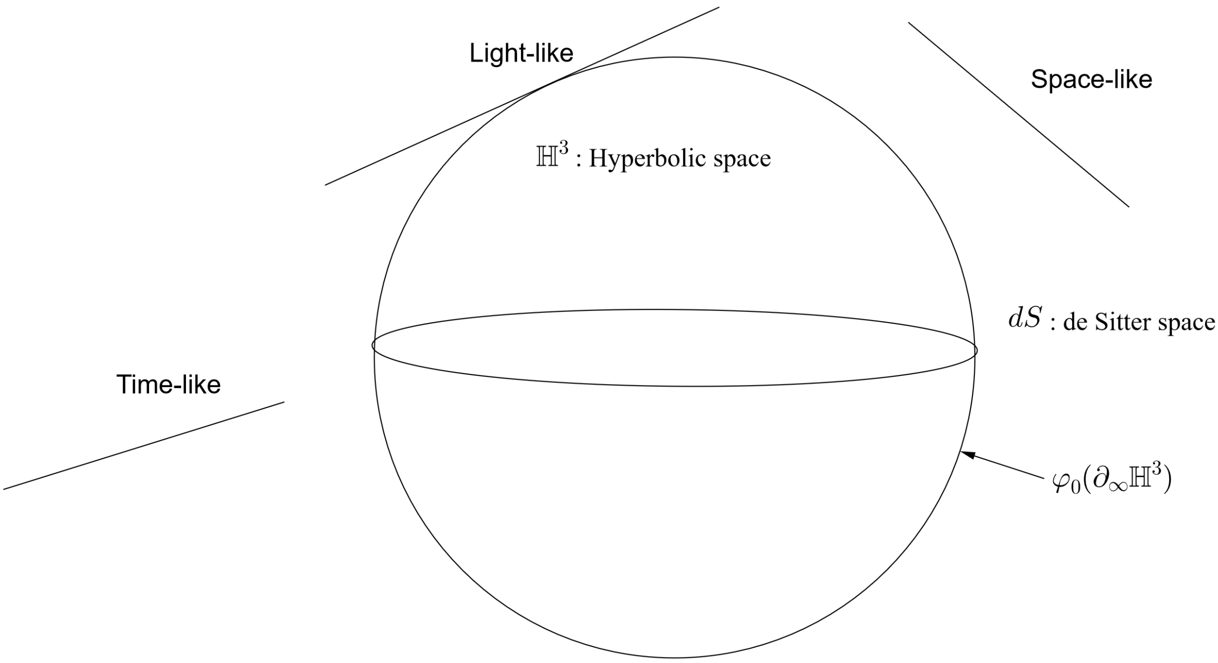

Figure 2.3: Geodesics of de Sitter space in the affine chart $\{x_0 \neq 0\}$.

- *Isometries of dS*: the isometry group of dS is the projective quotient of $O(3,1)$ denoted by $PO(3,1)$. It has two connected components preserving or reversing the orientation, and its identity component identifies with $PSO(3,1)$.

2.3.3 Anti-de Sitter space

In this section we recall the basic theory of Anti-de Sitter geometry, which is a Lorentzian analog of hyperbolic geometry. Among the most important references for this material, we cite the seminal parper of Geoffrey Mess in 1990 [Mes07], we cite also theses good references [LAB20, BS12, BS10, O'N83].

In the following we will describe a geometric model of Anti-de Sitter space (of dimension 3) we are most interested in, and illustrate some of its features.

We denote with $\mathbb{R}^{2,2}$ the vector space $\mathbb{R}^4$ endowed with the symmetric bilinear form of signature $(2,2)$,

$$\langle x, x\rangle_2^2 = -x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3.$$

Definition 2.3.2. *We define the quadratic model $\widehat{AdS^3}$ as follows,*

$$\widehat{AdS^3} = \{x \in \mathbb{R}^4 | \langle x, x\rangle_2^2 = -1\},$$

endowed with the Lorentzian metric induced by the restriction of the bilinear form b to its tangent spaces.

One can easily check that $\widehat{AdS^3}$ is a Lorentzian space of constant curvature -1, it geodesically complete, but not simply connected. The following properties can be easily checked:

- *Geodesics* in $\widehat{AdS^3}$ are given by straight lines. As usual there are three types of geodesics (up to isometry) classified according to the restriction of the bilinear form $\langle ., .\rangle_2^2$ on them. If it is positive (resp. negative, null) then it is called space-like (resp. time-like, light-like).

  Again, the totally geodesic planes are given by intersecting $\widehat{AdS^3}$ with hyperplanes of $\mathbb{R}^{2,2}$. Hence we say that a plane P is space-like (resp. time-like, light-like) if $\langle ., .\rangle_2^2|_P$ is positive (resp. negative, null).

- *Isometries*: The group of isometries of $\widehat{AdS^3}$ is the group $O(2, 2)$, while the group of orientation-preserving and time-preserving isometries is $SO_0(2, 2)$.

Klein model: As for the hyperbolic space and de Sitter space, there is the Klein model of anti-de Sitter space, obtained by projecting in $\mathbb{R}^{2,2}$ from the quadric to any tangent plane in the direction of the origin. In the following we will see that the anti-de Sitter space is sent to the interior of a one sheeted hyperboloid.

Now let $x, y \in \mathbb{R}^4$. We say that $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $x = \lambda y$.

Definition 2.3.3. *We define the Anti-de Sitter space of dimension* 3 *as follows:*

$$AdS^3 = \widehat{AdS^3}/\sim$$

endowed with the quotient metric.

It is easy to see that $\widehat{AdS^3}$ is a double cover of AdS^3. The pseudo-Riemannian metric induced on $\widehat{AdS^3}$ goes down to the quotient.

Image of AdS^3 in an affine chart: By definition AdS^3 is a subset of the projective space. In order to better visualize it, we look at its intersection with an affine chart and see its image in $\mathbb{R}^3$. Let $\varphi_0 : \mathbb{RP}^3 \setminus \{x_0 = 0\} \to \mathbb{R}^3$ be an affine chart of $\mathbb{RP}^3$ defined by:

$$\varphi_0([x_0, x_1, x_2, x_3]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{2.4}$$

Then $\varphi_0(AdS^3 \setminus \{x_0 = 0\})$ gives,

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 < 0 \Rightarrow -1 - \left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 + \left(\frac{x_3}{x_0}\right)^2 < 0$$

so in this affine chart AdS^3 fills the domain

$$-\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 < 1,$$

which is the interior of a one-sheeted hyperboloid. Notice that AdS^3 is not contained in a single affine chart. In the affine chart φ_0 we are missing a totally geodesic plane at infinity, corresponding to $\{x_0 = 0\}$.

In all the following, we will denote by $\mathbb{D}$, the disc $\begin{cases} \bar{x}_2^2 + \bar{x}_3^2 < 1 \\ \bar{x}_1 = 0 \end{cases}$ in the affine chart φ_0 (see Figure 2.4).

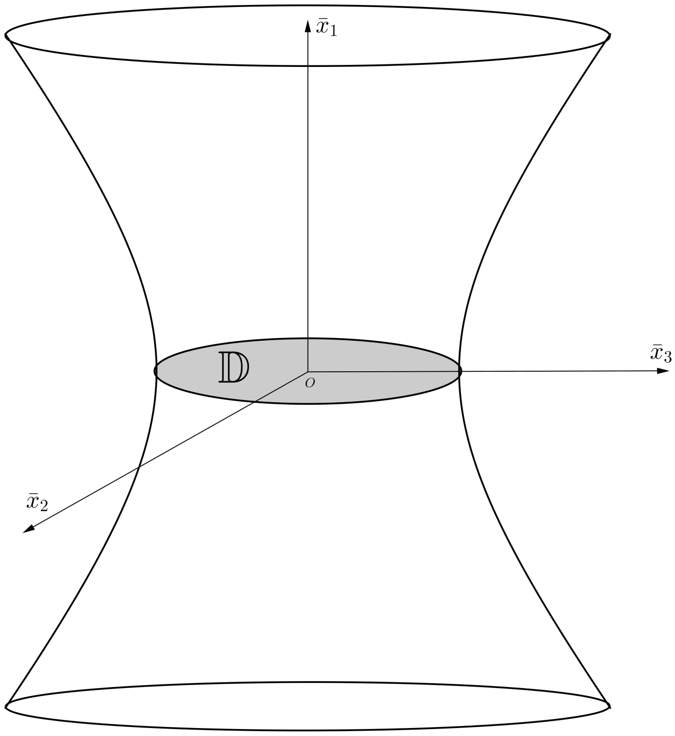

Figure 2.4: Image of AdS^3 in the affine chart φ_0.

Again, we give the following properties of the anti-de Sitter space:

- *Geodesics* are given by intersecting AdS^3 with projective lines of $\mathbb{RP}^3$. It is clear from the construction that in the affine chart φ_0, geodesics (resp. totally geodesic planes) are given by the intersection between affine lines (resp. affine planes) in $\mathbb{R}^3$ with the interior of the one sheeted hyperboloid described above. A plane P is space like if the restriction of the induced metric on P is positive-definite. A convex space-like surface in Anti-de Sitter space is a convex surface which has only space-like planes as support planes. Recall also that the ideal boundary at infinity $\partial_\infty AdS^3$ is given by

$$\partial_\infty AdS^3 = \{[x] \in \mathbb{RP}^3 : b(x,x) = 0\}/\sim .$$

We can distinguish the type of geodesics in Anti-de Sitter space as follows (see Figure 2.5):

- A geodesic in AdS^3 is space-like if it meets $\partial_\infty AdS^3$ in two different points.
- A geodesic in AdS^3 is time-like if it is strictly contained in AdS^3.
- A geodesic in AdS^3 is light-like if it meets $\partial_\infty AdS^3$ in only one point .

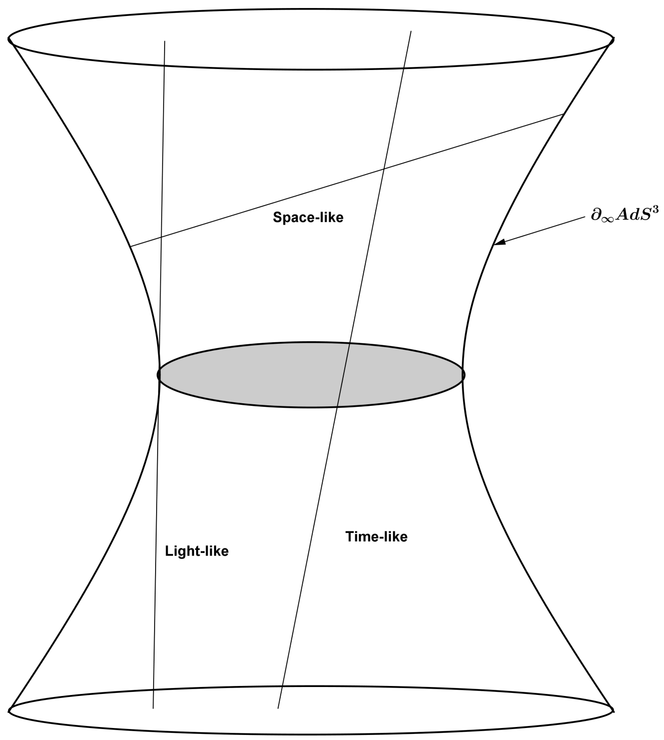

Figure 2.5: type of geodesics in AdS^3 space.

Therefore, a geodesic γ with initial point x and and tangent vector v is parametrized as follows:

- If γ is space-like then

$$\exp_x(tv) = \cosh(t)x + \sinh(t)v$$

– If γ is time-like then

$$\exp_x(tv) = \cos(t)x + \sin(t)v$$

– If γ is light-like then

$$\exp_x(tv) = x + tv$$

- *Isometries of AdS^3:* The isometry group of AdS^3 is the projective quotient of $O(2,2)$, this coincide with the group $PO(2,2)$.

  The group of orientation-preserving and time-preserving isometries of AdS^3 is then the projective quotient of $SO_0(2,2)$, it coincides with the group $PO_0(2,2)$.

Chapter 3

Approximation of metrics

3.1 Some of Alexandrov geometry

Alexandrov spaces with curvature $\leq k$ form a generalization of Riemannian manifolds with sectional curvature $\leq k$, where $k \in \mathbb{R}$. This generalization of curvature relies on the ability to compare a curved space with another space that has constant curvature k. This is done in general by comparing the geodesic triangles of the two spaces. We will explain that by details below. Now, given a real number k, in term of the previous notations (section 2.3) we will have:

- if $k < 0$ then $M_k^{+,2}$ is the hyperbolic space $\mathbb{H}^2$ with the distance function scaled by a factor of $1/\sqrt{-k}$.
- if $k = 0$ then $M_k^{+,2}$ is the Euclidean plane.
- if $k > 0$ then $M_k^{+,2}$ is the 2-sphere $\mathbb{S}^2$ with the metric scaled by a factor of $1/\sqrt{k}$.

In all the following (in this section) and for simplicity of writing, we will denote every space $M_k^{+,2}$ simply by M_k.

Let (X, d_0) be a geodesic metric space. A (geodesic) triangle $\Delta \in X$ consist of three points $x, y, z \in X$ and shortest paths $[x, y], [y, z]$ and $[z, x]$. A comparison triangle in M_k for Δ is a geodesic triangle $\tilde{\Delta}$ in M_k with vertices $\tilde{x}, \tilde{y}$ and $\tilde{z}$, such that $d_0(x, y) = d_{M_k}(\tilde{x}, \tilde{y})$, $d_0(y, z) = d_{M_k}(\tilde{y}, \tilde{z})$, $d_0(x, z) = d_{M_k}(\tilde{x}, \tilde{z})$. Note that if $k \leq 0$ then Δ always exists and if $k > 0$ then it exists provided the perimeter of Δ is less than $2\pi/\sqrt{k}$ in both cases it is unique up to isometry of M_k. The interior angle of comparison triangle $\tilde{\Delta}$ at a vertex $\tilde{x}$ is called the comparison angle between y and z at x of the triangle Δ.

