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Non-Normal Modal Logics: Neighbourhood Semantics and their Calculi

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Abstract

This thesis provides a proof-theoretical investigation of non-normal modal logics. Non-normal modal logics are modal logics that do not satisfy some of the modal principles of the weakest normal modal logic K. They have been studied since the very beginning of modern modal logic, and have found an interest in many areas, such as deontic, epistemic, and multi-agent reasoning. Moreover, they have been also studied as modal extensions of intuitionistic, rather than classical, logic.

Non-normal modal logics have been mainly investigated from the point of view of the semantics. In contrast, their proof theory is not as developed as their semantics.

In this thesis we consider non-normal modal logics based on both classical and intuitionistic propositional logic, pursuing two general aims: first, concerning non-normal modal logics based on classical logic, we aim to define proof systems that have "good" computational and semantic properties, such as providing decision procedures, allowing countermodel extraction for non-valid formulas, and at the same time suited for theorem-proving. Concerning non-normal modal logics based on intuitionistic logic, we aim to lay down a general framework for defining in a uniform way intuitionistic counterparts of classical non-normal modal logics, but also for capturing some relevant intuitionistic modal systems studied in the literature.

The thesis consists of three main parts. The first part (Chapters 1, 2, and 3) contains a general introduction to the axiomatisation and semantics of both classical and intuitionistic non-normal modal logics, as well as a general introduction to sequent calculi, together with a review of the existing proof systems for these logics.

The second part (Chapters 4, 5, and 6) presents our original results about classical non-normal modal logics: we first introduce a new semantics for these logics, and then we propose two new kinds of sequent calculi and investigate their properties. In addition, we also consider two specific logics, namely Elgesem's agency and ability logic and its coalitional extension by Troquard: we provide both a new semantics and calculi for them.

In the third part (Chapters 7 and 8) we define a family of intuitionistic non-normal modal logics and provide both sequent calculi and their semantic characterisation in terms of neighbourhood models. We also show how known logics from the literature, such as Wijesekera and Bellin and De Paiva's systems can be captured within our framework.

Résumé

Cette thèse présente une étude de théorie de la preuve des logiques modales non-normales. Ces logiques sont des logiques modales qui ne satisfont pas certains principes de la logique modale normale minimale K. Elles ont été étudiées depuis le début de la logique modale moderne et ont trouvé un intérêt dans de nombreux domaines, tels que le raisonnement déontique, le raisonnement épistémique, ainsi que le raisonnement dans les systèmes multi-agents. Elles ont été également considérées dans le cadre des extensions modales de la logique intuitionniste.

Les logiques modales non-normales ont été principalement étudiées du point de vue sémantique. En revanche, leur théorie de la preuve n'est pas aussi développée que leur sémantique.

Dans cette thèse, nous considérons des logiques modales non-normales basées sur la logique propositionnelle classique, ainsi que sur la logique propositionnelle intuitionniste, poursuivant deux objectifs généraux: D'abord, nous considérons les logiques modales non-normales avec une base classique et nous visons à définir des systèmes de preuve ayant de «bonnes» propriétés calculatoires et sémantiques, telles que le support de procédures de décision, l'extraction de contre-modèles des formules non-valides et en même temps adaptées à l'implantation dans des démonstrateurs. Concernant les logiques modales non-normales avec une base intuitionniste, notre objectif est celui d'établir un cadre général pour définir uniformément des versions intuitionnistes des logiques non-normales classiques, mais aussi pour capturer les systèmes intuitionnistes pertinents étudiés en littérature.

La thèse se compose de trois parties principales. La première partie (chapitres 1, 2, 3) contient une introduction générale à l'axiomatisation et à la sémantique des logiques modales non-normales classiques et intuitionnistes, une introduction générale aux calculs des séquents, ainsi qu'un état de l'art des systèmes de preuve existants pour ces logiques.

La deuxième partie (chapitres 4, 5, 6) présente les résultats originaux sur les logiques modales non-normales avec base classique : nous introduisons d'abord une nouvelle sémantique pour ces logiques, puis nous présentons deux types de calculs des séquents et nous étudions leurs propriétés. Nous considérons ultérieurement deux logiques modales non-normales spécifiques : la logique multi-agent de la «capabilité» d'Elgesem et son extension avec coalitions d'agents définie par Troquard, pour lesquelles nous développons à la fois une nouvelle sémantique et des calculs.

Dans la troisième partie (chapitres 7, 8), nous définissons une famille de logiques modales intuitionnistes non-normales et nous proposons des calculs des séquents et leur caractérisation sémantique en terme de modèles avec fonctions de voisinage. Nous montrons ultérieurement comment des logiques étudiées dans la littérature, telles que les systèmes de Wijesekera et celui de Bellin et De Paiva, peuvent être capturées dans notre cadre.

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Contents

Chapter 1

Introduction

1.1 General aims and motivation

This thesis provides a proof-theoretical investigation of non-normal modal logics. Non-normal modal logics are called in this way because they do not validate some of the modal principles of the weakest normal modal logic **K**. They have been studied since the very beginning of modern modal logic in the seminal works by C.I. Lewis, Lemmon, Kripke, Scott, Montague, Segerberg, and Chellas, and have found an interest in many areas. They are applied for instance in deontic logic, where they are used to handle with conflicting obligations and avoid many well-known paradoxes, or in epistemic logic, where they offer a (partial) solution to the problem of omniscience, but are also applied in multi-agent reasoning, reasoning about games, reasoning about probabilistic notions such as "true in most of the cases", and so on.

Non-normal modal logics have been mainly investigated from the point of view of the semantics. In contrast, their proof theory is not as developed as their semantics. Only in recent years the problem of finding suitable proof systems for this kind of logics has been more extensively addressed, and up to now the state of the art is not comparable with the one of proof systems for normal modal logics, for which there exist well-understood proof methods of many kinds.

Starting with the seminal works by Hilbert, Gentzen, and, for modal logics, Fitting, proof-theoretical investigations of logics have been carried out with many different purposes. To mention only two of them, by looking at the form of the proofs one can establish properties of the logic such as consistency, decidability, interpolation, and so on. Furthermore, suitable proof systems may be used for practical purposes, such as automated reasoning and theorem proving.

Depending on the aims, there can be several *desiderata* on proof systems. Here we are interested in calculi with the following properties. First, the calculi should provide a clear syntactic representation of the logics, as an alternative to their axiomatisation. In this respect, the rules should have a declarative reading and a semantic interpretation independent from any procedure. Moreover, similarly to Gentzen's sequent calculi, logical operators should be treated with separate left and right rules introducing each a single occurrence of a formula. Second, the calculi should have good structural and computational properties. Since all the considered logics are decidable, it should be possible to define a proof search strategy in the calculi that gives rise to a decision procedure for the respective logic, possibly of optimal complexity. In this light, we are interested in analytic calculi, meaning essentially that every derivation of a formula A only employs subformulas of A. In turn, this requires to define calculi with good structural properties, in particular the structural rules of weakening, contraction, and cut should be admissible. Moreover, in this case it should be possible to prove the admissibility of cut by means of a syntactic cut elimination procedure. Finally, we are interested in developing automated reasoning tools for the considered logics, whence the calculi should be also suitable for implementation.

Looking at the literature, it must be observed that none of the existing proof systems for non-normal modal logics is fully satisfactory with respect to these desiderata. The definition of proof systems satisfying our desiderata is one main goal of this thesis.

Non-normal modal logics have been also studied as modal extensions of intuitionistic, rather than classical, logic. There are two general motivations for such an analysis. On the one hand, from a theoretical perspective, there is a mathematical interest in combining these two different – but related – forms of logics. Moreover, the rejection of classical equivalences can allow for a finer analysis of the modalities. On the other hand, the combination of intuitionistic logic and non-normal modalities turn out to be very adequate for some applications of modal logic in computer science, such as for instance the formalisation of reasoning with partial information about concurrent transition systems and reasoning about correctness up to constraints in hardware verification. In addition, it is also considered for dealing with contextual reasoning in logic-based knowledge representation.

Despite their interest, the present state of the art of non-normal modal logics with intuitionistic basis only include systems designed for specific applications, such as Nerode and Wijesekera's Constructive Concurrent Dynamic Logic and so-called Constructive K. By contrast, differently from both *classical* non-normal modal logics and intuitionistic *normal* modal logics, to the best of our knowledge no systematic investigation of non-normal modalities with intuitionistic base has been carried out so far.

In this thesis we consider non-normal modal logics based on both classical and intuitionistic propositional logic, pursuing two main aims. First, concerning non-normal modal logics based on classical logic, we aim to define proof systems for them satisfying our desiderata. As for many other logics, this cannot be done within the basic framework of sequent calculi. For this reason, we consider extensions of the basic framework of sequent calculi of two kinds: *labelled* calculi, i.e., calculi defined on a language enriched with labels which express semantic information, and *structured* calculi, i.e., calculi where the structure of sequents is enriched with additional structural connectives, such as the bar "|" of hypersequents. We show that, despite their fundamental differences, both kinds of formalisms are adequate to define suitable proof systems for these logics. On the basis of the calculi we also provide alternative proofs of some fundamental properties of non-normal modal logics such as the finite model property, decidability, and complexity upper bounds. Moreover, by implementing the calculi we develop the first automated theorem provers for these logics that provide both derivations and countermodels. Our calculi (both labelled and unlabelled) are based on an original reformulation of the neighbourhood semantics of non-normal modal logics, called bi-neighbourhood semantics, that can be of interest also independently from the proof-theory.

Second, concerning non-normal modal logics based on intuitionistic logic, we aim to lay down a general framework for defining in a uniform way *intuitionistic counterparts* of classical non-normal modal logics, but also for capturing some intuitionistic modal systems of particular interest already studied in the literature. In order to define these logics we adopt a proof-theoretic approach: after formulating the relevant principles as sequent rules, we consider the existence of a cut-free calculus as the criterion for the acceptance of a logic among the family of the intuitionistic systems. Then, once the logics are defined we investigate them both semantically and proof-theoretically: we define for them a suitable semantic framework as well as alternative proof systems with different properties. In addition, on the basis of both the semantics and the proof systems we establish fundamental properties of these logics such as decidability and interpolation.

1.2 Structure of the thesis

This thesis consists of three main parts. The first part (Chapters 2 and 3) contains a general introduction to the logics that we shall study in this work. In particular,

- in Chapter 2 we present the axiomatisation and semantics of the family of classical non-normal modal logics considered in this work, as well as of two further non-normal systems, namely Elgesem's agency and ability logic [47] and its coalition extension by Troquard [165]. We also present the axiomatisation and semantics of some relevant intuitionistic modal logics, in particular Constructive K [14] and Wijesekera's Constructive Concurrent Dynamic Logic [170].

- Chapter 3 contains an introduction to sequent calculi and their properties, together with a review of the existing proof systems for non-normal modal logics.

The second part (Chapters 4, 5, and 6) presents our original results about classical non-normal modal logics.

- In Chapter 4 we introduce a new semantics for these logics, called bi-neighbourhood semantics, that generalises their standard neighbourhood semantics. On its basis we modularly characterise the whole family of the considered classical non-normal modal logics, as well as Elgesem's and Troquard's agency and ability logics [47, 165]. We also present an embedding of classical non-normal modal logics into monotonic logics with binary modalities.

- In Chapter 5 we present labelled sequent calculi for the whole family of the classical non-normal modal logics considered in this work. We prove that all calculi are sound and complete with respect to the corresponding axiomatic systems. Moreover, we prove that the structural rules are admissible, most importantly cut. We then propose an equivalent reformulation of the calculi in the form of tableaux systems. On their basis we define a terminating proof search strategy that provides a decision procedure for the derivability problem in the logics. Moreover, we show that from every failed proof it is possible to directly extract a countermodel of the non-valid/non-derivable formula in the bi-neighbourhood semantics. Finally, we show that our labelled calculi are of interest also for automated reasoning: we present a Prolog implementation of these calculi which provides the first theorem prover that covers these logics in a uniform way and computes both derivations and countermodels of non-valid formulas.

- In Chapter 6 we present hypersequent calculi for classical non-normal modal logics, as well as for Elgesem's and Troquard's agency and ability logics [47, 165]. The calculi can be seen as internal as they only employ the language of the logic, plus additional structural connectives. We show that the calculi are complete with respect to the logics by a syntactic proof of cut elimination. Then, we define a terminating backward proof search strategy based on the hypersequent calculi and show that it is optimal for coNP-complete logics. Moreover, we show that from every failed proof it is possible to directly extract a bi-neighbourhood countermodel of the non-derivable/non-valid formula. Finally, we present an alternative theorem prover for these logics based on a Prolog implementation of our hypersequent calculi, and compare its performance with that of the prover implementing the labelled calculi.

The third part (Chapters 7 and 8) presents our original results about intuitionistic non-normal modal logics.

- In Chapter 7 we define a family of intuitionistic non-normal modal logics which can be interpreted as intuitionistic counterparts of the systems of the classical cube. For these logics we provide both sequent calculi and an equivalent axiomatisation. We prove that all these systems are decidable and for some of them we also prove Craig interpolation. Finally, we define strictly terminating "à la Dyckhoff" calculi for our systems as well as for further intuitionistic modal logics studied in the literature, such as Constructive K and Wijesekera's system.

- In Chapter 8 we present a neighbourhood semantic framework that modularly characterises our intuitionistic non-normal modal logics as well as Constructive K and Wijesekera's system. We also prove that all these systems enjoy the finite model property. Moreover, basing on this semantics we present an embedding of intuitionistic non-normal modal logics into classical logics with multiple non-normal modalities. Finally, we present a tableaux calculus for a subclass of our intuitionistic logics that allows one to extract countermodels of non-valid formulas in the neighbourhood semantics defined in this chapter.

1.3 Publications

The main part of the results presented in this thesis have already been presented in the following publications:

[33] Tiziano Dalmonte, Nicola Olivetti, and Sara Negri. Non-normal modal logics: Bi-neighbourhood semantics and its labelled calculi. In: Proceedings of AiML 12 (Advances in Modal Logic), College Publications, pp. 159–178, 2018.

[35] Tiziano Dalmonte, Sara Negri, Nicola Olivetti, and Gian Luca Pozzato. PRONOM: Proof-search and countermodel generation for non-normal modal logics. In: Proceedings of AI*IA 2019 (18th International Conference of the Italian Association for Artificial Intelligence), Springer, LNAI 11946, pp. 165–179, 2019 (best student paper award).

[36] Tiziano Dalmonte, Charles Grellois, and Nicola Olivetti. Intuitionistic and classical non-normal modal logics: An embedding. In: TACL 2019, booklet of abstracts, pp. 67–68, 2019.

[37] Tiziano Dalmonte, Charles Grellois, and Nicola Olivetti. Intuitionistic non-normal modal logics: A general framework. Journal of Philosophical Logic, 49(5) (2020), pp. 833–882.

[38] Tiziano Dalmonte, Björn Lellmann, Nicola Olivetti, and Elaine Pimentel. Countermodel construction via optimal hypersequent calculi for non-normal modal logics. In: Proceedings of LFCS 2020 (International Symposium on Logical Foundations of Computer Science), Springer, LNCS 11972, pp. 27–46, 2020.

[39] Tiziano Dalmonte, Björn Lellmann, Nicola Olivetti, and Elaine Pimentel. Hypersequent calculi for non-normal modal and deontic logics: Countermodels and optimal complexity. arXiv preprint arXiv:2006.05436 (2020), submitted for publication.

[40] Tiziano Dalmonte, Nicola Olivetti, and Gian Luca Pozzato. HYPNO: Theorem proving with hypersequent calculi for non-normal modal logics. In: Proceedings of IJCAR 2020 (International Joint Conference on Automated Reasoning), part II, Springer, LNAI 12167, pp. 378–387, 2020.

[41] Tiziano Dalmonte, Charles Grellois, and Nicola Olivetti. Proof systems for the logics of bringing-it-about. In: Proceedings of DEON 2020/2021 (15th International Conference on Deontic Logic and Normative Systems), to appear.

In particular, in [33] we present the bi-neighbourhood semantics for the systems of the classical cube (Chapter 4) and their labelled sequent calculi (Chapter 5). In [35] we present the theorem prover PRONOM based on the Prolog implementation of the labelled calculi (Section 5.7). In [38] we present the hypersequent calculi for the systems of the classical cube (Chapter 6), and in [40] we present the theorem prover HYPNO based on these calculi (Section 6.7). Moreover, in [39] we extend the bi-neighbourhood semantics and the calculi of [38] to all classical systems containing the axioms T, P, D, and the rules RD_n^+. In [41] we present the bi-neighbourhood semantics and the hypersequent calculi for Elgesem's agency and ability logic and Troquard's coalition logic (Sections 4.6, 6.5, and 6.6). Furthermore, in [37] we present the sequent calculi and axiomatisations of intuitionistic non-normal modal logics (Chapter 7), as well as their neighbourhood semantics (Chapter 8). Finally, in [36] we present the embedding of intuitionistic non-normal modal logics into classical logics with multiple non-normal modalities (Section 8.6).

Part I
Background

Chapter 2

Non-normal modal logics

In this chapter, we present general motivations for non-normal modal logics based on several possible interpretations of the modalities. Then we present the axiomatisation and the standard semantics of the classical non-normal modal logics that we shall consider in this work. Finally, we consider non-normal systems based on intuitionistic logic, and present the axiomatisation and semantics of some relevant systems.

2.1 Non-normal modalities: what and why

Non-normal modal logics are called in this way because they do not validate some of the modal axioms and rules of the weakest normal modal logic **K**. They have been studied since the very beginning of modern modal logic starting with the seminal works by C.I. Lewis, Lemmon, Kripke, Scott, Montague, Segerberg, and Chellas [115, 114, 104, 157, 129, 158, 29] – non-normal modal logics are for instance C.I. Lewis' modal systems **S1**, **S2**, and **S3**, and Lemmon's sytems **E2** and **E3** –, and since then they have been extensively investigated.

As it is known, part of the success of modal logic is due to the possibility of interpreting the modalities in many different ways. At the same time, it has been widely recognised that several of these interpretations are incompatible with normal modalities, in the sense that their association with some principles of normal modal logics leads to counterintuitive or even unacceptable conclusions. This is one of the reasons why non-normal modal logics are considered. Among the several interpretations which are regarded as better described by non-normal modalities we mention the following examples.

Deontic logic. In deontic logic, formulas of the form $\Box A$ are intuitively interpreted as "A is obligatory", or "It ought to be the case that A". In the literature, the possibility to represent obligations by normal modalities has been widely criticised. Here we mention some of the most typical criticisms.

Problems for monotonicity. Perhaps the most problematic principle for deontic logic is the monotonicity principle, which states that if A implies B and A is obligatory, then B is also obligatory ($A \to B$ / $\Box A \to \Box B$). To see how this principle can be problematic, imagine a scenario (inspired by Asimov's novels) in which humans program some robots in such a way that they respect the precept that they must protect humanity. Imagine also that, by automatic reasoning, robots conclude that protecting humanity implies to deny humans' freedom, since humans represent a danger for themselves. Following the monotonicity principle, this conclusion would entail that robots must deny human freedom. But while it might be desirable to have robots that must protect humanity, it is clearly not acceptable that they must deny human freedom.

The same structure is at the basis of many paradoxes considered in the literature, such as the Ross' paradox [153], the paradox of free choice permissions [153], the good Samaritan paradox [4], and the paradox of gentle murder [59], among the others.[1] The latter paradox can be formulated as follows (cf. [142]): Consider a legal system containing the rules "Smith ought not to murder John", and "If Smith murders John, then Smith ought to murder John gently" (i.e., in such a way as to cause the least amount of pain). Moreover, assume that Smith murders John. The second and third premisses entail that Smith ought to murder John gently. In addition, it is clearly a valid statement that if Smith murders John gently, then Smith murders John. Then by monotonicity it follows from the last two sentences that Smith ought to murder John, in contradiction with the first rule.

Problems for agglomeration. In a deontic formulation, the agglomeration principle states that if someone has two obligations, then she is obliged to do both of them ($\Box A \land \Box B \to \Box(A \land B)$). Agglomeration leads to problematic consequences in presence of *normative conflicts*, that is, situations in which an agent ought to do two or more things but cannot do them all [72]. Normative conflicts are also known as *moral dilemmas* [113] when the obligation is generated by some moral commitment of the agent. Typical examples are incompatible promises: Suppose that Alice has promised Annie to meet her at the pub, but she has also promised her mother to stay at home, that is, not to go to the pub. If we accept the agglomeration principle, this implies for Alice the obligation to go to the pub and not to go to the pub, that is the obligation to perform a contradiction. The situation becomes even worse when agglomeration is combined with monotonicity, as the two principles together lead to the so-called problem of *deontic explosion* [72], or the *universal obligatoriness* problem [84]: since a contradiction implies every formula, from a normative conflict it would follow that everything is obligatory, so that for instance Alice's dilemma would imply for her the obligation to kill her mother.

Problems for necessitation. The necessitation principle states that whatever is valid is

[1] For a survey of paradoxes raising from the monotonicity principle in deontic logic see McNamara [124].

also obligatory ($A\ /\ \Box A$). It conflicts with the so-called principle of *deontic contingency*, which has been stated in [175] as "A tautologous act is not necessarily obligatory". The idea is that an obligation to realise a tautology like "You are morally required to either go to the pub or not go to the pub" does not make sense [84], or does not express a real prescription [176]. As an additional drawback, necessitation excludes the possibility of considering "*empty* normative systems" [60], that is systems without any obligation, since it implies that at least all tautologies are obligatory.

Epistemic logic. Epistemic logic interprets $\Box A$ roughly as "It is known that A", or "The agent knows that A". Under this interpretation, the principles of monotonicity, agglomeration, and necessitation express each a specific reasoning ability of the considered agent [142]. In particular, by the agglomeration principle the agent would know that $A \wedge B$ whenever she knows both A and B, whereas monotonicity expresses the stronger ability of inferring all the logical consequences of her knowledge. Finally, by necessitation the agent would know all "truths" (or validities) of the context in which she is involved.

These are very strong assumptions about the reasoning abilities of the agent. In particular, monotonicity and necessitation express two forms of *omniscience* (the former one has been called *logical omniscience* [89]). The following example is proposed in [18]: Assume that Peano's axioms entails Goldbach's conjecture (something that we actually do not know). In this case, monotonicity means that if the agent knows Peano's axioms, then she also knows that Goldbach's conjecture is true. An agent with this kind of abilities can be seen as a *perfect reasoner*, or as having unbounded reasoning power [130, 166]. As a consequence, normal modal logics are not adequate to describe fallible epistemic agents, or agents with human-like abilities.

Logic of agency and ability. Ability logics formalise the notion of ability of agents to perform some actions, or to obtain certain results. An example is Brown's ability logic [22, 23], where $\Box A$ is interpreted as "There is an action open to the agent, the execution of which would assure that A would be true". This interpretation conflicts with the principle of agglomeration. For instance, by a single move an agent might be able to draw a red card, and she might be able to draw a black card, but she might not be able to both draw a red card and draw a black card. A further example is Elgesem's agency and ability logic [47], also called the logic of *bringing-it-about* [165]. This logic provides a formalisation of agents' actions in terms of their results: that an agent "does something" is interpreted as the fact that the agent brings about something, for instance "John does a bank transfer" is interpreted as "John does that the bank transfer is done". Elgesem's characterisation of agency involves a notion of *responsibility*, so that an agent cannot realise something that would have happened independently from her action. From the logical point of view, this corresponds to reject necessitation.

Majority logic. Majority logic [9, 141] captures the notion of "true in most of the cases". A related logic is the logic of high probability [142], where $\Box A$ is interpreted as "It is highly probable that A". It is easy to see that in majority and high probability logic the agglomeration principle fails: to give a very simple example, imagine a group of three people where two are blond, two are tall, but just one is both blond and tall.

Non-normal modal logics offer a (partial) solution to the above difficulties by allowing us to drop the principles conflicting with the interpretation of the modalities under consideration. For instance, in order to avoid the conclusion that robots are obliged to deny human freedom we do not necessarily need to reject their obligation to protect humanity. On the contrary, we can drop the monotonicity principle and keep the second obligation without being committed to the first one. Similarly, we might need to describe situations where normative conflicts are admitted. But this does not imply that we also have to accept obligations to do something impossible, since by considering non-normal modalities we can drop the principle of agglomeration.

As we shall see, the principles of monotonicity, agglomeration, and necessitation correspond each to one of the axioms and rules that characterise the basic cube of classical non-normal modal logics, also called *classical cube* [110]. The classical cube is a lattice of eight systems: the weakest one does not satisfy any of the above principle, whereas the other systems satisfy some of these principles only if the corresponding axioms or rules explicitly belong to their axiomatisation.

Non-normal modal logics are clearly not restricted to the eight systems of the classical cube. Depending on the considered interpretation of the modality, it can be worth extending (some of) these eight systems with additional modal axioms or rules which are relevant for that specific interpretation. For instance, in epistemic logic one often consider the modal axioms 4 ($\Box A \to \Box\Box A$) and T ($\Box A \to A$) (see Figure 2.1, p. 17), which correspond to relevant assumptions about the properties of knowledge: axiom 4 represents the property of *introspection* of epistemic agents (if an agent knows that A, then she knows that she knows that A), whereas axiom T represents the so-called *factivity of knowledge* (something which is known must be true).

Further axioms are of interest in deontic logic. In this context one often considers the axioms D ($\neg(\Box A \land \Box\neg A)$) and P ($\neg\Box\bot$), which represent the impossibility of having, respectively, contradicting obligations and self-inconsistent obligations. Axiom P is also significant for agency logic, where it expresses the impossibility of realising something impossible. Besides axioms D and P, a relevant rule in deontic contexts is, for any $n \in \mathbb{N}$,

$$RD_n^+ \ \frac{\neg(A_1 \land ... \land A_n)}{\neg(\Box A_1 \land ... \land \Box A_n)} \ .$$

To the best of our knowledge, this family of rules has been only discussed – end rejected –

by Goble [72], in contrast we are not aware of any (non-normal) deontic system containing them. The rules RD_2^+, RD_3^+, RD_4^+, ... have a peculiar interest in deontic logic as they exclude the possibility of having 2, 3, 4, ... jointly incompatible obligations. Let us consider the following example, essentially from Hansson [85]: (1) I have to keep my mobile switched on (as I'm waiting for an urgent call), (2) I have to attend my child schoolplay, (3) being in the audience of a schoolplay I must keep my mobile switched off. Representing these three claims by $mobile\_on$, $schoolplay$, and $\neg(mobile\_on \wedge schoolplay)$, by using rule RD_3^+, it can be concluded that the three obligations are incompatible:

$$\neg(\Box mobile\_on \wedge \Box schoolplay \wedge \Box\neg(mobile\_on \wedge schoolplay))$$

This conclusion cannot be obtained in any non-normal modal logic without RD_3^+ or C ($\Box A \wedge \Box B \to \Box(A \wedge B)$), even if it contains both deontic axioms D and P.

A further relevant aspect of non-normal modal logics is the possibility to consider axioms or rules which are incompatible with normal modal logics, in the sense that their addition to normal systems trivialises the modality or even produces inconsistent systems. Typical examples are the anti-monotonicity rule and the axiom opposite to necessitation:

$$\frac{A \to B}{\Box B \to \Box A}\,, \qquad\qquad \neg\Box\top.$$

The anti-monotonicity rule is considered for instance by Dosen [42, 43] in order to provide a modal characterisation of negation and impossibility in intuitionistic contexts, whereas axiom $\neg\Box\top$ is considered for instance by Elgesem [48] in order to provide a logical characterisation of agency (cf. Section 2.4).

The solution of considering non-normal modalities has been actually adopted for the design of a wide range of modal systems. Besides Lewis' and Lemmon's systems we can mention the following illustrative examples. Starting with von Wright's seminal work [175], where a deontic system is proposed which rejects necessitation, non-normal modalities have become standard in deontic logic. A further example is the logic of weakest permission by Anglbereger et al. [3], in which the modal operator for obligations is non-monotonic, whereas several possible systems rejecting agglomeration are examined in Goble [72] for handling with the problem of normative conflicts. Further examples can be found in handbooks devoted to deontic logic such as [61]. Non-normal modalities have been also extensively considered in epistemic and doxastic logics, i.e., logics formalising the concepts of knowledge and belief. We can mention as examples Vardi's analysis of the problem logical omniscience [166], the epistemic and doxastic logics in Askounis et al. [9, 105] based on the concepts of weak filters and ultrafilters, the evidence logics in [16, 17, 13, 117], and the logical characterisation of the epistemic attitudes of non-omniscient agents in Balbiani et al. [12]

Moving to different contexts, examples of non-normal modal logics are also Brown's [22, 23] and Elgesem's [47] agency and ability logics, Pauly's [146] and Troquard's [165] coalition logics, alternating-time temporal logic by Alur et al. [2], and Parikh's logic of games [143] (Elgesem's and Troquard's logics are more extensively presented in Section 2.4). Moreover, non-normal modal logics have been also studied as extensions of first-order logic [7, 27, 100] and description logics [34, 160].[2] Furthermore, graded modalities [51, 52] as well as number restrictions in description logics [11, 90] can be seen as non-normal modalities.

This far-from-exhaustive list of non-normal modal systems witnesses the importance that non-normal modalities have acquired in the literature on modal logic. In this work we investigate non-normal modalities from a general perspective, without committing to any specific choice of axioms or any specific interpretation of the modalities.

2.2 Classical non-normal modal logics as axiomatic systems

We now define the family of classical non-normal modal logics that we shall consider in this work. By *classical non-normal modal logic* we understand any logic that extends classical propositional logic with non-normal modalities. They are distinguished from *intuitionistic non-normal modal logic* that in contrast are extensions of intuitionistic logic. We first introduce the language of the logics and some general definitions about axiomatic systems, and then move to the axiomatisations.

Syntactic preliminaries

Definition 2.2.1 (Language)**.** The language $\mathcal{L}$ of classical non-normal modal logics contains a set $Atm = \{p_1, p_2, p_3, ...\}$ of countably many propositional variables, and the primitive connectives $\bot$ and $\top$ (nullary), $\Box$ (unary), $\wedge$, $\vee$, and $\rightarrow$ (binary). The *formulas* of $\mathcal{L}$ are recursively defined as follows:

- p_i, $\bot$, and $\top$ are formulas of $\mathcal{L}$, where p_i is any element of Atm.
- If A and B are formulas of $\mathcal{L}$, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $\Box A$ are formulas of $\mathcal{L}$.

We call *atomic* the formulas of the form p_i, $\bot$, and $\top$, all others formulas are *compound*. We define the additional connectives $\neg$, $\leftrightarrow$, and $\Diamond$, as follows, where A and B are any formulas of $\mathcal{L}$:

- $\neg A := (A \rightarrow \bot)$,

[2] In particular, in [34] we have defined some modal description logics which can be seen as counterparts of the systems of the classical cube, and established complexity upper bounds for the satisfiability problem in these logics. I omit to present them here since they are not entirely in the scope of the present dissertation.

- $(A \leftrightarrow B) := ((A \rightarrow B) \wedge (B \rightarrow A))$, and
- $\Diamond A := \neg\Box\neg A$.

We use the usual conventions on the binding strength of connectives (from strongest to weakest): $\Box, \Diamond, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and omit brackets whenever possible. For instance, $p_1 \wedge p_2 \rightarrow p_3 \vee p_4$ stays for $((p_1 \wedge p_2) \rightarrow (p_3 \vee p_4))$. As usual, the connectives $\bot$ and $\top$ stay respectively for "falsity" and "truth", whereas the connectives $\Box$ and $\Diamond$ are called *modalities*. The above relation between the two modalities is expressed by saying that $\Box$ and $\Diamond$ are *dual*.

For the sake of simplicity, we use the symbol $\mathcal{L}$ to denote both the language and its set of formulas. In the following, we use A, B, C, D, E and p, q, r as metavariables for, respectively, arbitrary formulas and propositional variables of $\mathcal{L}$. We use $=$ to denote the identity relation between formulas. For every formula of $\mathcal{L}$, we define its weight and its subformulas as follows.

Definition 2.2.2 (Weight of formulas)**.** The function wg assigning to each formula A of $\mathcal{L}$ its weight $wg(A)$ is recursively defined as follows: for $p_i \in Atm$, $wg(p_i) = wg(\bot) = wg(\top) = 0$; for $\circ \in \{\wedge, \vee, \rightarrow\}$, $wg(A \circ B) = wg(A) + wg(B) + 1$; and $wg(\Box A) = wg(A) + 1$.

Definition 2.2.3 (Subformula, strict subformula)**.** For every formula A of $\mathcal{L}$, the set $sbf(A)$ of the *subformulas* of A is defined as follows: for $p_i \in Atm$, $sbf(p_i) = \{p_i\}$; $sbf(\bot) = \{\bot\}$; $sbf(\top) = \{\top\}$; for $\circ \in \{\wedge, \vee, \rightarrow\}$, $sbf(A \circ B) = sbf(A) \cup sbf(B) \cup \{A \circ B\}$; and $sbf(\Box A) = sbf(A) \cup \{\Box A\}$. We say that B is a *subformula* of A if it belongs to $sbf(A)$, and it is a *strict subformula* of A if it is a subformula of A and it is different from A.

We also consider the notion of schemata of formulas, which are defined on the basis of metavariables of formulas in the following way.

Definition 2.2.4 (Schema, substitution, instance)**.** A *schema* of formulas, or just schema, is an expression built by following the same grammar of the formulas of $\mathcal{L}$, but on the basis of metavariables rather than propositional variables of Atm. Moreover, we call *substitution* any function ς from schemata to formulas of $\mathcal{L}$ such that $\varsigma(p) \in Atm$, where p is any metavariable for propositional variables of $\mathcal{L}$, $\varsigma(\bot) = \bot$, $\varsigma(\top) = \top$, $\varsigma(A \circ B) = \varsigma(A) \circ \varsigma(B)$ for $\circ \in \{\wedge, \vee, \rightarrow\}$, and $\varsigma(\Box A) = \Box\varsigma(A)$. Finally, if A is a schema and ς is a substitution, then $\varsigma(A)$ is an *instance* of A.

For instance, the formula $\Box p_1 \wedge p_2 \rightarrow p_3 \vee (\Box p_1 \wedge p_2)$ is an instance of the schema $A \rightarrow B \vee A$, where the metavariables A and B are replaced respectively by the formulas $\Box p_1 \wedge p_2$ and p_3. Intuitively, a schema of formulas can be understood as a set of formulas of a certain form, whose elements are the instances of the schema.

Classical non-normal modal logics are defined in this section in the form of axiomatic systems. Before presenting their axiomatisation, we recall the basic definitions related to axiomatic systems.

Definition 2.2.5 (Axiomatic system)**.** An *axiomatic system*, also called Hilbert system, is a set of axiom schemata and rule schemata. An *axiom schema* is nothing but a schema of formulas, whereas a *rule schema* is a pair composed by a set of schemata A_1, ..., A_n, called *premisses*, and a schema A, called *conclusion*, and is written

$$R \, \frac{A_1 \quad \dots \quad A_n}{A} \; ,$$

where R is the name of the rule. Moreover, letting ς be a substitution, if A is an axiom schema, then $\varsigma(A)$ is an instance of the axiom schema, and if $\dfrac{B_1 \quad \dots \quad B_n}{B}$ is a rule schema, then $\dfrac{\varsigma(B_1) \quad \dots \quad \varsigma(B_n)}{\varsigma(B)}$ is an instance of the rule schema.

If the intended meaning is made clear by the context, in the following we use the generic terms *axiom* or *rule* to denote either an axiom or rule schema or an instance of the axiom or rule schema.

Definition 2.2.6 (Derivation of a formula, theorem)**.** Given a system $\mathbf{L}$ and a formula A, a *derivation* of A in $\mathbf{L}$ is a sequence of formulas ending with A in which each formula is an axiom of $\mathbf{L}$, or is obtained by previous formulas by the application of a rule of $\mathbf{L}$. We say that A is a *theorem* of $\mathbf{L}$, or is *derivable* in $\mathbf{L}$, if there exists a derivation of A in $\mathbf{L}$, in this case we write $\vdash_{\mathbf{L}} A$, or $\mathbf{L} \vdash A$. If in contrast A is not derivable in $\mathbf{L}$ we write $\nvdash_{\mathbf{L}} \mathcal{A}$. If A is a theorem of $\mathbf{L}$ we might say that $\mathbf{L}$ *contains* A. We denote by $Thm_{\mathbf{L}}$ the set $\{A \in \mathcal{L} \mid \mathbf{L} \vdash A\}$ of the theorems of $\mathbf{L}$.

Definition 2.2.7 (Derivation of a rule)**.** A *derivation* of a rule $R \, \dfrac{A_1 \quad \dots \quad A_n}{A}$ in the system $\mathbf{L}$ is a sequence of formulas ending with A in which each formula is an axiom of $\mathbf{L}$, or a premiss of R, or is obtained by previous formulas by the application of a rule of $\mathbf{L}$. We say that R is *derivable* in $\mathbf{L}$ if there exists a derivation of R in $\mathbf{L}$, in this case we write $\vdash_{\mathbf{L}} R$.

Definition 2.2.8. Let $\mathbf{L}$ and $\mathbf{L}'$ be two axiomatic systems. We say that

- $\mathbf{L}$ is *consistent* if $\mathbf{L} \nvdash \bot$, otherwise it is *inconsistent*.
- $\mathbf{L}$ is an *extension* of $\mathbf{L}'$ if $\mathbf{L} \supseteq \mathbf{L}'$, i.e., $\mathbf{L}$ is defined by adding some axioms or rules to the axioms and rules of $\mathbf{L}'$.
- $\mathbf{L}$ is *stronger* than $\mathbf{L}'$ (or, equivalently, $\mathbf{L}'$ is *weaker* than $\mathbf{L}$), if $Thm_{\mathbf{L}} \supseteq Thm_{\mathbf{L}'}$.
- $\mathbf{L}$ and $\mathbf{L}'$ are *equivalent*, written $\mathbf{L} \equiv \mathbf{L}'$, if $Thm_{\mathbf{L}} = Thm_{\mathbf{L}'}$.

Definition 2.2.9 (Logic)**.** Given an axiomatic system $\mathbf{L}$, we call *logic* the set $Thm_{\mathbf{L}}$ of its theorems. Conversely, given a logic $\mathbf{L}$, we call *axiomatisation* of $\mathbf{L}$ any axiomatic system $\mathbf{L}'$ such that $\mathbf{L} = Thm_{\mathbf{L}'}$.

$$RE\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B} \qquad RM\ \frac{A \to B}{\Box A \to \Box B} \qquad RN\ \frac{A}{\Box A} \qquad RD_n^+\ \frac{\neg(A_1 \wedge ... \wedge A_n)}{\neg(\Box A_1 \wedge ... \wedge \Box A_n)}\ n \geq 1$$

M	$\Box(A \wedge B) \to \Box A$	C	$\Box A \wedge \Box B \to \Box(A \wedge B)$	N	$\Box\top$
T	$\Box A \to A$	4	$\Box A \to \Box\Box A$	P	$\neg\Box\bot$
D	$\neg(\Box A \wedge \Box\neg A)$	K	$\Box(A \to B) \to (\Box A \to \Box B)$		

Figure 2.1: Modal axioms and rules.

With the present approach, logics and axiomatic systems are distinct but strongly related notions. When no ambiguity arises, in the following we use 'logic' and 'system' as interchangeable terms. Correspondingly, we use **L** to denote both the axiomatic system and the underlying logic.

Classical non-normal modal logics

Classical non-normal modal logics are Hilbert-style defined as extensions of classical propositional logic (**CPL**) formulated in the language $\mathcal{L}$. For **CPL** we consider the following axiomatisation, where the only rule (schema) is *modus ponens* (MP):

$$\begin{array}{llll} A \to (B \to A), & (A \to (B \to C)) \to ((A \to B) \to (A \to C)), & & \\ A \to A \vee B, & B \to A \vee B, & (A \to C) \to ((B \to C) \to (A \vee B \to C)), & \\ A \wedge B \to A, & A \wedge B \to B, & A \to (B \to A \wedge B), & \dfrac{A \to B \quad A}{B}\ . \\ \bot \to A, & A \to \top, & A \vee \neg A, & \end{array}$$

Classical non-normal modal logics are then obtained by extending **CPL** with additional axioms or rules for the modality $\Box$ (Figure 2.1). The minimal classical non-normal modal logic that we consider in this work is logic **E**, which is defined in language $\mathcal{L}$ by extending **CPL** with the *congruence* rule RE:

$$RE\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}\ .$$

Logics containing RE can be called *congruential*. All other classical systems considered in this work are extensions of **E**, whence they all are congruential.[3] Logic **E** is also the weakest system of the so-called *classical cube* [29, 110] (Figure 2.2), a lattice of eight systems which are obtained by extending **E** with any combination of axioms M, C, and N:

[3] There exist of course also non-congruential logics, examples are C.I. Lewis' systems **S1** and **S2** [115]. Starting with Lemmon [114], non-normal modal logics satisfying rule RE have been often called "classical" (cf. also [29]). Here we use the term "classical" to denote logics extending classical logic **CPL**, and we distinguish classical logics from intuitionistic logics, which are instead extensions of intuitionistic logic **IPL**. In any case, all the logics considered in this work contain rule RE, whence no ambiguity should arise.

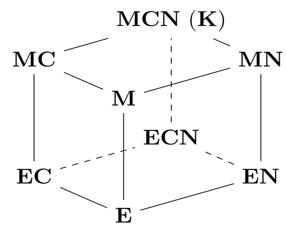

Figure 2.2: The classical cube.

$$M \quad \Box(A \wedge B) \to \Box A, \qquad C \quad \Box A \wedge \Box B \to \Box(A \wedge B), \qquad N \quad \Box\top.$$

As usual, the possibility to define classical modal logics by means of the only modality $\Box$ is due to the interdefinability between $\Box$ and $\Diamond$. The same systems could be equally defined by considering the $\Diamond$-versions of RE, M, C, and N, which are respectively the following:

$$\frac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}\,, \qquad \Diamond A \to \Diamond(A \vee B), \qquad \Diamond(A \vee B) \to \Diamond A \vee \Diamond B, \qquad \neg\Diamond\bot.$$

It can be shown that different combinations of axioms M, C, and N define different systems (with the present axiomatisation, a formula among M, C, and N is derivable in a system of the classical cube only if it explicitly belong to the set of its axioms). This means that the classical cube contains eight non-equivalent systems. A modal logic is called

- *monotonic* if it contains axiom M (and *non-monotonic* otherwise);
- *regular* if it contains both axioms M and C; and
- *normal* if it contains all the three axioms M, C, and N.

Indeed, the system obtained by extending **E** with axioms M, C, and N coincides with logic **K**, the minimal normal modal logic. However, for the sake of simplicity we use the term *non-normal modal logic* to uniformly denote all considered logics, including also those containing all M, C, and N.

We denote classical non-normal modal logics by **EX**, where **X** stays for the (possibly empty) list of axioms which are added to the basic system **E**. However, we adopt the convention of dropping the letter E from the name of monotonic systems, which are consequently denoted by **MX**. In addition, given a system **L**, we write $\mathbf{L}^*$ to indicate any extension of **L** with some of the axioms considered in this work (thus for instance $\mathbf{E}^*$ denotes any classical non-normal modal logic).

Examples of derivations in the systems of the classical cube are displayed in Figure 2.3 (see also Chellas [29] and Pacuit [142]). Observe that the axioms M and N are respectively equivalent to the rules of *monotonicity* RM and *necessitation* RN, in the sense that the

($\mathbf{M} \vdash RM$)

1.	$A \to B$	(assumption)
2.	$A \leftrightarrow B \wedge A$	(1, **CPL**)
3.	$\Box A \leftrightarrow \Box(B \wedge A)$	(2, RE)
4.	$\Box(B \wedge A) \to \Box B$	(M)
5.	$\Box A \to \Box B$	(3,4, **CPL**)

($\mathbf{EN} \vdash RN$)

1.	A	(assumption)
2.	$\top \leftrightarrow A$	(1, **CPL**)
3.	$\Box\top \leftrightarrow \Box A$	(2, RE)
4.	$\Box\top$	(N)
5.	$\Box A$	(3,4, MP)

($\mathbf{E} \cup \{RM\} \vdash M$)

1.	$A \wedge B \to A$	(theorem of **CPL**)
2.	$\Box(A \wedge B) \to \Box A$	(1, RM)

($\mathbf{E} \cup \{RN\} \vdash N$)

1.	$\top$	(theorem of **CPL**)
2.	$\Box\top$	(1, RN)

($\mathbf{MC} \vdash K$)

1.	$\Box(A \to B) \wedge \Box A \to \Box((A \to B) \wedge A)$	(C)
2.	$(A \to B) \wedge A \leftrightarrow (A \to B) \wedge A \wedge B$	(theorem of **CPL**)
3.	$\Box((A \to B) \wedge A) \leftrightarrow \Box((A \to B) \wedge A \wedge B)$	(2, RE)
4.	$\Box((A \to B) \wedge A \wedge B) \to \Box B$	(M)
5.	$\Box(A \to B) \wedge \Box A \to \Box B$	(1,3,4, **CPL**)
6.	$\Box(A \to B) \to (\Box A \to \Box B)$	(5, **CPL**)

($\mathbf{K} \vdash M$)

1.	$A \wedge B \to A$	(theorem of **CPL**)
2.	$\Box(A \wedge B \to A)$	(1, RN)
3.	$\Box(A \wedge B \to A) \to (\Box(A \wedge B) \to \Box A)$	(K)
4.	$\Box(A \wedge B) \to \Box A$	(2,3, MP)

($\mathbf{K} \vdash C$)

1.	$A \to (B \to A \wedge B)$	(theorem of **CPL**)
2.	$\Box(A \to (B \to A \wedge B))$	(1, RN)
3.	$\Box(A \to (B \to A \wedge B)) \to (\Box A \to \Box(B \to A \wedge B))$	(K)
4.	$\Box A \to \Box(B \to A \wedge B)$	(2,3, MP)
5.	$\Box(B \to A \wedge B) \to (\Box B \to \Box(A \wedge B))$	(K)
6.	$\Box A \to (\Box B \to \Box(A \wedge B))$	(4,5, **CPL**)
7.	$\Box A \wedge \Box B \to \Box(A \wedge B)$	(6, **CPL**)

Figure 2.3: Derivations of modal axioms and rules in the systems of the classical cube.

system $\mathbf{E} \cup \{RM\}$ is equivalent to $\mathbf{M}$, and the system $\mathbf{E} \cup \{RN\}$ is equivalent to $\mathbf{EN}$. Moreover, let the minimal normal modal logic $\mathbf{K}$ be defined as usual by extending $\mathbf{CPL}$ with the axiom K and the rule RN. On the basis of the derivations in Figure 2.3 we can observe that $\mathbf{MCN}$, i.e., the top system of the classical cube, coincides with the logic $\mathbf{K}$, since axiom K is derivable in $\mathbf{MC}$, N is equivalent to RN, and RE, M, and C are derivable in $\mathbf{K}$.

The systems of the classical cube represent the base of our investigation of classical non-normal modal logics. In addition to these eight systems we also consider their extensions with any combination of the axioms T, 4, D, P, and for every $n \in \mathbb{N}$, $n \geq 1$, the rule RD_n^+ (see Figure 2.1). We sometimes refer to the systems containing D, or P, or, for some n, RD_n^+, as *deontic* systems, since these axioms and rules are of interest in deontic logic. In particular, the rules RD_n^+ have a peculiar interest in deontic logic as they exclude the possibility of having n obligations that cannot be realised all together. While the rules RD_n^+ are entailed by the axioms D and P in normal modal logics, this is not the case in non-normal modal logics, therefore they must be considered explicitly. In the following, for every $n \in \mathbb{N}$ we denote by $\mathbf{ED_n^+}^*$ the system $\mathbf{E}^* \cup \{RD_n^+\}$.

Examples of derivations in deontic systems are displayed in Figure 2.4. It is easy to see that RD_n^+ is derivable in $\mathbf{ED_m^+}^*$ for every $m \geq n$; in the figure we show the case where $n = 2$ and $m = 3$. Notice also that the rule RD_1^+ is equivalent to axiom P. Moreover, it is also worth remarking that the axioms D and P are equivalent in normal modal logics but are not necessarily equivalent in non-normal ones. The two axioms become equivalent only in presence of both M and C. Finally, observe that all D, P, and, for every n, RD_n^+, are entailed by axiom T. The relations among the systems defined by adding D, P, and RD_n^+ to the systems of the classical cube are displayed in Figure 2.5. As we can see, these principles, which are equivalent in normal modal logics, define in contrast a rather complex family of non-normal modal logics.

Decidability and complexity

For every considered logic $\mathbf{L}$ we shall address the following *derivability problem*:

Given a formula A of $\mathcal{L}$, *establish whether A is derivable in* $\mathbf{L}$.

The derivability problem in a logic $\mathbf{L}$ is a *decision problem*, i.e., a problem that can be formulated as a yes-no question. A decision problem is decidable if there exists an effective method answering to the problem correctly; such a method is called decision procedure. In the following we say that:

- a logic $\mathbf{L}$ is *decidable* if the derivability problem for $\mathbf{L}$ is decidable;
- we call *decision procedure* for $\mathbf{L}$ any algorithm solving the derivability problem for $\mathbf{L}$;

($\mathbf{ED_3^+} \vdash RD_2^+$)

1.	$\neg(A \wedge B)$	(assumption)
2.	$\neg(A \wedge A \wedge B)$	(1, **CPL**)
3.	$\neg(\Box A \wedge \Box A \wedge \Box B)$	(2, RD_3^+)
4.	$\neg(\Box A \wedge \Box B)$	(3, **CPL**)

($\mathbf{ED_2^+} \vdash D$)

1.	$\neg(A \wedge \neg A)$	(theorem of **CPL**)
2.	$\neg(\Box A \wedge \neg \Box A)$	(1, RD_2^+)

($\mathbf{ED_1^+} \vdash P$)

1.	$\neg\bot$	(theorem of **CPL**)
2.	$\neg\Box\bot$	(1, RD_1^+)

($\mathbf{EP} \vdash RD_1^+$)

1.	$A \wedge B \to A$	(theorem of **CPL**)
2.	$\Box(A \wedge B) \to \Box A$	(1, RM)

($\mathbf{END} \vdash P$)

1.	$\top \leftrightarrow \neg\bot$	(theorem of **CPL**)
2.	$\Box\top \leftrightarrow \Box\neg\bot$	(1, RE)
3.	$\neg(\Box\bot \wedge \Box\neg\bot)$	(D)
4.	$\Box\top \to \neg\Box\bot$	(2,3, **CPL**)
5.	$\Box\top$	(N)
6.	$\neg\Box\bot$	(4,5, **CPL**)

($\mathbf{MD} \vdash RD_2^+$)

1.	$\neg(A \wedge B)$	(assumption)
2.	$A \to \neg B$	(1, **CPL**)
3.	$\Box A \to \Box\neg B$	(2, RM)
4.	$\neg(\Box B \wedge \Box\neg B)$	(D)
5.	$\neg(\Box A \wedge \Box B)$	(3,4, **CPL**)

($\mathbf{ECP} \vdash RD_n^+$)

1.	$\neg(A_1 \wedge ... \wedge A_n)$	(assumption)
2.	$A_1 \wedge ... \wedge A_n \leftrightarrow \bot$	(1, **CPL**)
3.	$\Box(A_1 \wedge ... \wedge A_n) \leftrightarrow \Box\bot$	(2, RE)
4.	$\neg\Box\bot$	(P)
5.	$\neg\Box(A_1 \wedge ... \wedge A_n)$	(3,4, **CPL**)
6.	$\Box A_1 \wedge \Box A_2 \to \Box(A_1 \wedge A_2)$	(C)
7.	$\Box A_1 \wedge \Box A_2 \wedge ... \wedge \Box A_n \to \Box(A_1 \wedge A_2) \wedge ... \wedge \Box A_n$	(6, **CPL**)
	$\vdots$	
8.	$\Box A_1 \wedge ... \wedge \Box A_n \to \Box(A_1 \wedge ... \wedge \Box A_n)$	(7, **CPL**)
9.	$\neg(\Box A_1 \wedge ... \wedge \Box A_n)$	(5,8, **CPL**)

($\mathbf{ET} \vdash RD_n^+$)

1.	$\neg(A_1 \wedge ... \wedge A_n)$	(assumption)
2.	$\Box A_1 \to A_1$	(T)
	$\vdots$	
3.	$\Box A_n \to A_n$	(T)
4.	$\Box A_1 \wedge ... \wedge \Box A_n \to A_1 \wedge ... \wedge A_n$	(2,3, **CPL**)
5.	$\neg\Box(A_1 \wedge ... \wedge A_n)$	(1,4, **CPL**)

Figure 2.4: Derivations of modal axioms and rules in deontic systems

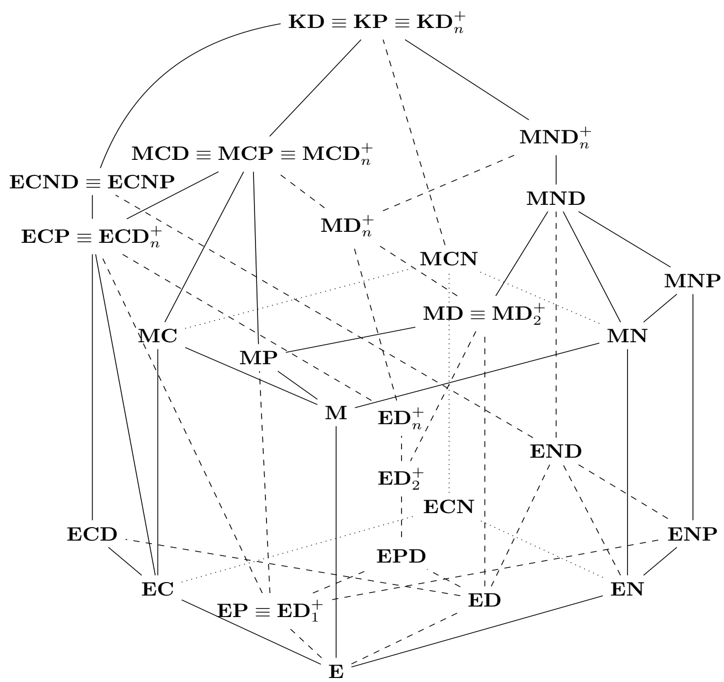

Figure 2.5: Diagram of deontic systems ("Pantheon").

- we call *complexity* of $\mathbf{L}$ the complexity of the derivability problem for $\mathbf{L}$.

Besides derivability, we can also consider the following related notion of deduction from assumptions:

Definition 2.2.10 (Deduction from assumptions)**.** Given a non-normal modal logic $\mathbf{L}$, we say that A is *deducible* from a set of formulas Φ, written $\Phi \vdash_{\mathbf{L}} A$, if there exists a sequence of formulas ending with A in which each formula is a theorem of $\mathbf{L}$, or belongs to Φ, or is obtained from previous formulas by an application of modus ponens (MP).

The one introduced by the above definition is a notion of *local* deduction, and must be distinguished from the different notion of global deduction, which instead allows one to apply modal rules of inference to the assumptions. For local deduction the following holds for every classical non-normal modal logic:

Theorem 2.2.1 (Deduction theorem)**.** Let Φ, A, and B be respectively a set of formulas of $\mathcal{L}$ and two formulas of $\mathcal{L}$. Then, $\Phi \cup \{A\} \vdash_{\mathbf{E}^*} B$ implies $\Phi \vdash_{\mathbf{E}^*} A \rightarrow B$.

As a consequence of the above theorem, whenever the set of assumptions $\{A_1, ..., A_n\}$ is finite, the assertion of deducibility $\{A_1, ..., A_n\} \vdash_{\mathbf{L}} B$ is equivalent to the assertion of derivability $\vdash_{\mathbf{L}} A_1 \rightarrow (... \rightarrow (A_n \rightarrow B)$, which is in turn equivalent to $\vdash_{\mathbf{L}} A_1 \wedge ... \wedge A_n \rightarrow B$.

All classical non-normal modal logics considered in this work are decidable. For some of them the following complexity bounds are established by Vardi [167]:

Theorem 2.2.2 (Decidability and complexity, [167])**.** The derivability problem for the logic **E** and any its extension with the axioms M, C, N, P, T, 4 is decidable. In particular, it is coNP-complete for the systems lacking the axiom C, and it is in PSPACE for the systems containing C.[4]

The cases excluded from the above theorem of the systems with the axiom D and the rules RD_n^+ (but without the axiom 4) are covered by a general result by Schröder and Pattinson [155] that uniformly states a PSPACE complexity upper bound for all the systems defined only by *non-iterative* axioms, i.e., the axioms not containing nested modalities:

Theorem 2.2.3 (Decidability and complexity, [155])**.** The derivability problem for the logic **E** and any its extension with the axioms M, C, N, P, T, D, and the rules RD_n^+ is in PSPACE.

2.3 Standard semantics of classical non-normal modal logics

We now present the standard semantics of non-normal modal logics. We shall consider three kinds of models: *standard neighbourhood* models, $\exists\forall$-*models* for monotonic systems, and *relational* models for regular systems.

Standard neighbourhood models

The standard semantic characterisation of classical non-normal modal logics is given in terms of *neighbourhood* models [142]. Neighbourhood models are also called *minimal* [29], or *Scott-Montague*, models, from the authors who independently introduced them in [157, 129]. In this work we call them *standard neighbourhood*, or just *standard*. Standard neighbourhood models are a generalisation of Kripke models for normal modal logics. They replace the binary relation of Kripke models with a so-called neighbourhood function, which assigns to each world a set of sets of worlds. Intuitively, the neighbourhood function assigns to each world the propositions that are necessary/obligatory/etc. in it. Standard neighbourhood models are defined as follows.

Definition 2.3.1 (Standard semantics)**.** A *standard neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{N}$ is a function $\mathcal{W} \longrightarrow \mathcal{PP}(\mathcal{W})$ – where $\mathcal{P}$ denotes the powerset, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a valuation function for propositional variables of $\mathcal{L}$. As usual, we call the elements of $\mathcal{W}$ *possible worlds*, and the function $\mathcal{N}$ *neighbourhood*

[4] In [167] complexity upper bounds for the satisfiability/derivability problem in the mentioned logics are proved. It is clear that for the logics in coNP this is also a lower bound since they contain **CPL**. By contrast, I am not aware of any proof that PSPACE is a lower bound for the non-normal logics with axiom C.

function. The forcing relation $\mathcal{M}, w \Vdash_{st} A$ is defined as follows, where $[\![A]\!]_{\mathcal{M}}$ denotes the set $\{v \in \mathcal{W} \mid \mathcal{M}, v \Vdash_{st} A\}$, also called the *truth set* of A:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash_{st} p_i & \text{iff} & w \in V(p_i); \\
\mathcal{M}, w \nVdash_{st} \bot; & & \\
\mathcal{M}, w \Vdash_{st} \top; & & \\
\mathcal{M}, w \Vdash_{st} A \wedge B & \text{iff} & \mathcal{M}, w \Vdash_{st} A \text{ and } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} A \vee B & \text{iff} & \mathcal{M}, w \Vdash_{st} A \text{ or } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} A \rightarrow B & \text{iff} & \mathcal{M}, w \nVdash_{st} A \text{ or } \mathcal{M}, w \Vdash_{st} B; \\
\mathcal{M}, w \Vdash_{st} \Box A & \text{iff} & [\![A]\!]_{\mathcal{M}} \in \mathcal{N}(w).
\end{array}
$$

From the above definition and the duality between $\Box$ and $\Diamond$ we can also extract the following truth condition for diamond formulas:

$$\mathcal{M}, w \Vdash_{st} \Diamond A \quad \text{iff} \quad \mathcal{W} \setminus [\![A]\!]_{\mathcal{M}} \notin \mathcal{N}(w);$$

or equivalently $\mathcal{M}, w \Vdash_{st} \Diamond A$ if and only if $[\![\neg A]\!]_{\mathcal{M}} \notin \mathcal{N}(w)$. We adopt the standard terminology of possible world semantics:

Definition 2.3.2 (True/satisfiable/valid formula, countermodel, semantic consequence)**.** Let A be a formula of $\mathcal{L}$. We say that

- A is *satisfied* by a world w of a model $\mathcal{M}$, or is true in w, if $\mathcal{M}, w \Vdash A$; otherwise we say that w *falsifies* A, or A is false in w, and we write $\mathcal{M}, w \nVdash A$.
- A is *satisfiable* in a model $\mathcal{M}$ if there is a world w of $\mathcal{M}$ such that $\mathcal{M}, w \Vdash A$.
- A is *valid* in a model $\mathcal{M}$, written $\mathcal{M} \models A$, if for every world w of $\mathcal{M}$, $\mathcal{M}, w \Vdash A$; otherwise we say that A is *false* in $\mathcal{M}$, and we write $\mathcal{M} \not\models A$. In the second case, $\mathcal{M}$ is called *countermodel* of A.
- A is *satisfiable* in a class of models $\mathcal{C}$ if there are a model $\mathcal{M} \in \mathcal{C}$ and a world w of $\mathcal{M}$ such that $\mathcal{M}, w \Vdash A$. Moreover, we say that A is *valid* in $\mathcal{C}$, written $\mathcal{C} \models A$, if for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models A$.
- A is a *semantic consequence* of a set of formulas Φ with respect to a class of models $\mathcal{C}$, written $\Phi \models_{\mathcal{C}} A$, if for every $\mathcal{M} \in \mathcal{C}$ and every world w of $\mathcal{M}$, if $\mathcal{M}, w \Vdash B$ for every $B \in \Phi$, then $\mathcal{M}, w \Vdash A$.

Definition 2.3.3 (Sound rule)**.** A rule R is *sound*, or valid, with respect to a model $\mathcal{M}$ (respectively a class of models $\mathcal{C}$) if in case all premisses of R are valid in $\mathcal{M}$ (respectively $\mathcal{C}$), then the conclusion of R is also valid in $\mathcal{M}$ (respectively $\mathcal{C}$).

As usual, we aim to put in correspondence the notion of derivability ($\vdash$) in a given logic with the notion of semantic consequence ($\models$) in a certain class of models:

Definition 2.3.4 (Soundness, completeness, characterisation)**.** Given a logic $\mathbf{L}$ and a class of models $\mathcal{C}$, we say that

- $\mathbf{L}$ is *sound* with respect to $\mathcal{C}$ if $\Gamma \vdash_{\mathbf{L}} A$ implies $\Gamma \models_{\mathcal{C}} A$.
- $\mathbf{L}$ is (strongly) *complete* with respect to $\mathcal{C}$ if $\Gamma \models_{\mathcal{C}} A$ implies $\Gamma \vdash_{\mathbf{L}} A$.
- $\mathbf{L}$ is *characterised* by $\mathcal{C}$ (or $\mathcal{C}$ characterises $\mathbf{L}$) if $\mathbf{L}$ is sound and complete with respect to $\mathcal{C}$.

The class of all standard models characterises the basic logic $\mathbf{E}$. In particular, it can be easily verified that the rule RE is sound in every standard model. On the contrary, the axioms M, C, and N (as well as all other considered modal axioms or rules) are not valid in standard models. For instance, the following model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W} = \{w\}$, $\mathcal{V}(p_1) = \{w\}$, $\mathcal{V}(p_2) = \emptyset$, and $\mathcal{N}(w) = \{\emptyset\}$, falsifies the formula $\Box(p_1 \wedge p_2) \rightarrow \Box p_1$, which is an instance of axiom M: $w \Vdash \Box(p_1 \wedge p_2)$ because $[\![p_1 \wedge p_2]\!] = \emptyset \in \mathcal{N}(w)$, but $w \not\Vdash \Box p_1$ because $[\![p_1]\!] = \{w\} \notin \mathcal{N}(w)$, then $w \not\Vdash \Box(p_1 \wedge p_2) \rightarrow \Box p_1$.

For the extensions of logic $\mathbf{E}$ a semantic characterisation in terms of standard models can be given by considering additional closure properties of the neighbourhood function. The properties associated to the axioms M, C, and N are the following:

(M)	If $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}(w)$.	(Supplementation)
(C)	If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$.	(Closure under intersection)
(N)	$\mathcal{W} \in \mathcal{N}(w)$.	(Containing the unit)

Accordingly, a standard model is *supplemented*, *closed under intersection*, or *contains the unit*, if it satisfies the corresponding property of supplementation, closure under intersection, or containing the unit. The semantic characterisation provided by these properties is *modular*, in the sense that the models for a system defined by a set of axioms among M, C, and N are obtained by adding together the conditions corresponding to each axiom.

By considering additional properties of the neighbourhood function one can also provide a modular characterisation of all systems defined in the previous section. The conditions are the following:

(T)	If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$.
(P)	$\emptyset \notin \mathcal{N}(w)$.
(D)	If $\alpha \in \mathcal{N}(w)$, then $\mathcal{W} \setminus \alpha \notin \mathcal{N}(w)$.
(RD_n^+)	If $\alpha_1, ..., \alpha_n \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.
(4)	If $\alpha \in \mathcal{N}(w)$, then $\{v \mid \alpha \in \mathcal{N}(v)\} \in \mathcal{N}(w)$.

In the following, for every logic $\mathbf{L}$ we denote with $\mathcal{C}^{st}_{\mathbf{L}}$ the class of standard models for $\mathbf{L}$.

Theorem 2.3.1 (Characterisation, [29])**.** Logic $\mathbf{E}^*$ is sound and strongly complete with respect to the corresponding class of standard neighbourhood models, that is: $\Phi \models_{\mathcal{C}^{st}_{\mathbf{E}^*}} A$ if and only if $\Phi \vdash_{\mathbf{E}^*} A$.

A proof of the above theorem can be found in Chellas [29] for the systems of the classical cube and their extensions with the axioms T, P, D, 4; the same proof can be easily extended to the systems with the rules RD^+_n.

Definition 2.3.5 (Finite model property)**.** We say that a possible-worlds model is *finite* if the set $\mathcal{W}$ of the possible worlds is finite. We say that a logic $\mathbf{L}$ enjoys the *finite model property* if every satisfiable formula in the class of models for $\mathbf{L}$ is also satisfiable in a finite model belonging to the same class, or, equivalently, if every non derivable formula in $\mathbf{L}$ has a finite countermodel in the class of models for $\mathbf{L}$.

Theorem 2.3.2 (Finite model property, [29])**.** The logics of the classical cube and their extensions with the axioms T, P, D, and the rules RD^+_n enjoy the finite model property.

For the systems of the classical cube a proof of the above theorem can be found in Chellas [29]. The proof can be easily extended to the systems containing T, P, D, and RD^+_n. By contrast, we are not aware of any proof of the finite model property for the logics with the axiom 4.

$\exists\forall$-models for monotonic logics

In addition to the one offered by supplemented standard models, an alternative – and simpler – semantic characterisation for monotonic systems can be given by considering the following reformulation of the forcing condition for boxed formulas in the standard semantics: in order to say that a world w satisfies a formula $\Box A$ we do not require that its neighbourhood contains exactly the truth set of A, but just that it contains a subset of it. The resulting semantics, called $\exists\forall$-semantics, is formally defined as follows.

Definition 2.3.6 ($\exists\forall$-semantics)**.** An $\exists\forall$-*neighbourhood model* is defined as a standard neighbourhood model (cf. Definition 2.3.1), except for the forcing condition of the boxed formulas, which is as follows:

$$\mathcal{M}, w \Vdash_{\exists\forall} \Box A \quad \text{iff} \quad \text{there is } \alpha \in \mathcal{N}(w) \text{ such that for every } v \in \alpha,\ \mathcal{M}, v \Vdash_{\exists\forall} A.$$

The above forcing condition can be equivalently rewritten as $\mathcal{M}, w \Vdash_{\exists\forall} \Box A$ if and only if there is $\alpha \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![A]\!]_{\mathcal{M}}$. As before, by duality with $\Box$ we can also obtain the forcing condition for diamond formulas, which is $\mathcal{M}, w \Vdash_{\exists\forall} \Diamond A$ if and only if for all $\alpha \in \mathcal{N}(w)$,

$\alpha \cap [\![A]\!]_{\mathcal{M}} \neq \emptyset$. Similarly to the standard semantics, extensions of the logic **M** with axioms C, N, T, P, D, 4, and rules RD_n^+, can be captured in the $\exists\forall$-semantics by considering additional properties of the neighbourhood function. The following result can be found in Pacuit [142].

Theorem 2.3.3 (Characterisation)**.** The logic **M** is sound and strongly complete with respect to the class of all $\exists\forall$-models.

Relational models for regular logics

As it is well-known, relational models provide a semantics for normal modal logics, as they validate both the rule of necessitation and axiom K. However, starting with Kripke [104], variants of relational models have been proposed for logics lacking the necessitation rule. In [104] Kripke introduces relational models with so-called non-normal worlds, with the aim of characterising a family of Lewis' and Lemmon's systems in which necessitation fails or is validated only in a restricted form. After Kripke's proposal, relational models with non-normal worlds have been reformulated in several ways (see e.g. [93, 119]), here we take into consideration the following definition considered by Fitting [56] and Priest [152]:

Definition 2.3.7 (Relational semantics)**.** A *relational model* with *non-normal worlds* is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\mathcal{W}^i \subseteq \mathcal{W}$ is the set of non-normal worlds, $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a binary relation, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a valuation function for the propositional variables of $\mathcal{L}$. The forcing relation $w \Vdash_r A$ is defined as in Definition 2.3.1, except for the boxed formulas, for which is defined as follows:

$$\mathcal{M}, w \Vdash_r \Box A \quad \text{iff} \quad w \notin \mathcal{W}^i \text{ and for all } v \in \mathcal{W}, \text{ if } w\mathcal{R}v \text{ then } \mathcal{M}, v \Vdash_r A.$$

By definition, non-normal worlds falsify every boxed formula. Validity is defined as usual (cf. Definition 2.3.2): we say that a formula is valid in a model if it is satisfied by all worlds (no matter if they are normal or non-normal).[5] It is easy to verify that non-normal relational models validate the axioms M and C but do not validate the axiom N. Notice also that in case $\mathcal{W}^i$ is empty, relational models with non-normal worlds collapse into standard Kripke models for normal modal logics. In the following, for any relational model and world w we denote with $\mathcal{R}(w)$ the set of worlds v such that $w\mathcal{R}v$.

[5]Actually, Priest [152] considers two notions of validity in non-normal relational models: the one considered here, and a second one according to which a formula is valid if it is satisfied by all *normal* worlds. The two notions validate different formulas. In particular, the second definition does not validate the congruence rule RE. Moreover, it validates the axiom N but does not validate the necessitation rule RN, which therefore are not equivalent (as a matter of fact, it is considered in order to characterise some Lewis' systems which are not congruential, and contain N but do not contain the necessitation rule in the general form). On the contrary, the definition considered here validates RE and makes equivalent, and both not valid, axiom N and rule RN. It is to notice that axiom N and rule RN are always equivalent in the presence of RE, so that in particular they are equivalent in all the systems considered in this work.

Theorem 2.3.4 (Characterisation, [56])**.** The logic **MC** is sound and strongly complete with respect to the class of all relational models with non-normal worlds.

2.4 Elgesem's and Troquard's agency and ability logics

In addition to the family of classical non-normal modal logics defined in Section 2.2, we also consider two specific non-normal modal logics studied in the literature, namely Elgesem's agency and ability logic [47] and its coalitional extension defined by Troquard [165].

Elgesem's logic, also called the logic of *bringing-it-about*, represents a standard reference in the literature on agency logic. It provides a formalisation of agents' actions in terms of their results: that an agent "does something" is interpreted as the fact that the agent brings about something. For instance, "John does a bank transfer" is interpreted as "John brings it about that the bank transfer is done". Elgesem's logic contains two modalities indexed by agents $\mathbb{E}_i$ and $\mathbb{C}_i$ (we adopt the notation of [78]). The former one expresses the agentive modality of bringing-it-about, whereas the latter one expresses capability: roughly speaking, $\mathbb{E}_{lucy}\, \mathit{BankTransfer}$ means that Lucy makes a bank transfer, whereas $\mathbb{C}_{lucy}\, \mathit{BankTransfer}$ means that Lucy *can* make a bank transfer. Elgesem's logic is also well-suited for formalising notions of control, power, and delegation. For instance, "Sara prevents Lucy from making a bank transfer" will be captured just by $\mathbb{E}_{sara}\, \neg\mathbb{E}_{lucy}\, \mathit{BankTransfer}$.

Elgesem's logic deals with actions of a single agent, who might be a human individual, or an institution or a group conceived as an indivisible entity. A natural extension of this logic is to handle groups or coalitions that act *jointly* to bring about an action. This has been proposed by Troquard [165] who has developed an extension of Elgesem's logic to handle "coalitions": individuals may gather in coalitions to bring about a joint action. In such a joint action, each participant must be involved, so that the logic rejects coalition monotonicity: $\mathbb{E}_g A \to \mathbb{E}_{g'} A$ whenever $g \subseteq g'$ is *not* assumed to be valid.

Besides their own interest, in the context of the present work Elgesem's and Troquard's logics represent significative examples of non-normal modal logics as (i) they are even *non-monotonic*, and (ii) they are *incomparable* with normal modal logics, as they contain the axiom $\neg\Box\top$, i.e., the negation of necessitation. In this section, we present their axiomatisation and their neighbourhood semantics.

Axiomatic systems

Let us formally introduce the axiomatisations of Elgesem's agency logic and Troquard's coalition logic, henceforth respectively called **ELG** and **COAL**. First, let $\mathcal{A} = \{a, b, c, ...\}$ be a set of *agents*. The logic **ELG** is then defined on a propositional language $\mathcal{L}_{Elg}$ that – instead of $\Box$ – contains, for every $i \in \mathcal{A}$, two unary modalities $\mathbb{E}_i$ and $\mathbb{C}_i$, respectively of "agency" and

$RE_{\mathbb{E}}$	$\dfrac{A \leftrightarrow B}{\mathbb{E}_i A \leftrightarrow \mathbb{E}_i B}$	$RE_{\mathbb{C}}$	$\dfrac{A \leftrightarrow B}{\mathbb{C}_i A \leftrightarrow \mathbb{C}_i B}$
$C_{\mathbb{E}}$	$\mathbb{E}_i A \wedge \mathbb{E}_i B \rightarrow \mathbb{E}_i(A \wedge B)$	$Q_{\mathbb{C}}$	$\neg \mathbb{C}_i \top$
$T_{\mathbb{E}}$	$\mathbb{E}_i A \rightarrow A$	$P_{\mathbb{C}}$	$\neg \mathbb{C}_i \bot$
$Int_{\mathbb{E}\mathbb{C}}$	$\mathbb{E}_i A \rightarrow \mathbb{C}_i A$		

Figure 2.6: Axiomatisation of Elgesem's agency and ability logic [47].

"ability". The formulas of $\mathcal{L}_{Elg}$ are defined by the following grammar:

$$A ::= p_i \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \mathbb{E}_i A \mid \mathbb{C}_i A,$$

where p_i is any propositional variable in Atm and i is any agent in $\mathcal{A}$. Formulas of the form $\mathbb{E}_i A$ and $\mathbb{C}_i A$ are respectively read as "the agent i brings it about that A", and "the agent i is capable of realising A". The logic **ELG** is defined by extending classical propositional logic (formulated in language $\mathcal{L}_{Elg}$) with the modal axioms and rules in Figure 2.6.[6]

Because of the presence of axiom $Q_{\mathbb{C}}$, Elgesem's logic is strictly non-normal. In this context, this axiom is used to formalise a peculiar aspect of Elgesem's account of agency, namely that agents are considered to realise (bring it about) something only if they are directly *responsible* of its realisation. As a consequence, according to Elgesem's account an agent cannot realise something that would have been the case also independently from her action, whence in particular tautologies. To make an example, an agent cannot bring it about that the Earth revolves around the sun.

Observe that $\neg\mathbb{E}_i\bot$ and $\neg\mathbb{E}_i\top$ are derivable (respectively from axiom $T_{\mathbb{E}}$, and from axioms $Int_{\mathbb{E}\mathbb{C}}$ and $Q_{\mathbb{C}}$). By contrast, the axioms C and T hold only for the modality $\mathbb{E}$, meaning respectively that if an agent realises two things, then she realises both, and that if A is brought about by some agent, then it is actually the case that A.

In Troquard's coalitional extension of Elgesem's logic [165], called **COAL**, single agents are replaced by *groups* of agents, the aim is to represent what agents do and can do when acting in coalitions. Correspondingly, the formulas of the language $\mathcal{L}_{coal}$ of **COAL** are defined as follows, where p_i is any propositional variable in Atm and g is any *subset* of $\mathcal{A}$.

$$A ::= p_i \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \mathbb{E}_g A \mid \mathbb{C}_g A.$$

The modal fomulas of $\mathcal{L}_{coal}$ are then indexed by groups of agents rather than by single agents.

The logic **COAL** is axiomatically defined by extending **CPL** with the modal axioms and rules in Figure 2.7. Apart from $F_{\mathbb{C}}$ and $Int^2_{\mathbb{E}\mathbb{C}}$, the axioms and rules of **COAL** are just

[6] A variant of Elgesem's logic not containing axiom $P_{\mathbb{C}}$ is considered in [78, 108]. All results presented in this work can be extended to this variant just by dropping the corresponding condition in the bi-neighbourhood semantics (Section 4.6) and the corresponding rule in the calculus (Section 6.5).

$RE_{\mathbb{E}}$	$\dfrac{A \leftrightarrow B}{\mathbb{E}_g A \leftrightarrow \mathbb{E}_g B}$	$RE_{\mathbb{C}}$	$\dfrac{A \leftrightarrow B}{\mathbb{C}_g A \leftrightarrow \mathbb{C}_g B}$
$C_{\mathbb{E}}$	$\mathbb{E}_g A \wedge \mathbb{E}_g B \rightarrow \mathbb{E}_g(A \wedge B)$	$Q_{\mathbb{C}}$	$\neg \mathbb{C}_g \top$
$T_{\mathbb{E}}$	$\mathbb{E}_g A \rightarrow A$	$P_{\mathbb{C}}$	$\neg \mathbb{C}_g \bot$
$Int^1_{\mathbb{E}\mathbb{C}}$	$\mathbb{E}_g A \rightarrow \mathbb{C}_g A$	$F_{\mathbb{C}}$	$\neg \mathbb{C}_{\emptyset} A$
$Int^2_{\mathbb{E}\mathbb{C}}$	$\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B \rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)$		

Figure 2.7: Axiomatisation of Troquard's coalition logic [165].

the coalition versions of the corresponding ones in **ELG**, with agents i replaced by groups g. The peculiar aspects of group agency are represented in **COAL** by the axioms $F_{\mathbb{C}}$ and $Int^2_{\mathbb{E}\mathbb{C}}$. In particular, the axiom $F_{\mathbb{C}}$ expresses that the empty group cannot realise anything, whereas the axiom $Int^2_{\mathbb{E}\mathbb{C}}$ expresses that if a group realises A and another group realises B, then by joining their forces they *could* realise both A and B. Observe that the axiom $Int^1_{\mathbb{E}\mathbb{C}}$ is derivable from $Int^2_{\mathbb{E}\mathbb{C}}$. Nevertheless, we keep it in the axiomatisation, as it is done in [165], in order to preserve a 1-1 the correspondence between the axioms of **COAL** and the rules of the hypersequent calculus the we shall define in Section 6.6, where a specific rule for $Int^1_{\mathbb{E}\mathbb{C}}$ will be needed.

Neighbourhood semantics

Elgesem's original semantics for the logic **ELG** is based on selection function models [47]. Here we consider the alternative neighbourhood semantics given by Governatori and Rotolo [78]. In this semantics, the models contain two neighbourhood functions $\mathcal{N}^{\mathbb{E}}$ and $\mathcal{N}^{\mathbb{C}}$ corresponding to the two modalities $\mathbb{E}$ and $\mathbb{C}$. The two functions are connected by a very simple relation: for every world w, $\mathcal{N}^{\mathbb{E}}(w)$ is included in $\mathcal{N}^{\mathbb{C}}(w)$. Moreover, each function satisfies the conditions corresponding to the $\mathbb{E}$- or $\mathbb{C}$-axioms in the standard semantics (cf. Section 2.3).

Definition 2.4.1 ([78])**.** A neighbourhood model for **ELG** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_i, \mathcal{N}^{\mathbb{C}}_i, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function, and for every agent i, $\mathcal{N}^{\mathbb{E}}_i$ and $\mathcal{N}^{\mathbb{C}}_i$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{PP}(\mathcal{W})$ satisfying the following conditions:

$(C_{\mathbb{E}})$	If $\alpha, \beta \in \mathcal{N}^{\mathbb{E}}_i(w)$, then $\alpha \cap \beta \in \mathcal{N}^{\mathbb{E}}_i(w)$.	$(Q_{\mathbb{C}})$	$\mathcal{W} \notin \mathcal{N}^{\mathbb{C}}_i(w)$.
$(T_{\mathbb{E}})$	If $\alpha \in \mathcal{N}^{\mathbb{E}}_i(w)$, then $w \in \alpha$.	$(P_{\mathbb{C}})$	$\emptyset \in \mathcal{N}^{\mathbb{C}}_i(w)$.
$(Int_{\mathbb{E}\mathbb{C}})$	$\mathcal{N}^{\mathbb{E}}_i(w) \subseteq \mathcal{N}^{\mathbb{C}}_i(w)$.		

The forcing relation $\Vdash$ is defined as usual for atomic formulas and boolean connectives, whereas for $\mathbb{E}$- and $\mathbb{C}$-formulas it is defined as follows:

$$\mathcal{M}, w \Vdash \mathbb{E}_i A \quad \text{iff} \quad [\![A]\!] \in \mathcal{N}^{\mathbb{E}}_i(w).$$
$$\mathcal{M}, w \Vdash \mathbb{C}_i A \quad \text{iff} \quad [\![A]\!] \in \mathcal{N}^{\mathbb{C}}_i(w).$$

The neighbourhood models for **ELG** have been reformulated by Troquard [165] in order to provide a semantic characterisation of the coalition logic **COAL**. The neighbourhood models for **COAL** are as follows:

Definition 2.4.2 ([165])**.** A neighbourhood model for **COAL** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_g^{\mathbb{E}}, \mathcal{N}_g^{\mathbb{C}}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function, and for every group of agents g, $\mathcal{N}_g^{\mathbb{E}}$ and $\mathcal{N}_g^{\mathbb{C}}$ are two bi-neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W}))$ satisfying the conditions $(C_{\mathbb{E}})$, $(T_{\mathbb{E}})$, $(Q_{\mathbb{C}})$, and $(P_{\mathbb{C}})$ of Definition 2.4.1 (but with $\mathcal{N}^{\mathbb{E}}$ and $\mathcal{N}^{\mathbb{C}}$ indexed by groups g instead of agents i), and also the following additional conditions:

$$
\begin{array}{ll}
(F_{\mathbb{C}}) & \mathcal{N}_{\emptyset}^{\mathbb{C}}(w) = \emptyset. \\
(Int_{\mathbb{E}\mathbb{C}}^2) & \text{If } \alpha \in \mathcal{N}_{g_1}^{\mathbb{E}}(w) \text{ and } \beta \in \mathcal{N}_{g_2}^{\mathbb{E}}(w), \text{ then } \alpha \cap \beta \in \mathcal{N}_{g_1 \cup g_2}^{\mathbb{C}}(w).
\end{array}
$$

The forcing relation $\Vdash$ is defined as in Definition 2.4.1, in particular:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \mathbb{E}_g A & \text{iff} & [\![A]\!] \in \mathcal{N}_g^{\mathbb{E}}(w). \\
\mathcal{M}, w \Vdash \mathbb{C}_g A & \text{iff} & [\![A]\!] \in \mathcal{N}_g^{\mathbb{C}}(w).
\end{array}
$$

Theorem 2.4.1 (Characterisation, [78, 165])**.** A formula of $\mathcal{L}_{Elg}$ is derivable in **ELG** if and only if it is valid in every neighbourhood model for **ELG**. Moreover, a formula of $\mathcal{L}_{coal}$ is derivable in **COAL** if and only if it is valid in every neighbourhood model for **COAL**.

2.5 Non-normal modalities with intuitionistic basis

The study of non-normal modalities is not restricted to the realm of classical logics. On the contrary, in the literature there exist also different non-normal modal logics with an intuitionistic basis. The general motivation for studying non-normal modalities with an intuitionistic basis is twofold. On the one hand, it is mathematically natural to combine these two forms of logic [161], considering in particular that both of them can be semantically arranged by means of possible worlds models. Moreover, the rejection of classical equivalences can allow for a finer analysis of the modalities. On the other hand, an intuitionistic basis is required for specific applications of modal logic in computer science [170, 50].

Without restricting to non-normal modalities, the study of modalities with an intuitionistic basis goes back to Fitch [55] in the late 1940s and has led to an important stream of research. We can very schematically identify two traditions: so-called *intuitionistic modal logics* versus *constructive modal logics*. The first tradition originated in the works by Bull [25], Fischer Servi [53, 54], Plotkin and Stirling [149], and Ewald [49], and the resulting logics have been further investigated and systematised by Simpson in his PhD thesis [161]. Despite considering different criteria, all three Fischer Servi, Ewald, and Simpson's aim was to identify correct intuitionistic analogues of certain classical modal logics. On the other hand, constructive modal logics are mainly motivated by their applications to computer science, such as the type-theoretic interpretations [14, 128] (Curry–Howard correspondence, typed lambda calculi), verification and knowledge representation [50, 102, 171], but also by their mathematical semantics [73] and their deduction systems [151].

Logics with non-normal modalities typically belong to the second tradition. The most prominent example is perhaps (the propositional fragment of) Nerode and Wijesekera's Constructive Concurrent Dynamic logic (**CCDL**) [170, 171], a logic for reasoning with partial information about the states of a concurrent transition system. **CCDL** is non-normal as it does not validate the distributivity of diamond over disjunction

$$C_\Diamond \quad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B.$$[7]

This logic has also an alternative epistemic interpretation in terms of internal/external observers proposed by Kojima [102]. Under this interpretation, each possible world w of a model represents an observer o_w, and $\Diamond(A \vee B)$ true at w means that o_w knows that $A \vee B$ is true at some world v. However, this does not entail that o_w can determine which disjunct among A and B is true at v, thus $\Diamond A \vee \Diamond B$ is not necessarily satisfied at w.

A system related to **CCDL** is the one known as Constructive K (**CK**), whose axiomatisation is presented in Bellin et al. [14]. This system not only rejects $C_\Diamond$, but also its nullary version

$$N_\Diamond \quad \Diamond\bot \supset \bot.$$

CK has been investigated in the context of type theory by Bellin et al. [14] and Mendler and Scheele [128]. Moreover, Mendler and de Paiva [126] have proposed a contextual interpretation of the modal operators in which $\Box A$ is read as "A holds in all contexts" and $\Diamond A$ as "A holds in some context".

A further example of intuitionistic logic with a non-normal modality is Propositional Lax Logic [125, 73, 50]. This is an intuitionistic logic for hardware verification containing a single modality $\bigcirc$, where $\bigcirc A$ can be interpreted as "for some constraint c, formula A holds under c." A peculiar aspect of this logic is that $\bigcirc$ satisfies some axioms that are typically associated to necessity ($\Box$), and also other axioms that are typically associated to possibility ($\Diamond$). But while $\bigcirc$ is normal if seen as a $\Box$-modality, it is instead non-normal if seen as a $\Diamond$-modality, as it validates neither $C_\Diamond$, nor $N_\Diamond$. Finally, further examples of intuitionistic modal logics rejecting $C_\Diamond$ are Fitch's logic [55] and Masini's system **I-2SC** [123].

Interestingly, while the axiom $C_\Diamond$ is rejected by all constructive systems, it is instead valid in all intuitionistic systems. As a consequence, we can identify $C_\Diamond$ as a cut-off point between the constructive and the intuitionistic tradition.

[7] In the language of intuitionistic logics we use the symbol $\supset$ instead of $\to$ to stress that the arrow must be interpreted as an intuitionistic implication.

Modal axioms and rules of intuitionistic non-normal modal logics

$$RE_\Box \ \frac{A \supset\subset B}{\Box A \supset\subset \Box B} \qquad RM_\Box \ \frac{A \supset B}{\Box A \supset \Box B} \qquad RN_\Box \ \frac{A}{\Box A}$$

$$RE_\Diamond \ \frac{A \supset\subset B}{\Diamond A \supset\subset \Diamond B} \qquad RM_\Diamond \ \frac{A \supset B}{\Diamond A \supset \Diamond B} \qquad RN_\Diamond \ \frac{\neg A}{\neg \Diamond A}$$

$M_\Box$	$\Box(A \wedge B) \supset \Box A$	$M_\Diamond$	$\Diamond A \supset \Diamond(A \vee B)$
$C_\Box$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$C_\Diamond$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
$N_\Box$	$\Box\top$	$N_\Diamond$	$\neg\Diamond\bot$

Duality axioms

$$Dual_\Diamond \quad \Box A \supset\subset \neg\Diamond\neg A \qquad Dual_\Box \quad \Diamond A \supset\subset \neg\Box\neg A$$

Intuitionistic versions of axiom K

$$K_\Box \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad K_\Diamond \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$$

Figure 2.8: Modal axioms and rules for intuitionistic systems.

Coming back to the systems **CK** and **CCDL**, we observe that while they have a non-normal modality $\Diamond$, they both have a normal $\Box$, in particular they satisfy the axioms

$$C_\Box \quad \Box A \wedge \Box B \supset \Box(A \wedge B) \qquad\qquad N_\Box \quad \Box\top,$$

which are the $\Box$-counterparts of the axioms $C_\Diamond$ and $N_\Diamond$. The possibility of having $\Box$ and $\Diamond$ satisfying different principles is a peculiar aspect of intuitionistic modal logics. This depends on the fact that, similarly to the other connectives, $\Box$ and $\Diamond$ are not interdefinable in these logics, in particular the duality axioms

$$Dual_\Diamond \quad \Box A \supset\subset \neg\Diamond\neg A \qquad\qquad Dual_\Box \quad \Diamond A \supset\subset \neg\Box\neg A$$

are not valid. As a consequence, in order to define intuitionistic logics with both modalities $\Box$ and $\Diamond$ it is necessary to explicitly state the principles which are satisfied by each of the two. For this reason, differently from classical modal logics, $\Diamond$-axioms must be explicitly considered. For instance, in Figure 2.8 we find the $\Diamond$-counterparts of the axioms and rules that characterise the systems of the classical cube (cf. Section 2.2). In the following, we call *non-normal* any intuitionistic modal logic which does not satisfy some of the modal axioms and rules in the first group of principles of Figure 2.8.

As we can see, in intuitionistic modal logics the $\Box$-axioms are not necessarily associated to their $\Diamond$-counterparts, and vice versa. This justifies the existence of logics that share the same $\Box$- or $\Diamond$-fragment but differ with respect to the other modality. A paradigmatic example is offered by the three systems **CK**, **CCDL**, and the so-called Intuitionistic K (**IK**) [161] (we shall more extensively consider these systems in the next section): while they are equivalent with respect to $\Box$, in each of them the modality $\Diamond$ validates different principles.

In addition to the motivations recalled above, an intuitionistic account of non-normal modalities may be justified by further interpretations of the modalities that would benefit from the non-interdefinability of $\Box$ and $\Diamond$. An example might be von Wright's distinction between *weak* and *strong* permissions [176], also widely discussed by Hansson [83] who distinguishes *explicit* and *implicit* permissions on one side (the strong ones) from *tacit* permissions on the other side (weak ones). In general one may require that the permission of A must be justified explicitly or positively (say by a proof from a corpus of norms) and not just established by the fact that $\neg A$ is not obligatory. As an example, a deontic logic where the modalities are both non-normal and non-interdefinable has been recently proposed in Anglberger et al. [3].

2.6 Intuitionistic versions of logic K

We now present the axiomatisation and semantics of the intuitionistic modal systems **IK**, **CCDL**, and **CK**. Apart from their specific motivations, in the context of the present work their interest relies on the fact that, though non-equivalent, they all can be regarded as intuitionistic counterparts of the same classical logic, namely the minimal normal modal logic **K**. Moreover, while they all have a normal modality $\Box$, the systems **CK** and **CCDL** have a non-normal $\Diamond$.

Intuitionistic modal logics are defined in the intuitionistic modal language $\mathcal{L}_i$, whose formulas are defined by the following grammar:

$$A ::= p_i \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A,$$

where p_i is any variable in Atm. In the language $\mathcal{L}_i$ of intuitionistic logics we use the arrow $\supset$ instead of $\rightarrow$ to stress that it must be interpreted as an intuitionistic implication. In addition, differently from the language $\mathcal{L}$ of classical modal logics, we explicitly add to $\mathcal{L}_i$ the symbol $\Diamond$, since it is not definable by means of $\Box$. Negation $\neg$ and double implication $\supset\subset$ are defined as $\neg A := A \supset \bot$ and $A \supset\subset B := (A \supset B) \wedge (B \supset A)$.

Axiomatic systems

Intuitionistic modal logics are axiomatically defined as extensions of intuitionistic propositional logic (**IPL**), for which we consider the following axiomatisation:

IK	CCDL	CK
Axiomatisation of **IPL**	Axiomatisation of **IPL**	Axiomatisation of **IPL**
$\dfrac{A}{\Box A}$	$\dfrac{A}{\Box A}$	$\dfrac{A}{\Box A}$
$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
$\neg\Diamond\bot$	$\neg\Diamond\bot$	
$\Diamond(A \lor B) \supset \Diamond A \lor \Diamond B$		
$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$		

Figure 2.9: Intuitionistic versions of **K**.

$$
\begin{array}{llll}
A \supset (B \supset A), & (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)), & & \\
A \supset A \lor B, & B \supset A \lor B, & (A \supset C) \supset ((B \supset C) \supset (A \lor B \supset C)), & \\
A \land B \supset A, & A \land B \supset B, & A \supset (B \supset A \land B), & \dfrac{A \supset B \quad A}{B}\ . \\
\bot \supset A, & A \supset \top, & &
\end{array}
$$

The systems **IK**, **CCDL** and **CK** are defined by extending **IPL** with the modal axioms and rules displayed in Figure 2.9. In the figure we find the original axiomatisations, respectively by Plotkin and Stirling [149], Wijesekera [170], and Bellin et al. [14] (Wijesekera's axiomatisation also includes the axiom $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$, but this is derivable from the other axioms, cf. e.g. [161], p. 48). However, the same systems could be equivalently axiomatised by replacing $RN_\Box$ and $K_\Box$ with the rule $RE_\Box$ and the axioms $M_\Box$, $C_\Box$, and $N_\Box$ (as the standard derivations are intuitionistically valid, cf. e.g. [127] for the alternative axiomatisation of **CK**), whence they all have a normal $\Box$.

The systems **IK**, **CCDL** and **CK** have a decreasing strength: **IK** $\supsetneq$ **CCDL** $\supsetneq$ **CK**. In particular, **IK** has a normal $\Diamond$ as it contains $C_\Diamond$ and $N_\Diamond$ as axioms, and $M_\Diamond$ is derivable:

1.	$A \supset (A \lor B)$	(theorem of **IPL**)
2.	$\Box(A \supset (A \lor B))$	(1, $RN_\Box$)
3.	$\Box(A \supset (A \lor B)) \supset (\Diamond A \supset \Diamond(A \lor B)$	($K_\Box$)
4.	$\Diamond A \supset \Diamond(A \lor B)$	(2,3, MP)

Axiom $M_\Diamond$ is derivable also in **CCDL** and **CK** by the same derivation. By contrast, $\Diamond$ is non-normal in **CCDL** and **CK** do to their lack of, respectively, $C_\Diamond$ and both $C_\Diamond$ and $N_\Diamond$.

Semantics

All **IK**, **CCDL**, and **CK** have been given a semantic characterisation in terms of relational models. Relational models for **IK** have been independently defined by Fischer Servi [54], Plotkin and Stirling [149], and Ewald [49] (for a survey of the different approaches that all led to the same semantics see [161]), whereas relational models for **CCDL** and **CK** have been defined respectively by Wijesekera [170] and Mendler and de Paiva [126]. The definitions are as follows.

Definition 2.6.1. A *relational model for* **IK** is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{R}$ is any binary relation on $\mathcal{W}$, and $\mathcal{V}$ is a *hereditary* valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$, that is

$$\text{if } w \preceq v, \text{ then } \mathcal{V}(w) \subseteq \mathcal{V}(v).$$

Moreover the following conditions connecting the relations $\preceq$ and $\mathcal{R}$ are satisfied:

(i) If $w\mathcal{R}v$ and $w \preceq u$, then there is $z \in \mathcal{W}$ such that $v \preceq z$ and $u\mathcal{R}z$.
(ii) If $w\mathcal{R}v$ and $v \preceq z$, then there is $u \in \mathcal{W}$ such that $w \preceq u$ and $u\mathcal{R}z$.

(i)
$$\begin{tikzcd} u \arrow[r, dotted, "\mathcal{R}"] & z \\ w \arrow[u, "\preceq"] \arrow[r, "\mathcal{R}"'] & v \arrow[u, dotted, "\preceq"'] \end{tikzcd}$$

(ii)
$$\begin{tikzcd} u \arrow[r, dotted, "\mathcal{R}"] & z \\ w \arrow[u, dotted, "\preceq"] \arrow[r, "\mathcal{R}"'] & v \arrow[u, "\preceq"'] \end{tikzcd}$$

The forcing relation $\mathcal{M}, w \Vdash A$ is defined as follows, where $v \succeq w$ holds if and only if $w \preceq v$:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p_i & \text{iff} & p_i \in \mathcal{V}(w); \\
\mathcal{M}, w \nVdash \bot; & & \\
\mathcal{M}, w \Vdash \top; & & \\
\mathcal{M}, w \Vdash A \wedge B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B; \\
\mathcal{M}, w \Vdash A \vee B & \text{iff} & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B; \\
\mathcal{M}, w \Vdash A \supset B & \text{iff} & \text{for all } v \succeq w,\ \mathcal{M}, v \Vdash A \text{ implies } \mathcal{M}, v \Vdash B; \\
\mathcal{M}, w \Vdash \Box A & \text{iff} & \text{for all } v \succeq w, \text{ for all } u \in \mathcal{W},\ v\mathcal{R}u \text{ implies } \mathcal{M}, u \Vdash A; \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff} & \text{there is } v \in \mathcal{W} \text{ such that } w\mathcal{R}v \text{ and } \mathcal{M}, v \Vdash A.
\end{array}$$

Definition 2.6.2. A *relational model for* **CCDL** is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{V}$ is a hereditary valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$, and $\mathcal{R}$ is a binary relation on $\mathcal{W}$. The forcing relation $w \Vdash A$ is defined as in Definition 2.6.1 for atomic formulas and propositional connectives. For the modal formulas it is as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash \Box A & \text{iff} & \text{for all } v \succeq w, \text{ for all } u \in \mathcal{W},\ v\mathcal{R}u \text{ implies } \mathcal{M}, u \Vdash A; \\
\mathcal{M}, w \Vdash \Diamond A & \text{iff} & \text{for all } v \succeq w, \text{ there is } u \in \mathcal{W} \text{ such that } v\mathcal{R}u \text{ and } \mathcal{M}, u \Vdash A.
\end{array}$$

Finally, relational models for **CK** are defined by Mendler and de Paiva [126] by enriching Wijesekera's models for **CCDL** with *inconsistent*, or "fallible", worlds, i.e., worlds satisfying $\bot$.

Definition 2.6.3. A *relational model for* **CK** is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{V}$ is a hereditary valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$, and $\mathcal{R}$ is a binary relation on $\mathcal{W}$. For every formula A of $\mathcal{L}_i$, the forcing relation $w \Vdash A$ is defined as in Definition 2.6.2, except for $\bot$, for which it is defined as follows:

if $\mathcal{M}, w \Vdash \bot$, then for every v, $w \preceq v$ or $w\mathcal{R}v$ implies $\mathcal{M}, v \Vdash \bot$;
if $\mathcal{M}, w \Vdash \bot$, then for every propositional variable $p \in Atm$, $\mathcal{M}, w \Vdash p$.

Relational models for **IK**, **CCDL**, and **CK** preserve the hereditary property of intuitionistic Kripke models (see Simpson [161] and Wijesekera [170]):

Proposition 2.6.1 (Hereditary property)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for **IK**, **CCDL**, or **CK**. Then for every world $w \in \mathcal{W}$ and every formula A of $\mathcal{L}_i$, if $\mathcal{M}, w \Vdash A$ and $w \preceq v$, then $\mathcal{M}, v \Vdash A$.

On the basis of the above proposition we can observe the following.

Proposition 2.6.2. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for **IK**. Then for every world $w \in \mathcal{W}$ and every formula A of $\mathcal{L}_i$,

$$\mathcal{M}, w \Vdash \Diamond A \quad \text{iff} \quad \text{for all } v \succeq w, \text{ there is } u \in \mathcal{W} \text{ such that } v\mathcal{R}u \text{ and } \mathcal{M}, u \Vdash A.$$

Proof. $\mathcal{M}, w \Vdash \Diamond A$ iff (by Proposition 2.6.1) for every $v \succeq w$, $\mathcal{M}, v \Vdash \Diamond A$ iff (by Definition 2.6.1) for every $v \succeq w$, there is $u \in \mathcal{W}$ such that $v\mathcal{R}u$ and $\mathcal{M}, u \Vdash A$. □

On the basis of Proposition 2.6.2, we see that the relational models for **IK**, **CCDL**, and **CK** are organised into a clear hierarchy: the relational models for **CCDL** are the particular cases of the relational models for **CK** without inconsistent worlds (i.e., worlds satisfying $\bot$), whereas the relational models for **IK** are the particular cases of the relational models for **CCDL** satisfying the conditions (i) and (ii) in Definition 2.6.1 connecting the relations $\preceq$ and $\mathcal{R}$. As a consequence, from the point of view of the relational semantics **CCDL** can be seen as the simplest logic among these three systems, since the definition of its relational models neither needs to resort to non-standard objects such as inconsistent worlds, nor it requires specific connections between the two relations $\preceq$ and $\mathcal{R}$. It is therefore interesting to notice that the extensions of **CCDL** are less studied than those of **IK** and **CK**. Indeed, while several extensions of **IK** and **CK** have been studied both semantically (see e.g. [54, 161, 1]) and proof-theoretically (see e.g. [64, 162, 127, 6, 86, 144]), we are not aware of any analogous investigation of possible extensions of **CCDL**.

In addition to the relational models, Kojima [102] defines neighbourhood models for **CCDL** as follows:

Definition 2.6.4. A Kojima neighbourhood model for **CCDL** is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in Definition 2.6.2, and $\mathcal{N}$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that $w \preceq v$ implies $\mathcal{N}(v) \subseteq \mathcal{N}(w)$, and for all $w \in \mathcal{W}$, $\mathcal{N}(w) \neq \emptyset$. The forcing relation $\Vdash_k$ is defined as in Definition 2.6.2, except for modal formulas, for which is as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash_k \Box A & \text{iff} & \text{for all } \alpha \in \mathcal{N}(w),\ \alpha \subseteq [\![A]\!]_{\mathcal{M}}; \\ \mathcal{M}, w \Vdash_k \Diamond A & \text{iff} & \text{for all } \alpha \in \mathcal{N}(w),\ \alpha \cap [\![A]\!]_{\mathcal{M}} \neq \emptyset. \end{array}$$

Theorem 2.6.3 (Characterisation, [54, 170, 126, 102])**.** The logics **IK**, **CCDL**, and **CK** are sound and complete with respect to the corresponding relational models. Moreover, **CCDL** is sound and complete with respect to Kojima's models.

Chapter 3

Sequent calculi and their properties

In this work we aim to investigate non-normal modal logics from a proof-theoretic perspective. With *proof theory* we understand the study of logics from the point of view of their proof systems. An essential part of proof theory is represented by the so-called *structural proof theory*, which originated with the works by Hilbert and Gentzen and consists in studying the proofs as mathematical objects, with the aim of establishing their structure and properties.

Proof-theoretical investigations of logics can be motivated by many different purposes. To mention just two general aims, on the one hand, by looking at the form of the proofs one can establish properties of the logic such as consistency, decidability, interpolation, and so on. On the other hand, suitable proof systems may be implemented and used for automated reasoning and theorem proving.

Besides the axiomatic systems already introduced in Chapter 2.2, in this work we shall mainly concentrate on sequent calculi, although other formalisms such as tableaux calculi will be also considered. In this chapter, we start to deal with sequent calculi for non-normal modal logics. We begin with some general definitions that will serve us throughout this thesis. We then present a list of desirable properties of proof systems. These properties shall be considered throughout this work in order to evaluate the different calculi taken into account. Finally, we make a brief account of the existing proof systems for classical and intuitionistic non-normal modal logics, and concentrate more in detail on their Gentzen calculi.

3.1 Sequents and sequent calculi

In this section we introduce the formalism of sequent calculi and give some general definitions that will serve us throughout the thesis.

Definition 3.1.1 (Multiset)**.** A *multiset* is a list without order, i.e., a structure where the number of occurrences of its elements counts but their order does not count. The multiset containing the formulas A_1, A_2, ..., and A_n is written $A_1, ..., A_n$. Multisets of formulas are denoted by capital Greek letters Γ, Δ, Σ, Π, Ω. Given a multiset Γ, we sometimes consider its *support* $\mathsf{set}(\Gamma)$, i.e., the set of its elements disregarding multiplicities.

In the following, if Γ is the multiset $A_1, ..., A_n$, we denote by $\Box\Gamma$ the multiset $\Box A_1, ..., \Box A_n$. If not differently specified, in the following when we write $\Box\Gamma$ we implicitly assume that Γ contains at least one formula.

Definition 3.1.2 (Sequent)**.** A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta,$$

where Γ and Δ are finite multisets of formulas. Γ and Δ are called respectively the *antecedent* and the *succedent* of the sequent (or, respectively, the 'left-hand side' and the 'right-hand side' of the sequent). We say that a sequent is in language $\mathcal{L}$ if all formulas occurring in the sequent belongs to $\mathcal{L}$.

Sequents represent in the object language the metalinguistic notion of derivability from assumptions. Symbol $\Rightarrow$ represents the consequence relation, comma (,) on the left represents metalinguistic conjunction, and comma on the right represents metalinguistic disjunction. Thus, the sequent $\Gamma \Rightarrow \Delta$ can be read as "from the set of assumptions in Γ it follows at least one formula in Δ".

Definition 3.1.3 (Formula interpretation)**.** We call *formula interpretation* a function *int* from sequents in language $\mathcal{L}$ to formulas of $\mathcal{L}$ such that

$$(\Gamma \Rightarrow \Delta)^{int} = \bigwedge \Gamma \rightarrow \bigvee \Delta,$$

where $\bigwedge A_1, ..., A_n$ and $\bigvee A_1, ..., A_n$ are abbreviations for, respectively, $A_1 \wedge ... \wedge A_n$ and $A_1 \vee ... \vee A_n$, and $\bigwedge \emptyset$ and $\bigvee \emptyset$ are interpreted respectively as $\top$ and $\bot$.[1]

We also consider the following *semantic interpretation* of sequents.

Definition 3.1.4 (Valid sequent)**.** Given a sequent $\Gamma \Rightarrow \Delta$, we say that

- $\Gamma \Rightarrow \Delta$ is *satisfied* by a world w of a possible-worlds model $\mathcal{M}$ if in case $w \Vdash A$ for every formula $A \in \Gamma$, then $w \Vdash B$ for some formula $B \in \Delta$.

- $\Gamma \Rightarrow \Delta$ is *valid* in a possible-worlds model $\mathcal{M}$, written $\mathcal{M} \models \Gamma \Rightarrow \Delta$, if it is satisfied by every world of $\mathcal{M}$.

[1]Provided that $\mathcal{L}$ contains the connectives $\wedge, \vee, \rightarrow$. Notice that the definition of function *int* is not fully precise because of the asymmetry between multisets, where order of formulas does not count, and conjunctions (resp. disjunctions), where the order counts. More precisely, $(\Gamma \Rightarrow \Delta)^{int}$ should be defined as $\bigwedge \Gamma' \rightarrow \bigvee \Delta'$, where Γ' and Δ' are *sequences* of formulas corresponding respectively to Γ and Δ. Then $(\Gamma \Rightarrow \Delta)^{int}$ is well-defined modulo logical equivalence (cf. [164]). Here we consider the standard "rough" definition as it is more intuitive and sufficient for the purposes of this work.

$$\mathsf{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{\Gamma \Rightarrow \top, \Delta}$$

$$\mathsf{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\mathsf{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \mathsf{R}\vee\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$\mathsf{L}\rightarrow\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \mathsf{R}\rightarrow\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}$$

Figure 3.1: Rules of **G3cp**.

- $\Gamma \Rightarrow \Delta$ is *valid* in a class of models $\mathcal{C}$, written $\mathcal{C} \models \Gamma \Rightarrow \Delta$, if it is valid in every model of $\mathcal{C}$.

We extend to sequents the following notions already introduced for formulas of $\mathcal{L}$.

Definition 3.1.5 (Complexity of sequents)**.** We define the *complexity* of a sequent $A_1, ..., A_n \Rightarrow B_1, ..., B_m$ as $wg(A_1)+...+wg(A_n)+wg(B_1)+...+wg(B_m)$, where wg is the weight of formulas as defined in Definition 2.2.2.

Definition 3.1.6 (Meta-sequent, substitution, instance)**.** A *meta-sequent* is a pair $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of schemata of formulas rather than multiset of formulas. If $A_1, ..., A_n \Rightarrow B_1, ..., B_m$ is a meta-sequent and ς is a substitution from schemata to formulas of $\mathcal{L}$, then ς is extended to $A_1, ..., A_n \Rightarrow B_1, ..., B_m$ as $\varsigma(A_1, ..., A_n \Rightarrow B_1, ..., B_m) = \varsigma(A_1), ..., \varsigma(A_n) \Rightarrow \varsigma(B_1), ..., \varsigma(B_m)$. The result of the substitution is called *instance* of the meta-sequent.

We now introduce sequent calculi, which are defined as sets of sequents rules as follows.

Definition 3.1.7 (Sequent calculus, sequent rule, initial sequent)**.** A *sequent calculus* is a set of sequent rules. A *sequent rule* is an ordered pair composed by a (possibly empty) set of meta-sequents $\Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$, called *premisses*, and a meta-sequent $\Gamma \Rightarrow \Delta$, called *conclusion*. A sequent rule is written

$$R\ \frac{\Gamma_1 \Rightarrow \Delta_1 \quad ... \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}\ ,$$

where R is the name of the rule. If $\frac{\Gamma_1 \Rightarrow \Delta_1 \quad ... \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is a rule and ς is a substitution, then $\frac{\varsigma(\Gamma_1 \Rightarrow \Delta_1) \quad ... \quad \varsigma(\Gamma_n \Rightarrow \Delta_n)}{\varsigma(\Gamma \Rightarrow \Delta)}$ is an instance of the rule. Instances of the conclusion of a rule with zero premisses are called *initial sequents*, or axioms.

$$\mathsf{Lwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \qquad \mathsf{Lctr}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rctr}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Figure 3.2: Structural rules and the cut rule.

As an example, the rules in Figure 3.1 define the sequent calculus **G3cp** for classical propositional logic [164]. As usual, we distinguish the rules of sequent calculi in two kinds:

- *logical rules* (cf. Figure 3.1), which introduce logical symbols, and
- *structural rules* (cf. Figure 3.2), which handle with the structure of sequents.

In addition, we consider the rule cut, which has a prominent role in structural proof theory. For every logical rule, we call

- *principal formula* the formula occurring in the conclusion and containing the logical symbol introduced by the rule;
- *active formulas* the formulas occurring in the premiss(es) which are subformulas of the principal formulas;
- *context* the formulas in Γ, Δ, that is, the formulas which are untouched by the rule.

For instance, in the rule $\mathsf{L}\wedge$, $A \wedge B$ is the principal formula, A and B are the active formulas, and Γ and Δ are the context. The logical rules are distinguished into *left* and *right* rules (except for init), depending whether the principal formula occurs in the antecedent or in the succedent of the conclusion.

The structural rules are called *weakening* and *contraction*. As the logical rules, they are distinguished into a left and a right rule. The principal formula of Lwk and Rwk is the formula which is introduced by the rule application, whereas the principal formula of Lctr and Rctr is the formula which is deleted by the rule application. Finally, the principal formula of cut, i.e., the one which is deleted by the application of the rule, is also called *cut formula*. As for logic formulas, the formulas in Γ, Δ are the context.

The rules of weakening, contraction, and cut correspond each to some property of the classical consequence relation. In particular, left weakening corresponds to monotonicity of $\vdash$, which means that if a formula is derivable from a set of assumption Γ, then it is derivable from any extension of Γ. Left contraction corresponds to the fact that every assumption can be used an arbitrary (finite) number of times. Finally, the cut rule reflects the fact that derivations can be done stepwise by going through "auxiliary lemmas".

We consider the *additive* version of cut, i.e., the version in which the two premisses share the same contexts Γ, Δ. Alternatively, one can consider the *multiplicative* formulation

$$\mathsf{cut}' \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ ;$$

in presence of weakening and contraction the two formulations are equivalent.

We now introduce the terminology related to derivations in sequent calculi.

Definition 3.1.8 (Derivation, endsequent, height)**.** A *derivation* in a sequent calculus $\mathbf{G}$ is a finite tree labelled with sequents such that the leaves are labelled by initial sequents, and each sequent at a node different from the leaves is obtained by the sequent(s) in the node(s) immediately above it by an application of a rule of $\mathbf{G}$. The sequent labelling the root is called *endsequent*, or *root sequent*. We define the *height* of a derivation as the lenght of longest path (i.e., the maximal number of nodes) between the endsequent and a leaf of the derivation.

Definition 3.1.9 (Derivable sequent)**.** A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in $\mathbf{G}$ if there exists a derivation in $\mathbf{G}$ with $\Gamma \Rightarrow \Delta$ as endsequent, in this case we write $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$. We sometimes write $\vdash_n \Gamma \Rightarrow \Delta$ to denote that $\Gamma \Rightarrow \Delta$ is derivable with a derivation of height at most n.

Definition 3.1.10 (Derivable rule)**.** A rule $\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad ... \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *derivable* in a sequent calculus $\mathbf{G}$ if there exists a derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G}$ such that every leaf is labelled by an initial sequent or by a sequent among $\Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$.

Definition 3.1.11 (Admissible rule)**.** A rule $\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad ... \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *admissible* in a sequent calculus $\mathbf{G}$ if in case all premisses $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$ are derivable in $\mathbf{G}$, then the conclusion is also derivable in $\mathbf{G}$. Moreover, a single-premiss rule $\dfrac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$ is *height-preserving* admissible if in case the premiss $\Gamma' \Rightarrow \Delta'$ is derivable with a derivation of height n, then the conclusion $\Gamma \Rightarrow \Delta$ is derivable with a derivation of height at most n.

Notice that every derivable rule is also admissible, but not vice versa. Notice also that in case a rule R is admissible in $\mathbf{G}$, then the set of sequents derivable in $\mathbf{G} \cup \{R\}$ (i.e., the calculus $\mathbf{G}$ extended with the rule R) coincides with the set of sequents derivable in $\mathbf{G}$ (that is, the rule R does not expand the set of the sequents that are derivable in $\mathbf{G}$).

Definition 3.1.12 (Invertible rule)**.** A rule $R \dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad ... \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *invertible* in a sequent calculus $\mathbf{G}$ if in case the conclusion is derivable in $\mathbf{G}$, then the premisses $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$ are also derivable in $\mathbf{G}$. Moreover, R is *height-preserving* admissible if in case $\Gamma \Rightarrow \Delta$ is derivable with a derivation of height n, then $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$ are derivable with derivations of height at most n. We say that a calculus is invertible if all rules of the calculus are invertible.

Observe that a rule $\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is invertible if and only if all rules $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma_1 \Rightarrow \Delta_1}$, ..., $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma_n \Rightarrow \Delta_n}$ are admissible.

An advantage of sequent calculi with respect to Hilbert systems is that in the former kind of proof systems it is much easier to find derivations. Derivations in sequent calculi can be constructed essentially in two ways. One can proceed *top-down*, starting with initial sequents and applying rules in the attempt to obtain the wanted sequent. But one can also proceed *bottom-up*, starting with the wanted sequent and applying rules backward in the attempt of ending up with initial sequents. We consider the following definitions.

