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Résolution numérique de quelques problèmes du type Helmholtz avec conditions au bord d'impédance ou des couches absorbantes (PML)

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Abstract

In this thesis, we propose wavenumber explicit convergence analyses of some finite element methods for time-harmonic Maxwell's equations with impedance boundary condition and for the Helmholtz equation with Perfectly Matched Layer (PML).

We first study the regularized formulation of time-harmonic Maxwell's equations with impedance boundary conditions (where we add a ∇ div-term to the original equation to have an elliptic problem) and keep the impedance boundary condition as an essential boundary condition. For a smooth domain, the well-posedness of this formulation is well-known. But the well-posedness for convex polyhedral domain has been not yet investigated. Hence, we start the first chapter with the proof of the well-posedness in this case, which is based on the fact that the variational space is embedded in H^1. In order to perform a wavenumber explicit error analysis of our problem, a wavenumber explicit stability estimate is mandatory. We then prove such an estimate for some particular configurations.

In the second chapter, we describe the corner and edge singularities for such problem. Then we deduce the regularity of the solution of the original and the adjoint problem, thus we have all ingredients to propose a explicit wavenumber convergence analysis for h-FEM with Lagrange element.

In the third chapter, we consider a non conforming hp-finite element approximation for domains with a smooth boundary. To perform a wavenumber explicit error analysis, we split the solution of the original problem (or its adjoint) into a regular but oscillating part and a rough component that behaves nicely for large frequencies. This result allows to prove convergence analysis for our FEM, again explicit in the wavenumber.

The last chapter is dedicated to the Helmholtz equation with PML. The Helmholtz equation in full space is often used to model time harmonic acoustic scattering problems, with Sommerfeld radiation condition at infinity. Adding a PML is a way to reduce the infinite domain to a finite one. It corresponds to add an artificial absorbing layer surrounding a computational domain, in which scattered wave will decrease very quickly. We first propose a wavenumber explicit stability result for such problem. Then, we propose two numerical discretizations: an hp-FEM and a multiscale method based on local subspace correction. The stability result is used to relate the choice of the parameters in the numerical methods to the wavenumber. A priori error estimates are shown.

At the end of each chapter, we perform numerical tests to confirm our theoritical results.

Keywords: Finite element method, Helmholtz equation, Pollution effect, Maxwell's equations, Impedance boundary condition, Perfectly Matched Layer (PML).

Résumé

Dans cette thèse, nous étudions la convergence de méthode de type éléments finis pour les équations de Maxwell en régime harmonique avec condition au bord d'impédance et l'équation de Helmholtz avec une couche parfaitement absorbante (PML).

On étudie en premier, la formulation régularisée de l'équation de Maxwell en régime harmonique avec condition au bord d'impédance (qui consiste à ajouter le term $\nabla \operatorname{div}$ à l'équation originale pour avoir un problème elliptique) et on garde la condition d'impédance comme une condition au bord essentielle. Pour des domaines à bord régulier, le caractère bien posé de cette formulation est bien connu mais cela n'est pas le cas pour des domaines polyédraux convexes. On commence alors le premier chapitre par la preuve du caractère bien posé dans le cas du polyèdre convexe, qui est basé sur le fait que l'espace variationnel est inclus dans H^1. Dans le but d'avoir des estimations explicites en le nombre d'onde k de ce problème, il est obligatoire d'avoir des résultats de stabilité explicites en ce nombre d'onde. C'est aussi proposé, pour quelques situations particulières, dans ce chapitre.

Dans le second chapitre on décrit les singularités d'arêtes et de coins pour notre problème. On peut alors déduire la régularité de la solution du problème original, ainsi que de son adjoint. On a tous les ingrédients pour proposer une analyse de convergence explicite en k pour une méthode d'éléments finis avec éléments de Lagrange.

Dans le troisième chapitre, on considère une méthode d'éléments finis hp non conforme pour un domaine à bord régulier. Pour obtenir des estimations explicites en k, on introduit un résultat de décomposition, qui sépare la solution du problème original (ou de son adjoint) en une partie régulière mais fortement oscillante et une partie moins régulière mais peu oscillante. Ce résultat permet de montrer des estimations explicites en k.

Le dernier chapitre est dédié à l'équation de Helmholtz avec une PML. L'équation de Helmholtz dans l'espace entier est souvent utilisée pour modéliser la diffraction d'onde acoustique (en régime harmonique), avec la condition de radiation à l'infini de Sommerfeld. L'ajout d'une PML est une façon pour passer d'un domaine infini à un domaine fini, elle correspond à l'ajout d'une couche autour du domaine de calcul qui absorbe très vite toutes les ondes sortantes. On propose en premier un résultat de stabilité explicite en k. On propose alors deux schémas numériques, une méthode d'éléments finis hp et une méthode multi-échelle basée sur un sous-espace local de correction. Le résultat de stabilité est utilisé pour mettre en relation de choix des paramètres des méthodes numériques considérées avec k. Nous montrons aussi des estimations d'erreur a priori. A la fin de ces chapitres, des tests numériques sont proposés pour confirmer nos résultats théoriques.

Mots clés: Méthode des éléments finis, équation de Helmholtz, effet de pollution, équations de Maxwell, condition au bord d'impédance, Perfectly Matched Layer (PML).

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Contents

Introduction

Time-harmonic wave equation (acoustic or electromagnetic) are widely used in physics, as for example for scattering problems which describe how a wave will bounce off of some obstacles or will be absorbed. One of the main parameter in such problems is the wavenumber k that become difficult to solve numerically, when k is large. In particular, finite element methods applied to such problems are well-known for their lack of stability: it appears that the numerical solution, if it exists, is possibly far from the best approximation (in the finite element space) of the exact solution for large wavenumber k. This phenomena is called the *pollution effect*. This lack of stability is due to the fact that the associated sesquilinear forms are not coercive. Consequently the quasi-optimality of the finite element solution is not guaranteed for arbitrary meshes, but is achieved only in an asymptotic range, i.e., for small enough mesh sizes, that depends on the wavenumber and the discretization order.

In this thesis, we analyse two problems, the first one is the time-harmonic Maxwell equation with impedance boundary condition, while the second one is the Helmholtz equation with a Perfectly Matched Layer.

The scattering problem for time-harmonic Maxwell equation is

$$\operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} \quad \text{in } \mathbb{R}^3 \setminus \mathcal{O},$$

with $\mathcal{O} \subset \mathbb{R}^3$ a bounded obstacle. Here $\mathbf{E}$ is the electric part and $\mathbf{H}$ is the magnetic part of the electromagnetic field, and the constant k corresponds to the wavenumber. The right hand side $\mathbf{J}$ is the current density which – in the absence of free electric charges – is divergence free, namely

$$\operatorname{div} \mathbf{J} = 0 \quad \text{in } \Omega.$$

For the sake of simplicity, we suppose that the domain is the vacuum, hence the relative permittivity and permeability are equal to 1 (for more details, see [55]). When the wavenumber k is different from 0, we can eliminate $\mathbf{H}$ by the first relation in this equation to have a second order system, i.e.

$$\operatorname{curl} \operatorname{curl} \mathbf{E} - k^2\mathbf{E} = ik\mathbf{J} \text{ in } \mathbb{R}^3 \setminus \mathcal{O}. \tag{1}$$

The standard scattering problem leads to find $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ solution of (1), where $\mathbf{E}^i$ is a given incident field (or waves) and $\mathbf{E}^s$ is the unknown scattered field. A typical incident field might be a plane wave, i.e.,

$$\mathbf{E}^i(x) = p \exp(ikx \cdot d),$$

with $d \in \mathbb{R}^3$ is a unit vector that gives the direction of propagation of the wave, $p \neq 0$ is called the polarization and must be orthogonal to d. We suppose that $\mathbf{E}^i$ satisfies:

$$\operatorname{curl} \operatorname{curl} \mathbf{E}^i - k^2 \mathbf{E}^i = \mathbf{J} \text{ in } \mathbb{R}^3,$$

where $\mathbf{J}$ is a given function describing the current source (in the plane wave case, $\mathbf{J} = 0$). To expect uniqueness of the solution, we need to impose a boundary condition on the obstacle and a radiation condition at infinity. The condition on the obstacle is dependent of the physical properties of the obstacle, it can be a perfect conductor ($\mathbf{E} \times n = 0$ and $\mathbf{H} \cdot n = 0$ on $\partial\mathcal{O}$) or an imperfectly conductor (it is also called impedance boundary condition), for example. The condition at infinity is called the Silver-Müller radiation condition,

$$\lim_{|x| \to \infty} |x| ((\operatorname{curl} \mathbf{E}^s) \times \frac{x}{|x|} - ik\mathbf{E}^s) = 0.$$

But, in order to use a finite element method for this problem, we must reduce the computational domain. Let $\Omega \subset \mathbb{R}^3$ be such $\overline{\mathcal{O}} \subset \Omega$, hence the computational domain will be $\Omega \setminus \mathcal{O}$. We need to add a boundary condition on $\partial\Omega$ and a standard way is formally to impose the Silver-Müller radition condition on $\partial\Omega$, i.e.

$$(\operatorname{curl} \mathbf{E}^s) \times n - ik\mathbf{E}^s = 0 \text{ on } \partial\Omega,$$

or equivalently

$$(\operatorname{curl} \mathbf{E}^s) \times n - ik\mathbf{E}^s_t = 0 \text{ on } \partial\Omega,$$

where n is the outward normal vector and $\mathbf{E}^s_t = (n \times \mathbf{E}^s) \times n$ is the tangential component of $\mathbf{E}^s$. This condition is also called Leontovich condition, or absorbing boundary condition and it is an impendance boundary condition. Obviously, there is no reason that the scattered field in $\mathbb{R}^3 \setminus \mathcal{O}$ and the scattered field in $\Omega \setminus \mathcal{O}$ should be equal but if Ω is large enough (and so the boundary of Ω is far from $\partial\mathcal{O}$) the difference will be small. The transparent boundary condition can also be used on $\partial\Omega$, but it provides a non-local operator on the boundary (Capacity or Calderon's operators, see [55, 59]). In this case, computing the finite element solution is quite expensive (as the associate linear system will be not sparse, more precisely, some blocks will be dense as each boundary nodes are connected to each other). S. Sauter and J.M. Melenk have recently studied this problem (without obstacle) in [53], and proposed a wavenumber explicit *hp*-FEM analysis for curl-conforming FEM (with Nédélec elements or also refered to Nédélec-Raviart-Thomas elements).

For this thesis, we consider the case with impedance boundary condition, but without obstacle.

$$\begin{cases} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \, \mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the impedance λ_{imp} is a smooth positive function. It is a more general boundary condition than the standard absorbing condition (when $\lambda_{\text{imp}} = 1$). Moreover, we study the case when Ω is either a polyhedron or a smooth domain [1].

As variational formulation of (2), a first attempt is to eliminate $\mathbf{H}$ by the relation $\mathbf{H} = \frac{1}{ik}\operatorname{curl}\mathbf{E}$, that transforms the impedance condition in the form

$$(\operatorname{curl}\mathbf{E}) \times \mathbf{n} - ik\lambda_{\text{imp}}\,\mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega.$$

Unfortunately such a boundary condition has no meaning in $\mathbf{H}(\text{curl}, \Omega)$, hence a solution is to introduce the subspace

$$\mathbf{H}_{\text{imp}}(\Omega) = \{\mathbf{u} \in \mathbf{H}(\text{curl}; \Omega) : \ \gamma_0 \mathbf{u}_t \in \mathbf{L}^2(\partial\Omega)\}.$$

Then eliminating $\mathbf{H}$ in the second identity of (2), and multiplying by a test function, we arrive at

$$\begin{aligned}(3) \qquad \int_\Omega (\operatorname{curl}\mathbf{E} \cdot \operatorname{curl}\bar{\mathbf{E}}' - k^2 \mathbf{E} \cdot \bar{\mathbf{E}}')\,dx - ik\int_{\partial\Omega} \lambda_{\text{imp}}\,\mathbf{E}_t \cdot \bar{\mathbf{E}}'_t\,d\sigma \\ = ik\int_\Omega \mathbf{J} \cdot \bar{\mathbf{E}}'\,dx, \ \forall \mathbf{E}' \in \mathbf{H}_{\text{imp}}(\Omega).\end{aligned}$$

Error analysis of (3) using Nédélec elements are available in [55, 31], but no explicit dependence with respect to k is proved. Note that a stability analysis has been performed in [36]. Moreover there is no hope to get easily regularity results of the solution by applying the theory of elliptic boundary value problems to the system associated with (3) because it is not elliptic (see [22, §4.5.d]).

A second attempt, proposed in [22, §4.5.d] for smooth boundaries and inspired from [59, §5.4.3], is to keep the full electromagnetic field and use the variational space

$$\mathbf{V} = \big\{(\mathbf{E}, \mathbf{H}) \in \big(\mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}(\text{div}, \Omega)\big)^2 : \ \mathbf{H} \times \mathbf{n} = \lambda_{\text{imp}}\mathbf{E}_t \ \text{ on } \partial\Omega\big\},$$

considering the impedance condition in (2) as an essential boundary condition. Hence the proposed variational formulation is: Find $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ such that

$$(4) \qquad \mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = \int_\Omega \Big(ik\mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \operatorname{curl}\bar{\mathbf{H}}'\Big)\,dx, \ \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V},$$

with the choice

$$\begin{aligned}\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) =& a_{k,s}(\mathbf{E}, \mathbf{E}') + a_{k,s}(\mathbf{H}, \mathbf{H}') \\ & - ik\int_{\partial\Omega} (\lambda_{\text{imp}}\,\mathbf{E}_t \cdot \bar{\mathbf{E}}'_t + \frac{1}{\lambda_{\text{imp}}}\mathbf{H}_t \cdot \bar{\mathbf{H}}'_t)\,d\sigma,\end{aligned}$$

with a positive real parameter s that may depend on k but is assumed to be in a fixed interval $[s_0, s_1]$ with $0 < s_0 \leq s_1 < \infty$ independ of k (see section 1.4 below for more details) and

$$a_{k,s}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\bar{\mathbf{v}} + s \operatorname{div}\mathbf{u} \operatorname{div}\bar{\mathbf{v}} - k^2 \mathbf{u} \cdot \bar{\mathbf{v}})\,dx.$$

[1] We mean by smooth domain a domain at least of class $\mathcal{C}^2$

The natural norm $\|\cdot\|_k$ of $\mathbf{V}$ associated with problem (4) is defined by

$$\|(\mathbf{E},\mathbf{H})\|_k^2 = \|\operatorname{curl}\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\mathbf{E}\|^2_{\mathrm{L}^2(\Omega)} + k^2\,\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl}\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\mathbf{H}\|^2_{\mathrm{L}^2(\Omega)} + k^2\,\|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)}\,.$$

This new formulation (4) has the advantage that its associated boundary value problem is an elliptic system (see [22, §4.5.d]), hence standard shift regularity results can be used. The *pollution effect* talked above is present here, so we need first to show the well-posedness of the problem (4).

The first chapter is devoted to the well-posedness of the problem (4). The smooth case is already known if we supose that $-\frac{k^2}{s}$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary condtion in Ω (cf. Lemma 4.5.9 of [22]). In this case, for $\mathbf{J} \in H(\mathrm{div};\Omega)$, the problem (4) has a unique solution and thus this solution is solution of the original problem (2). The proof of the well-posedness in this case is based on the fact that our variational space $\mathbf{V}$ is continously embedded into $(\mathbf{H}^1(\Omega))^2$ (and then $\mathbf{V}$ is compactly embedded in $(\mathbf{L}^2(\Omega))^2$), see for instance [3] or Lemma 4.5.5 of [22] and the Fredholm alternative. In the polyhedron case, we must adopt these arguments: then we first show that a similar embedding is still valid in this case (for the largest possible class of polyhedra, namely this embedding holds if and only if condition Ω is convex), this property correspond to Theorem 1.2.4. To use the Fredholm alternative, we need to show that the solution of (4) is unique, this is achieved by combining Lemma 1.3.3 and 1.3.4 (cf. proof of Theorem 1.3.5). One of the main result required to analyse the FEM is the stability estimate explicit in k of the problem (and its adjoint). This is performed in section 1.4 and the choice of s is also explained. For practical reason, we compute our tests on the TE formulation, it can be understood as two dimension Maxwell's equations, its description is given at the end of this chapter.

In the second chapter, we present an error estimate of the problem (4) for standard Lagrange finite element method in a polyhedral domain. Such error estimates are usually based on regularity results of the solution, but as our domain is not smooth, we need to determine the corner and edge singularities of our system. This is here done by adapting the techniques from [25, 21]. With the corner and edge singularities, we can determine the minimal regularity of the adjoint problem, and with stability results we can achieved a wavenumber explicit error bounds in an asymptotic range, in the same spirit of [15, 14] (with an expansion in power of k). Since the minimal regularity could be quite poor, this asymptotic range could be quite strong for quasi-uniform meshes, hence in the absence of edge singularities, we improve it by using adapted meshes, namely meshes refined near the corners of the domain. Some numerical tests, with the TE formulation, are proposed to confirm our theoretical analysis.

The third chapter is focused on error estimates of the problem (4) for a non-conforming Lagrange finite element method in a smooth domain. As the domain is smooth, we cannot impose the essential impedance boundary condition in the finite element space, this is why we add a penalization term on the boundary (corresponding to the impedance condition). In the first section, we present the discrete problem and we proved its well-posedness under a condition on the adjoint approximation quantity $\eta(\mathbf{V}_{h,p})$ (defined in (2.40), where $\mathbf{V}_{h,p}$ is the finite element space). After that, we analyse in two different ways the case of analytical boundary and the case of a boundary of class $\mathcal{C}^{\gamma+1,1}$. The analytical case is treated by following a similar approach as S. Sauter and J.M. Melenk have proposed in [48, 49, 51]. We split up the solution of (4) into a regular but oscillating part and a rough component that behaves nicely for large frequencies. This decomposition allows then to estimate $\eta(\mathbf{V}_{h,p})$, hence to get the well-posedness of the discrete problem under a condition on the meshsize and the polynomial degree, as well as to obtain some error estimates. Note that the estimation of the regular part heavily depends on analytic regularity of the solution of an elliptic system with lower order terms depending on the wavenumber k with bounds that explicitly depends on k. These bounds are obtained by combining analytic estimates of the same problem corresponding to $k = 0$ with bootstrapping and induction arguments. These analytical regularity results are postponed to an Appendix since we prove such results for general elliptic systems. For the case of a boundary of class $\mathcal{C}^{\gamma+1,1}$, to estimate the adjoint approximation quantity, we use an expansion of the solution in power of k, in the same spirit as S. Nicaise and T. Chaumont-Frelet did in [15, 14]. We propose some numerical tests that confirm our theoretical results.

The second problem discussed in this thesis is the Helmholtz equation with a Perfectly Matched Layer (PML). Let us first introduce time-harmonic acoustic scattering problem in a full space with a sound-soft obstacle, modelled through the Helmholtz equation subject to the Sommerfeld radiation condition [20]. This equation is

$$\begin{cases} -\Delta u - k^2 u = 0 \text{ in } \mathbb{R}^n \setminus \mathcal{O} \\ u^s = -u^i \text{ on } \partial\mathcal{O} \\ \lim\limits_{|x|\to\infty} |x|^{\frac{n-1}{2}}(\partial_{|x|}u^s(x) - iku^s(x)) = 0, \end{cases}$$

where $u = u^s + u^i$, u^i is an incident wave (for example, a plane wave) and u^s is the scattered wave (again, the "true unknown") and $n \in \{1, 2, 3\}$. To use a finite element method, we must reduce the computational domain and then add an artificial boundary. Then we take $\Omega \subset \mathbb{R}^n$ such that $\overline{\mathcal{O}} \subset \Omega$, and the equation becomes

$$\begin{cases} -\Delta u - k^2 u = 0 \text{ in } \Omega \setminus \mathcal{O}, \\ u^s = -u^i \text{ on } \partial\mathcal{O}, \\ \partial_n u^s = T(u^s) \text{ on } \partial\Omega, \end{cases}$$

with T an operator. The two usual ways to define T, corresponds either to the transparent boundary condition (T is called the Dirichlet-to-Neumman operator) or an absorbing boundary condition ($T(u^s) = iku^s$, this is a Robin type boundary condition). The main advantage of the Dirichlet-to-Neumman operator is that is a transparent condition, hence the solution of this problem is the same than the one in the full space restricted to Ω. But the principal disadvantage is that this operator has not a simple explicit form, hence computing a finite element solution is difficult [54]. Absorbing boundary condition is more effective numerically, as it is local. But it is an approximation of the Sommerfeld radiation condition hence the artificial boundary must be far enough from the obstacle and so the computational domain may be large. Another problem of this condition is that we can have reflective wave at the boundary, obviously without any physical sense (cf. [33]). Wavenumber explicit convergence analysis for FEM of these problems has been widely studied these last years. *hp*-FEM analysis can be found in [48, 49, 51] for analytic or convex polyhedral domain with Robin boundary condition or Dirichlet-to-Neumman operator. *h*-FEM for Robin boundary condition with $\mathcal{C}^{\gamma+1,1}$ domain is presented in [14], with polyhedral domain in [15] (both based on expansion of the solution in power of k). A different approach based on fine-scale correction techniques was proposed by [63]. It is based on low-order polynomials (in opposition of *hp*-FEM results), but the diameter of the support of the fine-scale corrections must grow logarithmically with k. This type of methods, called multiscale methods, have also been studied in [9, 30, 63] .

A different method to restrict the computational domain, introduced first by J.P Bérenger for Maxwell's equations in [5], is to add a Perfectly Matched Layer (PML). It is a absorbing layer surrounding Ω that has the remarkable property of being perfectly reflectionless, for a layer of infinite thickness. But it is well-known that we obtain an exponential decay for the wave inside a PML of finite thickness, this decay depending on the thickness of the layer.Hence for a PML with finite thickness, the property of being perfectly reflectionless is lost, but as the decay of the wave is very fast (in the PML), spurious reflections can be made exponentially small for large enough PML. Furthermore, the solution with the PML converges to the solution in the full space, restricted to the computational domain without PML, when the thickness of the layer tends to infinity (see for instance [40, 41, 8]). As this method is local (because it is a suitable change of variables), finite element methods are quite appropriate to approach the continuous problem.

In the fourth chapter, we analyse a *hp*-FEM and a multiscale method for two dimensional Helmholtz equation with PML. We start by giving the PML setting in polar coordinates (as in [19, 40]). The key ingredient to obtain analysis of a FEM explicit in k is to have the stability estimate explicit in k, this is given by Theorem 4.2.5. The proof is based on the combination of a direct estimate obtained in the PML region with a multiplier method (in the case of absorbing boundary conditions this last procedure corresponds to the choice of an appropriate test function, see [47]). The setting of our equation do not fit with those used in [14], to obtain an expansion in power of k of the solution, so we introduce an another equation with a sponge layer in place of a PML in which we applied the result from [14]. Hence, in Section 4.3 we compared the solution with the PML with those in presence of the sponge layer. The section 4.4 is devoted to the numerical discretization, with a *hp*-FEM and a multi-scale method. The asymptotic range for hp-FEM is obtained in the same way as in [14]. We also propose a pre-asymptotic error estimate, in the spirit of [28], by using an appropriate elliptic projection to get existence of the FEM solution under weaker condition than in the asymptotic range. A multiscale approach is analysed in the same way as in [40, 41, 8]. We propose some numerical tests in section 4.5. The appendix of this chapter lists a couple of elementary but important properties of the PML functions that we often refer to.

The majority of the numerical tests presented in this thesis are performed with the help of XLife++, a FEM library developed in C++ by P.O.E.M.S. (Ensta) and I.R.M.A.R. (Rennes) laboratories.

Let us finish this chapter with some notations used in the remainder of the thesis. For a bounded domain D, the usual norm and semi-norm of $H^t(D)$ $(t \geq 0)$ are denoted by $\|\cdot\|_{t,D}$ and $|\cdot|_{t,D}$, respectively. For $t = 0$, we will drop the index t. For shortness, we further write $\mathbf{H}^t(D) = H^t(D)^3$. Here and below γ_0 is a generic notation for the trace operator from $H^t(\mathcal{O})$ to $H^{t-\frac{1}{2}}(\partial\mathcal{O})$, for all $t > \frac{1}{2}$. The space of smooth functions with compact support in D is denoted by $\mathcal{D}(D)$. Furthermore, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means the existence of a positive constant C_1 (resp. C_2), which is independent of A, B, the wave number k, the parameter s and any mesh size h such that $A \leq C_1 B$ (resp. $A \geq C_2 B$). The notation $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

Chapter 1

General considerations for Maxwell's system

1.1 General setting

We are interested in the time-harmonic Maxwell equations for electromagnetic waves in a bounded, simply connected polyhedral domain Ω of $\mathbb{R}^3$ with a Lipschitz boundary (simply called polyhedron later on) or smooth domain [1] filled by an isotropic homogeneous material with an absorbing boundary condition (also called Leontovich condition) that takes the form

$$\text{(1.1)} \qquad \begin{cases} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega. \end{cases}$$

Here $\mathbf{E}$ is the electric part and $\mathbf{H}$ is the magnetic part of the electromagnetic field, and the constant k corresponds to the wave number or frequency and is, for the moment, supposed to be non-negative. The right hand side $\mathbf{J}$ is the current density which – in the absence of free electric charges – is divergence free, namely

$$\operatorname{div} \mathbf{J} = 0 \quad \text{in } \Omega.$$

As usual, $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ pointing outside Ω and $\mathbf{E}_t = \mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\,\mathbf{n}$ is the tangential component of $\mathbf{E}$. The impedance λ_{imp} is a smooth function [2] defined on $\partial\Omega$ satisfying

$$\text{(1.2)} \qquad \lambda_{\text{imp}} : \partial\Omega \to \mathbb{R}, \quad \text{such that} \quad \forall x \in \partial\Omega,\ \lambda_{\text{imp}}(x) \neq 0,$$

see for instance [59, 55]. The case $\lambda_{\text{imp}} \equiv 1$ is also called the Silver-Müller boundary condition [3].

In practice absorbing boundary conditions are used to reduce an unbounded domain of calculations into a bounded one, see [59, 55].

[1] We mean by smooth domain a domain at least of class $\mathcal{C}^2$

[2] $\lambda_{\text{imp}} \in C^{0,1}(\partial\Omega)$ is sufficient

As variational formulation, we use that which is proposed in [22, §4.5.d] for smooth boundaries and inspired from [59, §5.4.3], is to keep the full electromagnetic field and use the variational space

$$\mathbf{V} = \left\{(\mathbf{E}, \mathbf{H}) \in \left(\mathbf{H}(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)\right)^2 : \ \mathbf{H} \times \mathbf{n} = \lambda_{\mathrm{imp}} \mathbf{E}_t \ \text{ on } \ \partial\Omega\right\}, \tag{1.3}$$

considering the impedance condition in (1.1) as an essential boundary condition. Hence the proposed variational formulation is: Find $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ such that

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = \int_\Omega \left(ik\mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \mathrm{curl}\, \bar{\mathbf{H}}'\right) dx, \ \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}, \tag{1.4}$$

with the choice

$$\begin{aligned} \mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) =& a_{k,s}(\mathbf{E}, \mathbf{E}') + a_{k,s}(\mathbf{H}, \mathbf{H}') \\ & - ik \int_{\partial\Omega} (\lambda_{\mathrm{imp}}\, \mathbf{E}_t \cdot \bar{\mathbf{E}}'_t + \frac{1}{\lambda_{\mathrm{imp}}} \mathbf{H}_t \cdot \bar{\mathbf{H}}'_t)\, d\sigma, \end{aligned}$$

with a positive real parameter s that may depend on k but is assumed to be in a fixed interval $[s_0, s_1]$ with $0 < s_0 \leq s_1 < \infty$ independ of k (see section 1.4 below for more details) and

$$a_{k,s}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathrm{curl}\, \mathbf{u} \cdot \mathrm{curl}\, \bar{\mathbf{v}} + s\, \mathrm{div}\, \mathbf{u}\, \mathrm{div}\, \bar{\mathbf{v}} - k^2 \mathbf{u} \cdot \bar{\mathbf{v}})\, dx.$$

The natural norm $\|\cdot\|_k$ of V associated with problem (1.4) is defined by

$$\begin{aligned} \|(\mathbf{E}, \mathbf{H})\|_k^2 = & \|\mathrm{curl}\, \mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\, \mathbf{E}\|^2_{\mathrm{L}^2(\Omega)} + k^2 \|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} \\ & + \|\mathrm{curl}\, \mathbf{H}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\, \mathbf{H}\|^2_{\mathrm{L}^2(\Omega)} + k^2 \|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

This new formulation (1.4) has the advantage that its associated boundary value problem is an elliptic system (see [22, §4.5.d]), hense standard shift regularity results can be used.

1.2 Hidden regularity of the variational space

If $\partial\Omega$ is of class $\mathcal{C}^2$, it is well known that the continuous embedding

$$\mathbf{V} \hookrightarrow (\mathbf{H}^1(\Omega))^2 \tag{1.5}$$

holds, which means that $\mathbf{V} \subset (\mathbf{H}^1(\Omega))^2$ with the estimate

$$\begin{aligned} \|(\mathbf{E}, \mathbf{H})\|_{\mathbf{H}^1(\Omega)^2} \lesssim & \|\mathrm{curl}\, \mathbf{E}\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\, \mathbf{E}\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \\ & + \|\mathrm{curl}\, \mathbf{H}\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\, \mathbf{H}\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{H}\|_{\mathbf{L}^2(\Omega)}, \ \forall (\mathbf{E}, \mathbf{H}) \in \mathbf{V}. \end{aligned} \tag{1.6}$$

A proof of this result is available in [3] for a smooth boundary and in Lemma 4.5.5 of [22] for a $\mathcal{C}^2$ boundary. In both cases, the three main steps of the proof

are
1. The continuity of the trace operator

$$\mathbf{H}(\mathrm{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div};\partial\Omega) : \mathbf{U} \to \mathbf{U} \times \mathbf{n},$$

proved in [62] (see also [59, Theorem 5.4.2]).
2. The elliptic regularity of the Laplace-Beltrami operator $\Delta_{LB} = \mathrm{div}_t \nabla_t$ on a smooth manifold without boundary that implies that $\Delta_{LB} - I$ is an isomorphism from $H^{\frac{3}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$, see for instance [42].
3. The operator

$$H^2(\Omega) \to L^2(\Omega) \times H^{\frac{3}{2}}(\Gamma) : u \to (-\Delta u, \gamma_0 u),$$

is an isomorphism, see again [42].

If we want to extend this result to polyhedra, we then need to check if the three main points before are available. This is indeed the case, since point 1 can be found in [10], point 2 is proved in [12, Thm 8] under a geometrical assumption (see (1.9) below), while point 3 is a consequence of [25].

To be more precise, let us first introduce the following notations (see [10] or [60, Chap. 2]): as Ω is a polyhedron, its boundary Γ is a finite union of (open and disjoint) faces $\Gamma_j, j = 1, \cdots, N$ such that $\Gamma = \cup_{j=1}^N \bar{\Gamma}_j$. As usual, $\mathbf{n}$ is the unit outward normal vector to Ω and we will set $\mathbf{n}_i = \mathbf{n}_{|\Gamma_i}$ its restriction to Γ_i. When Γ_i and Γ_j are two adjacent faces, we denote by e_{ij} their common (open) edge and by τ_{ij} a unit vector parallel to e_{ij}. By convention, we assume that $\tau_{ij} = \tau_{ji}$. We further set $\mathbf{n}_{ij} = \tau_{ij} \times \mathbf{n}_i$. Note that the pair $(\mathbf{n}_{ij}, \tau_{ij})$ is an orthonormal basis of the plane generated by Γ_i and consequently $\mathbf{n}_{ij}$ is a normal vector to Γ_i along e_{ij}. For shortness, we introduce the set

$$\mathcal{E} = \{(i,j) : i < j \text{ and such that } \bar{\Gamma}_i \cap \bar{\Gamma}_j = \bar{e}_{ij}\}.$$

We denote by $\mathcal{C}$ the set of vertices of Γ (that are the vertices of Ω). Furthermore for any $c \in \mathcal{C}$, we denote by G_c the intersection between the infinite three-dimensional cone Ξ_c that coincides with Ω in a neighbourhood of c and the unit sphere centred at c and by ω_c the length of (in radians) of the boundary of G_c.

We first introduce the set

$$\mathbf{L}_t^2(\Gamma) = \{\mathbf{w} \in \mathbf{L}^2(\Gamma) : \mathbf{w} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

For a function $v \in L^2(\Gamma)$, we denote by v_j its restriction to Γ_j. As Γ is Lipschitz, we can define $H^1(\Gamma)$ via local charts, but we can notice that

$$\begin{aligned} H^1(\Gamma) \;=\; \{u \in L^2(\Gamma) : u_j \in H^1(\Gamma_j), \forall j = 1, \cdots, N \text{ satisfying} \\ \gamma_0 u_i = \gamma_0 u_j \text{ on } e_{ij}, \forall (i,j) \in \mathcal{E}\}. \end{aligned}$$

As Γ is only Lipschitz, we cannot directly define $H^t(\Gamma)$ for $t > 1$, but following [10] (or [12]), we define

$$H^{\frac{3}{2}}(\Gamma) = \{\gamma_0 u : u \in H^2(\Omega)\},$$

with

$$\|w\|_{\frac{3}{2},\Gamma} = \inf_{u\in H^2(\Omega):\gamma_0 u=w} \|u\|_{2,\Omega}.$$

Let us notice that according to Theorem 3.4 of [10], we have

$$H^{\frac{3}{2}}(\Gamma) = \{w \in H^1(\Gamma) : \nabla_t w \in \mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)\},$$

with

$$\|w\|_{\frac{3}{2},\Gamma} \sim \|w\|_{1,\Gamma} + \|\nabla_t w\|_{\|,\frac{1}{2},\Gamma},\ \forall w \in H^{\frac{3}{2}}(\Gamma),$$

where $\nabla_t u$ is the tangential gradient of u and $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)$ is defined by

$$\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma) = \left\{ \begin{array}{c} \mathbf{u} \in \mathbf{L}_t^2(\Gamma) : \mathbf{u}_i \in (H^{\frac{1}{2}}(\Gamma_i))^3,\ \forall i = 1,\cdots,N, \\ \text{and } \mathcal{N}_{ij}^{\|}(\mathbf{u}) < \infty,\ \forall (i,j) \in \mathcal{E} \end{array} \right\}, \tag{1.7}$$

where

$$\mathcal{N}_{ij}^{\|}(\mathbf{u}) = \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{u}_i(x)\cdot\tau_{ij} - \mathbf{u}_j(y)\cdot\tau_{ij}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y),$$

and finally

$$\|\mathbf{u}\|^2_{\|,\frac{1}{2},\Gamma} = \sum_{i=1}^N \|\mathbf{u}_i\|^2_{\frac{1}{2},\Gamma_i} + \sum_{(i,j)\in\mathcal{E}} \mathcal{N}_{ij}^{\|}(\mathbf{u}),\ \forall \mathbf{u} \in \mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma).$$

For further uses, we also introduce

$$\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma) = \left\{ \begin{array}{c} \mathbf{u} \in \mathbf{L}_t^2(\Gamma) : \mathbf{u}_i \in (H^{\frac{1}{2}}(\Gamma_i))^3,\ \forall i = 1,\cdots,N, \\ \text{and } \mathcal{N}_{ij}^{\perp}(\mathbf{u}) < \infty,\ \forall (i,j) \in \mathcal{E} \end{array} \right\}, \tag{1.8}$$

where

$$\mathcal{N}_{ij}^{\perp}(\mathbf{u}) = \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{u}_i(x)\cdot\mathbf{n}_{ij} - \mathbf{u}_j(y)\cdot\mathbf{n}_{ji}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y),$$

and finally

$$\|\mathbf{u}\|^2_{\perp,\frac{1}{2},\Gamma} = \sum_{i=1}^N \|\mathbf{u}_i\|^2_{\frac{1}{2},\Gamma_i} + \sum_{(i,j)\in\mathcal{E}} \mathcal{N}_{ij}^{\perp}(\mathbf{u}),\ \forall \mathbf{u} \in \mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma).$$

Let us also define (cf. [10]) $\mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma)$ as the dual of $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)$ (with pivot space $\mathbf{L}_t^2(\Gamma)$) and introduce the tangential divergence $\operatorname{div}_t : \mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma)$ as the adjoint of $-\nabla_t$, namely

$$\langle \operatorname{div}_t \mathbf{u}, \varphi\rangle_{H^{-\frac{3}{2}}(\Gamma)-H^{\frac{3}{2}}(\Gamma)} = -\langle \mathbf{u}, \nabla_t\varphi\rangle_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma)-\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)},\ \forall \mathbf{u} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma), \varphi \in H^{\frac{3}{2}}(\Gamma).$$

Finally, let us define

$$\mathbf{H}_{\|}^{-1/2}(\operatorname{div};\Gamma) = \{\mathbf{w} \in \mathbf{H}_{\|}^{-1/2}(\Gamma) : \operatorname{div}_t \mathbf{w} \in H^{-1/2}(\Gamma)\},$$

and recall the next result proved in [10, Theorem 3.9]:

Theorem 1.2.1. *[13, Thm 4.1] The trace mapping*

$$\mathbf{H}(\mathrm{curl}, \Omega) \to \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}; \Gamma) : \mathbf{U} \to \mathbf{U} \times \mathbf{n},$$

is linear, continuous, and surjective.

Theorem 1.2.2. *If Ω is a polyhedron satisfying*

$$\omega_c < 4\pi, \ \forall c \in \mathcal{C}, \tag{1.9}$$

then for any $h \in H^{-\frac{1}{2}}(\Gamma)$, there exists a unique $u \in H^{\frac{3}{2}}(\Gamma)$ such that

$$u - \mathrm{div}_t \nabla_t u = h \ \textit{in} \ H^{-\frac{1}{2}}(\Gamma), \tag{1.10}$$

with

$$\|u\|_{\frac{3}{2},\Gamma} \lesssim \|h\|_{-\frac{1}{2},\Gamma}. \tag{1.11}$$

Proof. Fix $h \in H^{-\frac{1}{2}}(\Gamma)$. Then there exists a unique solution $u \in H^1(\Gamma)$ of

$$\int_\Gamma (\nabla_t u \cdot \nabla_t \bar{v} + u\bar{v}) \, d\sigma(x) = \langle h, v\rangle, \ \forall v \in H^1(\Gamma).$$

This solution clearly satisfies (1.10). Furthermore owing to our assumption (1.9), Theorem 8 from [12] (with $t = \frac{1}{2}$, valid since $\frac{2\pi}{\omega_c} > \frac{1}{2}$ for all corners c) guarantees that $u \in H^{\frac{3}{2}}(\Gamma)$ since $h - u$ belongs to $H^{-\frac{1}{2}}(\Gamma)$.

To obtain the estimate (1.11), we take advantage of the closed graph theorem. Indeed introduce the mapping

$$T : \{v \in H^{\frac{3}{2}}(\Gamma) : \mathrm{div}_t \nabla_t v \in H^{-\frac{1}{2}}(\Gamma)\} \to H^{-\frac{1}{2}}(\Gamma) : u \to u - \mathrm{div}_t \nabla_t u,$$

that is well defined and continuous. Since the above arguments show that it is bijective, its inverse is also continuous, which yields

$$\|u\|_{\frac{3}{2},\Gamma} \lesssim \|u - \mathrm{div}_t \nabla_t u\|_{-\frac{1}{2},\Gamma},$$

and is exactly (1.11). □

Remark 1.2.3. Any convex polyhedron satisfies (1.9), since by [69, Problem 1.10.1], one always have $\omega_c < 2\pi$, for all $c \in \mathcal{C}$. But the class of polyhedra satisfying (1.9) is quite larger since the Fichera corner and any prism $D \times I$, where D is any polygon with a Lipschitz boundary and I is an interval satisfy (1.9).

Theorem 1.2.4. *If Ω is a convex polyhedron, then the continuous embedding* (1.5) *remains valid.*

Proof. The proof follows the one of Lemma 4.5.5 of [22] with the necessary adaptation. Let $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$. Let us prove that $\mathbf{E} \in \mathbf{H}^1(\Omega)$. The proof for $\mathbf{H}$ is similar.

By Theorems 2.17 and 3.12 of [1], there exists a vector potential $\mathbf{w} \in \mathbf{H}_T(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega)^3 : \mathbf{w} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}$ such that $\operatorname{div} \mathbf{w} = \mathbf{0}$ and

$$\operatorname{curl} \mathbf{w} = \operatorname{curl} \mathbf{E} \quad \text{in} \quad \Omega,$$

and satisfying

$$\|\mathbf{w}\|_{1,\Omega} \lesssim \|\operatorname{curl} \mathbf{E}\|_{\Omega}. \tag{1.12}$$

Thus, there exists a potential $\varphi \in H^1(\Omega)$ such that

$$\nabla \varphi = \mathbf{E} - \mathbf{w}, \tag{1.13}$$

with (by assuming that $\int_\Omega \varphi \, dx = 0$)

$$\|\varphi\|_{1,\Omega} \lesssim \|\mathbf{E}\|_\Omega + \|\mathbf{w}\|_\Omega \lesssim \|\mathbf{E}\|_{H(\mathrm{curl},\Omega)}.$$

Therefore, as a consequence of $\operatorname{div} \mathbf{E} \in L^2(\Omega)$ we find that

$$\operatorname{div} \nabla \varphi \in L^2(\Omega). \tag{1.14}$$

with

$$\|\operatorname{div} \nabla \varphi\|_\Omega \lesssim \|\operatorname{div} \mathbf{E}\|_\Omega. \tag{1.15}$$

By (1.13) the trace $\mathbf{E}_t$ coincides with $\mathbf{w}_t + \nabla_t \varphi$, i.e.,

$$\mathbf{E}_t = \mathbf{w}_t + \nabla_t \varphi \quad \text{on } \Gamma.$$

As $\mathbf{H}$ belongs to $\mathbf{H}(\mathrm{curl}, \Omega)$, by Theorem 1.2.1 its trace $\mathbf{H} \times \mathbf{n}$ belongs to $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}; \Gamma)$. By the impedance condition $\mathbf{H} \times \mathbf{n} = \lambda_{\mathrm{imp}} \mathbf{E}_t$, we deduce that $\lambda_{\mathrm{imp}} \mathbf{E}_t$ also belongs to $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}; \Gamma)$ with

$$\|\lambda_{\mathrm{imp}} \mathbf{E}_t\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div};\Gamma)} \lesssim \|H\|_{\mathbf{H}(\mathrm{curl},\Omega)}. \tag{1.16}$$

Likewise, as $\mathbf{w} \cdot \mathbf{n} = 0$ and $\mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, let us show that $\mathbf{w}_t$ also belongs $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}; \Gamma)$ with

$$\|\mathbf{w}_t\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div};\Gamma)} \lesssim \|\operatorname{curl} \mathbf{E}\|_\Omega. \tag{1.17}$$

Indeed the above properties imply that

$$\mathbf{w}_t = \mathbf{w} \in \mathbf{H}_{\perp}^{1/2}(\Gamma). \tag{1.18}$$

Namely to show that property we simply need to show that for any $(i, j) \in \mathcal{E}$, one has

$$\int_{\Gamma_i} \int_{\Gamma_j} \frac{|\mathbf{w}_i(x) \cdot \mathbf{n}_{ij} - \mathbf{w}_j(y) \cdot \mathbf{n}_{ji}|^2}{|x-y|^3} \, d\sigma(x) d\sigma(y) \lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \tag{1.19}$$

But for such a pair, $\mathbf{n}_{ij}$ is a linear combination of $\mathbf{n}_i$ and $\mathbf{n}_j$ and consequently

$$\begin{aligned}\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)|\cdot \mathbf{n}_{ij}^2}{|x-y|^3}\,d\sigma(x)d\sigma(y) &\lesssim \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot \mathbf{n}_j|^2}{|x-y|^3}\,d\sigma(x)d\sigma(y)\\ &= \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot \mathbf{n}_j-\mathbf{w}_j(y)\cdot\mathbf{n}_j|^2}{|x-y|^3}\,d\sigma(x)d\sigma(y)\end{aligned}$$

since $\mathbf{w}_i\cdot\mathbf{n}_i=0$ on Γ_i and $\mathbf{w}_j\cdot\mathbf{n}_j=0$ on Γ_j. This shows that

$$\begin{aligned}\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot \mathbf{n}_{ij}|^2}{|x-y|^3}\,d\sigma(x)d\sigma(y) &\lesssim \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)-\mathbf{w}_j(y)|^2}{|x-y|^3}\,d\sigma(x)d\sigma(y)\\ &\lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac12}(\Gamma)},\end{aligned}$$

as well as (by exchanging the role of Γ_i and Γ_j)

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_j\cdot \mathbf{n}_{ji}(y)|^2}{|x-y|^3}\,d\sigma(x)d\sigma(y) \lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac12}(\Gamma)}.$$

Hence (1.19) holds. As mentioned in [11, p. 39], Theorem 1.2.1, a density argument and a duality argument lead to the continuity of div_t from $\mathbf{H}_\perp^{\frac12}(\Gamma)$ to $H^{-\frac12}(\Gamma)$, and by (1.18) we deduce that

$$\operatorname{div}_t \mathbf{w}_t = \operatorname{div}_t \mathbf{w} \in H^{-\frac12}(\Gamma).$$

Altogether we finally obtain that $\lambda_{\mathrm{imp}}\nabla_t\varphi$ belongs to $\mathbf{H}_\parallel^{-1/2}(\operatorname{div};\Gamma)$ and since λ_{imp} is smooth and never 0 on Γ, we conclude that

$$\operatorname{div}_t \nabla_t\varphi \in H^{-\frac12}(\Gamma),$$

and since φ is in $H^{-\frac12}(\Gamma)$,

$$\varphi - \operatorname{div}_t \nabla_t\varphi \in H^{-\frac12}(\Gamma).$$

with

$$\|\varphi - \operatorname{div}_t \nabla_t\varphi\|_{-\frac12,\Gamma} \lesssim \|H\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)}. \tag{1.20}$$

By Theorem 1.2.2, we deduce that

$$\varphi_{|\Gamma} \in H^{\frac32}(\Gamma), \tag{1.21}$$

with

$$\|\varphi\|_{\frac32,\Gamma} \lesssim \|H\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)}. \tag{1.22}$$

Now, using the elliptic regularity for φ solution of the Dirichlet problem (1.14)-(1.21) in Ω (see [25, Corollary 18.19]), we find $\varphi \in H^2(\Omega)$ with

$$\begin{aligned}\|\varphi\|_{2,\Omega} &\lesssim \|\operatorname{div}\nabla\varphi\|_\Omega + \|\varphi\|_{\frac32,\Gamma}\\ &\lesssim \|\mathbf{H}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\operatorname{div}\mathbf{E}\|_\Omega.\end{aligned} \tag{1.23}$$

Coming back to (1.13), we have obtained that $\mathbf{E} \in \mathbf{H}^1(\Omega)$ with

$$\|\mathbf{E}\|_{1,\Omega} \leq \|\mathbf{w}\|_{1,\Omega} + \|\nabla\varphi\|_{1,\Omega}.$$

Hence taking into account (1.12) and (1.23) we arrive at the estimate

$$\|\mathbf{E}\|_{1,\Omega} \lesssim \|\mathbf{H}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\operatorname{div}\mathbf{E}\|_{\Omega}.$$

As said before, exchanging the role of $\mathbf{E}$ and $\mathbf{H}$ we can show that $\mathbf{H} \in \mathbf{H}^1(\Omega)$ with

$$\|\mathbf{H}\|_{1,\Omega} \lesssim \|\mathbf{H}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\operatorname{div}\mathbf{H}\|_{\Omega}.$$

The proof is then completed. $\square$

It turns out that the convexity condition is a necessary and sufficient condition that guarantees the continuous embedding (1.5), namely we have the

Corollary 1.2.5. *If Ω is a polyhedron. Then Ω is convex if and only if the continuous embedding* (1.5) *is valid.*

Proof. It suffices to prove that the convexity condition is a necessary condition. For that purpose, we use a contradiction argument. Assume that Ω is not convex, then by [25] (see also [21, §1]), there exists a (singular) function $\varphi \in H_0^1(\Omega)\setminus H^2(\Omega)$ such that

$$\Delta\varphi \in L^2(\Omega).$$

In that way the pair $(\nabla\varphi, \nabla\varphi)$ belongs to $\mathbf{V}$, but that cannot be in $\mathbf{H}^1(\Omega)^2$ since $\varphi \notin H^2(\Omega)$. This proves that (1.5) is not valid. $\square$

1.3 Well Posedness

Let us start with a coerciveness result for the sesquilinear form $a_{k,s}$.

Theorem 1.3.1. *If Ω is a convex polyhedron or a smooth domain, then the sesquilinear form $\mathbf{a}_{k,s}(\cdot,\cdot)$ is weakly coercive on $\mathbf{V}$, in the sense that there exists $c > 0$ independent of k and s such that for all $(\mathbf{E},\mathbf{H}) \in \mathbf{V}$*

$$\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{E},\mathbf{H}),(\mathbf{E},\mathbf{H})) \geq c\left(\|\mathbf{E}\|_{1,\Omega}^2 + \|\mathbf{H}\|_{1,\Omega}^2\right) - (k^2+1)\left(\|\mathbf{E}\|_{\Omega}^2 + \|\mathbf{H}\|_{\Omega}^2\right). \tag{1.24}$$

Proof. For the smooth case, the domain is at least $\mathcal{C}^2$, hence the weak coercivity is proven in Theorem 4.5.6 of [22]. In the same spirit of the smooth case, polyhedral case is a direct consequence of Theorem 1.2.4, recalling our assumption on λ_{imp} to be real valued. $\square$

Remark 1.3.2. *Under the assumptions of the previous Theorem, for $k \geq 1$, we have*

$$\|(\mathbf{E},\mathbf{H})\|_k \gtrsim \|(\mathbf{E},\mathbf{H})\|_{\mathbf{H}^1(\Omega)^2}.$$

The existence of a weak solution to (1.4) for $k > 0$ directly follows from this coerciveness and the next uniqueness result for problem (1.1).

Lemma 1.3.3. *Let* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ *be a solution of*

$$
\left\{\begin{array}{l}
\operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \\
\mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega,
\end{array}\right. \tag{1.25}
$$

for $k \in \mathbb{R}$ *and* Ω *a convex polyhedron or a smooth domain. Moreover if* $k = 0$, *we suppose that* Ω *is simply connected. Assume that* $\mathbf{E}$ *and* $\mathbf{H}$ *are divergence free. Then* $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$.

Proof. By Green's formula (see [32, Thm I.2.11]) we have

$$
\begin{aligned}
\int_\Omega (|\operatorname{curl} \mathbf{E}|^2 + |\operatorname{curl} \mathbf{H}|^2)\, dx &= ik \int_\Omega (\operatorname{curl} \mathbf{H} \cdot \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx \\
&= ik \int_\Omega (\mathbf{H} \cdot \operatorname{curl} \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx \\
&\quad -ik \int_{\partial\Omega} (\mathbf{H} \times \mathbf{n} \cdot \bar{\mathbf{E}})\, d\sigma(x).
\end{aligned}
$$

Hence using the impendance boundary condition in (1.25), we find that

$$
\begin{aligned}
\int_\Omega (|\operatorname{curl} \mathbf{E}|^2 + |\operatorname{curl} \mathbf{H}|^2)\, dx = ik &\int_\Omega (\mathbf{H} \cdot \operatorname{curl} \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx \\
- ik &\int_{\partial\Omega} \lambda_{\text{imp}} |\mathbf{E}_t|^2\, d\sigma(x).
\end{aligned}
$$

Taking the imaginary part of this identity we find that

$$
k \int_{\partial\Omega} \lambda_{\text{imp}} |\mathbf{E}_t|^2\, d\sigma(x) = 0.
$$

Hence if $k \neq 0$, we deduce that

$$
\mathbf{E}_t = \mathbf{0} \text{ on } \partial\Omega,
$$

as λ_{imp} is different from 0 on $\partial\Omega$. Again by the impendance boundary condition, $\mathbf{H}$ also satisfies

$$
\mathbf{H} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega.
$$

This means that we can extend $\mathbf{E}$ and $\mathbf{H}$ by zero outside Ω and that these extensions belong to $H(\text{curl}, \mathbb{R}^3)$. Owing to Theorem 4.13 of [55] we conclude that $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$.

For $k = 0$, we notice that (1.25) implies that $\mathbf{E}$ and $\mathbf{H}$ are curl free, hence as Ω is supposed to be simply connected, by Theorem I.2.6 of [32], there exist $\Phi_E, \Phi_H \in H^1(\Omega)$ such that

$$
\mathbf{E} = \nabla \Phi_E, \mathbf{H} = \nabla \Phi_H.
$$

Due to the H^1 regularity of $\mathbf{E}$ and $\mathbf{H}$, Φ_E and Φ_H both belong to $H^2(\Omega)$. Now using the impendance boundary condition, we have

$$
\operatorname{div}_t(\lambda_{\text{imp}} \nabla_t \Phi_E) = \operatorname{div}_t(\nabla \Phi_H \times \mathbf{n}) \text{ on } \partial\Omega,
$$

and by the standard property

$$\operatorname{div}_t(\mathbf{v}\times\mathbf{n}) = \operatorname{curl}\mathbf{v}\cdot\mathbf{n},$$

valid for all $\mathbf{v}\in H(\operatorname{curl},\Omega)$ (see [10, p.23]), we deduce that

$$\operatorname{div}_t(\lambda_{\text{imp}}\nabla_t\Phi_E) = 0 \text{ on } \partial\Omega.$$

By its definition (see [10, Def 3.3]), this property implies that

$$\int_{\partial\Omega} |\lambda_{\text{imp}}\nabla_t\Phi_E|^2\, d\sigma(x) = 0.$$

Consequently Φ_E is constant on the whole boundary. As $\mathbf{E}$ is divergence free, Φ_E is harmonic in Ω and consequently it is constant on the whole Ω, which guarantees that $\mathbf{E}=\mathbf{0}$. With this property and recalling the impedance boundary condition, we deduce that $\nabla_t\Phi_H = \mathbf{0}$ on the whole boundary. As $\mathbf{H}$ is also divergence free, Φ_H is harmonic in Ω and we conclude that $\mathbf{H}=\mathbf{0}$. □

Our next goal is to prove an existence and uniqueness result to problem (1.4), that can be formulated in the more general form

$$\mathbf{a}_{k,s}((\mathbf{E},\mathbf{H});(\mathbf{E}',\mathbf{H}')) = \langle \mathbf{F};(\mathbf{E}',\mathbf{H}')\rangle,\ \forall(\mathbf{E}',\mathbf{H}')\in\mathbf{V}, \tag{1.26}$$

with $\mathbf{F}\in\mathbf{V}'$. First, we need to show extra regularities of the divergence of any solution $(\mathbf{E},\mathbf{H})$ of this problem under the assumption that $\mathbf{F}$ belongs to $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ in the sense that

$$\langle \mathbf{F};(\mathbf{E}',\mathbf{H}')\rangle = \int_\Omega \left(\mathbf{f}_1\cdot\bar{\mathbf{E}}' + \mathbf{f}_2\cdot\bar{\mathbf{H}}'\right) dx, \tag{1.27}$$

with $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{L}^2(\Omega)$.