Upper angle

Let's recall the notion of "upper angle" that was first introduced by A. D. Alexandrov. Let γ, γ' be two non trivial geodesics issuing from the same point x. Let $\tilde{\angle}_k(\gamma(t)x\gamma'(t'))$ be the angle at $\tilde{x}$ of the comparison triangle $\tilde{\Delta}$ with vertices $\tilde{\gamma}(t), \tilde{x}$ and $\tilde{\gamma}'(t')$ in M_k corresponding to the triangle $\Delta(\gamma(t)x\gamma(t'))$ in X. Then the upper angle at x of γ and γ' denoted by $\bar{\angle}(\gamma, \gamma')$ is defined by

$$\bar{\angle}(\gamma, \gamma') = \limsup_{t,t' \longrightarrow 0} \tilde{\angle}_k(\gamma(t)x\gamma'(t')) \tag{3.1}$$

Note that the limit above is independent of the choice of k and that the upper angle lies in the interval $[0, \pi]$.

Definition 3.1.1. *We say that (X, d_0) is $CAT(k)$ if the upper angle between any couple of sides of every geodesic triangle with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in M_k.*

The terminology "CAT(k)" was introduced the first time by M. Gromov [Gro87]. The initials are in honour of E. Cartan, AD. Alexandrov and V.A. Toponogov.

Now, fix $k \in \mathbb{R}$. Let X be a geodesic metric space, the following proposition is worth mentioning, we will use those equivalent properties in the sequel. (when $k > 0$ we assume that the perimeter of each geodesic triangle considered is smaller than $2\pi/\sqrt{k}$).

Proposition 3.1.2. *([BH99], Proposition 1.7). The following conditions are equivalent.*

(1) *X is a $CAT(k)$ space.*

(2) *The Alexandrov angle between the sides of any geodesic triangle in X with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in M_k.*

(3) *Every geodesic triangle satisfies the $CAT(k)$ inequality that is: for every $x, y \in \Delta(pqr)$ and them comparison points $\tilde{x}, \tilde{y} \in \tilde{\Delta}(\tilde{p}\tilde{q}\tilde{r})$ we have*

$$d(x, y) \leq d(\tilde{x}, \tilde{y})$$

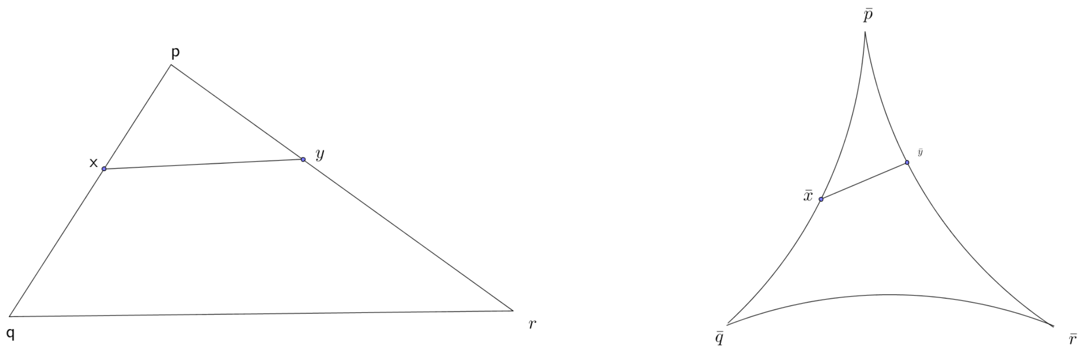

Figure 3.1: Comparison triangle and $CAT(k)$ inequality

Now, let $B_{d_0}(x,r)$ be the ball of center x and radius r in (X,d_0):

Definition 3.1.3. *A metric space (X,d_0) has a curvature $\leq k$ (in the Alexandrov sense), if for any x there exists r such that $B_{d_0}(x,r)$ endowed with the induced (intrinsic) metric is $CAT(k)$.*

Let us remind also the notion of bounded integral curvature [AZ67, Chapter I, p. 6].

A *simple triangle* is a triangle bounding an open set homeomorphic to a disc, consisting of three distinct points (the vertices of the triangle) and three shortest paths joining these points, and which is convex relative to the boundary, i.e. no two points of the boundary of the triangle, can be joined by a curve outside the triangle, which is shorter than a suitable part of the boundary joining the points, (see [AZ67] for more details). Two simple triangles are said to be nonoverlapping if they do not have common interior points.

Definition 3.1.4. *An intrinsic distance d_0 on a surface S is said to be of bounded integral curvature (in short, BIC), if (S,d_0) verifies the following property:*

For every $x \in S$ and every neighborhood N_x of x homeomorphic to an open disc, for any finite system $\mathcal{T}$ of pairwise non-overlapping simple triangles T belonging to

N_x, the sum of the excesses

$$\delta_0(T) = \bar{\alpha}_T + \bar{\beta}_T + \bar{\gamma}_T - \pi,$$

of the triangles $T \in \mathcal{T}$ with upper angles $(\bar{\alpha}_T, \bar{\beta}_T, \bar{\gamma}_T)$ is bounded from above by a number C depending only on the neighborhood N_x, i.e.

$$\sum_{T \in \mathcal{T}} \delta_0(T) \leq C.$$

Triangulation: Let (X, d) be an Alexandrov compact surface of curvature bounded from above. A triangulation is a finite family of triangles $(T_i)_{i \in I}$ such that:

- Each triangle is homeomorphic to an open disc and that the family $\{Ti\}_{i \in I}$ must cover X.
- The intersection of two faces is a disjoint union of edges and vertices.

This notion is then different from the usual notion of "triangulation" in topology.

Euler characteristic and Euler formula: The definition of the Euler characteristic is given by

$$\mathcal{X}(S) = (2 - 2g)$$

Where g is the genus of the compact surface S. Now, let $|T|$ be the number of triangles, E the number of edges and N the number of vertices in our geodesic triangulation. We have $E = \frac{3}{2}|T|$, and the Euler formula says

$$|T| - E + N = \mathcal{X}(S)$$

3.2 $CAT(k)$-surfaces are BIC

As we said in the introduction of this thesis, the general method to prove the realization of $CAT(k)$-metrics on surfaces, is by using approximation by "smooth" or "polyhedral" metrics. This method is based on the fact that some surfaces admit a triangulation by suitable triangles, these surfaces are called BIC-surfaces (see definition 3.1.4 above), we can resume that by stating this Alexandrov's Theorem.

Theorem 3.2.1 ([AZ67, Theorem 2 p.59])**.** *Let $\epsilon > 0$. A compact BIC surface admits a triangulation by a finite number of arbitrary non overlapping simple triangles of diameter $< \epsilon$.*

To use this Theorem for our $CAT(k)$ surfaces, we need to prove that these surfaces are BIC. Exactly we need to prove the following Theorem for all $k \in \mathbb{R}$.

Theorem 3.2.2. *A $CAT(k)$ surface is a BIC surface.*

For $k \leq 0$ the proof is obvious ([LAB20], Lemma 5.8) as the Alexandrov angles of a geodesic triangle in a $CAT(0)$ surface are less than the comparison angles in the Euclidian plane (Proposition 3.1.2), it follows immediately that for every triangle Δ on the surface we have

$$\delta_0(\Delta) \leq \delta_0(\tilde{\Delta}) = 0$$

where $\tilde{\Delta}$ is its comparison triangle in the Euclidiean plane and δ_0 is the excess of the triangle (see definition 3.1.4). This says that $CAT(0)$ surfaces are BIC. The proof follows by transitivity for all $k < 0$ because of the property that every $CAT(k)$ surface is a $CAT(0)$ surface for all $k < 0$ (see [BH99], Chapter II, page 165).

For $k > 0$ the proof is not obvious and need more delicate discussion. The idea repose on the result given by the following Theorem, whose the proof occupies the whole section 3.2.

Theorem 3.2.3. *For every geodesic triangle Δ we have*

$$\delta_0(\Delta) \leq kArea(\Delta)$$

where $Area(\Delta)$ is the 2-dimensional Hausdorff measure given by the $CAT(k)$ metric.

This Theorem was given without proof in ([MO01], Lemma 5.3). The authors say that the arguments for the proof are similar to the proof of the analog of Theorem 3.2.3 for spaces with curvature bounded from below (CBB) provided in ([Mac98], Theorem 2.0) and the principal argument is [(3) of Lemma 5.1 in [MO01]] which says (roughly speaking) that there are points with a neighborhood bi-Lipschitz to a Euclidean disc (Theorem 3.2.6). There is no proof of this Lemma in [MO01], the authors say that the proof can be done using some results of [OHon] which is a preprint currently not available.

3.2.1 Bi-Lipschitz equivalence

The aim of this section is to prove Theorem 3.2.6 below. For that, let's prepare some basic definitions and results that will be used in the proof.

Space of directions

In the following we will introduce the notion of *space of directions*, [LS97a, LS97b, KL97] this notion replaces the concept of the tangent space in the theory of smooth manifolds.

Let X be a complete length space, that is X is a connected complete metric space, and γ, γ' be two non trivial geodesics issuing from the same point x. We say that γ and γ' define the same direction at p if the Alexandrov angle between them is zero. We define a relation $\sim$ as follows: $\gamma \sim \gamma'$ if and only if $\bar{\angle}_x(\gamma, \gamma') = 0$ (the Alexandrov angle is null).

Using a triangle inequality for the upper angle it is easy to check that $\sim$ is an equivalence relation on $\Sigma'_{\underline{x}}$ (the set of equivalence classes of non-trivial geodesics issuing from p) and that $\angle_x$ induces a metric on Σ'_x hence a structure of a metric space $(\Sigma'_x, \bar{\angle}_x)$ is well defined. The completion of $(\Sigma'_x, \bar{\angle}_x)$ is called the space of directions at p and denoted by $(\Sigma_x, \bar{\angle}_x)$. An element of Σ_x is called a direction at x.

For a length surface X, the length $L(\Sigma_{\underline{x}})$ of Σ_x at $x \in X$ is by definition the one-dimensional Hausdorff measure of $(\Sigma_x, \bar{\angle}_x)$.

Singular points

A point $x \in X$ is called a positive (resp. negative) singular point if $L(\Sigma_x) < 2\pi$ (resp. $L(\Sigma_x) > 2\pi$). We denote the set of positive (resp. negative) singular points by $Sing^+(X)$ (resp. $Sing^-(X)$). For a metric space the set of singular points is the union of the sets of positives and negatives singular points. For Alexandrov spaces, an important remark that can be easily seen is that there are no negative singular points for Alexandrov spaces of curvature bounded from below and no positive ones on those of curvature bounded from above. A point $x \in X$ is said to be regular if it is not singular.

We will call a point $x \in X$ a δ-singular point if $L(\Sigma_x) \geq 2\pi + \delta$, or equivalently, the diameter of Σ_x is not less than $\pi + \delta/2$. We designate by $Sing_\delta X$ the set of δ-singular points in X and define $\text{Sing } X := \bigcup_{\delta>0} Sing_\delta X$ (the set of metrically singular points in X).

The following Lemma is given in ([MO01], Lemma 5.2) and the associated proof is a straightforward adaptation of the one given in ([Mac98], Lemma 1.3) for Alexandrov spaces with curvature bounded from below.

Lemma 3.2.4. *For any $\delta > 0$, the set of points whose space of directions is of length greater than $2\pi + 2\delta$ has no accumulation point. In particular, the number of singular points in a compact domain is countable.*

Proof. By contradiction we suppose that the set $Sing_\delta$ has an accumulation point x. Then there exists a sequence $\{x_n\} \subset Sing_\delta X$ such that $\lim_{n\to\infty} x_n = x$. Without loss of generality, we can take a subsequence, and suppose that for $n > m$, $|xx_n| \leq |pp_m|$. Since $\pi + \delta \leq \angle(xx_nx_m)$ (because the length of the space of direction is greater than $2\pi + 2\delta$) and that $\angle(xx_nx_m) \leq \tilde{\angle}(xx_nx_m)$ (because of the $CAT(k)$ properties), there is a positive number α (depending only on k and δ) such that $\angle(x_nxx_m) \leq \tilde{\angle}(x_nxx_m) \leq \alpha$ for arbitrary n, m with $n \neq m$. This leads to a contradiction because Σ_x is compact. The fact that the number of singular points in a compact domain is countable follows directly from the first result of the Lemma.

□

Tangent cone

In this section we will introduce quickly the notion of "tangent cone" in Alexandrov spaces, this notion will be used in the proof of Theorem 3.2.6.

Definition 3.2.5. *Let X be an Alexandrov space of curvature bounded from above, $x \in X$ and $\Sigma_x X$ be the space of directions of X at x. The tangent cone of X at x denoted by T_xX, is defined to be the Euclidean cone over the space of direction of X so that*

$$T_xX = C(\Sigma_x X)$$

Let's state here the main Theorem of the section 3.2.1.

Theorem 3.2.6. *Let $x \in X$ be a point satisfying $L(\Sigma_x) < 2\pi + \delta$ for sufficiently small δ. Then for sufficiently small $r > 0$, there is a positive number $\epsilon = \epsilon(\delta, r)$ with $\lim_{\delta,r\to 0} \epsilon(\delta, r) = 0$ such that the ball $B_x(r)$ (of center x and radius r) is bi-Lipschitz homeomorphic to a ball of radius r in $\mathbb{R}^2$ with the Lipschitz constant in $(1-\epsilon, 1+\epsilon)$.*

In the following we will give a sketch of the proof of Theorem 3.2.6, it will be done using ([BB98], Theorem 3.1) stated bellow as Theorem 3.2.9, whose the proof was done following a similar way as for ([BGP92], Theorem 5.4) proving the existence of some suitable maps associated to a collection of points called "distance frames" for CBA case (in some sense analog of the maps called "strainers" [Ber08] or "burst points, explosions" [BGP92]) in $\mathbb{R}^2$, the difference from the CBB case was (roughly speaking) only that a lower estimate for angles is based on the extendability of geodesics (see [BB98] form more details). For that in the sequel we will suppose that geodesics are extendable.

We recall that a geodesic $\gamma : [0, \beta] \longrightarrow X$ is said to be extandable if it is the restriction of a geodesic $\gamma' : [0, \beta'] \longrightarrow X$ with $\beta < \beta'$.

Chain of directions and iterated tangent cone

To prove the first part of Theorem 3.2.6 (existence of a bi-Lipschitz homeomerphism) we will use Theorem 3.2.9 and the iterated tangent cone below. The idea was taken from a part of the proof of ([BB98], Lemma 3.10).