Definition 3.1.13 (Backward proof search, failed proof)**.** We say that a rule R is *backwards applicable* to a sequent $\Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is the conclusion of an instance of R. We call *backward* (or root-first, or bottom-up) *proof search* for $\Gamma \Rightarrow \Delta$ the construction of a derivation tree from the root to the leaves such that the root is labelled by the sequent $\Gamma \Rightarrow \Delta$, and the branches are expanded by applying at each step a backwards applicable rule. We call *proof* of $\Gamma \Rightarrow \Delta$ the tree generated by a backward proof search for $\Gamma \Rightarrow \Delta$. A proof where some leaves are not initial sequents and that cannot be further expanded is called *failed proof*.

It is easy to see that a finite proof of $\Gamma \Rightarrow \Delta$ where all leaves are initial sequents is a derivation of $\Gamma \Rightarrow \Delta$.

3.2 Internal and external calculi

In the previous section we have defined the basic framework of sequent calculi. In the following, we call proof systems of this form *Gentzen calculi*. Several extensions of Gentzen calculi have been proposed in the literature in order to define proof systems for many logics. The resulting systems can be schematically distinguished into two categories: (i) Calculi that enrich the *language* of sequents: the language is enriched with *labels* which are used to import semantic information into the calculus. (ii) Calculi that enrich the *structure* of sequents: the calculi contain additional structural connectives (i.e., in addition to the sequent arrow "$\Rightarrow$" and the comma ","); typical examples are "|" and "[ ]", which are used to represent respectively sequences (or multisets) of sequents (called *hypersequents*, see e.g. [10]) and trees of sequents (called *nested sequents*, see e.g. [24, 150]). In the following, a calculus is called

- *internal*, if it only employs the language of the logic, possibly with additional structural connectives;
- *external*, if it is defined in an extended language with respect to the language of the logic.

$$
\mathsf{init}\ \frac{}{\Gamma, x : p \Rightarrow x : p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{\Gamma, x : \bot \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{\Gamma \Rightarrow x : \top, \Delta}
$$

$$
\mathsf{L}\wedge\ \frac{\Gamma, x : A, x : B \Rightarrow \Delta}{\Gamma, x : A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow x : A, \Delta \qquad \Gamma \Rightarrow x : B, \Delta}{\Gamma \Rightarrow x : A \wedge B, \Delta}
$$

$$
\mathsf{L}\rightarrow\ \frac{\Gamma \Rightarrow x : A, \Delta \qquad \Gamma, x : B \Rightarrow \Delta}{\Gamma, x : A \rightarrow B \Rightarrow \Delta} \qquad \mathsf{R}\rightarrow\ \frac{\Gamma, x : A \Rightarrow x : B, \Delta}{\Gamma \Rightarrow x : A \rightarrow B, \Delta}
$$

$$
\mathsf{L}\Box\ \frac{\Gamma, x : \Box A, x\mathcal{R}y, y : A, \Rightarrow \Delta}{\Gamma, x : \Box A, x\mathcal{R}y \Rightarrow \Delta} \qquad \mathsf{R}\Box\ \frac{\Gamma, x\mathcal{R}y \Rightarrow y : A, \Delta}{\Gamma \Rightarrow x : \Box A, \Delta}\ (y!)
$$

Figure 3.3: Labelled sequent calculus for **K** (Negri [133]).

To give an example, we consider the following well-known sequent calculi for the minimal normal modal logic **K**. The first one is the Genzten calculus for **K** (see for instance Wansing [169]), which is defined by extending the calculus **G3cp** for classical propositional logic (Figure 3.1) with the following modal rule K:

$$
\mathsf{K}\ \frac{\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta}\ (|\Sigma| \geq 0)\ .
$$

We call the resulting calculus **G3.K**. The calculus is internal as the sequents contain only formulas of $\mathcal{L}$. The second example is the labelled calculus by Negri [133] (Figure 3.3) – we call it **LS.K** –, in which the language of $\mathcal{L}$ is extended with a set of world labels $x, y, z, ...$, and the symbol $\mathcal{R}$. The calculus internalises the Kripke semantics for logic **K**: formulas of the form $x : A$ express the forcing relation $x \Vdash A$, whereas $\mathcal{R}$ represents the binary relation between worlds. The propositional rules are just the rules of **G3cp** enriched with world labels, whereas the rules $\mathsf{L}\Box$ and $\mathsf{R}\Box$ directly derive from the forcing condition of boxed formulas in the Kripke semantics (in rule $\mathsf{R}\Box$ the label y must satisfies the *eigenvariable* condition, i.e., it must not occur in the conclusion). Although they are different formalisms, the two calculi are equivalent, in particular both are proof systems for logic **K**, in the sense that, for every $A \in \mathcal{L}$,

$$
\vdash_{\mathbf{K}} A \qquad \text{iff} \qquad \vdash_{\mathbf{G3.K}} \Rightarrow A \qquad \text{iff} \qquad \vdash_{\mathbf{LS.K}} \Rightarrow x : A \text{ for every } x.
$$

It is worth remarking that the distinction between internal and external calculi is more an intuitive, operational classification rather than a formal one. Moreover, in some cases derivations in internal and external calculi are essentially isomorphic structures. Starting with Fitting [57] and Goré and Ramanayake [76], this has been shown for several modal logics and kinds of proof systems by means of constructive translations of derivations in one calculus into derivations in the other. Further examples are Ciabattoni et al. [31], Girlando et al. [69], Lellmann and Pimentel [109, 110], and Pimentel [147].

3.3 Desirable properties of sequent calculi: a discussion

Since their introduction by Gentzen [66, 67], sequent calculi have revealed very strong tools for the investigation of logics. Depending on the specific aims, a calculus might be required to satisfy some relevant properties. We discuss here some relevant properties that shall be considered throughout this work. For each calculus presented in this work we shall analyse which of these properties it satisfies.

Syntactic and semantic completeness. Given a logic $\mathbf{L}$ and a calculus $\mathbf{G}$, the minimal requirement to consider $\mathbf{G}$ as a proof system for $\mathbf{L}$ is that derivability in $\mathbf{G}$ coincides with derivability in $\mathbf{L}$, in the sense that

$$\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta \qquad \text{iff} \qquad \vdash_{\mathbf{L}} \bigwedge \Gamma \to \bigvee \Delta.$$

More precisely, we say that $\mathbf{G}$ is *sound* with respect to $\mathbf{L}$ if the left-to-right direction of the above equivalence holds, and it is *complete* with respect to $\mathbf{L}$ if the right-to-left direction holds. Thus, $\mathbf{G}$ is a *calculus for* $\mathbf{L}$ if it is sound and complete with respect to $\mathbf{L}$.

A possible way to prove that a calculus $\mathbf{G}$ is complete with respect to a logic $\mathbf{L}$ consists in showing, first, that all axioms of $\mathbf{L}$ are derivable in $\mathbf{G}$ (i.e., if A is an axiom of $\mathbf{L}$, then the sequent $\Rightarrow A$ is derivable in $\mathbf{G}$) and, second, that all rules of $\mathbf{L}$ are admissible in $\mathbf{G}$ (i.e., if $\dfrac{A_1 \quad \dots \quad A_n}{A}$ is a rule of $\mathbf{L}$, then the rule $\dfrac{\Rightarrow A_1 \quad \dots \quad \Rightarrow A_n}{\Rightarrow A}$ is admissible in $\mathbf{G}$). In this way, for every formula B derivable in $\mathbf{L}$ the corresponding sequent $\Rightarrow B$ will be derivable in $\mathbf{G}$. If this is the case, we say that the calculus is *syntactically complete* with respect to the axiomatisation.

In this respect, the rule of modus ponens has a particular relevance among the rules of Hilbert systems defined in Chapter 2.2. Typically, modus ponens is simulated in sequent calculi by means of the rule cut (with auxiliary applications of structural rules if needed). As an example, the following instance of MP:

$$\dfrac{A \qquad A \to B}{B}\, MP$$

is simulated by the the following derivation in $\mathbf{G}$, where the sequents $A \Rightarrow A, B$ and $A, B \Rightarrow B$ can be shown derivable for every A, B:

$$\dfrac{\mathsf{Rwk}\,\dfrac{\Rightarrow A}{\Rightarrow A, B} \qquad \dfrac{\mathsf{Rwk}\,\dfrac{\mathsf{Lwk}\,\dfrac{\Rightarrow A \to B}{A \Rightarrow A \to B}}{A \Rightarrow B, A \to B} \qquad \dfrac{A \Rightarrow A, B \qquad A, B \Rightarrow B}{A \to B, A \Rightarrow B}\,\mathsf{L}\to}{A \Rightarrow B}}{\Rightarrow B}\,\mathsf{cut}$$

As a consequence, in order to be syntactically complete with respect to an axiomatic system, it is crucial that a sequent calculus contains the rule cut, either as an explicit rule or as an admissible one.

A sequent calculus may be complete not only with respect to an axiomatic system, but also with respect to a semantics. We say that the calculus **G** is *sound* and *semantically complete* with respect to a class of models $\mathcal{C}$ if the sequents that are derivable in **G** coincide with the sequents that are valid in $\mathcal{C}$, that is:

$$\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta \qquad \text{iff} \qquad \models_{\mathcal{C}} \Gamma \Rightarrow \Delta.$$

Semantic completeness can be proved directly by showing, first, that every derivable sequent is valid (whence the calculus is sound) and, second, that every non-derivable sequent is not valid (whence the calculus is complete; cf. countermodel extraction below). Alternatively, semantic completeness may be also proved indirectly by showing that the calculus is complete with respect to an axiomatic system, which is in turn known to be complete with respect to the semantics.

Admissibility of structural rules. As we have seen, the structural rules and cut play a decisive role for the syntactic completeness of the calculi. At the same time, however, most of the "good" properties of the sequent calculi depend on the fact that the calculi do not contain these rules explicitly, that is, that the structural rules and cut are not part of their definition. These two needs can be reconciled by defining the calculi without the structural rules and cut and proving that these rules are admissible (equivalently, one can define the calculi with these rules and show that they are eliminable). For this reason, a main concern of structural proof theory consists in investigating the admissibility of this kind of rules.

In some cases, the admissibility of cut can be proved by means of a syntactic *cut elimination* procedure. By cut elimination we intend an effective procedure which transforms any derivation of a sequent $\Gamma \Rightarrow \Delta$ possibly containing some applications of cut into a derivation of the same sequent in which the rule cut is not used. This procedure provides a constructive proof of the admissibility of cut in the considered calculus. Analogous (and, for the calculi considered here, simpler) procedures can be applied to prove the admissibility of weakening and contraction. As usual, we call *cut-free* (respectively *contraction-free*) any calculus which does not contain explicitly the rule cut (respectively the rules Lctr and Rctr).

Analyticity. We say that a rule is *analytic* if all the formulas occurring in the premis(es) are subformulas of some formulas occurring in the conclusion. Moreover, we say that a rule is *strictly analytic* if all the formulas occurring in the premis(es) are *strict* subformulas of some formulas occurring in the conclusion (cf. Definition 2.2.3), and in addition the premisses have a smaller complexity than the conclusion (cf. Definition 3.1.5). Correspondingly, we say that a calculus is *analytic* (respectively *strictly analytic*) if all rules are analytic (respectively strictly analytic).

To make an example, the calculus **G3.K** in previous section is strictly analytic, because so are all its rules, i.e., the propositional rules and the rule K. In contrast, the labelled calculus

LS.K is not analytic, as the premisses of $\mathsf{L}\Box$ and $\mathsf{R}\Box$ contain formulas which do not occur in the conclusion. Nonetheless, **LS.K** is analytic with respect to language $\mathcal{L}$ (i.e., disregarding world labels and symbol $\mathcal{R}$), although not strictly analytic, since the formula $\Box A$ which is principal in $\mathsf{L}\Box$ occurs also in the premiss.

An immediate consequence of analyticity is the following: all formulas appearing in the derivation of a sequent are subformulas of the sequent itself. As we shall see, analyticity is a key property for the analysis of derivations in sequent calculi, and has in turn relevant consequences, such as the possibility of getting an immediate proof of consistency and ensuring the termination of backward proof search.

Terminating proof search. We are interested in using the calculi to establish the derivability/validity of formulas in the corresponding logics. To this purpose we shall consider backward proof search procedures in the calculi (cf. Definition 3.1.13). In general, we say that a proof search procedure is *terminating* if every proof generated by it is finite, and we say that a calculus is terminating if it allows for the definition of a terminating proof-search procedure.

It is easy to see that root-first proof search in a strictly analytic calculus like **G3cp** or **G3.K** is terminating. This depends on the fact that the complexity of sequents is reduced by any backward application of a rule, and since sequents are finite structures their complexity cannot be reduced indefinitely.

On the contrary, termination of root-first proof search is not ensured if the calculus is not analytic. For instance, in the labelled calculus **LS.K** the rule $\mathsf{L}\Box$ can be applied indefinitely because of the copy of the principal formulas into the premiss:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x : \Box p, x\mathcal{R}y, y : p, y : p, y : p \Rightarrow y : p \lor q}\,\mathsf{L}\Box}{x : \Box p, x\mathcal{R}y, y : p, y : p \Rightarrow y : p \lor q}\,\mathsf{L}\Box}{x : \Box p, x\mathcal{R}y, y : p \Rightarrow y : p \lor q}\,\mathsf{L}\Box}{x : \Box p, x\mathcal{R}y \Rightarrow y : p \lor q}$$

However, by considering restrictions to backward rule applications it might be nonetheless possible to define more refined proof search strategies that ensure termination and at the same time preserve the completeness of the calculus. For instance, in the example above one could consider a clause stating that $\mathsf{L}\Box$ is not applied to $x : \Box p$ and $x\mathcal{R}y$ if $y : p$ already occurs in the left-hand-side of a sequent in the proof, thus allowing only the first application of $\mathsf{L}\Box$ and preventing the subsequent ones.

Decision procedure by single proofs. The application of a terminating proof search procedure to a sequent $\Gamma \Rightarrow \Delta$ has two possible outcomes: either it provides a derivation of the sequent, or it returns a failed proof. In the first case we know that $\Gamma \Rightarrow \Delta$ is derivable, since we have a derivation of it as the result of proof search. By contrast, in the second case it is not necessarily guaranteed that $\Gamma \Rightarrow \Delta$ is not derivable.

As a matter of fact, in some cases backward proof search might be sensitive to the *order* in which the rules are applied. That is, given a sequent $\Gamma \Rightarrow \Delta$, there might exist sequences of bottom-up rule applications which provide a derivation of it, and different sequences which do not provide any derivation. For instance, consider the following proofs of $\Box p \Rightarrow \Box(p \vee q) \vee \Box q$ in **G3**.**K**. The first proof is failed, as the top sequent $p \Rightarrow q$ is not an initial sequent and no rule is backward applicable to it, whereas the second one has success.

$$\dfrac{\dfrac{p \Rightarrow q}{\Box p \Rightarrow \Box(p \vee q), \Box q}\,\mathsf{K}}{\Box p \Rightarrow \Box(p \vee q) \vee \Box q}\,\mathsf{R}\vee \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p, q}\,\mathsf{init}}{p \Rightarrow p \vee q}\,\mathsf{R}\vee}{\Box p \Rightarrow \Box(p \vee q), \Box q}\,\mathsf{K}}{\Box p \Rightarrow \Box(p \vee q) \vee \Box q}\,\mathsf{R}\vee$$

If this is the case, then the sequent is derivable (since a derivation exists), but it is actually derived only if the rules are applied in a right order. This means that in such a calculus a single failed proof is not enough to guarantee that a sequent is not derivable. As a consequence, in order to ensure the non-derivability of a sequent we need to reason over the whole space of possible derivations: a sequent is not derivable if *all* proofs of it are failed.

However, there are also calculi in which a *single* failed proof is sufficient for ensuring the non-derivability of a sequent. The crucial requirement is that all rules of the calculus are *invertible*. If this is the case, then for every derivation of a sequent $\Gamma \Rightarrow \Delta$ from assumptions $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$ it holds that $\Gamma \Rightarrow \Delta$ is derivable if and only if all assumptions $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$ are derivable. We consider as examples the following proofs of $x : \Box p \Rightarrow x : \Box(p \vee q) \vee \Box q$ in **LS**.**K**. Since the calculus is invertible, the outcome (i.e., either failure or success) is the same for every proof, in particular, differently from the proofs in **G3**.**K**, is not relevant which formula among $\Box(p \vee q)$ and $\Box q$ in the right-hand side of the conclusion is firstly processed.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : \Box p, x\mathcal{R}y, x : p, x\mathcal{R}z, z : p \Rightarrow z : p, z : q, y : q}\,\mathsf{init}}{x : \Box p, x\mathcal{R}y, x : p, x\mathcal{R}z, z : p \Rightarrow z : p \vee q, y : q}\,\mathsf{R}\vee}{x : \Box p, x\mathcal{R}y, x : p, x\mathcal{R}z \Rightarrow z : p \vee q, y : q}\,\mathsf{L}\Box}{x : \Box p, x\mathcal{R}y, x : p \Rightarrow x : \Box(p \vee q), y : q}\,\mathsf{R}\Box}{x : \Box p, x\mathcal{R}y \Rightarrow x : \Box(p \vee q), y : q}\,\mathsf{L}\Box}{x : \Box p \Rightarrow x : \Box(p \vee q), x : \Box q}\,\mathsf{R}\Box}{x : \Box p \Rightarrow x : \Box(p \vee q) \vee \Box q}\,\mathsf{R}\vee$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : \Box p, x\mathcal{R}y, y : p \Rightarrow y : p, y : q, x : \Box q}\,\mathsf{init}}{x : \Box p, x\mathcal{R}y, y : p \Rightarrow y : p \vee q, x : \Box q}\,\mathsf{R}\vee}{x : \Box p, x\mathcal{R}y \Rightarrow y : p \vee q, x : \Box q}\,\mathsf{L}\Box}{x : \Box p \Rightarrow x : \Box(p \vee q), x : \Box q}\,\mathsf{R}\Box}{x : \Box p \Rightarrow x : \Box(p \vee q) \vee \Box q}\,\mathsf{R}\vee$$

As a consequence, terminating proof search in a fully invertible calculus provides a direct *decision procedure* for the respective logic: given a formula A, every application of the proof search procedure to the sequent $\Rightarrow A$ ends after a finite number of steps and reveals whether A is derivable or not.

Optimal decision procedure. If a calculus allows for a decision procedure for the logic, we are interested in establishing the *complexity* of the procedure, which is *optimal* if it coincides with the complexity of the derivability problem in the logic.

Direct countermodel extraction. As we have seen, in some cases a single failed proof for a sequent $\Gamma \Rightarrow \Delta$ can ensure that the sequent is not derivable. We say that the calculus allows for *direct countermodel extraction* if in addition the failed proof provides sufficient information to define a countermodel of $\Gamma \Rightarrow \Delta$. To make a simple example, consider the following failed proof in **LS**.**K**:

$$\dfrac{\dfrac{\dfrac{x : \Box p, x\mathcal{R}y, y : p \Rightarrow y : q}{x : \Box p, x\mathcal{R}y \Rightarrow y : q}\,\mathsf{L}\Box}{x : \Box p \Rightarrow x : \Box q}\,\mathsf{R}\Box}{\Rightarrow x : \Box p \rightarrow \Box q}\,\mathsf{R}\rightarrow$$

By considering the standard semantics reading of sequents that interprets as true the formulas in the left-hand side of sequents, and as false the formulas on their right-hand side, basing on the information provided by the proof we obtain the relational model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W} = \{x, y\}$, $\mathcal{R}(x) = \{y\}$, $\mathcal{R}(y) = \emptyset$, $\mathcal{V}(p) = \{y\}$, and $\mathcal{V}(q) = \emptyset$, which is a countermodel of $\Box p \rightarrow \Box q$.

The decision procedures determined by terminating calculi allowing for direct countermodel extraction from failed proofs are *constructive*, since every answer to the derivability problem is certified either by a derivation, in the positive case, or by a countermodel, in the negative one.

Modularity. A family of calculi is defined *modularly* if the calculi for the stronger systems are defined by adding rules to the calculi for the weaker systems, without modifying the basic rules. For instance, labelled calculi for logics **KT**, **K4**, and **S4** can be defined simply by extending the basic calculus **G3**.**K** with the rule refl, or the rule trans, or both rules refl and trans below, respectively (cf. Negri [133]):

$$\mathsf{refl}\ \dfrac{x\mathcal{R}x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad\qquad \mathsf{trans}\ \dfrac{x\mathcal{R}z, x\mathcal{R}y, y\mathcal{R}z, \Gamma \Rightarrow \Delta}{x\mathcal{R}y, y\mathcal{R}z, \Gamma \Rightarrow \Delta}$$

Separate left and right rules. In the Gentzen calculus **G3cp** every connective is handled by a *left* and a *right* rule. Moreover, the rules introduce only a *single* occurrence of the principal connective. This kind of rules offer a declarative and purely syntactic account of the meaning of the connectives independent from any procedure.

Semantic interpretation. We are interested in establishing connections between syntactic formalisms and the semantics. For this reason, sequents and rules should have a semantic interpretation in the models of the logic.

$$\mathsf{E}\ \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \mathsf{M}\ \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \mathsf{N}\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta}$$

$$\mathsf{C}\ \frac{\Sigma \Rightarrow B \qquad \{B \Rightarrow A\}_{A \in \Sigma}}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta} \qquad \mathsf{MC}\ \frac{\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta}$$

Figure 3.4: Modal rules of Gentzen calculi for the systems of the classical cube.

3.4 Gentzen calculi for classical non-normal modal logics

In this section, we present Gentzen calculi for classical non-normal modal logics, and briefly discuss their properties. Gentzen calculi for the systems of the classical cube - except for **EN** and **ECN** - are defined by Lavendhomme and Lucas [107], the remaining cases are easy extensions and can be found in Indrzejczak [99] and Lellmann and Pimentel [110]. In [107] sequents are defined as pairs of sets, so that contraction is embedded into their structure. Here we consider the more common formulation with multisets. Moreover, differently from [97, 110], we formulate the modal rules with contexts in the conclusion in order to embed weakening in their application. The calculi **G3.E*** are defined by extending **G3cp** with the modal rules in Figure 3.4 in the following way:

G3.E := {E}; **G3.M** := {M}; **G3.EC** := {C}; **G3.MC** := {MC};
G3.EN := {E, N}; **G3.MN** := {M, N}; **G3.ECN** := {C, N}; **G3.MCN** := {MC, N}.

Recall that if $\Box\Gamma$ occurs in a rule schema, we implicitly understand that Γ contains at least one formula, thus differently from the rule K on page 45, in the rules C and MC we have $|\Sigma| \geq 1$ (thus in particular MC is not the same rule as K).

The definition of the calculi is not modular since we need to modify the rule for $\Box$ depending whether the logic is monotonic or contains the axiom C. The rules C and MC are the generalisation of the rules E and M to n principal formulas (instead of just one) in the left-hand-side of the conclusion. Observe that, differently from E and M, the rules C and MC introduce an arbitrary number of boxed formulas by a single application. The rule C has in addition a variable number of premisses which depends on the number of principal formulas in the rule application: if the rule handles n boxed formulas, then it has n premisses. An alternative way of looking at C and MC is to consider them as infinite sets of rules, each set containing a standard rule for any $n \geq 1$.

The calculi are proved syntactically complete with respect to the corresponding systems, in particular the structural rules can be proved admissible:

Theorem 3.4.1 (Completeness, [107, 97, 99, 110])**.** The structural rules and cut are admissible in the Gentzen calculi **G3.E*** for the logics of the classical cube. Moreover, the calculi are syntactically complete with respect to the corresponding systems.

The calculi **G3.E*** are strictly analytic, i.e., the complexity of the premisses of every rule is smaller than the complexity of the conclusion. As observed in the previous section, this has several good consequences, such as the possibility of getting an immediate proof of consistency and ensuring the termination of backward proof search.

By contrast, a drawback of calculi **G3.E*** is that the modal rules are not invertible. This can be easily seen by considering, e.g., the following instance of the rule M, where the conclusion is derivable but the premiss is not:

$$\dfrac{r \Rightarrow p}{\Box(p \wedge q), \Box r \Rightarrow \Box p}\ \mathsf{M}$$

Notice that the conclusion is derivable by chosing a different instance of M with $\Box(p \wedge q)$ and $\Box p$ as principal formulas:

$$\dfrac{\dfrac{\dfrac{}{p, q \Rightarrow p}\ \mathsf{init}}{p \wedge q \Rightarrow p}\ \mathsf{R}\wedge}{\Box(p \wedge q), \Box r \Rightarrow \Box p}\ \mathsf{M}$$

The non-invertibility of the modal rules is due to the fact that the context is deleted by their backward application. Essentially, when we backward apply a modal rule to a sequent, we have to *guess* a correct pair, or set, of modal formulas to which the rule is applied. As a consequence, the possible failure of a derivation is not necessarily due to the non-derivability of the sequent, but can be the consequence of a wrong choice of formulas in the application of the modal rules. As observed in the previous section, proof search in a non-invertible calculus requires to reason over the whole space of possible derivations. In such a calculus, proof search requires some form of backtracking that in case of a failed proof allows one to go back to the application of some modal rule and try to apply an alternative rule or to apply the same rule on different modal formulas. This should be repeated until a derivation is obtained or all possible applications are examined. The need of such a control is reflected into the complex proof search procedure defined in [107]. Furthermore, even when the non-derivability of a sequent is ensured, it is not obvious how to use the calculi to extract a countermodel. This difficulty is made clear also by the need of the additional use of analytic cut in the countermodel construction proposed in [107].

Extensions of calculi **G3.E*** to the systems with axioms T, 4, D, P are investigated in [97, 99, 110, 139, 140]. The calculi are defined by the rules in Figure 3.5. Similarly to the basic rules for $\Box$, some of the axioms have more than a corresponding rule depending whether the logic contains M and C, as summarised in Table 3.1.

For some combinations of axioms, additional rules are needed in the calculus in order to ensure cut elimination. In particular, for combinations of axiom D with axioms M or N we also need the rule P (notice that axiom P is derivable both from D and M and from D and N), and for the combination of D with 4 we need the rule $\mathsf{MD4}$ in the monotonic case and the rule $\mathsf{MCD4}$ in the regular case.

$$\mathsf{T}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{P}\ \frac{A \Rightarrow}{\Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{D}\ \frac{A, B \Rightarrow \qquad \Rightarrow A, B}{\Gamma, \Box A, \Box B \Rightarrow \Delta} \qquad \mathsf{4}\ \frac{\Box A \Rightarrow B \qquad B \Rightarrow \Box A}{\Gamma, \Box A \Rightarrow \Box B, \Delta}$$

$$\mathsf{MD}\ \frac{A, B \Rightarrow}{\Gamma, \Box A, \Box B \Rightarrow \Delta} \qquad \mathsf{CD}\ \frac{\Sigma, \Pi \Rightarrow \qquad \{\Rightarrow A, B\}_{A \in \Sigma, B \in \Pi}}{\Gamma, \Box\Sigma, \Box\Pi \Rightarrow \Delta} \qquad \mathsf{CP}\ \frac{\Sigma \Rightarrow}{\Gamma, \Box\Sigma \Rightarrow \Delta}$$

$$\mathsf{M4}\ \frac{\Box A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \qquad \mathsf{MC4}\ \frac{\Box\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B, \Delta} \qquad \mathsf{MD4}\ \frac{A, \Box B \Rightarrow}{\Gamma, \Box A, \Box B \Rightarrow \Delta} \qquad \mathsf{MCD4}\ \frac{\Sigma, \Box\Pi \Rightarrow}{\Gamma, \Box\Sigma, \Box\Pi \Rightarrow \Delta}$$

Figure 3.5: Rules of Gentzen calculi for the extensions of the classical cube.

	N	T	P	D	4
E	N	T	P	D	4
EC	N	T	CP	CD	?

	N	T	P	D	4
M	N	T	P	MD	M4
MC	N	T	CP	CP	MC4

Table 3.1: Gentzen calculi for the extensions of the classical cube.

Theorem 3.4.2 (Cut admissibility)**.** The rule cut is admissible in **G3.ET***, **G3.EP***, **G3.ED***, and **G3.M4***.

The above result is proved in [137] for **G3.MCT**, in [97, 99] for the extensions of **G3.E** and **G3.M** with T, D, and 4, and in [139, 140] for the calculi of the classical cube extended with P and D. All the remaining cases are found in [110].

Observe that **G3.E4** is excluded from the calculi listed in the above theorem. As a matter of fact, the rule cut is not admissible in **G3.E4**, as it is shown by the following example.

Example 3.4.1. The sequent $\Box p \Rightarrow \Box\Box\Box p$ is derivable in **G3.E4** $\cup$ {cut}, but it is not derivable in **G3.E4** without cut. The derivation with cut is as follows:

$$\dfrac{\dfrac{\dfrac{\Box p \Rightarrow \Box p \qquad \Box p \Rightarrow \Box p}{\Box p \Rightarrow \Box\Box p}\,\mathsf{4}}{\Box p \Rightarrow \Box\Box p, \Box\Box\Box p}\,\mathsf{Rwk} \qquad \dfrac{\dfrac{\Box\Box p \Rightarrow \Box\Box p \qquad \Box\Box p \Rightarrow \Box\Box p}{\Box\Box p \Rightarrow \Box\Box\Box p}\,\mathsf{4}}{\Box p, \Box\Box p \Rightarrow \Box\Box\Box p}\,\mathsf{Lwk}}{\Box p \Rightarrow \Box\Box\Box p}\,\mathsf{cut}$$

Let us now try to derive bottom up the sequent without using cut. At the first step the only backward applicable rules are E and 4. By applying E we obtain the premisses $p \Rightarrow \Box\Box p$ and $\Box\Box p \Rightarrow p$, both non derivable as they are neither initial sequents nor conclusions of any rule. By applying 4 instead of E we obtain $\Box p \Rightarrow \Box\Box p$ and $\Box\Box p \Rightarrow \Box p$. The first premiss is derivable. Concerning the second premiss, again we can apply either E or 4, obtaining either $\Box p \Rightarrow p$ and $p \Rightarrow \Box p$, or $\Box\Box p \Rightarrow p$ and $p \Rightarrow \Box\Box p$, all of them non derivable.

As it is shown in [99], cut is instead admissible in **G3.ET4**. By contrast, Gentzen calculi for non-monotonic systems with both axioms C and 4 are not investigated at all. A C-version of the rule 4 would very likely feature the same problem displayed in Example 3.4.1, but in this case it is not obvious how to obtain cut free calculi even in presence of the rule T.

Concerning the structural rules, only in [139, 140] the admissibility of contraction is explicitly investigated. It can be shown that contraction is admissible in most calculi, however there are some problematic cases. A first difficulty arises when one tries to give a purely syntactic proof of admissibility of contraction in the calculi with the rules for axiom D. This difficulty depends on the fact that these rules have (at least) two principal formulas in the left-hand side of the conclusion. For instance, suppose that a derivation contains an application of contraction to a boxed formula which is derived by an application of D:

$$\dfrac{\dfrac{A, A \Rightarrow \qquad \Rightarrow A, A}{\Gamma, \Box A, \Box A \Rightarrow \Delta}\,\mathsf{D}}{\Gamma, \Box A \Rightarrow \Delta}\,\mathsf{Lctr}$$

In such a situation, a standard proof of admissibility of contraction would require to reverse the order of the rule applications: we must first contract the premisses, and then apply D. However, the application of contraction to the sequents $A, A \Rightarrow$ and $\Rightarrow A, A$ would give $A \Rightarrow$ and $\Rightarrow A$, which are not premisses of D, whence D would not be applicable anymore. The solution adopted in [140] consists in adding one of the following two rules to the calculus, depending whether the rules are formulated for the axiom C:

$$\mathsf{D}'\ \dfrac{A \Rightarrow \qquad \Rightarrow A}{\Gamma, \Box A \Rightarrow \Delta} \qquad\qquad \mathsf{CD}'\ \dfrac{\Sigma \Rightarrow \qquad \{\Rightarrow A\}_{A \in \Sigma}}{\Gamma, \Box \Sigma \Rightarrow \Delta}$$

These rules solve the problem described above by allowing one to directly obtain the sequent $\Gamma, \Box A \Rightarrow \Delta$ from the premisses $A \Rightarrow$ and $\Rightarrow A$. Notice however that they are never applicable in consistent calculi, whence they do not extend the set of derivable sequents.

A calculus in which contraction is not admissible is **G3.MT**, as the following example shows.

Example 3.4.2. The sequent $\Box(p \land \neg\Box p) \Rightarrow$ is derivable in $\mathbf{G3.MT} \cup \{\mathsf{Lctr}\}$, but it is not derivable in **G3.MT** without Lctr.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, \neg\Box p \Rightarrow p}{p \land \neg\Box p \Rightarrow p}\,\mathsf{L}\land}{\Box(p \land \neg\Box p), p \Rightarrow \Box p}\,\mathsf{M}}{\Box(p \land \neg\Box p), p, \neg\Box p \Rightarrow}\,\mathsf{L}\neg}{\Box(p \land \neg\Box p), p \land \neg\Box p \Rightarrow}\,\mathsf{L}\land}{\Box(p \land \neg\Box p), \Box(p \land \neg\Box p) \Rightarrow}\,\mathsf{T}}{\Box(p \land \neg\Box p) \Rightarrow}\,\mathsf{Lctr} \qquad\qquad \dfrac{\dfrac{\dfrac{p \Rightarrow \Box p}{p, \neg\Box p \Rightarrow}\,\mathsf{L}\neg}{p \land \neg\Box p \Rightarrow}\,\mathsf{L}\land}{\Box(p \land \neg\Box p) \Rightarrow}\,\mathsf{T}$$

On the left we have a derivation of $\Box(p \land \neg\Box p) \Rightarrow$ in $\mathbf{G3.MT} \cup \{\mathsf{Lctr}\}$ (with an application of Lctr). By contrast, on the right we have a failed proof for $\Box(p \land \neg\Box p) \Rightarrow$ in **G3.MT**. The proof is failed because the top sequent $p \Rightarrow \Box p$ is not derivable, as it is neither an axiom nor the conclusion of a rule of **G3.MT**. Notice that at each step we have applied the only backwards applicable rule, which means that there exists no alternative possible choice of backward rule applications which provides a derivation of the sequent (this example is considered in Goré [74] in the context of tableaux calculi for the logic **KT**).

Nonetheless, admissibility of contraction in **G3.MT** can be recovered by replacing the rule T with the following one:

$$\mathsf{T}' \; \frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \; .$$

Given a rule of this form the calculus is not strictly analytic anymore, whence rough bottom-up proof search does not terminates. However, it is possible to regain the termination of proof search by considering straightforward detections of redundant applications of the rule T'.

3.5 Further proof systems for classical non-normal modal logics: state of the art

The proof theory of non-normal modal logics is not as developed as its semantics. Although a Gentzen calculus for a non-normal modal logic – in particular for the logic **MCT** – can be found already in Ohnishi and Matsumoto [137] in the 1950s, and the first tableaux calculus for the logic **M** goes back to Fitting [56] (where the logic **M** is called U) in the 1980s, only in recent years the problem of finding suitable proof systems for non-normal modal logics has been more extensively addressed. Up to now the study of proof systems for non-normal modal logics does not have a state of the art comparable with the one of proof systems for normal modal logics, for which there exist well-understood proof methods of many kinds. Moreover, the existing proof systems cover mainly monotonic logics, whereas non-monotonic logics have been more neglected despite their interest. We make here a brief account of these proof systems.

Besides Fitting's tableaux, further *tableaux calculi* for the logic **M** are defined in Hansen [82], where they are also extended to Pauly's coalition logic [146], as well as in Governatori and Luppi [77]. In the latter work, the extensions of **M** with the axiom C and both axioms C and N are also treated.

The first calculi covering also non-monotonic systems are the Gentzen calculi by Lavendhomme and Lucas [107] that we have presented in the previous section. As we have seen, Lavendhomme and Lucas' calculi have been extended beyond the classical cube by Indrzejczak [97, 99], Orlandelli [139, 140], and Lellmann and Pimentel [110]. Indrzejczak's calculi have been also reformulated as prefixed tableaux in Indrzejczak [98].

Fully modular proof systems for the whole classical cube are proposed by Gilbert and Maffezioli [68] in the form of *labelled sequent calculi*. The calculi are based on an embedding of non-normal modal logics into normal multimodal logics given by Kracht and Wolter [103] and Gasquet and Herzig [65]. This embedding allows one to design calculi by importing the (multi)relational semantics into the calculus by converting semantic conditions into rules in the way of Negri [133] and Viganò [168] (cf. the labelled calculus for logic **G3**.**K** in Figure 3.3). The same embedding of non-normal modal logics into normal multimodal logics is at the basis of the display calculi for monotonic systems proposed very recently by Chen et al. [30]. Semantic based labelled sequent calculi for the whole classical cube are also proposed in Negri [131]. As a difference with the calculi by Gilbert and Maffezioli, Negri's calculi directly import into the rules the standard neighbourhood semantics rather than the multirelational one. These calculi represent the starting point for the definition of our labelled calculi in Chapter 5.

Not only labelled calculi, but also structured calculi have been investigated for non-normal modal logics. In Lellmann and Pimentel [109, 110], so-called *linear nested sequent calculi* are modularly defined for the whole classical cube as well as for the extensions of monotonic systems with the axioms P, D, T, 4, and 5. Moreover, in Lellmann [111] a *nested sequent calculus* for Brown's ability logic [22] is proposed. This logic contains two modalities $[\forall\forall]$ and $[\exists\forall]$, the former is a normal **K**-modality whereas the latter is non-normal, in particular the fragment with the only modality $[\exists\forall]$ coincides with the logic **M**.

3.6 Gentzen calculi for intuitionistic non-normal modal logics

We now consider proof systems for intuitionistic non-normal modal logics. Several proof systems have been proposed for the logic **IK** and some extensions. For example, nested sequent calculi are proposed by Galmiche and Salhi [63, 64], Straßburger [162], and Marin and Straßburger [120, 121], whereas a labelled natural deduction systems is defined by Simpson [161]. Moreover, **IK** is also covered as a particular case by the more general calculi for bi-intuitionistic tense logic presented in Goré et al. [75].

There are in contrast less proof-theoretic investigations of the constructive systems **CK** and **CCDL**. Nonetheless, cut-free Gentzen calculi for both logics are defined respectively by Bellin et al. [14] and Wijesekera [170]. In addition to the Gentzen calculi, further proof systems have been defined for **CK**. In particular, a focused 2-sequent calculus is defined in Mendler and Scheele [127], and a nested sequent calculus is proposed in Arisaka et al. [6]. Both calculi cover also extensions of **CK** with (intuitionistic versions of) additional axioms among T, D, 4, 5, and B. Furthermore, focused sequent calculi for a constructive version of logic **S4** (i.e., an extension of **CK** with intuitionistic counterparts of axioms T and 4) are defined in Heilala and Pientka [86], and similar calculi for a constructive version of logic **S5** are defined in Park et al. [144].

We now present the Gentzen calculi for **CK** and **CCDL**. These calculi are defined in [14] and [170] as extensions of a suited calculus for intuitionistic logic. Here we take as base calculus the Gentzen calculus **G3ip** from Troelstra and Schwichtenberg [164], which is defined

$$\mathsf{init}\ \frac{}{\Gamma, p \Rightarrow p} \qquad \mathsf{L}\bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{\Gamma \Rightarrow \top}$$

$$\mathsf{L}\wedge\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\mathsf{L}\vee\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \mathsf{R}\vee_i\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i = 1, 2)$$

$$\mathsf{L}\supset\ \frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \mathsf{R}\supset\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Figure 3.6: Rules of **G3ip**.

$$\mathsf{MC}_\Box\ \frac{\Sigma \Rightarrow B}{\Gamma, \Box\Sigma \Rightarrow \Box B} \qquad \mathsf{N}_\Box\ \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \mathsf{M}_\Diamond\ \frac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathsf{N}_\Diamond\ \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta} \qquad \mathsf{W}\ \frac{\Sigma, B \Rightarrow C}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Diamond C} \qquad \mathsf{strC}\ \frac{\Sigma, B \Rightarrow}{\Gamma, \Box\Sigma, \Diamond B \Rightarrow \Delta}$$

Figure 3.7: Modal rules of Gentzen calculi for **CCDL** and **CK**.

by the rules in Figure 3.6 (this is not the one considered in [14, 170]). Our choice of **G3ip** is motivated by the fact all structural rules are admissible in this calculus. **G3ip** is the single-succedent version of the calculus **G3cp** for classical logic: in **G3ip** the right-hand-side of sequents can contain *at most one* formula. This requires to replace the rule $\mathsf{R}\vee$ with two rules $\mathsf{R}\vee_1$ and $\mathsf{R}\vee_2$, one for the left and one for the right disjunct of the principal formula. In addition, in order to ensure the admissibility of contraction, the rule $\mathsf{L}\supset$ is formulated in such a way that the principal formula $A \supset B$ is kept into the left premiss. The structural rules of **G3ip** are formulated as follows:

$$\mathsf{Lwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}$$

For the intuitionistic calculi we consider the multiplicative formulation of cut. This choice is motivated simply by the fact that the proofs of cut elimination are cleaner with this formulation, since they do not require auxiliary applications of the structural rules. As in the classical case, in the presence of weakening and contraction the multiplicative formulation of cut is equivalent to the additive formulation

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Gentzen calculi for **CK** and **CCDL** are defined by extending **G3ip** with the modal rules in Figure 3.7 in the following way:

$$\mathbf{G3.CK} := \{\mathsf{MC}_\Box, \mathsf{N}_\Box, \mathsf{M}_\Diamond, \mathsf{W}\}; \qquad \mathbf{G3.CCDL} := \{\mathsf{MC}_\Box, \mathsf{N}_\Box, \mathsf{M}_\Diamond, \mathsf{W}, \mathsf{N}_\Diamond, \mathsf{strC}\}.$$

As for the classical calculi, differently from [14, 170] we add the contexts in the conclusion of the modal rules in order to embed weakening in their application. The rules $\mathsf{MC}_\Box$ and $\mathsf{N}_\Box$ are the single-succedent versions of the rules MC and N of the classical calculi (see Figure 3.4), whereas the rules $\mathsf{M}_\Diamond$ and $\mathsf{N}_\Diamond$ are the single-succedent versions of the $\Diamond$-counterparts of the rules M and N. Furthermore, the rule W (we call it this way from "Wijesekera") corresponds to the axiom $K_\Diamond$. Differently from [14, 170], we require that $|\Sigma| \geq 1$, for this reason we must explicitly consider the rule $\mathsf{M}_\Diamond$ in addition to W ($\mathsf{M}_\Diamond$ is in contrast a particular case of W if $|\Sigma| = \emptyset$ is allowed). Finally, the rule strC does not correspond to any axiom of **CCDL**, but it is needed for cut elimination, as it is shown by the following example.

Example 3.6.1. The sequent $\Box p, \Diamond\neg p \Rightarrow$ is derivable in $\mathbf{G3.CCDL} \cup \{\mathsf{cut}\} \setminus \{\mathsf{strC}\}$, but it is not derivable in $\mathbf{G3.CCDL} \setminus \{\mathsf{strC}\}$ without cut. The derivation with cut is as follows:

$$\dfrac{\mathsf{W}\,\dfrac{p, \neg p \Rightarrow \bot}{\Box p, \Diamond\neg p \Rightarrow \Diamond\bot} \qquad \dfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow}\,\mathsf{N}_\Diamond}{\Box p, \Diamond\neg p \Rightarrow}\,\mathsf{cut}$$

In contrast, in the absence of cut the only backward applicable rule to $\Box p, \Diamond\neg p \Rightarrow$ is $\mathsf{N}_\Diamond$, but its premiss $\neg p \Rightarrow$ is not derivable. Finally, the sequent is trivially derivable by strC:

$$\dfrac{p, \neg p \Rightarrow}{\Box p, \Diamond\neg p \Rightarrow}\,\mathsf{strC}$$

The following result can be obtained by straightforwardly adapting the proofs in [14, 170]:

Theorem 3.6.1 (Completeness)**.** The structural rules and cut are admissible in **G3.CK** and **G3.CCDL**. Moreover, the two calculi are syntactically complete with respect to **CK** and **CCDL**, respectively.

Part II
Classical non-normal modal logics

Chapter 4

Bi-neighbourhood semantics

In this chapter, we present an alternative semantics for classical non-normal modal logics, that we call *bi-neighbourhood* semantics. First, we define bi-neighbourhood models for all classical non-normal modal logics considered in this work, and prove soundness and completeness of every system with respect to the corresponding class of models. Then, we analyse the relations between the bi-neighbourhood semantics and the standard semantics presented in Chapter 2, providing mutual transformations of models of the two kinds. We also define a bi-neighbourhood semantics for Elgesem's and Troquard's agency and ability logics. Finally, basing on the bi-neighbourhood semantics we show a syntactic embedding of classical non-normal modal logics into monotonic logics with dyadic modalities. Bi-neighbourhood semantics shall be the reference semantic framework in this work: it is at the basis of the design of our labelled calculi, and it is also the framework considered for the extraction of countermodels both from external and from internal calculi.

4.1 Bi-neighbourhood models

As recalled in Section 2.3, classical non-normal modal logics are characterised by standard neighbourhood models, and further semantics exist for monotonic and regular systems. Here we introduce an alternative semantics, that we call *bi-neighbourhood* semantics.

Bi-neighbourhood semantics differs from the standard one with respect to the neighbourhood function. Instead of a set of neighbourhoods, worlds in bi-neighbourhood models are equipped with a set of *pairs* of neighbourhoods, that we call *bi-neighbourhood pairs*. The intuition is that the two components of a pair provide, so to say, "positive" and "negative" support for a modal formula. Bi-neighbourhood models are formally defined as follows:

Definition 4.1.1 (Bi-neighbourhood semantics). A *bi-neighbourhood model* is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function $Atm \to \mathcal{P}(\mathcal{W})$, and $\mathcal{N}$ is a function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W}))$, that we call *bi-neighbourhood function*. The forcing relation $\mathcal{M}, w \Vdash_{bi} A$ is defined as in Definition 2.3.1 except for the modality, for which the clause is as follows:

$$\mathcal{M}, w \Vdash_{bi} \Box A \quad \text{iff} \quad \text{there is } (\alpha, \beta) \in \mathcal{N}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta.$$

By duality with $\Box$ we then obtain the following forcing condition for diamond formulas: $\mathcal{M}, w \Vdash_{bi} \Diamond A$ if and only if for every $(\alpha, \beta) \in \mathcal{N}(w)$, $[\![A]\!]_{\mathcal{M}} \cap \alpha \neq \emptyset$ or $[\![A]\!]_{\mathcal{M}} \cap \mathcal{W} \setminus \beta \neq \emptyset$.

Observe that the standard models (cf. Definition 2.3.1) can be seen as the particular cases of bi-neighbourhood models in which the components of all pairs are *complementary*. Moreover, it is easy to see that in classical logics the forcing condition for boxed formulas can be rewritten as $\mathcal{M}, w \Vdash_{bi} \Box A$ if and only if there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!]_{\mathcal{M}}$ and $\beta \subseteq [\![\neg A]\!]_{\mathcal{M}}$. Thus, bi-neighbourhood semantics essentially decomposes the forcing condition for boxed formulas of the standard semantics into *two components*, both of them *monotonic* (cf. Definition 2.3.6). This suggests a possible reduction of non-monotonic logics into monotonic logics with dyadic (binary) modalities, we develop this intuition in Section 4.7. In the following, we simply write $w \Vdash A$ and $[\![A]\!]$, i.e., without stating explicitly the model $\mathcal{M}$ and the kind of semantics (by the subscripts *bi*, *st*, $\exists\forall$, or *r*), if these are made clear by the context.

Motivation

To give an intuition of the bi-neighbourhood semantics, we present two examples based on two different possible interpretations of $\Box$. Firstly, let us consider an interpretation of $\Box A$ like "A is *lawful*", or "A *complies with the law*", where A intuitively represents some action or activity. We can say that in some cases actions or activities are ruled by specific legislations composed by two kinds of norms: obligations and prohibitions. In such a situation, an activity (described by the proposition A) is considered lawful ($\Box A$) if it fulfills all obligations and does not break any prohibition. A similar legislation can be described by a pair (α, β) of respectively obligations and prohibitions. The fact that the elements of bi-neighbourhood pairs are not required to be complementary might reflect that obligations and prohibitions usually do not exhaust the possible actions. On the contrary, there is usually the possibility of making choices among things that are neither obligations, nor prohibitions. For instance, a bartender must issue receipts (obligation) and must not serve alcohol to underage people (prohibition), but she is free to decide whether or not to serve pastis in her bar. In addition, different activities might be regulated by different norms, thus justifying the possible existence of more bi-neighbourhood pairs. For instance, the norms ruling bartending are not the same as the ones ruling, say, the management of a hospital.[1]

[1]Clearly, this interpretation would conflict with the idea of Standard Deontic Logic (SDL, see e.g. [124]) that obligations and prohibitions are interdefinable (A is prohibited if and only if $\neg A$ is obligatory). However,

The same structure can be found in completely different contexts. Imagine this time that a detective must solve a murder. Usually, a detective has two kinds of available information: things that are known to have happened (for instance, the victim has been killed with a knife) and things that are known not to have happened (for instance, the murderer did not enter the room through the main door). In such a situation, an explanation of the murder (described by the proposition A) is considered *plausible* ($\Box A$) if it explains all ascertained facts without requiring that something already ruled out should have happened (it explains how the murderer can have killed the victim with a knife without entering the room through the main door). In this context, different bi-neighbourhood pairs might represent different sources of information, for instance different witnesses.

In general, bi-neighbourhood semantics might be used to represent under-determined situations. While standard semantics can represent contexts where all information is explicitly given (for instance, all possible solutions to the murder are explicitly presented, or all lawful activities are fully described), bi-neighbourhood semantics can represent more liberal contexts, where all solutions are accepted which are consistent with the available information. As we shall see, the bi-neighbourhood semantics is also more suitable than the standard semantics for extracting countermodels from failed proofs in (different kinds of) sequent calculi. In particular, it is more suited for dealing with the partial information provided by the failed proofs.

Bi-neighbourhood models for extensions

As we shall see, the class of all bi-neighbourhood models characterises the basic logic **E**. In order to give a characterisation to the other systems we must consider additional properties of the bi-neighbourhood function. The semantics is defined modularly for all systems, which as usual means that each axiom has a corresponding semantic condition. We consider the following definition, which is needed to formulate the semantic condition for axiom 4.

Definition 4.1.2. Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a bi-neighbourhood model, and $\alpha, \beta \subseteq \mathcal{W}$. We define $\rotatebox[origin=c]{180}{\mathcal{M}}^+(\alpha, \beta)$ and $\rotatebox[origin=c]{180}{\mathcal{M}}^-(\alpha, \beta)$ as follows:

$$\rotatebox[origin=c]{180}{\mathcal{M}}^+(\alpha, \beta) = \{v \mid (\alpha, \beta) \in \mathcal{N}(v)\}; \text{ and}$$
$$\rotatebox[origin=c]{180}{\mathcal{M}}^-(\alpha, \beta) = \{v \mid \text{for all } (\gamma, \delta) \in \mathcal{N}(v), \alpha \cap \delta \neq \emptyset \text{ or } \beta \cap \gamma \neq \emptyset\}.$$

Intuitively, $\rotatebox[origin=c]{180}{\mathcal{M}}^+(\alpha, \beta)$ is the set of worlds containing the pair (α, β) in their neighbourhood, whereas $\rotatebox[origin=c]{180}{\mathcal{M}}^-(\alpha, \beta)$ is the set of worlds such that all their bi-neighbourhood pairs are incompatible with (α, β) (in particular, since $\alpha \cap \delta \neq \emptyset$ or $\beta \cap \gamma \neq \emptyset$, the pairs in $\rotatebox[origin=c]{180}{\mathcal{M}}^-(\alpha, \beta)$ and

it might be worth considering contexts where obligations and prohibitions are not assumed as interdefinable. As an example, the logical analysis of a normative code where obligations (prescriptions) and prohibitions are not interdefinable can be found in Gulisano [80].

(α, β) do not make the same boxed formulas true). We now present the semantic conditions corresponding to the considered modal axioms and rules.

Definition 4.1.3 (Semantic conditions for extensions)**.** Let X be any modal axiom or rule among M, C, N, T, P, D, RD_n^+, and 4. The semantic condition (X) corresponding to X in the bi-neighbourhood semantics is as follows:

(M)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\beta = \emptyset$.
(N)	There is $\alpha \subseteq \mathcal{W}$ such that for all $w \in \mathcal{W}$, $(\alpha, \emptyset) \in \mathcal{N}(w)$.
(C)	If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}(w)$.
(T)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $w \in \alpha$.
(P)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\alpha \neq \emptyset$.
(D)	If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$, then $\alpha \cap \gamma \neq \emptyset$ or $\beta \cap \delta \neq \emptyset$.
(RD_n^+)	If $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}(w)$, then $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.
(4)	If $(\alpha, \beta) \in \mathcal{N}(w)$, then there is $(\gamma, \delta) \in \mathcal{N}(w)$ s.t. $\gamma \subseteq \mathcal{V}^+(\alpha, \beta)$ and $\delta \subseteq \mathcal{V}^-(\alpha, \beta)$.

For every subset S of the above conditions, we call *S-model* any bi-neighbourhood model satisfying all conditions in S. For instance, a MC-model is any bi-neighbourhood satisfying both (M) and (C). The class of bi-neighbourhood models for a given non-normal modal logic **L** is determined by the conditions corresponding to the axioms of **L**. We denote by $\mathcal{C}^{bi}_{\mathbf{L}}$ the class of bi-neighbourhood models for **L**.

Possible alternative semantic conditions are examined in Section 4.4. It can be interesting to notice that some of these conditions have a clear meaning under the interpretations of $\Box A$ proposed above. For instance, under the deontic interpretation of $\Box A$ as "A is lawuful", the condition (N) expresses the existence of a legal system which does not state any prohibition, whereas (M) expresses the stronger statement that prohibitions are never stated at all. Moreover, the condition (P) states that every legislation must contain at least some obligation, and (RD_n^+) states that every n legislations have some common obligations.

In addition, we can observe that the bi-neighbourhood semantics for monotonic systems reduces to the $\exists\forall$-semantics (Definition 2.3.6) as the condition $[\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta$ of the satifaction clause of boxed formula is trivially satisfied in M-models, having $\beta = \emptyset$ for every bi-neighbourhood pairs (α, β). We shall more widely explore the relations between bi-neighbourhood and standard models in Section 4.3. Now we provide a direct proof of soundness and completeness of classical non-normal modal logics with respect to the corresponding bi-neighbourhood models.

4.2 Soundness and completeness

We prove that classical non-normal modal logics are sound with respect to their bi-neighbourhood models.

Theorem 4.2.1 (Soundness)**.** Every classical non-normal modal logic $\mathbf{E}^*$ is sound with respect to the corresponding class of bi-neighbourhood models: If $\Phi \vdash_{\mathbf{E}^*} A$, then $\Phi \models_{\mathcal{C}^{bi}_{\mathbf{E}^*}} A$.

Proof. We show that every axiom or rule X is valid in bi-neighbourhood X-models.

(RE) Assume $\mathcal{M} \models A \leftrightarrow B$, that is $[\![A]\!] = [\![B]\!]$. We have: $w \Vdash \Box A$ iff there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$ iff $\alpha \subseteq [\![B]\!] \subseteq \mathcal{W} \setminus \beta$ iff $w \Vdash \Box B$. Then $\mathcal{M} \models \Box A \leftrightarrow \Box B$.

(M) Let $\mathcal{M}$ be a M-model and assume $w \Vdash \Box(A \land B)$. Then there is $(\alpha, \emptyset) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A \land B]\!] \subseteq \mathcal{W} \setminus \emptyset$. Then $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \emptyset$, which implies $w \Vdash \Box A$.

(N) Let $\mathcal{M}$ be a N-model. By condition (N) there is $(\alpha, \emptyset) \in \mathcal{N}(w)$. Since trivially $\alpha \subseteq [\![\top]\!] \subseteq \mathcal{W} \setminus \emptyset$, we have $w \Vdash \Box\top$.

(C) Let $\mathcal{M}$ be a C-model and assume $w \Vdash \Box A \land \Box B$. Then there are $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$ and $\gamma \subseteq [\![B]\!] \subseteq \mathcal{W} \setminus \delta$. By condition (C) we have $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}(w)$, where $\alpha \cap \gamma \subseteq [\![A]\!] \cap [\![B]\!] = [\![A \land B]\!] \subseteq (\mathcal{W} \setminus \beta) \cap (\mathcal{W} \setminus \delta) = \mathcal{W} \setminus (\beta \cup \delta)$. Then $w \Vdash \Box(A \land B)$.

(T) Let $\mathcal{M}$ be a T-model and assume $w \Vdash \Box A$. Then there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. By condition (T), $w \in \alpha$, thus $w \Vdash A$.

(P) Let $\mathcal{M}$ be a P-model and assume by contradiction that $w \Vdash \Box\bot$. Then there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![\bot]\!] \subseteq \mathcal{W} \setminus \beta$. Thus $\alpha = \emptyset$, against condition (P). Therefore $w \Vdash \neg\Box\bot$.

(D) Let $\mathcal{M}$ be a D-model and assume by contradiction that $w \Vdash \Box A \land \Box\neg A$. Then there are $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$ and $\gamma \subseteq [\![\neg A]\!] \subseteq \mathcal{W} \setminus \delta$. Then $\alpha \cap \gamma = \emptyset$ and $\beta \cap \delta = \emptyset$, against condition (D). Therefore $w \Vdash \neg(\Box A \land \Box\neg A)$.

(RD^+_n) Let $\mathcal{M}$ be a RD^+_n-model and assume $\mathcal{M} \models \neg(A_1 \land ... \land A_n)$, that is $[\![A_1]\!] \cap ... \cap [\![A_n]\!] = \emptyset$. By contradiction, assume also $w \Vdash \Box A_1 \land ... \land \Box A_n$. Then there are $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}(w)$ such that $\alpha_1 \subseteq [\![A_1]\!] \subseteq \mathcal{W} \setminus \beta_1$, ..., $\alpha_n \subseteq [\![A_n]\!] \subseteq \mathcal{W} \setminus \beta_n$. Then $[\![\alpha_1]\!] \cap ... \cap [\![\alpha_n]\!] = \emptyset$, against condition ($\text{RD}^+_n$). Therefore $w \Vdash \neg(\Box A_1 \land ... \land \Box A_n)$.

(4) Let $\mathcal{M}$ be a 4-model and assume $w \Vdash \Box A$. Then there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. By condition (4), there is $(\gamma, \delta) \in \mathcal{N}(w)$ such that $\gamma \subseteq \mathcal{N}^+(\alpha, \beta)$ and $\delta \subseteq \mathcal{N}^-(\alpha, \beta)$. We show that $\mathcal{N}^+(\alpha, \beta) \subseteq [\![\Box A]\!] \subseteq \mathcal{W} \setminus \mathcal{N}^-(\alpha, \beta)$, which implies $\gamma \subseteq [\![\Box A]\!] \subseteq \mathcal{W} \setminus \delta$, whence $w \Vdash \Box\Box A$: If $w \in \mathcal{N}^+(\alpha, \beta)$, then $(\alpha, \beta) \in \mathcal{N}(w)$, which implies $w \Vdash \Box A$. If instead $w \in \mathcal{N}^-(\alpha, \beta)$, then for all $(\gamma, \delta) \in \mathcal{N}(w)$, $\gamma \cap \beta \neq \emptyset$ or $\delta \cap \alpha \neq \emptyset$, that is $\gamma \not\subseteq [\![A]\!]$ or $\delta \not\subseteq \mathcal{W} \setminus [\![A]\!]$, therefore $w \Vdash \neg\Box A$. $\square$

We now prove that classical non-normal modal logics are strongly complete with respect to the corresponding bi-neighbourhood models, that is $\Phi \models_{\mathcal{C}^{bi}_{\mathbf{E}^*}} A$ implies $\Phi \vdash_{\mathbf{E}^*} A$. The proofs are based on the canonical model construction. We adopt the usual terminology:

Definition 4.2.1 (Maximal consistent sets)**.** For every logic $\mathbf{L}$ in language $\mathcal{L}$ and every set Φ of formulas of $\mathcal{L}$, we say that

- Φ is $\mathbf{L}$-*consistent* if $\Phi \not\vdash_{\mathbf{L}} \bot$, and
- Φ is $\mathbf{L}$-*maximal consistent* (or just $\mathbf{L}$-maximal) if it is $\mathbf{L}$-consistent and for every formula A of $\mathcal{L}$, if $A \notin \Phi$, then $\Phi \cup \{A\}$ is not $\mathbf{L}$-consistent.

We denote by $Max_{\mathbf{L}}$ the class of all $\mathbf{L}$-maximal consistent sets of formulas of $\mathcal{L}$, and for every formula A we denote by $\uparrow A$ the set $\{\Phi \in Max_{\mathbf{L}} \mid A \in \Phi\}$.

Before defining canonical models, in the following Lemmas 4.2.2 and 4.2.3 we recall some basic properties of maximal consistent sets. The proofs are standard and can be found in any modal logic handbook, e.g. Chellas [29].

Lemma 4.2.2 (Lindenbaum lemma)**.** If $\Phi \not\vdash_{\mathbf{L}} A$, then there is $\Psi \in Max_{\mathbf{L}}$ such that $\Phi \subseteq \Psi$ and $A \notin \Psi$.

Proof. (Sketch) Let $B_0, B_1, B_2, ...$ be an enumeration of all formulas of $\mathcal{L}$. We construct a chain $\Psi_0, \Psi_1, \Psi_2, ...$ of sets of formulas of $\mathcal{L}$ as follows:

$$\Psi_0 = \Phi \cup \{\neg A\}.$$
$$\Psi_{n+1} = \begin{cases} \Psi_n \cup \{B_n\} & \text{if } \Psi_n \cup \{B_n\} \text{ is } \mathbf{L}\text{-consistent;} \\ \Psi_n & \text{otherwise.} \end{cases}$$

Moreover, we define $\Psi := \bigcup_{n\geq 0} \Psi_n$. By the construction of Ψ we have $\Phi \subseteq \Psi$. We show that Ψ is $\mathbf{L}$-maximal. First, Ψ_0 is $\mathbf{L}$-consistent: By contraposition, if $\Phi \cup \{\neg A\} \vdash_{\mathbf{L}} \bot$, then by the deduction theorem (Theorem 2.2.1), $\Phi \vdash_{\mathbf{L}} \neg A \to \bot = \neg\neg A$, then $\Phi \vdash_{\mathbf{L}} A$, against the hypothesis. Moreover, by the condition of construction it follows that, for every $n \in \mathbb{N}$, if Ψ_n is $\mathbf{L}$-consistent, then Ψ_{n+1} is $\mathbf{L}$-consistent; therefore Ψ_n is $\mathbf{L}$-consistent for every $n \in \mathbb{N}$. It follows that Ψ is $\mathbf{L}$-consistent. Now, let C be any formula such that $C \notin \Psi$. Then $C = B_i$ for some B_i in the enumeration of formulas. This means that at stage i of the construction, $\Psi_i \cup \{B_i\}$ is $\mathbf{L}$-inconsistent. Since $\Psi_i \subseteq \Psi$, we have that $\Psi \cup \{B_i\}$ is $\mathbf{L}$-inconsistent. Thus Ψ is $\mathbf{L}$-maximal. Finally, since Ψ is $\mathbf{L}$-consistent and $\neg A \in \Psi$, it follows that $A \notin \Psi$. $\square$

Lemma 4.2.3. Let Φ be an $\mathbf{L}$-maximal consistent set. (a) For all $A \in \mathcal{L}$, $A \in \Phi$ or $\neg A \in \Phi$. (b) If $\Psi \vdash_{\mathbf{L}} A$ and $\Psi \subseteq \Phi$, then $A \in \Phi$. (c) If $\vdash_{\mathbf{L}} A \leftrightarrow B$ and $\Box A \in \Phi$, then $\Box B \in \Phi$. (d) $\uparrow(A \wedge B) = \uparrow A \cap \uparrow B$. (e) $\uparrow(A \vee B) = \uparrow A \cup \uparrow B$. (f) $\uparrow A \subseteq \uparrow B$ iff $\vdash_{\mathbf{L}} A \to B$. (g) $\uparrow A = \uparrow B$ iff $\vdash_{\mathbf{L}} A \leftrightarrow B$.

Proof. (a) By contradiction, assume $A \notin \Phi$ and $\neg A \notin \Phi$. Then by definition, $\Phi \cup \{A\} \vdash_{\mathbf{L}} \bot$ and $\Phi \cup \{\neg A\} \vdash_{\mathbf{L}} \bot$. By Theorem 2.2.1, $\Phi \vdash_{\mathbf{L}} A \to \bot = \neg A$, and $\Phi \vdash_{\mathbf{L}} \neg A \to \bot$. Therefore, $\Phi \vdash_{\mathbf{L}} \bot$, against the consistency of Φ.

(b) If $\Psi \vdash_{\mathbf{L}} A$ and $\Psi \subseteq \Phi$, then $\Phi \vdash_{\mathbf{L}} A$. Thus, if $\neg A \in \Phi$, $\Phi \vdash_{\mathbf{L}} A \wedge \neg A$, then $\Phi \vdash_{\mathbf{L}} \bot$, againt the hypothesis. Therefore $\neg A \notin \Phi$, and by (a), $A \in \Phi$.

(c) Since $\mathbf{L}$ contains rule RE, $\vdash_{\mathbf{L}} \Box A \leftrightarrow \Box B$. Then by ($b$), $\Box A \leftrightarrow \Box B \in \Phi$. By contradiction, assume $\Box B \notin \Phi$. Then by (a), $\neg\Box B \in \Phi$. Thus $\Phi \vdash_{\mathbf{L}} \neg\Box A$, that since $\Box A \in \Phi$ implies $\Phi \vdash_{\mathbf{L}} \bot$, against the hypothesis.

(d) Let Ψ be any $\mathbf{L}$-maximal set. If $\Psi \in \uparrow(A \wedge B)$, i.e., $A \wedge B \in \Psi$, then $\Psi \vdash_{\mathbf{L}} A \wedge B$, thus $\Psi \vdash_{\mathbf{L}} A$ and $\Psi \vdash_{\mathbf{L}} B$. By ($b$), $A \in \Psi$ and $B \in \Psi$, that is $\Psi \in \uparrow A \cap \uparrow B$. The other direction is analogous. (e) is similar to (d).

(f) If $\uparrow A \subseteq \uparrow B$, then by Lemma 4.2.2, $\{A, \neg B\} \vdash_{\mathbf{L}} \bot$, otherwise there would be an $\mathbf{L}$-maximal set containing A and $\neg B$, that is $\uparrow A \not\subseteq \uparrow B$. Therefore $\vdash_{\mathbf{L}} A \wedge \neg B \to \bot$, which implies $\vdash_{\mathbf{L}} A \to B$. ($g$) is immediate from ($f$). $\Box$

In order to prove completeness of classical non-normal modal logics we consider the following definition of canonical model. In the definition of canonical models, the bi-neighbourhood function is defined in different ways for monotonic and non-monotonic systems. The need for this distinction is not a peculiarity of the bi-neighbourhood semantics, since an analogous situation arises in the completeness proof with respect to the standard semantics (cf. [29]). Canonical models are defined as follows.

Definition 4.2.2 (Canonical model)**.** Let $\mathbf{L}$ be a classical non-normal modal logic. The *canonical model* for $\mathbf{L}$ is the tuple $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{W}_{\mathbf{L}}, \mathcal{N}_{\mathbf{L}}, \mathcal{V}_{\mathbf{L}} \rangle$, where $\mathcal{W}_{\mathbf{L}} = Max_{\mathbf{L}}$; for every $p \in \mathcal{L}$, $\mathcal{V}_{\mathbf{L}}(p) = \{\Phi \in \mathcal{W}_{\mathbf{L}} \mid p \in \Phi\}$; and for every $\Phi \in \mathcal{W}_{\mathbf{L}}$,

$$\mathcal{N}_{\mathbf{L}}(\Phi) = \begin{cases} \{(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \mid \Box A \in \Phi\} & \text{if } \mathbf{L} \text{ is non-monotonic.} \\ \{(\uparrow A, \emptyset) \mid \Box A \in \Phi\} & \text{if } \mathbf{L} \text{ is monotonic.} \end{cases}$$

Lemma 4.2.4. Let $\mathbf{L}$ be a non-normal modal logic, and $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{W}_{\mathbf{L}}, \mathcal{N}_{\mathbf{L}}, \mathcal{V}_{\mathbf{L}} \rangle$ be the canonical model for $\mathbf{L}$. Then if $\mathbf{L}$ is non-monotonic we have $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$ if and only if $\Box A \in \Phi$, and if $\mathbf{L}$ is monotonic we have $(\uparrow A, \emptyset) \in \mathcal{N}_{\mathbf{L}}(\Phi)$ if and only if $\Box A \in \Phi$.

Proof. If $\mathbf{L}$ is non-monotonic: The right-to-left direction holds by definition of $\mathcal{N}_{\mathbf{L}}(\Phi)$. For the other direction: Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then there is $\Box B \in \Phi$ such that $\uparrow B = \uparrow A$. By Lemma 4.2.3 (g), $\vdash_{\mathbf{L}} B \leftrightarrow A$, and by Lemma 4.2.3 ($c$), $\Box A \in \Phi$. For monotonic systems the proof is analogous. $\Box$

Lemma 4.2.5 (Truth lemma)**.** Let $\mathbf{L}$ be a non-normal modal logic, and $\mathcal{M}_{\mathbf{L}}$ be the canonical model for $\mathbf{L}$. Then for every formula A of $\mathcal{L}$,

$$\mathcal{M}_{\mathbf{L}}, \Phi \Vdash A \quad \text{if and only if} \quad A \in \Phi.$$

Proof. By induction on A. For $A = p$ the claim holds by definition of $\mathcal{V}_{\mathbf{L}}$. For $A = \bot$, by definition $\Phi \nVdash \bot$, and by consistency of Φ, $\bot \notin \Phi$. For $A = \top$, $\Phi \Vdash \top$ and by maximality of Φ, $\top \in \Phi$. For $A = B \circ C$, with $\circ \in \{\wedge, \vee, \rightarrow\}$, the proof is immediate by applying the inductive hypothesis and the properties of maximal consistent sets. For $A = \Box B$, we distinguish between the monotonic and the non-monotonic case. If $\mathbf{L}$ is non-monotonic we have: If $\Box B \in \Phi$, then $(\uparrow B, \mathcal{W}_{\mathbf{L}} \setminus \uparrow B) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. By i.h., $\uparrow B = [\![B]\!]$, then $\Phi \Vdash \Box B$. If $\Phi \Vdash \Box B$, then there is $(\uparrow C, \mathcal{W}_{\mathbf{L}} \setminus \uparrow C) \in \mathcal{N}_{\mathbf{L}}(\Phi)$ such that $\uparrow C \subseteq [\![B]\!] \subseteq \uparrow C$ and $\Box C \in \Phi$. By i.h. $\uparrow C = \uparrow B$, and by Lemma 4.5.2, $\Box B \in \Phi$. If instead $\mathbf{L}$ is non-monotonic we have: If $\Box B \in \Phi$, then $(\uparrow B, \emptyset) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. By i.h. $\uparrow B = [\![B]\!]$, then $\Phi \Vdash \Box B$. If $\Phi \Vdash \Box B$, then there is $(\uparrow C, \emptyset) \in \mathcal{N}_{\mathbf{L}}(\Phi)$ such that $\uparrow C \subseteq [\![B]\!]$ and $\Box C \in \Phi$. By i.h. $\uparrow C \subseteq \uparrow B$, and by Lemma 4.5.2, $\vdash_{\mathbf{L}} C \rightarrow B$. Then by RM, $\vdash_{\mathbf{L}} \Box C \rightarrow \Box B$, thus $\Box C \rightarrow \Box B \in \Phi$, therefore $\Box B \in \Phi$. $\Box$

Lemma 4.2.6 (Model lemma)**.** Let $\mathbf{L}$ be a non-normal modal logic, $\mathcal{M}_{\mathbf{L}}$ be the canonical model for $\mathbf{L}$, and $X \in \{C, N, T, P, D, RD_n^+, 4\}$. If $\mathbf{L}$ contains X, then $\mathcal{M}_{\mathbf{L}}$ is a X-model.

Proof. We only consider non-monotonic systems, for monotonic systems we just need a slight simplification of this proof.

(C) Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A), (\uparrow B, \mathcal{W}_{\mathbf{L}} \setminus \uparrow B) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then by Lemma 4.5.2, $\Box A, \Box B \in \Phi$. By closure under derivation $\Box A \wedge \Box B \in \Phi$, and since Φ contains axiom C, $\Box(A \wedge B) \in \Phi$. Then $(\uparrow(A \wedge B), \mathcal{W}_{\mathbf{L}} \setminus \uparrow(A \wedge B)) = (\uparrow A \cap \uparrow B, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A \cup \mathcal{W}_{\mathbf{L}} \setminus \uparrow B) \in \mathcal{N}_{\mathbf{L}}(\Phi)$.

(N) $\Box \top \in \Phi$ for all $\Phi \in \mathcal{W}_{\mathbf{L}}$. Then $(\uparrow \top, \mathcal{W}_{\mathbf{L}} \setminus \uparrow \top) \in \mathcal{N}_{\mathbf{L}}(\Phi)$, where $\mathcal{W}_{\mathbf{L}} \setminus \uparrow \top = \emptyset$.

(T) Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then $\Box A \in \Phi$. By axiom T, $A \in \Phi$, that is $\Phi \in \uparrow A$.

(P) Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then $\Box A \in \Phi$. If $\uparrow A = \emptyset$, then $\uparrow A = \uparrow \bot$, which implies $\Box \bot \in \Phi$, against the fact that $\neg \Box \bot \in \Phi$ and Φ is $\mathbf{L}$-consistent. Therefore $\uparrow A \neq \emptyset$.

(D) Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A), (\uparrow B, \mathcal{W}_{\mathbf{L}} \setminus \uparrow B) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then $\Box A, \Box B \in \Phi$. By axiom D and the consistency of Φ it follows $\nvdash_{\mathbf{L}} B \leftrightarrow \neg A$. Then $\uparrow B \not\subseteq \uparrow \neg A$, or $\uparrow \neg A \not\subseteq \uparrow B$, that is $\uparrow A \cap \uparrow B \neq \emptyset$ or $\mathcal{W}_{\mathbf{L}} \setminus \uparrow A \cap \mathcal{W}_{\mathbf{L}} \setminus \uparrow B \neq \emptyset$.

(RD_n^+) Assume $(\uparrow A_1, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A_1), ..., (\uparrow A_n, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A_n) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then $\Box A_1, ..., \Box A_n \in \Phi$. By RD_n^+ and the consistency of Φ it follows $\nvdash_{\mathbf{L}} \neg(A_1 \wedge ... \wedge A_n)$. By Lemma 4.2.2 (b, d), there is $\Psi \in \mathcal{W}_{\mathbf{L}}$ such that $A_1 \wedge ... \wedge A_n \in \Psi$. Then $\Psi \in \uparrow(A_1 \wedge ... \wedge A_n)$, which implies $\uparrow A_1 \cap ... \cap \uparrow A_n \neq \emptyset$.

(4) Assume $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. Then $\Box A \in \Phi$, and since Φ contains axiom 4, $\Box\Box A \in \Phi$. Thus $(\uparrow \Box A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow \Box A) \in \mathcal{N}_{\mathbf{L}}(\Phi)$. We show that (i) $\uparrow \Box A = \mathcal{V}^+(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A)$ and (ii) $\mathcal{W}_{\mathbf{L}} \setminus \uparrow \Box A = \mathcal{V}^-(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A)$, whence $\mathcal{M}_{\mathbf{L}}$ is a 4-model. (i) $\Psi \in \uparrow \Box A$ iff $\Box A \in \Psi$ iff $(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{L}}(\Psi)$ iff $\Psi \in \mathcal{V}^+(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A)$. (ii) $\Psi \in \mathcal{W}_{\mathbf{L}} \setminus \uparrow \Box A$ iff $\Box A \notin \Psi$ iff

$(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \notin \mathcal{N}_{\mathbf{L}}(\Psi)$ iff for all $(\uparrow B, \mathcal{W}_{\mathbf{L}} \setminus \uparrow B) \in \mathcal{N}_{\mathbf{L}}(\Psi)$, $\uparrow B \neq \uparrow A$, that is $\uparrow B \cap \mathcal{W}_{\mathbf{L}} \setminus \uparrow A \neq \emptyset$ or $\uparrow A \cap \mathcal{W}_{\mathbf{L}} \setminus \uparrow B \neq \emptyset$, iff $\Psi \in \mathcal{W}^{-}(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A)$. $\square$

Theorem 4.2.7 (Completeness)**.** Every classical non-normal modal logic is strongly complete with respect to the corresponding class of bi-neighbourhood models, that is: If $\Phi \models_{\mathcal{C}^{bi}_{\mathbf{L}}} A$, then $\Phi \vdash_{\mathbf{L}} A$.

Proof. Assume $\Phi \not\vdash_{\mathbf{L}} A$. Then by Lemma 4.2.2, there is an $\mathbf{L}$-maximal consistent set Ψ such that $\Phi \cup \{\neg A\} \subseteq \Psi$. By definition, Ψ is a world in the canonical model $\mathcal{M}_{\mathbf{L}}$ for $\mathbf{L}$, and by Lemma 4.2.6, $\mathcal{M}_{\mathbf{L}} \in \mathcal{C}^{bi}_{\mathbf{L}}$. Finally, by Lemma 4.2.5, $\mathcal{M}_{\mathbf{L}}, \Psi \Vdash B$ for all $B \in \Phi$, and $\mathcal{M}_{\mathbf{L}}, \Psi \nVdash A$. Therefore $\Phi \not\models_{\mathcal{C}^{bi}_{\mathbf{L}}} A$. $\square$

4.3 Relations with the standard semantics

According to the results presented in the previous section, classical non-normal modal logics are characterised by their bi-neighbourhood models. Moreover, as recalled in Chapter 2.3, they are also characterised by their standard models. We can conclude that the two semantics are equivalent:

Theorem 4.3.1. A formula is valid in a class of bi-neighbourhood models if and only if it is valid in the corresponding class of standard models.

However, it is also worth showing this equivalence directly by model transformations. These transformations have also a practical interest: in the next chapters, given a failed proof in a sequent calculus for a classical non-normal modal logic, we shall extract a countermodel of the non-derivable formula in the bi-neighbourhood semantics. Then, by applying the transformations below to the extracted bi-neighbourhood countermodel we can also obtain an equivalent countermodel in the standard semantics.

From standard to bi-neighbourhood models

Given a standard model, an equivalent bi-neighbourhood model can be obtained as follows.

Proposition 4.3.2. Let $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be a standard model, and $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be the bi-neighbourhood model defined by taking the same $\mathcal{W}$ and $\mathcal{V}$ and, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{bi}(w) = \begin{cases} \{(\alpha, \mathcal{W} \setminus \alpha) \mid \alpha \in \mathcal{N}_{st}(w)\} & \text{if } \mathcal{M}_{st} \text{ is not supplemented.} \\ \{(\alpha, \emptyset) \mid \alpha \in \mathcal{N}_{st}(w)\} & \text{if } \mathcal{M}_{st} \text{ is supplemented.} \end{cases}$$

Then, for every formula A of $\mathcal{L}$ and every $w \in \mathcal{W}$, $\mathcal{M}_{bi}, w \Vdash A$ if and only if $\mathcal{M}_{st}, w \Vdash A$. Moreover, for every $X \in \{M,\ C,\ N,\ T,\ P,\ D,\ RD^{+}_{n},\ 4\}$, if $\mathcal{M}_{st}$ satisfies the condition corresponding to X in the standard semantics, then $\mathcal{M}_{bi}$ is a bi-neighbourhood X-model.

Proof. The equivalence is proved by induction on A. The basic cases $A = p, \bot, \top$ are trivial since the evaluation $\mathcal{V}$ is the same in the two models, and the inductive cases of boolean connectives are straightforward by applying the induction hypothesis. We consider the case $A = \Box B$. If $\mathcal{M}_{st}$ is not supplemented we have: $\mathcal{M}_{bi}, w \Vdash \Box B$ iff $(\llbracket B \rrbracket_{bi}, \mathcal{W} \setminus \llbracket B \rrbracket_{bi}) \in \mathcal{N}_{bi}(w)$ iff $\llbracket B \rrbracket_{bi} \in \mathcal{N}_{st}(w)$ iff (i.h.) $\llbracket B \rrbracket_{st} \in \mathcal{N}_{st}(w)$ iff $\mathcal{M}_{st}, w \Vdash \Box B$. If $\mathcal{M}_{st}$ is supplemented we have: $\mathcal{M}_{bi}, w \Vdash \Box B$ iff there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \llbracket B \rrbracket_{bi}$ iff $\alpha \in \mathcal{N}_{st}(w)$ and (i.h.) $\alpha \subseteq \llbracket B \rrbracket_{st}$ iff (by supplementation) $\llbracket B \rrbracket_{st} \in \mathcal{N}_{st}(w)$ iff $\mathcal{M}_{st}, w \Vdash \Box B$.

Now we show that $\mathcal{N}_{bi}$ satisfies the right properties. For axiom M the proof is immediate by definition of $\mathcal{N}_{bi}$. For the following conditions we just consider the non-supplemented case, the supplemented case is an easy simplification.

(N) $(\mathcal{W}, \emptyset) \in \mathcal{N}_{bi}(w)$ because $\mathcal{W} \in \mathcal{N}_{st}(w)$.

(C) If $(\alpha, \mathcal{W} \setminus \alpha), (\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$, then $\alpha, \beta \in \mathcal{N}_{st}(w)$, that implies $\alpha \cap \beta \in \mathcal{N}_{st}(w)$. Thus $(\alpha \cap \beta, \mathcal{W} \setminus (\alpha \cap \beta)) = (\alpha \cap \beta, \mathcal{W} \setminus \alpha \cup \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$.

(T) If $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$, then $\alpha \in \mathcal{N}_{st}(w)$, thus $w \in \alpha$.

(P) If $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$, then $\alpha \in \mathcal{N}_{st}(w)$, thus $\alpha \neq \emptyset$.

(D) If $(\alpha, \mathcal{W} \setminus \alpha), (\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(w)$, then $\alpha, \beta \in \mathcal{N}_{st}(w)$. Thus $\beta \neq \mathcal{W} \setminus \alpha$, that implies $\alpha \cap \beta \neq \emptyset$ or $\mathcal{W} \setminus \alpha \cap \mathcal{W} \setminus \beta \neq \emptyset$.

(RD_n^+) If $(\alpha_1, \mathcal{W} \setminus \alpha_1), ..., (\alpha_n, \mathcal{W} \setminus \alpha_n) \in \mathcal{N}_{bi}(w)$, then $\alpha_1, ..., \alpha_n \in \mathcal{N}_{st}(w)$, thus $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.

(4) If $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$, then $\alpha \in \mathcal{N}_{st}(w)$, therefore $\{v \mid \alpha \in \mathcal{N}_{st}(v)\} \in \mathcal{N}_{st}(w)$. Thus by definition $(\{v \mid \alpha \in \mathcal{N}_{st}(v)\}, \{v \mid \alpha \notin \mathcal{N}_{st}(v)\}) \in \mathcal{N}_{bi}(w)$. In addition, $\{v \mid \alpha \in \mathcal{N}_{st}(v)\} = \mathcal{Y}^+_{bi}(\alpha, \mathcal{W} \setminus \alpha)$ – since $\alpha \in \mathcal{N}_{st}(v)$ iff $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(v)$. Moreover, $\{v \mid \alpha \notin \mathcal{N}_{st}(v)\} = \mathcal{Y}^-_{bi}(\alpha, \mathcal{W} \setminus \alpha)$ – since $\alpha \notin \mathcal{N}_{st}(v)$ iff for all $\beta \in \mathcal{N}_{st}(v)$, $\alpha \neq \beta$ iff for all $(\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(v)$, $\alpha \neq \beta$ iff for all $(\beta, \mathcal{W} \setminus \beta) \in \mathcal{N}_{bi}(v)$, $\alpha \cap (\mathcal{W} \setminus \beta) \neq \emptyset$ or $(\mathcal{W} \setminus \alpha) \cap \beta \neq \emptyset$ iff $v \in \mathcal{Y}^-_{bi}(\alpha, \mathcal{W} \setminus \alpha)$. Thus $\mathcal{M}_{bi}$ is a 4-model. $\square$

From bi-neighbourhood to standard neighbourhood models

Fo the opposite direction we propose two transformations: a more general one, and a "finer" one which is relativised with respect to a set of formulas. The general transformation is as follows.

Proposition 4.3.3. Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood model, and $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be the standard model defined by taking the same $\mathcal{W}$ and $\mathcal{V}$ and, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{st}(w) = \{\gamma \subseteq \mathcal{W} \mid \text{there is } (\alpha, \beta) \in \mathcal{N}_{bi}(w) \text{ such that } \alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta\}.$$

Then, for every formula A of $\mathcal{L}$ and every $w \in \mathcal{W}$, $\mathcal{M}_{st}, w \Vdash A$ if and only if $\mathcal{M}_{bi}, w \Vdash A$. Moreover, for every $X \in \{M, C, N, T, P, D, RD_n^+, 4\}$, if $\mathcal{M}_{bi}$ is a bi-neighbourhood X-model, then $\mathcal{M}_{st}$ satisfies the condition corresponding to X in the standard semantics.

Proof. The proof of equivalence of the two models is by induction on A. As before, we only consider the inductive step where $A \equiv \Box B$. We have: $\mathcal{M}_{st}, w \Vdash \Box B$ iff $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff (i.h.) $[\![B]\!]_{bi} \in \mathcal{N}_{st}(w)$ iff there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq [\![B]\!]_{bi} \subseteq \mathcal{W} \backslash \beta$ iff $\mathcal{M}_{bi}, w \Vdash \Box B$.

Now we prove that $\mathcal{M}_{st}$ satisfies the right properties.

(M) Let $\mathcal{M}_{bi}$ be a M-model, and assume $\gamma \in \mathcal{N}_{st}(w)$ and $\gamma \subseteq \delta$. Then there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \emptyset$. Thus $\alpha \subseteq \delta \subseteq \mathcal{W} \setminus \emptyset$, which implies $\delta \in \mathcal{N}_{st}(w)$.

(N) Let $\mathcal{M}_{bi}$ be a N-model. Then there is $(\alpha, \emptyset) \in \mathcal{N}_{bi}(w)$. Since $\alpha \subseteq \mathcal{W} \subseteq \mathcal{W} \setminus \emptyset$, by definition $\mathcal{W} \in \mathcal{N}_{st}(w)$.

(C) Let $\mathcal{M}_{bi}$ be a C-model, and assume $\gamma, \delta \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}_{bi}(w)$ such that $\alpha_1 \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_1$, $\alpha_2 \subseteq \delta \subseteq \mathcal{W} \setminus \beta_2$. By condition (C), $(\alpha_1 \cap \alpha_2, \beta_1 \cup \beta_2) \in \mathcal{N}_{bi}(w)$, where $\alpha_1 \cap \alpha_2 \subseteq \gamma \cap \delta$, and $\gamma \cap \delta \subseteq \mathcal{W} \backslash \beta_1 \cap \mathcal{W} \backslash \beta_2 = \mathcal{W} \backslash \beta_1 \cup \beta_2$. Then $\gamma \cap \delta \in \mathcal{N}_{st}(w)$.

(T) Let $\mathcal{M}_{bi}$ be a T-model, and assume $\gamma \in \mathcal{N}_{st}(w)$. Then there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta$. By condition (T), $w \in \alpha$, then $w \in \gamma$.

(P) Let $\mathcal{M}_{bi}$ be a P-model, and assume by contradiction that $\emptyset \in \mathcal{N}_{st}(w)$. Then there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \emptyset \subseteq \mathcal{W} \setminus \beta$. Thus $\alpha = \emptyset$, against condition (P).

(D) Let $\mathcal{M}_{bi}$ be a D-model, and assume by contradiction that $\gamma, \mathcal{W} \setminus \gamma \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}_{bi}(w)$ such that $\alpha_1 \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_1$, $\alpha_2 \subseteq \mathcal{W} \setminus \gamma \subseteq \mathcal{W} \setminus \beta_2$. Then $\alpha_1 \cap \alpha_2 = \emptyset$ and $\beta_1 \cap \beta_2 = \emptyset$, against condition (D).

(RD_n^+) Let $\mathcal{M}_{bi}$ be a RD_n^+-model, and assume $\gamma_1, ..., \gamma_n \in \mathcal{N}_{st}(w)$. Then there are $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n) \in \mathcal{N}_{bi}(w)$ such that $\alpha_i \subseteq \gamma \subseteq \mathcal{W} \setminus \beta_i$ for all $1 \leq i \leq n$. By condition (RD_n^+), $\alpha_1 \cap ... \cap \alpha_n \neq \emptyset$. Then $\gamma_1 \cap ... \cap \gamma_n \neq \emptyset$.

(4) Let $\mathcal{M}_{bi}$ be a 4-model, and assume $\gamma \in \mathcal{N}_{st}(w)$. Then there is $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$ such that $\alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta$. By condition (4), there is $(\gamma, \delta) \in \mathcal{N}_{bi}(w)$ such that $\gamma \subseteq \mathcal{V}^+(\alpha, \beta)$ and $\delta \subseteq \mathcal{V}^-(\alpha, \beta)$. We show that (i) $\mathcal{V}^+(\alpha, \beta) \subseteq \{v \mid \gamma \in \mathcal{N}_{st}(v)\}$ and (ii) $\{v \mid \gamma \in \mathcal{N}_{st}(v)\} \subseteq \mathcal{W} \setminus \mathcal{V}^-(\alpha, \beta)$, which imply $\gamma \subseteq \{v \mid \gamma \in \mathcal{N}_{st}(v)\} \subseteq \mathcal{W} \setminus \delta$, thus $\{v \mid \gamma \in \mathcal{N}_{st}(v)\} \in \mathcal{N}_{st}(w)$. (i) If $v \in \mathcal{V}^+(\alpha, \beta)$, then $(\alpha, \beta) \in \mathcal{N}_{bi}(v)$, thus $\gamma \in \mathcal{N}_{st}(v)$. (ii) If $\gamma \in \mathcal{N}_{st}(v)$, then there is $(\alpha', \beta') \in \mathcal{N}_{bi}(v)$ such that $\alpha' \subseteq \gamma \subseteq \mathcal{W} \setminus \beta'$. Then $\alpha' \cap \beta = \emptyset$ and $\beta' \cap \alpha = \emptyset$. Therefore $v \notin \mathcal{V}^-(\alpha, \beta)$. $\Box$

Observe that, on the basis of the above transformation, elements of bi-neighbourhood pairs can be also seen as *lower and upper bounds* of neighbourhoods of standard models. For the non-monotonic case, a finer transformation can be also given by considering a set $\mathcal{S}$ of formulas which is closed under subformulas.

Proposition 4.3.4. Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood model and $\mathcal{S}$ be a set of formulas of $\mathcal{L}$ closed under subformulas. We define the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ with the same $\mathcal{W}$ and $\mathcal{V}$ and by taking, for all $w \in \mathcal{W}$,

$$\mathcal{N}_{st}(w) = \{[\![C]\!]_{bi} \mid \Box C \in \mathcal{S} \text{ and } \mathcal{M}_{bi}, w \Vdash \Box C\}.$$

Then for every formula $A \in \mathcal{S}$ and every world $w \in \mathcal{W}$, $\mathcal{M}_{st}, w \Vdash A$ if and only if $\mathcal{M}_{bi}, w \Vdash A$. Moreover, (N) if $\Box\top \in \mathcal{S}$ and $\mathcal{M}_{bi}$ is a N-model, then $\mathcal{M}_{st}$ contains the unit; (C) if $\Box A, \Box B \in \mathcal{S}$ implies $\Box(A \wedge B) \in \mathcal{S}$ and $\mathcal{M}_{bi}$ is a C-model, then $\mathcal{M}_{st}$ is closed under intersection; (T/P/D/RD$_n^+$) If $\mathcal{M}_{bi}$ is a T/P/D/RD$_n^+$-model, then $\mathcal{M}_{st}$ satisfies the corresponding condition in the standard semantics; (4$'$) if $\Box A \in \mathcal{S}$ implies $\Box\Box A \in \mathcal{S}$ and $\mathcal{M}_{bi}$ is a 4$'$-model, then $\mathcal{M}_{st}$ satisfies the condition corresponding to axiom 4 in the standard semantics

Proof. The equivalence is proved by induction on A. The basic cases are immediate. If $A \equiv B \circ C$, where $\circ \in \{\wedge, \vee, \rightarrow\}$, the claims holds by applying the inductive hypothesis since $B, C \in \mathcal{S}$ because $\mathcal{S}$ is closed under subformulas. If $A \equiv \Box B$, then $B \in \mathcal{S}$ and, by i.h., $[\![B]\!]_{st} = [\![B]\!]_{bi}$. Thus $\mathcal{M}_{st}, w \Vdash \Box B$ iff $[\![B]\!]_{st} \in \mathcal{N}_{st}(w)$ iff $[\![B]\!]_{bi} \in \mathcal{N}_{st}(w)$ iff there is $\Box C \in \mathcal{S}$ such that $[\![C]\!]_{bi} = [\![B]\!]_{bi}$ and $\mathcal{M}_{st}, w \Vdash \Box C$ iff $\mathcal{M}_{st}, w \Vdash \Box B$.

(N) Let $\mathcal{M}_{bi}$ be a N-model. Then $\mathcal{M}_{bi}, w \Vdash \Box\top$. Since $\Box\top \in \mathcal{S}$, by definition $[\![\top]\!]_{bi} = \mathcal{W} \in \mathcal{N}_{st}(w)$.

(C) Assume $\alpha, \beta \in \mathcal{N}_{st}(w)$. Then there are $\Box A, \Box B \in \mathcal{S}$ such that $\alpha = [\![A]\!]_{bi}$, $\beta = [\![B]\!]_{bi}$, and $\mathcal{M}_{bi}, w \Vdash \Box A$, $\mathcal{M}_{bi}, w \Vdash \Box B$, that is $\mathcal{M}_{bi}, w \Vdash \Box A \wedge \Box B$. Since $\mathcal{M}_{bi}$ is a C-model we have $\mathcal{M}_{bi}, w \Vdash \Box(A \wedge B)$. By the properties of $\mathcal{S}$, $\Box(A \wedge B) \in \mathcal{S}$. Then by definition $[\![A \wedge B]\!]_{bi} \in \mathcal{N}_{st}(w)$, where $[\![A \wedge B]\!]_{bi} = [\![A]\!]_{bi} \cap [\![B]\!]_{bi} = \alpha \cap \beta$.

(T) Assume $\alpha \in \mathcal{N}_{st}(w)$. Then $\alpha = [\![A]\!]_{bi}$ for some A such that $\Box A \in \mathcal{S}$ and $\mathcal{M}_{bi}, w \Vdash \Box A$. Since $\mathcal{M}_{bi}$ is a T-model, $\mathcal{M}_{bi}, w \Vdash A$, that is $w \in [\![A]\!]_{bi} = \alpha$.

(P) Assume by contradiction that $\emptyset \in \mathcal{N}_{st}(w)$. Then there is $\Box A \in \mathcal{S}$ such that $\mathcal{M}_{bi}, w \Vdash \Box A$ and $[\![A]\!]_{bi} = \emptyset = [\![\bot]\!]_{bi}$. Thus $\mathcal{M}_{bi}, w \Vdash \Box\bot$, against the soundness of axiom P with respect to P-models.

(D) Assume $\alpha, \mathcal{W} \setminus \alpha \in \mathcal{N}_{st}(w)$. Then there are $\Box A, \Box B \in \mathcal{S}$ such that $\alpha = [\![A]\!]_{bi}$, $\mathcal{W} \setminus \alpha = [\![B]\!]_{bi}$, and $\mathcal{M}_{bi}, w \Vdash \Box A$, $\mathcal{M}_{bi}, w \Vdash \Box B$. Then $[\![A]\!]_{bi} = \mathcal{W} \setminus [\![B]\!]_{bi} = [\![\neg B]\!]_{bi}$, that is $\mathcal{M}_{bi}, w \Vdash \Box\neg B$, against the soundness of axiom D with respect to D-models.

(RD$_n^+$) Assume $\alpha_1, ..., \alpha_n \in \mathcal{N}_{st}(w)$. Then there are $\Box A_1, ..., \Box A_n \in \mathcal{S}$ such that $\alpha_i = [\![A_i]\!]_{bi}$ and $\mathcal{M}_{bi}, w \Vdash \Box A_i$ for every $1 \leq i \leq n$, that is $\mathcal{M}_{bi}, w \Vdash \Box A_1 \wedge ... \wedge \Box A_n$. Then $\mathcal{M}_{bi} \not\models \neg(\Box A_1 \wedge ... \wedge \Box A_n)$, and since $\mathcal{M}_{bi}$ is a RD$_n^+$-model, $\mathcal{M}_{bi} \not\models \neg(A_1 \wedge ... \wedge A_n)$, that is $[\![A_1]\!]_{bi} \cap ... \cap [\![A_n]\!]_{bi} = \alpha_1 \cap ... \cap \alpha_n \neq \emptyset$.

(4) Assume $\alpha \in \mathcal{N}_{st}(w)$. Then $\alpha = [\![A]\!]_{bi}$ for some A such that $\Box A \in \mathcal{S}$ and $\mathcal{M}_{bi}, w \Vdash \Box A$. Since $\mathcal{M}_{bi}$ is a 4-model, $\mathcal{M}_{bi}, w \Vdash \Box\Box A$. In addition, $\Box\Box A \in \mathcal{S}$. Thus $[\![\Box A]\!]_{bi} \in \mathcal{N}_{st}(w)$, where $[\![\Box A]\!]_{bi} = [\![\Box A]\!]_{st} = \{v \mid [\![A]\!]_{st} \in \mathcal{N}_{st}(v)\} = \{v \mid [\![A]\!]_{bi} \in \mathcal{N}_{st}(v)\} = \{v \mid \alpha \in \mathcal{N}_{st}(v)\}$. $\Box$

The transformation in Proposition 4.3.4 is a proper refinement of the one in Proposition 4.3.3. Indeed, the definition of $\mathcal{N}_{st}(w)$ in the latter transformation could be equivalently rewritten as

$$\mathcal{N}_{st}(w) = \{\gamma \subseteq \mathcal{W} \mid \text{there is } (\alpha, \beta) \in \mathcal{N}_{bi}(w) \text{ such that } \alpha \subseteq \gamma \subseteq \mathcal{W} \setminus \beta \text{ and } \gamma = [\![C]\!]_{bi} \text{ for some } \Box C \in \mathcal{S}\}.$$

That is, instead of adding to $\mathcal{N}_{st}(w)$ every set γ lying between the elements of a pair $(\alpha, \beta) \in \mathcal{N}_{bi}(w)$, as it is done in the first transformation, we only add those sets that coincide with some relevant truth sets of formulas, thus obtaining smaller models from the point of view of the neighbourhood function. For the monotonic case, an analogous result could be obtained by considering, for every $w \in \mathcal{W}$, the *supplementation* of $\mathcal{N}_{st}(w)$ in Proposition 4.3.4, i.e., $\mathcal{N}'_{st}(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \Box C \in \mathcal{S} \text{ such that } \mathcal{M}_{bi}, w \Vdash \Box C \text{ and } [\![C]\!]_{bi} \subseteq \alpha\}$. However in this case the advantage of the finer transformation is not as relevant as is the non-monotonic case.

From bi-neighbourhood to relational models

As recalled in Section 2.3, regular logics, i.e., the logics containing both axioms M and C, have also a relational semantics (cf. Definition 2.3.7). We conclude this section by presenting transformations of *finite* bi-neighbourhood MC-models into equivalent relational ones.

Proposition 4.3.5. Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a *finite* MC-model, and let $\mathcal{N}^1(w)$ denote the set $\{\alpha \mid (\alpha, \emptyset) \in \mathcal{N}(w)\}$. We define the relational model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$ with the same $\mathcal{W}$ and $\mathcal{V}$ and by taking $\mathcal{W}^i = \{w \in \mathcal{W} \mid \mathcal{N}^1(w) = \emptyset\}$, and for all $w \in \mathcal{W}$, $\mathcal{R}(w) = \bigcap \mathcal{N}^1(w)$. Then for every formula A of $\mathcal{L}$ and every $w \in \mathcal{W}$, $\mathcal{M}_{bi}, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash A$. Moreover, if $\mathcal{M}_{bi}$ is a N/P/T/4-model, then $\mathcal{M}_r$ is a relational model for **MCN/P/T/4**.

Proof. The equivalence is proved by induction on A. The basic cases are straightforward since the two models share the same evaluation of atomic variables, and the inductive cases of boolean connectives are easy by applying the inductive hypothesis. We just consider the case $A = \Box B$: $\mathcal{M}_{bi}, w \Vdash \Box B$ iff there is $\alpha \in \mathcal{N}^1(w)$ s.t. $\alpha \subseteq [\![B]\!]_{bi}$ iff $\bigcap \mathcal{N}^1(w) \subseteq [\![B]\!]_{bi}$ and (since $\mathcal{M}$ is a finite MC-model) $\bigcap \mathcal{N}^1(w) \in \mathcal{N}^1(w)$ iff $w \notin \mathcal{W}^i$ and (by i.h.) $\mathcal{R}(w) \subseteq [\![B]\!]_r$ iff $\mathcal{M}_r, w \Vdash \Box B$.

The model conditions are proved as follows.

(Normality) If $\mathcal{M}_{bi}$ is a N-model, then for every $w \in \mathcal{W}$ there is $(\alpha, \emptyset) \in \mathcal{N}(w)$, that is $\mathcal{N}^1(w) \neq \emptyset$. Then $\mathcal{W}^i = \emptyset$.

(Seriality) Assume $w \in \mathcal{W} \setminus \mathcal{W}^i$. Then by condition (C), $(\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(w)$, thus by condition (P), $\bigcap \mathcal{N}^1(w) \neq \emptyset$. Then $\mathcal{R}(w) \neq \emptyset$.

(Reflexivity) Assume $w \in \mathcal{W} \setminus \mathcal{W}^i$. Then by condition (C), $(\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(w)$, thus by condition (T), $w \in \bigcap \mathcal{N}^1(w)$. Then $w \in \mathcal{R}(w)$, that is $w\mathcal{R}w$.

(Transitivity) Assume $w\mathcal{R}v$ and $v\mathcal{R}z$, where $w, v, z \in \mathcal{W} \setminus \mathcal{W}^i$. Then $v \in \bigcap \mathcal{N}^1(w)$ and $z \in \bigcap \mathcal{N}^1(v)$. By condition (C), $(\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(w)$, then by condition (4), there is $(\gamma, \emptyset) \in \mathcal{N}(w)$ such that $\gamma \subseteq \mathcal{V}^+(\bigcap \mathcal{N}^1(w), \emptyset) = \{y \mid (\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(y)\}$. So $\bigcap \mathcal{N}^1(w) \subseteq \gamma$, then $v \in \{y \mid (\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(y)\}$, that is $(\bigcap \mathcal{N}^1(w), \emptyset) \in \mathcal{N}(v)$. This implies $\bigcap \mathcal{N}^1(v) \subseteq \bigcap \mathcal{N}^1(w)$, thus $z \in \bigcap \mathcal{N}^1(w)$, therefore $w\mathcal{R}z$. $\square$

4.4 Alternative semantic conditions

For some of the considered modal axioms, it is possible to find alternative conditions that equally provide a characterisation of the axioms in the bi-neighbourhood semantics. We present in this section some examples of possible alternative conditions.

Proposition 4.4.1. The conditions (M) and (M′) below are equivalent, that is, a formula A of $\mathcal{L}$ is valid in the class of all M-models if and only if it is valid in the class of all M′-models:

(M) If $(\alpha, \beta) \in \mathcal{N}(w)$, then $\beta = \emptyset$.
(M′) If $(\alpha, \beta) \in \mathcal{N}(w)$, $\alpha \subseteq \gamma$ and $\delta \subseteq \beta$, then $(\gamma, \delta) \in \mathcal{N}(w)$.

Proof. We show that given a M-model we can define an equivalent M′-model, and vice versa. The considered transformations preserve conditions (N), (C), (T), (P), (D), (RD_n^+), and (4). First, let $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a M-model. We define the model $\mathcal{M}' = \langle \mathcal{W}, \mathcal{N}', \mathcal{V} \rangle$ by taking the same $\mathcal{W}$ and $\mathcal{V}$, and, for all $w \in \mathcal{W}$, $\mathcal{N}'(w) = \{(\alpha, \emptyset) \mid \text{there is } (\beta, \emptyset) \in \mathcal{N}(w) \text{ s.t. } \beta \subseteq \alpha\}$. It is easy to see that $\mathcal{M}'$ is a M′-model. We show by induction on A that $\mathcal{M}, w \Vdash A$ if and only if $\mathcal{M}', w \Vdash A$. We only consider the inductive case $A = \Box B$ as the other cases are immediate. We have: $\mathcal{M}', w \Vdash \Box B$ iff there is $(\alpha, \emptyset) \in \mathcal{N}'(w)$ such that $\alpha \subseteq [\![B]\!]_{\mathcal{M}'}$ iff there is $(\beta, \emptyset) \in \mathcal{N}(w)$ such that $\beta \subseteq \alpha$ iff, by i.h., $\beta \subseteq [\![B]\!]_{\mathcal{M}}$ iff $\mathcal{M}, w \Vdash \Box B$. Moreover, it is easy to see that the model conditions are preserved. For instance, assume $\mathcal{M}$ is a P-model and $(\alpha, \emptyset) \in \mathcal{N}'(w)$. Then there is $(\beta, \emptyset) \in \mathcal{N}(w)$ such that $\beta \subseteq \alpha$. By condition (P), $\beta \neq \emptyset$. Then $\alpha \neq \emptyset$.

For the opposite direction, let $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a M′-model. We define the model $\mathcal{M}^* = \langle \mathcal{W}, \mathcal{N}^*, \mathcal{V} \rangle$ by taking the same $\mathcal{W}$ and $\mathcal{V}$, and, for all $w \in \mathcal{W}$, $\mathcal{N}^*(w) = \{(\alpha, \emptyset) \mid \text{there is } \beta \subseteq \mathcal{W} \text{ s.t. } (\alpha, \beta) \in \mathcal{N}(w)\}$. Then $\mathcal{M}^*$ is a M-model. We show that $\mathcal{M}^*, w \Vdash A$ if and only if $\mathcal{M}, w \Vdash A$ for all $A \in \mathcal{L}$. We consider only the case $A = \Box B$. We have: $\mathcal{M}^*, w \Vdash \Box B$ iff there is $(\alpha, \emptyset) \in \mathcal{N}^*(w)$ such that $\alpha \subseteq [\![B]\!]_{\mathcal{M}^*}$ iff there is β such that

$(\alpha,\beta) \in \mathcal{N}(w)$ iff, by condition (M′), $(\alpha,\emptyset) \in \mathcal{N}(w)$, and by i.h., $\alpha \subseteq [\![B]\!]_{\mathcal{M}}$ iff $\mathcal{M}, w \Vdash \Box B$. As before, it is easy to show that the transformation preserves the model conditions. □

As remarked in Section 4.1, the condition (M) reduces the bi-neighbourhood semantics to the ∃∀-semantics for monotonic logics (cf. Definition 2.3.6). By contrast, M′-models can be put in correspondence with supplemented standard models (cf Section 2.3), in the sense that supplemented standard models can be seen as the particular cases of M′-models in which all bi-neighbourhood pairs are complementary.

A further example of alternative conditions are the following ones for axiom 4.

Proposition 4.4.2. The conditions (4) and (4′) below are equivalent in non-monotonic models (i.e., models not satisfying condition (M)), that is, a formula A of $\mathcal{L}$ is valid in the class of all non-monotonic 4-models if and only if it is valid in the class of all non-monotonic 4′-models. Moreover, the conditions (4) and (4m) below are equivalent in M-models:

(4)	If $(\alpha,\beta) \in \mathcal{N}(w)$, then there is $(\gamma,\delta) \in \mathcal{N}(w)$ s.t. $\gamma \subseteq \mathcal{W}^+(\alpha,\beta)$ and $\delta \subseteq \mathcal{W}^-(\alpha,\beta)$.
(4′)	If $(\alpha,\beta) \in \mathcal{N}(w)$, then $(\mathcal{W}^+(\alpha,\beta), \mathcal{W}^-(\alpha,\beta)) \in \mathcal{N}(w)$.
(4m)	If $(\alpha,\beta) \in \mathcal{N}(w)$, then $(\mathcal{W}^+(\alpha,\beta), \emptyset) \in \mathcal{N}(w)$.

Proof. Let us consider the conditions (4) and (4′) in non-monotonic models. We show that given satisfying one condition, it is possible to define an equivalent model satisfying the other condition. First, every 4′-model is also a 4-model, so there is nothing to do. For the other direction, let $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a 4-model. We define the model $\mathcal{M}' = \langle \mathcal{W}, \mathcal{N}', \mathcal{V} \rangle$ by taking the same $\mathcal{W}$ and $\mathcal{V}$, and, for every $w \in \mathcal{W}$, $\mathcal{N}'(w) = \{(\alpha,\beta) \mid$ there is $(\gamma,\delta) \in \mathcal{N}(w)$ such that $\gamma \subseteq \alpha$ and $\delta \subseteq \delta\}$. Then $\mathcal{M}'$ is a 4′-model: if $(\alpha,\beta) \in \mathcal{N}'(w)$, then there is $(\gamma,\delta) \in \mathcal{N}(w)$ such that $\gamma \subseteq \alpha$ and $\delta \subseteq \beta$. By (4), there is $(\epsilon,\zeta) \in \mathcal{N}(w)$ such that $\epsilon \subseteq \mathcal{W}^+(\gamma,\delta)$ and $\zeta \subseteq \mathcal{W}^-(\gamma,\delta)$. It is easy to verify that $\mathcal{W}^+(\gamma,\delta) \subseteq \mathcal{W}^{+\prime}(\alpha,\beta)$, and $\mathcal{W}^-(\gamma,\delta) \subseteq \mathcal{W}^{-\prime}(\alpha,\beta)$. Thus $\epsilon \subseteq \mathcal{W}^{+\prime}(\alpha,\beta)$, and $\zeta \subseteq \mathcal{W}^{-\prime}(\alpha,\beta)$, which imply $(\mathcal{W}^{+\prime}(\alpha,\beta), \mathcal{W}^{-\prime}(\alpha,\beta)) \in \mathcal{N}(w)$. We now show that, for every $A \in \mathcal{L}$, $\mathcal{M}', w \Vdash A$ if and only if $\mathcal{M}, w \Vdash A$. As usual, we only consider the inductive case $A = \Box B$. If $\mathcal{M}', w \Vdash \Box B$, then there is $(\alpha,\beta) \in \mathcal{N}'(w)$ such that $\alpha \subseteq [\![B]\!]_{\mathcal{M}'} \subseteq \mathcal{W} \setminus \beta$. By definition, there is $(\gamma,\delta) \in \mathcal{N}(w)$ such that $\gamma \subseteq \alpha$ and $\delta \subseteq \beta$. Then $\gamma \subseteq [\![B]\!]_{\mathcal{M}'} \subseteq \mathcal{W} \setminus \delta$. Moreover, by i.h., $[\![B]\!]_{\mathcal{M}'} = [\![B]\!]_{\mathcal{M}}$. Therefore $\mathcal{M}, w \Vdash \Box B$. For the oter direction, if $\mathcal{M}, w \Vdash \Box B$, then $(\alpha,\beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![B]\!]_{\mathcal{M}'} \overset{\text{i.h.}}{=} [\![B]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta$. Thus $(\alpha,\beta) \in \mathcal{N}'(w)$, therefore $\mathcal{M}', w \Vdash \Box B$. For the equivalence of conditions (4) and (4m) in M-models, the proof is a simplification of the one just given for the non-monotonic case. □

As a last example, it can be shown in an analogous way that the following conditions are equivalent with respect to the validity of formulas, whence they equally characterise the axiom N:

(N) There is $\alpha \subseteq \mathcal{W}$ such that for all $w \in \mathcal{W}$, $(\alpha, \emptyset) \in \mathcal{N}(w)$.
(N$'$) For all $w \in \mathcal{W}$, there is $\alpha \subseteq \mathcal{W}$ such that $(\alpha, \emptyset) \in \mathcal{N}(w)$.
(N$''$) For all $w \in \mathcal{W}$, $(\mathcal{W}, \emptyset) \in \mathcal{N}(w)$.

4.5 Bi-neighbourhood models for axiom 5

It would be interesting to extend the bi-neighbourhood semantics so to cover further systems defined by additional standard modal axioms. In this section, we consider the systems **E5** and **M5**, i.e., the extensions of **E** and **M** with the axiom

$$5 \quad \neg\Box A \to \Box\neg\Box A.$$

We present completeness results for these systems, as well as mutual transformations with standard models analogous to the ones in Section 4.3.

Similarly to the axioms T and 4, the axiom 5 is of interest in epistemic logic, where it expresses the so-called *negative introspection*: If the agent does not know that A, then she knows that she does not know that A. The reason for treating **E5** and **M5** separately from the other systems is that their completeness proof needs a different definition of canonical model than the one in Definition 4.2.2. Furthermore, this definition does not seem to be adequate for most of the other considered axioms, we let open the problem of finding a characterisation of extensions of **E5** and **M5** with further axioms.

Analogously to conditions (4$'$) and (4m) in previous section, in order to define bi-neighbourhood models for 5 we distinguish between the monotonic and the non-monotonic case:

Definition 4.5.1 (Semantic conditions for axiom 5)**.** We call 5-model any bi-neighbourhood model satisfying the condition (5) below. Moreover, we call M5m-model any bi-neighbourhood M-model satisfying the condition (5m) below.

(5) If $(\alpha, \mathcal{W} \setminus \alpha) \notin \mathcal{N}(w)$, then $(\curlyvee^-(\alpha, \mathcal{W} \setminus \alpha), \curlyvee^+(\alpha, \mathcal{W} \setminus \alpha)) \in \mathcal{N}(w)$.
(5m) If $(\alpha, \emptyset) \notin \mathcal{N}(w)$, then $(\curlyvee^-(\alpha, \emptyset), \emptyset) \in \mathcal{N}(w)$.

We now prove that the systems **E5** and **M5** are sound and complete with respect to the corresponding classes of bi-neighbourhood models.

Theorem 4.5.1 (Soundness)**.** Logics **E5** and **M5** are sound with respect to 5 and M5m-models, respectively.

Proof. We only show that axiom 5 is valid. Let $\mathcal{M}$ be a 5-model and assume $w \Vdash \neg\Box A$. Then for every α, β such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$, it holds $(\alpha, \beta) \notin \mathcal{N}(w)$. Thus in particular $([\![A]\!], [\![\neg A]\!]) \notin \mathcal{N}(w)$. By condition (5), $(\curlyvee^-([\![A]\!], [\![\neg A]\!]), \curlyvee^+([\![A]\!], [\![\neg A]\!])) \in \mathcal{N}(w)$. Moreover, $\curlyvee^-([\![A]\!], [\![\neg A]\!]) \subseteq [\![\neg\Box A]\!]$ and $\curlyvee^+([\![A]\!], [\![\neg A]\!]) \subseteq [\![\Box A]\!]$, then $w \Vdash \Box\neg\Box A$. For (5m) the proof is analogous. $\Box$

In order to prove the completeness of **E5** and **M5** we consider an alternative definition of canonical model, that following Chellas [29] we call *largest* canonical model.