Lemma 1.3.4. *If Ω is a convex polyhedron or a smooth domain, the impedance function λ_{imp} satisfies* (1.2) *and $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω, then for all $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{L}^2(\Omega)$, any solution $(\mathbf{E},\mathbf{H})\in\mathbf{V}$ to the problem*

$$\mathbf{a}_{k,s}((\mathbf{E},\mathbf{H});(\mathbf{E}',\mathbf{H}')) = \int_\Omega \left(\mathbf{f}_1\cdot\bar{\mathbf{E}}' + \mathbf{f}_2\cdot\bar{\mathbf{H}}'\right) dx,\ \forall(\mathbf{E}',\mathbf{H}')\in\mathbf{V}, \tag{1.28}$$

satisfies

$$\operatorname{div}\mathbf{E},\operatorname{div}\mathbf{H}\in H_0^1(\Omega),$$

with

$$\operatorname{div}\mathbf{E} = -(s\Delta+k^2)^{-1}\operatorname{div}\mathbf{f}_1,\quad \operatorname{div}\mathbf{H} = -(s\Delta+k^2)^{-1}\operatorname{div}\mathbf{f}_2.$$

Proof. For a convex polyhedron, we basically follow the proof of Lemma 4.5.8 of [22] (which corresponds to the smooth case) with a slight adaptation due to the change of right-hand side in (1.28) with respect to [22]. In (1.28) we first take test functions in the form $(\nabla\varphi, \mathbf{0})$ with an arbitrary $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. This directly implies that $(\nabla\varphi, \mathbf{0})$ belongs to $\mathbf{V}$, and therefore we get

$$s \int_\Omega \operatorname{div} \mathbf{E}\ \operatorname{div} \nabla\bar{\varphi}\, dx - k^2 \int_\Omega \mathbf{E} \cdot \nabla\bar{\varphi}\, dx = \int_\Omega \mathbf{f}_1 \cdot \nabla\bar{\varphi}\, dx.$$

Consequently, one deduces that

$$\int_\Omega \operatorname{div} \mathbf{E}\ (s\Delta + k^2)\varphi\, dx = -\langle \operatorname{div} \mathbf{f}_1; \varphi\rangle,\ \forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.29}$$

On the other hand, as $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in $H^2(\Omega)$, there exists a unique solution $q \in H_0^1(\Omega)$ to

$$(s\Delta + k^2)q = -\operatorname{div} \mathbf{f}_1.$$

Taking the duality with $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, after an integration by parts, we obtain equivalently that

$$\int_\Omega q\ (s\Delta + k^2)\varphi\, dx = -\langle \operatorname{div} \mathbf{f}_1; \varphi\rangle,\ \forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega).$$

Comparing this identity with (1.29), we find that

$$\int_\Omega (\operatorname{div} \mathbf{E} - q)\ (s\Delta + k^2)\varphi\, dx = 0,\ \forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega),$$

and since the range of $(s\Delta + \omega^2)$ is the whole $L^2(\Omega)$, one gets that $\operatorname{div} \mathbf{E} = q$, as announced.

The result for $\mathbf{H}$ follows in the same way by choosing test functions in the form $(\mathbf{0}, \nabla\bar{\varphi})$. □

We are now ready to prove an existence and uniqueness result to (1.26).

Theorem 1.3.5. *If Ω is a convex polyhedron or a smooth domain, the impedance function λ_{imp} satisfies* (1.2) *and $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω, then for any $\mathbf{F} \in \mathbf{V}'$, the problem* (1.26) *has a unique solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$.

Proof. We associate to problem (1.26) the continuous operator $A_{k,s}$ from $\mathbf{V}$ into its dual by

$$(A_{k,s}\mathbf{u})(\mathbf{v}) = \mathbf{a}_{k,s}(\mathbf{u}, \mathbf{v}),\ \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}.$$

Now according to Theorem 1.3.1, the sesquilinear form

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}, \mathbf{H})) + (k^2 + 1) \left(\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)} \right),$$

is strongly coercive in $\mathbf{V}$ and by Lax-Milgram lemma, the operator $A_{k,s} + (k^2+1)\mathbb{I}$ is an isomorphism $\mathbf{V}$ into its dual. As $\mathbf{V}$ is compactly embedded into $\mathbf{L}^2(\Omega)^6$, the operator $A_{k,s}$ is a Fredholm operator of index zero. Hence uniqueness implies existence and uniqueness.

So let us fix $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ be a solution of (1.26) with $\mathbf{F} = \mathbf{0}$. Then by Lemma 4.5.9 of [22] (valid due to Lemma 1.3.3 for the polyhedral case), we find that $(\mathbf{E}, \mathbf{H})$ is solution of the original problem (1.1) with $\mathbf{J} = \mathbf{0}$, namely (1.25). We further notice that Lemma 1.3.4 guarantees that $\mathbf{E}$ and $\mathbf{H}$ are divergence free (only useful for $k = 0$). As Lemma 1.3.3 yields that $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$, we conclude an existence and uniqueness result. $\square$

As already mentioned, for the particular choice

$$\langle \mathbf{F}; (\mathbf{E}', \mathbf{H}') \rangle = \int_\Omega \left(i\omega \mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \operatorname{curl} \bar{\mathbf{H}}' \right) dx,$$

with $\mathbf{J} \in \mathbf{L}^2(\Omega)$, problem (1.26) reduces to (1.4). Hence under the assumptions of Theorem 1.3.5 and if $\mathbf{J} \in \mathbf{L}^2(\Omega)$, this last problem has a unique solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$, that owing to Lemma 4.5.9 of [22] is moreover solution of the original problem (1.1) under the additional assumption that $\mathbf{J} \in \mathbf{H}(\operatorname{div}; \Omega)$.

Now under the assumptions of Theorem 1.3.5, given two functions $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, we denote by $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, the unique solution of (1.26) with $\mathbf{F}$ given by (1.27) or equivalently solution of (1.28). Note that the general considerations from [22, §4.5.d] implies that $(\mathbf{E}, \mathbf{H})$ is actually the solution of the boundary value elliptic system

$$\left\{ \begin{array}{ll} \left. \begin{array}{ll} L_{k,s}(\mathbf{E}) & = \mathbf{f}_1 \\ L_{k,s}(\mathbf{H}) & = \mathbf{f}_2 \end{array} \right\} & \text{in } \Omega, \\ \left. \begin{array}{ll} \operatorname{div} \mathbf{E} & = 0 \\ \operatorname{div} \mathbf{H} & = 0 \\ T(\mathbf{E}, \mathbf{H}) & = 0 \\ B_k(\mathbf{E}, \mathbf{H}) & = 0 \end{array} \right\} & \text{on } \partial\Omega, \end{array} \right. \tag{1.30}$$

where

$$\begin{aligned} L_{k,s}(\mathbf{E}) &= \operatorname{curl} \operatorname{curl} \mathbf{E} - s \nabla \operatorname{div} \mathbf{E} - k^2 \mathbf{E}, \\ T(\mathbf{E}, \mathbf{H}) &= \mathbf{H} \times \mathbf{n} - \lambda_{\mathrm{imp}} \mathbf{E}_t, \\ B_k(\mathbf{E}, \mathbf{H}) &= (\operatorname{curl} \mathbf{H}) \times \mathbf{n} + \frac{1}{\lambda_{\mathrm{imp}}} (\operatorname{curl} \mathbf{E})_t - \frac{ik}{\lambda_{\mathrm{imp}}} \mathbf{H}_t + ik \mathbf{E} \times \mathbf{n}. \end{aligned}$$

Remark 1.3.6. As suggested by its definition, under the assumptions of Theorem 1.3.5, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ depends on s, but if the data $\mathbf{f}_1$ and $\mathbf{f}_2$ are divergence free, then as Lemma 1.3.4 guarantees that each component of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ is divergence free, we deduce that

$$\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) = \mathbb{S}_{k,s'}(\mathbf{f}_1, \mathbf{f}_2),$$

for all $s' > 0$ such that $-k^2/s'$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω. In other words, in that case $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ does not depend on s and hence the parameter s can be chosen independent of k. This is of particular interest for practical applications (see problem (1.4)), since the data $\mathbf{f}_1$ and $\mathbf{f}_2$ are divergence free. The interest of considering non divergence free right-hand side will appear in the error analysis of our numerical schemes, see Remark 2.2.6.

Let us end up this section with an extra regularity result of the curl of each component of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ if $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ are divergence free.

Lemma 1.3.7. *Under the assumptions of Theorem 1.3.5, let* $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, *with* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ *such that*

$$\operatorname{div} \mathbf{f}_1 = \operatorname{div} \mathbf{f}_1 = 0.$$

Then $(\mathbf{U}, \mathbf{W}) = (\operatorname{curl} \mathbf{E} - ik\mathbf{H}, \operatorname{curl} \mathbf{H} + ik\mathbf{E})$ *belongs to* $\mathbf{V}$ *and satisfies the Maxwell system*

$$\operatorname{curl} \mathbf{U} + ik\mathbf{W} = \mathbf{f}_1 \quad \text{and} \quad \operatorname{curl} \mathbf{W} - ik\mathbf{U} = \mathbf{f}_2 \quad \text{in } \Omega. \tag{1.31}$$

Proof. According to Lemma 1.3.4, $\mathbf{E}$ and $\mathbf{H}$ are divergence free, hence $\mathbf{U}$ and $\mathbf{W}$ as well. Hence the identities (1.31) directly follows from the two first identities of (1.30). This directly furnishes the regularities

$$\operatorname{curl} \mathbf{U}, \operatorname{curl} \mathbf{W} \in \mathbf{L}^2(\Omega).$$

Finally the boundary conditions

$$\mathbf{W} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{U}_t = \mathbf{0} \quad \text{on } \partial\Omega,$$

directly follows from the last boundary conditions in (1.30). □

1.4 Wavenumber explicit stability analysis

The basic block for a wavenumber explicit error analysis of problem (1.30) (or (1.28)) is a so-called stability estimate at the energy level; for the Helmholtz equation, see [23, 27, 35]. Hence we make the following definition.

Definition 1.4.1. *We will say that system* (1.30) *satisfies the k-stability property with exponent $\alpha \geq 0$ (independent of k and s) if there exists $k_0 > 0$ such that for all $k \geq k_0$ and all* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, *the solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ *of* (1.28) *satisfies*

$$\|(\mathbf{E}, \mathbf{H})\|_k \lesssim k^\alpha (\|\mathbf{f}_1\|_{0,\Omega} + \|\mathbf{f}_2\|_{0,\Omega}). \tag{1.32}$$

Before going on, let us show that this property is valid for bounded domains with $\alpha = 2$. But for some domains, in particular it holds with $\alpha = 1$, it will be valid for rectangular cuboids of rational lengths, some tetrahedra and some prisms. To prove such a result, we first start with a similar property with divergence free data. In this case, our proof is a simple consequence of a result obtained in [61] for the time-dependent Maxwell system with impedance boundary conditions combined with the next result of functional analysis [65, 38].

Lemma 1.4.2. *A C_0 semigroup $(e^{t\mathcal{L}})_{t \geq 0}$ of contractions on a Hilbert space H is exponentially stable, i.e., satisfies*

$$\|e^{t\mathcal{L}} U_0\| \leq C\, e^{-\omega t} \|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{1.33}$$

and

$$\sup_{\beta\in\mathbb{R}} \|(i\beta\mathbb{I} - \mathcal{L})^{-1}\| < \infty, \tag{1.34}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Theorem 1.4.3. *In addition to the assumptions of Theorem 1.3.5, assume that Ω is star-shaped with respect to a point. Then for all $k \geq 0$ and all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \mathbf{f}_1 = \operatorname{div} \mathbf{f}_2 = 0$, the solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ of (1.28) satisfies (1.32) with $\alpha = 1$.*

Proof. As the data are divergence free, by Lemma 1.3.7, the auxiliary unknown $(\mathbf{U}, \mathbf{W}) = (\operatorname{curl} \mathbf{E} - ik\mathbf{H}, \operatorname{curl} \mathbf{H} + ik\mathbf{E})$ belongs to $\mathbf{V}$, is divergence free and satisfies the Maxwell system (1.31).

Now we notice that Theorem 4.1 of [61] (valid for star-shaped domain with a Lipschitz boundary) shows that the time-dependent Maxwell system

$$\begin{cases} \partial_t \mathbf{E} + \operatorname{curl} \mathbf{H} = \mathbf{0} \quad \text{and} \quad \partial_t \mathbf{H} - \operatorname{curl} \mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \, \mathbf{E_t} = \mathbf{0} \quad \text{on } \Xi, \end{cases}$$

is exponentially stable in $\mathcal{H} = \{(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0\}$. This equivalently means that the operator $\mathcal{L}$ defined by

$$\mathcal{L}(\mathbf{E}, \mathbf{H}) = (-\operatorname{curl} \mathbf{H}, \operatorname{curl} \mathbf{E}), \forall (\mathbf{E}, \mathbf{H}) \in D(\mathcal{L}),$$

with domain

$$D(\mathcal{L}) = \{(\mathbf{E}, \mathbf{H}) \in \mathbf{V} : \operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0\},$$

generates an exponentially stable C_0 semigroup in $\mathcal{H}$. Hence by Lemma 1.4.2, we deduce that its resolvent is bounded on the imaginary axis. This precisely implies that

$$\|\mathbf{U}\|_\Omega + \|\mathbf{W}\|_\Omega \lesssim \|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega, \tag{1.35}$$

for all $k \geq 0$. But coming back to the definition of $\mathbf{U}$ and $\mathbf{W}$, we can look at $(\mathbf{E}, \mathbf{H})$ as a solution in $D(\mathcal{L})$ of the Maxwell system

$$\operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{U}, \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{W}.$$

Hence the previous arguments show that

$$\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega \lesssim \|\mathbf{U}\|_\Omega + \|\mathbf{W}\|_\Omega.$$

By the estimate (1.35), we deduce that

$$\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega \lesssim \|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega. \tag{1.36}$$

Finally as

$$\|(\mathbf{E},\mathbf{H})\|_k \sim \|\operatorname{curl}\mathbf{E}\|_\Omega + \|\operatorname{curl}\mathbf{H}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega),$$

by the triangular inequality, we get that

$$\begin{aligned}\|(\mathbf{E},\mathbf{H})\|_k &\lesssim \|\operatorname{curl}\mathbf{E} - ik\mathbf{H}\|_\Omega + \|\operatorname{curl}\mathbf{H} + ik\mathbf{E}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega)\\ &\lesssim \|\mathbf{U}\|_\Omega + \|\mathbf{V}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega).\end{aligned}$$

By the estimates (1.35) and (1.36), we conclude that

$$\|(\mathbf{E},\mathbf{H})\|_k \lesssim k(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega),$$

as announced. □

Now we leave out the divergence free constraint on the data. Before let us denote by $\{\lambda_n\}_{n\in\mathbb{N}^*}$, the set of eigenvalues enumerated in increasing order (and not repeated according to their multiplicity) of the positive Laplace operator $-\Delta$ with Dirichlet boundary conditions in Ω. For each $n \in \mathbb{N}^*$, we also denote by $\varphi_{n,\ell}, \ell = 1,\cdots,m(n)$, the orthonormal eigenvectors associated with λ_n. For all $k > 0$ and each $s \in [1,2]$, let us define the unique integer $n(k,s)$ such that

$$\lambda_{n(k,s)} \le \frac{k^2}{s} < \lambda_{n(k,s)+1}, \tag{1.37}$$

and denote by

$$g_{n(k,s)} = \lambda_{n(k,s)+1} - \lambda_{n(k,s)},$$

the gap between these consecutive eigenvalues. Now we show that if $g_{n(k,s)}$ satifies some uniform lower bound, then the k-stability property holds.

Lemma 1.4.4. *In addition to the assumptions of Theorem 1.4.3, assume that there exists a non negative real number β and two positive real number γ_0 and k_1 such that*

$$\forall k \ge k_1\ \exists s \in [1,2] : g_{n(k,s)} \ge \gamma_0 k^{-2\beta}. \tag{1.38}$$

Then there exist two positive real numbers s_0, s_1 such that $s_0 < s_1$ (depending on β, γ_0 and k_1) and for an appropriate choice of $s \in [s_0, s_1]$ (but such that $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω), the k-stability property with exponent $\alpha = 2\beta + 1$ holds.

Proof. The first step is to reduce the problem to divergence free right-hand sides. For that purpose, for $i = 1$ or 2, we consider $u_i, \varphi_i \in H_0^1(\Omega)$ variational solutions of

$$\begin{aligned}\Delta u_i &= \operatorname{div}\mathbf{f}_i \text{ in } \Omega,\\ (\Delta\varphi_i + \frac{k^2}{s}\varphi_i) &= -s^{-1}u_i \text{ in } \Omega.\end{aligned}$$

Then simple calculations show that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = (\mathbf{E} - \nabla\varphi_1, \mathbf{H} - \nabla\varphi_2)$ belongs to $\mathbf{V}$ and is solution of (1.30) with divergence free right-hand side, namely

$$(1.39)\qquad \begin{cases} \left.\begin{aligned} L_{k,s}(\tilde{\mathbf{E}}) &= \tilde{\mathbf{f}}_1 = \mathbf{f}_1 - \nabla u_1, \\ L_{k,s}(\tilde{\mathbf{H}}) &= \tilde{\mathbf{f}}_2 = \mathbf{f}_2 - \nabla u_2, \end{aligned}\right\} \text{ in } \Omega, \\ \left.\begin{aligned} \operatorname{div} \tilde{\mathbf{E}} &= 0 \\ \operatorname{div} \tilde{\mathbf{H}} &= 0 \\ T(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) &= 0 \\ B(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) &= 0 \end{aligned}\right\} \text{ on } \partial\Omega, \end{cases}$$

In a first step we estimate the H^1-norm of φ_i. Since we assume that $\frac{k^2}{s}$ does not encounter the spectrum of the Laplace operator, by the spectral theorem, we can write

$$\varphi_i = -s^{-1} \sum_{n\in\mathbb{N}^*} (\frac{k^2}{s} - \lambda_n)^{-1} \sum_{\ell=1}^{m(n)} (u_i, \varphi_{n,\ell})_\Omega \varphi_{n,\ell}.$$

Consequently, we have

$$(1.40)\qquad \|\varphi_i\|^2_{1,\Omega} \sim s^{-2} \sum_{n\in\mathbb{N}^*} (\frac{k^2}{s} - \lambda_n)^{-2} \sum_{\ell=1}^{m(n)} |(u_i, \varphi_{n,\ell})_\Omega|^2 \lambda_n.$$

Hence our goal is to chosse s in an interval $[s_0, s_1]$ with s_0 and s_1 independent of k satisfying $0 < s_0 \leq s_1 < \infty$ and such that

$$(1.41)\qquad \left|\frac{k^2}{s} - \lambda_n\right| \gtrsim k^{-2\beta},\ \forall n \in \mathbb{N}^*,\ k \geq k_0,$$

with k_0 large enough. Indeed if this estimate is valid, then (1.40) can be transformed into

$$\|\varphi_i\|^2_{1,\Omega} \lesssim k^{4\beta} \sum_{n\in\mathbb{N}^*} \sum_{\ell=1}^{m(n)} |(u_i, \varphi_{n,\ell})_\Omega|^2 \lambda_n.$$

and therefore

$$\|\varphi_i\|_{1,\Omega} \lesssim k^{2\beta} \|u_i\|_{1,\Omega}.$$

As clearly

$$(1.42)\qquad \|u_i\|_{1,\Omega} \lesssim \|\mathbf{f}_i\|_\Omega,$$

we conclude that

$$(1.43)\qquad \|\varphi_i\|_{1,\Omega} \lesssim k^{2\beta} \|\mathbf{f}_i\|_\Omega.$$

As

$$(1.44)\qquad \|(\nabla\varphi_1, \nabla\varphi_2)\|_k \sim \sqrt{s}(\|\Delta\varphi_1\|_\Omega + \|\Delta\varphi_2\|_\Omega)) + k(\|\varphi_1\|_{1,\Omega} + \|\varphi_2\|_{1,\Omega}),$$

we need to estimate the L^2-norm of $\Delta\varphi_1$. But from its definition, we have

$$\Delta\varphi_i + \frac{k^2}{s}\varphi_i = -s^{-1}u_i,$$

and taking the L^2-inner product with φ_i, we get

$$(\Delta\varphi_i, \varphi_i)_\Omega + \frac{k^2}{s}\|\varphi_i\|_\Omega^2 = -s^{-1}(u_i, \varphi_i)_\Omega.$$

Using Cauchy-Schwarz's inequality, we get

$$\frac{k^2}{s}\|\varphi_i\|_\Omega^2 \le s^{-1}\|u_i\|_\Omega\|\varphi_i\|_\Omega + |\varphi_i|_{1,\Omega}^2.$$

With the help of (1.42) and (1.43), we obtain

$$k^2\|\varphi_i\|_\Omega^2 \lesssim \|\mathbf{f}_i\|_\Omega\|\varphi_i\|_\Omega + k^{4\beta}\|\mathbf{f}_i\|_\Omega^2.$$

Hence by Young's inequality, we get

$$k^2\|\varphi_i\|_\Omega^2 \lesssim k^{4\beta}\|\mathbf{f}_i\|_\Omega^2,$$

which proves that

$$\|\varphi_i\|_\Omega \lesssim k^{2\beta-1}\|\mathbf{f}_i\|_\Omega. \tag{1.45}$$

This directly implies that

$$\|\Delta\varphi_i\|_\Omega \le \frac{k^2}{s}\|\varphi_i\|_\Omega + s^{-1}\|u_i\|_\Omega \lesssim k^{2\beta+1}\|\mathbf{f}_i\|_\Omega.$$

Using this estimate and (1.43) in (1.44) leads to

$$\|(\nabla\varphi_1, \nabla\varphi_2)\|_k \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega). \tag{1.46}$$

At this stage, we use Theorem 1.4.3 that yields

$$\|(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})\|_k \lesssim k(\|\tilde{\mathbf{f}}_1\|_\Omega + \|\tilde{\mathbf{f}}_2\|_\Omega).$$

Hence by the definition of $\tilde{\mathbf{f}}_i$ and (1.42), we deduce that

$$\|(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})\|_k \lesssim k(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega).$$

As $(\mathbf{E}, \mathbf{H}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) + (\nabla\varphi_1, \nabla\varphi_2)$, the combination of this last estimate with (1.46) leads to

$$\|(\mathbf{E}, \mathbf{H})\|_k \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega), \tag{1.47}$$

which proves the stability estimate with $\alpha = 2\beta + 1$.

It remains to prove that (1.41) holds for an appropriate choice of s. This is done with the help of our assumption (1.38), by an eventual slight modification of s from this assumption. To be more precise, for all $k \ge k_1$, we fix one $s \in [1, 2]$

such that (1.38) holds and denote it by $s(k)$. We now distinguish between three cases.
a) If $\lambda_{n(k,s(k))} \leq \frac{k^2}{s(k)} \leq \lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}$, then we fix s such that

$$\frac{k^2}{s} = \lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}. \tag{1.48}$$

With this choice, we clearly have

$$\frac{k^2}{s} - \lambda_{n(k,s(k))} = \frac{\gamma_0}{3k^{2\beta}},$$

while

$$\lambda_{n(k,s(k))+1} - \frac{k^2}{s} = \lambda_{n(k,s(k))+1} - \lambda_{n(k,s(k))} - \frac{\gamma_0}{3k^{2\beta}} \geq \frac{2\gamma_0}{3k^{2\beta}},$$

which proves that (1.41) holds. Let us now show that s remains in a (uniformly) bounded interval. Indeed (1.48) is equivalent to

$$s = \frac{k^2}{\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}}.$$

As by assumption $k^2 \leq s(k)\left(\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}\right)$, we directly deduce that $s \leq s(k) \leq 2$. Conversely, from (1.37), we deduce that

$$\begin{aligned} \frac{k^2}{\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}} &\geq \frac{k^2}{\frac{k^2}{s(k)} + \frac{\gamma_0}{3k^{2\beta}}} \\ &\geq \frac{s(k)}{1 + \frac{\gamma_0 s(k)}{3k^{2(\beta+1)}}} \\ &\geq \frac{1}{1 + \frac{2\gamma_0}{3k_1^{2(\beta+1)}}}. \end{aligned}$$

b) If $\lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}} \leq \frac{k^2}{s(k)} \leq \lambda_{n(k,s(k))+1}$, then we fix s such that

$$\frac{k^2}{s} = \lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}.$$

We check exactly as in the first case that (1.41) holds. Furthermore, by assumption $s \geq 1$, while for the lower bound we see that

$$s = \frac{k^2}{\lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}} \leq \frac{k^2}{\frac{k^2}{s(k)} - \frac{\gamma_0}{3k^{2\beta}}} \leq \frac{s(k)}{1 - \frac{s(k)\gamma_0}{3k^{2\beta}}} \leq \frac{2}{1 - \frac{2\gamma_0}{3k^{2\beta}}}.$$

Hence $s \leq 3$ for $k \geq k_0$ with k_0 large enough.
c) If $\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}} < \frac{k^2}{s(k)} < \lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}$, then we fix $s = s(k)$. In such a case, we directly see that (1.41) holds since

$$k^2 - \lambda_{n(k,s(k))} \geq \frac{\gamma_0}{3k^{2\beta}}, \text{ and } \lambda_{n(k,s(k))+1} - k^2 \geq \frac{\gamma_0}{3k^{2\beta}}.$$

The proof is then complete. $\square$

Remark 1.4.5. The parameter s fixed in the previous Lemma clearly depends on k. Furthermore if β is positive, the quantity $\frac{k^2}{s}$ approaches the spectrum of $-\Delta$, and hence the norm of the resolvent operator $\Delta + \frac{k^2}{s}$ blows up, but the estimate (1.47) controls this blow up since it yields

$$\|\operatorname{div}\mathbf{E}\|_\Omega + \|\operatorname{div}\mathbf{H}\|_\Omega \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega).$$

Let us now show that (1.38) always holds with $\beta = \frac{1}{2}$.

Lemma 1.4.6. *For all bounded domain Ω (of $\mathbb{R}^3$), the assumption* (1.38) *holds with $\beta = \frac{1}{2}$.*

Proof. Assume that (1.38) does not hold with $\beta = \frac{1}{2}$, in other words

$$\forall \gamma_0 > 0,\ k_1 > 0\, \exists k \geq k_1\, \forall s \in [1,2] : g_{n(k,s)} < \gamma_0 k^{-1}. \tag{1.49}$$

We first fix γ_0 such that

$$\gamma_0 < \frac{1}{48\sqrt{2}c|\Omega|}, \tag{1.50}$$

where $|\Omega|$ is the measure of Ω and $c = \frac{1}{6\pi^2}$ is the universal constant such that Weyl's formula

$$\lim_{t\to\infty} \frac{N(t)}{c|\Omega|t^{\frac{3}{2}}} = 1, \tag{1.51}$$

holds, where $N(t)$ is the eigenvalue counting function of the positive Laplace operator $-\Delta$ with Dirichlet boundary conditions in Ω, i.e., the number of its eigenvalues, which are less than t. Then we fix k_1 large enough, namely $k_1^3 \geq 12\gamma_0$. Then for all $k \geq k_1$, we define the real numbers

$$s_i = 1 + \frac{3\gamma_0 i}{k^3}, \forall i = 1, \cdots, N_k,$$

where $N_k = \lfloor \frac{k^3}{6\gamma_0} \rfloor - 1$ (where $\lfloor x \rfloor$ is the integral part of any real number x, namely the unique integer such that $x \leq \lfloor x \rfloor < x + 1$). By our assumption N_k is larger than 1 and for k large it behaves like k^3. It is easy to see that all s_i belongs to $[1, \frac{3}{2}]$. Now we look at the intervals

$$I_i = \left[\frac{k^2}{s_i} - \frac{\gamma_0}{2k}, \frac{k^2}{s_i} + \frac{\gamma_0}{2k}\right], \forall i = 1, \cdots, N_k,$$

and show that they are disjoint, i.e.,

$$I_i \cap I_j = \emptyset, \forall i \neq j, \tag{1.52}$$

and included into the closed interval $\left[\frac{k^2}{2}, 2k^2\right]$:

$$I_i \subset \left[\frac{k^2}{2}, 2k^2\right], \forall i = 1, \cdots, N_k. \tag{1.53}$$

Indeed for the second assertion it suffices to show that

$$\frac{k^2}{s_i} - \frac{\gamma_0}{2k} \geq \frac{k^2}{2}, \tag{1.54}$$

and that

$$\frac{k^2}{s_i} + \frac{\gamma_0}{2k} \geq 2k^2. \tag{1.55}$$

This second estimate holds if and only if

$$\frac{k^2}{s_1} + \frac{\gamma_0}{2k} \geq 2k^2,$$

or equivalently

$$\frac{1}{s_1} \leq 2 - \frac{\gamma_0}{2k^3}.$$

Since $s_1 = 1 + \frac{3\gamma_0}{k^3}$, this holds if and only if

$$(2 - \frac{\gamma_0}{2k^3})(1 + \frac{3\gamma_0}{k^3}) \geq 1,$$

which means that $\frac{\gamma_0}{k^3}$ has to satisfy

$$\frac{11 - \sqrt{145}}{6} \leq \frac{\gamma_0}{k^3} \leq \frac{11 + \sqrt{145}}{6},$$

that is valid owing to our assumption on k_1 (and the fact that $k \geq k_1$).

In the same spirit, the estimate (1.54) holds if and only if

$$s_{N_k} \leq \frac{2}{1 + \frac{\gamma_0}{k^3}},$$

which holds because our assumption on k_1 implies that

$$\frac{3}{2} \leq \frac{2}{1 + \frac{2\gamma_0}{k^3}}.$$

Now to prove (1.52), it suffices to show that

$$I_i \cap I_{i+1} = \emptyset, \forall i = 1, \cdots, N_k - 1,$$

or

$$\frac{k^2}{s_{i+1}} + \frac{\gamma_0}{2k} < \frac{k^2}{s_i} - \frac{\gamma_0}{2k}, \forall i = 1, \cdots, N_k - 1.$$

By the definition of the s_i, this holds if and only if

$$s_i s_{i+1} < 3.$$

Since $s_i s_{i+1} \leq \frac{9}{4}$, we deduce that (1.52) is valid.

Since the length of I_i is exactly equal to $\frac{\gamma_0}{k}$ and due to our assumption (1.49), $\lambda_{n(k,s_i)}$ or $\lambda_{n(k,s_i)+1}$ belongs to I_i. Due to (1.52) and (1.53), for all $k \geq k_1$, we have found N_k distinct eigenvalues in the interval $[\frac{k^2}{2}, 2k^2]$. This implies that

$$N(2k^2) \geq N_k \geq \frac{k^3}{6\gamma_0} - 1 \geq \frac{k^3}{12\gamma_0}, \forall k \geq k_1.$$

But Weyl's formula (1.51) implies that there exists $k_2 > 0$ large enough such that

$$N(2k^2) \leq 2c|\Omega|(2k^2)^{\frac{3}{2}}, \forall k \geq k_2.$$

These two estimates yield

$$\gamma_0 \geq \frac{1}{48\sqrt{2}c|\Omega|},$$

which contradicts (1.50). □

We now notice that (1.38) may hold for $\beta \leq \frac{1}{2}$, in particular it holds with $\beta = 0$ once the next gap condition

$$\exists g_0 > 0 : \lambda_{n+1} - \lambda_n \geq g_0, \forall n \in \mathbb{N}^*, \tag{1.56}$$

holds.

Lemma 1.4.7. *Assume that* (1.56) *holds, then the assumption* (1.38) *is valid with* $\beta = 0$ *and* $\gamma_0 = g_0$.

Proof. If $\frac{k^2}{2}$ is different from $\lambda_{n(k,2)}$, then we take $s = 2$ and find

$$g_{n(k,2)} \geq g_0,$$

hence the result. On the contrary if $\frac{k^2}{2} = \lambda_{n(k,2)}$, then we choose $s = 2 - \varepsilon$ with $\varepsilon \in (0, 1)$ small enough such that

$$\frac{k^2}{2 - \varepsilon} < \lambda_{n(k,2)+1}.$$

Since $k^2 = 2\lambda_{n(k,2)}$, this means that we additionally require that

$$\varepsilon < 2\left(1 - \frac{\lambda_{n(k,2)}}{\lambda_{n(k,2)+1}}\right),$$

which is always possible since this right-hand side is positive. With this choice, we have that $n(k, s) = n(k, 2)$ and we conclude that $g_{n(k,s)} \geq g_0$. □

Corollary 1.4.8. *Assume that* $\Omega = (0, \sqrt{a_1}) \times (0, \sqrt{a_2}) \times (0, \sqrt{a_3})$, *with positive real numbers* a_i, $i = 1, 2, 3$ *such that* $\frac{a_i}{a_1}$ *is a rational number,* $i = 2, 3$. *Then the gap condition* (1.56) *holds with* $\beta = 0$ *and hence for an appropriate choice of* s, *the* k*-stability property with exponent* $\alpha = 1$ *holds.*

Proof. For such a cuboid, it is well known that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\pi^2(\frac{k_1^2}{a_1}+\frac{k_2^2}{a_2}+\frac{k_3^2}{a_3}),$$

for any $k_i\in\mathbb{N}^*, i=1,2,3$. Hence writting $\frac{a_i}{a_1}=\frac{n_i}{d}$, with $n_i, d\in\mathbb{N}^*$, the spectrum is equivalently characterized by the set of

$$\frac{\pi^2}{a_1n_2n_3}(k_1^2n_2n_3+k_2^2n_1n_3+k_3^2n_1n_2),$$

for any $k_i\in\mathbb{N}^*, i=1,2,3$. Since, in our situation, $k_1^2n_2n_3+k_2^2n_1n_3+k_3^2n_1n_2$ is a natural number, the spectrum is a subset of

$$g_0\mathbb{N}^*,$$

where $g_0=\frac{\pi^2}{a_1n_2n_3}$. Hence the distance between two consecutive different eigenvalues is at most larger than g_0. □

Remark 1.4.9. If the cuboid $\Omega=(0,\sqrt{a_1})\times(0,\sqrt{a_2})\times(0,\sqrt{a_3})$, with positive real numbers a_i, $i=1,2,3$ such that $\frac{a_2}{a_1}=\frac{a_3}{a_1}$ is an irrational number badly approximable. Then by the same arguments than before and the use of Proposition 2.1 of [7], the gap condition (1.56) holds with $\beta=1$ and hence for an appropriate choice of s, the k-stability property with exponent $\alpha=3$ holds.

Corollary 1.4.10. *Assume that Ω is a prism in the form $\Omega=T_a\times(0,\sqrt{h})$, with positive real numbers a and h such that $\frac{h}{a}$ is a rational number and T_a is an equilateral triangle of side of length $\sqrt{a}$. Then the gap condition* (1.56) *holds with $\beta=0$ and hence for an appropriate choice of s, the k-stability property with exponent $\alpha=1$ holds.*

Proof. For such a prism, using a separation of variables, a scaling argument and Theorem 1 of [64] (see also Theorem 3.2 of [37], case of type A_2), we deduce that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\frac{16\pi^2}{27a}(k_1^2+k_2^2+k_1k_2)+\frac{k_3^2\pi^2}{h},$$

for any $k_3\in\mathbb{N}^*$ and $k_1\in\mathbb{Z}^*, k_2\in\mathbb{Z}$ such that $k_1+k_2\neq 0$. Hence writting $\frac{h}{a}=\frac{n}{d}$ with $n, d\in\mathbb{N}^*$, the eigenvalues can be written as

$$\frac{\pi^2}{27an}((k_1^2+k_2+k_1k_2)n+27dk_3^2),$$

for the previous parameters k_i. As in the previous Corollary, this means that the distance between two consecutive different eigenvalues is at most larger than $g_0=\frac{\pi^2}{27an}$. □

Remark 1.4.11. By Theorem 3.2 of [37] (case of type C_2 or D_2, see also [4, Prop. 9]), Corollary 1.4.10 remains valid is T_a is an isosceles right triangle with two sides of lenght $\sqrt{a}$, with a positive number a.

Corollary 1.4.12. *Assume that* Ω *is a tetrahedron with vertices* $(0,0,0)$, $(\sqrt{a},0,0),(\sqrt{a}/2,\sqrt{a}/2,-\sqrt{a}/2)$, $(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$, *with a positive number* a. *Then the gap condition* (1.56) *holds with* $\beta = 0$ *and hence for appropriate choice of* s, *the* k*-stability property with exponent* $\alpha = 1$ *holds.*

Proof. For such a tetrahedron, by a scaling argument and Theorem 3.2 of [37] (case of type $A_3 = D_3$, see also [4, Prop. 9]) we deduce that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\frac{4\pi^2}{a}(k_1^2 + \frac{3}{4}(k_2^2 + k_3^2) + k_1k_2 + k_1k_3 + \frac{1}{2}k_2k_3),$$

for any $k_i \in \mathbb{N}^*, i = 1,2,3$. This means that the distance between two consecutive different eigenvalues is at most larger than $g_0 = \frac{\pi^2}{a}$. $\square$

Remark 1.4.13. By Theorem 3.2 of [37] (see also [4, Prop. 9]), Corollary 1.4.12 remains valid for a tetrahedron T_a with vertices $(0,0,0)$, $(\sqrt{a},0,0),(\sqrt{a}/2,\sqrt{a}/2,0),(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$ (case of type B_3) and for a tetrahedron T_a with vertices $(0,0,0)$, $(\sqrt{a}/2,0,0),(\sqrt{a}/2,\sqrt{a}/2,0),(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$ (case of type C_3), with a positive number a.

1.5 2D Maxwell's equations: the TE/TH formulation

In this part, we recall how to deduce a 2d formulation of Maxwell's equations, also called TE/TH formulation from the 3d-one. So we suppose that:

$$\Omega = D \times \mathbb{R} \text{ with } D \subset \mathbb{R}^2, \text{ a bounded domain.}$$

The outward normal along $\partial\Omega$ is then $n = \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix}$.

Also, we assume that the vector fields $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$ and $\mathbf{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$ are independent of the third variable, namely

$$\frac{\partial E_i}{\partial z} = \frac{\partial H_i}{\partial z} = 0 \text{ for } i \in \{1,2,3\}. \tag{1.57}$$

Hence, by simple calculations we have

$$\begin{aligned}\operatorname{curl}\mathbf{E} &= \begin{pmatrix} \partial_y E_3 - \partial_z E_2 \\ \partial_z E_1 - \partial_x E_3 \\ \partial_x E_2 - \partial_y E_1 \end{pmatrix} = \begin{pmatrix} \partial_y E_3 \\ -\partial_x E_3 \\ \partial_x E_2 - \partial_y E_1 \end{pmatrix} \\ &= \begin{pmatrix} \overrightarrow{\operatorname{curl}}(E_3) \\ \operatorname{curl}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \end{pmatrix},\end{aligned}$$

and

$$\mathbf{E} \times n = \begin{pmatrix} -n_2 E_3 \\ n_1 E_3 \\ n_2 E_1 - n_1 E_2 \end{pmatrix} = \begin{pmatrix} E_3 \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix} \\ - \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}_t \end{pmatrix},$$

and then

$$\begin{aligned} \mathbf{E}_t &= \mathbf{E} - (\mathbf{E} \cdot n) n \\ &= \begin{pmatrix} E_1 - (E_1 n_1^2 + E_2 n_2 n_1) \\ E_2 - (E_1 n_1 n_2 + E_2 n_2^2) \\ E_3 \end{pmatrix} \\ &= \begin{pmatrix} E_1 n_2^2 - E_2 n_2 n_1 \\ -E_1 n_1 n_2 + E_2 n_1^2 \\ E_3 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}_t \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix} \\ E_3 \end{pmatrix}. \end{aligned}$$

So, we can rewrite problem (1.1) in Ω:

$$\left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = 0 \text{ and } \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} & \text{in } \Omega \\ \mathbf{H} \times n - \lambda_{\text{imp}} \mathbf{E}_t = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.58}$$

in terms of $\overrightarrow{\text{curl}}$ and 2D-curl as follows

$$\begin{aligned} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = 0 &\Leftrightarrow \begin{pmatrix} \overrightarrow{\text{curl}}(E_3) \\ \operatorname{curl} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \end{pmatrix} - ik \begin{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \\ H_3 \end{pmatrix} = 0 \text{ in } \Omega \\ &\Leftrightarrow \left\{ \begin{array}{l} \overrightarrow{\text{curl}}(E_3) - ik \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = 0 \\ \operatorname{curl} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} - ikH_3 = 0 \end{array} \right. \quad \text{in } D. \end{aligned}$$

and

$$\operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{J} \Leftrightarrow \left\{ \begin{array}{l} \overrightarrow{\text{curl}}(H_3) + ik \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \\ \operatorname{curl} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + ikE_3 = J_3 \end{array} \right. \quad \text{in } D.$$

Similarly, for the essential boundary condition, we have

$$\begin{aligned}
\mathbf{H}\times n-\lambda_{\text{imp}}\mathbf{E}_t=0 &\Leftrightarrow \begin{pmatrix} H_3\begin{pmatrix}-n_2\\ n_1\end{pmatrix}\\ h_t\end{pmatrix}-\lambda_{\text{imp}}\begin{pmatrix}\begin{pmatrix}E_1\\ E_2\end{pmatrix}_t\begin{pmatrix}-n_2\\ n_1\end{pmatrix}\\ E_3\end{pmatrix}=0\\
&\Leftrightarrow \begin{cases}\left(H_3-\lambda_{\text{imp}}\begin{pmatrix}E_1\\ E_2\end{pmatrix}_t\right)\begin{pmatrix}n_2\\ -n_1\end{pmatrix}=0\\ \begin{pmatrix}H_1\\ H_2\end{pmatrix}_t-\lambda_{\text{imp}}E_3=0\end{cases}\\
&\Leftrightarrow \begin{cases}H_3-\lambda_{\text{imp}}\begin{pmatrix}E_1\\ E_2\end{pmatrix}_t=0\\ \begin{pmatrix}H_1\\ H_2\end{pmatrix}_t-\lambda_{\text{imp}}E_3=0.\end{cases}
\end{aligned}$$

This mean that problem (1.58) can be expressed in two independent boundary value problems, namely, it is equivalent to

$$\text{(1.59)}\qquad \begin{cases}\left.\begin{aligned}&\overrightarrow{\text{curl}}(E_3)-ik\begin{pmatrix}H_1\\ H_2\end{pmatrix}=0\\ &\text{curl}\begin{pmatrix}H_1\\ H_2\end{pmatrix}+ikE_3=J_3\end{aligned}\right\}\text{ in } D,\\ \begin{pmatrix}H_1\\ H_2\end{pmatrix}_t-\lambda_{\text{imp}}E_3=0 \qquad \text{on } \partial D,\end{cases}$$

and

$$\text{(1.60)}\qquad \begin{cases}\left.\begin{aligned}&\overrightarrow{\text{curl}}(H_3)+ik\begin{pmatrix}E_1\\ E_2\end{pmatrix}=\begin{pmatrix}J_1\\ J_2\end{pmatrix}\\ &\text{curl}\begin{pmatrix}E_1\\ E_2\end{pmatrix}-ikH_3=0\end{aligned}\right\}\text{ in } D,\\ H_3-\lambda_{\text{imp}}\begin{pmatrix}E_1\\ E_2\end{pmatrix}_t=0 \qquad \text{on } \partial D.\end{cases}$$

The first system is called the TH formutation and the second one the TE formulation.

As the two equations are independent and are identical by replacing k into $-k$ and exchanging $\mathbf{E}$ with $\mathbf{H}$, we will just study the TE formulation.

1.5.1 Weak formulation

We first recall Green's formula for curl in 2D: Let $H_3\in H^1(D)$ and $E\in H(\text{curl},D)$, then

$$\text{(1.61)}\qquad \int_D\left(H_3\,\text{curl}\,\overline{E}-\overrightarrow{\text{curl}}H_3\cdot\overline{E}\right)dx=\int_{\partial D}H\overline{E_t}\;d\sigma.$$

By taking the first equation of (1.60) and multiplying by $\overrightarrow{\mathrm{curl}} H_3'$ and using (1.61), we obtain

$$\begin{aligned}
\int_D \left(\overrightarrow{\mathrm{curl}} H_3 + ikE\right) \cdot \overrightarrow{\mathrm{curl}}\overline{H_3'} dx &= \int_D \nabla H_3 \cdot \nabla \overline{H_3'} + ikE \cdot \overrightarrow{\mathrm{curl}}\overline{H_3'} dx \\
&= \int_D \nabla H_3 \cdot \nabla \overline{H_3'} + ik \,\mathrm{curl}\, E \overline{H_3'} dx \\
&\quad - \int_{\partial D} ik E_t \overline{H_3'} \; d\sigma \\
&= \int_D \nabla H_3 \cdot \nabla \overline{H_3'} - k^2 H_3 \overline{H_3'} dx \\
&\quad - \int_{\partial D} ik\lambda_{\mathrm{imp}} E_t \overline{E_t'} \; d\sigma.
\end{aligned}$$

By taking the second equation of 1.60 and multiplying by $\mathrm{curl}\, E'$ and using (1.61), we get

$$\begin{aligned}
\int_D (\mathrm{curl}\, E - ikH_3)\, \mathrm{curl}\, \overline{E'} dx &= \int_D \mathrm{curl}\, E \,\mathrm{curl}\, \overline{E'} - ikH_3 \,\mathrm{curl}\, \overline{E'} dx \\
&= \int_D \mathrm{curl}\, E \,\mathrm{curl}\, \overline{E'} - ik\overrightarrow{\mathrm{curl}} H_3 \cdot \overline{E'} dx \\
&\quad - ik \int_{\partial D} H_3 \overline{E_t'} \; d\sigma \\
&= \int_D \mathrm{curl}\, E \,\mathrm{curl}\, \overline{E'} - k^2 E \cdot \overline{E'} dx \\
&\quad - \frac{ik}{\lambda_{\mathrm{imp}}} \int_{\partial D} H_3 \overline{H_3'} \; d\sigma.
\end{aligned}$$

Hence by summing the two previous equations and adding the divergence term, we get the following weak formulation: find $(E, H_3) \in \mathbf{V}$ such that:

$$a_{k,s}((E,H_3),(E',H_3')) = b((E',H_3')),\ \forall (E',H_3') \in \mathbf{V}, \tag{1.62}$$

with

$$\mathbf{V} = \left\{ \begin{array}{c} E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \in \mathrm{H}(\mathrm{curl}, D) \cap \mathrm{H}(\mathrm{div}, D) \text{ and } H_3 \in \mathrm{H}^1(D) \\ \text{avec } H_3 - \lambda_{\mathrm{imp}} E_t = 0 \text{ on } \partial D \end{array} \right\},$$

$$\begin{aligned}
a_{k,s}((E,H_3),(E',H_3')) &= \int_\Omega \left(\mathrm{curl}\, E \,\mathrm{curl}\, \overline{E'} + s \,\mathrm{div}\, E \,\mathrm{div}\, \overline{E'} - k^2 E \overline{E'}\right) dx \\
&\quad + \int_\Omega \left(\nabla H_3 \nabla \overline{H_3'} - k^2 H_3 \overline{H_3'}\right) dx \\
&\quad - ik \int_{\partial\Omega} \left(\lambda_{\mathrm{imp}} E_t \overline{E_t'} + \frac{1}{\lambda_{\mathrm{imp}}} H_3 \overline{H_3'}\right) dS,
\end{aligned}$$

and

$$b((E', H_3')) = \int_\Omega \left(ikJ\overline{E'} + \operatorname{curl} J\overline{H_3'}\right) dx - \int_{\partial\Omega} J_t \overline{H_3'} dS$$
$$= \int_\Omega f_1 \cdot \overline{E'} + f_2 \overline{H_3'} dx,$$

with $f_1 = ikJ$ and $f_2 = \operatorname{curl} J$ if we suppose that the support of J is strictly included in D and $\operatorname{div} J = 0$. In the more general case, we just suppose that $f_1 \in L^2(D)^2$ and $f_2 \in L^2(D)$.

Then, the continuous problem associated with the variational problem (1.62) is

$$\text{(1.63)} \qquad \begin{cases} \overrightarrow{\operatorname{curl}} \operatorname{curl} E - s\nabla \operatorname{div} E - k^2 E & = f_1 \text{ in } D, \\ -\Delta H_3 - k^2 H_3 & = f_2 \text{ in } D, \\ H_3 - \lambda_{\text{imp}} E_t & = 0 \text{ on } \partial D, \\ \operatorname{div} E & = 0 \text{ on } \partial D, \\ B_k(E, H_3) & = 0 \text{ on } \partial D, \end{cases}$$

with $B_k(E, H_3) = \frac{1}{\lambda_{\text{imp}}} \operatorname{curl} E - ikE_t + \partial_n H_3 - \frac{ik}{\lambda_{\text{imp}}} H_3$.

Now, we want to check that the solution of (1.63) is also solution of the original problem (1.60).

Theorem 1.5.1. *If J is divergence free and $-\frac{k^2}{s}$ is not an eigenvalue of the Laplacian with Dirichlet boundary condition in D, then E is also divergence free.*

Proof. Let $\phi \in H^2(D) \cap H_0^1(D)$, hence $(\nabla\phi, 0) \in V$ and by using (1.62), we get

$$a_{k,s}((E, H_3), (\nabla\phi, 0)) = s \int_D \operatorname{div} E \operatorname{div} \nabla\overline{\phi} - k^2 E \cdot \nabla\overline{\phi} dx$$
$$- ik \int_{\partial D} \lambda_{\text{imp}} E_t (\nabla\overline{\phi})_t \; d\sigma$$
$$= \int_D ikJ\nabla\overline{\phi} dx$$

Hence, by Green's formula, we obtain

$$\int_D \operatorname{div} E \left(-s\Delta - k^2\right) \overline{\phi} dx = 0,$$

and then $\operatorname{div} E = 0$ if $-\frac{k^2}{s}$ is not an eigenvalue of the Laplacian with Dirichlet boundary condition in D. □

Theorem 1.5.2. *Let $(R, W) \in \mathbf{V}$ such that*

$$\text{(1.64)} \qquad \begin{cases} \operatorname{curl} R - ikW & = 0 \textit{ in } D, \\ \overrightarrow{\operatorname{curl}} W + ikR & = 0 \textit{ in } D, \\ -\lambda_{\text{imp}} R_t + W & = 0 \textit{ on } \partial D, \end{cases}$$

then $R = W = 0$.

Proof. By Green's formula,

$$\begin{aligned}\int_D |\operatorname{curl} R|^2 + |\overrightarrow{\operatorname{curl}} W|^2 dx &= \int_D -ikW \operatorname{curl} \overline{R} + ikR\overrightarrow{\operatorname{curl}}\overline{W} dx \\ &= \int_D -ik\overrightarrow{\operatorname{curl}} W \overline{R} + ikR\overrightarrow{\operatorname{curl}}\overline{W} dx - ik \int_{\partial D} R_t \overline{W} \; d\sigma \\ &= 2k \int_D \operatorname{Im}(R\overrightarrow{\operatorname{curl}}\overline{W}) dx - \frac{ik}{\lambda_{\text{imp}}} \int_{\partial D} |R_t|^2 \; d\sigma.\end{aligned}$$

By taking the imaginary part, we get

$$R_t = 0 \text{ on } \partial D.$$

Hence, by the impedance boundary condition, $W = 0$ on ∂D. Moreover, the second equation of (1.64) allows us to have $(\overrightarrow{\operatorname{curl}} W)_t = \partial_n W = -ikR_t = 0$ on ∂D. Hence, by using the second equation of (1.64) to replace R in the first one, W verifies

$$\begin{cases} -\Delta W - k^2 W = 0 \text{ in } D, \\ \partial_n W = W = 0 \text{ on } \partial D. \end{cases}$$

Then, by Holmgren uniqueness theorem, $W = 0$ in D and $R = 0$ also. □

Theorem 1.5.3. *If J is divergence free, hence (E, H_3) solution of (1.63) is also solution of the original problem (1.60).*

Proof. By Theorem 1.5.1, (E, H_3) is solution of

$$\begin{cases} \overrightarrow{\operatorname{curl}} \operatorname{curl} E - k^2 E & = f_1 \text{ in } D, \\ -\Delta H_3 - k^2 H_3 & = f_2 \text{ in } D, \\ \operatorname{div} E & = 0 \text{ in } D, \\ H_3 + \lambda_{\text{imp}} E_t & = 0 \text{ on } \partial D, \\ B_k(E, H_3) & = 0 \text{ on } \partial D, \end{cases} \tag{1.65}$$

Let

$$\begin{aligned} R &= \overrightarrow{\operatorname{curl}}(H_3) + ikE - J, \\ W &= \operatorname{curl} E - ikH_3. \end{aligned}$$

Hence, by (1.65)

$$\begin{aligned} \operatorname{curl} R &= \operatorname{curl} \overrightarrow{\operatorname{curl}}(H_3) + ik \operatorname{curl} E - \operatorname{curl} J \\ &= k^2 H_3 + \operatorname{curl} J + ik \operatorname{curl} E - \operatorname{curl} J \\ &= k^2 H_3 + ik \operatorname{curl} E, \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{\operatorname{curl}} W &= \overrightarrow{\operatorname{curl}} \operatorname{curl} E - ik\overrightarrow{\operatorname{curl}} H_3 \\ &= k^2 E + ikJ - ik\overrightarrow{\operatorname{curl}} H_3. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{curl} R - ikW &= 0, \\ \overrightarrow{\operatorname{curl}} W + ikR &= 0. \end{aligned}$$

It remains to show that

$$-\lambda_{\text{imp}} R_t + W = 0 \text{ on } \partial D. \tag{1.66}$$

But

$$\begin{aligned} -\lambda_{\text{imp}} R_t + W &= -\lambda_{\text{imp}} \left(\overrightarrow{\operatorname{curl}} H_3 + ikE - J \right)_t + \operatorname{curl} E - ikH_3 \\ &= \lambda_{\text{imp}} \partial_n H_3 - ikH_3 - \lambda_{\text{imp}} ikE_t + \operatorname{curl} E \\ &= -\lambda_{\text{imp}} B_k(E, H_3) = 0. \end{aligned}$$

As

$$\begin{cases} \operatorname{curl} R - ikW & = 0 \text{ in } D, \\ \overrightarrow{\operatorname{curl}} W + ikR & = 0 \text{ in } D, \\ -\lambda_{\text{imp}} R_t + W & = 0 \text{ on } \partial D, \end{cases}$$

We conclude, by theorem 1.5.2, that

$$R = W = 0.$$

Then (E, H_3) is indeed solution of (1.60). □

Theorem 1.5.4. *Let D a convex polygon or a smooth domain, hence $\mathbf{V}$ is continuously embedded in $(H^1(D))^3$.*

Proof. Let $(E, H_3) \in \mathbf{V}$ and fix $\eta \in \mathcal{D}(]0,1[)$ a cut-off function. If D is a polygon, then we define $\Omega_{]0,1[} = D \times]0,1[$. Let

$$\mathbf{E} = \eta(x_3) \begin{pmatrix} E_1(x_1, x_2) \\ E_2(x_1, x_2) \\ 0 \end{pmatrix} \text{ and } \mathbf{H} = \eta(x_3) \begin{pmatrix} 0 \\ 0 \\ H_3(x_1, x_2) \end{pmatrix}$$

We can easly show that $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}(\Omega_{]0,1[})$, with

$$\mathbf{V}(\Omega_{]0,1[}) = \left\{ (\mathbf{E}, \mathbf{H}) \in \left(\mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega_{]0,1[}) \right)^2 : \mathbf{H} \times \mathbf{n} = \lambda_{\text{imp}} \mathbf{E}_t \text{ on } \partial\Omega_{]0,1[} \right\}.$$

Hence, we can apply Theorem 1.2.4 to prove that $(E, H_3) \in (H^1(D))^3$ (as $\mathbf{E}, \mathbf{H} \in (H^1(\Omega_{]0,1[}))^6$ and η is smooth). The continuity of the embedding directly follows from the continuity in Theorem 1.2.4. A similar approach allows us to show this result for a smooth domain.

