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On some problems of holomorphic analytic torsion

Par Siarhei FINSKI

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Dirigée par Xiaonan MA

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M. Jean-Michel BISMUT	Université Paris-Sud	rapporteur
M. José Ignacio BURGOS GIL	Institute of Mathematical Sciences	rapporteur
Mme Claire DEBORD	Université Paris Diderot	examinatrice
M. Colin GUILLARMOU	Université Paris-Sud	examinateur
M. Xiaonan MA	Université Paris Diderot	directeur
M. Gerard FREIXAS I MONTPLET	Université Pierre et Marie Curie	examinateur
M. Peter ZOGRAF	Steklov Mathematical Institute	examinateur

Institut de mathématiques de Jussieu-Paris
Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75252 Paris Cedex 05

Université Pierre et Marie Curie.
École doctorale de sciences
mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75252 Paris Cedex 05

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Titre: Sur quelques problèmes de la torsion analytique holomorphe.

Résumé: Le but de cette thèse est d'étudier la torsion analytique dans deux contextes différents.

Dans le premier contexte, on étudie l'asymptotique de la torsion analytique, quand un fibré vectoriel holomorphe hermitien est tordué par une puissance croissant du fibré en droites positif.

Dans le deuxième contexte, on généralise la théorie de la torsion analytique pour des surfaces de Riemann avec des pointes hyperboliques. Motivé par des singularités de la métrique complète de courbure scalaire constante -1 sur des surfaces de Riemann stables épointées, on demande que la métrique sur la surface de Riemann soit lisse seulement en dehors d'un nombre fini des points au voisinage auxquelles elle peut avoir des singularités comme la métrique de Poincaré sur un disque épointé. On fixe un fibré vectoriel holomorphe hermitien qui peut avoir au pire des singularites logarithmiques au voisinage des points marqués. Pour ces données, en renormalisant la trace de l'opérateur de la chaleur, on construit la torsion analytique et on étudie ces propriétés.

Puis on étudie des propriétés de la torsion analytique en famille: on démontre la théorème de la courbure, on étudie le comportement de la torsion analytique quand les pointes sont créées par dégénération et on donne quelques applications aux espaces de modules de courbes épointées.

Mots clefs : Cohomologie, Laplacian, spectre, noyau de chaleur, torsion analytique, surfaces hyperboliques, l'espace module de surfaces épointées, fibré de Hodge.

Title: On some problems of holomorphic analytic torsion.

Abstract: The goal of this thesis is to study the analytic torsion in two different contexts.

In the first context, we study the asymptotics of the analytic torsion, when a Hermitian holomorphic vector bundle is twisted by an increasing power of a positive line bundle.

In the second context, we generalize the theory of analytic torsion for surfaces with hyperbolic cusps. Motivated by singularities appearing in complete metrics of constant scalar curvature -1 on stable Riemann surfaces, we suppose that the metric on the surface is smooth outside a finite number points in the neighborhood of which it can to have singularities like Poincaré metric has on a punctured disc. We fix a Hermitian holomorphic vector bundle which has at worst logarithmic singularities in the neighborhood of the marked points. For these data, by renormalizing the trace of the heat operator, we construct the analytic torsion and study its properties.

Then we study the properties of the analytic torsion in family setting: we prove the curvature theorem, we study the behavior of the analytic torsion when the cusps are created by degeneration and we give some applications to the moduli spaces of pointed curves.

Keywords : Cohomology, Laplacian, spectre, heat kernel, analytic torsion, hyperbolic surfaces with cusps, moduli spaces of pointed surfaces, Hodge bundle.

Contents

Chapitre 1

Introduction

La torsion analytique holomorphe a été définie par Ray-Singer dans [103] comme l'analogue complexe de la torsion réelle, qui est un invariant analytique correspondant à la torsion de Reidemeister - le premier invariant topologique qui n'est pas un invariant homotopique. La torsion analytique holomorphe s'obtient comme le déterminant régularisé du laplacien de Kodaira d'un fibré vectoriel holomorphe sur une variété complexe compacte. La première application centrale du cet invariant se trouve dans les travaux de Bismut-Gillet-Soulé [21], [22], [23], où les auteurs établissent la théorème de la courbure, qu'on rappellera plus tard.

Aujourd'hui, la torsion analytique joue un rôle important dans de nombreux domaines des mathématiques. Dans la géométrie d'Arakelov, elle joue un rôle central dans les travaux de Gillet-Soulé [66], Gillet-Soulé-Rössler [63], Köhler-Rössler, [78], [79], [80], Freixas [59], [58]. Bershadsky-Cecotti-Ooguri-Vafa [14] ont trouvé ses applications en physique mathématique, et Fang-Lu-Yoshikawa [48] ont donné un traitement mathématique de leur théorie. Dans la théorie des formes automorphes et des surfaces K3, elle est la pierre angulaire du travail de Yoshikawa [119] concernant la torsion analytique et la forme automorphe de Borcherds...

Rappelons maintenant la définition de la torsion analytique. Soit X une variété complexe compacte munie d'une métrique riemannienne g^{TX}, qui est compatible avec la structure complexe. Soit (E, h^E) un fibré holomorphe hermitien sur X. Soit $\overline{\partial}^E$ l'opérateur de Dolbeaut sur l'espace vectorielle $\Omega^{(0,k)}(X, E)$, $k = 0, \ldots, \dim X$, des formes de type $(0, k)$ à valeurs dans E. L'espace $\Omega^{(0,k)}(X, E)$ est naturellement muni d'un produit scalaire L^2, obtenu comme

$$\langle \alpha, \alpha' \rangle_{L^2} := \int_X \langle \alpha(x), \alpha'(x) \rangle_h dv_X(x), \qquad \alpha, \alpha' \in \Omega^{(0,k)}(X, E), \tag{1.0.1}$$

où dv_X est la forme de volume riemannienne sur (X, g^{TX}) et $\langle \cdot, \cdot \rangle_h$ est un produit hermitien ponctuel induit par h^E et g^{TX}. On note $\overline{\partial}^{E,*}$ l'adjoint de l'opérateur $\overline{\partial}^E$ par rapport à $\langle \cdot, \cdot \rangle_{L^2}$. Soit

$$\square_k^E = \overline{\partial}^E \overline{\partial}^{E,*} + \overline{\partial}^{E,*} \overline{\partial}^E, \tag{1.0.2}$$

le laplacien de Kodaira associé, agissant sur $\Omega^{(0,k)}(X, E)$, $k = 1, \ldots, \dim X$. Comme X est une variété compacte et $\square_k^E$ sont des opérateurs elliptiques, les spectres de $\square_k^E$ sont les ensembles

discrèts. La torsion analytique $T(g^{TX}, h^E)$ est définie comme

$$T(g^{TX}, h^E) = \prod_{k\geq 0}^{\dim X} (\det{}' \square_k^E)^{k(-1)^k/2}, \tag{1.0.3}$$

où $\det{}'$ est le produit renormalisé de valeurs propres non-nulles.