Let's start by the following Theorem due to I. G. Nikolaev [Nik95].

Theorem 3.2.7. *([BH99], Theorem 3.19) Let $k \in \mathbb{R}$, and X a metric space with curvature $\leq k$ then the space of directions at each point $x \in X$ is a $CAT(1)$ space. And the tangent cone T_xX is $CAT(0)$ at every $x \in X$.*

A good remark, is that furthermore if X is with extandable geodesics then the space of direction will be with extandable geodesics too.

Now, as the space of directions is a space of curvature ≤ 1 then we can define a recurrence relation (space of directions of the space of directions) as follows

$$\begin{cases} \Sigma_{x,v_1,\dots,v_{m-1}}X = \Sigma_x X, \quad m = 1 \\ \Sigma_{x,v_1,\dots,v_m}X = \Sigma_{v_m}(\Sigma_{x,v_1,\dots,v_{m-1}}X) \text{ at } v_m \in \Sigma_{x,v_1,\dots,v_{m-1}}X, \ \forall m \in \mathbb{N} \end{cases}$$

The collection $\xi = (x; v_1, ..., v_m)$ is called the *chain of directions* at x, and the number $|\xi| = m$ is the length of ξ. It is clear that the chain ξ is maximal if and only if v_m is an isolated point of $\Sigma_{m,v_1,\dots,v_{m-1}}X$.

It is easy to prove that the direction space of the tangent cone T_xX at a point (v, t) is isometric to the spherical suspension[3] over $\Sigma_{x,v}$, thus $T_{(v,t)}(T_xX)$ is isometric to $C\Sigma_{x,v} \times \mathbb{R}$.

Iterating this operation for a chain $\xi = (x, v_1, .., v_m)$ and $t = (t_1, ..., t_m)$, $t_m \geq 0$, $\forall m \in \mathbb{N}$, we can define recurrently $T_{\xi t}X$ to be the tangent cone to $T_{\xi' t'}X$ at (v_m, t_m), where $\xi' = (x, v_1, .., v_{m-1})$ and $t' = (t_1, ..., t_{m-1})$. For a chain of direction $\xi = (x; v_1, ..., v_m)$ and a collection $t = (t_1, ..., t_m)$ of positive reals, we define recurrently $T_{\xi t}X$ as the tangent cone to $T_{\xi' t'}X$ at the point (v_m, t_m), where $t' = (t_1, ..., t_{m-1})$, $\xi' = (x; v_1, ..., v_{m-1})$. It follows directly that $T_{\xi t}X$ is isometric to $(C\Sigma_\xi) \times \mathbb{R}^m$. If ξ is maximal, then $T_{\xi t}X = \mathbb{R}^{|\xi|} = \mathbb{R}^m$.

3. Recall that the spherical suspension for a metric space X of diameter $\leq \pi$ is the quotient space $\Gamma(X) = X \times [0, \pi]/\sim$ where $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow a_1 = a_2 = 0$ or $a_1 = a_2 = \pi$ with the conical metric

$$\cos|\bar{x}_1\bar{x}_2| = cos(a_1)cos(a_2) + sin(a_1)sin(a_2)cos|x_1x_2|$$

where $\bar{x}_1 = (x_1, a_1), \bar{x}_2 = (x_2, a_2)$

If X is a surface of bounded curvature $\leq k$ then the length of the chain will be bounded ([BB98]) and not greater than 2, ($m \leq 2$).

Let's recall the following definition of the Gromov-Hausdorff distance.

Definition 3.2.8. *We define the distortion of a map $f : X \to Y$ between two metric spaces as follows:*

$$dis(f) = \sup_{x,x' \in X} |d_Y(f(x), f(x')) - d_X(x, x')|$$

Let A be the class of all maps $f : X \to Y$. If we put

$$\delta(X, Y) = \inf_{f \in A} dis(f)$$

Then the Gromov-Hosdorff distance between X and Y is defined by

$$d_{GH}(X, Y) = \max\{\delta(X, Y), \delta(Y, X)\}$$

A (compact) metric space X is the Gromov-hausdorff limit of a sequence of metric spaces $(X_n)_n$ if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$. (Of course this limit is unique up to isometry since d_{GH} is a metric).

Recall that for a metric space X and $\lambda > 0$, the metric space λX is the same set of points equipped with the original metric multiplied by λ.

If X is a compact surface of with extandable geodesics, then the tangent cone $T_x X$ is the Gromov-Hausdorff limit of pointed spaces $(\lambda X, x)$ when $\lambda \to \infty$ (see [BS07] or [BB98] for more details).

Since $T_x X$ is the Gromov-Hausdorff limit of the pointed spaces $(\lambda X, x)$ as $\lambda \to \infty$, then for every ϵ there exists $\lambda > 0$ and a small θ' such that

$$d_{GH}(B_{\theta'}(x) B_{\theta'}(o)) < \epsilon$$

for the balls $B_{\theta'}(x) \subset \lambda X$ and $B_{\theta'}(o) \subset T_x X$ where o is the vertex of the cone $T_x X$. Iterating this construction and using the fact that $T_{\xi t} X = \mathbb{R}^2$, we can find θ such that

$$d_{GH}(B_\theta(x) B_\theta(0)) < \epsilon$$

where $B_\theta(x) \subset \lambda X$ and $B_\theta(0) \subset \mathbb{R}^2$. Now we can apply the following Theorem due to B. Kleiner [Kle99], and proved in [BB98] to deduce the existence of a bi-Lipschitz homeomorphism to an open subset of $\mathbb{R}^2$.

Theorem 3.2.9. *([BB98], Theorem 3.1). Let X be geodesically complete surface of curvature $\leq k$ with extandable geodesics such that*

$$d_{GH}(B_\theta(x)B_\theta(0)) < \epsilon$$

where $B_\theta(x) \subset X$ and $B_\theta(0) \subset \mathbb{R}^2$. Then every ball $B_r(x) \subset B_\theta(x)$ is bi-Lipschitz homeomorphic to an open disc in $\mathbb{R}^2$.

Hence the first part of Theorem 3.2.6 (the existence of a bi-Lipschitz homeomorphism) is now proved. Now let's see how to prove the second part (the distortion coefficients of the homeomorphism are $(1-\epsilon, 1+\epsilon)$).

Before seeing that, we note that the proof of the Theorem 3.2.9 was done by proving that the associated distance map to the distance frames is a bi-Lipschitz homeomorphism (see [BB98] for more details). In the following we will introduce the homeomorphism in question (the homeomorphism verifying Theorem 3.2.9) and prove that under some suitable conditions is furthermore an almost[4] isometry (that is, the distortion coefficients are $(1-\epsilon, 1+\epsilon)$).

Distance frames

The notion of "distance frames" (see below) were introduced and used in [Ber75] (in the form of distance coordinates). Using this notion and Theorem 3.2.9, we will give a proof to the Theorem 3.2.6 by proving that the associated distance map to the distance frames is an ϵ-almost isometry.

Definition 3.2.10. *Let (X, d) be a locally compact space of upper bounded curvature with extendable geodesics. A collection $\Theta = \{x_0; a_i, b_i, i = 1, 2\}$ is called a distance frame if*

$$|\pi - \angle a_i x b_i| < \delta \qquad |\pi/2 - \angle a_i x a_j| < \delta$$
$$|\pi/2 - \angle a_i x b_j| < \delta \qquad |\pi/2 - \angle b_i x b_j| < \delta$$

for all $1 \leq i, j \leq 2, i \neq j$ and $x \in B_\delta(x_0)$

4. A surjective map between two metric spaces $f : X \to X'$ is said to be a ϵ_1-almost isometry if the following holds for any $x, x' \in X$

$$|d_{X'}(f(x), f(x')) - d_X(x, x')| < \epsilon_1 d_{X'}(x, x')$$

The associated distance map $f = f_\Theta : X \longrightarrow \mathbb{R}^2$ with the coordinate functions $f_i = f_{a_i,b_i}$ where $f_{a_i,b_i} : X \longrightarrow \mathbb{R}$ are given by $f_{a_i,b_i}(x) = \frac{1}{2}(d(b_i, x) - d(a_i, x))$

The following Theorem is an analogous of ([BGP92], Theorem 9.4) for CBB spaces.

Theorem 3.2.11. *Let X be a surface of curvature $\leq k$ and let $x_0 \in X$ have a distance frame $\{x_0; a_i, b_i, i = 1, 2\}$. Then the map $f : X \longrightarrow \mathbb{R}^2$ given by $f(q) = (f_{a_1,b_1}(q), f_{a_2,b_2}(q))$ is an almost isometry from a small neighborhood of the point x_0 onto a domain in $\mathbb{R}^2$.*

So we need to prove the following inequality

$$|\sum_{i=1}^{2} \frac{1}{4} \frac{((d(b_i, q) - d(b_i, r)) - (d(a_i, q) - d(a_i, r)))^2}{d(q, r)^2} - 1| < \epsilon(\delta, r)$$

The proof can be done using a suitable cosinus formula, we note that we didn't complete the proof yet. If we prove this inequality then the proof of Theorem 3.2.6 follows. (It can be done using an analog of ([BGP92], Lemma 9.3) which seems to be obvious for $n = 2$).

The quasi-total excess

Now, to prove the Theorem 3.2.3 we have to introduce the notion of "quasi-total excess".

Let Δ be a triangle Δ with bounds the geodesics $\gamma_1, \gamma_2, \gamma_3$, and vertices $V = \{\gamma_1(0), \gamma_2(0), \gamma_3(0)\}$ and let $\sum_x^\Delta$ be the space of directions at $x \in X$ of Δ, [MO01]. We define the quasi-total excess as follows

$$\delta_0'(\Delta) = 2\pi - \sum_{x \in V} (\pi - L(\textstyle\sum_x^\Delta))$$

This definition coicides with the definition of the total excess in Alexandrov surfaces of curvature bounded from below, and in such a case δ' depends only on the boundary $\partial\Delta$ of Δ and not on the choice of geodesics.

The total excess is then defined as follows

$$\delta_0(\Delta) = \sup\{\delta'(\Delta; \gamma_1, \gamma_2, \gamma_3)\}$$

where the supremum is taken over all pairs of geodesics which construct $\partial\Delta$. Now using Theorem 3.2.11, Lemma 3.2.4, and following the same technical way as in ([Mac98], Theorem 2.0) we can get the proof of Theorem 3.2.3.

In the end, using Theorem 3.2.3 then clearly Theorem 3.2.2 holds, one have just to take the bound C in the definition 3.1.4 as the sum of the areas of the triangles.

3.3 Approximation of $CAT(k)$-metrics by polyhedral metrics

In this section we will show how to approximate an Alexandrov metric of curvature $\leq k$ for $k \in \mathbb{R}$ on a compact surface S by a sequence of metrics with conical singularities of negative curvature.

In the following, we mean by a k-polyhedral metric on a surface S, a metric such that each point has a neighborhood isometric to the neighborhood of a point on a cone in M_k. The points having a neighborhood isometric to a neighborhood of the apex of a cone are the vertices of the metric. The curvature of a vertex is 2π minus the total angle around the corresponding cone vertex. For $k = 1$ we will call it simply a spherical polyhedral metric, and for $k = -1$ we will call it a hyperbolic polyhedral metric.

The convergence that will be used is the uniform convergence so let's recall its definition.

Definition 3.3.1. *We say that a sequence of metric spaces* $(S_n, d_n)_n$ converges uniformly *to the metric space* (S, d) *if there exist homeomorphisms* $f_n : S \longrightarrow S_n$ *such that*

$$\sup_{x,y \in S} |d_n(f_n(x), f_n(y)) - d(x, y)| \xrightarrow[n\to\infty]{} 0 \ .$$

If $S_n = S$ and $f_n = id$, then this is the usual definition of uniform convergence of distance functions.

In this section, we want to prove that a metric of curvature $\leq k$ on a closed surface S can be approximated (in the sense of the uniform convergence) by k-polyhedral metrics.

The main tool is [AZ67, Theorem 2 p. 59] it is given as follows:

Theorem 3.3.2. *Let $\epsilon > 0$. A compact BIC surface admits a covering by a finite number of arbitrary non overlapping simple triangles of diameter $< \epsilon$.*

We will also need the following result to prove that the sum of the angles in a cone point is not less than 2π, it corresponds to Theorem 11 in [AZ67, Chapter II, p. 47].

Lemma 3.3.3. *Let p be a point on a BIC surface such that there is at least one shortest arc containing p in its interior. Then for any decomposition of a neighborhood of p into sector convex relative to the boundary formed by geodesic rays issued from p such that the upper angles between the sides of these sectors exist and do not exceed π, the total sum of those angles is not less than 2π.*

To get a triangulation of our surface, we will use some properties of BIC surfaces, so let's consider the Theorem 3.2.2, we restate it here as follows

Theorem 3.3.4. *A $CAT(k)$ surface is a BIC surface.*

Note that for a BIC surface, the angle exists, that means that in (3.1) the limit exists in place of the limsup [AZ67]. So in the following we will speak about angles rather than upper angles.

Theorem 3.3.5. *Let (S, d) be a metric of curvature $\leq k$ on the closed surface S. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n is homeomorphic to S and d_n is the metric induced by a k-polyhedral metric on S_n.*

The remainder of this section is devoted to the proof of Theorem 3.3.5.

Applying Theorem 3.3.2 we obtain a triangulation $\mathcal{T}_\epsilon$ of our surface in which every simple triangle has diameter $< \epsilon$. Replace the interiors of the triangles of $\mathcal{T}_\epsilon$ by the interiors of the comparison triangles in M_k. We obtain $(\bar{S}_\epsilon, \bar{d}_\epsilon)$ which is a k-polyhedral metric with conical singularities, corresponding to the vertices of the triangles. By construction, $\bar{S}_\epsilon$ is endowed with a triangulation $\bar{\mathcal{T}}_\epsilon$.