□

Theorem 1.5.5. *If D is a convex polygon or a smooth domain, then the problem* (1.62) *is well-posedness.*

Proof. The proof follows the one of Theorem 1.3.5, based on the compact embedding of H^1 into L^2 and the Fredholm alternative. □

Chapter 2

Maxwell's system in polyhedral domains

In this chapter, we assume that Ω is a convex polyhedron.

2.1 Corner/edge singularities

Here for the sake of simplicity we assume that $\lambda_{\text{imp}} = 1$ and want to describe the regularity/singularity of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ with $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{H}^t(\Omega)$, for $t \geq 0$. As said before as the system (1.30) is an elliptic system, the shift property will be valid far from the corners and edges of Ω, in other words, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $\mathbf{H}^{t+2}(\Omega \setminus \mathcal{V}) \times \mathbf{H}^{t+2}(\Omega \setminus \mathcal{V})$, for any neighborhood $\mathcal{V}$ of the corners and edges.

We therefore need to determine the corner and edge singularities of system (1.30).

2.1.1 Corner singularities

For c be a corner of Ω, we recall that Ξ_c is the three-dimensional cone that coincides with Ω in a neighbourhood of c and that G_c is its section with the unit sphere. For shortness, if no confusion is possible, we will drop the index c. As usual denote by (r, ϑ) the spherical coordinates centred at c. The standard antsatz [25, 34, 39] is to look for the corner singularities $(\mathbf{E}, \mathbf{H})$ of problem (1.30) in the form

$$(\mathbf{E}, \mathbf{H}) = r^\lambda(\mathbf{U}(\vartheta), \mathbf{V}(\vartheta)), \tag{2.1}$$

with $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > -\frac{1}{2}$ and $\mathbf{U}, \mathbf{V} \in \mathbf{H}^1(G)$ that is solution of (as our system is invariant by translation)

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} = \mathbf{0} & \text{in } \Xi, \\ \operatorname{curl}\operatorname{curl}\mathbf{H} - s\nabla\operatorname{div}\mathbf{H} = \mathbf{0} & \text{in } Xi, \\ \operatorname{div}\,\mathbf{E} = \operatorname{div}\,\mathbf{H} = 0 & \text{on } \partial\Xi, \\ \mathbf{H} \times n - \mathbf{E}_t = (\operatorname{curl}\mathbf{H}) \times n + (\operatorname{curl}\mathbf{E})_t = \mathbf{0} & \text{on } \partial\Xi. \end{cases} \tag{2.2}$$

Remark 2.1.1. For the sake of simplicity, we consider here the spectral condition that is stronger than the notion of injectivity modulo the polynomials (from [25]) that consists in replacing the right-hand side in the two first identities of (2.2) by a polynomial of degree $\lambda - 2$. As a consequence, we eventually add some integer ≥ 2 in the set of corner singular exponent, that at least do not affect the regularity results up to $\frac{7}{2}$.

Inspired from [21], we introduce the auxiliary variables

$$q_E = \operatorname{div} \mathbf{E}, \quad q_H = \operatorname{div} \mathbf{H}, \psi_E = \operatorname{curl} \mathbf{E}, \quad \psi_H = \operatorname{curl} \mathbf{H},$$

and re-write the above system in the equivalent form

$$\left\{ \begin{array}{l} \Delta q_E = 0 \text{ in } \Xi, \\ q_E = 0 \text{ on } \partial\Xi, \end{array} \right. \quad \left\{ \begin{array}{l} \Delta q_H = 0 \text{ in } \Xi, \\ q_H = 0 \text{ on } \partial\Xi, \end{array} \right. \tag{2.3a}$$

$$\left\{ \begin{array}{l} \operatorname{curl} \psi_E = s\nabla q_E \text{ in } \Xi, \\ \operatorname{curl} \psi_H = s\nabla q_H \text{ in } \Xi, \\ \operatorname{div} \psi_E = \operatorname{div} \psi_H = 0 \text{ on } \partial\Xi, \\ \psi_H \times \mathbf{n} = -(\psi_E)_t \text{ on } \partial\Xi, \end{array} \right. \tag{2.3b}$$

$$\left\{ \begin{array}{l} \operatorname{curl} \mathbf{E} = \psi_E, \operatorname{div} \mathbf{E} = q_E \text{ in } \Xi, \\ \operatorname{curl} \mathbf{H} = \psi_H, \operatorname{div} \mathbf{H} = q_H \text{ in } \Xi, \\ \mathbf{H} \times n = \mathbf{E}_t \text{ on } \partial\Xi. \end{array} \right. \tag{2.3c}$$

Then three types of singularities appear:
Type 1: $(q_E, q_H) = (0, 0)$, $(\psi_E, \psi_H) = (\mathbf{0}, \mathbf{0})$ and $(\mathbf{E}, \mathbf{H})$ general non-zero solution of (2.3c).
Type 2: $(q_E, q_H) = (0, 0)$, (ψ_E, ψ_H) general non-zero solution of (2.3b) and $(\mathbf{E}, \mathbf{H})$ particular solution of (2.3c).
Type 3: (q_E, q_H) general non-zero solution of (2.3a), (ψ_E, ψ_H) particular solution of (2.3b) and $(\mathbf{E}, \mathbf{H})$ particular solution of (2.3c).

These singularities are different from those from [21] essentially due to the boundary conditions

$$\mathbf{H} \times n - \mathbf{E}_t = (\operatorname{curl} \mathbf{H}) \times n + (\operatorname{curl} \mathbf{E})_t = \mathbf{0} \text{ on } \partial\Xi.$$

Some singularities from [21] will be also singularities of our problem but not the converse, see below. To describe them, we recall the corner singularities of the Laplace operator with Dirichlet boundary conditions in Ξ, see [34, 25, 21] for instance. We first denote by L_G^{Dir} the positive Laplace-Beltrami operator with Dirichlet boundary conditions on G. Recall that L_G^{Dir} is a self-adjoint operators with a compact resolvent in $L^2(G)$, hence we denote its spectrum by $\sigma(L_G^{\text{Dir}})$. Then we make the following definition.

Definition 2.1.2. *The set $\Lambda_{\text{Dir}}(\Gamma)$ of corner singular exponents of the Laplace operator with Dirichlet boundary conditions in Ξ is defined as the set of $\lambda \in \mathbb{C}$ such that there exists a non-trivial solution $\varphi \in H_0^1(G)$ of*

$$\Delta(r^\lambda \varphi(\vartheta)) = 0. \tag{2.4}$$

We denote by Z_{Dir}^λ the set of such solutions.

Due to the relation

$$r^2\Delta = (r\partial_r)^2 + (r\partial_r) + \Delta_G,$$

for any $\lambda \in \mathbb{C}$ and $\varphi \in H^1(G)$, we have

$$\Delta(r^\lambda \varphi) = r^{\lambda-2}\mathcal{L}(\lambda)\varphi, \tag{2.5}$$

where

$$\mathcal{L}(\lambda)\varphi = \Delta_G\varphi + \lambda(\lambda+1)\varphi, \tag{2.6}$$

with Δ_G the Laplace-Beltrami operator on G. Consequently, the set $\Lambda_{\mathrm{Dir}}(\Gamma)$ is related to the spectrum $\sigma(L_G^{\mathrm{Dir}})$ of L_G^{Dir} as follows (see [21, Lemma 2.4]):

$$\Lambda_{\mathrm{Dir}}(\Gamma) = \{-\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}} : \mu \in \sigma(L_G^{\mathrm{Dir}})\}.$$

For $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma)$, the elements of Z^λ_{Dir} are related to the set $V_{\mathrm{Dir}}(\lambda)$ of eigenvectors of L_G^{Dir} associated with $\mu = \lambda(\lambda+1)$ via the relation

$$Z^\lambda_{\mathrm{Dir}} = \{r^\lambda\varphi : \varphi \in V_{\mathrm{Dir}}(\lambda)\}.$$

Recalling from the previous section that ω_c is the length of the network $\mathcal{R}_c$, we finally set

$$\Upsilon_c = \{\frac{2k\pi}{\omega_c} : k \in \mathbb{Z}\},$$

as well as

$$\Upsilon_c^* = \{\frac{2k\pi}{\omega_c} : k \in \mathbb{Z} \setminus \{0\}\}.$$

We are ready to consider our different types of singularities. We start with singularities of type 1.

Lemma 2.1.3. *Let $\lambda \in \mathbb{C}$ be different from -1. Then $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 1 if and only if $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$.*

Proof. $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 1 if and only if it satisfies

$$\begin{cases} \operatorname{curl} \mathbf{E} = \mathbf{0}, \operatorname{div} \mathbf{E} = 0 \text{ in } \Xi, \\ \operatorname{curl} \mathbf{H} = \mathbf{0}, \operatorname{div} \mathbf{H} = 0 \text{ in } \Xi, \\ \mathbf{H} \times n = \mathbf{E}_t \text{ on } \partial\Xi. \end{cases} \tag{2.7}$$

i) Since a singularity of type 1 from [21] is a vector field $\mathbf{E}_{CD}$ that satisfies

$$\begin{cases} \operatorname{curl} \mathbf{E}_{CD} = \mathbf{0}, \operatorname{div} \mathbf{E}_{CD} = 0 \text{ in } \Xi, \\ \mathbf{E}_{CD} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Xi, \end{cases}$$

by Lemma 6.4 of [21], we deduce that any $\lambda \in \mathbb{C}$ such that $\lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ induces a singularity of type 1 for our problem (pairs like $(\mathbf{E}_{CD}, \mathbf{0})$ for instance).

ii) We now show that other singular exponents appear. As $\lambda \neq -1$, by Lemma 6.1 of [21], the scalar fields

$$\Phi_E = \frac{1}{\lambda+1}\mathbf{E}\cdot\mathbf{x}, \Phi_H = \frac{1}{\lambda+1}\mathbf{H}\cdot\mathbf{x},$$

are scalar potentials of $\mathbf{E}$ and $\mathbf{H}$, namely

$$\mathbf{E} = \nabla\Phi_E, \mathbf{H} = \nabla\Phi_H \text{ in } \Xi. \tag{2.8}$$

Consequently by the divergence free property of $\mathbf{E}$ and $\mathbf{H}$, we deduce that

$$\Delta\Phi_E = \Delta\Phi_H = 0 \text{ in } \Xi. \tag{2.9}$$

Hence if we set

$$u_E(\vartheta) = \frac{1}{\lambda+1}\mathbf{E}(\vartheta)\cdot\vartheta, u_H(\vartheta) = \frac{1}{\lambda+1}\mathbf{H}(\vartheta)\cdot\vartheta,$$

we have

$$\Phi_E = r^{\lambda+1}u_E(\vartheta), \Phi_H = r^{\lambda+1}u_H(\vartheta), \tag{2.10}$$

and by the identity (2.5), we get

$$\mathcal{L}(\lambda+1)u_E = \mathcal{L}(\lambda+1)u_H = 0 \text{ in } G. \tag{2.11}$$

Now we come back to the boundary condition in (2.7) that can be written in polar coordinates (r,θ) in the form

$$\begin{cases} \partial_r\phi_H &= -\frac{1}{r}\partial_\theta\phi_E, \\ \frac{1}{r}\partial_\theta\phi_H &= \partial_r\phi_E. \end{cases}$$

Due to (2.10), in term of u_E and u_H, this is equivalent to

$$\begin{cases} u_H &= -\frac{1}{\lambda+1}\partial_\theta u_E, \\ \partial_\theta u_H &= (\lambda+1)u_E. \end{cases}$$

These two identities imply that u_H is known if u_E is (or the converse) and then u_E has to satisfy

$$\partial_\theta^2 u_E + (\lambda+1)^2 u_E = 0 \text{ on } \mathcal{R}_c. \tag{2.12}$$

In other words, u_E is an eigenvector of the positive Laplace operator on $\mathcal{R}_c$ of eigenvalue $(\lambda+1)^2$. As the set of such eigenvalue is precisely made of μ^2, with $\mu \in \Upsilon_c$, two alternatives occur:
a. $\lambda+1$ does not belong to Υ_c, hence in that case $u_E = u_H = 0$ and therefore

$$\Phi_E = \Phi_H = 0 \text{ on } \partial\Xi,$$

and we conclude as in Lemma 6.4 of [21] that $\lambda+1 \in \Lambda_{\text{Dir}}(\Gamma)$.
b. $\lambda+1$ belongs to Υ_c, hence a non trivial solution u_E of (2.12) exists (it is a

multiple of an associated eigenvector) and then $u_H = -\frac{1}{\lambda+1}\partial_\theta u_E$. This means that the trace of u_E and u_H are prescribed on ∂G (that is $\mathcal{R}_c$), call them φ_E and φ_H. Recalling (2.11), this means that u_E and u_H are respective solution of the following boundary value problems on G:

$$\begin{cases} \mathcal{L}(\lambda+1)u_E = 0 \text{ in } G, \\ u_E = \varphi_E \text{ on } \partial G, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}(\lambda+1)u_H = 0 \text{ in } G, \\ u_H = \varphi_H \text{ on } \partial G. \end{cases}$$

For both problems, either $\lambda + 1 \notin \Lambda_{\mathrm{Dir}}(\Gamma)$ and a solution exists, or $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ and no matter that a solution exists or not, because, by point i), this case already gives rise to a singular exponent. □

We go on with singularities of type 2.

Lemma 2.1.4. *Let $\lambda \in \mathbb{C}$. If $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 2, then $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$.*

Proof. If $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 2, then (see (2.3b)) (ψ_E, ψ_H) satisfies

$$\begin{cases} \operatorname{curl} \psi_E = \mathbf{0} \text{ in } \Xi, \\ \operatorname{curl} \psi_H = \mathbf{0} \text{ in } \Xi, \\ \operatorname{div} \psi_E = \operatorname{div} \psi_H = 0 \text{ on } \partial\Xi, \\ \psi_H \times \mathbf{n} = -(\psi_E)_t \text{ on } \partial\Xi. \end{cases}$$

If we compare this system with (2.7), we deduce equivalently that λ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$, recalling that (ψ_E, ψ_H) behaves like $r^{\lambda-1}$. Hence we have found that $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$ is a necessary condition. □

We end up with singularities of type 3.

Lemma 2.1.5. *Let $\lambda \in \mathbb{C}$. If $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 3, then $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$.*

Proof. If $(\mathbf{E}, \mathbf{H})$ in the form (2.1) is a singularity of type 3, then (q_E, q_H) is a solution of (2.3a), which means equivalently that $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ is a necessary condition. □

Among the corner singular exponents exhibited in the previous Lemmas, according to Lemma 1.3.4, we have to remove the ones for which

$$\operatorname{div} \mathbf{E} \notin H^1_{\mathrm{loc}}(\Xi) \text{ or } \operatorname{div} \mathbf{H} \notin H^1_{\mathrm{loc}}(\Xi).$$

No more constraint appears for singularities of type 1 or 2 since $\mathbf{E}$ and $\mathbf{H}$ are divergence free. On the contrary for singularities of type 3 as $\operatorname{div} \mathbf{E} = q_E$ (resp. $\operatorname{div} \mathbf{H} = q_H$), we get the restriction

$$\lambda - 1 > -\frac{1}{2}.$$

As Lemma 2.1.5 also says that $\lambda - 1 \in \Lambda_{\text{Dir}}(\Gamma)$ and as the set $\Lambda_{\text{Dir}}(\Gamma) \cap [-1, 0]$ is always empty, we get the final constraint

$$\lambda - 1 > 0.$$

In summary if we denote by Λ_c the set of corner singular exponents of the variational problem (1.30) (in $\mathbf{H}^1$), we have shown that

$$\Lambda_{c,1} \subset \Lambda_c \subset \Lambda_{c,1} \cup \Lambda_{c,2} \cup \Lambda_{c,3}, \tag{2.13}$$

where we have set

$$\Lambda_{c,1} = \{\lambda \in \mathbb{R} : \lambda > -\frac{1}{2} \text{ and } \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma) \cup \Upsilon_c^*\},$$
$$\Lambda_{c,2} = \{\lambda \in \mathbb{R} : \lambda > -\frac{1}{2} \text{ and } \lambda \in \Lambda_{\text{Dir}}(\Gamma) \cup \Upsilon_c^*\},$$
$$\Lambda_{c,3} = \{\lambda \in \mathbb{R} : \lambda > 1 \text{ and } \lambda - 1 \in \Lambda_{\text{Dir}}(\Gamma)\}.$$

Note that in the particular case of a cuboid, for all corners we have $\omega_c = \frac{3\pi}{2}$, while Proposition 18.8 of [25] yields

$$\Lambda_{\text{Dir}}(\Gamma) = \{3 + 2d : d \in \mathbb{N}\} \cup \{-(4 + 2d) : d \in \mathbb{N}\}.$$

Consequently, one easily checks that

$$\Lambda_{c,1} = \{2 + 2d : d \in \mathbb{N}\} \cup \{\frac{4k}{3} - 1 : k \in \mathbb{N}^*\},$$
$$\Lambda_{c,2} = \{3 + 2d : d \in \mathbb{N}^*\} \cup \{\frac{4k}{3} : k \in \mathbb{N}^*\},$$
$$\Lambda_{c,3} = \{4 + 2d : d \in \mathbb{N}\}.$$

Hence the smallest corner singular exponent is equal to $\frac{1}{3}$.

Similarly with the help of Lemma 18.7 of [25], the sets $\Lambda_{c,i}, i = 1, 2, 3$ can be characterized for any prism $D \times I$, where D is any polygon with a Lipschitz boundary and I is an interval.

2.1.2 Edge singularities

Our goal is to describe the edge singularities of problem (1.30). Let us then fix an edge e of Ω, then near an interior point of e, as our system (1.30) is invariant by translation and rotation (using a Piola transformation, that in this case corresponds to the covariant transformation), we may suppose that Ω behaves like $W_e = C_e \times \mathbb{R}$ where C_e is a two-dimensional cone centred at $(0, 0)$ of opening $\omega_e \in (0, 2\pi)$, with $\omega_e \neq \pi$. Here for the sake of generality, we do not assume that $\omega_e < \pi$. Below we will also use the polar coordinates (r, θ) in C_e centred at $(0, 0)$. Let us recall that the set $\Lambda_{\text{Dir}}(C_e)$ of singular exponents of the Laplace operator with Dirichlet boundary conditions in C_e is defined by

$$\Lambda_{\text{Dir}}(C_e) = \{\frac{k\pi}{\omega_e} : k \in \mathbb{Z} \setminus \{0\}\}.$$

Similarly we recall that the set of singular exponents of the Laplace operator with Neumann boundary conditions in C_e is defined by

$$\Lambda_{\mathrm{Neu}}(C) = \{\frac{k\pi}{\omega_e} : k \in \mathbb{Z}\}.$$

For convenience, when no confusion is possible, we will drop the index e. As usual, for $\lambda \in \mathbb{C}$, the edge singularities are obtained by looking for a non-polynomial solution $(\mathbf{E}, \mathbf{H})$ (independent of the x_3 variable) in the form of

$$(\mathbf{E}, \mathbf{H}) = r^\lambda \sum_{q=0}^{Q} (\ln r)^q (\mathbf{U}_q(\vartheta), \mathbf{V}_q(\vartheta)), \tag{2.14}$$

of

$$\left\{\begin{array}{ll} \operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla \operatorname{div}\mathbf{E} = \mathbf{F}_E & \text{in } W, \\ \operatorname{curl}\operatorname{curl}\mathbf{H} - s\nabla \operatorname{div}\mathbf{H} = \mathbf{F}_H & \text{in } W, \\ \operatorname{div}\ \mathbf{E} = \operatorname{div}\ \mathbf{H} = 0 & \text{on } \partial W, \\ \mathbf{H}\times n - \mathbf{E}_t = (\operatorname{curl}\mathbf{H})\times n + (\operatorname{curl}\mathbf{E})_t = \mathbf{0} & \text{on } \partial W, \end{array}\right. \tag{2.15}$$

$\mathbf{F}_E, \mathbf{F}_H$ being a polynomial in the x_1, x_2 variables. In that way, we see that the pair $\mathsf{E} = (E_1, E_2)$ made of the two first components of $\mathbf{E}$ and the third component $h := H_3$ of $\mathbf{H}$ satisfy

$$\left\{\begin{array}{ll} \operatorname{curl}\operatorname{curl}\mathsf{E} - s\nabla \operatorname{div}\mathsf{E} = \mathbf{F}_E & \text{in } C, \\ \Delta h = g & \text{in } C, \\ \operatorname{div}\ \mathsf{E} = 0 & \text{on } \partial C, \\ h + \mathsf{E}_t = \partial_n h - \operatorname{curl}\mathsf{E} = 0 & \text{on } \partial C, \end{array}\right. \tag{2.16}$$

F, g being a polynomial (in the x_1, x_2 variables) and as usual

$$\operatorname{curl}\mathsf{E} = \partial_1 E_2 - \partial_2 E_1,$$

and

$$\mathsf{E}_t = n_1 E_2 - n_2 E_1 \text{ on } \partial C,$$

if $\mathbf{n} = (n_1, n_2)$ on ∂C, further for a scalar field φ we have

$$\operatorname{curl}\varphi = \begin{pmatrix} \partial_2\varphi \\ -\partial_1\varphi \end{pmatrix}.$$

The pair (H_1, H_2) made of the two first components of $\mathbf{H}$ and $-E_3$, where E_3 is the third component of $\mathbf{E}$ satisfy the same system, hence we only need to characterize the singularities of (2.16).

Inspired from [21], the singularities of system (2.16) are obtained by introducing the scalar variables $q = \operatorname{div}\mathsf{E}$ and $\psi = \operatorname{curl}\mathsf{E}$. In this way, if

$\lambda \notin \mathbb{N}_2 := \{n \in \mathbb{N} : n \geq 2\}$ (or equivalently λ is not an integer or is an integer ≤ 1), we find the equivalent system

$$\left\{ \begin{array}{l} \Delta q = 0 \text{ in } C, \\ q = 0 \text{ on } \partial C, \end{array} \right. \tag{2.17a}$$

$$\left\{ \begin{array}{l} \operatorname{curl} \psi = s \nabla q \text{ in } C, \\ \Delta h = 0 \text{ in } C, \\ \partial_n h - \psi = 0 \text{ on } \partial C, \end{array} \right. \tag{2.17b}$$

$$\left\{ \begin{array}{l} \operatorname{curl} \mathsf{E} = \psi, \operatorname{div} \mathsf{E} = q \text{ in } C, \\ \mathsf{E}_t = -h \text{ on } \partial C. \end{array} \right. \tag{2.17c}$$

As before three types of singularities appear:
Type 1: $q = 0$, $\psi = 0$ and E general non-zero solution of (2.17c).
Type 2: $q = 0$, ψ general non-zero solution of (2.17b) and E particular solution of (2.17c).
Type 3: q general non-zero solution of (2.17a), ψ particular solution of (2.17b) and E particular solution of (2.17c).

The singularities of type 1 were treated in [21, §5c], where it is shown that $\lambda \notin \mathbb{N}_2$ is such that $\lambda + 1 \in \Lambda_{\text{Dir}}(C) \setminus \{2\}$.

Let us now look at singularities of type 2.

Lemma 2.1.6. *Let $\lambda \notin \mathbb{N}_2$ be such that $\operatorname{Re} \lambda > 0$. Then λ is a singularity of type 2 if and only if $\lambda \in \Lambda_{\text{Neu}}(C)$.*

Proof. If (E, h) in the form

$$\mathsf{E} = r^\lambda \sum_{q=0}^{Q} (\ln r)^q \mathsf{U}(\vartheta), h = r^\lambda \sum_{q=0}^{Q} (\ln r)^q v_q(\vartheta)), \tag{2.18}$$

is a singularity of type 2, then $\psi = \operatorname{curl} \mathsf{E}$ satisfies (see (2.17b))

$$\left\{ \begin{array}{l} \operatorname{curl} \psi = 0 \text{ in } C, \\ \Delta h = 0 \text{ in } C, \\ \partial_n h - \psi = 0 \text{ on } \partial C. \end{array} \right.$$

In this case, ψ is constant in the whole C. Hence we distinguish the case $\lambda = 1$ or not:
1. If $\lambda \neq 1$, then $\psi = 0$ and consequently h satisfies

$$\left\{ \begin{array}{l} \Delta h = 0 \text{ in } C, \\ \partial_n h = 0 \text{ on } \partial C, \end{array} \right. \tag{2.19}$$

which means that λ belongs to $\Lambda_{\text{Neu}}(C)$ and h is in the form

$$h = r^\lambda \cos(\lambda\theta).$$

2. If $\lambda = 1$, then there exists a constant c such that $\psi = c$ and consequently h satisfies

$$\left\{ \begin{array}{l} \Delta h = 0 \text{ in } C, \\ \partial_n h = c \text{ on } \partial C, \end{array} \right. \tag{2.20}$$

For two parameters c_1 and c_2, denote by

$$h_0 = c_1 x_1 + c_2 x_2 = r(c_1 \cos\theta + c_2 \sin\theta).$$

Clearly h_0 is harmonic and satisfies

$$\begin{aligned} \partial_n h_0(\theta = 0) &= -c_2, \\ \partial_n h_0(\theta = \omega) &= -c_1 \sin\omega + c_2 \cos\omega, \end{aligned}$$

hence it fulfils (2.20) if and only if (c_1, c_2) satisfies the 2×2 linear system

$$c_2 = -c, -c_1 \sin\omega + c_2 \cos\omega = c.$$

Since $\sin\omega$ is different from zero, such a solution exists and therefore $d = h - h_0$ satsifies (2.19). This would mean that 1 belongs to $\Lambda_{\mathrm{Neu}}(C)$, which is not possible.

Once ψ and h are found, we look for a particular solution E of (2.17c) with $q = 0$. From its curl free property, we look for E in the form

$$E = \nabla\Phi,$$

with

$$\Phi = r^{\lambda+1}\varphi(\theta),$$

where φ has to satisfy

$$\left\{ \begin{array}{l} \varphi'' + (\lambda+1)^2\varphi = 0 \text{ in } (0,\omega), \\ (\lambda+1)\varphi(0) = -1, (\lambda+1)\varphi(\omega) = -\cos(\lambda\omega). \end{array} \right.$$

As $\lambda + 1$ does not belong to $\Lambda_{\mathrm{Dir}}(C)$ and is different from zero, such a solution φ always exists. □

Lemma 2.1.7. *Let $\lambda \notin \mathbb{N}_2$ be such that $\operatorname{Re}\lambda > 0$. Then λ is a singularity of type 3 if and only if $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(C)$.*

Proof. If (E, h) in the form (2.18) is a singularity of type 3, then $q = \operatorname{div} \mathsf{E}$ satisfies (2.17a) and consequently $\lambda - 1$ belongs to $\Lambda_{\mathrm{Dir}}(C)$ and q is equal to

$$q = r^{\lambda-1}\sin((\lambda-1)\theta),$$

up to a non-zero multiplicative factor (that we then fix to be 1).

Now we look for (ψ, h) a particular solution of (2.17b). As simple calculations yield

$$\operatorname{curl}(r^{\lambda-1}\cos((\lambda-1)\theta)) = -\nabla r^{\lambda-1}\sin((\lambda-1)\theta),$$

we deduce that

$$\psi = -s r^{\lambda-1}\cos((\lambda-1)\theta) + k,$$

for some constant k, that we can fix to be zero since we look for particular solutions. Hence it remains to find h solution of

$$\left\{ \begin{array}{l} \Delta h = 0 \text{ in } C, \\ \partial_n h = -s r^{\lambda-1}\cos((\lambda-1)\theta) \text{ on } \partial C. \end{array} \right.$$

Such a h exists in the form

$$h = r^\lambda \eta(\theta),$$

since the previous problem is equivalent to

$$\begin{cases} \eta'' + \lambda^2 \eta = 0 \text{ in } (0,\omega), \\ \eta'(0) = s, \eta'(\omega) = \pm s(-1)^k, \end{cases}$$

when $\lambda = \frac{k\pi}{\omega}$ and this system has a unique solution since $\lambda \notin \Lambda_{\text{Neu}}(C)$.

Now we look for E a particular solution of (2.17c) with the functions q, ψ and h found before, which then takes the form

$$\begin{cases} \operatorname{curl} \mathsf{E} = -s r^{\lambda-1} \cos((\lambda-1)\theta) \text{ in } C, \\ \operatorname{div} \mathsf{E} = r^{\lambda-1} \sin((\lambda-1)\theta) \text{ in } C, \\ \mathsf{E}_t = -r^\lambda \eta(\theta) \text{ on } \partial C. \end{cases}$$

Hence we look for E in the form

$$\mathsf{E} = -\frac{s}{4\lambda} \operatorname{curl}(r^{\lambda+1} \cos((\lambda-1)\theta)) + \nabla\Phi.$$

As simple calculations yield

$$\operatorname{curl}\operatorname{curl}(r^{\lambda+1} \cos((\lambda-1)\theta)) = 4\lambda \cos((\lambda-1)\theta),$$

we deduce that the previous system in E is equivalent to

$$\begin{cases} \Delta\Phi = r^{\lambda-1} \sin((\lambda-1)\theta) \text{ in } C, \\ \partial_r \Phi(r,0) = c_0 r^\lambda, \partial_r \Phi(r,\omega) = c_\omega r^\lambda, \end{cases} \tag{2.21}$$

for two constants c_0 and c_ω. If $\lambda + 1 \notin \Lambda_{\text{Dir}}(C)$, then a solution Φ of this problem always exists in the form

$$r^{\lambda+1}\varphi(\theta),$$

since it is then equivalent to

$$\begin{cases} \varphi'' + (\lambda+1)^2 \varphi = \sin((\lambda-1)\theta) \text{ in } C, \\ \varphi(0) = \frac{c_0}{\lambda+1}, \partial_r \Phi(r,\omega) = \frac{c_\omega}{\lambda+1}. \end{cases}$$

On the contrary if $\lambda + 1 \in \Lambda_{\text{Dir}}(C)$ (that only occurs when $\omega = \frac{3\pi}{2}$), then we look for Φ in the form

$$r^{\lambda+1}(\varphi_0(\theta) + \log r \varphi_1(\theta)). \tag{2.22}$$

Since, in this particular choice, problem (2.21) is equivalent to

$$\begin{cases} \Delta\Phi = r^{\lambda-1} \sin((\lambda-1)\theta) \text{ in } C, \\ \Phi(r,0) = \frac{c_0}{\lambda+1} r^{\lambda+1}, \Phi(r,\omega) = \frac{c_\omega}{\lambda+1} r^{\lambda+1}, \end{cases}$$

by Theorem 4.22 of [60], we deduce that a solution Φ in the form (2.22) exists.

In both cases, a solution Φ exists, hence the existence of E. $\square$

As before among the edge singular exponents, we have to remove the ones for which

$$\operatorname{div} \mathbf{E} \notin H^1_{\mathrm{loc}}(W) \text{ or } \operatorname{div} \mathbf{H} \notin H^1_{\mathrm{loc}}(W).$$

No more constraint appears for singularities of type 1 or 2 since $\mathbf{E}$ and $\mathbf{H}$ are divergence free. On the contrary for singularities of type 3, we get the restriction

$$\lambda > 1.$$

In summary if we denote by Λ_e the set of edge singular exponents $\notin \mathbb{N}_2$ of the variational problem (1.30) (in $\mathbf{H}^1$, i.e., with $\operatorname{Re}\lambda > 0$), we have shown that

$$\Lambda_e = \Lambda_{e,1} \cup \Lambda_{e,2} \cup \Lambda_{e,3}, \tag{2.23}$$

where we have set

$$\begin{aligned} \Lambda_{e,1} &= \{\lambda \in \mathbb{R} : \lambda > 0 \text{ and } \lambda + 1 \in \Lambda_{\mathrm{Dir}}(C) \setminus \{2\}\}, \\ \Lambda_{e,2} &= \{\lambda \in \mathbb{R} : \lambda > 0 \text{ and } \lambda \in \Lambda_{\mathrm{Neu}}(C)\}, \\ \Lambda_{c,3} &= \{\lambda \in \mathbb{R} : \lambda > 1 \text{ and } \lambda - 1 \in \Lambda_{\mathrm{Dir}}(C)\}. \end{aligned}$$

Note that in the particular case of a cuboid, for all edges we have $\omega_e = \frac{\pi}{2}$, and consequently $\Lambda_e = \emptyset$ (recalling that the natural number in $\mathbb{N}_2$ are excluded from this set). Since one can show that $\lambda = 2$ is a singular exponent, the maximal regularity along the edge is $H^{3-\varepsilon}$, for any $\varepsilon > 0$.

In conclusion, for any polyhedral domain satisfying the assumption (1.9), there exists $t_\Omega \in (1, 2]$ such that for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $\mathbf{H}^t(\Omega)^2$, for all $t < t_\Omega$. For instance for a cuboid, we have $t_\Omega = \frac{11}{6}$.

2.2 h-finite element approximations

For the sake of simplicity, we here perform some error analyses when $\lambda_{\mathrm{imp}} = 1$, but for polyhedral domains satisfying the assumption (1.9) and for which the stability estimate is valid. Before stating some convergence results for different finite element approximations, we state some regularity results and a priori bounds.

2.2.1 Some regularity results and a priori bounds

Theorem 2.2.1. *Assume that $\lambda_{\mathrm{imp}} = 1$, and that Ω is a polyhedron satisfying the assumption* (1.9) *and that the k-stability property with exponent α holds. Then for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $\mathbf{H}^t(\Omega)^2$, for all $t < t_\Omega$ with*

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{t,\Omega} \lesssim (1 + k^{1+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_\Omega. \tag{2.24}$$

Proof. Since the regularity of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ was already stated in section 2.1, we only concentrate on the estimate (2.24). It indeed holds by looking at $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ as solution of (1.26) with $k = 0$ and a right-hand side defined by

$$\begin{aligned} \langle F, (\mathbf{E}', \mathbf{H}')\rangle &= \int_\Omega ((\mathbf{f}_1 + k^2\mathbf{E}) \cdot \bar{\mathbf{E}}' + (\mathbf{f}_2 + k^2\mathbf{H}) \cdot \bar{\mathbf{H}}'))\, dx \\ &+ ik \int_{\partial\Omega} (\mathbf{E}_t \cdot \bar{\mathbf{E}}'_t + \mathbf{H}_t \cdot \bar{\mathbf{H}}'_t)\, d\sigma. \end{aligned}$$

By elliptic regularity and the stability estimate (1.32), we obtain

$$\begin{aligned}\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{t,\Omega} &\lesssim \|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega + k^2\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega + k\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{\frac{1}{2},\partial\Omega}\\ &\lesssim (1+k^{1+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega,\end{aligned}$$

which proves (2.24). □

Now we show similar results in weighted Sobolev spaces (in the absence of edge singularities), namely for all $\ell \in \mathbb{N}, \ell \geq 2$, and all non-negative real numbers ν, if $r(\mathbf{x})$ is the distance from $\mathbf{x}$ to the corners of Ω, then we introduce the weighted space

$$H^{\ell,\nu}(\Omega) := \{v \in H^1(\Omega) : r^\alpha D^\beta v \in L^2(\Omega),\ \forall \beta \in \mathbb{N}^3 : 2 \leq |\beta| \leq \ell\},$$

which is a Hilbert space with its natural norm $\|\cdot\|_{\ell,\nu;\Omega}$.

Theorem 2.2.2. *In addition to the assumptions of Theorem 2.2.1, assume that $\omega_e \leq \frac{\pi}{2}$, for all edge e of Ω and that $\lambda \neq \frac{1}{2}$, for all $\lambda \in \Lambda_c$ and all corners c of Ω. Then for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ can be decomposed as follows:*

$$\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) = (\mathbf{E}_R,\mathbf{H}_R) + \sum_{c\in\mathcal{C}}\ \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda} r_c^\lambda(\varphi_{E,c,\lambda}(\vartheta_c),\varphi_{H,c,\lambda}(\vartheta_c)), \tag{2.25}$$

with $(\mathbf{E}_R,\mathbf{H}_R) \in \mathbf{H}^2(\Omega)^2$, $\mathcal{C}$ is the set of corners of Ω, (r_c,ϑ_c) are the spherical coordinates centred at c, $\kappa_{c,\lambda}$ is a constant and $\varphi_{E,c,\lambda}, \varphi_{H,c,\lambda}$ belongs to $\mathbf{H}^2(G_c)$. Furthermore we will have

$$\|(\mathbf{E}_R,\mathbf{H}_R)\|_{2,\Omega} + \sum_{c\in\mathcal{C}}\ \sum_{\lambda\in\Lambda_c:0<\lambda<\frac{1}{2}} |\kappa_{c,\lambda}| \lesssim (1+k^{1+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega. \tag{2.26}$$

In particular it holds $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$, for all $\nu > 2 - t_\Omega$ with

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{2,\nu;\Omega} \lesssim (1+k^{1+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega. \tag{2.27}$$

Proof. Since there is no edge singular exponent in the interval $[0,1]$, the results of section 2.1 and of section 8.2 of [39] (global regularity results in weighted Sobolev spaces for elliptic systems on domains with point singularities) allow to show that the splitting (2.25) and the estimate (2.26) hold. The regularity $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$, for all $\nu > 2 - t_\Omega$ and the estimate (2.27) directly follow from the fact that $r_c^\lambda(\varphi_{E,c,\lambda}(\vartheta_c),\varphi_{H,c,\lambda}(\vartheta_c))$ belongs to $H^{2,\nu}(\Omega)^6$, for all $\nu > 2 - t_\Omega$. □

Finally still in the absence of edge singularities, we want to improve the previous result for a regular part almost in H^3, namely we prove the next result.

Theorem 2.2.3. *Under the assumptions of Theorem 2.2.2, for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ can be decomposed as follows:*

$$\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) = \mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2) + (\mathbf{R}_{E,\mathrm{reg}},\mathbf{R}_{H,\mathrm{reg}}) + (\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}}), \tag{2.28}$$

with $\mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$*, for any* $\nu > 2 - t_\Omega$*, satisfying*

$$\|\mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{2,\nu;\Omega} \lesssim \|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega, \tag{2.29}$$

$(\mathbf{R}_{E,\mathrm{reg}},\mathbf{R}_{H,\mathrm{reg}}) \in \mathbf{H}^{3-\varepsilon}(\Omega)^2$ *and* $(\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}}) \in H^{3,\nu_0}(\Omega)^6$ *(for shortness their dependence in* s *is skipped), for any* $\varepsilon > 0$ *and any* $\nu_0 > 3 - t_\Omega$*, such that*

$$\|(\mathbf{R}_{E,\mathrm{reg}},\mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \|(\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}})\|_{3,\nu_0;\Omega} \lesssim (1+k^{2+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega. \tag{2.30}$$

Proof. In a first step, we split up $(\mathbf{E},\mathbf{H}) := \mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ (see [14] for a similar approach in domains with a smooth boundary) as follows:

$$\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) = \mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2) + (\mathbf{R}_E,\mathbf{R}_H), \tag{2.31}$$

where the remainder $(\mathbf{R}_E,\mathbf{R}_H) \in \mathbf{V}$ (for shortness it dependence in s is skipped) satisfies

$$\begin{aligned} \mathbf{a}_{0,s}((\mathbf{R}_E,\mathbf{R}_H),(\mathbf{E}',\mathbf{H}')) &= k^2 \int_\Omega (\mathbf{E}\cdot\bar{\mathbf{E}}' + \mathbf{H}\cdot\bar{\mathbf{H}}')\,dx \\ &- ik \int_{\partial\Omega} (\mathbf{E}_t\cdot\bar{\mathbf{E}}'_t + \mathbf{H}_t\cdot\bar{\mathbf{H}}'_t)\,d\sigma,\ \forall(\mathbf{E}',\mathbf{H}') \in \mathbf{V}. \end{aligned} \tag{2.32}$$

By Theorem 1.3.5, the existence and uniqueness of $\mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2)$ and of $(\mathbf{R}_E,\mathbf{R}_H)$ are guaranteed. Moreover from the estimate (2.27) (with $k = 0$), we see that $\mathbb{S}_{0,s}(\mathbf{f}_1,\mathbf{f}_2)$ belongs to $H^{2,\nu}(\Omega)^6$, for any $\nu > 2 - t_\Omega$ and that the estimate (2.29) holds. A similar result is valid for $(\mathbf{R}_E,\mathbf{R}_H)$, but we are interested in an improved regularity. More precisely, we want to show that

$$(\mathbf{R}_E,\mathbf{R}_H) = (\mathbf{R}_{E,\mathrm{reg}},\mathbf{R}_{H,\mathrm{reg}}) + (\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}}), \tag{2.33}$$

with $(\mathbf{R}_{E,\mathrm{reg}},\mathbf{R}_{H,\mathrm{reg}})$ and $(\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}})$ as stated in the Theorem. Indeed we first notice that the volumic term in the right-hand side of (2.32) has the appropriate regularity to obtain a decomposition of $(\mathbf{R}_E,\mathbf{R}_H)$ into a regular part in $\mathbf{H}^{3-\varepsilon}(\Omega)^2$ and a singular (corner) part. Unfortunately this is not the case for the boundary term, because $(\mathbf{E},\mathbf{H})$ is not in $\mathbf{H}^2(\Omega)^2$, but due to its splitting (2.25), we can use a lifting of the singular part. More precisely by using Lemma 6.1.13 of [39], for all corners c, and all $\lambda \in \cap(-\frac{1}{2},\frac{1}{2})$, there exists a field $(\mathbf{E}_{c,\lambda},\mathbf{H}_{c,\lambda})$ in the form

$$(\mathbf{E}_{c,\lambda},\mathbf{H}_{c,\lambda}) = r_c^{1+\lambda}\sum_{\ell=0}^{\kappa(\lambda)} \varphi_{c,\lambda,\ell}(\vartheta_c)(\ln r_c)^\ell,$$

with $\kappa(\lambda) \in \mathbb{N}$ and $\varphi_{c,\lambda,\ell} \in \mathbf{H}^{3-\varepsilon}(G_c)$ such that

$$\left\{\begin{array}{l} \left.\begin{array}{ll} L_{k,s}(\mathbf{E}_{c,\lambda}) &= \mathbf{0} \\ L_{k,s}(\mathbf{H}_{c,\lambda}) &= \mathbf{0} \end{array}\right\} \text{ in } \Xi_c, \\ \left.\begin{array}{ll} \operatorname{div}\mathbf{E}_{c,\lambda} &= 0 \\ \operatorname{div}\mathbf{H}_{c,\lambda} &= 0 \\ T(\mathbf{E}_{c,\lambda},\mathbf{H}_{c,\lambda}) &= 0 \\ B_0(\mathbf{E}_{c,\lambda},\mathbf{H}_{c,\lambda}) &= 2\varphi_{E,c,\lambda,t} \end{array}\right\} \text{ on } \partial\Xi_c. \end{array}\right.$$

Hence for any corner c by fixing a smooth cut-off function η_c equal to 1 near c and equal to 0 near the other corners, we introduce

$$(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H) = (\mathbf{R}_E, \mathbf{R}_H) - ik \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda}\eta_c(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}), \tag{2.34}$$

that still belongs to $\mathbf{V}$ and is solution of

$$\begin{aligned} \mathbf{a}_{0,s}((\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H), (\mathbf{E}', \mathbf{H}')) &= k^2 \int_\Omega (\mathbf{E}\cdot\bar{\mathbf{E}}' + \mathbf{H}\cdot\bar{\mathbf{H}}')\, dx \\ &- ikF(\mathbf{E}', \mathbf{H}'),\ \forall(\mathbf{E}', \mathbf{H}') \in \mathbf{V}, \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} F(\mathbf{E}', \mathbf{H}') &= \int_{\partial\Omega} (\mathbf{E}_t\cdot\bar{\mathbf{E}}'_t + \mathbf{H}_t\cdot\bar{\mathbf{H}}'_t)\, d\sigma \\ &\quad - \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda}\mathbf{a}_{0,s}(\eta_c(\mathbf{E}_c, \mathbf{H}_c), (\mathbf{E}', \mathbf{H}')) \\ &= \int_{\partial\Omega} (\mathbf{E}_{R,t}\cdot\bar{\mathbf{E}}'_t + \mathbf{H}_{R,t}\cdot\bar{\mathbf{H}}'_t)\, d\sigma \\ &\quad + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda} \int_{\partial\Omega} r_c^\lambda(1-\eta_c)(\varphi_{E,c,\lambda,t}\cdot\bar{\mathbf{E}}'_t + \varphi_{H,c,\lambda,t}\cdot\bar{\mathbf{H}}'_t)\, d\sigma \\ &\quad - \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda} \int_\Omega (L_{k,s}(\eta_c\mathbf{E}_{c,\lambda})\cdot\bar{\mathbf{E}}' + L_{k,s}(\eta_c\mathbf{H}_{c,\lambda})\cdot\bar{\mathbf{H}}')\, dx. \end{aligned}$$

Since $(1-\eta_c)\varphi_{E,c,\lambda,t}$, $(1-\eta_c)\varphi_{H,c,\lambda,t}$, $L_{k,s}(\eta_c\mathbf{E}_{c,\lambda})$, $L_{k,s}(\eta_c\mathbf{H}_{c,\lambda})$ are sufficiently regular, by the shift theorem, we deduce that $(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H)$ admits a decomposition into a regular part in $\mathbf{H}^{3-\varepsilon}(\Omega)^2$ for any $\varepsilon > 0$ and a singular part that corresponds to corner singularities, namely

$$(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H) = (\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} \kappa'_{\lambda,c}\mathbf{S}_c^\lambda, \tag{2.36}$$

where $(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) \in \mathbf{H}^{3-\varepsilon}(\Omega)^2$, $\mathbf{S}_c^\lambda$ is the singular function associated with λ, and $\kappa'_{\lambda,c} \in \mathbb{C}$. Furthermore we have the estimate

$$\begin{aligned} \|(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}| &\lesssim k^2\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_\Omega \\ &+ k\|(\mathbf{E}_R, \mathbf{H}_R)\|_{2,\Omega} + k\sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} |\kappa_{c,\lambda}|. \end{aligned}$$

Hence by the stability estimate (1.32) and the estimate (2.26), we get

$$\|(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}| \lesssim (1+k^{2+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_{0,\Omega}. \tag{2.37}$$

Coming back to the definition (2.34) of $(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H)$ and using its splitting (2.36), we find the decomposition (2.33) of $(\mathbf{R}_E, \mathbf{R}_H)$ with

$$\begin{aligned}(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}}) &= ik \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda}\eta_c(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}) \\ &\quad + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} \kappa'_{\lambda,c}\mathbf{S}_c^\lambda,\end{aligned}$$

that clearly belongs to $H^{3,\nu_0}(\Omega)^6$ for any $\nu_0 > 3 - t_\Omega$, with the estimate

$$\|(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}})\|_{3,nu_0;\Omega} \lesssim k \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} |\kappa_{c,\lambda}| + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}|.$$

Using the estimates (2.26) and (2.37), we conclude that (2.30) is valid. □

Obviously the same regularity results are valid for the solution $(\mathbf{E}^*, \mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F}, \mathbf{G})$ of the adjoint problem

$$\mathbf{a}_{k,s}((\mathbf{E}', \mathbf{H}'), (\mathbf{E}^*, \mathbf{H}^*)) = \int_\Omega (\bar{\mathbf{F}} \cdot \mathbf{E}' + \bar{\mathbf{G}} \cdot \mathbf{H}'),\ \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}. \tag{2.38}$$

Indeed as

$$\mathbf{a}_{k,s}((\mathbf{E}', \mathbf{H}'), (\mathbf{E}^*, \mathbf{H}^*)) = a_{k,s}(\bar{\mathbf{E}}^*, \bar{\mathbf{E}}') + a_{k,s}(\bar{\mathbf{H}}^*, \bar{\mathbf{H}}') + ik \int_{\partial\Omega} (\bar{\mathbf{E}}^*_t \cdot \mathbf{E}'_t + \bar{\mathbf{H}}^*_t \cdot \mathbf{H}'_t)\, d\sigma,$$

we deduce that

$$(\bar{\mathbf{E}}^*, \bar{\mathbf{H}}^*) = \mathbb{S}_{k,s}(\bar{\mathbf{F}}, \bar{\mathbf{G}}).$$

2.2.2 Wavenumber explicit error analyses

With the above regularity results from Theorems 2.2.1 or 2.2.2 in hands, we can perform some error analyses following a standard approach (see [47, Chap. 8] and [48, §4]), the differences with these references are the loss of regularity and/or the use of refined meshes. The situation from Theorem 2.2.3 is different and uses similar ideas than in [14].

$\mathbb{P}_1$-elements with regular meshes

We start with the simplest case where we approximate $\mathbf{V}$ by a subspace made of piecewise polynomials of degree 1 on a regular (in the Ciarlet sense) mesh $\mathcal{T}_h$ of Ω made of tetrahedra, namely we take

$$\mathbf{V}_h := \mathbf{V} \cap \mathbb{P}_{1,h},$$

where

$$\mathbb{P}_{1,h} := \{(\mathbf{E}_h, \mathbf{H}_h) \in \mathbf{L}^2(\Omega)^2 : \mathbf{E}_{h|T}, \mathbf{H}_{h|T} \in (\mathbb{P}_1(T))^3,\ \forall T \in \mathcal{T}_h\}.$$

At this stage, a finite element approximation of $(\mathbf{E},\mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) \in \mathbf{V}$ with $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ consists in looking for $(\mathbf{E}_h,\mathbf{H}_h) = \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2) \in \mathbf{V}_h$ solution of

$$\mathbf{a}_{k,s}((\mathbf{E}_h,\mathbf{H}_h);(\mathbf{E}',\mathbf{H}')) = \int_\Omega (\mathbf{f}_1 \cdot \bar{\mathbf{E}}'_h + \mathbf{f}_1 \cdot \bar{\mathbf{H}}'_h),\ \forall (\mathbf{E}'_h,\mathbf{H}'_h) \in \mathbf{V}_h. \tag{2.39}$$

To analyse the existence of such a solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ and the error between this approximated solution and $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$, according to a general principle (see for instance [48, 49] for the Helmholtz equation), we introduce the adjoint approximability

$$\eta(\mathbf{V}_h) = \sup_{(\mathbf{F},\mathbf{G})\in \mathbf{L}^2(\Omega)^2\setminus\{(\mathbf{0},\mathbf{0})\}} \inf_{(\mathbf{U}_h,\mathbf{V}_h)\in\mathbf{V}_h} \frac{\|\mathbb{S}^*_{k,s}(\mathbf{F},\mathbf{G}) - (\mathbf{U}_h,\mathbf{V}_h)\|_k}{\|(\mathbf{F},\mathbf{G})\|_\Omega}. \tag{2.40}$$

By Theorem 4.2 of [48] (that directly extends to our setting), the existence and uniqueness of a solution to (2.39) is guaranteed if $k\eta(\mathbf{V}_h)$ is small enough (stated precisely below).

To show such a result we will use the standard Lagrange interpolant. Namely for any $(\mathbf{E},\mathbf{H}) \in \mathbf{H}^t(\Omega)^2$, with $t > \frac{3}{2}$, by the Sobolev embedding theorem, its Lagrange interpolant $I_h(\mathbf{E},\mathbf{H})$ (defined as the unique element of $\mathbb{P}_{1,h}$ that coincides with $(\mathbf{E},\mathbf{H})$ at the nodes of the triangulation) has a meaning. If furthermore $(\mathbf{E},\mathbf{H})$ belongs to $\mathbf{V}$, then $I_h(\mathbf{E},\mathbf{H})$ will be also in $\mathbf{V}$, hence in $\mathbf{V}_h$, since the normal vector is constant along the faces of Ω.

Recall that for any $t > \frac{3}{2}$, we also have the error estimate

$$\|(\mathbf{E},\mathbf{H}) - I_h(\mathbf{E},\mathbf{H})\|_{\ell,\Omega} \lesssim h^{t-\ell}\|(\mathbf{E},\mathbf{H})\|_{t,\Omega}, \tag{2.41}$$

for $\ell = 0$ or 1, see [17, Thm 3.2.1] in the case $t \in \mathbb{N}$ and easily extended to non-integer t.

These estimates directly allow to bound $\eta(\mathbf{V}_h)$.

Lemma 2.2.4. *In addition to the assumptions of Theorem 2.2.1, assume that $t_\Omega > \frac{3}{2}$. Then for all $t \in (\frac{3}{2}, t_\Omega)$ and all $k \geq k_0$, we have*

$$\eta(\mathbf{V}_h) \lesssim k^{1+\alpha}h^{t-1}(1+kh). \tag{2.42}$$

Proof. Fix an arbitrary datum $(\mathbf{F},\mathbf{G}) \in \mathbf{L}^2(\Omega)^2$ and denote $(\mathbf{E}^*,\mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F},\mathbf{G})$. Then owing to (2.41), we have

$$\begin{aligned}\|(\mathbf{E}^*,\mathbf{H}^*) - I_h(\mathbf{E}^*,\mathbf{H}^*)\|_k &\lesssim k\|(\mathbf{E}^*,\mathbf{H}^*) - I_h(\mathbf{E}^*,\mathbf{H}^*)\|_{0,\Omega} \\ &\quad +\|(\mathbf{E}^*,\mathbf{H}^*) - I_h(\mathbf{E}^*,\mathbf{H}^*)\|_{1,\Omega} \\ &\lesssim (kh^t + h^{t-1})\|(\mathbf{E}^*,\mathbf{H}^*)\|_{t,\Omega}.\end{aligned}$$

The estimate (2.24) allows to obtain the result. □

Corollary 2.2.5. *Under the assumptions of Lemma 2.2.4, for any fixed $t \in (\frac{3}{2}, t_\Omega)$, there exists $C > 0$ (small enough and depending only on Ω and t) such that if*

$$k^{\frac{2+\alpha}{t-1}}h \leq C, \tag{2.43}$$

then for all $k \geq k_0$ and all $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, problem (2.39) has a unique solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ and the following error estimate holds

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \lesssim k^{1+\alpha}h^{t-1}. \tag{2.44}$$

Proof. We first notice that the assumption (2.43) is equivalent to

$$k^{2+\alpha}h^{t-1} \leq C^{t-1}$$

and also implies that

$$kh \leq C,$$

since $t \leq 2$. As (2.42) means that there exists $C_0 > 0$ (independent of k, s, and h) such that

$$k\eta(\mathbf{V}_h) \leq C_0 k^{2+\alpha}h^{t-1}(1+kh),$$

we deduce that

$$k\eta(\mathbf{V}_h) \leq C_0 k^{2+\alpha}h^{t-1}(1+kh) \leq C_0 C^{t-1}(1+C).$$

As mentioned before, the existence of $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ then follows from Theorem 4.2 of [48] if

$$C_0 C^{t-1}(1+C) \leq \frac{1}{4C_c},$$

where C_c is the continuity constant of $\mathbf{a}_{k,s}$ (that here is equal to $\max\{1, s_1\}$).