Definition 4.5.2 (Largest canonical model)**.** The *largest* canonical model for $\mathbf{L}$ is the tuple $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{W}_{\mathbf{L}}, \mathcal{N}_{\mathbf{L}}, \mathcal{V}_{\mathbf{L}} \rangle$, where $\mathcal{W}_{\mathbf{L}}$ and $\mathcal{V}_{\mathbf{L}}$ are as in Definition 4.2.2, and $\mathcal{N}_{\mathbf{L}}$ is defined as follows:

$$\mathcal{N}_{\mathbf{L}}(\Phi) = \begin{cases} \{(\alpha, \mathcal{W}_{\mathbf{L}} \setminus \alpha) \mid \alpha \subseteq \mathcal{W}_{\mathbf{L}}\} \setminus \{(\uparrow A, \mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \mid \Box A \notin \Phi\} & \text{if } \mathbf{L} \text{ is non-monotonic.} \\ \{(\alpha, \emptyset) \mid \alpha \subseteq \mathcal{W}_{\mathbf{L}}\} \setminus \{(\uparrow A, \emptyset) \mid \Box A \notin \Phi\} & \text{if } \mathbf{L} \text{ is monotonic.} \end{cases}$$

Lemma 4.5.2. If Φ is a **E5**-maximal set (respectively **M5**-maximal set), then $\Box A \in \Phi$ if and only if $(\uparrow A, \mathcal{W}_{\mathbf{E5}} \setminus \uparrow A) \in \mathcal{N}_{\mathbf{E5}}(\Phi)$ (respectively $(\uparrow A, \emptyset) \in \mathcal{N}_{\mathbf{M5}}(\Phi)$).

Proof. If $\Box A \notin \Phi$, then by definition $(\uparrow A, \mathcal{W}_{\mathbf{E5}} \setminus \uparrow A) \notin \mathcal{N}_{\mathbf{E5}}(\Phi)$. If $(\uparrow A, \mathcal{W}_{\mathbf{E5}} \setminus \uparrow A) \notin \mathcal{N}_{\mathbf{E5}}(\Phi)$, then there is $B \in \mathcal{L}$ such that $\uparrow B = \uparrow A$ and $\Box B \notin \Phi$. By Lemma 4.2.3 (g), $\vdash_{\mathbf{E5}} B \leftrightarrow A$, and by Lemma 4.2.3 (c), $\Box A \notin \Phi$. For **M5** the proof is analogous. $\square$

On the basis of the above lemma, similarly to Lemma 4.2.5 we can prove that for every formula A of $\mathcal{L}$ and every **E5**-maximal (respectively **M5**-maximal) set Φ, $A \in \Phi$ if and only if $\mathcal{M}_{\mathbf{E5}}, \Phi \Vdash A$ (respectively $\mathcal{M}_{\mathbf{M5}}, \Phi \Vdash A$). Moreover, the following holds.

Lemma 4.5.3 (Model lemma)**.** The largest canonical model for **E5** (respectively for **M5**) is a 5-model (respectively a M5m-model).

Proof. Assume $(\alpha, \mathcal{W} \setminus \alpha) \notin \mathcal{N}_{\mathbf{E5}}(\Phi)$. Then there is $A \in \mathcal{L}$ such that $\alpha = \uparrow A$ and $\Box A \notin \Phi$. By Lemma 4.2.3, $\neg\Box A \in \Phi$, and by axiom 5, $\Box\neg\Box A \in \Phi$. Thus $(\uparrow\neg\Box A, \uparrow\Box A) \in \mathcal{N}_{\mathbf{E5}}(\Phi)$. In the same way as in the proof of Lemma 4.2.6 point (4) we can prove that $\uparrow\Box A = \curlyvee^{+}(\uparrow A, \mathcal{W}_{\mathbf{E5}} \setminus \uparrow A)$ and $\mathcal{W}_{\mathbf{E5}} \setminus \uparrow\Box A = \curlyvee^{-}(\uparrow A, \mathcal{W}_{\mathbf{E5}} \setminus \uparrow A)$. Therefore $(\curlyvee^{-}(\alpha, \mathcal{W}_{\mathbf{E5}} \setminus \alpha), \curlyvee^{+}(\alpha, \mathcal{W}_{\mathbf{E5}} \setminus \alpha)) \in \mathcal{N}_{\mathbf{E5}}(\Phi)$. $\square$

Then, in the same way as for Theorem 4.2.7 we can prove the following theorem.

Theorem 4.5.4 (Completeness)**.** The logics **E5** and **M5** are strongly complete with respect to the class of all 5-models, and the class of all M5m-models, respectively.

We can also extend to **E5** and **M5** the model transformations presented in Section 4.3. As shown in Chellas [29], the axiom 5 is characterised in the standard neighbourhood semantics by the following condition

$$(5) \quad \text{If } \alpha \notin \mathcal{N}(w), \text{ then } \{v \mid \alpha \notin \mathcal{N}(v)\} \in \mathcal{N}(w).$$

Given a standard model for **E5** or **M5**, an equivalent bi-neighbourhood model can be obtained as follows.

Proposition 4.5.5. Let $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be a standard model satisfying the condition corresponding to the axiom 5 in the standard semantics, and let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be the bi-neighbourhood model defined on the basis of $\mathcal{M}_{st}$ as in Proposition 4.3.2. Then $\mathcal{M}_{bi}$ is a 5-model if $\mathcal{M}_{st}$ is not supplemented, and it is a 5m-model if $\mathcal{M}_{st}$ is supplemented.

Proof. (5) Assume $(\alpha, \mathcal{W} \setminus \alpha) \notin \mathcal{N}_{bi}(w)$. Then $\alpha \notin \mathcal{N}_{st}(w)$. By the condition corresponding to axiom 5 in the standard semantics, $\{v \mid \alpha \notin \mathcal{N}_{st}(v)\} \in \mathcal{N}_{st}(w)$. Thus by definition $(\{v \mid \alpha \notin \mathcal{N}_{st}(v)\}, \{v \mid \alpha \in \mathcal{N}_{st}(v)\}) \in \mathcal{N}_{bi}(w)$. We show that $\{v \mid \alpha \in \mathcal{N}_{st}(v)\} = \reflectbox{$\mathcal{N}$}^{+}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$, and $\{v \mid \alpha \notin \mathcal{N}_{st}(v)\} = \reflectbox{$\mathcal{N}$}^{-}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$. We have $\alpha \in \mathcal{N}_{st}(v)$ iff $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(v)$ iff $v \in \reflectbox{$\mathcal{N}$}^{+}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$. Moreover, $\alpha \notin \mathcal{N}_{st}(v)$ iff for all $\gamma \in \mathcal{N}_{st}(v)$, $\alpha \neq \gamma$ iff for all $(\gamma, \mathcal{W} \setminus \gamma) \in \mathcal{N}_{bi}(v)$, $\alpha \neq \gamma$, that is $\alpha \cap \mathcal{W} \setminus \gamma \neq \emptyset$ or $\mathcal{W} \setminus \alpha \cap \gamma \neq \emptyset$ iff $v \in \reflectbox{$\mathcal{N}$}^{-}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$. For the monotonic case the proof is analogous. $\square$

For the opposite direction, given a bi-neighbourhood model for **E5** or **M5** we obtain an equivalent standard model as follows.

Proposition 4.5.6. Let $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$ be a bi-neighbourhood 5- or 5m-model, and let $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$ be the standard model defined on the basis of $\mathcal{M}_{bi}$ as in Proposition 4.3.3. Then $\mathcal{M}_{st}$ satisfies the condition corresponding to 5 in the standard semantics.

Proof. Assume $\alpha \notin \mathcal{N}_{st}(w)$. Then in particular $(\alpha, \mathcal{W} \setminus \alpha) \notin \mathcal{N}_{bi}(w)$, thus by condition (5), $(\reflectbox{$\mathcal{N}$}^{-}_{bi}(\alpha, \mathcal{W} \setminus \alpha), \reflectbox{$\mathcal{N}$}^{+}_{bi}(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(w)$. We show that $\reflectbox{$\mathcal{N}$}^{+}_{bi}(\alpha, \mathcal{W} \setminus \alpha) \subseteq \{v \mid \alpha \in \mathcal{N}_{st}(v)\}$, and $\reflectbox{$\mathcal{N}$}^{-}_{bi}(\alpha, \mathcal{W} \setminus \alpha) \subseteq \{v \mid \alpha \notin \mathcal{N}_{st}(v)\}$, which implies $\{v \mid \alpha \notin \mathcal{N}_{st}(v)\} \in \mathcal{N}_{st}(w)$. If $v \in \reflectbox{$\mathcal{N}$}^{+}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$, then $(\alpha, \mathcal{W} \setminus \alpha) \in \mathcal{N}_{bi}(v)$, then $\alpha \in \mathcal{N}_{st}(v)$. If $v \in \reflectbox{$\mathcal{N}$}^{-}_{bi}(\alpha, \mathcal{W} \setminus \alpha)$, then for all $(\gamma, \delta) \in \mathcal{N}_{bi}(v)$, $\alpha \cap \delta \neq \emptyset$ or $\mathcal{W} \setminus \alpha \cap \gamma \neq \emptyset$, then $\gamma \not\subseteq \alpha$ or $\alpha \not\subseteq \mathcal{W} \setminus \delta$, therefore $\alpha \notin \mathcal{N}_{st}(v)$. For the monotonic case the proof is analogous. $\square$

4.6 Bi-neighbourhood semantics for agency and ability logics

In this section, we define the bi-neighbourhood models for Elgesem's agency and ability logic **ELG** [47] and its coalition extension **COAL** by Troquard [165] (see their axiomatisations and their neighbourhood semantics in Section 2.4).

Definition 4.6.1. A bi-neighbourhood model for **ELG** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_i, \mathcal{N}^{\mathbb{C}}_i, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\mathcal{V}$ is a valuation function, and for each agent i, $\mathcal{N}^{\mathbb{E}}_i$ and $\mathcal{N}^{\mathbb{C}}_i$ are two bi-neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W}))$ satisfying the following conditions:

($\mathrm{C}_{\mathbb{E}}$) If $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}^{\mathbb{E}}_i(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}^{\mathbb{E}}_i(w)$.
($\mathrm{T}_{\mathbb{E}}$) If $(\alpha, \beta) \in \mathcal{N}^{\mathbb{E}}_i(w)$, then $w \in \alpha$.
($\mathrm{Q}_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}^{\mathbb{C}}_i(w)$, then $\beta \neq \emptyset$.
($\mathrm{P}_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}^{\mathbb{C}}_i(w)$, then $\alpha \neq \emptyset$.
($\mathrm{Int}_{\mathbb{E}\mathbb{C}}$) $\mathcal{N}^{\mathbb{E}}_i(w) \subseteq \mathcal{N}^{\mathbb{C}}_i(w)$.

The forcing relation $\Vdash$ is defined as usual for atomic formulas and boolean connectives, whereas for $\mathbb{E}$- and $\mathbb{C}$-formulas it is defined as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \mathbb{E}_i A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta. \\ \mathcal{M}, w \Vdash \mathbb{C}_i A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta. \end{array}$$

As we shall see in Chapter 6, the bi-neighbourhood semantics for **ELG** will be convenient for the extraction of countermodels of non-valid formulas from failed proofs in sequent calculi. However, this semantics can also suggest an account of agency in terms of conditions enabling or preventing the realisation of actions, represented respectively by the elements α and β of bi-neighbourhood pairs. According to this interpretation, the conditions ($\mathrm{P}_{\mathbb{C}}$) and ($\mathrm{Q}_{\mathbb{C}}$), i.e., $\alpha \neq \emptyset$ and $\beta \neq \emptyset$, correspond to the fact that the results of actions can always be enabled (it is not possible to realise a contradiction) and prevented (it is not possible to realise a tautology). Notice also that, because of the validity of $\neg\mathbb{E}_i\top$ and of the axiom $T_{\mathbb{E}}$, formulas of the form $\mathbb{E}_i A$ are never valid in models for **ELG**, thus providing a semantic counterpart of the idea that actions can be always prevented.

As for classical non-normal modal logics (see Chapter 4) we can prove that **ELG** is sound and complete with respect to its bi-neighbourhood models.

Theorem 4.6.1 (Soundness and completeness)**.** A formula A of $\mathcal{L}_{Elg}$ is derivable in **ELG** if and only if it is valid in all bi-neighbourhood models for **ELG**.

Proof. As usual, the proof of soundness amounts to showing that all axioms are valid and all rules are validity-preserving. For axioms $C_{\mathbb{E}}$, $T_{\mathbb{E}}$, $P_{\mathbb{C}}$, and for rules $RE_{\mathbb{E}}$, $RE_{\mathbb{C}}$ we can refer to the proof of Theorem 4.2.1. For $Q_{\mathbb{C}}$ and $Int_{\mathbb{E}\mathbb{C}}$ the proof is as follows. ($Q_{\mathbb{C}}$) Assume by contradiction that $w \Vdash \mathbb{C}_i\top$. Then there is $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w)$ such that $\alpha \subseteq [\![\top]\!] \subseteq \mathcal{W} \setminus \beta$. Thus $\beta = \emptyset$, against condition ($\mathrm{Q}_{\mathbb{C}}$). Therefore $w \Vdash \neg\mathbb{C}_i\top$. ($Int_{\mathbb{E}\mathbb{C}}$) Assume $w \Vdash \mathbb{E}_i A$. Then there is $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. By condition ($\mathrm{Int}_{\mathbb{E}\mathbb{C}}$), $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(w)$, therefore $w \Vdash \mathbb{C}_i A$.

Completeness can be proved by the canonical model construction as it is done in Chapter 4 for classical non-normal modal logics. We define the canonical model for **ELG** as the tuple $\langle \mathcal{W}, \mathcal{N}_i^{\mathbb{E}}, \mathcal{N}_i^{\mathbb{C}}, \mathcal{V} \rangle$, where $\mathcal{W}$ is the class of **ELG**-maximal sets, $V(p) = \{\Phi \in \mathcal{W} \mid p \in \Phi\}$, and for every $i \in \mathcal{A}$ and $\mathbb{X} \in \{\mathbb{E}, \mathbb{C}\}$, $\mathcal{N}_i^{\mathbb{X}}(\Phi) = \{(\uparrow A, \mathcal{W} \setminus \uparrow A) \mid \mathbb{X}_i A \in \Phi\}$. We can prove that $\Phi \Vdash A$ if and only if $A \in \Phi$, (cf. the proof of Lemma 4.2.5), and that the canonical model is a bi-neighbourhood model for **ELG**. We only show that it satisfies the conditions ($\mathrm{Q}_{\mathbb{C}}$) and ($\mathrm{Int}_{\mathbb{E}\mathbb{C}}$), for the other conditions we refer to the proof of Lemma 4.2.6. ($\mathrm{Q}_{\mathbb{C}}$) Assume $(\uparrow A, \mathcal{W} \setminus \uparrow A) \in \mathcal{N}_i^{\mathbb{C}}(\Phi)$. Then $\mathbb{C}_i A \in \Phi$. If $\uparrow A = \mathcal{W}$, then $\uparrow A = \uparrow\top$, which implies $\mathbb{C}_i\top \in \Phi$, against the fact that $\neg\mathbb{C}_i\top \in \Phi$ and Φ is **ELG**-consistent. Therefore $\uparrow A \neq \mathcal{W}$, that is $\mathcal{W} \setminus \uparrow A \neq \emptyset$. ($\mathrm{Int}_{\mathbb{E}\mathbb{C}}$) Assume $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(\Phi)$. Then there is $\mathbb{E}_i A \in \Phi$ such that $\alpha = \uparrow A$ and $\beta = \mathcal{W} \setminus \uparrow A$. Since $\mathbb{E}_i A \to \mathbb{C}_i A \in \Phi$ and Φ is closed under derivation, $\mathbb{C}_i A \in \Phi$. Thus $(\uparrow A, \mathcal{W} \setminus \uparrow A) = (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(\Phi)$. □

It is easy to see that by applying the transformations in Section 4.3 to the bi-neighbourhood models for **ELG** it is possible to obtain equivalent models in the standard semantics for **ELG** defined in Governatori and Rotolo [78] (cf. Section 2.4).

We now consider the bi-neighbourhood models for **COAL**.

Definition 4.6.2. A bi-neighbourhood model for **COAL** is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_g^{\mathbb{E}}, \mathcal{N}_g^{\mathbb{C}}, \mathcal{V} \rangle$, where in particular for every group of agents g, $\mathcal{N}_g^{\mathbb{E}}$ and $\mathcal{N}_g^{\mathbb{C}}$ are two bi-neighbourhood functions satisfying the conditions $(C_{\mathbb{E}})$, $(T_{\mathbb{E}})$, $(Q_{\mathbb{C}})$, and $(P_{\mathbb{C}})$ of Definition 4.6.1 (but with $\mathcal{N}^{\mathbb{E}}$ and $\mathcal{N}^{\mathbb{C}}$ indexed by groups g instead of agents i), and also the following additional conditions:

$(\mathrm{F}_{\mathbb{C}})$ $\quad \mathcal{N}_{\emptyset}^{\mathbb{C}}(w) = \emptyset$.

$(\mathrm{Int}^2_{\mathbb{EC}})$ $\quad$ If $(\alpha, \beta) \in \mathcal{N}_{g_1}^{\mathbb{E}}(w)$ and $(\gamma, \delta) \in \mathcal{N}_{g_2}^{\mathbb{E}}(w)$, then $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}_{g_1 \cup g_2}^{\mathbb{C}}(w)$.

The forcing relation $\Vdash$ is defined as in Definition 4.6.1, in particular:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \mathbb{E}_g A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_g^{\mathbb{E}}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta. \\ \mathcal{M}, w \Vdash \mathbb{C}_g A & \text{iff} & \text{there is } (\alpha, \beta) \in \mathcal{N}_g^{\mathbb{C}}(w) \text{ s.t. } \alpha \subseteq [\![A]\!]_{\mathcal{M}} \subseteq \mathcal{W} \setminus \beta. \end{array}$$

Theorem 4.6.2 (Soundness and completeness)**.** A formula A of $\mathcal{L}_{coal}$ is derivable in **COAL** if and only if it is valid in all bi-neighbourhood models for **COAL**.

Proof. Essentially, we need to reformulate and extend the proof of Theorem 4.6.1. For the soundness we show the validity of axioms $F_{\mathbb{C}}$ and $Int^2_{\mathbb{EC}}$. $(F_{\mathbb{C}})$ Assume by contradiction that $w \Vdash \mathbb{C}_{\emptyset} A$. Then there is $(\alpha, \beta) \in \mathcal{N}_{\emptyset}^{\mathbb{C}}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. Thus $\mathcal{N}_{\emptyset}^{\mathbb{C}}(w) \neq \emptyset$, against condition $(\mathrm{F}_{\mathbb{C}})$. Therefore $w \Vdash \neg \mathbb{C}_{\emptyset} A$. $(Int^2_{\mathbb{EC}})$ Assume $w \Vdash \mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B$. Then there are $(\alpha, \beta) \in \mathcal{N}_{g_1}^{\mathbb{E}}(w)$ and $(\gamma, \delta) \in \mathcal{N}_{g_2}^{\mathbb{E}}(w)$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$ and $\gamma \subseteq [\![B]\!] \subseteq \mathcal{W} \setminus \delta$. Then by condition $(\mathrm{Int}^2_{\mathbb{EC}})$, $(\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}_{g_1 \cup g_2}^{\mathbb{C}}(w)$, where $\alpha \cap \gamma \subseteq [\![A \wedge B]\!] \subseteq \mathcal{W} \setminus (\beta \cup \delta)$. Therefore $w \Vdash \mathbb{C}_{g_1 \cup g_2}(A \wedge B)$.

For completeness, we define the canonical model for **COAL** as the tuple $\langle \mathcal{W}, \mathcal{N}_g^{\mathbb{E}}, \mathcal{N}_g^{\mathbb{C}}, \mathcal{V} \rangle$, where $\mathcal{W}$ is the class of **COAL**-maximal sets, $V(p) = \{\Phi \in \mathcal{W} \mid p \in \Phi\}$, and for every $g \subseteq \mathcal{A}$ and $\mathbb{X} \in \{\mathbb{E}, \mathbb{C}\}$, $\mathcal{N}_g^{\mathbb{X}}(\Phi) = \{(\uparrow A, \mathcal{W} \setminus \uparrow A) \mid \mathbb{X}_g A \in \Phi\}$. We can prove that $\Phi \Vdash A$ if and only if $A \in \Phi$, (cf. the proof of Lemma 4.2.5), and that the canonical model is a bi-neighbourhood model for **COAL**. Here we only show that it satisfies the conditions $(\mathrm{F}_{\mathbb{C}})$ and $(\mathrm{Int}^2_{\mathbb{EC}})$. $(\mathrm{F}_{\mathbb{C}})$ By contradiction, assume $(\uparrow A, \mathcal{W} \setminus \uparrow A) \in \mathcal{N}_{\emptyset}^{\mathbb{C}}(\Phi)$. Then $\mathbb{C}_{\emptyset} A \in \Phi$, against the fact that $\neg \mathbb{C}_{\emptyset} A \in \Phi$ and Φ is **COAL**-consistent. Therefore $\mathcal{N}_{\emptyset}^{\mathbb{C}}(\Phi) = \emptyset$. $(\mathrm{Int}^2_{\mathbb{EC}})$ Assume $(\alpha, \beta) \in \mathcal{N}_{g_1}^{\mathbb{E}}(w)$ and $(\gamma, \delta) \in \mathcal{N}_{g_2}^{\mathbb{E}}(w)$. Then there are $\mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B \in \Phi$ such that $\alpha = \uparrow A$, $\beta = \mathcal{W} \setminus \uparrow A$, $\gamma = \uparrow B$, and $\delta = \mathcal{W} \setminus \uparrow B$. Since $\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B \to \mathbb{C}_{g_1 \cup g_2}(A \wedge B) \in \Phi$, we have $\mathbb{C}_{g_1 \cup g_2}(A \wedge B) \in \Phi$. Thus $(\uparrow(A \wedge B), \mathcal{W} \setminus \uparrow(A \wedge B)) = (\alpha \cap \gamma, \beta \cup \delta) \in \mathcal{N}_{g_1 \cup g_2}^{\mathbb{C}}(\Phi)$. □

4.7 An embedding into monotonic dyadic logics

As observed in Section 4.1, the bi-neighbourhood semantics decomposes the forcing condition for boxed formulas of the standard semantics into two monotonic components. In this section, we show that the same argument can be also reformulated syntactically in the form of an embedding of classical non-normal modal logics into logics with a dyadic monotonic operator. The argument is as follows.

Let us consider a modal extension of **CPL**, that we call $\mathbf{M_2}$, formulated in a propositional modal language $\mathcal{L}_\heartsuit$ containing a dyadic modality $\heartsuit$ instead of $\Box$, whose formulas are defined by the following grammar, where p_i is any variable in Atm:

$$A ::= p_i \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \to A \mid \heartsuit(A/A),$$

We define the system $\mathbf{M_2}$ by extending **CPL** (formulated in language $\mathcal{L}_\heartsuit$) with the modal rule

$$RM_\heartsuit \; \frac{A \to C \qquad B \to D}{\heartsuit(A/B) \to \heartsuit(C/D)} \; .$$

On the basis of $RM_\heartsuit$, in the systems $\mathbf{M_2}$ the modality $\heartsuit$ is monotonic on both two arguments, in particular the two formulas $\heartsuit(A \wedge C/B) \to \heartsuit(A/B)$ and $\heartsuit(A/B \wedge C) \to \heartsuit(A/B)$ are derivable.

In addition, we also consider the following translation $\dagger : \mathcal{L} \longrightarrow \mathcal{L}_\heartsuit$ of the formulas of the language $\mathcal{L}$ of classical non-normal modal logics into formulas of $\mathcal{L}_\heartsuit$:

$$\begin{array}{l}
\dagger(p_i) = p_i, \text{ for every } p_i \in Atm; \\
\dagger(\bot) = \bot; \\
\dagger(\top) = \top; \\
\dagger(A \circ B) = \dagger(A) \circ \dagger(B), \text{ for } \circ \in \{\wedge, \vee, \to\}; \\
\dagger(\Box A) = \heartsuit(\dagger(A)/\neg \dagger(A)).
\end{array}$$

We show that the logic **E** can be simulated by $\mathbf{M_2}$ by means of the above translation, in the sense that a formula A of $\mathcal{L}$ is derivable in **E** if and only if $\dagger(A)$ is derivable in $\mathbf{M_2}$. We present the embedding only for the basic logic **E**, but the same argument could be extended to the other systems of the classical cube by considering extensions of $\mathbf{M_2}$ with the axioms

$$\begin{array}{ll}
\heartsuit(A/B) \to \heartsuit(A/\bot) & \text{(for systems with axiom } M\text{);} \\
\heartsuit(\top/\bot) & \text{(for systems with axiom } N\text{);} \\
\heartsuit(A/B) \wedge \heartsuit(C/D) \to \heartsuit(A \wedge C/B \vee D) & \text{(for systems with axiom } C\text{).}
\end{array}$$

The left-to-right direction of the claim (if A is derivable in **E**, then $\dagger(A)$ is derivable in $\mathbf{M_2}$) is proved directly by considering derivation in Hilbert systems, whereas the opposite direction is proved indirectly by a standard semantic argument, for which we need the following lemmas.

Lemma 4.7.1. $\mathbf{M_2}$ is sound with respect to bi-neighbourhood models under the following evaluation $\Vdash_{\heartsuit}$ of formulas of $\mathcal{L}_{\heartsuit}$, which is defined as $\Vdash_{bi}$ (Definition 4.1.1) for $A = p_i, \bot, \top, B \wedge C, B \vee C, B \to C$, and for $\heartsuit$-formulas is as follows:

$$\mathcal{M}, w \Vdash_{\heartsuit} \heartsuit(B/C) \text{ iff there is } (\alpha, \beta) \in \mathcal{N}(w) \text{ such that } \alpha \subseteq [\![B]\!]_{\mathcal{M}} \text{ and } \beta \subseteq [\![C]\!]_{\mathcal{M}}.$$

Proof. As usual, we have to show that all axioms of $\mathbf{M_2}$ are valid and that the rules of $\mathbf{M_2}$ preserve the validity. We only consider rule $RM_{\heartsuit}$: Assume that the premisses $A \to C$ and $B \to D$ are valid and that $w \Vdash \heartsuit(A/B)$. Then there is $(\alpha, \beta) \in \mathcal{N}(w)$ such that $\alpha \subseteq [\![A]\!]$ and $\beta \subseteq [\![B]\!]$. By the validity of the premisses, $[\![A]\!] \subseteq [\![C]\!]$ and $[\![B]\!] \subseteq [\![D]\!]$, which imply $\alpha \subseteq [\![C]\!]$ and $\beta \subseteq [\![D]\!]$. Then $w \Vdash \heartsuit(C/D)$. $\square$

Lemma 4.7.2. Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ be a bi-neighbourhood model. Then for every $w \in \mathcal{W}$ and every $A \in \mathcal{L}$, $w \Vdash_{bi} A$ if and only if $w \Vdash_{\heartsuit} \dagger(A)$.

Proof. By induction on A. The proof is immediate for $A = p_i, \bot, \top, B \wedge C, B \vee C, B \to C$. We consider the case $A = \Box B$: $w \Vdash \Box B$ iff there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![B]\!]_{bi} \subseteq \mathcal{W} \setminus \beta$ iff (i.h.) there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![\dagger(B)]\!]_{\heartsuit} \subseteq \mathcal{W} \setminus \beta$ iff $\alpha \subseteq [\![\dagger(B)]\!]_{\heartsuit}$ and $\beta \subseteq [\![\neg \dagger (B)]\!]_{\heartsuit}$ iff $w \Vdash_{\heartsuit} \heartsuit(\dagger(B)/\neg \dagger (B))$. $\square$

Theorem 4.7.3 (Embedding)**.** For every formula A of $\mathcal{L}$,

$$\mathbf{E} \vdash A \text{ if and only if } \mathbf{M_2} \vdash \dagger(A).$$

Proof. From left to right, assume $\mathbf{E} \vdash A$. The proof is by induction on the height h of the derivation of A in the system $\mathbf{E}$. If $h = 0$, then A is an instance of an axiom of **CPL**. It is immediate to verify that $\dagger(A)$ is an instance of the same axiom, whence it is derivable in $\mathbf{M_2}$. If $h \geq 1$, we consider the last rule applied in the derivation, which is either MP or RE. If the last rule applied is MP, then A is obtained from formulas B and $B \to A$ occurring in the same derivation at a smaller height. By i.h., $\dagger(B)$ and $\dagger((B \to A)) = \dagger(B) \to \dagger(A)$ are derivable in $\mathbf{M_2}$, then by MP, $\dagger(A)$ is derivable in $\mathbf{M_2}$. If the last rule applied is RE, then A has the form $\Box B \to \Box C$, and is obtained from formulas $B \to C$ and $C \to B$ occurring in the derivation at a smaller height. By i.h., $\dagger((B \to C)) = \dagger(B) \to \dagger(C)$ and $\dagger((C \to B)) = \dagger(C) \to \dagger(B)$ are derivable in $\mathbf{M_2}$. By propositional reasoning, $\neg \dagger (B) \to \neg \dagger (C)$ is derivable. Then by $RM_{\heartsuit}$ we obtain $\heartsuit(\dagger(B)/\neg \dagger (B)) \to \heartsuit(\dagger(C)/\neg \dagger (C)) = \dagger(A)$.

For the opposite direction, assume by contraposition that $\mathbf{E} \nvdash A$. Then by the completeness of $\mathbf{E}$ with respect to bi-neighbourhood models (Theorem 4.2.7), there are a bi-neighbourhood model $\mathcal{M}$ and a world w of $\mathcal{M}$ such that $w \nVdash_{bi} A$. By Lemma 4.7.2, this implies $w \nVdash_{\heartsuit} \dagger(A)$, therefore by Lemma 4.7.1, $\mathbf{M_2} \nvdash \dagger(A)$. $\square$

4.8 Discussion

In this chapter, we have provided an alternative semantic characterisation of all classical non-normal modal logics considered in this work. The semantics is based on so-called bi-neighbourhood models, these can be seen as a generalisation of standard neighbourhood models. As a difference with standard models, worlds in bi-neighbourhood models are equipped with pairs of neighbourhoods, rather than single neighbourhoods. Bi-neighbourhood semantics essentially decomposes the forcing condition for boxed formulas of the standard semantics into two monotonic components. Elements of bi-neighbourhood pairs can be also understood as lower and upper bounds of neighbourhoods in standard models. In general, bi-neighbourhood models can be used to represent reasoning with partial information. In this chapter, we have proved soundness and completeness of every system with respect to the corresponding bi-neighbourhood models, both directly by the canonical model construction and indirectly by mutual transformations with standard neighbourhood models. Moreover, we have partially extended the semantics to the systems containing the axiom 5, and we have also given a characterisation of Elgesem's and Troquard's agency logics in terms of bi-neighbourhood models. Finally, basing on the bi-neighbourhood semantics we have presented a syntactic embedding of classical non-normal modal logics into dyadic monotonic logics.

In future work, it would be worth extending the bi-neighbourhood semantics to combinations of axiom 5 with the other axioms considered in this work, as well as to systems containing additional standard modal axioms, such as B, 2, and Sahlqvist formulas. Considering the first task, it seems that neither the smallest nor the largest canonical model are adequate. Similarly to Chellas' completeness proof for **ET5** in the standard semantics [29], one probably needs to consider specific definitions of canonical model for every combination.

It is clear that the structure underlying the bi-neighbourhood semantics, i.e., essentially sets of pairs of neighbourhoods, can have an interest also independently from the systems considered in this work. For instance, we have seen in the previous section that it can be used to characterise systems with dyadic modalities, as it allows one to express forcing conditions like, e.g., $\mathcal{M}, w \Vdash \heartsuit(A/B)$ if and only if there is $(\alpha, \beta) \in \mathcal{N}(w)$ s.t. $\alpha \subseteq [\![A]\!]_{\mathcal{M}}$ and $\beta \subseteq [\![B]\!]_{\mathcal{M}}$. A similar structure is considered by Gulisano [80] in order to characterise dyadic non-normal modalities for the representation of conditional obligations and prohibitions. The same structure can be generalised for considering not only pairs of neighbourhoods but also arbitrary tuples of neighbourhoods. Models based on tuples of neighbourhoods are considered for instance by Calardo et al. [26, 28] in order to provide a semantic characterisation of a n-ary non-normal modal operator used for the representation of reparative obligations and preferences in social choice theory. In addition, in future work it would be interesting to investigate whether the bi-neighbourhood semantics can be suitable to model relevant notions involving positive and negative sides of information, such as the notion of bipolarity discussed in [44]. Similarly, the bi-neighbourhood semantics could be also of interest in evidence logics [16, 17, 13, 117], where a bi-neighbourhood pair could be seen as representing a source of information providing positive and negative evidence about a proposition.

As a further remark, we notice that the bi-neighbourhood semantics can recall Kracht and Wolter's [103] simulation of non-normal modal logics by means of normal multimodal logics. In particular, this simulation is based on a translation of modal formulas $\Box A$ into formulas of the form $\Diamond_1(\Box_2 A \wedge \Box_3 \neg A)$, which recalls the forcing condition of boxed formulas in the bi-neighbourhood semantics. The same similarity can be observed between the $\exists\forall$-semantics (cf. Section 2.3) and Kracht and Wolter's simulation of monotonic logics by means of normal bimodal logics, which is based on a translation of $\Box A$ into formulas of the form $\Diamond_1 \Box_2 A$. We do not think that this simulation can reduce in any way the interest of non-normal modal logics in general (exactly like Gödel's translation of intuitionistic logic into modal logic **S4** does not reduce the interest of intuitionistic logic; moreover, non-normal modal logics without axiom C have a weaker complexity than normal modal logics), and of the bi-neighbourhood semantics in particular. On the contrary, it would be interesting to study whether these similarities can be used to transfer results from the semantic level to the syntactic one, and vice versa.

We shall see in the next chapters that the bi-neighbourhood semantics offers significant *advantages for the proof theory*. On the one hand, labelled calculi based on the bi-neighbourhood semantics have a better behaviour in terms of modularity and termination of proof search. On the other hand, countermodels are easily and more efficiently extracted in this semantics rather than in the standard one, both from external and from internal calculi.

Chapter 5

Labelled calculi

In this chapter, we define labelled sequent calculi for all classical non-normal modal logics considered in this work. We prove that all calculi are sound and complete with respect to the corresponding axiomatic systems; in particular, completeness is proved by means of a syntactic proof of cut elimination. Then, we propose an equivalent reformulation of the calculi in the form of tableaux systems. Basing on the tableaux calculi we define a terminating proof search strategy that provides a decision procedure for the derivability problem in the corresponding logics. Moreover, we show that from every failed proof it is possible to directly extract a countermodel of the non-derivable formula, both in the bi-neighbourhood semantics and, for regular logics, also in the relational semantics. Finally, we present a theorem prover for non-normal modal logics computing both derivations and countermodels based on a Prolog implementation of our labelled calculi.

5.1 Labelled sequents and rules

The labelled sequent calculi $\mathbf{LS.E}^*$ are defined in the extended language $\mathcal{L}_{lab}$. We define the language $\mathcal{L}_{lab}$ by considering two denumerable (and disjoint) sets $\mathbb{WL} = \{x, y, z, ...\}$ and $\mathbb{NL} = \{a, b, c, ...\}$, respectively of *world labels* and of *neighbourhood labels*, and a distinct element $\tau \notin \mathbb{WL}, \mathbb{NL}$. On the basis of the neighbourhood labels and τ we build two kinds of *neighbourhood terms* (or just *terms*), i.e., positive and negative terms, as follows:

Definition 5.1.1 (Neighbourhood terms)**.** *Positive* neighbourhood terms are defined by the following grammar:

$$t, s ::= a \mid \tau \mid ts \mid J(t),$$

where a is any neighbourhood label, and ts is the multiset union of the terms t and s. Furthermore, if t is a positive term, then $\bar{t}$ is a *negative* term.

Observe that only positive terms can be joint for building complex terms, and that the operation of overlining a term cannot be iterated: it can be applied only once for turning a positive term into a negative one. For instance, $aJ(\tau bJ(c))$ and $\overline{aJ(\tau bJ(c))}$ are terms, whereas $\overline{a}J(\tau bJ(c))$, $aJ(\overline{\tau bJ(c)})$, and $\overline{\overline{a}}$ are not. We use t, s, r (respectively $\overline{t}, \overline{s}, \overline{r}$) as metavariables for positive (respectively negative) terms. Moreover, we use $\mathtt{t}$ as a metavariable for terms t or $\overline{t}$, no matter if positive or negative.

The exact interpretation of neighbourhood terms is given by the realisations in Definition 5.1.5. Intuitively, t and $\overline{t}$ represent the two members of a bi-neighbourhood pair. Moreover, the terms τ and $\overline{\tau}$, as well as the complex terms of kinds ts and $J(t)$, are used to import into the calculus the semantic conditions corresponding to some specific axioms, and in principle they are not needed in the calculi lacking the corresponding rules. In particular, $\overline{\tau}$ represents the *empty set* of worlds, and it is used to express the semantic condition for the axiom N. Terms ts and $\overline{ts}$ respectively represent the *intersection* of the sets represented by t and s, and the *union* of the sets represented by $\overline{t}$ and $\overline{s}$, and are used to import the semantic condition for the axiom C. Finally, if t and $\overline{t}$ correspond to the sets α and β, then $J(t)$ and $\overline{J(t)}$ respectively represent the sets $\curlywedge^+(\alpha, \beta)$ and $\curlywedge^-(\alpha, \beta)$ (cf. Definition 4.1.2).

On the basis of the world labels and neighbourhood terms, we define the labelled formulas of the extended language $\mathcal{L}_{lab}$ as follows:

Definition 5.1.2 (Labelled formulas)**.** The labelled formulas of $\mathcal{L}_{lab}$ have the forms

$$\phi ::= x : A \mid \mathtt{t} \Vdash^{\forall} A \mid \mathtt{t} \Vdash^{\exists} A \mid x \in \mathtt{t} \mid t \rhd x,$$

where A is any formula of $\mathcal{L}$, x is any world label, and $\mathtt{t}$ is any (positive or negative) neighbourhood term.

The semantic interpretation of labelled formulas is given in Definition 5.1.5. Intuitively, $x : A$ means that x forces A, $\mathtt{t} \Vdash^{\forall} A$ (respectively $\mathtt{t} \Vdash^{\exists} A$) means that every world in $\mathtt{t}$ (respectively some world in $\mathtt{t}$) forces A, $x \in \mathtt{t}$ means that x is a world in the neighbourhood $\mathtt{t}$, and $t \rhd x$ means that the pair $(t, \overline{t})$ is a bi-neighbourhood of x.

Sequents are defined as usual as pairs $\Gamma \Rightarrow \Delta$ of finite multisets of formulas, however in order to ensure admissibility of cut they must satisfy some restrictions.

Definition 5.1.3 (Sequents of $\mathbf{LS.E}^*$)**.** A *sequent* of $\mathbf{LS.E}^*$ is a pair $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas of $\mathcal{L}_{lab}$, that respect the following conditions: (1) Δ contains only formulas of the kinds $x : A$, $\mathtt{t} \Vdash^{\forall} A$ and $\mathtt{t} \Vdash^{\exists} A$ (whereas Γ may contain any kind of formula of $\mathcal{L}_{lab}$); (2) If Γ is non-empty, then all world labels and all neighbourhood labels occurring in Δ occur also in Γ.[1] (3) If Γ is empty, then Δ contains only formulas of the kind

[1] A neighbourhood label a occurs in (or belongs to) a labelled formula ϕ (set of formulas, sequent) if there is a (positive or negative) term containing a in ϕ.

$x : A$, and all these formulas are labelled by the same world label x. (4) If $x \in t$ is in Γ, then there is a world label y such that $t \rhd y$ is in Γ.

The calculi **LS.E*** are defined by the rules in Figure 5.1, where the notations (y!) and (a!) express the eigenvariable condition: the world label y must be fresh (i.e., it must not occur in the conclusion) in the application of the rules $\mathsf{R} \Vdash^{\forall}$, $\mathsf{L} \Vdash^{\exists}$, $\overline{\mathsf{J}}$, P, D_1, D_2, and for every $n \in \mathbb{N}$, D_n^+, whereas the neighbourhood label a must be fresh in the application of $\mathsf{L}\Box$.

In analogy with the calculi for normal modal logics based on relational semantics [133, 168] (cf. Section 3.2), the calculi have separate left and right rules for logical constants which are meaning conferring and directly derive from the semantic explanation of logical constants in the bi-neighbourhood semantics. The propositional rules and the rules for local forcing are standard. The first ones are G3-style rules (see Figure 3.1) enriched with world labels, whereas the second ones reflect the interpretation of symbols $\Vdash^{\forall}$ and $\Vdash^{\exists}$ as, respectively, universal and existential forcing. Moreover, the rules for box directly derive from the satisfaction clause of boxed formulas in the bi-neighbourhood semantics. Notice that in contrast to the labelled calculi based on the relational semantics, in these calculi the left-box rule has an eigenvariable condition whereas the right-box rule has not. This is due to the fact that the forcing condition for the boxed formulas is expressed in the (bi-)neighbourhood semantics by an existential claim (whereas in the relational semantics it is expressed by an universal claim).

The set composed by the propositional rules, the rules for local forcing, and the rules for box, defines the calculus **LS.E** for the basic classical non-normal modal logic **E**.

Modular extensions of the basic system are obtained by means of rules manipulating only labels, of which there are two kinds: First, the rules for neighbourhood terms fix the meaning of $\overline{\tau}$ and of the complex terms which are needed for some extensions. Furthermore, every modal axiom or rule has corresponding rules in the calculus which directly derive from the semantic conditions associated to the modal axioms in the bi-neighbourhood semantics (Definition 4.1.3). The only exception are the rules for 4, which are build on the basis of the alternative conditions (4$'$) and (4m) in Proposition 4.4.2.

For every classical non-normal modal logic **E***, the corresponding labelled sequent calculus **LS.E*** is defined according to the table in Figure 5.2. Essentially, for each modal axiom X we just need to consider the corresponding rule X in the calculus. The only exceptions are the axiom D, which is covered by the two rules D_1 and D_2, and, for every n, the rule RD_n^+, which is covered by the rules D_i^+ for every $1 \leq i \leq n$. In both cases this choice of rules is required for ensuring the admissibility of contraction. Observe in particular that the rule D_1 corresponds to the semantic condition $(\emptyset, \emptyset) \notin \mathcal{N}(w)$, which is satisfied by every D-model.

We can show that the characteristic axioms of classical non-normal modal logics are derivable in the labelled calculi containing the corresponding rules. As a preliminary step, we need to show that initial sequents init can be generalised to arbitrary labelled formulas ϕ that can

Propositional rules

$$\mathsf{init}\ \frac{}{x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \mathsf{L}\bot\ \frac{}{x : \bot, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{\Gamma \Rightarrow \Delta, x : \top}$$

$$\mathsf{L}\wedge\ \frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}$$

$$\mathsf{L}\rightarrow\ \frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\rightarrow\ \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \rightarrow B}$$

Rules for local forcing

$$\mathsf{L}\Vdash^{\forall}\ \frac{x \in \mathsf{t}, x : A, \mathsf{t} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in \mathsf{t}, \mathsf{t} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \qquad \mathsf{R}\Vdash^{\forall}\ \frac{y \in \mathsf{t}, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\forall} A}\ (y!)$$

$$\mathsf{L}\Vdash^{\exists}\ \frac{y \in \mathsf{t}, y : A, \Gamma \Rightarrow \Delta}{\mathsf{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ (y!) \qquad \mathsf{R}\Vdash^{\exists}\ \frac{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, x : A, \mathsf{t} \Vdash^{\exists} A}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\exists} A}$$

Rules for box

$$\mathsf{L}\Box\ \frac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{x : \Box A, \Gamma \Rightarrow \Delta}\ (a!)$$

$$\mathsf{R}\Box\ \frac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t \Vdash^{\forall} A \qquad t \rhd x, \bar{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}$$

Rules for neighbourhood terms

$$\mathsf{dec}\ \frac{x \in t, x \in s, x \in ts, \Gamma \Rightarrow \Delta}{x \in ts, \Gamma \Rightarrow \Delta} \qquad \overline{\mathsf{dec}}\ \frac{x \in \bar{t}, x \in \overline{ts}, \Gamma \Rightarrow \Delta \qquad x \in \bar{s}, x \in \overline{ts}, \Gamma \Rightarrow \Delta}{x \in \overline{ts}, \Gamma \Rightarrow \Delta}$$

$$\overline{\tau}^{\emptyset}\ \frac{}{x \in \overline{\tau}, \Gamma \Rightarrow \Delta} \qquad \mathsf{J}\ \frac{t \rhd x, x \in J(t), \Gamma \Rightarrow \Delta}{x \in J(t), \Gamma \Rightarrow \Delta}$$

$$\overline{\mathsf{J}}\ \frac{y \in t, y \in \bar{s}, x \in \overline{J(t)}, s \rhd x, \Gamma \Rightarrow \Delta \qquad y \in \bar{t}, y \in s, x \in \overline{J(t)}, s \rhd x, \Gamma \Rightarrow \Delta}{x \in \overline{J(t)}, s \rhd x, \Gamma \Rightarrow \Delta}\ (y!)$$

Rules for the classical cube and further extensions

$$\mathsf{M}\ \frac{}{t \rhd x, y \in \bar{t}, \Gamma \Rightarrow \Delta} \qquad \mathsf{N}\ \frac{\tau \rhd x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (x \text{ in } \Gamma \cup \Delta) \qquad \mathsf{C}\ \frac{ts \rhd x, t \rhd x, s \rhd x, \Gamma \Rightarrow \Delta}{t \rhd x, s \rhd x, \Gamma \Rightarrow \Delta}$$

$$\mathsf{T}\ \frac{t \rhd x, x \in t, \Gamma \Rightarrow \Delta}{t \rhd x, \Gamma \Rightarrow \Delta} \qquad \mathsf{P}\ \frac{t \rhd x, y \in t, \Gamma \Rightarrow \Delta}{t \rhd x, \Gamma \Rightarrow \Delta}\ (y!) \qquad \mathsf{4}\ \frac{J(t) \rhd x, t \rhd x, \Gamma \Rightarrow \Delta}{t \rhd x, \Gamma \Rightarrow \Delta}$$

$$\mathsf{D}_n^+\ \frac{t_1 \rhd x, ..., t_n \rhd x, y \in t_1, ..., y \in t_n, \Gamma \Rightarrow \Delta}{t_1 \rhd x, ..., t_n \rhd x, \Gamma \Rightarrow \Delta}\ (y!)\ (n \geq 1)$$

$$\mathsf{D}_1\ \frac{t \rhd x, y \in t, \Gamma \Rightarrow \Delta \qquad t \rhd x, y \in \bar{t}, \Gamma \Rightarrow \Delta}{t \rhd x, \Gamma \Rightarrow \Delta}\ (y!)$$

$$\mathsf{D}_2\ \frac{t \rhd x, s \rhd x, y \in t, y \in s, \Gamma \Rightarrow \Delta \qquad t \rhd x, s \rhd x, y \in \bar{t}, y \in \bar{s}, \Gamma \Rightarrow \Delta}{t \rhd x, s \rhd x, \Gamma \Rightarrow \Delta}\ (y!)$$

Figure 5.1: Rules of labelled sequent calculi $\mathbf{LS.E}^*$.

$\mathbf{LS.E} :=$ {propositional rules} $\cup$ {rules for local forcing} $\cup$ {rules for box}.	
$\mathbf{LS.M}^* := \mathbf{LS.E}^* \cup \{\mathsf{M}\}$.	$\mathbf{LS.EP}^* := \mathbf{LS.E}^* \cup \{\mathsf{P}\}$.
$\mathbf{LS.EN}^* := \mathbf{LS.E}^* \cup \{\mathsf{N}, \overline{\tau^{\emptyset}}\}$.	$\mathbf{LS.ED}^* := \mathbf{LS.E}^* \cup \{\mathsf{D}_1, \mathsf{D}_2\}$.
$\mathbf{LS.EC}^* := \mathbf{LS.E}^* \cup \{\mathsf{C}, \mathsf{dec}, \overline{\mathsf{dec}}\}$.	$\mathbf{LS.ED}_{\mathbf{n}}^{+*} := \mathbf{LS.E}^* \cup \{\mathsf{D}_1^+, \mathsf{D}_2^+, ..., \mathsf{D}_n^+\}$.
$\mathbf{LS.ET}^* := \mathbf{LS.E}^* \cup \{\mathsf{T}\}$.	$\mathbf{LS.E4}^* := \mathbf{LS.E}^* \cup \{\mathsf{4}, \mathsf{J}, \overline{\mathsf{J}}\}$.

Figure 5.2: Labelled sequent calculi $\mathbf{LS.E}^*$.

occur in both sides of sequents. The proof is based on the following definition of weight of labelled formulas.

Definition 5.1.4 (Weight of labelled formulas)**.** The *weight* $wg(\phi)$ of a labelled formula ϕ of the form $x : A$, $\mathsf{t} \Vdash^{\forall} A$ or $\mathsf{t} \Vdash^{\exists} A$ is the pair $(wg(f(\phi)), wg(l(\phi)))$, where $f(\phi)$ and $l(\phi)$ are, respectively, the $\mathcal{L}$ formula A and the world label or neighbourhood term occurring in ϕ. We define $wg(x) = 0$, $wg(a) = 1$, $wg(a_1...a_n) = wg(a_1) + ... + wg(a_n)$, $wg(J(t)) = wg(t) + 1$, $wg(\overline{t}) = wg(t)$. Moreover, we define $wg(\bot) = wg(\top) = wg(p) = 0$, $wg(A \circ B) = wg(A) + wg(B) + 1$ for $\circ \in \{\wedge, \vee, \rightarrow\}$, and $wg(\Box A) = wg(A) + 1$. For labelled formulas ϕ of the form $x \in \mathsf{t}$ or $t \rhd x$ we establish $wg(\phi) = (0, 0)$. We consider weights of formulas lexicographically ordered.

Proposition 5.1.1. Every sequent of the form $\phi, \Gamma \Rightarrow \Delta, \phi$ is derivable in $\mathbf{LS.E}^*$, where ϕ is any formula that can occur in both sides of sequents.

Proof. By induction on the weight of ϕ. If $wg(\phi) = (0, 0)$, then ϕ has the form $x : p$, or $x : \bot$, or $x : \top$. Then $\phi, \Gamma \Rightarrow \Delta, \phi$ is an initial sequent. If $wg(\phi) = (n, m)$, with $n > 0$ or $m > 0$, then by Definition 5.1.3, ϕ has the form $\mathsf{t} \Vdash^{\forall} A$, or $\mathsf{t} \Vdash^{\exists} A$, or $x : B \circ C$ with $\circ \in \{\wedge, \vee, \rightarrow\}$, or $x : \Box B$. We show some illustrative cases. If $\phi = t \Vdash^{\forall} A$ the derivation is as follows, where $y : A, y \in t, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, y : A$ is derivable by i.h. because $wg(y : A) < wg(t \Vdash^{\forall} A)$.

$$\dfrac{\dfrac{y : A, y \in t, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, y : A}{y \in t, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, y : A}\ \mathsf{L} \Vdash^{\forall}}{t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A}\ \mathsf{R} \Vdash^{\forall}$$

If $\phi = x : B \wedge C$ the derivation is as follows, where $x : B, x : C, \Gamma \Rightarrow \Delta, x : B$ is derivable by i.h. because $wg(x : B) < wg(x : B \wedge C)$.

$$\dfrac{\dfrac{x : B, x : C, \Gamma \Rightarrow \Delta, x : B \qquad x : B, x : C, \Gamma \Rightarrow \Delta, x : C}{x : B, x : C, \Gamma \Rightarrow \Delta, x : B \wedge C}\ \mathsf{R}\wedge}{x : B \wedge C, \Gamma \Rightarrow \Delta, x : B \wedge C}\ \mathsf{L}\wedge$$

If $\phi = x : \Box B$ the derivation is as follows, where $a \rhd x, a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta, x : \Box B, \overline{a} \Vdash^{\exists} B, a \Vdash^{\forall} B$ is derivable by i.h. because $wg(a \Vdash^{\forall} B) = wg(\overline{a} \Vdash^{\exists} B) < wg(x : \Box B)$.

$$\dfrac{\dfrac{a \rhd x, a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta, x : \Box B, \overline{a} \Vdash^{\exists} B, a \Vdash^{\forall} B \qquad a \rhd x, a \Vdash^{\forall} B, \overline{a} \Vdash^{\exists} B, \Gamma \Rightarrow \Delta, x : \Box B, \overline{a} \Vdash^{\exists} B}{a \rhd x, a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta, x : \Box B, \overline{a} \Vdash^{\exists} B}\,\mathsf{R}\Box}{x : \Box B, \Gamma \Rightarrow \Delta, x : \Box B}\,\mathsf{L}\Box$$

□

The derivations of the characteristic axioms and rules of classical non-normal modal logics are displayed in Figure 5.2. For the derivation of rule RE we assume that $y : A, \Gamma \Rightarrow \Delta, y : B$ and $y : B, \Gamma \Rightarrow \Delta, y : A$ are derivable in $\mathbf{LS}.\mathbf{E}^*$ for every world label y, and for the derivation of rule RD_n^+ we assume that $y : A_1, y : A_2, ..., y : A_n, \Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LS}.\mathbf{ED}_{\mathbf{n}}^{+*}$ for every world label y.

We now show that the calculi $\mathbf{LS}.\mathbf{E}^*$ are sound with respect to the corresponding classes of bi-neighbourhood models. In order to interpret the labelled sequents in bi-neighbourhood models we first need to interpret the world labels and the neighbourhood terms. For this purpose we introduce the notion of realisation.

Definition 5.1.5 (Realisation)**.** Given a bi-neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, a *realisation* is a pair of functions (ρ, σ), where $\rho : \mathbb{WL} \longrightarrow \mathcal{W}$, and $\sigma : \mathbb{NT} \longrightarrow \mathcal{P}(\mathcal{W})$. Moreover, σ satisfies the following conditions.

- $\sigma(\overline{\tau}) = \emptyset$; and $\sigma(\tau) = \alpha$ for a set α such that $(\alpha, \emptyset) \in \mathcal{N}(w)$ for all $w \in \mathcal{W}$.
- $\sigma(ts) = \sigma(t) \cap \sigma(s)$; and $\sigma(\overline{ts}) = \sigma(\overline{t}) \cup \sigma(\overline{s})$.
- $\sigma(J(t)) = \curlywedge^{+}(\sigma(t), \sigma(\overline{t}))$; and $\sigma(\overline{J(t)}) = \curlywedge^{-}(\sigma(t), \sigma(\overline{t}))$.

Moreover, for systems containing the rules for M and 4 we add the following condition:

- If $\mathcal{M}$ is a M4m-model, then $\sigma(\overline{t}) = \emptyset$ for every negative term $\overline{t}$.

Definition 5.1.6 (Semantic interpretation)**.** For every formula ϕ of $\mathcal{L}_{lab}$, the relation $\mathcal{M} \models_{\rho,\sigma} \phi$ is defined by cases as follows.

$$\begin{array}{lll}
\mathcal{M} \models_{\rho,\sigma} x \in \mathsf{t} & \text{iff} & \rho(x) \in \sigma(\mathsf{t}). \\
\mathcal{M} \models_{\rho,\sigma} x : A & \text{iff} & \mathcal{M}, \rho(x) \Vdash A. \\
\mathcal{M} \models_{\rho,\sigma} \mathsf{t} \Vdash^{\forall} A & \text{iff} & \text{for every } w \in \sigma(\mathsf{t}),\ \mathcal{M}, w \Vdash A. \\
\mathcal{M} \models_{\rho,\sigma} \mathsf{t} \Vdash^{\exists} A & \text{iff} & \text{there is } w \in \sigma(\mathsf{t}) \text{ such that } \mathcal{M}, w \Vdash A. \\
\mathcal{M} \models_{\rho,\sigma} t \rhd x & \text{iff} & (\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x)).
\end{array}$$

Then, given a sequent $\Gamma \Rightarrow \Delta$ we stipulate that $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if in case $\mathcal{M} \models_{\rho,\sigma} \phi$ for every formula ϕ in Γ, then $\mathcal{M} \models_{\rho,\sigma} \psi$ for some formula ψ in Δ. Moreover, we say that $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$ if for every realisation (ρ, σ), $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$.

Theorem 5.1.2 (Soundness)**.** If $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{LS}.\mathbf{E}^*$, then it is valid in the corresponding class of bi-neighbourhood models.

(RE)

$$
\begin{prooftree}
\AxiomC{$y : A, y \in a, a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A, y : B$}
\LeftLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$y \in a, a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A, y : B$}
\LeftLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A, a : B$}
\AxiomC{$y \in \overline{a}, y : B, a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A, y : A$}
\RightLabel{R $\Vdash^{\exists}$}
\UnaryInfC{$y \in \overline{a}, y : B, a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A$}
\RightLabel{L $\Vdash^{\exists}$}
\UnaryInfC{$\overline{a} \Vdash^{\exists} B, a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A$}
\RightLabel{R$\Box$}
\BinaryInfC{$a \rhd x, a \Vdash^{\forall} A \Rightarrow x : \Box B, \overline{a} \Vdash^{\exists} A$}
\RightLabel{L$\Box$}
\UnaryInfC{$x : \Box A \Rightarrow x : \Box B$}
\end{prooftree}
$$

(M)

$$
\begin{prooftree}
\AxiomC{$..., y : A, y : B, y \in a, a \Vdash^{\forall} A \wedge B \Rightarrow y : A, ...$}
\LeftLabel{L$\wedge$}
\UnaryInfC{$..., y : A \wedge B, y \in a, a \Vdash^{\forall} A \wedge B \Rightarrow y : A, ...$}
\LeftLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$..., y \in a, a \Vdash^{\forall} A \wedge B \Rightarrow y : A, ...$}
\LeftLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$..., a \Vdash^{\forall} A \wedge B \Rightarrow a \Vdash^{\forall} A, ...$}
\AxiomC{}
\RightLabel{M}
\UnaryInfC{$..., y \in \overline{a}, y : A, a \rhd x \Rightarrow ...$}
\RightLabel{L $\Vdash^{\exists}$}
\UnaryInfC{$..., \overline{a} \Vdash^{\exists} A, a \rhd x \Rightarrow ...$}
\RightLabel{R$\Box$}
\BinaryInfC{$a \rhd x, a \Vdash^{\forall} A \wedge B \Rightarrow x : \Box A, \overline{a} \Vdash^{\exists} A \wedge B$}
\RightLabel{L$\Box$}
\UnaryInfC{$x : \Box(A \wedge B) \Rightarrow x : \Box A$}
\end{prooftree}
$$

(N)

$$
\begin{prooftree}
\AxiomC{$\tau \rhd x, y \in \tau \Rightarrow x : \Box\top, y : \top$}
\LeftLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$\tau \rhd x \Rightarrow x : \Box\top, \tau : \top$}
\AxiomC{}
\RightLabel{$\overline{\tau}^{\emptyset}$}
\UnaryInfC{$\tau \rhd x, y \in \overline{\tau}, y : \top \Rightarrow x : \Box\top$}
\RightLabel{L $\Vdash^{\exists}$}
\UnaryInfC{$\tau \rhd x, \overline{\tau} \Vdash^{\exists} \top \Rightarrow x : \Box\top$}
\RightLabel{R$\Box$}
\BinaryInfC{$\tau \rhd x \Rightarrow x : \Box\top$}
\RightLabel{N}
\UnaryInfC{$\Rightarrow x : \Box\top$}
\end{prooftree}
$$

(C)

$$
\begin{prooftree}
\AxiomC{$..., y : A, y : B \Rightarrow x : \Box(A \wedge B), y : A$}
\AxiomC{$..., y : A, y : B \Rightarrow x : \Box(A \wedge B), y : B$}
\LeftLabel{R$\wedge$}
\BinaryInfC{$..., y : A, y : B, y \in a, y \in b, y \in ab, a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, ...$}
\LeftLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$..., y : A, y \in a, y \in b, y \in ab, a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, ...$}
\LeftLabel{L $\Vdash^{\forall}$}
\UnaryInfC{$..., y \in a, y \in b, y \in ab, a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, ...$}
\LeftLabel{dec}
\UnaryInfC{$..., y \in ab, a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, ...$}
\LeftLabel{R $\Vdash^{\forall}$}
\UnaryInfC{$..., a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow ab \Vdash^{\forall} A \wedge B, ...$}
\AxiomC{$..., y \in \overline{a}, y : A \Rightarrow \overline{a} \Vdash^{\exists} A, y : A...$}
\LeftLabel{R $\Vdash^{\exists}$}
\UnaryInfC{$..., y \in \overline{a}, y : A \Rightarrow \overline{a} \Vdash^{\exists} A...$}
\AxiomC{$..., y \in \overline{b}, y : B, \Rightarrow \overline{b} \Vdash^{\exists} B, y : B, ...$}
\RightLabel{R $\Vdash^{\exists}$}
\UnaryInfC{$..., y \in \overline{b}, y : B, \Rightarrow \overline{b} \Vdash^{\exists} B, ...$}
\RightLabel{dec}
\BinaryInfC{$..., y \in \overline{ab}, y : A, y : B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, ...$}
\RightLabel{L$\wedge$}
\UnaryInfC{$..., y \in \overline{ab}, y : A \wedge B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, ...$}
\RightLabel{L $\Vdash^{\exists}$}
\UnaryInfC{$..., \overline{ab} \Vdash^{\exists} A \wedge B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, ...$}
\RightLabel{R$\Box$}
\BinaryInfC{$ab \rhd x, a \rhd x, a \Vdash^{\forall} A, b \rhd x, b \Vdash^{\forall} B \Rightarrow x : \Box(A \wedge B), \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B$}
\RightLabel{C}
\UnaryInfC{$a \rhd x, a \Vdash^{\forall} A, b \rhd x, b \Vdash^{\forall} B \Rightarrow x : \Box(A \wedge B), \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B$}
\RightLabel{L$\Box$}
\UnaryInfC{$a \rhd x, a \Vdash^{\forall} A, x : \Box B \Rightarrow x : \Box(A \wedge B), \overline{a} \Vdash^{\exists} A$}
\RightLabel{L$\Box$}
\UnaryInfC{$x : \Box A, x : \Box B \Rightarrow x : \Box(A \wedge B)$}
\RightLabel{L$\wedge$}
\UnaryInfC{$x : \Box A \wedge \Box B \Rightarrow x : \Box(A \wedge B)$}
\end{prooftree}
$$

$$(T)\qquad \dfrac{\dfrac{\dfrac{x:A, x\in a, a\rhd x, a\Vdash^{\forall} A\Rightarrow x:A, \overline{a}\Vdash^{\exists} A}{x\in a, a\rhd x, a\Vdash^{\forall} A\Rightarrow x:A, \overline{a}\Vdash^{\exists} A}\;\mathsf{L}\Vdash^{\forall}}{a\rhd x, a\Vdash^{\forall} A\Rightarrow x:A, \overline{a}\Vdash^{\exists} A}\;\mathsf{T}}{x:\Box A\Rightarrow x:A}\;\mathsf{L}\Box$$

$$(P)\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{y:\bot, y\in a, a\rhd x, a\Vdash^{\forall}\bot\Rightarrow \overline{a}\Vdash^{\exists}\bot}\;\mathsf{L}\bot}{y\in a, a\rhd x, a\Vdash^{\forall}\bot\Rightarrow \overline{a}\Vdash^{\exists}\bot}\;\mathsf{L}\Vdash^{\forall}}{a\rhd x, a\Vdash^{\forall}\bot\Rightarrow \overline{a}\Vdash^{\exists}\bot}\;\mathsf{P}}{x:\Box\bot\Rightarrow}\;\mathsf{L}\Box$$

(D)

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{L}\Vdash^{\forall}\;\dfrac{\mathsf{L}\Vdash^{\forall}\;\dfrac{\mathsf{L}\neg\;\dfrac{y:A, y\in a, y\in b, a\rhd x, a\Vdash^{\forall} A, b\rhd x, b\Vdash^{\forall}\neg A\Rightarrow y:A, \ldots}{y:A, y:\neg A, y\in a, y\in b, a\rhd x, a\Vdash^{\forall} A, b\rhd x, b\Vdash^{\forall}\neg A\Rightarrow \ldots}}{y:A, y\in a, y\in b, a\rhd x, a\Vdash^{\forall} A, b\rhd x, b\Vdash^{\forall}\neg A\Rightarrow \ldots}}{y\in a, y\in b, a\rhd x, a\Vdash^{\forall} A, b\rhd x, b\Vdash^{\forall}\neg A\Rightarrow \ldots}\qquad \dfrac{\dfrac{\dfrac{\ldots, y:A, y\in \overline{a}, y\in\overline{b}\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists}\neg A, y:A}{\ldots, y\in \overline{a}, y\in\overline{b}\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists}\neg A, y:A, y:\neg A}\;\mathsf{R}\neg}{\ldots, y\in \overline{a}, y\in\overline{b}\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists}\neg A, y:A}\;\mathsf{R}\Vdash^{\exists}}{\ldots, y\in \overline{a}, y\in\overline{b}\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists}\neg A}\;\mathsf{R}\Vdash^{\exists}}{a\rhd x, a\Vdash^{\forall} A, b\rhd x, b\Vdash^{\forall}\neg A\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists}\neg A}\;\mathsf{D}_2}{a\rhd x, a\Vdash^{\forall} A, x:\Box\neg A\Rightarrow \overline{a}\Vdash^{\exists} A}\;\mathsf{L}\Box}{x:\Box A, x:\Box\neg A\Rightarrow}\;\mathsf{L}\Box}{x:\Box A\wedge\Box\neg A\Rightarrow}\;\mathsf{L}\wedge$$

$$(RD_n^+)\qquad \dfrac{\dfrac{\dfrac{\dfrac{y:A_1,\ldots,y:A_n, y\in a_1,\ldots,y\in a_n, a_1\rhd x,\ldots,a_n\rhd x, a_1\Vdash^{\forall} A_1,\ldots,a_n\Vdash^{\forall} A_n\Rightarrow \ldots}{y\in a_1,\ldots,y\in a_n, a_1\rhd x,\ldots,a_n\rhd x, a_1\Vdash^{\forall} A_1,\ldots,a_n\Vdash^{\forall} A_n\Rightarrow \ldots}\;\mathsf{L}\Vdash^{\forall}\times n}{a_1\rhd x,\ldots,a_n\rhd x, a_1\Vdash^{\forall} A_1,\ldots,a_n\Vdash^{\forall} A_n\Rightarrow \overline{a_1}\Vdash^{\exists} A_1,\ldots,\overline{a_n}\Vdash^{\exists} A_n}\;\mathsf{D}_n^+}{x:\Box A_1,\ldots,x:\Box A_n\Rightarrow}\;\mathsf{L}\Box\times n}{x:\Box A_1\wedge\ldots\wedge\Box A_n\Rightarrow}\;\mathsf{L}\wedge\times n$$

$$(4)\qquad \dfrac{\dfrac{\dfrac{\mathsf{R}\Vdash^{\forall}\;\dfrac{\mathsf{J}\;\dfrac{\mathsf{R}\Box\;\dfrac{\ldots, a\rhd y, a\Vdash^{\forall} A\Rightarrow y:\Box A, a\Vdash^{\forall} A,\ldots\qquad \ldots,\overline{a}\Vdash^{\exists} A, a\rhd y\Rightarrow y:\Box A, \overline{a}\Vdash^{\exists} A,\ldots}{\ldots, a\rhd y, y\in J(a), a\Vdash^{\forall} A\Rightarrow y:\Box A, \overline{a}\Vdash^{\exists} A,\ldots}}{\ldots, y\in J(a), a\Vdash^{\forall} A\Rightarrow y:\Box A, \overline{a}\Vdash^{\exists} A,\ldots}}{\ldots, a\Vdash^{\forall} A, \Rightarrow J(a)\Vdash^{\forall}\Box A, \overline{a}\Vdash^{\exists} A,\ldots}\qquad \dfrac{\dfrac{\dfrac{\mathsf{L}\Vdash^{\forall}\;\dfrac{\mathsf{R}\Vdash^{\exists}\;\dfrac{\ldots, z:A\Rightarrow z:A,\ldots}{\ldots, z:A, z\in\overline{b}\Rightarrow \overline{b}\Vdash^{\exists} A,\ldots}}{\ldots, z\in a, z\in\overline{b}, a\Vdash^{\forall} A\Rightarrow \overline{b}\Vdash^{\exists} A,\ldots}\qquad \dfrac{\dfrac{\ldots, z:A\Rightarrow z:A,\ldots}{\ldots, z\in b, b\Vdash^{\forall} A\Rightarrow z:A,\ldots}\;\mathsf{L}\Vdash^{\forall}}{\ldots, z\in\overline{a}, z\in b, b\Vdash^{\forall} A\Rightarrow \overline{a}\Vdash^{\exists} A,\ldots}\;\mathsf{R}\Vdash^{\exists}}{\ldots, y\in\overline{J(a)}, b\rhd y, b\Vdash^{\forall} A, a\Vdash^{\forall} A\Rightarrow \overline{a}\Vdash^{\exists} A, \overline{b}\Vdash^{\exists} A,\ldots}\;\overline{\mathsf{J}}}{\dfrac{\ldots, y\in\overline{J(a)}, y:\Box A, a\Vdash^{\forall} A\Rightarrow \overline{a}\Vdash^{\exists} A,\ldots}{\ldots, \overline{J(a)}\Vdash^{\exists}\Box A, a\Vdash^{\forall} A\Rightarrow \overline{a}\Vdash^{\exists} A,\ldots}\;\mathsf{L}\Vdash^{\exists}}\;\mathsf{L}\Box}{J(a)\rhd x, a\rhd x, a\Vdash^{\forall} A, \Gamma\Rightarrow\Delta, x:\Box\Box A, \overline{a}\Vdash^{\exists} A}\;\mathsf{R}\Box}{\dfrac{a\rhd x, a\Vdash^{\forall} A, \Gamma\Rightarrow\Delta, x:\Box\Box A, \overline{a}\Vdash^{\exists} A}{x:\Box A, \Gamma\Rightarrow\Delta, x:\Box\Box A}\;\mathsf{L}\Box}\;\mathsf{4}$$

Figure 5.2: Derivation of modal axioms and rules

Proof. We show that the initial sequents are valid and that every rule is sound in the corresponding class of models. The cases of propositional rules are standard. We consider $\mathsf{R} \Vdash^{\forall}$ as an example of rule for local forcing, and then we show the proof for the modal rules.

($\mathsf{R} \Vdash^{\forall}$) Assume the premiss $y \in \mathsf{t}, \Gamma \Rightarrow \Delta, y : A$ valid, $\mathcal{M} \models_{\rho,\sigma} \Gamma$, and, by contradiction, $\mathcal{M} \not\models_{\rho,\sigma} \phi$ for every ϕ in Δ and $\mathcal{M} \not\models_{\rho,\sigma} \mathsf{t} \Vdash^{\forall} A$. Then there is $w \in \sigma(\mathsf{t})$ such that $\mathcal{M}, w \nVdash A$. Since y is fresh in the application of $\mathsf{R} \Vdash^{\forall}$, we can extend ρ to ρ' by choosing $\rho'(y) = w$. Then $\mathcal{M} \models_{\rho',\sigma} y \in \mathsf{t}$ and, since y is not in Γ, $\mathcal{M} \models_{\rho',\sigma} \Gamma$. By the validity of the premiss, $\mathcal{M} \models_{\rho',\sigma} \phi$ for some ϕ in Δ, or $\mathcal{M} \models_{\rho',\sigma} y : A$. In the first case, $\mathcal{M} \models_{\rho,\sigma} \phi$ because y is not in ϕ. In the second case, $\rho'(y) = w \Vdash A$. Then we have a contradiction.

($\mathsf{L}\Box$) Assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} x : \Box A, \Gamma$. By definition there is a pair $(\alpha, \beta) \in \mathcal{N}(\rho(x))$ such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$. Since a is fresh in the application of $\mathsf{L}\Box$, we can extend σ to σ' by choosing $\sigma'(a) = \alpha$ and $\sigma'(\overline{a}) = \beta$. Thus $\mathcal{M} \models_{\rho,\sigma'} a \rhd x, a \Vdash^{\forall} A$ and $\mathcal{M} \not\models_{\rho,\sigma'} \overline{a} \Vdash^{\exists} A$. Moreover, since a is not in Γ, $\mathcal{M} \models_{\rho,\sigma'} \Gamma$. By the validity of the premiss we have either (i) $\mathcal{M} \models_{\rho,\sigma'} \overline{a} \Vdash^{\exists} A$, or (ii) $\mathcal{M} \models_{\rho,\sigma'} \phi$ for a formula ϕ in Δ. (i) gives a contradiction, then (ii) holds. Thus, since a is not in ϕ, we have $\mathcal{M} \models_{\rho,\sigma} \phi$.

($\mathsf{R}\Box$) Assume the premisses valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, \Gamma$. By the first premiss we have $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ, or $\mathcal{M} \models_{\rho,\sigma} x : \Box A$, or $\mathcal{M} \models_{\rho,\sigma} t \Vdash^{\forall} A$. In the first two cases we are done. In the third case, consider the following two alternatives. (i) $\sigma(\overline{t}) \cap [\![A]\!] = \emptyset$, and (ii) $\sigma(\overline{t}) \cap [\![A]\!] \neq \emptyset$. If (i), then since $\sigma(t) \subseteq [\![A]\!]$ and $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$, we have $\mathcal{M} \models_{\rho,\sigma} x : \Box A$. If (ii), then $\mathcal{M} \models_{\rho,\sigma} \overline{t} \Vdash^{\exists} A$, thus by the second premiss $\mathcal{M} \models_{\rho,\sigma} x : \Box A$ or $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

($\overline{\tau}^{\emptyset}$) For every $\mathcal{M}, \rho, \sigma$ we have $\mathcal{M} \not\models_{\rho,\sigma} x \in \overline{\tau}$, since by definition $\sigma(\overline{\tau}) = \emptyset$. Then $x \in \overline{\tau}, \Gamma \Rightarrow \Delta$ is valid.

(dec) Assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} x \in ts, \Gamma$. Then $\rho(x) \in \sigma(ts) = \sigma(t) \cap \sigma(s)$. Therefore $\mathcal{M} \models_{\rho,\sigma} x \in t, x \in s$. Then by the validity of the premiss we have $\mathcal{M} \models_{\rho,\sigma} \phi$ for a formula ϕ in Δ.

($\overline{\mathsf{dec}}$) Assume the premisses valid and $\mathcal{M} \models_{\rho,\sigma} x \in \overline{ts}, \Gamma$. Then $\rho(x) \in \sigma(\overline{ts}) = \sigma(\overline{t}) \cup \sigma(\overline{s})$, that is $\rho(x) \in \sigma(\overline{t})$ or $\rho(x) \in \sigma(\overline{s})$. Thus $\mathcal{M} \models_{\rho,\sigma} x \in \overline{t}$ or $\mathcal{M} \models_{\rho,\sigma} x \in \overline{s}$. Then by the validity of the premisses we have $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

(J) Assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} x \in J(t), \Gamma$. Then $\rho(x) \in \sigma(J(t))$. By definition, this means $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$. Thus $\mathcal{M} \models_{\rho,\sigma} t \rhd x$. Then by the validity of the premiss we have $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

($\overline{\mathsf{J}}$) Assume the premisses valid and $\mathcal{M} \models_{\rho,\sigma} x \in \overline{J(t)}, s \rhd x, \Gamma$. Then $(\sigma(s), \sigma(\overline{s})) \in \mathcal{N}(\rho(x))$, and $\rho(x) \in \mathcal{N}^{-}(\sigma(t), \sigma(\overline{t}))$, that is for all $(\gamma, \delta) \in \mathcal{N}(\rho(x))$, $\sigma(t) \cap \delta \neq \emptyset$ or $\sigma(\overline{t}) \cap \gamma \neq \emptyset$. Then in particular $\sigma(t) \cap \sigma(\overline{s}) \neq \emptyset$ or $\sigma(\overline{t}) \cap \sigma(s) \neq \emptyset$, that is there is $v \in \mathcal{W}$ such that $v \in \sigma(t)$ and $v \in \sigma(\overline{s})$, or $v \in \sigma(\overline{t})$ and $v \in \sigma(s)$. Since y is fresh we can extend ρ to

ρ' by choosing $\rho'(y) = v$. We then have $\mathcal{M} \models_{\rho',\sigma} y \in t, y \in \overline{s}, x \in \overline{J(t)}, s \rhd x, \Gamma$ or $\mathcal{M} \models_{\rho',\sigma} y \in \overline{t}, y \in s, x \in \overline{J(t)}, s \rhd x, \Gamma$. By the validity of the premisses this implies $\mathcal{M} \models_{\rho',\sigma} \phi$ for a ϕ in Δ, and since y is not in ϕ, $\mathcal{M} \models_{\rho,\sigma} \phi$.

(M) For every $\mathcal{M}, \rho, \sigma$ we have $\mathcal{M} \not\models_{\rho,\sigma} t \rhd x, y \in \overline{t}$, otherwise we would have $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$ and $\rho(y) \in \sigma(\overline{t})$, against condition (M). Then $t \rhd x, y \in \overline{t}, \Gamma \Rightarrow \Delta$ is valid.

(N) Suppose $\mathcal{M}$ is a N-model. Moreover, assume the premiss $\tau \rhd x, \Gamma \Rightarrow \Delta$ valid and $\mathcal{M} \models_{\rho,\sigma} \Gamma$. By condition (N), and due to the definition of $\sigma(\tau)$, there is a pair $(\alpha, \emptyset) \in \mathcal{N}(\rho(x))$ such that $\alpha = \sigma(\tau)$. Thus $\mathcal{M} \models_{\rho,\sigma} \tau \rhd x$, then by the validity of the premiss $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

(C) Suppose $\mathcal{M}$ is a C-model. Moreover, assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, s \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\overline{t})), (\sigma(s), \sigma(\overline{s})) \in \mathcal{N}(\rho(x))$. By condition (C), $(\sigma(t) \cap \sigma(s), \sigma(\overline{t}) \cup \sigma(\overline{s})) \in \mathcal{N}(\rho(x))$, that is $(\sigma(ts), \sigma(\overline{ts})) \in \mathcal{N}(\rho(x))$. Then by the validity of the premiss we have $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

(T) Suppose $\mathcal{M}$ is a T-model. Moreover, assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$. By condition (T), $\rho(x) \in \sigma(t)$. Thus $\mathcal{M} \models_{\rho,\sigma} x \in t$, and by the validity of the premiss, $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ.

(P) Suppose $\mathcal{M}$ is a P-model. Moreover, assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$. By condition (P), there is $w \in \sigma(t)$. Since y is fresh we can extend ρ to ρ' by choosing $\rho'(y) = w$. Then $\mathcal{M} \models_{\rho',\sigma} y \in t$ and, since y is fresh, $\mathcal{M} \models_{\rho',\sigma} t \rhd x, \Gamma$. By the validity of the premiss, this implies $\mathcal{M} \models_{\rho',\sigma} \phi$ for a ϕ in Δ. Then, since y is not in ϕ, we also have $\mathcal{M} \models_{\rho,\sigma} \phi$.

(D_1) Suppose $\mathcal{M}$ is a D-model. Moreover, assume the premisses valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\overline{t})) \in \mathcal{N}(\rho(x))$. As a consequence of condition (D) we have that $(\emptyset, \emptyset) \notin \mathcal{N}(\rho(x)$, that is, there is $w \in \mathcal{W}$ such that $w \in \sigma(t)$ or $w \in \sigma(\overline{t})$. Since y is fresh we can extend ρ to ρ' by choosing $\rho'(y) = w$. Then depending on the case we have $\mathcal{M} \models_{\rho',\sigma} y \in t$ or $\mathcal{M} \models_{\rho',\sigma} y \in \overline{t}$ and, since y is fresh, $\mathcal{M} \models_{\rho',\sigma} t \rhd x, \Gamma$. By the validity of the premisses we then obtain $\mathcal{M} \models_{\rho',\sigma} \phi$ for a ϕ in Δ, and since y is not in ϕ, $\mathcal{M} \models_{\rho,\sigma} \phi$.

(D_2) Suppose $\mathcal{M}$ is a D-model. Moreover, assume the premisses valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, s \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\overline{t})), (\sigma(s), \sigma(\overline{s})) \in \mathcal{N}(\rho(x))$. By condition (D), there is $w \in \mathcal{W}$ such that $w \in \sigma(t) \cap \sigma(s)$ or $w \in \sigma(\overline{t}) \cap \sigma(\overline{s})$. Since y is fresh we can extend ρ to ρ' by choosing $\rho'(y) = w$. Then depending on the case we have $\mathcal{M} \models_{\rho',\sigma} y \in t, y \in s$ or $\mathcal{M} \models_{\rho',\sigma} y \in \overline{t}, y \in \overline{s}$ and, since y is fresh, $\mathcal{M} \models_{\rho',\sigma} t \rhd x, s \rhd x, \Gamma$. By the validity of the premisses we then obtain $\mathcal{M} \models_{\rho',\sigma} \phi$ for a ϕ in Δ, and since y is not in ϕ, $\mathcal{M} \models_{\rho,\sigma} \phi$.

(D_n^+) Suppose $\mathcal{M}$ is a RD_n^+-model. Moreover, assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} t_1 \rhd x, ..., t_n \rhd x, \Gamma$. Then $(\sigma(t_1), \sigma(\overline{t_1}))$, ..., $(\sigma(t_n), \sigma(\overline{t_n})) \in \mathcal{N}(\rho(x))$. By condition ($\mathrm{RD}_n^+$), there is $w \in \mathcal{W}$ such that $w \in \sigma(t_1) \cap ... \cap \sigma(t_n)$. Since y is fresh we can extend ρ to ρ' by choosing $\rho'(y) = w$. Then $\mathcal{M} \models_{\rho',\sigma} y \in t_1, ..., y \in t_n$, and, since y is fresh, $\mathcal{M} \models_{\rho',\sigma} t_1 \rhd x, ..., t_n \rhd x, \Gamma$. By the validity of the premiss we then obtain $\mathcal{M} \models_{\rho',\sigma} \phi$ for a ϕ in Δ, and since y is not in ϕ, $\mathcal{M} \models_{\rho,\sigma} \phi$.

(4) Suppose $\mathcal{M}$ is a $4'$-model. Moreover, assume the premiss valid and $\mathcal{M} \models_{\rho,\sigma} t \rhd x, \Gamma$. Then $(\sigma(t), \sigma(\bar{t})) \in \mathcal{N}(\rho(x))$. If $\mathcal{M}$ is not a M-model, then by condition $(4')$, $(\mathcal{V}^+(\sigma(t), \sigma(\bar{t})), \mathcal{V}^-(\sigma(t), \sigma(\bar{t}))) \in \mathcal{N}(\rho(x))$, whereas if $\mathcal{M}$ is a M-model, then by condition (4m), $(\mathcal{V}^+(\sigma(t), \sigma(\bar{t})), \emptyset) \in \mathcal{N}(\rho(x))$. In both cases $(\sigma(J(t)), \sigma(\overline{J(t)})) \in \mathcal{N}(\rho(x))$, since $\sigma(\overline{J(t)}) = \emptyset$ in M4m-models. Thus $\mathcal{M} \models_{\rho,\sigma} J(t) \rhd x$. Then by the validity of the premiss, $\mathcal{M} \models_{\rho,\sigma} \phi$ for a ϕ in Δ. $\square$

Observe that all rules are also sound in standard models in which $\bar{t}$ is interpreted as the complement of t, with the exception of rule M, which is incompatible with such an interpretation. In what follows, we prove the main structural properties of our labelled calculi, most importantly admissibility of cut, from which we obtain the syntactic completeness of the calculi.

5.2 Structural properties and syntactic completeness

In this section we investigate the structural properties of the labelled calculi $\mathbf{LS.E}^*$, in particular we show that the structural rules of weakening, contraction and cut are admissible. As a preliminary step, we consider the following operations of substitution of world labels and substitution of neighbourhood terms, and show that they preserve derivability of sequents.

Definition 5.2.1 (Substitutions of world labels and neighbourhood terms)**.** Substitution of world labels is defined as follows:

$$x(y/z) = \begin{cases} y & \text{if } z = x, \\ x & \text{if } z \neq x. \end{cases}$$

Moreover, the operation of substitutions of a *positive* term for a neighbourhood label inside a term is recursively defined as follows:

$$a(t/b) = \begin{cases} t & \text{if } b = a, \\ a & \text{if } b \neq a; \end{cases} \qquad \tau(t/b) = \tau; \qquad (sr)(t/b) = (s(t/b))(r(t/b));$$

$$(J(s))(t/b) = J(s(t/b)); \qquad \bar{s}(t/b) = \overline{s(t/b)}.$$

Substitutions of world labels and of neighbourhood terms are extended to formulas of $\mathcal{L}_{lab}$ and to multisets of formulas in the obvious way, for instance $(x : A)(y/z) = x(y/z) : A$; $(s \Vdash^{\forall} A)(t/b) = s(t/b) \Vdash^{\forall} A$; and $(\phi_1, ..., \phi_n)(y/z) = \phi_1(y/z), ..., \phi_n(y/z)$.

Recall that a rule $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}$ is *height-preserving* admissible if in case the premiss $\Gamma \Rightarrow \Delta$ is derivable, then the conclusion $\Gamma' \Rightarrow \Delta'$ is derivable with a derivation of at most the same height. We write $\vdash_n \Gamma \Rightarrow \Delta$ to denote that $\Gamma \Rightarrow \Delta$ has a derivation with height at most n. We might sometimes abbreviate "height-preserving" with "hp-".

Proposition 5.2.1. Substitution of world labels and substitution of neighbourhood terms are height-preserving admissible in $\mathbf{LS.E}^*$, that is, for every x, y, a, t, if $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(x/y) \Rightarrow \Delta(x/y)$, and $\vdash_n \Gamma(t/a) \Rightarrow \Delta(t/a)$.

Proof. The two claims are proved by induction in the height n of the derivation of $\Gamma \Rightarrow \Delta$. If $n = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$. Then it is easy to verify that both $\Gamma(x/y) \Rightarrow \Delta(x/y)$ and $\Gamma(t/a) \Rightarrow \Delta(t/a)$ are initial sequents or conclusions of M or $\overline{\tau}^{\emptyset}$. If $n \geq 1$, we consider the last rule applied in the derivation of $\Gamma \Rightarrow \Delta$. In most cases the proof is straightforward, we show as examples the cases where the last rule applied is $\mathsf{R} \Vdash^{\forall}$ or P for the substitution (x/y), and $\mathsf{L}\Box$, C, dec, $\overline{\mathsf{dec}}$, or J for the substitution (t/a).

• The last rule applied is $\mathsf{R} \Vdash^{\forall}$. If y does not occur in $\Gamma \Rightarrow \Delta$, then the substitution (x/y) is vacuous and $\Gamma(x/y) \Rightarrow \Delta(x/y)$ is derivable with height n by hypothesis. If instead y occurs in the conclusion we have

$$\dfrac{z \in t, \Gamma \Rightarrow \Delta', z : A}{\Gamma \Rightarrow \Delta', t \Vdash^{\forall} A}\ \mathsf{R} \Vdash^{\forall}$$

Since z is fresh in the application of $\mathsf{R} \Vdash^{\forall}$ and y occurs in the conclusion, z is different from y. In order to avoid clash of variables we use the inductive hypothesis twice and make two applications of hp-substitution to the premiss: the first one is to replace z with a world label u different form x and fresh with respect to the conclusion. The second one is to replace y with x. Since label u is fresh, we can successively apply $\mathsf{R} \Vdash^{\forall}$ and derive $\Gamma(x/y) \Rightarrow \Delta'(x/y), t \Vdash^{\forall} A$.

• The last rule applied is P. If y does not occur in $\Gamma \Rightarrow \Delta$, then the substitution (x/y) is vacuous and $\Gamma(x/y) \Rightarrow \Delta(x/y)$ is derivable with height n by hypothesis. If instead y occurs in the conclusion we have

$$\dfrac{t \rhd z, u \in t, \Gamma' \Rightarrow \Delta}{t \rhd z, \Gamma' \Rightarrow \Delta}\ \mathsf{P}$$

Since z is fresh in the application of P and y occurs in the conclusion, z is different from y. We proceed as before: first we replace u with a world label v different form x and fresh with respect to the conclusion. Then we replace y with x. Since label v is fresh, we can successively apply P and derive $t \rhd z(x/y), \Gamma'(x/y) \Rightarrow \Delta(x/y)$.

• The last rule applied is $\mathsf{L}\Box$. If a does not occur in $\Gamma \Rightarrow \Delta$, then the substitution (t/a) is vacuous and $\Gamma(t/a) \Rightarrow \Delta(t/a)$ is derivable with height n by hypothesis. If instead a occurs in the conclusion we have

$$\frac{b \rhd x, b \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta', \overline{b} \Vdash^{\exists} A}{x : \Box A, \Gamma' \Rightarrow \Delta} \textsf{L}\Box$$

Since b is fresh in the application of $\textsf{L}\Box$ and a occurs in the conclusion, b is different from a. By using the inductive hypothesis twice we make two applications of hp-substitution to the premiss. The first one is to replace b with a label c fresh both with respect to the conclusion and with respect to t, in order to avoid clash of variables in case b occurs in t. The second one is to replace a with t. Since label c is fresh, we can successively apply $\textsf{L}\Box$ and derive $x : \Box A, \Gamma'(t/a) \Rightarrow \Delta(t/a)$.

- The last rule applied is $\textsf{C}$.

$$\frac{sr \rhd x, s \rhd x, r \rhd x, \Gamma' \Rightarrow \Delta}{s \rhd x, r \rhd x, \Gamma' \Rightarrow \Delta} \textsf{C}$$

By inductive hypothesis $(sr)(t/a) \rhd x, s(t/a) \rhd x, r(t/a) \rhd x, \Gamma'(t/a) \Rightarrow \Delta(t/a)$ is derivable in $n-1$ steps. Since $(sr)(t/a) = s(t/a)r(t/a)$, we can apply $\textsf{C}$ and obtain $s(t/a) \rhd x, r(t/a) \rhd x, \Gamma'(t/a) \Rightarrow \Delta(t/a)$.

- The last rule applied is $\textsf{dec}$.

$$\frac{x \in s, x \in r, x \in sr, \Gamma' \Rightarrow \Delta}{x \in sr, \Gamma' \Rightarrow \Delta} \textsf{dec}$$

By inductive hypothesis $x \in s(t/a), x \in r(t/a), x \in (sr)(t/a), \Gamma'(t/a) \Rightarrow \Delta(t/a)$ is derivable in $n-1$ steps. Since $(sr)(t/a) = s(t/a)r(t/a)$, we can apply $\textsf{dec}$ and obtain $x \in sr(t/a), \Gamma'(t/a) \Rightarrow \Delta(t/a)$.

- The last rule applied is $\overline{\textsf{dec}}$.

$$\frac{x \in \overline{s}, x \in \overline{sr}, \Gamma' \Rightarrow \Delta \qquad x \in \overline{r}, x \in \overline{sr}, \Gamma' \Rightarrow \Delta}{x \in \overline{sr}, \Gamma' \Rightarrow \Delta} \overline{\textsf{dec}}$$

By inductive hypothesis $x \in \overline{s}(t/a), x \in \overline{sr}(t/a), \Gamma'(t/a) \Rightarrow \Delta(t/a)$ and $x \in \overline{r}(t/a), x \in \overline{sr}(t/a), \Gamma'(t/a) \Rightarrow \Delta(t/a)$ are derivable in $n-1$ steps. Since $\overline{sr}(t/a) = \overline{s(t/a)r(t/a)}$, we can apply $\overline{\textsf{dec}}$ and obtain $x \in \overline{\textsf{sr}}(t/a), \Gamma'(t/a) \Rightarrow \Delta(t/a)$.

- The last rule applied is $\textsf{J}$.

$$\frac{s \rhd x, x \in J(s), \Gamma' \Rightarrow \Delta}{s \rhd x, \Gamma' \Rightarrow \Delta} \textsf{J}$$

By inductive hypothesis $s(t/a) \rhd x, x \in (J(s))(t/a), \Gamma(t/a)' \Rightarrow \Delta(t/a)$ is derivable in $n-1$ steps. Since $(J(s))(t/a) = J(s(t/a))$, we can apply $\textsf{J}$ and obtain $s(t/a) \rhd x, \Gamma'(t/a) \Rightarrow \Delta(t/a)$. $\Box$

Lemma 5.2.2. The following rules of left and right weakening are height-preserving admissible in $\mathbf{LS.E}^*$, where ψ is any formula such that $\Gamma \Rightarrow \Delta, \psi$ respects the restrictions of Definition 5.1.3:

$$\mathsf{Lwk}\ \frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \qquad\qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \psi}$$

Proof. For both rules the proof is by induction on the height n of the derivation of the premiss. If $n = 0$, i.e., the premiss is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$, then the conclusion is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$. If $n \geq 1$, we consider the last rule applied in the derivation, let it be R. We modify the derivation as follows: first we apply the inductive hypothesis to the premiss of R, and then we apply R. For rules with fresh variable conditions we might need to apply height-preserving substitution of world labels or neighbourhood terms to the premiss of R before applying the inductive hypothesis. For instance, we consider the derivation below on the left, where y is fresh as a condition for the application of P, and $\mathcal{D}$ represents the derivation of $t \rhd x, y \in t, \Gamma \Rightarrow \Delta$. The derivation on the left is converted into the derivation on the right, where z is a fresh world label different from y, and $\mathcal{D}(z/y)$ represents the derivation obtained from $\mathcal{D}$ by replacing all occurrences of y with z.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \nabla \\ t \rhd x, y \in t, \Gamma \Rightarrow \Delta\end{array}}{t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{P}}{y : A, t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{Lwk} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}(z/y)\\ \nabla \\ t \rhd x, z \in t, \Gamma \Rightarrow \Delta\end{array}}{y : A, t \rhd x, z \in t, \Gamma \Rightarrow \Delta}\ \mathsf{Lwk}}{y : A, t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{P}$$

$\square$

Lemma 5.2.3. All rules of $\mathbf{LS.E}^{*}$ are height-preserving invertible.

Proof. For cumulative rules, i.e., rules where all formulas occurring in the conclusion also occur in the premiss(es), height-preserving invertibility is an immediate consequence of the height-preserving admissibility of weakening. For non-cumulative rules, i.e., the propositional rules and $\mathsf{R} \Vdash^{\forall}$, $\mathsf{L} \Vdash^{\exists}$, and $\mathsf{L}\Box$, the proof is by induction on the height of the derivation of the conclusion. We show as an example the proof for $\mathsf{L}\Box$. We have to show that

$$\frac{x : \Box A, \Gamma \Rightarrow \Delta}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}$$

is height-preserving admissible. The proof is by induction on the hight n of the derivation of $x : \Box A, \Gamma \Rightarrow \Delta$. If $n = 0$, then $a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A$ is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$. If $n \geq 1$ we consider the last rule application R in the derivation of $x : \Box A, \Gamma \Rightarrow \Delta$. If $x : \Box A$ is not principal in R, then we first apply the inductive hypothesis to the premiss of R and then we apply R. If instead $x : \Box A$ is principal, then R is $\mathsf{L}\Box$, and we have

$$\frac{b \rhd x, b \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{b} \Vdash^{\exists} A}{x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box$$

with b is fresh in the application of $\mathsf{L}\Box$. Then by height-preserving substitution of neighbourhood terms we replace b with a and obtain a derivation of height $n-1$ of $a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A$. $\Box$

Proposition 5.2.4. The following rules of left and right contraction are height-preserving admissible in $\mathbf{LS.E}^*$:

$$\mathsf{Lctr}\ \frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ , \qquad\qquad \mathsf{Rctr}\ \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\ .$$

Proof. We simultaneously prove height-preserving admissibility of Lctr and Rctr by mutual induction on the height of the derivation of their premiss. Let R be Lctr or Rctr. If the derivation of the premiss of R has height $n = 0$, i.e., the premiss is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$, then the conclusion is an initial sequent or a conclusion of M or $\overline{\tau}^{\emptyset}$. If $n \geq 1$, then let R' be the last rule applied in the derivation of the premiss of R. If the contracted formula ϕ is not principal in the application of R', then we modify the derivation by first applying the inductive hypothesis to the premiss(es) of R' and then applying R'. If in contrast ϕ is principal in the application of R', we distinguish two cases according whether R' is a cumulative rule.

• If R' is a cumulative we consider the following subcases. (i) If R' different from D_2 and D_n^+, then we modify the derivation by first applying the inductive hypothesis to the premiss(es) of R' and then applying R'. (ii) If R is D_2 we have

$$\dfrac{\dfrac{t \rhd x, t \rhd x, y \in t, y \in t, \Gamma \Rightarrow \Delta \qquad t \rhd x, t \rhd x, y \in \overline{t}, y \in \overline{t}, \Gamma \Rightarrow \Delta}{t \rhd x, t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{D}_2}{t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}$$

We transform the derivation as follows, with two applications of the i.h. to each premiss and an application of D_1 instead of D_2:

$$\dfrac{\mathsf{Lctr}{\times}2\ \dfrac{t \rhd x, t \rhd x, y \in t, y \in t, \Gamma \Rightarrow \Delta}{t \rhd x, y \in t, \Gamma \Rightarrow \Delta} \qquad \dfrac{t \rhd x, t \rhd x, y \in \overline{t}, y \in \overline{t}, \Gamma \Rightarrow \Delta}{t \rhd x, y \in \overline{t}, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}{\times}2}{t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{D}_1$$

(iii) If R is D_n^+ we proceed similarly to D_2 by considering the rule D_{n-1}^+.

• If instead R' is a non-cumulative rule, then we modify the derivation by first considering the height-preserving invertibility of the premiss(es) of R', then applying the inductive hypothesis, and finally applying R'. For instance, consider the case where the last rule applied in the derivation is $\mathsf{L}\Box$. We have:

$$\dfrac{\dfrac{x : \Box A, a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{x : \Box A, x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box}{x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}$$

The derivation is converted as follows:

$$\dfrac{\dfrac{\dfrac{x : \Box A, a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{a \rhd x, a \Vdash^{\forall} A, a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A, \overline{a} \Vdash^{\exists} A}\ \mathsf{invL}\Box}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}\ \mathsf{ctr}\times 3}{x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box$$

□

We now move to prove the admissibility of the following cut rule, where ϕ is any formula of $\mathcal{L}_{lab}$ that can occur on both sides of sequents (notice that every application of cut preserves the restrictions on sequents of Definition 5.1.3).

$$\mathsf{cut}\ \dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem 5.2.5 (Cut elimination)**.** The rule cut is admissible in $\mathbf{LS.E}^*$.

Proof. Given a derivation of a sequent possibly containing some applications of cut, we show how to remove any such application and obtain a derivation of the same sequent without cut. The proof is by double induction, with primary induction on the weight of the cut formula and secondary induction on the cut height. We recall that, for any application of cut, the cut formula is the formula which is deleted by that application, while the cut height is the sum of the heights of the derivations of the premisses of cut. On the basis of Definition 5.1.3, cut formulas can only be of the kinds $x : A$, $\mathsf{t} \Vdash^{\forall} A$ and $\mathsf{t} \Vdash^{\exists} A$, since formulas of kinds $x \in \mathsf{t}$ and $t \rhd x$ never occur in the right-hand side of sequents. As usual, we consider several cases depending how the premisses of cut have been derived.

(i) At least one of the two premisses of cut is derived by a zero-premiss rule, i.e., init, $\mathsf{L}\bot$, $\mathsf{R}\top$, $\overline{\tau}^{\emptyset}$, or M. If a premiss is derived by init, $\mathsf{L}\bot$, or $\mathsf{R}\top$, then the proof is standard. For instance, if the right premiss is derived by $\mathsf{R}\top$ we have:

$$\dfrac{\Gamma \Rightarrow \Delta, x : \top, \phi \qquad \dfrac{}{\phi, \Gamma \Rightarrow \Delta, x : \top}\ \mathsf{R}\top}{\Gamma \Rightarrow \Delta, x : \top}\ \mathsf{cut}$$

where the conclusion of cut is derivable by $\mathsf{R}\top$. The situation is similar if the left premiss is derived by $\mathsf{R}\top$ and the cut formula is not $x : \top$. If instead the cut formula is $x : \top$ we have:

$$\dfrac{\mathsf{R}\top\ \dfrac{}{\Gamma \Rightarrow \Delta, x : \top} \qquad x : \top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

Then we must consider the last rule applied in the derivation of $x : \top, \Gamma \Rightarrow \Delta$ and cut $x : \top$ away from its premiss(es). The situation is analogous to the point ($ii.ii$) below so we do not show any example.

If a premiss of cut is instead derived by $\overline{\tau}^{\emptyset}$ or M, observe that the cut formula is not $x \in \overline{\tau}$, or $x \in \overline{t}$, or $t \rhd x$, since these formulas do not occur in the right-hand side of sequents. Then the conclusion of cut is derivable by $\overline{\tau}^{\emptyset}$ or M. For instance if we have

$$\dfrac{t \rhd x, y \in \bar{t}, \Gamma \Rightarrow \Delta, \phi \qquad \dfrac{}{\phi, t \rhd x, y \in \bar{t}, \Gamma \Rightarrow \Delta}\,\mathsf{M}}{t \rhd x, y \in \bar{t}, \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

where the conclusion of cut is derivable by M.

(ii) None of the premisses is derived by a zero-premiss rule. We distinguish three subcases.

($ii.i$) The cut formula is not principal in the last rule application in the derivation of the left premiss of cut. As an example we consider the case where the rule last rule applied is N. We have the derivation on the left, which is converted into the derivation on the right.

$$\mathsf{cut}\,\dfrac{\mathsf{N}\,\dfrac{\tau \rhd x, \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi} \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\tau \rhd x, \Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, \Gamma \Rightarrow \Delta}{\tau \rhd x, \phi, \Gamma \Rightarrow \Delta}\,\mathsf{Lwk}}{\tau \rhd x, \Gamma \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma \Rightarrow \Delta}\,\mathsf{N}$$

Notice that having both $\Gamma \Rightarrow \Delta, \phi$ and $\phi, \Gamma \Rightarrow \Delta$ derivable, and since the calculus is sound, we have that $\Gamma \cup \Delta$ is non-empty. By Definition 5.1.3, if $\Gamma \neq \emptyset$, then x occurs in Γ, and if $\Gamma = \emptyset$, then x occurs in Δ. Then the last application or N is possible because the side condition $x \in \Gamma \cup \Delta$ is satisfied.

($ii.ii$) The cut formula is not principal in the last rule application in the derivation of the right premiss of cut. As an example we consider the case where the last rule applied is $\mathsf{L}\Box$. We have:

$$\dfrac{x : \Box A, \Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{\phi, x : \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{L}\Box}{x : \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

The derivation is converted as follows, with an application of height-preserving invertibility of $\mathsf{L}\Box$ and an application of cut with smaller height.

$$\dfrac{\dfrac{\mathsf{invL}\Box\,\dfrac{x : \Box A, \Gamma \Rightarrow \Delta, \phi}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A, \phi} \qquad \phi, a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}\,\mathsf{cut}}{x : \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{L}\Box$$

($ii.iii$) The cut formula is principal in the last rule applications in the derivations of both premisses of cut. There are four subcases.

• The cut formula is of the form $x : A \wedge B$, or $x : A \vee B$, or $x : A \to B$. The proof is standard. For instance if we have

$$\dfrac{\mathsf{R}\wedge\,\dfrac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B} \qquad \dfrac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\,\mathsf{L}\wedge}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

then the proof is converted as follows, with two applications of cut with a cut formula of smaller weight:

$$\mathsf{cut}\ \dfrac{\Gamma \Rightarrow \Delta, x:B \qquad \dfrac{\mathsf{Lwk}\ \dfrac{\Gamma \Rightarrow \Delta, x:A}{x:B, \Gamma \Rightarrow \Delta, x:A} \qquad x:A, x:B, \Gamma \Rightarrow \Delta}{x:B, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{\Gamma \Rightarrow \Delta}$$

• The cut formula is $\mathsf{t} \Vdash^{\forall} A$. The derivation is as follows, where $\mathcal{D}$ represents the derivation of the sequent $x \in \mathsf{t}, y \in \mathsf{t}, \Gamma \Rightarrow \Delta, y : A$.

$$\dfrac{\mathsf{R}\Vdash^{\forall}\ \dfrac{\begin{matrix}\mathcal{D}\\ \nabla\\ x \in \mathsf{t}, y \in \mathsf{t}, \Gamma \Rightarrow \Delta, y:A\end{matrix}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\forall} A} \qquad \dfrac{x \in \mathsf{t}, \mathsf{t} \Vdash^{\forall} A, x:A, \Gamma \Rightarrow \Delta}{x \in \mathsf{t}, \mathsf{t} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\forall}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

with y fresh in the application of $\mathsf{R}\Vdash^{\forall}$. The derivation is converted as follows, where $\mathcal{D}(x/y)$ represents the derivation obtained from $\mathcal{D}$ by replacing all occurrences of y with x.

$$\dfrac{\mathsf{Lctr}\ \dfrac{\begin{matrix}\mathcal{D}(x/y)\\ \nabla\\ x \in \mathsf{t}, x \in \mathsf{t}, \Gamma \Rightarrow \Delta, x:A\end{matrix}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, x:A} \qquad \dfrac{\mathsf{Lwk}\ \dfrac{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\forall} A}{x \in \mathsf{t}, x:A, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\forall} A} \qquad x \in \mathsf{t}, \mathsf{t} \Vdash^{\forall} A, x:A, \Gamma \Rightarrow \Delta}{x \in \mathsf{t}, x:A, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

The new derivation contains two applications of cut, the first one has a smaller height and the second one has a cut formula of smaller weight.

• The cut formula is $\mathsf{t} \Vdash^{\exists} A$. We have:

$$\dfrac{\mathsf{R}\Vdash^{\exists}\ \dfrac{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\exists} A, x:A}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\exists} A} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \nabla\\ x \in \mathsf{t}, y \in \mathsf{t}, y:A, \Gamma \Rightarrow \Delta\end{matrix}}{\mathsf{t} \Vdash^{\exists} A, x \in \mathsf{t}, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\exists}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

with y fresh in the application of $\mathsf{L}\Vdash^{\exists}$. The derivation is converted as follows:

$$\dfrac{\mathsf{cut}\ \dfrac{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, \mathsf{t} \Vdash^{\exists} A, x:A \qquad \dfrac{\mathsf{t} \Vdash^{\exists} A, x \in \mathsf{t}, \Gamma \Rightarrow \Delta}{\mathsf{t} \Vdash^{\exists} A, x \in \mathsf{t}, \Gamma \Rightarrow \Delta, x:A}\ \mathsf{Rwk}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta, x:A} \qquad \dfrac{\begin{matrix}\mathcal{D}(x/y)\\ \nabla\\ x \in \mathsf{t}, x \in \mathsf{t}, x:A, \Gamma \Rightarrow \Delta\end{matrix}}{x \in \mathsf{t}, x:A, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}}{x \in \mathsf{t}, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

• The cut formula has the form $x : \Box A$. We have:

$$\dfrac{\mathsf{R}\Box\ \dfrac{\begin{matrix}t \rhd x, \Gamma \Rightarrow \Delta, x:\Box A, t \Vdash^{\forall} A\\ \vdots \qquad t \rhd x, \bar{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x:\Box A\end{matrix}}{t \rhd x, \Gamma \Rightarrow \Delta, x:\Box A} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \nabla\\ a \rhd x, a \Vdash^{\forall} A, t \rhd x, \Gamma \Rightarrow \Delta, \bar{a} \Vdash^{\exists} A\end{matrix}}{x:\Box A, t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box}{t \rhd x, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

with a fresh in the application of $\mathsf{L}\Box$. The derivation is converted as follows, where the application of cut is replaced by four applications of cut, each of them with a smaller height or with a cut formula of smaller weight.

$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\mathcal{D}(t/a)\\ \nabla\end{array}}{t \rhd x, t \Vdash^{\forall} A, t \rhd x, \Gamma \Rightarrow \Delta, \bar{t} \Vdash^{\exists} A}
}{t \rhd x, t \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \bar{t} \Vdash^{\exists} A}\,\mathsf{Lctr}
\quad
\begin{array}{c}
\dfrac{
\dfrac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t \Vdash^{\forall} A \quad \dfrac{t \rhd x, x : \Box A, \Gamma \Rightarrow \Delta}{t \rhd x, x : \Box A, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A}\,\mathsf{Rwk}}{t \rhd x, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A}\,\mathsf{cut}
}{t \rhd x, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A, \bar{t} \Vdash^{\exists} A}\,\mathsf{Rwk}\\
\vdots\\
\vdots\\
\vdots
\end{array}
}{t \rhd x, \Gamma \Rightarrow \Delta, \bar{t} \Vdash^{\exists} A}\,\mathsf{cut}
\quad
\dfrac{
t \rhd x, \bar{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A
\quad
\begin{array}{c}
\dfrac{t \rhd x, x : \Box A, \Gamma \Rightarrow \Delta}{t \rhd x, \bar{t} \Vdash^{\exists} A, x : \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{Lwk}\\
\vdots
\end{array}
}{t \rhd x, \bar{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}
}{t \rhd x, \Gamma \Rightarrow \Delta}\,\mathsf{cut}
$$

$\Box$

Having shown that cut is admissible in $\mathbf{LS.E}^*$, we can prove the syntactic completeness of the calculi with respect to the corresponding axiomatic systems.

Theorem 5.2.6 (Syntactic completeness)**.** If A is derivable in $\mathbf{E}^*$, then $\Rightarrow x : A$ is derivable in $\mathbf{LS.E}^*$ for every world label x.

Proof. We have to show that all axioms of $\mathbf{E}^*$ are derivable in $\mathbf{LS.E}^*$, and that all rules of $\mathbf{E}^*$ are admissible in $\mathbf{LS.E}^*$. For propositional axioms we can consider their standard derivations in G3-style calculi, whereas the derivations of the modal axioms and rules are displyed in Figure 5.2. For the derivation of rule RE we assume that $y : A, \Gamma \Rightarrow \Delta, y : B$ and $y : B, \Gamma \Rightarrow \Delta, y : A$ are derivable for every world label y, and for the derivation of rule RD_n^+ we assume that $y : A_1, y : A_2, ..., y : A_n, \Gamma \Rightarrow \Delta$ is derivable for every y. Finally, MP is simulated by cut in the usual way (cf. Section 3.3). $\Box$

5.3 Further admissible rules

We have seen that the structural rules and cut are admissible in $\mathbf{LS.E}^*$. In this section, we present further admissible rules, namely the rules for the modality $\Diamond$ and some simplified rules for $\Box$ that turn out to be admissible in monotonic calculi.

Rules for $\Diamond$

Analogously to the axiomatic systems, the labelled calculi $\mathbf{LS.E}^*$ are defined by considering only the rules for $\Box$, and not the rules for $\Diamond$. Nonetheless, the rules for $\Diamond$ could be equivalently obtained by converting the satisfaction clause of diamond formulas in the bi-neighbourhood semantics into sequent rules. The resulting left and right rules for $\Diamond$ are as follows:

$$\mathsf{L}\Diamond\ \dfrac{t \rhd x, x : \Diamond A, t \Vdash^{\exists} A, \Gamma \Rightarrow \Delta \qquad t \rhd x, x : \Diamond A, \Gamma \Rightarrow \Delta, \overline{t} \Vdash^{\forall} A}{t \rhd x, x : \Diamond A, \Gamma \Rightarrow \Delta}$$

$$\mathsf{R}\Diamond\ \dfrac{a \rhd x, \overline{a} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A}\ (a!)$$

It is easy to verify that $\mathsf{L}\Diamond$ and $\mathsf{R}\Diamond$ are sound in the bi-neighbourhood semantics. Basing on the admissibility of cut and the definition of $\Diamond A$ as $\neg\Box\neg A$, we now show that the rules $\mathsf{L}\Diamond$ and $\mathsf{R}\Diamond$ are admissible in $\mathbf{LS.E}^*$. The rule $\mathsf{R}\Diamond$ can be shown admissible as follows.

$$\dfrac{\dfrac{a \rhd x, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\forall} A, \overline{a} \Vdash^{\exists} \neg A \qquad \dfrac{\mathsf{Rwk}\ \dfrac{\mathsf{Lwk}\ \dfrac{a \rhd x, \overline{a} \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}}{a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, \overline{a} \Vdash^{\exists} \neg A} \qquad \dfrac{\dfrac{\dfrac{y \in A, y : A, a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A, y : A}{y \in A, y : A, a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, y : \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A}\ \mathsf{L}\neg}{y \in A, y : A, a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A}\ \mathsf{L}\Vdash^{\forall}}{a \Vdash^{\exists} A, a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A}\ \mathsf{L}\Vdash^{\exists}}{a \rhd x, \overline{a} \Vdash^{\forall} A, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A}\ \mathsf{cut}}{\dfrac{a \rhd x, a \Vdash^{\forall} \neg A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} \neg A}{x : \Box\neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box}\ \mathsf{cut}}{\Gamma \Rightarrow \Delta, x : \neg\Box\neg A}\ \mathsf{R}\neg$$

Moreover, the rule $\mathsf{L}\Diamond$ can be shown admissible as follows.

$$\dfrac{t \rhd x, \overline{t} \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A, x : \neg\Box\neg A \qquad \dfrac{\mathsf{Rwk}\ \dfrac{\mathsf{Lwk}\ \dfrac{t \rhd x, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, \overline{t} \Vdash^{\forall} A}{t \rhd x, \overline{t} \Vdash^{\exists} \neg A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, \overline{t} \Vdash^{\forall} A}}{t \rhd x, \overline{t} \Vdash^{\exists} \neg A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, \overline{t} \Vdash^{\forall} A, x : \Box\neg A} \qquad \dfrac{\dfrac{t \rhd x, y \in \overline{t}, y : \neg A, y : A, \overline{t} \Vdash^{\forall} A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A}{t \rhd x, y \in \overline{t}, y : \neg A, \overline{t} \Vdash^{\forall} A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A}\ \mathsf{L}\Vdash^{\forall}}{t \rhd x, \overline{t} \Vdash^{\exists} \neg A, \overline{t} \Vdash^{\forall} A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A}\ \mathsf{L}\Vdash^{\exists}}{t \rhd x, \overline{t} \Vdash^{\exists} \neg A, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A}\ \mathsf{cut}}{②\ \ t \rhd x, \overline{t} \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x : \Box\neg A}\ \mathsf{cut}$$

$$\dfrac{\dfrac{\dfrac{t \rhd x, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A, x : \Box\neg A, x : \neg\Box\neg A \qquad \dfrac{\mathsf{R}\Vdash^{\forall}\ \dfrac{\mathsf{R}\Vdash^{\exists}\ \dfrac{t \rhd x, x : \neg\Box\neg A, y \in t, \Gamma \Rightarrow \Delta, t \Vdash^{\exists} A, y : A, y : \neg A, x : \Box\neg A}{t \rhd x, x : \neg\Box\neg A, y \in t, \Gamma \Rightarrow \Delta, t \Vdash^{\exists} A, y : \neg A, x : \Box\neg A}}{t \rhd x, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, t \Vdash^{\exists} A, t \Vdash^{\forall} \neg A, x : \Box\neg A} \qquad \dfrac{\dfrac{t \rhd x, x : \neg\Box\neg A, t \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{t \rhd x, x : \neg\Box\neg A, t \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A}\ \mathsf{Rwk}}{t \rhd x, x : \neg\Box\neg A, t \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A, x : \Box\neg A}\ \mathsf{Rwk}}{t \rhd x, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A, x : \Box\neg A}\ \mathsf{cut}}{t \rhd x, \Gamma \Rightarrow \Delta, t \Vdash^{\forall} A, x : \Box\neg A}\ \mathsf{cut} \qquad ②}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box\neg A}\ \mathsf{R}\Box}{t \rhd x, x : \neg\Box\neg A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg$$

Simplified rules for monotonic systems

In Negri [131], labelled calculi for monotonic logics are defined by considering alternative rules for $\Box$ expressing the forcing condition of boxed formulas in the $\exists\forall$-semantics. The rules, rewritten with the present notation, are as follows:

$$\mathsf{L\Box m}\ \frac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\ (a!) \qquad\qquad \mathsf{R\Box m}\ \frac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t \Vdash^{\forall} A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}$$

It can be shown that the rules $\mathsf{L\Box m}$ and $\mathsf{R\Box m}$ are sound both in $\exists\forall$-models and in bi-neighbourhood M-models. In the first case, formulas $t \rhd x$ must be interpreted as $\sigma(t) \in \mathcal{N}(\rho(x))$, whereas in the second case they must be interpreted as $(\sigma(t), \emptyset) \in \mathcal{N}(\rho(x))$ (in both cases the interpretation of $t \rhd x$ is different from the one given in Definition 5.1.5).