Lemma 3.3.6. *The total angles around the conical singularities of $\bar{d}_\epsilon$ are not less than 2π.*

Proof. By a property of the CAT(k) spaces, we have that every vertex of $\mathcal{T}_\epsilon$ lies in the interior of some geodesic in (S, d) [BH99, II.5.12]. Applying Lemma 3.3.3 we immediately get that the sum of the sector angles α_i at any vertex V of the triangulation $\mathcal{T}_\epsilon$ in (S, d) is not less than 2π. By definition of the CAT(k) spaces, we have that the angles $\alpha_{k,i}$ of the comparison triangles in M_k are not less than the corresponding angles at every vertex V in the triangulation $\mathcal{T}_\epsilon$ in (S, d). It follows that

$$2\pi \leq \sum_i \alpha_i \leq \sum_i \alpha_{k,i} \ .$$

□

We want to prove that the finer the triangulation is, the closer $\bar{d}_\epsilon$ is from d (for the uniform convergence between metric spaces). This relies on a series of lemmas.

Lemma 3.3.7. *Let α be the angle at a vertex of a triangle T in a surface of curvature $\leq k$, and α_k be the corresponding angle in a comparison triangle T_k in M_k then,*

$$\alpha_k - \alpha \leq karea(T_k) - \delta_0(T).$$

Proof. If β and λ are the angles of T and β_k, λ_k the corresponding angles in T_k, then we have

$$\alpha_k - \alpha \leq \alpha_k - \alpha + \beta_k - \beta + \lambda_k - \lambda = \delta_0(T_k) - \delta_0(T) = karea(T_k) - \delta_0(T).$$

□

Lemma 3.3.8. *If $\mathcal{T}$ is a triangulation of a compact surface (S, d) with curvature $\leq k$ by non overlapping simple triangles, then*

$$\sum_{T \in \mathcal{T}} \delta_0(T) \geq 2\pi \mathcal{X}(S),$$

with $\mathcal{X}(S)$ the Euler characteristic of S.

Proof. Let $|T|$ be the number of triangles, E the number of edges and N the number of vertices in our geodesic triangulation. We have $E = \frac{3}{2}|T|$, so that the Euler formula

$$|T| - E + N = \mathcal{X}(S)$$

implies

$$2N - |T| = 2\mathcal{X}(S). \tag{3.2}$$

We have also that the sum of all the angles of all the triangles is equal to the sum of all the cone angles i.e. if we denote by α_i the sum of the angles around each vertex v_i then using (3.2) it follows that

$$\sum_{T \in \mathcal{T}} \delta_0(T) = \sum_{i=1}^{N} \alpha_i - |T|\pi = \sum_{i=1}^{N} (\alpha_i - 2\pi) + 2\pi \mathcal{X}(S)$$

Now as $\alpha_i - 2\pi \geq 0$ for all i, the proof follows.

□

Lemma 3.3.9. *Let T be an isosceles triangle in M_k with diameter less than a given ϵ and with edges length x, x, l and θ the angle opposite to the edge of length l, then*

$$l \leq f_k(\epsilon)\theta.$$

where $f_{-1} = \sinh$, $f_0 = id$, $f_1 = \sin$

Proof. $k=-1$. By the hyperbolic cosine law,

$$\cosh(l) = \cosh^2(x) - \sinh^2(x)\cos(\theta),$$

that is equivalent to

$$1 + 2\sinh^2(\frac{l}{2}) = \cosh^2(x) - \sinh^2(x)(1 - 2\sin^2(\frac{\theta}{2})),$$

so

$$\frac{l}{2} \leq \sinh(\frac{l}{2}) = \sinh(x)\sin(\frac{\theta}{2}) \leq \sinh(\epsilon)\frac{\theta}{2}.$$

$k=0$. By the Euclidean cosine law,

$$l^2 = 2x^2 - 2x^2\cos(\theta) = 2x^2\left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right) = 4x^2\sin^2\left(\frac{\theta}{2}\right),$$

hence

$$l = 2x\sin\left(\frac{\theta}{2}\right) \leq 2\epsilon\frac{\theta}{2}$$

$k=1$. By the spherical cosine law,

$$\cos(l) = \cos^2(x) + \sin^2(x)\cos(\theta)$$

that is equivalent to

$$1 - 2\sin^2\left(\frac{l}{2}\right) = \cos^2(x) + \sin^2(x)\left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)$$

hence

$$\sin\left(\frac{l}{2}\right) = \sin(x)\sin\left(\frac{\theta}{2}\right)$$

the result follows because for $0 < \lambda < 1$ we have $\arcsin(\lambda y) \leq \lambda \arcsin(y)$. □

Lemma 3.3.10. *Let $\epsilon > 0$. Let T be a simple triangle in (S,d) of diameter $< \epsilon$ with vertices OXY. Let A (rep B) be on the edge OX (rep. OY) and at distance a (resp. b) from O. Let T_k^1 be a comparison triangle for T in M_k, with vertices $O'X'Y'$. Let A' (resp. B') be the corresponding point of A (resp. B) (i.e on the edge $O'X'$ (resp. $O'Y'$) and at distance a (resp. b) from O'), then*

$$0 \leq d_{M_k}(A',B') - d(A,B) \leq -\delta_0(T)f_k(\epsilon), \quad k \leq 0$$

$$0 \leq d_{M_k}(A', B') - d(A, B) \leq (o(\epsilon) - \delta_0(T))f_k(\epsilon), \quad k > 0$$

where o is a function such that $o(\epsilon) \underset{\epsilon \to 0}{\longrightarrow} 0$.

Proof. For $k \leq 0$: The left hand inequality comes from the fact that we are in $CAT(k)$ neighborhood (see Proposition 3.1.2). Let T_k^2 be the comparison triangle for OAB in M_k drawn such that the edge of length a is identified with $O'A'$ (see Figure 3.2). Let B'' be the corresponding comparison point for B in T_k^2 (i.e. B'' satisfies $d(A, B) = d_{M_k}(A', B'')$ and $d(O, B) = d_{M_k}(O', B'')$). By triangle inequality we have

$$d_{M_k}(A', B') - d(A, B) = d_{M_k}(A', B') - d_{M_k}(A', B'') \leq d_{M_k}(B', B''). \tag{3.3}$$

Let θ_1 be the angle at O of T_k^1 (i.e. the angle at O' of $O'A'B'$), and let θ_2 be the angle at O' of T_k^2 (i.e. $O'A'B''$).

We have that $\theta_1 - \theta_2$ is the angle at O of $OB'B''$ which is isosceles so by inequality (3.3) and Lemma 3.3.9 it follows that

$$d_{M_k}(A', B') - d(A, B) \leq f_k(\epsilon)(\theta_1 - \theta_2) \tag{3.4}$$

If β is the angle of T at O, then both θ_1 and θ_2 are angles corresponding to β in comparison triangles, so by Lemma 3.3.7

$$\theta_1 - \theta_2 = \theta_1 - \beta + \beta - \theta_2 \leq \theta_1 - \beta \leq karea(T_k^1) - \delta_0(T) \tag{3.5}$$

hence it is not greater than $-\delta_0(T)$, for $k \leq 0$. Now replacing in (3.4) the proof follows.

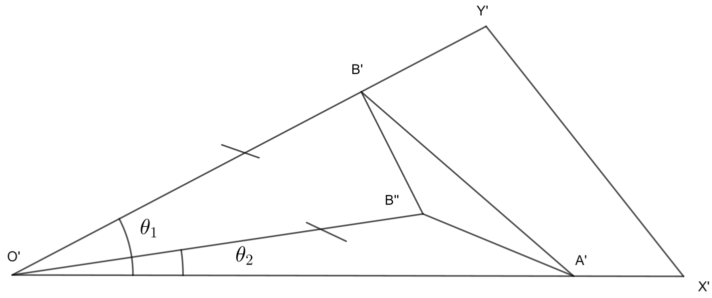

Figure 3.2: Notations for the proof of Lemma 3.3.10.

For $k > 0$: The left hand inequality comes from the fact that we are in $CAT(k)$ neighborhood (see Proposition 3.1.2). For the right hand side, if T_k^1 is a comparison triangle with edges of lengths not greater than ϵ. The result follows from a majoration and a taylor expansion from L'Huilier's Theorem which says that: if T_k^1 has edge length a, b, c then

$$area(T_k^1) = \frac{4}{k} arctan \sqrt{\tan \frac{a+b+c}{2} \tan \frac{a+b}{2} \tan \frac{a+c}{2} \tan \frac{b+c}{2}}$$

then using inequality (3.5) which can be found directly following the same way as for the case $k \leq 0$, the proof follows.

$\square$

Now, let's describe a homeomorphism between (S, d) and $(\bar{S}_\epsilon, \bar{d}_\epsilon)$ in the following way. The triangle $\bar{T}_i$ do not degenerate into segments, since the sum of every two sides is greater than the third. Therefore, the triangles T_i can be mapped homeomorphically onto the corresponding triangles $\bar{T}_i$, such that the vertices are sent to vertices and the homeomorphism restricts to an isometry along the edges. We consider any homeomorphism from the interior of the triangles that extend the homeomorphism on the boundary. As the surfaces are triangulated by such triangles, this gives a homeomorphism from S to $\bar{S}_\epsilon$.

For two points H and J on S, we denote by H', J' the correponding points on $\bar{S}_\epsilon$.

Fact 3.3.11. *For $k \leq 0$, with the notations above, $-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - 2\pi\mathcal{X}(S) f_k(\epsilon)$.*

Proof. The idea of this proof is the same as [Ale06, Lemma 2, page 263]. Let's prove the first inequality. Let $H', J' \in \bar{S}_\epsilon$ and γ' a shortest path joining H' and J' and γ be a path joining H and J such that the intersection with every triangle T is a shortest path (i.e. each connected piece of γ' meeting a triangle T' from a point A' to a point B' on the boundary of T' is associated in T the shortest path joining the corresponding (in the sense of Lemma 3.3.10) points A and B).

Let us denote by γ'_i, $i = 0, \dots, m+1$ the decomposition of γ' given by the triangles it crosses, and by $l(\gamma'_i)$ their lengths.

As (S, d) is CAT(k), the length of a connected component of the intersection of γ' with T' joining two points of the boundary is greater than the length of the corresponding component of γ in T ([BH99], page 158). Now, because the diameters are not greater than ϵ then $l(\gamma_0) + l(\gamma_{m+1}) \leq 2\epsilon$ and $l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon$. It follows that

$$d(H, J) \leq \sum_{i=1}^{m} l(\gamma_i) + 2\epsilon \leq \sum_{i=1}^{m} l(\gamma'_i) + 2\epsilon \leq \bar{d}_\epsilon(H', J') + 2\epsilon$$

then

$$-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J).$$

The first inequality is now proved.

Let's now prove the second inequality. For that, consider a shortest path γ joining H and J in S and γ' be a path in $\bar{S}_\epsilon$ joining H' and J' such that the intersection with every triangle T' is a shortest path (i.e. each connected piece of γ meeting a triangle T from a point A to a point B on the boundary of T is associated in T' the shortest path joining the corresponding (in the sense of Lemma 3.3.10) points A' and B').

Let us denote by γ_i, $i = 0, \dots, m+1$ the decomposition of γ given by the triangles it crosses, and by $l(\gamma_i)$ their lengths, we find

$$\bar{d}_\epsilon(H', J') - d(H, J) \leq l(\gamma'_0) + l(\gamma'_{m+1}) + \sum_{i=1}^{m} l(\gamma'_i) - l(\gamma_i).$$

Since $l(\gamma'_0)$ and $l(\gamma'_{m+1})$ are not greater than ϵ then $l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon$. By Lemma 3.3.10 it follows that

$$\bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - \sum_{i=1}^{m} \delta_0(T_i) f_k(\epsilon),$$

But $\delta_0(T_i)$ are non positive and moreover the triangles T are relative convex, so γ meets each triangle at most once (because, if the shortest path γ meets the (geodesic) triangle more than once, then there will be two points on the boundary of the triangle joined by a shortest path lying outside of the triangle, that contradicts the fact that the triangles are convex relative to the boundary), so $-\sum_{i=1}^{m}\delta_0(T_i)$ is less than $-\sum_T \delta_0(T)$ for all the triangles of the triangulation of S, which is less than $-2\mathcal{X}(S)$ by Lemma 3.3.8. The second inequality is now proved.

□

This fact is now proved.

Fact 3.3.12. *For $k > 0$, and S is the sphere $-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - o'(\epsilon) f_k(\epsilon)$*

Proof. Following the same way as in fact 3.3.11 and using Lemma 3.3.10 we have that

$$-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon + \sum_{i=1}^{m}(karea(T_k^1) - \delta_0(T_i)) f_k(\epsilon),$$

hence

$$-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon + \sum_{i=1}^{m}(o(\epsilon) - \delta_0(T_i)) f_k(\epsilon),$$

□

The lemmas above imply the uniform convergence. Theorem 3.3.5 is now proved.

3.4 Approximation of polyhedral metrics by smooth metrics

In this section we will approximate a hyperbolic metric with conical singularities of negative curvature.

Proposition 3.4.1. *Let d be the metric induced by a hyperbolic metric with conical singularities of negative curvature on the closed surface S. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n is homeomorphic to S, d_n is metric induced by a Riemannian metric of sectional curvature < -1.*

We use the same method as that in [Slu13, Lemma 3.9], but we choose the cone in anti-de Sitter space (Figure 3.3), rather than the hyperbolic space $\mathbb{H}^3$.

Proof. Let $p \in S$ be a singular point of the polyhedral hyperbolic metric d. Consider a neighborhood U_p of p in S which doesn't contain any other singular point of d. As the curvature is supposed to be negative, the neighborhood U_p equipped with the restriction of the metric d will be isometric to the neighborhood of a space-like circular cone C_p in the affine model of the anti-de Sitter space, such that the singularity p corresponds to the apex of C_p. Consider a sequence of smooth convex functions, whose graphs coincide with the cone C_p outside a neighborhood of the apex, and converging to C_p (this is very classical, see e.g. [Slu13, Lemma 3.9]).

Using Gauss formula, one can easily check that the sectional curvature for the induced metric on the smooth approximating surfaces is ≤ -1. We can multiply those metrics by any constant $\lambda > 1$ to get the sectional curvature < -1. As the surfaces differ only on a compact set, and as the approximating sequence is smooth, it follows from (4.7) that the induced distances are uniformly bi-Lipschitz to the hyperbolic metric. From this and Proposition 4.1.8, it is classical to deduce that the induced distances converges locally uniformly (hence uniformly in this case), see e.g. the proof of Proposition 3.12 in [FS18].

The proposition follows by applying this procedure simultaneously to all singular points of the metric d.