Now, we use the arguments from Theorem 4.2 of [48]. Namely, we notice that

$$\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})) \geq \min\{1,s_0\}\|(\mathbf{U},\mathbf{W})\|_k^2 - 2k^2\left(\|\mathbf{U}\|_\Omega^2 + \|\mathbf{W}\|_{\Omega)}^2\right),$$

where for shortness we write $(\mathbf{U},\mathbf{W}) = \mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$. Therefore by (2.38), one has

$$\begin{aligned}
&\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}))\\
&\qquad = \operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})) + 2k^2\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}))\\
&\qquad = \operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})) + 2k^2\left(\|\mathbf{U}\|_\Omega^2 + \|\mathbf{W}\|_\Omega^2\right),
\end{aligned}$$

and by the previous estimate we deduce that

$$\min\{1,s_0\}\|(\mathbf{U},\mathbf{W})\|_k^2 \leq \operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})).$$

By Galerkin orthogonality, we can transform the right-hand side of this estimate as follows:

$$\begin{aligned}
&\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}))\\
&\qquad = \operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h))\\
&\qquad\quad + 2k^2\operatorname{Re}\mathbf{a}_{k,s}((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})) - (\mathbf{U}_h,\mathbf{W}_h)),
\end{aligned}$$

for any $(\mathbf{U}_h,\mathbf{W}_h),(\mathbf{Y}_h,\mathbf{Z}_h) \in \mathbf{V}_h$. By the continuity of the sesquilinear form a with respect to the norm $\|\cdot\|_k$, the previous estimate and identity yield

$$\|(\mathbf{U},\mathbf{W})\|_k^2 \lesssim \|(\mathbf{U},\mathbf{W})\|_k(\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k^2\|\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}) - (\mathbf{U}_h,\mathbf{W}_h)\|_k.$$

As $(\mathbf{U}_h, \mathbf{W}_h)$ and $(\mathbf{Y}_h, \mathbf{Z}_h)$ are arbitrary in $\mathbf{V}_h$, by taking the infimum, we deduce that

$$\begin{aligned}\|(\mathbf{U},\mathbf{W})\|_k &\lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)-(\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k^2\eta(\mathbf{V}_h)\|(\mathbf{U},\mathbf{W})\|_\Omega \\ &\lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)-(\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k\eta(\mathbf{V}_h)\|(\mathbf{U},\mathbf{W})\|_k.\end{aligned}$$

Hence for $k\eta(\mathbf{V}_h)$ small enough we deduce that

$$\|(\mathbf{U},\mathbf{W})\|_k \lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)-(\mathbf{Y}_h,\mathbf{Z}_h)\|_k. \tag{2.45}$$

By the estimates (2.24) and (2.41), we conclude that

$$\|(\mathbf{U},\mathbf{W})\|_k \lesssim (kh^t + h^{t-1})k^{1+\alpha} = k^{1+\alpha}h^{t-1}(1+kh) \lesssim k^{1+\alpha}h^{t-1}.$$

□

Remark 2.2.6. The interest of considering non divergence free right-hand side in problem (1.28) appears in the definition of $\eta(\mathbf{V}_h)$ (and its estimate) and in the above proof. In both cases, the problem comes from the fact that even for divergence free fields $\mathbf{f}_1, \mathbf{f}_2$, each component of $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ is not divergence free. As a consequence, $\mathbb{S}^*_{k,s}(\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2))$ depends on s, but this plays no rule in the estimate (2.44), except that s has to be fixed so that the stability estimate holds. Consequently at least theoretically $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ has to be computed with such an s, even if $\mathbb{S}_{k,h}(\mathbf{f}_1,\mathbf{f}_2)$ is independent of s in case of divergence free fields $\mathbf{f}_1, \mathbf{f}_2$, while practically (see below) it is fixed by comparing k^2 with the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition in Ω (or an approximation of it).

Remark 2.2.7. For the unit cuboid, as $\alpha = 1$ (see Corollary 1.4.8) and t can be as close as we want to $\frac{11}{6}$, the condition (2.43) is mostly $k^{\frac{18}{5}}h$ small enough.

Remark 2.2.8. Let us notice that the estimate (2.45) is valid under the above assumptions, but if $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ belongs to $\mathbf{H}^{p+1}(\Omega)^2$ and polynomials of degree p will be used to define $\mathbf{V}_h$, then the rate of convergence in h in the estimate (2.44) will be improved, passing from h^{t-1} to h^p.

$\mathbb{P}_1$-elements with refined meshes

Here we assume that the assumptions of Theorem 2.2.2 hold and want to take advantage of the regularity of $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ in $H^{2,\nu}(\Omega)^6$, for any $\nu > 2 - t_\Omega$ (see estimate (2.27)). More precisely following the arguments from [44, Thm 3.3] (see also [2]) using a family of refined meshes $\mathcal{T}_h$ satisfying the refined rules

$$h_T \lesssim h \inf_{\mathbf{x}\in T} r(\mathbf{x})^\nu \quad \text{if } T \text{ is far away from the corners of } \Omega, \tag{2.46}$$

$$h_T \lesssim h^{\frac{1}{1-\nu}} \quad \text{if } T \text{ has a corners of } \Omega \text{ as vertex}, \tag{2.47}$$

with a fixed but arbitrary $\nu \in (2 - t_\Omega, 1)$ (as close as we want from $2 - t_\Omega$), we have that

$$\|(\mathbf{E},\mathbf{H}) - I_h(\mathbf{E},\mathbf{H})\|_{\ell,\Omega} \lesssim h^{2-\ell}\|(\mathbf{E},\mathbf{H})\|_{2,\nu;\Omega},$$

for $\ell = 0$ or 1. Consequently as in the previous subsection, for $\mathbf{V}_h$ build on such meshes, there exists a positive constant C (independent of k, s and h) such that if

$$k^{2+\alpha} h \leq C,$$

then for all $k \geq k_0$ and all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, problem (2.39) has a unique solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ and the following error estimate holds

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)\|_k \lesssim k^{1+\alpha} h. \tag{2.48}$$

$\mathbb{P}_2$-elements with refined meshes

Under the assumptions of Theorem 2.2.2 we can improve the previous orders of convergence and reduce the constraint between k and h. For that purposes, we use the splitting (2.31) of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ and the estimates (2.29) and (2.30) (recalling (2.33)). Then as in the previous subsection, we need to use a family of refined meshes $\mathcal{T}_h$ satisfying the refined rules

$$h_T \lesssim h \inf_{\mathbf{x} \in T} r(\mathbf{x})^{\frac{\nu_0}{2}} \quad \text{if } T \text{ is far away from the corners of } \Omega, \tag{2.49}$$

$$h_T \lesssim h^{\frac{2}{2-\nu_0}} \quad \text{if } T \text{ has a corners of } \Omega \text{ as vertex,} \tag{2.50}$$

with a fixed but arbitrary $\nu_0 \in (3 - t_\Omega, 2)$. In such a situation, again by (2.41) and by [44, Thm 3.3] we have

$$\|(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}}) - I_h(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}})\|_{\ell,\Omega} \lesssim h^{3-\varepsilon-\ell} \|(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}})\|_{3-\varepsilon,\Omega}, \tag{2.51}$$

$$\|(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}}) - I_h(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}})\|_{\ell,\Omega} \lesssim h^{3-\ell} \|(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}})\|_{3,\nu_0;\Omega}, \tag{2.52}$$

for $\ell = 0$ or 1.

Let us now show that (2.49) (resp. (2.50)) guarantees that (2.46) (resp. (2.47)) holds with $\nu = \nu_0 - 1$. In the first case, we simply notice that

$$r(\mathbf{x})^{\frac{\nu_0}{2}} = r(\mathbf{x})^{\frac{\nu+1}{2}},$$

and therefore

$$r(\mathbf{x})^{\frac{\nu+1}{2}} \lesssim r(\mathbf{x})^{\nu}$$

if and only if

$$r(\mathbf{x})^{\nu+1} \lesssim r(\mathbf{x})^{2\nu}.$$

This last estimate is valid for any $x \in T$ because ν belongs to $(0, 1)$ and $r(\mathbf{x})$ is bounded. The second implication is a simple consequence of the fact that

$$h^{\frac{2}{2-\nu_0}} = h^{\frac{2}{1-\nu}} \lesssim h^{\frac{1}{1-\nu}}.$$

Since our family of meshes then satisfies (2.46) and (2.47) with $\nu = \nu_0 - 1 > 2 - t_\Omega$, we deduce that

$$\|\mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2) - I_h \mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2)\|_{\ell,\Omega} \lesssim h^{2-\ell} \|\mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2)\|_{2,\nu;\Omega}, \tag{2.53}$$

for $\ell = 0$ or 1. With such estimates in hand, we can estimate the adjoint approximability.

Lemma 2.2.9. *For $\mathbf{V}_h$ build on meshes satisfying* (2.49) *and* (2.50), *we have*

$$\eta(\mathbf{V}_h) \lesssim (1+kh)\left(h + k^3 h^{2-\varepsilon}\right). \tag{2.54}$$

Proof. Fix an arbitrary datum $(\mathbf{F},\mathbf{G}) \in \mathbf{L}^2(\Omega)^2$, we denote $(\mathbf{E}^*,\mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F},\mathbf{G})$. Then we use its splitting

$$(\mathbf{E}^*,\mathbf{H}^*) = \mathbb{S}^*_0(\mathbf{F},\mathbf{G}) + (\mathbf{R}^*_{E,\mathrm{reg}},\mathbf{R}^*_{H,\mathrm{reg}}) + (\mathbf{R}^*_{E,\mathrm{sing}},\mathbf{R}^*_{H,\mathrm{sing}}).$$

Owing to (2.51), (2.52), and (2.53), we have

$$\begin{aligned}\|(\mathbf{E}^*,\mathbf{H}^*) - I_h(\mathbf{E}^*,\mathbf{H}^*)\|_k &\lesssim (1+kh)h\|\mathbb{S}_0(\mathbf{f}_1,\mathbf{f}_2)\|_{2,\nu;\Omega} \\ &+ (1+kh)h^{2-\varepsilon}\|(\mathbf{R}^*_{E,\mathrm{reg}},\mathbf{R}^*_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} \\ &+ (1+kh)h^2\|(\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}})\|_{3,\nu_0;\Omega}.\end{aligned}$$

The estimates (2.29) and (2.30) allow to obtain the result. □

Consequently as in the previous subsection, for $\mathbf{V}_h$ build on such meshes, there exists a positive constant C (independent of k, s and h) such that if

$$k^4 h^{2-\varepsilon} \le C,$$

then for all $k \ge k_0$ and all $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, problem (2.39) has a unique solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ with the error estimate

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \lesssim k^3 h^{2-\varepsilon}.$$

Remark 2.2.10. Note that the impedance boundary conditions are imposed as essential boundary conditions. As we are dealing with polyhedral domains, Lagrange elements can be used to construct conforming subspaces $\mathbf{V}_h$. The extension to curved domains seems to be difficult, but a penalisation technique can be used (cf. Chapter 3).

2.2.3 Some numerical tests

For the sake of simplicity, we restrict ourselves to the TE/TH polarization of the problem (1.30). In other words, we take

$$\Omega = D \times \mathbb{R},$$

where D is a two-dimensional polygon and assume that the solution of our problem is independent of the third variable. In such a case, the original problem splits up into a TE polarization problem in (E_1,E_2,H_3) in D (correspond to (1.60)), and a TH polarization one in (H_1,H_2,E_3) in D (correspond to (1.59)). We restrict ourselves to the TE polarization here, as the TH is fully similar. The variational form is given by (1.62).
Furthermore the singularities of such problems correspond to the edge singularities of the original one.

We first use a toy experiment in the unit square $D = (0,1)^2$ to illustrate our results. In such a case, as exact solution, we take

$$\begin{aligned} E_1(x_1,x_2) &= -\ell\pi\cos(\ell\pi x_1)\sin(\ell\pi x_2), \\ E_2(x_1,x_2) &= \ell\pi\sin(\ell\pi x_1)\cos(\ell\pi x_2), \\ H_3(x_1,x_2) &= \sin(\ell\pi x_1)\sin(\ell\pi x_2), \end{aligned}$$

where $\ell \in \mathbb{N}^*$. With such a choice, we notice that (E_1, E_2) is divergence free, that

$$\Delta E_1 + k^2 E_1 = \Delta E_1 + k^2 E_2 = \Delta H_3 + k^2 H_3 = 0,$$

with $k^2 = 2\ell^2\pi^2$ and that they satisfy the impedance boundary condition for all λ_{imp} satisfying (1.2), then we choose $\lambda_{\text{imp}} = -1$ for this test. We then compute the right-hand side of (1.26) accordingly (where only a boundary term occurs). In our numerical experiments, we have chosen either $\ell = 2$, 5, 8, 10, 15 or 29 and $s = 14.3$. This choice of s is made because it yields satisfactory numerical results, but it is also in accordance with the condition that $-\frac{k^2}{s}$ is different from the eigenvalues of the Laplace operator Δ with Dirichlet boundary conditions in D, which in this case means that

$$\frac{k^2}{s} \neq (\ell_1^2 + \ell_2^2)\pi^2, \tag{2.55}$$

for all positive integers ℓ_1, ℓ_2. Indeed in the first case $\ell = 2$, the ratio $\frac{k^2}{s}$ is smaller than the smallest eigenvalue $2\pi^2$, while in the other cases, it is strictly between two eigenvalues.

In Figures 2.1 to 2.3, we have depicted the different orders of convergence for different values of h, k, and $p = 1, 2$, and 4. From these figures, we see that if polynomials of order p are used, then in the asymptotic regime, the convergence rate is p for h small enough as theoretically expected, since the solution is smooth (see Remark 2.2.8).

The second main result from subsections 2.2.2 and 2.2.2 states that if $k^{p+2}h^p \lesssim 1$ with $p = 1$ or 2 (up to ε for $p = 2$), then

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \lesssim \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_k, \tag{2.56}$$

where $\mathbb{P}_h$ is the orthogonal projection on $\mathbf{V}_h$ for the inner product associated with the norm $\|\cdot\|_k$, namely for $(\mathbf{U},\mathbf{V}) \in \mathbf{V}$, $\mathbb{P}_h(\mathbf{U},\mathbf{V})$ is the unique solution of

$$(\mathbb{P}_h(\mathbf{U},\mathbf{V}), (\mathbf{U}'_h,\mathbf{V}'_h))_k = ((\mathbf{U},\mathbf{V}), (\mathbf{U}'_h,\mathbf{V}'_h))_k, \ \forall (\mathbf{U}'_h,\mathbf{V}'_h) \in \mathbf{V}_h,$$

where

$$\begin{aligned} ((\mathbf{U},\mathbf{V}), (\mathbf{U}',\mathbf{V}'))_k &= \int_\Omega (\operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\bar{\mathbf{U}}' + s\operatorname{div}\mathbf{U}\operatorname{div}\bar{\mathbf{U}}' + k^2\mathbf{U}\cdot\bar{\mathbf{U}}')\,dx \\ &+ \int_\Omega (\operatorname{curl}\mathbf{V}\cdot\operatorname{curl}\bar{\mathbf{V}}' + s\operatorname{div}\mathbf{V}\operatorname{div}\bar{\mathbf{V}}' + k^2\mathbf{V}\cdot\bar{\mathbf{V}}')\,dx. \end{aligned}$$

In order to see if this bound is sharp or not, we compute $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ and $\mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ for different values of h, p, and k. For each k and p, we denote by $h^\star(k)$ the greatest value h_0 such that

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)\|_k \leq 2\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_k, \quad \forall h \leq h_0. \tag{2.57}$$

The value of $h^\star(k)$ for a given k is obtained by inspecting the ratio

$$\frac{\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)\|_k}{\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_k}.$$

Condition (2.57) state that the finite element solution must be quasi optimal in the $\|\cdot\|_k$ norm, uniformly in k (with the arbitrary constant 2).

The graph of $h^\star(k)$ is represented in Figure 2.4(a), 2.4(b) and 2.4(c) for $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_4$ elements, respectively. We observe that in both cases $h^\star(k) \sim k^{-1-1/p}$, which is better than the condition $k^{p+2}h^p \lesssim 1$ that would furnish $h^\star(k) \sim k^{-1-2/p}$. Indeed, it means that quasi-optimality in the sense of (2.57) is achieved under the condition that $h \leq h^\star(k) \sim k^{-1-1/p}$, which is equivalent to $k^{p+1}h^p \leq k^{p+1}\left[h^\star(k)\right]^p \lesssim 1$, that is better than $k^{p+2}h^p \lesssim 1$. We thus conclude that our stability condition seems to be not sharp and can probably be improved. Note that our experiments indicate that this stability condition remains valid for values of p larger than the theoretical one, that is here equal to 2.

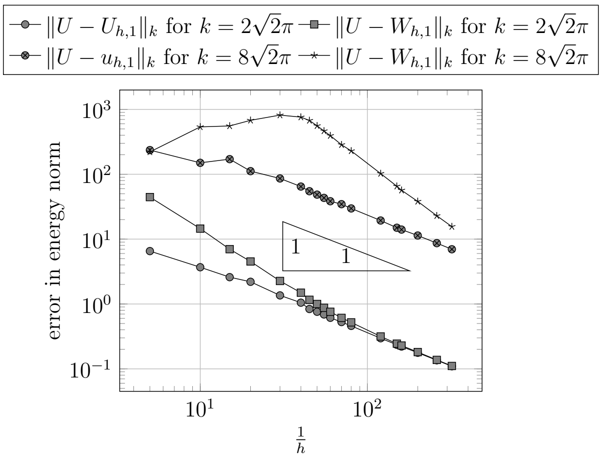


Figure 2.1: Rates of convergence for $p = 1, k = 2\sqrt{2}\pi$ or $8\sqrt{2}\pi$ ($U = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2), U_{h,p} = \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2), W_{h,p} = \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$).

As a second example, we take on the square $(-1, 1)^2$ the exact solution given by

$$\begin{aligned} E_1(x_1, x_2) &= x_2 e^{ikx_1}, \\ E_2(x_1, x_2) &= -x_1 e^{ikx_1}, \\ H_3(x_1, x_2) &= \lambda_{\mathrm{imp}} e^{ikx_1}, \end{aligned}$$

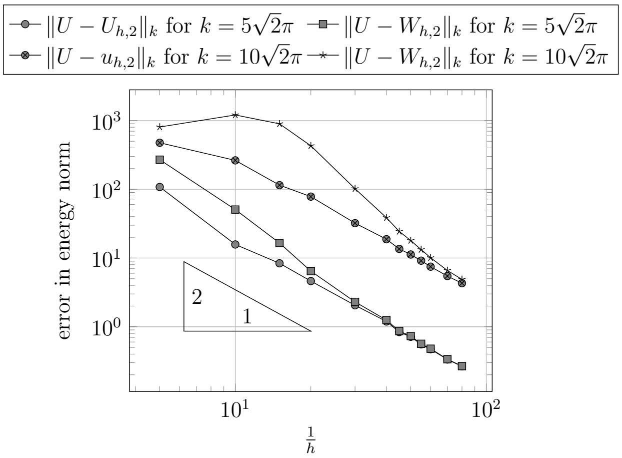


Figure 2.2: Rates of convergence for $p = 2, k = 5\sqrt{2}\pi$ or $10\sqrt{2}\pi$.

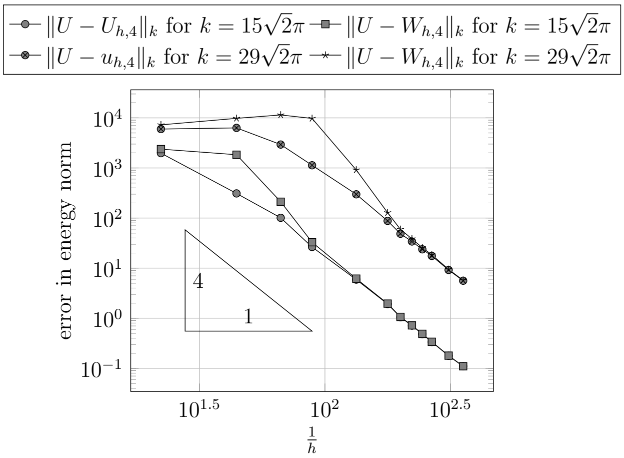


Figure 2.3: Rates of convergence for $p = 4, k = 15\sqrt{2}\pi$ or $29\sqrt{2}\pi$.

that satisfies the homogeneous impedance boundary condition

$$H_3 - \lambda_{\text{imp}}\mathbf{E_t} = 0 \text{ on } \partial D.$$

We have computed the numerical approximation of this solution for $k = 30$, the choice $s = 14.3$ (again with this choice, $\frac{k^2}{s}$ is smaller than the smallest eigenvalue $2\pi^2$), and for different values of λ_{imp}, namely we have chosen $\lambda_{\text{imp}} = -1, -10, -50$, and -100. In Figure 2.5, we have depicted the different orders of convergence for $p = 1, 2$, and 4 and different values of h. Again since the solution is regular, the rate of convergence p is observed in the asymptotic regime and seems not to be affected by the variation of λ_{imp}.

Finally, we have tested the case when a corner singularity appears. Namely on the L-shaped domain $L = (-1, 1)^2 \setminus ((0, 1) \times (-1, 0))$, we take as exact solution (written in polar coordinates (r, θ) centred at $(0, 0)$)

$$\mathbf{E}(r, \theta) = \nabla \left(r^{\frac{4}{3}} \sin(\frac{4\theta}{3}) e^{ikr} \right),$$
$$H_3(r, \theta) = 0.$$

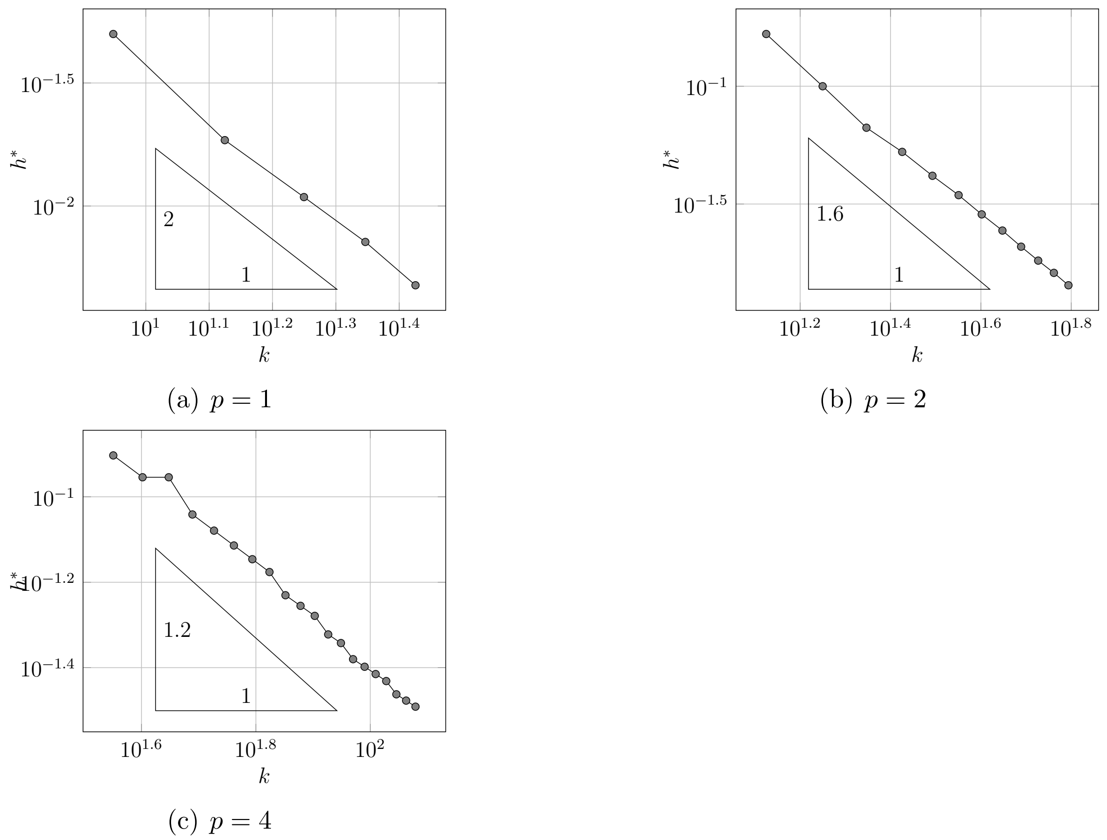


Figure 2.4: Asymptotic range of $h^*(k)$ for $p = 1, 2, 4$.

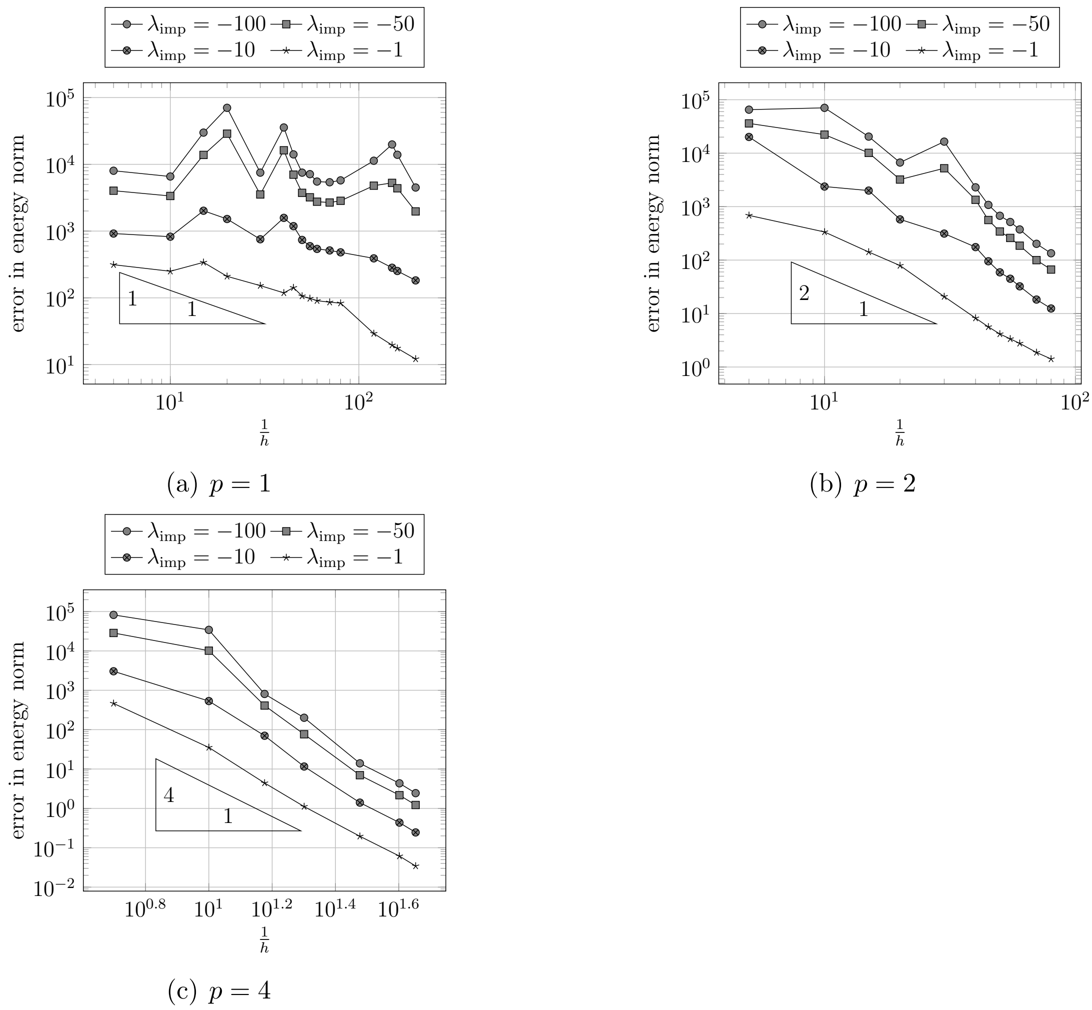


Figure 2.5: Rates of convergence for $\lambda_{imp} = -1, -10, -50, -100$ with $p = 1, 2, 4$.

This solution exhibits the typical edge singularity of our Maxwell system described in subsection 2.1.2.

This solution does not satisfies the homogeneous impedance boundary condition (with $\lambda_{\text{imp}} = -1$), hence we have imposed to our numerical solutions $(\mathbf{E}_h, H_{3h})$ to satisfy

$$H_{3h}(v) + \mathbf{E}_{h,\mathbf{t}}(v) = \mathbf{E}_{\mathbf{t}}(v),$$

at all nodes of the boundary of L. The convergence rates for $k = 1, 50$ and 100 are presented in Figures 2.6 and 2.7 for different values of h and p. There we observe, in the asymptotic regime, that for $k = 1$, the use of quasi-uniform meshes affects the rate of convergence since for $p = 1$ it is equal to $\frac{1}{3}$, while the use of refined meshes restores the optimal rate of convergence 1 (as theoreticaly expected). On the contrary for $k = 50$ or 100, we see, again in the asymptotic range, that the rate of convergence is p. This observation is in accordance with a recent result proved in [15] for Helmholtz problems in polygonal domains, which shows that in high frequency the dominant part of the solution is the regular part of the solution (which in our case is zero). Note that we have also chosen $s = 14.3$. Indeed

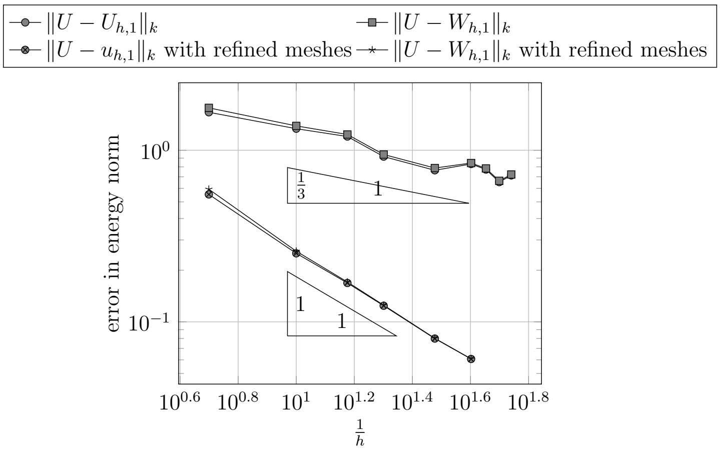
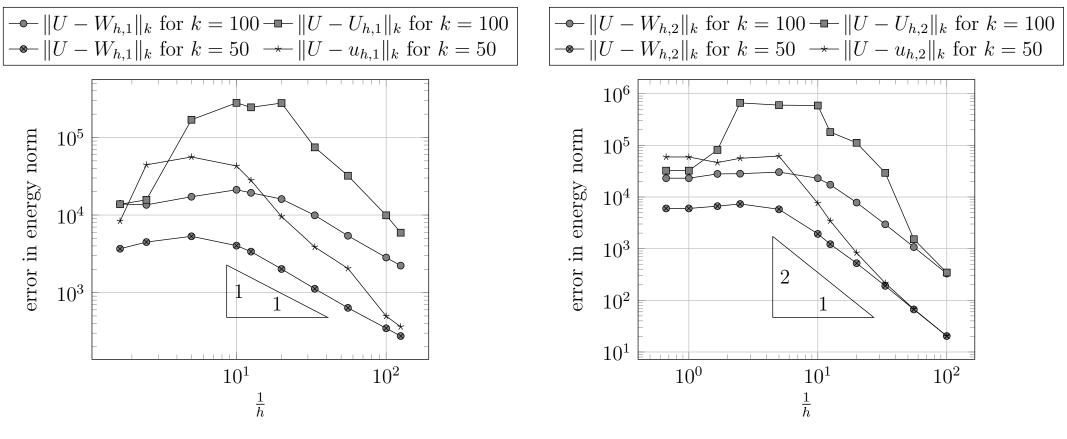


Figure 2.6: Rates of convergence for the singular solution in the L-shaped domain for $k = 1$ with uniform and refined meshes for $p = 1$.



(a) $p = 1$

(b) $p = 2$

Figure 2.7: Rates of convergence for the singular solution in the L-shaped domain for $k = 50$ or 100 with $p = 1$ (left) and $p = 2$ (right).

for $k = 1$, the spectral condition on $\frac{k^2}{s}$ holds since the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions in L is approximatively equal to 9.6387, see [29, 68]. We are not able to check if the spectral condition is valid for $k = 50$ or 100 since the approximated values of the eigenvalues of the Laplace operator with Dirichlet boundary conditions in L seem to be only available up to 97, see [68, Table 1], but since our numerical results are satisfactory, we suppose that it is satisfied.

Chapter 3

Maxwell's system in smooth domains

3.1 The discrete problem

3.1.1 The hp-nonconforming finite element method

To approximate problem (1.28) we will use a nonconforming finite element method, because we can not impose the impedance boundary condition (the essential boundary condition) in the finite element space. Futhermore we cannot build an interpolation operator which preserves the essential condition. So, we have decided to penalize this condition.

Let $\mathcal{T}_h$ be a partition of Ω into "simplicial" elements which are the image of the reference tetrahedron, denoted by $\hat{K}$, via an element map $F_K : \hat{K} \to K$ that satisfies (see Assumption 5.1 in [50]) the next assumption:

Hypothesis 3.1.1. *(Quasi-uniform regular triangulation) For each $K \in \mathcal{T}_h$, there exist mappings R_K and A_K which verify $F_K = R_K \circ A_K$, $\tilde{K} = A_K(K)$ with (recalling that J_f is the Jacobian of f)*

- *A_K is an affine transformation and R_K is a $\mathcal{C}^\infty$ transformation,*
- $\|J_{A_K}\|_{\infty,\hat{K}} \le C_{\mathit{affine}} h, \quad \|(J_{A_K})^{-1}\|_{\infty,\hat{K}} \le C_{\mathit{affine}} h^{-1},$
- $\|(J_{R_K})^{-1}\|_{\infty,\tilde{K}} \le C_{\mathit{metric}}, \quad \|\nabla^n R_K\|_{\infty,\tilde{K}} \le C_{\mathit{metric}} \beta^n n!, \ \forall n \in \mathbb{N},$

with C_{affine}, C_{metric}, $\beta > 0$ independent of the maximal meshsize $h = \max\limits_{K \in \mathcal{T}_h} h_K$, where h_K is the diamenter of the element K.

Let $\mathbf{S}_{h,p}$ be the hp-FEM space (without constraint on the boundary)

$$\mathbf{S}_{h,p} = \mathcal{S}_{h,p}(\Omega)^6, \tag{3.1}$$

with

$$\mathcal{S}_{h,p}(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; v_{|_K} \circ F_K \in \mathbb{P}^p \,, \forall K \in \mathcal{T}_h \right\}. \tag{3.2}$$

As we cannot add the essential boundary condition to our finite element space, we will use a discrete sesquilinear form, where we penalyse this boundary condition. Therefore we define the discrete sesquilinear form $\mathbf{a}_{k,s,h,p}(\cdot,\cdot) : H^1(\Omega)^6 \times H^1(\Omega)^6 \to \mathbb{C}$ as follow

$$\begin{aligned}\mathbf{a}_{k,s,h,p}(\mathbf{u},\mathbf{v}) = \mathbf{a}_{k,s}(\mathbf{u},\mathbf{v}) &- \int_{\partial\Omega} (\operatorname{curl}\mathbf{E}\times\mathbf{n} + ik\mathbf{E}_t)\,\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma \\ &- \int_{\partial\Omega} (\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n})\overline{(\operatorname{curl}\mathbf{E}'\times\mathbf{n} + ik\mathbf{E}'_t)}\ d\sigma \\ &+ \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f \int_f (\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n})\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma,\end{aligned}$$

with $\mathbf{u} = (\mathbf{E},\mathbf{H})$ and $\mathbf{v} = (\mathbf{E}',\mathbf{H}')$, and where $\mathcal{E}^B$ is the set of faces of the triangulation included into $\partial\Omega$. Note that the last term of this right-hand side is a penalization term, while the two other added ones are introduced to guaranbee the consistency of the approximation scheme. The parameters α_f are positive constants that will be fixed large enough to ensure the coercivity of the form $\mathbf{a}_{k,s,h,p}$ (cf. (3.4) below).

Let us first check the consistency of the formulation, that is

Lemma 3.1.2. *Let $\mathbf{f} \in \mathrm{L}^2(\Omega)^6$ and $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ (i.e., solution of (1.28)), then*

$$\mathbf{a}_{k,s,h,p}(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}),\ \forall \mathbf{v} \in \mathbf{H}^1(\Omega)^2.$$

Proof. Indeed, as $\mathbf{u} = (\mathbf{E},\mathbf{H})$ satisfies $\mathbf{H}\times\mathbf{n} - \lambda_{\text{imp}}\mathbf{E}_t = 0$ on $\partial\Omega$, one has

$$\mathbf{a}_{k,s,h,p}(\mathbf{u},\mathbf{v}) = \mathbf{a}_{k,s}(\mathbf{u},\mathbf{v}) - \int_{\partial\Omega} (\operatorname{curl}\mathbf{E}\times\mathbf{n} + ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma.$$

As $\mathbf{f} \in \mathrm{L}^2(\Omega)^6$ then $(\mathbf{E},\mathbf{H}) \in \mathbf{H}^2(\Omega)^2$ (cf. [22]) and by Green's formula,

$$\begin{aligned}&\int_\Omega \left(\operatorname{curl}\mathbf{E}\cdot\operatorname{curl}\overline{\mathbf{E}'} + s\operatorname{div}\mathbf{E}\ \operatorname{div}\overline{\mathbf{E}'} - k^2\mathbf{E}\cdot\overline{\mathbf{E}'}\right)\ dx \\ &\qquad = \int_\Omega L_{k,s}\mathbf{E}\cdot\overline{\mathbf{E}'}\ dx + \int_{\partial\Omega}\left(\operatorname{curl}\mathbf{E}\times\mathbf{n}\cdot\overline{\mathbf{E}'_t} + s\operatorname{div}\mathbf{E}\,\overline{\mathbf{E}'}\cdot\mathbf{n}\right)\ d\sigma.\end{aligned}$$

Applying the previous identity to $\mathbf{E}$ and $\mathbf{H}$, noticing that $\operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}
\mathbf{a}_{k,s,h,p}(\mathbf{u},\mathbf{v}) &= \int_\Omega \left(L_{k,s}(\mathbf{E})\cdot\overline{\mathbf{E}'} + L_{k,s}(\mathbf{H})\cdot\overline{\mathbf{H}'}\right)\ dx \\
&\quad - \int_{\partial\Omega} ((\operatorname{curl}\mathbf{E}\times\mathbf{n}) + ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma \\
&\quad + \int_{\partial\Omega} \left((\operatorname{curl}\mathbf{E}\times\mathbf{n})\cdot\overline{\mathbf{E}'_t} + s\operatorname{div}\mathbf{E}\,\overline{E'\cdot n}\right)\ d\sigma \\
&\quad + \int_{\partial\Omega} \left((\operatorname{curl}\mathbf{H}\times\mathbf{n})\cdot\overline{H'_t} + s\operatorname{div}\mathbf{H}\,\overline{\mathbf{H}'\cdot\mathbf{n}}\right)\ d\sigma \\
&\quad - ik\int_{\partial\Omega}\left(\lambda_{\text{imp}}\mathbf{E}_t\cdot\overline{\mathbf{E}'_t} + \frac{1}{\lambda_{\text{imp}}}\mathbf{H}_t\cdot\overline{\mathbf{H}'_t}\right)\ d\sigma \\
&= \int_\Omega L_{k,s}(\mathbf{E})\cdot\overline{\mathbf{E}'}\ dx + \int_\Omega L_{k,s}(\mathbf{H})\cdot\overline{\mathbf{H}'}\ dx + \int_{\partial\Omega} B_k(\mathbf{E},\mathbf{H})\cdot\overline{\mathbf{H}'_t}\ d\sigma.
\end{aligned}$$

As $B_k(\mathbf{E},\mathbf{H}) = 0$, we conclude the consistency of the problem. $\square$

The discrete norm (related to the space $\mathbf{S}_{h,p}$) associated with the discrete sesquilinear form $\mathbf{a}_{k,s,h,p}$ is

$$\|\mathbf{u}\|^2_{k,h,p} = \|\mathbf{u}\|^2_k + \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\left\|\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n}\right\|^2_f.$$

Remark 3.1.3. *We can remark that for all $\mathbf{v}\in\mathbf{V}$, $\|\mathbf{v}\|_{k,h,p} = \|\mathbf{v}\|_k$.*

In order to compensate the negative term in $\mathbf{a}_{k,s,h,p}(\cdot,\cdot)$, we introduce the sesquilinear form $\mathbf{b}_{k,s,h,p}(\cdot,\cdot) = \mathbf{a}_{k,s,h,p}(\cdot,\cdot) + 2k^2(\cdot,\cdot)$, which turns to be continuous and coercive. Before proving these properties, we introduce a useful technical lemma.

Lemma 3.1.4. *Let $\mathbf{E},\mathbf{E}',\mathbf{H}'\in\mathcal{S}_{h,p}(\Omega)^3$, then*

$$\begin{aligned}
&\left|\int_{\partial\Omega}(-\operatorname{curl}\mathbf{E}\times\mathbf{n} + ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma\right| \\
&\quad\lesssim \frac{p}{\sqrt{h}}\left(\int_\Omega(|\operatorname{curl}\mathbf{E}|^2 + k^2|\mathbf{E}|^2)\ dx\right)^{\frac12}\times\left(\sum_{f\in\mathcal{E}^B}\left\|\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n}\right\|^2_f\ d\sigma\right)^{\frac12}.
\end{aligned}$$

Proof. First, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\int_{\partial\Omega}(-\operatorname{curl}\mathbf{E}\times\mathbf{n} + ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n})}\ d\sigma\right| \\
&\quad\lesssim \sum_{f\in\mathcal{E}^B}\left[\left(\int_f(|\operatorname{curl}\mathbf{E}\times\mathbf{n}|^2 + k^2|\mathbf{E}_t|^2)\ d\sigma\right)^{\frac12}\times\left(\left\|\mathbf{E}'_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}'\times\mathbf{n}\right\|^2_f\right)^{\frac12}\right].
\end{aligned}$$

By using a covariant transformation, which preserves the curl, namely

$$\operatorname{curl}\mathbf{E}(x) = \frac{DF_K(\hat{x})}{J_{F_K}(\hat{x})}\hat{\operatorname{curl}}\,\hat{\mathbf{E}}(\hat{x}), \text{ for } x = F_K(\hat{x}), \tag{3.3}$$

with an inverse trace inequality (cf. Lemma 4.3 of [51]), we have

$$\int_f \left(|\operatorname{curl}\mathbf{E}\times\mathbf{n}|^2 + k^2|\mathbf{E}_t|^2\right)\, d\sigma \lesssim \frac{p^2}{h}\int_{K_f}\left(|\operatorname{curl}\mathbf{E}|^2 + k^2|\mathbf{E}|^2\right)\, dx,$$

where $K_f \in \mathcal{T}_h$ is the unique tetrahedron such that $f \subset \partial K_f$. The conclusion follows from the two above inequalities. $\square$

Now, we can show the coercivity of $\mathbf{b}_{k,s,h,p}$. Let $\mathbf{u} = (\mathbf{E}, \mathbf{H}) \in \mathbf{S}_{h,p}$ be fixed. Then

$$\begin{aligned}\operatorname{Re}(\mathbf{b}_{k,s,h,p}(\mathbf{u},\mathbf{u})) = \|\mathbf{u}\|_k^2 - 2\operatorname{Re}\left(\int_{\partial\Omega}(\operatorname{curl}\mathbf{E}\times\mathbf{n} - ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n})}\, d\sigma\right)\\ + \frac{p^2}{h}\operatorname{Re}\left(\sum_{f\in\mathcal{E}^B}\alpha_f\int_f\left|\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n}\right|^2\, d\sigma\right).\end{aligned}$$

We then need to estimate $A = \operatorname{Re}\left(\int_{\partial\Omega}(-\operatorname{curl}\mathbf{E}\times\mathbf{n} + ik\mathbf{E}_t)\cdot\overline{(\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n})}\, d\sigma\right)$. But Lemma 3.1.4 and Young's inequality yield

$$\begin{aligned}A &\lesssim \frac{p}{\sqrt{h}}\left(\int_\Omega(|\operatorname{curl}\mathbf{E}|^2 + k^2|\mathbf{E}|^2)\, dx\right)^{\frac{1}{2}}\left(\sum_{f\in\mathcal{E}^B}\left\|\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n}\right\|_f^2\right)^{\frac{1}{2}}\\ &\lesssim \frac{\epsilon}{2}\int_\Omega\left(|\operatorname{curl}\mathbf{E}|^2 + k^2|\mathbf{E}|^2\, dx\right) + \frac{1}{2\epsilon}\frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\left\|\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n}\right\|_f^2,\end{aligned}$$

for all $\epsilon > 0$. Hence there exists a positive constant C such that

$$\begin{aligned}\operatorname{Re}(\mathbf{b}_{k,s,h,p}(\mathbf{u},\mathbf{u})) \geq \|\mathbf{u}\|_k^2 - C\epsilon(\|\operatorname{curl}\mathbf{E}\|_\Omega^2 + k^2\|\mathbf{E}\|_\Omega)\\ + \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}(\alpha_f - \frac{C}{\epsilon})\left\|\mathbf{E}_t - \frac{1}{\lambda_{\text{imp}}}\mathbf{H}\times\mathbf{n}\right\|_f,\end{aligned}$$

for all $\epsilon > 0$. We then fix $\epsilon = \frac{1}{2C}$ and therefore by choosing $\alpha_f > 0$ large enough such that $\alpha_f \geq \frac{2C}{\epsilon} = 4C^2$, we deduce that

$$\operatorname{Re}(\mathbf{b}_{k,s,h,p}(\mathbf{u},\mathbf{u})) \gtrsim \|\mathbf{u}\|_{k,h,p}^2. \tag{3.4}$$

The continuity of $\mathbf{b}_{k,s,h,p}$, namely

$$|\mathbf{b}_{k,s,h,p}(\mathbf{u},\mathbf{v})| \lesssim \|\mathbf{u}\|_{k,h,p}\|\mathbf{v}\|_{k,h,p}, \forall\mathbf{u},\mathbf{v}\in\mathbf{S}_{h,p}, \tag{3.5}$$

directly follows from the continuity of $\mathbf{a}_{k,s}$ and Lemma 3.1.4. Note that this argument also allows to show the continuity of $\mathbf{a}_{k,s,h,p}$.

Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, we define the following approximated problem: Find $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ such that

$$\mathbf{a}_{k,s,h,p}(\mathbf{u}_{h,p}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall \mathbf{v} \in \mathbf{S}_{h,p}. \tag{3.6}$$

Such $\mathbf{u}_{h,p}$, if it exists, is called a Galerkin solution.
We will now show that under an appropriate condition, (3.6) has an unique solution $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ and give some error estimates.

Lemma 3.1.5. *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ and if $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ is a solution of* (3.6)*, then we have*

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + k \sup_{\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}} \frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_\Omega}. \tag{3.7}$$

Proof. Let $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$ be arbitrary, then by the triangle inequality, we have

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \leq \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p}.$$

Moreover

$$\begin{aligned}\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|^2_{k,h,p} &\lesssim \mathcal{R}(\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})) \\ &\lesssim |\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| + |\mathbf{b}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|.\end{aligned}$$

By the fact that $\mathbf{b}_{k,s,h,p} = \mathbf{a}_{k,s,h,p} + 2k^2(\cdot,\cdot)$ and the Galerkin orthogonality, we have

$$\begin{aligned}\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|^2_{k,h,p} &\lesssim |\mathbf{b}_{k,s,h,p}(\mathbf{v}_{h,p} - \mathbf{u}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| + 2k^2|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})| \\ &\lesssim \|\mathbf{v}_{h,p} - \mathbf{u}\|_{k,h,p} \|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_{k,h,p} + k^2|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|.\end{aligned}$$

We then have

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p} + k\frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|}{\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_\Omega}.$$

We conclude by the bound

$$\frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{v}_{h,p} - \mathbf{u}_{h,p})|}{\|\mathbf{v}_{h,p} - \mathbf{u}_{h,p}\|_\Omega} \leq \sup_{\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}} \frac{(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_\Omega},$$

and then by taking the infimum on $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$. □

In order to control the second term of the right-hand side of (3.7), we introduce the quantity $\eta(\mathbf{S}_{h,p})$, called adjoint approximation quantity (cf. [48, 52, 14]):

$$\eta(\mathbf{S}_{h,p}) = \sup_{\mathbf{f} \in L^2(\Omega)^6} \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \frac{\left\|\mathbb{S}^*_{k,s}(\mathbf{f}) - \mathbf{v}_{h,p}\right\|_{k,h,p}}{\|\mathbf{f}\|_\Omega}, \tag{3.8}$$

where $\mathbb{S}^*_{k,s}(\mathbf{f}) = \overline{\mathbb{S}_{k,s}(\bar{\mathbf{f}})}$ is the adjoint operator of $\mathbb{S}_{k,s}(\mathbf{f})$.

Now we will use the Schatz argument (Aubin-Nitsche trick for the Helmholtz equation) [66] in order to bring out $\eta(\mathbf{S}_{h,p})$ and $\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}$ in (3.7) and obtain the following theorem.

Theorem 3.1.6. *There exists a positive constant C such that if $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, then for any $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, if $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ and if $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ is a solution of* (3.6)*, then*

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{v \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}\|_{k,h,p}, \tag{3.9}$$

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega} \lesssim \eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}. \tag{3.10}$$

Proof. Let $\boldsymbol{\phi} = \mathbb{S}^*_{k,s}(\mathbf{w}_{h,p})$, with $\mathbf{w}_{h,p} \in \mathbf{S}_{h,p}$, then for any $\boldsymbol{\phi}_{h,p} \in \mathbf{S}_{h,p}$ owing to the continuity of $\mathbf{a}_{k,s,h,p}$ and the Galerkin orthogonality, one has

$$\begin{aligned}
|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})| &= |\mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \boldsymbol{\phi})| \\
&= |\mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \boldsymbol{\phi} - \boldsymbol{\phi}_{h,p})| \\
&\lesssim \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \|\boldsymbol{\phi} - \boldsymbol{\phi}_{h,p}\|_{k,h,p}.
\end{aligned}$$

By the definition of $\eta(\mathbf{S}_{h,p})$ we can conclude that

$$k\frac{|(\mathbf{u} - \mathbf{u}_{h,p}, \mathbf{w}_{h,p})|}{\|\mathbf{w}_{h,p}\|_{\Omega}} \lesssim k\eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p}. \tag{3.11}$$

We obtain by Lemma 3.1.5 and (3.11) the existence of a constant $C > 0$ such that

$$(1 - Ck\eta(\mathbf{S}_{h,p})) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}} \|\mathbf{u} - \mathbf{v}_{h,p}\|_{k,h,p}.$$

This means that (3.9) holds as soon as $1 - Ck\eta(\mathbf{S}_{h,p})$ is positive.

It remains to estimate the L^2 norm. First by the definition of $\mathbb{S}^*_{k,s}$ and the Galerkin orthogonality, one has

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h,p}\|^2_{\Omega} &= \mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbb{S}^*_{k,s}(\mathbf{u} - \mathbf{u}_{h,p})) \\
&= \mathbf{a}_{k,s,h,p}(\mathbf{u} - \mathbf{u}_{h,p}, \mathbb{S}^*_{k,s}(\mathbf{u} - \mathbf{u}_{h,p}) - \mathbf{v}_{h,p}),
\end{aligned}$$

for all $\mathbf{v}_{h,p} \in \mathbf{S}_{h,p}$. By the continuity of $\mathbf{a}_{k,s,h,p}$ and the defintion of $\eta(\mathbf{S}_{h,p})$, we conclude that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h,p}\|^2_{\Omega} &\leq C_c \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \|\mathbb{S}^*_{k,s}(\mathbf{u} - \mathbf{u}_{h,p}) - \mathbf{v}_{h,p}\|_{k,h,p} \\
&\leq C_c \|\mathbf{u} - \mathbf{u}_{h,p}\|_{k,h,p} \eta(\mathbf{S}_{h,p}) \|\mathbf{u} - \mathbf{u}_{h,p}\|_{\Omega}.
\end{aligned}$$

which proves (3.10). □

Corollary 3.1.7. *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$ and $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$. If $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, then problem* (3.6) *has a unique solution $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$.*

Proof. As $\mathbf{S}_{h,p}$ is finite-dimensional, problem (3.6) is a linear system. So, we just need to prove uniqueness to have existence. Let $\mathbf{u}_{h,p} \in \mathbf{S}_{h,p}$ be such that $\mathbf{a}_{k,s,h,p}(\mathbf{u}_{h,p}, \mathbf{v}) = 0$, $\forall \mathbf{v} \in \mathbf{S}_{h,p}$. By Theorem 3.1.6 and if $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, we have (since $\mathbf{0}$ is the unique solution of (1.28) with $\mathbf{f} = \mathbf{0}$)

$$\|\mathbf{u}_{h,p}\|_{k,h,p} \lesssim \inf_{\mathbf{v} \in \mathbf{S}_{h,p}} \|\mathbf{v}\|_{k,h,p} = 0,$$

which shows the uniqueness. □

We have shown that under the condition $\eta(\mathbf{S}_{h,p}) < \frac{1}{kC}$, there exists a unique (discrete) solution $\mathbf{u}_{h,p}$ to (3.6), this solution may then be called $\mathbb{S}_{k,s,h,p}(\mathbf{f})$. In the next sections, we will give reasonable conditions between k, h and p such that this condition holds. But before, we recall some interpolation error estimates.

3.1.2 Some interpolation error estimates

We will use the same interpolation operators as in the papers [52] and [48]. These operators are built from the following definition:

Definition 3.1.8. *(element-by-element construction, from [52])*
Let $\hat{K}$ be the reference simplex of $\mathbb{R}^3$. A polynomial Π is said to permit an element-by-element construction of polynomial of degree p for $u \in H^s(\hat{K})$, $s > \frac{3}{2}$, if

(i) $\Pi(V) = u(V)$ *for each vertices of $\hat{K}$,*

(ii) for each edge e of $\hat{K}$, $\Pi_{|e} \in \mathcal{P}_p$ is the unique minimizer of

$$\Pi \to p^{\frac{1}{2}} \|u - \Pi\|_e + \|u - \Pi\|_{H_{00}^{\frac{1}{2}}(e)},$$

where Π verifies (i) and $\|v\|^2_{H_{00}^{\frac{1}{2}}(e)} = \|v\|^2_{\frac{1}{2},e} + \left\| \frac{v}{\sqrt{dist(.,\partial e)}} \right\|^2_e$,

(iii) for each face f of $\hat{K}$, $\Pi_{|f} \in \mathcal{P}_p$ is the unique minimizer of

$$\Pi \to p \|u - \Pi\|_f + \|u - \Pi\|_{1,f},$$

where Π verifies (i) and (ii).

J. M. Melenk and S. Sauter propose in [52] (see [48] for more details) two interpolants satisfying the conditions (i) to (iii) from Definition 3.1.8, the first one for general $H^s(\Omega)$ functions ($s > \frac{3}{2}$) and the second one more specific for analytic functions.