Here we show that the rules $\mathsf{L\Box m}$ and $\mathsf{R\Box m}$ are derivable in our monotonic calculi $\mathbf{LS.M}^*$. We also show that our rules $\mathsf{L\Box}$ and $\mathsf{R\Box}$ are derivable from $\mathsf{L\Box m}$, $\mathsf{R\Box m}$ and M. Therefore, in presence of rule M, our basic rules for $\Box$ and the above monotonic rules for $\Box$ are equivalent.

- $\mathsf{L\Box}$ is derivable from M and $\mathsf{L\Box m}$:

$$\dfrac{\dfrac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A \qquad \dfrac{\dfrac{}{a \rhd x, y \in \overline{a}, y : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{M}\ (a, y \notin \Gamma, \Delta)}{a \rhd x, \overline{a} \Vdash^{\exists} A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Vdash^{\exists}}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{L\Box m}$$

- $\mathsf{L\Box m}$ is derivable from $\mathsf{L\Box}$:

$$\dfrac{\dfrac{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \rhd x, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}\ \mathsf{Rwk}}{x : \Box A, \Gamma \Rightarrow \Delta}\ \mathsf{L\Box}$$

- $\mathsf{R\Box m}$ is derivable from M and $\mathsf{R\Box}$:

$$\dfrac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t : A \qquad \dfrac{\dfrac{}{t \rhd x, y \in \overline{t}, y : A, \Gamma \Rightarrow \Delta, x : \Box A}\ \mathsf{M}\ (y \notin \Gamma, \Delta)}{t \rhd x, \overline{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A}\ \mathsf{L}\Vdash^{\exists}}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}\ \mathsf{R\Box}$$

- $\mathsf{R\Box}$ is derivable from $\mathsf{R\Box m}$:

$$\dfrac{\mathsf{Rwk}\ \dfrac{\mathsf{R\Box m}\ \dfrac{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, t : A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A, \overline{t} \Vdash^{\exists} A} \qquad t \rhd x, \overline{t} \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x : \Box A}{t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A}\ \mathsf{cut}$$

The calculus obtained by adding the monotonic rules for box $\mathsf{L}\Box\mathsf{m}$ and $\mathsf{R}\Box\mathsf{m}$ to the propositional rules and the rules for local forcing is an alternative calculus for logic $\mathbf{M}$. We can show that cut is admissible in this calculus (cf. [131]), the proof is essentially a simplification of the one of Theorem 5.2.5. This calculus has the advantage of not requiring negative terms at all, by contrast, in this way the definition of the calculi is not modular between the monotonic and the non-monotonic case.

5.4 Tableaux calculi and decision procedure

In this section, we present a reformulation of the labelled sequent calculi $\mathbf{LS.E}^*$ in the form of labelled tableaux calculi, except for the calculi containing the rules for axiom 4 which are considered later in Section 5.6. We then define a simple terminating proof search strategy in the tableaux calculi that provides a decision procedure for the corresponding logics.

We point out that all results presented in this and the next sections could be equivalently achieved by using the sequent calculi $\mathbf{LS.E}^*$. However, since the calculi $\mathbf{LS.E}^*$ have all the rules invertible they can be naturally looked as tableaux systems. This alternative formulation has the advantage of offering a simpler formalism to display failed proofs and derivations. As a matter of fact, since the backward applications of the sequent rules preserve the context as well as some principal formulas, the proofs in our sequent calculi quickly become very large, by contrast the replication of formulas is avoided in the tableaux formulation that we consider in the following. Moreover, we now consider refinements of some rules which avoid the creation of unnecessary neighbourhood terms (see the rules $\neg\Box_{\mathbf{T}}$, $\mathsf{dec}_{\mathbf{T}}$, $\overline{\mathsf{dec}}_{\mathbf{T}}$, $\mathsf{C}_{\mathbf{T}}$ below). Although these refined rules would break the possibility to give a direct proof of cut elimination in the sequent calculi, they are still adequate to build countermodels, whence to prove the semantic completeness of the calculi. Finally, the tableaux systems allow us to encode a terminating proof search strategy in a straightforward way. We see all this in the following.

The tableaux calculi $\mathbf{LT.E}^*$ are defined by the rules in Figure 5.3, as summarised in Figure 5.4. For the propositional connectives and $\Box$, the calculi contain a "positive" and a "negative" rule, that correspond respectively to left and right rules of sequent calculi. Moreover, the zero-premisses rules of sequent calculi are reformulated as closure conditions of tableaux branches, in order to express these conditions we use a special symbol $\mathbf{f}$. Observe that, as a difference with the rule C of the sequent calculus, the rule $\mathsf{C}_{\mathbf{T}}$ builds n-ary neighbourhood terms directly from n atomic terms. Analogously, the rules $\mathsf{dec}_{\mathbf{T}}$ and $\overline{\mathsf{dec}}_{\mathbf{T}}$ directly decompose n-ary terms into n atomic terms. As we shall see, while a stepwise decomposition of neighbourhood terms (as it is done by dec and $\overline{\mathsf{dec}}$ in the sequent calculus) is needed for obtaining a syntactic proof of cut elimination, this is not necessary for a proof of semantic completeness by countermodel extraction for non-valid formulas. We consider the following definitions.

Rules of the basic calculus LT.E

$$\mathsf{init}_{\mathbf{T}}\ \frac{\begin{array}{c} x : A \\ x : \neg A \end{array}}{\mathbf{f}} \qquad \bot_{\mathbf{T}}\ \frac{x : \bot}{\mathbf{f}} \qquad \neg\top_{\mathbf{T}}\ \frac{x : \neg\top}{\mathbf{f}}$$

$$\wedge_{\mathbf{T}}\ \frac{x : A \wedge B}{\begin{array}{c} x : A \\ x : B \end{array}} \qquad \neg\wedge_{\mathbf{T}}\ \frac{x : \neg(A \wedge B)}{x : \neg A \mid x : \neg B} \qquad \rightarrow_{\mathbf{T}}\ \frac{x : A \rightarrow B}{x : \neg A \mid x : B} \qquad \neg\rightarrow_{\mathbf{T}}\ \frac{x : \neg(A \rightarrow B)}{\begin{array}{c} x : A \\ x : \neg B \end{array}}$$

$$\Vdash^{\forall}{}_{\mathbf{T}}\ \frac{\begin{array}{c} \mathsf{t} \Vdash^{\forall} A \\ x \in \mathsf{t} \end{array}}{x : A} \qquad \Box_{\mathbf{T}}\ \frac{x : \Box A}{\begin{array}{c} a \rhd x \\ a \Vdash^{\forall} A \\ \overline{a} \Vdash^{\forall} \neg A \end{array}}\ (a!) \qquad \neg\Box_{\mathbf{T}}\ \frac{\begin{array}{c} x : \neg\Box A \\ t \rhd x \end{array}}{\begin{array}{c} y \in t \\ y : \neg A \end{array} \ \Bigg|\ \begin{array}{c} y \in \overline{t} \\ y : A \end{array}}\ (y!)$$

Rules for neighbourhood terms

$$\overline{\tau}^{\emptyset}{}_{\mathbf{T}}\ \frac{x \in \overline{\tau}}{\mathbf{f}} \qquad \mathsf{dec}_{\mathbf{T}}\ \frac{x \in a_1 ... a_n}{\begin{array}{c} x \in a_1 \\ \vdots \\ x \in a_n \end{array}} \qquad \overline{\mathsf{dec}}_{\mathbf{T}}\ \frac{x \in \overline{a_1 ... a_n}}{x \in \overline{a_1} \mid ... \mid x \in \overline{a_n}}$$

Rules for extensions

$$\mathsf{M}_{\mathbf{T}}\ \frac{\begin{array}{c} t \rhd x \\ x \in \overline{t} \end{array}}{\mathbf{f}} \qquad \mathsf{N}_{\mathbf{T}}\ \frac{\mathcal{B}}{\tau \rhd x}\ (x \text{ in } \mathcal{B}) \qquad \mathsf{T}_{\mathbf{T}}\ \frac{t \rhd x}{x \in t} \qquad \mathsf{P}_{\mathbf{T}}\ \frac{t \rhd x}{y \in t}\ (y!)$$

$$\mathsf{C}_{\mathbf{T}}\ \frac{\begin{array}{c} a_1 \rhd x \\ \vdots \\ a_n \rhd x \end{array}}{a_1 ... a_n \rhd x} \qquad \mathsf{D}_{1\mathbf{T}}\ \frac{t \rhd x}{y \in t \mid y \in \overline{t}}\ (y!) \qquad \mathsf{D}_{2\mathbf{T}}\ \frac{\begin{array}{c} t \rhd x \\ s \rhd x \end{array}}{\begin{array}{c} y \in t \\ y \in s \end{array} \ \Bigg|\ \begin{array}{c} y \in \overline{t} \\ y \in \overline{s} \end{array}}\ (y!) \qquad \mathsf{D}^{+}_{n}{}_{\mathbf{T}}\ \frac{\begin{array}{c} t_1 \rhd x \\ \vdots \\ t_n \rhd x \end{array}}{\begin{array}{c} y \in t_1 \\ \vdots \\ y \in t_n \end{array}}\ (y!)$$

Figure 5.3: Rules of labelled tableaux calculi $\mathbf{LT.E}^*$.

Definition 5.4.1 (Tableau, derivation)**.** Let Φ and A be respectively a set of n labelled formulas of $\mathcal{L}_{lab}$ and a formula of $\mathcal{L}$.

- A *tableau* for Φ in $\mathbf{LT.E}^*$ is a tree such that the first n nodes starting from the root are labelled each by a different formula of Φ, end every succedent node is obtained by the application of a rule of $\mathbf{LT.E}^*$ to formulas occurring in the same branch at smaller depths.

- A branch of a tableau is *closed* if it contains $\mathbf{f}$, otherwise it is *open*. A tableau is *closed* if all its branches are closed.

- A branch $\mathcal{B}$ *can be closed* if there exists a closed tableau obtained by expanding $\mathcal{B}$ with applications of rules of $\mathbf{LT.E}^*$.

$\mathbf{LT.E} := \{\text{propositional rules}, \Vdash^{\forall}_{\mathbf{T}}, \Box_{\mathbf{T}}, \neg\Box_{\mathbf{T}}\}$.	$\mathbf{LT.ET}^* := \mathbf{LT.E}^* \cup \{\mathsf{T}_{\mathbf{T}}\}$.
$\mathbf{LT.M}^* := \mathbf{LT.E}^* \cup \{\mathsf{M}_{\mathbf{T}}\}$.	$\mathbf{LT.EP}^* := \mathbf{LT.E}^* \cup \{\mathsf{P}_{\mathbf{T}}\}$.
$\mathbf{LT.EN}^* := \mathbf{LT.E}^* \cup \{\mathsf{N}_{\mathbf{T}}, \overline{\tau}^{\emptyset}{}_{\mathbf{T}}\}$.	$\mathbf{LT.ED}^* := \mathbf{LT.E}^* \cup \{\mathsf{D}_{1\mathbf{T}}, \mathsf{D}_{2\mathbf{T}}\}$.
$\mathbf{LT.EC}^* := \mathbf{LT.E}^* \cup \{\mathsf{C}_{\mathbf{T}}, \mathsf{dec}_{\mathbf{T}}, \overline{\mathsf{dec}}_{\mathbf{T}}\}$.	$\mathbf{LT.ED}^{+*}_{\mathbf{n}} := \mathbf{LT.E}^* \cup \{\mathsf{D}^{+}_{m\mathbf{T}} \mid 1 \leq m \leq n\}$.

Figure 5.4: Labelled tableaux calculi $\mathbf{LT.E}^*$.

1.	$x : \neg(\Box(A \wedge B) \rightarrow \Box A)$	
2.	$x : \Box(A \wedge B)$	$(1, \neg\!\rightarrow_{\mathbf{T}})$
3.	$x : \neg\Box A$	$(1, \neg\!\rightarrow_{\mathbf{T}})$
4.	$a \rhd x$	$(2, \Box_{\mathbf{T}})$
5.	$a \Vdash^{\forall} A \wedge B$	$(2, \Box_{\mathbf{T}})$
6.	$\overline{a} \Vdash^{\forall} \neg(A \wedge B)$	$(2, \Box_{\mathbf{T}})$

Left branch:

7.	$a \Vdash^{\exists} \neg A$	$(3, 4, \neg\Box_{\mathbf{T}})$
8.	$y \in a$	$(7, \Vdash^{\exists}_{\mathbf{T}})$
9.	$y : \neg A$	$(7, \Vdash^{\exists}_{\mathbf{T}})$
10.	$y : A \wedge B$	$(5, 8, \Vdash^{\forall}_{\mathbf{T}})$
11.	$y : A$	$(10, \wedge_{\mathbf{T}})$
12.	$y : B$	$(10, \wedge_{\mathbf{T}})$
	$\mathbf{f}$	$(9, 11, \mathsf{init}_{\mathbf{T}})$

Right branch:

7.	$\overline{a} \Vdash^{\exists} A$	$(3, 4, \neg\Box_{\mathbf{T}})$
8.	$y \in \overline{a}$	$(7, \Vdash^{\exists}_{\mathbf{T}})$
9.	$y : A$	$(7, \Vdash^{\exists}_{\mathbf{T}})$
	$\mathbf{f}$	$(4, 8, \mathsf{M}_{\mathbf{T}})$

Figure 5.5: Derivation of axiom M in $\mathbf{LT.M}$.

- A *derivation* of Φ in $\mathbf{LT.E}^*$ is a finite closed tableau for Φ.
- A *derivation* of $A \in \mathcal{L}$ is a derivation of $x_0 : \neg A$.

As an example, in Figure 5.5 it is displayed a derivation of axiom M in the calculus $\mathbf{LT.M}$. Since the tableaux calculi $\mathbf{LT.E}^*$ are essentially a reformulation of the sequent calculi $\mathbf{LS.E}^*$, it is possible to prove the following result:

Theorem 5.4.1 (Syntactic equivalence)**.** For every formula A of $\mathcal{L}$, A is derivable in $\mathbf{LT.E}^*$ if and only if A is derivable in $\mathbf{E}^*$.

We omit cumbersome details. The idea is that every derivation in the sequent calculus $\mathbf{LS.E}^*$ can be transformed into an equivalent derivation in $\mathbf{LT.E}^*$. The transformation is easy as there is a 1-1 correspondence between rule applications in sequent proofs and rules applications in tableaux proofs. The only exceptions are the rules $\mathsf{C}_{\mathbf{T}}$, $\mathsf{dec}_{\mathbf{T}}$, and $\overline{\mathsf{dec}}_{\mathbf{T}}$, which correspond to multiple consecutive applications of the analogous rules in the sequent calculus, and the rule $\neg\Box_{\mathbf{T}}$, which correspond to an application of $\mathsf{R}\Box$ and a subsequent application of $\mathsf{R}\Vdash^{\forall}$ or $\mathsf{L}\Vdash^{\exists}$. Notice that because of the invertibility of the sequent calculi $\mathbf{LS.E}^*$, a derivation in $\mathbf{LS.E}^*$ can be always converted into an equivalent derivation where the rule applications are ordered in a suitable way (e.g., applications of C are conveniently grouped so to correspond to an application of $\mathsf{C}_{\mathbf{T}}$). Furthermore, by the same argument, or by a direct proof analogous to the one of Theorem 5.1.2, we can also prove that the calculi $\mathbf{LT}.\mathbf{E}^*$ are sound with respect to the bi-neighbourhood models for $\mathbf{E}^*$.

We are interested in using the tableaux calculus $\mathbf{LS}.\mathbf{E}^*$ to define a decision procedure for the logic $\mathbf{E}^*$. To this aim, it is useful to characterise the branches that can belong to a proof of $x_0 : \neg A$.

Definition 5.4.2 (Adequate branch)**.** Let $\mathcal{B}$ be a branch of a proof in $\mathbf{LT}.\mathbf{E}^*$. We say that $\mathcal{B}$ is *adequate* if it satisfies the following conditions: (i) If a neighbourhood label a occurs in $\mathcal{B}$, then there is exactly one world label x such that $a \rhd x$ is in $\mathcal{B}$, and there is exactly one formula A such that $a \Vdash^{\forall} A$ in $\mathcal{B}$. (ii) $a \Vdash^{\forall} A$ is in $\mathcal{B}$ if and only if $\overline{a} \Vdash^{\forall} \neg A$ is in $\mathcal{B}$. (iii) If $\mathsf{t} \Vdash^{\forall} A$ is in $\mathcal{B}$, then t is an atomic term a or $\overline{a}$ different from $\tau, \overline{\tau}$. (iv) If $a_1...a_n \rhd x$ is in $\mathcal{B}$, then $a_1 \rhd x$, ..., $a_n \rhd x$ are in $\mathcal{B}$. (v) If $a_1...a_n$ or $\overline{a_1...a_n}$ occurs in $\mathcal{B}$, then all atomic terms a_1, ..., a_n occur in $\mathcal{B}$.

Lemma 5.4.2. Every branch of a proof of A in $\mathbf{LT}.\mathbf{E}^*$ is adequate.

Proof. The branch consisting only in the formula $x_0 : \neg A$ is adequate. Moreover, it is easy to verify that all rules of $\mathbf{LT}.\mathbf{E}^*$ preserve the adequacy of branches, that is, if a branch is adequate, then its expansion by the application of any rule of $\mathbf{LT}.\mathbf{E}^*$ is also adequate. $\square$

We now define the proof search strategy for formulas A of $\mathcal{L}$ in $\mathbf{LT}.\mathbf{E}^*$. Starting with $x_0 : \neg A$, the strategy essentially amounts to applying the rules of $\mathbf{LT}.\mathbf{E}^*$ as much as possible but avoiding redundant rule applications, where the application of a rule can be considered redundant if, roughly, it only adds information which is already contained by the branch. Redundancy of rule applications is formally defined by relying on the following notion of saturation.

Definition 5.4.3 (Saturated branch)**.** Let $\mathcal{B}$ be a branch of a tableaux proof for $x_0 : A$ in $\mathbf{LT}.\mathbf{E}^*$. The saturation conditions associated to the application of rules of $\mathbf{LT}.\mathbf{E}^*$ are as follows. ($\mathsf{init}_{\mathbf{T}}$) $x : B$ is not in $\mathcal{B}$ or $x : \neg B$ is not in $\mathcal{B}$. ($\bot_{\mathbf{T}}$) $x : \bot$ is not in $\mathcal{B}$. ($\neg\top_{\mathbf{T}}$) $x : \neg\top$ is not in $\mathcal{B}$. ($\wedge_{\mathbf{T}}$) If $x : B \wedge C$ is in $\mathcal{B}$, then $x : B$ and $x : C$ are in $\mathcal{B}$. ($\neg\wedge_{\mathbf{T}}$) If $x : \neg(B \wedge C)$ is in $\mathcal{B}$, then $x : \neg B$ or $x : \neg C$ is in $\mathcal{B}$. ($\Vdash^{\forall}_{\mathbf{T}}$) If $\mathsf{t} \Vdash^{\forall} B$ and $x \in \mathsf{t}$ are in $\mathcal{B}$, then $x : B$ is in $\mathcal{B}$. ($\Box_{\mathbf{T}}$) If $x : \Box B$ is in $\mathcal{B}$, then for a neighbourhood label a, $a \rhd x$, $a \Vdash^{\forall} B$, and $\overline{a} \Vdash^{\forall} \neg B$ are in $\mathcal{B}$. ($\neg\Box_{\mathbf{T}}$) If $x : \neg\Box B$ and $t \rhd x$ are in $\mathcal{B}$, then there is a world label y such that $y \in t$ and $y : \neg B$ are in $\mathcal{B}$, or $y \in \overline{t}$ and $y : B$ are in $\mathcal{B}$ ($\mathsf{M}_{\mathbf{T}}$) $t \rhd x$ is not in $\mathcal{B}$ or $y \in \overline{t}$ is not in $\mathcal{B}$. ($\mathsf{N}_{\mathbf{T}}$) If x is a world label occurring in $\mathcal{B}$, then $\tau \rhd x$ is in $\mathcal{B}$. ($\overline{\tau}^{\emptyset}{}_{\mathbf{T}}$) $x \in \overline{\tau}$ is not in $\mathcal{B}$. ($\mathsf{C}_{\mathbf{T}}$) If $a_1 \rhd x$, ..., $a_n \rhd x$ are in $\mathcal{B}$, then there is $s \rhd x$ in $\mathcal{B}$ such that $\mathsf{set}(s) = \mathsf{set}(a_1...a_n)$. ($\mathsf{dec}_{\mathbf{T}}$) If $x \in a_1...a_n$ is in $\mathcal{B}$, then $x \in a_1$, ..., $x \in a_n$ are in $\mathcal{B}$. ($\overline{\mathsf{dec}_{\mathbf{T}}}$) If $x \in \overline{a_1...a_n}$ is in $\mathcal{B}$, then for some $1 \leq i \leq n$, $x \in \overline{a_i}$ is in $\mathcal{B}$. ($\mathsf{T}_{\mathbf{T}}$) If $t \rhd x$ is in $\mathcal{B}$, then $x \in t$ is in $\mathcal{B}$. ($\mathsf{P}_{\mathbf{T}}$) If $t \rhd x$ is in

$\mathcal{B}$, then there is a world label y such that $y \in t$ is in $\mathcal{B}$. ($\mathsf{D}_{1\mathbf{T}}$) If $t \rhd x$ is in $\mathcal{B}$, then there is a world label y such that $y \in t$ is in $\mathcal{B}$, or $y \in \bar{t}$ is in $\mathcal{B}$. ($\mathsf{D}_{2\mathbf{T}}$) If $t \rhd x$, $s \rhd x$ are in $\mathcal{B}$, then there is a world label y such that $y \in t$ and $y \in s$ are in $\mathcal{B}$, or $y \in \bar{t}$ and $y \in \bar{s}$ are in $\mathcal{B}$. ($\mathsf{D}^{+}_{n}{}_{\mathbf{T}}$) If $t_1 \rhd x$, ..., $t_n \rhd x$ are in $\mathcal{B}$, then there is a world label y such that $y \in t_1$, ..., $y \in t_n$ are in $\mathcal{B}$.

We say that $\mathcal{B}$ is saturated with respect to an application of a rule $R_{\mathbf{T}}$ if it satisfies the saturation condition ($R_{\mathbf{T}}$) for that particular rule application, and that it is saturated with respect to $\mathbf{LT}.\mathbf{E}^*$ if it is saturated with respect to all possible applications of any rule of $\mathbf{LT}.\mathbf{E}^*$.

Observe that every saturated branch is not closed. The proof search strategy in $\mathbf{LT}.\mathbf{E}^*$ is then defined as follows.

Definition 5.4.4 (Proof search strategy, failed proof)**.** Given a formula A of $\mathcal{L}$, a proof of A in $\mathbf{LT}.\mathbf{E}^*$ is constructed as follows. Firstly, the root node is labelled with $x_0 : \neg A$. Then, every branch of the proof is expanded by backward applications of rules of $\mathbf{LT}.\mathbf{E}^*$ in the respect of the following two conditions: (i) No rule can be applied to a closed branch. (ii) The application of a rule in a branch is not allowed if the branch is already saturated with respect to that particular rule application. A branch is expanded until there are no possible additional rule applications in the respect of conditions (i) and (ii). We call *failed proof* of A any tableau for A which is constructed in accordance with the strategy and contains some saturated branch, i.e., some branch that is open and that cannot be further expanded.

We now prove that every tableau built in accordance with the strategy is finite. The proof consists in showing that every tableau built in accordance with the strategy can contain only finitely many labels, and that as a consequence it can contain only finitely many labelled formulas.

Definition 5.4.5. Let $\mathcal{B}$ be a branch of a proof of $x_0 : A$ in $\mathbf{LT}.\mathbf{E}^*$. We denote by $n(i)$ the formula at the node of $\mathcal{B}$ at depth i, and, for every world label x and neighbourhood term t, we denote by $d(x)$ (respectively $d(\mathsf{t})$) the smallest i such that x (respectively t) occurs in $n(i)$. We define three relations $\hookrightarrow_1 \subseteq \mathbb{WL} \times \mathbb{NT}$, $\hookrightarrow_2 \subseteq \mathbb{NT} \times \mathbb{WL}$, and $\hookrightarrow_w \subseteq \mathbb{WL} \times \mathbb{WL}$ as follows:

$$
\begin{array}{rll}
x \hookrightarrow_1 \mathsf{t} & \text{iff} & \begin{cases} \mathsf{t} = t \neq \tau \text{ and for some } i \in \mathbb{N},\ d(t) = i \text{ and } n(i) = t \rhd x; \\ \mathsf{t} = \tau \text{ and } x = x_0; \\ \mathsf{t} = \bar{t} \text{ and } x \hookrightarrow_1 t. \end{cases} \\
\mathsf{t} \hookrightarrow_2 x & \text{iff} & \text{for some } i \in \mathbb{N},\ d(x) = i \text{ and } n(i) = x \in \mathsf{t}. \\
x \hookrightarrow_w y & \text{iff} & \text{for some term } \mathsf{t},\ x \hookrightarrow_1 \mathsf{t} \text{ and } \mathsf{t} \hookrightarrow_2 y.
\end{array}
$$

We say that x *generates* t if if $x \hookrightarrow_1 \mathsf{t}$, that t generates x if $\mathsf{t} \hookrightarrow_2 x$, and that x generates y if $x \hookrightarrow_w y$. We denote by $\mathcal{T}_w$ the graph determined by x_0 and the relation $\hookrightarrow_w$.

Lemma 5.4.3. Given a branch $\mathcal{B}$ of a proof for $x_0 : A$ in $\mathbf{LT}.\mathbf{E}^*$ built in accordance with the strategy, we have that (a) the graph $\mathcal{T}_w$ determined by x_0 and the relation $\hookrightarrow_w$ is a tree with root x_0, and (b) every world label occurring in $\mathcal{B}$ is a node of $\mathcal{T}_w$.

Proof. (a) As immediate consequences of the definitions we have that $x_1 \hookrightarrow_1 t$ and $x_2 \hookrightarrow_1 t$ implies $x_1 = x_2$, and $t_1 \hookrightarrow_2 y$ and $t_2 \hookrightarrow_2 y$ implies $t_1 = t_2$, thus $x_1 \hookrightarrow_w y$ and $x_2 \hookrightarrow_w y$ implies $x_1 = x_2$. Moreover, $x \hookrightarrow_w y$ implies $d(x) < d(y)$, therefore $x \hookrightarrow_w y$ implies $y \not\hookrightarrow_w x$.

(b) By complete induction on $d(x)$. If $d(x) = 1$, then $x = x_0$, thus x is in $\mathcal{T}_w$ by definition. If $d(x) > 1$, then x is a world label occurring in $\mathcal{B}$ different from x_0. Then x is introduced in $\mathcal{B}$ by a formula $x \in \mathsf{t}$ where t is a preexisting term in $\mathcal{B}$, thus $\mathsf{t} \hookrightarrow_2 x$ and $d(t) < d(x)$. In turn, t is introduced in $\mathcal{B}$ by a formula $t \rhd y$, where y is a preexisting term in $\mathcal{B}$, thus $y \hookrightarrow_1 \mathsf{t}$ and $d(y) < d(t)$. It follows $y \hookrightarrow_w x$ and $d(y) < d(x)$. By inductive hypothesis, y is in $\mathcal{T}_w$, therefore x is also in $\mathcal{T}_w$. $\square$

Lemma 5.4.4. Given a branch $\mathcal{B}$ of a proof for $x_0 : A$ in $\mathbf{LT}.\mathbf{E}^*$ built in accordance with the strategy, every branch of the tree $\mathcal{T}_w$ determined by the relation $\hookrightarrow_w$ is finite.

Proof. For every world label x occurring in $\mathcal{B}$, we define its *modal degree* as $md(x) = max\{md(A) \mid x : A \text{ is in } \mathcal{B}\}$. We prove that $x \hookrightarrow_w y$ implies $md(y) < md(x)$. Since $md(x_0)$ is finite, this implies that every branch of $\mathcal{T}_w$ is finite.

Assume that $y : A$ is in $\mathcal{B}$, and $x \hookrightarrow_w y$, that is there is t such that $x \hookrightarrow_1 \mathsf{t} \hookrightarrow_2 y$. We prove by induction on the depth of the node of $\mathcal{B}$ labelled by $y : A$ that $md(A) < md(x)$. Notice that $y : A$ can be obtained in the tableau by: (i) an application of a propositional rule, or (ii) an application of $\neg\Box_{\mathbf{T}}$, or (iii) an application of $\Vdash^{\forall}{}_{\mathbf{T}}$.

(i) If $y : A$ is obtained by an application of a propositional rule, then $y : A$ is obtained from a formula $y : B$ occurring in $\mathcal{B}$ at a smaller depth, where B has the same modal degree of A. Then by inductive hypothesis $md(A) = md(B) < md(x)$.

(ii) If $y : A$ is obtained by an application of $\neg\Box_{\mathbf{T}}$, then it is obtained from formulas $z : \neg\Box B$ and $s \rhd z$, with $A = \neg B$ or $A = B$. Since y is fresh in the application of $\neg\Box_{\mathbf{T}}$, $\mathsf{s} \hookrightarrow_2 y$, then $\mathsf{s} = \mathsf{t}$. Moreover, since $x \hookrightarrow_1 \mathsf{t} = \mathsf{s}$, by Definition 5.4.2 (i), $z = x$. Then $x : \neg\Box B$ is in $\mathcal{B}$, therefore $md(A) < md(x)$.

(iii) If $y : A$ is obtained by an application of $\Vdash^{\forall}{}_{\mathbf{T}}$, then it is obtained from formulas $\mathsf{s} \Vdash^{\forall} A$ and $y \in \mathsf{s}$. By Definition 5.4.2 (iii), since $\mathsf{s} \Vdash^{\forall} A$ is in $\mathcal{B}$ we have $\mathsf{s} = a$ or $\mathsf{s} = \overline{a}$ for some atomic term a. Then since $y \in \mathsf{s}$ is in $\mathcal{B}$ we have the following possibilities: $\mathsf{s} \hookrightarrow_2 y$, or $b_1...b_n \hookrightarrow_2 y$ and $\mathsf{s} = b_i$, or $\overline{b_1...b_n} \hookrightarrow_2 y$ and $\mathsf{s} = \overline{b_i}$, or $s \rhd y$ is in $\mathcal{B}$. If $\mathsf{s} \hookrightarrow_2 y$, then $\mathsf{s} = \mathsf{t}$, thus $x \hookrightarrow_1 \mathsf{s}$. Then $\mathsf{s} \Vdash^{\forall} A$ is obtained from a formula $x : \Box B$, with $A = B$ or $A = \neg B$. Then $md(A) < md(x)$. If $b_1...b_n \hookrightarrow_2 y$ or $\overline{b_1...b_n} \hookrightarrow_2 y$, then $x \hookrightarrow_1 b_1...b_n$ or $x \hookrightarrow_1 \overline{b_1...b_n}$, thus $b_1...b_n \rhd x$ is in $\mathcal{B}$. By Definition 5.4.2 (iv), $b_i \rhd x$ is in $\mathcal{B}$, then $x \hookrightarrow_1 \mathsf{s}$. As before, $x : \Box B$ is in $\mathcal{B}$, with $A = B$ or $A = \neg B$. Then $md(A) < md(x)$. If $s \rhd y$ is in $\mathcal{B}$, then $s \rhd y$ is obtained together with $s \Vdash^{\forall} B$ and $\overline{s} \Vdash^{\forall} \neg B$ by an application of $\Box_{\mathbf{T}}$ to a formula $y : \Box B$. Then $A = B$ or $A = \neg B$. Since $y : \Box B$ occurs in $\mathcal{B}$ at a smaller depth, by i.h. we have that $md(A) < md(\Box B) < md(x)$. $\square$

Lemma 5.4.5. Given a branch $\mathcal{B}$ of a proof for $x_0 : A$ in $\mathbf{LT.E}^*$ built in accordance with the strategy, the tree $\mathcal{T}_w$ determined by the relation $\hookrightarrow_w$ has finitely many branches. Moreover, every world label occurring in $\mathcal{B}$ generates finitely many neighbourhood terms.

Proof. We show that (a) every world label generates finitely many neighbourhood terms, and (b) every neighbourhood term generates finitely many world labels. It follows that every world label generates finitely many world labels.

(a) First, a world label x generates finitely many (positive) atomic terms a. Indeed, a is generated by x by means of an application of $\Box_{\mathbf{T}}$ to a formula $x : \Box B$. By its saturation condition, rule $\Box_{\mathbf{T}}$ can be applied to $x : \Box B$ at most once. Moreover, all formulas $\Box B$ such that $x : \Box B$ is in $\mathcal{B}$ are subformulas of the formula A at the root of the tableau, whence they are finitely many. Furthermore, if x generates n atomic terms, then by the saturation condition of $\mathsf{C}_{\mathbf{T}}$ it generates at most $2^n - 1$ (positive) complex terms.

(b) A term t generates a world label y by an application of a rule among $\neg\Box_{\mathbf{T}}$, $\mathsf{P}_{\mathbf{T}}$, $\mathsf{D}_{1\mathbf{T}}$, $\mathsf{D}_{2\mathbf{T}}$, and $\mathsf{D}^{+}_{n}{}_{\mathbf{T}}$, to a formula $t \rhd x$, or a pair $t \rhd x$, $s \rhd x$, or a tuple $t \rhd x, s_1 \rhd x, ..., s_n \rhd x$ (all of them with the same world label x). By the saturation conditions, every rule among $\neg\Box_{\mathbf{T}}$, $\mathsf{P}_{\mathbf{T}}$, $\mathsf{D}_{1\mathbf{T}}$, $\mathsf{D}_{2\mathbf{T}}$, and $\mathsf{D}^{+}_{n}{}_{\mathbf{T}}$, can be applied only once to the same formula $t \rhd x$, or pair $t \rhd x$, $s \rhd x$, or tuple $t \rhd x, s_1 \rhd x, ..., s_n \rhd x$. Then the problem is reduced to counting how many of these formulas can occur in $\mathcal{B}$. But by (a), we know that the world label x generates finitely many terms, then the formulas of the form $r \rhd x$ occurring in $\mathcal{B}$ are finitely many. $\square$

Theorem 5.4.6 (Termination of proof search)**.** Every branch of a proof of $x : \neg A$ in $\mathbf{LT.E}^*$ built in accordance with the strategy is finite. Thus, for every formula A the proof search procedure in $\mathbf{LT.E}^*$ is terminating. Moreover, every branch of the proof is either closed or saturated.

Proof. Let $\mathcal{B}$ be a branch of a proof of $x : \neg A$. By Lemmas 5.4.4 and 5.4.5 and König's Lemma, tree $\mathcal{T}_w$ determined by the relation $\hookrightarrow_w$ is finite. Moreover, by Lemma 5.4.3, every world label occurring in $\mathcal{B}$ also occurs in $\mathcal{T}_w$, then $\mathcal{B}$ contains finitely many world labels. Furthermore, by Lemma 5.4.5, every world label generates finitely many neighbourhood terms, then $\mathcal{B}$ also contains finitely many neighbourhood terms. In addition, the formulas of $\mathcal{L}$ occurring in $\mathcal{B}$ are subformulas (or negated subformulas) of A, whence they are in a finite number. Considering that only finitely many labelled formulas can be built by combining finitely many world labels, neighbourhood terms, and formulas of $\mathcal{L}$, and that multiple occurrences of the same labelled formulas are prevented by the saturation conditions of the rules of $\mathbf{LT.E}^*$, we conclude that the branch $\mathcal{B}$ is finite. $\square$

5.5 Countermodel extraction and semantic completeness

In the previous section we have defined a terminating proof search strategy in $\mathbf{LT.E}^*$ that for every formula of $\mathcal{L}$ provides either a derivation or a failed proof. Here we show that from every saturated branch of a failed proof it is possible to directly extract a countermodel of the non-derivable formula. It follows that $\mathbf{LT.E}^*$ is semantically complete with respect to the corresponding bi-neighbourhood models, since every non-derivable formula is not valid. Countermodels are defined in the bi-neighbourhood semantics, however for regular logics we show that it is also possible to directly extract countermodels in the relational semantics.

Given a failed proof for A, a bi-neighbourhood countermodel of A is defined as follows.

Definition 5.5.1 (Countermodel extraction)**.** Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{LT.E}^*$. On the basis of $\mathcal{B}$, we define the bi-neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V}\rangle$ as follows:

- $\mathcal{W}$ = the set of world labels occurring in $\mathcal{B}$.
- For every term $a_1...a_n$, $\overline{a_1...a_n}$ occurring in $\mathcal{B}$, where every a_i is a neighbourhood label or τ,

  $\alpha_{a_1...a_n} = \{x \in \mathcal{W} \mid x \in a_1, ..., x \in a_n \text{ are in } \mathcal{B}\}$, and

  $\alpha_{\overline{a_1...a_n}} = \{x \in \mathcal{W} \mid x \in \overline{a_1} \text{ is in } \mathcal{B}, \text{ or } ..., \text{ or } x \in \overline{a_n} \text{ is in } \mathcal{B}\}$.
- For every $x \in \mathcal{W}$, $\mathcal{N}(x) = \{(\alpha_t, \alpha_{\overline{t}}) \mid t \rhd x \text{ is in } \mathcal{B}\}$.
- For every $p \in \mathcal{L}$, $\mathcal{V}(p) = \{x \in \mathcal{W} \mid x : p \text{ is in } \mathcal{B}\}$.

We prove the following lemma.

Lemma 5.5.1. Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{LT.E}^*$, $\mathcal{M}$ be the model defined on the basis of $\mathcal{B}$ as in Definition 5.5.1, and (ρ, σ) be the realisation defined by taking $\rho(x) = x$ for every world label x occurring in $\mathcal{B}$, and $\sigma(\mathsf{t}) = \alpha_{\mathsf{t}}$ for every neighbourhood term t occurring in $\mathcal{B}$. Then for every $\phi \in \mathcal{L}_{lab}$,

$$\text{if } \phi \text{ is in } \mathcal{B}, \text{ then } \mathcal{M} \models_{\rho,\sigma} \phi.$$

Moreover, for every $X \in \{M, C, N, T, P, D, RD_n^+\}$, if $\mathbf{LT.E}^*$ contains the rules for X, then $\mathcal{M}$ is a bi-neighbourhood X-model.

Proof. First, notice that function σ is well-defined: by saturation of $\overline{\tau}^{\emptyset}{}_{\mathbf{T}}$, $\sigma(\overline{\tau}) = \alpha_{\overline{\tau}} = \emptyset$, and by saturation of $\mathsf{N}_{\mathbf{T}}$, $\tau \rhd x$ is in $\mathcal{B}$ for all x occurring in $\mathcal{B}$, then by definition $(\alpha_\tau, \emptyset) \in \mathcal{N}(x)$ for all $x \in \mathcal{W}$. Moreover, for $t = a_1...a_n$ and $s = b_1...b_m$, we have $\sigma(ts) = \alpha_{ts} = \alpha_{a_1...a_n b_1...b_m} =$

$\alpha_{a_1} \cap ... \cap \alpha_{a_n} \cap \alpha_{b_1} \cap ... \cap \alpha_{b_m} = \alpha_{a_1...a_n} \cap \alpha_{b_1...b_m} = \alpha_t \cap \alpha_s = \sigma(t) \cap \sigma(s)$, and $\sigma(\overline{ts}) = \alpha_{\overline{ts}} = \alpha_{\overline{a_1...a_n b_1...b_m}} = \alpha_{\overline{a_1}} \cup ... \cup \alpha_{\overline{a_n}} \cup \alpha_{\overline{b_1}} \cup ... \cup \alpha_{\overline{b_m}} = \alpha_{\overline{a_1...a_n}} \cup \alpha_{\overline{b_1...b_m}} = \alpha_{\overline{t}} \cup \alpha_{\overline{s}} = \sigma(\overline{t}) \cup \sigma(\overline{s})$.

The first claim (truth lemma) is proved by cases and by induction on the weight of ϕ.

($\phi = t \rhd x$) By definition, $(\sigma(t), \sigma(\overline{t})) = (\alpha_t, \alpha_{\overline{t}}) \in \mathcal{N}(x)$, then $\mathcal{M} \models_{\rho,\sigma} t \rhd x$.

($\phi = x \in \mathsf{t}$) Let $\mathsf{t} = a_1...a_n$. Then by saturation of $\mathsf{dec}_{\mathbf{T}}$, $x \in a_1$, ..., $x \in a_n$ are in $\mathcal{B}$. Then by definition $\rho(x) = x \in \alpha_{a_1...a_n} = \alpha_{\mathsf{t}} = \sigma(\mathsf{t})$, thus $\mathcal{M} \models_{\rho,\sigma} x \in \mathsf{t}$. If $\mathsf{t} = \overline{a_1...a_n}$ the proof is analogous by considering saturation of $\overline{\mathsf{dec}}_{\mathbf{T}}$.

($\phi = x : p$) By definition, $\rho(x) = x \in \mathcal{V}(p)$, then $\rho(x) \Vdash p$, thus $\mathcal{M} \models_{\rho,\sigma} x : p$.

($\phi = x : \neg p$) By saturation of $\mathsf{init}_{\mathbf{T}}$, $x : p$ is not in $\mathcal{B}$, then $x \notin \mathcal{V}(p)$, then $\rho(x) \Vdash \neg p$, thus $\mathcal{M} \models_{\rho,\sigma} x : \neg p$.

($\phi = x : \top$) $\rho(x) \Vdash \top$, then $\mathcal{M} \models_{\rho,\sigma} x : \top$.

($\phi = x : \neg\top$) By saturation of $\neg\top_{\mathbf{T}}$, $x : \neg\top$ is not in $\mathcal{B}$.

($\phi = x : \bot$) By saturation of $\bot_{\mathbf{T}}$, $x : \bot$ is not in $\mathcal{B}$.

($\phi = x : \neg\bot$) $\rho(x) \Vdash \neg\bot$, then $\mathcal{M} \models_{\rho,\sigma} x : \neg\bot$.

($\phi = x : B \wedge C$) By saturation of $\wedge_{\mathbf{T}}$, $x : B$ and $x : C$ are in $\mathcal{B}$. Then by i.h. $\mathcal{M} \models_{\rho,\sigma} x : B$ and $\mathcal{M} \models_{\rho,\sigma} x : C$. Thus $\mathcal{M} \models_{\rho,\sigma} x : B \wedge C$.

($\phi = x : \neg(B \wedge C)$) By saturation of $\neg\wedge_{\mathbf{T}}$, $x : \neg B$ is in $\mathcal{B}$ or $x : \neg C$ is in $\mathcal{B}$. Then by i.h. $\mathcal{M} \models_{\rho,\sigma} x : \neg B$ or $\mathcal{M} \models_{\rho,\sigma} x : \neg C$. Thus $\mathcal{M} \models_{\rho,\sigma} x : \neg(B \wedge C)$.

($\phi = x : B \vee C, x : B \to C, x : \neg(B \vee C), x : \neg(B \to C)$) Similar to ($\phi = x : B \wedge C$) or ($\phi = x : \neg(B \wedge C)$).

($\phi = \mathsf{t} \Vdash^{\forall} B$) Since the branch $\mathcal{B}$ is adequate, by Definition 5.4.2 the term t is atomic. Then for any $x \in \alpha_{\mathsf{t}}$, $x \in \mathsf{t}$ is in $\mathcal{B}$. Thus by saturation of $\Vdash^{\forall}{}_{\mathbf{T}}$, $x : B$ is in $\mathcal{B}$, and by i.h., $x \Vdash B$. Therefore $\mathcal{M} \models_{\rho,\sigma} \mathsf{t} \Vdash^{\forall} B$.

($\phi = x : \Box B$) By saturation of $\Box_{\mathbf{T}}$, there are $a \rhd x$, $a \Vdash^{\forall} B$, and $\overline{a} \Vdash^{\forall} \neg B$ in $\mathcal{B}$. Then by definition $(\alpha_a, \alpha_{\overline{a}}) \in \mathcal{N}(x)$, and by i.h., $\alpha_a \subseteq [\![B]\!]$ and $\alpha_{\overline{a}} \subseteq [\![\neg B]\!]$. Thus $x \Vdash \Box B$, therefore $\mathcal{M} \models_{\rho,\sigma} x : \Box B$.

($\phi = x : \neg\Box B$) Assume $(\gamma, \delta) \in \mathcal{N}(x)$. Then there are terms $t, \overline{t}$ such that $\gamma = \alpha_t$, $\delta = \alpha_{\overline{t}}$, and $t \rhd x$ is in $\mathcal{B}$. Thus by saturation of $\neg\Box_{\mathbf{T}}$, there are $y \in t$ and $y : \neg B$ in $\mathcal{B}$, or there are $y \in \overline{t}$ and $y : B$ in $\mathcal{B}$. By i.h., $y \in \alpha_t$ and $y \Vdash \neg B$, or $y \in \alpha_{\overline{t}}$ and $y \Vdash B$. Then $\alpha_t \not\subseteq [\![B]\!]$, or $[\![B]\!] \not\subseteq \mathcal{W} \setminus \alpha_{\overline{t}}$. Therefore $x \nVdash \Box B$, so $\mathcal{M} \models_{\rho,\sigma} x : \neg\Box B$.

We now show that $\mathcal{M}$ satisfies condition (X) if $\mathbf{LT.E}^*$ contains the rules for the axiom or rule X.

(M) Assume $(\gamma, \delta) \in \mathcal{N}(x)$. Then there are terms $t, \bar{t}$ such that $\alpha_t = \gamma$, $\alpha_{\bar{t}} = \delta$, and $t \rhd x$ is in $\mathcal{B}$. Let $t = a_1...a_n$. Then since $\mathcal{B}$ is adequate, by definition $a_1 \rhd x$, ..., $a_n \rhd x$ are in $\mathcal{B}$. Thus by saturation of $\mathsf{M}_{\mathbf{T}}$, there is no y such that $y \in \overline{a_i}$ is in $\mathcal{B}$. Then $\alpha_{\overline{a_1...a_n}} = \delta = \emptyset$.

(N) By saturation of $\mathsf{N}_{\mathbf{T}}$, for every x in $\mathcal{B}$, $\tau \rhd x$ is in $\mathcal{B}$, then $(\alpha_\tau, \alpha_{\overline{\tau}}) \in \mathcal{N}(x)$. Moreover, by saturation of $\overline{\tau}^{\emptyset}{}_{\mathbf{T}}$, there is no $y \in \overline{\tau}$ in $\mathcal{B}$, thus $\alpha_{\overline{\tau}} = \emptyset$.

(C) Assume $(\gamma_1, \delta_1), (\gamma_2, \delta_2) \in \mathcal{N}(x)$. Then there are terms $a_1...a_n$ and $b_1...b_m$ such that $\alpha_{a_1...a_n} = \gamma_1$, $\alpha_{\overline{a_1...a_n}} = \delta_1$, $\alpha_{b_1...b_m} = \gamma_2$, $\alpha_{\overline{b_1...b_m}} = \delta_2$, and $a_1...a_n \rhd x$, $b_1...b_m \rhd x$ are in $\mathcal{B}$. By definition of adequate branch, $a_i \rhd x$ and $b_j \rhd x$ are in $\mathcal{B}$ for every $1 \leq i \leq n$, $1 \leq j \leq m$. By saturation of C, $t \rhd x$ is in $\mathcal{B}$ for a term t such that $\mathsf{set}(t) = \mathsf{set}(a_1...a_nb_1...b_m)$. Then $(\alpha_t, \alpha_{\bar{t}}) = (\alpha_{a_1...a_nb_1...b_m}, \alpha_{\overline{a_1...a_nb_1...b_m}}) = (\alpha_{a_1...a_n} \cap \alpha_{b_1...b_m}, \alpha_{\overline{a_1...a_n}} \cup \alpha_{\overline{b_1...b_m}}) = (\gamma_1 \cap \gamma_2, \delta_1 \cup \delta_2) \in \mathcal{N}(x)$.

(T) Assume $(\gamma, \delta) \in \mathcal{N}(x)$. Then there is a term t such that $\alpha_t = \gamma$ and $t \rhd x$ is in $\mathcal{B}$. By saturation of $\mathsf{T}_{\mathbf{T}}$, $x \in t$ is in $\mathcal{B}$, then $x \in \alpha_t = \gamma$.

(P) Assume $(\gamma, \delta) \in \mathcal{N}(x)$. Then there is a term t such that $\alpha_t = \gamma$ and $t \rhd x$ is in $\mathcal{B}$. By saturation of $\mathsf{P}_{\mathbf{T}}$, there is y such that $y \in t$ is in $\mathcal{B}$. Then $y \in \alpha_t = \gamma$, thus $\gamma \neq \emptyset$.

(D) Assume $(\gamma_1, \delta_1), (\gamma_2, \delta_2) \in \mathcal{N}(x)$. If $\gamma_1 \neq \gamma_2$ or $\delta_1 \neq \delta_2$, then there are terms t, s such that $\alpha_t = \gamma_1$, $\alpha_{\bar{t}} = \delta_1$, $\alpha_s = \gamma_2$, $\alpha_{\bar{s}} = \delta_2$, and $t \rhd x$, $s \rhd x$ are in $\mathcal{B}$. Thus by saturation of $\mathsf{D}_{2\mathbf{T}}$, there is y such that $y \in t$ and $y \in s$ are in $\mathcal{B}$, or $y \in \bar{t}$ and $y \in \bar{s}$ are in $\mathcal{B}$. Then $y \in \alpha_t$ and $y \in \alpha_s$, or $y \in \alpha_{\bar{t}}$ and $y \in \alpha_{\bar{s}}$, therefore $\gamma_1 \cap \gamma_2 \neq \emptyset$ or $\delta_1 \cap \delta_2 \neq \emptyset$. If instead $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$, then there is a term t such that $\alpha_t = \gamma_1$, $\alpha_{\bar{t}} = \delta_1$, and $t \rhd x$ is in $\mathcal{B}$. By saturation of $\mathsf{D}_{1\mathbf{T}}$, there is y such that $y \in t$ is in $\mathcal{B}$, or $y \in \bar{t}$ is in $\mathcal{B}$. Thus $\gamma_1 \neq \emptyset$ or $\delta_1 \neq \emptyset$.

(RD_n^+) Let $(\gamma_1, \delta_1), ..., (\gamma_m, \delta_m)$ be any $m \leq n$ different bi-neighbourhood pairs belonging to $\mathcal{N}(x)$. Then there are terms $t_1, ..., t_m$ such that $t_1 \rhd x$, ..., $t_m \rhd x$ are in $\mathcal{B}$, and for every $1 \leq i \leq m$, $\alpha_{t_1} = \gamma_i$ and $\alpha_{\overline{t_i}} = \delta_i$. By saturation of rule $\mathsf{D}^+_{m}{}_{\mathbf{T}}$ (that by definition belongs to the calculus $\mathbf{LT.ED^{+*}_n}$), there is y such that $y \in t_1$, ..., $y \in t_m$ are in $\mathcal{B}$. Then $y \in \alpha_{t_1} \cap ... \cap \alpha_{t_m}$, thus $\alpha_{t_1} \cap ... \cap \alpha_{t_m} = \gamma_1 \cap ... \cap \gamma_m \neq \emptyset$.

Although not necessary for calculi laking the rules for 4, we can also show that in the monotonic case the interpretation of every negative term turns out to be empty, i.e., not only the interpretation of the negative terms $\bar{t}$ such that $t \rhd x$ is in $\mathcal{B}$ for some x: If $\overline{a_1...a_n}$ is in $\mathcal{B}$, then by definition of adequate branch there is x such that $a_1 \rhd x$, ..., $a_n \rhd x$ are in $\mathcal{B}$. Then by saturation of $\mathsf{M}_{\mathbf{T}}$, there is no y such that $y \in a_i$ for some $1 \leq i \leq n$, thus $\alpha_{\overline{a_1...a_n}} = \emptyset$. □

As a consequence of this lemma we can prove that the calculi $\mathbf{LT.E}^*$ are complete with respect to the corresponding class of bi-neighbourhood models.

Theorem 5.5.2 (Semantic completeness)**.** If A is valid in every bi-neighbourhood X-model, then A is derivable in $\mathbf{LT.EX}^*$.

$$
\begin{array}{lll}
\mathbf{1.} & x : \neg(\Box(p \wedge q) \to \Box p) & \\
\mathbf{2.} & x : \Box(p \wedge q) & (1,\ \neg{\to}_{\mathbf{T}}) \\
\mathbf{3.} & x : \neg\Box p & (1,\ \neg{\to}_{\mathbf{T}}) \\
\mathbf{4.} & a \rhd x & (2,\ \Box_{\mathbf{T}}) \\
\mathbf{5.} & a \Vdash^{\forall} p \wedge q & (2,\ \Box_{\mathbf{T}}) \\
\mathbf{6.} & \overline{a} \Vdash^{\forall} \neg(p \wedge q) & (2,\ \Box_{\mathbf{T}})
\end{array}
$$

Left branch:

$$
\begin{array}{lll}
7. & y \in a & (3,\ 4,\ \neg\Box_{\mathbf{T}}) \\
8. & y : \neg p & (3,\ 4,\ \neg\Box_{\mathbf{T}}) \\
9. & y : p \wedge q & (5,\ 7,\ \Vdash^{\forall}{}_{\mathbf{T}}) \\
10. & y : p & (9,\ \wedge_{\mathbf{T}}) \\
11. & y : q & (9,\ \wedge_{\mathbf{T}}) \\
 & \mathbf{f} \quad (8,\ 10,\ \mathsf{init}_{\mathbf{T}}) &
\end{array}
$$

Right branch:

$$
\begin{array}{lll}
\mathbf{7.} & y \in \overline{a} & (3,\ 4,\ \neg\Box_{\mathbf{T}}) \\
\mathbf{8.} & y : p & (3,\ 4,\ \neg\Box_{\mathbf{T}}) \\
\mathbf{9.} & y : \neg(p \wedge q) & (6,\ 7,\ \Vdash^{\forall}{}_{\mathbf{T}})
\end{array}
$$

$$
\begin{array}{lll}
10. & y : \neg p & (9,\ \neg\wedge_{\mathbf{T}}) \\
 & \mathbf{f} \quad (8,\ 10,\ \mathsf{init}_{\mathbf{T}}) &
\end{array}
\qquad
\begin{array}{lll}
\mathbf{10.} & y : \neg q & (9,\ \neg\wedge_{\mathbf{T}}) \\
 & \textbf{saturated} &
\end{array}
$$

Figure 5.6: Failed proof of axiom M in $\mathbf{LT.E}$.

Proof. If A is not derivable in $\mathbf{LT.EX}^*$, then it has a failed proof containing a saturated branch $\mathcal{B}$. By Lemma 5.5.1, we can construct an X-model $\mathcal{M}$ satisfying all labelled formulas occurring in $\mathcal{B}$. Then, since $x : \neg A$ is in $\mathcal{B}$, in particular we have $\mathcal{M}, x \Vdash \neg A$, whence $\mathcal{M}, x \nVdash A$, therefore A is not valid in the class of all X-models. $\Box$

Observe that the combination of termination of proof search and countermodel extraction from failed proofs (Theorem 5.4.6 and Lemma 5.5.1) provides an alternative proof of the *finite model property* for the considered classical non-normal modal logics, that is, every non-derivable/non-valid formula has a finite countermodel. We now present two examples of failed proofs in the tableaux calculi and the bi-neighbourhood countermodels extracted from their saturated branches. We also present the standard models obtained by the transformation in Proposition 4.3.4.

Example 5.5.1 (Axiom M is not derivable in logic $\mathbf{E}$)**.** In Figure 5.6 we find a failed proof of $\Box(p \wedge q) \to \Box p$ in $\mathbf{LT.E}$. The countermodels are as follows.

Bi-neighbourhood countermodel. Following Definition 5.5.1, from the saturated branch we obtain the following bi-neighbourhood model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$: $\mathcal{W} = \{x, y\}$. $\mathcal{N}_{bi}(x) = \{(\alpha_a, \alpha_{\overline{a}})\} = \{(\emptyset, \{y\})\}$ and $\mathcal{N}_{bi}(y) = \emptyset$. $\mathcal{V}(p) = \{y\}$ and $\mathcal{V}(q) = \emptyset$. Then $\mathcal{M}_{bi}$ is a countermodel of $\Box(p \wedge q) \to \Box p$: $x \Vdash \Box(p \wedge q)$ because $\emptyset \subseteq [\![p \wedge q]\!] = \emptyset \subseteq \mathcal{W} \setminus \{y\}$, but $x \nVdash \Box p$ because $[\![p]\!] = \{y\} \not\subseteq \mathcal{W} \setminus \{y\}$.

Neighbourhood countermodel. We consider the set $\mathcal{S} = \{\Box(p \wedge q) \to \Box p, \Box(p \wedge q), \Box p, p \wedge q, p, q\}$ of the subformulas of $\Box(p \wedge q) \to \Box p$. By applying the transformation in Proposition 4.3.4 to the bi-neighbourhood model $\mathcal{M}_{bi}$, we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, and $\mathcal{N}_{st}(x) = \{\emptyset\}$, since $\mathcal{N}_{st}(x) = \{[\![p \wedge q]\!]_{\mathcal{M}_{bi}}\}$ and $[\![p \wedge q]\!]_{\mathcal{M}_{bi}} = \emptyset$.

Example 5.5.2 (Axiom K is not derivable in logic **EC**)**.** In Figure 5.7 we find a failed proof of $\Box(p \to q) \to (\Box p \to \Box q)$ in **LT**.**EC**. The countermodels are as follows.

Bi-neighbourhood countermodel. Following Definition 5.5.1, from the saturated branch we obtain the following model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$: $\mathcal{W} = \{x, y, z\}$. $\mathcal{V}(p) = \emptyset$ and $\mathcal{V}(q) = \{y\}$. $\mathcal{N}_{bi}(y) = \mathcal{N}_{bi}(z) = \emptyset$. $\mathcal{N}_{bi}(x) = \{(\{z\}, \emptyset), (\emptyset, \{y\})\}$, because $\mathcal{N}_{bi}(x) = \{(\alpha_a, \alpha_{\overline{a}}), (\alpha_b, \alpha_{\overline{b}}), (\alpha_{ab}, \alpha_{\overline{ab}})\}$, and $\alpha_a = \{z\}$, $\alpha_{\overline{a}} = \emptyset$, $\alpha_b = \emptyset$, $\alpha_{\overline{b}} = \{y\}$, $\alpha_{ab} = \emptyset$, $\alpha_{\overline{ab}} = \{y\}$. We have $x \Vdash \Box(p \to q)$ because $\{z\} \subseteq [\![p \to q]\!] = \mathcal{W} \subseteq \mathcal{W} \setminus \emptyset$; and $x \Vdash \Box p$ because $\emptyset \subseteq [\![p]\!] = \emptyset \subseteq \mathcal{W} \setminus \{y\}$; but $x \nVdash \Box q$ because $\{z\} \not\subseteq [\![q]\!] = \{y\}$ and $[\![q]\!] = \{y\} \not\subseteq \mathcal{W} \setminus \{y\}$, whence $x \nVdash \Box(p \to q) \to (\Box p \to \Box q)$. Observe that $\mathcal{M}_{bi}$ is a C-model since $(\{z\} \cap \emptyset, \emptyset \cup \{y\}) = (\emptyset, \{y\})$.

Neighbourhood countermodel. By logical equivalence we can restrict the considered set of formulas $\mathcal{S}$ to $\{\Box(p \to q), \Box p, \Box q, p \to q, p, q, \Box((p \to q) \wedge q), \Box(p \wedge q)\}$. By the transformation in Proposition 4.3.4, from $\mathcal{M}_{bi}$ we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, $\mathcal{N}_{st}(y) = \mathcal{N}_{st}(z) = \emptyset$, and $\mathcal{N}_{st}(x) = \{[\![p \to q]\!]_{\mathcal{M}_{bi}}, [\![p]\!]_{\mathcal{M}_{bi}}, [\![p \wedge q]\!]_{\mathcal{M}_{bi}}\} = \{\mathcal{W}, \emptyset\}$.

Relational countermodels for regular logics

As recalled in Section 2.3, in addition to the different kinds of neighbourhood semantics, regular logics (i.e., **MC** and its extensions) have also a relational semantics (see Definition 2.3.7). Moreover, we have seen in Section 4.3 that every finite bi-neighbourhood MC-model can be transformed into an equivalent relational model (see Proposition 4.3.5). Thus, in principle one could obtain relational countermodels of non-valid formulas by first extracting bi-neighbourhood models from failed proofs in **LT**.**MC***, and then converting the bi-neighbourhood models into relational ones. However, we now show that it is also possible to directly extract the relational countermodels from the failed proofs in **LT**.**MC***. To this purpose we consider the following notion of *maximal* terms.

Definition 5.5.2 (Maximal term)**.** Let $\mathcal{B}$ be a saturated branch of a failed proof in **LT**.**MC***, and let t and x be respectively a neighbourhood term and a world label occurring in $\mathcal{B}$. We say that t is *maximal* for x if the formula $t \rhd x$ is in $\mathcal{B}$, and for every term s such that $s \rhd x$ is in $\mathcal{B}$, $\mathsf{set}(s) \subseteq \mathsf{set}(t)$.

We observe that, if present, the maximal term for x in $\mathcal{B}$ is unique: assume t and r are two distinct maximal terms for x. Then $\mathsf{set}(t) = \mathsf{set}(r) = \{a_1, ..., a_n\}$ for some neighbourhood labels $a_1, ..., a_n$. Thus, both terms are generated by an application of $\mathsf{C}_\mathbf{T}$ to $a_1 \rhd x$, ..., $a_n \rhd x$.

1. $x : \neg(\Box(p \to q) \to (\Box p \to \Box q))$
2. $x : \Box(p \to q)$ — $(1, \neg{\to}_{\mathbf{T}})$
3. $x : \neg(\Box p \to \Box q)$ — $(1, \neg{\to}_{\mathbf{T}})$
4. $x : \Box p$ — $(3, \neg{\to}_{\mathbf{T}})$
5. $x : \neg\Box q$ — $(3, \neg{\to}_{\mathbf{T}})$
6. $a \rhd x$ — $(2, \Box_{\mathbf{T}})$
7. $a \Vdash^{\forall} p \to q$ — $(2, \Box_{\mathbf{T}})$
8. $\overline{a} \Vdash^{\forall} \neg(p \to q)$ — $(2, \Box_{\mathbf{T}})$
9. $b \rhd x$ — $(4, \Box_{\mathbf{T}})$
10. $b \Vdash^{\forall} p$ — $(4, \Box_{\mathbf{T}})$
11. $\overline{b} \Vdash^{\forall} \neg p$ — $(4, \Box_{\mathbf{T}})$
12. $ab \rhd x$ — $(6, 9, \mathsf{C}_{\mathbf{T}})$

Left branch:

13. $y \in ab$ — $(5, 12, \neg\Box_{\mathbf{T}})$
14. $y : \neg q$ — $(5, 12, \neg\Box_{\mathbf{T}})$
15. $y \in a$ — $(13, \mathsf{dec}_{\mathbf{T}})$
16. $y \in b$ — $(13, \mathsf{dec}_{\mathbf{T}})$
17. $y : p \to q$ — $(7, 15, \Vdash^{\forall}{}_{\mathbf{T}})$
18. $y : p$ — $(10, 16, \Vdash^{\forall}{}_{\mathbf{T}})$

- 19. $y : \neg p$ — $(17, \to_{\mathbf{T}})$
  f $(18, 19, \mathsf{init}_{\mathbf{T}})$
- 19. $y : q$ — $(17, \to_{\mathbf{T}})$
  f $(14, 19, \mathsf{init}_{\mathbf{T}})$

Right branch:

13. $y \in \overline{ab}$ — $(5, 12, \neg\Box_{\mathbf{T}})$
14. $y : q$ — $(5, 12, \neg\Box_{\mathbf{T}})$

- 15. $y \in \overline{a}$ — $(13, \overline{\mathsf{dec}}_{\mathbf{T}})$
  16. $y : \neg(p \to q)$ — $(8, 15, \Vdash^{\forall}{}_{\mathbf{T}})$
  17. $y : p$ — $(16, \neg{\to}_{\mathbf{T}})$
  18. $y : \neg q$ — $(16, \neg{\to}_{\mathbf{T}})$
  f $(14, 18, \mathsf{init}_{\mathbf{T}})$
- $(*)$

$(*)$

15. $y \in \overline{b}$ — $(13, \overline{\mathsf{dec}}_{\mathbf{T}})$
16. $y : \neg p$ — $(11, 15, \Vdash^{\forall}{}_{\mathbf{T}})$

- **17.** $z \in a$ — $(5, 6, \neg\Box_{\mathbf{T}})$
  18. $z : \neg q$ — $(5, 6, \neg\Box_{\mathbf{T}})$
  19. $z : p \to q$ — $(7, 17, \Vdash^{\forall}{}_{\mathbf{T}})$
  - **20.** $z : \neg p$ — $(19, \to_{\mathbf{T}})$
    saturated
  - 20. $z : q$ — $(19, \to_{\mathbf{T}})$
    f $(18, 20, \mathsf{init}_{\mathbf{T}})$
- 17. $z \in \overline{a}$ — $(5, 6, \neg\Box_{\mathbf{T}})$
  18. $z : q$ — $(5, 6, \neg\Box_{\mathbf{T}})$
  19. $z : \neg(p \to q)$ — $(8, 18, \Vdash^{\forall}{}_{\mathbf{T}})$
  20. $z : p$ — $(19, \neg{\to}_{\mathbf{T}})$
  21. $z : \neg q$ — $(19, \neg{\to}_{\mathbf{T}})$
  f $(18, 21, \mathsf{init}_{\mathbf{T}})$

Figure 5.7: Failed proof of axiom K in **LT.EC**.

But one term is necessarily generated before the other one, then the generation of the second is prevented by the saturation condition of $\mathsf{C}_{\mathbf{T}}$

Lemma 5.5.3. Let $\mathcal{B}$ be a saturated branch of a failed proof for A in $\mathbf{LT}.\mathbf{MC}^*$. Then, (a) for every world label x occurring in $\mathcal{B}$, if there are formulas $s \rhd x$ is in $\mathcal{B}$, then there is a maximal term for x in $\mathcal{B}$. Moreover, (b) if $\mathbf{LT}.\mathbf{MC}^*$ contains the rules for N, then for every worl label x occurring in $\mathcal{B}$ there is a maximal term for x in $\mathcal{B}$.

Proof. (a) Assume that $s_1 \rhd x$, ..., $s_n \rhd x$ are in $\mathcal{B}$. By termination of proof search, there are only finitely many formulas of the form $s \rhd x$ in $\mathcal{B}$. It is easy to see that, by saturation of rule $\mathsf{C}_{\mathbf{T}}$, there is a term t such that $\mathsf{set}(s_1), ..., \mathsf{set}(s_n) \subseteq \mathsf{set}(t)$ and $t \rhd x$ is in $\mathcal{B}$, whence t is maximal (b) By saturation of rule N, for every world label x occurring in $\mathcal{B}$, $\tau \rhd x$ is in $\mathcal{B}$. Then by (a) there is a maximal term for x in $\mathcal{B}$. $\square$

Definition 5.5.3 (Relational countermodel)**.** Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{LT}.\mathbf{E}^*$. The relational model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W}$, $\mathcal{V}$, and, for every term t in $\mathcal{B}$, α_t, are defined as in Definition 5.5.1.
- $\mathcal{W}^i = \{x \in \mathcal{W} \mid \text{there is no term } t \text{ such that } t \rhd x \text{ is in } \mathcal{B}\}$.
- $x\mathcal{R}y$ iff $y \in a$ is in $\mathcal{B}$ for every $a \in \mathsf{set}(t)$, where t is the maximal term for x in $\mathcal{B}$.

Lemma 5.5.4. Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{LT}.\mathbf{E}^*$, and $\mathcal{M}_r$ be the model defined on the basis of $\mathcal{B}$ as in Definition 5.5.3. We define the realisation (ρ, σ) by taking $\rho(x) = x$ for every world label x in $\mathcal{B}$, and $\sigma(\mathsf{t}) = \alpha_{\mathsf{t}}$ for every term t in $\mathcal{B}$. Then for every $\phi \in \mathcal{L}_{lab}$, if ϕ is in $\mathcal{B}$, then $\mathcal{M}_r \models_{\rho,\sigma} \phi$. Moreover, if $\mathbf{LT}.\mathbf{MC}^*$ contains the rules for axiom N, then $\mathcal{M}_r$ is a standard Kripke model for the normal modal logic $\mathbf{K}$; if $\mathbf{LT}.\mathbf{MCN}^*$ contains the rule for axiom T, then the relation $\mathcal{R}$ is reflexive; and if $\mathbf{LT}.\mathbf{MCN}^*$ contains a rule among P, D, and RD_n^+ (which are equivalent in $\mathbf{LT}.\mathbf{MCN}^*$), then $\mathcal{R}$ is serial.

Proof. The truth lemma is proved by induction on ϕ. In most cases the proof is the same as for Lemma 5.5.1. Here we only consider modal formulas.

($\phi = x : \Box B$) By saturation of $\Box_{\mathbf{T}}$, there is a neighbourhood label a such that $a \rhd x$, $a \Vdash^{\forall} A$ and $\overline{a} \Vdash^{\forall} \neg A$ are in $\mathcal{B}$. Then by Lemma 5.5.3 there is a maximal term t for x in $\mathcal{B}$, and by definition $a \in \mathsf{set}(t)$. Now assume $x\mathcal{R}y$. By definition of $\mathcal{R}$, $y \in a$ is in $\mathcal{B}$. Then by saturation of $\Vdash^{\forall}{}_{\mathbf{T}}$, $y : A$ is in $\mathcal{B}$, and by i.h., $y \Vdash A$. Therefore $x \Vdash \Box A$.

($\phi = x : \neg\Box B$) If there is no formula $s \rhd x$ in $\mathcal{B}$, then by definition $x \in \mathcal{W}^i$, thus $x \nVdash \Box B$. Otherwise, let $t = a_1 ... a_n$ be the maximal term for x in $\mathcal{B}$. Then $t \rhd x$ is in $\mathcal{B}$. By saturation of $\neg\Box_{\mathbf{T}}$, there is y such that $y \in t$ and $y : \neg B$ are in $\mathcal{B}$, or $y \in \overline{t}$ and $y : B$ are in $\mathcal{B}$. But the second possibility is excluded by saturation of $\mathsf{M_T}$. Then by saturation of $\mathsf{dec_T}$, $y \in a_1$, ..., $y \in a_n$ are in $\mathcal{B}$, thus $x\mathcal{R}y$. But by i.h., $y \Vdash \neg B$. Therefore $x \Vdash \neg\Box B$.

The model properties are shown as follows.

($\mathcal{W}^i = \emptyset$) By saturation of $\mathsf{N_T}$, for every world label x, $\tau \rhd x$ is in $\mathcal{B}$. Then by definition $x \notin \mathcal{W}^i$.

(Reflexivity) By Lemma 5.5.3 (b), there is a maximal term for x in $\mathcal{B}$, let it be t. Then $t \rhd x$ is in $\mathcal{B}$, and by saturation of $\mathsf{T_T}$, $x \in t$ is in $\mathcal{B}$. Then by saturation of $\mathsf{dec_T}$, $x \in a$ is in $\mathcal{B}$ for every $a \in \mathsf{set}(t)$, thus $x\mathcal{R}x$.

(Seriality) Assume **LT**.**MCN*** contains rule $\mathsf{P_T}$, and let t be the maximal term for x in $\mathcal{B}$. Then $t \rhd x$ is in $\mathcal{B}$, and by saturation of $\mathsf{P_T}$, there is $y \in t$ in $\mathcal{B}$. Then by saturation of $\mathsf{dec_T}$, $y \in a$ is in $\mathcal{B}$ for every $a \in \mathsf{set}(t)$, thus $x\mathcal{R}y$. The proof is similar if **LT**.**MCN*** contains the rules for D or RD_n^+. □

As for the bi-neighbourhood semantics, we obtain the following completeness theorem.

Theorem 5.5.5 (Semantic completeness)**.** If A is valid in every relational model for **MC** or **MCN***, then it is derivable in **LT**.**MC** or **LT**.**MCNX***, respectively.

We conclude this section by presenting two examples of failed proofs of axiom 4 in the calculi **LT**.**MC** and **LT**.**MCNT** and the countermodels directly extracted from the saturated branches.

Example 5.5.3 (Axiom 4 is not derivable in **MC**)**.** In Figure 5.8 we find a failed proof of $\Box p \to \Box\Box p$ in **LT**.**MC**. We obtain the following countermodels.

Bi-neighbourhood countermodel. Following Definition 5.5.1, from the saturated branch we obtain the following bi-neighbourhood model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$: $\mathcal{W} = \{x, y\}$, $\mathcal{V}(p) = \{y\}$, $\mathcal{N}(y) = \emptyset$, and $\mathcal{N}(x) = \{(\alpha_a, \alpha_{\overline{a}})\} = \{(\{y\}, \emptyset)\}$. We have $x \Vdash \Box p$ because $\{y\} \subseteq [\![p]\!] \subseteq \mathcal{W} \setminus \emptyset$; but $y \not\Vdash \Box p$ because $\mathcal{N}(y) = \emptyset$, then $\{y\} \not\subseteq [\![\Box p]\!]$, thus $x \not\Vdash \Box\Box p$.

Relational countermodel. If in contrast we consider Definition 5.5.3 we obtain the relational model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, $\mathcal{W}^i = \{y\}$, and $\mathcal{R}(x) = \alpha_a = \{y\}$. Then $x \Vdash \Box p$ because $x\mathcal{R}w$ implies $w = y$, and $y \Vdash p$. Moreover, since $y \in \mathcal{W}^i$, by definition $y \not\Vdash \Box p$. Then since $x\mathcal{R}y$ we have $x \not\Vdash \Box\Box p$.

Example 5.5.4 (Axiom 4 is not derivable in **MCNT**)**.** In Figure 5.8 we find a failed proof of $\Box p \to \Box\Box p$ in **LT**.**MCNT**. We obtain the following countermodels.

Bi-neighbourhood countermodel. Following Definition 5.5.1, from the saturated branch we obtain the following bi-neighbourhood model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$: $\mathcal{W} = \{x, y, z\}$, $\mathcal{V}(p) = \{x, y\}$, $\mathcal{N}(x) = \{(\alpha_a, \alpha_{\overline{a}}), (\alpha_\tau, \alpha_{\overline{\tau}}), (\alpha_{a\tau}, \alpha_{\overline{a\tau}})\} = \{(\{x, y\}, \emptyset), (\{x, y, z\}, \emptyset)\}$. $\mathcal{N}(y) = \mathcal{N}(z) =$

$\{(\alpha_{\mathcal{T}}, \alpha_{\overline{\mathcal{T}}})\} = \{(\{x, y, z\}, \emptyset)\}$. We have $x \Vdash \Box p$ because $\{x, y\} \subseteq [\![p]\!] = \{x, y\} \subseteq \mathcal{W} \setminus \emptyset$. But $y \not\Vdash \Box p$ and $z \not\Vdash \Box p$ because $\{x, y, z\} \not\subseteq [\![p]\!]$. Then $z \not\Vdash \Box\Box p$ because $\{x, y\} \not\subseteq [\![\Box p]\!] = \{x\}$. It is easy to verify that $\mathcal{M}$ is a MCT-model.

Relational countermodel. If in contrast we consider the construction in Definition 5.5.3 we obtain the relational model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, $\mathcal{W}^i = \{\emptyset\}$, $\mathcal{R}(x) = \alpha_{a\mathcal{T}} = \{x, y\}$, and $\mathcal{R}(y) = \mathcal{R}(z) = \alpha_{\mathcal{T}} = \{x, y, z\}$. Then $x \Vdash \Box p$ because $x\mathcal{R}w$ implies $w = z$ or $w = y$, and $x \Vdash p$ and $y \Vdash p$. But $z \not\Vdash p$, then $y \not\Vdash \Box p$, therefore $x \not\Vdash \Box\Box p$. Notice that $\mathcal{M}_r$ is a relational model for **MCNT**, and it is not transitive since $x\mathcal{R}y$, $y\mathcal{R}z$, but not $x\mathcal{R}z$.

Observe that in the Examples 5.5.3 and 5.5.4 above, the relational models directly extracted from the saturated hypersequents are the same models that we obtain by applying the transformation in Proposition 4.3.5 to the bi-neighbourhood models.

5.6 Tableaux calculi for logics with axiom 4

In this section, we conclude the presentation of our labelled tableaux calculi by considering the calculi for the systems with axiom 4. Differently from the tableaux calculi presented in previous sections, at present we have not obtained a direct proof of semantic completeness of these calculi with respect their bi-neighbourhood models. Here we prove that these calculi are semantically complete with respect to their *standard* semantics. As before, the proof consists in showing how to directly extract countermodels from failed proofs.

For every logic **E4***, we define the corresponding labelled tableaux calculus **LT.E4*** by extending **LT.E*** (cf. Figure 5.3) with the rules in Figure 5.9, namely the rules for 4 and the rules for complements. The rules for 4 are just the tableaux reformulation of the corresponding sequent rules in Figure 5.1. In addition, the rules for complements $\mathsf{cmp}_{\cap\mathbf{T}}$ and $\mathsf{cmp}_{\cup\mathbf{T}}$ respectively express that positive and negative terms must be disjoint, and that their union must coincide with the whole set of world labels, and are used to force the interpretation of negative terms as complements of the positive ones. While these rules can be shown admissible in the calculi without the rules for 4, at present we could not prove the same in presence of the rules for 4, for this reason we add them explicitly to the calculi.

In order to prove the semantic completeness of the calculi **LT.E4*** we extend the list of saturation conditions in Definition 5.4.3 with the conditions for the rules for complements and for axioms 4.

Definition 5.6.1 (Saturated branch)**.** Let $\mathcal{B}$ be a branch of a tableau in **LT.E4***. The saturation conditions associated to the rules for 4 are as follows: ($\mathsf{cmp}_{\cap\mathbf{T}}$) $x \in t$ and $x \in \bar{t}$ are not both in $\mathcal{B}$. ($\mathsf{cmp}_{\cup\mathbf{T}}$) $x \in t$ is in $\mathcal{B}$ or $x \in \bar{t}$ is in $\mathcal{B}$. ($4_{\mathbf{T}}$) If $t \rhd x$ is in $\mathcal{B}$, then $J(t) \rhd x$ is

LT.MC

1.	$x : \neg(\Box p \to \Box\Box p)$	
2.	$x : \Box p$	$(1, \neg{\to}_{\mathbf{T}})$
3.	$x : \neg\Box\Box p$	$(1, \neg{\to}_{\mathbf{T}})$
4.	$a \rhd x$	$(2, \Box_{\mathbf{T}})$
5.	$a \Vdash^{\forall} p$	$(2, \Box_{\mathbf{T}})$
6.	$\overline{a} \Vdash^{\forall} \neg p$	$(2, \Box_{\mathbf{T}})$

Left branch:

7.	$y \in a$	$(3, 4, \neg\Box_{\mathbf{T}})$
8.	$y : \neg\Box p$	$(3, 4, \neg\Box_{\mathbf{T}})$
9.	$y : p$	$(5, 7, \Vdash^{\forall}{}_{\mathbf{T}})$
	saturated	

Right branch:

7.	$y \in \overline{a}$	$(3, 4, \neg\Box_{\mathbf{T}})$
8.	$y : \Box p$	$(3, 4, \neg\Box_{\mathbf{T}})$
	$\mathbf{f}$	$(4, 7, \mathsf{M}_{\mathbf{T}})$

LT.MCNT

1.	$x : \neg(\Box p \to \Box\Box p)$	
2.	$x : \Box p$	$(1, \neg{\to}_{\mathbf{T}})$
3.	$x : \neg\Box\Box p$	$(1, \neg{\to}_{\mathbf{T}})$
4.	$a \rhd x$	$(2, \Box_{\mathbf{T}})$
5.	$a \Vdash^{\forall} p$	$(2, \Box_{\mathbf{T}})$
6.	$\overline{a} \Vdash^{\forall} \neg p$	$(2, \Box_{\mathbf{T}})$
7.	$\tau \rhd x$	$(\mathsf{N}_{\mathbf{T}})$
8.	$a\tau \rhd x$	$(4, 7, \mathsf{C}_{\mathbf{T}})$
9.	$x \in a\tau$	$(8, \mathsf{T}_{\mathbf{T}})$
10.	$x \in a$	$(9, \mathsf{dec}_{\mathbf{T}})$
11.	$x \in \tau$	$(9, \mathsf{dec}_{\mathbf{T}})$
12.	$x : p$	$(5, 10, \Vdash^{\forall}{}_{\mathbf{T}}$

Left branch:

13.	$y \in a\tau$	$(3, 8, \neg\Box_{\mathbf{T}})$
14.	$y : \neg\Box p$	$(3, 8, \neg\Box_{\mathbf{T}})$
15.	$y \in a$	$(13, \mathsf{dec}_{\mathbf{T}})$
16.	$y \in \tau$	$(13, \mathsf{dec}_{\mathbf{T}})$
17.	$y : p$	$(5, 15, \Vdash^{\forall}{}_{\mathbf{T}})$
18.	$\tau \rhd y$	$(\mathsf{N}_{\mathbf{T}})$

Left-left branch:

19.	$z \in \tau$	$(14, 18, \neg\Box_{\mathbf{T}})$
20.	$z : \neg p$	$(14, 18, \neg\Box_{\mathbf{T}})$
21.	$\tau \rhd z$	$(\mathsf{N}_{\mathbf{T}})$
	saturated	

Left-right branch:

19.	$z \in \overline{\tau}$	$(14, 18, \neg\Box_{\mathbf{T}})$
20.	$z : p$	$(14, 18, \neg\Box_{\mathbf{T}})$
	$\mathbf{f}$	$(19, \overline{\tau}^{\emptyset}{}_{\mathbf{T}})$

Right branch:

13.	$y \in \overline{a\tau}$	$(3, 8, \neg\Box_{\mathbf{T}})$
14.	$y : \Box p$	$(3, 4, \neg\Box_{\mathbf{T}})$
	$\mathbf{f}$	$(8, 13, \mathsf{M}_{\mathbf{T}})$

Figure 5.8: Failed proofs of axiom 4 in **LT**.**MC** and **LT**.**MCNT**.

Rules for axiom 4

$$\mathsf{4_T}\ \frac{t \rhd x}{J(t) \rhd x} \qquad \mathrm{J_T}\ \frac{x \in J(t)}{t \rhd x} \qquad \overline{\mathrm{J}}_{\mathbf{T}}\ \frac{x \in \overline{J(t)} \quad s \rhd x}{y \in t,\ y \in \overline{s} \mid y \in \overline{t},\ y \in s}\ (y!)$$

Rules for complements

$$\mathsf{cmp}_{\cap\mathbf{T}}\ \frac{x \in t \quad x \in \overline{t}}{\mathbf{f}} \qquad \mathsf{cmp}_{\cup\mathbf{T}}\ \frac{\mathcal{B}}{x \in t \mid x \in \overline{t}}\ (x, \mathsf{t} \text{ in } \mathcal{B})$$

Figure 5.9: Rules of labelled tableaux calculi **LT.E4***

in $\mathcal{B}$. ($\mathrm{J_T}$) If $x \in J(t)$ is in $\mathcal{B}$, then $t \rhd x$ is in $\mathcal{B}$. ($\overline{\mathrm{J}}_{\mathbf{T}}$) If $x \in \overline{J(t)}$ and $s \rhd x$ are in $\mathcal{B}$, then there is y such that $y \in t$ and $y \in \overline{s}$ are in $\mathcal{B}$, or $y \in \overline{t}$ and $y \in s$ are in $\mathcal{B}$.

We say that $\mathcal{B}$ is saturated with respect to **LT.E4*** if it is saturated with respect to all possible applications of any rule of **LT.E4***.

Then, from every saturated branch in **LT.E4*** we can directly extract a standard countermodel as follows.

Definition 5.6.2 (Neighbourhood countermodel)**.** Let $\mathcal{B}$ be a saturated branch of a failed proof of A in **LT.E4***. The standard neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W}$ and $\mathcal{V}$ are defined as in Definition 5.5.1.
- For every t occurring in $\mathcal{B}$, $\alpha_{\mathsf{t}} = \{x \in \mathcal{W} \mid x \in \mathsf{t} \text{ is in } \mathcal{B}\}$.
- $\mathcal{N}(x) = \{\alpha_t \mid t \rhd x \text{ is in } \mathcal{B}\}$.

Lemma 5.6.1. Let $\mathcal{B}$ be a saturated branch of a failed proof of A in **LT.E4***, and $\mathcal{M}$ be the model defined on the basis of $\mathcal{B}$ as in Definition 5.6.2. We define the realisation (ρ, σ) by taking $\rho(x) = x$ for every world label x in $\mathcal{B}$, and $\sigma(\mathsf{t}) = \alpha_{\mathsf{t}}$ for every term t in $\mathcal{B}$. Then for every $\phi \in \mathcal{L}_{lab}$, if ϕ is in $\mathcal{B}$, then $\mathcal{M} \models_{\rho,\sigma} \phi$. Moreover, $\mathcal{M}$ satisfies the condition for axiom 4 in the standard semantics, and for every $X \in \{C, N, T, P, D, RD_n^+\}$, if **LT.E4*** contains the rules for X, then $\mathcal{M}$ satisfies the condition associated to X in the standard semantics (cf. the semantic conditions in Section 2.3).

Proof. First, we show that $\alpha_{\overline{t}} = \mathcal{W} \setminus \alpha_t$. By saturation of $\mathsf{cmp}_{\cap\mathbf{T}}$, $x \in t$ and $x \in \overline{t}$ are not both in $\mathcal{B}$, thus $\alpha_t \cap \alpha_{\overline{t}} = \emptyset$. Moreover, by saturation of $\mathsf{cmp}_{\cap\mathbf{T}}$, $x \in t$ is in $\mathcal{B}$ or $x \in \overline{t}$ is in $\mathcal{B}$, thus $\alpha_t \cup \alpha_{\overline{t}} = \mathcal{W}$.

For the first claim (truth lemma) we can refer to the proof of Lemma 5.5.1. We prove that $\mathcal{M}$ satisfies the condition for axiom 4 in the standard semantics: Assume that $\gamma \in \mathcal{N}(x)$. Then there is a term t such that $\alpha_t = \gamma$ and $t \triangleright x$ is in $\mathcal{B}$. By saturation of $\mathsf{4}_{\mathbf{T}}$, $J(t) \triangleright x$ is in $\mathcal{B}$. Then by definition, $\sigma(J(t)) \in \mathcal{N}(x)$. We show that $\sigma(J(t)) = \{y \mid \alpha_t \in \mathcal{N}(y)\} = \{y \mid \gamma \in \mathcal{N}(y)\}$: If $y \in \sigma(J(t))$, where $\sigma(J(t)) = \alpha_{J(t)}$, then $y \in J(t)$ is in $\mathcal{B}$. Thus by saturation of $\mathrm{J}_{\mathbf{T}}$, $t \triangleright y$ is in $\mathcal{B}$. Then $\alpha_t \in \mathcal{N}(y)$. Now assume $\alpha_t \in \mathcal{N}(y)$. Then there is a term s such that $s \triangleright y$ is in $\mathcal{B}$, and $\alpha_s = \alpha_t$. By saturation of $\mathsf{4}_{\mathbf{T}}$, $J(s) \triangleright y$ is in $\mathcal{B}$. Moreover, assume by contradiction that $y \notin \sigma(J(t))$. Then $y \in J(t)$ is not in $\mathcal{B}$, and, by saturation of $\mathsf{cmp}_{\cup\mathbf{T}}$, $y \in \overline{J(t)}$ is in $\mathcal{B}$. Thus by saturation of $\overline{\mathrm{J}}_{\mathbf{T}}$, there is a world label z such that $z \in t$ and $z \in \overline{s}$ are in $\mathcal{B}$, or $z \in \overline{t}$ and $z \in s$ are in $\mathcal{B}$. In both cases $\alpha_t \neq \alpha_s$, which gives a contradiction.

Finally, for the other model conditions the proof is essentially a simplification of the one in Lemma 5.5.1, we present as examples the following two cases.

(N) By saturation of $\mathsf{N}_{\mathbf{T}}$, $\tau \triangleright x$ is in $\mathcal{B}$ for every x in $\mathcal{B}$. Then $\alpha_\tau \in \mathcal{N}(x)$. Moreover, by saturation of $\overline{\tau}^{\emptyset}{}_{\mathbf{T}}$, there is no $y \in \overline{\tau}$ in $\mathcal{B}$. Then $\alpha_{\overline{\tau}} = \emptyset$, therefore $\alpha_\tau = \mathcal{W}$.

(D) Assume by contradiction $\gamma \in \mathcal{N}(x)$ and $\mathcal{W} \setminus \gamma \in \mathcal{N}(x)$. Then there are terms t, s such that $\alpha_t = \gamma$, $\alpha_s = \mathcal{W} \setminus \gamma$, and $t \triangleright x$, $s \triangleright x$ are in $\mathcal{B}$. By saturation of rule $\mathsf{D}_{2\mathbf{T}}$, there is y such that $y \in t$, $y \in s$ are in $\mathcal{B}$, or $y \in \overline{t}$, $y \in \overline{s}$ are in $\mathcal{B}$. Then $y \in \alpha_t \cap \alpha_s$, or $y \in \alpha_{\overline{t}} \cap \alpha_{\overline{s}} = (\mathcal{W} \setminus \alpha_t) \cap (\mathcal{W} \setminus \alpha_s)$. Then $\alpha_s \neq \mathcal{W} \setminus \alpha_t$, which gives a contradiction. $\square$

On the basis of the above lemma we obtain the following result in the standard way.

Theorem 5.6.2 (Semantic completeness)**.** If A is valid in all standard neighbourhood models for $\mathbf{E4}^*$, then A is derivable in $\mathbf{LT}.\mathbf{E4}^*$.

5.7 Implementation

The labelled calculi presented in this chapter are not only useful for a proof-theoretical analysis of classical non-normal modal logics, but are also suitable for implementation. Not many theorem provers for non-normal modal logics have been developed so far, here is a brief account: In Giunchiglia et al. [71] optimal decision procedures are presented for the whole classical cube; these procedures reduce a validity/satisfiability checking in the logics to a set of SAT problems and then call an efficient SAT solver. For this reason they probably outperform any (implementation of) specific calculi for these logics, but they do not provide explicitly derivations, nor countermodels. A theorem prover for the logic $\mathbf{M}$ based on a tableaux calculus is presented in Hansen [82]. This system also handles more complex Pauly's coalition logic [146] and Alternating Time Temporal logic by Alur et al. [2], and it is implemented in ELAN, an environment for rewriting systems. Finally, Lellmann [111] presents a Prolog implementation of Brown's ability logic [22], a non-normal modal logics containing two modalities $[\forall\forall]$ and $[\exists\forall]$, where the fragment with only $[\exists\forall]$ coincides with the logic $\mathbf{M}$.

In this section, we present a Prolog implementation of our labelled sequent calculi $\mathbf{LS.E}^*$ for the systems of the classical cube. We call it PRONOM (theorem PROver for NOn-normal Modal logics).[2] PRONOM implements both the proof search and the countermodel extraction in the labelled calculi: given an input formula, it firstly searches for a derivation, then in case of failure it produces a countermodel of it. PRONOM comprises a set of clauses, each of them implementing a sequent rule or an initial sequent of $\mathbf{LS.E}^*$. The proof search is provided for free by the mere depth-first search mechanism of Prolog, without any additional ad hoc mechanism. In this section we present a description of the implementation without considering its performance. We postpone the analysis of the performance to Section 6.7, where a different theorem prover for the same logics is presented.

PRONOM represents a sequent with Prolog lists `Spheres`, `Gamma` and `Delta`. Lists `Gamma` and `Delta` represent the left-hand side and the right-hand side of the sequent, respectively. Elements of `Gamma` and `Delta` are labelled formulas, implemented by Prolog lists with two, three or four elements, as follows:

- standard formulas are pairs `[x,f]`, where `x` is a label and `f` is a formula;
- formulas of the form either $x \in t$ or $x \in \bar{t}$ are triples `[x,0,t]` (respectively `[x,1,t]`), where `x` is a label and `t` represents term t; the inner value, either 0 or 1, is used to distinguish between positive and negative terms, t and $\bar{t}$, respectively;
- formulas of the form $t \Vdash^{\exists} A$, or $t \Vdash^{\forall} A$, or $\bar{t} \Vdash^{\exists} A$, or $\bar{t} \Vdash^{\forall} A$ are represented by quadruples `[exists,t,0,a]`, `[forall,t,0,a]`, `[exists,t,1,a]`, `[forall,t,1,a]`, respectively.

The list `Spheres` contains pairs of the form `[x,Items]`, where `Items` is the list of terms t such that $t \rhd x$ is in the sequent. Symbols $\top$ and $\bot$ are represented by constants `true` and `false`, respectively, whereas connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\Box$ are represented by `-`, `^`, `?`, `->`, and `box`. Propositional variables are represented by Prolog atoms. As an example, the Prolog lists

```
[ [x,[t]] ]
[ [y,1,t], [y,a], [forall,t,0,a^b] ]
[ [exists,t,1,a^b], [x,box(a)] ]
```

are used to represent the sequent

$$t \rhd x, y \in \bar{t}, y : A, t \Vdash^{\forall} A \wedge B \Rightarrow \bar{t} \Vdash^{\exists} A \wedge B, x : \Box A.$$

Given a formula of $\mathcal{L}$ represented by the Prolog term `f`, PRONOM executes the main predicate of the prover, called `prove`, whose only two clauses implement the functioning of

[2] PRONOM is available for free usage and download at http://193.51.60.97:8000/pronom/.

PRONOM: the first clause checks whether the formula is valid and, in case of failure, the second one computes a countermodel. In detail, the predicate `prove` first checks whether the formula is valid by executing the predicate:

```
terminating_proof_search(Spheres,Gamma,Delta,ProofTree,RBox,RExist,LAll).
```

This predicate succeeds if and only if the sequent represented by the lists `Spheres, Gamma` and `Delta` is derivable. When it succeeds, the output term `ProofTree` matches with a representation of the derivation found by the prover. Further arguments `RBox`, `RExist`, and `LAll` are used to control the application of the cumulative rules $\mathsf{R}\Box$, $\mathsf{R} \Vdash^{\exists}$, and $\mathsf{L} \Vdash^{\forall}$ in order to avoid redundant rule applications. For instance, by a backward application of rule $\mathsf{R}\Box$ to a sequent of the form $t \rhd x, \Gamma \Rightarrow \Delta, x : \Box A$, both principal formulas $t \rhd x$ and $x : \Box A$ are copied into the premisses. Therefore, in order to ensure termination of proof search we need to prevent further applications of the same rule to the same formulas, whence we need to control the applications of $\mathsf{R}\Box$. Such a control is made by means of the list `RBox`, which contains triples of the form `[x,a,t]` that are used to keep trace of the pairs of formulas to which $\mathsf{R}\Box$ has been already applied. Thus, the application of $\mathsf{R}\Box$ is restricted by instantiating a Prolog variable `T` such that `[X,A,T]` does not belong to `RBox`. The lists `RExist` and `LAll` are used in analogous ways in order to control the applications of, respectively, $\mathsf{R} \Vdash^{\exists}$ and $\mathsf{L} \Vdash^{\forall}$.

To make an example, in order to test whether the sequent

$$x : \Box(A \wedge (B \vee C)) \Rightarrow x : \Box((A \wedge B) \vee (A \wedge C))$$

is derivable in $\mathbf{E}$, we query PRONOM with the goal:

```
terminating_proof_search([x, [  ]], [[x, (box (a ^ (b ?  c)))]], [[x, (box
        ((a ^ b) ?  (a ^ c)))]], ProofTree, [ ], [ ], [ ]).
```

Each clause of `terminating_proof_search` implements an axiom or rule of the sequent calculi $\mathbf{LS.E}^*$. To search for a derivation of a sequent $\Gamma \Rightarrow \Delta$, PRONOM proceeds as follows. First, if $\Gamma \Rightarrow \Delta$ is an initial sequent, then the goal will immediately succeed by using one of the following clauses:

```
terminating_proof_search(Spheres,Gamma,Delta,tree(axiom,[Spheres,Gamma,Delta],no,no),_,_,_):-
    member([X,A],Gamma),
    member([X,A],Delta),!.
terminating_proof_search(Spheres,Gamma,Delta,tree(axiom,[Spheres,Gamma,Delta],no,no),_,_,_):-
    member([_,false],Gamma),!.
terminating_proof_search(Spheres,Gamma,Delta,tree(axiom,[Spheres,Gamma,Delta],no,no),_,_,_):-
    member([_,true],Delta),!.
```

If in contrast $\Gamma \Rightarrow \Delta$ is not an initial sequent, then the first applicable rule will be chosen. For instance, if `Spheres` contains an element `[X, List]` such that `List` contains `T`, representing that $t \rhd x$ is in the left-hand side of the sequent, and `Delta` contains a formula `[X,box A]`, representing that $x : \Box A$ is in the right-hand side of the sequent, then the clause implementing the R$\Box$ rule will be chosen, and PRONOM will be recursively invoked on the premisses of such a rule. PRONOM proceeds in a similar way for the other rules. The ordering of the clauses is such that the application of the branching rules is postponed as much as possible. As an example, the clause implementing R$\Box$ is as follows:

```
1.  terminating_proof_search(Spheres,Gamma,Delta,
                  tree(rbox,LeftTree,RightTree),RBox,RExist,LAll):-
2.      member([X,box A],Delta),
3.      member([X,SpOfX],Spheres),
4.      member(T,SpOfX),
5.      \+member([X,A,T],RBox),
6.      !,
7.      terminating_proof_search(Spheres,Gamma,[[forall,T,0,A]|Delta],
                                     LeftTree,[[X,A,T]|RBox],RExist,LAll),
8.      terminating_proof_search(Spheres,[[exists,T,1,A]|Gamma],Delta,
                                     RightTree,[[X,A,T]|RBox],RExist,LAll).
```

Line 5 implements the restriction on the application of the rule described above, and is used in order to ensure termination of proof search: given an instantiation of the Prolog variable `T`, the rule is applied only if it has not been already applied with the same `T` and the formula $x : \Box A$ in the current branch, that is, `[X,A,T]` does not belong to `RBox`. Since the rule is invertible, Prolog cut `!` is used in line 6 in order to block backtracking.

When the predicate `terminating_proof_search` fails, whence the initial formula is not valid, PRONOM extracts a countermodel from a saturated branch. The countermodel is computed by executing the predicate:

```
build_saturate_branch(Spheres,Gamma,Delta,Model,RBox,RExist,LAll).
```

This predicate has the same arguments as `terminating_proof_search`, except for the fourth one: here the variable `Model` matches a description of an open branch obtained by applying the rules of the calculi to the initial formula. Since the very objective of this predicate is to build a saturated branch in the sequent calculus, its clauses are essentially the same as the ones for the predicate `terminating_proof_search`, however the rules introducing a new branch in the backward proof search are implemented by *pairs* of (disjoint) clauses, each one representing an attempt to build a saturated branch. As an example, the following clauses implement the saturation in presence of a formula $x : \Box A$ in the right-hand side of a sequent:

```
build_saturate_branch(Spheres,Gamma,Delta,Model,RBox,RExist,LAll):-
```

```
2.        member([X,box A],Delta),
3.        member([X,SpOfX],Spheres),
4.        member(T,SpOfX),
5.        \+member([X,A,T],RBox),
6.        build_saturate_branch(Spheres,Gamma,[[forall,T,0,A]|Delta],Model,
                                 [[X,A,T]|RBox],RExist,LAll).
7.    build_saturate_branch(Spheres,Gamma,Delta,Model,RBox,RExist,LAll):-
8.        member([X,box A],Delta),
9.        member([X,SpOfX],Spheres),
10.       member(T,SpOfX),
11.       \+member([X,A,T],RBox),
12.       build_saturate_branch(Spheres,[[exists,T,1,A]|Gamma],Delta,Model,
                                [[X,A,T]|RBox],RExist,LAll).
```

PRONOM will first try to build a countermodel by considering the left premiss of the rule R$\Box$, corresponding to recursively invoking the predicate `build_saturate_branch` on the premiss introducing $t \Vdash^{\forall} A$ in the right-hand side of the sequent in line 6. In case of failure, the saturation process is completed by considering the right premiss of R$\Box$ introducing $\overline{t} \Vdash^{\exists} A$ by the recursive call of line 12.

Clauses implementing initial sequents for the predicate `terminating_proof_search` are replaced by the last clause, checking whether the current sequent represents an open and saturated branch.

```
build_saturate_branch(Spheres,Gamma,Delta,model(Spheres,Gamma,Delta),_,_,_):-
    \+instanceOfAnAxiom(Spheres,Gamma,Delta).

instanceOfAnAxiom(Gamma,Delta):-member([X,A],Gamma),member([X,A],Delta),!.
instanceOfAnAxiom(Gamma,_):-member([_,false],Gamma),!.
instanceOfAnAxiom(_,Delta):-member([_,true],Delta),!.
```

Since this is the very last clause of the predicate `build_saturate_branch`, it is considered by PRONOM only if no other clause/rule is applicable, therefore the branch is saturated. The auxiliary predicate `instanceOfAnAxiom` checks whether the branch is open by proving that it is not an instance of an axioms. The third argument matches a term `model` representing the countermodel extracted from the lists `Spheres, Gamma`, and `Delta`.

The implementation of the calculi for extensions of **E** is very similar: given the modularity of the calculi **LS**.**E***, the systems implementing the extensions are easily obtained by adding clauses for both the predicates `terminating_proof_search` and `build_saturate_branch` corresponding to the additional rules of the extensions under consideration. For instance, the implementation of logic **M** contains the following additional clause corresponding to rule M:

```
terminating_proof_search(Spheres,Gamma,Delta,tree(m),_,_,_):-
   member([_,List],Spheres),
   member(T,List),
   member([_,1,T],Gamma),!.
```

For logic **M** we give both the modular version in Figure 5.1 and an optimised version containing the rule $\mathsf{R}\Box\mathsf{m}$ instead of $\mathsf{R}\Box$ (see Section 5.3). For the extensions of **M** we only propose the simpler version with $\mathsf{R}\Box\mathsf{m}$.

5.8 Discussion

In this chapter, we have presented semantic-based labelled calculi for the whole family of classical non-normal modal logics considered in this work, i.e., the basic logic **E** and all its extensions with combinations of the axioms M, C, N, T, P, 4, and the rules RD_n^+. The calculi are based on the bi-neighbourhood semantics, their language contains labels which are used to import semantic information into the calculus. In particular, the rules for $\Box$ directly derive from the forcing clause of boxed formulas in the bi-neighbourhood semantics, whereas the rules for the extensions are given by expressing the properties of the bi-neighbourhood function in the corresponding models. The definition of the calculi is fully modular: for every extension of **E**, the calculus is obtained simply by adding the rules corresponding to the additional axioms, without any modification of the basic calculus.

We have firstly presented labelled sequent calculi, for which we have proved the admissibility of the structural rules and cut. As a consequence, we have obtained a syntactic proof of completeness of the calculi with respect to the axiomatic systems. As far as we know, these are the first *cut-free* calculi for the logic **E4** and its extensions without the axioms M or T.

Then, we have proposed a reformulation of the calculi in the form of labelled tableaux systems, and for every system not containing the rules for 4 we have defined a terminating proof search strategy. Moreover, we have shown that from every saturated branch of every failed proof it is possible to directly extract a (finite) countermodel of the non-valid formula in the bi-neighbourhood semantics, and for regular logics also in the relational semantics. This provided an alternative proof of the *finite model property* of classical non-normal modal logics, and at the same time also a *constructive* proof of their *decidability*, since for every formula the procedure returns either a derivation, if the formula is derivable/valid, or a countermodel, if the formula is not valid.

As pointed out, the tableaux reformulation of the calculi is not strictly necessary, as the same results could be also obtained with the sequent calculi. However, while the sequent calculi have the advantage to allow for a purely syntactic proof of the admissibility of some rules, such as the simplified rules for monotonic systems $\mathsf{L}\Box\mathsf{m}$ and $\mathsf{R}\Box\mathsf{m}$ (cf. Section 5.3) and cut, thus also allowing for a purely syntactic proof of the completeness of the calculi, the tableaux calculi offer a cleaner formalism to display derivations and failed proofs.

Furthermore, we have considered tableaux systems for the logics with axiom 4, and we have shown that the calculi containing the rules for 4 as well as $\mathsf{cmp}_{\cap\mathbf{T}}$ and $\mathsf{cmp}_{\cup\mathbf{T}}$ are complete with respect to the corresponding standard models. In particular, from every failed proof one can directly extract a standard countermodel of the non-derivable formula.

Finally, we have presented the prover PRONOM, a Prolog implementation of our labelled sequent calculi for the systems of the classical cube. The prover implements both the proof search and the countermodel extraction. To the best of our knowledge, PRONOM is the first theorem prover that provides derivations and countermodels for all the systems of the classical cube. In future work we plan to extend the prover so to cover all systems treated in this chapter.

We conclude this chapter by briefly discussing two main open problems. A first problem concerns the possibility to define a terminating proof search strategy for the tableaux calculi with the rules for axiom 4. These calculi contain the rules for neighbourhood terms of the form $J(t)$. Intuitively, these terms can be understood as representing truth sets of nested modal formulas, i.e., if a represents the truth set of A, then $J(a)$ represents the truth set of $\Box A$. Given a term t, the rule $\mathsf{4}_{\mathbf{T}}$ always allows one to build the complex term $J(t)$, whence the calculi are in principle not terminating. For a terminating proof search strategy we therefore need to block the creation of terms of the form $J(t)$ in a suitable way. The basic intuition is that the creation of terms $J(t)$ must be bounded by the modal degree of the root formula, that is, no term $\underbrace{J...J}_{n}(t)$ should be needed in the proof of a formula A with modal degree $n-1$. In addition, for an exhaustive treatment of the systems with axiom 4 we also aim to directly extract bi-neighbourhood countermodels of non-valid formulas by avoiding the use of rules $\mathsf{cmp}_{\cap\mathbf{T}}$ and $\mathsf{cmp}_{\cup\mathbf{T}}$, thus directly proving the semantic completeness of the calculi with respect to the bi-neighbourhood semantics.

A second problem concerns the possibility to further extend our labelled calculi to non-normal modal logics defined by additional modal axioms. Since the calculi are based on the bi-neighbourhood semantics, this firstly requires to find suitable bi-neighbourhood models for these systems. Nonetheless, the definition of the labelled calculi is not necessarily straightforward even when the models are found. A paradigmatic case is the one of logic **E5**: although we have given a semantic characterisation of **E5** with bi-neighbourhood models (cf. Section 4.5), we have not found yet a cut-free labelled calculus for this systems. We aim to search for a suitable labelled calculus for this system, as well as for other extensions, in future work.

Chapter 6

Hypersequent calculi

In this chapter, we present some hypersequent calculi for all the systems of the classical cube and their extensions with the axioms T, P, D, and, for every $n \geq 1$, the rule RD_n^+. The calculi are internal as they only employ the language of the logic, plus additional structural connectives. We show that the calculi are complete with respect to the corresponding axiomatisation by a syntactic proof of cut elimination. Then, we define a terminating root-first proof search strategy based on the hypersequent calculi and show that it is optimal for coNP-complete logics. Moreover, we show that from every saturated leaf of a failed proof it is possible to define a countermodel of the root hypersequent in the bi-neighbourhood semantics, and for regular logics also in the relational semantics. We then present hypersequent calculi for two specific classical non-normal modal logics, namely Elgesem's agency and ability logic [47] and its coalition extension proposed by Troquard [165]. Finally, we present a second theorem prover for non-normal modal logics based on a Prolog implementation of our hypersequent calculi, and compare its performance with that of the prover obtained by the implementation of the labelled calculi.

6.1 Blocks, hypersequents and rules

In order to define our calculi we extend the structure of sequents in two ways. First, we establish that sequents can contain so-called *blocks* in addition to formulas of $\mathcal{L}$. Second, we use hypersequents rather than simple sequents. We consider the following definitions.

Definition 6.1.1 (Blocks, sequents, hypersequents)**.** A *block* is a structure $\langle\Sigma\rangle$, where Σ is a finite multiset of formulas of $\mathcal{L}$. A *sequent* of $\mathbf{H.E}^*$ is a pair $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset of formulas and blocks, and Δ is a finite multiset of formulas. A *hypersequent* is a finite multiset of sequents, and is written

$$\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n.$$

$\mathbf{H.E} := \{\text{propositional rules}, \mathsf{L}\Box, \mathsf{R}\Box\}$.	$\mathbf{H.M} := \{\text{propositional rules}, \mathsf{L}\Box, \mathsf{R}\Box\mathsf{m}\}$.
$\mathbf{H.EN}^* := \mathbf{H.E}^* \cup \{\mathsf{N}\}$.	$\mathbf{H.MN}^* := \mathbf{H.M}^* \cup \{\mathsf{N}\}$.
$\mathbf{H.EC}^* := \mathbf{H.E}^* \cup \{\mathsf{C}\}$.	$\mathbf{H.MC}^* := \mathbf{H.M}^* \cup \{\mathsf{C}\}$.
$\mathbf{H.ET}^* := \mathbf{H.E}^* \cup \{\mathsf{T}\}$.	$\mathbf{H.MT}^* := \mathbf{H.M}^* \cup \{\mathsf{T}\}$.
$\mathbf{H.EP}^* := \mathbf{H.E}^* \cup \{\mathsf{P}\}$.	$\mathbf{H.MP}^* := \mathbf{H.M}^* \cup \{\mathsf{P}\}$.
$\mathbf{H.ED}^* := \mathbf{H.E}^* \cup \{\mathsf{D}_1, \mathsf{D}_2\}$.	$\mathbf{H.MD}^* := \mathbf{H.M}^* \cup \{\mathsf{D}_1^+, \mathsf{D}_2^+\}$.
$\mathbf{H.ED}_\mathbf{n}^{+*} := \mathbf{H.E}^* \cup \{\mathsf{D}_i^+ \mid 1 \leq i \leq n\}$.	$\mathbf{H.MD}_\mathbf{n}^{+*} := \mathbf{H.M}^* \cup \{\mathsf{D}_i^+ \mid 1 \leq i \leq n\}$.

Table 6.1: Hypersequent calculi $\mathbf{H.E}^*$.

Given a hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$, we call *components* of H the sequents $\Gamma_1 \Rightarrow \Delta_1$, ..., $\Gamma_n \Rightarrow \Delta_n$.

Intuitively, the symbol "|" of hypersequents represents a meta-logical disjunction of valid sequents. Observe that blocks can occur only in the antecedent of sequents (and not in the succedent). Both blocks and sequents, but not hypersequents, can be interpreted as formulas of $\mathcal{L}$. The *formula interpretation* of sequents is as follows:

$$i(A_1, ..., A_n, \langle\Sigma\rangle_1, ..., \langle\Sigma\rangle_m \Rightarrow B_1, ..., B_k) = A_1 \wedge ... \wedge A_n \wedge \Box \bigwedge \Sigma_1 \wedge ... \wedge \Box \bigwedge \Sigma_m \to B_1 \vee ... \vee B_k.$$

By contrast, there is no formula interpretation of hypersequents. The reason is that non-normal modalities are not *strong enough* to express the structural connective "|". In principle, a formula interpretation of hypersequents would be possible in a richer language containing a normal **S5**-modality, see e.g. Avron [10]. This can be compared with the nested calculi in [111], where the connective "[ ]" for nesting is translated into a normal **K**-modality.

The semantic interpretation of sequents and hypersequents is as follows.

Definition 6.1.2 (Valid hypersequent)**.** We say that a sequent S is *valid* in a possible-worlds model $\mathcal{M}$ (written $\mathcal{M} \models S$) if for every world w of $\mathcal{M}$, $\mathcal{M}, w \Vdash i(S)$. We say that a hypersequent H is *valid* in $\mathcal{M}$ if for some component S of H, $\mathcal{M} \models S$.

For every logic $\mathbf{E}^*$, the corresponding hypersequent calculus $\mathbf{H.E}^*$ is defined by a subset of the rules in Figure 6.1, as summarised in Table 6.1. The rules are given in their cumulative, or *kleene'd*, versions, i.e., the principal formulas or blocks are copied into the premiss(es). The propositional rules are just the hypersequent versions of kleene'd rules of sequent calculi.

We make clear that the hypersequent structure *is not needed* to obtain a sound and complete calculus for the logics under investigation, as it is shown by the Gentzen calculi discussed in Section 3.4. Moreover, it can be checked that whenever a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ is derivable, then there is some component $\Gamma_i \Rightarrow \Delta_i$ which is derivable. The choice of both cumulative rules and the hypersequent structure is motivated by the possibility of *directly* obtain countermodels of non-valid formulas. In particular, the hypersequent structure allows us to make all rules invertible. In this respect, observe that

Propositional rules

$$\mathsf{init}\ \frac{}{G \mid \Gamma, p \Rightarrow p, \Delta} \qquad \mathsf{L}\bot\ \frac{}{G \mid \Gamma, \bot \Rightarrow \Delta} \qquad \mathsf{R}\top\ \frac{}{G \mid \Gamma \Rightarrow \top, \Delta}$$

$$\mathsf{L}\rightarrow\ \frac{G \mid \Gamma, A \rightarrow B \Rightarrow A, \Delta \qquad G \mid \Gamma, A \rightarrow B, B \Rightarrow \Delta}{G \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \qquad \mathsf{R}\rightarrow\ \frac{G \mid \Gamma, A \Rightarrow B, A \rightarrow B, \Delta}{G \mid \Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$\mathsf{L}\wedge\ \frac{G \mid \Gamma, A \wedge B, A, B \Rightarrow \Delta}{G \mid \Gamma, A \wedge B \Rightarrow \Delta} \qquad \mathsf{R}\wedge\ \frac{G \mid \Gamma \Rightarrow A, A \wedge B, \Delta \qquad G \mid \Gamma \Rightarrow B, A \wedge B, \Delta}{G \mid \Gamma \Rightarrow A \wedge B, \Delta}$$

$$\mathsf{L}\vee\ \frac{G \mid \Gamma, A \vee B, A \Rightarrow \Delta \qquad G \mid \Gamma, A \vee B, B \Rightarrow \Delta}{G \mid \Gamma, A \vee B \Rightarrow \Delta} \qquad \mathsf{R}\vee\ \frac{G \mid \Gamma \Rightarrow A, B, A \vee B, \Delta}{G \mid \Gamma \Rightarrow A \vee B, \Delta}$$

Modal rules for the classical cube

$$\mathsf{L}\Box\ \frac{G \mid \Gamma, \Box A, \langle A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{R}\Box\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B \qquad \{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta}$$

$$\mathsf{R}\Box\mathsf{m}\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta} \qquad \mathsf{N}\ \frac{G \mid \Gamma, \langle \top \rangle \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{C}\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

Modal rules for extensions

$$\mathsf{T}\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta} \qquad \mathsf{P}\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta} \qquad \mathsf{D}_n^+\ \frac{G \mid \Gamma, \langle \Sigma_1 \rangle, ..., \langle \Sigma_n \rangle \Rightarrow \Delta \mid \Sigma_1, ..., \Sigma_n \Rightarrow}{G \mid \Gamma, \langle \Sigma_1 \rangle, ..., \langle \Sigma_n \rangle \Rightarrow \Delta}$$

$$\mathsf{D}_2\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta \mid \Sigma, \Pi \Rightarrow \qquad \{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta \mid \Rightarrow A, B\}_{A \in \Sigma, B \in \Pi}}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

$$\mathsf{D}_1\ \frac{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow \qquad \{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta}$$

Figure 6.1: Rules of hypersequent calculi $\mathbf{H.E}^*$.

backward applications of the rules $\mathsf{R}\Box$, $\mathsf{R}\Box\mathsf{m}$, P, D_1, D_2, and D_n^+ create new components, but the principal component in the conclusion is kept into the premiss in order to let potential alternative rule applications still possible. As observed in Section 3.3, the full invertibility of the calculus entails that the order of the rule applications in the construction of the proofs is not relevant, that is, modulo the order of rule applications every formula has a single derivation, or a single failed proof. Furthermore, cumulative rules allow us to keep all information about the branch at each step of backward proof search. As we shall see, this entails that every saturated *leaf* of every failed proof contains all information needed to define a countermodel.