□

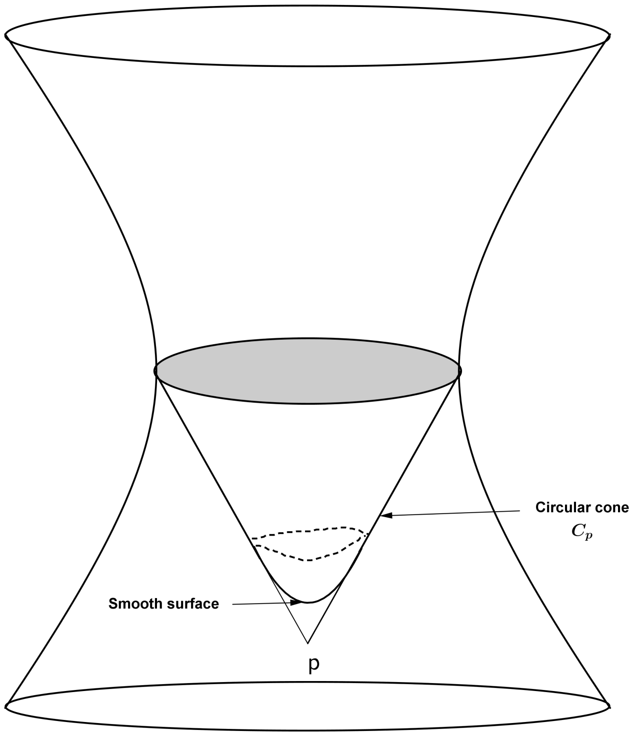

Figure 3.3: Smooth surface and circular cone in Anti-de Sitter space.

Let d be any metric of curvature ≤ -1 on a compact surface S. We obtain a sequence $(d_n)_n$ from Theorem 3.3.5, and for each d_n, a sequence $(d_{n_k})_k$ from Proposition 3.4.1.

The following Theorem follows directly from a diagonal argument.

Theorem 3.4.2. *Let (S, d) be a metric of curvature ≤ -1. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n are homeomorphic to S and d_n are induced by Riemannian metrics with sectional curvature < -1 .*

Chapter 4

Realization of metrics

4.1 Realization of metrics with curvature ≤ -1

4.1.1 Definition of the problem and outline of the proof

In all the following, S is a closed connected oriented surface and when we speak about a metric with curvature $\leq k$, this means that S is endowed with a distance d satisfying a curvature bound in the sense of A.D. Alexandrov, see e.g. [BBI01] or Section 3. We want to prove the following Theorem.

Theorem 4.1.1. *Let d be a metric with curvature ≤ -1 on a closed surface S of genus > 1. Then there exists a Lorentzian manifold L of sectional curvature -1 homeomorphic to $S \times \mathbb{R}$ which contains a space-like convex surface whose induced metric is isometric to (S, d).*

The proof of Theorem 4.1.1 will be given by a classical approximation procedure, following the main lines of [FS18]. The proof relies on the smooth analogue of Theorem 4.1.1 proved by F. Labourie and J.-M. Schlenker, see Theorem 4.1.27. We will prove Theorem 4.1.1 showing that the universal cover of (S, d) is isometric to a convex surface in anti-de Sitter space, invariant under the action of a discrete group of isometries leaving invariant a totally geodesic hyperbolic surface. Such groups are usually called *Fuchsian*, and the quotient of a suitable part of anti-de Sitter space by such a group may be called a Fuchsian anti-de Sitter manifold. The main issues in our case, comparing to [FS18], is that we lost the vector space structure given by the Minkowski space —it is the Lorentzian analogue of the problem to go from Euclidean space to hyperbolic space.

4.1.2 Coordinates for a region of AdS^3

In all the following, we will be interested in surfaces invariant under the action of a Fuchsian group. We will see that theses surfaces are living in a cylinder. In the following we introduce this cylinder and endow it with a suitable metric (a metric in term of two suitable coordinates).

Note that $AdS^3 \cap \{x \in \mathbb{R}^4 | x_1 = 0\} =: H_0$ is isometric to the hyperbolic plane. We use this fact to define the following map. Let $\tilde{\Psi} : \mathbb{H}^2 \times \mathbb{R} \longrightarrow AdS^3$ be the map defined by $\tilde{\Psi}(x,t) = \exp_x(tV)$ where

- $\tilde{\Psi}(\mathbb{H}^2, 0) = H_0$, and $x \mapsto \tilde{\Psi}(x,0)$ is an isometry,

- V is a choice of a unit vector field orthogonal to H_0, for the anti-de Sitter metric.

Indeed, we have $\tilde{\Psi}(x,t) = \cos(t)x + \sin(t)V$ with $V = (0,-1,0,0)$. For a given x, $t \mapsto \tilde{\Psi}(x,t)$ is a time-like geodesic loop with time-length 2π. We will call *AdS cylinder* the cylinder $\mathbb{H}^2 \times [0, \pi/2[$ endowed with the Lorentzian metric g_{AdS}, which is the pull back of the anti-de Sitter metric by $\tilde{\Psi}$. Let us denote $AdS^3 \cap \{x \in \mathbb{R}^4 | x_1 = r\} =: H_r$. The induced metric onto H_r is homothetic to the hyperbolic metric with factor $(1-r^2)$, and clearly $\tilde{\Psi}(\mathbb{H}^2, t) = H_{\sin(t)}$. In turn,

$$g_{AdS}(x,t) = \cos^2(t) g_{\mathbb{H}^2}(x) - dt^2$$

where $g_{\mathbb{H}^2}$ is the metric on the hyperbolic plane.

It will be suitable to work with the image of $\tilde{\Psi}$ in the affine chart considered above. Let us denote $\Psi = \varphi_0 \circ \tilde{\Psi}$. The set $\Psi(\mathbb{H}^2 \times [0, \pi/2[)$ is indeed a Euclidean half-cylinder in $\mathbb{R}^3$ (see Figure 4.1). We have $\Psi(\mathbb{H}^2, 0) = \mathbb{D}$ and for $x \in \mathbb{H}^2$, $t \mapsto \tilde{\Psi}(x,t)$ is a vertical half line from $\mathbb{D}$. We will call *affine AdS cylinder* the image of $\mathbb{H}^2 \times [0, \pi/2[$ by Ψ. For convexity reasons, we will need only to consider a half cylinder.

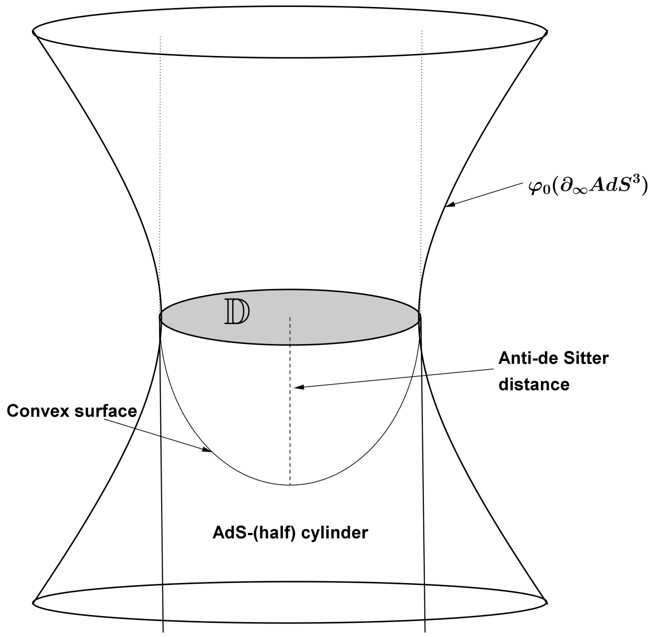

Figure 4.1: The AdS cylinder and a convex surface inside.

4.1.3 Convex functions

For a function $u : \mathbb{H}^2 \to [0, \pi/2[$, we denote

$$S_u = \{(x, u(x)) | x \in \mathbb{H}^2\} \ .$$

For every $x \in \mathbb{H}^2$ we denote by $\bar{x} = \Psi(x, 0)$ the corresponding point on the disc $\mathbb{D}$, where Ψ it the map introduced in the previous section. The image of S_u in the affine AdS cylinder is the graph of a function over $\mathbb{D}$, that we will denote by $\bar{u}$. We will denote by $S_{\bar{u}}$ the image of S_u. Hence, $\bar{u} : \mathbb{D} \to \mathbb{R}$ and

$$(\bar{x}, \bar{u}(\bar{x})) = \Psi(x, u(x)) \ .$$

For a point $\bar{x} \in \mathbb{D}$ we use the notation $\bar{x} = (\bar{x}_2, \bar{x}_3)$ for its Euclidean coordinates, and its Euclidean norm is $\| \bar{x} \| = \sqrt{\bar{x}_2^2 + \bar{x}_3^2}$. We obtain the following relation.

Lemma 4.1.2. *With the notations above* $\bar{u}(\bar{x}) = -\tan(u(x))\sqrt{1 - \| \bar{x} \|^2}$.

Proof. We may use the quadratic model (Definition 2.3.2). Let's consider the surface given by the intersection of the Anti-de Sitter space and the set $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 | x_1 = r\}$ where r is a constant. The equation of the intersection surface is given by

$$-x_0^2 + x_2^2 + x_3^2 = -1 + r^2.$$

Its image in the affine chart φ_0 is given by the following simple computation,

$$-1 + \bar{x}_2^2 + \bar{x}_3^2 = (-1 + r^2)\tfrac{1}{x_0^2},$$

it follows that

$$\frac{1 - r^2}{r^2}\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = 1 \tag{4.1}$$

and this is an ellipsoid of parameters $(\sqrt{\frac{r^2}{1-r^2}}, 1, 1)$.

Also the intersection of AdS^3 with the set $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 | x_2 = x_3 = 0\}$ is the circle (see Figure 4.2) whose equation is:

$$x_0^2 + x_1^2 = 1.$$

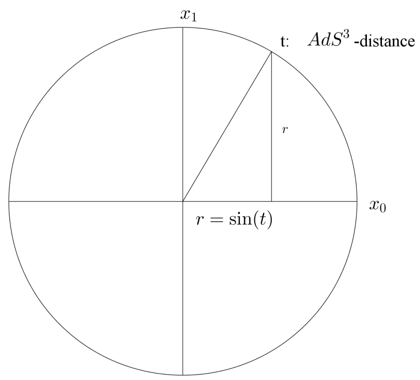

Figure 4.2: We look at the induced AdS^3 distance on this intersection.

From the computation above, we have that $r = \sin(t)$ where t is the anti-de Sitter distance from $\mathbb{H}^2$. (It is straightforward to see that the set of points with constant Anti-de Sitter distance from $\mathbb{H}^2$ is an ellipsoid). From equation (4.1) it follows that

$$\frac{1}{\tan^2(t)}\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = 1 \tag{4.2}$$

Let $(x, u(x))$ be a point on the ellipsoid (the AdS distance from $\mathbb{H}^2$ is constant) of equation (4.2) and $(\bar{x}, \bar{u}(\bar{x}))$ be its corresponding point. Considering the lower side of the ellipsoid, then for all $x \in \mathbb{H}^2$ we have

$$\bar{u}(0,0) = -\tan(u(x)) \tag{4.3}$$

Now, let's consider in the affine chart (in the Euclidean cylinder) the lower half ellipsoid with parameters $(\bar{u}(0,0), 1, 1)$, its intersection with the perpendicular line passing through the point $(0, \bar{x}_2, \bar{x}_3)$ verifies,

$$\frac{\bar{u}^2(\bar{x}_2, \bar{x}_3)}{\bar{u}^2(0,0)} + \bar{x}_2^2 + \bar{x}_3^2 = 1 \tag{4.4}$$

From (4.3) and (4.4) we deduce that

$$\bar{u}(\bar{x}_2, \bar{x}_3) = -\tan(u(x))\sqrt{1 - \bar{x}_2^2 - \bar{x}_3^2} \tag{4.5}$$

□

The Lemma above clarify the notion of convexity in our model, in fact a convex function u corresponds to a function $\bar{u}$ which is actually convex in $\mathbb{R}^3$.

Definition 4.1.3. *Let $u : \mathbb{H}^2 \to \mathbb{R}$ be a function. We say that u is* C-convex *if*

- *$u \geq 0$ and there is $R < \pi/2$ such that $u \leq R < \pi/2$;*
- *the corresponding function $\bar{u}$ is convex.*

It is worth noting that for $R \geq 0$, if $u = R$, then the graph of the map defined by $\bar{u}(\bar{x}) = -\tan(R)\sqrt{1- \parallel \bar{x} \parallel^2}$ is a half ellipsoid. Also, $|\bar{u}(\bar{x})| \leq \tan(R)\sqrt{1- \parallel \bar{x} \parallel^2}$. It follows that if u is C-convex, then $\bar{u}$ is bounded and satisfies $\bar{u}|_{\partial\mathbb{D}} = 0$.

It is also clear that a bounded convex function $\bar{u} : \mathbb{D} \to \mathbb{R}$ vanishes everywhere if it vanishes in a point of the open disc $\mathbb{D}$. So we have $\bar{u} \leq 0$ by definition, and $\bar{u} < 0$ or $\bar{u} = 0$.

Let us note the following.

Lemma 4.1.4. *In the image of AdS^3 by φ_0,*

1. *Every time-like line passes through the disc $\mathbb{D}$.*
2. *Every light-like line which doesn't pass through the boundary $\partial_\infty\mathbb{D}$ must pass through the disc $\mathbb{D}$.*

3. *A cone with basis the disc $\mathbb{D}$ and with apex in the affine cylinder is a convex space-like surface.*

Proof. The proof is very easy, as we know that times-like lines are strictly contained in the hyperboloid, and that light-like lines meet $\partial_\infty AdS^3$ at most once.