Lemma 3.1.9. *Let $v \in H^m(\Omega)$ with $m > 0$, and h_K the diameter of an element K, then we have*

$$|v|_{m,K} \lesssim h_K^{\frac{d}{2}(1-m)} |\hat{v}|_{m,\hat{K}},$$
$$|\hat{v}|_{m,\hat{K}} \lesssim h_K^{\frac{d}{2}(m-1)} |v|_{m,K},$$

and, for $\hat{\mathbf{u}} \in \mathbf{H}^t(\hat{K})^2$, with $p + 1 \geq t > \frac{3}{2}$, there exists $\hat{\Pi}_p \hat{\mathbf{u}} \in \mathbf{S}_{h,p}$ (satisfying the conditions (i) to (iii) from Definition 3.1.8), such that

$$\left\| \hat{\mathbf{u}} - \hat{\Pi}_p \hat{\mathbf{u}} \right\|_{t',\hat{K}} \lesssim p^{-(t-t')} |\hat{\mathbf{u}}|_{t,\hat{K}}, \ \forall t' \in [0, t],$$
$$\left\| \hat{\mathbf{u}} - \hat{\Pi}_p \hat{\mathbf{u}} \right\|_{t',\hat{f}} \lesssim p^{-(t-1/2-t')} |\hat{\mathbf{u}}|_{t,\hat{K}}, \ \forall t' \in [0, t - 1/2].$$

Combining the two above results, for all $\mathbf{u} \in \mathbf{H}^t(\Omega)^2$*, we obtain*

$$\|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',K} \lesssim \left(\frac{h}{p}\right)^{t-t'} |\mathbf{u}|_{t,K}, \ \forall t' \in [0,t],$$
$$\|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',f} \lesssim \left(\frac{h}{p}\right)^{t-t'-1/2} |\mathbf{u}|_{t,K}, \ \forall t' \in [0,t-1/2],$$

as well as

$$\|\mathbf{u} - \Pi_p \mathbf{u}\|_{t',\Omega} \lesssim \left(\frac{h}{p}\right)^{t-t'} |\mathbf{u}|_{t,\Omega}, \ \forall t' \in [0,t].$$

Proof. The proof of this lemma can be found in [48, Theorem B.4] (applied to each component of the vector fields). □

Lemma 3.1.10. *For* $\beta > 0$*, there exists* $\sigma > 0$ *such that for all analytic function* $\mathbf{u}_{\mathcal{A}}$ *satisfying*

$$|\mathbf{u}_{\mathcal{A}}|_{n,K} \leq (2\beta \max(n,k))^n C_K, \ \forall n \in \mathbb{N} : n \geq 2,$$

for all $K \in \mathcal{T}_h$ *and some* $C_K > 0$ *(independent of* n *and* k*), there exists* $\Pi_p \mathbf{u}_{\mathcal{A}} \in \mathbf{S}_{h,p}$ *(which respect to Definition 3.1.8) such that for* $q \in \{0,1,2\}$*,*

$$\|\mathbf{u}_{\mathcal{A}} - \Pi_p \mathbf{u}_{\mathcal{A}}\|_{q,K} \lesssim h^{-q} C_K \left(\left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right).$$

Proof. With a scaling argument, we can apply Lemma C.3 of [48] to each component. □

3.2 The analytical case

Here, following the approach from [48, 49], we will split up the solution of the adjoint problem (appearing in the definition of $\eta(\mathbf{S}_{h,p})$) in a H^2-part and an analytical part. This decomposition allows to give an estimate of $k\eta(\mathbf{S}_{h,p})$, which depends on k, h and p and obtain some error estimates.

3.2.1 A splitting lemma

The aim of this part is to split the solution $\mathbf{u} = (\mathbf{E}, \mathbf{H})$ of problem (1.30) in two parts: an analytical part but strongly oscillating and a part only in $\mathbf{H}^2(\Omega)^2$ but weakly oscillating.

We start by introducing some technical tools:

- First, a frequency splitting, based on Fourier transform, which will be applied to the right-hand side $\mathbf{f}_i$ ($i \in \{1,2\}$). More precisely, we will split up $\mathbf{f}_i$ in two parts, one part just in L^2 and the other one being analytic.
- Second, we will introduce two auxiliary problems and give a stability result for these problems.

Frequency splitting

The frequency splitting is done with the help of the Fourier transform and an extension operator. We recall that for a compactly supported function $u \in L^2(\mathbb{R}^3)$, its Fourier transform is

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\ dx,$$

and this mapping can be extended into an isometry from $L^2(\mathbb{R}^3)$ into itself. Hence we denote by $\mathcal{F}^{-1}$ its inverse transformation.

Let $\eta > 0$, we denote by $\chi_{\eta k}$ the indicator function of the ball $B_{\eta k}(0)$. Then, we define the low-pass frequency projection

$$L_{\mathbb{R}^d}(f) = \mathcal{F}^{-1}\left(\chi_{\eta k}\mathcal{F}(f)\right), \tag{3.12}$$

and the high-pass frequency projection

$$H_{\mathbb{R}^d}(f) = \mathcal{F}^{-1}\left((1-\chi_{\eta k})\mathcal{F}(f)\right), \forall f \in L^2(\mathbb{R}^d). \tag{3.13}$$

For $f \in \mathrm{L}^2(\Omega)$, we set

$$E_\Omega(f) = \begin{cases} f \text{ in } \Omega, \\ 0 \text{ outside } \Omega. \end{cases}$$

as well as

$$\begin{aligned} L_\Omega(f) &= L_{\mathbb{R}^d}(E_\Omega(f))|_\Omega, \\ H_\Omega(f) &= H_{\mathbb{R}^d}(E_\Omega(f))|_\Omega. \end{aligned}$$

Theorem 3.2.1. *Let $\eta > 0$ be the parameter which is in the definition of $H_{\mathbb{R}^d}$ and $L_{\mathbb{R}^d}$, then for all $0 \le t' \le t$, $p \in \mathbb{N}^*$, and for each $f \in H^t(\mathbb{R}^3)$, we have*

$$\begin{aligned} \|H_{\mathbb{R}^3}(f)\|_{t',\mathbb{R}^3} &\lesssim (\eta k)^{t'-t} \|f\|_{t,\mathbb{R}^3}, \\ |L_{\mathbb{R}^3}(f)|_{p,\mathbb{R}^3} &\le (\eta k)^p \|f\|_{\mathbb{R}^3}, \end{aligned}$$

while for all $f \in L^2(\Omega)$, we have

$$\begin{aligned} \|H_\Omega(f)\|_\Omega &\lesssim \|f\|_\Omega, \\ |L_\Omega(f)|_{p,\Omega} &\lesssim (\eta k)^p \|f\|_\Omega, \end{aligned}$$

with a constant independent of p.

Proof. Cf. Lemmas 4.2 and 4.3 of [52]. □

Auxiliary problems

We will introduce two well-known problems which are useful for our splitting of $\mathbf{u}$.

The first problem is to consider $\mathbf{E} = N_k(\mathbf{f})$ solution of

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} - k^2\mathbf{E} = \mathbf{f} \text{ in } \mathbb{R}^3. \tag{3.14}$$

As usual, $N_k(\mathbf{f})$ is obtained with the help of the Green function G (here, it is a matrix) of this problem, namely the distribution that satifies

$$\operatorname{curl}\operatorname{curl} G(x) - s\nabla \operatorname{div} G(x) - k^2 G(x) = \delta_x \mathrm{Id}_3.$$

Applying the Fourier transfom to this identity, direct calculations show that $\hat{G}$ satisfies

$$M(\xi)\hat{G}(\xi) = \mathrm{Id}_3,$$

with

$$M(\xi) = \begin{pmatrix} |\xi|^2 - k^2 - (1-s)\xi_1^2 & -(1-s)\xi_1\xi_2 & -(1-s)\xi_1\xi_3 \\ -(1-s)\xi_1\xi_2 & |\xi|^2 - k^2 - (1-s)\xi_2^2 & -(1-s)\xi_2\xi_3 \\ -(1-s)\xi_1\xi_3 & -(1-s)\xi_2\xi_3 & |\xi|^2 - k^2 - (1-s)\xi_3^2 \end{pmatrix}.$$

Therefore

$$\hat{G}(\xi) = M(\xi)^{-1}\mathrm{Id}_3.$$

By direct calculations, we check that the eigenvalues of $M(\xi)^{-1}$ are $\frac{1}{s|\xi|^2-k^2}$ and $\frac{1}{|\xi|^2-k^2}$. Recalling that $s \in [1,2]$, we get

$$\|M(\xi)^{-1}\| = \frac{1}{|\xi|^2 - k^2} \text{ if } |\xi| > k. \tag{3.15}$$

For $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}^3)$, we define $N_k(\mathbf{f})$ as the convolution product of G with $\mathbf{f}$, namely

$$N_k(\mathbf{f})(x) = (G * \mathbf{f})(x) = \int_{\mathbb{R}^3} G(x-y)\mathbf{f}(y)dy,$$

which verifies (3.14).

Now we want to estimate the norm of $N_k(H_\Omega \mathbf{f})$.

Lemma 3.2.2. *Let* $\mathbf{f} \in \mathrm{L}^2(\Omega)^3$, *if* $\mathbf{E} = N_k(H_\Omega \mathbf{f})$ *then for all* $q \in (0,1)$, *there exists* $\eta > 0$ *(appearing in the definition of* L_Ω*) such that*

$$\|\mathbf{E}\|_k \le qk^{-1} \|\mathbf{f}\|_\Omega, \quad \|\mathbf{E}\|_{2,\Omega} \lesssim \|\mathbf{f}\|_\Omega. \tag{3.16}$$

Proof. We recall that $\mathbf{E} = G * (H_\Omega \mathbf{f})$ and fix $\eta > 1$. We start by estimating the L^2 norm of $\mathbf{E}$:

$$\begin{aligned} \|\mathbf{E}\|_{\mathbb{R}^3} &= \|\mathcal{F}(G * H_\Omega f)\|_{\mathbb{R}^3} \\ &= \left\|\hat{G}(1-\chi_{\eta k})\hat{\mathbf{f}}\right\|_{\mathbb{R}^3} \\ &= \left(\int_{\mathbb{R}^3} \left|M(\xi)^{-1}(1-\chi_{\eta k}(\xi))\hat{\mathbf{f}}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\le \left(\int_{\mathbb{R}^3\setminus B(\eta k)} \left|\frac{1}{|\xi|^2-k^2}\right|^2 \left|\hat{\mathbf{f}}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}, \end{aligned}$$

this last estimate following from (3.15). As $|\xi| \geq \eta k$ on $\mathbb{R}^3 \backslash B(\eta k)$, we deduce that

$$\|\mathbf{E}\|_{\mathbb{R}^3} \leq \frac{1}{\eta^2 - 1} k^{-2} \|\mathbf{f}\|_\Omega .$$

Now, we estimate the H^1 norm of $\mathbf{E}$:

$$\begin{aligned} |\mathbf{E}|_{1,\mathbb{R}^3} &= \left(\sum_{i=1}^3 \int_{\mathbb{R}^3} \left| \xi_i M(\xi)^{-1} (1 - \chi_{\eta k}(\xi)) \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^3 \int_{\mathbb{R}^3 \backslash B(\eta k)} \left| \frac{\xi_i}{|\xi|^2 - k^2} \right|^2 \left| \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} . \end{aligned}$$

As before we deduce that

$$|\mathbf{E}|_{1,\mathbb{R}^3} \leq \frac{1}{\eta - \frac{1}{\eta}} k^{-1} \|\mathbf{f}\|_\Omega .$$

We end up with the H^2 norm of $\mathbf{E}$:

$$\begin{aligned} |\mathbf{E}|_{2,\mathbb{R}^3} &= \left(\sum_{i,j=1}^3 \int_{\mathbb{R}^3} \left| \xi_i \xi_j M(\xi)^{-1} (1 - \chi_{\eta k}(\xi)) \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i,j=1}^3 \int_{\mathbb{R}^3 \backslash B(\eta k)} \left| \frac{\xi_i \xi_j}{|\xi|^2 - k^2} \right|^2 \left| \hat{\mathbf{f}}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} . \end{aligned}$$

And again we obtain

$$|\mathbf{E}|_{2,\mathbb{R}^3} \leq \frac{1}{1 - \frac{1}{\eta^2}} \|\mathbf{f}\|_\Omega .$$

Hence, we have proved (3.16), for η large enough. □

Now, we will study the second problem, namely: For $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, we consider $(\mathbf{V}_1, \mathbf{V}_2) = \mathbb{S}^+_{k,s}(\mathbf{f})$ solution of

$$\left\{ \begin{array}{l} \left. \begin{array}{ll} L^+_{k,s}(\mathbf{V}_1) & = L^+_{k,s}(N_k(H_\Omega \mathbf{f}_1)) \\ L^+_{k,s}(\mathbf{V}_2) & = L^+_{k,s}(N_k(H_\Omega \mathbf{f}_2)) \end{array} \right\} \text{ in } \Omega, \\ \left. \begin{array}{ll} \operatorname{div} \mathbf{V}_1 & = 0 \\ \operatorname{div} \mathbf{V}_2 & = 0 \\ T(\mathbf{V}_1, \mathbf{V}_2) & = 0 \\ B_k(\mathbf{V}_1, \mathbf{V}_2) & = 0 \end{array} \right\} \text{ on } \partial\Omega. \end{array} \right. \tag{3.17}$$

where $L^+_{k,s}(\mathbf{E}) = -\Delta \mathbf{E} + (1-s) \nabla \operatorname{div} \mathbf{E} + k^2 \mathbf{E}$. The existence of this solution as well as norm estimates are the goal of the next lemma.

Lemma 3.2.3. *Let $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, then problem* (3.17) *has a unique solution and for all $q \in]0,1[$, there exists $\eta > 0$ such that*

$$\left\| \mathbb{S}^+_{k,s}(\mathbf{f}) \right\|_k \lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}\|_\Omega , \tag{3.18}$$

$$\left\| \mathbb{S}^+_{k,s}(\mathbf{f}) \right\|_{2,\Omega} \lesssim \|\mathbf{f}\|_\Omega . \tag{3.19}$$

Proof. We first notice that the variational formulation of problem (3.17) is

$$\mathbf{b}_{k,s}(\mathbb{S}^+_{k,s}(\mathbf{f}),\mathbf{v}') = (\left(L^+_k(N_k(H_\Omega \mathbf{f}_1)), L^+_k(N_k(H_\Omega \mathbf{f}_2))\right), \mathbf{v}'),\ \forall \mathbf{v}' = (\mathbf{E}',\mathbf{H}') \in \mathbf{V},$$

with

$$\begin{aligned}
\mathbf{b}_{k,s}(\mathbf{v},\mathbf{v}') = &\int_\Omega \left(\operatorname{curl}\mathbf{V}_1 \cdot \overline{\operatorname{curl}\mathbf{E}'} + s\operatorname{div}\mathbf{V}_1\,\overline{\operatorname{div}\mathbf{E}'} + k^2\mathbf{V}_1\cdot\overline{\mathbf{E}'}\right)\,dx \\
&+ \int_\Omega \left(\operatorname{curl}\mathbf{V}_2 \cdot \overline{\operatorname{curl}\mathbf{H}'} + s\operatorname{div}\mathbf{V}_2\,\overline{\operatorname{div}(\mathbf{H}')} + k^2\mathbf{V}_2\cdot\overline{\mathbf{H}'}\right)\,dx \\
&+ ik\int_{\partial\Omega} \left(\lambda_{\text{imp}}(\mathbf{V}_1)_t\cdot\overline{\mathbf{E}'_t} + \frac{1}{\lambda_{\text{imp}}}(\mathbf{V}_2)_t\cdot\overline{\mathbf{H}'_t}\right)\,d\sigma,
\end{aligned}$$

for all $\mathbf{v} = (\mathbf{V}_1,\mathbf{V}_2), \mathbf{v}' = (\mathbf{E}',\mathbf{H}') \in \mathbf{V}$.

The existence and uniqueness of a solution follows from Lax-Milgram lemma since the sesquilinear form $\mathbf{b}_{k,s}$ is coercive and continuous on $\mathbf{V}$.

Now by taking $\mathbf{v}' = \mathbb{S}^+_{k,s}(\mathbf{f}) = (\mathbf{V}_1,\mathbf{V}_2)$ and the real part, we have:

$$\begin{aligned}
\left\|\mathbb{S}^+_{k,s}(\mathbf{f})\right\|_k^2 &= \operatorname{Re}(\mathbf{b}_{k,s}(\mathbb{S}^+_{k,s}(\mathbf{f}),\mathbb{S}^+_{k,s}(\mathbf{f}))) \\
&= \operatorname{Re}\left(\int_\Omega \left(L^+_{k,s}(N_k(H_\Omega\mathbf{f}_1))\cdot\overline{\mathbf{V}_1} + L^+_{k,s}(N_k(H_\Omega\mathbf{f}_2))\cdot\overline{\mathbf{V}_2}\right)\,dx\right).
\end{aligned}$$

But by Green's formula, for $i = 1$ or 2, we notice that

$$\begin{aligned}
&\left|\int_\Omega L^+_{k,s}(N_k(H_\Omega\mathbf{f}_i))\cdot\overline{\mathbf{V}_i}\,dx\right| \\
&= \left|\int_\Omega \left(\operatorname{curl}N_k(H_\Omega\mathbf{f}_i)\cdot\operatorname{curl}\overline{\mathbf{V}_i} + s\operatorname{div}N_k(H_\Omega\mathbf{f}_i)\,\operatorname{div}\overline{\mathbf{V}_i}\right)\,dx\right. \\
&\quad + \int_\Omega k^2N_k(H_\Omega\mathbf{f}_i)\cdot\overline{\mathbf{V}_i}\,dx + \int_{\partial\Omega}\operatorname{curl}(N_k(H_\Omega\mathbf{f}_i))\times\mathbf{n}\cdot\overline{\mathbf{V}_i}\,d\sigma \\
&\quad \left. + \int_{\partial\Omega}\operatorname{div}N_k(H_\Omega\mathbf{f}_i)\,\overline{\mathbf{V}_i}.\mathbf{n}\,d\sigma\right| \\
&\lesssim \|N_k(H_\Omega\mathbf{f}_i)\|_k\left\|\mathbb{S}^+_{k,s}(\mathbf{f})\right\|_k \\
&\quad + \left|\int_{\partial\Omega}\operatorname{div}N_k(H_\Omega\mathbf{f}_i)\,\overline{\mathbf{V}_i}.\mathbf{n}\,d\sigma\right| + \left|\int_{\partial\Omega}\operatorname{curl}N_k(H_\Omega\mathbf{f}_i)\times\mathbf{n}\cdot\overline{\mathbf{V}_i}\,d\sigma\right|.
\end{aligned}$$

Now, we must estimate the boundary term. First Cauchy-Schwarz's inequality yields

$$\left|\int_{\partial\Omega}\operatorname{div}N_k(H_\Omega\mathbf{f}_i)\,\overline{\mathbf{V}_i}.\mathbf{n}\,d\sigma\right| \lesssim \|\operatorname{div}N_k(H_\Omega\mathbf{f}_i)\|_{\partial\Omega}\|\mathbf{V}_i\|_{\partial\Omega},$$

$$\left|\int_{\partial\Omega}\operatorname{curl}N_k(H_\Omega\mathbf{f}_i)\times\mathbf{n}\cdot\overline{\mathbf{V}_i}\,d\sigma\right| \lesssim \|\operatorname{curl}N_k(H_\Omega\mathbf{f}_i)\|_{\partial\Omega}\|\mathbf{V}_i\|_{\partial\Omega}.$$

Second by a trace estimate and Young's inequality, we have

$$\begin{aligned}
\|\mathbf{V}_i\|_{\partial\Omega} &\lesssim \|\mathbf{V}_i\|_\Omega^{\frac12}\|\mathbf{V}_i\|_{1,\Omega}^{\frac12} \\
&\lesssim k^{-\frac12}\left(k\|\mathbf{V}_i\|_\Omega + \|\mathbf{V}_i\|_{1,\Omega}\right) \\
&\lesssim k^{-\frac12}\left\|\mathbb{S}^+_{k,s}(\mathbf{f})\right\|_k.
\end{aligned}$$

Thirdly by Lemma 3.2.2, we also get

$$
\begin{aligned}
\|\operatorname{curl} N_k(H_\Omega \mathbf{f}_i)\|_{\partial\Omega} &\lesssim \|\operatorname{curl} N_k(H_\Omega \mathbf{f}_i)\|_{\Omega}^{\frac{1}{2}} \|\operatorname{curl} N_k(H_\Omega \mathbf{f}_i)\|_{1,\Omega}^{\frac{1}{2}} \\
&\lesssim \|N_k(H_\Omega \mathbf{f}_i)\|_{k}^{\frac{1}{2}} \|N_k(H_\Omega \mathbf{f}_i)\|_{2,\Omega}^{\frac{1}{2}} \\
&\lesssim q^{\frac{1}{2}} k^{-\frac{1}{2}} \|\mathbf{f}_i\|_{\Omega}.
\end{aligned}
$$

In the same way, we obtain

$$
\|\operatorname{div} N_k(H_\Omega \mathbf{f}_i)\|_{\partial\Omega} \lesssim q^{\frac{1}{2}} k^{-\frac{1}{2}} \|\mathbf{f}_i\|_{\Omega}.
$$

These estimates lead to

$$
\begin{aligned}
\left|\int_{\partial\Omega} \operatorname{div} N_k(H_\Omega \mathbf{f}_i)\, \overline{\mathbf{V}_i}.\mathbf{n}\ d\sigma\right| &\lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}_i\|_{\Omega} \left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{k}, \\
\left|\int_{\partial\Omega} \operatorname{curl} N_k(H_\Omega \mathbf{f}_i) \times \mathbf{n} \cdot \overline{\mathbf{V}_i}\ d\sigma\right| &\lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}_i\|_{\Omega} \left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{k}.
\end{aligned}
$$

Hence, by the previous estimates and Lemma 3.2.2, we have

$$
\left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{k}^{2} \lesssim q^{\frac{1}{2}} k^{-1} \|\mathbf{f}\|_{\Omega} \left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{k},
$$

which proves (3.18).

To estimate the H^2 norm of $\mathbb{S}_{k,s}^{+}(\mathbf{f})$, we apply Theorem 2.D of [22] (the constant being independent of s since the ellipticity of $L_{k,s}^{+}$ is continuous in $s \in [1,2]$) to get

$$
\left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{2,\Omega} \lesssim k^2 \left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{\Omega} \lesssim k \left\|\mathbb{S}_{k,s}^{+}(\mathbf{f})\right\|_{k},
$$

which proves (3.19) owing to (3.18). □

The splitting result

Now, we can state the main result of this part, namely the following decomposition theorem:

Theorem 3.2.4. *Assume that the k-stability property (1.32) holds with $\alpha \geq 1$. Let $\mathbf{u} = (\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f})$, where $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$, then there exist $\mathbf{u}_{\mathcal{A}}$ an analytical function and $\mathbf{u}_{H^2}$ a H^2 function such that:*

$$
\mathbf{u} = \mathbf{u}_{\mathcal{A}} + \mathbf{u}_{H^2},
$$

with

$$
\|\mathbf{u}_{\mathcal{A}}\|_{k} \lesssim k^{\alpha} \|\mathbf{f}\|_{\Omega}, \tag{3.20}
$$

$$
|\mathbf{u}_{\mathcal{A}}|_{p,\Omega} \lesssim K^p \max(p,k)^p k^{\alpha-1} \|\mathbf{f}\|_{\Omega}, \ \forall p \in \mathbb{N}, p \geq 2, \tag{3.21}
$$

$$
\|\mathbf{u}_{H^2}\|_{k} \lesssim k^{-1} \|\mathbf{f}\|_{\Omega}, \tag{3.22}
$$

$$
\|\mathbf{u}_{H^2}\|_{2,\Omega} \lesssim \|\mathbf{f}\|_{\Omega}, \tag{3.23}
$$

for some constant $K \geq 1$.

To prove this theorem, we will need the following lemma:

Lemma 3.2.5. *Under the assumption of Theorem 3.2.4, let* $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathrm{L}^2(\Omega)^6$. *Then* $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$ *admits the splitting*

$$\mathbf{u} = \mathbf{u}_{H^2} + \mathbf{u}_{\mathcal{A}} + \tilde{\mathbf{u}},$$

where $\tilde{\mathbf{u}} = \mathbb{S}_{k,s}(\tilde{\mathbf{f}})$ *for some* $\tilde{\mathbf{f}} \in \mathrm{L}^2(\Omega)^6$ *with*

$$\left\|\tilde{\mathbf{f}}\right\|_\Omega \leq q' \left\|\mathbf{f}\right\|_\Omega,$$

for some $q' \in (0,1)$ *and the following estimates hold*

$$\begin{aligned}
\|\mathbf{u}_{\mathcal{A}}\|_k &\lesssim k^\alpha \|\mathbf{f}\|_\Omega, \\
|\mathbf{u}_{\mathcal{A}}|_{p,\Omega} &\lesssim K^p \max(p,k)^p k^{\alpha-1} \|\mathbf{f}\|_\Omega, \ \forall p \in \mathbb{N} : p \geq 2, \\
\|\mathbf{u}_{H^2}\|_k &\lesssim k^{-1} \|\mathbf{f}\|_\Omega, \\
\|\mathbf{u}_{H^2}\|_{2,\Omega} &\lesssim \|\mathbf{f}\|_\Omega.
\end{aligned}$$

Proof. We set

$$\mathbf{u}_{\mathcal{A}} = \mathbb{S}_{k,s}(\mathrm{L}_\Omega(\mathbf{f})) \text{ and } \mathbf{u}_{H^2} = \mathbb{S}^+_{k,s}(\mathbf{f}).$$

Then, we see that

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{\mathcal{A}} - \mathbf{u}_{H^2}$$

verifies

$$\left\{\begin{array}{lll}
\left.\begin{array}{ll} L_{k,s}(\tilde{\mathbf{E}}) & = \tilde{\mathbf{f}}_1, \\ L_{k,s}(\tilde{\mathbf{H}}) & = \tilde{\mathbf{f}}_2, \end{array}\right\} & \text{in } \Omega, \\
\left.\begin{array}{ll} \operatorname{div} \tilde{\mathbf{E}} & = 0 \\ \operatorname{div} \tilde{\mathbf{H}} & = 0 \\ T(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) & = 0 \\ B(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) & = 0 \end{array}\right\} & \text{on } \partial\Omega.
\end{array}\right. \tag{3.24}$$

where $\tilde{\mathbf{f}} = 2k^2(\mathbb{S}^+_{k,s}(\mathbf{f}) - N_k(H_\Omega(\mathbf{f}))$.

Now, we will estimate the different norms. First the estimate on the norms of $\mathbf{u}_{H^2}$ directly follows from Lemma 3.2.3. Secondly by Lemmas 3.2.3 and 3.2.2, we have

$$\left\|\tilde{\mathbf{f}}\right\|_\Omega = 2k^2 \left(\left\|\mathbb{S}^+_{k,s}(\mathbf{f})\right\|_\Omega + \|N_k(H_\Omega(\mathbf{f})\|_\Omega \right) \leq C q^{\frac{1}{2}} \|\mathbf{f}\|_\Omega \leq q' \|\mathbf{f}\|_\Omega,$$

where $q' = Cq^{\frac{1}{2}}$ that belongs to $]0,1[$ for q small enough.

To estimate $\|\mathbf{u}_{\mathcal{A}}\|_k$, we simply use the k-stability property (1.32) to get

$$\|\mathbf{u}_{\mathcal{A}}\|_k \lesssim k^\alpha \|L_\Omega(\mathbf{f})\|_\Omega \lesssim k^\alpha \|\mathbf{f}\|_\Omega. \tag{3.25}$$

To estimate $|\mathbf{u}_{\mathcal{A}}|_{p,\Omega}$ with $p \geq 2$, we apply Theorem 3.5.1 below and (3.25) to get

$$\begin{aligned}
|\mathbf{u}_{\mathcal{A}}|_{p,\Omega} &\lesssim K^p \max(p,k)^p \left(\|\mathbf{f}\|_\Omega + k^{-1} \|\mathbf{u}_{\mathcal{A}}\|_k \right) \\
&\lesssim K^p \max(p,k)^p k^{\alpha-1} \|\mathbf{f}\|_\Omega.
\end{aligned}$$

Lemma 3.2.3 directly furnishes the estimate of the norms of $\mathbf{u}_{H^2}$. □

Now, we can prove Theorem 3.2.4.

Proof of Theorem 3.2.4. Let $\mathbf{u} = \mathbb{S}_{k,s}(\mathbf{f})$, we apply the previous lemma, and obtain that there exists $q' \in (0,1)$ such that

$$\mathbf{u} = \mathbf{u}_{\mathcal{A}}^1 + \mathbf{u}_{H^2}^1 + \tilde{\mathbf{u}}^1,$$

where $\tilde{\mathbf{u}}^1 = \mathbb{S}_{k,s}(\tilde{\mathbf{f}}^1)$ with $\left\|\tilde{\mathbf{f}}^1\right\|_\Omega \leq q' \left\|\mathbf{f}\right\|_\Omega$.

We iterate this procedure to get

$$\begin{aligned}\mathbf{u} &= \sum_{i=1}^{\infty} \mathbf{u}_{\mathcal{A}}^i + \sum_{i=1}^{\infty} \mathbf{u}_{H^2}^i \\ &= \mathbf{u}_{\mathcal{A}} + \mathbf{u}_{H^2}.\end{aligned}$$

We then have the right estimates by the previous lemma and the fact that $q' < 1$ (so that the associated geometric series converge). ■

3.2.2 Estimation of $k\eta(\mathbf{S}_{h,p})$

The approximation quatity $\eta(\mathbf{S}_{h,p})$ will be estimated by using the decomposition theorem applied to the adjoint problem.

Theorem 3.2.6. *Assume that the k-stability property* (1.32) *holds with* $\alpha \geq 1$ *and that* $\frac{kh}{p} \lesssim 1$. *Let* $\mathbf{S}_{h,p}$ *be previously defined, then we have*

$$k\eta(\mathbf{S}_{h,p}) \lesssim \left(\frac{kh}{\sqrt{p}} + k^\alpha \left(p\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right)\right). \tag{3.26}$$

Proof. For any $\mathbf{f} \in \mathrm{L}^2(\Omega)^6$, we apply the decomposition theorem 3.2.4 to $\mathbf{u} = \mathbb{S}_{k,s}^*(\mathbf{f})$ and obtain

$$\mathbf{u} = \mathbf{u}_{H^2} + \mathbf{u}_{\mathcal{A}}.$$

The analytical part highly dependent on k, while the H^2 part is less dependent on k, so we will estimate separately the two parts.

For $\mathbf{u}_{H^2}$, we use the same construction as in Theorem B.4 of [48] (Lemma 3.1.9), hence there exists $\mathbf{w}_{H^2} (= \Pi_p \mathbf{u}_{H^2}) \in \mathbf{S}_{h,p}$ such that

$$\left\|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\right\|_{t,\Omega} \lesssim \left(\frac{h}{p}\right)^{2-t} \left\|\mathbf{u}_{H^2}\right\|_{2,\Omega},$$

for all $0 \leq t < 2$. Hence

$$k\left\|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\right\|_k \lesssim \left(\frac{hk}{p} + \left(\frac{hk}{p}\right)^2\right) \left\|\mathbf{f}\right\|_\Omega.$$

We now have to estimate the boundary term in $\|\mathbf{u}_{H^2}-\mathbf{w}_{H^2}\|_{k,h,p}$. This essentially follows from Lemma 3.1.9 and the estimate (3.23).

(3.27)
$$\begin{aligned}\frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\left\|(\mathbf{E}-\mathbf{w}^1_{H^2})_t-\frac{1}{\lambda_{\mathrm{imp}}}(\mathbf{H}-\mathbf{w}^2_{H^2})\times\mathbf{n}\right\|_f^2 &\lesssim \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\|\mathbf{u}_{H^2}-\mathbf{w}_{H^2}\|_f^2\\ &\lesssim \frac{p^2}{h}\left(\frac{h}{p}\right)^3\sum_{f\in\mathcal{E}^B}\alpha_f|\mathbf{u}_{H^2}|^2_{K_f}\\ &\lesssim \left(\frac{h^2}{p}\right)\sum_{f\in\mathcal{E}^B}\alpha_f|\mathbf{u}_{H^2}|^2_{2,K_f}\\ &\lesssim \left(\left(\frac{h}{\sqrt{p}}\right)\|\mathbf{f}\|_\Omega\right)^2.\end{aligned}$$

We hence obtain

$$k\,\|\mathbf{u}_{H^2}-\mathbf{w}_{H^2}\|_{k,h,p}\lesssim\left(\frac{kh}{\sqrt{p}}\right)\|\mathbf{f}\|_\Omega\,. \tag{3.28}$$

We now estimate the analytical part. The estimate (3.21) gives us

$$|\mathbf{u}_{\mathcal{A}}|_{n,\Omega}\le C(\gamma\max(n,k))^n k^{\alpha-1}\|\mathbf{f}\|_\Omega\,,\ \forall n\in\mathbb{N},n\ge 2.$$

We then define C_K by

$$C_K^2=\sum_{n\in\mathbb{N}:n\ge2}\frac{\|\nabla^n\mathbf{u}_{\mathcal{A}}\|_K^2}{(2\gamma\max\{n,k\})^{2n}},$$

to have

$$|\mathbf{u}_{\mathcal{A}}|_{n,K}\le(2\gamma\max\{n,k\})^n C_K,\ \forall n\in\mathbb{N}:n\ge2,$$

but also

$$\sum_{K\in\mathcal{T}}C_K^2\le Ck^{2(\alpha-1)}\|\mathbf{f}\|^2_\Omega\,. \tag{3.29}$$

We use Lemma 3.1.10 (cf. Lemma C.3 of [48]), to get, for $\sigma>0$, the following estimate, for $q=0,1,2$, with $\mathbf{w}_{\mathcal{A}}=\Pi_{\mathcal{A}}\mathbf{u}_{\mathcal{A}}$:

$$\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\|_{q,K}\le Ch^{-q}C_K\left(\left(\frac{h}{h+\sigma}\right)^{p+1}+\left(\frac{kh}{\sigma p}\right)^{p+1}\right). \tag{3.30}$$

This estimate for $q=0$ and 1 leads to

$$\begin{aligned}k^2\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\|_k^2&=k^2\sum_{K\in\mathcal{T}}(|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}|^2_{1,K}+k^2\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\|_K^2)\\&\lesssim k^2\left(h^{-1}+k\right)^2\left(\left(\frac{h}{h+\sigma}\right)^{p+1}+\left(\frac{kh}{\sigma p}\right)^{p+1}\right)^2\left(\sum_{K\in\mathcal{T}}C_K^2\right).\end{aligned}$$

Simple calculations yield

$$
\begin{aligned}
(h^{-1}+k)&\left(\left(\frac{h}{h+\sigma}\right)^{p+1}+\left(\frac{kh}{\sigma p}\right)^{p+1}\right)\\
&\lesssim (1+kh)\left(\frac{h}{h+\sigma}\right)^{p}+\left(\frac{k}{p}+\frac{k^2h}{p}\right)\left(\frac{kh}{\sigma p}\right)^{p}\\
&\lesssim \left(\frac{1}{p}+\frac{kh}{p}\right)\left(p\left(\frac{h}{h+\sigma}\right)^{p}+k\left(\frac{kh}{\sigma p}\right)^{p}\right)\\
&\lesssim p\left(\frac{h}{h+\sigma}\right)^{p}+k\left(\frac{kh}{\sigma p}\right)^{p},
\end{aligned}
$$

recalling that $\frac{kh}{p} \lesssim 1$. These two estimates and (3.29) give

$$
k\left\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\right\|_k \lesssim \left(p\left(\frac{h}{h+\sigma}\right)^{p}+k\left(\frac{kh}{\sigma p}\right)^{p}\right)k^{\alpha}\left\|\mathbf{f}\right\|_{\Omega}. \tag{3.31}
$$

As before we need to estimate the boundary term in $\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\|_{k,h,p}$:

$$
B=\frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\left\|(\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}})_t-\frac{1}{\lambda_{\text{imp}}}(\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}})\times\mathbf{n}\right\|_f^2.
$$

By using the trace estimate

$$
\|v\|_{\partial K}^2 \le C\left(\|v\|_K |v|_{1,K}+h^{-1}\|v\|_K^2\right),
$$

we get

$$
\begin{aligned}
B&\lesssim \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\left\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\right\|_{\partial K_f}^2\\
&\lesssim \frac{p^2}{h}\sum_{f\in\mathcal{E}^B}\alpha_f\left(\left\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\right\|_{K_f}\left|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\right|_{1,K_f}+h^{-1}\left\|\mathbf{u}_{\mathcal{A}}-\mathbf{w}_{\mathcal{A}}\right\|_{K_f}^2\right).
\end{aligned}
$$

By (3.30) with $q=0$ or 1, we obtain

$$
B\lesssim \frac{p^2}{h^2}\left(\left(\frac{h}{h+\sigma}\right)^{p+1}+\left(\frac{kh}{\sigma p}\right)^{p+1}\right)^2\left(\sum_{f\in\mathcal{E}^B}C_{K_f}^2\right).
$$

Again simple calculations yield

$$
\frac{p}{h}\left(\left(\frac{h}{h+\sigma}\right)^{p+1}+\left(\frac{kh}{\sigma p}\right)^{p+1}\right)\lesssim p\left(\frac{h}{h+\sigma}\right)^{p}+k\left(\frac{kh}{\sigma p}\right)^{p}.
$$

These two estimates and (3.29) give

$$
B\lesssim \left(p\left(\frac{h}{h+\sigma}\right)^{p}+k\left(\frac{kh}{\sigma p}\right)^{p}\right)^2 k^{2(\alpha-1)}\left\|\mathbf{f}\right\|_{\Omega}^2.
$$

Combining this estimate with (3.31), we get

$$k \left\|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\right\|_{k,h,p} \lesssim \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) k^\alpha \left\|\mathbf{f}\right\|_\Omega . \tag{3.32}$$

We can now estimate $k\eta(\mathbf{S}_{h,p})$, indeed the triangle inequality yields

$$k \left\|\mathbf{u} - \mathbf{w}_{H^2} - \mathbf{w}_{\mathcal{A}}\right\|_{k,h,p} \leq k \left\|\mathbf{u}_{H^2} - \mathbf{w}_{H^2}\right\|_{k,h,p} + k \left\|\mathbf{u}_{\mathcal{A}} - \mathbf{w}_{\mathcal{A}}\right\|_{k,h,p} .$$

By (3.28) and (3.32), we deduce that

$$k \left\|\mathbf{u} - \mathbf{w}_{H^2} - \mathbf{w}_{\mathcal{A}}\right\|_{k,h,p} \lesssim \left(\frac{kh}{\sqrt{p}} + k^\alpha \left(p \left(\frac{h}{h+\sigma} \right)^p + k \left(\frac{kh}{\sigma p} \right)^p \right) \right) \left\|\mathbf{f}\right\|_\Omega ,$$

which proves (3.26) because $\mathbf{w}_{H^2} + \mathbf{w}_{\mathcal{A}}$ belongs to $\mathbf{S}_{h,p}$. □

Remark 3.2.7. In the previous proof, we can see that the term $\frac{kh}{\sqrt{p}}$ in the right-hand side of (3.26) appears because of the penalisation term (see (3.27)). Since this term is, up to the factor $\frac{h}{\sqrt{p}}$, bounded by the H^2-norm of $\mathbf{u}_{H^2}$ in a neighborhood of the boundary, we beleave that this penalisation term is neglectable and that the term $\frac{kh}{\sqrt{p}}$ can be replaced by $\frac{kh}{p}$. This fact is confirmed by our numerical experiments.

In the same manner, we obtain the following convergence result:

Theorem 3.2.8. *Assume that $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$. Let $\mathbf{u}$ be the solution of* (1.30) *and $\mathbf{u}_{h,p}$ the solution of* (3.6). *Then, we have*

$$\left\|\mathbf{u} - \mathbf{u}_{h,p}\right\|_{k,h,p} \lesssim \frac{h}{\sqrt{p}} + k^{\alpha-1} p \left(\frac{h}{h+\sigma} \right)^p + k^\alpha \left(\frac{kh}{\sigma p} \right)^p ,$$

$$\left\|\mathbf{u} - \mathbf{u}_{h,p}\right\|_{0,\Omega} \lesssim \left(\frac{h}{\sqrt{p}} + k^{\alpha-1} p \left(\frac{h}{h+\sigma} \right)^p + k^\alpha \left(\frac{kh}{\sigma p} \right)^p \right)^2 .$$

Proof. We use Theorem 3.1.6 and the same decomposition technique as for the estimate of $\eta(\mathbf{S}_{h,p})$. □

For practical purposes, we formulate explicit conditions that guarantee $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$ (compare with [48, Corollary 5.6]).

Theorem 3.2.9. *There exist three positive constants C_1, C_2 and k_0, such that if $k > k_0$ and*

$$\frac{kh}{\sqrt{p}} \leq C_1 \text{ and } \ln k \leq C_2 p, \tag{3.33}$$

then $k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}$.

Proof. We just need to find some positive constants C_1 and C_2, such that

$$\begin{cases} \frac{kh}{\sqrt{p}} \leq C_1 \\ \ln k \leq C_2 p \end{cases} \Rightarrow k\eta(\mathbf{S}_{h,p}) \leq \frac{1}{C}. \tag{3.34}$$

By (3.26), it is sufficient to control its right-hand side, namely to show that

$$\frac{kh}{\sqrt{p}} + k^{\alpha}\left(p\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right) \leq \frac{1}{C} = C'. \tag{3.35}$$

We will first show that for all $C > 0$, $\gamma \in (0,1)$ and $\delta \geq 0$, there exist $\beta > 0$ and $k_0 > 0$ such that if

$$\beta \ln k \leq \frac{p}{2} \text{ and } k > k_0, \tag{3.36}$$

then we have

$$k^{\alpha} p^{\delta} \gamma^p \leq C. \tag{3.37}$$

First we want to find $\beta > 0$ such that

$$\frac{\alpha}{|\ln\gamma|}\ln k - \frac{\ln C}{|\ln\gamma|} \leq \beta \ln k, \tag{3.38}$$

or equivalently

$$-\frac{\ln C}{|\ln\gamma|} \leq \left(\beta - \frac{\alpha}{|\ln\gamma|}\right)\ln k.$$

Consequently for $\beta > \frac{\alpha}{|\ln\gamma|} + 1$ and $k_0 \geq e^{-\frac{\ln C}{|\ln\gamma|}}$, (3.38) is valid.
Second, there exists $p_0 \geq 0$ such that for $p \geq p_0$ we have

$$\frac{p}{2} \leq p - \frac{\delta \ln p}{|\ln\gamma|}. \tag{3.39}$$

By (3.36), (3.38) and (3.39), we obtain

$$\frac{\alpha}{|\ln\gamma|}\ln k - \frac{\ln C}{|\ln\gamma|} \leq p - \frac{\delta \ln p}{|\ln\gamma|}.$$

And, since $\ln\gamma < 0$,

$$\alpha \ln k + (\ln\gamma)p + \delta \ln p \leq \ln C.$$

By taking the exponential, we get (3.37) with $\beta > \max\left(\frac{\alpha}{|\ln\gamma|} + 1, \frac{p_0}{2\ln k_0}\right)$.

Now, we can control each term of the left hand-side of (3.35):

1. $\frac{kh}{\sqrt{p}} \leq \frac{C'}{3}$.

2. there exist $C_3 > 0$ and $k_{0,1} \geq e^{-\frac{\ln \frac{C'}{3}}{|\ln\gamma|}}$ with $\gamma = \frac{\mathrm{diam}(\Omega)}{\mathrm{diam}(\Omega)+\sigma}$ and $\delta = 1$ such that if $C_3 \ln k \leq p$ and $k > k_{0,1}$, then we have

$$k^{\alpha} p\left(\frac{h}{h+\sigma}\right)^p \leq k^{\alpha} p\left(\frac{\mathrm{diam}(\Omega)}{\mathrm{diam}(\Omega)+\sigma}\right)^p \leq \frac{C'}{3}.$$

3. there exist $C_4 > 0$ and $k_{0,2} \geq e^{-\frac{\ln \frac{C'}{3}}{|\ln \gamma|}}$ with $\gamma = \frac{1}{2}$ and $\delta = 0$, such that if $C_4 \ln k \leq p$ and $k > k_{0,2}$, then we have

$$k^{\alpha+1}\left(\frac{1}{2}\right)^p \leq \frac{C'}{3}.$$

Hence if $\frac{kh}{\sigma p} \leq \frac{kh}{\sigma\sqrt{p}} \leq \frac{1}{2}$, then

$$k^{\alpha+1}\left(\frac{kh}{\sigma p}\right)^p \leq \frac{C'}{3}.$$

Hence, (3.34) holds with $C_1 = \min\left(\frac{C'}{3}, \frac{\sigma}{2}\right)$, $\frac{1}{C_2} = \max\left(C_3, C_4\right)$ and $k_0 \geq \max(k_{0,1}, k_{0,2})$. □

Remark 3.2.10. From the above proof, we see that C_1 and k_0 depend on $\frac{1}{C}$ (in such a way that if C is large, then C_1 is small and k_0 is large), while C_2 depends only on α, $\mathrm{diam}(\Omega)$ and σ.

3.3 The case of a boundary of class $\mathcal{C}^{\gamma+1,1}$

In this part, we suppose that Ω is of class $\mathcal{C}^{\gamma+1,1}$. To treat this case, we take advantage of a recent paper from S. Nicaise and T. Chaumont-Frelet [14] which proposes a decomposition of the solution of general wave propagation problems into a sum of functions which are more and more regular and dependent of k. As their method is built for general elliptic second order operator but with standard boundary condition (Dirichlet or Neumann/Robin), we need to check if the method can be applied to our setting.

In order to estimate $k\eta(\mathbf{S}_{h,p})$, we first prove a decomposition result.

3.3.1 Expansion of u in power of k

We want to decompose $\mathbf{u} = (\mathbf{E}, \mathbf{H}) = S_{k,s}(\mathbf{f})$, in a serie of powers of k. First, we recall that

$$\begin{aligned} L_{0,s}(\mathbf{u}) :=& (L_{0,s}(\mathbf{E}), L_{0,s}(\mathbf{H})) \\ =& (\operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla\operatorname{div}\mathbf{E}, \operatorname{curl}\operatorname{curl}\mathbf{H} - s\nabla\operatorname{div}\mathbf{H}). \end{aligned}$$

In order to simplify the notation, we write here

$$\mathcal{B}(\mathbf{u}) := \begin{pmatrix} \operatorname{div}\mathbf{E} \\ \operatorname{div}\mathbf{H} \\ B_0(\mathbf{E},\mathbf{H}) \end{pmatrix} \text{ and } \mathbf{G}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ -i\frac{\lambda}{\lambda}\mathbf{E}_t - \frac{i}{\lambda}\mathbf{H}_t \end{pmatrix}.$$

We recall that $\mathbf{u} = S_{k,s}(\mathbf{f})$ is the solution of

$$\begin{cases} L_{0,s}(\mathbf{u}) &= \mathbf{f} + k^2\mathbf{u} & \text{in } \Omega, \\ T(\mathbf{u}) &= 0 & \text{on } \partial\Omega, \\ \mathcal{B}(\mathbf{u}) &= k\mathbf{G}\mathbf{u} & \text{on } \partial\Omega. \end{cases}$$

Idea of the construction

Assume that $\mathbf{u}$ admits the formal expansion $\mathbf{u} = \sum_{j\geq 0} k^j \mathbf{u}_j$, thus the identity $L_{0,s}(\mathbf{u}) = \mathbf{f} + k^2\mathbf{u}$ is equivalent to

$$\begin{aligned} L_{0,s}(\mathbf{u}) = L_{0,s}(\sum_{j\geq 0} k^j \mathbf{u}_j) &= \sum_{j\geq 0} k^j L_{0,s}(\mathbf{u}_j) \\ &= \mathbf{f} + k^2 \sum_{j\geq 0} k^j \mathbf{u}_j. \end{aligned}$$

By identification of powers of k, we get

$$\begin{aligned} &L_{0,s}(\mathbf{u}_0) = \mathbf{f}, \\ &L_{0,s}(\mathbf{u}_1) = 0, \\ &L_{0,s}(\mathbf{u}_j) = \mathbf{u}_{j-2}, \quad \forall j \geq 2. \end{aligned}$$

In the same way,

$$T(\mathbf{u}_j) = 0, \quad \forall j \geq 0,$$

and

$$\begin{aligned} &\mathcal{B}(\mathbf{u}_0) = 0, \\ &\mathcal{B}(\mathbf{u}_j) = \mathbf{G}\mathbf{u}_{j-1}, \quad \forall j \geq 1. \end{aligned}$$

In summary, we have

$$\left\{ \begin{array}{lcll} L_{0,s}(\mathbf{u}_j) & = & \mathbf{u}_{j-2} & \text{in } \Omega, \\ T(\mathbf{u}_j) & = & 0 & \text{on } \partial\Omega, \\ \mathcal{B}(\mathbf{u}_j) & = & \mathbf{G}\mathbf{u}_{j-1} & \text{on } \partial\Omega, \end{array} \right. \tag{3.40}$$

for all $j \geq 0$, with $\mathbf{u}_{-2} = \mathbf{u}$ and $\mathbf{u}_{-1} = \mathbf{f}$.
Hence, we can get the following theorem:

Theorem 3.3.1. *Let $\mathbf{u}_j$ verify* (3.40)*, for $j \in \{0, \ldots, l-1\}$, with $l \leq \gamma - 1$ and $\mathbf{r}_l := \mathbf{u} - \sum_{j=0}^{l-1} k^j \mathbf{u}_j$. We also assume that the k-stability property with exponent $\alpha \geq 0$ holds. Then $\mathbf{u}_j \in \mathbf{H}^{j+2}(\Omega)^2$, $\mathbf{r}_l \in \mathbf{H}^{l+2}(\Omega)^2$ and we have*

$$\begin{aligned} \|\mathbf{u}_j\|_{j+2,\Omega} &\lesssim \|\mathbf{f}\|_{\Omega}, \\ \|\mathbf{r}_l\|_{l+2,\Omega} &\lesssim k^{\alpha+l+1} \|\mathbf{f}\|_{\Omega}. \end{aligned}$$

Proof. We first state a shift theorem.

Theorem 3.3.2. *Let $\mathbf{u}$ be a solution of*

$$\left\{ \begin{array}{lcll} L_{0,s}(\mathbf{u}) & = & \mathbf{f} & \textit{in } \Omega, \\ T(\mathbf{u}) & = & 0 & \textit{on } \partial\Omega, \\ \mathcal{B}(\mathbf{u}) & = & \mathbf{g} & \textit{on } \partial\Omega, \end{array} \right. \tag{3.41}$$

with Ω of class $\mathcal{C}^{j+1,1}$, $\mathbf{f} \in H^j(\Omega)^2$ and $\mathbf{g} \in H^{j+\frac{1}{2}}(\partial\Omega)$. Then $\mathbf{u} \in \mathbf{H}^{j+2}(\Omega)^2$ with

$$\|\mathbf{u}\|_{j+2,\Omega} \lesssim \|\mathbf{f}\|_{j,\Omega} + \|\mathbf{g}\|_{j+\frac{1}{2},\partial\Omega}.$$

Proof. (3.41) is an elliptic boundary value problem with an ellipticity constant bounded independently from s, since $s \in [1,2]$. Then, the result directly follows from a standard elliptic regularity result (cf. [22])), with a constant independent of s. □

Initialization is done directly from the definition of $\mathbf{u}_0$ and $\mathbf{u}_1$. For $j \geq 2$, one has

$$\begin{aligned}
L_{0,s}(\mathbf{r}_j) &= L_{0,s}(\mathbf{u}) - \sum_{0\leq l\leq j-1} k^l L_{0,s}(\mathbf{u}_l) \\
&= L_{0,s}(\mathbf{u}) - L_{0,s}(\mathbf{u}_0) - L_{0,s}(\mathbf{u}_1) - \sum_{2\leq l\leq j-1} k^l L_{0,s}(\mathbf{u}_l) \\
&= f + k^2\mathbf{u} - f - \sum_{2\leq l\leq j-1} k^l \mathbf{u}_{l-2} \\
&= k^2(\mathbf{u} - \sum_{0\leq l\leq j-3} k^l \mathbf{u}_l) \\
&= k^2 \mathbf{r}_{j-2}
\end{aligned}$$

and

$$\begin{aligned}
T(\mathbf{r}_j) &= T(\mathbf{u}) - T(\mathbf{u}_0) - \sum_{1\leq l\leq j-1} k^l T(\mathbf{u}_l) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}(\mathbf{r}_j) &= \mathcal{B}(\mathbf{u}) - \mathcal{B}(\mathbf{u}_0) - \sum_{1\leq l\leq j-1} k^l \mathcal{B}(\mathbf{u}_l) \\
&= k\mathbf{G}\mathbf{u} - k \sum_{0\leq l\leq j-2} k^l \mathbf{G}\mathbf{u}_l \\
&= k\mathbf{G}\mathbf{r}_{j-1}.
\end{aligned}$$

Hence, we conclude that

$$\left\{\begin{array}{lcll}
L_{0,s}(\mathbf{r}_j) & = & k^2\mathbf{r}_{j-2} & \text{in } \Omega \\
T(\mathbf{r}_j) & = & 0 & \text{on } \partial\Omega, \\
\mathcal{B}(\mathbf{r}_j) & = & k\mathbf{G}\mathbf{r}_{j-1} & \text{on } \partial\Omega,
\end{array}\right. \tag{3.42}$$

for $j \geq 1$, with $\mathbf{r}_{-1} = \mathbf{r}_0 = \mathbf{u}$.
By the shift Theorem 3.3.2 and the k-stability property, we get

$$\begin{aligned}
\|\mathbf{r}_1\|_{3,\Omega} &\lesssim k^2 \|\mathbf{u}\|_{1,\Omega} + k \|\mathbf{G}\mathbf{u}\|_{1+\frac{1}{2},\partial\Omega} \\
&\lesssim k^2 \|\mathbf{u}\|_{1,\Omega} + k \|\mathbf{u}\|_{2,\Omega} \\
&\lesssim k^{2+\alpha} \|\mathbf{f}\|_{\Omega}.
\end{aligned}$$

And, similarly

$$\begin{aligned}\|\mathbf{r}_2\|_{\mathrm{H}^4(\Omega)} &\lesssim k^2 \|\mathbf{u}\|_{2,\Omega} + k \|\mathbf{G}\mathbf{r}_1\|_{2+\frac{1}{2},\partial\Omega} \\ &\lesssim k^2 \|\mathbf{u}\|_{1,\Omega} + k \|\mathbf{r}_1\|_{3,\Omega} \\ &\lesssim k^{3+\alpha} \|\mathbf{f}\|_{\Omega} .\end{aligned}$$

Hence, for $j > 2$,

$$\begin{aligned}\|\mathbf{r}_j\|_{j+2,\Omega} &\lesssim k^2 \|\mathbf{r}_{j-2}\|_{j,\Omega} + k \|\mathbf{G}\mathbf{r}_{j-1}\|_{j+\frac{1}{2},\partial\Omega} \\ &\lesssim k^{j+1+\alpha} \|\mathbf{f}\|_{\Omega} .\end{aligned}$$

□

3.3.2 Estimation of $k\eta(\mathbf{S}_{h,p})$.

We hence assume that $p \leq \gamma + 1$. In this part, the constants depend on p, so we analyse only h-FEM.
Let $\mathbf{f} \in \mathrm{L}^2(\Omega)^6$, by Theorem 3.3.1, we can split $S^*_{k,s}(\mathbf{f})$ in the following way

$$S^*_{k,s}(\mathbf{f}) = \sum_{j=0}^{p-2} k^j \mathbf{u}_j + \mathbf{r}_{p-1}. \tag{3.43}$$

So,

$$\begin{aligned}\inf_{\mathbf{v}\in\mathbf{S}_{h,p}} \|S^*_{k,s}(\mathbf{f}) - \mathbf{v}\|_{k,h,p} &\leq \left\| \sum_{j=0}^{p-2} \left(k^j \mathbf{u}_j - \Pi_p k^j \mathbf{u}_j\right) + \mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1} \right\|_{k,h,p} \\ &\leq \sum_{j=0}^{p-2} k^j \|\mathbf{u}_j - \Pi_p \mathbf{u}_j\|_{k,h,p} + \|\mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1}\|_{k,h,p} . \qquad (3.44)\end{aligned}$$

By Theorem 3.3.1, the definition of Π_p and as $hk < 1$, we have

$$\|\mathbf{u}_j - \Pi_p \mathbf{u}_j\|_k \lesssim (1 + hk) h^{j+1} \|\mathbf{u}_j\|_{j+2,\Omega} \lesssim h^{j+1} k^j \|\mathbf{f}\|_{\Omega} . \tag{3.45}$$

In the same way, we get

$$\|\mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1}\|_k \lesssim h^p k^{p+\alpha} \|\mathbf{f}\|_{\Omega} . \tag{3.46}$$

It remains to estimate the "discrete part" of the norm, namely by Lemma 3.1.9

$$\begin{aligned}\sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} \left\| (\mathbf{H}_j - \Pi_p \mathbf{E}_j)_t - \frac{1}{\lambda} (\mathbf{E}_j - \Pi_p \mathbf{H}_j) \times \mathbf{n} \right\|_f^2 &\lesssim \sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} \|\mathbf{u}_j - \Pi_p \mathbf{u}_j\|_f^2 \\ &\lesssim \sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} \left(\frac{h}{p}\right)^{2j+3} \|\mathbf{u}_j\|_{j+2,K_f}^2 \\ &\lesssim \sum_{f\in\mathcal{E}^B} h^{2(j+1)} \|\mathbf{u}_j\|_{j+2,K_f}^2 \\ &\lesssim h^{2(j+1)} \|\mathbf{f}\|_{\Omega}^2 .\end{aligned}$$

In the same way, writing $\mathbf{r}_{p-1} = (\mathbf{E}_{p-1}, \mathbf{H}_{p-1})$, we have

$$\begin{aligned}
\sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} & \left\| (\mathbf{E}_{p-1} - \Pi_p \mathbf{E}_{p-1})_t - \frac{1}{\lambda}(\mathbf{H}_{p-1} - \Pi_p \mathbf{H}_{p-1}) \times \mathbf{n} \right\|_f^2 \\
&\lesssim \sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} \left\| \mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1} \right\|_f^2 \\
&\lesssim \sum_{f\in\mathcal{E}^B} \alpha_f \frac{p^2}{h} \left(\frac{h}{p}\right)^{2p+1} \left\| \mathbf{r}_{p-1} \right\|_{p+1,\Omega}^2 \\
&\lesssim h^{2p} k^{2(p+\alpha)} \left\| \mathbf{f} \right\|_\Omega^2 .
\end{aligned}$$

In summary, we obtain

$$\begin{aligned}
\left\| \mathbf{u}_j - \Pi_p \mathbf{u}_j \right\|_{k,h,p} &\lesssim h^{j+1} \left\| \mathbf{f} \right\|_\Omega , \\
\left\| \mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1} \right\|_{k,h,p} &\lesssim k^{p+\alpha} h^p \left\| \mathbf{f} \right\|_\Omega .
\end{aligned}$$

Then, by (3.44), we get

$$\begin{aligned}
\inf_{\mathbf{v}\in\mathbf{S}_{h,p}} \left\| S_{k,s}^*(\mathbf{f}) - \mathbf{v} \right\|_{k,h,p} &\lesssim \sum_{j=0}^{p-2} k^j \left\| \mathbf{u}_j - \Pi_p \mathbf{u}_j \right\|_{k,h,p} + \left\| \mathbf{r}_{p-1} - \Pi_p \mathbf{r}_{p-1} \right\|_{k,h,p} \\
&\lesssim \left(\sum_{j=0}^{p-2} k^j h^{j+1} + k^{p+\alpha} h^p \right) \left\| \mathbf{f} \right\|_\Omega \\
&\lesssim (h + k^{p+\alpha} h^p) \left\| \mathbf{f} \right\|_\Omega .
\end{aligned}$$

Thanks to Theorem 3.1.6, we can conclude that

Theorem 3.3.3. *If $\partial\Omega$ is of class $\mathcal{C}^{\gamma+1,1}$ and $p \le \gamma+1$, then for hk small enough, we have*

$$k\eta(\mathbf{S}_{h,p}) \lesssim hk + k^{p+1+\alpha} h^p. \tag{3.47}$$

And if $hk + k^{p+1+\alpha} h^p$ is small enough, then problem (3.6) has a unique solution $\mathbf{u}_{\mathbf{S}_{h,p}} \in \mathbf{S}_{h,p}$ and we have

$$\left\| \mathbf{u} - \mathbf{u}_{\mathbf{S}_{h,p}} \right\|_{k,h,p} \lesssim h + k^{p+\alpha} h^p. \tag{3.48}$$

3.4 Some numerical tests

For the sake of simplicity, we restrict ourselves to the TE/TH polarization of the problem (1.30). In other words, we take

$$\Omega = D \times \mathbb{R},$$

where D is a two-dimensional disk and assume that the solution of our problem is independent of the third variable. In such a case, the original problem splits up into a TE polarization problem in (E_1, E_2, H_3) in D (correspond to (1.60)), and a TH polarization one in (H_1, H_2, E_3) in D (correspond to (1.59)). We restrict ourselves to the TE polarization here. The variational formulation for the continuous problem is given by (1.62). The discrete formulation of this problem is:

$$\begin{aligned} a_{k,s,h,p}((E,H_3),(E',H_3')) &= a_{k,s}((E,H_3),(E',H_3')) \\ &+ \int_{\partial D} (-\frac{1}{\lambda_{\text{imp}}} \operatorname{curl} E + ikE_t)\overline{(\lambda_{\text{imp}}E_t' - H_3')}\, d\sigma \\ &+ \int_{\partial D} (\lambda_{\text{imp}}E_t - H_3)\overline{(-\frac{1}{\lambda_{\text{imp}}} \operatorname{curl} E' - ikE_t')}\, d\sigma \\ &+ \frac{p^2\alpha}{h} \int_{\partial D} (\lambda_{\text{imp}}E_t - H_3)\overline{(\lambda_{\text{imp}}E_t' - H_3')}\, d\sigma. \end{aligned}$$

In our tests, we take $D = B(0,1)$ and use meshes built with the help of quadrangles of order 2. We choose $\lambda_{\text{imp}} = -1$, hence the impedance boundary condition is then:

$$H_3 + E_t = 0 \text{ on } \partial D.$$

As the discrete space, we take $\mathcal{S}_{h,p}(D)^3$. To illustrate our results, we consider two exact solutions, the first one is given by

$$E_{ex}(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix} H_{3,ex}(x,y) \text{ and } H_{3,ex}(x,y) = e^{ik(x^2+y^2)^{\frac{1,1}{2}}},$$

that belongs to $H^2(D)$ but is not in $H^3(D)$, while as second example we consider

$$E_{ex}(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix} H_{3,ex}(x,y) \text{ and } H_{3,ex}(x,y) = e^{ikx},$$

that, in this case, is analytical. In both cases, we compute the right-hand side of (1.28) accordingly. In our numerical experiments, we have chosen $s = 14.3$ and $\alpha_f = 10$, because they yield satisfactory numerical results. Figure 3.1 corresponds to the tests for the first solution, while Figure 3.2 corresponds to the tests for the one.