Le but de cette thèse est d'étudier la torsion analytique dans deux contextes différents.

Dans le premier contexte, on étudie l'asymptotique de la torsion analytique, quand le fibré vectoriel holomorphe hermitien (E, h^E) est tordue par une puissance p, $p \to \infty$ du fibré en droites positif (L, h^L).

Dans le deuxième contexte, on généralise la théorie de la torsion analytique pour des surfaces avec des pointes hyperboliques. Motivé par des singularités de la métrique complète de courbure scalaire constante -1 sur une surface stable épointée, on ne demande plus que la métrique g^{TX} soit lisse sur X tout entier, mais seulement en dehors d'un nombre fini des points au voisinage auxquelles g^{TX} peut avoir des singularités comme la métrique de Poincaré sur le disque épointé. La difficulté ici réside dans le fait que le spectre de l'opérateur $\square_i^E$ n'est plus discret car la variété n'est plus compacte, donc la définition (1.0.3) n'est pas valable et on doit définir la torsion analytique par une autre façon. On le fait par la renormalisation de trace de l'opérateur de la chaleur, puis on étudie des propriétés du cet invariant et ces applications à l'espace de modules de courbes épointées.

Les deux chapitres composant cette thèse sont indépendants, et on donne ci-dessous une introduction plus détaillée à chacun d'entre eux. Insistons cependant sur le fait que le point de vue qu'on adopte tout au long de cette thèse est inspiré de la théorie de l'indice locale. Les objets centraux de cette thèse sont le laplacien, la torsion analytique et le noyau de la chaleur.

1.1 L'asymptotique de la torsion analytique

Dans [27], Bismut-Vasserot ont calculé l'asymptotique de la torsion analytique associée à des puissances croissantes d'un fibré en droites positif. Cette asymptotique a joué un rôle important dans un résultat d'amplitude arithmétique de Gillet et Soulé [64] (voir aussi [106, Chapter VIII]).

Récemment, Klevtsov-Ma-Marinescu-Wiegmann en cas de surfaces de Riemann, [73], ont relié cette asymptotique avec l'asymptotique de la fonction génératrice qui apparait dans l'étude de l'effet de Hall quantique entier quand le flux du champ magnétique tends vers l'infini. À partir de ces résultats, ils ont conjecturé [73, p.839] la formule pour ce terme de l'asymptotique de Bismut-Vasserot. On calcule explicitement le terme suivante pour une variété kählérienne de dimension

quelconque et ainsi on démontre la vérité de la conjecture pour le terme logarithmique et sa fausseté pour le terme constante. On donne aussi la forme générale de l'asymptotique de la torsion analytique dans le cadre plus générale d'orbifold.

Plus précisément, soit (X, g^{TX}, Θ) une variété hermitienne de la dimension complexe n. Soit (E, h^E) un fibré vectoriel holomorphe hermitien sur X de rang $\mathrm{rk}(E)$. On note $c_1(E) \in H^2(X, \mathbb{Z})$ la première classe de Chern de E. Soit (L, h^L) un fibré en droites holomorphe positif sur X. On note ω la $(1,1)$-forme sur X, défini par

$$\omega := c_1(L, h^L) := \frac{\sqrt{-1}}{2\pi} R^L, \tag{1.1.1}$$

où R^L est la courbure de la connexion de Chern sur (L, h^L). On définit $\mathring{R}^L \in \mathrm{End}(T^{(1,0)}X)$ par

$$g^{TX}(\mathring{R}^L u, \overline{v}) = R^L(u, \overline{v}), \quad u, v \in T^{(1,0)}X. \tag{1.1.2}$$

On note $T(g^{TX}, h^{L^p \otimes E})$ la torsion analytique, (1.0.3), de $L^p \otimes E$ par rapport à g^{TX}, h^L, h^E. Rappelons brièvement les points essentiels dans la définition formelle de $T(g^{TX}, h^{L^p \otimes E})$.

On note les spectres des opérateurs $\square_k^E$ par λ_{kj}, $k \in 0, \ldots, \dim X$, $j \in \mathbb{N}$. Par la loi de Weyl, la fonction zêta associée

$$\zeta_X^{E,k}(s) := \sum_{j \in \mathbb{N}, \lambda_{kj} \neq 0} \lambda_{kj}^{-s}, \tag{1.1.3}$$

est bien définie et holomorphe pour $s \in \mathbb{C}$, $\mathrm{Re}(s) > \dim X/2$. De plus, l'identité suivante est vraie

$$\zeta_X^{E,i}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\big[\exp^{\perp}(-t\square_i^E)\big] t^s \frac{dt}{t}, \tag{1.1.4}$$

où $\exp^{\perp}(-t\square_i^E)$ est la projection spectrale de l'opérateur de la chaleur $\exp(-t\square_i^E)$ sur l'espace propre correspondant aux valeurs propres non-nulles.

À partir de l'identité (1.1.4) et des propriétés de la trace de l'opérateur $\exp^{\perp}(-t\square_i^E)$, la fonction $\zeta_X^{E,i}(s)$ s'étend méromorphiquement sur $\mathbb{C}$ et holomorphe en 0. D'après Ray-Singer, [103], on définit

$$T(g^{TX}, h^E) := \exp\Big(\sum_{i=0}^{\dim X} i \cdot (-1)^i \cdot (\zeta_X^{E,i})'(0)/2\Big). \tag{1.1.5}$$

Par l'identité (1.1.3), on voit que (1.1.5) correspond formellement à la définition (1.0.3).

Théorème 1.1.1 ([53, Theorem 1.1]). *Il existe des coefficients locaux, c'est-à-dire qui peuvent être exprimés comme intégrales de densités définies localement, $\alpha_i, \beta_i \in \mathbb{R}, i \in \mathbb{N}$ tels que pour chaque $k \in \mathbb{N}$, on a l'asymptotique suivante*

$$-2\log T(g^{TX}, h^{L^p \otimes E}) = \textstyle\sum_{i=0}^{k} p^{n-i}(\alpha_i \log p + \beta_i) + o(p^{n-k}), \tag{1.1.6}$$

lorsque $p \to +\infty$. De plus, les coefficients α_i ne dépendent pas de g^{TX}, h^L, h^E.