1) In the affine chart φ_0, let D be a time-like line. The general equation of D passing by a point $(\bar{x}_1^A, \bar{x}_2^A, \bar{x}_3^A)$ is,

$$\begin{cases} \bar{x}_1 = a_1 t + \bar{x}_1^A \\ \bar{x}_2 = a_2 t + \bar{x}_2^A \\ \bar{x}_3 = a_3 t + \bar{x}_3^A \end{cases}$$

If $a_1 = 0$ then D is parallel to the plane $(o\bar{x}_2\bar{x}_3)$, in this case it's easy to see that it meets $\partial_\infty AdS^3$ in two points, D is then space-like line. Now let's suppose that $a_1 \neq 0$, by a property of time-like lines, D is strictly contained in the one sheeted hyperboloid.

$$-(a_1 t + \bar{x}_1^A)^2 + (a_2 t + \bar{x}_2^A)^2 + (a_3 t + \bar{x}_3^A)^2 < 1,$$

the intersection with the plane $\{\bar{x}_1 = 0\}$ gives $t = -\frac{\bar{x}_1^A}{a_1}$ hence the intersection of D with the plane $\{\bar{x}_1 = 0\}$ which is the point

$$(0, -a_2\frac{\bar{x}_1^A}{a_1} + \bar{x}_2^A, -a_3\frac{\bar{x}_1^A}{a_1} + \bar{x}_3^A),$$

verifies

$$(-a_2\tfrac{\bar{x}_1^A}{a_1} + \bar{x}_2^A)^2 + (-a_3\tfrac{\bar{x}_1^A}{a_1} + \bar{x}_3^A)^2 < 1,$$

it follows that D passes through $\mathbb{H}^2$.

2) A light-like line meets the boundary $\partial_\infty AdS^3$ in at most one point. Then with the same computation, we have for all t, $(-a_2\frac{\bar{x}_1^A}{a_1} + \bar{x}_2^A)^2 + (-a_3\frac{\bar{x}_1^A}{a_1} + \bar{x}_3^A)^2 \leq 1$ as the line doesn't pas by $\partial_\infty\mathbb{D}$ then $(-a_1\frac{z_A}{a_2} + \bar{x}_2^A)^2 + (-a_2\frac{z_A}{a_2} + \bar{x}_3^A)^2 < 1$ then it must pass trough the disc $\mathbb{H}^2$.

3) For the third point, either a support plane of the cone does not meet the closure of $\mathbb{D}$, hence it is space-like, or a support plane of the cone contains a half-line of the cone, then it meets the boundary of the disc, but by assumption this half-line is not vertical, hence not light-like, so the plane is space-like.

□

Lemma 4.1.5. *Let $u : \mathbb{H}^2 \to [0, \pi/2[$ be a C-convex function. Then the surface S_u is space-like.*

Proof. Let p be a point on the image of S_u in the affine half cylinder, and let C_p be the cone with basis the disc $\mathbb{D}$ and apex p. By definition, this cone is contained in the affine half cylinder. By convexity, a support plane to the surface at p is a support plane of the cone, so by Lemma 4.1.4 it must be space-like. □

We say that a sequence $(u_n)_n$ of C-convex functions is *uniformly bounded* if there is $R < \pi/2$ such that for any n, $u_n < R$.

Lemma 4.1.6. *Let $(u_n)_n$ be a sequence of uniformly bounded C-convex functions. Up to extracting a subsequence, $(u_n)_n$ converges to a C-convex function u, uniformly on compact sets.*

Proof. This is a classical property of the corresponding convex functions $\bar{u}_n$, [Roc97, Theorem 10.9], in the special case when the surfaces vanish on the boundary of the disc $\mathbb{D}$. □

Let u_n, $n > 1$, be uniformly bounded C-convex functions converging to a C-convex function $u = u_0$. Let $c : I \to \mathbb{H}^2$ be a Lipschitz curve and $\bar{c} : I \to \mathbb{D}$ be its image by Ψ. Then $\bar{u} \circ \bar{c}$, $\bar{u}_n \circ \bar{c}$ are Lipschitz —the Lipschitz nature of $\bar{c}$ is independent of a choice of a Riemannian metric on the disc. By Rademacher Theorem, there exists a set I_0 of Lebesgue measure 0 in I such that for all $n \in \mathbb{N}$, $\bar{u}_n$ is differentiable on $I \setminus I_0$.

Lemma 4.1.7. *Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ be uniformly bounded C-convex functions converging to a C-convex function u, and let $c : I \to \mathbb{H}^2$ be a Lipschitz curve. Up to extracting a subsequence, for almost all t,*

$$(u_n \circ c)'(t) \to (u \circ c)'(t) \ .$$

Proof. The following proof is a straightforward adaptation of [FS18, Lemma 3.6]. We first prove the Lemma for the corresponding functions $\bar{u}_n$ and $\bar{u}$, then we deduce the proof for u_n and u using continuity and Lemma 4.1.2. We consider that $\bar{c}$ is parameterized by arc-length.

Let $\langle\cdot,\cdot\rangle$ be the the Euclidean metric on the affine cylinder, and we use the notation $\begin{pmatrix} a \\ b \end{pmatrix}$, with $a \in \mathbb{D}$ and $b \in \mathbb{R}$. Let t be such that the derivatives exist. Let X be the unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and Y the unit vector $\begin{pmatrix} \bar{c}'(t) \\ 0 \end{pmatrix}$, we have $\langle X, Y\rangle$=0. The tangent vector to the curve $\begin{pmatrix} \bar{c} \\ \bar{u}_n \circ \bar{c} \end{pmatrix}$ at every point $\begin{pmatrix} \bar{c}(t) \\ (\bar{u}_n \circ \bar{c})(t) \end{pmatrix}$ is given by

$$V_n = (\bar{u}_n \circ \bar{c}(t))' X + Y$$

and in the plane P spanned by X and Y, the vector

$$N_n = (\bar{u}_n \circ \bar{c}(t))' Y - X$$

is orthogonal to V_n for $\langle \cdot, \cdot \rangle$. Now because $\bar{u}_n$ and $\bar{u}$ are equi-Lipschitz on any compact set of $\mathbb{D}$ (see [Roc97, Theorem 10.6]) then there exists k such that $|(\bar{u}_n \circ \bar{c})'(t)| \leq k$ for all $n \in \mathbb{N}$, then

$$\| N_n \| \leq |(\bar{u}_n \circ \bar{c})'(t)| \| Y \| + \| X \| \leq |(\bar{u}_n \circ \bar{c})'(t)| + 1 \leq k + 1$$

so $\| N_n \|$ are uniformly bounded. Hence, up to extracting a subsequence $(N_n)_n$ converges to a vector N. Note that N is not the zero vector, otherwise $\langle N_n, X \rangle$ would converge to 0, that is impossible because $\langle N_n, X \rangle = -1$.

Let T_n be the intersection of the convex surface $S_{\bar{u}_n}$ defined by $\bar{u}_n$ and the plane P. The set T_n is a convex set in P, and V_n is a tangent vector, hence by convexity for any $\bar{y} \in \mathbb{D} \cap P$,

$$\langle N_n, \begin{pmatrix} \bar{c}(t) \\ \bar{u}_n \circ \bar{c}(t) \end{pmatrix} - \begin{pmatrix} \bar{y} \\ \bar{u}_n(\bar{y}) \end{pmatrix} \rangle \geq 0,$$

and passing to the limit we get

$$\langle N, \begin{pmatrix} \bar{c}(t) \\ \bar{u} \circ \bar{c}(t) \end{pmatrix} - \begin{pmatrix} \bar{y} \\ \bar{u}(\bar{y}) \end{pmatrix} \rangle \geq 0,$$

this says that N is a normal vector to T (the intersection of $S_{\bar{u}}$ with P), hence

$$\langle N, (\bar{u} \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \bar{c}'(t) \\ 0 \end{pmatrix} \rangle = 0.$$

So there exists λ such that

$$(\bar{u} \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \bar{c}'(t) \\ 0 \end{pmatrix} = \lambda \lim_{n \to \infty} (\bar{u}_n \circ \bar{c})'(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \bar{c}'(t) \\ 0 \end{pmatrix}.$$

By identification it follows that $\lambda = 1$, hence $(\bar{u}_n \circ \bar{c})'(t)$ must converge to $(\bar{u} \circ \bar{c})'(t)$. The functions $\bar{u}_n \circ \bar{c}$ and $u_n \circ c$ are defined from $I \subset \mathbb{R}$ to $\mathbb{R}$, by Lemma 4.1.2

$$u_n \circ c(t) = \arctan \left(\frac{\bar{u}_n \circ \bar{c}(t)}{h(t)} \right)$$

where $h(t) = -\sqrt{1 - \| \bar{c}(t) \|^2}$, hence $u_n \circ c$ is clearly differentiable almost everywhere for all $n \in \mathbb{N}$ and

$$(u_n \circ c)'(t) = \frac{(\bar{u}_n \circ \bar{c})'(t)h(t) - (\bar{u}_n \circ \bar{c})(t)h'(t)}{h^2(t) + (\bar{u}_n \circ \bar{c})^2(t)} \tag{4.6}$$

also we have (by hypothesis) for almost all t, that

$$(u_n \circ c)(t) \underset{n\to\infty}{\longrightarrow} (u \circ c)(t)$$

hence by continuity (in the relation given by Lemma 4.1.2) it is clear that,

$$(\bar{u}_n \circ \bar{c})(t) \underset{n\to\infty}{\longrightarrow} (\bar{u} \circ \bar{c})(t)$$

then by the preceding arguments and by continuity again in (4.6) and passing to the limit, it follows that $(u_n \circ c)'(t)$ converge to $(u \circ c)'(t)$. □

Let $u : \mathbb{H}^2 \to \mathbb{R}$ be a C-convex function. For $c : [0, 1] \to \mathbb{H}^2$ a Lipschitz curve, $(c, u \circ c)$ is a curve on S_u, and its length for the anti-de Sitter metric is

$$\mathcal{L}_u(c) = \int_0^1 \sqrt{\cos^2(u \circ c(t))\|c'(t)\|^2_{\mathbb{H}^2} - (u \circ c)'^2(t)}dt \ . \tag{4.7}$$

By Lemma 4.1.7 above and using the dominated convergence Theorem, we get the following proposition.

Proposition 4.1.8. *Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ be uniformly bounded C-convex functions converging to a C-convex function u, and let $c : I \to \mathbb{H}^2$ be a Lipschitz curve. Up to extracting a subsequence, $\mathcal{L}_{u_n}(c) \to \mathcal{L}_u(c)$.*

The induced (intrinsic) metric d_{S_u} on S_u is the pseudo-distance induced by $\mathcal{L}_u$: for $x, y \in S_u$, $d_{S_u}(x, y)$ is the infimum of the lengths of Lipschitz curves between x and y contained in S_u. Note that as the AdS cylinder has a Lorentzian metric, the induced distance between two distinct points on S_u may be equal to 0, that is a major difference with the case of induced metrics on surfaces in a Riemannian space.

Definition 4.1.9. *We denote by d_u the pull-back of d_{S_u} on $\mathbb{H}^2$, so that for every point $x, y \in \mathbb{H}^2$*

$$d_u(x, y) = d_{S_u}((x, u(x)), (y, u(y))).$$

From (4.7), as $\cos \leq 1$, we clearly have the following.

Lemma 4.1.10. *With the notations above, for $x, y \in \mathbb{H}^2$, $d_u(x, y) \leq d_{\mathbb{H}^2}(x, y)$.*

Proof. Let $c:[0,1]\to\mathbb{H}^2$ be a Lipschitz curve such that $c(0)=x$ and $c(1)=y$. The length of the curve $\begin{pmatrix} c \\ u\circ c\end{pmatrix}$ is given by

$$\mathcal{L}_u(c)=\int_0^1\sqrt{\cos^2(u\circ c(t))\|c'(t)\|^2_{\mathbb{H}^2}-(u\circ c)'^2(t)}dt,$$

it is clear that

$$\mathcal{L}_u(c)\leq\int_0^1\sqrt{\cos^2(u\circ c(t))\parallel c'(t)\parallel^2_{\mathbb{H}^2}}dt\leq\int_0^1\parallel c'(t)\parallel_{\mathbb{H}^2}dt,$$

it follows that $d_u\leq d_{\mathbb{H}^2}$. □

4.1.4 Fuchsian invariance

Convergence of surfaces implies convergence of metrics

The aim of this section is to state Proposition 4.1.19. The arguments are quite general and close to the ones of [FS18]. The main point is Lemma 4.1.14 below, that is the AdS analogue of Corollary 3.11 in [FS18].

Recall that a Fuchsian group is a discrete group of orientation preserving isometries acting on the hyperbolic plane. In the present article, we will restrict this definition to the groups acting moreover freely and cocompactly.

Definition 4.1.11. *A* Fuchsian C-convex function *is a couple* (u,Γ)*, where* u *is a C-convex function and* Γ *is a Fuchsian group such that for all* $\sigma\in\Gamma$ *we have* $u\circ\sigma=u$*.*

We will often abuse terminology, speaking about Fuchsian for a single function u, so that the Fuchsian group will remain implicit.

Definition 4.1.12. *Let* $(\Gamma_n)_n$ *be a sequence of discrete groups.* $(\Gamma_n)_n$ *converges to a group* Γ *if there exist isomorphisms* $\tau_n:\Gamma\to\Gamma_n$ *such that for all* $\sigma\in\Gamma,\tau_n(\sigma)$ *converge to* σ*.*

Definition 4.1.13. *We say that a sequence of Fuchsian C-convex functions* $(u_n,\Gamma_n)_n$ *converges to a pair* (u,Γ)*, if* u *is a C-convex function,* Γ *is a Fuchsian group such that* $(u_n)_n$ *converges to* u *and* $(\Gamma_n)_n$ *converges to* Γ*.*

It is easy to see that if (u_n,Γ_n) is a sequence of Fuchsian C-convex functions that converges to a pair (u,Γ), then (u,Γ) is a Fuchsian C-convex function, see e.g. [FS18, Lemma 3.17]. Recall the definition of the distance d_u from Definition 4.1.9. Recall also that a C-convex function is differentiable almost everywhere. At a point where u is differentiable, we denote by $\|\cdot\|_u$ the norm induced by the ambient anti-de Sitter metric on the tangent of S_u at this point.

Lemma 4.1.14. *Let u be a C-convex function. Let $K := \inf(\| v \|_u / \| v \|_{\mathbb{H}^2})$, and let $d_{\mathbb{H}^2}$ be the distance given by the hyperbolic metric (for instance, $d_{\mathbb{H}^2} = d_u$ for $u = 0$). Then $d_u(x,y) \geq K d_{\mathbb{H}^2}(x,y)$.*

Moreover, if u is Fuchsian, then $K > 0$.