First to validate our method, we have computed the error in the norm $\|\cdot\|_{k,h,p}$ and compare it with the projection error $\|u - \mathbb{P}_{h,p}u\|_{k,h,p}$, where $\mathbb{P}_{h,p}$ is the orthogonal projection on $\mathbf{S}_{h,p}$ for the inner product associated with the norm $\|\cdot\|_{k,h,p}$, namely for $(E',H_3') \in \mathbf{V}$, $\mathbb{P}_{h,p}(E',H_3') \in \mathbf{S}_{h,p}$ is the unique solution of

$$(\mathbb{P}_h(E',H_3'),(U,q))_{k,h,p} = ((E',H_3'),(U,q))_{k,h,p},\ \forall (U,q) \in \mathbf{S}_{h,p},$$

where

$$\begin{aligned} ((E',H_3'),(U,q))_{k,h,p} &= \int_D (\operatorname{curl} E \cdot \operatorname{curl} \bar{U} + s \operatorname{div} E \operatorname{div} \bar{U} + k^2 E \cdot \bar{U})\, dx \\ &+ \int_D (\nabla H_3 \cdot \nabla \bar{q} + k^2 H_3 \cdot \bar{q})\, dx \\ &+ \frac{10p^2}{h} \int_{\partial D} (E_t + H_3)(\bar{U}_t + \bar{q})\, d\sigma. \end{aligned}$$

In Figures 3.1(a) and 3.1(b), we have depicted the two errors for the non smooth solution with $k = 30$ or 60, $h = \frac{\pi}{10}$ or $h = \frac{\pi}{20}$ and various values of p, there we see that for p large enough we enter in the asymptotic regime (since both errors are almost equal) and the convergence rate is around 1.1 as theoretocally expected. Similarly for the analytical solution, we can see in Figures 3.2(a) and 3.2(b), the convergence rate seems to be exponential. Let us notice that in the asymptotic regime, the error seems to reach a lower bound for the largest degrees of freedom. This can be explained by the fact that the error due to the variational crime (caused by the nonconformity of our meshes) becomes predominant with respect to the approximation error.

The second main result from section 3.2 states that if (3.33) holds, then

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h,p}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p} \lesssim \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p}. \tag{3.49}$$

In order to see if this bound is sharp or not, we compute $\mathbb{S}_{k,s,h,p}(\mathbf{f}_1,\mathbf{f}_2)$ and $\mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ for different values of h, p, and k. For different values of k, h, and p, we denote by $p^\star$ the smallest value p_0 such that

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,p,h}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p} \leq 2\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p}, \quad \forall p \leq p_0. \tag{3.50}$$

The value of $p^\star$ for a given pair (k,h) is obtained by inspecting the ratio

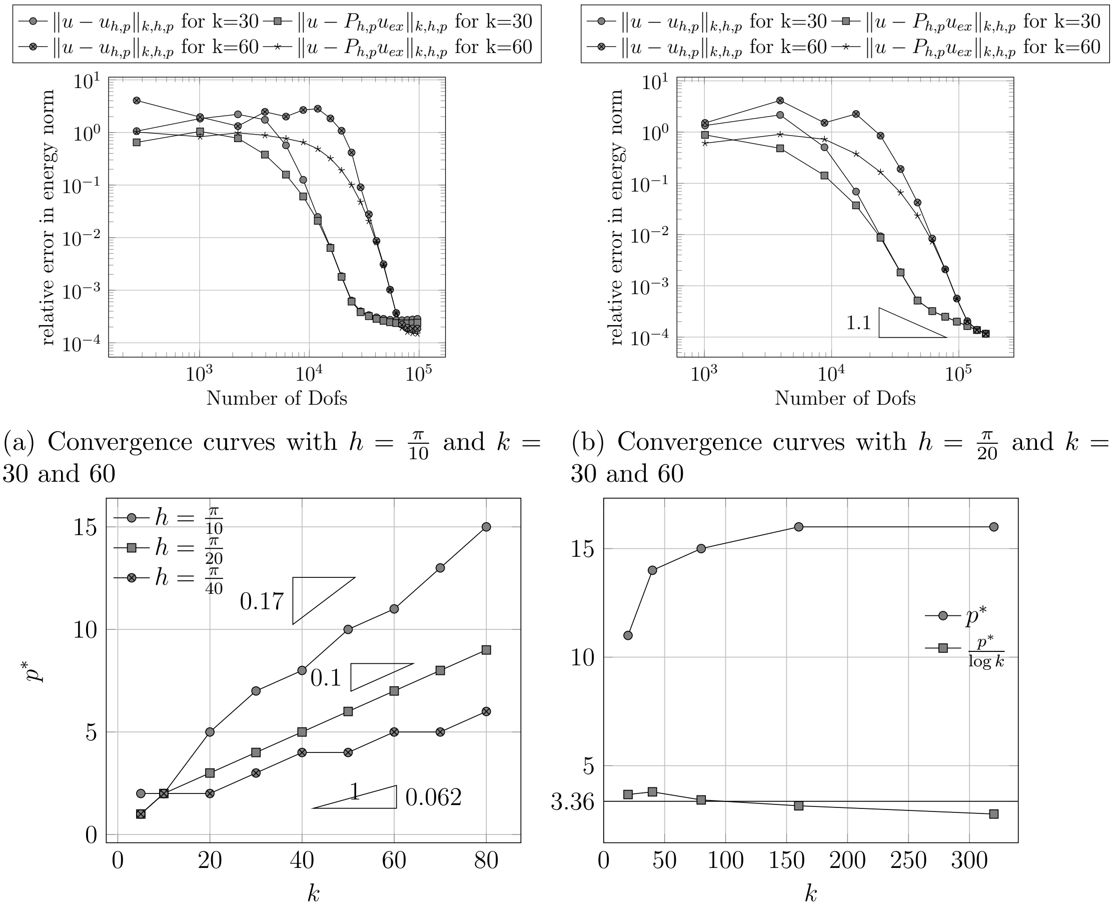
$$\frac{\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,p,h}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p}}{\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_{h,p}\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{k,h,p},}.$$

Condition (3.50) state that the finite element solution must be quasi optimal in the $\|\cdot\|_{k,h,p}$ norm, uniformly in k (with the arbitrary constant 2).

We have compute $p^*(k)$ in two different ways: First, we have chosen the mesh size h independent of k. So, three values of the meshsize $h = \frac{\pi}{10}, \frac{\pi}{20}$ and $\frac{\pi}{40}$ have been fixed and we have computed the value of $p^\star$ for k varying from 5 to 80. The graph of $p^\star(k)$ is represented in Figures 3.1(c) and 3.2(c). There we observe that in both cases $p^\star(k) \sim k$, which is better than conditions (3.33) since for h bounded from below, these conditions are equivalent to $p \geq Ck^2$ for C large enough (but in accordance with our conjecture from Remark 3.2.7). Moreover, the slope seems to depend linearly on h, in other words, the condition on $p^*(k)$ seems to be $p^*(k) = Chk$. Secondly, we fix the product kh to be constant (equal to 5π) with k varying from 20 to 320 for the non smooth solution (and 160 for the smooth solution) and again compute $p^\star$ as before. In that case, the conditions (3.33) are satisfied if $p \geq C \ln k$ for C large enough. This is confirmed experimentally since Figures 3.1(d) and 3.2(d) show a behavior of $p^\star$ of the order of $\ln k$.

3.5 Appendix: Analytic regularity with bounds explicit in the wavenumber

In this section, we will prove the analytical regularity for the solution of the problem (1.30) with estimates explicit in the wavenumber k. For that purpose,



(a) Convergence curves with $h=\frac{\pi}{10}$ and $k=$ 30 and 60

(b) Convergence curves with $h=\frac{\pi}{20}$ and $k=$ 30 and 60

(c) p^* for different values of k and h.

(d) p^* for different values of k, with $kh=$ 5π

Figure 3.1: First experiment with $\mathbf{E}_{ex}(x,y)=(y,-x)q_{ex}(x,y)$ and $q_{ex}(x,y)=e^{ikr^{1.1}}$

the right-hand side $\mathbf{f}$ is supposed to be an analytic function such that

$$|\mathbf{f}|_{p,\Omega} \le C_{\mathbf{f}}\lambda_{\mathbf{f}}^p \max(p,k)^p,\ \forall p \in \mathbb{N}. \tag{3.51}$$

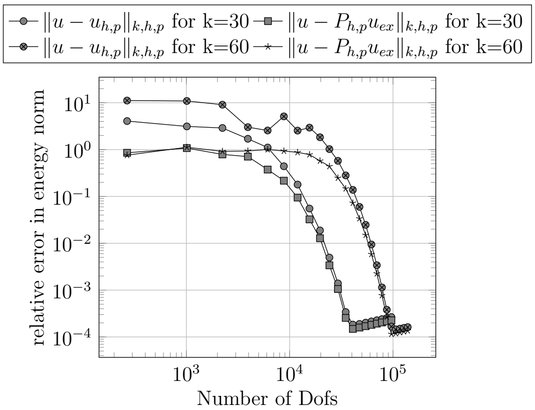
Theorem 3.5.1. *Let $\Omega \subset \mathbb{R}^n$, $n \le 3$, be a bounded domain with an analytical boundary, and (L,D,B) an elliptic system in the sense of Definition 2.2.27 of [22] with L (resp. D and B) a $N \times N$ (resp. $N_0 \times N$ and $N_0 \times N$ with $N_0, N_1 \in \mathbb{N}^*$ such that $N_0+N_1=N$) system of differential operators of order 2 (resp. 0 and 1) with $N \in \mathbb{N}^*$ and $k>1$. Let $\mathbf{f}$ be an analytical function which verifies hypothesis* (3.51) *and $\mathbf{G}$ a matrix with analytical coefficients. If $\mathbf{u}$ is a solution of*

$$\begin{cases} L(\mathbf{u}) &= \mathbf{f}+k^2\mathbf{u} & \text{in } \Omega,\\ D(\mathbf{u}) &= 0 & \text{on } \partial\Omega,\\ B(\mathbf{u}) &= k\mathbf{G}\mathbf{u} & \text{on } \partial\Omega, \end{cases} \tag{3.52}$$

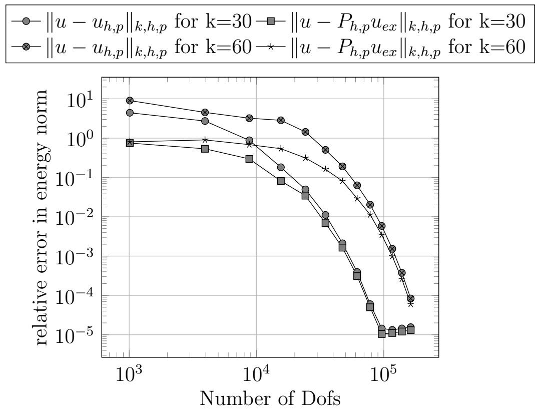
then we have

$$|\mathbf{u}|_{p,\Omega} \le C_{\mathbf{u}}K^p \max(p,k)^p,\ \forall p \in \mathbb{N}, p \ge 2,$$

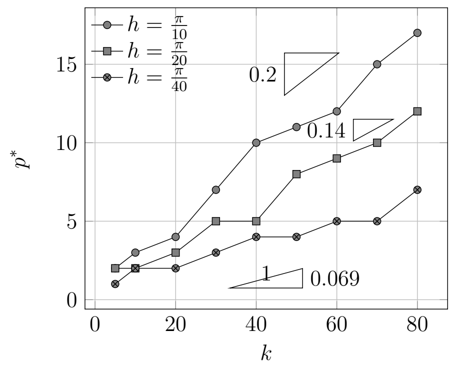
with $C_{\mathbf{u}} = C(C_{\mathbf{f}} + \|\mathbf{u}\|_{\Omega} + k^{-1}\|\mathbf{u}\|_{1,\Omega})$.



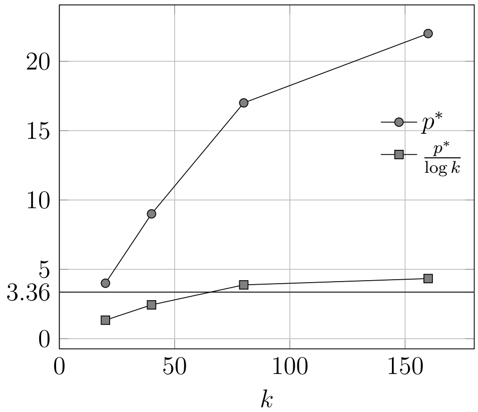
(a) Convergence curves with $h = \frac{\pi}{10}$ and $k =$ 30 and 60



(b) Convergence curves with $h = \frac{\pi}{20}$ and $k =$ 30 and 60



(c) p^* for different values of k and h.



(d) p^* for different values of k, with $kh = 5\pi$

Figure 3.2: Second experiment with $\mathbf{E}_{ex} = (y, -x)q_{ex}(x, y)$ and $q_{ex}(x, y) = e^{ikx}$

Corollary 3.5.2. *When $\Omega \subset \mathbb{R}^3$ is a bounded domain with an analytical boundary and if we take $L = (L_{0,s}, L_{0,s})$, $D = T$, $B = (\mathrm{div}, \mathrm{div}, B_0)$ and* $\mathbf{Gu} = \begin{pmatrix} 0 \\ 0 \\ -i\mathbf{E} \times \mathbf{n} - \frac{i}{\lambda_{\mathrm{imp}}}\mathbf{H}_t \end{pmatrix}$*, then if $\mathbf{u}$ is a solution of* (1.30) *with $\mathbf{f}$ verifying the same hypothesis as in Theorem 3.5.1. Then we have*

$$|\mathbf{u}|_{p,\Omega} \leq C_{\mathbf{u}} K^p \max(p, k)^p, \ \forall p \in \mathbb{N}, p \geq 2,$$

with $C_{\mathbf{u}} = C(C_{\mathbf{f}} + \|\mathbf{u}\|_{\Omega} + k^{-1} \|\mathbf{u}\|_{1,\Omega})$.

Proof. The proof is the same as the previous Theorem, but in this case, L depends on s, which in pratice depends on k. As s is supposed to be in the compact set $[1, 2]$, the ellipticity constant can be bounded independently from s. Hence, the estimates (3.64) and (3.71) below (standard elliptic regularity results in balls or half balls) remain valid with some constants independent of k. □

Remark 3.5.3. Theorem 3.5.1 is applicable for the Helmholtz equation with the standard absorbing boundary conditions (of Robin type), see [49, p. 1225]. But it is also applicable for the time-harmonic elastodynamic system in an isotropic medium with the so-called Lysmer-Kuhlemeyer absorbing boundary conditions [45, 18, 24].

In order to prove this theorem, we will first introduce two auxiliary lemmas which give us regularity results in half balls with a boundary condition on the flat part (Lemma 3.5.8) and balls without boundary condition (Lemma 3.5.10).

By a covering of Ω by some well chosen balls, we can apply these two auxiliary lemmas to obtain Theorem 3.5.1.

3.5.1 Analytic regularity near the boundary

Let $B_R^+ = B(0,R) \cap \{x | x_n > 0\}$ and $\Gamma_R = \{x \in \overline{B_R^+} | x_n = 0\}$, with $R \in (0,1]$.
Let $\mathbf{f}$ be an analytical function and $\mathbf{G}$ a matrix with analytical coefficients defined in $\overline{B_R^+}$ such that

$$\|\partial^\alpha \mathbf{f}\|_{\overline{B_R^+}} \leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^{|\alpha|} \max(|\alpha|, k)^{|\alpha|}, \forall \alpha \in \mathbb{N}^n, \tag{3.53}$$

$$\|\partial^\alpha \mathbf{G}\|_{\infty,\overline{B_R^+}} \leq C_{\mathbf{G}} \lambda_G^{|\alpha|} |\alpha|!, \forall \alpha \in \mathbb{N}^n, \tag{3.54}$$

for some $k \geq 1$ for some positive constants $C_{\mathbf{f}}$, $\lambda_{\mathbf{f}}$, $C_{\mathbf{G}}$ and $\lambda_{\mathbf{G}}$ independent of k.

Let $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ be a solution of

$$\begin{cases} L(\mathbf{u}) &= \mathbf{f} + k^2\mathbf{u} & \text{in } B_R^+, \\ D(\mathbf{u}) &= 0 & \text{on } \Gamma_R, \\ B(\mathbf{u}) &= k\mathbf{G}\mathbf{u} & \text{on } \Gamma_R, \end{cases} \tag{3.55}$$

where (L, D, B) is an elliptic system with analytical coefficients (in the above sense), with T (resp. B) an operator of order 0 (resp. 1).

For further purposes, we define a few norms

$$\begin{aligned}
|\mathbf{u}|_{p,q,B_R^+} &:= \max_{\substack{|\alpha|=p \\ \alpha_n \leq q}} \|\partial^\alpha \mathbf{u}\|_{B_R^+}, \\
[\![\mathbf{u}]\!]_{p,q,B_R^+} &:= \max_{0\leq\rho\leq\frac{R}{2p}} \rho^p |\mathbf{u}|_{p,q,B_{R-p\rho}^+}, \text{ for all } p > 0, \\
[\![\mathbf{u}]\!]_{0,0,B_R^+} &:= \|\mathbf{u}\|_{B_R^+}, \\
\rho_*^2 [\![\mathbf{u}]\!]_{p,q,B_R^+} &:= \max_{0\leq\rho\leq\frac{R}{2(p+1)}} \rho^{p+2} |\mathbf{u}|_{p,q,B_{R-(p+1)\rho}^+}, \\
|\mathbf{u}|_{p,\frac{1}{2},\Gamma_R} &:= \max_{\alpha'\in\mathbb{N}^{n-1}:|\alpha'|=p} \|\partial^{\alpha'} \mathbf{u}\|_{\frac{1}{2},\Gamma_R}, \\
\rho_*^{\frac{3}{2}} [\![\mathbf{u}]\!]_{p,\frac{1}{2},\Gamma_R} &:= \max_{0\leq\rho\leq\frac{R}{2(p+1)}} \rho^{p+\frac{3}{2}} |\mathbf{u}|_{p,\frac{1}{2},\Gamma_{R-(p+1)\rho}},
\end{aligned}$$

for all $p, q \in \mathbb{N}$, $q \leq p$.

We will first estimate the norm of the tangential derivatives (and the normal derivative up to 2) by using standard analytic regularity of elliptic systems. Then, we will be able to estimate the complete norm $[\![\mathbf{u}]\!]_{p,q,B_R^+}$. So we start with an estimation of the norm of tangental derivatives $[\![\mathbf{u}]\!]_{p,2,B_R^+}$. Before let us prove the next technical results that allow to pass from a sum on the multi-indices into a sum on their lengths.

Lemma 3.5.4. *Let h be a mapping from $\mathbb{N}$ into $[0,\infty)$ and a multi-index $\alpha' \in \mathbb{N}^{n-1}$, for $n = 2$ or 3. Then we have*

$$\sum_{\beta' \in \mathbb{N}^{n-1} : \beta' \leq \alpha'} h(|\beta'|) \leq 2 \sum_{p=0}^{|\alpha'|} h(p) e^{|\alpha'|-p}. \tag{3.56}$$

Proof. The extimate (3.56) being trivial for $n = 2$, we only need to consider the case $n = 3$. In this case, without loss of generality, we can assume that $\alpha' = (\alpha_1, \alpha_2)$ is such that $\alpha_2 \leq \alpha_1$. Now since in the left-hand side of (3.56) the summand depends only on the length of β', we may write

$$\sum_{\beta' \in \mathbb{N}^2 : \beta' \leq \alpha'} h(|\beta'|) = \sum_{p=0}^{|\alpha'|} h(p) N_p, \tag{3.57}$$

where N_p is the number of pairs $\beta' = (\beta_1, \beta_2) \leq \alpha'$ of length p that can be explicitly computed:

$$N_p = \begin{cases} p+1 & \text{if } 0 \leq p \leq \alpha_2, \\ \alpha_2 + 1 & \text{if } \alpha_2 \leq p \leq \alpha_1, \\ |\alpha'| - p + 1 & \text{if } \alpha_1 \leq p \leq |\alpha'|. \end{cases}$$

Since

$$x \leq e^x, \ \forall x \in [0, \infty),$$

we easily see that

$$N_p \leq 2e^{|\alpha'|-p}, \ \forall p \in \{0, \cdots, |\alpha'|\}.$$

This estimate and (3.57) yield (3.56). □

Corollary 3.5.5. *Let h be a mapping from $\mathbb{N}$ into $[0,\infty)$ and a multi-index $\alpha \in \mathbb{N}^n$ with $\alpha_n \leq 1$. Then we have*

$$\sum_{\beta \in \mathbb{N}^n : \beta \leq \alpha} h(|\beta|) \leq 2(1 + \frac{1}{e}) \sum_{p=0}^{|\alpha|} h(p) e^{|\alpha|-p}. \tag{3.58}$$

Proof. If $n = 1$, (3.58) is direct, so we assume that $n = 2$ or 3. If $\alpha_n = 0$, the assertion is a direct consequence of (3.56), while if $\alpha_n = 1$, we write

$$\sum_{\beta \in \mathbb{N}^n : \beta \leq \alpha} h(|\beta|) = \sum_{\beta = (\beta', 0) \in \mathbb{N}^n : \beta' \leq \alpha'} h(|\beta'|) + \sum_{\beta = (\beta', 1) \in \mathbb{N}^n : \beta' \leq \alpha'} h(|\beta'| + 1).$$

Then we apply the estimate (3.56) to each term of this right-hand side to get

$$\sum_{\beta \in \mathbb{N}^n : \beta \leq \alpha} h(|\beta|) = 2 \sum_{p=0}^{|\alpha'|} h(p) e^{|\alpha'|-p} + 2 \sum_{p=0}^{|\alpha'|} h(p+1) e^{|\alpha'|-p}.$$

We conclude by performing a simple change of unknowns in the second sum of this right-hand side and adding some non negative terms. □

Lemma 3.5.6. *There exist a positive constant C (depending on n), a positive constant $C_{tr,R}$ (depending only on $R \leq 1$), and a positive constant $\lambda'_{\mathbf{G}} \geq \lambda_{\mathbf{G}}$ such that for all $l \in \mathbb{N}$ and any $\mathbf{u} \in \mathbf{H}^{l+1}(B_R^+)$, we have*

$$\rho_*^{\frac{3}{2}} [\![\mathbf{G}\mathbf{u}]\!]_{l,\frac{1}{2},\Gamma_R} \leq C C_{tr,R} C_{\mathbf{G}} \sum_{p=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p} \max(l+1,k)^{l+1-p} [\![\mathbf{u}]\!]_{p,2,B_R^+}. \tag{3.59}$$

Proof. $\mathbf{G}$ being a matrix with analytical coefficients defined in B_R^+, by a standard trace theorem, there exists a positive constant $C_{tr,R}$ depending only on $R \leq 1$ such that

$$\rho_*^{\frac{3}{2}} [\![\mathbf{G}\mathbf{u}]\!]_{l,\frac{1}{2},\Gamma_R} \leq C_{tr,R} \left([\![\mathbf{G}\mathbf{u}]\!]_{l,0,B_R^+} + [\![\mathbf{G}\mathbf{u}]\!]_{l+1,1,B_R^+} \right). \tag{3.60}$$

We now estimate each term of this right-hand side. First for any $|\alpha| \leq l+1$, Leibniz's rule and the assumption (3.54) yields

$$\begin{aligned} \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_R^+} &\leq n \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left\|\partial^{\alpha-\beta}\mathbf{G}\right\|_{\infty B_R^+} \left\|\partial^\beta \mathbf{u}\right\|_{B_R^+} \\ &\leq n C_{\mathbf{G}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} (|\alpha|-|\beta|)! \left\|\partial^\beta \mathbf{u}\right\|_{B_R^+}. \end{aligned}$$

As one easily checks that

$$\frac{p!}{q!} \leq p^{p-q}, \ \forall p, q \in \mathbb{N} : q \leq p, \tag{3.61}$$

together with the combinatorial inequality (that can be shown using the combinatorial interpretation of binomial coefficients, see [16, p. 328] or [22, p. 48])

$$\frac{\beta!}{\gamma!(\beta-\gamma)!} \leq \frac{|\beta|!}{|\gamma|!\,(|\beta|-|\gamma|)!},$$

we deduce that

$$\|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_R^+} \leq n C_{\mathbf{G}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} \max(|\alpha|,k)^{|\alpha|-|\beta|} \left\|\partial^\beta \mathbf{u}\right\|_{B_R^+}. \tag{3.62}$$

Therefore, we may write

$$\begin{aligned} [\![\mathbf{G}\mathbf{u}]\!]_{l,0,B_R^+} &= \max_{0 \leq \rho \leq \frac{R}{2l}} \rho^l \max_{\substack{\alpha_n = 0 \\ |\alpha| = l}} \|\partial^\alpha \mathbf{G}\mathbf{u}\|_{B_{R-l\rho}^+} \\ &\leq n C_{\mathbf{G}} \max_{0 \leq \rho \leq \frac{R}{2l}} \rho^l \max_{\substack{\alpha_n = 0 \\ |\alpha| = l}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l,k)^{l-|\beta|} \left\|\partial^\beta \mathbf{u}\right\|_{B_{R-l\rho}^+}. \end{aligned}$$

As $R \leq 1$ and as $|\beta| \leq l$, we have $\rho^l \leq \rho^{|\beta|}$, and then

$$[\![\mathbf{G}\mathbf{u}]\!]_{l,0,B_R^+} \leq n C_{\mathbf{G}} \max_{\substack{\alpha_n = 0 \\ |\alpha| = l}} \sum_{\beta \leq \alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l,k)^{l-|\beta|} \max_{0 \leq \rho \leq \frac{R}{2|\beta|}} \rho^{|\beta|} \left\|\partial^\beta \mathbf{u}\right\|_{B_{R-|\beta|\rho}^+}.$$

In the above estimate as $\beta \le \alpha$ and $\alpha_n = 0$, β_n is trivially equal to zero, and we deduce that

$$[\![\mathbf{Gu}]\!]_{l,0,B_R^+} \le nC_{\mathbf{G}} \max_{\substack{\alpha_n=0\\|\alpha|=l}} \sum_{\beta\le\alpha} \lambda_{\mathbf{G}}^{l-|\beta|} \max(l,k)^{l-|\beta|} [\![\mathbf{u}]\!]_{|\beta|,0,B_R^+}.$$

Applying Lemma 3.5.4 to the sum on β (recalling that $\alpha_n = 0$), we deduce that

$$[\![\mathbf{Gu}]\!]_{l,0,B_R^+} \le 2nC_{\mathbf{G}} \sum_{p=0}^{l} (e\lambda_{\mathbf{G}})^{l-p} \max(l,k)^{l-p} [\![\mathbf{u}]\!]_{p,0,B_R^+}. \tag{3.63}$$

Similarly, using (3.62) we have

$$\begin{aligned}
[\![\mathbf{Gu}]\!]_{l+1,1,B_R^+} &\le \max_{0\le\rho\le\frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n\le 1\\|\alpha|=l+1}} \|\partial^\alpha \mathbf{Gu}\|_{B^+_{R-(l+1)\rho}} \\
&\le nC_{\mathbf{G}} \max_{0\le\rho\le\frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n\le 1\\|\alpha|=l+1}} \sum_{\beta\le\alpha} \lambda_{\mathbf{G}}^{|\alpha|-|\beta|} K^{|\alpha|-|\beta|} \max(|\alpha|,k)^{|\alpha|-|\beta|} \left\|\partial^\beta \mathbf{u}\right\|_{B^+_{R-(l+1)\rho}}.
\end{aligned}$$

Since $|\alpha| = l+1$, we get as before

$$\begin{aligned}
&[\![\mathbf{Gu}]\!]_{l+1,1,B_R^+} \\
&\quad\le nC_{\mathbf{G}} \max_{0\le\rho\le\frac{R}{2(l+1)}} \rho^{l+1} \max_{\substack{\alpha_n\le 1\\|\alpha|=l+1}} \sum_{\beta\le\alpha} \lambda_{\mathbf{G}}^{l+1-|\beta|} \max(l+1,k)^{l+1-|\beta|} \left\|\partial^\beta \mathbf{u}\right\|_{B^+_{R-(l+1)\rho}} \\
&\quad\le nC_{\mathbf{G}} \max_{\substack{\alpha_n\le 1\\|\alpha|=l+1}} \sum_{\beta\le\alpha} \lambda_{\mathbf{G}}^{l+1-|\beta|} \max(l+1,k)^{l+1-|\beta|} [\![\mathbf{u}]\!]_{|\beta|,1,B_R^+}.
\end{aligned}$$

Applying Corollary 3.5.5 to the summation on β, we conclude that

$$[\![\mathbf{Gu}]\!]_{l+1,1,B_R^+} \le 2(1+\frac{1}{e})nC_{\mathbf{G}} \sum_{p=0}^{l+1} (e\lambda_{\mathbf{G}})^{l+1-p} \max(l+1,k)^{l+1-p} [\![\mathbf{u}]\!]_{p,1,B_R^+}.$$

This estimate and (3.63) in the estimate (3.60) lead to (3.59) with $\lambda'_{\mathbf{G}} = e\lambda_{\mathbf{G}}$ (as $[\![\mathbf{u}]\!]_{|\beta|,j,B_R^+} \le [\![\mathbf{u}]\!]_{|\beta|,2,B_R^+}$, for $j = 0$ or 1). □

Now we can estimate the different derivatives.

Lemma 3.5.7. *Let* $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ *be a solution of* (3.55) *with* $\mathbf{f}$ *and* $\mathbf{G}$ *analytic and satisfying* (3.53)-(3.54). *Then there exist* $K > 1$ *and* $C_R > 1$ *such that for all* $p \ge 2$,

$$[\![\mathbf{u}]\!]_{p,2,B_R^+} \le C_{\mathbf{u}}(B_R^+) K^p \max(p,k)^p,$$

with $C_{\mathbf{u}}(B_R^+) = C_R(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1}\|\mathbf{u}\|_{1,B_R^+})$.

Proof. We will prove this result by induction, by applying a standard analytic regularity result (i.e. Proposition 2.6.6 of [22]), which gives us a real number $A \ge 1$ such that for all $p \ge 2$

(3.64)

$$[\![\mathbf{u}]\!]_{p,2,B_R^+} \le \sum_{l=0}^{p-2} A^{p-1-l} \left(\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} + \rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R}\right) + A^{p-1} \sum_{l=0}^{1} [\![\mathbf{u}]\!]_{l,l,B_R^+}.$$

Initialization: For $p = 2$, by (3.64), we have

$$
\begin{aligned}
[\![\mathbf{u}]\!]_{2,2,B_R^+} &\le A\left(\rho_*^2 [\![L(\mathbf{u})]\!]_{0,0,B_R^+} + \rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{0,\frac{1}{2},\Gamma_R}\right) + A\sum_{l=0}^{1} [\![\mathbf{u}]\!]_{l,l,B_R^+} \\
&\le A\left(\rho_*^2 [\![\mathbf{f} + k^2\mathbf{u}]\!]_{0,0,B_R^+} + \rho_*^{\frac{3}{2}} [\![k\mathbf{G}\mathbf{u}]\!]_{0,\frac{1}{2},\Gamma_R}\right) + A\sum_{l=0}^{1} [\![\mathbf{u}]\!]_{l,l,B_R^+} \\
&\le A\left(\|\mathbf{f}\|_{B_R^+} + k^2 \|\mathbf{u}\|_{B_R^+} + k \|\mathbf{G}\mathbf{u}\|_{\frac{1}{2},\Gamma_R}\right) + A\sum_{l=0}^{1} \|\mathbf{u}\|_{l,B_R^+} \\
&\le A\left(\|\mathbf{f}\|_{B_R^+} + (k^2+1) \|\mathbf{u}\|_{B_R^+} + kC_{tr,R} \|\mathbf{G}\mathbf{u}\|_{1,B_R^+} + \|\mathbf{u}\|_{1,B_R^+}\right),
\end{aligned}
$$

with the positive constant $C_{tr,R}$ introduced before. By noticing that

$$
kC_{tr,R} \|\mathbf{G}\mathbf{u}\|_{1,B_R^+} \le CkC_{tr,R}kC_{\mathbf{G}}(\|\mathbf{u}\|_{1,B_R^+} + \lambda_{\mathbf{G}} \|\mathbf{u}\|_{B_R^+}),
$$

we then have

$$
\begin{aligned}
[\![\mathbf{u}]\!]_{2,2,B_R^+} \le A\Big(&\|\mathbf{f}\|_{B_R^+} + (k^2 + 1 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}k) \|\mathbf{u}\|_{B_R^+} \\
&+ (CC_{tr,R}C_{\mathbf{G}}k + 1) \|\mathbf{u}\|_{1,B_R^+}\Big) \\
\le\ & Ak^2\left(C_{\mathbf{f}} + (2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}) \|\mathbf{u}\|_{B_R^+} + (CC_{tr,R}C_{\mathbf{G}} + 1)k^{-1} \|\mathbf{u}\|_{1,B_R^+}\right) \\
\le\ & Ak^2 \max(2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}, CC_{tr,R}C_{\mathbf{G}} + 1)(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1} \|\mathbf{u}\|_{1,B_R^+}) \\
\le\ & C_{\mathbf{u}}(B_R^+) \max(2,k)^2 \le C_{\mathbf{u}}(B_R^+) K^2 \max(2,k)^2,
\end{aligned}
$$

with $C_R \ge A \max(2 + CC_{tr,R}C_{\mathbf{G}}\lambda_{\mathbf{G}}, CC_{tr,R}C_{\mathbf{G}} + 1) \ge 1$ and since $K \ge 1$.
Induction hypothesis: For all $2 \le p' \le p$, we have

$$
[\![\mathbf{u}]\!]_{p',2,B_R^+} \le C_{\mathbf{u}}(B_R^+) K^{p'} \max(p',k)^{p'}. \tag{3.65}
$$

We will show this estimate for $p + 1$: Using (3.64), we may write

$$
[\![\mathbf{u}]\!]_{p+1,2,B_R^+} \le \sum_{l=0}^{p-1} A^{p-l}\left(\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} + \rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R}\right) + A^p \sum_{l=0}^{1} [\![\mathbf{u}]\!]_{l,l,B_R^+}. \tag{3.66}
$$

Now we need to estimate each term of this right-hand side. We start by estimating $\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+}$ for $l \le p - 1$: First we notice that

$$
\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} \le [\![\mathbf{f} + k^2\mathbf{u}]\!]_{l,0,B_R^+} \le [\![\mathbf{f}]\!]_{l,0,B_R^+} + k^2 [\![\mathbf{u}]\!]_{l,2,B_R^+}.
$$

By the induction hypothesis (3.65), we then have

$$
\begin{aligned}
\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} &\le C_{\mathbf{f}} \lambda_{\mathbf{f}}^l \max(l,k)^l + k^2 C_{\mathbf{u}}(B_R^+) K^l \max(l,k)^l \\
&\le k^2 \max(l,k)^l C_{\mathbf{u}}(B_R^+) K^l \left(\frac{\lambda_{\mathbf{f}}^l}{k^2 K^l} + 1\right).
\end{aligned}
$$

As $l+2 \le p+1$, this estimate directly implies that

$$\begin{aligned}\rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} &\le \max(p+1,k)^{p+1} C_{\mathbf{u}}(B_R^+) K^l \left(\left(\frac{\lambda_{\mathbf{f}}}{K} \right)^l + 1 \right) \\ &\le 2 \max(p+1,k)^{p+1} C_{\mathbf{u}}(B_R^+) K^l,\end{aligned}$$

for $K > \lambda_{\mathbf{f}}$. Multiplying this estimate by A^{p-l} and summing on l, one gets

$$\begin{aligned}\sum_{l=0}^{p-1} A^{p-l} \rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} &\le C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1,k)^{p+1} \frac{2}{K} \sum_{l=0}^{p-1} A^{p-l} K^{l-p} \\ &\le C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1,k)^{p+1} \frac{2}{K} \sum_{l=0}^{p-1} \left(\frac{A}{K} \right)^{p-l}.\end{aligned}$$

If $K \ge 2A$, then $\sum\limits_{l=0}^{p-1} \left(\frac{A}{K}\right)^{p-l} \le \sum\limits_{l=1}^{\infty} \left(\frac{A}{K}\right)^l \le 1$, which yields

$$\sum_{l=0}^{p-1} A^{p-l} \rho_*^2 [\![L(\mathbf{u})]\!]_{l,0,B_R^+} \le C_{\mathbf{u}}(B_R^+) K^{p+1} \max(p+1,k)^{p+1} \frac{2}{K}. \tag{3.67}$$

Estimation of $\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R}$: By the boundary condition on $\mathbf{u}$, we have

$$\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} = k \rho_*^{\frac{3}{2}} [\![\mathbf{G}\mathbf{u}]\!]_{l,\frac{1}{2},\Gamma_R},$$

and by the estimate (3.59), we get

$$\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \le k C C_{tr,R} C_{\mathbf{G}} \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} \max(l+1,k)^{l+1-p'} [\![\mathbf{u}]\!]_{p',2,B_R^+}. \tag{3.68}$$

The induction hypothesis (3.65) then leads to

$$\begin{aligned}&\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \\ &\quad \le k C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} K^{p'} \max(l+1,k)^{l+1-p'} \max(p',k)^{p'} \\ &\quad \le k C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) \max(l+1,k)^{l+1} K^{l+1} \sum_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K} \right)^{l+1-p'}.\end{aligned}$$

Hence for $K \ge 2\lambda'_{\mathbf{G}}$ (recalling that $l+2 \le p+1$ and that $\sum\limits_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K}\right)^{l+1-p'} \le 2$), we deduce

$$\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \le 2 C C_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K^{l+1} \max(p+1,k)^{p+1}.$$

Multiplying this estimate by A^{p-l} and summing on l, we get

$$\sum_{l=0}^{p-1} A^{p-l}\rho_*^{\frac{3}{2}}[\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \leq C_{\mathbf{u}}(B_R^+)K^{p+1}\max(p+1,k)^{p+1}2CC_{tr,R}C_{\mathbf{G}}\sum_{l=0}^{p-1}A^{p-l}K^{l-p}$$

$$\leq C_{\mathbf{u}}(B_R^+)K^{p+1}\max(p+1,k)^{p+1}\frac{2CC_{tr,R}C_{\mathbf{G}}A}{K}\sum_{l=0}^{p-1}\left(\frac{A}{K}\right)^l.$$

Again, for $K \geq 2A$, we arrive at

$$\sum_{l=0}^{p-1} A^{p-l}\rho_*^{\frac{3}{2}}[\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \leq C_{\mathbf{u}}(B_R^+)K^{p+1}\max(p+1,k)^{p+1}\frac{4CC_{tr,R}C_{\mathbf{G}}A}{K}. \tag{3.69}$$

Finally using the definition of $C_{\mathbf{u}}(B_R^+)$, we directly check that

$$\sum_{l=0}^{1}[\![\mathbf{u}]\!]_{l,l,B_R^+} \leq \frac{k}{C_R}C_{\mathbf{u}}(B_R^+), \tag{3.70}$$

and therefore (since we assume that $K \geq 2A$)

$$A^p\sum_{l=0}^{1}[\![\mathbf{u}]\!]_{l,l,B_R^+} \leq \frac{1}{C_R}C_{\mathbf{u}}(B_R^+)K^p\max(p+1,k)^{p+1}.$$

In summary, using this estimate, (3.67), and (3.69) in (3.66), we have obtained that

$$[\![\mathbf{u}]\!]_{p+1,2,B_R^+} \leq C_{\mathbf{u}}(B_R^+)K^{p+1}\max(p+1,k)^{p+1}\frac{\left(2+4CC_{tr,R}C_{\mathbf{G}}A+\frac{1}{C_R}\right)}{K}.$$

This yields (3.65) for $p+1$ if

$$K \geq \max\left(\lambda_{\mathbf{f}}, 2A, 2\lambda'_{\mathbf{G}}, 2+4CC_{tr,R}C_{\mathbf{G}}A+\frac{1}{C_R}\right).$$

□

Now, we will show an equivalent lemma but which also estimates the norm of the normal derivatives of higher order.

Lemma 3.5.8. *Let $\mathbf{u} \in \mathbf{H}^2(B_R^+)$ be a solution of* (3.55) *with $\mathbf{f}$ and $\mathbf{G}$ analytic and satisfying* (3.53)-(3.54). *Then there exist $K_1, K_2 \geq 1$ such that for all $p, q \geq 2$ with $q \leq p$, we have*

$$[\![\mathbf{u}]\!]_{p,q,B_R^+} \leq C_{\mathbf{u}}(B_R^+)K_1^pK_2^q\max(p,k)^p,$$

with $C_{\mathbf{u}}(B_R^+) = C_R(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_R^+} + k^{-1}\|\mathbf{u}\|_{1,B_R^+})$.

Proof. Again, we will show this lemma by induction and by using a standard analytical regularity result for elliptic problem (i.e. proposition 2.6.7 of [22]), which gives us

(3.71)
$$\llbracket \mathbf{u} \rrbracket_{p,q,B_R^+} \leq \sum_{l=0}^{p-2} A^{p-1-l} \left\{ \sum_{\nu=0}^{\min(l,q-2)} B^{q-1-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} + B^{q-1} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right\} + A^{p-1} B^{q-1} \sum_{l=0}^{1} \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+},$$

for some positive constants A and $B \geq 1$. The induction is done on q, the initialization step $q = 2$ is obtained from Lemma 3.5.7, by taking $K_1 \geq K$ and $K_2 \geq 1$. The induction hypothesis is: For all $p \geq 3$, $2 \leq q' \leq q \leq p-1$, it holds

$$\llbracket \mathbf{u} \rrbracket_{p,q',B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q'} \max(p,k)^p. \tag{3.72}$$

We use the estimate (3.71) to get

(3.73)
$$\llbracket \mathbf{u} \rrbracket_{p,q+1,B_R^+} \leq \sum_{l=0}^{p-2} A^{p-1-l} \left\{ \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} + B^{q} \rho_*^{\frac{3}{2}} \llbracket B(\mathbf{u}) \rrbracket_{l,\frac{1}{2},\Gamma_R} \right\} + A^{p-1} B^{q} \sum_{l=0}^{1} \llbracket \mathbf{u} \rrbracket_{l,l,B_R^+}.$$

We start with the estimate of $\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+}$. By the induction hypothesis (3.72), we may write

$$\begin{aligned}
\rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} &\leq \llbracket \mathbf{f} \rrbracket_{l,\nu,B_R^+} + k^2 \llbracket \mathbf{u} \rrbracket_{l,\nu,B_R^+} \\
&\leq C_{\mathbf{f}} \lambda_{\mathbf{f}}^l \max(l,k)^l + k^2 C_{\mathbf{u}}(B_R^+) K_1^l K_2^\nu \max(l,k)^l \\
&\leq C_{\mathbf{u}}(B_R^+) K_1^l K_2^\nu k^2 \max(l,k)^l \left(\left(\frac{\lambda_{\mathbf{f}}}{K_1} \right)^l \frac{1}{k^2 K_2^\nu} + 1 \right) \\
&\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \frac{2}{K_1 K_2} K_1^{l-p+1} K_2^{\nu-q},
\end{aligned}$$

if $K_1 \geq \lambda_{\mathbf{f}}$. Multiplying this estimate by $A^{p-1-l} B^{q-\nu}$ and summing on ν and l, one gets

$$\begin{aligned}
&\sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 \llbracket L(\mathbf{u}) \rrbracket_{l,\nu,B_R^+} \\
&\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \frac{2}{K_1 K_2} \sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} K_1^{l-p+1} K_2^{\nu-q} \\
&\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \frac{2}{K_1 K_2} \sum_{l=0}^{p-2} \left(\frac{A}{K_1} \right)^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} \left(\frac{B}{K_2} \right)^{q-\nu}.
\end{aligned}$$

Choosing $K_1 \geq 2A$ and $K_2 \geq 2B$, we conclude that
(3.74)
$$\sum_{l=0}^{p-2} A^{p-1-l} \sum_{\nu=0}^{\min(l,q-1)} B^{q-\nu} \rho_*^2 [\![L(\mathbf{u})]\!]_{l,\nu,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \frac{8}{K_1 K_2}.$$

Estimation of $\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R}$ for $l \leq p-2$: We use the estimate (3.59) and the induction hypothesis (3.72) to get

$$\begin{aligned}\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} &\leq kCC_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \\ &\quad \times \sum_{p'=0}^{l+1} (\lambda'_{\mathbf{G}})^{l+1-p'} K_1^{p'} \max(l+1,k)^{l+1-p'} \max(p',k)^{p'}.\end{aligned}$$

In the above right-hand side as $l+2 \leq p$ and $p' \leq p-1$, we obtain

$$\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \leq CC_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \max(p,k)^p K_1^{l+1} \sum_{p'=0}^{l+1} \left(\frac{\lambda'_{\mathbf{G}}}{K_1}\right)^{l+1-p'}.$$

For $K_1 \geq 2\lambda'_{\mathbf{G}}$, we deduce that

$$\rho_*^{\frac{3}{2}} [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \leq 2CC_{tr,R} C_{\mathbf{G}} C_{\mathbf{u}}(B_R^+) K_2^2 \max(p,k)^p K_1^{l+1}. \tag{3.75}$$

Multiplying this estimate by $A^{p-1-l} B^q$ and summing on l, as before one gets (since $K_1 \geq 2A$ and $K_2 \geq 2B$)

$$\sum_{l=0}^{p-2} A^{p-1-l} B^q \rho^{\frac{3}{2}} * [\![B(\mathbf{u})]\!]_{l,\frac{1}{2},\Gamma_R} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \left(\frac{4CC_{tr,R} C_{\mathbf{G}} A K_2}{K_1}\right),$$

Finally using (3.70), one has

$$A^{p-1} B^q \sum_{l=0}^{1} [\![\mathbf{u}]\!]_{l,l,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p \frac{1}{C_R K_1 K_2}.$$

Using this estimate and the estimates (3.74)-(3.75) into (3.66), we can conclude that

$$\begin{aligned}[\![\mathbf{u}]\!]_{p,q+1,B_R^+} &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \left(\frac{8}{K_1 K_2} + \frac{4CC_{tr,R} C_{\mathbf{G}} K_2}{K_1} + \frac{1}{C_R K_1 K_2}\right) \\ &\leq C_{\mathbf{u}}(B_R^+) K_1^p K_2^{q+1} \max(p,k)^p,\end{aligned}$$

for K_1 and K_2 large enough. □

Remark 3.5.9. *In Lemma 3.5.8, if we take $p = q$, we obtain*

$$[\![\mathbf{u}]\!]_{p,p,B_R^+} \leq C_{\mathbf{u}}(B_R^+) K^p \max(p,k)^p,$$

with $K = K_1 K_2$.

3.5.2 Interior analytic regularity

Let $B_R = B(0,R)$, L an elliptic system of order 2 defined in B_R, and $k>1$. Here we consider a solution $\mathbf{u}$ of

$$L(\mathbf{u}) = \mathbf{f} + k^2\mathbf{u} \text{ in } B_R. \tag{3.76}$$

We now define the following semi-norms

$$\begin{aligned}
[\![\mathbf{u}]\!]_{p,B_R} &:= \max_{0<\rho<\frac{R}{2p}} \max_{|\alpha|=p} \rho^p \left\|\partial^\alpha \mathbf{u}\right\|_{B_{R-p\rho}},\\
\rho_*^2[\![\mathbf{u}]\!]_{p,B_R} &:= \max_{0<\rho<\frac{R}{2p}} \max_{|\alpha|=p} \rho^{p+2} \left\|\partial^\alpha \mathbf{u}\right\|_{B_{R-p\rho}}.
\end{aligned}$$

We suppose that $\mathbf{f}$ is analytic with

$$\left\|\partial^\alpha \mathbf{f}\right\|_{B_R} \le C_{\mathbf{f}} \lambda_{\mathbf{f}}^p \max(|\alpha|,k)^{|\alpha|},\ \ \forall \alpha \in \mathbb{N}^n, \tag{3.77}$$

for some positive constants $C_{\mathbf{f}}$ and $\lambda_{\mathbf{f}}$ independent of k.

Lemma 3.5.10. *Let $\mathbf{u} \in \mathbf{H}^2(B_R)$ be a solution of* (3.76) *with f satisfying* (3.77)*. Then there exists $K \ge 1$ such that*

$$[\![\mathbf{u}]\!]_{p,B_R} \le C_{\mathbf{u}}(B_R) K^p \max(p,k)^p,$$

with $C_{\mathbf{u}}(B_R) = C_R(C_{\mathbf{f}} + \left\|\mathbf{u}\right\|_{B_R} + k^{-1}\left\|\mathbf{u}\right\|_{1,B_R})$, for $C_R \ge 1$.