Dans [27, Theorem 8], Bismut-Vasserot ont démontré Théorème 1.1.1 pour $k = 0$. Ils ont montré

$$\alpha_0 = \frac{n\,\mathrm{rk}(E)}{2}\int_X \frac{\omega^n}{n!}, \quad \beta_0 = \frac{\mathrm{rk}(E)}{2}\int_X \log\Big(\det\frac{\mathring{R}^L}{2\pi}\Big)\frac{\omega^n}{n!}. \tag{1.1.7}$$

Théorème 1.1.2 ([53, Theorem 1.3]). *Sous l'hypothèse $\Theta = \omega$, on a*

$$\alpha_1 = \frac{(3n+1)\mathrm{rk}(E)}{12}\int_X c_1(TX)\frac{\omega^{n-1}}{(n-1)!} + \frac{n}{2}\int_X c_1(E)\frac{\omega^{n-1}}{(n-1)!}, \tag{1.1.8}$$

$$\beta_1 = \frac{\mathrm{rk}(E)}{24}(24\zeta'(-1) + 2\log(2\pi) + 7)\int_X c_1(TX)\frac{\omega^{n-1}}{(n-1)!} + \frac{1}{2}\int_X c_1(E)\frac{\omega^{n-1}}{(n-1)!}. \tag{1.1.9}$$

Remarque 1.1.3. *La valeur explicite de α_1, β_1 a été conjecturée par Klevtsov-Ma-Marinescu-Wiegmann dans [73, p.839] pour les surfaces de Riemann. On déduit du Théorème 1.1.2 que leur conjecture est vraie pour α_1, mais fausse pour β_1.*

Notre dernier résultat principal du première partie est une généralisation du Théorème 1.1.1 dans un cadre d'orbifold. Plus précisément, soit $(\mathcal{M}, g^{T\mathcal{M}}, \Theta)$ un orbifold hermitien compact et effectif. On note $\Sigma\mathcal{M}$ sa strate. On note $\Sigma\mathcal{M}^{[j]}$, $j \in J$, les composantes connexes de $\Sigma\mathcal{M}$, et n_j ses dimensions respectives. Soit $(\mathcal{E}, h^{\mathcal{E}})$ un fibré vectoriel holomorphe propre et hermitien sur $\mathcal{M}$ et soit $(\mathcal{L}, h^{\mathcal{L}})$ un fibré en droites holomorphe propre et positif sur $\mathcal{M}$. Dans ce cadre, la définition de la torsion analytique $T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}})$ est analogue à la definition dans le cadre des variétés. Pour plus de détails, voir Ma [83].

Théorème 1.1.4 ([53, Theorem 1.5]). *Il existe des coefficients locaux, $\widetilde{\alpha}_i, \widetilde{\beta}_i \in \mathbb{R}$ et $m_j \in \mathbb{N}, \gamma_{j,i}, \kappa_{j,i} \in \mathbb{R}, j \in J, i \in \mathbb{N}$ tels que pour chaque $k \in \mathbb{N}$, on a l'asymptotique suivante,*

$$\begin{aligned} -2\log T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}}) &= \sum_{i=0}^{k} p^{n-i}\big(\widetilde{\alpha}_i\log p + \widetilde{\beta}_i\big) \\ &\quad + \sum_{i=0}^{k+n_j-n}\sum_{j\in J}\frac{p^{n_j-i}}{m_j}e^{\sqrt{-1}\theta_j p}\big(\gamma_{j,i}\log p + \kappa_{j,i}\big) + o(p^{n-k}), \end{aligned} \tag{1.1.10}$$

lorsque $p \to +\infty$. Les valeurs $\theta_j, \gamma_{j,i}, \kappa_{j,i}, m_j$ ne dépendent que de la géométrie locale de lieu singulier de $\mathcal{M}$, et

$$\widetilde{\alpha}_0 = \frac{n\,\mathrm{rk}(\mathcal{E})}{2}\int_{\mathcal{M}}\frac{\omega^n}{n!}, \quad \widetilde{\beta}_0 = \frac{\mathrm{rk}(\mathcal{E})}{2}\int_{\mathcal{M}}\log\Big(\det\frac{\mathring{R}^{\mathcal{L}}}{2\pi}\Big)\frac{\omega^n}{n!}, \tag{1.1.11}$$

où $\tilde{\omega}$, $\mathring{R}^{\mathcal{L}}$ sont des analogues de (1.1.1) et (1.1.2) dans un cadre d'orbifold. De plus, il existe des constantes $c_j \neq 0$ telles que

$$\gamma_{j,0} = \begin{cases} c_j\int_{\Sigma\mathcal{M}^{[j]}}\frac{\tilde{\omega}^{n-1}}{(n-1)!}, & \textit{si } \mathrm{codim}\,\Sigma\mathcal{M}^{[j]} = 1, \\ 0, & \textit{sinon.} \end{cases} \tag{1.1.12}$$

Comme dans le cas des variétés, les constantes $\widetilde{\alpha}_i$, $\gamma_{j,i}$ ne dépendent pas de $g^{T\mathcal{M}}$,$h^{\mathcal{L}}$, $h^{\mathcal{E}}$. Si $\Theta = \omega$, $\widetilde{\alpha}_1, \widetilde{\beta}_1$ sont données par (1.1.8) et (1.1.9), ou l'intégration est faite sur $\mathcal{M}$.

Corollaire 1.1.5. *L'ensemble* $\{T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}}) : p \in \mathbb{N}\}$ *détecte les singularités de codimension 1.*

Théorème 1.1.4 donne le raffinement du théorème principal de l'article [68] par Hsiao-Huang, lorsque l'orbifold $\mathcal{M}$ est obtenu comme le quotient de l'action CR transversallement libre du S^1 sur une variété CR.