Similarly to the propositional connectives, the boxed formulas are handled in the calculus by separate left and right rules. Observe that the rule $\mathsf{R}\Box$ has a non-fixed number of premisses, but for every application of $\mathsf{R}\Box$ the number of the premisses is determined by the cardinality of the principal block $\langle \Sigma \rangle$. The rule $\mathsf{R}\Box\mathsf{m}$ is a right rule for $\Box$ which replaces $\mathsf{R}\Box$ in the definition of monotonic calculi. Apart from the distinction between monotonic and non-monotonic calculi, the calculi are modular; in particular, extensions of $\mathbf{H.E}$ and $\mathbf{H.M}$ do not require to modify the basic rules for $\Box$, and are simply defined by adding the rules corresponding to the additional axioms. Every axiom has a corresponding rule, with the only exceptions of the axiom D and the rules RD_n^+: axiom D needs both D_1 and D_2 in the non-monotonic case, whereas it needs D_1^+ and D_2^+ in the monotonic case. Moreover, the rules RD_n^+ need D_m^+ for every $1 \leq m \leq n$. Concerning the calculi $\mathbf{H.MD}^*$ and $\mathbf{H.ED_n^+}^*$, this choice of rule is required in order to ensure the admissibility of contraction. Analogously, the rule D_1 seems required in the non-monotonic calculi $\mathbf{H.ED}^*$ in order to give a purely syntactic proof of admissibility of contraction (however we do not dispose of any concrete example showing that D_1 is necessary for the admissibility of contraction, this leaves open the problem whether the calculus is complete also without this rule). For the axiom D a similar solution is adopted in [140] in the context of standard Gentzen calculi. However, differently from the rule given there (see the rule D' in Section 3.4 of this thesis), our rule D_1 is in principle applicable. Moreover, as we shall see in the countermodel extraction, the rule D_1 is the syntactic counterpart of the property $(\emptyset, \emptyset) \notin \mathcal{N}(w)$, which is satisfied by every bi-neighbourhood D-model.

Blocks have a central role in all modal rules. Notice that the only rule which expands blocks is C, so that in the absence of this rule the blocks occurring in a proof for a single formula contain only one formula. The possibility of collecting formulas by means of blocks allows us to avoid rules with n principal boxed formulas, in contrast to the rules C e MC of Gentzen calculi (see Section 3.4). As we shall see, by using blocks we can also easily define the bi-neighbourhood function when building countermodels from failed proofs.

Derivations of modal axioms and rules are displayed in Figure 6.2. Notice that the simulations of the rules make use of the external weakening rule Ewk, which we show to be admissible in Proposition 6.2.1. In the derivations we implicitly make use of the following lemma, which states that initial hypersequents can be generalised to arbitrary formulas.

Proposition 6.1.1. $G \mid \Gamma, A \Rightarrow A, \Delta$ is derivable in $\mathbf{H.E}^*$ for every A, Γ, Δ, G.

Proof. By induction on A. If $A = p, \bot, \top$, then $G \mid \Gamma, A \Rightarrow A, \Delta$ is an initial hypersequent, whence it is derivable. If $A = B \wedge C$ we consider the following derivation

$$\dfrac{\dfrac{G \mid \Gamma, B \wedge C, B, C \Rightarrow B, B \wedge C, \Delta \qquad G \mid \Gamma, B \wedge C, B, C \Rightarrow C, B \wedge C, \Delta}{G \mid \Gamma, B \wedge C, B, C \Rightarrow B \wedge C, \Delta}\ \mathsf{R}\wedge}{G \mid \Gamma, B \wedge C \Rightarrow B \wedge C, \Delta}\ \mathsf{L}\wedge$$

where the premisses are derivable by i.h.. The cases $A = B \vee C$ or $A = B \wedge C$ are analogous. If $A = \Box B$ we consider the following derivation

$$\dfrac{\dfrac{G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow B \qquad G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta \mid B \Rightarrow B}{G \mid \Gamma, \Box B, \langle B \rangle \Rightarrow \Box B, \Delta}\ \mathsf{R}\Box}{G \mid \Gamma, \Box B \Rightarrow \Box B, \Delta}\ \mathsf{L}\Box$$

(RE)
$$\dfrac{\dfrac{\mathsf{Ewk}\,\dfrac{A \Rightarrow B}{\Box A, \langle A\rangle \Rightarrow \Box B \mid A \Rightarrow B} \qquad \dfrac{B \Rightarrow A}{\Box A, \langle A\rangle \Rightarrow \Box B \mid B \Rightarrow A}\,\mathsf{Ewk}}{\Box A, \langle A\rangle \Rightarrow \Box B}\,\mathsf{R}\Box}{\Box A \Rightarrow \Box B}\,\mathsf{L}\Box$$

(M)
$$\dfrac{\dfrac{\dfrac{\Box(A\wedge B),\langle A\wedge B\rangle \Rightarrow \Box A \mid A\wedge B, A, B \Rightarrow A}{\Box(A\wedge B),\langle A\wedge B\rangle \Rightarrow \Box A \mid A\wedge B \Rightarrow A}\,\mathsf{L}\wedge}{\Box(A\wedge B),\langle A\wedge B\rangle\Rightarrow\Box A}\,\mathsf{R}\Box\mathsf{m}}{\Box(A\wedge B)\Rightarrow\Box A}\,\mathsf{L}\Box$$

(N)
$$\dfrac{\dfrac{\langle\top\rangle \Rightarrow \Box\top \mid \top\Rightarrow\top \qquad \langle\top\rangle \Rightarrow \Box\top \mid \top\Rightarrow\top}{\langle\top\rangle\Rightarrow\Box\top}\,\mathsf{R}\Box}{\Rightarrow\Box\top}\,\mathsf{N}$$

(C)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{...,\langle A,B\rangle \Rightarrow \Box(A\wedge B) \mid A,B\Rightarrow A\wedge B \qquad ... \mid A\wedge B\Rightarrow A \qquad ... \mid A\wedge B \Rightarrow B}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle, \langle B\rangle, \langle A,B\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{R}\Box}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle, \langle B\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{C}}{\Box A\wedge\Box B, \Box A, \Box B, \langle A\rangle \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\Box}{\Box A\wedge\Box B, \Box A, \Box B \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\Box}{\Box A\wedge\Box B \Rightarrow \Box(A\wedge B)}\,\mathsf{L}\wedge$$

(T)
$$\dfrac{\dfrac{\Box A, \langle A\rangle, A \Rightarrow A}{\Box A, \langle A\rangle \Rightarrow A}\,\mathsf{T}}{\Box A \Rightarrow A}\,\mathsf{L}\Box$$

(P)
$$\dfrac{\dfrac{\Box\bot, \langle\bot\rangle \Rightarrow \mid \bot \Rightarrow}{\Box\bot, \langle\bot\rangle \Rightarrow}\,\mathsf{P}}{\Box\bot \Rightarrow}\,\mathsf{L}\Box$$

(D)
$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{L}\neg\,\dfrac{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A \Rightarrow A}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A, \neg A \Rightarrow} \qquad \dfrac{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid A \Rightarrow A}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow \mid \Rightarrow A, \neg A}\,\mathsf{R}\neg}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle, \langle\neg A\rangle \Rightarrow}\,\mathsf{D}_2}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A, \langle A\rangle \Rightarrow}\,\mathsf{L}\Box}{\Box A\wedge\Box\neg A, \Box A, \Box\neg A \Rightarrow}\,\mathsf{L}\Box}{\Box A\wedge\Box\neg A \Rightarrow}\,\mathsf{L}\wedge$$

(RD_n^+)
$$\dfrac{\dfrac{\dfrac{\dfrac{A_1, ..., A_n \Rightarrow}{\Box A_1\wedge ...\wedge \Box A_n, \Box A_1, ..., \Box A_n, \langle A_1\rangle, ..., \langle A_n\rangle \Rightarrow \mid A_1, ..., A_n \Rightarrow}\,\mathsf{Ewk}}{\Box A_1\wedge ...\wedge \Box A_n, \Box A_1, ..., \Box A_n, \langle A_1\rangle, ..., \langle A_n\rangle \Rightarrow}\,\mathsf{D}_n^+}{\Box A_1\wedge ...\wedge \Box A_n, \Box A_1, ..., \Box A_n \Rightarrow}\,\mathsf{L}\wedge\times n}{\Box A_1\wedge ...\wedge \Box A_n \Rightarrow}\,\mathsf{L}\wedge\times n$$

Figure 6.2: Derivations of modal axioms and rules.

where the premisses are derivable by i.h.. $\square$

We can prove that the hypersequent calculi are sound with respect to the corresponding bi-neighbourhood models.

Theorem 6.1.2 (Soundness)**.** If H is derivable in $\mathbf{H.EX}^*$, then it is valid in all X-models.

Proof. The initial hypersequents are clearly valid. We show that all rules are sound with respect to the corresponding bi-neighbourhood models. For propositional rules the proof is standard, we just consider the modal rules.

($\mathsf{L}\square$) Assume $\mathcal{M} \models G \mid \Gamma, \square A, \langle A \rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$, or $\mathcal{M} \models \Gamma, \square A, \langle A \rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \square A, \langle A \rangle \Rightarrow \Delta) = i(\Gamma, \square A, \square A \Rightarrow \Delta)$, which is equivalent to $i(\Gamma, \square A \Rightarrow \Delta)$.

($\mathsf{R}\square$) Assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \square B, \Delta \mid \Sigma \Rightarrow B$ and $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \square B, \Delta \mid B \Rightarrow A$ for all $A \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle \Sigma \rangle \Rightarrow \square B, \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow B$ and $\mathcal{M} \models B \Rightarrow A$ for all $A \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \rightarrow B$ and $\mathcal{M} \models B \rightarrow A$ for all $A \in \Sigma$, that is $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow B$. Since RE is valid, $\mathcal{M} \models \square \bigwedge \Sigma \rightarrow \square B = i(\langle \Sigma \rangle \Rightarrow \square B)$. Thus $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle \Rightarrow \square B, \Delta)$.

($\mathsf{R}\square\mathsf{m}$) Analogous to $\mathsf{R}\square$, by considering that in M-models $\mathcal{M} \models \bigwedge \Sigma \rightarrow B$ implies $\mathcal{M} \models \square \bigwedge \Sigma \rightarrow \square B$.

(N) Suppose $\mathcal{M}$ is a N-model and assume $\mathcal{M} \models G \mid \Gamma, \langle \top \rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$, or $\mathcal{M} \models \Gamma, \langle \top \rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle \top \rangle \Rightarrow \Delta)$, which is equivalent to $\square \top \rightarrow i(\Gamma \Rightarrow \Delta)$. Since $\square \top$ is valid in $\mathcal{M}$, $\mathcal{M} \models \Gamma \Rightarrow \Delta$.

(C) Suppose $\mathcal{M}$ is a C-model and assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle, \langle \Pi \rangle, \langle \Sigma, \Pi \rangle \Rightarrow \Delta) = i(\Gamma, \square \bigwedge \Sigma, \square \bigwedge \Pi, \square(\bigwedge \Sigma \wedge \bigwedge \Pi) \Rightarrow \Delta)$. This is equivalent to $\square \bigwedge \Sigma \wedge \square \bigwedge \Pi \wedge \square(\bigwedge \Sigma \wedge \bigwedge \Pi) \rightarrow i(\Gamma \Rightarrow \Delta)$, and since axiom C is valid in $\mathcal{M}$, this is equivalent to $\square \bigwedge \Sigma \wedge \square \bigwedge \Pi \rightarrow i(\Gamma \Rightarrow \Delta)$. Thus $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta)$.

(T) Suppose $\mathcal{M}$ is a T-model and assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle \Sigma \rangle, \Sigma \Rightarrow \Delta) = \square \bigwedge \Sigma \wedge \bigwedge \Sigma \rightarrow i(\Gamma \Rightarrow \Delta)$. Since axiom T is valid in $\mathcal{M}$, this is equivalent to $\square \bigwedge \Sigma \rightarrow i(\Gamma \Rightarrow \Delta)$. Then $\mathcal{M} \models \Gamma, \langle \Sigma \rangle \Rightarrow \Delta$.

(P) Suppose $\mathcal{M}$ is a P-model and assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta \mid \Sigma \Rightarrow$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle \Sigma \rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \rightarrow \bot$. and by the validity of axiom P, $\mathcal{M} \models \square \bigwedge \Sigma \rightarrow \bot = i(\langle \Sigma \rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle \Sigma \rangle \Rightarrow \Delta$.

(D_n^+) Suppose $\mathcal{M}$ is a RD_n^+-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma_1\rangle, ..., \langle\Sigma_n\rangle \Rightarrow \Delta \mid \Sigma_1, ..., \Sigma_n \Rightarrow$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma_1\rangle, ..., \langle\Sigma_n\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma_1, ..., \Sigma_n \Rightarrow$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \neg(\bigwedge \Sigma_1 \wedge ... \wedge \bigwedge \Sigma_n)$. and by the soundness of rule RD_n^+, $\mathcal{M} \models \neg(\Box \bigwedge \Sigma_1 \wedge ... \wedge \Box \bigwedge \Sigma_n) = i(\langle\Sigma_1\rangle, ..., \langle\Sigma_n\rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma_1\rangle, ..., \langle\Sigma_n\rangle \Rightarrow \Delta$.

(D_2) Suppose $\mathcal{M}$ is a D-model and assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta \mid \Sigma, \Pi \Rightarrow$, and $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta \mid \Rightarrow A, B$ for all $A \in \Sigma, B \in \Pi$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma, \Pi \Rightarrow$ and $\mathcal{M} \models \Rightarrow A, B$ for all $A \in \Sigma, B \in \Pi$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \wedge \bigwedge \Pi \to \bot$ and $\mathcal{M} \models A \vee B$ for all $A \in \Sigma, B \in \Pi$. Thus $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow \neg \bigwedge \Pi$. By the soundness of axiom D, $\mathcal{M} \models \Box \bigwedge \Sigma \wedge \Box \bigwedge \Pi \to \bot = i(\langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma\rangle, \langle\Pi\rangle \Rightarrow \Delta$.

(D_1) Assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma \Rightarrow$, and $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A$ for all $A \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$, or (iii) $\mathcal{M} \models \Sigma \Rightarrow$ and $\mathcal{M} \models \Rightarrow A$ for all $A \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma \to \bot$ and $\mathcal{M} \models \bigwedge \Sigma$, which is impossible. Then (i) or (ii) holds. □

6.2 Structural properties and syntactic completeness

We now investigate the structural properties of our calculi. We first show that weakening and contraction are height-preserving admissible, both in their internal and in their external variants, and that all rules are invertible. Then, we prove that the cut rule is admissible, which allows us to directly prove the completeness of the calculi with respect to the corresponding axiomatisations. In the proofs we use the following definition of weight of formulas and blocks.

Definition 6.2.1. The weight wg of a formula or block is recursively defined as $wg(\bot) = wg(\top) = wg(p) = 0$; for $\circ \in \{\wedge, \vee, \to\}$, $wg(A \circ B) = wg(A) + wg(B) + 1$; $wg(\langle A_1, ..., A_n\rangle) = max\{wg(A_1), ..., wg(A_n)\} + 1$; $wg(\Box A) = wg(A) + 2$.

Proposition 6.2.1. The following structural rules are height-preserving admissible in $\mathbf{H.E}^*$, where ϕ is any formula A or block $\langle\Sigma\rangle$. Moreover, all rules of $\mathbf{H.E}^*$ are height-preserving invertible.

$$\mathsf{Lwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow A, \Delta} \qquad \mathsf{Ewk}\ \frac{G}{G \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{Lctr}\ \frac{G \mid \Gamma, \phi, \phi \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta} \qquad \mathsf{Rctr}\ \frac{G \mid \Gamma \Rightarrow A, A, \Delta}{G \mid \Gamma \Rightarrow A, \Delta} \qquad \mathsf{Ectr}\ \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

$$\mathsf{Bctr}\ \frac{G \mid \Gamma, \langle\Sigma, A, A\rangle \Rightarrow \Delta}{G \mid \Gamma, \langle\Sigma, A\rangle \Rightarrow \Delta} \qquad \mathsf{Bmgl}\ \frac{G \mid \Gamma, \langle\Sigma, A\rangle \Rightarrow \Delta}{G \mid \Gamma, \langle\Sigma, A, A\rangle \Rightarrow \Delta}$$

Proof. For every rule R, the proof is straightforward by induction on the height of the derivation of the premiss, only rules Lctr and Rctr are simultaneously proved admissible by mutual induction on the height of the derivation on the premisses. Moreover, admissibility of Sctr and Smgl rely on height-preserving admissibility of contraction and weakening on formulas outside blocks, respectively.

For every rule R, if the premiss of R is an initial hypersequent, then so is the conclusion. For the inductive step we consider the last rule applied in the derivation of the premiss of R, let it be R'. Then we transform the derivation by applying first R to the premiss(es) of R' (which is allowed by i.h.), and then we apply R'. For instance suppose to have

$$\dfrac{\dfrac{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta}\,\mathsf{R\Box m}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta}\,\mathsf{Ectr}$$

Then we transform the derivation into

$$\dfrac{\dfrac{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}\,\mathsf{Ectr}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta}\,\mathsf{R\Box m}$$

The only exceptions are the cases where R is Lctr and R' is D_2 or D_n^+, in this cases we must consider the rules D_1 or D_{n-1}^+, respectively. For instance suppose to have

$$\dfrac{\dfrac{G \mid \Gamma, \langle\Sigma\rangle, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma, \Sigma \Rightarrow \qquad \{G \mid \Gamma, \langle\Sigma\rangle, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A, B\}_{A,B\in\Sigma}}{G \mid \Gamma, \langle\Sigma\rangle, \langle\Sigma\rangle \Rightarrow \Delta}\,\mathsf{D}_2}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta}\,\mathsf{Lctr}$$

The derivation is converted as follows, where n is the cardinality of Σ.

$$\dfrac{\mathsf{Lctr}{\times}(n+1)\,\dfrac{G \mid \Gamma, \langle\Sigma\rangle, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma, \Sigma \Rightarrow}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Sigma \Rightarrow} \qquad \left\{\dfrac{\dfrac{G \mid \Gamma, \langle\Sigma\rangle, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A, A}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A, A}\,\mathsf{Lctr}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta \mid \Rightarrow A}\,\mathsf{Rctr}\right\}_{A\in\Sigma}}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta}\,\mathsf{D}_1$$

Finally, notice that since all rules are cumulative, the height-preserving invertibility of all rules in an immediate consequence of the height-preserving admissibility of weakening. For instance, invertibility of rule $\mathsf{R\Box m}$ is proved as follows:

$$\dfrac{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta}{G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Box B, \Delta \mid \Sigma \Rightarrow B}\,\mathsf{Ewk}$$

□

Due to the fact that the rule $\mathsf{R\Box}$ isolates single formulas from block in its right premisses, in the non-monotonic case the full-blown weakening inside blocks (from $G \mid \Gamma, \langle\Sigma\rangle \Rightarrow \Delta$ derives $G \mid \Gamma, \langle\Sigma, A\rangle \Rightarrow \Delta$) is *not* admissible. However, the weaker rule Bmgl of *mingle* inside blocks is admissible.

The proof of admissibility of cut is a bit more intricate and deserves more attention. In the hypersequent framework we formulate the rule cut as follows:

$$\mathsf{cut}\ \frac{G \mid \Gamma \Rightarrow A, \Delta \qquad G \mid \Gamma, A \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

In order to prove the admissibility of cut we need to prove simultaneously the admissibility of cut and of the following rule sub, which states that a formula A inside one or more blocks can be replaced by any equivalent multiset of formulas Σ:

$$\mathsf{sub}\ \frac{G \mid \Sigma \Rightarrow A \qquad \{G \mid A \Rightarrow B\}_{B \in \Sigma} \qquad G \mid \Gamma, \langle A^{n_1}, \Pi_1 \rangle, ..., \langle A^{n_k}, \Pi_k \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma^{n_1}, \Pi_1 \rangle, ..., \langle \Sigma^{n_k}, \Pi_k \rangle \Rightarrow \Delta}$$

where A^{n_i} (resp. Σ^{n_i}) is a compact way to denote n_i occurrences of A (resp. Σ). In the monotonic case we need to consider, instead of sub, the rule

$$\mathsf{sub_M}\ \frac{G \mid \Sigma \Rightarrow A \qquad G \mid \Gamma, \langle A^{n_1}, \Pi_1 \rangle, ..., \langle A^{n_k}, \Pi_k \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma^{n_1}, \Pi_1 \rangle, ..., \langle \Sigma^{n_k}, \Pi_k \rangle \Rightarrow \Delta}$$

Observe that $\mathsf{sub_M}$ is essentially a cut on formula inside blocks. Although the formulation of the cut rule could be generalised so to cover also $\mathsf{sub_M}$, we keep the two rules cut and $\mathsf{sub_M}$ distinguished to make the proof of cut elimination uniform in the monotonic and the non-monotonic case.

Theorem 6.2.2 (Cut elimination)**.** If $\mathbf{H.E}^*$ is non-monotonic, then the rules cut and sub are admissible in $\mathbf{H.E}^*$, otherwise cut and $\mathsf{sub_M}$ are admissible in $\mathbf{H.E}^*$.

Proof. We prove that cut and sub are admissible in non-monotonic $\mathbf{H.E}^*$; the proof in the monotonic case is analogous. Recall that, for an application of cut, the *cut formula* is the formula which is deleted by that application, while the *cut height* is the sum of the heights of the derivations of the premisses of cut. The theorem is a consequence of the following claims, where $Cut(c, h)$ means that all applications of cut of height h on a cut formula of weight c are admissible, and $Sub(c)$ means that all applications of sub where the principal formula A has weight c are admissible (for any $\Sigma, \Pi_1, ..., \Pi_k$): **(A)** $\forall c.Cut(c, 0)$. **(B)** $\forall h.Cut(0, h)$. **(C)** $\forall c.(\forall h.Cut(c, h) \rightarrow Sub(c))$. **(D)** $\forall c.\forall h.((\forall c' < c.(Sub(c') \wedge \forall h'.Cut(c', h')) \wedge \forall h'' < h.Cut(c, h'')) \rightarrow Cut(c, h))$.

(A) If the cut height is 0, then cut is applied to initial hypersequents $G \mid \Gamma \Rightarrow A, \Delta$ and $G \mid \Gamma, A \Rightarrow \Delta$. We show that the conclusion of cut $G \mid \Gamma \Rightarrow \Delta$ is an initial hypersequent, whence it is derivable without cut. If G is an initial hypersequent we are done. Otherwise $\Gamma \Rightarrow A, \Delta$ and $\Gamma, A \Rightarrow \Delta$ are initial sequents. For the first sequent there are three possibilities: (i) $\Gamma \Rightarrow \Delta$ is an initial sequent, or (ii) $A = \top$, or (iii) $A = p$ and $\Gamma = \Gamma', p$. If (ii), then the second sequent is $\Gamma, \top \Rightarrow \Delta$, which implies that $\Gamma \Rightarrow \Delta$ is an initial sequent. If (iii), then the second sequent is $\Gamma', p, p \Rightarrow \Delta$. Then $\Gamma' \Rightarrow \Delta$ is an initial sequent, or $\Delta = p, \Delta'$, which implies that $\Gamma', p \Rightarrow p, \Delta' = \Gamma \Rightarrow \Delta$ is an initial sequent.

(B) If the cut formula has weight 0, then it is $\bot$, $\top$, or a propositional variable p. For all three possibilities the proof is by complete induction on h. The basic case $h = 0$ is a particular case of **(A)**. For the inductive step, we distinguish three cases.

(i) The cut formula $\bot$, $\top$, or p is not principal in the last rule applied in the derivation of the left premiss. By examining all possible rule applications, we show that the application of cut can be replaced by one or more applications of cut at a smaller height. For instance, assume that the last rule applied is $\mathsf{L}\Box$.

$$\dfrac{\mathsf{L}\Box\,\dfrac{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta, \bot}{G \mid \Box A, \Gamma \Rightarrow \Delta, \bot} \qquad G \mid \bot, \Box A, \Gamma \Rightarrow \Delta}{G \mid \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

The derivation is transformed as follows, with a height-preserving application of Lwk and an application of cut of smaller height.

$$\dfrac{\dfrac{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta, \bot \qquad \dfrac{G \mid \bot, \Box A, \Gamma \Rightarrow \Delta}{G \mid \bot, \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{Lwk}}{G \mid \langle A\rangle, \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{cut}}{G \mid \Box A, \Gamma \Rightarrow \Delta}\,\mathsf{L}\Box$$

The situation is similar if the last rule in the derivation of the left premiss is applied to some sequent in G.

(ii) The cut formula $\bot$, $\top$, or p is not principal in the last rule applied in the derivation of the right premiss. The case is analogous to (i). As an example, suppose that the last rule applied is $\mathsf{R}\Box\mathsf{m}$.

$$\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot \qquad \dfrac{G \mid \bot, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}{G \mid \bot, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{R}\Box\mathsf{m}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{cut}$$

The derivation is converted into

$$\dfrac{\dfrac{\mathsf{Ewk}\,\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B, \bot \mid \Sigma \Rightarrow B} \qquad G \mid \bot, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B}\,\mathsf{cut}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\,\mathsf{R}\Box\mathsf{m}$$

where cut is applied at a smaller height.

(iii) The cut formula $\bot$, $\top$, or p is principal in the last rule applied in the derivation of both premisses. Then the cut formula is p, as $\bot$ (resp. $\top$) is never principal on the right-hand side (resp. the left-hand side) of the conclusion of any rule application. This means that both premisses are derived by init, which implies that $h = 0$. Then we are back to case **(A)**.

(C) Assume $\forall h Cut(c, h)$. We prove that all applications of sub where A has weight c are admissible. The proof is by induction on the height m of the derivation of $G \mid$

$\langle A^{n_1}, \Pi_1\rangle, ..., \langle A^{n_k}, \Pi_k\rangle, \Gamma \Rightarrow \Delta$. If $m = 0$ or no block among $\langle A, \Pi_1\rangle, ..., \langle A, \Pi_k\rangle$ is principal in the last rule application, then the proof proceeds similarly to previous cases. If $m > 0$ and at least one block among $\langle A, \Pi_1\rangle, ..., \langle A, \Pi_k\rangle$ is principal in the last rule application we have the following possibilities.

- The last rule applied is $\mathsf{R}\Box$:

$$\dfrac{\overset{①}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid A^{n_i}, \Pi_i \Rightarrow D} \qquad \{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow A\}_1^{n_i} \quad \overset{\{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow C\}_{C \in \Pi_i}}{\vdots}}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D}\ \mathsf{R}\Box$$

The derivation is converted as follows. First we derive:

$$\dfrac{\dfrac{G \mid \Sigma \Rightarrow A}{G \mid \Sigma \Rightarrow A \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{Ewk} \quad \left\{\dfrac{G \mid A \Rightarrow B}{G \mid A \Rightarrow B \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{Ewk}\right\}_{B \in \Sigma} \quad ①}{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid A^{n_i}, \Pi_i \Rightarrow D}\ \mathsf{sub}$$

(where rule sub possibly applies to further blocks inside Γ). Then by applying Ewk to $G \mid \Sigma \Rightarrow A$ we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma \Rightarrow A$. By auxiliary applications of wk we can cut A and get $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma, A^{n_i-1}, \Pi_i \Rightarrow D$. Then with further applications of cut (each time with auxiliary applications of wk) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid \Sigma^{n_i}, \Pi_i \Rightarrow D$. By doing the same with the other premisses of $\mathsf{R}\Box$ in the initial derivation we obtain also $\{\{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow B\}_{B \in \Sigma}\}_1^{n_1}$ and $\{G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D \mid D \Rightarrow C\}_{C \in \Pi_i}$. Finally by $\mathsf{R}\Box$ we derive the conclusion of sub $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta, \Box D$.

- The last rule applied is C:

$$\dfrac{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \langle A^{n_i}, A^{n_j}, \Pi_i, \Pi_j\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_i}, \Pi_i\rangle, \langle A^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

By applying sub to the premiss we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \langle \Sigma^{n_j}, \Pi_j\rangle, \langle \Sigma^{n_i}, \Sigma^{n_j}, \Pi_i, \Pi_j, \rangle, \Gamma \Rightarrow \Delta$, then by C we derive $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \langle \Sigma^{n_j}, \Pi_j\rangle, \Gamma \Rightarrow \Delta$.

- The last rule applied is T:

$$\dfrac{G \mid A^{n_i}, \Pi_i, \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{T}$$

By applying the inductive hypothesis to the premiss we obtain $G \mid A^{n_i}, \Pi_i, \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$. Then, from this and $G \mid \Sigma \Rightarrow A$, by several applications of cut (each time with auxiliary applications of wk) we obtain $G \mid \Sigma^{n_i}, \Pi_i, \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$. Finally, by T we derive $G \mid \langle \Sigma^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta$.

- The last rule applied is P:

$$\dfrac{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, \Pi_i \Rightarrow}{G \mid \langle A^{n_i}, \Pi_i\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{P}$$

By applying the inductive hypothesis to the premiss (after auxiliary applications of Ewk to the other premisses of sub) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i \rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, \Pi_i \Rightarrow$. Then, from this and $G \mid \Sigma \Rightarrow A$, by several applications of cut (each time with auxiliary applications of wk) we obtain $G \mid \langle \Sigma^{n_i}, \Pi_i \rangle, \Gamma \Rightarrow \Delta \mid \Sigma^{n_i}, \Pi_i \Rightarrow$. Finally, by P we derive $G \mid \langle \Sigma^{n_i}, \Pi_i \rangle, \Gamma \Rightarrow \Delta$.

• The last rule applied is D_2: Then $G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta$ has been derived by the following premisses. $G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta \mid A^{n_i}, A^{n_j}, \Pi_i, \Pi_j \Rightarrow$; $\{G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta \mid \Rightarrow A, A\}_1^{n_i+n_j}$; $\{\{G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta \mid \Rightarrow A, C\}_{C \in \Pi_i}\}_1^{n_j}$; $\{\{G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta \mid \Rightarrow A, D\}_{D \in \Pi_j}\}_1^{n_i}$; and $\{G \mid \langle A^{n_i}, \Pi_i \rangle, \langle A^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta \mid \Rightarrow A, A\}_{C \in \Pi_i, D \in \Pi_j}$. We consider the other premisses of sub and apply cut many times (each time with auxiliary applications of wk) so to replace all occurrences of A with formulas in Σ. As final step we can apply D_2 and obtain $G \mid \langle \Sigma^{n_i}, \Pi_i \rangle, \langle \Sigma^{n_j}, \Pi_j \rangle, \Gamma \Rightarrow \Delta$.

• The lacking cases D_n^+ and D_1 are similar to the previous ones.

(D) Assume $\forall c' < c. (Sub(c') \wedge \forall h'. Cut(c', h'))$ and $\forall h'' < h. Cut(c, h'')$. We show that all applications of cut of height h on a cut formula of weight c can be replaced by different applications of cut, either of smaller height or on a cut formula of smaller weight. We can assume $h, c > 0$ as the cases $h = 0$ and $c = 0$ have been already considered in **(A)** and **(B)**. We distinguish two cases.

(i) The cut formula is not principal in the last rule application in the derivation of at least one of the two premisses of cut. This case is analogous to (i) or (ii) in **(B)**.

(ii) The cut formula is principal in the last rule application in the derivation of both premisses. Then the cut formula is either $B \circ C$, with $\circ \in \{\wedge, \vee, \to\}$, or $\Box B$.

• The case of boolean connective is standard. We consider as an example $B \to C$. We have:

$$\dfrac{\mathsf{R}\!\to\dfrac{\overset{\textcircled{1}}{G \mid B, \Gamma \Rightarrow \Delta, B \to C, C}}{G \mid \Gamma \Rightarrow \Delta, B \to C} \qquad \dfrac{\overset{\textcircled{2}}{G \mid B \to C, \Gamma \Rightarrow \Delta, B} \qquad \overset{\textcircled{3}}{G \mid C, B \to C, \Gamma \Rightarrow \Delta}}{G \mid B \to C, \Gamma \Rightarrow \Delta}\mathsf{L}\!\to}{G \mid \Gamma \Rightarrow \Delta}\;\mathsf{cut}$$

The derivation is converted into the following one:

$$\dfrac{\mathsf{cut}\;\dfrac{\mathsf{Rwk}\;\dfrac{\mathsf{cut}\;\dfrac{\mathsf{Rwk}\;\dfrac{G \mid \Gamma \Rightarrow \Delta, B \to C}{G \mid \Gamma \Rightarrow \Delta, B \to C, B} \qquad \textcircled{2}}{G \mid \Gamma \Rightarrow \Delta, B}}{G \mid \Gamma \Rightarrow \Delta, B, C} \qquad \dfrac{\textcircled{1} \qquad \dfrac{\dfrac{G \mid B \to C, \Gamma \Rightarrow \Delta}{G \mid B, B \to C, \Gamma \Rightarrow \Delta}\;\mathsf{Lwk}}{G \mid B, B \to C, \Gamma \Rightarrow \Delta, C}\;\mathsf{Rwk}}{G \mid B, \Gamma \Rightarrow \Delta, C}\;\mathsf{cut}}{G \mid \Gamma \Rightarrow \Delta, C} \qquad \mathsf{cut}\;\dfrac{\mathsf{Lwk}\;\dfrac{G \mid \Gamma \Rightarrow \Delta, B \to C}{G \mid C, \Gamma \Rightarrow \Delta, B \to C} \qquad \textcircled{3}}{G \mid C, \Gamma \Rightarrow \Delta}}{G \mid \Gamma \Rightarrow \Delta}$$

• If the cut formula is $\Box B$ we have

$$G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B$$

$$\dfrac{\mathsf{R}\Box\ \dfrac{\vdots \quad \{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid B \Rightarrow C\}_{C\in\Sigma}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B} \qquad \dfrac{G \mid \langle B\rangle, \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

The derivation is converted as follows, with several applications of cut of smaller height and an admissible application of sub.

$$\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B \qquad \dfrac{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\ \mathsf{Ewk}}{④\ G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B}\ \mathsf{cut}$$

$$\dfrac{\dfrac{\dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}{G \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}\ \mathsf{Lwk} \qquad G \mid \langle B\rangle, \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{cut}}{⑤\ G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle B\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{Ewk}$$

$$\dfrac{\dfrac{\dfrac{④ \qquad \left\{ \dfrac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid B \Rightarrow C \qquad \dfrac{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}{G \mid \Box B, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\ \mathsf{Ewk}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid B \Rightarrow C}\ \mathsf{cut} \right\}_{C\in\Sigma} \qquad ⑤}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle, \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{sub}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{Lctr}}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta}\ \mathsf{Ectr}$$

□

Given the admissibility of the structural rules and cut we can prove that the calculi are syntactically complete with respect to the corresponding axiomatic systems.

Theorem 6.2.3 (Syntactic completeness)**.** If A is derivable in $\mathbf{E}^*$, then $\Rightarrow A$ is derivable in $\mathbf{H}.\mathbf{E}^*$.

Proof. As usual, we have to show that all axioms of $\mathbf{E}^*$ are derivable in $\mathbf{H}.\mathbf{E}^*$, and that all rules of $\mathbf{E}^*$ are admissible in $\mathbf{H}.\mathbf{E}^*$. The derivations of the modal axioms and rules are displayed in Figure 6.2. For the derivations of the axioms we implicitly consider Proposition 6.1.1. For the derivation of rule RE we assume that $A \to B$ and $B \to A$ are derivable in $\mathbf{E}^*$, and for the derivation of rule RD_n^+ we assume that $\neg(A_1, ..., A_n)$ is derivable in $\mathbf{ED_n^+}^*$. Finally, MP is simulated by cut in the usual way. □

6.3 Complexity of proof search

In this section, we analyse the complexity of proof search in our hypersequent calculi. Recall that, as proved by Vardi [167], the considered classical non-normal modal logics are coNP-complete in the absence of axiom C, and are in PSPACE with C (cf. Section 2.2). Here we firstly present a terminating proof search strategy in our calculi and show that it provides a coNP-*optimal* decision procedure for the logics without C. In this way we also establish coNP-complexity of the logics with the rules RD_n^+, which are not covered in [167]. The we briefly comment on the case where C is present.

Extensions without axiom C

The decision procedures for the logics without the axiom C implement backwards proof search on a polynomially bounded nondeterministic Turing machine with universal choices to handle the branching caused by rules with several premisses. Since all the rules are invertible, we can fix an order in which the rules are applied. To prevent loops, we employ a *local loop checking* strategy, stating that a rule is not applied (bottom-up) to a hypersequent G, if at least one of its premisses is trivial in the sense that each of its components can be derived from a component of the conclusion using only weakening and contraction. The formal definition is as follows.

Definition 6.3.1. An application of a hypersequent rule with premisses $H_1, \ldots, H_n$ and conclusion G satisfies the *local loop checking condition* if for each premiss H_i there exists a component $\Gamma \Rightarrow \Delta$ in H_i such that for no component $\Sigma \Rightarrow \Pi$ of the conclusion G we have: for all $A \in \Gamma$ also $A \in \Sigma$; and for all $\langle \Theta \rangle \in \Gamma$ there is a $\langle \Xi \rangle \in \Sigma$ with $\mathsf{set}(\Theta) = \mathsf{set}(\Xi)$; and $\mathsf{set}(\Delta) \subseteq \mathsf{set}(\Pi)$.

Since the rules are cumulative, every application of a rule satisfying the local loop checking condition adds in each of its premisses at least one new block or formula to an existing component, or adds a new component, which is not subsumed by a component of the conclusion. The following proposition shows that local loop checking preserves the completeness of the calculus.

Proposition 6.3.1. If a hypersequent is derivable in $\mathbf{H}.\mathbf{E}^*$ with a derivation of height n, then it is derivable using a derivation of height n in which every rule application satisfies the local loop checking condition.

Proof. By induction on the height n of the derivation. The zero-premisses rules trivially satisfy the local loop checking condition. If $n \geq 1$, consider the bottom-most rule application. If it satisfies the local loop checking condition, then we apply the induction hypothesis to each of its premisses and we are done. Otherwise, there is a premiss such that for each of its components $\Gamma, \langle \Theta_1 \rangle, \ldots, \langle \Theta_m \rangle \Rightarrow \Delta$ (where Γ does not contain any block) there is a component $\Sigma \Rightarrow \Pi$ of the conclusion G of the derivation with $\mathsf{set}(\Gamma) \subseteq \mathsf{set}(\Sigma)$, and for every $i \leq m$ there is a $\langle \Theta_i' \rangle \in \Sigma$ with $\mathsf{set}(\Theta_i) = \mathsf{set}(\Theta_i')$, and $\mathsf{set}(\Delta) \subseteq \mathsf{set}(\Pi)$. By considering the height-preserving admissibility of the structural rules (Proposition 6.2.1) we thus obtain a

Algorithm 1: Decision procedure for the derivability problem in $\mathbf{H.E}^*$

Input: A hypersequent G and the code of a logic $\mathbf{L}$

Output: "yes" if G is derivable in $\mathbf{H.L}$, a hypersequent if it is not.

if *there is a component* $\Gamma \Rightarrow \Delta$ *in* G *with* $\bot \in \Gamma$*, or* $\top \in \Delta$*, or* $\Gamma \cap \Delta \neq \emptyset$ **then**
  return "yes" and halt ;
else if *there is an applicable rule* **then**
  pick the first applicable rule;
  universally choose a premiss H of this rule application;
  check recursively whether H is derivable, output the answer and halt;
else
  return G and halt;
end

derivation of G of height n, and an appeal to the induction hypothesis yields a derivation of height n where every rule application satisfies the local loop checking condition. □

Notice that in the proof of this proposition, no new rule applications are added to a derivation, and that the order of the rule applications is preserved in the proof of admissibility of the structural rules (Proposition 6.2.1). Thus, given a derivation of a hypersequent, we can first adjust the ordering of the rules using invertibility, and then remove all the rule applications that violate the local loop checking condition. This yields completeness of proof search under these constraints:

Corollary 6.3.2. Proof search in $\mathbf{H.E}^*$ with local loop checking and a fixed order on the applications of the rules is complete.

The proof search algorithm thus applies the rules backwards in an arbitrary but fixed order, universally chooses one of their premisses and then recursively checks whether this premiss is derivable. The procedure is shown in Algorithm 1. In order to facilitate the countermodel construction for non-derivable hypersequents in the next section, we show termination for all considered logics, even those containing axiom C:

Theorem 6.3.3. Algorithm 1 terminates for all calculi $\mathbf{H.E}^*$.

Proof. Due to the subformula property of the rules, every formula occurring in a hypersequent in a run of Algorithm 1 is a subformula of the input. Moreover, local loop checking prevents the duplication of formulas, blocks and components. Thus, every component occurring in a run of the algorithm contains a subset of (occurrences of) subformulas of the input both on its antecedent and succedent, together with a set of blocks, each containing a subset of (occurrences of) subformulas of the input. Since there are only finitely many of these, the number of possible components is finite, and then also the number of hypersequents occurring in a run of the algorithm. Since every rule application satisfying local loop checking strictly increases the size of the hypersequent, each run of the algorithm thus halts after finitely many steps. $\square$

For the logics without axiom C, a closer analysis of the run time yields the optimal complexity bound:

Theorem 6.3.4. For the logics without C, Algorithm 1 runs in coNP, whence for these logics the calculi provide a complexity-optimal decision procedure.

Proof. Since the procedure is in the form of a non-deterministic Turing machine with universal choices, it suffices to show that every computation of this machine has polynomial length. Every application of a rule adds either a subformula of its conclusion or a new block to one of the components, or adds a new component. Due to local loop checking it never adds a formula, block or component which is already in the conclusion, so it suffices to calculate the maximal size of a hypersequent occurring in proof search for G. Suppose that the size of G is n. Then both the number of components in G and the number of subformulas of G are bounded by n. Since the local loop check prevents the duplication of formulas, each component contains at most n formulas in the antecedent and n formulas in the succedent. Moreover, since we only consider logics without the axiom C, every newly created block contains exactly one formula. Again, due to the local loop checking condition no block is duplicated, so every component contains at most n blocks. Thus every component has size at most $3n$. The procedure creates new components from a block and a formula of an already existing component using one of the rules $\mathsf{R}\Box$ and $\mathsf{R}\Box\mathsf{m}$, or from ℓ components using one of the rules P, D_2, D_1, D^+_ℓ, with $\ell \leq k$ for a fixed k depending on the logic. Hence there are at most $n^2 + k \cdot n^k$ many different components which can be created without violating the local loop checking condition. Thus every hypersequent occurring in the proof contains at most $n + n^2 + k \cdot n^k$ many components, each of size at most $3n$, giving a total size and thus running time of $\mathcal{O}(n^3)$, resp. $\mathcal{O}(n^{k+1})$ for $k > 2$. $\square$

As noted above, Algorithm 1 works properly also for logics with the axiom C, ensuring in particular termination. However, hypersequents occurring in a proof of H can be exponentially large with respect to the size of H. This is due to the presence of the rule C that, given n formulas $\Box A_1, \ldots, \Box A_n$, allows one to build a block for every subset of $\{A_1, \ldots, A_n\}$. In this respect, this decision procedure does not match the PSPACE complexity upper bound established for these systems by Vardi [167]. However, this is not really unexpected, since one of the main appeals of the hypersequent calculi is that they can be used to directly construct countermodels for unprovable hypersequents, and in some logics with C it is possible to force exponentially large countermodels, in particular in normal modal logic **K** [21]. Hence for these logics the hypersequents will need to be of exponential size, suggesting that we need to modify the hypersequent calculi to obtain complexity-optimal decision procedures.

Logics with axiom C

In order to obtain a PSPACE decision procedure for logics with axiom C we must adopt a different strategy. Since already the standard sequent calculi could be used to obtain complexity-optimal decision procedures in a standard way, we only sketch the ideas. Instead of the rules in Figure 6.1, we consider their *unkleene'd* – and non-invertible – version, i.e. the ones with all principal formulas and structures deleted from the premisses. For instance $\mathsf{R}\Box\mathsf{m}$, $\mathsf{R}\Box$ and C are replaced respectively with

$$\frac{G \mid \Gamma \Rightarrow \Delta \,|\, \Sigma \Rightarrow B}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta} \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow B \quad \{G \mid \Gamma \Rightarrow \Delta \mid B \Rightarrow A\}_{A \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Box B, \Delta} \qquad \frac{G \mid \Gamma, \langle \Sigma, \Pi \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta}$$

Call the resulting calculus $\mathbf{H.E}^*_-$. Backwards proof search is then implemented on an alternating Turing machine by existentially guessing the last applied rule except for N, and universally checking that all of its premisses are derivable. To ensure that N is applied if it is present in the system, we stipulate that it is applied once to every component of the input, and that if the existentially guessed rule creates a new component, the rule N is applied immediately afterwards to each of its premisses. Since no rule application keeps the principal formulas in the premisses, and since the rule N if present is applied exactly once to every component, there is no need for any loop checking condition.

Ona can show that the calculi $\mathbf{H.E}^*_-$ are sound and complete. Soundness is obvious, since we can add the missing formulas and structures and recover derivations in $\mathbf{H.E}^*$. Completeness can be proved by a cut elimination argument similar to the one in Theorem 6.2.2, or alternatively by simulating the standard sequent calculi by Lavendhomme and Lucas [107] (cf. Section 3.4). We can show that the calculi $\mathbf{H.E}^*_-$ give a PSPACE upper bound:

Theorem 6.3.5. Backwards proof search in $\mathbf{H.E}^*_-$ is in PSPACE.

Proof. We need to show that every run of the procedure terminates in polynomial time. Assume that the size of the input is n. Let the *weight* of a component in a hypersequent be the sum of the weights of the formulas and blocks occurring in it according to Definition 6.2.1, and suppose that the maximal weight of components in the input is m. Then every rule apart from N decreases the weight of the component active in its conclusion. Moreover, a new component is only introduced in place of at least one subformula of the input, hence any hypersequent occurring in the proof search has at most $n + n$ components. The weight of each of these components is at most the maximal weight of a component of the input (plus one in the cases with N). Since the rule N is applied at most once to each component, it is thus applied at most n times in the total proof search. Thus the runtime in total is $\mathcal{O}(n^2 \cdot m)$, hence polynomial in the size of the input. Thus the procedure runs in alternating polynomial time, and thus in PSPACE. $\square$

In brief, the situation of logics with axiom C can be summarised as follows. On the one hand, we have the fully invertible calculi $\mathbf{H.EC}^*$ which are terminating but not optimal. As we shall see in the next section, these calculi allow for a direct extraction of countermodels from single failed proofs. On the other hand, we have the calculi $\mathbf{H.EC}^*_-$, which are optimal but contain non-invertible rules, whence it is not possible to define a countermodel only on the basis of a single failed proof. As for many other logics, this illustrates the existence of a necessary *trade-off* between the optimal complexity of the calculi and the direct extraction of countermodels.

6.4 Countermodel extraction and semantic completeness

In this section, we show that from every failed proof in $\mathbf{H.E}^*$ we can directly extract a countermodel of the non-derivable hypersequent. This allows us to obtain a semantic proof of completeness of our calculi, i.e., every valid hypersequent is derivable. The countermodels extracted from the failed proofs are defined in the bi-neighbourhood semantics, as it turns out to be more adequate than the standard one. The reason is that a failed proof only provides a partial model that does not specify exactly the truth sets of formulas, as required by the standard semantics.

To see this, recall that if we want a world w to satisfy a modal formula $\Box A$ in the standard semantics, we have to make sure that the truth set $[\![A]\!]$ belongs to the neighbourhood of w, thus $[\![A]\!]$ must be computed. In order to define countermodels on the basis of the information provided by the failed proofs, we consider the natural semantic reading of hypersequents according to which every component corresponds to a *world* in the model, every formula in the antecedent of a component is *true* in the corresponding world, and every formula in the succedent of a component is *false* in that world. Thus, in order to determine the extension of $[\![A]\!]$ basing only on the information provided by the failed proof we need that A occurs either on the left or on the right of *every* component. But this is hardly ever the case: we will more often find components where A does not occur *neither* on the left, *nor* in the right, whence the determination of $[\![A]\!]$ is not possible. In contrast, this situation strictly reflects the structure of the bi-neighbourhood semantics. As we shall see, suited bi-neighbourhood pairs (α, β) such that $\alpha \subseteq [\![A]\!] \subseteq \mathcal{W} \setminus \beta$ can be extracted from the proof even without knowing exactly the extension of $[\![A]\!]$.

In order to prove the semantic completeness of the calculus we make use of the backwards proof search strategy based on local loop checking already considered in Section 6.3 (Algorithm 1). This strategy amounts to consider the following notion of saturation, stating that no bottom-up rule application is allowed to initial sequents, and that a bottom-up application of a rule R is not allowed to a hypersequent H if H already fulfills the corresponding saturation condition.

Definition 6.4.1 (Saturated hypersequent)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent occurring in a proof for H'. The saturation conditions associated to each application of a rule of $\mathbf{H.E}^*$ are as follows: (init) $\Gamma_i \cap \Delta_i = \emptyset$. $(\mathsf{L}\bot)$ $\bot \notin \Gamma_i$. $(\mathsf{R}\top)$ $\top \notin \Delta_i$. $(\mathsf{L} \rightarrow)$ If $A \rightarrow B \in \Gamma_i$, then $A \in \Delta_i$ or $B \in \Gamma_i$. $(\mathsf{R} \rightarrow)$ If $A \rightarrow B \in \Delta_i$, then $A \in \Gamma_i$ and $B \in \Delta_i$. $(\mathsf{L}\wedge)$ If $A \wedge B \in \Gamma_i$, then $A \in \Gamma_i$ and $B \in \Gamma_i$. $(\mathsf{R}\wedge)$ If $A \wedge B \in \Delta_i$, then $A \in \Delta_i$ or $B \in \Delta_i$. $(\mathsf{L}\Box)$ If $\Box A \in \Gamma_i$, then $\langle A \rangle \in \Gamma_i$. $(\mathsf{R}\Box)$ If $\langle \Sigma \rangle, \Gamma \Rightarrow \Delta, \Box B$ is in H, then there is $\Gamma' \Rightarrow \Delta', B$ in H such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$, or there is $B, \Gamma' \Rightarrow \Delta', A$ in H for some $A \in \Sigma$. $(\mathsf{R}\Box\mathsf{m})$ If $\langle \Sigma \rangle, \Gamma \Rightarrow \Delta, \Box B$ is in H, then there is $\Gamma' \Rightarrow \Delta', B$ in H such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$. (N) $\langle \top \rangle \in \Gamma_i$. (C) If $\langle \Sigma \rangle, \langle \Pi \rangle \in \Gamma_i$, then there is $\langle \Omega \rangle \in \Gamma_i$ such that $\mathsf{set}(\Sigma, \Pi) = \mathsf{set}(\Omega)$. (T) If $\langle \Sigma \rangle \in \Gamma_n$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_n$. (P) If $\Gamma, \langle \Sigma \rangle \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow \Delta'$ in H such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$. (D_1) If $\Gamma, \langle \Sigma \rangle \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow \Delta'$ in H such that $\mathsf{set}(\Sigma) \subseteq \Gamma'$, or there is $\Gamma' \Rightarrow \Delta', A$ in H for some $A \in \Sigma$. (D_2) If $\Gamma, \langle \Sigma \rangle, \langle \Pi \rangle \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow \Delta'$ in H such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma'$, or there is $\Gamma' \Rightarrow \Delta', A, B$ in H for some $A \in \Sigma, B \in \Pi$. (D_n^+) If $\Gamma, \langle \Sigma_1 \rangle, ..., \langle \Sigma_n \rangle \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow \Delta'$ in H such that $\mathsf{set}(\Sigma_1, ..., \Sigma_n) \subseteq \Gamma'$.

We say that H is *saturated* with respect to an application of a rule R if it satisfies the saturation condition (R) for that particular rule application, and that it is saturated with respect to $\mathbf{H.E}^*$ if it is saturated with respect to all possible applications of any rule of $\mathbf{H.E}^*$.

Then the strategy is simply defined as follows:

Definition 6.4.2 (Proof search strategy, failed proof)**.** Given a hypersequent H, a proof of H in $\mathbf{H.E}^*$ is constructed bottom-up starting with H and applying backwards rules of $\mathbf{H.E}^*$. The rule applications must respect the following two conditions: (i) No rule can be applied to an initial hypersequent. (ii) The application of a rule to a hypersequent is not allowed if the hypersequent is already saturated with respect to that particular rule application. The construction of the proof tree terminates when no additional rule applications are possible in the respect of conditions (i) and (ii). We call *failed proof* of A any tree which is constructed in accordance with the strategy and contains some saturated hypersequent.

The strategy essentially amounts to avoiding applications of rules that do not add any additional information to the hypersequents. As shown in Theorem 6.3.3, the proof search procedure for H always terminates. Moreover, every branch ends either with an initial hypersequent or a saturated one. We now show that, given a saturated hypersequent H, one can directly construct a countermodel of H in the bi-neighbourhood semantics.

Definition 6.4.3 (Countermodel construction)**.** Let H be a saturated hypersequent occurring in a proof for H'. Moreover, let $e : \mathbb{N} \longrightarrow H$ be an enumeration of the components of H.

Given e, we can write H as $\Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$. The model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W} = \{n \mid \Gamma_n \Rightarrow \Delta_n \in H\}$.
- $\mathcal{V}(p) = \{n \mid p \in \Gamma_n\}$.
- For every block $\langle \Sigma \rangle$ appearing in a component $\Gamma_m \Rightarrow \Delta_m$ of H,

  $\Sigma^+ = \{n \mid \mathsf{set}(\Sigma) \subseteq \Gamma_n\}$ and $\Sigma^- = \{n \mid \Sigma \cap \Delta_n \neq \emptyset\}$.
- The definition of $\mathcal{N}$ depends whether the calculus is or not monotonic:
  - Non-monotonic case: $\mathcal{N}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle \Sigma \rangle \in \Gamma_n\}$.
  - Monotonic case: $\mathcal{N}(n) = \{(\Sigma^+, \emptyset) \mid \langle \Sigma \rangle \in \Gamma_n\}$.

Lemma 6.4.1. Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent, and $\mathcal{M}$ be the model defined on the basis of H as in Definition 6.4.3. Then for every A, $\langle \Sigma \rangle$ and every $n \in \mathcal{W}$, we have:

– if $A \in \Gamma_n$, then $\mathcal{M}, n \Vdash A$;
– if $\langle \Sigma \rangle \in \Gamma_n$, then $\mathcal{M}, n \Vdash \Box \bigwedge \Sigma$; and
– if $A \in \Delta_n$, then $\mathcal{M}, n \nVdash A$.

Moreover, if the proof is in calculus $\mathbf{H.EX}^*$, then $\mathcal{M}$ is a X-model.

Proof. The first claim is proved by mutual induction on A and $\langle \Sigma \rangle$.

($p \in \Gamma_n$) By definition, $n \in \mathcal{V}(p)$. Then $n \Vdash p$.

($p \in \Delta_n$) By saturation of init, $p \notin \Gamma_n$. Then $n \notin \mathcal{V}(p)$, thus $n \nVdash p$.

($B \wedge C \in \Gamma_n$) By saturation of $\mathsf{L}\wedge$, $B \in \Gamma_n$ and $C \in \Gamma_n$. Then by i.h., $n \Vdash B$ and $n \Vdash C$, thus $n \Vdash B \wedge C$.

($B \wedge C \in \Delta_n$) By saturation of $\mathsf{R}\wedge$, $B \in \Delta_n$ or $C \in \Delta_n$. Then by i.h., $n \nVdash B$ or $n \nVdash C$, thus $n \nVdash B \wedge C$.

($B \vee C, B \to C \in \Gamma_n$, $B \vee C, B \to C \in \Delta_n$) Analogous to ($B \wedge C \in \Gamma_n$) and ($B \wedge C \in \Delta_n$), respectively.

($\langle \Sigma \rangle \in \Gamma_n$) In the non-monotonic case we have: By definition $(\Sigma^+, \Sigma^-) \in \mathcal{N}(n)$. We show that $\Sigma^+ \subseteq [\![\bigwedge \Sigma]\!]$ and $\Sigma^- \subseteq [\![\neg \bigwedge \Sigma]\!]$, which implies $n \Vdash \Box \bigwedge \Sigma$. If $m \in \Sigma^+$, then $\mathsf{set}(\Sigma) \subseteq \Gamma_m$. By i.h. $m \Vdash A$ for all $A \in \Sigma$, then $m \Vdash \bigwedge \Sigma$. If $m \in \Sigma^-$, then there is $B \in \Sigma \cap \Delta_m$. By i.h. $m \nVdash B$, then $m \nVdash \bigwedge \Sigma$. In the monotonic case the proof is analogous.

($\Box B \in \Gamma_n$) By saturation of $\mathsf{L}\Box$, $\langle B \rangle \in \Gamma_n$. Then by i.h. $n \Vdash \Box B$.

($\Box B \in \Delta_n$) In the non-monotonic case, assume $(\alpha, \beta) \in \mathcal{N}(n)$. Then there is $\langle\Sigma\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{R}\Box$, there is $m \in \mathcal{W}$ such that $\Sigma \subseteq \Gamma_m$ and $B \in \Delta_m$, or there is $m \in \mathcal{W}$ such that $\Sigma \cap \Delta_m \neq \emptyset$ and $B \in \Gamma_m$. In the first case, $m \in \Sigma^+ = \alpha$ and by i.h. $m \not\Vdash B$, thus $\alpha \not\subseteq [\![B]\!]$. In the second case, $m \in \Sigma^- = \beta$ and by i.h. $m \Vdash B$, thus $\beta \cap [\![B]\!] \neq \emptyset$, i.e., $[\![B]\!] \not\subseteq \mathcal{W} \setminus \beta$. Therefore $n \not\Vdash \Box B$. The monotonic case is analogous.

Now we prove that if the failed proof is in $\mathbf{H.EX}^*$, then $\mathcal{M}$ satisfies condition (X).

(M) By definition, $\beta = \emptyset$ for every $(\alpha, \beta) \in \mathcal{N}(n)$.

(N) By saturation of rule N, $\langle\top\rangle \in \Gamma_n$ for all $n \in \mathcal{W}$, thus $(\top^+, \top^-) \in \mathcal{N}(n)$. Moreover, by saturation of $\mathsf{R}\top$, $\top^- = \emptyset$.

(C) Assume that $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(n)$. Then there are $\langle\Sigma\rangle, \langle\Pi\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$, $\Sigma^- = \beta$, $\Pi^+ = \gamma$ and $\Pi^- = \delta$. By saturation or rule C, there is $\langle\Omega\rangle \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi)$, thus $(\Omega^+, \Omega^-) \in \mathcal{N}(n)$. We show that (i) $\Omega^+ = \alpha \cap \gamma$ and (ii) $\Omega^- = \beta \cup \delta$. (i) $m \in \Omega^+$ iff $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$ iff $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$ iff $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$ iff $m \in \alpha \cap \gamma$. (ii) $m \in \Omega^-$ iff $\Omega \cap \Delta_m \neq \emptyset$ iff $\Sigma, \Pi \cap \Delta_m \neq \emptyset$ iff $\Sigma \cap \Delta_m \neq \emptyset$ or $\Pi \cap \Delta_m \neq \emptyset$ iff $m \in \Sigma^- = \beta$ or $m \in \Pi^- = \delta$ iff $m \in \beta \cup \delta$.

(T) If $(\alpha, \beta) \in \mathcal{N}(n)$, then there is $\langle\Sigma\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule T, $\mathsf{set}(\Sigma) \subseteq \Gamma_n$, then $n \in \Sigma^+ = \alpha$.

(P) If $(\alpha, \beta) \in \mathcal{N}(n)$, then there is $\langle\Sigma\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule P, there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$, then $m \in \Sigma^+ = \alpha$, that is $\alpha \neq \emptyset$.

(D) Assume $(\alpha, \beta), (\gamma, \delta) \in \mathcal{N}(n)$. If $(\alpha, \beta) \neq (\gamma, \delta)$, then there are $\langle\Sigma\rangle, \langle\Pi\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha, \Sigma^- = \beta, \Pi^+ = \gamma$ and $\Pi^- = \delta$. If the calculus is non-monotonic, then by saturation of rule D_2 there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$ or there is $m \in \mathcal{W}$ such that $A, B \in \Delta_m$ for $A \in \Sigma$ and $B \in \Pi$. In the first case, $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$, thus $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$, that is $\alpha \cap \gamma \neq \emptyset$. In the second case, $m \in \Sigma^- = \beta$ and $m \in \Pi^- = \delta$, that is $\beta \cap \delta \neq \emptyset$. If in contrast the calculus is monotonic, by saturation of $\mathsf{D_M}$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma, \Pi) \subseteq \Gamma_m$. Then $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $\mathsf{set}(\Pi) \subseteq \Gamma_m$, thus $m \in \Sigma^+ = \alpha$ and $m \in \Pi^+ = \gamma$, that is $\alpha \cap \gamma \neq \emptyset$. The other possibility is that $(\alpha, \beta) \neq (\gamma, \delta)$. Then there is $\langle\Sigma\rangle \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. In the non-monotonic case, by saturation of D_1 there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ or there is $m \in \mathcal{W}$ such that $A \in \Delta_m$ for some $A \in \Sigma$. Then $m \in \Sigma^+ = \alpha$, that is $\alpha \neq \emptyset$, or $m \in = \Sigma^- = \beta$, that is $\beta \neq \emptyset$. In the monotonic case we can consider saturation of P and conclude that $\Sigma^+ = \alpha \neq \emptyset$.

(RD_n^+) Assume $(\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)$ be any $m \leq n$ different bi-neighbourhood pairs belonging to $\mathcal{N}(n)$. Then there are $\langle\Sigma_1\rangle, ..., \langle\Sigma_m\rangle \in \Gamma_n$ such that $\Sigma_i^+ = \alpha_i$ and $\Sigma_i^- = \beta_i$ for every $1 \leq i \leq m$. By saturation of rule D_m^+ (that by definition belongs to the calculus $\mathbf{H.ED_n^{+*}}$), there is $\ell \in \mathcal{W}$ such that $\mathsf{set}(\Sigma_1, ..., \Sigma_m) \subseteq \Gamma_\ell$. Then $\ell \in \Sigma_1^+ = \alpha_1$, ..., $\ell \in \Sigma_m^+ = \alpha_m$, that is $\alpha_1 \cap ... \cap \alpha_m \neq \emptyset$. $\Box$

Observe that, since all rules are cumulative, $\mathcal{M}$ is also a countermodel of the root hypersequent H'. Moreover, since every proof built in accordance with the strategy either provides a derivation of the root hypersequent or contains a saturated hypersequent, this allows us to prove the following theorem.

Theorem 6.4.2 (Semantic completeness)**.** If H is valid in all bi-neighbourhood models for $\mathbf{E}^*$, then it is derivable in $\mathbf{H}.\mathbf{E}^*$.

Proof. Assume H not derivable in $\mathbf{H}.\mathbf{E}^*$. Then there is a failed proof of H in $\mathbf{H}.\mathbf{E}^*$ containing some saturated hypersequent H'. By Lemma 6.4.1, we can construct a bi-neighbourhood countermodel of H', whence a countermodel of H, that satisfies all properties of bi-neighbourhood models for $\mathbf{E}^*$. Therefore H is not valid in every bi-neighbourhood model for $\mathbf{E}^*$. □

Since the countermodels constructed for non-derivable hypersequents are based on the saturated hypersequents returned by Algorithm 1, and since the latter are finite, we immediately obtain the finite model property for all the logics. For the logics without C we can further bound the *size* of the models, defined in the following way.

Definition 6.4.4 (Size of models)**.** The *size* of a bi-neighbourhood or standard model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as $\mathsf{size}(\mathcal{M}) := |\mathcal{W}| + \sum_{w \in \mathcal{W}} |\mathcal{N}(w)|$.

Corollary 6.4.3 (Polysize model property)**.** Every classical non-normal modal logic $\mathbf{E}^*$ without axiom C has the *polysize model property* with respect to bi-neighbourhood models, i.e., there is a polynomial p such that if a formula A of size n is satisfiable, then it is satisfiable in a bi-neighbourhood model of size at most $p(n)$.

Proof. Given a non-derivable formula of size n, from the proof of Theorem 6.3.4 we obtain that the saturated hypersequent used for constructing the countermodel has $\mathcal{O}(n^k)$ many components, each containing $\mathcal{O}(n)$ many blocks for k depending only on the logic. Since the worlds of the countermodel correspond to the components, and the neighbourhoods for each world are constructed from the blocks occurring in that component, this model has at most $\mathcal{O}(n^k)$ many worlds, each with a neighbourhood of size at most $\mathcal{O}(n)$. Therefore the size of the model is $\mathcal{O}(n^{k+1})$. □

As the above construction shows, we can directly extract a bi-neighbourhood countermodel from any failed proof. If we want to obtain a countermodel in the standard semantics we then need to apply the transformations presented in Section 2.3. In principle, the rough transformation (Proposition 4.3.3) can be embedded into the countermodel construction in order to directly construct a neighbourhood model, we just need to modify the definition of $\mathcal{N}(n)$ in Definition 6.4.3 as follows:

$$\mathcal{N}(n) = \{\gamma \mid \text{there is } \langle \Sigma \rangle \in \Gamma_n \text{ such that } \Sigma^+ \subseteq \gamma \subseteq \mathcal{W} \setminus \Sigma^-\}.$$

However, in this way we might obtain a model with a larger neighbourhood function than needed. In contrast, there is no obvious way to integrate the finer transformation of Proposition 4.3.4 into the countermodel construction, since it relies on the evaluation of formulas in an already existing model. But it does lead to smaller models:

Corollary 6.4.4. Every classical non-normal modal logic $\mathbf{E}^*$ without axiom C has the *polysize model property* with respect to standard models.

Proof. Given a satisfiable formula of size n, from Corollary 6.4.3 we obtain a bi-neighbourhood model with $\mathcal{O}(n)$ worlds. Since the transformation of Proposition 4.3.4 constructs neighbourhoods from sets of truth sets of subformulas of the input, the size of $\mathcal{N}_{st}(w)$ is at most n for each world w. Then the total size of the standard model is polynomial in the size n of the formula. $\square$

An alternative way of obtaining countermodels in the standard neighbourhood semantics is proposed in [107]. It basically consists in forcing the proof search procedure to determine exactly the truth set of each formula. To this aim, whenever a sequent representing a new world is created, the sequent is saturated with respect to all disjunctions $A \vee \neg A$ such that A is a subformula of the root sequent. This solution is equivalent to using *analytic cut* and makes the proof search procedure significantly more complex than the one given here.

We now show some examples of countermodel extraction from failed proofs, both in the bi-neighbourhood and in the standard neighbourhood semantics, the latter kind of models are obtained by applying the transformation in Proposition 4.3.4.

Example 6.4.1 (Proof search for axiom M in $\mathbf{H}.\mathbf{E}$ and countermodels)**.** The following is a failed proof of $\Box(p \wedge q) \Rightarrow \Box p$ in $\mathbf{H}.\mathbf{E}$.

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\text{derivable}\\ \langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \wedge q \Rightarrow p\end{matrix} \quad \dfrac{\begin{matrix}\text{derivable}\\ \dots \mid p \Rightarrow p \wedge q, p\end{matrix} \quad \begin{matrix}\textbf{saturated}\\ \langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \Rightarrow p \wedge q, q\end{matrix}}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p \mid p \Rightarrow p \wedge q}\,\mathsf{R}\wedge}{\langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p}\,\mathsf{R}\Box}{\Box(p \wedge q) \Rightarrow \Box p}\,\mathsf{L}\Box
$$

Bi-neighbourhood countermodel. Let us consider the following enumeration of the components of the saturated hypersequent H: $1 \mapsto \langle p \wedge q\rangle, \Box(p \wedge q) \Rightarrow \Box p$; and $2 \mapsto p \Rightarrow p \wedge q, q$. According to the construction in Definition 6.4.3, from H we obtain the following countermodel $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$. $\mathcal{V}(p) = \{2\}$ and $\mathcal{V}(q) = \emptyset$. $\mathcal{N}_{bi}(2) = \emptyset$ and $\mathcal{N}_{bi}(1) = \{(\emptyset, \{2\})\}$, as $\mathcal{N}_{bi}(1) = \{(p \wedge q^+, p \wedge q^-)\}$ and $p \wedge q^+ = \emptyset$, $p \wedge q^- = \{2\}$. We have $1 \Vdash \Box(p \wedge q)$ because $\emptyset \subseteq [\![p \wedge q]\!] = \emptyset \subseteq \mathcal{W} \setminus \{2\}$, and $1 \nVdash \Box p$ because $[\![p]\!] = \{2\} \nsubseteq \mathcal{W} \setminus \{2\}$. Then $1 \nVdash \Box(p \wedge q) \to \Box p$.

Neighbourhood countermodel. We consider the set $\mathcal{S} = \{\Box(p \wedge q) \to \Box p, \Box(p \wedge q), \Box p, p \wedge q, p, q\}$ of the subformulas of $\Box(p \wedge q) \to \Box p$. By applying the transformation in Proposition 4.3.4 to the bi-neighbourhood model $\mathcal{M}_{bi}$, we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, and $\mathcal{N}_{st}(1) = \{\emptyset\}$, since $\mathcal{N}_{st}(1) = \{[\![p \wedge q]\!]_{\mathcal{M}_{bi}}\}$ and $[\![p \wedge q]\!]_{\mathcal{M}_{bi}} = \emptyset$.

Example 6.4.2 (Proof search for axiom K in **H.EC** and countermodels)**.** If Figure 6.3 we find a failed proof of $\Box(p \to q) \to (\Box p \to \Box q)$ in **H.EC**. The countermodels are as follows.

Bi-neighbourhood countermodel. We consider the following enumeration of the compontents of the saturated hypersequent H:

$$\begin{array}{lll} 1 & \mapsto & \Box(p \to q), \Box p, \langle p \to q \rangle, \langle p \rangle, \langle p \to q, p \rangle \Rightarrow \Box q. \\ 2 & \mapsto & q \Rightarrow p. \\ 3 & \mapsto & p \to q \Rightarrow q, p. \end{array}$$

According to the construction in Definition 6.4.3, from H we obtain the following countermodel $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}_{bi}, \mathcal{V} \rangle$: $\mathcal{W} = \{1,2,3\}$. $\mathcal{V}(p) = \emptyset$ and $\mathcal{V}(q) = \{2\}$. $\mathcal{N}_{bi}(2) = \mathcal{N}_{bi}(3) = \emptyset$, and $\mathcal{N}_{bi}(1) = \{(\emptyset, \{2,3\}), (\{3\}, \emptyset)\}$, as $\mathcal{N}_{bi}(1) = \{(p^+, p^-), (p \to q^+, p \to q^-), (p, p \to q^+, p, p \to q^-)\}$ and $p^+ = \emptyset$, $p^- = \{2,3\}$, $p \to q^+ = \{3\}$, $p \to q^- = \emptyset$, $p, p \to q^+ = \emptyset$, $p, p \to q^- = \{2,3\}$.

Then we have $1 \Vdash \Box(p \to q)$ because $\{3\} \subseteq [\![p \to q]\!] = \mathcal{W} \subseteq \mathcal{W} \setminus \emptyset$; and $x \Vdash \Box p$ because $\emptyset \subseteq [\![p]\!] = \emptyset \subseteq \mathcal{W} \setminus \{2,3\}$; but $x \not\Vdash \Box q$ because $\{3\} \not\subseteq [\![q]\!] = \{2\}$ and $[\![q]\!] = \{2\} \not\subseteq \mathcal{W} \setminus \{2,3\}$, whence $x \not\Vdash \Box(p \to q) \to (\Box p \to \Box q)$. Observe that $\mathcal{M}_{bi}$ is a C-model since $(\emptyset \cap \{3\}, \{2,3\} \cup \emptyset) = (\emptyset, \{2,3\})$.

Neighbourhood countermodel. By logical equivalence we can restrict the considered set of formulas $\mathcal{S}$ to $\{\Box(p \to q), \Box p, \Box q, p \to q, p, q, \Box((p \to q) \wedge q), \Box(p \wedge q)\}$. By the transformation in Proposition 4.3.4, from $\mathcal{M}_{bi}$ we obtain the standard model $\mathcal{M}_{st} = \langle \mathcal{W}, \mathcal{N}_{st}, \mathcal{V} \rangle$, where $\mathcal{W}$ and $\mathcal{V}$ are as in $\mathcal{M}_{bi}$, and $\mathcal{N}_{st}(1) = \{[\![p \to q]\!]_{\mathcal{M}_{bi}}, [\![p]\!]_{\mathcal{M}_{bi}}, [\![p \wedge q]\!]_{\mathcal{M}_{bi}}\} = \{\mathcal{W}, \emptyset\}$.

Finally, the next example shows the need of rule D_1 for the calculus **H.ED** and its non-monotonic extensions from the point of view of the countermodel extraction.

Example 6.4.3 (Proof search for $\neg\Box\top$ in **H.ED** and countermodel)**.** Let us consider the following failed proof of $\Box\top \Rightarrow$ in **H.ED**.

$$\dfrac{\dfrac{\overset{\textbf{saturated}}{\Box\top, \langle\top\rangle \Rightarrow \mid \top \Rightarrow} \qquad \dfrac{}{\Box\top, \langle\top\rangle \Rightarrow \mid \Rightarrow \top}\,\mathsf{R}\top}{\Box\top, \langle\top\rangle \Rightarrow}\,\mathsf{D}_1}{\Box\top \Rightarrow}\,\mathsf{L}\Box$$

Consider the saturated hypersequent and establish $1 \mapsto \Box\top, \langle\top\rangle \Rightarrow$, and $2 \mapsto \top \Rightarrow$. We obtain the bi-neighbourhood countermodel $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W} = \{1,2\}$; $\mathcal{N}(1) = \{(\top^+, \top^-)\} = \{(\{2\}, \emptyset)\}$; and $\mathcal{N}(2) = \emptyset$. This is a D-model and falsifies $\neg\Box\top$, as $1 \Vdash \Box\top$.

Now imagine that the rule D_1 does not belong to the calculus **H.ED**. In this case the proof would end with $\Box\top, \langle\top\rangle \Rightarrow$, as no other rule is backwards applicable to it. From this we would get the model $\mathcal{M}' = \langle \mathcal{W}', \mathcal{N}', \mathcal{V}' \rangle$, where $\mathcal{W}' = \{1\}$ and $\mathcal{N}'(1) = \{(\emptyset, \emptyset)\}$, which falsifies $\neg\Box\top$ but is *not* a D-model.

derivable

$$\ldots \mid p \to q, q \Rightarrow q$$

derivable

$$\ldots \mid q \Rightarrow p \to q$$

saturated

$$\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q, p \qquad \ldots \mid p \to q, q \Rightarrow q}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q}\ \mathsf{L}\to$$

$$\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \mid p \to q \Rightarrow q \qquad \ldots \mid q \Rightarrow p \to q}{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p}\ \mathsf{R}\Box$$

derivable

$$\ldots \mid p \to q, p \Rightarrow q$$

derivable

$$\ldots \mid q \Rightarrow p \to q$$

$$\dfrac{\ldots \mid p \to q, p \Rightarrow q \qquad \Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q \mid q \Rightarrow p \qquad \ldots \mid q \Rightarrow p \to q}{\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle, \langle p \to q, p\rangle \Rightarrow \Box q}{\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle, \langle p\rangle \Rightarrow \Box q}{\dfrac{\Box(p \to q), \Box p, \langle p \to q\rangle \Rightarrow \Box q}{\Box(p \to q), \Box p \Rightarrow \Box q}\ \mathsf{L}\Box}\ \mathsf{L}\Box}\ \mathsf{C}}\ \mathsf{R}\Box$$

Figure 6.3: Failed proof of axiom K in **H**.**EC**.

Relational countermodels for regular logics

We now consider the possibility to define relational countermodels of formulas which are not derivable in the calculi $\mathbf{H.MC}^*$ for regular logics (Definition 2.3.7). In principle, this could be done as follows: we first extract a bi-neighbourhood countermodel, and then apply the transformation in Proposition 4.3.5. We now show that, analogously to the labelled calculi (cf. Section 5.5), relational countermodels can be also extracted directly from failed proofs in $\mathbf{H.MC}^*$. This possibility not only makes the definition of the relational models more efficient (as it prevents to go through the transformation of previously extracted bi-neighbourhood models), but also shows the independency of the calculus from any specific semantic choice. Relational models are extracted from failed proofs in $\mathbf{H.MC}^*$ as follows.

Definition 6.4.5 (Relational countermodel)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof for H' in $\mathbf{H.MC}^*$. For every $1 \leq n \leq k$, we say that a block $\langle\Sigma\rangle$ is *maximal* for n if $\langle\Sigma\rangle \in \Gamma_n$, and for every $\langle\Pi\rangle \in \Gamma_n$, $\mathsf{set}(\Pi) \subseteq \mathsf{set}(\Sigma)$. It is easy to see that by saturation of rule C every component either contains a maximal block or does not contain any block at all. On the basis of H we define the relational model $\mathcal{M} = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$, as follows.

- $\mathcal{W}$, $\mathcal{V}$, and for every block $\langle\Sigma\rangle$, Σ^+, are defined as in Definition 6.4.3.
- $\mathcal{W}^i$ is the set of worlds n such that Γ_n does not contain any block.
- For every $n \in \mathcal{W} \setminus \mathcal{W}^i$, $\mathcal{R}(n) = \Sigma^+$, where $\langle\Sigma\rangle$ is a maximal block for n.

Observe that if $\langle\Sigma\rangle$ and $\langle\Pi\rangle$ are two maximal blocks for n, then $\Sigma^+ = \Pi^+$, whence $\mathcal{R}(n)$ is unique for every n.

Lemma 6.4.5. Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof for H' in $\mathbf{H}.\mathbf{MC}^*$, and $\mathcal{M}$ be the model defined on the basis of H as in Definition 6.4.5. Then for every formula A and block $\langle\Sigma\rangle$ we have: if $A \in \Gamma_n$, then $\mathcal{M}, n \Vdash A$; if $\langle\Sigma\rangle \in \Gamma_n$, then $\mathcal{M}, n \Vdash \Box \bigwedge \Sigma$; and if $A \in \Delta_n$, then $\mathcal{M}, n \nVdash A$. Moreover, $\mathcal{M}$ is a relational model for $\mathbf{MC}$, and if $\mathbf{H}.\mathbf{MC}^*$ contains rule N, then $\mathcal{M}$ is a standard Kripke model for normal modal logic $\mathbf{K}$.

Proof. The truth lemma is proved by mutual induction on A and $\langle\Sigma\rangle$. As usual we only consider modal formulas and blocks.

($\langle\Sigma\rangle \in \Gamma_n$) Then $n \in \mathcal{W} \setminus \mathcal{W}^i$. Moreover, given a block $\langle\Pi\rangle$ maximal for n, $\mathsf{set}(\Sigma) \subseteq \mathsf{set}(\Pi)$. We show that $\mathcal{R}(n) = \Pi^+ \subseteq [\![\bigwedge \Sigma]\!]$, which implies $n \Vdash \Box \bigwedge \Sigma$. If $m \in \Pi^+$, then $\mathsf{set}(\Pi) \subseteq \Gamma_m$, then for all $A \in \Pi$, $A \in \Gamma_m$, and by i.h., $m \Vdash A$. Thus for all $A \in \Sigma$, $m \Vdash A$, that is $m \Vdash \bigwedge \Sigma$.

($A \in \Gamma_n$) By saturation of $\mathsf{L}\Box$, $\langle A\rangle \in \Gamma_n$. Then by i.h., $n \Vdash \Box A$.

($A \in \Delta_n$) If there is no block in Γ_n, then $n \in \mathcal{W}^i$, and by definition $n \nVdash \Box A$. Otherwise, let $\langle\Sigma\rangle$ be a maximal block for n. Then by saturation of rule $\mathsf{R}\Box\mathsf{m}$ there is $m \in \mathcal{W}$ such that $\mathsf{set}(\Sigma) \subseteq \Gamma_m$ and $A \in \Delta_m$. Thus $m \in \Sigma^+ = \mathcal{R}(n)$, and by i.h., $m \nVdash A$, therefore $\mathcal{R}(n) \not\subseteq [\![A]\!]$, which implies $n \nVdash \Box A$. $\Box$

As examples, we show failed proofs of axiom 4 in $\mathbf{H}.\mathbf{MC}$ and $\mathbf{H}.\mathbf{MCN}$ and the extracted countermodels.

Example 6.4.4 (Proof search for axiom 4 in $\mathbf{H}.\mathbf{MC}$ and countermodels)**.** A failed proof of 4 in $\mathbf{H}.\mathbf{MC}$ is as follows.

$$\dfrac{\dfrac{\begin{array}{c}\textbf{saturated}\\ \Box p, \langle p\rangle \Rightarrow \Box\Box p \mid p \Rightarrow \Box p\end{array}}{\Box p, \langle p\rangle \Rightarrow \Box\Box p}\,\mathsf{R}\Box\mathsf{m}}{\Box p \Rightarrow \Box\Box p}\,\mathsf{L}\Box$$

Let: $1 \mapsto \Box p, \langle p\rangle \Rightarrow \Box\Box p$; and $2 \mapsto p \Rightarrow \Box p$.

<u>Bi-neighbourhood countermodel</u>. From Definition 6.4.3 we obtain the following model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$. $\mathcal{V}(p) = \{2\}$. $\mathcal{N}(1) = \{(p^+, p^-)\} = \{(\{2\}, \emptyset)\}$, and $\mathcal{N}(2) = \emptyset$. We have $\{2\} \subseteq [\![p]\!] = \{2\} \subseteq \mathcal{W} \setminus \emptyset$, then $1 \Vdash \Box p$, but $\{2\} \not\subseteq [\![\Box p]\!] = \{1\}$, then $1 \nVdash \Box\Box p$.

<u>Relational countermodel</u>. From Definition 6.4.5 we obtain the following model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$: $\mathcal{W} = \{1, 2\}$ and $\mathcal{W}^i = \{2\}$. $\mathcal{V}(p) = \{2\}$. $\mathcal{R}(1) = p^+ = \{2\}$. Since $2 \Vdash p$ we have $1 \Vdash \Box p$. Moreover, since $2 \in \mathcal{W}^i$, by definition $2 \nVdash \Box p$, then $1 \nVdash \Box\Box p$.

Example 6.4.5 (Proof search for axiom 4 in $\mathbf{H}.\mathbf{MCNT}$ and countermodels)**.** A failed proof of 4 in $\mathbf{H}.\mathbf{MCNT}$ is as follows.

$$
\begin{array}{cl}
\textbf{saturated} & \\
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p \mid \top, \langle\top\rangle \Rightarrow p & \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p \mid \top \Rightarrow p & \mathsf{N} \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top, \langle\top\rangle \Rightarrow \Box p & \mathsf{R\Box m} \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p \mid p, \top \Rightarrow \Box p & \mathsf{N} \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p & \mathsf{R\Box m} \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle \Rightarrow \Box\Box p & \mathsf{T} \\
\hline
\Box p, \langle p\rangle, \langle\top\rangle \Rightarrow \Box\Box p & \mathsf{C} \\
\hline
\Box p, \langle p\rangle \Rightarrow \Box\Box p & \mathsf{N} \\
\hline
\Box p \Rightarrow \Box\Box p & \mathsf{L\Box}
\end{array}
$$

Let: $1 \mapsto \Box p, \langle p\rangle, \langle\top\rangle, \langle p,\top\rangle, p, \top \Rightarrow \Box\Box p$. $2 \mapsto p, \top, \langle\top\rangle \Rightarrow \Box p$. $3 \mapsto \top, \langle\top\rangle \Rightarrow p$.

Bi-neighbourhood countermodel. From Definition 6.4.3 we obtain the following model $\mathcal{M}_{bi} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V}\rangle$: $\mathcal{W} = \{1,2,3\}$. $\mathcal{V}(p) = \{1,2\}$. $\mathcal{N}(1) = \{(p^+, p^-), (\top^+, \top^-), (p, \top^+; p, \top^-)\} = \{(\{1,2\},\{3\}), (\{1,2,3\},\emptyset)\}$. $\mathcal{N}(2) = \mathcal{N}(3) = \{(\top^+, \top^-)\} = \{(\{1,2,3\},\emptyset)\}$. It is easy to see that $\mathcal{M}_{bi}$ is a MCNT-model. Moreover, $1 \Vdash \Box p$ and $1 \not\Vdash \Box\Box p$, then $1 \not\Vdash \Box p \to \Box\Box p$.

Relational countermodel. From Definition 6.4.5 we obtain the following model $\mathcal{M}_r = \langle \mathcal{W}, \mathcal{W}^i, \mathcal{R}, \mathcal{V}\rangle$: $\mathcal{W} = \{1,2,3\}$ and $\mathcal{W}^i = \emptyset$. $\mathcal{V}(p) = \{1,2\}$. $\mathcal{R}(1) = (p,\top)^+ = \{1,2\}$; and $\mathcal{R}(2) = \mathcal{R}(3) = \top^+ = \{1,2,3\}$. Since $1 \Vdash p$ and $2 \Vdash p$, $1 \Vdash \Box p$. But $3 \not\Vdash p$, then $2 \not\Vdash \Box p$, thus $1 \not\Vdash \Box\Box p$. Then we have $1 \not\Vdash \Box p \to \Box\Box p$. Notice that $\mathcal{R}$ is reflexive but is not transitive, as $1\mathcal{R}2$, $2\mathcal{R}3$, but not $1\mathcal{R}3$.

Observe that in the above Examples 6.4.4 and 6.4.5, the relational models directly extracted from the saturated hypersequents are the same models that one obtains by applying the transformation in Proposition 4.3.5 to the bi-neighbourhood models.

6.5 Hypersequent calculus for agency and ability logic

In this section, we present a hypersequent calculus for Elgesem's agency and ability logic [47] (see the axiomatisation in Section 2.4). To this aim, we take advantage of the modularity of our hypersequent calculi $\mathbf{H.E}^*$ for classical non-normal modal logics. As a matter of fact, apart from the axioms $Int_{\mathbb{E}\mathbb{C}}$ and $Q_{\mathbb{C}}$, the modal axioms and rules of **ELG** are already covered by the calculi $\mathbf{H.E}^*$. Thus, in order to define the calculus for **ELG** we consider the set of hypersequent rules corresponding to these axioms, and give additional rules for the remaining axioms. In this section, we present the hypersequent calculus **H.ELG** for **ELG** and prove that it is complete with respect to the axiomatisation. Moreover, we prove that the calculus is semantically complete by directly extracting bi-neighbourhood countermodels for Elgesem's logic (cf. Section 4.6) from failed proofs.

In order to define the hypersequent calculus **H.ELG** for **ELG**, we consider blocks of the form $\langle\Sigma\rangle_i^{\mathbb{E}}$ or $\langle\Sigma\rangle_i^{\mathbb{C}}$, where Σ is a multiset of formulas of $\mathcal{L}_{Elg}$ and i is an agent. As in the

$$\mathsf{L}_{\mathbb{E}}\ \frac{G \mid \Gamma, \mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}} \Rightarrow \Delta}{G \mid \Gamma, \mathbb{E}_i A \Rightarrow \Delta} \qquad \mathsf{L}_{\mathbb{C}}\ \frac{G \mid \Gamma, \mathbb{C}_i A, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \Delta}{G \mid \Gamma, \mathbb{C}_i A \Rightarrow \Delta} \qquad \mathsf{Int}_{\mathbb{E}\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

$$\mathsf{R}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{E}_i A, \Delta}$$

$$\mathsf{R}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A, \Delta}$$

$$\mathsf{C}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}}, \langle \Sigma, \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \langle \Pi \rangle_i^{\mathbb{E}} \Rightarrow \Delta} \qquad \mathsf{T}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}}, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{E}} \Rightarrow \Delta}$$

$$\mathsf{Q}_{\mathbb{C}}\ \frac{\{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta \mid \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta} \qquad \mathsf{P}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta}$$

Figure 6.4: Modal rules of **H.ELG**.

calculi **H.E***, sequents are pairs $\Gamma \Rightarrow \Delta$, where Γ is a multiset of formulas and blocks, and Δ is a multiset of formulas, whereas hypersequents are multisets of sequents. The formula interpretation of sequents of **H.ELG** is as follows:

$$i(A_1, ..., A_n, \langle \Sigma_1 \rangle_{a_1}^{\mathbb{E}}, ..., \langle \Sigma_m \rangle_{a_m}^{\mathbb{E}}, \langle \Pi_1 \rangle_{b_1}^{\mathbb{C}}, ..., \langle \Pi_k \rangle_{b_k}^{\mathbb{C}} \Rightarrow B_1, ..., B_\ell)$$
$$=$$
$$\bigwedge_{i \leq n} A_i \wedge \bigwedge_{j \leq m} \mathbb{E}_{a_j} \bigwedge \Sigma_j \wedge \bigwedge_{s \leq k} \mathbb{C}_{a_s} \bigwedge \Pi_s \to \bigvee_{t \leq \ell} B_t.$$

The calculus **H.ELG** is defined by the propositional rules of hypersequent calculi in Figure 6.1, plus the modal rules in Figure 6.4. For every axiom of **ELG** there is a corresponding rule in the calculus. Most rules have been already considered in the calculi **H.E***, the only new rules are $\mathsf{Int}_{\mathbb{E}\mathbb{C}}$ and $\mathsf{Q}_{\mathbb{C}}$. Observe that $\mathbb{E}$-blocks can be merged by means of the rule $\mathsf{C}_{\mathbb{E}}$, but there is no analogous rule for $\mathbb{C}$-blocks. However, once complex $\mathbb{E}$-blocks are created, they can be converted into $\mathbb{C}$-blocks by means of the rule $\mathsf{Int}_{\mathbb{E}\mathbb{C}}$. In general, blocks allow us to encode in a simple (and analytic) way the relation between the modalities $\mathbb{E}$ and $\mathbb{C}$. For the derivations of most axioms we can refer to Figure 6.2, whereas axioms $Int_{\mathbb{E}\mathbb{C}}$ and $Q_{\mathbb{C}}$ are derivable as follows.

$$\dfrac{\dfrac{\dfrac{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A \mid A \Rightarrow A \qquad \mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A \mid A \Rightarrow A}{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}}, \langle A \rangle_i^{\mathbb{C}} \Rightarrow \mathbb{C}_i A}\ \mathsf{R}_{\mathbb{C}}}{\mathbb{E}_i A, \langle A \rangle_i^{\mathbb{E}} \Rightarrow \mathbb{C}_i A}\ \mathsf{Int}_{\mathbb{E}\mathbb{C}}}{\mathbb{E}_i A \Rightarrow \mathbb{C}_i A}\ \mathsf{L}_{\mathbb{E}} \qquad \dfrac{\dfrac{\dfrac{}{\mathbb{C}_i \top, \langle \top \rangle_i^{\mathbb{C}} \Rightarrow \ \mid\ \Rightarrow \top}\ \mathsf{R}\top}{\mathbb{C}_i \top, \langle \top \rangle_i^{\mathbb{C}} \Rightarrow}\ \mathsf{Q}_{\mathbb{C}}}{\mathbb{C}_i \top \Rightarrow}\ \mathsf{L}_{\mathbb{E}}$$

We can prove that **H.ELG** is sound with respect to the bi-neighbourhood models for **ELG**.

Theorem 6.5.1 (Soundness)**.** If H is derivable in **H.ELG**, then it is valid in all bi-neighbourhood models for **ELG**.

Proof. As usual, we have to show that the initial sequents are valid, and that whenever the premiss(es) of a rule are valid, so is the conclusion. The proof proceeds as for Theorem 6.1.2, here we only consider the rules $\mathsf{Int}_{\mathbb{E}\mathbb{C}}$ and $\mathsf{Q}_{\mathbb{C}}$ which are not already covered by the hypersequent calculi for classical non-normal modal logics.

($\mathsf{Int}_{\mathbb{E}\mathbb{C}}$) Assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models \Gamma, \langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle_i^{\mathbb{E}}, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta)$, which is equivalent to $\mathbb{E}_i \bigwedge \Sigma \wedge \mathbb{C}_i \bigwedge \Sigma \to i(\Gamma \Rightarrow \Delta)$. By the validity of axiom $Int_{\mathbb{E}\mathbb{C}}$, this is in turn equivalent to $\mathbb{E}_i \bigwedge \Sigma \to i(\Gamma \Rightarrow \Delta)$. Therefore $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle_i^{\mathbb{E}} \Rightarrow \Delta)$.

($\mathsf{Q}_{\mathbb{C}}$) Assume $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta \mid \Rightarrow B$ for all $B \in \Sigma$. Then (i) $\mathcal{M} \models G$, or (ii) $\mathcal{M} \models \Gamma, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta$, or (iii) $\mathcal{M} \models\ \Rightarrow B$ for all $B \in \Sigma$. If (i) or (ii) we are done. If (iii), then $\mathcal{M} \models \bigwedge \Sigma$, that is $\mathcal{M} \models \bigwedge \Sigma \leftrightarrow \top$. By axiom $Q_{\mathbb{C}}$, $\mathcal{M} \models \mathbb{C}_i \bigwedge \Sigma \to \bot = i(\langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow)$. Then $\mathcal{M} \models \Gamma, \langle\Sigma\rangle_i^{\mathbb{C}} \Rightarrow \Delta$. □

We now move to prove the syntactic completeness of the calculus. Similarly to $\mathbf{H.E}^*$, we can prove that weakening and contraction, both in their internal and external variants, as well as cut are admissible in **H.ELG**. To this purpose we must consider the following definition of weight of formulas and blocks of **H.ELG**.

Definition 6.5.1. The weight wg of a formula or block is recursively defined as $wg(\bot) = wg(\top) = wg(p) = 0$; for $\circ \in \{\wedge, \vee, \to\}$, $wg(A \circ B) = wg(A) + wg(B) + 1$; $wg(\langle A_1, ..., A_n\rangle_i^{\mathbb{E}}) = wg(\langle A_1, ..., A_n\rangle_j^{\mathbb{C}}) = max\{wg(A_1), ..., wg(A_n)\} + 1$; $wg(\Box A) = wg(A) + 2$.

The proofs essentially extend the ones in Section 6.2. Here we concentrate on the admissibility of cut. As before, for admissibility of cut we must consider a rule for substitution of equivalent formulas inside blocks, which is now formulated as follows.

$$\mathsf{sub}_{\mathsf{ELG}}\ \frac{G \mid \Sigma \Rightarrow A \qquad \{G \mid A \Rightarrow B\}_{B \in \Sigma} \qquad G \mid \overrightarrow{\langle A^n, \Pi\rangle_i^{\mathbb{E}}}, \overrightarrow{\langle A^m, \Omega\rangle_j^{\mathbb{C}}}, \Gamma \Rightarrow \Delta}{G \mid \overrightarrow{\langle \Sigma^n, \Pi\rangle_i^{\mathbb{E}}}, \overrightarrow{\langle \Sigma^m, \Omega\rangle_j^{\mathbb{C}}}, \Gamma \Rightarrow \Delta}$$

where for instance $\overrightarrow{\langle A^n, \Pi\rangle_i^{\mathbb{E}}}$ stays for $\langle A^{n_1}, \Pi_1\rangle_{i_1}^{\mathbb{E}}, ..., \langle A^{n_k}, \Pi_k\rangle_{i_k}^{\mathbb{E}}$, and A^{n_ℓ} is a compact way to denote n_ℓ occurrences of A.

Theorem 6.5.2 (Admissibility of structural rules and cut elimination)**.** The structural rules of weakening and contraction are height-preserving admissible in **H.ELG**. Moreover, the rules cut and $\mathsf{sub}_{\mathsf{ELG}}$ are admissible in **H.ELG**.

Proof. We concentrate on the admissibility of cut. Let $Cut(c, h)$ mean that all applications of cut of height h on a cut formula of weight c are admissible, and $Sub(c)$ mean that all applications of $\mathsf{sub}_{\mathsf{ELG}}$ where A has weight c are admissible. Then the theorem is a consequence of the following claims: **(A)** $\forall c.Cut(c, 0)$; **(B)** $\forall h.Cut(0, h)$; **(C)** $\forall c.(\forall h.Cut(c, h) \to Sub(c))$;

(D) $\forall c.\forall h.\ ((\forall c' < c.(Sub(c') \wedge \forall h'.Cut(c', h')) \wedge \forall h'' < h.Cut(c, h'')) \rightarrow Cut(c, h))$. The proof is analogous to the one of Theorem 6.2.2. Here we only show **(C)** in the cases where the last rule applied is $\mathsf{Int}_{\mathbb{EC}}$ or $\mathsf{Q}_{\mathbb{C}}$, as these two rules do not have a counterpart in calculi $\mathbf{H.E}^*$.

- The last rule applied is $\mathsf{Int}_{\mathbb{EC}}$:

$$\frac{G \mid \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{E}}, \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta}{G \mid \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{E}}, \Gamma \Rightarrow \Delta} \ \mathsf{Int}_{\mathbb{EC}}$$

By applying the inductive hypothesis to the premiss we obtain $G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{E}}, \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta$. Then by $\mathsf{Int}_{\mathbb{EC}}$ we derive $G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta$.

- The last rule applied is $\mathsf{Q}_{\mathbb{C}}$:

$$\frac{\{G \mid \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta \mid \Rightarrow A\}_1^{n_k} \qquad \{G \mid \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta \mid \Rightarrow C\}_{C \in \Pi_i}}{G \mid \langle A^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta} \ \mathsf{Q}_{\mathbb{C}}$$

By applying the inductive hypothesis to the premisses (after auxiliary applications of Ewk to the other premisses of $\mathsf{sub}_{\mathsf{ELG}}$) we obtain $\{G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta \mid \Rightarrow A\}_1^{n_k}$ and $\{G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta \mid \Rightarrow C\}_{C \in \Pi_i}$. By considering $\Gamma \mid A \Rightarrow B$ for all $B \in \Sigma$, by several applications of cut (each time with auxiliary applications of wk) we obtain $\{\{G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta \mid \Rightarrow B\}_{B \in \Sigma}\}_1^{n_k}$. Finally, by $\mathsf{Q}_{\mathbb{C}}$ we derive $G \mid \langle \Sigma^{n_k}, \Pi_k \rangle_i^{\mathbb{C}}, \Gamma \Rightarrow \Delta$. □

As a consequence of admissibility of cut we can prove the following completeness theorem.

Theorem 6.5.3 (Syntactic completeness)**.** If A is derivable in **ELG**, then $\Rightarrow A$ is derivable in **H.ELG**.

Proof. All modal axioms and rules of **ELG** are derivable in **H.ELG**: on p. 162 we have shown the derivations of axioms $Int_{\mathbb{EC}}$ and $Q_{\mathbb{C}}$, whereas for the other modal axioms and rules we can refer to Figure 6.2. Finally, MP is simulated by cut, which has been proved admissible, in the usual way. □

We adopt for **H.ELG** the proof search strategy already considered in Section 6.4 for the calculi $\mathbf{H.E}^*$ (Definition 6.4.2). To this purpose, we consider the following saturation conditions for the rules of **H.ELG**:

Definition 6.5.2 (Saturated hypersequent)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent occurring in a proof for H'. We say that H is *saturated* with respect to an application of a rule R of **H.ELG** if it satisfies the saturation condition (R) below, and that it is saturated with respect to **H.ELG** if it is saturated with respect to all possible applications of any rule of **H.ELG**. For the propositional rules and the rules $\mathsf{L}_{\mathbb{E}}$, $\mathsf{L}_{\mathbb{C}}$, $\mathsf{R}_{\mathbb{E}}$, $\mathsf{R}_{\mathbb{C}}$, $\mathsf{C}_{\mathbb{E}}$, $\mathsf{T}_{\mathbb{E}}$, and $\mathsf{P}_{\mathbb{C}}$, the saturation conditions are as in Definition 6.4.1 (but properly formulated with blocks $\langle \Sigma \rangle_i^{\mathbb{E}}$ or $\langle \Sigma \rangle_i^{\mathbb{C}}$), whereas for $\mathsf{Q}_{\mathbb{C}}$ and $\mathsf{Int}_{\mathbb{EC}}$ they are as follows: $(\mathsf{Q}_{\mathbb{C}})$ If $\Gamma, \langle \Sigma \rangle_i^{\mathbb{C}} \Rightarrow \Delta$ is in H, then there is $\Gamma' \Rightarrow B, \Delta'$ in H for some $B \in \Sigma$; and ($\mathsf{Int}_{\mathbb{EC}}$) If $\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$, then there is $\langle\Omega\rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$.

We can show that proof search in **H.ELG** always terminates.

Proposition 6.5.4. Every branch of a proof of a hypersequent H in **H.ELG** built in accordance with the strategy is finite, whence the proof search procedure for H always terminates. Moreover, every branch ends either with an initial hypersequent or a saturated one.

Proof. Let $\mathfrak{P}$ be a proof of H in **H.ELG**. Then all formulas occurring in $\mathfrak{P}$ (both inside and outside blocks) are subformulas of formulas of H, so they are finitely many. Moreover, saturation conditions prevent duplications of the same formulas (both inside and outside blocks) and same blocks. Therefore every branch of $\mathfrak{P}$ can contain only finitely many hypersequents. □

Similarly to the calculi **H.EC***, because of the presence of the rule $\mathsf{C}_{\mathbb{E}}$ hypersequents occurring in a proof of H can be exponentially large with respect to the size of H. In this respect, our decision procedure does not match the PSPACE complexity upper bound established for Elgesem's logic by Schröder and Pattinson [155] as a particular case of non-iterative modal logic, and also by Troquard [165]. An optimal calculus could be obtained by considering a non-invertible formulation of the rules (cf. the calculi **H.E**$^*_-$ in Section 6.3), but in this way we would lose the possibility to directly extract a countermodel from every single failed proof.

Given a saturated hypersequent occurring in a failed proof of H', a countermodel of H' can be directly extracted as follows.

Definition 6.5.3 (Countermodel construction)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof for H'. The model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{W}$, $\mathcal{V}$, and, for every block $\langle\Sigma\rangle_i^{\mathbb{E}}$ or $\langle\Sigma\rangle_i^{\mathbb{C}}$, Σ^+ and Σ^-, are defined as in Definition 6.4.3.
- For every agent $i \in \mathcal{A}$ and every world $n \in \mathcal{W}$,
  $\mathcal{N}_i^{\mathbb{E}}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n\}$ and $\mathcal{N}_i^{\mathbb{C}}(n) = \{(\Sigma^+, \Sigma^-) \mid \langle\Sigma\rangle_i^{\mathbb{C}} \in \Gamma_n\}$.

Lemma 6.5.5. Let $\mathcal{M}$ be defined as in Definition 6.5.3. Then for every A, $\langle\Sigma\rangle_i^{\mathbb{E}}$, $\langle\Pi\rangle_j^{\mathbb{C}}$ and every $n \in \mathcal{W}$, we have: If $A \in \Gamma_n$, then $n \Vdash A$; if $\langle\Sigma\rangle_i^{\mathbb{E}} \in \Gamma_n$, then $n \Vdash \mathbb{E}_i \bigwedge \Sigma$; if $\langle\Pi\rangle_j^{\mathbb{C}} \in \Gamma_n$, then $n \Vdash \mathbb{C}_j \bigwedge \Pi$; and if $A \in \Delta_n$, then $n \nVdash A$. Moreover, $\mathcal{M}$ is a bi-neighbourhood model for **ELG**.

Proof. The proof is as for Lemma 6.4.1, with the new conditions proved as follows.

($Q_{\mathbb{C}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(n)$, then there is $\langle\Sigma\rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{Q}_{\mathbb{C}}$, there is $m \in \mathcal{W}$ such that $\Sigma \cap \Delta_m \neq \emptyset$. Then $m \in \Sigma^- = \beta$, that is $\beta \neq \emptyset$.

($Int_{\mathbb{EC}}$) If $(\alpha, \beta) \in \mathcal{N}_i^{\mathbb{E}}(n)$, then there is $\langle \Sigma \rangle_i^{\mathbb{E}} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. By saturation of rule $\mathsf{Int}_{\mathbb{EC}}$, then there is $\langle \Omega \rangle_i^{\mathbb{C}} \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$. Then $(\Omega^+, \Omega^-) = (\Sigma^+, \Sigma^-) = (\alpha, \beta) \in \mathcal{N}_i^{\mathbb{C}}(n)$. $\square$

We then obtain the following completeness theorem.

Theorem 6.5.6 (Semantic completeness)**.** If H is valid in all bi-neighbourhood models for **ELG**, then it is derivable in **H.ELG**.

As an example of countermodel extraction we show the failure of *delegation* in Elgesem's logic. The treatment of delegation represents a main difference between Elgesem's account of agency and other accounts, such as for instance the one formalised by STIT-logic [15, 91]. It is explicitly rejected by Elgesem [47]: "a person is normally not considered the agent of some consequence of his action if another agent interferes in the causal chain." For instance, we can say that having the car repaired is not the same as repairing the car by yourself, as shown in the example below.

Example 6.5.1 (Failure of delegation)**.** Let us represent Anna by a, Beatrice by b, and "repairing the car" by p. Then by using our calculus we automatically obtain the following countermodel to "If Anna gets Beatrice to repair her car, then Anna repairs her car".

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\ldots \mid \langle p \rangle_b^{\mathbb{E}}, p, \mathbb{E}_b p \Rightarrow p$}
\LeftLabel{$\mathsf{T}_{\mathbb{E}}$}
\UnaryInfC{$\ldots \mid \langle p \rangle_b^{\mathbb{E}}, \mathbb{E}_b p \Rightarrow p$}
\LeftLabel{$\mathsf{L}_{\mathbb{E}}$}
\UnaryInfC{$\ldots \mid \mathbb{E}_b p \Rightarrow p$}
\AxiomC{\textbf{saturated}}
\noLine
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \langle p \rangle_b^{\mathbb{E}}, \langle \mathbb{E}_b p \rangle_a^{\mathbb{C}}, \langle p \rangle_b^{\mathbb{C}}, p, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p \mid \Rightarrow p$}
\RightLabel{$\mathsf{Q}_{\mathbb{C}}$}
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \langle p \rangle_b^{\mathbb{E}}, \langle \mathbb{E}_b p \rangle_a^{\mathbb{C}}, \langle p \rangle_b^{\mathbb{C}}, p, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p$}
\doubleLine
\RightLabel{$\mathsf{Int}_{\mathbb{EC}} \times 2$}
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \langle p \rangle_b^{\mathbb{E}}, p, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p$}
\RightLabel{$\mathsf{T}_{\mathbb{E}}$}
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \langle p \rangle_b^{\mathbb{E}}, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p$}
\RightLabel{$\mathsf{L}_{\mathbb{E}}$}
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p$}
\RightLabel{$\mathsf{T}_{\mathbb{E}}$}
\UnaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p \mid p \Rightarrow \mathbb{E}_b p$}
\RightLabel{$\mathsf{R}_{\mathbb{E}}$}
\BinaryInfC{$\langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p$}
\RightLabel{$\mathsf{L}_{\mathbb{E}}$}
\UnaryInfC{$\mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p$}
\end{prooftree}
$$

We consider the following enumeration of the components of the saturated hypersequent:

$$1 \mapsto \langle \mathbb{E}_b p \rangle_a^{\mathbb{E}}, \langle p \rangle_b^{\mathbb{E}}, \langle \mathbb{E}_b p \rangle_a^{\mathbb{C}}, \langle p \rangle_b^{\mathbb{C}}, p, \mathbb{E}_b p, \mathbb{E}_a \mathbb{E}_b p \Rightarrow \mathbb{E}_a p. \qquad 2 \mapsto p \Rightarrow \mathbb{E}_b p. \qquad 3 \mapsto\ \Rightarrow p.$$

According to the construction in Definition 6.5.3, we obtain the following countermodel $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}_i^{\mathbb{E}}, \mathcal{N}_i^{\mathbb{C}}, \mathcal{V} \rangle$: $\mathcal{W} = \{1, 2, 3\}$. $\mathcal{V}(p) = \{1, 2\}$. $\mathcal{N}_a^{\mathbb{E}}(1) = \mathcal{N}_a^{\mathbb{C}}(1) = \{(\{1\}, \{2\})\}$, because $\mathcal{N}_a^{\mathbb{E}}(1) = \mathcal{N}_a^{\mathbb{C}}(1) = \{(\mathbb{E}_b p^+, \mathbb{E}_b p^-)\}$, $\mathbb{E}_b p^+ = \{1\}$, and $\mathbb{E}_b p^- = \{2\}$. $\mathcal{N}_b^{\mathbb{E}}(1) = \mathcal{N}_b^{\mathbb{C}}(1) = \{(\{1, 2\}, \{3\})\}$, because $\mathcal{N}_b^{\mathbb{E}}(1) = \mathcal{N}_b^{\mathbb{C}}(1) = \{(p^+, p^-)\}$, $p^+ = \{1, 2\}$, and $p^- = \{3\}$. $\mathcal{N}_i^{\mathbb{E}}(n) = \mathcal{N}_i^{\mathbb{C}}(n) = \emptyset$ for $i = a, b$ and $n = 2, 3$.

We have $1 \Vdash \mathbb{E}_b p$ because $(\{1, 2\}, \{3\}) \in \mathcal{N}_b^{\mathbb{E}}(1)$ and $\{1, 2\} \subseteq [\![p]\!] \subseteq \mathcal{W} \setminus \{3\}$; moreover, $2 \nVdash \mathbb{E}_b p$ and $3 \nVdash \mathbb{E}_b p$ because $\mathcal{N}_b^{\mathbb{E}}(2) = \mathcal{N}_b^{\mathbb{E}}(3) = \emptyset$, thus $[\![\mathbb{E}_b p]\!] = \{1\}$. Then we have $1 \Vdash \mathbb{E}_a \mathbb{E}_b p$ because $(\{1\}, \{2\}) \in \mathcal{N}_a^{\mathbb{E}}(1)$ and $\{1\} \subseteq [\![\mathbb{E}_p]\!] \subseteq \mathcal{W} \setminus \{2\}$. But $1 \nVdash \mathbb{E}_a p$ because $[\![p]\!] \not\subseteq \mathcal{W} \setminus \{2\}$. Therefore $1 \nVdash \mathbb{E}_a \mathbb{E}_b p \to \mathbb{E}_a p$.

$$\mathsf{L}_{\mathbb{E}}\ \frac{G \mid \Gamma, \mathbb{E}_g A, \langle A \rangle^{\mathbb{E}}_g \Rightarrow \Delta}{G \mid \Gamma, \mathbb{E}_g A \Rightarrow \Delta} \qquad \mathsf{L}_{\mathbb{C}}\ \frac{G \mid \Gamma, \mathbb{C}_g A, \langle A \rangle^{\mathbb{C}}_g \Rightarrow \Delta}{G \mid \Gamma, \mathbb{C}_g A \Rightarrow \Delta} \qquad \mathsf{Int}^1_{\mathbb{EC}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g \Rightarrow \Delta}$$

$$\mathsf{R}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g \Rightarrow \mathbb{E}_g A, \Delta}$$

$$\mathsf{R}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta \mid \Sigma \Rightarrow A \qquad \{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta \mid A \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \mathbb{C}_g A, \Delta}$$

$$\mathsf{C}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g, \langle \Pi \rangle^{\mathbb{E}}_g, \langle \Sigma, \Pi \rangle^{\mathbb{E}}_g \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g, \langle \Pi \rangle^{\mathbb{E}}_g \Rightarrow \Delta} \qquad \mathsf{T}_{\mathbb{E}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g, \Sigma \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_g \Rightarrow \Delta}$$

$$\mathsf{Q}_{\mathbb{C}}\ \frac{\{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \Delta \mid \ \Rightarrow B\}_{B \in \Sigma}}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \Delta} \qquad \mathsf{P}_{\mathbb{C}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \Delta \mid \Sigma \Rightarrow}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_g \Rightarrow \Delta}$$

$$\mathsf{F}_{\mathbb{C}}\ \frac{}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{C}}_\emptyset \Rightarrow \Delta} \qquad \mathsf{Int}^2_{\mathbb{EC}}\ \frac{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_{g_1}, \langle \Pi \rangle^{\mathbb{E}}_{g_2}, \langle \Sigma, \Pi \rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_{g_1}, \langle \Pi \rangle^{\mathbb{E}}_{g_2} \Rightarrow \Delta}$$

Figure 6.5: Modal rules of **H.COAL**.

6.6 Hypersequent calculus for coalition logic

In this section we present the hypersequent calculus **H.COAL** for Troquard's coalition logic **COAL** [165] (see the axiomatisation in Section 2.4). We show that the calculus is semantically complete with respect to the bi-neighbourhood models for **COAL** (cf. Definition 4.6.2) by directly extracting bi-neighbourhood countermodels of non-derivable hypersequents from failed proofs.

The hypersequent calculus **H.COAL** is defined by the propositional rules in Figure 6.1 and the modal rules in Figure 6.5. As for **H.ELG** (cf. Section 6.5), each axiom of **COAL** has a corresponding rule in the calculus. We show as an example the derivation of axiom $Int^2_{\mathbb{EC}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{..., \langle A, B \rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B) \mid A, B \Rightarrow A \wedge B \qquad ... \mid A \wedge B \Rightarrow A \qquad ... \mid A \wedge B \Rightarrow B}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B, \langle A \rangle^{\mathbb{E}}_{g_1}, \langle B \rangle^{\mathbb{E}}_{g_2}, \langle A, B \rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{R}_{\mathbb{C}}}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B, \langle A \rangle^{\mathbb{E}}_{g_1}, \langle B \rangle^{\mathbb{E}}_{g_2} \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{Int}^2_{\mathbb{EC}}}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B, \mathbb{E}_{g_1} A, \mathbb{E}_{g_2} B \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{L}_{\mathbb{E}} \times 2}{\mathbb{E}_{g_1} A \wedge \mathbb{E}_{g_2} B \Rightarrow \mathbb{C}_{g_1 \cup g_2}(A \wedge B)}\ \mathsf{L}\wedge$$

Theorem 6.6.1 (Soundness)**.** If H is derivable in **H.COAL**, then it is valid in all bi-neighbourhood models for **COAL**.

Proof. We only consider the rules $\mathsf{F}_{\mathbb{C}}$ and $\mathsf{Int}^2_{\mathbb{EC}}$.

($\mathsf{Int}^2_{\mathbb{EC}}$) Assume $\mathcal{M} \models G \mid \Gamma, \langle \Sigma \rangle^{\mathbb{E}}_{g_1}, \langle \Pi \rangle^{\mathbb{E}}_{g_2}, \langle \Sigma, \Pi \rangle^{\mathbb{C}}_{g_1 \cup g_2} \Rightarrow \Delta$. Then $\mathcal{M} \models G$ or $\mathcal{M} \models$

$\Gamma, \langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2}, \langle\Sigma,\Pi\rangle^{\mathbb{C}}_{g_1\cup g_2} \Rightarrow \Delta$. In the first case we are done. In the second case, $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2}, \langle\Sigma,\Pi\rangle^{\mathbb{C}}_{g_1\cup g_2} \Rightarrow \Delta)$, which is equivalent to $\mathbb{E}_{g_1} \bigwedge\Sigma \wedge \mathbb{E}_{g_2} \bigwedge\Pi \wedge \mathbb{C}_{g_1\cup g_2}(\bigwedge\Sigma \wedge \bigwedge\Pi) \to i(\Gamma \Rightarrow \Delta)$. By the validity of axiom $Int^2_{\mathbb{EC}}$, this is in turn equivalent to $\mathbb{E}_{g_1} \bigwedge\Sigma \wedge \mathbb{E}_{g_2} \bigwedge\Pi \to i(\Gamma \Rightarrow \Delta)$. Therefore $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2} \Rightarrow \Delta)$.