Proof. The proof is exactly the same as the one of Lemma 3.5.7 when we use Proposition 1.6.3 of [22] (a standard interior regularity result) instead of Proposition 2.6.6 of [22]. □

3.5.3 Proof of Theorem 3.5.1

The first step of the proof is to consider a covering of Ω by some balls, which verifies

$$\Omega \subset \cup_{j=1}^N \hat{B}_j \subset \cup_{j=1}^N B_j,$$

where $B_j = B(x_j,\xi_j)$ and $\hat{B}_j = B(x_j, \frac{\xi_j}{2})$, with $\xi_j > 0$ small enough such that $\overline{B}(x_j,\xi_j) \subset \Omega$ if $x_j \in \Omega$. This yields

$$\begin{aligned}
|\mathbf{u}|_{p,\Omega} &\lesssim \sum_{i=1}^N |\mathbf{u}|_{p,\hat{B}_i\cap\Omega}\\
&\lesssim \sum_{1\le i\le N: x_i\in\Omega} |\mathbf{u}|_{p,\hat{B}_i} + \sum_{1\le i\le N: x_i\in\partial\Omega} |\mathbf{u}|_{p,\hat{B}_i\cap\Omega}.
\end{aligned}$$

In the case of an interior ball, namely for i such that $x_i \in \Omega$, we simply perform a translation to apply Lemma 3.5.10. Hence, the operator L does not change and we directly have

$$|\mathbf{u}|_{p,\hat{B}_i} \lesssim p^n [\![\mathbf{u}]\!]_{p,B_i} \lesssim p^n C_{\mathbf{u}}(B_i) K^p \max(p,k)^p.$$

By the definition of $C_{\mathbf{u}}(B_i)$, we then arrive at

$$|\mathbf{u}|_{p,\hat{B}_i} \lesssim \left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i} + k^{-1}\|\mathbf{u}\|_{1,B_i}\right)(eK)^p \max(p,k)^p. \tag{3.78}$$

In the case when a ball intersects the boundary of Ω, namely for each i such that $x_i \in \partial\Omega$, we apply a change of variables which allow to pass from $B_i \cap \Omega$ to $B^+_{\xi_i}$. First thanks to a Faà-di-Bruno formula, we obtain (see [22, (1.b)])

$$|\mathbf{u}|_{p,\hat{B}_i\cap\Omega} \lesssim c_i^{p+1} \sum_{l=0}^{p} \frac{k!}{l!} |\hat{\mathbf{u}}|_{l,B^+_{\xi_i/2}},$$

with a positive constant c_i which depends only on the transformation that allows to pass from $B_i \cap \Omega$ to $B^+_{\xi_i}$. Then we can can apply Lemma 3.5.8 (see Remark 3.5.9) and get

$$|\mathbf{u}|_{p,\hat{B}_i\cap\Omega} \lesssim e^p c_i^{p+1}\, C_{\hat{\mathbf{u}}}(B^+_{\xi_i}) \sum_{l=0}^{p} \frac{k!}{l!} K^l \max(l,k)^l.$$

Using (3.61), and a change of variables (in $C_{\hat{\mathbf{u}}}(B^+_{\xi_i})$ and again Faà-di-Bruno formula) we obtain

$$|\mathbf{u}|_{p,\hat{B}_i\cap\Omega} \lesssim e^p c_i^{p+1}\left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i\cap\Omega} + k^{-1}\|\mathbf{u}\|_{1,B_i\cap\Omega}\right)\max(p,k)^p \sum_{l=0}^{p} K^l.$$

This yields

$$|\mathbf{u}|_{p,\hat{B}_i\cap\Omega} \lesssim \frac{c_i K}{K-1}\left(C_{\mathbf{f}} + \|\mathbf{u}\|_{B_i\cap\Omega} + k^{-1}\|\mathbf{u}\|_{1,B_i\cap\Omega}\right)(c_i e K)^p \max(p,k)^p.$$

The combination of this estimate with (3.78) yield the result.

Chapter 4

Helmholtz equation with Perfectly Matched Layer (PML)

4.1 The scattering problem with a polar PML

We consider the Helmholtz equation set in the outside of a smooth, star-shaped sound-soft obstacle $\mathcal{O} \subset \mathbb{R}^2$. In order to approximate this problem with finite elements, it is required to truncate the computational domain. Here, we propose to analyze the Helmholtz equation when a Perfectly Matched Layer (PML) is employed.

Without losing generality, we select the coordinate system so that $\mathcal{O}$ is star-shaped with respect to the origin. We introduce two positive numbers $a < b$ such that $\mathcal{O}$ is contained in $B(0,a)$, the ball of $\mathbb{R}^2$ centered at 0 with radius a, and we employ the notation $\Omega_0 = B(0,a) \setminus \mathcal{O}$. In addition, we assume that the computational domain Ω is convex and contains $B(0,b)$. We also introduce the notation $\Gamma = \{|\mathbf{x}| = a\} = \partial B(0,a)$. The geometric setting is displayed in Figure 4.1. The relevant definitions and properties of the involved functions are listed in Appendix 4.7.

As usual we denote by (ρ, θ) the polar coordinates centred at 0. According to [19, §3] and using the notations from Appendix 4.7, for an arbitrary real number k, we consider the boundary value problem

$$\begin{aligned} k^2 d\tilde{d}u + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(q\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{q\rho^2}\frac{\partial^2 u}{\partial\theta^2} &= d\tilde{d}f \text{ in } \Omega, && (4.1)\\ u &= 0 \text{ on } \partial\Omega, && (4.2)\end{aligned}$$

where the datum f is supposed to be in $L^2(\Omega)$. As $d = \tilde{d} = 1$ in Ω_0, the problem reduces to the Helmholtz equation in Ω_0, the PML being situated in $\Omega \setminus \Omega_0$. Multiplying the partial differential equation by q, we obtain the equivalent problem

$$\begin{aligned} k^2 \tilde{d}^2 u + \frac{q}{\rho}\frac{\partial}{\partial\rho}\left(q\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} &= \tilde{d}^2 f \text{ in } \Omega, && (4.3)\\ u &= 0 \text{ on } \partial\Omega. && (4.4)\end{aligned}$$

The variational formulation of this problem is obtained by multiplying the partial differential equation by a test-function $\bar{v} \in H_0^1(\Omega)$ and by using formal integration

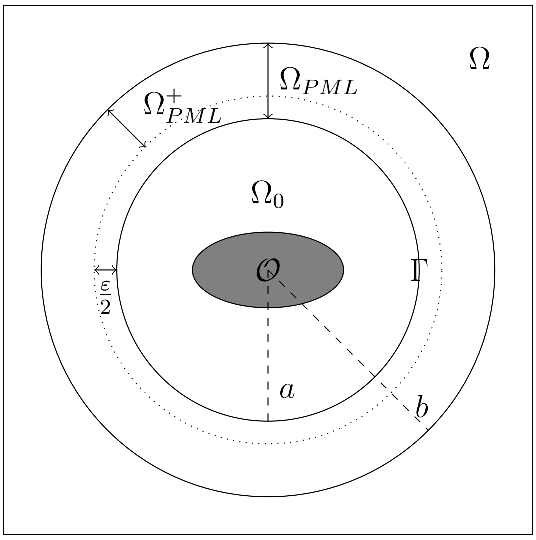


Figure 4.1: Illustration of the geometric setting.

by parts. Hence we look for $u \in H_0^1(\Omega)$ solution of

$$\int_\Omega \left\{ q\frac{\partial u}{\partial \rho}\frac{\partial}{\partial \rho}(q\bar{v}) + \frac{1}{\rho^2}\frac{\partial u}{\partial \theta}\frac{\partial \bar{v}}{\partial \theta} \right\} dx + k^2 \int_\Omega \tilde{d}^2 u\bar{v} dx = \int_\Omega \tilde{d}^2 f\bar{v} dx, \forall v \in H_0^1(\Omega). \tag{4.5}$$

By Leibniz's rule, this formulation is equivalent to

$$b_k(u,v) = -\int_\Omega \tilde{d}^2 f\bar{v} dx, \forall v \in H_0^1(\Omega), \tag{4.6}$$

where the sesquilinear form b is defined by

$$b_k(u,v) = \int_\Omega \left\{ q^2\frac{\partial u}{\partial \rho}\frac{\partial \bar{v}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial u}{\partial \theta}\frac{\partial \bar{v}}{\partial \theta} + q\frac{\partial q}{\partial \rho}\frac{\partial u}{\partial \rho}\bar{v} - k^2\tilde{d}^2 u\bar{v} \right\} dx, \forall u,v \in H_0^1(\Omega).$$

By Theorem 2 of [19], this problem has a unique solution for all real numbers k except possibly a discrete set. For this exceptional discrete set, as we are in a Fredholm setting, uniqueness of a solution is equivalent to existence and uniqueness.

4.2 The stability estimate

Let us start with the following definition.

Definition 4.1. *We will say that system* (4.6) *satisfies the k-stability property if there exists $k_0 > 0$ large enough such that for all $k \geq k_0$ and all $f \in L^2(\Omega)$ the solution $u \in H_0^1(\Omega)$ of* (4.6) *satisfies*

$$k\|u\|_\Omega + |u|_{1,\Omega} \lesssim \|f\|_\Omega, \tag{4.7}$$

for all $k \geq k_0$.

According to this definition, the k-stability property directly implies that for $k \geq k_0$, problem (4.6) is well-posed since the only solution u of problem (4.6) with $f = 0$ is zero.

Let us further remark that once we assume that the k-stability property holds, then the best constant in the right-hand side of (4.7) is equivalent to 1. More precisely, we can prove the next result.

Lemma 4.2.1. *Assume that* (4.7) *holds for all* $k \geq k_0 > 0$ *and introduce*

$$C_{\text{opt}}(k) := \sup_{f \in L^2(\Omega): f \neq 0} \frac{k\|u_f\|_\Omega + |u_f|_{1,\Omega}}{\|f\|_\Omega},$$

where $u_f \in H_0^1(\Omega)$ *is the unique solution of* (4.6)*. Then one has*

$$C_{\text{opt}}(k) \sim 1, \forall k \geq k_0. \tag{4.8}$$

Proof. The bound $C_{\text{opt}}(k) \lesssim 1$ being trivial since (4.7) is assumed, we only concentrate on the converse estimate. For that purpose, fix a non zero real valued function $\chi \in \mathcal{D}(\Omega)$ that vanishes in the PML region Ω_{PML}. Then for all $k \geq k_0$, define

$$u(\mathbf{x}) = e^{ikx_1}\chi(\mathbf{x}), \forall \mathbf{x} \in \Omega,$$

where x_1 is the first component of $\mathbf{x}$, that is considered as solution of (4.6) with $f = \Delta u + k^2 u$ (as u is zero in the PML). Then direct calculations yield

$$\|f\|_\Omega \sim \|\Delta\chi\|_\Omega + k\|\partial_1\chi\|_\Omega,$$

and

$$k\|u\|_\Omega + |u|_{1,\Omega} \sim k\|\chi\|_\Omega + |\chi|_{1,\Omega}.$$

Consequently as $\|\chi\|_\Omega > 0$, we find

$$\frac{k\|u\|_\Omega + |u|_{1,\Omega}}{\|f\|_\Omega} \gtrsim 1,$$

which proves that $C_{\text{opt}}(k) \gtrsim 1$, for all $k \geq k_0$. □

Let us also notice that any solution $u \in H_0^1(\Omega)$ of (4.6) satisfies

$$k^2 \tilde{d}^2 u + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(q^2\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} = \tilde{d}^2 f + q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho} \text{ in } \mathcal{D}'(\Omega), \tag{4.9}$$

which is equivalent to (4.3) in the distributional sense. As q tends to 1 as k goes to infinity (cf. Lemma 4.7.2), we deduce that the system

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(q^2\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}$$

is strongly elliptic (uniformly in k) for k large enough. By elliptic regularity, we deduce that, for k large enough, any solution $u \in H_0^1(\Omega)$ of (4.6) belongs to $H^2(\Omega)$ with the estimate

$$\|u\|_{2,\Omega} \lesssim \|f\|_\Omega + k^2\|u\|_\Omega. \tag{4.10}$$

Combined with (4.7), we obviously deduce that

$$\|u\|_{2,\Omega} \lesssim k\|f\|_{\Omega}, \tag{4.11}$$

for k large enough. Note finally that in such a case (4.9) holds strongly, i.e., as an equality in $L^2(\Omega)$.

The goal of this section is to prove the k-stability property. This will be made in different steps.

Lemma 4.2.2. *For k large enough, we have*

$$\int_{\Omega^+_{PML}} \left|\frac{\partial u}{\partial \rho}\right|^2 dx + \int_{\Omega_{PML}} \tilde{\sigma} k^2 |u|^2 dx \lesssim k\|f\|_{\Omega}\|u\|_{\Omega} + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}. \tag{4.12}$$

Proof. In (4.6), we take $v = u$ and the imaginary part to obtain

$$\int_{\Omega} \left\{ -\operatorname{Im} q^2 \left|\frac{\partial u}{\partial \rho}\right|^2 + k^2 \operatorname{Im} \tilde{d}^2 |u|^2 \right\} dx = \operatorname{Im} \int_{\Omega} \tilde{d}^2 f \bar{u} dx + \operatorname{Im} \int_{\Omega} q \frac{\partial q}{\partial \rho}\frac{\partial u}{\partial \rho} \bar{u} dx.$$

By Cauchy-Schwarz inequality, the fact that $q = \tilde{d} = 1$ in Ω_0 and Lemma 4.7.3, we find

$$\int_{\Omega_{PML}} \left\{ -\operatorname{Im} q^2 \left|\frac{\partial u}{\partial \rho}\right|^2 + k^2 \operatorname{Im} \tilde{d}^2 |u|^2 \right\} dx \lesssim \|f\|_{\Omega}\|u\|_{\Omega} + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}.$$

By the identities (4.75) to (4.77), the previous estimate can be equivalently written

$$\int_{\Omega_{PML}} \left\{ \frac{2\gamma k \rho \tilde{\sigma}'(\rho)}{k^2+\sigma^2(\rho)} \left|\frac{\partial u}{\partial \rho}\right|^2 + 2k\tilde{\sigma}|u|^2 \right\} dx \lesssim \|f\|_{\Omega}\|u\|_{\Omega} + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}. \tag{4.13}$$

Since $\tilde{\sigma}'$ and σ are positive in Ω_{PML}, in the left-hand side of this estimate, we can reduce the integral over the first summand to Ω^+_{PML}, namely

$$\int_{\Omega^+_{PML}} \frac{2\gamma k \rho \tilde{\sigma}'(\rho)}{k^2+\sigma^2(\rho)} \left|\frac{\partial u}{\partial \rho}\right|^2 dx + \int_{\Omega_{PML}} 2k\tilde{\sigma}|u|^2 dx \lesssim \|f\|_{\Omega}\|u\|_{\Omega} + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}.$$

By (4.71) and the fact that γ tends to 1 as k tends to infinity, we conclude that (4.12) holds for k large enough. □

Lemma 4.2.3. *For k large enough, we have*

$$\int_{\Omega} |\nabla u|^2 \, dx \lesssim k^2\|u\|^2_{\Omega} + \|f\|_{\Omega}\|u\|_{\Omega} \lesssim k^2\|u\|^2_{\Omega} + \|f\|^2_{\Omega}. \tag{4.14}$$

Proof. In (4.6), we take $v = u$ and the real part to obtain

$$\begin{aligned}\int_{\Omega} &\left\{ \operatorname{Re} q^2 \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right\} dx \\ &= k^2 \int_{\Omega} \operatorname{Re} \tilde{d}^2 |u|^2 dx - \operatorname{Re} \int_{\Omega} \tilde{d}^2 f \bar{u} dx - \operatorname{Re} \int_{\Omega} q \frac{\partial q}{\partial \rho}\frac{\partial u}{\partial \rho} \bar{u} dx.\end{aligned}$$

By Cauchy-Schwarz's inequality, the boundedness of $\tilde{d}$ and q for k large (see Lemma 4.7.2) and Lemma 4.7.3, we obtain

$$\int_\Omega \left\{ \operatorname{Re} q^2 \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right\} dx \lesssim k^2 \|u\|_\Omega^2 + \|f\|_\Omega \|u\|_\Omega + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}}.$$

As q tends to 1 as k tends to infinity (see Lemma 4.7.2), for k large enough, we get

$$\int_\Omega \left\{ \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right\} dx \lesssim k^2 \|u\|_\Omega^2 + \|f\|_\Omega \|u\|_\Omega + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}}.$$

By Young's inequality, we can absorb the last term of this right-hand side, namely

$$\int_\Omega \left\{ \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right\} dx \le Ck^2 \|u\|_\Omega^2 + C\|f\|_\Omega \|u\|_\Omega + \frac{1}{2k^2}\|\frac{\partial u}{\partial \rho}\|_\Omega^2 + C^2\|u\|_\Omega^2.$$

for some $C > 0$ independent of k. Consequently we get

$$(1 - \frac{1}{2k^2}) \int_\Omega \left\{ \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right\} dx \le Ck^2 \|u\|_\Omega^2 + C\|f\|_\Omega \|u\|_\Omega + C^2\|u\|_\Omega^2,$$

which yields (4.14) for k large enough since $|\nabla u|^2 = \left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{\rho^2}\left|\frac{\partial u}{\partial \theta}\right|^2$. $\square$

In view of this Lemma, we see that the k-stability property will be proved if we can estimate $k\|u\|_\Omega$. Since Lemma 4.2.2 gives an estimate of this quantity in Ω_{PML}^+, it remains to estimate it in $\Omega \setminus \Omega_{PML}^+$. This is made via a multiplier method originally introduced by [56, 57, 58]. For the cut-off function η fixed in the Appendix 4.7, let us introduce the multiplier

$$m(x) = x\eta(\rho), \forall x \in \Omega,$$

the functions (depending only on the radial variable ρ)

$$\alpha = \eta'(\bar{q}^2 - 2\operatorname{Re} q^2) + 2\eta\bar{q}\frac{\partial \bar{q}}{\partial \rho}, \tag{4.15}$$

$$\beta = 2\tilde{d}^2\eta + \rho\frac{\partial}{\partial \rho}\left(\tilde{d}^2\eta\right), \tag{4.16}$$

as well as the expressions

$$\Sigma = \int_\Omega (q^2 - \bar{q}^2)\eta(\rho)\frac{\partial u}{\partial \rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \bar{u}}{\partial \rho}\right) dx, \tag{4.17}$$

$$\Sigma_1 = \int_\Omega (\bar{\tilde{d}}^2 - \tilde{d}^2)\eta(\rho)\rho\frac{\partial u}{\partial \rho}\bar{u}\, dx. \tag{4.18}$$

With these notations, we can prove the following identity with multiplier:

Lemma 4.2.4. *The next identity holds*

$$\int_\Omega \left(-k^2\beta|u|^2+\rho\eta'\left|\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right|^2-\alpha\rho\left|\frac{\partial u}{\partial\rho}\right|^2\right)dx+\int_{\partial\mathcal{O}}|\nabla u\cdot n|^2 x\cdot n\,d\sigma(x) \tag{4.19}$$
$$=\Sigma-k^2\Sigma_1+2\,\mathrm{Re}\int_\Omega(\tilde{d}^2 f+q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho})\eta\rho\frac{\partial\bar{u}}{\partial\rho}\,dx.$$

Proof. For shortness, let us set $f_1=\tilde{d}^2f+q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho}$, then as already said before u satisfies (4.9) or equivalently

$$k^2\tilde{d}^2u+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(q^2\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\theta^2}=f_1.$$

Multiplying this identity by $m\cdot\nabla\bar{u}=\eta\rho\frac{\partial\bar{u}}{\partial\rho}$ and integrating in Ω (meaningful as $u\in H^2(\Omega)$), we find

$$k^2I_a+k^2I+J_a+J_{rad}+J_{ang}=\int_\Omega f_1\eta\rho\frac{\partial\bar{u}}{\partial\rho}\,dx, \tag{4.20}$$

where we have set

$$\begin{aligned}
J_{rad} &= \int_{\Omega\setminus\Omega_0}\frac{\partial}{\partial\rho}\left(q^2\rho\frac{\partial u}{\partial\rho}\right)\eta\frac{\partial\bar{u}}{\partial\rho}\,dx,\\
J_{ang} &= \int_{\Omega\setminus\Omega_0}\frac{1}{\rho}\frac{\partial^2u}{\partial\theta^2}\eta\frac{\partial\bar{u}}{\partial\rho}\,dx,\\
I &= \int_{\Omega\setminus\Omega_0}\tilde{d}^2u\eta\rho\frac{\partial\bar{u}}{\partial\rho}\,dx,\\
I_a &= \int_{\Omega_0}u(m\cdot\nabla\bar{u})\,dx,\\
J_a &= \int_{\Omega_0}\Delta u(m\cdot\nabla\bar{u})\,dx.
\end{aligned}$$

We now transform these expressions by using some integrations by parts.

a) **Transformation of** I: As η is zero outside $B(0,b)$, we have

$$I=\int_0^{2\pi}\int_a^b\tilde{d}^2u\eta\rho^2\frac{\partial\bar{u}}{\partial\rho}\,d\rho d\theta.$$

By integration by parts in ρ, we have

$$I=-\int_0^{2\pi}\int_a^b\frac{\partial}{\partial\rho}\left(\tilde{d}^2u\eta\rho^2\right)\bar{u}\,d\rho d\theta-\int_\Gamma a|u|^2d\sigma(x),$$

the boundary term being zero since $\eta(b)=0$. By Leibniz's rule, we deduce that

$$\begin{aligned}
I &= -\int_0^{2\pi}\int_a^b\frac{\partial}{\partial\rho}\left(\tilde{d}^2\eta\rho^2\right)|u|^2\,d\rho d\theta\\
&- \int_0^{2\pi}\int_a^b\tilde{d}^2\eta\rho^2\frac{\partial u}{\partial\rho}\bar{u}\,d\rho d\theta-\int_\Gamma a|u|^2d\sigma(x).
\end{aligned}$$

The second term of this right-hand side would be equal to $-\bar{I}$ if $\tilde{d}^2$ were real, hence by introducing Σ_1, we find that

$$2\,\mathrm{Re}\, I = -\int_{\Omega\setminus\Omega_0} \beta|u|^2\,dx + \Sigma_1 - \int_\Gamma a|u|^2 d\sigma(x). \tag{4.21}$$

b) **Transformation of I_a**: By the Green formula, we have

$$\begin{aligned} 2\,\mathrm{Re}\, I_a &= 2\,\mathrm{Re}\int_{\Omega_0} u(m\cdot\nabla\bar{u})\,dx \\ &= \int_{\Omega_0} m\cdot\nabla|u|^2\,dx \\ &= -\int_{\Omega_0} 2|u|^2\,dx + \int_{\partial\Omega_0} m\cdot n|u|^2\,d\sigma(x). \end{aligned}$$

Since $u=0$ on $\partial\mathcal{O}$, we have

$$2\,\mathrm{Re}\, I_a = -\int_{\Omega_0} \beta|u|^2\,dx + \int_\Gamma a|u|^2 d\sigma(x). \tag{4.22}$$

c) **Transformation of J_{ang}**: As before we have

$$J_{ang} = \int_0^{2\pi}\int_a^b \frac{\partial^2 u}{\partial\theta^2}\eta\frac{\partial\bar{u}}{\partial\rho}\,d\rho d\theta,$$

and by integration by parts in θ, we find

$$J_{ang} = -\int_0^{2\pi}\int_a^b \frac{\partial u}{\partial\theta}\eta\frac{\partial^2\bar{u}}{\partial\theta\partial\rho}\,d\rho d\theta.$$

Since

$$\frac{\partial}{\partial\rho}\left|\frac{\partial u}{\partial\theta}\right|^2 = 2\,\mathrm{Re}\left(\frac{\partial u}{\partial\theta}\frac{\partial^2\bar{u}}{\partial\theta\partial\rho}\right),$$

we then have

$$2\,\mathrm{Re}\, J_{ang} = -\int_0^{2\pi}\int_a^b \eta\frac{\partial}{\partial\rho}\left|\frac{\partial u}{\partial\theta}\right|^2\,d\rho d\theta.$$

By integration by parts in ρ, we deduce that

$$2\,\mathrm{Re}\, J_{ang} = \int_{\Omega_0} \rho\eta'\left|\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right|^2\,dx + \int_\Gamma a\left|\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right|^2\,d\sigma(x). \tag{4.23}$$

d) **Transformation of J_{rad}**: As before we have

$$J_{rad} = \int_0^{2\pi}\int_a^b \frac{\partial}{\partial\rho}\left(q^2\rho\frac{\partial u}{\partial\rho}\right)\eta\rho\frac{\partial\bar{u}}{\partial\rho}\,d\rho d\theta,$$

and an integration by parts in ρ yields

$$\begin{aligned} J_{rad} &= -\int_0^{2\pi}\int_a^b q^2\rho\frac{\partial u}{\partial\rho}\frac{\partial}{\partial\rho}\left(\eta\rho\frac{\partial\bar{u}}{\partial\rho}\right)d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x) \\ &= -\int_0^{2\pi}\int_a^b q^2\eta'\left|\rho\frac{\partial u}{\partial\rho}\right|^2 d\rho d\theta \\ &- \int_0^{2\pi}\int_a^b q^2\eta\rho\frac{\partial u}{\partial\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\bar{u}}{\partial\rho}\right)d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x). \end{aligned}$$

This can be equivalently written as

$$J_{rad} = -K - \int_0^{2\pi}\int_0^b q^2\eta'\left|\rho\frac{\partial u}{\partial\rho}\right|^2 d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x), \tag{4.24}$$

where we have set

$$K := \int_0^{2\pi}\int_0^b q^2\eta w\frac{\partial\bar{w}}{\partial\rho}\,d\rho d\theta \qquad \text{and} \qquad w := \rho\frac{\partial u}{\partial\rho}.$$

Introducing Σ, we see that

$$K = \Sigma + \int_0^{2\pi}\int_0^b \bar{q}^2\eta w\frac{\partial\bar{w}}{\partial\rho}\,d\rho d\theta,$$

hence integrating by parts in ρ in the second term of this right-hand side, we get

$$\begin{aligned} K &= \Sigma - \int_0^{2\pi}\int_0^b \frac{\partial}{\partial\rho}\left(\bar{q}^2\eta w\right)\bar{w}\,d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x) \\ &= \Sigma - \bar{K} - \int_0^{2\pi}\int_0^b \frac{\partial}{\partial\rho}\left(\bar{q}^2\eta\right)|w|^2\,d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x). \end{aligned}$$

This yields

$$2\,\mathrm{Re}\,K = \Sigma - \int_0^{2\pi}\int_0^b \frac{\partial}{\partial\rho}\left(\bar{q}^2\eta\right)|w|^2\,d\rho d\theta - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x).$$

Taking the real part of the identity (4.24), we conclude that

$$2\,\mathrm{Re}\,J_{rad} = -\Sigma - \int_\Omega \alpha\rho\left|\frac{\partial u}{\partial\rho}\right|^2 dx - \int_\Gamma a\left|\frac{\partial u}{\partial\rho}\right|^2 d\sigma(x). \tag{4.25}$$

e) **Transformation of J_a**: By integration by parts, we have

$$\begin{aligned} J_a &= \int_{\Omega_0}\Delta u(m\cdot\nabla\bar{u})\,dx \\ &= -\int_{\Omega_0}\nabla u\cdot\nabla(m\cdot\nabla\bar{u})\,dx + \int_\Gamma \nabla u\cdot n(m\cdot\nabla\bar{u})\,d\sigma(x) \\ &\quad + \int_{\partial\mathcal{O}}\nabla u\cdot n(m\cdot\nabla\bar{u})\,d\sigma(x), \end{aligned}$$

We recall that $m = x$ in Ω_0. In addition, since $u = 0$ on $\partial\mathcal{O}$, we also have $\nabla u \cdot t = 0$ on $\partial\mathcal{O}$ for the unit tangent vector t. It follows that

$$m \cdot \nabla \bar{u} = m \cdot n \nabla \bar{u} \cdot n + m \cdot t \nabla \bar{u} \cdot t = \nabla \bar{u} \cdot n x \cdot n,$$

and

$$\nabla u \cdot n (m \cdot \nabla \bar{u}) = |\nabla u \cdot n|^2 x \cdot n,$$

on $\partial\mathcal{O}$. On the other hand, Rellich's identity yields that

$$2 \operatorname{Re} \int_{\Omega_0} \nabla u \cdot \nabla (m \cdot \nabla \bar{u}) = \int_{\partial\Omega_0} |\nabla u|^2 x \cdot n = \int_{\Gamma} |\nabla u|^2 x \cdot n + \int_{\partial\mathcal{O}} |\nabla u \cdot n|^2 x \cdot n.$$

Recalling that $m = x$ in Ω_0 and $\nabla u \cdot t = 0$ on $\partial\mathcal{O}$, and using Rellich's identity, we find that

$$\begin{aligned} 2 \operatorname{Re} J_a &= \int_{\partial\mathcal{O}} |\nabla u \cdot n|^2 x \cdot n - \int_{\Gamma} |\nabla u|^2 x \cdot n + 2 \int_{\Gamma} \nabla u \cdot n (m \cdot \nabla \bar{u}) \\ &= \int_{\partial\mathcal{O}} |\nabla u \cdot n|^2 x \cdot n + \int_{\Gamma} a \left| \frac{\partial u}{\partial \rho} \right|^2 - \int_{\Gamma} a \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 . \end{aligned} \tag{4.26}$$

Coming back to (4.20), taking the real part and using (4.21), (4.22), (4.23), (4.25) and (4.26), we arrive at (4.19). □

The previous Lemmas allow to conclude the

Theorem 4.2.5. *System* (4.6) *satisfies the k-stability property.*

Proof. We first look at the behavior of β as k is large. By Leibniz's rule, we have

$$\beta = \beta_0 + \rho \tilde{d}^2 \eta',$$

with

$$\beta_0 = 2(\tilde{d}^2 + \rho \tilde{d} \frac{\partial \tilde{d}}{\partial \rho}) \eta.$$

With this splitting, (4.19) implies that

$$\begin{aligned} &\int_{\Omega} \left(k^2 \beta_0 |u|^2 - \rho \eta' \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 \right) dx \\ &\qquad \leq -\Sigma + k^2 \Sigma_1 - 2 \operatorname{Re} \int_{\Omega} f_1 \eta \rho \frac{\partial \bar{u}}{\partial \rho} \, dx \\ &\qquad\quad - \int_{\Omega} \alpha \rho \left| \frac{\partial u}{\partial \rho} \right|^2 dx - k^2 \int_{\Omega^+_{PML}} \rho \tilde{d}^2 \eta' |u|^2 \, dx. \end{aligned} \tag{4.27}$$

Since $\tilde{d}^2$ tends to 1 as k goes to infinity and $\frac{\partial \tilde{d}}{\partial \rho} = \frac{i \tilde{\sigma}'}{k}$ tends to 0 as k goes to infinity, we directly see that

$$\operatorname{Re} \beta_0 \geq \eta, \text{ for } k \text{ large enough.} \tag{4.28}$$

Using this property, the boundedness of $\tilde{d}$ and the fact that $\eta' \leq 0$ in (4.27), we find that

$$k^2 \int_\Omega \eta |u|^2 \, dx \lesssim |\Sigma| + k^2|\Sigma_1| + \|f_1\|_\Omega \|\nabla u\|_\Omega + \int_\Omega |\alpha|\rho \left|\frac{\partial u}{\partial \rho}\right|^2 dx + k^2 \int_{\Omega^+_{PML}} |u|^2 \, dx, \tag{4.29}$$

for k large enough. Now by the definition of α and Lemmas 4.7.2 and 4.7.3, we have

$$\int_\Omega |\alpha|\rho \left|\frac{\partial u}{\partial \rho}\right|^2 dx + k^2 \int_{\Omega^+_{PML}} |u|^2 \, dx \lesssim \int_{\Omega^+_{PML}} \left(\left|\frac{\partial u}{\partial \rho}\right|^2 + k^2|u|^2 \right) dx + \frac{1}{k} \int_{\Omega_{PML}} \left|\frac{\partial u}{\partial \rho}\right|^2 dx.$$

With the help of (4.12), we then obtain

$$\int_\Omega |\alpha|\rho \left|\frac{\partial u}{\partial \rho}\right|^2 dx + k^2 \int_{\Omega^+_{PML}} |u|^2 \, dx \lesssim k\|f\|_\Omega \|u\|_\Omega + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}}.$$

This estimate in (4.29) leads to

$$k^2 \int_\Omega \eta |u|^2 \, dx \lesssim |\Sigma| + k^2|\Sigma_1| + \|f_1\|_\Omega \|\nabla u\|_\Omega + k\|f\|_\Omega \|u\|_\Omega + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}}, \tag{4.30}$$

for k large enough.

It then remains to estimate $|\Sigma|$ and $k^2|\Sigma_1|$.

i) By the definition (4.18) and (4.81), it holds

$$k^2|\Sigma_1| \leq 4 \int_{\Omega_{PML}} \tilde{\sigma}^{1/2} \left|\frac{\partial u}{\partial \rho}\right| k\tilde{\sigma}^{1/2}|u| \, dx.$$

Cauchy-Schwarz's inequality and the boundedness of $\tilde{\sigma}^{1/2}$ then lead to

$$k^2|\Sigma_1| \lesssim \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|k\tilde{\sigma}^{1/2} u\|_{\Omega_{PML}}.$$

Using Young's inequality (with an arbitrary $\lambda > 0$) and (4.12) we infer

$$\begin{aligned} k^2|\Sigma_1| &\lesssim \lambda \|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}} + \frac{1}{\lambda} \int_{\Omega_{PML}} \tilde{\sigma} k^2 |u|^2 dx \\ &\lesssim \lambda \|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}} + \frac{1}{\lambda} \left(k\|f\|_\Omega \|u\|_\Omega + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} \right). \end{aligned} \tag{4.31}$$

For the second term of this right-hand side using Young's inequality we find

$$\begin{aligned}
|\Sigma| &\lesssim \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}} \\
&+ \frac{1}{\delta k\sqrt{k}}\int_{\Omega_{PML}} \tilde{\sigma}' \left|\frac{\partial u}{\partial \rho}\right|^2 dx \\
&+ \frac{\delta}{\sqrt{k}}\int_{\Omega_{PML}} \left|\frac{\partial^2 u}{\partial \rho^2}\right|^2 dx,
\end{aligned}$$

for all $\delta > 0$. Using (4.13), the fact that γ tends to 1 as k goes to infinity and the property $\frac{k}{k^2+\sigma^2} \geq \frac{1}{k}$ valid for k large enough, we find

$$\begin{aligned}
|\Sigma| &\lesssim \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}} \\
&+ \frac{1}{\delta\sqrt{k}}(\|f\|_\Omega\|u\|_\Omega + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}) \\
&+ \frac{\delta}{\sqrt{k}}\int_{\Omega_{PML}} \left|\frac{\partial^2 u}{\partial \rho^2}\right|^2 dx,
\end{aligned}$$

for all $\delta > 0$ and for k large enough. For the last term of this right-hand side, using the estimate (4.10), we arrive at

$$\begin{aligned}
|\Sigma| &\lesssim \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|^2_{\Omega_{PML}} \\
&+ \frac{1}{\delta\sqrt{k}}(\|f\|_\Omega\|u\|_\Omega + \frac{1}{k}\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}}) \\
&+ \frac{\delta}{\sqrt{k}}(\|f\|^2_\Omega + k^4\|u\|^2_\Omega).
\end{aligned}$$

This estimate and (4.31) in (4.30)

$$\begin{aligned}
k^2\int_\Omega \eta|u|^2\,dx &\lesssim \|f_1\|_\Omega\|\nabla u\|_\Omega \\
&+ k\|f\|_\Omega\|u\|_\Omega + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}} + (\lambda + \frac{1}{k})\|\nabla u\|^2_{\Omega_{PML}} \\
&+ (\frac{k}{\lambda} + \frac{1}{\delta\sqrt{k}})\|f\|_\Omega\|u\|_\Omega + (\frac{1}{\lambda} + \frac{1}{\delta k^{\frac{3}{2}}})\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}}\|u\|_{\Omega_{PML}} \\
&+ \frac{\delta}{\sqrt{k}}\|f\|^2_\Omega + \delta k^{\frac{7}{2}}\|u\|^2_\Omega,
\end{aligned}$$

for k large enough.

Comparing this estimate with (4.12) and recalling that $\eta = 1$ in $\Omega \setminus \Omega^+_{PML}$ and

(4.71) for large k, we have shown that

$$\begin{aligned} k^2 \int_\Omega |u|^2\,dx &\leq C\Big(\|f_1\|_\Omega \|\nabla u\|_\Omega \\ &+ k\|f\|_\Omega \|u\|_\Omega + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} + (\lambda + \frac{1}{k})\|\nabla u\|^2_{\Omega_{PML}} \\ &+ (\frac{k}{\lambda} + \frac{1}{\delta\sqrt{k}})\|f\|_\Omega \|u\|_\Omega + (\frac{1}{\lambda} + \frac{1}{\delta k^{\frac{3}{2}}})\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} \\ &+ \frac{\delta}{\sqrt{k}}\|f\|^2_\Omega + \delta k^{\frac{7}{2}}\|u\|^2_\Omega\Big), \end{aligned}$$

for k large enough and some positive constant C independent of k. We now chose $\delta > 0$ so that

$$C\delta k^{\frac{7}{2}} = \frac{k^2}{2},$$

or equivalently

$$\delta = \frac{k^{-\frac{3}{2}}}{2C}.$$

With this choice we find

$$\begin{aligned} k^2 \int_\Omega |u|^2\,dx &\lesssim \|f_1\|_\Omega \|\nabla u\|_\Omega \\ &+ k(1 + \frac{1}{\lambda})\|f\|_\Omega \|u\|_\Omega + (1 + \frac{1}{\lambda})\|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} \\ &+ (\lambda + \frac{1}{k})\|\nabla u\|^2_{\Omega_{PML}} + \|\frac{\partial u}{\partial \rho}\|_{\Omega_{PML}} \|u\|_{\Omega_{PML}} + \frac{1}{k^2}\|f\|^2_\Omega, \end{aligned}$$

for k large enough. Recalling the definition of f_1 and Lemma 4.7.3, we get

$$\|f_1\|_\Omega \lesssim \|f\|_\Omega + \frac{1}{k}\|\nabla u\|_\Omega,$$

and consequently

$$\begin{aligned} k^2 \int_\Omega |u|^2\,dx &\lesssim \|f\|_\Omega \|\nabla u\|_\Omega + (\lambda + \frac{1}{k})\|\nabla u\|^2_\Omega \\ &+ k(1 + \frac{1}{\lambda})\|f\|_\Omega \|u\|_\Omega + (1 + \frac{1}{\lambda})\|\nabla u\|_\Omega \|u\|_\Omega + \|f\|^2_\Omega. \end{aligned}$$

for k large enough. By Young's inequality, this estimate implies that

$$\begin{aligned} k^2 \int_\Omega |u|^2\,dx &\leq \frac{C}{\mu}\|f\|^2_\Omega + \mu\|\nabla u\|^2_\Omega + C(\lambda + \frac{1}{k})\|\nabla u\|^2_\Omega \\ &+ \frac{C}{\mu_1}(1 + \frac{1}{\lambda})\|f\|^2_\Omega + \mu_1(1 + \frac{1}{\lambda})k^2\|u\|^2_\Omega \\ &+ \frac{C}{\mu_2 k^2}(1 + \frac{1}{\lambda})\|\nabla u\|^2_\Omega + \mu_2(1 + \frac{1}{\lambda})k^2\|u\|^2_\Omega + C\|f\|^2_\Omega, \end{aligned}$$

for k large enough, for any positive real numbers μ, μ_1 and μ_2 and a positive constant C independent of k (and μ, μ_1 and μ_2). Choosing $\mu_1 = \mu_2 = (4(1 + 1/\lambda))^{-1}$, we find that

$$k^2 \int_\Omega |u|^2 \, dx \leq C(1 + \frac{1}{\mu}(1 + \frac{1}{\lambda})^2)\|f\|^2_\Omega + \Big(\mu + C\lambda + \frac{C}{k}(1 + \frac{1}{\lambda})^2\Big)\|\nabla u\|^2_\Omega,$$

for k large enough, for any positive real numbers μ, λ and a positive constant C independent of k, μ, λ. At this stage we take advantage of (4.14) to obtain

$$k^2 \int_\Omega |u|^2 \, dx \leq C\Big((1 + \frac{1}{\mu}(1 + \frac{1}{\lambda})^2 + \mu + C\lambda + \frac{C}{k}(1 + \frac{1}{\lambda})^2\Big)\|f\|^2_\Omega$$
$$+ (\mu + C\lambda)k^2\|u\|^2_\Omega + kC(1 + \frac{1}{\lambda})^2\|u\|^2_\Omega.$$

Choosing $\mu = \frac{1}{4}$ and $\lambda = \frac{1}{4C}$, we find that

$$k^2 \int_\Omega |u|^2 \, dx \leq C\|f\|^2_\Omega + Ck\|u\|^2_\Omega,$$

for k large enough and a positive constant C independent of k. As for k large enough $Ck \leq \frac{k^2}{2}$, we have proved that

$$k\|u\|_\Omega \lesssim \|f\|_\Omega,$$

for k large enough. Coming back to (4.14), we conclude that

$$\int_\Omega |\nabla u|^2 \; dx \lesssim \|f\|^2_\Omega,$$

for k large enough. □

4.3 Comparison with a sponge layer

The boundary value problem corresponding to a sponge layer consists in looking at u_{sponge} solution of

$$L_{\text{sponge}} u_{\text{sponge}} = f \text{ in } \Omega, \tag{4.32}$$
$$u_{\text{sponge}} = 0 \text{ on } \partial\Omega, \tag{4.33}$$

where the operator L_{sponge} is defined by

$$L_{\text{sponge}} v = \Delta v + (k^2 + 2i\tilde{\sigma}k)v = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial v}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2} + (k^2 + 2i\tilde{\sigma}k)v.$$

This problem (4.32) enters in the framework developed recently in [14] if the boundary of Ω is $C^{1,1}$ or if it is a convex polygon, since it satisfies the assumption of Section 2 of that [14] (with the choices $\mathcal{L}_0 = -Id, \mathcal{L}_1 = -2\tilde{\sigma}Id,$ and $\mathcal{L}_2 = -\Delta$), and since its variational formulation is given by

$$a_{\text{sponge}}(u_{\text{sponge}}, v) = -\int_\Omega f\bar{v} \, dx, \forall v \in H^1_0(\Omega), \tag{4.34}$$

where the sesquilinear form $a_{\text{sponge}}(\cdot,\cdot)$ is defined by

$$a_{\text{sponge}}(v,w) = \int_\Omega \left(\nabla v \cdot \nabla \bar{w} - (k^2 + 2i\tilde{\sigma}k)v\bar{w}\right)\, dx, \forall v, w \in H_0^1(\Omega).$$

This sesquilinear form trivially satisfies

$$|a_{\text{sponge}}(v,w)| \lesssim |||v|||\,|||w|||, \forall v, w \in H_0^1(\Omega),$$

where

$$|||v||| = \left(k^2\|v\|_\Omega^2 + |v|_{1,\Omega}^2\right)^{\frac{1}{2}},$$

and

$$\operatorname{Re} a_{\text{sponge}}(v,v) \geq |v|_{1,\Omega}^2 - k^2\|v\|_\Omega^2, \forall v \in H_0^1(\Omega).$$

Consequently the associated operator A_{sponge} is a Fredholm operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, therefore it is an isomorphism if and only if it is injective. But the injectivity is not difficult to show because $u \in H_0^1(\Omega)$ solution of (4.34) with $f = 0$ satisfies in particular

$$a_{\text{sponge}}(u,u) = 0,$$

and taking the imaginary part we get

$$u = 0 \text{ on } \Omega_{PML}.$$

Since u also satisfies

$$\Delta u + (k^2 + 2i\tilde{\sigma}k)u = 0 \text{ in } \Omega,$$

by Holmgrem's theorem we deduce that $u = 0$.

In order to compare (4.32) with (4.9), we rewrite (4.9) as

$$L_{\text{PML}}u = \tilde{d}^2 f,$$

with

$$L_{\text{PML}}v = k^2\tilde{d}^2 v + \frac{q^2}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial v}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\theta^2} + q\frac{\partial q}{\partial\rho}\frac{\partial v}{\partial\rho},$$

We can look at u as solution of

$$L_{\text{sponge}}u = f^{(k)} \text{ in } \Omega, \tag{4.35}$$

$$u = 0 \text{ on } \partial\Omega, \tag{4.36}$$

where $f^{(k)} = L_{\text{sponge}}u - L_{\text{PML}}u + \tilde{d}^2 f$ and consequently

$$f^{(k)} = \tilde{d}^2 f + (1-q^2)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \left((k^2 + 2i\tilde{\sigma}k) - k^2\tilde{d}^2\right)u - q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho}. \tag{4.37}$$

Let us now estimate the L^2-norm of $f^{(k)}$.

Lemma 4.3.1. *For k large enough, it holds*

$$\|f^{(k)}\|_\Omega \lesssim \|f\|_\Omega. \tag{4.38}$$

Proof. As $\tilde{d}$ is uniformly bounded in Ω, it suffices to estimate the L^2-norm of the three other terms of the right-hand side of (4.37). For the second term of (4.37), by (4.84), we have

$$\|(1-q^2)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)\|_\Omega \lesssim \frac{1}{k}\|\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)\|_{\Omega_{PML}} \lesssim \frac{1}{k}\|u\|_{2,\Omega_{PML}}.$$

By (4.11), we conclude that

$$\|(1-q^2)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)\|_\Omega \lesssim \|f\|_\Omega, \tag{4.39}$$

for k large enough.

The definition of $\tilde{d}$ shows

$$(k^2+2i\tilde{\sigma}k)-k^2\tilde{d}^2=\tilde{\sigma}^2.$$

This identity and the bound (4.7) show that the the third term of (4.37) satisfies

$$\|\left((k^2+2i\tilde{\sigma}k)-k^2\tilde{d}^2\right)u\|_\Omega \lesssim \|u\|_\Omega \lesssim \|f\|_\Omega, \tag{4.40}$$

for k large enough.

For the last term of (4.37), using Lemma 4.7.3 and again (4.7), we directly conclude that

$$\|q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho}\|_\Omega \lesssim \frac{1}{k}|u|_{1,\Omega} \lesssim \|f\|_\Omega,$$

for k large enough. This estimate, (4.39), and (4.40) lead to the asserted estimate. □

At this stage, we can look at $u \in H_0^1(\Omega)$ as the unique solution of (4.32) with a datum $f^{(k)}$ instead of f. The L^2 norm of $f^{(k)}$ is uniformly bounded in k. Consequently applying Theorem 1 of [14], we directly get the next result.

Theorem 4.3.2. *Let γ be a natural number and assume that the boundary of Ω is of class $C^{\gamma+1,1}$. Then for k large enough, for all $\ell \in \{0,\cdots,\gamma\}$, the unique solution $u \in H_0^1(\Omega)$ of (4.6) admits the splitting*

$$u=\sum_{j=0}^{\ell-1}k^j u_j^{(k)}+r_\ell^{(k)}, \tag{4.41}$$

where $u_j^{(k)} \in H^{j+2}(\Omega)$ with

$$\|u_j^{(k)}\|_{j+2,\Omega} \lesssim \|f\|_\Omega, \tag{4.42}$$

for $0 \le j \le \ell-1$ and $r_\ell^{(k)} \in H^{\ell+2}(\Omega)$ with

$$\|r_\ell^{(k)}\|_{\ell+2,\Omega} \lesssim k^{\ell+1}\|f\|_\Omega. \tag{4.43}$$

Remark 4.3.3. *This result remains valid for a convex polygon with $\gamma = 0$.*

4.4 Finite element discretizations

4.4.1 hp-FEM

Here we want to take advantage of the splitting from Theorem 4.3.2 to derive stability conditions and error estimates for hp finite element discretizations of (4.6).

We look for a finite element approximation $u_{h,p}$ to u. To this end, we consider a family of regular (in Ciarlet's sense) meshes $\{\mathcal{T}_h\}_h$ of Ω, where each mesh is made of triangular elements K. To simplify the analysis, we assume that the boundary of Ω is exactly triangulated, and therefore, we consider curved Lagrange finite elements [6]. Also, for each element K, we denote by $\mathcal{F}_K$ the mapping taking the reference element $\hat{K}$ to K.

Then, for all $p \leq \gamma + 1$, the finite element approximation space $V_{h,p}$ is defined as

$$V_{h,p} = \left\{ v_{h,p} \in H_0^1(\Omega) \mid v_{h,p}|_K \circ \mathcal{F}_K^{-1} \in \mathbb{P}_p(\hat{K}) \; \forall K \in \mathcal{T}_h \right\},$$

where $\mathbb{P}_p(\hat{K})$ stands for the set of polynomials of total degree less than or equal to p.

As the family of meshes is regular, for each $v \in H^{l+1}(\Omega) \cap H_0^1(\Omega)^S$ $(0 \leq l \leq p)$, there exists an element $\mathcal{I}_{h,p} v \in V_{h,p}$ such that

$$|v - \mathcal{I}_{h,p} v|_{j,\Omega} \lesssim h^{l+1-j} \|v\|_{l+1,\Omega}, \quad (0 \leq j \leq l). \tag{4.44}$$

We refer the reader to [6, Corollary 5.2] (see also [17]).

Then a finite element approximation of u is obtained by looking for $u_{h,p} \in V_{h,p}$ such that

$$b_k(u_{h,p}, v_{h,p}) = -\int_\Omega \tilde{d}^2 f \bar{v}_{h,p} dx, \quad \forall v_{h,p} \in V_{h,p}. \tag{4.45}$$

Asymptotic error estimate

Now we are ready to prove a convergence result in an appropriate asymptotic range.

Theorem 4.4.1. *Assume that the boundary of Ω is of class $C^{\gamma+1,1}$ for some natural number γ (or a convex polygon) and let $f \in L^2(\Omega)$. Then there exists k_0 large enough and $\delta > 0$ small enough such that if $k \geq k_0$, $kh \leq \delta$ and $k^{p+1}h^p \leq \delta$ with $p \leq \gamma + 1$ ($p = 1$ if Ω is a convex polygon), there exists a unique finite element solution $u_{h,p} \in V_{h,p}$ to (4.45), and the estimate*

$$|||u - u_{h,p}||| \lesssim \inf_{\phi_{h,p} \in V_{h,p}} |||u - \phi_{h,p}||| \tag{4.46}$$

holds. Furthermore, we have

$$|||u - u_{h,p}||| \lesssim kh\|f\|_\Omega. \tag{4.47}$$

Proof. The proof of this Theorem is exactly the same as the one of Theorem 2 from [14], by using Theorem 4.3.2 and the fact that the sesquilinear form b_k satisfies Assumption 1 from [14]. Indeed the continuity property

$$|b_k(v,w)| \lesssim |||v|||\,|||w|||, \forall v, w \in H_0^1(\Omega),$$

is a direct consequence of Cauchy-Schwarz's inequality. Let us now prove the Gårding inequality

$$\operatorname{Re} b_k(u,u)| \gtrsim |u|_{1,\Omega}^2 - k^2\|u\|_\Omega^2, \quad \forall u \in H_0^1(\Omega). \tag{4.48}$$

Fix an arbitrary $u \in H_0^1(\Omega)$. First by the properties (4.78), (4.80) and (4.82), for k large enough we have

$$\operatorname{Re} b_k(u,u)| \geq \frac{1}{2}|u|_{1,\Omega}^2 - 2k^2\|u\|_\Omega^2 - \frac{C}{k}\int_{\Omega_{PML}} |\nabla u||u|\,dx,$$

for some $C > 0$ independent of k. Cauchy-Schwarz's inequality and Young's inequality then lead to

$$\operatorname{Re} b_k(u,u) \geq \frac{1}{4}|u|_{1,\Omega}^2 - (2k^2 + \frac{C^2}{k^2})\|u\|_\Omega^2.$$

This proves (4.48). □

Pre-asymptotic error estimate

In this part, we aim at giving a pre-asymptotic error estimate for the problem (4.5). As in [28], we use an appropriate elliptic projection, in order to obtain the existence of a solution $u_{h,p}$ to (4.45) under a weaker condition than in the asymptotic range.