Décrivons maintenant l'histroire des problèmes reliées et proposons des directions dans lesquelles nos résultats pourront être utiles. Dans l'article [28], Bismut-Vasserot ont généralisé [27]. Ils ont calculé l'asymptotique de $\log T(g^{TM}, h^{E\otimes \mathrm{Sym}^p\zeta})$, quand $p \to +\infty$, où (ζ, g^ζ) est un fibré hermitien positif au sens de Griffith et (E, h^E) est un fibré vectoriel hermitien. Récemment, Puchol [101] a obtenu une généralisation de ce résultat en famille. On rappelle que dans [19, §3], Bismut a généralisé la définition des formes de torsion pour des fibrations holomorphes (qui ne sont pas nécessairement Kähler). Puchol a obtenu l'asymptotique des formes de torsion de Bismut associé aux puissances croissantes de fibré en droites positif le long de fibres. Il est naturel de penser qu'on peut combiner notre résultat avec [19, §3] et [101] pour obtenir l'asymptotique générale des formes de torsion pour une fibration holomorphe.

1.2 Torsion analytique pour des surfaces aux pointes et l'espaces de modules

Le deuxième partie de la thèse est consacrée à l'introduction et à l'étude de la torsion analytique pour des surfaces aux pointes. Plus précisément, soit $\overline{M}$ une surface de Riemann compacte, et soit $D_M = \{P_1^M, \dots, P_m^M\}$ un ensemble fini de points distincts dans $\overline{M}$. Soit g^{TM} une métrique kählérienne sur la surface de Riemann épointée $M := \overline{M} \setminus D_M$.

Pour $\epsilon \in]0, 1]$, $i = 1, \dots, m$, on fixe $z_i^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon) := \{z \in \mathbb{C} : |z| \leq \epsilon\}$ des coordonnées locales holomorphes, centrées en P_i^M. On note

$$V_i^M(\epsilon) := \{x \in M : |z_i^M(x)| < \epsilon\}. \tag{1.2.1}$$

On dit que g^{TM} est *compatible au sens de Poincaré* avec les coordonnées $z_1^M, \dots, z_m^M$ si pour chaque $i = 1, \dots, m$, il existe $\eta > 0$ telle que $g^{TM}|_{V_i^M(\eta)}$ est induite par la forme kählérienne

$$\frac{\sqrt{-1}dz_i^M d\overline{z}_i^M}{\left|z_i^M \log |z_i^M|\right|^2}. \tag{1.2.2}$$

On dit que g^{TM} est une *métrique cuspidale* si g^{TM} est compatible au sens de Poincaré avec des coordonnées holomorphes près de D_M. Une *surface aux pointes* est un triplet $(\overline{M}, D_M, g^{TM})$ constitué de la surface de Riemann $\overline{M}$, de l'ensemble de points $D_M \subset \overline{M}$ et de la métrique cuspidale g^{TM} sur M (cf. [93]).

Par exemple, si la surface épointée $(\overline{M}, D_M)$ est stable, c'est-à-dire le genre $g(\overline{M})$ de $\overline{M}$ satisfait la condition suivante

$$2g(\overline{M}) - 2 + m > 0, \tag{1.2.3}$$

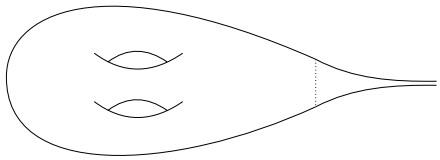

Figure 1.1: Une surface aux pointes.

alors, par le théorème d'uniformisation (cf. [49, Chapter IV], [9, Lemma 6.2]), il existe une unique métrique g^{TM}_{hyp} de courbure scalaire constante -1 sur M, appelé la métrique *canonique hyperbolique*. Encore une fois, par le théorème d'uniformisation il existe des coordonnées locales holomorphes z_i^M centrées en P_i^M, $i = 1, \dots, m$, telles que g^{TM}_{hyp} est induit par la forme kählérienne (1.2.2) au voisinage de D_M. C'est-à-dire que le triplet $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ est une surface aux pointes.

On définit le *fibré en droites canonique tordu* par

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{1.2.4}$$

La métrique g^{TM} induit la norme hermitienne $\|\cdot\|_M$ sur $\omega_M(D)$ par l'isomorphisme canonique $\omega_M(D) \to \omega_{\overline{M}}$ au-dessus de M. On fixe un fibré hermitien (ξ, h^ξ) sur $\overline{M}$. Le premier but dans cette partie de la thèse est de définir $T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ comme l'analogue cuspidale de (1.0.3). Pour cela on exprime la torsion analytique à l'aide de la noyau de la chaleur. On régularise la définition de la trace de l'opérateur de la chaleur en soustrayant au voisinage de la pointe une contribution universelle calculée sur $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ et en prenant la limite de la quantité obtenue quand on s'approche de la pointe.

On s'intéresse plutôt ici à la métrique de Quillen qui contient dans son définition la torsion analytique. Plus précisément, pour $n \leq 0$, on définit le produit scalaire suivant

$$\langle \alpha, \alpha' \rangle_{L^2} := \frac{1}{2\pi} \int_M \langle \alpha(x), \alpha'(x) \rangle_h dv_M(x), \qquad \alpha, \alpha' \in \Omega^{(0,i)}(\overline{M}, \xi \otimes \omega_M(D)^n). \tag{1.2.5}$$

Par analogie avec la la théorie de Hodge dans le cas compact, on peut plonger les espaces vectoriels $H^i(\overline{M}, \xi \otimes \omega_M(D)^n)$, $i = 0, 1$, dans les espaces $\Omega^{(0,i)}(\overline{M}, \xi \otimes \omega_M(D)^n)$, $i = 0, 1$. Cette injection, avec le produit scalaire (1.2.5), induisent une norme hermitienne sur la droite complexe

$$\lambda(\xi \otimes \omega_M(D)^n) := \left(\Lambda^{\max} H^0(\overline{M}, \xi \otimes \omega_M(D)^n)\right)^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, \xi \otimes \omega_M(D)^n), \tag{1.2.6}$$

qu'on note $\|\cdot\|_{L^2}(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$. La métrique de Quillen sur la ligne complexe (1.2.6) est définie par l'identité suivante

$$\|\cdot\|_Q (g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) = T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) \cdot \|\cdot\|_{L^2} (g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}). \tag{1.2.7}$$

Le but de cette partie de la thèse est d'étendre la théorie de la métrique de Quillen pour des surfaces aux pointes.

Tout d'abord, expliquons pourquoi on formule toujours nos résultats en termes de la métrique de Quillen (1.2.7) et pas en termes de la torsion analytique. Considérons une famille de surfaces

de Riemann compactes. Comme la dimension des espaces de cohomologie varient selon la fibre, la version en famille de la métrique L^2 n'est pas forcement continue. La beauté et la magie de la métrique de Quillen réside dans le fait que le produit de la torsion analytique avec la métrique L^2 n'est pas seulement continue, mais lisse par un résultat de Bismut-Gillet-Soulé, [23]. Il y a de plus une formule explicite pour la courbure de la connexion de Chern associée et elle correspond a la forme différentielle représentant le côté droit de la formule de Riemann-Roch-Grothendieck. La métrique de Quillen est donc mieux adaptée à la situation en famille.