Proof. Let c be a Lipschitz curve between two points $x, y \in \mathbb{H}^2$. Let v be the tangent vector field of $(c, u \circ c)$ whenever it exists. We have

$$\mathcal{L}_u(c) = \int_a^b \| v \|_u \geq K \int_a^b \| v \|_{\mathbb{H}^2} \geq K d_{\mathbb{H}^2}(x,y)$$

and the first result follows as by definition $d_u(x,y)$ is an infimum of lengths.

Now let us suppose that u is Fuchsian. Let us suppose that $K = 0$, i.e. there is a sequence $(x_n)_n$ such that u is differentiable at each x_n, and $v_n \neq 0$ in $T_{x_n}\mathbb{H}^2$ such that $\| v_n \|_u / \| v_n \|_{\mathbb{H}^2} \to 0$. Without loss of generality, let us consider that $\| v_n \|_{\mathbb{H}^2} = 1$. Let σ_n be isometries of $\mathbb{H}^2$ that send (x_n, v_n) to a given pair (x, v), and let $u_n := u \circ \sigma_n$. As u is Fuchsian, there exists $\beta < \pi/2$ such that $u \leq \beta$, and in turn $u_n \leq \beta$. By Lemma 4.1.6, up to consider a subsequence, $(u_n)_n$ converges to a C-convex function u_0. As we supposed that $\| v_n \|_u \to 0$, then S_{u_0} must have a light-like support plane, that contradicts Lemma 4.1.5. □

Note that Lemma 4.1.14 indicates that in the Fuchsian case, d_u is a distance and not only a pseudo-distance.

Let us recall the following classical result, see e.g. Lemma 3.14 in [FS18]. The homeomorphisms in the statement below could also be constructed by hand, for example using canonical polygons as fundamental domains for the Fuchsian groups, see Section 6.7 in [Bus10].

Lemma 4.1.15. *Let $(\Gamma_n)_n$ be a sequence of Fuchsian groups converging to a group Γ and τ_n the isomorphisms given in Definition 4.1.12. There exist homeomorphisms $\phi_n : \mathbb{H}^2/\Gamma \longrightarrow \mathbb{H}^2/\Gamma_n$ whose lifts $\tilde{\phi}_n$ satisfy for any $\sigma \in \Gamma$,*

$$\tilde{\phi}_n \circ \sigma = \tau_n(\sigma) \circ \tilde{\phi}_n$$

and such that $(\tilde{\phi}_n)_n$ converges to the identity map uniformly on compact sets i.e

$$\forall x \in \mathbb{H}^2, \tilde{\phi}_n(x) \underset{n\to\infty}{\longrightarrow} x$$

Now, let u be a C-convex function and S_u the surface described by u. The length structure $\mathcal{L}_u$ given by (4.7) induces a (pseudo-)distance d_{S_u}. In turn, d_{S_u} induces a length structure denoted by $L_{d_{S_u}}$ defined in the following way: the length of a curve $(c, u \circ c) : [0,1] \to S_u$ is defined as

$$\begin{aligned} L_{d_{S_u}}(c, u \circ c) &= \sup_{\delta} \sum_{i=1}^{n} d_{S_u}((c(t_i), u \circ c(t_i)), (c(t_{i+1}), u \circ c(t_{i+1}))), \\ &= \sup_{\delta} \sum_{i=1}^{n} d_u(c(t_i), c(t_{i+1})) = L_{d_u}(c) \qquad \text{(see definition 4.1.9)} \end{aligned}$$

where the sup is taken over all the decompositions

$$\delta = \{(t_1 \dots t_n) | t_1 = 0 \leq t_2 \leq ... \leq t_n = 1\},$$

We have the following proposition

Proposition 4.1.16. *Let $(u_n)_n$ be a sequence of convex functions such that:*

- *d_{u_n} is a complete distance with Lipschitz shortest paths,*
- *$\mathcal{L}_{u_n} = L_{d_{u_n}}$ on the set of Lipschitz curves,*
- *There exists $0 < R < \pi/2$ with $0 \leq u_n < R$,*

Then, up to extracting a subsequence, $(u_n)_n$ converges to a convex function u and $(d_{u_n})_n$ converges to d_u uniformly on compact sets.

Proof. The proof of this proposition is similar as the one done in [FS18, Proposition 3.12]. The proof was done using proposition 4.1.8, the only difference is to use Lemma 4.1.10 and 4.1.14 instead of [FS18, corollary 3.11].

□

We recall that in this paper we are using approximation by smooth surfaces. We note also that by Lemma 4.1.14 and Lemma 4.1.10, d_{u_n} are complete distances on $\mathbb{H}^2$, also we have $\mathcal{L}_{u_n} = L_{d_{u_n}}$ (because of smoothness, see [Bur15] for more details), we deduce the following

Lemma 4.1.17. *Let (u_n, Γ_n) be Fuchsian C-convex functions such that:*

- *$(u_n, \Gamma_n)_n$ converges to a pair (u, Γ),*
- *There exist $0 < R < \pi/2$ with $0 \leq u_n < R$,*

- *d_{u_n} are distances with Lipschitz shortest paths,*
- *d_{u_n} converge to d_u, uniformly on compact sets.*

Then on any compact set of $\mathbb{H}^2$, $d_{u_n}(\tilde{\phi}_n(.), \tilde{\phi}_n(.))$ uniformly converge to d_u, where $\tilde{\phi}_n$ is given by Lemma 4.1.15.

Proof. By Lemma 4.1.14 and Lemma 4.1.10, the topology induced by d_u onto $\mathbb{H}^2$ is the topology for the hyperbolic metric. It follows that for the maps $\tilde{\phi}_n$ of Lemma 4.1.15, we have that on compact sets, the maps $x \mapsto d_{u_n}(\tilde{\phi}_n(x), x)$ uniformly converge to 0. By the triangle inequality we have,

$$d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_u(x, y) \leq d_{u_n}(\tilde{\phi}_n(x), x) + d_{u_n}(\tilde{\phi}_n(y), y) + d_{u_n}(x, y) - d_u(x, y)$$

by the preceding arguments and proposition 4.1.16, for n sufficiently large the right-hand side is uniformly less than any $\epsilon > 0$. On the other hand, by triangle inequality again we have,

$$\begin{aligned} d_u(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) = \; & d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) \\ \leq \; & d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, \tilde{\phi}_n(x)) + d_{u_n}(y, \tilde{\phi}_n(y)) \\ + \; & d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) \end{aligned}$$

which is uniformly less than any $\epsilon > 0$ for n sufficiently large (by the same arguments).

□

By definition, if (u, Γ) is a Fuchsian C-convex function, then Γ acts by isometries on d_u. In turn, d_u defines a distance on the compact surface $\mathbb{H}^2/\Gamma$.

Definition 4.1.18. *For a Fuchsian C-convex function (u, Γ), we denote by $\bar{d}_u$ the distance defined by d_u on $\mathbb{H}^2/\Gamma$.*

The reason to introduce the maps $\tilde{\phi}_n$ from Lemma 4.1.15 is the following Corollary of Lemma 4.1.17. Its proof is formally the same as the one of Proposition 3.19 in [FS18]. (The definition of uniform convergence of metric spaces is recalled in Definition 3.3.1.)

Proposition 4.1.19. *Let (u_n, Γ_n) be Fuchsian C-convex functions converging to a pair (u, Γ). Up to extracting a subsequence, $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ uniformly converges to $(\mathbb{H}^2/\Gamma, \bar{d}_u)$.*

Convergence of metrics implies convergence of groups

The aim of this section is to prove Proposition 4.1.20, that may be seen as a kind of converse of Proposition 4.1.19. The distance $\bar{d}_u$ was defined in Definition 4.1.18.

Proposition 4.1.20. *Let (S, d) be a metric of curvature ≤ -1 and let (u_n, Γ_n) be smooth Fuchsian C-convex functions, such that the sequence $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ uniformly converges to (S, d). Up to extracting a subsequence,*

- *$(\Gamma_n)_n$ converges to a Fuchsian group Γ;*
- *there exists $0 < \beta < \pi/2$ such that $0 \leq u_n < \beta$.*

Under the hypothesis of Proposition 4.1.20, let's first prove the convergence of groups. We first have a consequence of simple hyperbolic geometry, see [FS18, Corollary 4.2].

Lemma 4.1.21. *There exists $G > 0$ and $N > 0$ such that for any $n > N$, for any $x \in \mathbb{H}^2$, for every element $\sigma_n \in \Gamma_n \setminus \{0\}$*

$$d_{u_n}(x, \sigma_n(x)) \geq G.$$

Proposition 4.1.22. *Under the hypothesis of Proposition 4.1.20, up to extracting a subsequence, the sequence $(\Gamma_n)_n$ converges to a Fuchsian group Γ.*

Proof. First by Lemma 4.1.10 we have that for all $x, y \in \mathbb{H}^2$,

$$d_{u_n}(x, y) \leq d_{\mathbb{H}^2}(x, y),$$

and by Lemma 4.1.21, we have that there exists $G > 0$ and $N > 0$ such that for any $n > N$ and for any $x \in \mathbb{H}^2$:

$$G \leq d_{u_n}(x, \sigma_n(x)) \leq d_{\mathbb{H}^2}(x, \sigma_n(x)),$$

in particular if

$$L_{\sigma_n} = \min_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(x, \sigma_n(x)),$$

we have

$$G \leq L_{\sigma_n}.$$

The length is uniformly bounded from below, hence by a classical result of Mumford [Mum71] we can deduce that up to extracting a subsequence, the sequence of groups converges. □

Lemma 4.1.23. *Under the assumptions of Proposition 4.1.20, there exists $M < \pi/2$ such that for all n, there is $x_n \in \mathbb{H}^2$ such that $u_n(x_n) < M$.*

Proof. Suppose that the result is false: for a sequence $M_k \to \pi/2$, there is n_k such that $u_{n_k} \geq M_k$. By the definition of the length structure (4.7), it follows that $d_{u_{n_k}} \leq \cos M_k d_{\mathbb{H}^2}$. In turn, $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ has a subsequence converging to 0, that is a contradiction. □

Proposition 4.1.24. *Under the hypothesis of Proposition 4.1.20, there exists $0 < \beta < \pi/2$ such that, for any $n \in \mathbb{N}$, for any $x \in \mathbb{H}^2$,*

$$u_n(x) < \beta.$$

To prove this proposition, let us consider the affine model of anti-de Sitter space. As the sequence of groups converges, there exists a compact set $C \subset \mathbb{D}$, which contains a fundamental domain for Γ_n for all n. Hence the points x_n given by Lemma 4.1.25 can be chosen to all belong to C. The result follows because the convex maps $\bar{u}_n$ on the disc are zero on the boundary, so for any compact set C in the interior of the disc, the difference between the minimum and the maximum of $\bar{u}_n$ on C cannot be arbitrary large.

The following two facts are just a detailed proof of the previous paragraph.

Fact 4.1.25. $\exists M \in [0, \pi/2[, \exists x \in C, \forall n \in \mathbb{N}, u_n(x) < M$

Proof. By contradiction, we suppose that

$$\forall M \in [0, \pi/2[, \forall x \in C, \exists n \in \mathbb{N}, u_n(x) \geq M.$$

is true.

Let x', y' be two points in $\mathbb{H}^2$ and $c : [0,1] \to \mathbb{H}^2$ be a Lipschitz curve, such that $c(0) = x'$ and $c(1) = y'$.

Let $u_n : \mathbb{H}^2 \to [0, \pi/2[$ be a convex function, we denote by $\mathcal{L}_{u_n}(c)$ the length of the curve $\begin{pmatrix} c \\ u_n \circ c \end{pmatrix}$ between the two points $\begin{pmatrix} x \\ u_n(x) \end{pmatrix}$ and $\begin{pmatrix} y \\ u_n(y) \end{pmatrix}$ of the surface defined by the function u_n. We have

$$\mathcal{L}_{u_n}(c) = \int_0^1 \sqrt{\cos^2(u_n \circ c(t)) \parallel c'(t) \parallel^2_{\mathbb{H}^2} - (u_n \circ c)'^2(t)} dt.$$

Let's consider the ellipsoid of parameter $(M, 1, 1)$ so the length of the curve between the two points $\begin{pmatrix} x \\ M \end{pmatrix}$, $\begin{pmatrix} y \\ M \end{pmatrix}$ will be given by

$$\mathcal{L}_M(x,y) = \int_0^1 \sqrt{\cos^2(M) \parallel c'(t) \parallel^2_{\mathbb{H}^2}} dt.$$

Now we suppose that $\forall x \in \mathbb{H}^2, \exists n \in \mathbb{N}, u_n(x) \geq M$ It follows that for each $t \in [0,1]$

$$\cos(u_n \circ c(t)) \leq \cos(M)$$

then

$$\int_0^1 \sqrt{\cos^2(u_n \circ c(t)) \parallel c'(t) \parallel^2_{\mathbb{H}^2} - (u_n \circ c)'^2(t)} dt \leq \int_0^1 \sqrt{\cos^2(M) \parallel c'(t) \parallel^2_{\mathbb{H}^2}} dt.$$

hence

$$\mathcal{L}_{u_n}(c) \leq \cos(M)\mathcal{L}_{\mathbb{H}^2}(c),$$

then

$$d_{u_n}(x,y) \leq \cos(M) d_{\mathbb{H}^2}(x,y).$$

If $M \to \pi/2$ then $\cos(M) \to 0$ and this gives a contradiction with the hypothesis of proposition 4.1.20 (the distances d_{u_n} converge to a non-zero distance d). □

Fact 4.1.26. *Let S_u be the graph of a convex function u, and M, N any two points of S_u then the geodesic passing through the two points is space-like.*

Proof. Let's consider the cone Δ with vertex M and base the disc $\mathbb{D}$. Because of convexity, it is clear that the line passing through the two points is out of the cone Δ, and so out of the disc $\mathbb{D}$. By Lemma 4.1.4 the proof follows.

□

Proof of proposition 4.1.24: By the fact 4.1.25 we have,

$$\exists M \in [0, \pi/2[, \exists x_0 \in C, \forall n \in \mathbb{N}, u_n(x_0) < M \tag{4.8}$$

We want to prove that

$$\exists \beta \in [0, \pi/2[, \forall n \in \mathbb{N}, \forall x \in C, u_n(x) < \beta$$

Let's suppose that the converse,

$$\forall \beta \in [0, \pi/2[, \exists n_1 \in \mathbb{N}, \exists x_1 \in C, u_{n_1}(x_1) \geq \beta \tag{4.9}$$

is true.