First, we define:

$$\begin{aligned} L_q(u) &:= q^2\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{q^2}{\rho}\frac{\partial u}{\partial\rho} + q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho} \\ &= \frac{q}{\rho}\frac{\partial}{\partial\rho}\left(q\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} \\ &= \Delta u + (1-q^2)\frac{\partial^2 u}{\partial\rho^2} + (1-q^2)\frac{1}{\rho}\frac{\partial u}{\partial\rho} + q\frac{\partial q}{\partial\rho}\frac{\partial u}{\partial\rho}. \end{aligned}$$

Then, we look at the following problem: find $u \in H_0^1(\Omega) \cap H^2(\Omega)$ solution of

$$\begin{cases} L_q(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The variational form of this problem is: Find $u \in H_0^1(\Omega)$ such that

$$a_k(u,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \tag{4.49}$$

with $f \in L^2(\Omega)$ and

$$\begin{aligned} a_k(u,v) &:= \int_\Omega \left(q \frac{\partial u}{\partial \rho} \frac{\partial (q\overline{v})}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial u}{\partial \theta} \frac{\partial \overline{v}}{\partial \theta} \right) dx \\ &= \int_\Omega \left(q^2 \frac{\partial u}{\partial \rho} \frac{\partial \overline{v}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial u}{\partial \theta} \frac{\partial \overline{v}}{\partial \theta} + q \frac{\partial q}{\partial \rho} \frac{\partial u}{\partial \rho} \overline{v} \right) dx. \end{aligned}$$

Lemma 4.4.2. *There exist a unique solution $u \in H_0^1(\Omega)$ to problem (4.49), further we have $u \in H^2(\Omega)$ with*

$$\|u\|_{2,\Omega} \lesssim \|f\|_\Omega . \tag{4.50}$$

Proof. We first prove that a_k is continuous and coercive. Indeed one trivially has

$$|a_k(u,v)| \lesssim (\|q^2\|_\infty + 1) \|u\|_{1,\Omega} \|v\|_{1,\Omega} + \left\| q \frac{\partial q}{\partial \rho} \right\|_\infty \|\nabla u\|_\Omega \|v\|_\Omega , \forall u, v \in H_0^1(\Omega).$$

Hence, with Lemma 4.7.2, 4.7.3, we have the existence of a constant independent from k such that

$$|a_k(u,v)| \lesssim \|u\|_{1,\Omega} \|v\|_{1,\Omega} , \forall u, v \in H_0^1(\Omega).$$

On the other hand, if k is large enough, we have

$$\begin{aligned} \operatorname{Re} a_k(u,u) &\geq \min(\operatorname{Re} q, 1) \|\nabla u\|_\Omega^2 - \left\| q \frac{\partial q}{\partial \rho} \right\|_\infty \|\nabla u\|_\Omega \|v\|_\Omega \\ &\geq C_1 \|\nabla u\|_\Omega^2 - \frac{C_2}{k} \|u\|_{1,\Omega}^2 \\ &\geq \left(C_1 - \frac{C_2}{k} \right) \|u\|_{1,\Omega}^2 \\ &\gtrsim \|u\|_{1,\Omega}^2 . \end{aligned}$$

Then, since a_k is continuous and coercive, by Lax-Milgram Lemma, we have the existence and uniqueness of a solution $u \in H_0^1(\Omega)$ to (4.49). The strong ellipticity of L_q gives us the $H^2(\Omega)$ regularity of u. So, $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and we have

$$\begin{aligned} \|u\|_{2,\Omega} &\lesssim \|\Delta u\|_\Omega \\ &\lesssim \|L_q(u)\|_\Omega + \|1-q^2\|_\infty \|u\|_{2,\Omega} + \|1-q^2\|_\infty \|u\|_{1,\Omega} + \left\| q \frac{\partial q}{\partial \rho} \right\|_\infty \|u\|_{1,\Omega} \\ &\lesssim \|f\|_\Omega + \frac{1}{k} \|u\|_{2,\Omega} + \|u\|_{1,\Omega} , \end{aligned}$$

hence for k large enough, we obtain (4.50). □

Lemma 4.4.3. *We define the projections $\mathcal{P}_{h,p} u \in V_{h,p}$ and $\mathcal{P}^*_{h,p} u \in V_{h,p}$ as unique solutions to*

$$\begin{aligned} a_k(\mathcal{P}_{h,p} u, v_{h,p}) &= a_k(u, v_{h,p}) \qquad \forall v_{h,p} \in V_{h,p}, \\ a_k(v_h, \mathcal{P}^*_{h,p} u) &= a_k(v_h, u) \qquad \forall v_{h,p} \in V_{h,p}. \end{aligned}$$

If $u_\phi \in H_0^1(\Omega)$ solves $b_k(u_\phi, v) = (\phi, v)$ for all $v \in H_0^1(\Omega)$ for some $\phi \in L^2(\Omega)$, then we have

$$\left\|u_\phi - \mathcal{P}^*_{h,p} u_\phi\right\|_\Omega \lesssim (h^2 + k^p h^{p+1}) \left\|\phi\right\|_\Omega$$

and

$$\left\|u_\phi - \mathcal{P}^*_{h,p} u_\phi\right\|_{1,\Omega} \lesssim (h + (kh)^p) \left\|\phi\right\|_\Omega .$$

In addition, we have

$$\left\|u_\phi - \mathcal{I}_{h,p} u_\phi\right\|_{1,\Omega} \lesssim (h + (kh)^p) \left\|\phi\right\|_\Omega . \tag{4.51}$$

Proof. The existence and uniqueness of $\mathcal{P}_{h,p} u$ and of $\mathcal{P}^*_{h,p} u$ comes from the coercivity and continuity of a_k. We recall that, by Theorem 4.3.2 (with $\ell = p-1$), we have

$$u_\phi = \sum_{j=0}^{p-2} k^j u_\phi^{(j)} + r_\phi$$

with

$$\left\|u_\phi^{(j)}\right\|_{j+2,\Omega} \lesssim \left\|\phi\right\|_\Omega \tag{4.52}$$

$$\left\|r_\phi\right\|_{p+1,\Omega} \lesssim k^p \left\|\phi\right\|_\Omega . \tag{4.53}$$

By Céa's lemma, we have

$$\left\|u_\phi - \mathcal{P}^*_{h,p} u_\phi\right\|_{1,\Omega} \lesssim \inf_{v_{h,p} \in V_{h,p}} \left\|u_\phi - v_{h,p}\right\|_{1,\Omega} \lesssim \left\|u_\phi - \mathcal{I}_{h,p} u_\phi\right\|_{1,\Omega} .$$

To estimate this right-hand side, we use (4.52) and (4.53) and (4.44) to obtain

$$\begin{aligned}
\left\|u_\phi - \mathcal{I}_{h,p} u_\phi\right\|_{1,\Omega} &\lesssim \sum_{j=0}^{p-2} k^j \left\|u_\phi^{(j)} - \mathcal{I}_{h,p} u_\phi^{(j)}\right\|_{1,\Omega} + \left\|r_\phi - \mathcal{I}_{h,p} r_\phi\right\|_{1,\Omega} \\
&\lesssim \sum_{j=0}^{p-2} k^j h^{j+1} \left\|u_\phi^{(j)}\right\|_{j+2,\Omega} + h^p \left\|r_\phi\right\|_{p+1,\Omega} \\
&\lesssim h \sum_{j=0}^{p-2} k^j h^j \left\|u_\phi^{(j)}\right\|_{j+2,\Omega} + (kh)^p \left\|\phi\right\|_\Omega .
\end{aligned}$$

This proves (4.51), and hence

$$\left\|u_\phi - \mathcal{P}^*_{h,p} u_\phi\right\|_{1,\Omega} \lesssim (h + (kh)^p) \left\|\phi\right\|_\Omega ,$$

Similarly we can show that

$$\left\|u_\phi - \mathcal{I}_{h,p} u_\phi\right\|_\Omega \lesssim h(h + (kh)^p) \left\|\phi\right\|_\Omega . \tag{4.54}$$

This estimate cannot be used to bound the L^2-norm of $u_\phi - \mathcal{P}^*_{h,p} u_\phi$, hence we use an Aubin-Nitsche trick. For this, we introduce $\xi \in H_0^1(\Omega)$ solution to $a_k(\xi, v) =$

$(u_\phi - \mathcal{P}^*_{h,p} u_\phi, v)$, for all $v \in H^1_0(\Omega)$. The existence and uniqueness of ξ follow from the properties of a_k and we have

$$\begin{aligned}
\left\| u_\phi - \mathcal{P}^*_{h,p} u_\phi \right\|^2_\Omega &= a_k(\xi, u_\phi - \mathcal{P}^*_{h,p} u_\phi) \\
&= a_k(\xi - \mathcal{P}_{h,p}\xi, u_\phi - \mathcal{P}^*_{h,p} u_\phi) \\
&\leq \left\| u_\phi - \mathcal{P}^*_{h,p} u_\phi \right\|_{1,\Omega} \left\| \xi - \mathcal{P}_{h,p}\xi \right\|_{1,\Omega} \\
&\lesssim (h + (kh)^p) \left\| \phi \right\|_\Omega h \left\| u_\phi - \mathcal{P}^*_{h,p} u_\phi \right\|_\Omega \\
&\lesssim (h^2 + k^p h^{p+1}) \left\| \phi \right\|_\Omega \left\| u_\phi - \mathcal{P}^*_{h,p} u_\phi \right\|_\Omega .
\end{aligned}$$

□

Now, since we have introduced the elliptic projection and its approximation properties in Lemma 4.4.3, we can follow [28] to produce a pre-asymptotic error estimate.

Theorem 4.4.4. *Assume that $k^{p+2}h^{p+1}$ is small enough, then there exists a unique solution $u_{h,p} \in V_{h,p}$ of problem (4.45) and it holds*

$$|||u - u_{h,p}||| \lesssim (kh + k^{2p+1}h^{2p}) \left\| f \right\|_\Omega . \tag{4.55}$$

Proof. We use Aubin-Nitsche's trick, that is why we introduce $\xi \in H^1_0(\Omega)$, which verifies $b_k(v, \xi) = (v, u - u_h)$, for all $v \in H^1_0(\Omega)$. Hence we have, by the above lemma,

$$\begin{aligned}
\left\| u - u_{h,p} \right\|^2_\Omega &= b_k(u - u_{h,p}, \xi) = b_k(u - u_{h,p}, \xi - \mathcal{P}^*_{h,p}\xi) \\
&= -k^2(\tilde{d}^2(u - u_{h,p}), \xi - \mathcal{P}^*_{h,p}\xi) + a_k(u - u_{h,p}, \xi - \mathcal{P}^*_{h,p}\xi) \\
&= -k^2(\tilde{d}^2(u - u_{h,p}), \xi - \mathcal{P}^*_{h,p}\xi) + a_k(u - \mathcal{I}_{h,p}u, \xi - \mathcal{P}^*_{h,p}\xi) \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|_\Omega \left\| \xi - \mathcal{P}^*_{h,p}\xi \right\|_\Omega + \left\| u - \mathcal{I}_{h,p}u \right\|_{1,\Omega} \left\| \xi - \mathcal{P}^*_{h,p}\xi \right\|_{1,\Omega} \\
&\lesssim ((kh)^2 + k^{p+2}h^{p+1}) \left\| u - u_{h,p} \right\|^2_\Omega + (h^2 + (kh)^{2p}) \left\| f \right\|_\Omega \left\| u - u_{h,p} \right\|_\Omega .
\end{aligned}$$

Then, if $k^{p+2}h^{p+1}$ and kh are small enough,

$$\left\| u - u_{h,p} \right\|_\Omega \lesssim (h^2 + (kh)^{2p}) \left\| f \right\|_\Omega . \tag{4.56}$$

This allows to estimate the energy norm of $u - u_{h,p}$ as follows:

$$\begin{aligned}
|||u - u_{h,p}|||^2 &\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |u - u_{h,p}|^2_{1,\Omega} \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |a_k(u - u_{h,p}, u - u_{h,p})| \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |a_k(u - u_{h,p}, u - u_{h,p}) - k^2(\tilde{d}^2(u - u_{h,p}), u - u_{h,p})| \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |b_k(u - u_{h,p}, u - u_{h,p})| \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |b_k(u - u_{h,p}, u - \mathcal{I}_{h,p}u)| \\
&\lesssim k^2 \left\| u - u_{h,p} \right\|^2_\Omega + |||u - u_{h,p}||| \cdot |||u - \mathcal{I}_{h,p}u|||.
\end{aligned}$$

Young's inequality gives us

$$|||u - u_{h,p}||| \lesssim k \left\| u - u_{h,p} \right\|_\Omega + |||u - \mathcal{I}_{h,p}u|||.$$

By (4.56), (4.51) and (4.54), we deduce that

$$|||u - u_{h,p}||| \lesssim (k(h^2 + (kh)^{2p}) + h + (kh)^p) \, \|f\|_\Omega \, ,$$

which proves (4.55) as $kh^2 \lesssim kh$ and $h + (kh)^p \lesssim kh$.

□

4.4.2 A multiscale approach

An alternative to high-order polynomials for achieving stability is the computation of subscale corrections in a multiscale fashion. The approach was first used for numerical homogenization problems [46] and later applied to Helmholtz problems by [63]. A Petrov–Galerkin variant of this approach is studied in [30], while [9] discusses the case of variable coefficients, which is closely related to the present case of a PML. In order to state the PML setting in the framework of [9], it is convenient to reformulate the original boundary-value problem (4.1) in Cartesian coordinates as follows

$$-\nabla \cdot A \nabla u - k^2 d\tilde{d} u = -d\tilde{d} f \quad \text{in } \Omega \qquad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

The resulting coefficient matrix A has been provided by [19] and reads

$$A(\rho, \theta) = \begin{pmatrix} q\cos^2\theta + q^{-1}\sin^2\theta & (q - q^{-1})\cos\theta\sin\theta \\ (q - q^{-1})\cos\theta\sin\theta & q\sin^2\theta + q^{-1}\cos^2\theta \end{pmatrix}$$

where it is understood that $q = q(\rho)$. This problem is equivalent to (4.1) (and thereby to (4.3)) in the sense that they have the same unique solution u. The reason why the multiscale method is stated for this version of the equation it has the structure of a standard Helmholtz equation with a nontrivial diffusion coefficient. For this case, stability and error estimates have been formulated in [9, 30, 63], and they immediately apply to the present situation. As the equations are equivalent on the PDE level, the stability results from Section 4.2 remain valid. The corresponding alternative variational formulation (equivalent to (4.1) or (4.3)) reads: find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}_k(u, v) = (\tilde{f}, v)_{L^2(\Omega)} \tag{4.57}$$

where $\tilde{f} := -d\tilde{d} f$ and the sesquilinear form $\mathcal{A}_k$ is defined by

$$\mathcal{A}_k(v, w) := (A\nabla v, \nabla w)_{L^2(\Omega)} - k^2(d\tilde{d}u, v)_{L^2(\Omega)} \quad \text{for any } v, w \in H_0^1(\Omega).$$

With help of the results from Section 4.2 it can be shown that $\mathcal{A}_k$ satisfies the following inf-sup condition.

Lemma 4.4.5. *The sesquilinear form $\mathcal{A}_k$ satisfies*

$$\gamma(k) \lesssim \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\operatorname{Re} \mathcal{A}_k(v, w)}{|||v||| \, |||w|||} \tag{4.58}$$

where $\gamma(k) > 0$ satisfies $\gamma(k)^{-1} \sim k$.

Proof. Let $v \in H_0^1(\Omega)$ be given. We follow the approach of [47] and denote by $z \in H_0^1(\Omega)$ the solution to the following dual problem

$$\mathcal{A}_k(\eta, z) = 2k^2(\eta, v)_{L^2(\Omega)} \quad \text{for all } \eta \in H_0^1(\Omega).$$

The form $\mathcal{A}_k$ is symmetric (but not self-adjoint) and so the stability bound from Theorem 4.2.5 applies to z and reads

$$|||z||| \lesssim \|k^2 v\|_{L^2(\Omega)} \lesssim k|||v|||.$$

After setting $w := v + z$ one concludes

$$\mathcal{A}_k(v, w) = \mathcal{A}_k(v, v) + \mathcal{A}_k(v, z) = \mathcal{A}_k(v, v) + 2k^2\|v\|_{L^2(\Omega)}^2 = |||v|||^2$$

as well as

$$|||w||| \leq |||v||| + |||z||| \lesssim (1 + k)|||v||| \lesssim k|||v|||$$

for k large enough. The combination of these estimates yields

$$\operatorname{Re} \mathcal{A}_k(v, w) = |||v|||^2 \gtrsim k^{-1}|||v|||\ |||w|||$$

which implies the claimed stability condition with $\gamma(k)^{-1} \lesssim k$.

Conversely, if we assume that (4.58) holds, then we have

$$\gamma(k)|||u||| \lesssim \sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\operatorname{Re} \mathcal{A}_k(u, w)}{|||w|||}$$

for the solution $u \in H_0^1(\Omega)$ of (4.57) with $f \in L^2(\Omega)$. Consequently by Cauchy-Schwarz's inequality one gets

$$\gamma(k)|||u||| \lesssim \frac{\|f\|_{0,\Omega}}{k},$$

or equivalently

$$|||u||| \lesssim \frac{1}{k\gamma(k)}\|f\|_{0,\Omega}.$$

According to the definition of $C_{\mathrm{opt}}(k)$ from Lemma 4.2.1, we deduce that

$$\frac{1}{k\gamma(k)} \gtrsim C_{\mathrm{opt}}(k),$$

which proves the converse bound for $\gamma(k)^{-1}$ due to the equivalence (4.8). □

The numerical method is based on a coarse quasi-uniform finite element grid $\mathcal{T}_H$ and first-order conforming finite elements $V_{H,1}$. The mesh size is indicated by the symbol H because h will refer to the fine-scale discretization parameter in the two-scale method. Let $\mathcal{J}_H : H_0^1(\Omega) \to V_{H,1}$ denote a quasi-interpolation operator satisfying the usual first-order approximation and stability property

$$H^{-1}\|v - \mathcal{J}_H v\|_{L^2(T)} + \|\nabla \mathcal{J}_H v\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(N(T))},$$

for all $T \in \mathcal{T}_H$ and all $v \in H_0^1(\Omega)$. Here, $N(T) = N^1(T)$ is the union of all elements from $\mathcal{T}_H$ that have a nonempty intersection with T. More generally, we define $N^0(T) := T$ and

$$N^m(T) := \bigcup\{K \in \mathcal{T}_H : K \cap N^{m-1}(T) \neq \emptyset\}$$

for any positive integer m. On quasi-uniform meshes, the cardinality of $N^m(T)$ grows polynomially with m.

Let h denote the fine-scale mesh parameter and consider the finite element space $V_{h,1}$ related to the mesh $\mathcal{T}_h$. It is supposed that $\mathcal{T}_h$ is sufficiently fine such that the finite element method over $V_{h,1}$ is stable in the sense that

$$\gamma(k) \lesssim \inf_{v_h \in V_{h,1}\setminus\{0\}} \sup_{w_h \in V_{h,1}\setminus\{0\}} \frac{\operatorname{Re} \mathcal{A}_k(v_h, w_h)}{|||v_h|||\, |||w_h|||} \tag{4.59}$$

where $\gamma(k)$ is the inf-sup constant of $\mathcal{A}_k$ from Lemma 4.4.5. More precisely, if we assume that k^2h is small enough, then (4.59) holds. Indeed let us introduce

$$\eta(V_{h,1}) = \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v_h \in V_{h,1}} \frac{|||\mathbb{S}_k^* f - v_h|||}{\|f\|_{0,\Omega}},$$

where $\mathbb{S}_k^* f \in H_0^1(\Omega)$ is the solution of the adjoint problem of (4.57) with a right-hand side f. Then by standard interpolation estimates and the H^2 regularity of $\mathbb{S}_k^* f$, we can see that

$$\eta(V_{h,1}) \lesssim kh.$$

Consequently by using the arguments of [48, Thm 4.2] and the stability bound from Theorem 4.2.5, we deduce that (4.59) as soon as k^2h is small enough.

Since global computations with $\mathcal{T}_h$ are too costly, only certain functions from $V_{h,1}$ with quasi-local support will be utilized to stabilize a scheme over $\mathcal{T}_H$. The stabilization is as follows. The kernel of $\mathcal{J}_H$ reads

$$W_h = \{v_h \in V_{h,1} : \mathcal{J}_H v_h = 0\}.$$

Given $T \in \mathcal{T}_H$ and $v_H \in V_{H,1}$, its so-called element correction $\mathcal{C}_T v_h \in W_h$ is defined as the solution to the following variational problem

$$\mathcal{A}_k(w_h, \mathcal{C}_T v_H) = \mathcal{A}_{k,T}(w_h, v_H) \quad \text{for all } w_h \in W_h. \tag{4.60}$$

Here and throughout this section, the notation $\mathcal{A}_{k,\omega}$ indicates the spatial restriction of the form $\mathcal{A}_k$ to a subdomain ω. Problem (4.60) is well-posed due to the next result.

Lemma 4.4.6. *Provided $Hk \lesssim 1$, we have the coercivity*

$$\|\nabla w_h\|^2_{L^2(\Omega)} \lesssim \operatorname{Re} \mathcal{A}_k(w_h, w_h) \quad \textit{for all } w_h \in W_h.$$

The constants involved in "$\lesssim$" only depend on the bounds of the coercivity and continuity constant of A as well as on the maximal modulus of $d\tilde{d}$.

Proof. The proof almost verbatim follows [9, Lemma 1]. □

This result readily implies boundedness of $\mathcal{C}_T$,

$$|||\mathcal{C}_T v_H||| \lesssim |||v_H|||_T \quad \text{for all } v_H \in V_H.$$

By linearity, one can see that the "global corrector" $\mathcal{C} v_H := \sum_{T\in\mathcal{T}_H} \mathcal{C}_T v_H$ solves

$$\mathcal{A}_k(w_h, \mathcal{C} v_H) = \mathcal{A}_k(w_h, v_H) \quad \text{for all } w_h \in W_h,$$

and thus satisfies the continuity

$$|||\mathcal{C} v_H||| \lesssim |||v_H||| \quad \text{for all } v_H \in V_H.$$

As mentioned above, the correctors from (4.60) shall serve as an additive stabilizing component to the coarse finite element basis functions. But at this stage (4.60) defines a global fine-scale problem and, thus, $\mathcal{C}_T v_H$ is not computationally available. The key observation from [46] is that such computations can be localized to certain neighbourhoods of T. Let $\ell \in \mathbb{N}$ be a localization (or oversampling) parameter and define

$$\Omega_T := \operatorname{int} N^\ell(T)$$

and

$$W_h(\Omega_T) := \{w_h \in W_h : w_h = 0 \text{ outside } \Omega_T\}.$$

These objects depend on the parameter ℓ, which will, however, be suppressed for convenient notation. Problem (4.60) is now approximated by seeking $\mathcal{C}_{T,\ell} v_H \in W_h(\Omega_T)$ such that

$$\mathcal{A}_{k,\Omega_T}(w_h, \mathcal{C}_{T,\ell} v_H) = \mathcal{A}_{k,T}(w_h, v_H) \quad \text{for all } w_h \in W_h. \tag{4.61}$$

Note that the numerical computation of each of the problems (4.61) is feasible (with $\mathcal{O}(\ell H/h)^2$ vertices in 2D) as long as ℓ is of moderate size. The global localized version of $\mathcal{C}$ is defined as

$$\mathcal{C}_\ell v_H := \sum_{T\in\mathcal{T}_H} \mathcal{C}_{T,\ell} v_H.$$

The localized approximation is justified by the following exponential decay result.

Theorem 4.4.7. *Provided $kH \lesssim 1$, there exists $0 < \beta < 1$ such that any $v_H \in V_H$, any $T \in \mathcal{T}_H$, and any $\ell \in \mathbb{N}$ satisfy*

$$\begin{aligned} \|\nabla(\mathcal{C}_T - \mathcal{C}_{T,\ell}) v_H\|_{L^2(\Omega)} &\lesssim \beta^\ell \|\nabla v_H\|_{L^2(T)}, \\ \|\nabla(\mathcal{C} - \mathcal{C}_\ell) v_H\|_{L^2(\Omega)} &\lesssim C(\ell)\beta^\ell \|\nabla v_H\|_{L^2(T)}, \end{aligned}$$

with a constant $C(\ell)$ that grows not faster than polynomially with ℓ.

Proof. For a proof we refer to [30]. See also [9, Theorem 4]. □

The multiscale scheme is a Petrov–Galerkin method and referred to as multiscale Petrov–Galerkin scheme (MSPG). It seeks $u_H^{(\ell)} \in V_{H,1}$ such that

$$\mathcal{A}_k(u_H^{(\ell)}, (1-\mathcal{C}_\ell)v_H) = (\tilde{f}, (1-\mathcal{C}_\ell)v_H)_{L^2(\Omega)} \quad \text{for all } v_H \in V_{H,1}. \tag{4.62}$$

Well-posedness of (4.62) is ensured through an appropriate parameter choice that will be described in the following. Suppose the fine-scale mesh size h is small enough such that (4.59) is satisfied. The important property of the multiscale method is that it suffices to relate the oversampling lengths *logarithmically* to the wave number k.

Theorem 4.4.8. *Suppose $kH \lesssim 1$ and (4.59) as well as*

$$\ell \gtrsim |\log \gamma(k)| \Big/ |\log \beta|. \tag{4.63}$$

Then, the Petrov–Galerkin bilinear form from (4.62) satisfies

$$\gamma(k) \lesssim \inf_{v_H \in V_{H,1}\setminus\{0\}} \sup_{w_H \in V_{H,1}\setminus\{0\}} \frac{\operatorname{Re}\mathcal{A}_k(v_H, (1-\mathcal{C}_\ell)w_H)}{|||v_H|||\,|||w_H|||}.$$

Proof. For a proof we refer to [30]. See also [9, Theorem 5]. □

As in [30, Thm 3], it can be shown that

$$|||u_h - u_H^{(\ell)}||| \lesssim \inf_{v_H \in V_{H,1}} |||u_h - v_H|||.$$

Thus, the triangle inequality and classical approximation properties together with the H^2 bound (4.11) show for h sufficiently small that

$$|||u - u_H^{(\ell)}||| \lesssim H\|u\|_{H^2(\Omega)} \lesssim Hk\|f\|_{L^2(\Omega)}.$$

In particular, this means that the standard resolution condition $kH \lesssim 1$ for approximation is also sufficient for stability of the multiscale scheme.

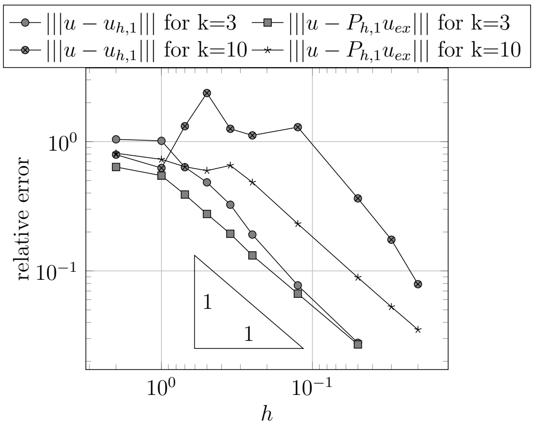
4.5 Some numerical examples

4.5.1 A first example

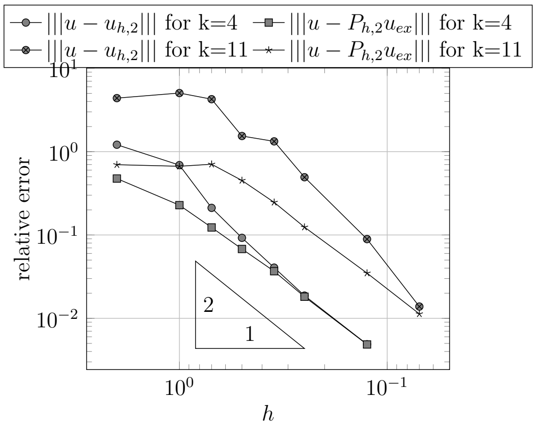
For the first test, we have taken $\Omega = [-6,6]^2 \setminus B(0,1)$, the fictious absorption coefficient σ and the exact solution u_{ex} as follows:

$$\sigma(\rho) = \begin{cases} 0 & \text{if } \rho \le 4 \\ \frac{(\rho-4)^2}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad u_{ex}(x,y) = (x^2-36)(y^2-36)e^{ikx}.$$

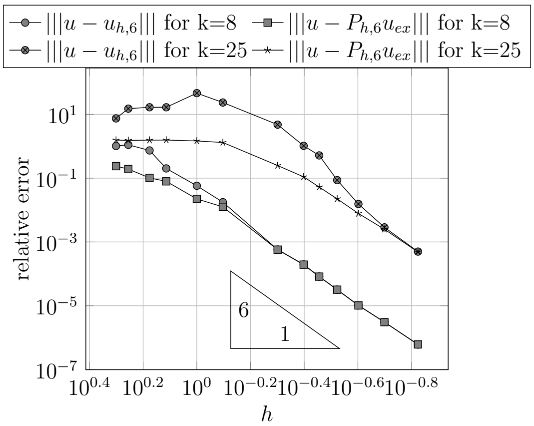
In Figure 4.2, we have depicted the rates of convergence for different values of h and k, for $p = 1$ and 2. We can see that, when h is small enough, the order of convergence is p, as expected from the theory. From these plots we can



(a) Rates of convergence for $k = 3$ or 10 with $p = 1$

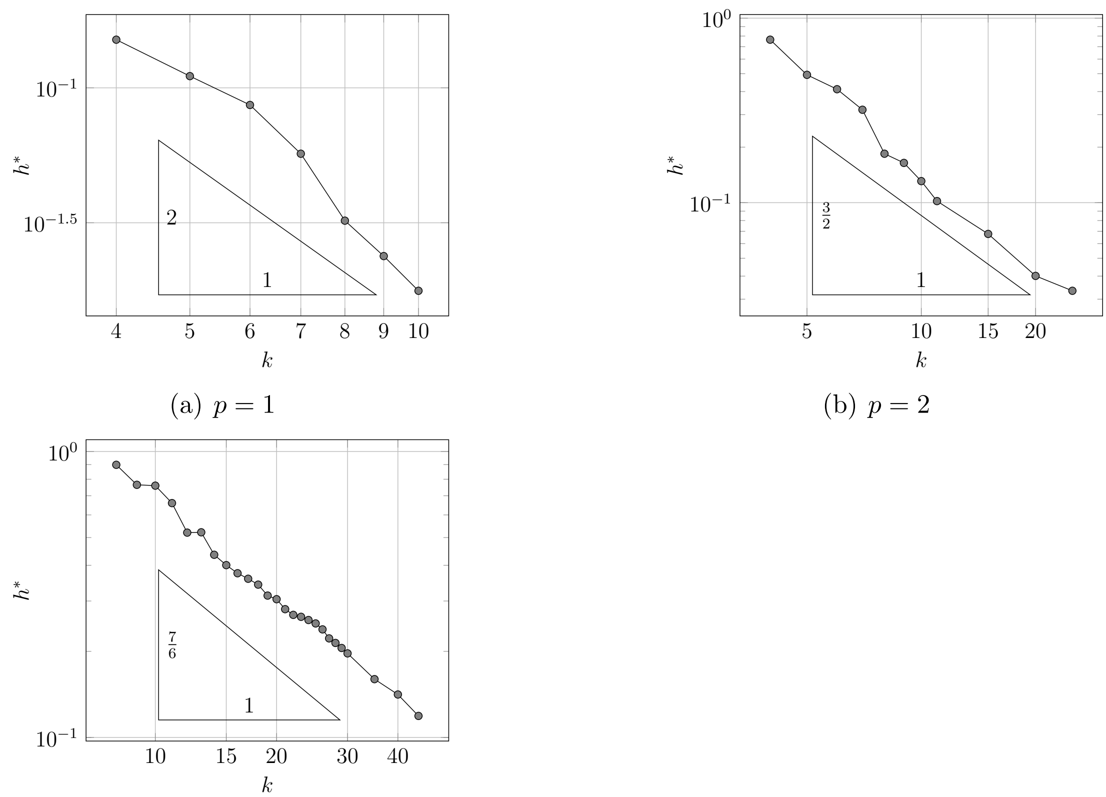


(b) Rates of convergence for $k = 4$ or 11 with $p = 2$



(c) Rates of convergence for $k = 8$ or 25 with $p = 6$

Figure 4.2: First experiment with hp FEM: convergence curves for different values of k and p.



(a) $p = 1$

(b) $p = 2$

(c) $p = 6$

Figure 4.3: First experiment with hp FEM: Asymptotic range of $h^*(k)$ for $p = 1$, 2 and 6.

observe three states of convergence: no convergence range/ pre-asymptotic range / asymptotic range.

Theorem 4.4.1 states that, provided $k^{p+1}h^p \lesssim 1$, the following error bound holds

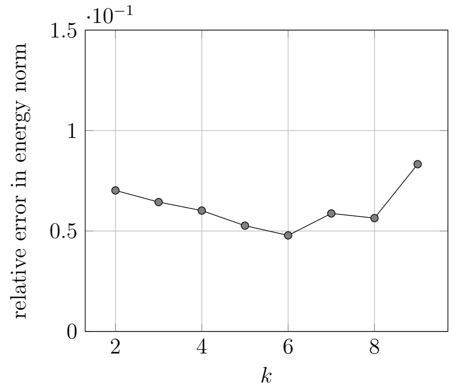
$$|||u_{ex} - u_{h,p}||| \lesssim |||u_{ex} - P_{h,p}u_{ex}|||,$$

where $P_{h,p}u_{ex}$ the orthogonal projection of u_{ex} on $V_{h,p}$ for the inner product associated with the norm $|||\cdot|||$. For different values of k, h and p, we compute $u_{h,p}$ and $P_{h,p}u_{ex}$, and denote by $h^*(k)$ the greatest value of h such that

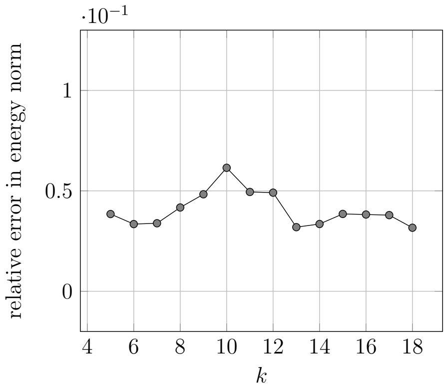
$$|||u_{ex} - u_{h,p}||| \leq 2|||u_{ex} - P_{h,p}u_{ex}|||.$$

Figure 4.3 displays the graph of $h^*(k)$ (in a log-log scale) for $p = 1$ and 2. In both cases, we observe that $h^*(k) \sim k^{-1-1/p}$, which means that the condition $k^{p+1}h^p \lesssim 1$ is optimal. Figure 4.4 displays the relative errors in the preasymptotic range dependent on the wavenumber k, while k and h are coupled (depending on p) as in Theorem 4.4.4. As predicted by the theory, the relative error stays constant, which means that the discretization is stable with that choice of h and p.

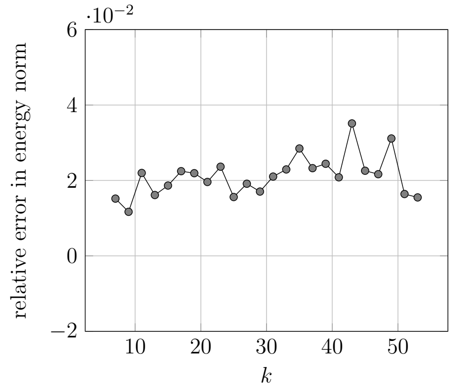
Next, we report numerical results for the multiscale scheme. We consider Q_1 (bilinear) finite elements on a sequence of uniformly refined square meshes of mesh



(a) $p = 1$ with k^3h^2 constant

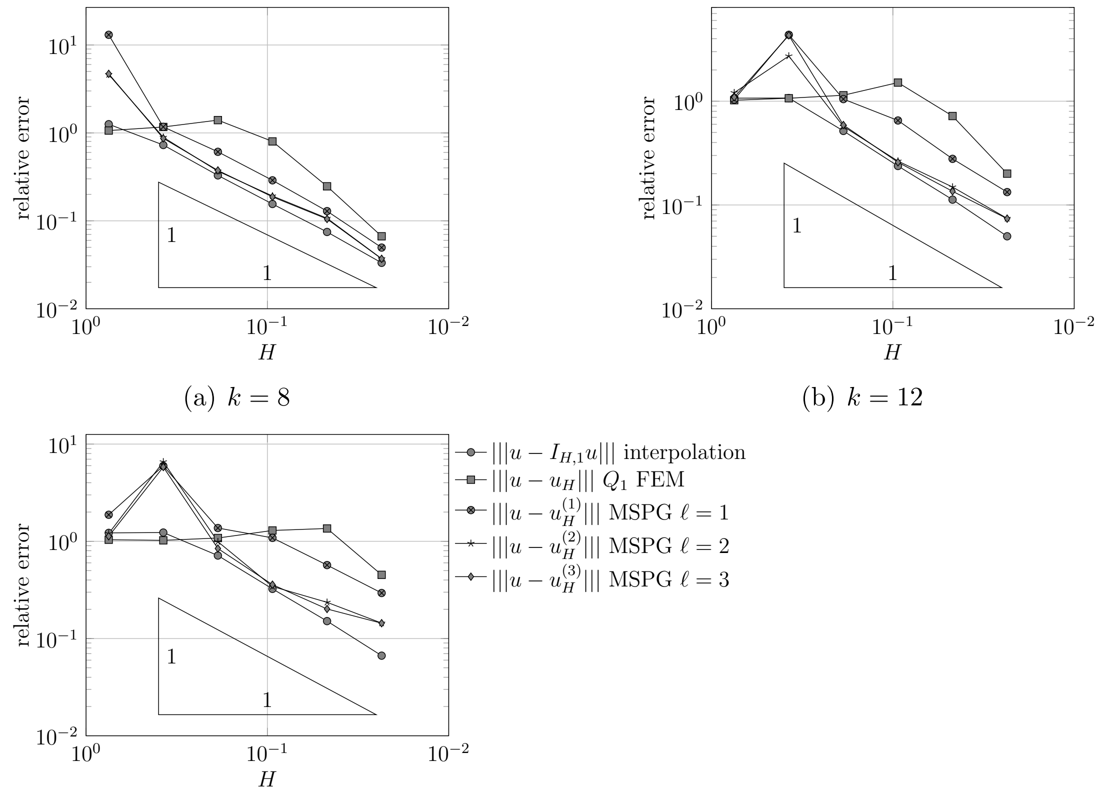


(b) $p = 2$ and $k^5h^4 = C$



(c) $p = 6$ and $k^{13}h^{12} = C$

Figure 4.4: First experiment with hp FEM: preasymptotic range.



(a) $k = 8$

(b) $k = 12$

(c) $k = 16$

Figure 4.5: First experiment. Relative error plots for the nodal interpolation $I_{H,1}u$, the Q_1 FEM, and the multiscale Petrov–Galerkin method ('MSPG') with oversampling parameter $\ell = 1, 2, 3$.

size $H = 3/4, 3/8, \ldots, 3/128$. The reference mesh has the mesh size $h = 3/256$. The very regular structure of square meshes allows a quite efficient numerical implementation [30] of the method in which the correctors $\mathcal{C}_\ell$ outside the PML are computed on a reference patch and re-used where the same configuration occurs. For simplicity, we disregard the possibility of resolving the curved boundary within the corrector problems, although this can be done in principle [26, 67]. We do not further analyze the error caused by this geometric perturbation. For wave numbers $k = 8, 12, 16$, Figure 4.5 compares the relative errors in the energy norm $|||\cdot|||$, namely the nodal interpolation by Q_1 finite element functions, the Q_1-FEM error, and the error of the MSPG method where the oversampling parameter varies from $\ell = 1$ to $\ell = 3$. For the FEM, pollution is clearly visible, while the MSPG scheme produces smaller errors that are close to the best approximation for appropriate ℓ. Especially in the case $k = 16$, the choice of $\ell = 1$ seems to be insufficient, while $\ell = 2, 3$ lead to better results. This indicates the necessity of the coupling $\ell \sim \log k$. Since the accuracy of the MSPG method is limited by that of the FEM on the reference mesh, the last two mesh refinements for $k = 16$ do no provide a reasonable improvement. We finally mention that the mesh resolution condition "hk^2 small" is not fully satisfied for $k = 16$, but we empirically observe that this choice of h seems to be sufficient.

4.5.2 A scattering problem

Here we want to show the efficiency of our method by approaching a real scattering problem. Namely as obstacle $\mathcal{O}$ we take the unit disc and take

$$u_{\mathrm{scat}}(\theta,\rho) = \sum_{j=-\infty}^{\infty} i^j \left(\frac{J_j(k)}{J_j(k)+iY_j(k)}\right)(J_j(k\rho)+iY_j(k\rho))\, e^{ij\theta}$$

as exact solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \mathcal{O}$, which corresponds to the scattered solution of the incidence wave e^{ikx_1} (see [43, (3.3)] or [20]). As fictitious absorption coefficient, we choose

$$\sigma(\rho) = \begin{cases} 0 & \text{if } \rho \le a \\ \frac{\beta(\rho-a)^2}{(b-a)^2} & \text{otherwise} \end{cases},$$

with $\beta > 0$. Now, consider the solution u_b of (compare with (4.3))

$$\begin{cases} k^2 \tilde{d}^2 u_b + \frac{q}{\rho}\frac{\partial}{\partial\rho}\left(q\rho\frac{\partial u_b}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u_b}{\partial\theta^2} = 0 & \text{in } \Omega, \\ u_b = e^{ikx_1} & \text{on } \partial\mathcal{O}, \\ u_b = 0 & \text{on } \partial\Omega \setminus \partial\mathcal{O}, \end{cases} \tag{4.64}$$

where $\Omega = B(0,b)$ (see section 4.1) with $1 < a < b$. It is well-known (see for instance [40, 41, 8]) that u_b converges to u_{scat} (even exponentially but the constant being dependent of the wave number k) in $H^1(B(0,a))$ as b goes to infinity. For our tests, we take $a = 3$ and $b = 6$.

As an approximation we compute

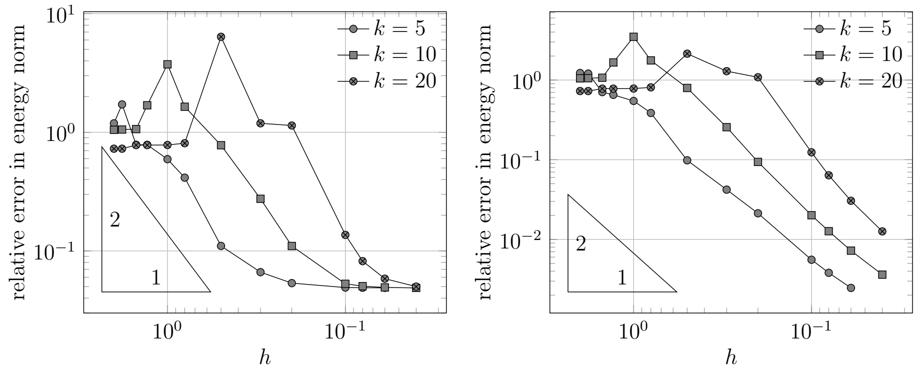
$$u_{h,p} \in \tilde{V}_{h,p} = \left\{ v_{h,p} \in H^1(\Omega) \mid v_{h,p}|_K \circ \mathcal{F}_K^{-1} \in \mathbb{P}_p(\hat{K})\ \forall K \in \mathcal{T}_h \right\},$$

the FEM solution of (4.64).

As u_b is unknown, we compare the FEM solution $u_{h,p}$ with u_{scat}, and the relative error in energy norm means that we compute $\frac{|||u_{h,p}-u_{\mathrm{scat}}|||_{\Omega_a}}{|||u_{\mathrm{scat}}|||_{\Omega_a}}$. The full error clearly satisfies

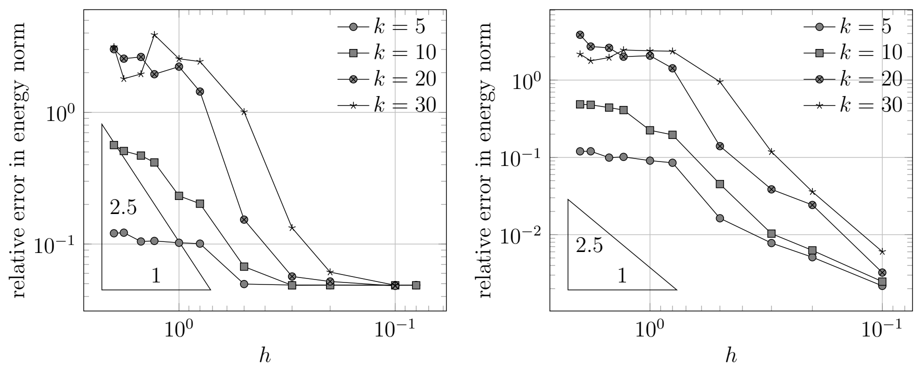
$$|||u_{h,p} - u_{\mathrm{scat}}|||_{\Omega_a} \le |||u_{h,p} - u_b||| + |||u_b - u_{\mathrm{scat}}|||_{\Omega_a} \tag{4.65}$$

Figure 4.6 shows convergence curves for different values of k, given in the relative energy norm by using polynomials of degree 2. On the left, we have chosen $\beta = 3$ small enough so that the error $|||u_b - u_{\mathrm{scat}}|||$ is not negligible. Accordingly, the error does not tend to 0 when h is small. On the right, with $\beta = 6$, the term $|||u_b - u_{\mathrm{scat}}|||$ is negligible compared to the FEM error. As $\sigma \in \mathcal{C}^2(\Omega)$, we know that u_b is at least $H^3(\Omega)$, which is the reason why we have 2 for the convergence rate. Figure 4.7 shows for polynomials of degree 6 that the empirical convergence rate is not higher than 2.5, which indicates that the solution u_b might not be smoother than $H^{7/2}$. In comparison with the case $p = 2$, in the case $\beta = 6$, the term $|||u_b - u_{\mathrm{scat}}|||$ seems here more dominant as the rate of convergence deteriorates more rapidly.



(a) $\beta = 3$ (b) $\beta = 6$

Figure 4.6: Second experiment with *hp* FEM: Convergence curves for different values of k and β, with $p = 2$.



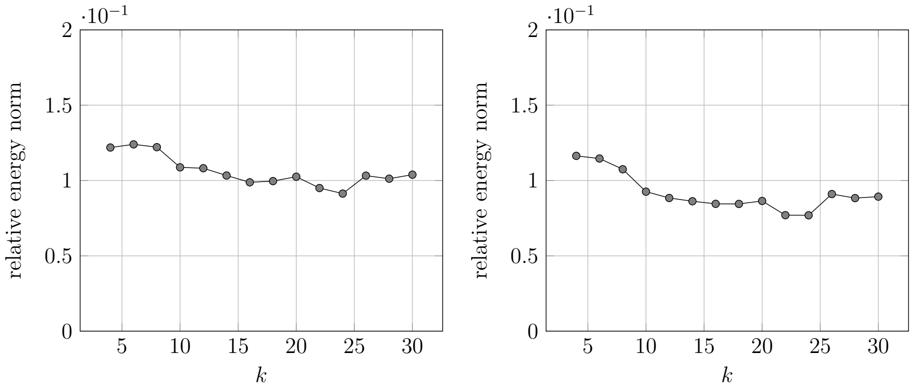
(a) $\beta = 3$ (b) $\beta = 6$

Figure 4.7: Second experiment with *hp* FEM: Convergence curves for different values of k and β, with $p = 6$.

We also made a pre-asymptotic test (see Figure 4.8) with $p = 2$ and $\beta = 3$ or 6. We observe that when k^5h^4 is constant, the relative error in energy norm is constant too, which is in accordance with the estimate (4.65) since in the pre-asymptotic range the second term of the right-hand side is negligible, while the first one is constant due to Theorem 4.4.4.

Figure 4.9 displays the real part of u_{scat} and $u_{h,p}$, for $k = 20$, $p = 6$ and $\beta = 10$, where we see a good agreement between the exact solution and its approximation in Ω_a.

The computational results obtained by the MSPG method are displayed in Figure 4.10. The parameters H, h, ℓ, and k are chosen as in the first experiment, and $\beta = 10$. As in the first experiment, the FEM suffers from pollution, which is mitigated by the MSPG method. The precision increases with larger ℓ.



(a) $\beta = 3$ (b) $\beta = 6$

Figure 4.8: The pre-asymptotic examples with $p = 2$.

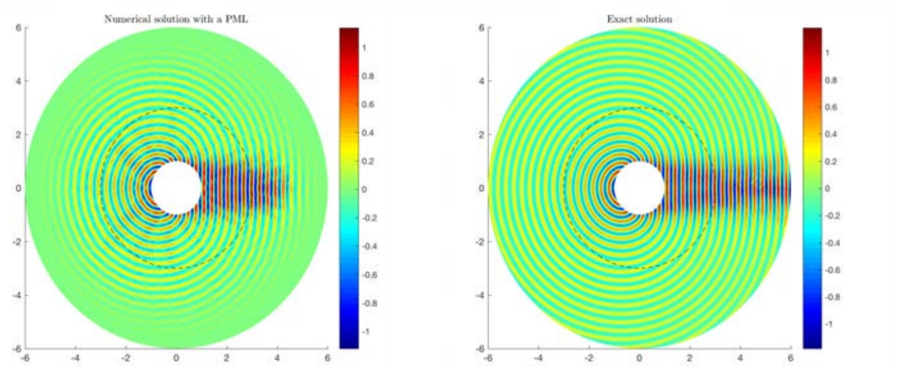
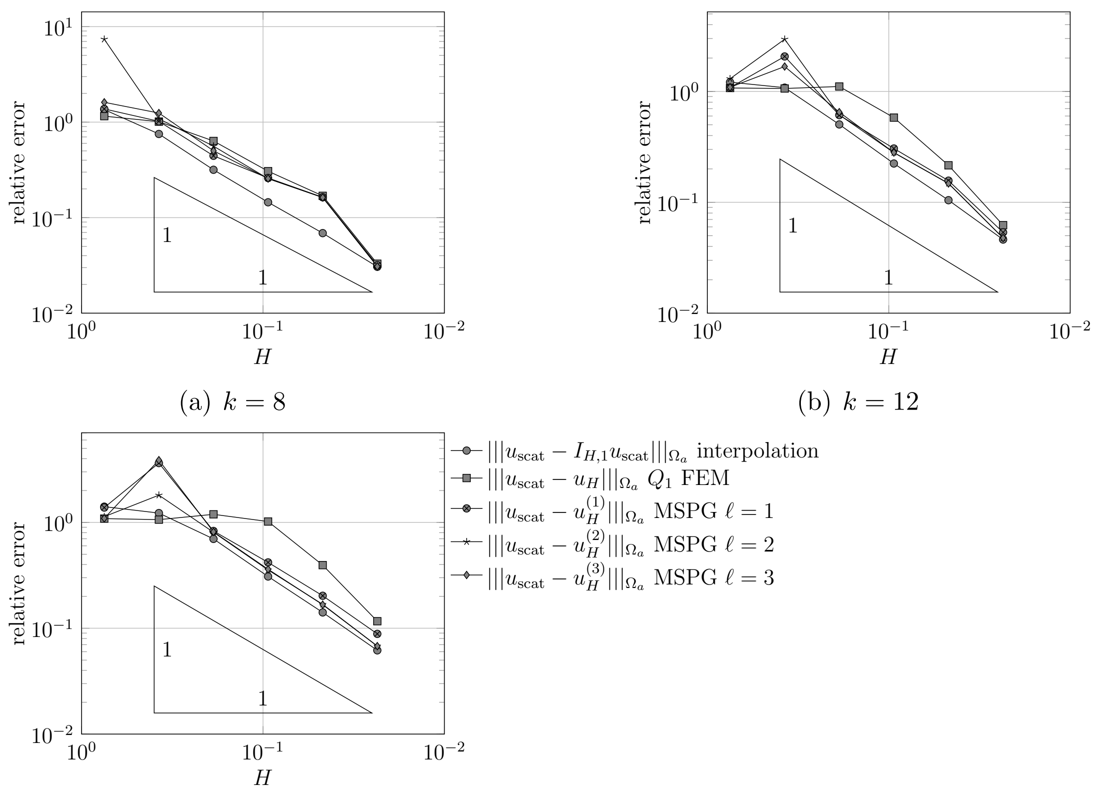


Figure 4.9: Second experiment with hp FEM: Real part of the exact solution and the computed PML solution ($p = 6$ and $\beta = 10$) with $k = 20$.



(a) $k = 8$

(b) $k = 12$

(c) $k = 16$

Figure 4.10: Second experiment. Relative error plots for the nodal interpolation $I_{H,1}u$, the Q_1 FEM, and the multiscale Petrov–Galerkin method ('MSPG') with oversampling parameter $\ell = 1, 2, 3$.

4.6 Conclusion

We have shown that the PML model problem satisfies the k-stability property. This result enables is key to the numerical analysis of the two schemes presented in this work. The numerical results underline that the stability conditions for the numerical methods are sharp. Instead of comparing the two proposed schemes, we rather mention that they are are designed for different types of applications: the *hp* FEM is of high order for smooth domains, while the multiscale scheme is pollution-free without smoothness, but restricted to first order.

4.7 Appendix: Useful properties of the PML functions

We recall from [19] that the fictious absorption coefficient σ is supposed to be a non decreasing function in $C^1(0,\infty)$ such that

$$\sigma(\rho) = \begin{cases} = 0, \forall \rho \leq a, \\ > 0, \forall \rho > a. \end{cases} \tag{4.66}$$

Then we define $\tilde{\sigma} \in C[0,\infty)$ as follows

$$\tilde{\sigma}(\rho) = \begin{cases} = 0, \forall \rho \leq a, \\ \frac{1}{\rho}\int_a^\rho \sigma(s)\,ds, \forall \rho > a. \end{cases} \tag{4.67}$$

From this expression, we deduce that

$$\rho\tilde{\sigma}(\rho) = \int_a^\rho \sigma(s)\,ds, \forall \rho > a,$$

and therefore

$$\sigma(\rho) = (\rho\tilde{\sigma})'(\rho), \forall \rho > a.$$

By Leibniz' rule, we get

$$\rho\tilde{\sigma}'(\rho) = \sigma(\rho) - \tilde{\sigma}(\rho), \forall \rho > a. \tag{4.68}$$

In addition, as σ is non decreasing, (4.67) directly implies

$$\tilde{\sigma}(\rho) \leq \frac{\rho - a}{\rho}\sigma(\rho) < \sigma(\rho), \forall \rho > a. \tag{4.69}$$

These two estimates directly lead to

$$\tilde{\sigma}'(\rho) > 0, \forall \rho > a, \tag{4.70}$$

and therefore $\tilde{\sigma}$ is a also a non decreasing function. Furthermore $\tilde{\sigma} \in C^1[0,\infty)$ because from (4.68) and the continuity at a of σ and $\tilde{\sigma}$, one has

$$\tilde{\sigma}'(\rho) = \frac{\sigma(\rho) - \tilde{\sigma}(\rho)}{\rho} \to 0, \text{ as } \rho \to a.$$

From (4.66), (4.67) and (4.70) and the C^1 property of $\tilde{\sigma}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(\rho) \geq \delta, \tilde{\sigma}(\rho) \geq \delta, \tilde{\sigma}'(\rho) \geq \delta, \forall \rho \geq a + \frac{\varepsilon}{2}. \tag{4.71}$$

We then fix $\varepsilon > 0$ small enough such that $a + \varepsilon < b$ and fix a cut-off function $\eta \in \mathcal{D}(\mathbb{R})$ with $\eta' \leq 0$ such that

$$\eta(\rho) = \begin{cases} 1, \forall \rho \leq a + \frac{\varepsilon}{2}, \\ 0, \forall \rho \geq a + \varepsilon. \end{cases}$$

For convenience, we denote by Ω_{PML}, the PML region, i. e.,

$$\Omega_{PML} = \{x \in \Omega : |x| > a\}.$$

We also set

$$\Omega_{PML}^{+} = \{x \in \Omega : |x| > a + \frac{\varepsilon}{2}\}.$$

Lemma 4.7.1. *We always have*

$$\sigma \leq \tilde{\sigma}' \text{ in } \Omega_{PML}. \tag{4.72}$$

Proof. By (4.69), one has

$$0 \leq \lim_{\rho \to a+} \frac{\tilde{\sigma}(\rho)}{\sigma(\rho)} \leq \lim_{\rho \to a+} \frac{\rho - a}{\rho} = 0,$$

which shows that

$$\lim_{\rho \to a+} \frac{\tilde{\sigma}(\rho)}{\sigma(\rho)} = 0.$$

Using (4.68), we then have

$$\lim_{\rho \to a+} \frac{\tilde{\sigma}'(\rho)}{\sigma(\rho)} = \frac{1}{a}.$$

Consequently for $\rho > a$ but close to a, we trivially have (4.72). On the other hand, for $\rho \in [a + \varepsilon_1, b]$, with $\varepsilon_1 > 0$ as small as we want, (4.71) and the continuity of σ directly yield (4.72). The proof is then complete. □

As in [19], we set

$$d = 1 + \frac{i\sigma}{k}, \quad \text{and } \tilde{d} = 1 + \frac{i\tilde{\sigma}}{k}. \tag{4.73}$$

Let us also define

$$q = \frac{\tilde{d}}{d}. \tag{4.74}$$

Lemma 4.7.2. *The next properties hold*

$$\operatorname{Im} q(\rho) = -\frac{k\rho\tilde{\sigma}'(\rho)}{k^2+\sigma^2(\rho)} \leq 0, \tag{4.75}$$

$$\operatorname{Im} q^2 = 2\gamma \operatorname{Im} q, \tag{4.76}$$

$$\operatorname{Im} \tilde{d}^2 = \frac{2\tilde{\sigma}}{k} \geq 0, \tag{4.77}$$

$$q \to 1, \ \textit{as } k \to \infty, \tag{4.78}$$

$$d \to 1, \ \textit{as } k \to \infty, \tag{4.79}$$

$$\tilde{d} \to 1, \ \textit{as } k \to \infty, \tag{4.80}$$

$$\overline{\tilde{d}^2} - \tilde{d}^2 = -\frac{4i}{k}\tilde{\sigma} \tag{4.81}$$

where $0 < \gamma = \frac{1+\frac{\sigma\tilde{\sigma}}{k^2}}{1+\frac{\sigma^2}{k^2}}$ *that tends to 1 as k goes to infinity.*

Proof. The properties (4.77) to (4.81) are direct. To prove (4.75) and (4.76), we notice that q admits the writing

$$q = \gamma + \frac{ik}{k^2+\sigma^2}(\tilde{\sigma}-\sigma),$$

which directly yields the results recalling (4.68). □

Lemma 4.7.3. *We have*

$$|\frac{\partial}{\partial\rho}q| \lesssim \frac{1}{k} \ \textit{in } \Omega_{PML}, \tag{4.82}$$

$$q = 1 \ \textit{in } \Omega_0, \tag{4.83}$$

$$|q-1| \lesssim \frac{1}{k} \ \textit{in } \Omega_{PML}. \tag{4.84}$$

Proof. The second identity being immediate, let us concentrate on the two other ones. By direct calculations, we see that

$$\frac{\partial}{\partial\rho}q = \frac{i}{k}(\frac{\tilde{\sigma}'}{d} - \frac{\tilde{d}\sigma'}{d^2}).$$

The estimate (4.82) follows as $|d| \geq 1$ as well as $|\tilde{d}| \geq 1$ and since σ' and $\tilde{\sigma}'$ are bounded.

Concerning the last one, we see that

$$q - 1 = \frac{1}{k}\frac{i(\tilde{\sigma}-\sigma)}{1+\frac{i\sigma}{k}}.$$

Hence the estimate (4.84) holds because $|1+\frac{i\sigma}{k}| \geq 1$ and because σ and $\tilde{\sigma}$ are bounded. □

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