Cette partie de la thèse est décomposée en trois sujets liés.

Dans le premier sujet, on donne la définition de la torsion analytique pour une surface aux pointes. Ceci n'est pas une tâche triviale, car le spectre du laplacien de Kodaira n'est plus discret, et la définition (1.0.3) n'a donc aucun sens. Comme on a dit avant, pour cela on exprime la torsion analytique à l'aide de la noyau de la chaleur. On régularise la définition de la trace de l'opérateur de la chaleur en soustrayant au voisinage de la pointe une contribution universelle calculée sur $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ et en prenant la limite de la quantité obtenue quand on s'approche de la pointe.

On prouve deux résultats qui permettent de calculer cette torsion analytique. Le premier résultat, qu'on appelle aussi *le théorème de la perturbation compacte*, exprime le quotient de deux métriques de Quillen associées aux surfaces avec le même nombre de pointes par le quotient de deux métriques de Quillen associées aux surfaces compactes construites comme des "aplanissements" des métriques cuspidales. Le deuxième résultat, qu'on appelle *la formule d'anomalie*, exprime en termes des classes de Bott-Chern le changement de la métrique de Quillen induit par le changement de la métrique hermitienne h^{ξ} et par le changement conforme de la métrique g^{TM}.

Dans le deuxième sujet on considère une famille de surfaces de Riemann aux pointes hyperboliques. On muni le fibré determinant de la métrique de Quillen et on démonte le théorème de la courbure associé. Donc on donne un raffinement de la théorème de Riemann-Roch-Grothendieck au niveau des courants avec une contribution explicite de courbes singulières. Ce théorème généralise le théorème de courbure de Bismut-Bost [20] (qui généralise le théorème de courbure de Bismut-Gillet-Soulé unidimensionnel, [21], [22], [23] en permettant des fibres singulières) et le théorème de courbure de Takhtajan-Zograf [107]. Pour démontrer que la première forme de Chern du fibré determinant, muni de la métrique de Quillen, est bien définie en tant que courant, on étudie le comportement de la métrique de Quillen renormalisée près de diviseur de courbes singulières. On donne aussi des conditions suffisantes pour que cette métrique s'étende continument sur le diviseur singulier, qui motive notre troisième sujet.

Comme consequence facile de notre théorie, on obtient que la métrique de Weil-Petersson admet un potentiel continu. Ce résultat a été déjà prouvé par Wolpert [112] dans sa preuve analytique de la projectivité de $\overline{\mathscr{M}}_{g,m}$.

Comme autre conséquence facile, on obtient aussi que les volumes de Weil-Petersson des espaces de modules de courbes épointées sont des multiples rationnels de puissances de π. À l'origine, ce théorème est aussi du à Wolpert [113].

Dans notre démonstration, ce résultat est une conséquence du fait que la forme de volume est égale, modulo multiplication par une puissance de π, à une puissance maximale de la forme de Chern d'un fibré hermitien muni d'une métrique qui est "good" au sens de Mumford. En revanche, Wolpert a utilisé la formule donnant ω_{WP} en termes de coordonnées de Fenchel-Nielsen

et il a explicitement calculé l'intersection de ω_{WP} avec une famille de 2-cycles analytiques qui engendrent $H^{6g-8}(\overline{\mathscr{M}}_{g,0},\mathbb{R})$. Sa preuve est donc très différente de la notre.

En troisième sujet on considère une famille de surfaces de Riemann aux pointes hyperboliques. Il s'agit d'étudier le comportement de la métrique de Quillen lorsque des pointes sont crées par dégénérescence. Plus précisément, par le résultat de deuxième partie on sait que la métrique de Quillen se prolonge par continuité sur le lieu singulier. On démontre que restriction de la métrique sur le lieu singulier coïncide, à une constante explicite près, avec la métrique de Quillen sur la normalisée. Ce sujet est relié a plusieurs travaux, voir par exemple Bismut [18], Freixas [59], [60], Wolpert [115].

On démontre aussi que dans le cas special de la métrique hyperbolique de courbure scalaire constante -1, notre définition de la torsion analytique coïncide avec la définition de la torsion analytique de Takhtajan-Zograf, [107, (6)], définie à l'aide de l'ensemble de longueurs de géodésiques fermées. Ce résultat généralise le théorème de Phong-D'Hoker [44, (7.30)], [45, (3.6)] pour des surfaces non-compactes.

Si on applique le théorème principal du troisième partie de la thèse à des espaces de modules de courbes épointées, on obtient la compatibilité entre la métrique de Quillen et les morphismes de "clutching", construits par Knudsen [74], [75].

Ci-dessous on va donner l'introduction plus détaillée à chacun de ces sujets.

1.2.1 Théorème de la perturbation compacte relative et la formule d'anomalie

Le premier but dans cette partie de la thèse est d'étendre la définition (1.0.3) de la torsion analytique et de l'étudier. Puis on exprime la métrique de Quillen d'une surface aux pointes en fonction d'une métrique de Quillen d'une surface compacte et d'une partie purement locale. Ensuite on utilise ce résultat pour obtenir une généralisation de formule de Polyakov. Les pointes apparaissent explicitement dans notre formule sous forme de masses de Dirac.

Pour expliquer le façon dont on étend la définition de la torsion analytique, rappelons les points essentielles dans la définition de (1.0.3). On conserve la notation de la Section 1.2.