Let Δ be the light cone of origin the point $\begin{pmatrix} x_0 \\ u_{n_1}(x_0) \end{pmatrix}$, $S_{u_{n_1}}$ be the graph described by the convex function u_{n_1} and for any $x \in C$ let $L(x)$ be the Anti-de Sitter distance from the point $\begin{pmatrix} x \\ 0 \end{pmatrix}$ to (the lower side of the light cone) Δ. Now if we suppose that (4.9) is true, then using (4.8) and taking $\beta = \sup_{x \in C} L(x)$, the point $\begin{pmatrix} x_1 \\ u_{n_1}(x_1) \end{pmatrix}$ of the surface $S_{u_{n_1}}$ will be in the lower side of the light cone Δ, hence the segment joining the two points, $\begin{pmatrix} x_0 \\ u_{n_1}(x_0) \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ u_{n_1}(x_1) \end{pmatrix}$ of $S_{u_{n_1}}$ will be time-like. This gives a contradiction with the fact 4.1.26 because $S_{u_{n_1}}$ is space-like.

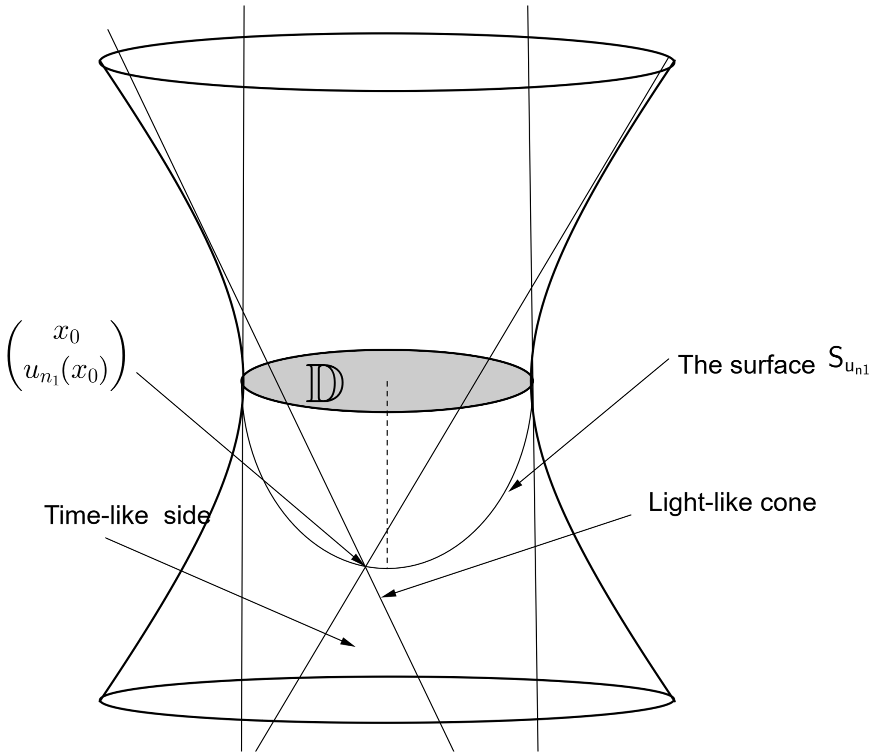

Figure 4.3: Bounded surface and light cone

Proposition 4.1.20 is now proved.

4.1.5 Proof of Theorem 4.1.1

The proof relies on the following result.

Theorem 4.1.27 ([LS00])**.** *Let (S,d) be a metric induced by a Riemannian metric of sectional curvature < -1. Then there exists a C^∞ Fuchsian C-convex $u : \mathbb{H}^2 \to [0,\pi/2[$ such that $\bar{d}_u$ is isometric to d.*

and Theorem 4.1.28 (we restate it here),

Theorem 4.1.28. *Let (S,d) be a metric of curvature ≤ -1. Then there exists a sequence (S_n,d_n) converging uniformly to (S,d), where S_n are homeomorphic to S and d_n are induced by Riemannian metrics with sectional curvature < -1 .*

Note that we are not aware if the analogue of Theorem 4.1.28 holds for metrics of curvature ≤ 1.

Let d be a metric of curvature ≤ -1 on S. From Theorem 4.1.28, there exists a sequence $(d_n)_n$ of metrics induced by Riemannian metrics with sectional curvature < -1 on S that converges uniformly to d. By Theorem 4.1.27, for each $n \in \mathbb{N}$ there exists a Fuchsian C-convex pair (u_n,Γ_n) such that $\bar{d}_{u_n}$ is isometric to d_n and u_n is smooth. By Proposition 4.1.20 there is a subsequence of $(\Gamma_n)_n$ converging to a Fuchsian group Γ, and $\beta < \pi/2$ such that $0 \leq u_n < \beta$.

So Lemma 4.1.6 and Proposition 4.1.19 applies: up to extracting a subsequence, there is a function u such that the induced distance on $\bar{d}_u$ (the quotient of d_u by Γ) is the uniform limit of $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})$, i.e. the uniform limit of (S,d_n). The limit for uniform convergence is unique, up to isometries [BBI01], so $\bar{d}_u$ is isometric to d. Theorem 4.1.1 is proved, with L the quotient of the AdS cylinder of Section 4.1.2 by Γ.

4.1.6 The open question

"Global hyperbolicity" [HE73],[BEE96],[Ger70] is a term derived from the Einstein's theory of general relativity in which the "space-time" is modelled on a Lorentzian manifold. To resolve some problems in the theory of relativity this variety must be globally hyperbolic.

In mathematics, a Lorentzian manifold is globally hyperbolic, if it contains an embedded space-like surface (called Cauchy surface) which intersects every inextensible non space-like line exactly in one point. Note that the Anti-de Sitter space is not globally hyperbolic because time-like geodesics are loops, however the AdS-cylinder that we consider in this paper is globally hyperbolic because we don't consider the point at infinity.

In the beginning of the paper we asked the following question 1.5.4.

Question 4.1.29. *Let g, h be two metrics with curvature ≤ -1 on a compact surface S. Is there a globally hyperbolic Anti-de Sitter manifold with boundary, such that the boundary is space-like convex and isometric to g and h?*

This question is a generalisation of Tamburelli's Theorem [Tam18], it is still open. The Theorem 4.1.1 proved in this paper is a particular answer to it. In fact, if the metrics g and h are equal then we can take two isometric surfaces in the AdS-Cylinder that could be described by u_1 and $-u_1$ where u_1 is a Fuchsian convex function.

The quotient of the two surfaces by the Fuchsian group Γ will be isometric to (S, g).

4.2 Realisation of metrics with curvature ≤ 1

4.2.1 Convex surfaces in de Sitter space

Recall that we are most interested in the Klein model of de Sitter (section 2.3). In the following we will give quickly a description of convexe surfaces in this model of de Sitter space. Let's start by the following definition.

Recall that in this model de Sitter space is the exterior of the ball $\mathbb{H}^3$. Moreover geodesics are straight lines (section 2.3). S,

We define a support space of a surface S to be a half space caontaining S and bounded by an affine hyperplane H. A support plane of S is then the intersection $K \cap S$ if it is not empty.

Definition 4.2.1. *A space-like closed convex surface in de Sitter space is a surface such that (up to a global isometry of de Sitter space) is a closed convex surface in the Klein model of dS with only space-like support planes.*

It is easy to see that in the Klein model, a space-like closed convex surface S is a convex surface which contains the closed ball in its interior (Figure 4.4).

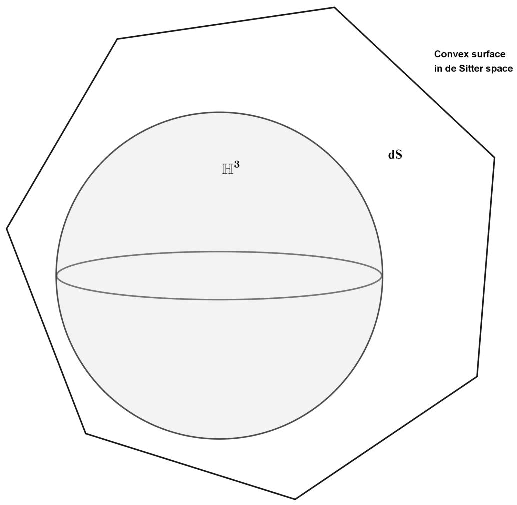

Figure 4.4: Convexe surface in de Sitter space containing the ball.

Recalling also that a spherical polyhedral surface is a surface endowed with a metric of sectional curvature equal to 1 everywhere except at a discrete set of points with conical singularities of angles not less 2π.

A polyhedral geodesic is then a curve on a polyhedral surface which corresponds to a geodesic for the induced metric, in its restriction to each face of the polyhedral surface is a segment of a geodesic of de Sitter space.

4.2.2 Hyperbolic-de Sitter duality

There is a well known classical duality [Sch03, Riv86, Riv93] between hyperbolic space and de Sitter space. This duality associats to each point x of the hyperbolic space $\mathbb{H}^3$ a space-like totally geodesic plane in de Sitter space dS in the following way:

For $x \in \mathbb{H}^3$, we take the line of $Min(3,1)$ passing through x and 0, this line is clearly time-like. Now, we take the hyperplane orthogonal to this line, it is clear that this hyperplane is space-like passing through 0, its intersection with dS denoted by x^* is the dual of x.

Conversely and in the same way, we can associate to each point of de Sitter space, its dual, an oriented totally geodesic plane in the hyperbolic space.

Duality for smooth surfaces: Given a smooth surface S in the hyperbolic space $\mathbb{H}^3$ such that S is convex. For each point $x \in S$, we consider the oriented tangent plane to S in x, and x^* the dual point of this tangent plane. When x travels S, then x^* travels a dual surface S^*, briefly we write

$$S^* = \{x \in dS : \ x^{\perp} \text{ is a plane in } \mathbb{H}^3 \text{ tangent to } S\}$$

It very easy to check that the dual surface S^* is a space-like, convex and smooth surface in de Sitter space.

Duality for noon-smooth surfaces: The notion of duality can be extended to non smooth surfaces, considering the support plane of the surface rather than the tangent plane. Hence we define the dual surface for a non-smooth surface S in $\mathbb{H}^3$ as follows

$$S^* = \{x \in dS : \ x^{\perp} \text{ is a support plane of } S \text{ in } \mathbb{H}^3\}$$

Duality for polyhedral surfaces: In the same way and following the same construction, it is very easy to check the following points:

- The dual of a unit vector (a point on a pseudosphere) is its orthogonal hyperplane
- The dual of a plane is its orthogonal plane
- The dual of a vertex is a face.
- The dual of a face is a vertex.
- The dual of an edge is an edge.

hence the dual of a polyhedral convex surface surface is clearly a polyhedral convex surface which is furthermore space-like.

4.2.3 Definition of the problem and outline of the proof

As always when we speak about a metric with curvature ≤ 1, this means that S is endowed with a distance d satisfying a curvature bound in the sense of A.D. Alexandrov (Chapter 3).

The realization of metrics of curvature ≤ 1 on surfaces in de Sitter space follows a similar way as in the case of metrics with curvature ≤ -1 in anti-de Sitter space ([LAB20], or section 4.1). But here we will use approximation by polyhedral metrics rather than smooth metrics. So that this repose on the approximation result (Theorem 3.3.5) found in section 3.3 for $k = 1$ which says that

Theorem 4.2.2. *Let (S, d) be a metric of curvature ≤ 1 on the closed surface S. Then there exists a sequence (S_n, d_n) converging uniformly to (S, d), where S_n is homeomorphic to S and d_n is the metric induced by a spherical polyhedral metric on S_n.*

4.2.4 For the sphere

We want to prove the following Theorem

Theorem 4.2.3. *Let d be a metric of curvature ≤ 1 on a sphere $\mathbb{S}^2$ such that all closed contractible geodesics on $(\mathbb{S}^2, d)$ have length greater than 2π. Then there exists a convex space-like surface in de Sitter space such that the induced metric is isometric to $(\mathbb{S}^2, d)$.*

The proof repose on the Theorem 4.2.2, and of course the Theorem of I. Rivin and D. Hodgson below that prove the posibility of realizing of a spherical polyhedral metric on a convex space-like surface in de Sitter space, we give the theorem as follows

Theorem 4.2.4. *([Riv86],[HR93]) Let d be a spherical metric with conical singularities on the sphere $\mathbb{S}^2$ such that:*

1. *All cone angles of $(\mathbb{S}^2, d)$ are greater than 2π.*
2. *All closed contractible geodesics on $(\mathbb{S}^2, d)$ have length greater than 2π.*

Then there exists a convex polyhedral surface in de Sitter space such that the induced metric is isometric to $(\mathbb{S}^2, d)$.

Sketch of the proof of Theorem 4.2.3: Let's consider the statement of Theorem 4.2.3. Let d be a metric of curvature ≤ 1, by Theorem 4.2.2, there exists a sequence of spherical polyhedral metrics d_n converging uniformly to d. By Theorem 4.2.4, for each n there is a convex space-like polyhedral surface S_n in de Sitter space with the induced metric d_n. Then we have to prove that:

Convergence of metrics implies convergence of convex surfaces: We have to prove that when the metrics converge then surfaces S_n converge to a surface S. This will rely as usual on Azrela–Ascoli Theorem, but a subtle thing in de Sitter situation is to use the hypothesis about the length of closed geodesics to provide the surfaces to go to the boundary at infinity of the ambiant space. Howevere, the arguments are already given in [HR93].

Finally: one has to check that convergence convex surfaces implies convergence of the induced metrics, that is not so different from Minkowski or anti-de Sitter cases.

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For higher genius: Analogously to the sphere, we wish to prove the following Theorem for a surface with genius ≥ 1

Theorem 4.2.5. *Let d be a metric of curvature ≤ 1 on a compact surface S such that all closed contractible geodesics on (S, d) have length greater than 2π. Then there exists a convex space-like surface in de Sitter space such that the induced metric is isometric to (S, d).*

The proof follows a similar way as for the sphere, (we use approximation by polyhedral metrics), the group convergence is handled as in the other cases.

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