$(\mathsf{F}_{\mathbb{C}})$ Due to the validity of axiom $F_{\mathbb{C}}$, it is never the case that $w \Vdash \mathbb{C}_\emptyset \bigwedge\Sigma$. Then $\mathcal{M} \models i(\Gamma, \langle\Sigma\rangle^{\mathbb{C}}_\emptyset \Rightarrow \Delta)$ for every Γ, Δ. Therefore $\mathcal{M} \models G \mid \Gamma, \langle\Sigma\rangle^{\mathbb{C}}_\emptyset \Rightarrow \Delta$. □

We consider a proof search strategy in **H.COAL** analogous to the one in the calculi **H.E*** and **H.ELG** (cf. Definition 6.4.2). To this purpose, we consider the following saturation conditions for the rules of **H.COAL**:

Definition 6.6.1 (Saturated hypersequent)**.** Let $H = \Gamma_1 \Rightarrow \Delta_1 \mid ... \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent occurring in a proof for H'. We say that H is *saturated* with respect to an application of a rule R of **H.COAL** if it satisfies the corresponding saturation condition (R), and that it is saturated with respect to **H.COAL** if it is saturated with respect to all possible applications of any rule of **H.COAL**. We consider the saturation considitions already considered for the rules of **H.ELG** in Definition 6.5.2 (but properly formulated with groups g instead of agents i), plus the following conditions for $\mathsf{F}_{\mathbb{C}}$ and $\mathsf{Int}^2_{\mathbb{EC}}$. $(\mathsf{F}_{\mathbb{C}})$ $\langle\Sigma\rangle^{\mathbb{C}}_\emptyset \notin \Gamma_n$, and $(\mathsf{Int}^2_{\mathbb{EC}})$ if $\langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2} \in \Gamma_n$, then $\langle\Omega\rangle^{\mathbb{C}}_{g_1\cup g_2} \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma,\Pi)$.

In the same way as for **H.ELG**, we can prove that proof search always terminates, whence it provides a decision procedure for the logic **COAL**. As before, proof search is not optimal since the derivations can have an exponential size whereas the logic is in PSPACE, as it has been proved by Troquard [165].

Proposition 6.6.2. Every branch of a proof of a hypersequent H in **H.COAL** built in accordance with the strategy is finite, whence the proof search procedure for H always terminates. Moreover, every branch ends either with an initial hypersequent or a saturated one.

Proof. The proof is analogous to the one of Proposition 6.5.4. We observe in addition that only finitely many groups can be created starting from finitely many groups, each of them containing finitely many agents. □

We now prove that the calculus is semantically complete. As usual, the proof consists in showing that we can extract a countermodel of every non-derivable hypersequent on the basis of the information provided by the failed proof.

Theorem 6.6.3 (Semantic completeness)**.** If H is valid in all bi-neighbourhood models for **COAL**, then it is derivable in **H.COAL**.

Proof. Given a saturated hypersequent H we define a model $\mathcal{M}$ as in Definition 6.5.3 (replacing agents i with groups g). We can prove that formulas and blocks in the left-hand side of the components are satisfied in the corresponding worlds, and that formulas in the right-hand side are falsified, whence $\mathcal{M}$ is a countermodel of H. Moreover, we can prove that $\mathcal{M}$ is a bi-neighbourhood model for **COAL**. The proofs are as in Lemma 6.5.5. We only consider the following two conditions.

($Int^2_{\mathbb{EC}}$) Assume $(\alpha,\beta) \in \mathcal{N}^{\mathbb{E}}_{g_1}(n)$ and $(\gamma,\delta) \in \mathcal{N}^{\mathbb{E}}_{g_2}(n)$. If $(\alpha,\beta) \neq (\gamma,\delta)$ or $g_1 \neq g_2$, then there are $\langle\Sigma\rangle^{\mathbb{E}}_{g_1}, \langle\Pi\rangle^{\mathbb{E}}_{g_2} \in \Gamma_n$ such that $\Sigma^+ = \alpha$, $\Sigma^- = \beta$, $\Pi^+ = \gamma$ and $\Pi^- = \delta$. By saturation or rule $\mathsf{Int}^2_{\mathbb{EC}}$, there is $\langle\Omega\rangle^{\mathbb{C}}_{g_1 \cup g_2} \in \Gamma_n$ such that $\mathsf{set}(\Omega) = \mathsf{set}(\Sigma,\Pi)$, thus $(\Omega^+,\Omega^-) \in \mathcal{N}^{\mathbb{C}}_{g_1\cup g_2}(n)$, where, as shown in the proof of Lemma 6.5.5 case ($C_{\mathbb{E}}$), $\Omega^+ = \alpha \cap \gamma$ and $\Omega^- = \beta \cup \delta$. If instead $(\alpha,\beta) = (\gamma,\delta)$ and $g_1 = g_2$, then there is $\langle\Sigma\rangle^{\mathbb{E}}_{g_1} \in \Gamma_n$ such that $\Sigma^+ = \alpha$ and $\Sigma^- = \beta$. Then by saturation of rule $\mathsf{Int}_{\mathbb{EC}}$ there is $\langle\Omega\rangle^{\mathbb{C}}_i \in \Gamma_n$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Omega)$. Then $(\Omega^+,\Omega^-) = (\Sigma^+,\Sigma^-) = (\alpha,\beta) \in \mathcal{N}^{\mathbb{C}}_i(n)$.

($F_{\mathbb{C}}$) By saturation of $\mathsf{F}_{\mathbb{C}}$, there is no block $\langle\Sigma\rangle^{\mathbb{C}}_{\emptyset} \in \Gamma_n$, then $\mathcal{N}^{\mathbb{C}}_{\emptyset}(n) = \emptyset$. □

A peculiar aspect of Troquard's characterisation of coalitional agency is that in joint actions every single participant must be involved, so that the logic rejects *coalition monotonicity*: $\mathbb{E}_g A \to \mathbb{E}_{g'} A$ whenever $g \subseteq g'$ is *not* considered as valid. We conclude this section by showing a failed proof of coalition monotonicity in **H.COAL** and the extracted countermodel.

Example 6.6.1 (Failure of coalition monotonicity)**.** The formula $\mathbb{E}_{\{a\}}p \to \mathbb{E}_{\{a,b\}}p$ is not valid in **COAL**. A failed proof is as follows.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{saturated}}{\langle p\rangle^{\mathbb{E}}_{\{a\}}, \langle p\rangle^{\mathbb{C}}_{\{a\}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p \mid\ \Rightarrow p}}{\langle p\rangle^{\mathbb{E}}_{\{a\}}, \langle p\rangle^{\mathbb{C}}_{\{a\}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\,\mathsf{Q}_{\mathbb{C}}}{\langle p\rangle^{\mathbb{E}}_{\{a\}}, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\,\mathsf{Int}_{\mathbb{EC}}}{\langle p\rangle^{\mathbb{E}}_{\{a\}}, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\,\mathsf{T}_{\mathbb{E}}}{\mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p}\,\mathsf{L}_{\mathbb{E}}
$$

Let $1 \mapsto \langle p\rangle^{\mathbb{E}}_a, \langle p\rangle^{\mathbb{C}}_a, p, \mathbb{E}_{\{a\}}p \Rightarrow \mathbb{E}_{\{a,b\}}p$, and $2 \mapsto\ \Rightarrow p$. We obtain the model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}^{\mathbb{E}}_g, \mathcal{N}^{\mathbb{C}}_g, \mathcal{V}\rangle$, where $\mathcal{W} = \{1,2\}$; $\mathcal{V}(p) = \{1\}$; $\mathcal{N}^{\mathbb{E}}_{\{a\}}(1) = \mathcal{N}^{\mathbb{C}}_{\{a\}}(1) = \{(p^+,p^-)\} = \{(\{1\},\{2\})\}$; and $\mathcal{N}^{\mathbb{E}}_g(k) = \mathcal{N}^{\mathbb{C}}_g(k) = \emptyset$ for $g \neq \{a\}$ or $k \neq 1$. Then $1 \Vdash \mathbb{E}_{\{a\}}p$ because $\{1\} \subseteq [\![p]\!] \subseteq \mathcal{W} \setminus \{2\}$, but $1 \not\Vdash \mathbb{E}_{\{a,b\}}p$ because $\mathcal{N}^{\mathbb{E}}_{\{a,b\}}(1) = \emptyset$.

6.7 Implementation

In this section, we present HYPNO (HYpersequent Prover for NOn-normal modal logics), a Prolog implementation of our hypersequent calculi $\mathbf{H.E}^*$ for the systems of the classical cube.[1] The architecture of HYPNO is similar to the one of PRONOM (cf. Section 5.7): the program comprises a set of clauses, each of them implementing a sequent rule or an axiom of $\mathbf{H.E}^*$, and proof search is provided for free by the mere depth-first search mechanism of Prolog, without any additional ad hoc mechanism. Moreover, similarly to PRONOM, HYPNO implements two separate procedures: a first one that searches for a proof of an input formula, and a second one that builds a countermodel in case of failure of proof search. The proof search procedure is implemented by a predicate `terminating_proof_search` which tries to generate a derivation of the given input formula. In case of failure, on demand by the user, another predicate `build_saturated_branch` is invoked that computes a saturated hypersequent from which a countermodel is extracted.

The calculi implemented by HYPNO are a minor variant of the ones in Figure 6.1. They contain an additional arrow $\Rrightarrow$, which is used to represent that the formulas on the left of $\Rrightarrow$ entails the *conjunction* (rather than the disjunction) of the formulas on its right. Moreover, the rule $\mathsf{R}\Box$ of calculi $\mathbf{H.E}^*$ is replaced by the following three rules.

$$\mathsf{R}\Box_{\Rrightarrow}\ \frac{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid \Sigma \Rightarrow B \qquad G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B \mid B \Rrightarrow \Sigma}{G \mid \langle\Sigma\rangle, \Gamma \Rightarrow \Delta, \Box B}$$

$$\Rrightarrow_1\ \frac{G \mid A \Rightarrow B}{G \mid A \Rrightarrow B} \qquad\qquad \Rrightarrow_2\ \frac{G \mid A \Rightarrow B \qquad G \mid A \Rrightarrow \Sigma}{G \mid A \Rrightarrow B, \Sigma}\ |\Sigma| \geq 1$$

By this modification, all rules of the calculi are at most binary, which is more convenient for implementation. We point out that no rule different from $\Rrightarrow_1$ and $\Rrightarrow_2$ can be applied to sequents of the form $A \Rrightarrow \Sigma$. The equivalence of the modified calculi with the original ones is straightforward. Essentially, an application of $\mathsf{R}\Box$ in calculi $\mathbf{H.E}^*$ corresponds here to an application of $\mathsf{R}\Box_{\Rrightarrow}$ and subsequent applications of $\Rrightarrow_2$, which terminate when only one formula occurs on the right of $\Rrightarrow$. In this way the "right" premisses of $\mathsf{R}\Box$ are created once at a time.

HYPNO represents a hypersequent with a Prolog list whose elements are Prolog terms of the form `singleSeq([Gamma,Delta],Additional)`, each one representing a sequent in the hypersequent. `Gamma`, `Delta`, and `Additional` are in turn Prolog lists: `Gamma` and `Delta` represent the left side and the right side of the single sequent, respectively, whereas `Additional` keeps track of the rules already applied to each sequent in order to ensure termination by avoiding multiple redundant applications of the same rule to a given hypersequent. Elements

[1] HYPNO, as well as all the Prolog source files, including those used for the performance evaluation, are available for free usage and download at http://193.51.60.97:8000/HYPNO/.

of Gamma and Delta are either formulas or Prolog lists representing blocks. Logical symbols are represented as in PRONOM: symbols $\top$ and $\bot$ are represented by constants true and false, respectively, whereas connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\Box$ are represented by -, ^, ?, ->, and box. Moreover, the new arrow $\Rrightarrow$ is represented by =>. As an example, the Prolog list

```
[singleSeq( [[box (a ^ c), [true], [a,c]], [a, b, a -> b, box b]],
 [n, right(a -> b), apdR([a,c],b)]), singleSeq ([[d], [d]] ,[ ])]
```

is used to represent the hypersequent

$$\Box(A \wedge C), \langle\top\rangle, \langle A, C\rangle \Rightarrow A, B, A \vee B, \Box B \mid D \Rightarrow D,$$

to which the rules N, R$\vee$ and R$\Box$ have been already applied, the last one by using the block $\langle A, C\rangle$ and the formula $\Box B$ as the principal formulas. In turn, no rule has been applied to $D \Rightarrow D$ (the list Additional is empty).

Given a formula of $\mathcal{L}$ represented by the Prolog term f, HYPNO executes the main predicate of the prover, called prove, whose only two clauses implement the functioning of HYPNO: the first clause checks whether the formula is valid and, in case of failure, the second one enables the graphical interface to invoke a predicate called counter to compute a countermodel. In detail, the predicate prove first checks whether the formula is valid by executing the predicate:

```
terminating_proof_search(Hyper, ProofTree).
```

This predicate succeeds if and only if the hypersequent represented by the list Hyper is derivable in $\mathbf{H.E}^*$. When it succeeds, the output term ProofTree matches with a representation of the derivation found by the prover. As an example, in order to prove that the sequent

$$\Box(A \wedge (B \vee C)) \Rightarrow \Box((A \wedge B) \vee (A \wedge C))$$

is valid in $\mathbf{E}$, one queries HYPNO with the goal:

```
terminating_proof_search([singleSeq([[box (a ^ (b ?  c))], [box ((a ^ b) ?  (a ^ c))]], [
                                   ]), ProofTree).
```

Each clause of terminating_proof_search implements an axiom or rule of the sequent calculi $\mathbf{H.E}^*$. To search for a derivation of a sequent $\Gamma \Rightarrow \Delta$, HYPNO proceeds as follows. First, if $\Gamma \Rightarrow \Delta$ is an initial sequent, then the goal will succeed immediately by using one of the clauses implementing the zero-premisses rules. As an example, the clause implementing init is as follows:

```
terminating_proof_search(Hyper,tree(axiom,PrintableHyper,no,no)):-
   member(singleSeq([Gamma,Delta],_),Hyper),
   member(P,Gamma), member(P,Delta),!,
   extractPrintableSequents(Hyper,PrintableHyper).
```

The auxiliary predicate `extractPrintableSequents` is used just for a graphical rendering of the hypersequent. If $\Gamma \Rightarrow \Delta$ is not an instance of the axioms, then the first applicable rule will be chosen, e.g. if `Gamma` contains a list `Sigma` representing a block $\langle\Sigma\rangle \in \Gamma$, and `Delta` contains `box b` representing that $\Box B \in \Delta$, then the clause for R$\Box$ will be chosen, and HYPNO will be recursively invoked on its premisses. HYPNO proceeds in a similar way for the other rules. The ordering of the clauses is such that the application of branching rules is postponed as much as possible. As an example, here is the clause implementing R$\Box$:

```
1. terminating_proof_search(Hyper,tree(rbox,PrintableHyper,Sub1,Sub2)):-
2.   select(singleSeq([Gamma,Delta],Additional),Hyper,NewHyper),
3.   member(Sigma,Gamma), is_list(Sigma),member(box B,Delta),
4.   list_to_ord_set(Sigma,SigmaOrd), \+member(apdR(SigmaOrd,B),Additional),!,
5.   terminating_proof_search([singleSeq([Sigma,[B]],[])|
      [singleSeq([Gamma,Delta],[apdR(SigmaOrd,B)|Additional])|NewHyper]],Sub1),
6.   terminating_proof_search([singleSeq([[],[B => Sigma]],[])|
      [singleSeq([Gamma,Delta],[apdR(SigmaOrd,B)|Additional])|NewHyper]],Sub2),
7.   extractPrintableSequents(Hyper,PrintableHyper).
```

Line 3 checks whether `Gamma` contains an item `Sigma` which is a list representing a block and if a box formula `box B` belongs to the list `Delta`. Line 4 implements the restriction on the application of the rule used in order to ensure a terminating proof search: if the `Additional` list contains the Prolog term `apdR(SigmaOrd,B)`[2], this means that the rule R$\Box$ has been already applied to that sequent by using $\Box B$ and the block Σ, and HYPNO does no longer apply it. Otherwise, the predicate `terminating_proof_search` is recursively invoked on the two premisses of the rule (lines 5 and 6), by introducing $\Sigma \Rightarrow B$ and $B \Rrightarrow \Sigma$ respectively. Since the rule is invertible, Prolog cut `!` is used in line 4 to eventually block backtracking.

When the predicate `terminating_proof_search` fails, HYPNO can extract a countermodel following the countermodel extraction described in Section 6.4. The model is computed by executing the predicate:

```
build_saturated_branch(Hyper, Model).
```

When this predicate succeeds, the variable `Model` matches a description of a saturated hypersequent obtained by backward applying the rules of $\mathbf{H.E}^*$ to the initial formula. Since the very objective of this predicate is to build a saturated hypersequent, its clauses are essentially the same as the ones for the predicate `terminating_proof_search`, however rules introducing a branching in a backward proof search are implemented by pairs of (disjoint) clauses, each one attempting to build an open saturated hypersequent from the corresponding premiss. As an example, the following clauses implement the saturation in presence of a block Σ in the left hand side and of a boxed formula $\Box B$ in the right hand side of a sequent:

[2] The predicate `list_to_ord_set` is used in order to check the applicability of the rule by ignoring the order of the formulas in the block.

```
build_saturated_branch(Hyper,Model):-
  select(singleSeq([Gamma,Delta],Additional),Hyper,NewHyper),
  member(Sigma,Gamma),is_list(Sigma),member(box B,Delta),
  list_to_ord_set(Sigma,SigmaOrd),\+member(apdR(SigmaOrd,B),Additional),
  build_saturated_branch([singleSeq([Sigma,[B]],[])|
   [singleSeq([Gamma,Delta],[apdR(SigmaOrd,B)|Additional])|NewHyper]],Model).
build_saturated_branch(Hyper,Model):-
  select(singleSeq([Gamma,Delta],Additional),Hyper,NewHyper),
  member(Sigma,Gamma),is_list(Sigma),member(box B,Delta),
  list_to_ord_set(Sigma,SigmaOrd),\+member(apdR(SigmaOrd,B),Additional),
  build_saturated_branch([singleSeq([[],[B => Sigma]],[])|
   [singleSeq([Gamma,Delta],[apdR(SigmaOrd,B)|Additional])|NewHyper]],Model).
```

HYPNO will first try to build a countermodel by considering the left premiss of R$\Box$, whence recursively invoking the predicate `build_saturated_branch` on the premiss with the sequent $\Sigma \Rightarrow B$. In case of a failure, it will carry on the saturation process by using the right premiss of R$\Box$ with the sequent $B \Rrightarrow \Sigma$.

Clauses implementing axioms for the predicate `terminating_proof_search` are replaced by the last clause, checking whether the current hypersequent is saturated:

```
build_saturated_branch(Hyper,model(Hyper)):-\+instanceOfAnAxiom(Hyper).
```

Since this is the very last clause of `build_saturated_branch`, it is considered by HYPNO only if no other clause is applicable, then the hypersequent is saturated. The auxiliary predicate `instanceOfAnAxiom` checks whether the hypersequent is open by proving that it is not an instance of the axioms. The second argument matches a term `model` representing the countermodel extracted from `Hyper`.

Apart from distinguishing between monotonic and non-monotonic calculi, the implementation of the extensions is fully modular and reflects the modularity of the calculi $\mathbf{H.E}^*$: each system is obtained by just adding clauses for both the predicates `terminating_proof_search` and `build_saturated_branch` corresponding to the specific additional rules.

Performance

We have compared the performance of HYPNO with that of PRONOM (Section 5.7), obtaining promising results. We have tested it by running SWI-Prolog, version 7.6.4, on an Apple MacBook Pro, 2.7 GHz Intel Core i7, 8GB RAM machine. First, we have tested HYPNO over hundred valid formulas in $\mathbf{E}$ and considered extensions obtained by generalizing schemas of valid formulas by varying some crucial parameters, like the modal degree. For instance, we have considered the schemas (valid in all systems):

$$(\Box(\Box(A_1 \wedge (B_1 \vee C_1)) \wedge \cdots \wedge \Box(A_n \wedge (B_n \vee C_n)))) \to (\Box(\Box((A_1 \wedge B_1) \vee (A_1 \wedge C_1)) \wedge \cdots \wedge \Box((A_n \wedge B_n) \vee (A_n \wedge C_n)))$$

System	0,1 ms	1ms	100ms	1s	5s
HYPNO	91,50 %	78,91 %	28,23 %	9,52 %	5,78 %
PRONOM	85,71 %	77,55 %	57,82 %	31,16 %	19,80 %

Table 6.2: Percentage of timeouts over valid formulas in **E**.

Vars/Depth	**1ms**	**10ms**	**1s**	**10s**
3 vars - depth 5	4 - 5,58 %	0,78 - 1,76 %	0,02 - 0,48 %	0 - 0,22 %
3 vars - depth 7	23,78 - 25,18 %	10,86 - 20,16 %	3,16 - 14,40 %	2,02 - 12 %
7 vars - depth 10	45,22 - 44,94 %	34,36 - 42,36 %	19,06 - 30,30 %	16,06 - 20,34 %

Table 6.3: Percentage of timeouts in 5000 random tests (system **E**).

$$(\Box^n C_1 \land \cdots \land \Box^n C_j \land \Box^n A) \to (\Box^n A \lor \Box^n D_1 \lor \cdots \lor \Box^n D_k)$$

obtaining encouraging results: Table 6.2 reports the number of timeouts of HYPNO and PRONOM over a set of valid formulas in system **E**. HYPNO is able to answer in less than one second on more than the 90% of the tests, whereas PRONOM is not even if we extend the time limit to 5 s.

We have also tested HYPNO on randomly generated formulas, fixing different time limits, numbers of propositional variables, and levels of nesting of connectives. We have compared the performances of HYPNO with those of PRONOM, obtaining the results in Table 6.3: in each pair, the first number is the percentage of timeouts of HYPNO, the second number is the percentage of timeouts of PRONOM given the fixed time limit. Also in case of formulas generated from 3 different atomic variables and with a higher level of nesting (7), HYPNO is able to answer in more than 96% of the cases within 1 s, against the 85% of PRONOM. We have repeated the experiments also for all the extensions of **E** considered by HYPNO: complete results can be found at http://193.51.60.97:8000/HYPNO/#experiments.

6.8 Discussion

In this chapter, we have presented internal calculi for the basic cube of classical non-normal modal logics and their extensions with the axioms T, P, D, and the rules RD_n^+. The calculi are defined by extending standard sequents with the structures of blocks and hypersequents. Apart from the distinction between monotonic and non-monotonic systems, the calculi are modular. They also have a natural “almost internal” interpretation, as each component of a hypersequent can be read as a formula of the language. We have shown that the hypersequent calculi have good structural properties, in particular they enjoy cut elimination, from which we have obtained a syntactic proof of completeness. Moreover, the calculi provide a decision procedure of optimal complexity for the logics without axiom C, and from a failed proof we can easily extract a countermodel (of polynomial size for logics without axiom C) in the bi-neighbourhood semantics, whence by an easy transformation also in the standard one. In general, the hypersequent formulation turned out to be very adequate for non-normal modal logics, as it ensures good semantic, computational, as well as structural properties.

At the same time, we have seen that the bi-neighbourhood semantics is more adequate than the standard semantics for the direct extraction of countermodels from failed proofs, as the bi-neighbourhood pairs are well-suited to directly express, and reason with, the partial information provided by the failed proofs.

In addition, we have also presented hypersequent calculi for Elgesem's agency and ability logic [47] and Troquard's coalition logic [165]. For the definition of these calculi we have taken advantage of the modularity of our hypersequent calculi: for most axioms we have simply considered the corresponding rule in the calculus, moreover, the structure of blocks allowed us to represent the relations between the ability and agency modalities by means of simple and analytic rules. We have shown that the calculi are syntactically complete with respect to the logics, and that they allow for countermodel extraction of non-valid formulas in the bi-neighbourhood semantics.

Finally, we have presented HYPNO, a Prolog implementation of the hypersequent calculi for the systems of the classical cube. Similarly to the theorem prover PRONOM based on labelled calculi (Section 5.7), for every formula HYPNO provides either a proof in the calculus or a countermodel, directly built from an open saturated hypersequent. On the basis of our tests, the performance of HYPNO seems promising, in particular it outperforms PRONOM. In future work we aim to analyse the performance of both provers more comprehensively with tests analogous to the ones considered in Giunchiglia et al. [71].

As we made clear in Section 6.1, to the purpose of having sound and complete calculi for non-normal modal logics the hypersequent framework is not necessary, as for instance the Gentzen calculi discussed in Section 3.4 show. Moreover, for the calculi $\mathbf{H.E}^*$ it holds that whenever a hypersequent is derivable there is a component which is derivable. But as we have seen, the hypersequent framework is very adequate to extract countermodels from a single failed proof, ensuring at the same time good computational and structural properties. In particular, hypersequents allow one to easily compute countermodels as they can represent *all* worlds of a model by means of a rather simple structure. Moreover, differently from other possible structures such as, e.g., nested sequents, the *flat* structure of hypersequents makes possible for every world to have direct access to all other worlds, thus avoiding any possible redundancy in the construction of the models.

A similar remark can be done for the theorem prover. It is clear that to the purpose of obtaining an automated tool answering to the derivability/satisfiability problem for non-normal modal logics one could also implement their Gentzen calculi (Section 3.4). As an advantage, the implementation of the Gentzen calculi would not be only simpler, but would likely provide a more efficient prover than HYPNO. In particular, the backtracking needed for the backward proof search in the Gentzen calculi (cf. Section 3.4) would be made automatically by Prolog. However, differently from the Gentzen calculi, the hypersequent calculi $\mathbf{H.E}^*$ have the advantage of offering a simple countermodel extraction procedure which is very suitable to be implemented. It would be in contrast overly complicated to implement the complex countermodel construction defined by Lavendhomme and Lucas [107] for the Gentzen calculi, moreover, its implementation would in any case require a data structure which is not simpler than the one needed for the representation of the hypersequents. In addition, given the modularity of the calculi $\mathbf{H.E}^*$, the implementation of both the proof search and the countermodel construction of HYPNO can be extended to all systems covered by the hypersequent calculi simply by adding clauses corresponding to the additional rules. The same would not be equally easy for the implementation of the Gentzen calculi, since their definition is not modular, and in addition the construction of countermodels for the systems beyond the classical cube is still unexplored.

Concerning Elgesem's and Troquard's logics, to our knowledge the only calculus for Elgesem's logic was proposed by Lellmann [108] in the form of a Gentzen calculus, but apart from our calculus no other proof system is known which connects syntax and semantics and allows for countermodel extraction of non-valid formula. Moreover, no other calculus at all is known for Troquard's coalition logic. Troquard [165] has also developed a decision procedure for Elgesem's as well as his coalition logic that reduces validity checking to a set of SAT problems, similarly to Vardi [167] and Giunchiglia et al. [71]. This algorithm based on SAT-reduction is efficient but does not provide neither derivations, nor countermodels. In contrast, our decision procedure based on the hypersequent calculus is *constructive*, as for every formula returns either a derivation or a countermodel.

Possible further extensions

The syntactic framework based on hypersequents and blocks behaves particularly well for *non-iterative* logics, that is, logics defined only by axioms not containing nested modalities. In contrast, we have excluded from this chapter the logics with axiom 4, which are the only non-iterative non-normal modal logics among the ones considered in Section 2.2. In this work (Chapter 5) we have presented cut-free labelled calculi for the logic **E4** and its extensions. By contrast, as recalled in Section 3.4, to the best of our knowledge no cut-free internal, Gentzen-style calculus exists for the logic **E4**. In future work we aim to extend our hypersequent calculi to the logics with axiom 4, and possibly others. Here we limit ourselves to the following remarks.

At present, it seems possible to cover the logics with axiom 4 by extending our calculi with one of the rules below:

$$\mathsf{4}\ \frac{G \mid \Gamma, \langle \Sigma \rangle, \langle \langle \Sigma \rangle \rangle \Rightarrow \Delta}{G \mid \Gamma, \langle \Sigma \rangle \Rightarrow \Delta} \qquad\qquad \mathsf{L}\Box_{\mathsf{4}}\ \frac{G \mid \Gamma, \langle \Box A \rangle \Rightarrow \Delta}{G \mid \Gamma, \Box A \Rightarrow \Delta}$$

The rule on the right, which is given by analogy with calculi for normal modal logics (see for instance [87]), is essentially a second left-box rule. By contrast, the rule on the left follows the general idea at the basis of the design of our hypersequent calculi of modularly defining extensions by means of structural rules handling blocks. Given the rule $\mathsf{4}$, a derivation of axiom 4 would be as follows:

$$\cfrac{\cfrac{\cfrac{\mathsf{R}\Box\ \cfrac{... \mid A \Rightarrow A \qquad ... \mid A \Rightarrow A}{\Box A, \langle A \rangle, \langle\langle A \rangle\rangle \Rightarrow \Box\Box A \mid \langle A \rangle \Rightarrow \Box A} \qquad \cfrac{\Box A, \langle A \rangle, \langle\langle A \rangle\rangle \Rightarrow \Box\Box A \mid \Box A, \langle A \rangle \Rightarrow \langle A \rangle}{\Box A, \langle A \rangle, \langle\langle A \rangle\rangle \Rightarrow \Box\Box A \mid \Box A \Rightarrow \langle A \rangle}\ \mathsf{L}\Box}{\Box A, \langle A \rangle, \langle\langle A \rangle\rangle \Rightarrow \Box\Box A}\ \mathsf{R}\Box}{\Box A, \langle A \rangle \Rightarrow \Box\Box A}\ \mathsf{4}}{\Box A \Rightarrow \Box\Box A}\ \mathsf{L}\Box$$

As it is made clear by the above derivation, this solution would require to extend our syntactic framework by allowing both *nesting of blocks* and the presence of *blocks in the right-hand-side* of sequents. This represents a significant modification of the formalism that, if accepted, would likely entail the need of additional investigation of the structural properties of the calculi, for instance one might need to examine applications of cut over bocks. All this requires further analysis that we would like to carry on in future work.

Part III

Intuitionistic non-normal modal logics

Chapter 7

A sequent framework for intuitionistic non-normal modal logics

In this chapter, we define a family of intuitionistic non-normal modal logics that can bee seen as intuitionistic counterparts of the systems of the classical cube (cf. Section 2.2). We first consider monomodal logics, which contain only $\Box$ or $\Diamond$. We then consider the more important case of bimodal logics, which contain both modal operators. In this case we define several interactions between the two modalities of increasing strength, although weaker than duality. We thereby obtain a lattice of 24 distinct bimodal logics. For all logics we provide both a Hilbert axiomatisation and a cut-free sequent calculus. Moreover, on the basis of the sequent calculi we prove the decidability of all defined systems and also investigate the property of Craig's interpolation. Finally, we present strictly terminating sequent calculi for our systems as well as for the intuitionistic non-normal modal logics **CK** and **CCDL** (cf. Section 2.6).

7.1 The strategy

As explained in Section 2.5, differently from both *classical* non-normal modal logics and intuitionistic *normal* modal logics, no general investigation of non-normal modalities with an intuitionistic basis has been carried out in the literature. By contrast, their study is limited to specific systems such as the ones presented in Sections 2.5 and 2.6. In this chapter, we aim to provide a general framework for non-normal modal logics with an intuitionistic basis. In the following, under "intuitionistic modal logics" we understand any modal logic **L** that extends intuitionistic propositional logic (**IPL**) and satisfies the following requirements:

(R1) **L** is conservative over **IPL**: its non-modal fragment coincides with **IPL**.

(R2) **L** satisfies the disjunction property: if $A \vee B$ is derivable, then at least one formula between A and B is also derivable.

We shall consider *monomodal* logics, i.e., logics containing only one modality, either $\Box$ or $\Diamond$, as well as the more interesting case of *bimodal* logics, i.e., logics containing both $\Box$ and $\Diamond$. For the logics containing both $\Box$ and $\Diamond$, we require that the two modalities are not interdefinable; this is coherent with the non-interdefinability of connectives in intuitionistic logic. In particular, the duality axioms

$$Dual_{\Diamond} \quad \Box A \supset\subset \neg\Diamond\neg A, \qquad\qquad Dual_{\Box} \quad \Diamond A \supset\subset \neg\Box\neg A,$$

should not be valid. We take this lack of a duality as an additional requirement for the definition of intuitionistic non-normal bimodal logics:

(R3) $\Box$ and $\Diamond$ are not interdefinable.

The above requirements (R1), (R2), (R3) are part of the requirements considered by Simpson [161] as general features of any intuitionistic modal logic. In contrast, we move away from Simpson's criteria by considering the following requirement:

(R4) **L** does not contain $C_{\Diamond}$, neither as an axiom, nor as a theorem.

By assuming (R4), we ideally put our investigation of intuitionistic non-normal modal logics into the *constructive* tradition, which originated the intuitionistic systems with non-normal modalities already studied in the literature (cf. Section 2.5).

We have seen in Section 2.6 three systems – namely, **CK**, **CCDL**, and **IK** – that, though not equivalent, can all be seen as intuitionistic counterparts of the classical normal modal logic **K**. In this chapter, we are interested in defining logics that can be seen as *intuitionistic counterparts of the non-normal systems of the classical cube*. To this aim, we consider the intuitionistic versions of the characteristic modal axioms and rules of the systems of the classical cube (Figure 2.8, p. 33). We first define monomodal logics, that by analogy with the classical systems are Hilbert-style defined by extending **IPL** with the congruence rule plus combinations of the other axioms (although $\Diamond$-logics do not contain $C_{\Diamond}$). We then move to the definition of bimodal logics. In this case, we distinguish *monotonic* and *non-monotonic* logics by requiring that the systems either contain both $M_{\Box}$ and $M_{\Diamond}$, or do not contain either of them. In addition, logics with both $\Box$ and $\Diamond$ must contain some form of interaction between the two modalities, although always weaker than duality. To this purpose, we consider interactions that can be seen as "weak duality principles", and are determined by answering the following question, for any two formulas A and B:

under which conditions are $\Box A$ and $\Diamond B$ jointly inconsistent?

We distinguish three degrees of increasing strength: $\Box A$ and $\Diamond B$ are jointly inconsistent when

$RE_\Box\ \dfrac{A \supset\subset B}{\Box A \supset\subset \Box B}$	$RM_\Box\ \dfrac{A \supset B}{\Box A \supset \Box B}$	$RN_\Box\ \dfrac{A}{\Box A}$
$RE_\Diamond\ \dfrac{A \supset\subset B}{\Diamond A \supset\subset \Diamond B}$	$RM_\Diamond\ \dfrac{A \supset B}{\Diamond A \supset \Diamond B}$	$RN_\Diamond\ \dfrac{\neg A}{\neg\Diamond A}$
$M_\Box$ $\Box(A \wedge B) \supset \Box A$	$M_\Diamond$ $\Diamond A \supset \Diamond(A \vee B)$	
$C_\Box$ $\Box A \wedge \Box B \supset \Box(A \wedge B)$	$C_\Diamond$ $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$	
$N_\Box$ $\Box\top$	$N_\Diamond$ $\neg\Diamond\bot$	

Figure 7.1: Modal axioms and rules of intuitionistic non-normal modal logics.

(i) one of the two is $\top$ and the other is $\bot$.

(ii) A is equivalent to $\neg B$, or B is equivalent to $\neg A$.

(iii) A and B are jointly inconsistent.

In order to put some order into the picture of the possible systems defined by the above ctriteria, we base our definition of intuitionistic non-normal bimodal logics on cut-free Gentzen calculi: First, we formulate sequent rules corresponding to the $\Box$- and $\Diamond$-axioms (these shall be analogous to the rules by Lavendhomme and Lucas [107] for the systems of the classical cube, cf. Section 3.4), as well as sequent rules corresponding to the three considered degrees of interaction between $\Box$ and $\Diamond$. Then, we investigate the admissibility of `cut` in the systems defined by any combination of the considered rules. The existence of a cut-free calculus shall be our criterion to identify meaningful systems: *A system is accepted if and only if the combination of its sequent rules provides a cut-free calculus.*

7.2 Intuitionistic non-normal monomodal logics

In this section, we begin with the definition of intuitionistic non-normal modal logics by considering monomodal systems, that is systems containing only one modality, either $\Box$ or $\Diamond$. We first define the axiomatic systems, and then present their sequent calculi.

Hilbert systems

By analogy with the definition of classical non-normal modal logics (cf. Section 2.2), we define over **IPL** two families of intuitionistic non-normal monomodal logics, that depend on the considered modal operator, and are called therefore the $\Box$- and the $\Diamond$-family. The systems are defined by the $\Box$- or $\Diamond$-counterparts of the characteristic axioms and rules of the systems of the classical cube (cf. Section 2.2). We already considered these axioms and rules in Section 2.5, however for the sake of readability we display them again in Figure 7.1.

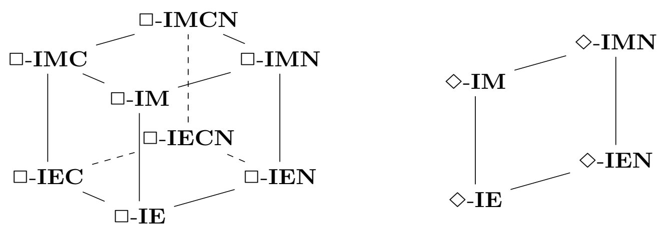

Figure 7.2: The lattices of intuitionistic non-normal monomodal logics.

$\Box$- and $\Diamond$-logics are respectively defined in the monomodal languages $\mathcal{L}_\Box := \mathcal{L}_i \setminus \{\Diamond\}$ and $\mathcal{L}_\Box := \mathcal{L}_i \setminus \{\Box\}$, where $\mathcal{L}_i$ is the intuitionistic modal language with both $\Box$ and $\Diamond$ defined in Section 2.6. Moreover, for **IPL** we consider the axiomatisation presented in the same section.

Definition 7.2.1 ($\Box$- and $\Diamond$-logics)**.** An intuitionistic non-normal monomodal $\Box$-logic is any logic in language $\mathcal{L}_\Box := \mathcal{L}_i \setminus \{\Diamond\}$ that extends **IPL** with the rule $RE_\Box$ and a (possible empty) combination of axioms among $M_\Box$, $C_\Box$ and $N_\Box$ in Figure 7.1. Moreover, an intuitionistic non-normal monomodal $\Diamond$-logic is any logic in language $\mathcal{L}_\Diamond := \mathcal{L}_i \setminus \{\Box\}$ that extends **IPL** with the rule $RE_\Diamond$ and a (possible empty) combination of axioms among $M_\Diamond$ and $N_\Diamond$.

Recall that we are not considering intuitionistic non-normal modal logics containing axiom $C_\Diamond$. We denote the resulting logics by, respectively, $\Box$-**IE*** and $\Diamond$-**IE***, where **E*** replaces any system of the classical cube (for $\Diamond$-logics, any system not containing $C_\Diamond$).

Notice that, having rejected the definability of the lacking modality, $\Box$- and $\Diamond$-logics are distinct, as $\Box$ and $\Diamond$ behave differently. Moreover, as a consequence of the fact that the systems in the classical cube are pairwise non-equivalent, we have that the $\Box$-family contains eight distinct logics, whereas the $\Diamond$-family contains four distinct logics (something not derivable in a classical system is clearly not derivable in the corresponding intuitionistic system). We then obtain the two lattices in Figure 7.2. It is also worth noticing that, as it happens in the classical case, the axioms $M_\Box$, $M_\Diamond$ and $N_\Box$ are interderivable, respectively, with the rules $RM_\Box$, $RM_\Diamond$ and $RN_\Box$, and that $K_\Box$ is derivable from $M_\Box$ and $C_\Box$, as the standard derivations are intuitionistically valid (see e.g. the derivations in Figure 2.3, where the used propositional axioms and rules are valid also in **IPL**).

Sequent calculi

We now present sequent calculi for intuitionistic non-normal monomodal logics. The calculi are defined as modal extensions of a given sequent calculus for **IPL**. We take **G3ip** as the base calculus (Figure 3.6 on page 57), and extend it with suitable combinations of the modal rules in Figure 7.3. The $\Box$-rules can be compared with the rules given by Lavendhomme and Lucas [107] for the classical cube of non-normal modal logics (see Section 3.4). As a difference with the rules in Figure 3.4, the rules in Figure 7.3 for intuitionistic calculi have at most

$$\mathsf{E}_\Box \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{E}_\Diamond \frac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathsf{M}_\Box \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{M}_\Diamond \frac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B}$$

$$\mathsf{N}_\Box \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \mathsf{N}_\Diamond \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta}$$

$$\mathsf{C}_\Box \frac{A_1, ..., A_n \Rightarrow B \qquad B \Rightarrow A_1 \ \ ... \ \ B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B} \ (n \geq 1)$$

$$\mathsf{MC}_\Box \frac{A_1, ..., A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n \Rightarrow \Box B} \ (n \geq 1)$$

Figure 7.3: Modal rules of Gentzen calculi for intuitionistic non-normal modal logics.

G3.$\Box$-**IE**	:=	$\{\mathsf{E}_\Box\}$	**G3**.$\Box$-**IEC**	:=	$\{\mathsf{C}_\Box\}$
G3.$\Box$-**IM**	:=	$\{\mathsf{M}_\Box\}$	**G3**.$\Box$-**IMC**	:=	$\{\mathsf{MC}_\Box\}$
G3.$\Box$-**IEN**	:=	$\{\mathsf{E}_\Box, \mathsf{N}_\Box\}$	**G3**.$\Box$-**IECN**	:=	$\{\mathsf{C}_\Box, \mathsf{N}_\Box\}$
G3.$\Box$-**IMN**	:=	$\{\mathsf{M}_\Box, \mathsf{N}_\Box\}$	**G3**.$\Box$-**IMCN**	:=	$\{\mathsf{MC}_\Box, \mathsf{N}_\Box\}$
G3.$\Diamond$-**IE**	:=	$\{\mathsf{E}_\Diamond\}$			
G3.$\Diamond$-**IM**	:=	$\{\mathsf{M}_\Diamond\}$			
G3.$\Diamond$-**IEN**	:=	$\{\mathsf{E}_\Diamond, \mathsf{N}_\Diamond\}$			
G3.$\Diamond$-**IMN**	:=	$\{\mathsf{M}_\Diamond, \mathsf{N}_\Diamond\}$			

Table 7.1: Gentzen calculi for monomodal logics.

one formula in the right-hand side of sequents. This restriction is adopted in order to have single-succedent calculi (as it is **G3ip**).

As usual, we consider the sequent calculi to be defined by the modal rules that are added to **G3ip**. For every monomodal logic **L**, the corresponding Gentzen calculus **G3.L** is defined as displayed in Table 7.1. Notice that, analogously to the classical calculi, the axiom $C_\Box$ is captured by modifying the rules $\mathsf{E}_\Box$ and $\mathsf{M}_\Box$. In particular, these rules are replaced by $\mathsf{C}_\Box$ and $\mathsf{MC}_\Box$, respectively, that are the generalisations of $\mathsf{E}_\Box$ and $\mathsf{M}_\Box$ with n principal formulas (instead of just one) in the left-hand side of sequents.

We now prove the admissibility of the structural rules, and then show the equivalence between the sequent calculi and the associated Hilbert systems.

Proposition 7.2.1. The following weakening and contraction rules are height-preserving admissible in any monomodal calculus:

$$\mathsf{Lwk} \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rwk} \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr} \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}.$$

Proof. We show that whenever the premiss of an application of Lwk, Rwk or ctr is derivable, then its conclusion has a derivation of at most the same height. As usual, the proof is by induction on the height n of the derivation of the premiss. If $n = 0$, i.e., the premiss is an initial sequent, then so is the conclusion. If $n \geq 1$, we consider the last rule applied in the derivaiton of the premiss. For left and right weakening, if the last rule applied is a rule of **G3ip**, then the proof is standard. If it is a modal rule, then the proof is easy. For instance, if the premiss of Rwk is derived by $\mathsf{N}_\Diamond$, then we have the derivation of the left, which is converted into the derivation on the right containing a different application of $\mathsf{N}_\Diamond$.

$$\dfrac{\dfrac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow}\,\mathsf{N}_\Diamond}{\Gamma, \Diamond A \Rightarrow B}\,\mathsf{Rwk} \quad \rightsquigarrow \quad \dfrac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow B}\,\mathsf{N}_\Diamond$$

For contraction, the proof is known if the last rule applied is a rule of **G3ip**. If this is a modal rule, then the proof is easy. As an example, consider the case where the premiss of ctr is derived by $\mathsf{MC}_\Box$. We have the derivation on the left, which is converted into the derivation on the right containing an application of ctr at a smaller height, which is admissible by i.h..

$$\dfrac{\dfrac{A_1, ..., A_n, B, B \Rightarrow C}{\Gamma, \Box A_1, ..., \Box A_n, \Box B, \Box B \Rightarrow \Box C}\,\mathsf{MC}_\Box}{\Gamma, \Box A_1, ..., \Box A_n, \Box B \Rightarrow \Box C}\,\mathsf{ctr} \quad \rightsquigarrow \quad \dfrac{\dfrac{A_1, ..., A_n, B, B \Rightarrow C}{A_1, ..., A_n, B \Rightarrow C}\,\mathsf{ctr}}{\Gamma, \Box A_1, ..., \Box A_n, \Box B \Rightarrow \Box C}\,\mathsf{MC}_\Box$$

□

We now show that the cut rule

$$\mathsf{cut}\,\dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}$$

is admissible in every monomodal calculus. The proof makes use of the following definition of weight of formulas, which is different from the one of weight of formulas of the classical language $\mathcal{L}$ (Definition 2.2.2):

Definition 7.2.2 (Weight of formulas)**.** The function wg_i assigning to each formula A of $\mathcal{L}_i$ its weight $wg_i(A)$ is defined as follows: $wg_i(\bot) = wg_i(\top) = 0$; $wg_i(p_i) = 1$ for every $p_i \in Atm$; $wg_i(A \circ B) = wg_i(A) + wg_i(B) + 1$ for $\circ \equiv \wedge, \vee, \supset$; and $wg_i(\Box A) = wg_i(\Diamond A) = wg_i(A) + 2$.

Observe in particular that, given the present definition, $\neg A$ has a smaller weight than $\Box A$ and $\Diamond A$. Although irrelevant to the next theorem, this shall be used in Section 7.3 for the proof of cut elimination in bimodal calculi.

Theorem 7.2.2 (Cut elimination)**.** The rule cut is admissible in every monomodal calculus.

Proof. Given a derivation of a sequent with some applications of cut, we show how to remove any such application and obtain a derivation of the same sequent without cut. The proof is by double induction, with primary induction on the weight of the cut formula and secondary induction on the cut height. We recall that, for any application of cut, the cut formula is the formula which is deleted by that application, while the cut height is the sum of the heights of the derivations of the premisses of cut.

We just consider the cases in which the cut formula is principal in the last rule applied in the derivation of both premisses of cut. Moreover, we treat explicitly only the cases in which both premisses are derived by modal rules, as the non-modal cases are already considered in the proof of cut admissibility for **G3ip**, and because modal and non-modal rules do not interact in any relevant way.

• $(\mathsf{C}_\Box;\ \mathsf{C}_\Box)$. Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$. The first derivation is converted into the second one, which contains several applications of cut on a cut formula of smaller weight.

$$\dfrac{\mathsf{C}_\Box\,\dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \dots\ B\Rightarrow A_n}{\Gamma, \Box\Gamma_1, \Rightarrow \Box B} \qquad \dfrac{B,\Gamma_2 \Rightarrow D \quad D\Rightarrow B \quad D \Rightarrow C_1 \ \dots\ D\Rightarrow C_m}{\Gamma', \Box B, \Box\Gamma_2 \Rightarrow \Box D}\,\mathsf{C}_\Box}{\Gamma,\Gamma',\Box\Gamma_1,\Box\Gamma_2 \Rightarrow \Box D}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{C}_\Box\,\dfrac{\mathsf{cut}\,\dfrac{\Gamma_1\Rightarrow B \quad B,\Gamma_2\Rightarrow D}{\Gamma_1,\Gamma_2\Rightarrow D} \qquad \left(\mathsf{cut}\,\dfrac{D\Rightarrow B \quad B\Rightarrow A_i}{D\Rightarrow A_i}\right)_{i=1}^{n} \qquad D\Rightarrow C_1\ \dots\ D\Rightarrow C_m}{\Gamma,\Gamma',\Box\Gamma_1,\Box\Gamma_2\Rightarrow\Box D}$$

• $(\mathsf{MC}_\Box;\ \mathsf{MC}_\Box)$ is analogous to $(\mathsf{C}_\Box;\ \mathsf{C}_\Box)$. $(\mathsf{E}_\Box;\ \mathsf{E}_\Box)$ and $(\mathsf{M}_\Box;\ \mathsf{M}_\Box)$ are the particular cases where $n, m = 1$.

• $(\mathsf{N}_\Box;\ \mathsf{C}_\Box)$. Let $\Gamma_1 = B_1, ..., B_n$. The first derivation is converted into the second one, which has an application of cut on a cut formula of smaller weight.

$$\dfrac{\mathsf{N}_\Box\,\dfrac{\Rightarrow A}{\Gamma\Rightarrow\Box A} \qquad \dfrac{A,\Gamma_1\Rightarrow C \quad C\Rightarrow A \quad C\Rightarrow B_1\ \dots\ C\Rightarrow B_n}{\Gamma',\Box A,\Box\Gamma_1\Rightarrow\Box C}\,\mathsf{C}_\Box}{\Gamma,\Gamma',\Box\Gamma_1\Rightarrow\Box C}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Rightarrow A \quad A,\Gamma_1\Rightarrow C}{\Gamma_1\Rightarrow C} \qquad C\Rightarrow B_1\ \dots\ C\Rightarrow B_n}{\Gamma,\Gamma',\Box\Gamma_1\Rightarrow\Box C}\,\mathsf{C}_\Box$$

• $(\mathsf{N}_\Box;\ \mathsf{MC}_\Box)$ is analogous to $(\mathsf{N}_\Box;\ \mathsf{C}_\Box)$. $(\mathsf{N}_\Box;\ \mathsf{E}_\Box)$ and $(\mathsf{N}_\Box;\ \mathsf{M}_\Box)$ are the particular cases where $n = 1$.

• $(\mathsf{E}_\Diamond;\ \mathsf{E}_\Diamond)$ and $(\mathsf{M}_\Diamond;\ \mathsf{M}_\Diamond)$ are analogous to $(\mathsf{E}_\Box;\ \mathsf{E}_\Box)$ and $(\mathsf{M}_\Box;\ \mathsf{M}_\Box)$, respectively.

• $(\mathsf{E}_\Diamond;\ \mathsf{N}_\Diamond)$. The derivation on the left is converted into the derivation on the right.

$$\dfrac{\mathsf{E}_\Diamond\,\dfrac{A\Rightarrow B \quad B\Rightarrow A}{\Gamma,\Diamond A\Rightarrow\Diamond B} \qquad \dfrac{B\Rightarrow}{\Gamma',\Diamond B\Rightarrow\Delta}\,\mathsf{N}_\Diamond}{\Gamma,\Gamma',\Diamond A\Rightarrow\Delta}\,\mathsf{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{A\Rightarrow B \quad B\Rightarrow}{A\Rightarrow}\,\mathsf{cut}}{\Gamma,\Gamma',\Diamond A\Rightarrow\Delta}\,\mathsf{N}_\Diamond$$

• $(\mathsf{M}_\Diamond;\ \mathsf{N}_\Diamond)$ is analogous to $(\mathsf{E}_\Diamond;\ \mathsf{N}_\Diamond)$. $\Box$

$$(RE_{\Box})\qquad \cfrac{\cfrac{\cfrac{\Rightarrow A \supset B \qquad A, A \supset B \Rightarrow B}{A \Rightarrow B}\,\mathsf{cut} \qquad \cfrac{\Rightarrow B \supset A \qquad B, B \supset A \Rightarrow A}{B \Rightarrow A}\,\mathsf{cut}}{\Box A \Rightarrow \Box B}\,\mathsf{E}_{\Box}}{\Rightarrow \Box A \supset \Box B}\,\mathsf{R}{\supset}$$

$$(M_{\Box})\qquad \cfrac{\cfrac{A \wedge B \Rightarrow A}{\Box(A \wedge B) \Rightarrow \Box A}\,\mathsf{M}_{\Box}}{\Rightarrow \Box(A \wedge B) \supset \Box A}\,\mathsf{R}{\supset} \qquad\qquad (C_{\Box})\qquad \cfrac{\cfrac{\cfrac{A, B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}\,\mathsf{C}_{\Box}}{\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\,\mathsf{L}{\wedge}}{\Rightarrow \Box A \wedge \Box B \supset \Box(A \wedge B)}\,\mathsf{R}{\supset}$$

$$(N_{\Box})\qquad \cfrac{\Rightarrow \top}{\Rightarrow \Box \top}\,\mathsf{N}_{\Box} \qquad\qquad (N_{\Diamond})\qquad \cfrac{\cfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow}\,\mathsf{N}_{\Diamond}}{\Rightarrow \neg\Diamond\bot}\,\mathsf{R}{\neg} \qquad\qquad (M_{\Diamond})\qquad \cfrac{\cfrac{A \Rightarrow A \vee B}{\Diamond A \Rightarrow \Diamond(A \vee B)}\,\mathsf{M}_{\Diamond}}{\Rightarrow \Diamond A \supset \Diamond(A \vee B)}\,\mathsf{R}{\supset}$$

Figure 7.4: Derivations of modal axioms and rules of $\Box$-**IE*** and $\Diamond$-**IE***.

On the basis of the admissibility of cut we can prove that the sequent calculi and the axiomatic systems are equivalent.

Theorem 7.2.3 (Syntactic equivalence)**.** Let **L** be any intuitionistic non-normal monomodal logic. Then the calculus **G3**.**L** is equivalent to the system **L**. that is

$$\vdash_{\mathbf{G3.L}} \Gamma \Rightarrow \Delta \;\text{ if and only if }\; \vdash_{\mathbf{L}} \bigwedge \Gamma \supset \bigvee \Delta.$$

Proof. As usual, for the right-to-left direction we have to show that the axioms of **L** are derivable in **G3**.**L**, and that the rules of **L** are admissible in **G3**.**L**. For the axioms of **IPL** we can consider their derivations in **G3ip**, whereas MP is simulated by cut in the usual way. Moreover, in Figure 7.9 we show the derivations of the modal axioms and rules by the corresponding sequent rules, the lacking derivation of $RE_{\Diamond}$ is analogous to the one of $RE_{\Box}$.

For the other direction, we prove that the rules of **G3**.**L** are derivable in **L**. As before, we restrict our attention to the modal rules, and consider the following illustrative derivations:

- If **L** contains $N_{\Box}$, then $\mathsf{N}_{\Box}$ is derivable: Assume $\vdash_{\mathbf{L}} A$. Then by $RN_{\Box}$ (which is equivalent to $N_{\Box}$), $\vdash_{\mathbf{L}} \Box A$.

- If **L** contains $N_{\Diamond}$, then $\mathsf{N}_{\Diamond}$ is derivable: Assume $\vdash_{\mathbf{L}} A \supset \bot$. Since $\vdash_{\mathbf{L}} \bot \supset A$, by $\mathsf{E}_{\Diamond}$ we obtain $\vdash_{\mathbf{L}} \Diamond A \supset \Diamond\bot$. Then $\vdash_{\mathbf{L}} \neg\Diamond\bot \supset \neg\Diamond A$, and, since $\vdash_{\mathbf{L}} \neg\Diamond\bot$, we have $\vdash_{\mathbf{L}} \neg\Diamond A$.

- If **L** contains $C_{\Box}$, then $\mathsf{C}_{\Box}$ is derivable: Assume $\vdash_{\mathbf{L}} A_1 \wedge ... \wedge A_n \supset B$ and $\vdash_{\mathbf{L}} B \supset A_i$ for all $1 \leq i \leq n$. Then $\vdash_{\mathbf{L}} B \supset A_1 \wedge ... \wedge A_n$. By $RE_{\Box}$, $\vdash_{\mathbf{L}} \Box(A_1 \wedge ... \wedge A_n) \supset \Box B$. In addition, by several applications of $C_{\Box}$, $\vdash_{\mathbf{L}} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box(A_1 \wedge ... \wedge A_n)$. Therefore $\vdash_{\mathbf{L}} \Box A_1 \wedge ... \wedge \Box A_n \supset \Box B$. $\Box$

$$\mathsf{weak_a}\ \frac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad \mathsf{weak_b}\ \frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{neg_a}\ \frac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad \mathsf{neg_b}\ \frac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{str}\ \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

Figure 7.5: Interaction rules for sequent calculi.

7.3 Intuitionistic non-normal bimodal logics

In this section, we define intuitionistic non-normal bimodal logics, i.e., with both $\Box$ and $\Diamond$, by following the proof-theoretic approach described in Section 7.1. In practice, we realise our list of desiderata as follows. As before, we take **G3ip** (Figure 3.6) as the base calculus for intuitionistic logic. This is extended with combinations of the characteristic rules of intuitionistic non-normal monomodal logics given in Figure 7.3. As a difference with monomodal calculi, the calculi for bimodal logics contain both some rules for $\Box$ and some rules for $\Diamond$. In order to distinguish monotonic and non-monotonic logics, we require that the calculi contain either both $\mathsf{E}_\Box$ and $\mathsf{E}_\Diamond$ (in this case the corresponding logic will be non-monotonic), or both $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$ (corresponding to monotonic logics). In addition, the calculi shall contain some rules for interaction between $\Box$ and $\Diamond$ corresponding to the three degrees of interactions considered in Section 7.1. The interaction rules are displayed in Figure 7.5. We require that the calculi contain either both $\mathsf{weak_a}$ and $\mathsf{weak_b}$ (corresponding to the first lever of interaction), or both $\mathsf{neg_a}$ and $\mathsf{neg_b}$ (corresponding to the second level), or str (corresponding to the third level).

In the following, we present the sequent calculi for intuitionistic non-normal bimodal logics obtained by following this methodology. Then, for each cut-free sequent calculus we define an equivalent axiomatisation.

Sequent calculi for logics without $C_\Box$

In the first part, we focus on sequent calculi for logics containing only axioms among $M_\Box$, $M_\Diamond$, $N_\Box$ and $N_\Diamond$ — in other words, we do not consider the axiom $C_\Box$. The calculi are obtained by adding to **G3ip** (Figure 3.6) suitable combinations of the modal rules in Figures 7.3 and 7.5. Although in principle any combination of rules could define a calculus, we accept only the calculi that satisfy the restrictions mentioned above. In particular, this entails that we investigate cut elimination. As usual, the first step towards the study of cut elimination is to prove the admissibility of the other structural rules.

$\mathbf{G3.IE_1}$	$:=$	$\{\mathsf{E}_\Box, \mathsf{E}_\Diamond, \mathsf{weak_a}, \mathsf{weak_b}\}$			
$\mathbf{G3.IE_2}$	$:=$	$\{\mathsf{E}_\Box, \mathsf{E}_\Diamond, \mathsf{neg_a}, \mathsf{neg_b}\}$	$\mathbf{G3.IEN_\Diamond}^*$	$:=$	$\mathbf{G3.IE}^* \cup \{\mathsf{N}_\Diamond\}$
$\mathbf{G3.IE_3}$	$:=$	$\{\mathsf{E}_\Box, \mathsf{E}_\Diamond, \mathsf{str}\}$	$\mathbf{G3.IEN_\Box}^*$	$:=$	$\mathbf{G3.IE}^* \cup \{\mathsf{N}_\Diamond, \mathsf{N}_\Box\}$
$\mathbf{G3.IM}$	$:=$	$\{\mathsf{M}_\Box, \mathsf{M}_\Diamond, \mathsf{str}\}$			

Table 7.2: Gentzen calculi for intuitionistic non-normal bimodal logics without $C_\Box$.

Proposition 7.3.1. Weakening and contraction are height-preserving admissible in each sequent calculus defined by any combination of modal rules in Figures 7.3 and 7.5.

Proof. The proposition is proved by extending the proof of Proposition 7.2.1 with an examination of the interaction rules in Figure 7.5. Due to their form, however, it is easy to verify that if the premiss of wk or ctr is derivable by any interaction rule, then the conclusion is derivable by the same rule. □

On the basis of the above result and the definition of $\neg A$ as $A \supset \bot$, we can also prove the admissibility of the following rules for negation, that we shall use in the following in order to abbreviate the derivations:

$$\mathsf{L}\neg\ \frac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow} \qquad\qquad \mathsf{R}\neg\ \frac{\Gamma, A \Rightarrow}{\Gamma \Rightarrow \neg A}$$

These rules are shown admissible in the following standard way.

$$\dfrac{\mathsf{Lwk}\ \dfrac{\Gamma \Rightarrow A}{\Gamma, A \supset \bot \Rightarrow A} \qquad \dfrac{}{\Gamma, \bot \Rightarrow}\ \mathsf{L}\bot}{\Gamma, A \supset \bot \Rightarrow}\ \mathsf{L}\supset \qquad\qquad \dfrac{\dfrac{\Gamma, A \Rightarrow}{\Gamma, A \Rightarrow \bot}\ \mathsf{Rwk}}{\Gamma \Rightarrow A \supset \bot}\ \mathsf{R}\supset$$

We can now examine the admissibility of the rule cut. As stated by the following theorem, our methodology leads to consideration of 12 sequent calculi for intuitionistic non-normal bimodal logics.

Theorem 7.3.2 (Cut elimination)**.** The cut rule is admissible in every calculus $\mathbf{G3.IE}^*$ in Table 7.2.

Proof. The structure of the proof is similar to the one of Theorem 7.2.2. As before, we consider only the cases where the cut formula is principal in the last rule applied in the derivation of both premisses, with the further restriction that the last rules are modal ones.

The combinations between $\Box$-rules, or between $\Diamond$-rules, have been already considered in the proof of Theorem 7.2.2. Here we only consider the possible combinations of $\Box$- or $\Diamond$-rules with rules for interaction. For each case below, the derivation on the left is transformed into the derivation on the right.

- ($\mathsf{E}_\Box$; $\mathsf{weak_a}$).

$$\dfrac{\mathsf{E}_\Box\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \qquad C \Rightarrow}{\Gamma', \Box B, \Diamond C \Rightarrow \Delta}\,\mathsf{weak_a}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \mathsf{weak_a}\,\dfrac{\mathsf{cut}\,\dfrac{\Rightarrow B \qquad B \Rightarrow A}{\Rightarrow A} \qquad C \Rightarrow}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}$$

- ($\mathsf{E}_\Diamond$; $\mathsf{weak_a}$).

$$\dfrac{\mathsf{E}_\Diamond\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{\Rightarrow C \qquad B \Rightarrow}{\Gamma', \Box C, \Diamond B \Rightarrow \Delta}\,\mathsf{weak_a}}{\Gamma, \Gamma', \Diamond A, \Box C \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\Rightarrow C \qquad \dfrac{A \Rightarrow B \qquad B \Rightarrow}{A \Rightarrow}\,\mathsf{cut}}{\Gamma, \Gamma', \Diamond A, \Box C \Rightarrow \Delta}\,\mathsf{weak_a}$$

- ($\mathsf{E}_\Box$; $\mathsf{weak_b}$).

$$\dfrac{\mathsf{E}_\Box\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{B \Rightarrow \qquad \Rightarrow C}{\Gamma', \Box B, \Diamond C \Rightarrow \Delta}\,\mathsf{weak_b}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \mathsf{weak_b}\,\dfrac{\mathsf{cut}\,\dfrac{A \Rightarrow B \qquad B \Rightarrow}{A \Rightarrow} \qquad \Rightarrow C}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}$$

- ($\mathsf{E}_\Diamond$; $\mathsf{weak_b}$).

$$\dfrac{\mathsf{E}_\Diamond\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{C \Rightarrow \qquad \Rightarrow B}{\Gamma', \Box C, \Diamond B \Rightarrow \Delta}\,\mathsf{weak_b}}{\Gamma, \Gamma', \Diamond A, \Box C \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\Rightarrow C \qquad \dfrac{\Rightarrow B \qquad B \Rightarrow A}{\Rightarrow A}\,\mathsf{cut}}{\Gamma, \Gamma', \Diamond A, \Box C \Rightarrow \Delta}\,\mathsf{weak_b}$$

- ($\mathsf{E}_\Diamond$; $\mathsf{neg_a}$).

$$\dfrac{\mathsf{E}_\Box\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \dfrac{C, B \Rightarrow \qquad \neg B \Rightarrow C}{\Gamma', \Box C, \Diamond B \Rightarrow \Delta}\,\mathsf{neg_b}}{\Gamma, \Gamma', \Box C, \Diamond A \Rightarrow \Delta}\,\mathsf{cut}$$

$$\wr$$

$$\dfrac{\mathsf{cut}\,\dfrac{A \Rightarrow B \qquad C, B \Rightarrow}{C, A \Rightarrow} \qquad \dfrac{\mathsf{R}\neg\,\dfrac{\mathsf{L}\neg\,\dfrac{B \Rightarrow A}{B, \neg A \Rightarrow}}{\neg A \Rightarrow \neg B} \qquad \neg B \Rightarrow C}{\neg A \Rightarrow C}\,\mathsf{cut}}{\Gamma, \Gamma', \Box C, \Diamond A \Rightarrow \Delta}\,\mathsf{neg_b}$$

Observe that the second derivation has two applications of cut, both of them with a cut formula of smaller weight; in particular $wg_i(\neg B) < wg_i(\Diamond B)$ (cf. Definition 7.2.2).

- ($\mathsf{E}_\Box$; $\mathsf{neg_b}$).

$$\dfrac{\mathsf{E}_\Box\,\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{B, C \Rightarrow \qquad \neg B \Rightarrow C}{\Gamma', \Box B, \Diamond C \Rightarrow \Delta}\,\mathsf{neg_b}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{cut}$$

$$\wr$$

$$\dfrac{\mathsf{cut}\,\dfrac{A \Rightarrow B \qquad B, C \Rightarrow}{A, C \Rightarrow} \qquad \dfrac{\mathsf{R}\neg\,\dfrac{\mathsf{L}\neg\,\dfrac{B \Rightarrow A}{B, \neg A \Rightarrow}}{\neg A \Rightarrow \neg B} \qquad \neg B \Rightarrow C}{\neg A \Rightarrow C}\,\mathsf{cut}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{neg_b}$$

- ($\mathsf{M}_\Box$; str).

$$\mathsf{M}_\Box\,\dfrac{\dfrac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \dfrac{B, C \Rightarrow}{\Gamma', \Box B, \Diamond C \Rightarrow \Delta}\,\mathsf{str}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{A \Rightarrow B \qquad B, C \Rightarrow}{A, C \Rightarrow}\,\mathsf{cut}}{\Gamma, \Gamma', \Box A, \Diamond C \Rightarrow \Delta}\,\mathsf{str}$$

- ($\mathsf{N}_\Box$; str).

$$\mathsf{N}_\Box\,\dfrac{\dfrac{\Rightarrow A}{\Gamma \Rightarrow \Box A} \qquad \dfrac{A, B \Rightarrow}{\Gamma', \Box A, \Diamond B \Rightarrow \Delta}\,\mathsf{str}}{\Gamma, \Gamma', \Diamond B \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow A \qquad A, B \Rightarrow}{B \Rightarrow}\,\mathsf{cut}}{\Gamma, \Gamma', \Diamond B \Rightarrow \Delta}\,\mathsf{N}_\Diamond$$

The lacking combinations can be easily treated in similar ways. Observe that $\mathsf{N}_\Diamond$ does not interact significantly with any interaction rule, since the principal formula $\Diamond B$ occurs in the left-hand side of the conclusion. $\Box$

It can be shown that all combinations of rules excluded from Theorem 7.3.2 do not give a cut-free calculus. In particular, cut elimination fails if we take the rule $\mathsf{N}_\Box$ and we do not take the rule $\mathsf{N}_\Diamond$, or if we combine the monotonic rules for $\Box$ and $\Diamond$ with interaction rules different from str, as it is shown by the following examples.

Example 7.3.1. Sequent $\Diamond\bot \Rightarrow$ is derivable from $\mathsf{N}_\Box + \mathsf{weak_a} + \mathsf{weak_b} + \mathsf{cut}$ (without $\mathsf{N}_\Diamond$), but it is not derivable from $\mathsf{N}_\Box + \mathsf{weak_a} + \mathsf{weak_b}$ without cut. With cut a possible derivation is the following:

$$\mathsf{N}_\Box\,\dfrac{\dfrac{\Rightarrow \top}{\Rightarrow \Box\top} \qquad \dfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\,\mathsf{weak_a}}{\Diamond\bot \Rightarrow}\,\mathsf{cut}$$

In contrast, without cut the only rule with a conclusion of the form $\Diamond\bot \Rightarrow$ is $\mathsf{N}_\Diamond$, whence the sequent does not have any cut-free derivation without $\mathsf{N}_\Diamond$.

Example 7.3.2. Sequent $\Box\neg p, \Diamond(p \wedge q) \Rightarrow$ is derivable from $\mathsf{M}_\Box + \mathsf{neg_a} + \mathsf{neg_b} + \mathsf{cut}$, but it is not derivable from $\mathsf{M}_\Box + \mathsf{neg_a} + \mathsf{neg_b}$ without cut. A possible derivation is as follows:

$$\mathsf{M}_\Box\,\dfrac{\dfrac{\neg p \Rightarrow \neg(p \wedge q)}{\Box\neg p \Rightarrow \Box\neg(p \wedge q)} \qquad \dfrac{\neg(p \wedge q), p \wedge q \Rightarrow \qquad \neg(p \wedge q) \Rightarrow \neg(p \wedge q)}{\Box\neg(p \wedge q), \Diamond(p \wedge q) \Rightarrow}\,\mathsf{neg_a}}{\Box\neg p, \Diamond(p \wedge q) \Rightarrow}\,\mathsf{cut}$$

Let us now try to derive bottom-up the sequent without using cut. The bottom-most rule can only be $\mathsf{neg_a}$ or $\mathsf{neg_b}$, as these are the only rules with a conclusion of the right form. In the first case, the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg\neg p \Rightarrow p \wedge q$; while in the second case the premisses would be $\neg p, p \wedge q \Rightarrow$, and $\neg(p \wedge q) \Rightarrow \neg p$. It is clear, however, that in both cases the second premiss is not derivable.

Sequent calculi for logics with $C_\Box$

We now consider sequent calculi for logics containing the axiom $C_\Box$. We have seen in the case of monomodal $\Box$-logics that the rules for congruence and monotonicity of $\Box$ must be generalised to n principal boxed formulas in order to obtain cut-free calculi which capture $C_\Box$. For the same reason, interaction rules need to be generalised in an analogous way. In this regard, observe that the rules in Figure 7.5 do not provide cut-free calculi if combined with $\mathsf{C}_\Box$ or $\mathsf{MC}_\Box$, as the following example shows.

Example 7.3.3. The sequent $\Box p, \Box\neg p, \Diamond\top \Rightarrow$ is derivable by $\mathsf{MC}_\Box + \mathsf{weak_a} + \mathsf{weak_b} + \mathsf{cut}$, but is not derivable by $\mathsf{MC}_\Box + \mathsf{weak_a} + \mathsf{weak_b}$ without cut. The derivation with cut is as follows:

$$\dfrac{\mathsf{MC}_\Box\,\dfrac{p, \neg p \Rightarrow \bot}{\Box p, \Box\neg p \Rightarrow \Box\bot} \qquad \dfrac{\bot \Rightarrow \qquad \Rightarrow \top}{\Box\bot, \Diamond\top \Rightarrow}\,\mathsf{weak_b}}{\Box p, \Box\neg p, \Diamond\top \Rightarrow}\,\mathsf{cut}$$

In contrast, the sequent is not derivable without cut, as the only applicable rule would be $\mathsf{weak_b}$, but neither p nor $\neg p$ is a contradiction.

Suitable generalisations of rules $\mathsf{weak_b}$, $\mathsf{neg_a}$, str are displayed in Figure 7.6. Observe that the rule $\mathsf{weak_a}$ has not been modified, and more interestingly, that there is no rule corresponding to $\mathsf{neg_b}$. Concerning $\mathsf{weak_a}$, as a difference with other rules, the boxed formula which is principal in an application of $\mathsf{weak_a}$ occurs unboxed only in the right-hand side of the premiss, for this reason the rule does not need to be modified (as it is shown in the proof of Theorem 7.3.4).

Concerning $\mathsf{neg_b}$, its generalisation to n principal formulas would be as follows:

$$\mathsf{neg_b C}\,\dfrac{A_1, ..., A_n, B \Rightarrow \qquad \neg(A_1 \wedge ... \wedge A_n) \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}\ .$$

This rule is not analytic as the right premiss contains a conjunction which does not occur in the conclusion. In contrast with the case of the rules $\mathsf{C}_\Box$ and $\mathsf{neg_a}$, it is not possible to decompose the right premiss into simpler premisses. In particular, notice that taking the n premisses $\neg A_1 \Rightarrow B$, ..., $\neg A_n \Rightarrow B$ is not the same as taking $\neg(A_1 \wedge ... \wedge A_n) \Rightarrow B$, since $\neg(A_1 \wedge ... \wedge A_n) \supset \neg A_1 \vee ... \vee \neg A_n$ is not valid in intuitionistic logic. At present it is an open problem whether by adopting the rule $\mathsf{neg_b C}$ we would still obtain a cut-free calculus. For this reason we exclude this rule from the calculi for $C_\Box$, and we stipulate that the calculi $\mathbf{G3.IE_2C}^*$ contain only rule $\mathsf{neg_a C}$. As a consequence, the calculi $\mathbf{G3.IE_2C}^*$ are not proper extensions of $\mathbf{G3.IE_2}^*$.

As before, it can be easily proved that weakening and contraction are height-preserving admissible in the considered systems.

$$\mathsf{weak_a}\ \dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad\qquad \mathsf{weak_bC}\ \dfrac{A_1, ..., A_n \Rightarrow \qquad \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{neg_aC}\ \dfrac{A_1, ..., A_n, B \Rightarrow \qquad \neg B \Rightarrow A_1 \quad ... \quad \neg B \Rightarrow A_n}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{strC}\ \dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}$$

Figure 7.6: Modified interaction rules for $C_\Box$. For each rule we have $n \geq 1$.

G3.IE₁C	:=	$\{\mathsf{C}_\Box, \mathsf{E}_\Diamond, \mathsf{weak_a}, \mathsf{weak_bC}\}$			
G3.IE₂C	:=	$\{\mathsf{C}_\Box, \mathsf{E}_\Diamond, \mathsf{neg_aC}\}$	$\mathbf{G3.IECN}_\Diamond{}^*$	:=	$\mathbf{G3.IEC}^* \cup \{\mathsf{N}_\Diamond\}$
G3.IE₃C	:=	$\{\mathsf{C}_\Box, \mathsf{E}_\Diamond, \mathsf{strC}\}$	$\mathbf{G3.IECN}_\Box{}^*$	:=	$\mathbf{G3.IEC}^* \cup \{\mathsf{N}_\Diamond, \mathsf{N}_\Box\}$
G3.IMC	:=	$\{\mathsf{MC}_\Box, \mathsf{M}_\Diamond, \mathsf{strC}\}$			

Table 7.3: Gentzen calculi for intuitionistic non-normal bimodal logics with $C_\Box$.

Proposition 7.3.3. Weakening and contraction are height-preserving admissible in each sequent calculus defined by any combination of modal rules in Figures 7.3 and 7.6.

Following our methodology, we obtain again 12 sequent calculi, as stated by the following theorem:

Theorem 7.3.4. The rule cut is admissible in every calculus $\mathbf{G3.IEC}^*$ in Table 7.3.

Proof. As before, we only present some relevant cases.

- $(\mathsf{C}_\Box; \mathsf{weak_a})$. Let Γ_1 be the multiset $A_1, ..., A_n$, and $\Box\Gamma_1$ be $\Box A_1, ..., \Box A_n$.

$$\dfrac{\mathsf{C}_\Box\ \dfrac{\Gamma_1 \Rightarrow B \qquad B \Rightarrow A_1 \quad ... \quad B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{\Rightarrow B \qquad C \Rightarrow}{\Gamma', \Box B, \Diamond C \Rightarrow \Delta}\ \mathsf{weak_a}}{\Gamma, \Gamma', \Box\Gamma_1, \Diamond C \Rightarrow \Delta}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\ \dfrac{\Rightarrow B \qquad B \Rightarrow A_1}{\Rightarrow A_1} \qquad C \Rightarrow}{\Gamma, \Gamma', \Box A_1, \Box A_2, ..., \Box A_n, \Diamond C \Rightarrow \Delta}\ \mathsf{weak_a}$$

- $(\mathsf{C}_\Box; \mathsf{neg_aC})$. Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$.

$$\dfrac{\mathsf{C}_\Box\ \dfrac{\Gamma_1 \Rightarrow B \qquad B \Rightarrow A_1 \quad ... \quad B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow \qquad \neg D \Rightarrow B \qquad \neg D \Rightarrow C_1 \quad ... \quad \neg D \Rightarrow C_m}{\Gamma', \Box B, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}\ \mathsf{neg_bC}}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}\ \mathsf{cut}$$

$$\rightsquigarrow$$

$$\mathsf{C}_\Box\ \dfrac{\mathsf{cut}\ \dfrac{\Gamma_1 \Rightarrow B \qquad B, \Gamma_2, D \Rightarrow}{\Gamma_1, \Gamma_2, D \Rightarrow} \qquad \left(\mathsf{cut}\ \dfrac{\neg D \Rightarrow B \qquad B \Rightarrow A_i}{\neg D \Rightarrow A_i}\right)_{i=1}^{n} \qquad \neg D \Rightarrow C_1 \quad ... \quad \neg D \Rightarrow C_m}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}$$

$weak_a$	$\neg(\Box\top \wedge \Diamond\bot)$	$weak_b$	$\neg(\Box\bot \wedge \Diamond\top)$	str	$\dfrac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)}$
neg_a	$\neg(\Box\neg A \wedge \Diamond A)$	neg_b	$\neg(\Box A \wedge \Diamond\neg A)$		

Figure 7.7: Hilbert axioms and rules for interactions between $\Box$ and $\Diamond$.

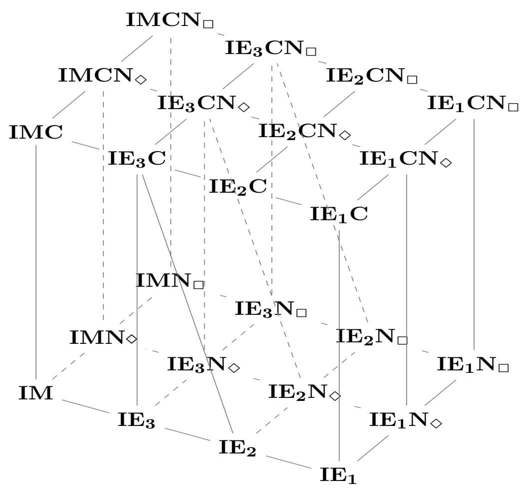

Figure 7.8: The lattice of intuitionistic non-normal bimodal logics **IE***.

- $(\mathsf{C}_\Box; \mathsf{strC})$. Let $\Gamma_1 = A_1, ..., A_n$ and $\Gamma_2 = C_1, ..., C_m$. We have:

$$\dfrac{\mathsf{C}_\Box\,\dfrac{\Gamma_1 \Rightarrow B \quad B \Rightarrow A_1 \ \dots \ B \Rightarrow A_n}{\Gamma, \Box\Gamma_1 \Rightarrow \Box B} \qquad \dfrac{B, \Gamma_2, D \Rightarrow}{\Gamma', \Box B, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}\,\mathsf{strC}}{\Gamma, \Gamma', \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow B \qquad B, \Gamma_2, D \Rightarrow}{\Gamma_1, \Gamma_2, D \Rightarrow}\,\mathsf{cut}}{\Gamma, \Box\Gamma_1, \Box\Gamma_2, \Diamond D \Rightarrow \Delta}\,\mathsf{strC}$$

$\Box$

Notably, the cut-free calculi in Theorem 7.3.4 are the $C_\Box$-versions of the cut-free calculi in Theorem 7.3.2, with the only exception of the calculi **G3.IE$_2$C*** which do not contain any rule corresponding to $\mathsf{neg_b}$. This means that, once the interaction rules are conveniently modified, the generalisation of the modal rules to n principal formulas preserves cut elimination.

Hilbert systems

For each sequent calculus, we now define an equivalent Hilbert system. To this purpose, in addition to the formulas in Figure 2.8, we also consider the interaction axioms and rules displayed in Figure 7.7. The Hilbert systems are defined by the set of modal axioms and rules that are added to **IPL**, as summarised in Table 7.4.

$$
\begin{array}{ll}
\mathbf{IE_1} := \{RE_\Box,\, RE_\Diamond,\, weak_a,\, weak_b\} & \mathbf{IE_1C} := \mathbf{IE_1} \cup \{C_\Box\} \\
\mathbf{IE_2} := \{RE_\Box,\, RE_\Diamond,\, neg_a,\, neg_b\} & \mathbf{IE_2C} := \{RE_\Box,\, RE_\Diamond,\, neg_a,\, C_\Box\} \\
\mathbf{IE_3} := \{RE_\Box,\, RE_\Diamond,\, str\} & \mathbf{IE_3C} := \mathbf{IE_3} \cup \{C_\Box\} \\
\mathbf{IM} := \{RE_\Box,\, RE_\Diamond,\, M_\Box,\, M_\Diamond,\, str\} & \mathbf{IMC} := \mathbf{IM} \cup \{C_\Box\}
\end{array}
$$

$$
\begin{array}{l}
\mathbf{IEN_\Diamond}^* := \mathbf{IE}^* \cup \{N_\Diamond\} \\
\mathbf{IEN_\Box}^* := \mathbf{IE}^* \cup \{N_\Box\}
\end{array}
$$

Table 7.4: Axiomatisations of intuitionistic non-normal bimodal logics $\mathbf{IE}^*$.

$$
(weak_a)\quad \cfrac{\cfrac{\cfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Box\top, \Diamond\bot \Rightarrow}\,\mathsf{weak_a}}{\Box\top \wedge \Diamond\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\top \wedge \Diamond\bot)}\,\mathsf{R}\neg
\qquad
(weak_b)\quad \cfrac{\cfrac{\cfrac{\Rightarrow \top \qquad \bot \Rightarrow}{\Diamond\top, \Box\bot \Rightarrow}\,\mathsf{weak_b}}{\Diamond\top \wedge \Box\bot \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Diamond\top \wedge \Box\bot)}\,\mathsf{R}\neg
$$

$$
(neg_a)\quad \cfrac{\cfrac{\cfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box\neg A, \Diamond A \Rightarrow}\,\mathsf{neg_a}}{\Box\neg A \wedge \Diamond A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box\neg A \wedge \Diamond A)}\,\mathsf{R}\neg
\qquad
(neg_b)\quad \cfrac{\cfrac{\cfrac{A, \neg A \Rightarrow \qquad \neg A \Rightarrow \neg A}{\Box A, \Diamond\neg A \Rightarrow}\,\mathsf{neg_b}}{\Box A \wedge \Diamond\neg A \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond\neg A)}\,\mathsf{R}\neg
$$

$$
(str)\quad \cfrac{\cfrac{\cfrac{\mathsf{cut}\,\cfrac{\Rightarrow \neg(A\wedge B) \qquad A, B, \neg(A\wedge B) \Rightarrow}{A, B \Rightarrow}}{\Box A, \Diamond B \Rightarrow}\,\mathsf{str}}{\Box A \wedge \Diamond B \Rightarrow}\,\mathsf{L}\wedge}{\Rightarrow \neg(\Box A \wedge \Diamond B)}\,\mathsf{R}\neg
$$

Figure 7.9: Derivations of interaction axioms and rules of $\mathbf{IE}^*$.

The relations among the different systems are depicted in Figure 7.8. Notice in particular that the systems $\mathbf{IE_2C}$, $\mathbf{IE_2CN_\Diamond}$, and $\mathbf{IE_2CN_\Box}$ are not extensions of, respectively, $\mathbf{IE_2}$, $\mathbf{IE_2N_\Diamond}$ and $\mathbf{IE_2N_\Box}$, as explained for the corresponding calculi on p. 193.

Theorem 7.3.5 (Syntactic equivalence)**.** Let $\mathbf{G3.L}$ be any sequent calculus for intuitionistic non-normal bimodal logics. Then $\mathbf{G3.L}$ is equivalent to the system $\mathbf{L}$, that is

$$\vdash_{\mathbf{G3.L}} \Gamma \Rightarrow \Delta \;\text{ if and only if }\; \vdash_{\mathbf{L}} \bigwedge\Gamma \supset \bigvee\Delta.$$

Proof. In Figure 7.9, we show that every axiom of $\mathbf{L}$ is derivable in $\mathbf{G3.L}$, and that every rule of $\mathbf{L}$ is admissible in $\mathbf{G3.L}$. We only consider the interactions between the modalities, since the derivations of the other axioms have been already shown in the proof of Theorem 7.2.3.

Conversely, we prove that every rule of $\mathbf{G3.L}$ is derivable in $\mathbf{L}$. As before, we only consider the interaction rules, we show the following illustrative derivations:

- If $\mathbf{L}$ contains the axiom $weak_a$, then the rule $\mathsf{weak_a}$ is derivable. Assume that $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} B \supset \bot$. Then $\vdash_{\mathbf{L}} \top \supset A$ and, since $\vdash_{\mathbf{L}} A \supset \top$, by $RE_\Box$ we have $\vdash_{\mathbf{L}} \Box A \supset \Box\top$. Moreover, since $\vdash_{\mathbf{L}} \bot \supset B$, by $RE_\Diamond$ we have $\vdash_{\mathbf{L}} \Diamond B \supset \Diamond\bot$, hence $\vdash_{\mathbf{L}} \neg\Diamond\bot \supset \neg\Diamond B$. By $weak_a$ we also have $\vdash_{\mathbf{L}} \Box\top \supset \neg\Diamond\bot$. Thus $\vdash_{\mathbf{L}} \Box A \supset \neg\Diamond B$, which gives $\vdash_{\mathbf{L}} \neg(\Box A \wedge \Diamond B)$.

• If $\mathbf{L}$ contains the axioms $C_\Box$ and $weak_b$, then the rule $\mathsf{weak_bC}$ is derivable. Assume $\vdash_\mathbf{L} A_1 \land ... \land A_n \supset \bot$ and $\vdash_\mathbf{L} B$. Then $\vdash_\mathbf{L} A_1 \land ... \land A_n \supset\subset \bot$ and $\vdash_\mathbf{L} B \supset\subset \top$. By $RE_\Box$, $\vdash_\mathbf{L} \Box(A_1 \land ... \land A_n) \supset \Box\bot$, and by considering axiom $C_\Box$ $n-1$ times, $\vdash_\mathbf{L} \Box A_1 \land ... \land \Box A_n \supset \Box(A_1 \land ... \land A_n)$. Moreover, by $RE_\Diamond$we have $\vdash_\mathbf{L} \Diamond B \supset\subset \Diamond\top$, whence $\vdash_\mathbf{L} \neg\Diamond\top \supset\subset \neg\Diamond B$. Then $\vdash_\mathbf{L} \Box A_1 \land ... \land \Box A_n \land \Diamond B \supset \bot$.

• If $\mathbf{L}$ contains the axiom neg_b, then the rule $\mathsf{neg_b}$ is derivable. Assume $\vdash_\mathbf{L} \neg(A \land B)$ – that is $\vdash_\mathbf{L} B \supset \neg A$ – and $\vdash_\mathbf{L} \neg B \supset A$. Then, by $RE_\Diamond$, $\vdash_\mathbf{L} \Diamond B \supset \Diamond\neg A$. By neg_b we have $\vdash_\mathbf{L} \Diamond\neg A \supset \neg\Box A$. Thus $\vdash_\mathbf{L} \Diamond B \supset \neg\Box A$, which gives $\vdash_\mathbf{L} \neg(\Box A \land \Diamond B)$.

• If $\mathbf{L}$ contains the axioms $C_\Box$ and neg_a, then the rule $\mathsf{neg_aC}$ is derivable. Assume $\vdash_\mathbf{L}$ $A_1 \land ... \land A_n \land B \supset \bot$ and $\vdash_\mathbf{L} \neg B \supset A_1$, ..., $\vdash_\mathbf{L} \neg B \supset A_n$. Then $\vdash_\mathbf{L} A_1 \land ... \land A_n \supset \neg B$ and $\vdash_\mathbf{L} \neg B \supset A_1 \land ... \land A_n$. By $RE_\Box$, $\vdash_\mathbf{L} \Box(A_1 \land ... \land A_n) \supset \Box\neg B$, and by considering axiom $C_\Box$ $n-1$ times, $\vdash_\mathbf{L} \Box A_1 \land ... \land \Box A_n \supset \Box(A_1 \land ... \land A_n)$. Moreover, by neg_a we have $\vdash_\mathbf{L} \Box\neg B \supset \neg\Diamond B$. Then $\vdash_\mathbf{L} \Box A_1 \land ... \land \Box A_n \land \Diamond B \supset \bot$. □

7.4 Decidability, and other consequences of cut elimination

In this section, we take advantage of the admissibility of cut in all sequent calculi defined in Sections 7.2 and 7.3 in order to prove additional properties of the corresponding logics, most importantly decidability. Looking at the shape of the rules, we first observe that all calculi satisfy all the requirements on intuitionistic non-normal modal logics that we have initially assumed, i.e., that they are conservative over **IPL** (R1); that they satisfy the disjunction property (R2); that the duality principles $Dual_\Box$ and $Dual_\Diamond$ are not derivable (R3); and that the axiom $C_\Diamond$ is not derivable (R4). In a similar way, we prove that all calculi are pairwise distinct, hence the lattices of intuitionistic non-normal modal logics contain, respectively, 8 distinct monomodal $\Box$-logics, 4 distinct monomodal $\Diamond$-logics, and 24 distinct bimodal logics.

Proposition 7.4.1. Every intuitionistic non-normal modal logic defined in Sections 7.2 and 7.3 satisfies the requirements (R1), (R2), (R3), and (R4) (the third one being only relevant for bimodal logics).

Proof. (R1) Every logic is conservative over **IPL**: the non-modal rules of each sequent calculus are exactly the rules of **G3ip**.

(R2) Every logic satisfies the disjunction property: Assume $\vdash_\mathbf{L} A \lor B$. Then, by Theorems 7.2.3 and 7.6.6, $\vdash_{\mathbf{G3.L}} \Rightarrow A \lor B$. Moreover, we observe that the only rule of **G3.L** with a conclusion of this form is $\mathsf{R}\lor$, whence this is necessarily the last rule applied in the derivation of $\Rightarrow A \lor B$. This has premiss $\Rightarrow A$ or $\Rightarrow B$, which in turn is derivable. Thus $\vdash_\mathbf{L} A$ or $\vdash_\mathbf{L} B$.

(R3) For any system $\mathbf{L}$, the axioms $Dual_\Box$ and $Dual_\Diamond$ are not derivable in $\mathbf{L}$ for an arbitrary A. In particular, neither $\neg\Box\neg p \supset \Diamond p$, nor $\neg\Diamond\neg p \supset \Box p$ (instances of the right-to-left implication of $Dual_\Box$ and $Dual_\Diamond$) is derivable. For instance, if we try to derive bottom-up the sequent $\neg\Box\neg p \Rightarrow \Diamond p$ in **G3**.**L**, the only applicable rule would be $\mathsf{L}\supset$. This has premiss $\neg\Box\neg p \Rightarrow \Box\neg p$. Again, $\mathsf{L}\supset$ is the only applicable rule, with the same sequent as premiss (or, if contained by **G3**.**L**, we could apply $\mathsf{N}_\Box$ and get the non-derivable sequent $\Rightarrow \neg p$). Since $\neg\Box\neg p \Rightarrow \Box\neg p$ is not an initial sequent, we have that $\neg\Box\neg p \Rightarrow \Diamond p$ is not derivable. The situation is analogous for $\neg\Diamond\neg p \Rightarrow \Box p$.

(R4) For any system **L**, $C_\Diamond$ is not derivable: Let us consider the sequent $\Rightarrow \Diamond(p\vee q) \supset \Diamond p\vee\Diamond q$. The only backward applicable rule is $\mathsf{R}\supset$, which gives $\Diamond(p \vee q) \Rightarrow \Diamond p \vee \Diamond q$. The backward applicable rules are now (i) $\mathsf{N}_\Diamond$ (if is contained by the calculus), which gives the non-derivable sequent $p\vee q \Rightarrow$, or (ii) $\mathsf{R}\vee$, which gives either $\Diamond(p\vee q) \Rightarrow \Diamond p$, or $\Diamond(p\vee q) \Rightarrow \Diamond q$. Considering $\Diamond(p\vee q) \Rightarrow \Diamond p$, we can apply $\mathsf{N}_\Diamond$, falling again into case (i); alternatively, we can apply $\mathsf{E}_\Box$ (if the calculus is non-monotonic), or $\mathsf{M}_\Box$ (if the calculus is monotonic). In both cases we obtain the non-derivable premiss $p \vee q \Rightarrow p$. The analysis for $\Diamond(p \vee q) \Rightarrow \Diamond q$ is analogous. □

Theorem 7.4.2. The lattice of intuitionistic non-normal bimodal logics contains 24 *distinct* systems.

Proof. Given two logics $\mathbf{L_1}$ and $\mathbf{L_2}$ of the lattice, we can always find some formulas (or rules) that are derivable in $\mathbf{L_1}$ and not in $\mathbf{L_2}$, or vice versa. This can be easily done by considering the corresponding calculi **G3**.$\mathbf{L_1}$ and **G3**.$\mathbf{L_2}$. In particular, if $\mathbf{L_1}$ is stronger than $\mathbf{L_2}$, then some characteristic axioms or rule of $\mathbf{L_1}$ are not derivable in $\mathbf{L_2}$. If instead $\mathbf{L_1}$ and $\mathbf{L_2}$ are incomparable, then they both have some characteristic axioms (or rules) that are not derivable in the other. For instance, Example 7.3.2 shows that the rule *str* is not derivable in $\mathbf{IE_2}$. □

On the basis of the cut-free calculi we can also compare our systems with the intuitionistic logics **CK** and **CCDL** (cf. the axiomatisations in Section 2.6 and the Gentzen calculi in Section 3.6). First, we can observe that the axiom $K_\Diamond$, which belongs to both **CK** and **CCDL**, is not derivable even in the strongest calculus **G3**.$\mathbf{IMCN}_\Box$: in bottom-up proof search for $\Rightarrow \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$ we can only apply twice $\mathsf{R}\supset$ and obtain the sequent $\Box(p \supset q), \Diamond p \Rightarrow \Diamond q$. Then at this stage we can apply either strC, thus obtaining $p \supset q, p \Rightarrow$, or $\mathsf{N}_\Diamond$, thus obtaining $p \Rightarrow$, both of them non-derivable. By contrast, all the rules of **G3**.$\mathbf{IMCN}_\Box$ also belong to **G3**.**CCDL**. Therefore we have:

Proposition 7.4.3. Every intuitionistic non-normal modal logic $\mathbf{IE}^*$ in Table 7.4 is strictly weaker of **CCDL**, that is: $Thm_{\mathbf{IE}^*} \subsetneq Thm_{\mathbf{CCDL}}$.

By contrast, **CK** is not comparable even with the weakest logic $\mathbf{IE_1}$, since the axioms $weak_a$ and $weak_b$ are not derivable in **G3**.**CK**. For instance, in oder to derive $weak_a$, the calculus should derive the sequent $\Box\top, \Diamond\bot \Rightarrow$, but this is neither an initial sequent, nor the conclusion of any rule of **G3**.**CK**. Then we have:

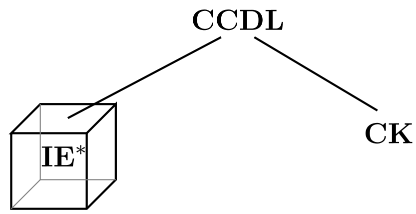

Figure 7.10: Extended lattice of intuitionistic non-normal modal logics.

Proposition 7.4.4. Every intuitionistic non-normal modal logic $\mathbf{IE}^*$ in Table 7.4 is incomparable with $\mathbf{CK}$, that is: $Thm_{\mathbf{IE}^*} \not\subseteq Thm_{\mathbf{CK}}$ and $Thm_{\mathbf{CK}} \not\subseteq Thm_{\mathbf{IE}^*}$.

The relations among the intuitionistic systems stated by the above propositions can be schematised as in Figure 7.10.

On the basis of the cut-free calculi we can also prove that the logics are decidable. As usual, this is a consequence of the subformula property. However, for the calculi containing the rules $\mathsf{neg_a}$ (or $\mathsf{neg_a C}$) and $\mathsf{neg_b}$, we need to slightly relax the property by considering $\neg A$ as a "subformula" of $\Box A$ and $\Diamond A$. We extend Definition 2.2.3 as follows:

Definition 7.4.1 (Strict subformula and negated subformula)**.** For any formulas A and B, we say that A is a *strict subformula* of B if A is a subformula of B and $A \neq B$. Moreover, we say that A is a *negated subformula* of B if there is a formula C such that C is a strict subformula of B and $A = \neg C$.

Definition 7.4.2 (Subformula property and negated subformula property)**.** We say that a sequent calculus $\mathbf{G3.L}$ enjoys the *subformula property* if all formulas in any derivation are subformulas of the endsequent. We say that $\mathbf{G3.L}$ enjoys the *negated subformula property* if all formulas in any derivation are either subformulas or negated subformulas of the endsequent.

The following result is an immediate consequence of cut elimination:

Theorem 7.4.5. Every sequent calculus different from $\mathbf{G3.IE_2(C/N_\Diamond/N_\Box)}$ enjoys the subformula property. Moreover, the calculi $\mathbf{G3.IE_2(C/N_\Diamond/N_\Box)}$ enjoy the negated subformula property.

Given the subformula property we can easily define a terminating proof search procedure in our calculi. Notice however that due to the presence of the rule $\mathsf{L\supset}$ which copies the principal formula $A \supset B$ into the left premiss, the calculi are not strictly analytic, meaning as usual that the complexity of the premisses is not strictly smaller than the complexity of the conclusion (cf. Section 3.3) As a consequence, some restriction in the backward application of the rules is needed in order to ensure termination. Nonetheless, since $\mathsf{L\supset}$ is the only non strictly analytic rule (in particular, all modal rules are strictly analytic), it suffices to consider the standard loop checking in G3-style calculi for intuitionistic logic, such as the one defined in Troelstra and Schwichtenberg [164], p. 109. We formulate it as follows:

Definition 7.4.3 (Proof search strategy). A proof of $\Gamma_0 \Rightarrow \Delta_0$ is constructed bottom-up by applying the rules of **G3.L** backwards respecting the following condition: given a branch $\mathcal{B} = \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$ of the proof, a rule R is not applied to $\Gamma_n \Rightarrow \Delta_n$ if (i) $\Gamma_n \Rightarrow \Delta_n$ is an initial sequent, or (ii) there is a premiss $\Sigma \Rightarrow \Pi$ of R and a sequent $\Gamma_i \Rightarrow \Delta_i$ in $\mathcal{B}$ such that $\mathsf{set}(\Sigma) = \mathsf{set}(\Gamma_i)$ and $\Pi = \Delta_i$.

This strategy ensures termination of proof search as it prevents loops that can be created by unrestricted applications of $\mathsf{L}\supset$. For instance, applications of $\mathsf{L}\supset$ like the following are not allowed:

$$\frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow A}{\Gamma, A \supset B \Rightarrow A}\ \mathsf{L}\supset$$

By adopting this strategy, given a root sequent $\Gamma \Rightarrow \Delta$, every proof of $\Gamma \Rightarrow \Delta$ is finite. Notice that, since the calculi are not invertible, a single failed proof is not sufficient to ensure the non-derivability of the root sequent. However, because of the subformula property every sequent has only a finite number of possible proofs. Then the decision procedure will trivially consist in checking all possible proofs. We obtain as a consequence the following result:

Theorem 7.4.6 (Decidability). For every intuitionistic non-normal modal logic defined in Sections 7.2 and 7.3, it is decidable whether a given formula is derivable.

7.5 Craig's interpolation

In this section, we take advantage of the admissibility of cut in our calculi **G3.IE*** to investigate the property of Craig's interpolation in the corresponding logics.

Definition 7.5.1 (Craig's interpolation). A logic **L** enjoys *Craig's interpolation* if for every $A, B \in \mathcal{L}_i$, if $\vdash_{\mathbf{L}} A \supset B$, then there is $I \in \mathcal{L}_i$ such that $\vdash_{\mathbf{L}} A \supset I$, and $\vdash_{\mathbf{L}} I \supset B$, and $var(I) \subseteq var(A) \cap var(B)$.

We are able to prove Craig's interpolation for the systems $\mathbf{IE_1}$, $\mathbf{IE_2}$, **IM**, **IMC**, and their extensions with the axioms $N_\Diamond$ or $N_\Box$, by means of a methodology based on cut-free sequent calculi introduced by Maehara [118] (in contrast this methodology does not seem to be adequate for the non-monotonic calculi with the rules for $C_\Box$). For every derivable sequent $A \Rightarrow B$, Maehara's method allows one to find a suitable intepolant I such that $A \Rightarrow I$ and $I \Rightarrow B$ are derivable, thus providing a constructive proof of interpolation. The same method is used to prove Craig's interpolation for **CCDL** in Wijesekera [170] and for the non-normal modal logics of the classical cube in Orlandelli [140]. The crucial statement to be proved, formulated for our systems, is the following.

Proposition 7.5.1. Let **L** be any of the systems $\mathbf{IE_1}$, $\mathbf{IE_2}$, **IM**, **IMC**, or their extensions with $N_\Diamond$ or $N_\Box$. If $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ is derivable in **G3.L**, then there is $I \in \mathcal{L}_i$ such that $\Gamma_1 \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable in **G3.L**, and $var(I) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \Delta)$.

Proof. Assume $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ derivable in **G3.L**. We show how to construct the interpolant I by induction on the height n of the derivation of $\Gamma_1, \Gamma_2 \Rightarrow \Delta$. In the proof we write $\Gamma_1', A; B, \Gamma_2' \Rightarrow \Delta$ to denote the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ where $\Gamma_1 = \Gamma_1', A$ and $\Gamma_2 = B, \Gamma_2'$.

If $n = 0$, then $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ is an initial sequent, and has one of the forms below on the left. For each possibility, we establish the interpolant I and show that $\Gamma_1 \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable.

(i) $\bot, \Gamma_1'; \Gamma_2 \Rightarrow \Delta \quad \rightsquigarrow \quad I = \bot$: $\dfrac{}{\bot, \Gamma_1' \Rightarrow \bot}\,\mathsf{L}\bot \qquad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\Gamma_1; \bot, \Gamma_2' \Rightarrow \Delta \quad \rightsquigarrow \quad I = \top$: $\dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \qquad \dfrac{}{\top, \bot, \Gamma_2' \Rightarrow \Delta}\,\mathsf{L}\bot$

(iii) $\Gamma_1, \Gamma_2 \Rightarrow \top \quad \rightsquigarrow \quad I = \top$: $\dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \qquad \dfrac{}{\top, \Gamma_2 \Rightarrow \top}\,\mathsf{R}\top$

(iv) $p, \Gamma_1'; \Gamma_2 \Rightarrow p \quad \rightsquigarrow \quad I = p$: $\dfrac{}{p, \Gamma_1' \Rightarrow p}\,\mathsf{init} \qquad \dfrac{}{p, \Gamma_2 \Rightarrow p}\,\mathsf{init}$

(v) $\Gamma_1; p, \Gamma_2' \Rightarrow p \quad \rightsquigarrow \quad I = \top$: $\dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \qquad \dfrac{}{p, \Gamma_2' \Rightarrow p}\,\mathsf{init}$

In all these cases, it is easy to see that $var(I) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \Delta)$.

If $n \geq 1$, we consider the last rule R applied in the derivation of $\Gamma_1, \Gamma_2 \Rightarrow \Delta$. The cases of propositional rules are standard: we show as examples the cases $R = \mathsf{L}\wedge$, and $R = \mathsf{R}\wedge$, and then we consider the modal rules.

• If the last rule applied is $\mathsf{L}\wedge$, then we must consider two cases, depending whether the principal formula $A \wedge B$ belongs to Γ_1 or Γ_2.

(i) $$\dfrac{A, B, \Gamma_1'; \Gamma_2 \Rightarrow \Delta}{A \wedge B, \Gamma_1'; \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\wedge$$

By i.h., there is I such that $A, B, \Gamma_1' \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable, and $var(I) \subseteq var(A, B, \Gamma_1') \cap var(\Gamma_2, \Delta)$. By applying $\mathsf{L}\wedge$ to the first sequent we obtain $A \wedge B, \Gamma_1' \Rightarrow I$. Moreover, $var(I) \subseteq var(A \wedge B, \Gamma_1') \cap var(\Gamma_2, \Delta)$. Then we have the claim.

(ii) $$\dfrac{\Gamma_1; A, B, \Gamma_2' \Rightarrow \Delta}{\Gamma_1; A \wedge B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{L}\wedge$$

By i.h., there is I such that $\Gamma_1 \Rightarrow I$ and $I, A, B, \Gamma_2' \Rightarrow \Delta$ are derivable, and $var(I) \subseteq var(\Gamma_1) \cap var(A, B, \Gamma_2', \Delta)$. By applying $\mathsf{L}\wedge$ to the second sequent we obtain $A \wedge B, \Gamma_2' \Rightarrow \Delta$. Moreover, $var(I) \subseteq var(\Gamma_1) \cap var(A \wedge B, \Gamma_2', \Delta)$. Then we have the claim.

• If the last rule applied is $\mathsf{R}\wedge$, we have

$$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow A \qquad \Gamma_1, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B}\,\mathsf{R}\wedge$$

By i.h., there are I_1, I_2 such that (i) $\Gamma_1 \Rightarrow I_1$, (ii) $I_1, \Gamma_2 \Rightarrow A$, (iii) $\Gamma_1 \Rightarrow I_2$, and (iv) $I_2, \Gamma_2 \Rightarrow B$ are derivable, and $var(I_1) \subseteq var(\Gamma_1) \cap var(\Gamma_2, A)$ and $var(I_2) \subseteq var(\Gamma_1) \cap var(\Gamma_2, B)$. We establish $I = I_1 \wedge I_2$ and consider the following derivations:

$$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow I_1 \qquad \Gamma_1, \Gamma_2 \Rightarrow I_2}{\Gamma_1, \Gamma_2 \Rightarrow I_1 \wedge I_2}\,\mathsf{R}\wedge \qquad\qquad \dfrac{\mathsf{L}\wedge\,\dfrac{\mathsf{Rwk}\,\dfrac{I_1, \Gamma_2 \Rightarrow A}{I_1, I_2, \Gamma_2 \Rightarrow A}}{I_1 \wedge I_2, \Gamma_2 \Rightarrow A} \qquad \dfrac{\dfrac{I_2, \Gamma_2 \Rightarrow B}{I_1, I_2, \Gamma_2 \Rightarrow B}\,\mathsf{Rwk}}{I_1 \wedge I_2, \Gamma_2 \Rightarrow B}\,\mathsf{L}\wedge}{I_1 \wedge I_2, \Gamma_2 \Rightarrow A \wedge B}\,\mathsf{R}\wedge$$

where $var(I_1 \wedge I_2) = var(I_1) \cup var(I_2) \subseteq var(\Gamma_1) \cap (var(\Gamma_2, A) \cup var(\Gamma_2, B)) = var(\Gamma_1) \cap var(\Gamma_2, A \wedge B)$

• If the last rule applied is $\mathsf{E}_\Box$ we must consider two cases, depending whether the principal formula $\Box A$ in the left-hand side of the conclusion belongs to Γ_1 or Γ_2.

(i) $\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma_1', \Box A; \Gamma_2 \Rightarrow \Box B}\,\mathsf{E}_\Box$

By i.h., there are I_1 and I_2 such that $\vdash_{\mathbf{G3.L}} A \Rightarrow I_1$; and $\vdash_{\mathbf{G3.L}} I_1 \Rightarrow B$; and $\vdash_{\mathbf{G3.L}} B \Rightarrow I_2$; and $\vdash_{\mathbf{G3.L}} I_2 \Rightarrow A$; where $var(I_1), var(I_2) \subseteq var(A) \cap var(B)$. We establish I as folllows, and show that $\Gamma_1 \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable.

$$\rightsquigarrow \quad I = \Box I_1: \quad \dfrac{A \Rightarrow I_1 \qquad \dfrac{I_1 \Rightarrow B \qquad B \Rightarrow A}{I_1 \Rightarrow A}\,\mathsf{cut}}{\Gamma_1', \Box A \Rightarrow \Box I_1}\,\mathsf{E}_\Box \;; \qquad \dfrac{I_1 \Rightarrow B \qquad \dfrac{B \Rightarrow A \qquad A \Rightarrow I_1}{B \Rightarrow I_1}\,\mathsf{cut}}{\Box I_1, \Gamma_2 \Rightarrow \Box B}\,\mathsf{E}_\Box$$

(ii) $\dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\Gamma_1; \Box A, \Gamma_2' \Rightarrow \Box B}\,\mathsf{E}_\Box \quad \rightsquigarrow \quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \;; \quad \dfrac{A \Rightarrow B \qquad B \Rightarrow A}{\top, \Box A, \Gamma_2' \Rightarrow \Box B}\,\mathsf{E}_\Box$

• If the last rule applied is $\mathsf{E}_\Diamond$, $\mathsf{M}_\Box$, or $\mathsf{M}_\Diamond$, the proof is similar to the case $\mathsf{E}_\Box$.

• If the last rule applied is $\mathsf{N}_\Box$, we have:

$$\dfrac{\Rightarrow A}{\Gamma_1, \Gamma_2 \Rightarrow \Box A}\,\mathsf{N}_\Box \quad \rightsquigarrow \quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \;; \quad \dfrac{\Rightarrow A}{\top, \Gamma_2 \Rightarrow \Box A}\,\mathsf{N}_\Box$$

• If the last rule applied is $\mathsf{N}_\Diamond$, we have the following two possibilities.

(i) $\dfrac{A \Rightarrow}{\Gamma_1', \Diamond A; \Gamma_2 \Rightarrow \Delta}\,\mathsf{N}_\Diamond \quad \rightsquigarrow \quad I = \bot: \quad \dfrac{A \Rightarrow}{\Gamma_1', \Diamond A \Rightarrow \bot}\,\mathsf{N}_\Diamond \;; \quad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{A \Rightarrow}{\Gamma_1; \Diamond A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{N}_\Diamond \quad \rightsquigarrow \quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top \;; \quad \dfrac{A \Rightarrow}{\top, \Diamond A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{N}_\Diamond$

• If the last rule applied is $\mathsf{weak_a}$ we have the following four possibilities. For each case we establish I and show that $\Gamma_1 \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable.

(i) $\dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma_1', \Box A, \Diamond B; \Gamma_2 \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \bot: \quad \dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma_1', \Box A, \Diamond B \Rightarrow \bot}\,\mathsf{weak_a}\ ; \qquad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma_1; \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \qquad \dfrac{\Rightarrow A \qquad B \Rightarrow}{\top, \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_a}$

(iii) $\dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma_1', \Box A; \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \Box\top: \quad \dfrac{\mathsf{R}\top\,\dfrac{}{A \Rightarrow \top} \qquad \dfrac{\Rightarrow A}{\top \Rightarrow A}\,\mathsf{Lwk}}{\Gamma_1', \Box A \Rightarrow \Box\top}\,\mathsf{E}_\Box\ ; \qquad \dfrac{\mathsf{R}\top\,\dfrac{}{\Rightarrow \top} \qquad B \Rightarrow}{\Box\top, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_a}$

(iv) $\dfrac{\Rightarrow A \qquad B \Rightarrow}{\Gamma_1', \Diamond B; \Box A, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \Diamond\bot: \quad \dfrac{\mathsf{Rwk}\,\dfrac{B \Rightarrow}{B \Rightarrow \bot} \qquad \dfrac{}{\bot \Rightarrow B}\,\mathsf{L}\bot}{\Gamma_1', \Diamond B \Rightarrow \Diamond\bot}\,\mathsf{E}_\Diamond\ ; \qquad \dfrac{\Rightarrow A \qquad \dfrac{}{\bot \Rightarrow}\,\mathsf{L}\bot}{\Diamond\bot, \Box A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_a}$

• If the last rule applied is $\mathsf{weak_b}$ we have the following four possibilities. For each case we establish I and show that $\Gamma_1 \Rightarrow I$ and $I, \Gamma_2 \Rightarrow \Delta$ are derivable. In all cases $var(I) = \emptyset$, whence $var(I) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \Delta)$.

(i) $\dfrac{A \Rightarrow \qquad \Rightarrow B}{\Gamma_1', \Box A, \Diamond B; \Gamma_2 \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \bot: \quad \dfrac{A \Rightarrow \qquad \Rightarrow B}{\Gamma_1', \Box A, \Diamond B \Rightarrow \bot}\,\mathsf{weak_b}\ ; \qquad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{A \Rightarrow \qquad \Rightarrow B}{\Gamma_1; \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \qquad \dfrac{A \Rightarrow \qquad \Rightarrow B}{\top, \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_b}$

(iii) $\dfrac{A \Rightarrow \qquad \Rightarrow B}{\Gamma_1', \Box A; \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \Box\bot: \quad \dfrac{\mathsf{Rwk}\,\dfrac{A \Rightarrow}{A \Rightarrow \bot} \qquad \dfrac{}{\bot \Rightarrow A}\,\mathsf{L}\bot}{\Gamma_1', \Box A \Rightarrow \Box\bot}\,\mathsf{E}_\Box\ ; \qquad \dfrac{\mathsf{L}\bot\,\dfrac{}{\bot \Rightarrow} \qquad \Rightarrow B}{\Box\bot, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_b}$

(iv) $\dfrac{A \Rightarrow \qquad \Rightarrow B}{\Gamma_1', \Diamond B; \Box A, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \Diamond\top: \quad \dfrac{\mathsf{R}\top\,\dfrac{}{B \Rightarrow \top} \qquad \dfrac{\Rightarrow B}{\top \Rightarrow B}\,\mathsf{Lwk}}{\Gamma_1', \Diamond B \Rightarrow \Diamond\top}\,\mathsf{E}_\Diamond\ ; \qquad \dfrac{A \Rightarrow \qquad \dfrac{}{\Rightarrow \top}\,\mathsf{R}\top}{\Diamond\top, \Box A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{weak_b}$

• If the last rule applied is $\mathsf{neg_a}$ we have the following four possibilities.

(i) $\dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma_1', \Box A, \Diamond B; \Gamma_2 \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \bot: \quad \dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma_1', \Box A, \Diamond B \Rightarrow \bot}\,\mathsf{neg_b}\ ; \qquad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma_1; \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad\rightsquigarrow\quad I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \qquad \dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\top, \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{neg_b}$

(iii) $\dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma_1', \Box A; \Diamond B, \Gamma_2' \Rightarrow \Delta}$

By i.h., there are I_1 and I_2 such that $A \Rightarrow I_1$, and $I_1, B \Rightarrow$, and $\neg B \Rightarrow I_2$, and $I_2 \Rightarrow A$ are derivable, and $var(I_1), var(I_2) \subseteq var(A) \cap var(B)$. We establish $I = \Box I_1$ and consider the following derivations.

$$\dfrac{A \Rightarrow I_1 \qquad \dfrac{\mathsf{R}\neg\,\dfrac{I_1, B \Rightarrow}{I_1 \Rightarrow \neg B} \qquad \neg B \Rightarrow A}{I_1 \Rightarrow A}\,\mathsf{cut}}{\Gamma_1', \Box A \Rightarrow \Box I_1}\,\mathsf{E}_\Box \qquad\qquad \dfrac{I_1, B \Rightarrow \qquad \dfrac{\neg B \Rightarrow A \qquad A \Rightarrow I_1}{\neg B \Rightarrow I_1}\,\mathsf{cut}}{\Box I_1, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{neg_a}$$

(iv) $\dfrac{A, B \Rightarrow \qquad \neg B \Rightarrow A}{\Gamma_1', \Diamond B; \Box A, \Gamma_2' \Rightarrow \Delta}$

By i.h., there are I_1 and I_2 such that $A \Rightarrow I_1$, and $I_1, B \Rightarrow$, and $\neg B \Rightarrow I_2$, and $I_2 \Rightarrow A$ are derivable, and $var(I_1), var(I_2) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \Delta)$. We establish $I = \neg\Box I_1$ and consider the following derivations.

$$\dfrac{\dfrac{I_1, B \Rightarrow \qquad \dfrac{\neg B \Rightarrow A \qquad A \Rightarrow I_1}{\neg B \Rightarrow I_1}\,\mathsf{cut}}{\Gamma_1', \Diamond B, \Box I_1 \Rightarrow}\,\mathsf{neg_a}}{\Gamma_1', \Diamond B \Rightarrow \neg\Box I_1}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow I_1 \qquad \dfrac{\mathsf{R}\neg\,\dfrac{I_1, B \Rightarrow}{I_1 \Rightarrow \neg B} \qquad \neg B \Rightarrow A}{I_1 \Rightarrow A}\,\mathsf{cut}}{\Box A, \Gamma_2' \Rightarrow \Box I_1}\,\mathsf{E}_\Box}{\neg\Box I_1, \Box A, \Gamma_2' \Rightarrow}\,\mathsf{L}\neg}{\neg\Box I_1, \Box A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{Rwk}$$

• If the last rule applied is $\mathsf{neg_b}$ we have the following four possibilities.