On suppose tout d'abord $m = 0$, c'est-à-dire g^{TM} n'a pas de points hyperboliques. Dans ce cas le différentiel de Dolbeaut donne bijection entre les spectres non nulles de $\square^{\xi\otimes\omega_M(D)^n} := \square_0^{\xi\otimes\omega_M(D)^n}$ et de $\square_1^{\xi\otimes\omega_M(D)^n}$. On peut donc interpréter la torsion analytique comme

$$T(g^{TM}, h^{\xi}\otimes\|\cdot\|_M^{2n}) = \big(\det{}'\big(\square^{\xi\otimes\omega_M(D)^n}\big)\big)^{1/2}. \tag{1.2.8}$$

Plus formellement, soit $\zeta_M(s)$, $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, est une fonction zêta de $\square^{\xi\otimes\omega_M(D)^n}$. C'est-à-dire $\zeta_M(s)$ s'exprime pour $\mathrm{Re}(s)>1$ comme une somme infini de puissances $-s$ des valeurs propres non nulles de $\square^{\xi\otimes\omega_M(D)^n}$. C'est facile a voir que la fonction zeta s'exprime en fonction de la projection spectrale $\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})$ de l'opérateur de la chaleur $\exp(-t\square^{\xi\otimes\omega_M(D)^n})$ sur l'espace propre correspondant aux valeurs propres non-nulles par la formule suivante

$$\zeta_M(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\mathrm{Tr}\big[\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})\big]t^s\frac{dt}{t}. \tag{1.2.9}$$

D'après Ray-Singer, [103], on définit la torsion analytique pour des surfaces compactes par

$$T(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta'_M(0)/2). \tag{1.2.10}$$

Maintenant si $m > 0$, c'est-à-dire la metrique g^{TM} admet des points hyperboloques, le spectre de l'opérateur $\square^{\xi\otimes\omega_M(D)^n}$ n'est plus discret (voir Müller [93]), donc l'interprétation (1.0.3) n'a pas de sens. De plus, comme l'opérateur $\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})$ est majoré par une projection spectrale de dimension infini, il n'a pas de trace, et la formule (1.2.9) n'a pas de sens non plus... L'idée de notre approche est de régulariser la trace de l'opérateur $\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})$ et de l'utiliser en place de la trace usuel dans (1.2.9) pour donner la *définition* de la fonction zêta. Puis on définit la torsion analytique par la même formule (1.2.10).

Pour régulariser la trace de l'opérateur $\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})$, au voisinage de la pointe, on soustrait de la noyau de la chaleur une contribution universelle calculée sur $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ et on prends la limite de la quantité obtenue quand on s'approche de la pointe.

Autrement dit, on définit la trace régularisée $\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})\big]$ comme "une partie fini" de l'intégrale du noyau de la chaleur.

Puis on démontre que cette trace régularisée a des propriétés similaires aux propriétés de la trace de l'opérateur de la chaleur associé à la variété compacte. Pour étudier le comportement du ce trace quand $t \to \infty$, il nous faut un résultat qui garantissent l'existence d'un trou spectral. Pour $n = 0$, on utilise le résultat de Müller, et pour $n < 0$ on utilise l'inégalité de Nakano. Pour contrôler le comportement du ce trace quand $t \to 0$, on compare des noyaux de la chaleur de deux surfaces aux pointes près de pointes. Pour éliminer des effets de (ξ, h^{ξ}), qui n'est pas forcement triviale au voisinage de pointes, on utilise la construction de parametrix du noyau de la chaleur et puis on utilise des techniques de localisation de Bismut-Lebeau [25, §11].

Donc pour $m \in \mathbb{N}$, on peut définir la fonction zêta régularisée par l'identité suivante

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{\xi\otimes\omega_M(D)^n})\big] t^s \frac{dt}{t}. \tag{1.2.11}$$

Comme dans le cas compact, la fonction $\zeta_M(s)$ admet une extension méromorphe en $\mathbb{C}$ et holomorphe en 0. On peut donc définir la torsion analytique par la même formule (1.2.10)[1] en utilisant la définition de la fonction zêta de (1.2.11).

Notre définition est similaire à la définition de Jorgenson-Lundelius de la torsion analytique relative, qui est donné pour (ξ, h^{ξ}) trivial et $n = 0$, voir [81], [70], [71]. Les techniques qu'on utilise sont différentes des ceux de Jorgenson-Lundelius, et l'apparence de (ξ, h^{ξ}), qui n'est pas forcément triviale autour de pointes, rends la problème beaucoup plus difficile.

La question naturelle à se poser c'est *comment peut-on calculer la torsion analytique définit comme ci-dessus*? Avant d'expliquer le premier théorème dans cette direction, on va introduire quelques notions.

[1]En fait, on renormalisé cette définition en multipliant par une constante, qui dépend de la géométrie du $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$, muni de la métrique hyperbolique à courbure scalaire constante -1, voir [54, Definition 2.17]. On le fait pour que notre définition soit compatible avec la définition de Takhtajan-Zograf.

Définition 1.2.1 (Un aplanissement de la métrique)**.** *Soit* $(\overline{M}, D_M, g^{TM})$ *une surface aux pointes. On dit que la métrique* g_{f}^{TM} *sur* $\overline{M}$ *est un aplanissement de* g^{TM} *s'il existe* $\nu > 0$ *telle que* g^{TM} *est induite par (1.2.2) sur* $V_i^M(\nu)$*, et*

$$g_{\mathrm{f}}^{TM}|_{M\setminus(\cup_i V_i^M(\nu))} = g^{TM}|_{M\setminus(\cup_i V_i^M(\nu))}. \tag{1.2.12}$$

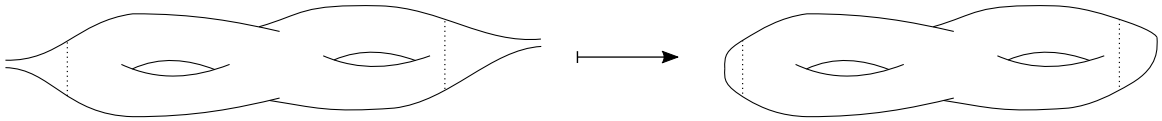

Figure 1.2: Un exemple de l'aplanissement.

On fixe un *aplanissement* g_{f}^{TN} de g^{TN}. On dit que des aplanissements g_{f}^{TM}, g_{f}^{TN} sont *compatibles*, s'il existe $\nu > 0$, satisfaisant (1.2.12) et

$$g_{\mathrm{f}}^{TN}|_{N\setminus(\cup_i V_i^N(\nu))} = g^{TN}|_{N\setminus(\cup_i V_i^N(\nu))}. \tag{1.2.13}$$

telle que pour chaque $i = 1, \ldots, m$, on a

$$((z_i^N)^{-1} \circ z_i^M)^*(g_{\mathrm{f}}^{TM}|_{V_i^M(\nu)}) = g_{\mathrm{f}}^{TN}|_{V_i^N(\nu)}. \tag{1.2.14}$$

Ici z_i^M, z_i^N sont des coordonnées compatibles au sens de Poincaré avec g^{TM}, g^{TN} respectivement.