(i) $\dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma_1', \Box A, \Diamond B; \Gamma_2 \Rightarrow \Delta} \quad \rightsquigarrow \quad I = \bot\text{:} \quad \dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma_1', \Box A, \Diamond B \Rightarrow \bot}\,\mathsf{neg_b}\ ; \quad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma_1; \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad \rightsquigarrow \quad I = \top\text{:} \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \quad \dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\top, \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{neg_b}$

(iii) $\dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma_1', \Box A; \Diamond B, \Gamma_2' \Rightarrow \Delta}$

By i.h., there are I_1 and I_2 such that $A \Rightarrow I_1$, and $I_1, B \Rightarrow$, and $\neg A \Rightarrow I_2$, and $I_2 \Rightarrow B$ are derivable, and $var(I_1), var(I_2) \subseteq var(A) \cap var(B)$. We establish $I = \neg\Diamond I_2$ and consider the following derivations.

$$\dfrac{\dfrac{\mathsf{cut}\,\dfrac{I_2 \Rightarrow B \qquad A, B \Rightarrow}{A, I_2 \Rightarrow} \qquad \neg A \Rightarrow I_2}{\Gamma_1', \Box A, \Diamond I_2 \Rightarrow}\,\mathsf{neg_b}}{\Gamma_1', \Box A \Rightarrow \neg\Diamond I_2}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{\dfrac{\mathsf{cut}\,\dfrac{\mathsf{R}\neg\,\dfrac{A, B \Rightarrow}{B \Rightarrow \neg A} \qquad \neg A \Rightarrow I_2}{B \Rightarrow I_2} \qquad I_2 \Rightarrow B}{\Diamond B, \Gamma_2' \Rightarrow \Diamond I_2}\,\mathsf{E}_\Diamond}{\neg\Diamond I_2, \Diamond B, \Gamma_2' \Rightarrow}\,\mathsf{L}\neg}{\neg\Diamond I_2, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{Rwk}$$

(iv) $\dfrac{A, B \Rightarrow \qquad \neg A \Rightarrow B}{\Gamma_1', \Diamond B; \Box A, \Gamma_2' \Rightarrow \Delta}$

By i.h., there are I_1 and I_2 such that $A \Rightarrow I_1$, and $I_1, B \Rightarrow$, and $\neg A \Rightarrow I_2$, and $I_2 \Rightarrow B$ are derivable, and $var(I_1), var(I_2) \subseteq var(A) \cap var(B)$. We establish $I = \Diamond I_2$ and consider the following derivations.

$$\dfrac{\mathsf{cut}\,\dfrac{\mathsf{R}\neg\,\dfrac{A, B \Rightarrow}{B \Rightarrow \neg A} \qquad \neg A \Rightarrow I_2}{B \Rightarrow I_2} \qquad I_2 \Rightarrow B}{\Gamma_1', \Diamond B \Rightarrow \Diamond I_2}\,\mathsf{E}_\Diamond \qquad \dfrac{\mathsf{cut}\,\dfrac{I_2 \Rightarrow B \qquad A, B \Rightarrow}{A, I_2 \Rightarrow} \qquad \neg A \Rightarrow I_2}{\Diamond I_2, \Box A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{neg_b}$$

• If the last rule applied is str we have the following four possibilities.

(i) $\dfrac{A, B \Rightarrow}{\Gamma_1', \Box A, \Diamond B; \Gamma_2 \Rightarrow \Delta} \quad \rightsquigarrow \quad I = \bot\text{:} \quad \dfrac{A, B \Rightarrow}{\Gamma_1', \Box A, \Diamond B \Rightarrow \bot}\,\mathsf{str}\ ; \quad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$

(ii) $\dfrac{A, B \Rightarrow}{\Gamma_1; \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta} \quad \rightsquigarrow \quad I = \top\text{:} \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \quad \dfrac{A, B \Rightarrow}{\top, \Box A, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{str}$

(iii) $\dfrac{A, B \Rightarrow}{\Gamma_1', \Box A; \Diamond B, \Gamma_2' \Rightarrow \Delta}$

By i.h., there is I' such that $A \Rightarrow I'$ and $I', B \Rightarrow$ are derivable, and $var(I') \subseteq var(A) \cap var(B)$. We establish $I = \Box I'$ and consider the following derivations.

$$\dfrac{A \Rightarrow I'}{\Gamma_1', \Box A \Rightarrow \Box I'}\,\mathsf{M}_\Box \qquad \dfrac{I', B \Rightarrow}{\Box I', \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{str}$$

(iv) $\dfrac{A, B \Rightarrow}{\Gamma_1', \Diamond B; \Box A, \Gamma_2' \Rightarrow \Delta}$

By i.h., there is I' such that $B \Rightarrow I'$ and $I', A \Rightarrow$ are derivable, and $var(I') \subseteq var(A) \cap var(B)$. We establish $I = \Diamond I'$ and consider the following derivations.

$$\dfrac{B \Rightarrow I'}{\Gamma_1', \Diamond B \Rightarrow \Diamond I'}\,\mathsf{M}_\Diamond \qquad \dfrac{I', A \Rightarrow}{\Diamond I', \Box A, \Gamma_2' \Rightarrow \Delta}\,\mathsf{str}$$

- If the last rule applied is $\mathsf{MC}_\Box$, we have the following three possibilities.

(i) $\dfrac{A_1, ..., A_n \Rightarrow B}{\Gamma_1', \Box A, ..., \Box A_n; \Gamma_2 \Rightarrow \Box B}\,\mathsf{MC}_\Box$

By i.h., there is I' such that $A_1, ..., A_n \Rightarrow I'$ and $I' \Rightarrow B$ are derivable, and $var(I') \subseteq var(A_1, ..., A_n) \cap var(B)$.

$$\rightsquigarrow \quad I = \Box I'\text{:} \qquad \dfrac{A_1, ..., A_n \Rightarrow I'}{\Gamma_1', \Box A_1, ..., \Box A_n \Rightarrow \Box I'}\,\mathsf{MC}_\Box \qquad \dfrac{I' \Rightarrow B}{\Box I', \Gamma_2 \Rightarrow \Box B}\,\mathsf{MC}_\Box$$

(ii) $\dfrac{A_1, ..., A_n \Rightarrow B}{\Gamma_1; \Box A, ..., \Box A_n, \Gamma_2' \Rightarrow \Box B}\,\mathsf{MC}_\Box \ \rightsquigarrow \ I = \top\text{:} \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \quad \dfrac{A_1, ..., A_n \Rightarrow B}{\top, \Box A, ..., \Box A_n, \Gamma_2' \Rightarrow \Box B}\,\mathsf{MC}_\Box$

(iii) $\dfrac{A_1, ..., A_i, A_{i+1}, ..., A_n \Rightarrow B}{\Gamma_1', \Box A, ... \Box A_1; \Box A_{i+1}, ..., \Box A_n, \Gamma_2' \Rightarrow \Box B}\,\mathsf{MC}_\Box$

By i.h., there is I' such that $A_1, ..., A_i \Rightarrow I'$ and $I', A_{i+1}, ..., A_n \Rightarrow B$ are derivable, and $var(I') \subseteq var(A_1, ..., A_i) \cap var(A_{i+1}, ..., A_n, B)$.

$$\rightsquigarrow \quad I = \Box I'\text{:} \qquad \dfrac{A_1, ..., A_n \Rightarrow I'}{\Gamma_1', \Box A_1, ..., \Box A_n \Rightarrow \Box I'}\,\mathsf{MC}_\Box \qquad \dfrac{I', A_{i+1}, ..., A_n \Rightarrow B}{\Box I', \Box A_{i+1}, ..., \Box A_n, \Gamma_2' \Rightarrow \Box B}\,\mathsf{MC}_\Box$$

• If the last rule applied is strC we have the following six possibilities.

(i) $$\dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma_1', \Box A_1, ..., \Box A_n, \Diamond B; \Gamma_2 \Rightarrow \Delta} \rightsquigarrow I = \bot: \quad \dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma_1', \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \bot}\,\mathsf{str}\ ; \quad \dfrac{}{\bot, \Gamma_2 \Rightarrow \Delta}\,\mathsf{L}\bot$$

(ii) $$\dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma_1; \Box A_1, ..., \Box A_n, \Diamond B, \Gamma_2' \Rightarrow \Delta} \rightsquigarrow I = \top: \quad \dfrac{}{\Gamma_1 \Rightarrow \top}\,\mathsf{R}\top\ ; \quad \dfrac{A_1, ..., A_n, B \Rightarrow}{\top, \Box A_1, ..., \Box A_n, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{str}$$

(iii) $$\dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma_1', \Box A_1, ...\Box A_n; \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

By i.h., there is I' such that $A_1, ..., A_n \Rightarrow I'$ and $I', B \Rightarrow$ are derivable, and $var(I') \subseteq var(A_1, ..., A_n) \cap var(B)$.

$$\rightsquigarrow \quad I = \Box I': \quad \dfrac{A_1, ..., A_n \Rightarrow I'}{\Gamma_1', \Box A_1, ..., \Box A_n \Rightarrow \Box I'}\,\mathsf{MC}_\Box \qquad \dfrac{I', B \Rightarrow}{\Box I', \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

(iv) $$\dfrac{A_1, ..., A_n, B \Rightarrow}{\Gamma_1', \Diamond B; \Box A_1, ...\Box A_n, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

By i.h., there is I' such that $B \Rightarrow I'$ and $I', A_1, ..., A_n \Rightarrow$ are derivable, and $var(I') \subseteq var(A_1, ..., A_n) \cap var(B)$.

$$\rightsquigarrow \quad I = \Diamond I': \quad \dfrac{B \Rightarrow I'}{\Gamma_1', \Diamond B \Rightarrow \Diamond I'}\,\mathsf{M}_\Diamond \qquad \dfrac{I', A_1, ..., A_n \Rightarrow}{\Diamond I', \Box A_1, ..., \Box A_n, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

(v) $$\dfrac{A_1, ..., A_i, A_{i+1}, ..., A_n, B \Rightarrow}{\Gamma_1', \Box A_1, ..., \Box A_i, \Diamond B; \Box A_{i+1}, ..., \Box A_n, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

By i.h., there is I' such that $A_{i+1}, ..., A_n \Rightarrow I'$ and $I', A_1, ..., A_i, B \Rightarrow$ are derivable, and $var(I') \subseteq var(A_{i+1}, ..., A_n) \cap var(A_1, ..., A_i, B)$.

$$\rightsquigarrow \quad I = \neg\Box I': \quad \dfrac{\dfrac{I', A_1, ..., A_i, B \Rightarrow}{\Gamma_1', \Box I', \Box A_1, ..., \Box A_i, \Diamond B \Rightarrow}\,\mathsf{strC}}{\Gamma_1', \Box A_1, ..., \Box A_i, \Diamond B \Rightarrow \neg\Box I'}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{A_{i+1}, ..., A_n \Rightarrow I'}{\Box A_{i+1}, ..., \Box A_n, \Gamma_2' \Rightarrow \Box I'}\,\mathsf{MC}_\Box}{\neg\Box I', \Box A_{i+1}, ..., \Box A_n, \Gamma_2' \Rightarrow \Delta}\,\mathsf{L}\neg$$

(vi) $$\dfrac{A_1, ..., A_i, A_{i+1}, ..., A_n, B \Rightarrow}{\Gamma_1', \Box A_1, ..., \Box A_i; \Box A_{i+1}, ..., \Box A_n, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

By i.h., there is I' such that: $A_1, ..., A_i \Rightarrow I'$ and $I', A_{i+1}, ..., A_n, B \Rightarrow$ are derivable, and $var(I') \subseteq var(A_1, ..., A_i) \cap var(A_{i+1}, ..., A_n, B)$.

$$\rightsquigarrow \quad I = \Box I': \quad \dfrac{A_1, ..., A_i \Rightarrow I'}{\Gamma_1', \Box A_1, ..., \Box A_i \Rightarrow \Box I'}\,\mathsf{MC}_\Box \qquad \dfrac{I', A_{i+1}, ..., A_n, B \Rightarrow}{\Box I', \Box A_{i+1}, ..., \Box A_n, \Diamond B, \Gamma_2' \Rightarrow \Delta}\,\mathsf{strC}$$

$\Box$

Theorem 7.5.2. The logics $\mathbf{IE_1}$, $\mathbf{IE_2}$, **IM**, **IMC**, and their extensions with $N_\Diamond$ or $N_\Box$ enjoy Craig's interpolation.

$$\mathsf{L0}\supset \frac{\Gamma, p, B \Rightarrow \Delta}{\Gamma, p, p \supset B \Rightarrow \Delta} \qquad \mathsf{L}\wedge\supset \frac{\Gamma, C \supset (D \supset B) \Rightarrow \Delta}{\Gamma, (C \wedge D) \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\vee\supset \frac{\Gamma, C \supset B, D \supset B \Rightarrow \Delta}{\Gamma, (C \vee D) \supset B \Rightarrow \Delta} \qquad \mathsf{L}\supset\supset \frac{\Gamma, C, D \supset B \Rightarrow D \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, (C \supset D) \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\top\supset \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, \top \supset B \Rightarrow \Delta}$$

Figure 7.11: Left implication rules of **G4ip**.

Proof. Let **L** be any of these systems, and assume $\vdash_{\mathbf{L}} A \supset B$. Then by Theorem 7.6.6, $\vdash_{\mathbf{G3.L}} A \Rightarrow B$. Moreover, by Proposition 7.5.1 there is $I \in \mathcal{L}_i$ such that $\vdash_{\mathbf{G3.L}} A \Rightarrow I$, $\vdash_{\mathbf{G3.L}} I \Rightarrow B$, and $var(I) \subseteq var(A) \cap var(B)$. Therefore $\vdash_{\mathbf{L}} A \supset I$ and $\vdash_{\mathbf{L}} I \supset B$. □

7.6 Strictly terminating calculi

In this section, we present strictly terminating calculi for our intuitionistic non-normal modal logics **IE*** as well as for **CK** and **CCDL**. As remarked in Section 7.4, G3-style calculi for intuitionistic logics are not strictly terminating because of the copy of the principal implication $A \supset B$ into the left premiss of $\mathsf{L}\supset$. For **IPL**, alternative, strictly terminating calculi are defined by Hudelmaier [92] and Dyckhoff [45]. These calculi are defined by replacing $\mathsf{L}\supset$ with several rules, one for every possible outermost connective in the antecedent A of the principal implication $A \supset B$. Here we consider Dyckhoff's calculus **G4ip** in the formulation given in Dyckhoff and Negri [46]. The calculus **G4ip** is defined as **G3ip** (Figure 3.6), with the difference that the rule $\mathsf{L}\supset$ is replaced by the five rules in Figure 7.11. As a difference with [46], we also take the rule $\mathsf{L}\top\supset$ since we are considering $\top$ as a primitive symbol of the language. Dyckhoff's [45] original completeness proof of **G4ip** is indirect: essentially, it consists in showing how to transform a derivation in **G3ip** in order to obtain a derivation of the same sequent in **G4ip**. An alternative completeness proof of **G4ip** based on a direct syntactic proof of cut elimination is given in Dyckhoff and Negri [46]. This second proof is more suitable for studying extensions of **G4ip** as it can be modularly extended with the analysis of the additional rules, for this reason we consider here this second kind of proof.

A modal extension of Dyckhoff's calculus is presented in Iemhoff [94], where strictly terminating calculi for intuitionistic monomodal versions of the classical modal logics **K** and **KD** are presented. The calculi are defined by considering the following additional left implication rule for the case where the antecedent of the principal implication is a modal formula $\Box D$:

$$\mathsf{L}\Box\supset \frac{\Gamma' \Rightarrow D \qquad \Gamma, \Box\Gamma', B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Delta} \; |\Gamma'| \geq 0$$

$$\mathsf{L}\Box_{\mathsf{E}}\supset \frac{C \Rightarrow D \qquad D \Rightarrow C \qquad \Gamma, \Box C, B \Rightarrow \Delta}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta} \qquad \mathsf{L}\Box_{\mathsf{M}}\supset \frac{C \Rightarrow D \qquad \Gamma, \Box C, B \Rightarrow \Delta}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\Box_{\mathsf{C}}\supset \frac{\Gamma' \Rightarrow D \qquad D \Rightarrow C_1 \quad ... \quad D \Rightarrow C_n \qquad \Gamma, \Box\Gamma', B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Delta}\ (\Gamma' = C_1, ..., C_n)$$

$$\mathsf{L}\Box_{\mathsf{MC}}\supset \frac{\Gamma' \Rightarrow D \qquad \Gamma, \Box\Gamma', B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Delta} \qquad \mathsf{L}\Box_{\mathsf{N}}\supset \frac{\Rightarrow D \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, \Box D \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\Diamond_{\mathsf{E}}\supset \frac{C \Rightarrow D \qquad D \Rightarrow C \qquad \Gamma, \Diamond C, B \Rightarrow \Delta}{\Gamma, \Diamond C, \Diamond D \supset B \Rightarrow \Delta} \qquad \mathsf{L}\Diamond_{\mathsf{M}}\supset \frac{C \Rightarrow D \qquad \Gamma, \Diamond C, B \Rightarrow \Delta}{\Gamma, \Diamond C, \Diamond D \supset B \Rightarrow \Delta}$$

$$\mathsf{L}\Diamond_{\mathsf{W}}\supset \frac{\Gamma', E \Rightarrow D \qquad \Gamma, \Box\Gamma', \Diamond E, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}$$

Figure 7.12: Left implication rules of modal extensions of **G4ip**.

Modal rule	$\mathsf{E}_\Box$	$\mathsf{C}_\Box$	$\mathsf{M}_\Box$	$\mathsf{MC}_\Box$	$\mathsf{N}_\Box$	$\mathsf{E}_\Diamond$	$\mathsf{M}_\Diamond$	W
Implication rule	$\mathsf{L}\Box_{\mathsf{E}}\supset$	$\mathsf{L}\Box_{\mathsf{C}}\supset$	$\mathsf{L}\Box_{\mathsf{M}}\supset$	$\mathsf{L}\Box_{\mathsf{MC}}\supset$	$\mathsf{L}\Box_{\mathsf{N}}\supset$	$\mathsf{L}\Diamond_{\mathsf{E}}\supset$	$\mathsf{L}\Diamond_{\mathsf{M}}\supset$	$\mathsf{L}\Diamond_{\mathsf{W}}\supset$

Table 7.5: Correspondence between modal rules and left implication rules in calculi **G4.L**.

We take here a similar strategy and define G4-style calculi for intuitionistic non-normal modal logics. Similarly to Iemhoff's rule, these calculi contain suitable left implication rules taking care of the $\Box$- or $\Diamond$-formulas in the antecedent of implications. The relevant rules are displayed in Figure 7.12 (as before, when $\Box\Gamma$ occurs in a rule (schema), we implicitly understand that Γ contains at least one formula).

Definition 7.6.1 (G4-style calculi)**.** Let **L** be any system among **IE***, **CK**, or **CCDL**. Then the calculus **G4.L** for **L** contains

- the propositional rules of **G3.L** *different* from $\mathsf{L}\supset$ (cf. Tables 7.2 and 7.3 and Section 3.6);
- the modal rules of **G3.L**;
- the left implication rules in Figure 7.11;
- for every rule among $\mathsf{E}_\Box$, $\mathsf{C}_\Box$, $\mathsf{M}_\Box$, $\mathsf{MC}_\Box$, $\mathsf{N}_\Box$, $\mathsf{E}_\Diamond$, $\mathsf{M}_\Diamond$, and W contained by **G3.L**, **G4.L** contains the corresponding modal left implication rule in Figure 7.12, as summarised in Table 7.5.

Observe that the calculi **G4.L** do not properly satisfy the subformula property, in particular the principal formulas in the premiss of $\mathsf{L}\wedge\supset$, $\mathsf{L}\vee\supset$, and the left premiss of $\mathsf{L}\supset\supset$ are not subformula of any formula in the conclusion. However, it is possible to define an ordering of sequents according to which the premisses of every rule of **G4.L** are strictly smaller than the conclusion (cf. [45, 46, 94]). To this purpose we need to re-define the weight of formulas as follows:

Definition 7.6.2 (Weight of formulas)**.** The function wg_4 assigning to each formula A of $\mathcal{L}_i$ its weight $wg_4(A)$ is recursively defined as follows: $wg_4(\bot) = wg_4(\top) = 0$; $wg_4(p_i) = 1$ for every $p_i \in Atm$; $wg_4(A \supset B) = wg_4(A) + wg_4(B) + 1$; $wg_4(A \wedge B) = wg_4(A) + wg_4(B) + 2$; $wg_4(A \vee B) = wg_4(A) + wg_4(B) + 3$; and $wg_4(\Box A) = wg_4(\Diamond A) = wg_4(A) + 2$.

Then, basing on the above notion of weight of formulas we consider the multiset ordering of sequents, which is defined as follows:

Definition 7.6.3 (Multiset ordering of sequents)**.** Given two multisets Σ and Π of formulas of $\mathcal{L}_i$, we define $\Sigma \ll \Pi$ if and only if Π is the result of replacing one or more formulas in Σ by zero or more formulas of lower weight according to Definition 7.6.2. Moreover, given two sequents $\Gamma \Rightarrow \Delta$ and $\Gamma' \Rightarrow \Delta'$, we define $\Gamma \Rightarrow \Delta \ll \Gamma' \Rightarrow \Delta'$ if and only if $\Gamma, \Delta \ll \Gamma', \Delta'$.

The above relation $\ll$ is a well-ordering on sequents. As a consequence, since the premisses of every rule of **G4.L** are strictly smaller than the conclusion according to $\ll$, every application of backward proof search for a sequent $\Gamma \Rightarrow \Delta$ comes to an end after a finite number of steps, whence the calculi **G4.L** are strictly terminating.

We now investigate the property of the calculi **G4.L**. In particular, we first show that the structural rules of weakening and contraction are admissible, and then prove the equivalence between the calculi **G4.L** and the corresponding calculi **G3.L**. Finally, we present a direct proof of cut elimination in **G4.L**.

Lemma 7.6.1. Every left implication rule in Figure 7.12 is height-preserving invertible with respect to the rightmost premiss, i.e., the following rules are height-preserving admissible in the corresponding calculi:

$$\frac{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}{\Gamma, \Box C, B \Rightarrow \Delta} \qquad \frac{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Delta}{\Gamma, \Box\Gamma', B \Rightarrow \Delta} \qquad \frac{\Gamma, \Box D \supset B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}$$

$$\frac{\Gamma, \Diamond C, \Diamond D \supset B \Rightarrow \Delta}{\Gamma, \Diamond C, B \Rightarrow \Delta} \qquad \frac{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, B \Rightarrow \Delta}$$

Proof. The proof is by induction on the height n of the derivation $\mathcal{D}$ of the premiss. If $n = 0$, that is the premiss is an initial sequent, then the conclusion is also an initial sequent. If $n \geq 1$, then we distinguish two cases, depending whether $\Box D \supset B$ or $\Diamond D \supset B$ is principal in the last rule application in $\mathcal{D}$: If $\Box D \supset B$ (respectively $\Diamond D \supset B$) is principal, then the conclusion of the rule coincides with the rightmost premiss of the last rule application in $\mathcal{D}$, whence it is derivable with a derivation of smaller height by hypothesis. If it is not principal, then either we apply the inductive hypothesis and then use the last rule applied, or, in case the last rule applied is a modal rule, we simply consider a different application of the same rule. □

Proposition 7.6.2. For every $\mathbf{L} \in \{\mathbf{IE}^*, \mathbf{CK}, \mathbf{CCDL}\}$, the structural rules Lwk and Rwk are height-preserving admissible in $\mathbf{G4.L}$. Moreover, ctr is admissible in $\mathbf{G4.L}$.

Proof. The proof of height-preserving admissibility of weakening is standard by induction on the height of the derivation of the premiss (see e.g. the proof of Proposition 7.2.1). Here we prove the admissibility of contraction.

As usual, the proof is by induction on the height n of the derivation $\mathcal{D}$ of the premiss of ctr. If $n = 0$, that is the premiss is an initial sequent, then the conclusion is also an initial sequent. If $n \geq 1$ and the contracted formula is not principal in the last rule applied in $\mathcal{D}$, the the proof is standard. If in contrast the contracted formula is principal in the last rule application, then we consider the following four possibilities: If the last rule applied is a propositional rule different from left implication rules, then the proof is standard. If the last rule applied is a propositional left implication rule (Figure 7.11), then we can refer to the proof in Dyckhoff and Negri [46]. If the last rule applied is a modal rule, then we can refer to the proofs of Propositions 7.2.1, 7.3.1, and 7.3.3 in previous sections. The remaining case is that the last rule applied is a modal left implication rule (Figure 7.12). We consider as examples the cases where the last rule applied is $\mathsf{L}\Box_{\mathsf{E}}\supset$ or $\mathsf{L}\Diamond_{\mathsf{W}}\supset$.

• The last rule applied is $\mathsf{L}\Box_{\mathsf{E}}\supset$. There are two possibilities depending whether the contracted formula is $\Box C$ or $\Box D \supset B$. If the contracted formula is $\Box C$ we have the following derivation, which is converted into the derivation below, where ctr is applied at a smaller height.

$$\dfrac{\dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \Gamma, \Box C, \Box C, B \Rightarrow \Delta}{\Gamma, \Box C, \Box C, \Box D \supset B \Rightarrow \Delta}\ \mathsf{L}\Box_{\mathsf{E}}\supset}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}\ \mathsf{ctr}$$

$$\rightsquigarrow$$

$$\dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \dfrac{\Gamma, \Box C, \Box C, B \Rightarrow \Delta}{\Gamma, \Box C, B \Rightarrow \Delta}\ \mathsf{ctr}}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}\ \mathsf{L}\Box_{\mathsf{E}}\supset$$

If the contracted formula is $\Box D \supset B$ we have the following derivation, which is converted into the derivation below, containing an application of height-preserving invertibility of $\mathsf{L}\Box_{\mathsf{E}}\supset$ (Lemma 7.6.1) and an application of ctr at a smaller height.

$$\dfrac{\dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \Gamma, \Box C, B, \Box D \supset B \Rightarrow \Delta}{\Gamma, \Box C, \Box D \supset B, \Box D \supset B \Rightarrow \Delta}\ \mathsf{L}\Box_{\mathsf{E}}\supset}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}\ \mathsf{ctr}$$

$$\rightsquigarrow$$

$$
\dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \dfrac{\dfrac{\Gamma, \Box C, B, \Box D \supset B \Rightarrow \Delta}{\Gamma, \Box C, B, B \Rightarrow \Delta}\,\textsf{inv-L}\Box_{\textsf{E}}\!\supset}{\Gamma, \Box C, B \Rightarrow \Delta}\,\textsf{ctr}}{\Gamma, \Box C, \Box D \supset B \Rightarrow \Delta}\,\textsf{L}\Box_{\textsf{E}}\!\supset
$$

• The last rule applied is $\textsf{L}\Diamond_{\textsf{W}}\!\supset$. There are three possibilities depending whether the contracted formula is $\Box A$, or $\Diamond C$, or $\Diamond D \supset B$. If the contracted formula is $\Box A$ we have:

$$
\dfrac{\dfrac{\Gamma', A, A, E \Rightarrow D \qquad \Gamma, \Box\Gamma', \Box A, \Box A, \Diamond E, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box A, \Box A, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset}{\Gamma, \Box\Gamma', \Box A, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{ctr}
$$

$$\rightsquigarrow$$

$$
\dfrac{\textsf{ctr}\,\dfrac{\Gamma', A, A, E \Rightarrow D}{\Gamma', A, E \Rightarrow D} \qquad \dfrac{\Gamma, \Box\Gamma', \Box A, \Box A, \Diamond E, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box A, \Diamond E, B \Rightarrow \Delta}\,\textsf{ctr}}{\Gamma, \Box\Gamma', \Box A, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset
$$

If the contracted formula is $\Diamond E$ we have:

$$
\dfrac{\dfrac{\Gamma', E \Rightarrow D \qquad \Gamma, \Box\Gamma', \Diamond E, \Diamond E, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset}{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{ctr}
$$

$$\rightsquigarrow$$

$$
\dfrac{\Gamma', E \Rightarrow D \qquad \dfrac{\Gamma, \Box\Gamma', \Diamond E, \Diamond E, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, B \Rightarrow \Delta}\,\textsf{ctr}}{\Gamma, \Box\Gamma', \Box A, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset
$$

If the contracted formula is $\Diamond D \supset B$ we have the following derivation, which is converted into the derivation below, containing an application of height-preserving invertibility of $\textsf{L}\Diamond_{\textsf{W}}\!\supset$ (Lemma 7.6.1) and an application of $\textsf{ctr}$ at a smaller height.

$$
\dfrac{\dfrac{\Gamma', E \Rightarrow D \qquad \Gamma, \Box\Gamma', \Diamond E, B, \Diamond D \supset B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset}{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{ctr}
$$

$$\rightsquigarrow$$

$$
\dfrac{\Gamma', E \Rightarrow D \qquad \dfrac{\dfrac{\Gamma, \Box\Gamma', \Diamond E, B, \Diamond D \supset B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond E, B, B \Rightarrow \Delta}\,\textsf{inv-L}\Diamond_{\textsf{W}}\!\supset}{\Gamma, \Box\Gamma', \Diamond E, B \Rightarrow \Delta}\,\textsf{ctr}}{\Gamma, \Box\Gamma', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\textsf{L}\Diamond_{\textsf{W}}\!\supset
$$

$\Box$

We now prove that every calculus **G4.L** is equivalent to the corresponding calculus **G3.L**, whence also to the logic **L**. The proof is based on the following two propositions.

Proposition 7.6.3. For every $\mathbf{L} \in \{\mathbf{IE}^*, \mathbf{CK}, \mathbf{CCDL}\}$, the following rule $\textsf{L}\!\supset$ is admissible in **G4.L**:

$$\mathsf{L}\supset \frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \ .$$

Proof. We first prove that the following rule is admissible in **G4.L**:

$$\mathsf{L}\supset' \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \ .$$

The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow A$. If $n = 0$, then $\Gamma \Rightarrow A$ is an initial sequent. There are three possibilities: (i) If $\Gamma = \Gamma', \bot$, then $\Gamma', \bot, A \supset B \Rightarrow \Delta$ is derivable by $\mathsf{L}\bot$. (ii) If $A = \top$, then $\Gamma, \top \supset B \Rightarrow \Delta$ is derivable from the right premiss $\Gamma, B \Rightarrow \Delta$ by $\mathsf{L}\top\supset$. (iii) If $\Gamma \Rightarrow A = \Gamma', p \Rightarrow p$, then $\Gamma', p, p \supset B \Rightarrow \Delta$ is derivable from the right premiss $\Gamma', p, B \Rightarrow \Delta$ by $\mathsf{L0}\supset$.

In $n \geq 1$, then we consider the last rule applied in the derivation of $\Gamma \Rightarrow A$. If this is a propositional rule of **G4ip**, then we can refer to the proof in [46]. If the last applied rule R is $\mathsf{N}_\Diamond$ or an interaction rule in Figure 7.5 or 7.6, then A is not principal in the rule application. Then, $\Gamma, A \supset B \Rightarrow \Delta$ can be derived from the premiss of R by a different application of R. For the other rules we show the following three illustrative examples.

• If the last rule applied is $\mathsf{E}_\Box$, then $\Gamma \Rightarrow A$ has the form $\Gamma', \Box C \Rightarrow \Box D$ and is derived from $C \Rightarrow D$ and $D \Rightarrow C$. Moreover, $\Gamma, B \Rightarrow \Delta$ has the form $\Gamma', \Box C, B \Rightarrow \Delta$. Then by applying $\mathsf{L}\Box_\mathsf{E}\supset$ to $C \Rightarrow D$, $D \Rightarrow C$, and $\Gamma', \Box C, B \Rightarrow \Delta$ we derive $\Gamma', \Box C, \Box D \supset B \Rightarrow \Delta$, which is the conclusion of $\mathsf{L}\supset'$.

• If the last rule applied is W, then $\Gamma \Rightarrow A$ has the form $\Gamma', \Box\Gamma'', \Diamond C \Rightarrow \Diamond D$, and is derived from $\Gamma'', C \Rightarrow D$. Moreover, $\Gamma, B \Rightarrow \Delta$ has the form $\Gamma', \Box\Gamma'', \Diamond C, B \Rightarrow \Delta$. Then by applying $\mathsf{L}\Diamond_\mathsf{W}\supset$ to $\Gamma'', C \Rightarrow D$ and $\Gamma', \Box\Gamma'', \Diamond C, B \Rightarrow \Delta$ we derive $\Gamma', \Box\Gamma'', \Diamond C, \Diamond D \supset B \Rightarrow \Delta$, which is the conclusion of $\mathsf{L}\supset'$.

• If the last rule applied is $\mathsf{L}\Box_\mathsf{E}\supset$, then $\Gamma \Rightarrow A$ has the form $\Gamma', \Box C, \Box D \supset E \Rightarrow A$, and is derived from $C \Rightarrow D$, $D \Rightarrow C$, and $\Gamma', \Box C, E \Rightarrow A$. Moreover, $\Gamma, B \Rightarrow \Delta$ has the form $\Gamma', \Box C, \Box D \supset E, B \Rightarrow \Delta$. Then by Lemma 7.6.1, $\Gamma', \Box C, E, B \Rightarrow \Delta$ is derivable. Thus we can obtain the conclusion of $\mathsf{L}\supset'$ as follows, where the application of $\mathsf{L}\supset'$ is admissible by i.h.:

$$\dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \dfrac{\Gamma', \Box C, E \Rightarrow A \qquad \Gamma', \Box C, E, B \Rightarrow \Delta}{\Gamma', \Box C, E, A \supset B \Rightarrow \Delta}\,\mathsf{L}\supset'}{\Gamma', \Box C, \Box D \supset E, A \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_\mathsf{E}\supset$$

Now, having shown that $\mathsf{L}\supset'$ is admissible, and that weakening and contraction are also admissible in **G4.L** (Proposition 7.6.2), we can show the admissibility of $\mathsf{L}\supset$ as follows:

$$\dfrac{\dfrac{\Gamma, A \supset B \Rightarrow A \qquad \dfrac{\Gamma, B \Rightarrow \Delta}{\Gamma, B, A \supset B \Rightarrow \Delta}\,\mathsf{Lwk}}{\Gamma, A \supset B, A \supset B \Rightarrow \Delta}\,\mathsf{L}\supset'}{\Gamma, A \supset B \Rightarrow \Delta}\,\mathsf{ctr}$$

$\Box$

Proposition 7.6.4. For every $\mathbf{L} \in \{\mathbf{IE}^*, \mathbf{CK}, \mathbf{CCDL}\}$, every left implication rule of $\mathbf{G4.L}$ is admissible in $\mathbf{G3.L}$.

Proof. As before, we only consider modal left implication rules, whereas for propositional rules we refer to [46]. We present the following illustrative cases.

- Rule $\mathsf{L}\Box_{\mathsf{C}}\supset$ is derivable in $\mathbf{G3.IE}^*$ as follows, where $\Gamma' = C_1, ..., C_n$:

$$\dfrac{\dfrac{\Gamma' \Rightarrow D \qquad D \Rightarrow C_1 \quad ... \quad D \Rightarrow C_n}{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Box D}\,\mathsf{C}_\Box \qquad \Gamma, \Box\Gamma', B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\supset$$

- Rule $\mathsf{L}\Diamond_{\mathsf{M}}\supset$ is derivable in $\mathbf{G3.IM}^*$ as follows:

$$\dfrac{\dfrac{C \Rightarrow D}{\Gamma, \Diamond C, \Diamond D \supset B \Rightarrow \Diamond D}\,\mathsf{M}_\Diamond \qquad \Gamma, \Diamond C, B \Rightarrow \Delta}{\Gamma, \Diamond C, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\supset$$

- Rule $\mathsf{L}\Diamond_{\mathsf{W}}\supset$ is derivable in $\mathbf{G3.CCDL}$ and $\mathbf{G3.CK}$ as follows:

$$\dfrac{\dfrac{\Gamma', F \Rightarrow D}{\Gamma, \Box\Gamma', \Diamond F, \Diamond D \supset B \Rightarrow \Diamond D}\,\mathsf{W} \qquad \Gamma, \Box\Gamma', \Diamond F, B \Rightarrow \Delta}{\Gamma, \Box\Gamma', \Diamond F, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\supset$$

□

Then, from Propositions 7.6.3 and 7.6.4 we obtain the following result.

Theorem 7.6.5. For every $\mathbf{L} \in \{\mathbf{IE}^*, \mathbf{CK}, \mathbf{CCDL}\}$, a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3.L}$ if and only if it is derivable in $\mathbf{G4.L}$.

Proof. $\Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3.L}$ iff (by Proposition 7.6.4) it is derivable in $\mathbf{G3.L}$ plus the left implication rules of $\mathbf{G4.L}$ iff (by Definition 7.6.1) it is derivable in $\mathbf{G4.L} \cup \{\mathsf{L}\supset\}$ iff (by Proposition 7.6.3) it is derivable in $\mathbf{G4.L}$. □

Moreover, from this and the syntactic completeness of the calculi $\mathbf{G3.L}$ (Theorems 7.6.6 and 3.6.1) we obtain as an immediate corollary:

Theorem 7.6.6 (Syntactic completeness)**.** For every $\mathbf{L} \in \{\mathbf{IE}^*, \mathbf{CK}, \mathbf{CCDL}\}$,

$$\vdash_{\mathbf{G4.L}} \Gamma \Rightarrow \Delta \;\text{ if and only if }\; \vdash_{\mathbf{L}} \bigwedge \Gamma \supset \bigvee \Delta.$$

Furthemore, since cut is admissible in the calculi $\mathbf{G3.L}$ (cf. Section 7.3 and Theorem 3.6.1), from the above equivalence it also follows that cut is admissible in the calculi $\mathbf{G4.L}$. This means as usual that every derivable sequent has a cut-free derivation in the calculus. It is nonetheless interesting to prove cut elimination directly. As an advantage, such a proof gives an effective procedure to convert every derivation containing applications of cut into an equivalent derivation not containing any of them, thus showing the admissibility of cut in a constructive way. Moreover, this provides also a completeness proof for the calculi **G4.L** which is independent from the one for the calculi **G3.L**.

Theorem 7.6.7 (Cut elimination)**.** The rule cut is admissible in **G4.IE*** as well as in **G4.CK** and **G4.CCDL**.

Proof. As usual, the proof is by double induction, with primary induction on the weight of the cut formula and secondary induction on the cut height. As for previous proofs of cut elimination, we consider only the cases where the cut formula is principal in the last rule applied in the derivation of both premisses. Moreover, we only consider relevant combinations with left implication rules of modal G4-style calculi. We present some significant cases, the lacking combinations are analogous to some of the ones below.

- $(\mathsf{R}{\supset};\ \mathsf{L}\Box_{\mathsf{C}}{\supset})$ Let $\Gamma'' = C_1, ..., C_n$. We have the following derivation, which is converted into the derivation below, containing two applications of cut on cut formulas of smaller weight and n application of contraction, which has been proved admissible (Proposition 7.6.2).

$$\dfrac{\mathsf{R}{\supset}\,\dfrac{\Gamma, \Box D \Rightarrow B}{\Gamma \Rightarrow \Box D \supset B} \qquad \dfrac{\Gamma'' \Rightarrow D \quad D \Rightarrow C_1 \quad ... \quad D \Rightarrow C_n \quad \Gamma', \Box\Gamma'', B \Rightarrow \Delta}{\Gamma', \Box\Gamma'', \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{C}}{\supset}}{\Gamma, \Gamma', \Box\Gamma'' \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\mathsf{cut}\,\dfrac{\mathsf{C}_{\Box}\,\dfrac{\Gamma'' \Rightarrow D \quad D \Rightarrow C_1 \quad ... \quad D \Rightarrow C_n}{\Box\Gamma'' \Rightarrow \Box D} \qquad \Gamma, \Box D \Rightarrow B}{\Gamma, \Box\Gamma'' \Rightarrow B}\quad\mathsf{cut} \qquad \Gamma', \Box\Gamma'', B \Rightarrow \Delta}{\Gamma, \Gamma', \Box\Gamma'', \Box\Gamma'' \Rightarrow \Delta}}{\Gamma, \Gamma', \Box\Gamma'' \Rightarrow \Delta}\,\mathsf{ctr}{\times}n$$

- $(\mathsf{C}_{\Box};\ \mathsf{L}\Box_{\mathsf{C}}{\supset})$ Let $\Gamma' = A_1, ..., A_n$ and $\Gamma''' = C_1, ..., C_m$.

$$\dfrac{\mathsf{C}_{\Box}\,\dfrac{\Gamma' \Rightarrow E \quad (E \Rightarrow A_i)_{i=1}^n}{\Gamma, \Box\Gamma' \Rightarrow \Box E} \qquad \dfrac{E, \Gamma''' \Rightarrow D \quad D \Rightarrow E \quad (D \Rightarrow C_j)_{j=1}^m \quad \Gamma'', \Box E, \Box\Gamma''', B \Rightarrow \Delta}{\Gamma'', \Box E, \Box\Gamma''', \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{C}}{\supset}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Box D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Gamma' \Rightarrow E \quad E, \Gamma''' \Rightarrow D}{\Gamma', \Gamma''' \Rightarrow D} \quad \left(\mathsf{cut}\,\dfrac{D \Rightarrow E \quad E \Rightarrow A_i}{D \Rightarrow A_i}\right)_{i=1}^n \quad (D \Rightarrow C_j)_{j=1}^m \quad \dfrac{\Gamma, \Box\Gamma' \Rightarrow \Box E \quad \Gamma'', \Box E, \Box\Gamma''', B \Rightarrow \Delta}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{C}}{\supset}$$

- $(\mathsf{MC}_{\Box};\ \mathsf{L}\Box_{\mathsf{MC}}{\supset})$

$$\dfrac{\mathsf{MC}_{\Box}\,\dfrac{\Gamma' \Rightarrow A}{\Gamma, \Box\Gamma' \Rightarrow \Box A} \qquad \dfrac{\Gamma''', A, E \Rightarrow D \qquad \Gamma'', \Box\Gamma''', \Box A, \Diamond E, B \Rightarrow \Delta}{\Gamma'', \Box\Gamma''', \Box A, \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{MC}}{\supset}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Gamma' \Rightarrow A \qquad \Gamma''', A, E \Rightarrow D}{\Gamma', \Gamma''', E \Rightarrow D} \qquad \dfrac{\Gamma, \Box\Gamma' \Rightarrow \Box A \qquad \Gamma'', \Box\Gamma''', \Box A, \Diamond E, B \Rightarrow \Delta}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{MC}}{\supset}$$

- (W; $\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$)

$$\dfrac{\mathsf{W}\,\dfrac{\Gamma', A \Rightarrow E}{\Gamma, \Box\Gamma', \Diamond A \Rightarrow \Diamond E} \qquad \dfrac{\Gamma''', E \Rightarrow D \qquad \Gamma'', \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}{\Gamma'', \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond A, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Gamma', A \Rightarrow E \qquad \Gamma''', E \Rightarrow D}{\Gamma', \Gamma''', A \Rightarrow D} \qquad \dfrac{\Gamma, \Box\Gamma', \Diamond A \Rightarrow \Diamond E \qquad \Gamma'', \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond A, B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond A, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$$

- ($\mathsf{MC}_\Box$; $\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$)

$$\dfrac{\mathsf{MC}_\Box\,\dfrac{\Gamma' \Rightarrow A}{\Gamma, \Box\Gamma' \Rightarrow \Box A} \qquad \dfrac{A, \Gamma''', E \Rightarrow D \qquad \Gamma'', \Box A, \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}{\Gamma'', \Box A, \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Gamma' \Rightarrow A \qquad A, \Gamma''', E \Rightarrow D}{\Gamma', \Gamma''', E \Rightarrow D} \qquad \dfrac{\Gamma, \Box\Gamma' \Rightarrow \Box A \qquad \Gamma'', \Box A, \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma'', \Box\Gamma', \Box\Gamma''', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$$

- ($\mathsf{M}_\Diamond$; $\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$)

$$\dfrac{\mathsf{M}_\Diamond\,\dfrac{A \Rightarrow E}{\Gamma, \Diamond A \Rightarrow \Diamond E} \qquad \dfrac{\Gamma'', E \Rightarrow D \qquad \Gamma', \Box\Gamma'', \Diamond E, B \Rightarrow \Delta}{\Gamma', \Box\Gamma'', \Diamond E, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}}{\Gamma, \Gamma', \Box\Gamma'', \Diamond A, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{A \Rightarrow E \qquad \Gamma'', E \Rightarrow D}{\Gamma'', A \Rightarrow D} \qquad \dfrac{\Gamma, \Diamond A \Rightarrow \Diamond E \qquad \Gamma', \Box\Gamma'', \Diamond E, B \Rightarrow \Delta}{\Gamma, \Gamma', \Box\Gamma'', \Diamond A, B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma', \Box\Gamma'', \Diamond A, \Diamond D \supset B \Rightarrow \Delta}\,\mathsf{L}\Diamond_{\mathsf{W}}{\supset}$$

- ($\mathsf{N}_\Box$; $\mathsf{L}\Box_{\mathsf{E}}{\supset}$)

$$\dfrac{\mathsf{N}_\Box\,\dfrac{\Rightarrow C}{\Gamma \Rightarrow \Box C} \qquad \dfrac{C \Rightarrow D \qquad D \Rightarrow C \qquad \Gamma', \Box C, B \Rightarrow \Delta}{\Gamma', \Box C, \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{E}}{\supset}}{\Gamma, \Gamma', \Box D \supset B \Rightarrow \Delta}\,\mathsf{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\mathsf{cut}\,\dfrac{\Rightarrow C \qquad C \Rightarrow D}{\Rightarrow D} \qquad \dfrac{\Gamma \Rightarrow \Box C \qquad \Gamma', \Box C, B \Rightarrow \Delta}{\Gamma, \Gamma', B \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma', \Box D \supset B \Rightarrow \Delta}\,\mathsf{L}\Box_{\mathsf{N}}{\supset}$$

□

We have shown that, for every logic **L** among our intuitionistic systems **IE*** and **CK** and **CCDL**, the calculus **G4.L** is strictly terminating and complete with respect to the logic. As a consequence of this results, we obtain an alternative proof of decidability of the respective logics, with the advantage that – differently from the G3-calculi – proof search does not require loop checking: for every logic **L** and formula A, the decision procedure trivially consists in checking all possible proofs of $\Rightarrow A$ in **G4.L**, which are in a finite number. Furthermore, this result partially solves the problem left open in Iemhoff [94] concerning the possibility to define strictly terminating G4-style calculi for intuitionistic logics with both modalities $\Box$ and $\Diamond$.

7.7 Discussion

The results presented in this chapter represent the initial step towards a general investigation of non-normal modalities with an intuitionistic base. We have defined a new family of intuitionistic non-normal modal logics that can be seen as intuitionistic counterparts of classical non-normal modal logics. In particular, we have defined 12 monomodal logics – 8 logics with $\Box$ modality and 4 logics with $\Diamond$ modality – and 24 bimodal logics. For each of them we have provided both a Hilbert axiomatisation and a cut-free sequent calculus, and on its basis we have proved that all systems are decidable. Furthermore, we have proved that the systems $\mathbf{IE_1}$, $\mathbf{IE_2}$, **IM**, **IMC**, and their extensions with axioms $N_\Diamond$ or $N_\Box$, enjoy Craig's interpolation. Finally, we have presented strictly terminating G4-style calculi for all our systems as well as for **CK** and **CCDL**.

All intuitionistic non-normal modal logics defined in this chapter contain some of the modal axioms characterising the classical cube. In addition, bimodal logics contain interactions between the modalities that can be seen as "weak duality principles", and express under which conditions two formulas $\Box A$ and $\Diamond B$ are jointly inconsistent. On the basis of the different strength of such interactions, we identify different intuitionistic counterparts of a given classical logic. Since there are three degrees of interaction, to every classical non-monotonic logic $\mathbf{E}^*$ are associated at least three intuitionistic counterparts of increasing strength.

The notion of "intuitionistic counterpart of a classical system" considered here is not a formal one, but rather an informal notion which is used for the classification of intuitionistic systems. However, one could intuitively identify at least two conditions that an intuitionistic modal logic **IL** should satisfy in order to be interpreted as an intuitionistic counterpart of a classical modal logic **L**: first, it should contain the characteristic modal axioms and rules of **L**, and second, it should be weaker than **L**, the latter because **IPL** is weaker than **CPL**. In this sense, it would not be true that $\mathbf{IE_3}$ is a counterpart of classical **E**. Indeed, the rule *str* is classically equivalent to the rule RM, whence it allows one to derive formulas which are not derivable in $\mathbf{E}$. For instance, $\neg(\Box(p \wedge q) \wedge \Diamond\neg p)$ is derivable in $\mathbf{IE_3}$ but is *not* derivable in $\mathbf{E}$, as it is shown by the proofs below: on the left we see a derivation of $\Box(p \wedge q) \wedge \Diamond\neg p \Rightarrow$ in $\mathbf{G3.IE_3}$, while on the right we see a failed proof of $\Box(p \wedge q) \wedge \neg\Box p \Rightarrow$ in the hypersequent calculus $\mathbf{H.E}$ (cf. Chapter 6).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, q \Rightarrow p}\,\mathsf{init}}{p, q, \neg p \Rightarrow}\,\mathsf{L}\neg}{p \wedge q, \neg p \Rightarrow}\,\mathsf{L}\wedge}{\Box(p \wedge q), \Diamond\neg p \Rightarrow}\,\mathsf{str}}{\Box(p \wedge q) \wedge \Diamond\neg p \Rightarrow}\,\mathsf{L}\wedge
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dots \mid p \wedge q \Rightarrow p \qquad \dfrac{\dots \mid p \Rightarrow p \wedge q, p \qquad \overset{\text{saturated}}{\dots \mid p \Rightarrow p \wedge q, q}}{\dots \mid p \Rightarrow p \wedge q}\,\mathsf{R}\wedge}{\Box(p \wedge q) \wedge \neg\Box p, \Box(p \wedge q), \neg\Box p, \langle p \wedge q \rangle \Rightarrow \Box p}\,\mathsf{R}\Box}{\Box(p \wedge q) \wedge \neg\Box p, \Box(p \wedge q), \neg\Box p \Rightarrow \Box p}\,\mathsf{L}\Box}{\Box(p \wedge q) \wedge \neg\Box p, \Box(p \wedge q), \neg\Box p \Rightarrow}\,\mathsf{R}\neg}{\Box(p \wedge q) \wedge \neg\Box p \Rightarrow}\,\mathsf{L}\wedge
$$

At the same time, however, it would be unnatural to consider $\mathbf{IE_3}$ as corresponding to classical $\mathbf{M}$, as neither $M_\Box$ nor $M_\Diamond$ is derivable.

We see therefore that the picture of systems that emerge from a certain set of logic principles is richer in the intuitionistic case than in the classical one. In particular, assuming an intuitionistic base not only allows us to make subtle distinctions between principles that are not distinguishable in classical logic, but also gives us the possibility to investigate systems that in a sense lie between two different classical logics, and do not correspond essentially to any of the two.

The results presented in this chapter can be extended in several ways, here we highlight some possible directions.

Interpolation. First of all, in Section 7.5 we have proved Craig's interpolation for a subset of our intuitionistic non-normal modal logics by means of a general methodology based on cut-free Gentzen calculi. This methodology does not seem to be adequate for the non-monotonic calculi with the rules for $C_\Box$ (the same problem was encountered for classical non-normal modal logics in Orlandelli [140]). In future work we aim to study the possibility to find a syntactic proof of interpolation for these logics by means of different kind of calculi. Furthermore, it is known that both intuitionistic logic and the non-normal modal logics of the classical cube enjoy the stronger property of *uniform* interpolation (for the systems of the classical cube this was uniformly proved by Pattinson [145] and for monotonic logics also by Santocanale and Venema [154], while the first proof of uniform interpolation for **IPL** was given by Pitts [148] on the basis of a G4-style calculus). Therefore, it is natural to ask whether this property is satisfied also by our systems. For some intuitionistic *normal* modal logics, uniform interpolation has been recently proved by Iemhoff [95] on the basis of G4-style calculi. In Section 7.6 we have presented G4-style calculi for all systems $\mathbf{IE}^*$. In future work we aim to study the possibility to extend Iemhoff's proof so to cover also our systems.

A further direction of research can consist in studying additional extensions of our family of intuitionistic non-normal modal logics. We conclude this chapter with some remarks in this direction.

Non-monotonic systems with $C_\Box$ and negative interactions a and b. We have seen in Section 7.3 that the combination of rule $\mathsf{C}_\Box$ with $\mathsf{neg_a C}$ – i.e., the generalisation to n principal formulas of rule $\mathsf{neg_a}$ – provides a cut-free calculus, and that the admissibility of cut is preserved by the addition of rules $\mathsf{N}_\Diamond$ and $\mathsf{N}_\Box$. By contrast, the addition of a proper generalisation of rule $\mathsf{neg_b}$ is problematic. As remarked in Section 7.3, a natural rule would be the following

$$\mathsf{neg_b C}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg(A_1 \land ... \land A_n) \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}\ ;$$

but at present it is an open problem whether this rule would give a cut-free calculus.

Alternatively, one could consider the rule

$$\mathsf{neg_b C'}\ \frac{A_1, ..., A_n, B \Rightarrow \qquad \neg A_1 \Rightarrow B \quad ... \quad \neg A_n \Rightarrow B}{\Gamma, \Box A_1, ..., \Box A_n, \Diamond B \Rightarrow \Delta}\ .$$

It can be shown that this rule gives cut-free calculi. However, since $\neg A_1 \lor ... \lor \neg A_n$ is not intuitionistically equivalent to $\neg(A_1 \land ... \land A_n)$, it is not obvious to see how these calculi would be related with the other systems of the lattice. The addition of this rule would be instead natural in a non-normal modal extension of a suitable intermediate logic.

Combinations of monotonic and non-monotonic modalities. In all systems considered in this chapter $\Box$ and $\Diamond$ are either both monotonic or both non-monotonic. However from a combinatorial perspective, and possibly under certain interpretation of the modalities, it makes sense to consider the cases in which one modality is monotonic and the other one is non-monotonic.

Let us consider first the cases in which one modality is characterised only by the congruence rule and the other modality is characterised only by the monotonicity rule (i.e., there are no axioms for N and C). It can be shown that the only rule for interaction which gives a cut-free calculus is

$$\mathsf{str}\ \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}\ .$$

In calculi defined by adding the other two interactions ($\mathsf{weak_a} + \mathsf{weak_b}$ and $\mathsf{neg_b} + \mathsf{neg_a}$) the cut rule is not admissible and it is possible to find counterexamples to cut elimination. For instance, the counterexample presented in Example 7.3.2 still holds when $\Diamond$ is non-monotonic. Concerning extensions with rules for N and C, we remark that $\mathsf{weak_a}$ is derivable from $\mathsf{N}_\Diamond$, whence in principle there might be more combinations of rules enjoying cut admissibility. We leave to future investigation the study of these combinations and the semantic properties of the resulting systems.

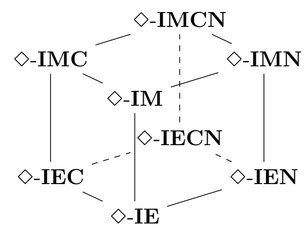

Figure 7.13: Extended lattice of $\Diamond$-logics.

Systems containing $C_\Diamond$. In this chapter we have restricted the analysis to systems not containing axiom $C_\Diamond$. This axiom is of particular significance in the intuitionistic context since it can be seen as a cut-off point between the constructive and the intuitionistic tradition. In future work we aim to extend our framework to cover also such systems, here we limit ourselves to some preliminary remarks.

Let us extend the $\Diamond$-family of monomodal logics to the systems containing $C_\Diamond$: $\Diamond$-logics are now defined by adding to **IPL** the congruence rule $RE_\Diamond$ and any combination of axioms $M_\Diamond$, $N_\Diamond$, and $C_\Diamond$. We obtain the picture in Figure 7.13, which contains 8 non equivalent systems. From the point of view of sequent calculi, $\Diamond$-logics containing $C_\Diamond$ could be covered by modifying the rules $\mathsf{E}_\Diamond$ and $\mathsf{M}_\Diamond$ in the following way, where Δ is a finite multiset of formulas of $\mathcal{L}_i$ without further restrictions on its cardinality:

$$\mathsf{C}_\Diamond \ \frac{A \Rightarrow B_1, ..., B_n \qquad B_1 \Rightarrow A \quad ... \quad B_1 \Rightarrow A}{\Gamma, \Diamond A \Rightarrow \Diamond B_1, ..., \Diamond B_n, \Delta} \qquad \mathsf{MC}_\Diamond \ \frac{A \Rightarrow B_1, ..., B_n}{\Gamma, \Diamond A \Rightarrow \Diamond B_1, ..., \Diamond B_n, \Delta}$$

If compared with the other rules considered in this work, rules for $C_\Diamond$ have the crucial difference of containing multiple formulas on the right-hand side of sequents. In order to admit these rules we have to take as base calculus instead of **G3ip** a multisuccedent calculus for intuitionistic logic, as for instance the propositional fragment of **m-G3i** in Troelstra and Schwichtenberg [164]. The sequent calculi for $\Diamond$-systems containing $C_\Diamond$ would then be defined by extending the basic multisuccedent calculus for **IPL** with modal rules as follows:

G3.$\Diamond$-**IEC**	:=	$\{\mathsf{C}_\Diamond\}$	**G3**.$\Diamond$-**IECN**	:=	$\{\mathsf{C}_\Diamond, \mathsf{N}_\Diamond\}$
G3.$\Diamond$-**IMC**	:=	$\{\mathsf{MC}_\Diamond\}$	**G3**.$\Diamond$-**IMCN**	:=	$\{\mathsf{MC}_\Diamond, \mathsf{N}_\Diamond\}$

Similarly to the calculi in Sections 7.2 and 7.3, one can show that the calculi defined in this way enjoy cut elimination. This result can be trivially extended to logics with $\Box$ and $\Diamond$ but without interactions (after rewriting the rules for $\Box$ in their multi-succedent versions). We leave to future work the investigation of interactions between the modalities which give cut-free calculi in presence of $C_\Diamond$.

Chapter 8

Intuitionistic neighbourhood semantics

In this chapter, we present a semantic framework for intuitionistic non-normal modal logics. This framework is defined in terms of neighbourhood models. On its basis, we provide a modular characterisation of all systems $\mathbf{IE}^*$ defined in the previous chapter, as well as of further intuitionistic non-normal modal logics such as **CK** and **CCDL** (cf. Section 2.6). Then, by applying the filtration technique we show that all these systems enjoy the finite model property. Moreover, basing on the semantics, we present an embedding of intuitionistic non-normal modal logics into classical non-normal multimodal logics. Finally, we present a prefixed tableaux calculus for our intuitionistic monotonic logics that allows one to extract countermodels of non-valid formulas in the neighbourhood semantics introduced in this chapter.

8.1 Coupled intuitionistic neighbourhood models

In this section, we present neighbourhood models for all the systems $\mathbf{IE}^*$ defined in Chapter 7. These models can be regarded as intuitionistic counterparts of the standard neighbourhood models for classical non-normal modal logics (cf. Section 2.3). Since we want to deal with logics containing both $\Box$ and $\Diamond$, we consider neighbourhood models endowed with *two* distinct neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$. We call these models *Coupled Intuitionistic Neighbourhood Models* (abbreviated as CINMs in the following). The two neighbourhood functions can be related by natural conditions that correspond to the different forms of interaction between the two modal operators. We also consider further closure conditions of neighbourhoods, that are analogous to the ones characterising the neighbourhood function in classical models. Furthermore, as in standard intuitionistic models, CINMs also contain a partial order on worlds which is used to deal with the intuitionistic implication. CINMs are defined as follows.

Definition 8.1.1 (Intuitionistic neighbourhood models for bimodal logics)**.** A *Coupled Intuitionistic Neighbourhood Model* (CINM) is a tuple $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set, $\preceq$ is a preorder over $\mathcal{W}$, $\mathcal{V}$ is a valuation function $\mathcal{W} \longrightarrow \mathcal{P}(Atm)$ satisfying the following hereditary condition:

$$\text{if } w \preceq v, \text{ then } \mathcal{V}(w) \subseteq \mathcal{V}(v);$$

and $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are two neighbourhood functions $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ such that:

$$\text{if } w \preceq v, \text{ then } \mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v) \text{ and } \mathcal{N}_\Diamond(w) \supseteq \mathcal{N}_\Diamond(v). \qquad (\mathsf{hp})$$

Functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are *supplemented*, *closed under intersection*, or *contain the unit*, if they satisfy the following conditions, where $\mathcal{N}_\circ \in \{\mathcal{N}_\Box, \mathcal{N}_\Diamond\}$:

If $\alpha \in \mathcal{N}_\circ(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\circ(w)$.	(Supplementation)
If $\alpha, \beta \in \mathcal{N}_\circ(w)$, then $\alpha \cap \beta \in \mathcal{N}_\circ(w)$.	(Closure under intersection)
$\mathcal{W} \in \mathcal{N}_\circ(w)$.	(Containing the unit)

Furthermore, let us define

$$-\alpha = \{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}.$$

Then, we say that $\mathcal{M}$ is $\mathsf{weakInt}$, or $\mathsf{negInt_b}$, or $\mathsf{negInt_a}$, or strInt, if $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are related by the corresponding condition below:

For all $w \in \mathcal{W}$, $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$.	(Weak interaction ($\mathsf{weakInt}$))
If $-\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus \alpha \in \mathcal{N}_\Diamond(w)$.	(Negation closure interaction-a ($\mathsf{negInt_a}$))
If $\alpha \in \mathcal{N}_\Box(w)$, then $\mathcal{W} \setminus -\alpha \in \mathcal{N}_\Diamond(w)$.	(Negation closure interaction-b ($\mathsf{negInt_b}$))
If $\alpha \in \mathcal{N}_\Box(w)$ and $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}_\Diamond(w)$.	(Strong interaction (strInt))

Finally, the forcing relation $\mathcal{M}, w \Vdash A$ associated to CINMs is as follows, where $[\![B]\!] = \{v \in \mathcal{W} \mid \mathcal{M}, v \Vdash B\}$:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in \mathcal{V}(w); \\
\mathcal{M}, w \nVdash \bot; & & \\
\mathcal{M}, w \Vdash \top; & & \\
\mathcal{M}, w \Vdash B \wedge C & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B; \\
\mathcal{M}, w \Vdash B \vee C & \text{iff} & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B; \\
\mathcal{M}, w \Vdash B \supset C & \text{iff} & \text{for all } v \succeq w, \mathcal{M}, v \Vdash B \text{ implies } \mathcal{M}, v \Vdash C; \\
\mathcal{M}, w \Vdash \Box B & \text{iff} & [\![B]\!] \in \mathcal{N}_\Box(w); \\
\mathcal{M}, w \Vdash \Diamond B & \text{iff} & \mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(w).
\end{array}
$$

The notion of satisfiability, validity, and semantic consequence are defined in the usual way (see Definition 2.3.2). As usual, we omit to specify the model $\mathcal{M}$, and just write $w \Vdash A$, when the considered model is made clear by the context.

In the above definition, we are considering for $\supset$-formulas their standard satisfaction clause in intuitionistic Kripke models, whereas for $\Box$- and $\Diamond$-formulas we are considering their satisfaction clauses in classical standard neighbourhood models (cf. Definition 2.3.1), although as a difference with standard models, $\Box$- and $\Diamond$-formulas are evaluated on the basis of distinct neighbourhood functions. By means of the condition hp which connects the neighbourhood functions $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ to the order $\preceq$, we ensure that CINMs preserve the hereditary property of intuitionistic Kripke models:

Proposition 8.1.1 (Hereditary property)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM. Then for every $A \in \mathcal{L}_i$ and $w, v \in \mathcal{W}$, if $w \Vdash A$ and $w \preceq v$, then $v \Vdash A$.

Proof. By induction on A. For $A = \bot, \top$ is immediate. For the other cases we have:

$(A = p)$ If $w \Vdash p$, then $p \in \mathcal{V}(w)$, then since $\mathcal{V}(w) \subseteq \mathcal{V}(v)$, $p \in \mathcal{V}(v)$, thus $v \Vdash p$.

$(A = B \wedge C)$ If $w \Vdash B \wedge C$, then $w \Vdash B$ and $w \Vdash C$, thus by i.h., $v \Vdash B$ and $v \Vdash C$, therefore $v \Vdash B \wedge C$.

$(A = B \vee C)$ If $w \Vdash B \vee C$, then $w \Vdash B$ or $w \Vdash C$, thus by i.h., $v \Vdash B$ or $v \Vdash C$, therefore $v \Vdash B \vee C$.

$(A = B \supset C)$ If $w \Vdash B \supset C$, then for all $z \geq w$, $z \Vdash B$ implies $z \Vdash C$, thus for all $z \geq v$, $z \Vdash B$ implies $z \Vdash C$, therefore $v \Vdash B \supset C$.

$(A = \Box B)$ If $w \Vdash \Box B$, then $[\![B]\!] \in \mathcal{N}_\Box(w)$, then by hp, $[\![B]\!] \in \mathcal{N}_\Box(v)$, therefore $v \Vdash \Box B$.

$(A = \Diamond B)$ If $w \Vdash \Diamond B$, then $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(w)$, then by hp, $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(v)$, therefore $v \Vdash \Diamond B$. $\square$

By simplifying Definition 8.1.1, we can also define intuitionistic neighbourhood models for monomodal $\Box$- and $\Diamond$-logics, henceforth called $\Box$-intuitionistic neighbourhood model and $\Diamond$-intuitionistic neighbourhood model, and abbreviated respectively as $\Box$-INMs and $\Diamond$-INMs.

Definition 8.1.2 (Intuitionistic neighbourhood models for monomodal logics)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a coupled intuitionistic neighbourhood model. Then its reduct $\langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{V} \rangle$ is a $\Box$-intuitionistic neighbourhood model, whereas its reduct $\langle \mathcal{W}, \preceq, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ is a $\Diamond$-intuitionistic neighbourhood model.

Essentially, models for monomodal $\Box$- and $\Diamond$-logics are defined by removing from Definition 8.1.1 the neighbourhood function and the forcing condition for the lacking modality. For every intuitionistic mono- or bimodal logic, we associate a class of intuitionistic neighbourhood models as follows.

$M_\Box$	$\mathcal{N}_\Box$ is supplemented
$N_\Box$	$\mathcal{N}_\Box$ contains the unit
$C_\Box$	$\mathcal{N}_\Box$ is closed under $\cap$
$M_\Diamond$	$\mathcal{N}_\Diamond$ is supplemented
$N_\Diamond$	$\mathcal{N}_\Diamond$ contains the unit

$weak_a + weak_b$	weakInt
neg_a	$\mathsf{negInt_a}$
neg_b	$\mathsf{negInt_b}$
str	strInt

Table 8.1: Semantic conditions associated to intuitionistic modal axioms.

Definition 8.1.3. Let $\mathbf{L}$ be any intuitionistic non-normal bi- or monomodal logic. Then, a model for $\mathbf{L}$ is any CINM (respectively $\Box$- or $\Diamond$-INM) where $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ (respectively $\mathcal{N}_\Box$ or $\mathcal{N}_\Diamond$) satisfy the conditions associated to every axiom of $\mathbf{L}$ according to Table 8.1. For every logic $\mathbf{L}$, we denote with $\mathcal{C}_\mathbf{L}$ the class of CINMs (respectively $\Box$- or $\Diamond$-INMs) for $\mathbf{L}$.

It is easy to see that for monotonic CINMs – that is, when both $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented – the two conditions weakInt and strInt are equivalent. For the sake of simplicity we therefore consider weakInt as the semantic condition corresponding to rule *str* in monotonic logics.

We can compare our models with other models already studied in the literature. For monomodal $\Box$-logics, our models essentially coincide with Goldblatt's neighbourhood spaces [73], although in that work the property of containing the unit is not considered. As a difference, in Goldblatt's spaces the neighbourhoods are assumed to be closed with respect to the order, that is, the following condition is assumed to hold:

$$\text{If } \alpha \in \mathcal{N}_\Box(w),\, v \in \alpha \text{ and } v \preceq u, \text{ then } u \in \alpha. \qquad (\preceq\text{-closure})$$

However, as already observed by Goldblatt, this property is irrelevant from the point of view of the validity of formulas, in the sense that a formula is valid in $\Box$-INMs (that are supplemented, closed under intersection, contain the unit) if and only if it is valid in the subclass of the corresponding $\Box$-INMs that in addition satisfy the $\preceq$-closure. Goldblatt's spaces have been considered in order to semantically characterise further intuitionistic monomodal logics: they have been considered by Goldblatt [73] and Fairtlough and Mendler [50] to provide a semantic characterisation of Propositional Lax Logic, an intuitionistic monomodal logic motivated by hardware verification which can be seen as non-normal as it fails to validate the axiom $C_\Diamond$ (cf. Section 2.5). Furthermore, they have been reformulated and extended to an intuitionistic version of logic $\mathbf{ET}$ by Witczak [172].

Concerning the use of distinct neighbourhood functions taking care of the different modal operators, in the context of classical modal logics an analogous solution is adopted e.g. in Anglberger et al. [3] in order to separately characterise obligations and permissions. As we have seen in Section 2.4, this solution is also adopted in the context of agency logics by Governatori and Rotolo [78] and Troquard [165] in order to separately characterise realisation of actions and capability.

8.2 Soundness and completeness

In this section, we prove that the intuitionistic non-normal modal logics $\mathbf{IE}^*$ are sound and complete with respect to the corresponding CINMs. We present all results explicitly only for bimodal logics, whereas for the simpler case of monomodal logics analogous results can be obtained by simplifying the proofs given here. Soundness is proved as follows.

Theorem 8.2.1 (Soundness)**.** Every intuitionistic non-normal modal logic $\mathbf{IE}^*$ is sound with respect to the corresponding CINMs: If $\Phi \vdash_{\mathbf{IE}^*} A$, then $\Phi \models_{\mathcal{C}_{\mathbf{IE}^*}} A$.

Proof. As usual, we have to show that every axiom or rule of $\mathbf{IE}^*$ is valid in the corresponding class of models. The proof is standard or propositional axioms and rules, and it is easy for $\Box$- and $\Diamond$-axioms. We consider here the interaction axioms and rules.

($weak_a$) Let $\mathcal{M}$ be $\mathsf{weakInt}$, and assume $x \Vdash \Box\top$. Then $[\![\top]\!] = \mathcal{W} \in \mathcal{N}_\Box(w)$. Thus by $\mathsf{weakInt}$, $\mathcal{W} \setminus [\![\bot]\!] = \mathcal{W} \in \mathcal{N}_\Diamond(w)$, then $x \not\Vdash \Diamond\bot$. Therefore $\mathcal{M} \models \neg(\Box\top \wedge \Diamond\bot)$.

($weak_b$) Let $\mathcal{M}$ be $\mathsf{weakInt}$, and assume $x \Vdash \Box\bot$. Then $[\![\bot]\!] = \emptyset \in \mathcal{N}_\Box(w)$. Thus by $\mathsf{weakInt}$, $\mathcal{W} \setminus [\![\top]\!] = \emptyset \in \mathcal{N}_\Diamond(w)$, then $x \not\Vdash \Diamond\top$. Therefore $\mathcal{M} \models \neg(\Box\bot \wedge \Diamond\top)$.

(neg_a) Let $\mathcal{M}$ be $\mathsf{negInt_a}$, and assume $x \Vdash \Box\neg A$. Then $[\![\neg A]\!] \in \mathcal{N}_\Box(w)$. We have $[\![\neg A]\!] = \{v \mid \text{for all } u \geq v, u \not\Vdash A\} = \{v \mid \text{for all } u \geq v, u \notin [\![A]\!]\} = -[\![A]\!]$. Thus by $\mathsf{negInt_a}$, $\mathcal{W} \setminus [\![A]\!] \in \mathcal{N}_\Diamond(w)$, then $x \not\Vdash \Diamond A$. Therefore $\mathcal{M} \models \neg(\Box\neg A \wedge \Diamond A)$.

(neg_b) Let $\mathcal{M}$ be $\mathsf{negInt_b}$, and assume $x \Vdash \Box A$. Then $[\![A]\!] \in \mathcal{N}_\Box(w)$, thus by $\mathsf{negInt_b}$, $\mathcal{W} \setminus -[\![A]\!] \in \mathcal{N}_\Diamond(w)$. As before, $-[\![A]\!] = [\![\neg A]\!]$. Then $\mathcal{W} \setminus [\![\neg A]\!] \in \mathcal{N}_\Diamond(w)$, which implies $x \not\Vdash \Diamond\neg A$. Therefore $\mathcal{M} \models \neg(\Box A \wedge \Diamond\neg A)$.

(str) Let $\mathcal{M}$ be strInt, and assume $\mathcal{M} \models \neg(A \wedge B)$. Then $[\![A]\!] \cap [\![B]\!] = \emptyset$, i.e., $[\![A]\!] \subseteq \mathcal{W} \setminus [\![B]\!]$. Now assume $x \Vdash \Box A$. Then $[\![A]\!] \in \mathcal{N}_\Box(w)$, thus by strInt, $\mathcal{W} \setminus [\![B]\!] \in \mathcal{N}_\Diamond(w)$, which implies $x \not\Vdash \Diamond B$. Therefore $\mathcal{M} \models \neg(\Box A \wedge \Diamond B)$. $\square$

We now move to the completeness of systems $\mathbf{IE}^*$. The completeness proofs are based on the canonical model construction. We consider the following standard definition.

Definition 8.2.1 (Prime sets)**.** Let $\mathbf{L}$ be any intuitionistic non-normal modal logic and $\mathcal{L}_i$ be the corresponding language. We call $\mathbf{L}$-*prime* any set Φ of formulas of $\mathcal{L}_i$ which is:

- consistent: $\Phi \not\vdash_{\mathbf{L}} \bot$;
- closed under derivation: if $\Phi \vdash_{\mathbf{L}} A$, then $A \in \Phi$;
- and satisfies the disjunction property: if $(A \vee B) \in \Phi$, then $A \in \Phi$ or $B \in \Phi$.

Moreover, for every $A \in \mathcal{L}_i$, we denote with $\uparrow A$ the class of prime sets Φ such that $A \in \Phi$.

The standard properties of prime sets hold, in particular:

Lemma 8.2.2. (a) If $\Phi \nvdash_{\mathbf{L}} A \supset B$, then there is a $\mathbf{L}$-prime set Ψ such that $\Phi \cup \{A\} \subseteq \Psi$ and $B \notin \Psi$. (b) For every $A, B \in \mathcal{L}$, $\uparrow A \subseteq \uparrow B$ implies $\vdash_{\mathbf{L}} A \supset B$.

Proof. The proof of claim (a) is standard: for the propositional fragment it can be found e.g. in [19], and for modal extensions it does not require major modifications. We present however a sketch of the proof. Let $C_0, C_1, C_2, ...$ be an enumeration of all formulas of $\mathcal{L}_i$. We construct a chain $\Psi_0, \Psi_1, \Psi_2, ...$ of sets of formulas of $\mathcal{L}_i$ as follows:

$$\begin{array}{rcl}\Psi_0 & = & \Phi \cup \{A\}. \\ \Psi_{n+1} & = & \begin{cases} \Psi_n \cup \{C_n\} & \text{if } \Psi_n \cup \{C_n\} \nvdash_{\mathbf{L}} B; \\ \Psi_n & \text{otherwise.}\end{cases}\end{array}$$

Moreover, we define $\Psi := \bigcup_{n\geq 0} \Psi_n$.

By construction we have $\Phi \cup \{A\} \subseteq \Psi$. Moreover, by construction and inductive hypothesis it is immediate to see that $\Psi_n \nvdash_{\mathbf{L}} B$ for every n. This implies $\Psi \nvdash_{\mathbf{L}} B$, therefore $B \notin \Psi$. We can show that Ψ is a $\mathbf{L}$-prime set: (Consistency) $\Psi \nvdash_{\mathbf{L}} \bot$ follows from $\Psi \nvdash_{\mathbf{L}} B$. (Closure under derivation) Assume $\Psi \vdash_{\mathbf{L}} D$. Then $\Psi \cup \{D\} \nvdash_{\mathbf{L}} B$. Moreover, $D = C_i$ for some C_i in the enumeration. Since $\Psi_i \subseteq \Psi$, $\Psi_i \cup \{D\} \nvdash_{\mathbf{L}} B$, then $D \in \Psi_{i+1}$ by construction. Therefore $D \in \Psi$. (Disjunction property) Assume by contradiction $D \vee E \in \Psi$ and $D \notin \Psi$, $E \notin \Psi$. Then $D = C_i$, $E = C_j$ for some C_i, C_j in the enumeration. Assume $i \geq j$. Then $D \notin \Psi_i$, $E \notin \Psi_i$, and by construction hypothesis, $\Psi_i \cup \{D\} \vdash_{\mathbf{L}} B$ and $\Psi_i \cup \{E\} \vdash_{\mathbf{L}} B$. This implies $\Psi_i \cup \{D \vee E\} \vdash_{\mathbf{L}} B$, and since $\Psi_i \cup \{D \vee E\} \subseteq \Psi$, $\Psi \vdash_{\mathbf{L}} B$, which was shown not to be the case.

(b) If $\nvdash_{\mathbf{L}} A \supset B$, then by (a) there is a $\mathbf{L}$-prime set Ψ such that $A \in \Psi$ and $B \notin \Psi$. Then $\Psi \in \uparrow A$ and $\Psi \notin \uparrow B$, therefore $\uparrow A \not\subseteq \uparrow B$. $\square$

As in the classical case – cf. Chellas [29] for the proof with the standard semantics, and Section 4.2 for the bi-neighbourhood semantics – in order to prove completeness we need to consider separately monotonic and non-monotonic systems. We first consider canonical models for non-monotonic systems, then we define canonical models enjoying supplementation for monotonic ones.

Definition 8.2.2 (Canonical models for non-monotonic systems)**.** Let $\mathbf{L}$ be any system not containing axioms $M_\Box$ and $M_\Diamond$. The *canonical model* $\mathcal{M}_{\mathbf{L}}$ for $\mathbf{L}$ is defined as the tuple $\langle \mathcal{W}_{\mathbf{L}}, \preceq_{\mathbf{L}}, \mathcal{N}_\Box^{\mathbf{L}}, \mathcal{N}_\Diamond^{\mathbf{L}}, \mathcal{V}_{\mathbf{L}} \rangle$, where:

- $\mathcal{W}_{\mathbf{L}}$ is the class of $\mathbf{L}$-prime sets;
- for every $\Phi, \Psi \in \mathcal{W}_{\mathbf{L}}$, $\Phi \preceq_{\mathbf{L}} \Psi$ if and only if $\Phi \subseteq \Psi$;

- $\mathcal{N}^{\mathbf{L}}_{\Box}(\Phi) = \{\uparrow A \mid \Box A \in \Phi\}$;
- $\mathcal{N}^{\mathbf{L}}_{\Diamond}(\Phi) = \mathcal{P}(\mathcal{W}_{\mathbf{L}}) \setminus \{\mathcal{W}_{\mathbf{L}} \setminus \uparrow A \mid \Diamond A \in \Phi\}$;
- $\mathcal{V}_{\mathbf{L}}(\Phi) = \{p \in \mathcal{L} \mid p \in \Phi\}$.

First of all, observe that the canonical model $\mathcal{M}_{\mathbf{L}}$ is a CINM: in particular, it follows from the definition that $\Phi \preceq_{\mathbf{L}} \Psi$ implies both $\mathcal{N}^{\mathbf{L}}_{\Box}(\Phi) \subseteq \mathcal{N}^{\mathbf{L}}_{\Box}(\Psi)$ and $\mathcal{N}^{\mathbf{L}}_{\Diamond}(\Phi) \supseteq \mathcal{N}^{\mathbf{L}}_{\Diamond}(\Psi)$. We prove the following lemma.

Lemma 8.2.3. Let $\mathbf{L}$ be any non-monotonic system, and $\mathcal{M}_{\mathbf{L}} = \langle \mathcal{W}_{\mathbf{L}}, \preceq_{\mathbf{L}}, \mathcal{N}^{\mathbf{L}}_{\Box}, \mathcal{N}^{\mathbf{L}}_{\Diamond}, \mathcal{V}_{\mathbf{L}} \rangle$ be the canonical model for $\mathbf{L}$. Then, for every $\Phi \in \mathcal{W}_{\mathbf{L}}$ and $A \in \mathcal{L}_i$ we have

$$\Phi \Vdash A \text{ if and only if } A \in \Phi.$$

Moreover: (i) If $\mathbf{L}$ contains $N_{\Box}$, then $\mathcal{N}^{\mathbf{L}}_{\Box}$ contains the unit.
(ii) If $\mathbf{L}$ contains $C_{\Box}$, then $\mathcal{N}^{\mathbf{L}}_{\Box}$ is closed under intersection.
(iii) If $\mathbf{L}$ contains $N_{\Diamond}$, then $\mathcal{N}^{\mathbf{L}}_{\Diamond}$ contains the unit.
(iv) If $\mathbf{L}$ contains $weak_a$ and $weak_b$, then $\mathcal{M}_{\mathbf{L}}$ is $\mathsf{weakInt}$.
(v) If $\mathbf{L}$ contains neg_a, then $\mathcal{M}_{\mathbf{L}}$ is $\mathsf{negInt_a}$.
(vi) If $\mathbf{L}$ contains neg_b, then $\mathcal{M}_{\mathbf{L}}$ is $\mathsf{negInt_b}$.
(vii) If $\mathbf{L}$ contains str, then $\mathcal{M}_{\mathbf{L}}$ is strInt.

Proof. By induction on A, we prove that $\Phi \Vdash A$ if and only if $A \in \Phi$. If $A \equiv p, \ \bot, \top, \ B \wedge C, \ B \vee C$ the proof is immediate. We consider the cases $A = B \supset C, \Box B, \Diamond B$.

($A = B \supset C$) If $\Phi \Vdash B \supset C$, then for every $\Psi \in \mathcal{W}$ such that $\Phi \preceq_{\mathbf{L}} \Psi$, $\Psi \Vdash B$ implies $\Psi \Vdash C$. Then by definition of $\preceq_{\mathbf{L}}$ and i.h., for every $\Psi \in \mathcal{W}$ such that $\Phi \subseteq \Psi$, $B \in \Psi$ implies $C \in \Psi$. Then by Lemma 8.2.2, $\Phi \vdash_{\mathbf{L}} B \supset C$ (otherwise there would be a $\mathbf{L}$-prime set $\Upsilon \in \mathcal{W}_{\mathbf{L}}$ such that $\Phi \subseteq \Upsilon$, $B \in \Upsilon$, and $C \notin \Upsilon$). Then by closure under derivation, $B \supset C \in \Phi$. For the other direction, if $B \supset C \in \Phi$, then for all $\Psi \supseteq \Phi$, $B \supset C \in \Psi$. By closure under derivation, for all $\Psi \supseteq \Phi$, $B \in \Psi$ implies $C \in \Psi$. Then by definition of $\preceq_{\mathbf{L}}$ and i.h., for all $\Phi \in \mathcal{W}_{\mathbf{L}}$ such that $\Phi \preceq_{\mathbf{L}} \Psi$, $\Psi \Vdash B$ implies $\Psi \Vdash C$. Therefore $\Phi \Vdash B \supset C$.

($A = \Box B$) For the converse implication, assume $\Box B \in \Phi$. Then by definition $\uparrow B \in \mathcal{N}^{\mathbf{L}}_{\Box}(\Phi)$, and by inductive hypothesis, $\uparrow B = [\![B]\!]_{\mathcal{M}_{\mathbf{L}}}$, therefore $\Phi \Vdash \Box B$. For the direct implication, assume $\Phi \Vdash \Box B$. Then we have $[\![B]\!]_{\mathcal{M}_{\mathbf{L}}} \in \mathcal{N}^{\mathbf{L}}_{\Box}(\Phi)$, and, by inductive hypothesis, $[\![B]\!]_{\mathcal{M}_{\mathbf{L}}} =\uparrow B$. By definition, this means that there is $C \in \mathcal{L}_i$ such that $\Box C \in \Phi$ and $\uparrow C =\uparrow B$. Then, by Lemma 8.2.2, $\vdash_{\mathbf{L}} C \supset B$ and $\vdash_{\mathbf{L}} B \supset C$. Thus by $RE_{\Box}$, $\vdash_{\mathbf{L}} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in \Phi$.

($A = \Diamond B$) For the converse implication, assume $\Diamond B \in \Phi$. Then by definition $\mathcal{W}_{\mathbf{L}} \setminus \uparrow B \notin \mathcal{N}^{\mathbf{L}}_{\Diamond}(\Phi)$, and by inductive hypothesis, $\uparrow B = [\![B]\!]_{\mathcal{M}_{\mathbf{L}}}$, therefore $\Phi \Vdash \Diamond B$. For the direct implication, assume $\Phi \Vdash \Diamond B$. Then we have $\mathcal{W}_\mathbf{L} \setminus [\![B]\!]_{\mathcal{M}_\mathbf{L}} \notin \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$, and, by inductive hypothesis, $\mathcal{W}_\mathbf{L} \setminus \uparrow B \notin \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$. This means that there is $C \in \mathcal{L}_i$ such that $\Diamond C \in \Phi$ and $\uparrow C = \uparrow B$. Thus, $\vdash_\mathbf{L} C \supset B$ and $\vdash_\mathbf{L} B \supset C$, therefore by $RE_\Diamond$, $\vdash_\mathbf{L} \Diamond C \supset \Diamond B$. By closure under derivation, we obtain that $\Diamond B \in \Phi$.

Claims (i)–(vii) are proved as follows: (i) $\Box\top \in \Phi$ for every $\Phi \in \mathcal{W}_\mathbf{L}$. Then by definition $\mathcal{W}_\mathbf{L} = \uparrow\top \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$.

(ii) Assume $\alpha, \beta \in \mathcal{N}_\mathbf{L}(\Phi)$. Then there are $A, B \in \mathcal{L}$ such that $\Box A, \Box B \in \Phi$, $\alpha = \uparrow A$ and $\beta = \uparrow B$. By closure under derivation, $\Box(A \wedge B) \in \Phi$, and, by definition, $\uparrow(A \wedge B) \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$, where $\uparrow(A \wedge B) = \uparrow A \cap \uparrow B = \alpha \cap \beta$.

(iii) $\neg\Diamond\bot \in \Phi$ for every $\Phi \in \mathcal{W}_\mathbf{L}$, thus by consistency, $\Diamond\bot \notin \Phi$. If $\mathcal{W}_\mathbf{L} \setminus \uparrow\bot \notin \mathcal{N}^\mathbf{L}_\Box(\Phi)$, then there is $A \in \mathcal{L}_i$ such that $\uparrow A = \uparrow\bot$ and $\Diamond A \in \Phi$, that implies $\Diamond\bot \in \Phi$. Therefore $\mathcal{W}_\mathbf{L} = \mathcal{W}_\mathbf{L} \setminus \uparrow\bot \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$.

(iv) Assume by contradiction that $\alpha \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$ and $\alpha \notin \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow A$, $\alpha = \mathcal{W}_\mathbf{L} \setminus \uparrow B$, and $\Box A, \Diamond B \in \Phi$, therefore $\uparrow A = \mathcal{W}_\mathbf{L} \setminus \uparrow B$. By the properties of prime sets, this implies that $\vdash_\mathbf{L} \neg(A \wedge B)$ and $\vdash_\mathbf{L} A \vee B$, and by the disjunction property, $\vdash_\mathbf{L} A$ or $\vdash_\mathbf{L} B$. If we assume $\vdash_\mathbf{L} A$, then $\vdash_\mathbf{L} A \supset\subset \top$ and $\vdash_\mathbf{L} B \supset\subset \bot$. Therefore by $RE_\Box$ and $RE_\Diamond$, $\vdash_\mathbf{L} \Box A \supset \Box\top$ and $\vdash_\mathbf{L} \Diamond B \supset \Diamond\bot$, thus by closure under derivation, $\Box\top, \Diamond\bot \in \Phi$. But $\neg(\Box\top \wedge \Diamond\bot) \in \Phi$, in contradiction with the consistency of prime sets. If we now assume $\vdash_\mathbf{L} B$, then $\vdash_\mathbf{L} B \supset\subset \top$ and $\vdash_\mathbf{L} A \supset\subset \bot$. We obtain an analogous contradiction considering $\neg(\Diamond\top \wedge \Box\bot)$.

(v) By contraposition, assume that $\mathcal{W}_\mathbf{L} \setminus \alpha \notin \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$. Then there is $A \in \mathcal{L}_i$ such that $\mathcal{W}_\mathbf{L} \setminus \alpha = \mathcal{W}_\mathbf{L} \setminus \uparrow A$ and $\Diamond A \in \Phi$. Thus $\alpha = \uparrow A$, and by neg_a, $\Box\neg A \notin \Phi$. Therefore $\uparrow\neg A \notin \mathcal{N}^\mathbf{L}_\Box(\Phi)$ — otherwise there would be $\Box B \in \Phi$ such that $\uparrow\neg A = \uparrow B$, which implies $\Box\neg A \in \Phi$. Since $\uparrow\neg A = - \uparrow A = -\alpha$, the claim holds.

(vi) Assume $\alpha \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$. Then there is $A \in \mathcal{L}_i$ such that $\alpha = \uparrow A$ and $\Box A \in \Phi$. Thus, by neg_b and consistency of Φ, $\Diamond\neg A \notin \Phi$. Therefore $\mathcal{W}_\mathbf{L} \setminus \uparrow\neg A \in \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$ (otherwise there would be $B \in \mathcal{L}$ such that $\uparrow B = \uparrow\neg A$ and $\Diamond B \in \Phi$, which implies $\Diamond\neg A \in \Phi$). Since $\uparrow\neg A = - \uparrow A$ ($\uparrow\neg A = [\![\neg A]\!]_{\mathcal{M}_\mathbf{L}} = -[\![A]\!]_{\mathcal{M}_\mathbf{L}} = - \uparrow A$) and $- \uparrow A = -\alpha$, the claim holds.

(vii) Assume by contradiction that $\alpha \in \mathcal{N}^\mathbf{L}_\Box(\Phi)$, $\alpha \subseteq \beta$, and $\beta \notin \mathcal{N}^\mathbf{L}_\Diamond(\Phi)$. Then there are $A, B \in \mathcal{L}$ such that $\alpha = \uparrow A$, $\beta = \mathcal{W}_\mathbf{L} \setminus \uparrow B$ and $\Box A, \Diamond B \in \Phi$. Moreover, $\uparrow A \subseteq \mathcal{W}_\mathbf{L} \setminus \uparrow B$, which implies $\uparrow A \cap \uparrow B = \emptyset$. Thus $\vdash_\mathbf{L} \neg(A \wedge B)$; and by *str* we have $\vdash_\mathbf{L} \neg(\Box A \wedge \Diamond B)$, in contradiction with the consistency of Φ. $\Box$

We now define canonical models and prove an analogous lemma for the monotonic systems. We shorten the proof by considering, instead of the axioms $M_\Box$ and $M_\Diamond$, the syntactically equivalent rules $RM_\Box$ and $RM_\Diamond$.

Definition 8.2.3 (Canonical models for monotonic systems)**.** Let $\mathbf{L}$ be any system containing axioms $M_\Box$ and $M_\Diamond$. The *canonical model* $\mathcal{M}^+_\mathbf{L}$ for $\mathbf{L}$ is the tuple $\langle \mathcal{W}_\mathbf{L}, \preceq_\mathbf{L}, \mathcal{N}^+_\Box, \mathcal{N}^+_\Diamond, \mathcal{V}_\mathbf{L}\rangle$, where $\mathcal{W}_\mathbf{L}, \preceq_\mathbf{L}, \mathcal{V}_\mathbf{L}$ are defined as in Definition 8.2.2, and:

$\mathcal{N}^+_\Box(\Phi) = \{\alpha \subseteq \mathcal{W}_\mathbf{L} \mid \text{there is } A \in \mathcal{L}_i \text{ such that } \Box A \in \Phi \text{ and } \uparrow A \subseteq \alpha\}$.
$\mathcal{N}^+_\Diamond(\Phi) = \mathcal{P}(\mathcal{W}_\mathbf{L}) \setminus \{\alpha \subseteq \mathcal{W}_\mathbf{L} \mid \text{there is } A \in \mathcal{L}_i \text{ such that } \Diamond A \in \Phi \text{ and } \alpha \subseteq \mathcal{W}_\mathbf{L} \setminus \uparrow A\}$.

Lemma 8.2.4. Let $\mathbf{L}$ be any monotonic system and $\mathcal{M}^+_\mathbf{L} = \langle \mathcal{W}_\mathbf{L}, \preceq_\mathbf{L}, \mathcal{N}^+_\Box, \mathcal{N}^+_\Diamond, \mathcal{V}_\mathbf{L}\rangle$ be the canonical model for $\mathbf{L}$. Then $\Phi \Vdash A$ if and only if $A \in \Phi$. Moreover, claims (i)–(iii) of Lemma 8.2.3 still hold. Finally: (iv), if $\mathbf{L}$ contains *str*, then $\mathcal{M}^+_\mathbf{L}$ is $\mathsf{weakInt}$.

Proof. Observe that both $\mathcal{N}^+_\Box$ and $\mathcal{N}^+_\Diamond$ are supplemented. The proof is by induction on A, we only show the modal cases.

($A = \Box B$) For the converse implication, assume that $\Box B \in \Phi$. Then by definition $\uparrow B \in \mathcal{N}^+_\Box(\Phi)$, and by inductive hypothesis, $\uparrow B = [\![B]\!]_{\mathcal{M}^+_\mathbf{L}}$, therefore $\Phi \Vdash \Box B$. For the direct implication, assume that $\Phi \Vdash \Box B$. Then we have $[\![B]\!]_{\mathcal{M}^+_\mathbf{L}} \in \mathcal{N}^+_\Box(\Phi)$, and, by inductive hypothesis, $[\![B]\!]_{\mathcal{M}^+_\mathbf{L}} = \uparrow B$. By definition, this means that there is $C \in \mathcal{L}_i$ such that $\Box C \in \Phi$ and $\uparrow C \subseteq \uparrow B$, which then implies $\vdash_\mathbf{L} C \supset B$. Thus, by $RM_\Box$, $\vdash_\mathbf{L} \Box C \supset \Box B$, and, by closure under derivation, $\Box B \in \Phi$.

($A = \Diamond B$) For the converse implication, assume that $\Diamond B \in \Phi$. Then by definition $\mathcal{W}_\mathbf{L} \setminus \uparrow B \notin \mathcal{N}^+_\Diamond(\Phi)$, and by inductive hypothesis, $\uparrow B = [\![B]\!]_{\mathcal{M}^+_\mathbf{L}}$, therefore $\Phi \Vdash \Diamond B$. For the direct implication, assume $\Phi \Vdash \Diamond B$. Then we have $\mathcal{W}_\mathbf{L} \setminus [\![B]\!]_{\mathcal{M}_\mathbf{L}} \notin \mathcal{N}^+_\Diamond(\Phi)$, and, by inductive hypothesis, $\mathcal{W}_\mathbf{L} \setminus \uparrow B \notin \mathcal{N}^+_\Diamond(\Phi)$. This means that there is $C \in \mathcal{L}_i$ such that $\Diamond C \in \Phi$ and $\mathcal{W}_\mathbf{L} \setminus B \subseteq \mathcal{W}_\mathbf{L} \setminus C$, that is $\uparrow C \subseteq \uparrow B$. Thus, $\vdash_\mathbf{L} C \supset B$, therefore by $RE_\Diamond$, $\vdash_\mathbf{L} \Diamond C \supset \Diamond B$. By closure under derivation we then have $\Diamond B \in \Phi$.

Claims (i)–(iii) are proved similarly to the claims (i)–(iii) in Lemma 8.2.3. For (iv): By contradiction, assume that $\alpha \in \mathcal{N}^+_\Box(\Phi)$ and $\alpha \notin \mathcal{N}^+_\Diamond(\Phi)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow A \subseteq \alpha$, $\alpha \subseteq \mathcal{W}_\mathbf{L} \setminus \uparrow B$, and $\Box A, \Diamond B \in \Phi$. Therefore $\uparrow A \subseteq \mathcal{W}_\mathbf{L} \setminus \uparrow B$, which implies $\vdash_\mathbf{L} \neg(A \wedge B)$. By *str* we then have $\neg(\Box A \wedge \Diamond B) \in \Phi$, in contradiction with the consistency of Φ. $\Box$

Theorem 8.2.5 (Completeness)**.** Every intuitionistic non-normal modal logic $\mathbf{IE}^*$ is complete with respect to the corresponding CINMs: If $\Phi \models_{\mathcal{C}_{\mathbf{IE}^*}} A$, then $\Phi \vdash_{\mathbf{IE}^*} A$.

Proof. Assume $\Phi \not\vdash_{\mathbf{IE}^*} A$. Then $\Phi \not\vdash_{\mathbf{IE}^*} \top \supset A$. Thus by Lemma 8.2.2, there is a $\mathbf{IE}^*$-prime set Ψ such that $\Phi \subseteq \Psi$ and $A \notin \Psi$. By definition, Ψ is a world of the canonical model $\mathcal{M}_{\mathbf{IE}^*}$ for $\mathbf{IE}^*$ (respectively $\mathcal{M}^+_{\mathbf{IE}^*}$ if $\mathbf{IE}^*$ is monotonic). Moreover, by Lemma 8.2.3, $\mathcal{M}_{\mathbf{IE}^*}, \Psi \Vdash B$ for every $B \in \Phi$, and $\mathcal{M}_{\mathbf{IE}^*}, \Psi \not\Vdash A$. Then, since $\mathcal{M}_{\mathbf{IE}^*}$ is a CINM for $\mathbf{IE}^*$, we have $\Phi \not\models_{\mathcal{C}_{\mathbf{IE}^*}} A$. $\Box$

It can be easily verified that by removing $\mathcal{N}_{\Diamond}^{\mathbf{L}}$ (resp. $\mathcal{N}_{\Diamond}^{+}$) or $\mathcal{N}_{\Box}^{\mathbf{L}}$ (resp. $\mathcal{N}_{\Box}^{+}$) from the definition of $\mathcal{M}_{\mathbf{L}}$ (resp. $\mathcal{M}_{\mathbf{L}}^{+}$), we obtain analogous results for monomodal logics. Therefore we have:

Theorem 8.2.6 (Completeness)**.** Every intuitionistic non-normal monomodal $\Box$- or $\Diamond$-logic is sound and complete with respect to the corresponding $\Box$- or $\Diamond$-INMs, respectively.

8.3 Finite model property

We have seen that all intuitionistic non-normal modal logics defined in Section 7.2 and 7.3 are sound and complete with respect to the corresponding models. In this section, we prove that all logics enjoy the finite model property, meaning that if a formula is satisfiable in the class of models for a given logic, then it is satisfiable in a finite model belonging to the same class. Since these logics are recursively axiomatisable, the finite model property implies that the logics are decidable (see e.g. Blackburn et al. [21], p. 339), thus providing a semantic proof of decidability which is alternative to the syntactic one presented in Section 7.4.

Our proof of the finite model property is based on the filtration technique. Given a model, this technique allows one to define a finite model which is equivalent to the initial one with respect to a finite set of formulas. The proofs are given explicitly for bimodal logics, while the simpler proofs for monomodal logics can be easily extracted. We consider the following definitions.

Definition 8.3.1. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}, \mathcal{V}\rangle$ be a CINM and Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas. We define the equivalence relation $\sim$ on $\mathcal{W}$ as follows:

$$w \sim v \quad \text{iff} \quad \text{for all } A \in \Phi,\ w \Vdash A \text{ iff } v \Vdash A.$$

Moreover, for every $w \in \mathcal{W}$ and $\alpha \subseteq \mathcal{W}$, we denote with $w_\sim$ the set $\{v \in \mathcal{W} \mid v \sim w\}$, i.e., the equivalence class containing w, and with $\alpha^\sim$ the set $\{w_\sim \mid w \in \alpha\}$ (thus in particular $[\![A]\!]_{\mathcal{M}}^{\sim}$ is the set $\{w_\sim \mid w \in [\![A]\!]_{\mathcal{M}}\}$).

Definition 8.3.2 (Filtration)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}, \mathcal{V}\rangle$ be a CINM and Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas. A *filtration* of $\mathcal{M}$ through Φ (or Φ-filtration) is any CINM $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}_{\Box}^*, \mathcal{N}_{\Diamond}^*, \mathcal{V}^*\rangle$ such that:

- $\mathcal{W}^* = \{w_\sim \mid w \in \mathcal{W}\}$;
- $w_\sim \preceq^* v_\sim$ if and only if for all $A \in \Phi$, $\mathcal{M}, w \Vdash A$ implies $\mathcal{M}, v \Vdash A$;
- for every $\Box A \in \Phi$, $[\![A]\!]_{\mathcal{M}}^{\sim} \in \mathcal{N}_{\Box}^*(w_\sim)$ if and only if $[\![A]\!]_{\mathcal{M}} \in \mathcal{N}_{\Box}(w)$;
- for every $\Diamond A \in \Phi$, $\mathcal{W}^* \setminus [\![A]\!]_{\mathcal{M}}^{\sim} \in \mathcal{N}_{\Diamond}^*(w_\sim)$ if and only if $\mathcal{W} \setminus [\![A]\!]_{\mathcal{M}} \in \mathcal{N}_{\Diamond}(w)$;

- for every $p \in \Phi$, $p \in \mathcal{V}^*(w_\sim)$ if and only if $p \in \mathcal{V}(w)$.

Observe that the above model $\mathcal{M}^*$ is well-defined, in particular for every $\Box A, \Diamond B, p \in \Phi$, $w \sim v$ implies that $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ if and only if $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(v_\sim)$, $\mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(w_\sim)$ if and only if $\mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Diamond(v_\sim)$, and $p \in \mathcal{V}^*(w_\sim)$ if and only if $p \in \mathcal{V}^*(v_\sim)$. For instance, if $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$, then $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then $w \Vdash \Box A$, thus, since $\Box A \in \Phi$, $v \Vdash A$, therefore $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(v)$, hence $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(v_\sim)$. Moreover, it is easy to see that $\mathcal{M}^*$ satisfies the properties of CINMs.

Lemma 8.3.1 (Filtration lemma)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM, Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas, and $\mathcal{M}^*$ be a Φ-filtration of $\mathcal{M}$. Then, for every formula $A \in \Phi$,

$$\mathcal{M}^*, w_\sim \Vdash A \text{ if and only if } \mathcal{M}, w \Vdash A.$$

Proof. This is equivalent to prove that $[\![A]\!]_{\mathcal{M}^*} = [\![A]\!]^\sim_\mathcal{M}$. The proof is by induction on A. For $A = p, \bot, \top, B \land C$, or $B \lor C$, the proof is immediate. For the other cases we have:

($A = B \supset C$) For Assume $\mathcal{M}, w \nVdash B \supset C$. Then there is $v \succeq w$ such that $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$. By inductive hypothesis $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. Moreover, by definition of $\preceq^*$ and the fact that $\mathcal{M}$ satisfies the hereditary property, $w_\sim \preceq^* v_\sim$. Therefore $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Now assume $\mathcal{M}^*, w_\sim \nVdash B \supset C$. Then there is $v_\sim \in \mathcal{W}^*$ such that $w_\sim \preceq^* v_\sim$, $\mathcal{M}^*, v_\sim \Vdash B$ and $\mathcal{M}^*, v_\sim \nVdash C$. By inductive hypothesis $\mathcal{M}, v \Vdash B$ and $\mathcal{M}, v \nVdash C$, thus $\mathcal{M}, v \nVdash B \supset C$. By definition of $\preceq^*$ we then have $\mathcal{M}, w \nVdash B \supset C$.

($A = \Box B$) $\mathcal{M}^*, w_\sim \Vdash \Box B$ iff $[\![B]\!]_{\mathcal{M}^*} \in \mathcal{N}^*_\Box(w_\sim)$ iff (i.h.) $[\![B]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$ iff $[\![B]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$ iff $\mathcal{M}, w \Vdash \Box B$.

($A = \Diamond B$) $\mathcal{M}^*, w_\sim \Vdash \Diamond B$ iff $\mathcal{W}^* \setminus [\![B]\!]_{\mathcal{M}^*} \notin \mathcal{N}^*_\Diamond(w_\sim)$ iff (i.h.) $\mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M} \notin \mathcal{N}^*_\Diamond(w_\sim)$ iff $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$ iff $\mathcal{M}, w \Vdash \Diamond B$. $\Box$

We show that the general notion of filtration allows us to prove the finite model property for the logics which are not monotonic and do not contain $C_\Box$.

Lemma 8.3.2. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM, Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas, and $\mathcal{M}^*$ be a Φ-filtration of $\mathcal{M}$. Then: (i) If $\mathcal{N}_\Box$ contains the unit and $\Box\top \in \Phi$, then $\mathcal{N}^*_\Box$ contains the unit. (ii) If $\mathcal{N}_\Diamond$ contains the unit and $\Diamond\bot \in \Phi$, then $\mathcal{N}^*_\Diamond$ contains the unit. (iii) If $\mathcal{M}$ is $\mathsf{weakInt}$, then $\mathcal{M}^*$ is $\mathsf{weakInt}$. (iv) If $\mathcal{M}$ is strInt, then $\mathcal{M}^*$ is strInt. (v) If $\mathcal{M}$ is $\mathsf{negInt_a}$ and Φ is such that $\neg A \in \Phi$ for all $\Diamond A \in \Phi$, then $\mathcal{M}^*$ is $\mathsf{negInt_a}$. (vi) If $\mathcal{M}$ is $\mathsf{negInt_b}$ and Φ is such that $\neg A \in \Phi$ for all $\Box A \in \Phi$, then $\mathcal{M}^*$ is $\mathsf{negInt_b}$.

Proof. (i, ii) The claims follow from Definition 8.3.2 and Lemma 8.3.1, for instance if $\mathcal{W} = [\![\top]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then, since $\Box\top \in \Phi$, we have $[\![\top]\!]^\sim_\mathcal{M} = [\![\top]\!]_{\mathcal{M}^*} = \mathcal{W}^* \in \mathcal{N}^*_\Box(w_\sim)$.

(iii) Assume by contradiction that $\alpha \in \mathcal{N}^*_\Box(w_\sim)$ and $\alpha \notin \mathcal{N}^*_\Diamond(w_\sim)$. Then $\alpha = [\![A]\!]^\sim_\mathcal{M}$ for a $A \in \mathcal{L}_i$ such that $\Box A \in \Phi$ and $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover $\alpha = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[\![A]\!]^\sim_\mathcal{M} = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$, which implies $[\![A]\!]_\mathcal{M} = \mathcal{W} \setminus [\![B]\!]_\mathcal{M}$ ($w \in [\![A]\!]_\mathcal{M}$ iff $w_\sim \in [\![A]\!]^\sim_\mathcal{M}$ iff $w_\sim \in \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ iff $w \in \mathcal{W} \setminus [\![B]\!]_\mathcal{M}$). Then, since $\mathcal{M}$ is $\mathsf{weakInt}$, $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(iv) Assume by contradiction that $\alpha \in \mathcal{N}^*_\Box(w_\sim)$, $\alpha \subseteq \beta$ and $\beta \notin \mathcal{N}^*_\Diamond(w_\sim)$. Then $\alpha = [\![A]\!]^\sim_\mathcal{M}$ for a $A \in \mathcal{L}_i$ such that $\Box A \in \Phi$ and $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover $\beta = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ for a $B \in \mathcal{L}$ such that $\Diamond B \in \Phi$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[\![A]\!]^\sim_\mathcal{M} \subseteq \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$, which implies $[\![A]\!]_\mathcal{M} \subseteq \mathcal{W} \setminus [\![B]\!]_\mathcal{M}$. Then, since $\mathcal{M}$ is strInt, $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(v) Assume by contradiction that $-\alpha \in \mathcal{N}^*_\Box(w_\sim)$ and $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}^*_\Diamond(w_\sim)$. Then there is $\Box A \in \Phi$ such that $-\alpha = [\![A]\!]^\sim_{\mathcal{M}^*}$ and $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. In addition there is $\Diamond B \in \Phi$ such that $\mathcal{W}^* \setminus \alpha = \mathcal{W}^* \setminus [\![B]\!]^\sim_{\mathcal{M}^*}$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence we have $[\![A]\!]^\sim_{\mathcal{M}^*} = -[\![B]\!]^\sim_{\mathcal{M}^*} = [\![\neg B]\!]^\sim_{\mathcal{M}^*}$. Having $\neg B \in \Phi$, by the filtration lemma we obtain $[\![A]\!]_\mathcal{M} = [\![\neg B]\!]_\mathcal{M}$. Then $[\![\neg B]\!]_\mathcal{M} = -[\![B]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Finally, by $\mathsf{negInt_a}$ $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

(vi) Assume by contradiction that $\alpha \in \mathcal{N}^*_\Box(w_\sim)$ and $\mathcal{W}^* \setminus -\alpha \notin \mathcal{N}^*_\Diamond(w_\sim)$. Then there is $\Box A \in \Phi$ s.t. $\alpha = [\![A]\!]^\sim_{\mathcal{M}^*}$ and $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. In addition there is $\Diamond B \in \Phi$ s.t. $\mathcal{W}^* \setminus -\alpha = \mathcal{W}^* \setminus [\![B]\!]^\sim_{\mathcal{M}^*}$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence we have $[\![B]\!]^\sim_{\mathcal{M}^*} = -[\![A]\!]^\sim_{\mathcal{M}^*} = [\![\neg A]\!]^\sim_{\mathcal{M}^*}$. Having $\neg A \in \Phi$, by the filtration lemma we obtain $[\![B]\!]_\mathcal{M} = [\![\neg A]\!]_\mathcal{M}$. Since $\mathcal{M}$ is $\mathsf{negInt_b}$, we have $\mathcal{W} \setminus -[\![A]\!]_\mathcal{M} = \mathcal{W} \setminus [\![\neg A]\!]_\mathcal{M} = \mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction. □

In contrast, in order to prove the finite model property for models satisfying supplementation or closure under intersection we must consider a finer notion that, following Chellas [29], we call *finest* filtration.

Definition 8.3.3 (Finest filtration, supplementation, intersection closure, quasi-filtering)**.** Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM, and Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas. We call *finest* Φ-filtration of $\mathcal{M}$ any Φ-filtration $\mathcal{M}^*$ of $\mathcal{M}$ such that:

$$\mathcal{N}^*_\Box(w_\sim) = \{[\![A]\!]^\sim_\mathcal{M} \mid \Box A \in \Phi \text{ and } [\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)\};$$
$$\mathcal{N}^*_\Diamond(w_\sim) = \mathcal{P}(\mathcal{W}^*) \setminus \{\mathcal{W}^* \setminus [\![A]\!]^\sim_\mathcal{M} \mid \Diamond A \in \Phi \text{ and } \mathcal{W} \setminus [\![A]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)\}.$$

Moreover, let $\mathcal{M}^\circ = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}^\circ_\Box, \mathcal{N}^\circ_\Diamond, \mathcal{V}^* \rangle$ be a CINM where $\mathcal{W}^*$, $\preceq^*$ and $\mathcal{V}^*$ are as in $\mathcal{M}^*$. We say that:

- $\mathcal{M}^\circ$ is the *supplementation* of $\mathcal{M}^*$ if:

$$\alpha \in \mathcal{N}^\circ_\Box(w_\sim) \text{ iff there is } \beta \in \mathcal{N}^*_\Box(w_\sim) \text{ such that } \beta \subseteq \alpha, \text{ and}$$
$$\alpha \in \mathcal{N}^\circ_\Diamond(w_\sim) \text{ iff for every } \beta \subseteq \mathcal{W}^*, \text{ if } \alpha \subseteq \beta, \text{ then } \beta \in \mathcal{N}^*_\Diamond(w_\sim).$$

- $\mathcal{M}^\circ$ is the *intersection closure* of $\mathcal{M}^*$ if:

  $\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ iff there are $\beta_1, ..., \beta_n \in \mathcal{N}^*_\Box(w_\sim)$ such that $\beta_1 \cap ... \cap \beta_n = \alpha$, and
  $\alpha \in \mathcal{N}^\circ_\Diamond(w_\sim)$ iff $\alpha \in \mathcal{N}^*_\Diamond(w_\sim)$.

- $\mathcal{M}^\circ$ is the *quasi-filtering* of $\mathcal{M}^*$ if:

  $\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ iff there are $\beta_1, ..., \beta_n \in \mathcal{N}^*_\Box(w_\sim)$ such that $\beta_1 \cap ... \cap \beta_n \subseteq \alpha$, and
  $\alpha \in \mathcal{N}^\circ_\Diamond(w_\sim)$ iff for every $\beta \subseteq \mathcal{W}^*$, if $\alpha \subseteq \beta$, then $\beta \in \mathcal{N}^*_\Diamond(w_\sim)$.

It is easy to verify that the supplementation of a model is supplemented, its intersection closure is closed under intersection, and its quasi-filtering is both supplemented and closed under intersection.

Lemma 8.3.3. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ and $\mathcal{M}^\circ = \langle \mathcal{W}^\circ, \preceq^\circ, \mathcal{N}^\circ_\Box, \mathcal{N}^\circ_\Diamond, \mathcal{V}^\circ\rangle$ be two CINMs, Φ be a set of formulas of $\mathcal{L}_i$ closed under subformulas, and $\mathcal{M}^*$ be a Φ-filtration of $\mathcal{M}$. Then:

(i) If $\mathcal{M}$ is supplemented and $\mathsf{weakInt}$, and $\mathcal{M}^\circ$ is the supplementation of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is $\mathsf{weakInt}$ and is a Φ-filtration of $\mathcal{M}$.

(ii) If $\mathcal{M}$ is closed under intersection and $\mathsf{weakInt}$, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is $\mathsf{weakInt}$ and is a Φ-filtration of $\mathcal{M}$.

(iii) If $\mathcal{M}$ is supplemented, closed under intersection, and $\mathsf{weakInt}$, and $\mathcal{M}^\circ$ is the quasi-filtering of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is $\mathsf{weakInt}$ and is a Φ-filtration of $\mathcal{M}$.

(iv) If $\mathcal{M}$ is closed under intersection and strInt, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, then $\mathcal{M}^\circ$ is strInt and is a Φ-filtration of $\mathcal{M}$.

(v) If $\mathcal{M}$ is closed under intersection and $\mathsf{negInt_a}$, and $\mathcal{M}^\circ$ is the intersection closure of $\mathcal{M}^*$, and Φ is such that $\neg A \in \Phi$ for all $\Diamond A \in \Phi$, then $\mathcal{M}^\circ$ is $\mathsf{negInt_a}$ and is a Φ-filtration of $\mathcal{M}$.