Dans une manière similaire, on définit les notions *d'aplanissement* pour la norme hermitienne $\|\cdot\|_M$ sur $\omega_M(D)$ induit par g^{TM}. On définit aussi la notion des *aplanissements compatibles* $\|\cdot\|_M^{\mathrm{f}}$,$\|\cdot\|_N^{\mathrm{f}}$ pour les normes hermitiennes $\|\cdot\|_M$,$\|\cdot\|_N$ sur $\omega_M(D)$ et $\omega_N(D)$ respectivement.

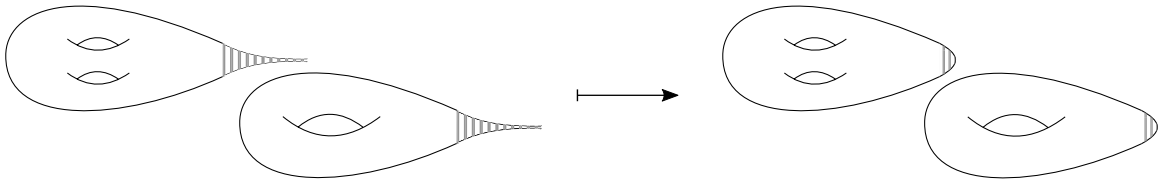

Figure 1.3: Un exemple des aplanissements compatibles. Les régions rayées sont isomorphes.

Théorème 1.2.1 (Théorème de la perturbation compacte relative, [54, Theorem A])**.** *Soit* $(\overline{M}, D_M, g^{TM})$*,* $(\overline{N}, D_N, g^{TN})$ *deux surfaces avec le même nombre de pointes. Soit* (ξ, h^ξ) *un fibré hermitienne au-dessus de* $\overline{M}$ *de rang* $\mathrm{rk}(\xi)$*. On note par* $\|\cdot\|_M$*,* $\|\cdot\|_N$ *les normes induites par* g^{TM}*,* g^{TN} *sur* $\omega_M(D)$ *et* $\omega_N(D)$ *au-dessus de* M *et* N *respectivement. Soit* $g_{\mathrm{f}}^{TM}, g_{\mathrm{f}}^{TN}$,$\|\cdot\|_M^{\mathrm{f}}$,$\|\cdot\|_N^{\mathrm{f}}$ *des aplanissements compatibles de* g^{TM}, g^{TN},$\|\cdot\|_M\|\cdot\|_N$ *respectivement. Pour chaque* $n \in \mathbb{Z}, n \leq 0$*, on a l'identité suivant*

$$2\log\left(\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)/\|\cdot\|_Q\left(g_{\mathrm{f}}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\right)\right)$$

$$
\begin{aligned}
&- 2\mathrm{rk}(\xi)\log\Big(\|\cdot\|_Q\left(g^{TN},\|\cdot\|_N^{2n}\right)/\|\cdot\|_Q\left(g_{\mathrm{f}}^{TN},(\|\cdot\|_N^{\mathrm{f}})^{2n}\right)\Big)\\
&\qquad = \int_M c_1(\xi,h^\xi)\Big(2n\log(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)+\log(g_{\mathrm{f}}^{TM}/g^{TM})\Big). \quad (1.2.15)
\end{aligned}
$$

En d'autres termes, la norme de Quillen relative peut être calculée par une perturbation compacte.

Remarque 1.2.2. *Pour $n = 0$ et (ξ, h^ξ) trivial, Théorème 1.2.1 a été prouvé par Jorgenson-Lundelius dans [71, Theorem 7.3] et Albin-Aldana-Rochon dans [3, Theorem 5.2], où les auteurs utilisent substantiellement que la géométrie de (M, g^{TM}) et (N, g^{TN}) coïncide près de pointes. Ceci n'est plus vrai dans notre cas en vue de présence de (ξ, h^ξ). Les techniques qu'on utilise sont très différents mémé si (ξ, h^ξ) est trivial et $n = 0$.*

On va décrire le deuxième résultat qui est une généralisation des formule de Polyakov et qui exprime en termes des classes de Bott-Chern le changement de la métrique de Quillen induit par le changement de g^{TM} et h^ξ.

On rappelle que par [21, Theorem 1.27], des classes de Bott-Chern de fibré vectoriel ξ avec des métriques hermitiennes h_1^ξ, h_2^ξ sont des formes différentielles (en réalité, les classes des formes différentielles, mais cette distinction ne serai pas important ici), qui satisfont des identités suivantes

$$
\begin{aligned}
\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{Td}}(\xi,h_1^\xi,h_2^\xi) &= \mathrm{Td}(\xi,h_1^\xi)-\mathrm{Td}(\xi,h_2^\xi),\\
\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{ch}}(\xi,h_1^\xi,h_2^\xi) &= \mathrm{ch}(\xi,h_1^\xi)-\mathrm{ch}(\xi,h_2^\xi),
\end{aligned}
\quad (1.2.16)
$$

où Td, ch sont des formes de Todd et Chern. Par [21, Theorem 1.27], on a des identités suivantes

$$
\widetilde{\mathrm{ch}}(\xi,h_1^\xi,h_2^\xi)^{[0]} = 2\widetilde{\mathrm{Td}}(\xi,h_1^\xi,h_2^\xi)^{[0]} = \log\big(\det(h_1^\xi/h_2^\xi)\big). \quad (1.2.17)
$$

Si de plus, $\xi := L$ est un fibré en droites, on a l'identité suivante

$$
\widetilde{\mathrm{ch}}(L,h_1^L,h_2^L)^{[2]} = 6\widetilde{\mathrm{Td}}(L,h_1^L,h_2^L)^{[2]} = \log(h_1^L/h_2^L)\Big(c_1(L,h_1^L)+c_1(L,h_2^L)\Big)/2, \quad (1.2.18)
$$

où $c_1(L, h_i^L)$ est la première forme de Chern.