Proof. The proofs of (i)–(iv) are very similar to each other. We show as an example the proof of (iii), and then we prove (v).

(iii) First, we show by contradiction that $\mathcal{M}^\circ$ is $\mathsf{weakInt}$. Assume $\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ and $\alpha \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$; and there is $\beta \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\alpha \subseteq \beta$. By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [\![A_1]\!]^\sim_\mathcal{M}$, ..., $\alpha_n = [\![A_n]\!]^\sim_\mathcal{M}$, and $[\![A_1]\!]_\mathcal{M}, ..., [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Moreover, there is $\Diamond B \in \Phi$ s.t. $\beta = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. As a consequence, we also have $[\![A_1]\!]^\sim_\mathcal{M} \cap ... \cap [\![A_n]\!]^\sim_\mathcal{M} \subseteq \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, by the filtration lemma this implies $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \subseteq \mathcal{W} \setminus [\![B]\!]_\mathcal{M}$. Then by intersection closure of $\mathcal{N}_\Box$, $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, and by its supplementation, $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Finally, since $\mathcal{M}$ is $\mathsf{weakInt}$, $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction.

We now prove that $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$. Let $\Box A \in \Phi$. If $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^*_\Box(w_\sim)$, and also $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^\circ_\Box(w_\sim)$. Now assume that $[\![A]\!]^\sim_\mathcal{M} \in \mathcal{N}^\circ_\Box(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq [\![A]\!]^\sim_\mathcal{M}$. By definition, this means that there are $\Box A_1, ..., \Box A_n \in \Phi$ s.t. $\alpha_1 = [\![A_1]\!]^\sim_\mathcal{M}$, ..., $\alpha_n = [\![A_n]\!]^\sim_\mathcal{M}$, and $[\![A_1]\!]_\mathcal{M}, ..., [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$. Then, since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \subseteq [\![A]\!]_\mathcal{M}$. By intersection closure of $\mathcal{N}_\Box$, $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then by supplementation, $[\![A]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$.

Now let $\Diamond A \in \Phi$. If $\mathcal{W} \setminus [\![A]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$, then $\mathcal{W}^* \setminus [\![A]\!]^\sim_\mathcal{M} \notin \mathcal{N}^*_\Diamond(w_\sim)$, and also $\mathcal{W}^* \setminus [\![A]\!]^\sim_\mathcal{M} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Now assume $\mathcal{W}^* \setminus [\![A]\!]^\sim_\mathcal{M} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there is $\beta \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\mathcal{W}^* \setminus [\![A]\!]^\sim_\mathcal{M} \subseteq \beta$. By definition, $\beta = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ for a $\Diamond B \in \Phi$ s.t. $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$, we have $\mathcal{W} \setminus [\![A]\!]_\mathcal{M} \subseteq \mathcal{W} \setminus [\![B]\!]_\mathcal{M}$. Then by supplementation, $\mathcal{W} \setminus [\![A]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$.

(v) Assume by contradiction that $-\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ and $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n = -\alpha$; in addition $\mathcal{W}^* \setminus \alpha \notin \mathcal{N}^*_\Diamond(w_\sim)$. By definition there are $\Box A_1, ..., \Box A_n, \Diamond B \in \Phi$ s.t. $\alpha_1 = [\![A_1]\!]^\sim_\mathcal{M}$, ..., $\alpha_n = [\![A_n]\!]^\sim_\mathcal{M}$, and $[\![A_1]\!]_\mathcal{M}, ..., [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$; moreover $\mathcal{W}^* \setminus \alpha = \mathcal{W}^* \setminus [\![B]\!]^\sim_\mathcal{M}$ and $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. Thus $[\![A_1]\!]^\sim_\mathcal{M} \cap ... \cap [\![A_n]\!]^\sim_\mathcal{M} = -[\![B]\!]^\sim_\mathcal{M} = [\![\neg B]\!]^\sim_\mathcal{M}$. Since $\mathcal{M}^*$ is a Φ-filtration of $\mathcal{M}$ and $\neg B \in \Phi$, by the filtration lemma this implies $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} = [\![\neg B]\!]_\mathcal{M} = -[\![B]\!]_\mathcal{M}$. But by intersection closure of $\mathcal{N}_\Box$, $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$, then by $\mathsf{negInt_a}$, $\mathcal{W} \setminus [\![B]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which gives a contradiction. Similarly to (iii) we can also prove that $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$. □

Theorem 8.3.4. If a formula A of $\mathcal{L}_i$ is satisfiable in a CINM $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$, then A is satisfiable in a CINM $\mathcal{M}' = \langle \mathcal{W}', \preceq', \mathcal{N}'_\Box, \mathcal{N}'_\Diamond, \mathcal{V}' \rangle$, where $\mathcal{N}'_\Box$ and $\mathcal{N}'_\Diamond$ satisfy the same properties of $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$, and $\mathcal{W}'$ is finite.

Proof. The proof is standard, by taking $\Phi = sbf(A) \cup \Psi$, where $sbf(A)$ is the set of subformulas of A, and Ψ depends on the properties of $\mathcal{M}$. In particular, Ψ contains $\Diamond\bot, \bot$ if $\mathcal{N}_\Diamond$ contains the unit; it contains $\Box\top, \top, \bot$ if $\mathcal{N}_\Box$ contains the unit; it contains $\neg B$ for all $\Diamond B \in sbf(A)$ if $\mathcal{M}$ is $\mathsf{negInt_a}$ (and not strInt); and it contains $\neg B$ for all $\Box B \in sbf(A)$ if $\mathcal{M}$ is $\mathsf{negInt_b}$ (and not strInt). Moreover, depending on the properties of $\mathcal{M}$ we consider the right transformation $\mathcal{M}'$ of $\mathcal{M}$. Observe that the set Φ is always finite, which implies that any Φ-filtration $\mathcal{M}'$ of $\mathcal{M}$ is a finite model. □

As an immediate corollary we obtain the following result.

Theorem 8.3.5 (Finite model property)**.** Every intuitionistic non-normal mono- or bimodal logic enjoys the finite model property.

8.4 Coupled intuitionistic neighbourhood models for CK and CCDL

In this section, we show that the framework of CINMs is *general enough* to cover two additional intuitionistic non-normal modal logics studied in the literature, namely **CK** [14] and **CCDL**, the latter being the propositional fragment of Wijesekera's first-order modal logic [170] (see their axiomatisations in Section 2.6). In particular, we show that the two systems can be captured in our framework by considering a very simple additional property. This result is particularly significant for **CK** since it provides a semantics for it without the need of inconsistent worlds. In the following, we first define CINMs for **CK** and **CCDL**, and prove the soundness and completeness of both systems. Then, by the filtration technique we prove that the two systems enjoy the finite model property.

Definition 8.4.1 (Coupled intuitionistic neighbourhood models for **CK** and **CCDL**)**.** A CINM for **CK** – or **CK**-*model* – is any CINM $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ in which $\mathcal{N}_\Box$ is supplemented, closed under intersection, and contains the unit; $\mathcal{N}_\Diamond$ is supplemented; and $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are related by the following condition:

$$\text{If } \alpha \in \mathcal{N}_\Box(w) \text{ and } \beta \in \mathcal{N}_\Diamond(w), \text{ then } \alpha \cap \beta \in \mathcal{N}_\Diamond(w). \qquad (\mathsf{WInt})$$

Moreover, a CINM for **CCDL** – or **CCDL**-*model* – is any CINM for **CK** that also satisfies the condition $\mathsf{weakInt}$, i.e., for every $w \in \mathcal{W}$, $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Diamond(w)$.

Notice that, as a consequence of the above definition, the function $\mathcal{N}_\Diamond$ in **CCDL**-models contains the unit, whereas this is not necessarily the case in **CK**-models (more precisely, for every **CK**-model $\mathcal{M}$ it holds that $\mathcal{M}$ is $\mathsf{weakInt}$ if and only if $\mathcal{N}_\Diamond$ contains the unit). We now prove that the logics **CK** and **CCDL** are sound and complete with respect to the corresponding models.

Theorem 8.4.1 (Soundness)**.** The logics **CK** and **CCDL** are sound with respect to **CK**- and **CCDL**-models, respectively.

Proof. We just show that the axiom $K_\Diamond$ is valid. Let $\mathcal{M}$ be a **CK**- or **CCDL**-model, w be a world of $\mathcal{M}$, and assume $w \Vdash \Box(A \supset B)$ and $w \not\Vdash \Diamond B$. Then $[\![A \supset B]\!] \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [\![B]\!] \in \mathcal{N}_\Diamond(w)$. By WInt, $[\![A \supset B]\!] \cap (\mathcal{W} \setminus [\![B]\!]) \in \mathcal{N}_\Diamond(w)$. Since $[\![A \supset B]\!] \cap (\mathcal{W} \setminus [\![B]\!]) \subseteq (\mathcal{W} \setminus [\![A]\!])$, by supplementation we have $\mathcal{W} \setminus [\![A]\!] \in \mathcal{N}_\Diamond(w)$; then $w \not\Vdash \Diamond A$. Therefore $\mathcal{M} \models \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$. ∎

Completeness is proved as for systems $\mathbf{IE}^*$ (cf. Section 8.2) by the canonical model construction.

Lemma 8.4.2. Let the canonical models $\mathcal{M}_{\mathbf{CK}}$ for **CK**, and $\mathcal{M}_{\mathbf{CCDL}}$ for **CCDL**, be defined as in Definition 8.2.3. Then $\mathcal{M}_{\mathbf{CK}}$ and $\mathcal{M}_{\mathbf{CCDL}}$ are, respectively, a **CK**-model and a **CCDL**-model.

Proof. We show that both $\mathcal{M}_{\mathbf{CK}}$ and $\mathcal{M}_{\mathbf{CCDL}}$ satisfy the condition of WInt. Assume $\alpha \in \mathcal{N}_{\Box}^{+}(X)$ and $\alpha \cap \beta \notin \mathcal{N}_{\Diamond}^{+}(X)$. Then there are $A, B \in \mathcal{L}$ such that $\uparrow A \subseteq \alpha$, $\alpha \cap \beta \subseteq \mathcal{W}_{\mathbf{L}} \setminus \uparrow B$ and $\Box A, \Diamond B \in X$. As a consequence, $\uparrow A \cap \beta \subseteq \mathcal{W}_{\mathbf{L}} \setminus \uparrow B$, that by standard properties of set inclusion implies $\beta \subseteq (\mathcal{W}_{\mathbf{L}} \setminus \uparrow A) \cup (\mathcal{W}_{\mathbf{L}} \setminus \uparrow B) = \mathcal{W}_{\mathbf{L}} \setminus \uparrow (A \wedge B)$. Moreover, since $(\Box A \wedge \Diamond B) \supset \Diamond(A \wedge B)$ is derivable (from $A \supset (B \supset A \wedge B)$, by $RM_{\Box}$ and $K_{\Diamond}$), we have $\Diamond(A \wedge B) \in X$. Thus, by definition, $\beta \notin \mathcal{N}_{\Diamond}^{+}(X)$. In addition, by Lemma 8.2.4 (iv), $\mathcal{M}_{\mathbf{CCDL}}$ is also $\mathsf{weakInt}$, as *str* is derivable in **CCDL**. $\Box$

On the basis of the above lemma, with a proof analogous to the one of Theorem 8.2.5 we obtain the following result.

Theorem 8.4.3 (Completeness)**.** Logics **CK** and **CCDL** are complete with respect to **CK**- and **CCDL**-models, respectively.

We now prove that **CK** and **CCDL** enjoy the finite model property by applying the filtration technique to **CK**- and **CCDL**-models. For **CK**, a proof of the finite model property with respect to the original relational semantics (cf. Section 2.6) can be found in Mendler and de Paiva [126].

Lemma 8.4.4. Let $\mathcal{M}$ and $\mathcal{M}^*$ be CINMs, where $\mathcal{M}^*$ is a finest Φ-filtration of $\mathcal{M}$ for a set Φ of formulas that is closed under subformulas and contains $\Box\top, \Diamond\bot$. We call WInt *closure* of $\mathcal{M}^*$ any CINM $\mathcal{M}^\circ = \langle \mathcal{W}^*, \preceq^*, \mathcal{N}_{\Box}^{\circ}, \mathcal{N}_{\Diamond}^{\circ}, \mathcal{V}^* \rangle$ such that

$$
\begin{array}{lll}
\alpha \in \mathcal{N}_{\Box}^{\circ}(w_\sim) & \text{iff} & \text{there are } \alpha_1, ..., \alpha_n \in \mathcal{N}_{\Box}^{*}(w_\sim) \text{ such that } \alpha_1 \cap ... \cap \alpha_n \subseteq \alpha; \\
\alpha \in \mathcal{N}_{\Diamond}^{\circ}(w_\sim) & \text{iff} & \text{for all } \beta_1, ..., \beta_n, \gamma \subseteq \mathcal{W}^*, \text{ if } \alpha \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma \\
 & & \text{and } \beta_1, ..., \beta_n \in \mathcal{N}_{\Box}^{*}(w_\sim), \text{ then } \gamma \in \mathcal{N}_{\Diamond}^{*}(w_\sim).
\end{array}
$$

The following hold: (i) If $\mathcal{M}$ is a **CK**-model, then $\mathcal{M}^\circ$ is a **CK**-model.

(ii) If $\mathcal{M}$ is a **CCDL**-model, then $\mathcal{M}^\circ$ is a **CCDL**-model.

(iii) If $\mathcal{M}$ is a **CK**- or a **CCDL**-model, then $\mathcal{M}^\circ$ is a Φ-filtration of $\mathcal{M}$.

Proof. (i) Clearly $\mathcal{N}_{\Box}^{\circ}$ is supplemented and closed under intersection, and it is immediate to check that $\mathcal{N}_{\Diamond}^{\circ}$ is supplemented. By Lemma 8.3.2 we also have that $\mathcal{N}_{\Box}^{\circ}$ contains the unit. Here we show that $\mathcal{M}^\circ$ satisfies WInt: assume $\alpha \in \mathcal{N}_{\Box}^{\circ}(w_\sim)$ and $\alpha \cap \beta \notin \mathcal{N}_{\Diamond}^{\circ}(w_\sim)$. By definition, there are $\alpha_1, ..., \alpha_n \in \mathcal{N}_{\Box}^{*}(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$. Moreover, there are $\beta_1, ..., \beta_k \in \mathcal{N}_{\Box}^{*}(w_\sim)$ and $\gamma \notin \mathcal{N}_{\Diamond}^{*}(w_\sim)$ s.t. $(\alpha \cap \beta) \cap \beta_1 \cap ... \cap \beta_k \subseteq \gamma$. This implies that $\alpha_1 \cap ... \cap \alpha_n \cap \beta \cap \beta_1 \cap ... \cap \beta_k \subseteq \gamma$. Therefore $\beta \notin \mathcal{N}_{\Diamond}^{\circ}(w_\sim)$.

(ii) In addition to the properties of (i), we prove here that $\mathcal{M}^\circ$ is also $\mathsf{weakInt}$. Assume by contradiction that $\alpha \in \mathcal{N}^\circ_\Box(w_\sim)$ and $\alpha \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\alpha_1, ..., \alpha_n \in \mathcal{N}^*_\Box(w_\sim)$ s.t. $\alpha_1 \cap ... \cap \alpha_n \subseteq \alpha$. Moreover, there are $\beta_1, ..., \beta_n \in \mathcal{N}^*_\Box(w_\sim)$ and $\gamma \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\alpha \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma$. This implies that there are $\Box A_1, ..., \Box A_n, \Box B_1, ..., \Box B_k, \Diamond C \in \Phi$ s.t. $\alpha_1 = [\![A_1]\!]^\sim_{\mathcal{M}^*}, ..., \alpha_n = [\![A_n]\!]^\sim_{\mathcal{M}^*}, \beta_1 = [\![B_1]\!]^\sim_{\mathcal{M}^*}, ..., \beta_k = [\![B_k]\!]^\sim_{\mathcal{M}^*}$, and $\gamma = \mathcal{W}^* \setminus [\![C]\!]^\sim_{\mathcal{M}^*}$. In addition, $[\![A_1]\!]_\mathcal{M}, ..., [\![A_n]\!]_\mathcal{M}, [\![B_1]\!]_\mathcal{M}, ..., [\![B_k]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [\![C]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. By the filtration lemma, we obtain $[\![A_1]\!]_\mathcal{M} \cap ... \cap [\![A_n]\!]_\mathcal{M} \cap [\![B_1]\!]_\mathcal{M} \cap ... \cap [\![B_k]\!]_\mathcal{M} \subseteq \mathcal{W} \setminus [\![C]\!]_\mathcal{M}$. Finally, since $\mathcal{N}_\Box$ is supplemented and closed under intersection, and $\mathcal{M}$ is $\mathsf{weakInt}$, we have $\mathcal{W} \setminus [\![C]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which leads to a contradiction.

(iii) For $\Box A \in \Phi$, the proof is exactly as in Lemma 8.3.3. Let $\Diamond A \in \Phi$. If $\mathcal{W} \setminus [\![A]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$, then $\mathcal{W}^* \setminus [\![A]\!]^\sim_{\mathcal{M}^*} \notin \mathcal{N}^*_\Diamond(w_\sim)$. Thus, since by Lemma 8.3.2 $\mathcal{W}^* \in \mathcal{N}_\Box(w_\sim)$, we have $\mathcal{W}^* \setminus [\![A]\!]^\sim_{\mathcal{M}^*} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Now assume that $\mathcal{W}^* \setminus [\![A]\!]^\sim_{\mathcal{M}^*} \notin \mathcal{N}^\circ_\Diamond(w_\sim)$. Then there are $\beta_1, ..., \beta_n \in \mathcal{N}^*_\Box(w_\sim)$ and $\gamma \notin \mathcal{N}^*_\Diamond(w_\sim)$ s.t. $\mathcal{W}^* \setminus [\![A]\!]^\sim_{\mathcal{M}^*} \cap \beta_1 \cap ... \cap \beta_n \subseteq \gamma$. Hence, by definition, there exist $\Box A_1, ..., \Box A_n, \Box B_1, ..., \Box B_k, \Diamond C \in \Phi$ s.t. $\beta_1 = [\![B_1]\!]^\sim_{\mathcal{M}^*}, ..., \beta_k = [\![B_k]\!]^\sim_{\mathcal{M}^*}$, and $\gamma = \mathcal{W}^* \setminus [\![C]\!]^\sim_{\mathcal{M}^*}$. In addition, $[\![B_1]\!]_\mathcal{M}, ..., [\![B_k]\!]_\mathcal{M} \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [\![C]\!]_\mathcal{M} \notin \mathcal{N}_\Diamond(w)$. By contradiction, assume that $\mathcal{W} \setminus [\![A]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$. Then, by intersection closure of $\mathcal{N}_\Box$ and WInt, $[\![B_1]\!]_\mathcal{M} \cap ... \cap [\![B_k]\!]_\mathcal{M} \cap \mathcal{W} \setminus [\![A]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$. Moreover, by the filtration lemma, we have that $[\![B_1]\!]_\mathcal{M} \cap ... \cap [\![B_k]\!]_\mathcal{M} \cap \mathcal{W} \setminus [\![A]\!]_\mathcal{M} \subseteq \mathcal{W} \setminus [\![C]\!]_\mathcal{M}$. Thus, by supplementation of $\mathcal{N}_\Diamond$, we obtain that $\mathcal{W} \setminus [\![C]\!]_\mathcal{M} \in \mathcal{N}_\Diamond(w)$, which leads to a contradiction. $\Box$

As before, we obtain the following theorem:

Theorem 8.4.5. **CK** and **CCDL** enjoy the finite model property.

8.5 Direct equivalence with pre-existing semantics for CK and CCDL

From the results of the previous section, it follows that **CK** and **CCDL** are equally characterised by our neighbourhood semantics and by their pre-existing semantics, namely Wijesekera's relational models [170] and Kojima's neighbourhood models [102] for **CCDL**, and Mendler and de Paiva's relational models [126] for **CK** (see Definitions 2.6.2, 2.6.3, and 2.6.4 in Section 2.6). It is instructive, however, to prove the equivalence directly by mutual transformations of models. For both **CK** and **CCDL**, the transformations from CINMs to the models in the pre-existing semantics only hold for finite models. The reason is that we have to consider a closure property involving the intersection of all neighbourhoods that is ensured by the closure under (finitary) intersection of $\mathcal{N}_\Box$ only for finite models; in order to obtain analogous transformations holding for arbitrary models we should consider the stronger property of augmentation $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Box(w)$ in CINMs (cf. Chellas [29], p. 220). The equivalence between CINMs and the pre-existing models is then a consequence of the finite model property of **CK** and **CCDL** with respect to their CINMs shown in the previous section. We begin with system **CCDL**, considering both Kojima's and Wijesekera's models.

Semantic equivalence for CCDL

A proof of equivalence between Kojima's and relational models is given in Kojima [102]. Here we prove directly the equivalence of Kojima's and CINMs for **CCDL**. In particular, we show that every Kojima model can be transformed into an equivalent CINM for **CCDL**, and that every *finite* CINM for **CCDL** can be transformed into an equivalent Kojima model. By combining these results with the transformations given by Kojima we also obtain direct transformations between CINMs and relational models. Furthermore, considering also the finite model property of **CCDL** with respect to the corresponding CINMs (cf. Theorem 8.4.5), this provides an alternative proof of equivalence of the three semantics.

In the proof of some of the next lemmas we shall make use of the following property, which is satisfied by any finite model for **CCDL** and **CK**, and is an easy consequence of WInt and the intersection closure of $\mathcal{N}_\Box$.

Lemma 8.5.1. Every finite CINM for **CCDL** or for **CK** satisfies the following property:

$$\text{For every } \alpha \in \mathcal{N}_\Diamond(w), \text{ there is } \beta \in \mathcal{N}_\Diamond(w) \text{ such that } \beta \subseteq \alpha \text{ and } \beta \subseteq \bigcap \mathcal{N}_\Box(w). \qquad (\mathsf{WInt}')$$

First, given a Kojima model, we obtain an equivalent CINM for **CCDL** as follows.

Lemma 8.5.2. Let $\mathcal{M}_k = \langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$ be a Kojima model for **CCDL**, and let $\mathcal{M}_n$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_k$, and:

$$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \bigcup \mathcal{N}_k(w) \subseteq \alpha\};$$
$$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } \beta \in \mathcal{N}_k(w) \text{ s.t. } \beta \subseteq \alpha\}.$$

Then $\mathcal{M}_n$ is a CINM for **CCDL**. Moreover, for every $A \in \mathcal{L}_i$ and $w \in \mathcal{W}$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_k, w \Vdash_k A$.

Proof. It is immediate to verify that $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented and contain the unit; that $\mathcal{N}_\Box$ is closed under intersection; and that $w \preceq v$ implies $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$ and $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$. We show that $\mathcal{M}_n$ satisfies the other properties of **CCDL**-models.

($\mathsf{weakInt}$) Assume $\alpha \in \mathcal{N}_\Box(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$, and, since $\mathcal{N}_k(w) \neq \emptyset$, there is $\beta \in \mathcal{N}_k(w)$ such that $\beta \subseteq \alpha$. Therefore $\alpha \in \mathcal{N}_\Diamond(w)$.

(WInt) Assume $\alpha \in \mathcal{N}_\Box(w)$ and $\beta \in \mathcal{N}_\Diamond(w)$. Then $\bigcup \mathcal{N}_k(w) \subseteq \alpha$ and there is $\gamma \in \mathcal{N}_k(w)$ such that $\gamma \subseteq \beta$. Thus $\gamma \subseteq \bigcup \mathcal{N}_k(w)$, which implies $\gamma \subseteq \alpha \cap \beta$. Therefore $\alpha \cap \beta \in \mathcal{N}_\Diamond(w)$.

By induction on A, we now prove that $\mathcal{M}_n$ and $\mathcal{M}_k$ are pointwise equivalent. Since the two models share the same order and evaluation of propositional variables, we only consider the inductive cases $A = \Box B, \Diamond B$.

($A = \Box B$) $\mathcal{M}_n, w \Vdash \Box B$ iff $[\![B]\!]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\bigcup \mathcal{N}_k(w) \subseteq [\![B]\!]_{\mathcal{M}_n}$ iff (i.h.) $\bigcup \mathcal{N}_k(w) \subseteq [\![B]\!]_{\mathcal{M}_k}$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [\![B]\!]_{\mathcal{M}_k}$ iff $\mathcal{M}_k, w \Vdash_k \Box B$.

($A = \Diamond B$) $\mathcal{M}_n, w \Vdash \Diamond B$ iff $\mathcal{W} \setminus [\![B]\!]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$ iff (i.h.) for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_k} \neq \emptyset$ iff $\mathcal{M}_k, w \Vdash_k \Diamond B$. □

For the opposite transformation, given a *finite* CINM for **CCDL**, we obtain an equivalent Kojima model as follows.

Lemma 8.5.3. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a finite CINM for **CCDL**, and let $\mathcal{M}_k$ be the model $\langle \mathcal{W}, \preceq, \mathcal{N}_k, \mathcal{V} \rangle$ where $\mathcal{W}$, $\preceq$ and $\mathcal{V}$ are as in $\mathcal{M}_n$, and:

$$\mathcal{N}_k(w) = \{\alpha \in \mathcal{N}_\Diamond(w) \mid \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}.$$

Then $\mathcal{M}_k$ is a Kojima model for **CCDL**. Moreover, for every $A \in \mathcal{L}_i$ and $w \in \mathcal{W}$, $\mathcal{M}_k, w \Vdash_k A$ if and only if $\mathcal{M}_n, w \Vdash A$.

Proof. First, notice that $\mathcal{M}_k$ is a Kojima model: by intersection closure, we have that $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Box(w)$, hence by $\mathsf{weakInt}$, $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$. Thus $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$, which implies $\mathcal{N}_k(w) \neq \emptyset$. Moreover, assume that $w \preceq v$ and $\alpha \in \mathcal{N}_k(v)$. It follows that $\alpha \in \mathcal{N}_\Diamond(v)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(v)$. Since $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$ and $\mathcal{N}_\Box(w) \subseteq \mathcal{N}_\Box(v)$, we have both $\alpha \in \mathcal{N}_\Diamond(w)$ and $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, therefore $\alpha \in \mathcal{N}_k(w)$.

We prove by induction on A that, for every $A \in \mathcal{L}_i$ and $w \in \mathcal{W}$, $\mathcal{M}_k, w \Vdash_k A$ if and only if $\mathcal{M}_n, w \Vdash A$. As before, we only consider the inductive cases $A \equiv \Box B, \Diamond B$.

($A = \Box B$) $\mathcal{M}_k, w \Vdash_k \Box B$ iff for all $\alpha \in \mathcal{N}_k(w)$, $\alpha \subseteq [\![B]\!]_{\mathcal{M}_k}$ iff (since $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_k(w)$) $\bigcap \mathcal{N}_\Box(w) \subseteq [\![B]\!]_{\mathcal{M}_k}$ iff (i.h.) $\bigcap \mathcal{N}_\Box(w) \subseteq [\![B]\!]_{\mathcal{M}_n}$ iff (by properties of $\mathcal{N}_\Box(w)$) $[\![B]\!]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff $\mathcal{M}_n, w \Vdash \Box B$.

($A = \Diamond B$) Assume $\mathcal{M}_k, w, \Vdash_k \Diamond B$. Then for every $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_k} \neq \emptyset$, and, by inductive hypothesis, $\alpha \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Thus for every $\alpha \in \mathcal{N}_\Diamond(w)$ s.t. $\alpha \subseteq \bigcap \mathcal{N}_\Box(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Let β be any neighbourhood in $\mathcal{N}_\Diamond(w)$. By WInt', there is $\gamma \subseteq \beta$ s.t. $\gamma \in \mathcal{N}_\Diamond(w)$ and $\gamma \subseteq \bigcap \mathcal{N}_\Box(w)$. Then $\gamma \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$, which implies $\beta \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume that $\mathcal{M}_n, w \Vdash \Diamond B$. Then for every $\alpha \in \mathcal{N}_\Diamond(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Thus for every $\alpha \in \mathcal{N}_k(w)$, $\alpha \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$, and, by i.h., $\alpha \cap [\![B]\!]_{\mathcal{M}_k} \neq \emptyset$. Therefore $\mathcal{M}_k, w \Vdash_k \Diamond B$. □

Theorem 8.5.4 (Semantic equivalence)**.** A formula A of $\mathcal{L}_i$ is valid in all Kojima models for **CCDL** if and only if it is valid in all CINMs for **CCDL**.

Proof. If A is falsified in a Kojima model, then by Lemma 8.5.2, there is a CINM for **CCDL** that falsifies A. Vice versa, if A is falsified in a CINM for **CCDL**, then by Theorem 8.4.5, there is a finite CINM for **CCDL** that falsifies A, and consequently by Lemma 8.5.3, there is a Kojima model that falsifies A. $\square$

By combining the above transformations with those between Kojima models and relational model for **CCDL** given in Kojima [102], we can obtain the following direct transformations between CINMs and relational models for **CCDL**:

Lemma 8.5.5. Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V} \rangle$ be a relational model for **CCDL**, and let $\mathscr{R}(w) = \{v \mid w\mathcal{R}v\}$. We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ by taking $\mathcal{W}$, $\preceq$, $\mathcal{V}$ as in $\mathcal{M}_r$, and the following neighbourhood functions:

$$\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\};$$
$$\mathcal{N}_\Diamond(w) = \{\alpha \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha\}.$$

Then $\mathcal{M}_n$ is a CINM for **CCDL**. Moreover, for every $A \in \mathcal{L}_i$ and every $w \in \mathcal{W}$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash_r A$.

Lemma 8.5.6. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a finite CINM for **CCDL**. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:

- $\mathcal{W}^* = \{(w, \alpha) \mid w \in \mathcal{W},\ \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$;
- $(w, \alpha) \preceq^* (v, \beta)$ iff $w \preceq v$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ iff $v \in \alpha$;
- $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$ for all $w \in \mathcal{W}$.

Then $\mathcal{M}^*$ is a relational model for **CCDL**. Moreover, for all $A \in \mathcal{L}_i$ and $w \in \mathcal{W}$, the following claims are equivalent:

1) $\mathcal{M}_n, w \Vdash A$.

2) For every $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

3) There is $(w, \alpha) \in \mathcal{W}^*$ such that $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

Theorem 8.5.7 (Semantic equivalence)**.** A formula A of $\mathcal{L}_i$ is valid in all relational models for **CCDL** if and only if it is valid in all CINMs for **CCDL**.

The above transformations between CINMs and relational models for **CCDL** are particular cases of the forthcoming transformations for models of **CK**, to see this we just need to consider the set on inconsistent worlds in the relational models for **CK** to be empty. For this reason we do not give the proofs of Lemmas 8.5.5 and 8.5.6, but refer to the ones of the next Lemmas 8.5.8 and 8.5.9.

Semantic equivalence for CK

We now directly prove the equivalence of relational and neighbourhood semantics for **CK**. As for **CCDL**, we can prove that every relational model can be transformed into an equivalent CINM for **CK**, and that every *finite* CINM for **CK** can be transformed into an equivalent relational model. The equivalence of the two semantics is then a consequence of the finite model property of **CK** with respect to its CINMs.

Given a relational model for **CK**, we can define an equivalent CINM as follows.

Lemma 8.5.8. Let $\mathcal{M}_r = \langle \mathcal{W}, \preceq, \mathcal{R}, \mathcal{V}\rangle$ be a relational model for **CK**. We denote with $\mathcal{W}^+$ the set $\{w \in \mathcal{W} \mid \mathcal{M}_r, w \nVdash_r \bot\}$ of the consistent worlds of $\mathcal{M}_r$, and, for all $\alpha \subseteq \mathcal{W}$, we denote with α^+ the set $\alpha \cap \mathcal{W}^+$. We define the neighbourhood model $\mathcal{M}_n = \langle \mathcal{W}^+, \preceq^+, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}^+\rangle$, where $\preceq^+$ and $\mathcal{V}^+$ are the restrictions of $\preceq$ and $\mathcal{V}$ to $\mathcal{W}^+$, and $\mathcal{N}_\Box$, $\mathcal{N}_\Diamond$ are the following neighbourhood functions:

$$\mathcal{N}_\Box(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{for all } v \succeq w, \mathscr{R}(v) \subseteq \alpha\};$$
$$\mathcal{N}_\Diamond(w) = \{\alpha^+ \subseteq \mathcal{W} \mid \text{there is } v \succeq w \text{ s.t. } \mathscr{R}(v) \subseteq \alpha^+\}.$$

Then, $\mathcal{M}_n$ is a CINM for **CK**. Moreover, for every $A \in \mathcal{L}_i$ and every $w \in \mathcal{W}^+$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash_r A$.

Proof. It is easy to verify that $\mathcal{M}_n$ is a CINM for **CK**. In particular, for WInt, assume that $\alpha^+ \in \mathcal{N}_\Box(w)$ and $\beta^+ \in \mathcal{N}_\Diamond(w)$. Then there is $v \succeq w$ s.t. $\mathscr{R}(v) \subseteq \beta^+$; thus $\mathscr{R}(v) \subseteq \alpha$. Then $\mathscr{R}(v) \subseteq \alpha \cap \beta^+ = (\alpha \cap \beta)^+$. Therefore $(\alpha \cap \beta)^+ = \alpha^+ \cap \beta^+ \in \mathcal{N}_\Diamond(w)$.

We now prove that for every $w \in \mathcal{W}^+$, $\mathcal{M}_n, w \Vdash A$ if and only if $\mathcal{M}_r, w \Vdash_r A$. This is equivalent to stating that $[\![A]\!]_{\mathcal{M}_n} = [\![A]\!]^+_{\mathcal{M}_r}$. As usual, we only consider the modal cases.

$(A = \Box B)$ Let $w \in \mathcal{W}^+$. $\mathcal{M}_n, w \Vdash \Box B$ iff $[\![B]\!]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$ iff (i.h.) $[\![B]\!]^+_{\mathcal{M}_r} \in \mathcal{N}_\Box(w)$ iff for all $v \succeq w$, $\mathscr{R}(v) \subseteq [\![B]\!]_{\mathcal{M}_r}$ iff $\mathcal{M}_r, w \Vdash_r \Box B$.

$(A = \Diamond B)$ Assume that $\mathcal{M}_r, w \Vdash_r \Diamond B$ and $w \in \mathcal{W}^+$. Then for every $v \succeq w$, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $\mathcal{M}_r, u \Vdash_r B$. Thus for every $v \succeq w$, $\mathcal{R}(v) \nsubseteq \mathcal{W} \setminus [\![B]\!]_{\mathcal{M}_r}$, which in particular implies that $\mathcal{R}(v) \nsubseteq (\mathcal{W} \setminus [\![B]\!]_{\mathcal{M}_r})^+$. Moreover, $(\mathcal{W} \setminus [\![B]\!]_{\mathcal{M}_r})^+ = \mathcal{W}^+ \setminus [\![B]\!]^+_{\mathcal{M}_r} =$ (i.h.) $\mathcal{W}^+ \setminus [\![B]\!]_{\mathcal{M}_n}$. Then $\mathcal{W}^+ \setminus [\![B]\!]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$, therefore $\mathcal{M}_n, w \Vdash \Diamond B$. Now assume that $\mathcal{M}_n, w \Vdash \Diamond B$. Then $\mathcal{W}^+ \setminus [\![B]\!]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$. This implies that for every $v \succeq w$, $\mathcal{R}(v) \nsubseteq \mathcal{W}^+ \setminus [\![B]\!]_{\mathcal{M}_n}$; that is, there is $u \in \mathcal{W}$ s.t. $v\mathcal{R}u$ and $u \notin \mathcal{W}^+ \setminus [\![B]\!]_{\mathcal{M}_n}$. Thus $u \notin \mathcal{W}^+$ or $u \in [\![B]\!]_{\mathcal{M}_n}$. If $u \notin \mathcal{W}^+$, then $\mathcal{M}_r, u \Vdash_r \bot$, hence $\mathcal{M}_r, u \Vdash_r B$. If $u \in [\![B]\!]_{\mathcal{M}_n}$, by inductive hypothesis $u \in [\![B]\!]^+_{\mathcal{M}_r}$, thus $\mathcal{M}_r, u \Vdash_r B$. Therefore $\mathcal{M}_r, w \Vdash_r \Diamond B$. $\square$

For the opposite direction, given a finite CINM for **CK** we can obtain an equivalent relational model as follows. Notice that in the next definition we have to add a specific world denoted $(f, \{f\})$ playing the role of a fallible world.

Lemma 8.5.9. Let $\mathcal{M}_n = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a finite CINM for **CK**, and $f \notin \mathcal{W}$. The relational model $\mathcal{M}^* = \langle \mathcal{W}^*, \preceq^*, \mathcal{R}^*, \mathcal{V}^* \rangle$ is defined as follows:

- $\mathcal{W}^* = \quad \{(w, \alpha) \mid w \in \mathcal{W}, \mathcal{N}_\Diamond(w) \neq \emptyset, \alpha \in \mathcal{N}_\Diamond(w), \text{ and } \alpha \subseteq \bigcap \mathcal{N}_\Box(w)\}$
  $\cup \; \{(v, \bigcap \mathcal{N}_\Box(v) \cup \{f\}) \mid v \in \mathcal{W} \text{ and } \mathcal{N}_\Diamond(v) = \emptyset\}$
  $\cup \; \{(f, \{f\})\}$;
- $(w, \alpha) \preceq^* (v, \beta)$ if and only if $w \preceq v$ or $w, v = f$;
- $(w, \alpha)\mathcal{R}^*(v, \beta)$ if and only if $v \in \alpha$;
- for every $w \in \mathcal{W}$, $\mathcal{V}^*((w, \alpha)) = \{p \mid p \in \mathcal{V}(w)\}$;
- $\mathcal{V}^*((f, \{f\})) = Atm$;
- $\mathcal{M}^*, (f, \{f\}) \Vdash_r \bot$.

Then $\mathcal{M}^*$ is a relational model for **CK**. Moreover, for every $A \in \mathcal{L}_i$ and $w \in \mathcal{W}$, the following claims are equivalent:

1) $\mathcal{M}_n, w \Vdash A$.
2) For every $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.
3) There is $(w, \alpha) \in \mathcal{W}^*$ such that $\mathcal{M}^*, (w, \alpha) \Vdash_r A$.

Proof. It is easy to check that $\mathcal{M}^*$ is a relational model for **CK**, in particular that the conditions on inconsistent worlds are satisfied. We prove by induction on A that 1), 2) and 3) are equivalent. As usual we only consider the inductive cases $A = \Box B, \ \Diamond B$.

- $A = \Box B$.
  - 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Box B$. Then $[\![B]\!]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$, that implies $\bigcap \mathcal{N}_\Box(w) \subseteq [\![B]\!]_{\mathcal{M}_n}$. Let $(w, \alpha) \in \mathcal{W}^*$, and $(w, \alpha) \preceq^* (v, \beta)$. Then $w \preceq v$, so $\bigcap \mathcal{N}_\Box(v) \subseteq \bigcap \mathcal{N}_\Box(w)$. We distinguish two cases:

    (*a*) $f \in \beta$. Then $(v, \beta)\mathcal{R}^*(u, \gamma)$ implies $u \in \bigcap \mathcal{N}_\Box(v)$ or $u = f$.
    If $u = f$, then $(u, \gamma) = (f, \{f\})$, so $\mathcal{M}^*, (u, \gamma) \Vdash_r B$.
    If $u \in \bigcap \mathcal{N}_\Box(v)$, then $u \in [\![B]\!]_{\mathcal{M}_n}$. By inductive hypothesis we have $\mathcal{M}^*, (u, \gamma) \Vdash_r B$ for all γ s.t. $(u, \gamma) \in \mathcal{W}^*$.

    (*b*) $f \notin \beta$. Then $\beta \subseteq \bigcap \mathcal{N}_\Box(v)$, thus $\beta \subseteq [\![B]\!]_{\mathcal{M}_n}$. Let $(v, \beta)\mathcal{R}^*(u, \gamma)$. Then $u \in \beta$, so $\mathcal{M}_n, u \Vdash B$. By inductive hypothesis we have $\mathcal{M}^*, (u, \gamma) \Vdash_r B$.

    By (*a*) and (*b*) we have that for all $(v, \beta) \succeq^* (w, \alpha)$ and all (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Therefore for all α s.t. $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r \Box B$.

- 2) implies 3). Immediate because for every $w \in \mathcal{W}$ there is α such that $(w, \alpha) \in \mathcal{W}^*$.

- 3) implies 1). Assume $\mathcal{M}^*, (w, \alpha) \Vdash_r \Box B$ for an α s.t. $(w, \alpha) \in \mathcal{W}^*$. Then for every $(v, \beta) \succeq^* (w, \alpha)$ and everys (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Thus, in particular, for every δ s.t. $(w, \delta) \in \mathcal{W}^*$, for every (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$, $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Take any world $z \in \bigcap \mathcal{N}_\Box(w)$. There exists γ s.t. $(z, \gamma) \in \mathcal{W}^*$. Then $(w, \bigcap \mathcal{N}_\Box(w))\mathcal{R}^*(z, \gamma)$ or $(w, \bigcap \mathcal{N}_\Box(w) \cup \{f\})\mathcal{R}^*(z, \gamma)$ (depending on whether $\mathcal{N}_\Diamond(w) \neq \emptyset$ or $\mathcal{N}_\Diamond(w) = \emptyset$; in the first case $\bigcap \mathcal{N}_\Box(w) \in \mathcal{N}_\Diamond(w)$). Thus $\mathcal{M}^*, (z, \gamma) \Vdash_r B$; and by inductive hypothesis, $\mathcal{M}_n, z \Vdash B$. So $\bigcap \mathcal{N}_\Box(w) \subseteq [\![B]\!]_{\mathcal{M}_n}$, which implies $[\![B]\!]_{\mathcal{M}_n} \in \mathcal{N}_\Box(w)$. Therefore $\mathcal{M}_n, w \Vdash \Box B$.

- $A = \Diamond B$.

  - 1) implies 2). Assume $\mathcal{M}_n, w \Vdash \Diamond B$, and let $(w, \alpha) \in \mathcal{W}^*$ and $(w, \alpha) \preceq^* (v, \beta)$. We distinguish two cases:

    (*a*) $f \in \beta$. Then $(v, \beta)\mathcal{R}^*(f, \{f\})$, and $\mathcal{M}^*, (f, \{f\}) \Vdash_r B$.

    (*b*) $f \notin \beta$. Then $w \preceq v$ and $\beta \in \mathcal{N}_\Diamond(v)$, thus since $\mathcal{N}_\Diamond(v) \subseteq \mathcal{N}_\Diamond(w)$, $\beta \in \mathcal{N}_\Diamond(w)$. By $\mathcal{M}_n, w \Vdash \Diamond B$, we have that $\mathcal{W} \setminus [\![B]\!]_{\mathcal{M}_n} \notin \mathcal{N}_\Diamond(w)$. Then by supplementation, for every $\gamma \in \mathcal{N}_\Diamond(w)$, $\gamma \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$; thus in particular $\beta \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Then there is $u \in \beta$ s.t. $\mathcal{M}_n, u \Vdash B$. By inductive hypothesis, for every δ s.t. $(u, \delta) \in \mathcal{W}^*$, $\mathcal{M}^*, (u, \delta) \Vdash_r B$. Moreover, there is ϵ s.t. $(u, \epsilon) \in \mathcal{W}^*$. Thus $(v, \beta)\mathcal{R}^*(u, \epsilon)$ and $\mathcal{M}^*, (u, \epsilon) \Vdash_r B$.

    By (*a*) and (*b*) we have that for every $(v, \beta) \succeq^* (w, \alpha)$, there is (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Therefore, for every α s.t. $(w, \alpha) \in \mathcal{W}^*$, $\mathcal{M}^*, (w, \alpha) \Vdash_r \Diamond B$.

  - 2) implies 3). Immediate because for every $w \in \mathcal{W}$ there is α such that $(w, \alpha) \in \mathcal{W}^*$.

  - 3) implies 1). Assume $\mathcal{M}^*, (w, \alpha) \Vdash_r \Diamond B$ for a α s.t. $(w, \alpha) \in \mathcal{W}^*$. Then for every $(v, \beta) \succeq^* (w, \alpha)$, there is (u, γ) s.t. $(v, \beta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. Thus in particular, for every δ s.t. $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}^*, (u, \gamma) \Vdash_r B$. We distinguish two cases:

    (*a*) $f \in \delta$ for a $(w, \delta) \in \mathcal{W}^*$. Then $\mathcal{N}_\Diamond(w) = \emptyset$, so $\mathcal{M}_n, w \Vdash \Diamond B$.

    (*b*) $f \notin \delta$ for every $(w, \delta) \in \mathcal{W}^*$. Then by inductive hypothesis we have that for every $(w, \delta) \in \mathcal{W}^*$, there is (u, γ) s.t. $(w, \delta)\mathcal{R}^*(u, \gamma)$ and $\mathcal{M}_n, u \Vdash B$. So $u \in \delta$. This means that for every $\delta \in \mathcal{N}_\Diamond(w)$ s.t. $\delta \subseteq \bigcap \mathcal{N}_\Box(w)$, $\delta \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Then by $\mathsf{WInt'}$, we have that for every $\epsilon \in \mathcal{N}_\Diamond(w)$, $\epsilon \cap [\![B]\!]_{\mathcal{M}_n} \neq \emptyset$. Therefore $\mathcal{M}_n, w \Vdash \Diamond B$.

$\Box$

$weak^e$	$\boxdot A \to \boxed{\diamond} A$	neg_a^e	$\boxdot\boxed{\leq} A \to \boxed{\diamond} A$	neg_b^e	$\boxdot A \to \boxed{\diamond}\Diamond A$
str^e	$\boxdot(A \wedge B) \to \boxed{\diamond} A$	$K_\diamond^e$	$\boxdot A \wedge \Diamond B \to \Diamond(A \wedge B)$		

Figure 8.1: Connecting axioms of classical multimodal logics.

Theorem 8.5.10 (Semantic equivalence)**.** A formula A of $\mathcal{L}_i$ is valid in all relational models for **CK** if and only if it is valid in all CINMs for **CK**.

Proof. Assume A is not valid in all relational models for **CK**. Then there are a relational model $\mathcal{M}_r$ and a world w such that $\mathcal{M}_r, w \not\Vdash_r A$. The world w is consistent (i.e., $\mathcal{M}_r, w \not\Vdash_r \bot$) as inconsistent worlds satisfy all formulas. Then by Lemma 8.5.8, there is a CINM $\mathcal{M}_n$ for **CK** such that $\mathcal{M}_n, w \not\Vdash A$. Now, assume A is not valid in all CINMs for **CK**. Then by Theorem 8.4.5, there are a finite model $\mathcal{M}_n$ and a world w such that $\mathcal{M}_n, w \not\Vdash A$. Therefore by Lemma 8.5.9, there are a relational model $\mathcal{M}^*$ and a world (w, α) such that $\mathcal{M}^*, (w, \alpha) \not\Vdash_r A$. Then A is not valid in all CINMs for **CK**. $\square$

8.6 Embedding into classical non-normal multimodal logics

In this section, we present an embedding of our intuitionistic non-normal modal logics **IE***, as well as of **CK** and **CCDL**, into classical multimodal logics of the form (**S4**, $\mathbf{cL_2}$, $\mathbf{cL_3}$), where **S4** is the normal modal logics defined by extending **K** with the axioms T and 4, and $\mathbf{cL_2}$ and $\mathbf{cL_3}$ are two non-normal modal logics of the classical cube (cf. Section 2.2). The logics (**S4**, $\mathbf{cL_2}$, $\mathbf{cL_3}$) are defined on a propositional modal language $\mathcal{L}_3$ containing three modalities $\boxed{\leq}$, $\boxdot$, $\boxed{\diamond}$. Correspondingly, the formulas of $\mathcal{L}_3$ are defined as follows, where p_i is any variable in Atm:

$$A ::= p_i \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \to A \mid \boxed{\leq} A \mid \boxdot A \mid \boxed{\diamond} A.$$

As usual, we define $\neg A := A \to \bot$. Moreover, the diamond-modalities can be defined by duality with box, for instance, $\Diamond A := \neg \boxed{\diamond} \neg A$.

Definition 8.6.1. Let **L** be any logic among our intuitionistic non-normal modal logics **IE*** and **CK** and **CCDL**. We denote by $\mathbf{L}^e$ the classical multimodal logic $(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3)$ which is defined in language $\mathcal{L}_3$ by extending **CPL** (formulated in language $\mathcal{L}_3$) with the following axioms for the modalities $\boxed{\leq}$, $\boxdot$, and $\boxed{\diamond}$:

- the modal axioms and rules of **S4** for $\boxed{\leq}$;
- the rule RE for $\boxdot$ and $\boxed{\diamond}$;

- if **L** contains $M_\Box$, or $N_\Box$, or $C_\Box$, then $\mathbf{L}^e$ contains the axiom M, or N, or C, for $\boxed{\Box}$, respectively;
- if **L** contains $M_\Diamond$ or $N_\Diamond$, then $\mathbf{L}^e$ contains the axiom M or N for $\boxed{\Diamond}$, respectively;
- if **L** contains $weak_a$ and $weak_b$, or neg_a, or neg_b, or str, or $K_\Diamond$, then $\mathbf{L}^e$ contains the axiom $weak^e$, or neg_a^e, or neg_b^e, or str^e, or $K_\Diamond^e$, respectively (Figure 8.1).

We consider the following translation of formulas of $\mathcal{L}_i$ into formulas of $\mathcal{L}_3$.

Definition 8.6.2. The translation $\dagger : \mathcal{L}_i \longrightarrow \mathcal{L}_3$ is recursively defined as follows:

$$
\begin{aligned}
&\dagger(p_i) = \boxed{\leq}p_i, \text{ for every } p_i \in Atm;\\
&\dagger(\bot) = \bot;\\
&\dagger(\top) = \top;\\
&\dagger(A \circ B) = \dagger(A) \circ \dagger(B), \text{ for } \circ \in \{\wedge, \vee\};\\
&\dagger(A \supset B) = \boxed{\leq}(\dagger(A) \to \dagger(B));\\
&\dagger(\Box A) = \boxed{\leq}\boxed{\Box}\dagger(A);\\
&\dagger(\Diamond A) = \boxed{\leq}\Diamond\mkern-14mu\diamond\,\dagger(A).
\end{aligned}
$$

For instance, we have

$$
\begin{aligned}
&\dagger(\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)) =\\
&\boxed{\leq}(\boxed{\leq}\boxed{\Box}\boxed{\leq}(\dagger(A) \to \dagger(B)) \to \boxed{\leq}(\boxed{\leq}\Diamond\mkern-14mu\diamond\,\dagger(A) \to \boxed{\leq}\Diamond\mkern-14mu\diamond\,\dagger(B))).
\end{aligned}
$$

Our goal is to show that every intuitionistic non normal modal logic **L** can be simulated by the corresponding classical non-normal multimodal logic $\mathbf{L}^e$ by means of the above translation, in the sense that a formula A of $\mathcal{L}_i$ is derivable in **L** if and only if $\dagger(A)$ is derivable in $\mathbf{L}^e$. One direction can be proved syntactically as follows:

Theorem 8.6.1. Let **L** be an intuitionistic non-normal modal logic. Then for every formula A of $\mathcal{L}_i$, $\mathbf{L} \vdash A$ implies $\mathbf{L}^e \vdash \dagger(A)$.

Proof. We can show that the translations of all axioms and rules of **L** are derivable in $\mathbf{L}^e$, i.e., if B is an axiom of **L**, then $\dagger(B)$ is derivable in $\mathbf{L}^e$, and if $\dfrac{B_1 \ \ldots \ B_n}{B}$ is a rule of **L**, then $\dfrac{\dagger(B_1) \ \ldots \ \dagger(B_n)}{\dagger(B)}$ is derivable in $\mathbf{L}^e$. Illustrative derivations of some modal axioms and rules of **L** are displayed in Figure 8.2. $\square$

For the other direction we proceed as follows. First, we define an evaluation of formulas of $\mathcal{L}_3$ in CINMs, and show that $\mathbf{L}^e$ is sound with respect to the CINMs for **L** under this evaluation. Then we show that in CINMs every formula A of $\mathcal{L}_i$ is equivalent to its translation $\dagger(A)$. This allows us to semantically prove that if a formula A is not derivable in **L**, then $\dagger(A)$ is not derivable in $\mathbf{L}^e$.

$\mathbf{IE_1N_\Diamond}^e \vdash \dagger(N_\Diamond)$

1.	$\boxed{\Diamond}\top$	$(N_{\boxed{\Diamond}})$
2.	$\langle\Diamond\rangle\bot \to \bot$	(1, **CPL**)
3.	$\boxed{\leq}\langle\Diamond\rangle\bot \to \langle\Diamond\rangle\bot$	$(T_{\boxed{\leq}})$
4.	$\boxed{\leq}\langle\Diamond\rangle\bot \to \bot$	(2, 3, **CPL**)
5.	$\boxed{\leq}(\boxed{\leq}\langle\Diamond\rangle\bot \to \bot)$	(4, $RN_{\boxed{\leq}}$)

$\mathbf{IE_1}^e \vdash \dagger(weak_a)$

1.	$\boxed{\Diamond}\top \wedge \neg\boxed{\Diamond}\top \to \bot$	(theorem of **CPL**)
2.	$\boxed{\Diamond}\top \wedge \langle\Diamond\rangle\bot \to \bot$	(1, duality $\boxed{\Diamond}, \langle\Diamond\rangle$)
3.	$\boxed{\Box}\top \to \boxed{\Diamond}\top$	$(weak^e)$
4.	$\boxed{\Box}\top \wedge \langle\Diamond\rangle\bot \to \bot$	(2, 3, **CPL**)
5.	$\boxed{\leq}\boxed{\Box}\top \wedge \boxed{\Box}\langle\Diamond\rangle\bot \to \bot$	(4, $T_{\boxed{\leq}}$, **CPL**)
6.	$\boxed{\leq}(\boxed{\leq}\boxed{\Box}\top \wedge \boxed{\Box}\langle\Diamond\rangle\bot \to \bot)$	(5, $RN_{\boxed{\leq}}$)

$\mathbf{IE_1}^e \vdash \dagger(RE_\Box)$

1.	$\boxed{\leq}(A \to B)$	(assumption)
2.	$\boxed{\leq}(B \to A)$	(assumption)
3.	$A \to B$	(1, $T_{\boxed{\leq}}$)
4.	$B \to A$	(2, $T_{\boxed{\leq}}$)
5.	$\boxed{\Box}A \to \boxed{\Box}B$	(3, 4, $RE_{\boxed{\Box}}$)
6.	$\boxed{\leq}\boxed{\Box}A \to \boxed{\leq}\boxed{\Box}B$	(5, $RM_{\boxed{\leq}}$)
7.	$\boxed{\leq}(\boxed{\leq}\boxed{\Box}A \to \boxed{\leq}\boxed{\Box}B)$	(5, $RN_{\boxed{\leq}}$)

$\mathbf{IE_3}^e \vdash \dagger(str)$

1.	$\boxed{\leq}(A \wedge B \to \bot)$	(assumption)
2.	$A \wedge B \to \bot$	(1, $T_{\boxed{\leq}}$, **CPL**)
3.	$A \leftrightarrow A \wedge \neg B$	(2, **CPL**)
4.	$\boxed{\Box}A \to \boxed{\Box}(A \wedge \neg B)$	(3, $RE_{\boxed{\Box}}$)
5.	$\boxed{\Box}(A \wedge \neg B) \to \boxed{\Diamond}\neg B$	(str^e)
6.	$\boxed{\Box}A \to \boxed{\Diamond}\neg B$	(4, 5, **CPL**)
7.	$\boxed{\Box}A \wedge \langle\Diamond\rangle B \to \bot$	(6, **CPL**)
8.	$\boxed{\leq}\boxed{\Box}A \wedge \boxed{\leq}\langle\Diamond\rangle B \to \bot$	(7, $T_{\boxed{\leq}}$, **CPL**)
9.	$\boxed{\leq}(\boxed{\leq}\boxed{\Box}A \wedge \boxed{\leq}\langle\Diamond\rangle B \to \bot)$	(8, $RN_{\boxed{\leq}}$)

$\mathbf{CK}^e/\mathbf{CCDL}^e \vdash \dagger(K_\Diamond)$

1.	$(A \to B) \wedge A \to B$	(**CPL**)
2.	$\langle\Diamond\rangle((A \to B) \wedge A) \to \langle\Diamond\rangle B$	(1, $RM_{\langle\Diamond\rangle}$)
3.	$\boxed{\Box}(A \to B) \wedge \langle\Diamond\rangle A \to \langle\Diamond\rangle((A \to B) \wedge A)$	$(K^e_\Diamond)$
4.	$\boxed{\Box}(A \to B) \to (\langle\Diamond\rangle A \to \langle\Diamond\rangle B)$	(2, 3, **CPL**)
5.	$\boxed{\leq}(A \to B) \to (A \to B)$	$(T_{\boxed{\leq}})$
6.	$\boxed{\Box}\boxed{\leq}(A \to B) \to \boxed{\Box}(A \to B)$	(5, $RM_{\boxed{\Box}}$)
7.	$\boxed{\Box}\boxed{\leq}(A \to B) \to (\langle\Diamond\rangle A \to \langle\Diamond\rangle B)$	(4, 6, **CPL**)
8.	$\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B) \to (\boxed{\leq}\langle\Diamond\rangle A \to \boxed{\leq}\langle\Diamond\rangle B)$	(7, $RM_{\boxed{\leq}}$, $K_{\boxed{\leq}}$, **CPL**)
9.	$\boxed{\leq}\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B) \to \boxed{\leq}(\boxed{\leq}\langle\Diamond\rangle A \to \boxed{\leq}\langle\Diamond\rangle B)$	(8, $RM_{\boxed{\leq}}$)
10.	$\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B) \to \boxed{\leq}\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B)$	$(4_{\boxed{\leq}})$
11.	$\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B) \to \boxed{\leq}(\boxed{\leq}\langle\Diamond\rangle A \to \boxed{\leq}\langle\Diamond\rangle B)$	(9, 10, **CPL**)
12.	$\boxed{\leq}(\boxed{\leq}\boxed{\Box}\boxed{\leq}(A \to B) \to \boxed{\leq}(\boxed{\leq}\langle\Diamond\rangle A \to \boxed{\leq}\langle\Diamond\rangle B))$	(11, $RN_{\boxed{\leq}}$)

Figure 8.2: Derivations in classical multimodal logics.

Definition 8.6.3. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ be a CINM, w be a world of $\mathcal{M}$, and A be a formula of $\mathcal{L}_3$. The forcing relation $w \Vdash_e A$ is defined as $w \Vdash_{st} A$ in Definition 2.3.1 for $A = \bot, \top, A \wedge B, A \vee B, A \to B$, whereas for $A = p_i, \boxed{\leq}B, \boxed{\Box}B, \boxed{\Diamond}B$ it is defined as follows:

$$
\begin{array}{lll}
w \Vdash_e p_i & \text{iff} & p \in \mathcal{V}(w); \\
w \Vdash_e \boxed{\leq}B & \text{iff} & \text{for every } v \succeq w,\ v \Vdash_e B; \\
w \Vdash_e \boxed{\Box}B & \text{iff} & [\![B]\!] \in \mathcal{N}_\Box(w); \\
w \Vdash_e \boxed{\Diamond}B & \text{iff} & [\![B]\!] \in \mathcal{N}_\Diamond(w).
\end{array}
$$

Lemma 8.6.2. Let **L** be an intuitionistic non-normal modal logic. Then $\mathbf{L}^e$ is sound with respect to CINMs for **L** under the forcing relation $\Vdash_e$.

Proof. Assume $\mathcal{M}$ is a CINM for $\mathbf{L}$. As usual, we have to show that all axioms and rules of $\mathbf{L}^e$ are valid according to the considered evaluation of formulas. For the axioms and rules of **CPL**, as well as for the characteristic axioms and rules of **S4** and of classical non-normal modal logics, the proof is standard. We consider the different axioms connecting the three modalities.

• If $\mathcal{M}$ is $\mathsf{weakInt}$, then $\mathcal{M} \models \boxbox A \to \boxdiamond A)$ If $w \Vdash_e \boxbox A$, then $[\![A]\!] \in \mathcal{N}_\Box(w)$, thus by $\mathsf{weakInt}$, $[\![A]\!] \in \mathcal{N}_\Diamond(w)$, therefore $w \Vdash_e \boxdiamond A$.

• If $\mathcal{M}$ is $\mathsf{negInt_a}$, then $\mathcal{M} \models \boxbox\boxleq A \to \boxdiamond A)$ If $w \Vdash_e \boxbox\boxleq A$, then $[\![\boxleq A]\!] \in \mathcal{N}_\Box(w)$, where $[\![\boxleq A]\!] = \{v \mid \text{for all } u \succeq v, u \Vdash_e A\} = \{v \mid \text{for all } u \succeq v, u \notin [\![\neg A]\!]\} = -[\![\neg A]\!]$. Then by $\mathsf{negInt_a}$, $\mathcal{W} \setminus [\![\neg A]\!] = [\![A]\!] \in \mathcal{N}_\Diamond(w)$. Therefore $w \Vdash_e \boxdiamond A$.

• If $\mathcal{M}$ is $\mathsf{negInt_b}$, then $\mathcal{M} \models \boxbox A \to \boxdiamond\diamondleq A)$ If $w \Vdash_e \boxbox A$, then $[\![A]\!] \in \mathcal{N}_\Box(w)$. Thus by $\mathsf{negInt_b}$, $\mathcal{W} \setminus -[\![A]\!] \in \mathcal{N}_\Diamond(w)$. We have $-[\![A]\!] = \{v \mid \text{for all } u \succeq v, u \notin [\![A]\!]\} = \{v \mid \text{for all } u \succeq v, u \Vdash_e \neg A\} = [\![\boxleq\neg A]\!]$. Thus $\mathcal{W} \setminus -[\![A]\!] = [\![\neg\boxleq\neg A]\!] = [\![\diamondleq A]\!]$. Therefore $w \Vdash_e \boxdiamond\diamondleq A$.

• If $\mathcal{M}$ is strInt, then $\mathcal{M} \models \boxbox(A \wedge B) \to \boxdiamond A)$ If $w \Vdash_e \boxbox(A \wedge B)$, then $[\![A \wedge B]\!] \in \mathcal{N}_\Box(w)$. Since $[\![A \wedge B]\!] \subseteq [\![A]\!]$, by strInt, $[\![A]\!] \in \mathcal{N}_\Diamond(w)$, therefore $w \Vdash_e \boxdiamond A$.

• If $\mathcal{M}$ is WInt, then $\mathcal{M} \models \boxbox A \wedge \diamondleq B \to \diamondleq(A \wedge B))$ Assume $w \Vdash_e \boxbox A$ and $w \nVdash \diamondleq(A \wedge B)$. Then $[\![A]\!] \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [\![A \wedge B]\!] \in \mathcal{N}_\Diamond(w)$. Then by WInt, $[\![A]\!] \cap (\mathcal{W} \setminus [\![A \wedge B)]\!] = [\![A]\!] \cap (\mathcal{W} \setminus [\![B]\!]) \in \mathcal{N}_\Diamond(w)$. Thus by supplementation of $\mathcal{N}_\Diamond(w)$, $\mathcal{W} \setminus [\![B]\!] \in \mathcal{N}_\Diamond(w)$, therefore $w \nVdash_e \diamondleq B$. $\square$

Lemma 8.6.3. Let $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$ be a CINM. Then for every $w \in \mathcal{W}$ and every $A \in \mathcal{L}_i$, $w \Vdash A$ if and only if $w \Vdash_e \dagger(A)$.

Proof. By induction on A. For $A = p, \bot, \top, B \wedge C, B \vee C$ the proof is immediate. For the other cases it is as follows.

$(A = B \supset C)$ If $w \Vdash B \supset C$ iff for all $v \succeq w$, $v \nVdash B$ or $v \Vdash C$ iff (i.h.) for all $v \succeq w$, $v \nVdash_e \dagger(B)$ or $v \Vdash_e \dagger(C)$ iff for all $v \succeq w$, $v \Vdash_e \dagger(B) \to \dagger(C)$ iff $w \Vdash_e \boxleq(\dagger(B) \to \dagger(C))$.

$(A = \Box B)$ $w \Vdash \Box B$ iff $[\![B]\!] \in \mathcal{N}_\Box(w)$ iff (i.h.) $[\![\dagger(B)]\!]_e \in \mathcal{N}_\Box(w)$ iff for all $v \succeq w$, $[\![\dagger(B)]\!]_e \in \mathcal{N}_\Box(v)$ iff for all $v \succeq w$, $v \Vdash_e \boxbox\dagger(B)$ iff $w \Vdash_e \boxleq\boxbox\dagger(B)$.

$(A = \Diamond B)$ $w \Vdash \Diamond B$ iff $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(w)$ iff (i.h.) $\mathcal{W} \setminus [\![\dagger(B)]\!]_e \notin \mathcal{N}_\Diamond(w)$ iff for all $v \succeq w$, $\mathcal{W} \setminus [\![\dagger(B)]\!]_e \notin \mathcal{N}_\Diamond(v)$ iff for all $v \succeq w$, $v \Vdash_e \diamondleq\dagger(B)$ iff $w \Vdash_e \boxleq\diamondleq\dagger(B)$. $\square$

Theorem 8.6.4. Let $\mathbf{L}$ be an intuitionistic non-normal modal logic. Then for every formula A of $\mathcal{L}_i$, $\mathbf{L}^e \vdash \dagger(A)$ implies $\mathbf{L} \vdash A$.

Proof. By contraposition, assume $\mathbf{L} \nvdash A$. Then by the completeness of $\mathbf{L}$ (Theorem 8.2.5), there are a CINM $\mathcal{M}$ for $\mathbf{L}$ and a world w of $\mathcal{M}$ such that $w \nVdash A$. Therefore by Lemma 8.6.3, $w \nVdash \dagger(A)$, finally by Lemma 8.6.2, $\mathbf{L}^e \nvdash \dagger(A)$. $\square$

$$\begin{array}{rcl}
\mathbf{T.IM} & := & \{\text{propositional rules}, \mathsf{hp}_{\langle\rangle}, \mathsf{T}\Box, \mathsf{M}_\Box, \mathsf{M}_\Diamond, \mathsf{str}\} \\
\mathbf{T.IMC}^* & := & \mathbf{T.IM}^* \cup \{\mathsf{C}_\Box\}. \\
\mathbf{T.IMN}_\Box{}^* & := & \mathbf{T.IM}^* \cup \{\mathsf{N}_\Box\}. \\
\mathbf{T.IMN}_\Diamond{}^* & := & \mathbf{T.IM}^* \cup \{\mathsf{N}_\Diamond\}.
\end{array}$$

Table 8.2: Prefixed tableaux calculi $\mathbf{T.IM}^*$.

From Theorems 8.6.1 and 8.6.4 we then obtain our result:

Corollary 8.6.5 (Embedding)**.** $\mathbf{L} \vdash A$ if and only if $\mathbf{L}^e \vdash \dagger(A)$.

8.7 Prefixed tableaux

Similarly to classical non-normal modal logics, we are interested in defining calculi for intuitionistic non-normal modal logics allowing for direct countermodel extraction for non-valid formulas. We conclude this chapter by presenting some preliminary results about proof systems of this kind. In particular, we present some tableaux calculi for monotonic logics $\mathbf{IM}^*$: we show that the calculi are sound and semantically complete with respect to the corresponding classes of CINMs. In particular, we show that from every failed proof we can extract a countermodel of the non-derivable formula.

We define proof systems for the logics $\mathbf{IM}^*$ in the form of prefixed tableaux. For the design of these calculi we adopt the solution already used for the hypersequent calculi $\mathbf{H.E}^*$ (cf. Chapter 6) of collecting $\Box$-formulas by means of blocks. Do to the absence of axiom $C_\Diamond$, no analogous structure is needed for $\Diamond$-formulas.

Definition 8.7.1 (Prefixed formulas)**.** A *block* is a structure $\langle\Sigma\rangle$, where Σ is a finite set of formulas of $\mathcal{L}_i$. A *prefixed formula* is a triple

$$\sigma\ X\ \phi,$$

where σ is a finite sequence of natural numbers, called *prefix*, $X \in \{T, F\}$, and ϕ is a formula of $\mathcal{L}_i$ or a block.

Intuitively, the prefixes σ, ρ, ... represent worlds of CINMs. Moreover, their sequential structure incorporates the ordering among worlds: in general, a world represented by a prefix σ will be related through $\preceq$ to any world represented by $\sigma.\sigma'$. Formulas $\sigma\ T\ \phi$ or $\sigma\ F\ \phi$ intuitively represent that ϕ is satisfied (respectively falsified) in world σ. Furthermore, similarly to the calculi $\mathbf{LT.E}^*$ presented in Section 5.4, we use a special symbol $\mathbf{f}$ to denote branch closure.

We take as base calculus for **IPL** Fitting's tabeaux system [58] (see the propositional rules in Figure 8.3). For every logic $\mathbf{IM}^*$, the corresponding calculus is defined by extending it with suitable modal rules from Figure 8.3, as summarised in Table 8.2. As usual, in the definition

Propositional rules

$$\mathsf{init}\ \frac{\begin{array}{c}\sigma\ T\ A\\ \sigma\ F\ A\end{array}}{\mathbf{f}} \qquad \mathsf{T}\bot\ \frac{\sigma\ T\ \bot}{\mathbf{f}} \qquad \mathsf{F}\top\ \frac{\sigma\ F\ \top}{\mathbf{f}}$$

$$\mathsf{T}\wedge\ \frac{\sigma\ T\ A\wedge B}{\begin{array}{c}\sigma\ T\ A\\ \sigma\ T\ B\end{array}} \qquad \mathsf{F}\wedge\ \frac{\sigma\ F\ A\wedge B}{\sigma\ F\ A \mid \sigma\ F\ B} \qquad \mathsf{T}\vee\ \frac{\sigma\ T\ A\vee B}{\sigma\ T\ A \mid \sigma\ T\ B} \qquad \mathsf{F}\vee\ \frac{\sigma\ F\ A\vee B}{\begin{array}{c}\sigma\ F\ A\\ \sigma\ F\ B\end{array}}$$

$$\mathsf{T}\supset\ \frac{\sigma\ T\ A\rightarrow B}{\sigma\ F\ A \mid \sigma\ T\ B} \qquad \mathsf{F}\supset\ \frac{\sigma\ F\ A\supset B}{\begin{array}{c}\sigma.n\ T\ A\\ \sigma.n\ F\ B\end{array}}\ (\sigma.n!) \qquad \mathsf{hp}\ \frac{\sigma\ T\ A}{\sigma.n\ T\ A}\ (\sigma.n\text{¡})$$

Modal rules of basic monotonic calculus T.IM

$$\mathsf{T}\Box\ \frac{\sigma\ T\ \Box A}{\sigma\ T\ \langle A\rangle} \qquad \mathsf{hp}_{\langle\rangle}\ \frac{\sigma\ T\ \langle\Sigma\rangle}{\sigma.n\ T\ \langle\Sigma\rangle}\ (\sigma.n\text{¡}) \qquad \mathsf{M}_\Box\ \frac{\begin{array}{c}\sigma\ T\ \langle A_1,...,A_n\rangle\\ \sigma\ F\ \Box B\end{array}}{\begin{array}{c}n\ T\ A_1\\ \vdots\\ n\ T\ A_n\\ n\ F\ B\end{array}}\ (n!) \qquad \mathsf{str}\ \frac{\begin{array}{c}\sigma\ T\ \langle A_1,...,A_n\rangle\\ \sigma\ T\ \Diamond B\end{array}}{\begin{array}{c}n\ T\ A_1\\ \vdots\\ n\ T\ A_n\\ n\ T\ B\end{array}}\ (n!)$$

$$\mathsf{M}_\Diamond\ \frac{\begin{array}{c}\sigma\ T\ \Diamond A\\ \sigma\ F\ \Diamond B\end{array}}{\begin{array}{c}n\ T\ A\\ n\ F\ B\end{array}}\ (n!)$$

Modal rules for extensions

$$\mathsf{C}_\Box\ \frac{\begin{array}{c}\sigma\ T\ \langle\Sigma\rangle\\ \sigma\ T\ \langle\Pi\rangle\end{array}}{\sigma\ T\ \langle\Sigma,\Pi\rangle} \qquad \mathsf{N}_\Box\ \frac{\mathcal{B}}{\sigma\ T\ \langle\top\rangle}\ (\sigma\text{¡}) \qquad \mathsf{N}_\Diamond\ \frac{\sigma\ T\ \Diamond A}{n\ T\ A}\ (n!)$$

Figure 8.3: Rules of prefixed tableaux calculi **T.IM***.

of the rules $(\sigma!)$ means that the prefix σ must be fresh in the application of the rule, i.e., it does not occur in the branch to which the rule is applied. By contrast, $(\sigma\text{¡})$ means that σ must be *old*, i.e., it already occurs in the branch to which the rule is applied. Derivations are defined in the standard way:

Definition 8.7.2 (Derivation)**.** A *tableau* for a prefixed formula $\sigma\ X\ \phi$ in **T.IM*** is a tree where the root is labelled by $\sigma\ X\ \phi$, end every succedent node is obtained by the application of a rule of **T.IM*** to formulas occurring in the same branch at smaller depths. Moreover, we say that a branch of a tableau is *closed* if it contains $\mathbf{f}$, otherwise it is *open*, and that a tableau is *closed* if all its branches are closed. Finally, a *derivation* of $A \in \mathcal{L}_i$ is a closed tableau for $1\ F\ A$.

As an example, we show in Figure 8.4 the derivation in **T.IM** of $\Box(A\wedge B) \supset \neg\Diamond\neg A$ (that in **IM** is equivalent to the rule *str*).

Definition 8.7.3. Let $\mathcal{B}$ be a branch of a tableau in **T.IM***, and $\mathcal{B}_\sigma$ be the set of prefixes occurring in $\mathcal{B}$. Then $\mathcal{B}$ is *satisfiable* in a CINM $\mathcal{M} = \langle\mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ if $\mathcal{B}$ does not contain

$$
\begin{array}{rll}
1. & 1\ F\ \Box(A \wedge B) \supset (\Diamond(A \supset \bot) \supset \bot) & \\
2. & 1.1\ T\ \Box(A \wedge B) & (1, \mathsf{F} \supset) \\
3. & 1.1\ F\ \Diamond(A \supset \bot) \supset \bot & (1, \mathsf{F} \supset) \\
4. & 1.1.1\ T\ \Diamond(A \supset \bot) & (3, \mathsf{F} \supset) \\
5. & 1.1.1\ F\ \bot & (3, \mathsf{F} \supset) \\
6. & 1.1.1\ T\ \Box(A \wedge B) & (2, \mathsf{hp}) \\
7. & 1.1.1\ T\ \langle A \wedge B \rangle & (6, \mathsf{T}\Box) \\
8. & 2\ T\ A \wedge B & (4{,}7, \mathsf{str}) \\
9. & 2\ T\ A \supset \bot & (4{,}7, \mathsf{str}) \\
10. & 2\ T\ A & (8, \mathsf{T}\wedge) \\
11. & 2\ T\ B & (8, \mathsf{T}\wedge)
\end{array}
$$

$$
\begin{array}{rcl}
12. & 2\ F\ A & (9, \mathsf{T} \supset) \\
13. & \mathbf{f} & (10{,}12, \mathsf{init})
\end{array}
\qquad
\begin{array}{rcl}
12. & 2\ T\ \bot & (9, \mathsf{T} \supset) \\
13. & \mathbf{f} & (12, \mathsf{T}\bot)
\end{array}
$$

Figure 8.4: Derivation of $\Box(A \wedge B) \supset \neg\Diamond\neg A$ in **T.IM**.

$\mathbf{f}$, and there is a function $\varrho : \Phi_\sigma \longrightarrow \mathcal{W}$ such that:

- if ρ and $\rho.n$ are in $\mathcal{B}_\sigma$, then $\varrho(\rho) \preceq \varrho(\rho.n)$;
- if $\rho\ T\ A \in \mathcal{B}$, then $\mathcal{M}, \varrho(\rho) \Vdash A$;
- if $\rho\ F\ A \in \mathcal{B}$, then $\mathcal{M}, \varrho(\rho) \nVdash A$;
- if $\rho\ T\ \langle\Sigma\rangle \in \mathcal{B}$, then $\mathcal{M}, \varrho(\rho) \Vdash \Box \bigwedge \Sigma$.

Moreover, we say that a tableau is *satisfiable* if a branch of the tableau is satisfiable.

Notice that on the basis of the above definition, no closed tableau is satisfiable. We now show that the calculi **T.IM*** are sound with respect to the corresponding CINMs.

Theorem 8.7.1 (Soundness)**.** If A is derivable in **T.IM***, then A is valid in every CINM for **IM***.

Proof. If A is derivable in **T.IM***, then there is a closed tableau for $1\ F\ A$ in **T.IM***. We show that whenever a tableau in **T.IM*** is satisfiable in a CINM $\mathcal{M}$ for **IM***, then the tableau obtained by extending it with the application of any rule of **T.IM*** is satisfiable in $\mathcal{M}$ as well. Since a closed tableau is not satisfiable, this implies that $1\ F\ A$ is not satisfiable either, therefore A is valid. For the propositional rules the proof is standard, in particular the soundness of hp is a consequence of the hereditary property of CINMs (Proposition 8.1.1). Here we consider the modal rules.

($\mathsf{M}_\Box$) Assume $\mathcal{B} = \mathcal{B}' \cup \{\sigma\ T\ \langle A_1, ..., A_n\rangle, \sigma\ F\ \Box B\}$ satisfiable in $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$. Then in particular there are ϱ and a world $w \in \mathcal{W}$ such that $\varrho(\sigma) = w$, $w \Vdash \Box(A_1 \wedge ... \wedge A_n)$, and $w \not\Vdash \Box B$. Then $[\![A_1 \wedge ... \wedge A_n]\!] \in \mathcal{N}_\Box(w)$ and $[\![B]\!] \notin \mathcal{N}_\Box(w)$. Since $\mathcal{N}_\Box(w)$ is supplemented, this means that $[\![A_1 \wedge ... \wedge A_n]\!] \not\subseteq [\![B]\!]$, then there is $v \in \mathcal{W}$ such that $v \Vdash A_1$, ..., $v \Vdash A_n$, and $v \not\Vdash B$. Since the prefix n is fresh in the application of $\mathsf{M}_\Box$, we can extend ϱ to ϱ' by choosing $\varrho'(n) = v$. Then $\mathcal{B} \cup \{n\ T\ A_1, ..., n\ T\ A_n, n\ F\ B\}$ is satisfiable in $\mathcal{M}$ under ϱ'.

($\mathsf{M}_\Diamond$) Assume $\mathcal{B} = \mathcal{B}' \cup \{\sigma\ T\ \Diamond A, \sigma\ F\ \Diamond B\}$ satisfiable in $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$. Then in particular there are ϱ and a world $w \in \mathcal{W}$ such that $\varrho(\sigma) = w$, $w \Vdash \Diamond A$, and $w \not\Vdash \Diamond B$. Then $\mathcal{W} \setminus [\![A]\!] \notin \mathcal{N}_\Diamond(w)$ and $\mathcal{W} \setminus [\![B]\!] \in \mathcal{N}_\Diamond(w)$. Since $\mathcal{N}_\Diamond(w)$ is supplemented, this means that $\mathcal{W} \setminus [\![B]\!] \not\subseteq \mathcal{W} \setminus [\![A]\!]$, i.e., $\setminus [\![A]\!] \not\subseteq \setminus [\![B]\!]$. then there is $v \in \mathcal{W}$ such that $v \Vdash A$, and $v \not\Vdash B$. Since the prefix n is fresh in the application of $\mathsf{M}_\Diamond$, we can extend ϱ to ϱ' by choosing $\varrho'(n) = v$. Then $\mathcal{B} \cup \{n\ T\ A, n\ F\ B\}$ is satisfiable in $\mathcal{M}$ under ϱ'.

(str) Assume $\mathcal{B} = \mathcal{B}' \cup \{\sigma\ T\ \langle A_1, ..., A_n\rangle, \sigma\ T\ \Diamond B\}$ satisfiable in $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$. Then in particular there are ϱ and a world $w \in \mathcal{W}$ such that $\varrho(\sigma) = w$, $w \Vdash \Box(A_1 \wedge ... \wedge A_n)$, and $w \Vdash \Diamond B$. Then $[\![A_1 \wedge ... \wedge A_n]\!] \in \mathcal{N}_\Box(w)$ and $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(w)$. Since $\mathcal{M}$ is $\mathsf{weakInt}$, $[\![A_1 \wedge ... \wedge A_n]\!] \in \mathcal{N}_\Diamond(w)$, then since $\mathcal{N}_\Diamond(w)$ is supplemented, $[\![A_1 \wedge ... \wedge A_n]\!] \not\subseteq \mathcal{W} \setminus [\![B]\!]$. It follows that there is $v \in \mathcal{W}$ such that $v \Vdash A_1$, ..., $v \Vdash A_n$, and $v \Vdash B$. Since the prefix n is fresh in the application of str, we can extend ϱ to ϱ' by choosing $\varrho'(n) = v$. Then $\mathcal{B} \cup \{n\ T\ A_1, ..., n\ T\ A_n, n\ T\ B\}$ is satisfiable in $\mathcal{M}$ under ϱ'.

($\mathsf{C}_\Box$) Assume $\mathcal{B} = \mathcal{B}' \cup \{\sigma\ T\ \langle\Sigma\rangle, \sigma\ T\ \langle\Pi\rangle\}$ satisfiable in a model $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ for $\mathbf{IMC}^*$. Then there are ϱ and a world $w \in \mathcal{W}$ such that $\varrho(\sigma) = w$, $w \Vdash \Box \bigwedge \Sigma$, and $w \Vdash \Box \bigwedge \Pi$, then $[\![\bigwedge \Sigma]\!], [\![\bigwedge \Pi]\!] \in \mathcal{N}_\Box(w)$. Since $\mathcal{N}_\Box(w)$ is closed under intersection, $[\![\bigwedge \Sigma]\!] \cap [\![\bigwedge \Pi]\!] = [\![\bigwedge \Sigma \wedge \bigwedge \Pi]\!] \in \mathcal{N}_\Box(w)$. Then $w \Vdash \Box(\bigwedge \Sigma \wedge \bigwedge \Pi)$, therefore $\mathcal{B} \cup \{\sigma\ T\ \langle\Sigma, \Pi\rangle\}$ is satisfiable.

($\mathsf{N}_\Box$) Assume $\mathcal{B}$ satisfiable under ϱ in a model $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ for $\mathbf{IMN}_\Box{}^*$, and σ occurs in $\mathcal{B}$. Since $\mathcal{N}_\Box(\varrho(\sigma))$ contains the unit, $\varrho(\sigma) \Vdash \Box\top$, then $\mathcal{B} \cup \{\sigma\ T\ \langle\top\rangle\}$ is satisfiable.

($\mathsf{N}_\Diamond$) Assume $\mathcal{B} = \mathcal{B}' \cup \{\sigma\ T\ \Diamond A\}$ satisfiable in a model $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ for $\mathbf{IMN}_\Diamond{}^*$. Then there are ϱ and $w \in \mathcal{W}$ such that $\varrho(\sigma) = w$ and $w \Vdash \Diamond A$, that is $\mathcal{W} \setminus [\![A]\!] \notin \mathcal{N}_\Diamond(w)$. Since $\mathcal{N}_\Diamond(w)$ contains the unit, $\mathcal{W} \setminus [\![A]\!] \neq \mathcal{W}$, i.e., $[\![A]\!] \neq \emptyset$. Then there is $v \in \mathcal{W}$ such that $v \Vdash A$. Since the prefix n is fresh in the application of $\mathsf{N}_\Diamond$, we can extend ϱ to ϱ' by choosing $\varrho'(n) = v$. Then $\mathcal{B} \cup \{n\ T\ A\}$ is satisfiable in $\mathcal{M}$ under ϱ'. □

We now prove that the calculi $\mathbf{T.IM}^*$ are semantically complete with respect to the corresponding CINMs. As usual, the proof consists in showing that every non-derivable formula has a countermodel. In particular, we show that the countermodel can be directly extracted from the failed proof.

Definition 8.7.4 (Countermodel)**.** Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{LT.E}^*$. On the basis of $\mathcal{B}$, we define the CINM $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V}\rangle$ as follows:

- $\mathcal{W}$ = the set of prefixes occurring in $\mathcal{B}$.

- $\sigma \preceq \rho$ iff there is a (possibly empty) σ' such that $\rho = \sigma.\sigma'$.
- $\mathcal{V}(\sigma) = \{p \mid \sigma\ T\ p \in \mathcal{B}\}$.
- For every formula $A \in \mathcal{L}_i$ end every block $\langle A_1, ..., A_n \rangle$,

  $A^+ = \{\sigma \in \mathcal{W} \mid \sigma\ T\ A \in \mathcal{B}\}$ and $(A_1, ..., A_n)^+ = A_1^+ \cap ... \cap A_n^+$.
- $\mathcal{N}_\Box(\sigma) = \{\alpha \subseteq \mathcal{W} \mid$ there is $\sigma\ T\ \langle \Sigma \rangle \in \mathcal{B}$ such that $\Sigma^+ \subseteq \alpha\}$.
- $\mathcal{N}_\Diamond(\sigma) = \mathcal{P}(\mathcal{W}) \setminus \{\alpha \subseteq \mathcal{W} \mid$ there is $\sigma\ T\ \Diamond A \in \mathcal{B}$ such that $A^+ \cap \alpha = \emptyset\}$.

Lemma 8.7.2. Let $\mathcal{B}$ be a saturated branch of a failed proof of A in $\mathbf{T.IM}^*$, and $\mathcal{M}$ be the model defined on the basis of $\mathcal{B}$ as in Definition 8.7.4. Then $\mathcal{M}$ is a CINM for $\mathbf{IM}^*$. Moreover, for every $A \in \mathcal{L}_i$ and every block $\langle \Sigma \rangle$,

– if $\sigma\ T\ A \in \mathcal{B}$, then $\sigma \Vdash A$;
– if $\sigma\ F\ A \in \mathcal{B}$, then $\sigma \nVdash A$; and
– if $\sigma\ T\ \langle \Sigma \rangle \in \mathcal{B}$, then $\sigma \Vdash \Box \bigwedge \Sigma$.

Proof. We first prove that $\mathcal{M}$ is a CINM. By the definition of $\preceq$ it is immediate that $\preceq$ is reflexive and transitive. Moreover, assume $\sigma \preceq \rho$. Then $\rho = \sigma.\sigma'$ for some σ'. We have:

• $\mathcal{V}(\sigma) \subseteq \mathcal{V}(\rho)$: If $p \in \mathcal{V}(\sigma)$, then by definition $\sigma\ T\ p \in \mathcal{B}$. Then by saturation of hp, $\rho\ T\ p \in \mathcal{B}$, therefore $p \in \mathcal{V}(\rho)$.

• $\mathcal{N}_\Box(\sigma) \subseteq \mathcal{N}_\Box(\rho)$: If $\alpha \in \mathcal{N}_\Box(\sigma)$, then by definition there is $\sigma\ T\ \langle \Sigma \rangle \in \mathcal{B}$ such that $\Sigma^+ \subseteq \alpha$. Then by saturation of $\mathsf{hp}_{\langle\rangle}$, $\rho\ T\ \langle \Sigma \rangle \in \mathcal{B}$, therefore $\alpha \in \mathcal{N}_\Box(\rho)$.

• $\mathcal{N}_\Diamond(\sigma) \supseteq \mathcal{N}_\Diamond(\rho)$: If $\alpha \notin \mathcal{N}_\Diamond(\sigma)$, then by definition there is $\sigma\ T\ \Diamond A \in \mathcal{B}$ such that $A^+ \cap \alpha = \emptyset$. Then by saturation of hp, $\rho\ T\ \Diamond A \in \mathcal{B}$, therefore $\alpha \notin \mathcal{N}_\Diamond(\rho)$.

We now show that $\mathcal{M}$ satisfies the conditions of CINMs for $\mathbf{IM}^*$:

• $\mathcal{M}$ is $\mathsf{weakInt}$: Assume $\alpha \in \mathcal{N}_\Box(\sigma)$. Then there is $\sigma\ T\ \langle \Sigma \rangle \in \mathcal{B}$ such that $\Sigma^+ \subseteq \alpha$. By contradiction, assume $\alpha \notin \mathcal{N}_\Diamond(\sigma)$. Then there is $\sigma\ T\ \Diamond B \in \mathcal{B}$ such that $B^+ \cap \alpha = \emptyset$. By saturation of str, there is n such that $n\ T\ A \in \mathcal{B}$ for every $A \in \Sigma$, and $n\ T\ B \in \mathcal{B}$. Then by definition, $n \in \Sigma^+$ and $n \in B^+$. Thus $\Sigma^+ \cap B^+ \neq \emptyset$, therefore $\alpha \cap B^+ \neq \emptyset$, which gives a contradiction. Therefore $\alpha \in \mathcal{N}_\Diamond(\sigma)$.

• $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ are supplemented: Immediate by definition of $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$.

• If $\mathbf{T.IM}^*$ contains rule $\mathsf{N}_\Diamond$, then $\mathcal{N}_\Diamond$ contains the unit: By contradiction, assume $\mathcal{W} \notin \mathcal{N}_\Diamond$. Then there is $\sigma\ T\ \Diamond A \in \mathcal{B}$ such that $A^+ \cap \mathcal{W} = \emptyset$, that is $A^+ = \emptyset$. By saturation of $\mathsf{N}_\Diamond$, there is n such that $n\ T\ A \in \mathcal{B}$. Then $n \in A^+$, thus $A^+ \neq \emptyset$, which gives a contradiction.

• If **T.IM*** contains rule $\mathsf{N}_\Box$, then $\mathcal{N}_\Box$ contains the unit: By saturation of $\mathsf{N}_\Box$, for every $\sigma \in \mathcal{W}$, $\sigma\ T\ \langle\top\rangle \in \mathcal{B}$. Moreover, $\top^+ \subseteq \mathcal{W}$, then by definition $\mathcal{W} \in \mathcal{N}_\Box(\sigma)$.

• If **T.IM*** contains rule $\mathsf{C}_\Box$, then $\mathcal{N}_\Box$ is closed under intersection: Assume $\alpha, \beta \in \mathcal{N}_\Box(\sigma)$. Then there are $\sigma\ T\ \langle\Sigma\rangle, \sigma\ T\ \langle\Pi\rangle \in \mathcal{B}$ such that $\Sigma^+ \subseteq \alpha$ and $\Pi^+ \subseteq \beta$. By saturation of $\mathsf{C}_\Box$, $\sigma\ T\ \langle\Sigma,\Pi\rangle \in \mathcal{B}$. Moreover, $(\Sigma,\Pi)^+ = \Sigma^+ \cap \Pi^+$: $\rho \in (\Sigma,\Pi)^+$ iff for every $B \in \Sigma, \Pi$, $\rho\ T\ B \in \mathcal{B}$ iff for every $B \in \Sigma$, $\rho\ T\ B \in \mathcal{B}$ and for every $B \in \Pi$, $\rho\ T\ B \in \mathcal{B}$ iff $\rho \in \Sigma^+$ and $\rho \in \Pi^+$ iff $\rho \in \Sigma^+ \cap \Pi^+$. Then $(\Sigma,\Pi)^+ \subseteq \alpha \cap \beta$, which implies $\alpha \cap \beta \in \mathcal{N}_\Box(\sigma)$.

Finally, we prove the truth lemma by induction on A and $\langle\Sigma\rangle$:

($\sigma\ T\ p \in \mathcal{B}$) By definition, $p \in \mathcal{V}(\sigma)$, then $\sigma \Vdash p$.

($\sigma\ F\ p \in \mathcal{B}$) By saturation of init, $\sigma\ T\ p \notin \mathcal{B}$. Then $p \notin \mathcal{V}(\sigma)$, therefore $\sigma \not\Vdash p$.

($\sigma\ T\ \bot \in \mathcal{B}$) Impossible by saturation of $\mathsf{T}\bot$.

($\sigma\ F\ \bot \in \mathcal{B}$) By definition, $\sigma \not\Vdash \bot$.

($\sigma\ T\ \top \in \mathcal{B}$; $\sigma\ F\ \top \in \mathcal{B}$) Analogous to cases $\sigma\ T\ \bot \in \mathcal{B}$ and $\sigma\ F\ \bot \in \mathcal{B}$.

($\sigma\ T\ B \wedge C \in \mathcal{B}$) By saturation of $\mathsf{T}\wedge$, $\sigma\ T\ B \in \mathcal{B}$ and $\sigma\ T\ C \in \mathcal{B}$. Then by i.h., $\sigma \Vdash B$ and $\sigma \Vdash C$. Therefore $\sigma \Vdash B \wedge C$.

($\sigma\ F\ B \wedge C \in \mathcal{B}$) By saturation of $\mathsf{F}\wedge$, $\sigma\ F\ B \in \mathcal{B}$ or $\sigma\ F\ C \in \mathcal{B}$. Then by i.h., $\sigma \not\Vdash B$ or $\sigma \not\Vdash C$. Therefore $\sigma \not\Vdash B \wedge C$.

($\sigma\ T\ B \vee C \in \mathcal{B}$; $\sigma\ F\ B \vee C \in \mathcal{B}$) Analogous to cases $\sigma\ T\ B \wedge C \in \mathcal{B}$ and $\sigma\ F\ B \wedge C \in \mathcal{B}$.

($\sigma\ T\ B \supset C \in \mathcal{B}$) Let $\sigma \preceq \rho$. Then $\rho = \sigma.\sigma'$. Thus by saturation of hp, $\rho\ T\ B \supset C \in \mathcal{B}$. Moreover, by saturation of $\mathsf{T}\supset$, $\rho\ F\ B \in \mathcal{B}$ or $\rho\ T\ C \in \mathcal{B}$. Then by i.h., $\rho \not\Vdash B$ or $\rho \Vdash C$. Therefore $\sigma \Vdash B \supset C$.

($\sigma\ F\ B \supset C \in \mathcal{B}$) By saturation of $\mathsf{F}\supset$, there is $\sigma.n$ such that $\sigma.n\ T\ B \in \mathcal{B}$ and $\sigma.n\ F\ C \in \mathcal{B}$. Then by definition $\sigma \preceq \sigma.n$, and by i.h., $\sigma.n \Vdash B$ and $\sigma.n \not\Vdash C$. Therefore $\sigma \not\Vdash B \supset C$.

($\sigma\ T\ \Box B \in \mathcal{B}$) By saturation of $\mathsf{T}\Box$, $\sigma\ T\ \langle B\rangle \in \mathcal{B}$. Then by i.h., $\sigma \Vdash \Box B$.

($\sigma\ T\ \langle\Sigma\rangle \in \mathcal{B}$) First, notice that $\Sigma^+ \subseteq [\![\bigwedge\Sigma]\!]$: if $\rho \in \Sigma^+$, then $\rho\ T\ B \in \mathcal{B}$ for all $B \in \Sigma$, thus by i.h., $\rho \Vdash B$ for all $B \in \Sigma$, that is $\rho \Vdash \bigwedge\Sigma$. Then by definition, $[\![\bigwedge\Sigma]\!] \in \mathcal{N}_\Box(\sigma)$, therefore $\sigma \Vdash \Box\bigwedge\Sigma$.

($\sigma\ F\ \Box B \in \mathcal{B}$) Assume $\alpha \in \mathcal{N}_\Box(\sigma)$. Then there is $T\ \langle\Sigma\rangle \in \mathcal{B}$ such that $\Sigma^+ \subseteq \alpha$. Then by saturation of $\mathsf{M}_\Box$, there is n such that $n\ T\ C \in \mathcal{B}$ for all $C \in \Sigma$, and $n\ F\ B \in \mathcal{B}$. Then $n \in \Sigma^+$ and, by ih, $n \not\Vdash B$. Thus $\Sigma^+ \not\subseteq [\![B]\!]$, which implies $\alpha \neq [\![B]\!]$. Therefore $[\![B]\!] \notin \mathcal{N}_\Box(\sigma)$, which implies $\sigma \not\Vdash \Box B$.

($\sigma\ T\ \Diamond B \in \mathcal{B}$) As before, we have $B^+ \subseteq [\![B]\!]$. Then $B^+ \cap \mathcal{W} \setminus [\![B]\!] = \emptyset$, thus $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(\sigma)$, which gives $\sigma \Vdash \Diamond B$.

($\sigma\ F\ \Diamond B \in \mathcal{B}$) By contradiction, assume $\mathcal{W} \setminus [\![B]\!] \notin \mathcal{N}_\Diamond(\sigma)$. Then there is $\sigma\ T\ \Diamond C \in \mathcal{B}$ such that $C^+ \cap \mathcal{W} \setminus [\![B]\!] = \emptyset$. By saturation of $\mathsf{M}_\Diamond$, there is n such that $n\ T\ C \in \mathcal{B}$ and $n\ F\ B \in \mathcal{B}$. Then $n \in C^+$, and by i.h., $n \nVdash B$, i.e., $n \notin [\![B]\!]$, which implies $C^+ \cap \mathcal{W} \setminus [\![B]\!] \neq \emptyset$. Therefore $\mathcal{W} \setminus [\![B]\!] \in \mathcal{N}_\Diamond(\sigma)$, thus $\sigma \nVdash \Diamond B$. $\square$

Theorem 8.7.3 (Semantic completeness)**.** For every formula A of $\mathcal{L}_i$, if A is valid in the class of CINMs for **IM***, then A is derivable in **T.IM***.

Proof. If A is not derivable in **T.IM***, then it has a failed proof. By Lemma 8.7.2, from this we obtain a countermodel of A which is a CINM for **IM***, therefore A is not valid in the class of CINMs for **IM***. $\square$

From the completeness of systems **IM*** with respect to the corresponding CINMs we then obtain the following result:

Theorem 8.7.4. For every formula A of $\mathcal{L}_i$, A is derivable in **T.IM*** if and only if it is derivable in **IM***.

Proof. A is derivable in **T.IM*** iff (by Theorems 8.7.1 and 8.7.3) A is valid in the class of CINMs for **IM*** iff (by Theorems 8.2.1 and 8.2.5) A is derivable in **IM***. $\square$

We conclude this section with an example of a failed proof in **T.IMCN**$_\Box$ and the extracted countermodel.

Example 8.7.1 (Axiom $K_\Diamond$ is not derivable in **T.IMCN**$_\Box$)**.** In Figure 8.5 we find a failed proof of $\Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$ in **T.IMCN**$_\Box$. Following Definition 8.7.4, from the saturated branch we obtain the following CINM $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_\Box, \mathcal{N}_\Diamond, \mathcal{V} \rangle$:

- $\mathcal{W} = \{1, 1.1, 1.1.1, 2, 3\}$;
- $\preceq$ is the transitive and reflexive closure of $1 \preceq 1.1 \preceq 1.1.1$;
- $\mathcal{V}(1) = \mathcal{V}(1.1) = \mathcal{V}(1.1.1) = \emptyset$; $\mathcal{V}(2) = \{p, q\}$; and $\mathcal{V}(3) = \{p\}$;
- for every $w \in \mathcal{W}$, $\mathcal{N}_\Box(w) = \{\alpha \subseteq \mathcal{W} \mid \{2\} \subseteq \alpha\}$;
- $\mathcal{N}_\Diamond(1.1.1) = \mathcal{P}(\mathcal{W}) \setminus \{\emptyset, \{1\}, \{1.1\}, \{1.1.1\}, \{1, 1.1\}, \{1, 1.1.1\}, \{1.1, 1.1.1\}, \{1, 1.1, 1.1.1\}\}$;
- for every $w \neq 1.1.1$, $\mathcal{N}_\Diamond(w) = \mathcal{P}(\mathcal{W})$.

We have $\mathcal{M} \not\models \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$, since:

- $1.1.1 \Vdash \Box(p \supset q)$, since $[\![p \supset q]\!] = \mathcal{P}(\mathcal{W}) \setminus \{3\} \in \mathcal{N}_\Box(1.1.1)$.
- $1.1.1 \Vdash \Diamond p$, since $\mathcal{W} \setminus [\![p]\!] = \{1, 1.1, 1.1.1\} \notin \mathcal{N}_\Diamond(1.1.1)$.
- $1.1.1 \nVdash \Diamond p$, since $\mathcal{W} \setminus [\![q]\!] = \{1, 1.1, 1.1.1, 3\} \in \mathcal{N}_\Diamond(1.1.1)$.

It is easy to verify that $\mathcal{M}$ satisfies the conditions of CINMs for **T.IMCN**$_\Box$. Moreover, it does not satisfy WInt, since $\{2\} \in \mathcal{N}_\Box(1.1.1)$, $\{3\} \in \mathcal{N}_\Diamond(1.1.1)$, but $\{2\} \cap \{3\} = \emptyset \notin \mathcal{N}_\Diamond(1.1.1)$.

1.	$1\ F\ \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$	
2.	$1\ T\ \langle \top \rangle$	$(1, \mathsf{N}_{\Box})$
3.	$1.1\ T\ \Box(p \supset q)$	$(1, \mathsf{F} \supset)$
4.	$1.1\ F\ \Diamond p \supset \Diamond q$	$(1, \mathsf{F} \supset)$
5.	$1.1\ T\ \langle \top \rangle$	$(3, \mathsf{N}_{\Box})$
6.	$1.1\ T\ \langle p \supset q \rangle$	$(3, \mathsf{T}\Box)$
7.	$1.1\ T\ \langle \top, p \supset q \rangle$	$(5{,}6, \mathsf{C}_{\Box})$
8.	$1.1.1\ T\ \Diamond p$	$(4, \mathsf{F} \supset)$
9.	$1.1.1\ F\ \Diamond q$	$(4, \mathsf{F} \supset)$
10.	$1.1.1\ T\ \langle \top \rangle$	$(8, \mathsf{N}_{\Box})$
11.	$1.1.1\ T\ \Box(p \supset q)$	$(3{,}8, \mathsf{hp})$
12.	$1.1.1\ T\ \langle p \supset q \rangle$	$(11, \mathsf{T}\Box)$
13.	$1.1.1\ T\ \langle \top, p \supset q \rangle$	$(10{,}12, \mathsf{C}_{\Box})$
14.	$2\ T\ \top$	$(8{,}13, \mathsf{str})$
15.	$2\ T\ p \supset q$	$(8{,}13, \mathsf{str})$
16.	$2\ T\ p$	$(8{,}13, \mathsf{str})$

17.	$2\ F\ p$	$(15, \mathsf{T} \supset)$
18.	**f**	$(16{,}17, \mathsf{init})$

17.	$2\ T\ q$	$(15, \mathsf{T} \supset)$
18.	$2\ T\ \langle \top \rangle$	$(17, \mathsf{N}_{\Box})$
19.	$3\ T\ p$	$(8{,}9, \mathsf{M}_{\Diamond})$
20.	$3\ F\ q$	$(8{,}9, \mathsf{M}_{\Diamond})$
21.	$3\ T\ \langle \top \rangle$	$(19, \mathsf{N}_{\Box})$
	saturated	

Figure 8.5: Failed proof of axiom $K_{\Diamond}$ in $\mathbf{T.IMCN}_{\Box}$.

8.8 Discussion

In this chapter, we have presented a semantic framework for intuitionistic non-normal modal logics defined in terms of Coupled Intuitionistic Neighbourhood Models. On its basis, we have provided a modular semantic characterisation of all the systems introduced in Chapter 7, as well as of pre-existing intuitionistic non-normal modal logics **CK** and **CCDL**. The models contain an order relation and two neighbourhood functions handling the modalities $\Box$ and $\Diamond$ separately. The two functions can be supplemented, closed under intersection, or contain the unit. Moreover, they can be related in different ways reflecting the possible interactions between $\Box$ and $\Diamond$. Through a filtration argument we have also proved that all logics enjoy the finite model property. Our semantics turns out to be a versatile tool to analyse intuitionistic non-normal modal logics, and can capture further well-known logics such as Constructive K and the propositional fragment of Wijesekera's Constructive Concurrent Dynamic Logic. For these two systems we have proved completeness both directly by the canonical model construction and indirectly by mutual transformations with models in their original semantics. Furthermore, we have shown an embedding of intuitionistic non-normal modal logics into classical logics with multiple non-normal modalities. Finally, we have presented tableaux calculi for the monotonic logics $\mathbf{IE}^*$ that allow one to directly extract countermodels of non-valid formulas from failed proofs. These calculi can be reformulated for monotonic monomodal $\Box$- and $\Diamond$-logics simply by removing the rules for the lacking modality.

The fact that **CK** and **CCDL** fit in our framework is interesting for two reasons. On the one hand, it shows the power of our neighbourhood semantics, that can accommodate in a natural way many systems. On the other hand, it shows that **CK** and **CCDL** can be obtained as extensions of weaker logics in a modular way. As a further advantage, differently from relational models, coupled intuitionistic neighbourhood models provide a standard semantics for **CK** which does not need to resort to inconsistent worlds.

In future work, we aim to study semantic characterisation with CINMs of further extensions of our systems with additional standard modal axioms. As a preliminary remark, we observe that restricting the analysis to systems without interaction between the modalities, a semantic characterisation of axiom $C_\Diamond$ can be given by requiring that $\mathcal{N}_\Diamond$ is closed under intersection. Completeness of systems of this form can be proved by slightly modifying Definition 8.2.2 of canonical models, that is defining $\mathcal{N}_\Diamond^{\mathbf{L}}(X) = \{\mathcal{W}_{\mathbf{L}} \setminus \uparrow A \mid \Diamond A \notin X\}$. This result can be extended to logics with both $\Box$ and $\Diamond$ but without interactions between the modalities. On the contrary, as a consequence of the modification of the definition of canonical models, for the logics with interactions between $\Box$ and $\Diamond$ the completeness proof presented in Section 8.2 does not work anymore. Further investigation is required to establish whether in presence of $C_\Diamond$ we can preserve the semantic conditions connecting $\mathcal{N}_\Box$ and $\mathcal{N}_\Diamond$ that we considered in this work, or whether we need to consider different connections instead. In addition, it would be interesting to investigate whether our semantic framework is suitable to cover also stronger systems studied in the literature such as for instance the intuitionistic epistemic logic in Artemov and Protopopescu [8].

Concerning the tableaux calculi, we recall that for *classical* non-normal modal logics prefixed tableaux calculi are presented in Indrzejczak [98]. However, differently from Indrzejczak's calculi, in our calculi the sequential structure of prefixes takes care of the intuitionistic order $\preceq$, which is determined by the applications of the rule $\mathsf{F} \supset$, and *not* of the generation of worlds by means of the application of the modal rules. As a matter of fact, for the semantic completeness of the calculi it is not necessary to keep track of the latter relation, neither in the classical case, as it is also witnessed by the possibility to extract countermodels from hypersequent calculi (cf. Chapter 6), nor in the intuitionistic one.

A lot of work has still to be done for a fully satisfactory proof-theoretic account of the considered logics. First of all, similarly to other calculi for intuitionistic logic, the tableaux calculi $\mathbf{T.IM}^*$ are not strictly terminating (cf. Section 7.4). In future work we intend to define proof search strategies that ensure termination and at the same time preserve the completeness of the calculi. Furthermore, we aim to extend both the systems and the countermodel extraction to all the intuitionistic non-normal modal logics defined in Chapter 7 as well as to **CK** and **CCDL**.

Chapter 9

Conclusion

Summary of main results

In this thesis we have carried out a proof-theoretical investigation of non-normal modal logics. We have considered non-normal modal logics based on both classical and intuitionistic propositional logic. Concerning non-normal modal logics based on classical logic, we have firstly defined a new semantics, the we called *bi-neighbourhood semantics* (Chapter 4), for all the systems of the classical cube and their extensions with the axioms T, 4, D, P, and the rules RD_n^+. Moreover, we have shown that this semantics can also characterise Elgesem's and Troquard's agency and ability logics. This semantics generalises the standard neighbourhood semantics and can be used to represent under-determined situations. The bi-neighbourhood semantics essentially decomposes the forcing condition of boxed formulas in the standard semantics into two monotonic components. We have shown that from a syntactic point of view, this observation corresponds to an embedding of classical non-normal modal logics into logics with binary monotonic modalities.

Moreover, we have presented two kinds of calculi for these logics: *labelled calculi* (Chapter 5) and *hypersequent calculi* (Chapter 6). The first calculi are defined by extending the language of the logics with labels which are used to import semantic information into the calculus. Modular extensions of the basic calculus are obtained by means of rules that directly express properties of bi-neighbourhood models for the corresponding logics. The second kind of calculi are defined by extending the structure of sequents: they employ hypersequents (i.e., multisets of sequents) and blocks, which are used to collect modal formulas. We have proved that both labelled and hypersequent calculi enjoy admissibility of structural rules, and that they are equivalent to the axiomatisations, the latter by means of a syntactic proof of cut elimination. In addition, we have shown that both kinds of calculi allow for simple terminating proof search strategies, and in case of failed proofs it is always possible to extract countermodels of the non-valid/non-derivable formulas in the bi-neighbourhood semantics. All this provided an alternative proof of the finite model property of classical non-normal modal logics, as well as a constructive proof of their decidability, since for every formula the proof search procedures return either a derivation, if the formula is derivable/valid, or a countermodel, otherwise. We have also presented Prolog implementations of both the proof search and the countermodel extraction in the two calculi, these provide the first theorem provers that uniformly cover non-normal modal logics and compute both derivations and countermodels of non-valid formulas. Furthermore, we have extended the hypersequent calculi to Elgesem's and Troquard's agency and ability logics. In conclusion, both kinds of extension of the basic framework of sequent calculi (i.e., extension of the language and extension of the structure of sequents) turned out to be adequate to define proof systems for classical non-normal modal logics satisfying our desiderata.

It is also worth highlighting some relevant differences between the two kinds of calculi. First of all, while the hypersequent calculi essentially need to distinguish between monotonic and non-monotonic systems, the labelled calculi allow for a *fully modular* treatment of the whole family of the considered logics. The possibility to uniformly cover monotonic and non-monotonic systems essentially depends on the fact that, by directly converting model conditions into rules, the calculus preserves the modularity of the bi-neighbourhood semantics. Moreover, within the labelled framework we have defined *cut-free* calculi for all the logics containing axiom 4. To the best of our knowledge, these are the first cut-free calculi for logic **E4** and its extensions without axioms M or T. On the other hand, while both kinds of calculi allow for terminating decision procedures, we have shown that the hypersequent calculi have very good computational properties, a fact that is also witnessed by the better performance of the prover HYPNO (which implements the hypersequent calculi) compared with that of PRONOM (which implements the labelled calculi). In particular, hypersequent calculi allow for a *complexity optimal* decision procedure for coNP-complete logics, i.e., all covered logics not containing axiom C. Moreover, the calculi provide a constructive proof of the *polysize model property* for these logics, since for every non-derivable formula the procedure returns a countermodel which is polynomial with respect to the size of the input formula.

Concerning non-normal modal logics with an intuitionistic basis, in Chapter 7 we have defined a family of systems that can be interpreted as *intuitionistic counterparts* of the logics of the classical cube. Every system contains some of the characteristic modal axioms and rules of classical non-normal modal logics, plus some axioms connecting the two modalities $\Box$ and $\Diamond$ which can be seen as "weak duality principles". As we observed, assuming an intuitionistic basis allows us to do finer distinctions between principles that are not distinguishable in classical logic. As a consequence, the picture of systems that emerge from a certain set of logic principles is richer in the intuitionistic case than in the classical one: while the classical cube contains 8 logics, the intuitionistic lattice features 24. In addition, as we observed for the logic $\mathbf{IE_3}$ (cf. Section 7.7), assuming an intuitionistic basis gives us also the possibility to define systems containing combinations of principles which are not jointly compatible in classical logics.

For every intuitionistic system we have provided both a cut-free sequent calculus and an equivalent axiomatisation. Moreover, we have proved that all systems are decidable and that some of them enjoy Craig interpolation. In addition, we have also defined strictly terminating "à la Dyckhoff" calculi for all our intuitionistic systems as well as for **CK** and Wijesekera's **CCDL**. These calculi allow for an alternative proof of decidability of the respective logics without needing any loop checking mechanism.

We then moved to the investigation of the semantics of the intuitionistic systems: in Chapter 8 we have defined a general semantic framework in terms of so-called *coupled intuitionistic neighbourhood models* which modularly captures all the systems defined in Chapter 7 as well as **CK** and **CCDL**. For the latter systems, we have also shown direct transformations between coupled intuitionistic neighbourhood models and models in their original semantics. Moreover, we have proved that all these systems enjoy the finite model property, and that they can be embedded into classical non-normal modal logics with multiple modalities. Finally, we have presented tableaux calculi for intuitionistic monotonic systems that allow one to directly extract countermodels in their neighbourhood semantics from failed proofs.

Open problems and future work

We have discussed open problems and possible extensions of the presented results at the end of each chapter, here we briefly recapitulate some of the main issues.

Concerning extensions of the presented results, first of all we can investigate the possibility to further extend the bi-neighbourhood semantics as well as the labelled and the hypersequent calculi to systems defined by additional standard modal axioms, such as for instance 5, B, and Sahlqvist formulas. As remarked, this seems a non-trivial task especially when axioms with nested modalities are concerned. For instance, in Section 4.5 we have proved the completeness of the systems **E** and **M** extended with axiom 5, but we have not obtained yet analogous results for systems containing 5 together with other modal axioms, such as D, P, etc. Moreover, although we have defined cut-free labelled calculi for the systems with axiom 4 (Chapter 5), we have not found yet a proof search procedure in these calculi that ensures termination and at the same time preserves the completeness of the calculi. We aim to search for such a terminating procedure in future work, thus ideally complementing the decidability result established for these systems by Vardi [167].

We have shown that the proof systems defined in this thesis are also suitable to cover specific logical systems studied in the literature, in particular Elgesem's agency logic and Troquard's coalition logic. In future work we aim to explore the possibility to cover further systems, such as for instance Pauly's coalition logic [146]. Furthermore, we aim to extend the implementations of both the labelled and the hypersequent calculi to all the logics captured by the proof systems, whence also Elgesem's and Troquard's logics. Moreover, we aim to improve the performance of the provers, for instance by establishing finer restrictions on the order of the rule applications, or by using more efficient data structures.

Furthermore, in Section 7.5 we have proved Craig's interpolation for a subclass of intuitionistic non-normal modal logics by applying Maehara's method based on cut-free Gentzen calculi. Interestingly, this method does not seem to work for non-monotonic logics with axiom C, neither in the classical, nor in the intuitionistic case (see also Orlandelli [140]). In several papers, Kuznets has presented proof-theoretical proofs of interpolation for modal and related logics based on different kinds of proof systems, both labelled and structured (see e.g. [106]). It would be worth investigating whether our labelled or hypersequent calculi are suitable to prove Craig's interpolation for these logics, thus complementing the semantic proof provided for the classical systems by Pattinson [145]. Moreover, we also aim to study whether Iemhoff's [95] proof of uniform interpolation for intuitionistic modal logics based on strictly terminating G4-style calculi can be extended to our intuitionistic systems as well as to **CK** and **CCDL**, for which we have defined G4-style calculi in Section 7.6.

Finally, we have presented preliminary results about proof systems for intuitionistic non-normal modal logics allowing for direct countermodel extraction of non-valid formulas. These results must be understood as initial steps toward a more comprehensive investigation. In this respect, we aim to explore the possibility to define proof systems for these logics analogous to the labelled or hypersequent calculi for classical non-normal modal logics presented in this work; to this purpose we also aim to define bi-neighbourhood models for the intuitionistic systems. Ideally, similarly to the proof systems for classical non-normal modal logics, these calculi should be also suitable for implementation. A theorem prover which computes derivations and countermodels in **IK** and some extensions has been recently described in Girlando and Straßburger [70], but as far as we know no analogous tool exists for constructive systems. Furthermore, it would be worth investigating further extensions of constructive modal logics. In particular, differently from both **IK** and **CK**, we are not aware of any semantical or proof-theoretical study of extensions of **CCDL**. Finally, it would be interesting to see whether our intuitionistic logics, similarly to **CK** (cf. Bellin et al. [14]), can be given a type-theoretical interpretation by a suitable extension of the typed lambda-calculus. All this will be object of future investigation.

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