Définition 1.2.3. *Pour une surface aux pointes $(\overline{M}, D_M, g^{TM})$, la norme $\|\cdot\|_i^W$ sur les droites complexes $\omega_{\overline{M}}|_{P_i^M}$, $i = 1, \ldots, m$, est définie par $\|dz_i^M\|_i^W = 1$. Ces normes induisent la norme de Wolpert $\|\cdot\|^W$ sur la ligne complexe $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i^M}$.*

Remarque 1.2.4. *Puisque les coordonnées compatibles au sens de Poincaré sont définies de manière unique, jusqu'à une multiplication par un constante unimodulaire, les normes $\|\cdot\|_i^W$ sont bien définis. À l'origine les normes $\|\cdot\|_i^W$ ont été définis par Wolpert dans [117, Definition 1] pour des surfaces hyperboliques à courbure scalaire constante -1.*

Théorème 1.2.2 (La formule d'anomalie pour des surfaces aux pointes, [54, Theorem B])**.** *Soit g^{TM}, g_0^{TM} deux métriques sur M telles qye des triplets $(\overline{M}, D_M, g^{TM})$, $(\overline{M}, D_M, g_0^{TM})$ sont des surfaces aux pointes. On note par $\|\cdot\|_M$,$\|\cdot\|_M^0$ les normes induites par g^{TM}, g_0^{TM} on $\omega_M(D)$, et par $\|\cdot\|^W$, $\|\cdot\|_0^W$ les normes de Wolpert associées. Soit h^ξ, h_0^ξ deux métriques hermitiennes sur ξ au-dessus de $\overline{M}$. Alors le côté droit de l'équation suivante est fini, et*

$$\begin{aligned}
&2\log\Big(\|\cdot\|_Q\left(g_0^{TM}, h_0^\xi\otimes(\|\cdot\|_M^0)^{2n}\right)/\|\cdot\|_Q\left(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}\right)\Big)\\
&\quad=\int_M\Big[\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1},\|\cdot\|_M^{-2},(\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi,h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n,\|\cdot\|_M^{2n}\big)\\
&\qquad+\mathrm{Td}\big(\omega_M(D)^{-1},(\|\cdot\|_M^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi,h^\xi,h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n,\|\cdot\|_M^{2n}\big)\\
&\qquad+\mathrm{Td}\big(\omega_M(D)^{-1},(\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi,h_0^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n,\|\cdot\|_M^{2n},(\|\cdot\|_M^0)^{2n}\big)\Big]^{[2]}\\
&\quad-\frac{\mathrm{rk}(\xi)}{6}\log\left(\|\cdot\|^W/\|\cdot\|_0^W\right)+\frac{1}{2}\sum\log\Big(\det(h^\xi/h_0^\xi)|_{P_i^M}\Big).
\end{aligned}\tag{1.2.19}$$

Remarque 1.2.5. *a) La formule d'anomalie a été prouvé par Polyakov dans [100] pour $m=0$, $n=0$ et (ξ,h^ξ) trivial. Elle a été généralisé par Bismut-Gillet-Soulé [23, Theorem 1.23] pour $m=0$ en dimension quelconque. Pour $m=0$, Fay dans [50], a donné une démonstration alternative de (1.2.19), qui n'utilise pas le noyau de la chaleur. Notre preuve est basé sur Théorème 1.2.1 et la formule d'anomalie pour $m=0$.*

b) On définit la fonction $\phi: M\to\mathbb{R}$ par l'identité $g^{TM}=e^{2\phi}g_0^{TM}$. Si ϕ est de support compact dans M, Théorème 1.2.2 est une conséquence directe de la formule d'anomalie de Bismut-Gillet-Soulé et du Théorème 1.2.1. La différence entre Théorème 1.2.2 et le théorème de Bismut-Gillet-Soulé est dans les deux dernières termes de (1.2.19):

$$-\frac{\mathrm{rk}(\xi)}{6}\log\left(\|\cdot\|^W/\|\cdot\|_0^W\right)+\frac{1}{2}\sum\log\Big(\det(h^\xi/h_0^\xi)|_{P_i^M}\Big).\tag{1.2.20}$$

Pour $n=0$ et (ξ,h^ξ) trivial, Albin-Aldana-Rochon dans [2, Theorem 2.9] en améliorant un résultat de la thèse de Aldana [6, Theorem 4.5] ont obtenue la version de Théorème 1.2.2. Là les auteurs ne supposent pas que ϕ est à support compact, mais ils supposent que le comportement de ϕ près de D_M est comme $(\log|z|)^{-2}$, ou z est une coordonne holomorphe centré en une point de D_M (voir [2, (2.11)]). Les transformations conformes avec ϕ, satisfaisant ce type des hypothèses n'alterne pas la métrique de Wolpert, donc les termes (1.2.20) n'apparaissent pas dans ses formules, voir [2, Theorem 2.9], [6, Theorem 4.5].

Dans tous nos applications, on utilise substantivement que par la formule d'anomalie, on peut trivialiser les coordonnées compatibles au sens de Poincaré au voisinage de pointes. En particulier, on change la métrique de Wolpert par ces trivialisations. L'apparence de termes (1.2.20) est donc d'importance capitale.

c) Si (ξ,h^ξ) est trivial et $n=0$, un théorème similaire a apparu dans l'article de Lundelius [81, Theorem 1.1]. Pourtant, on n'est pas d'accord avec son résultat, comme il ne contient pas de termes (1.2.20).

En fait, on peut combiner les deux résultats principales dans un seul théorème. Ce théorème décrit une relation explicite entre la métrique de Quillen associée à une métrique en pointe et la métrique de Quillen associée à une métrique sur la surface de Riemann compactifié.

Pour le préciser, on va définir *l'intégral régularisé* sur une surface aux pointes. Soit $(\overline{M}, D_M, g^{TM})$ est une surface avec des pointes. Soit $\alpha \in \mathscr{C}^\infty(M, \wedge^2 T\overline{M})$. On suppose que pour tout $P_i \in D_M$, il y a des coordonnées holomorphes z_i autour de $P_i \in D_M$, tel que pour certains $\epsilon > 0$ assez petit, il y a $C \in \mathbb{C}$, $l \in \mathbb{N}$ telle que

$$\alpha|_{\{|z_i|<\epsilon\}} = \frac{C \cdot dz_i d\overline{z}_i}{|z_i|^2 |\log |z_i||} + O\Big(\frac{\log|\log|z_i||^l dz_i d\overline{z}_i}{|z_i \log|z_i||^2}\Big). \tag{1.2.21}$$

On définit $\int_M^{\mathbf{r}} \alpha \in \mathbb{C}$ par la limite suivante

$$\int_M^{\mathbf{r}} \alpha = \lim_{\epsilon \to 0} \Big(\int_{M \setminus (\cup\{|z_i|<\epsilon\})} \alpha - 4C\pi \cdot \log|\log \epsilon| \Big). \tag{1.2.22}$$

En d'autres termes, $\int_M^{\mathbf{r}} \alpha$ est la partie finie de $\int_{M\setminus(\cup\{|z_i|<\epsilon\})} \alpha$ pour $\epsilon \to 0$.

Théorème 1.2.6 (Théorème de perturbation compacte)**.** *Pour n'importe lequel $n \in \mathbb{Z}, n \leq 0$, il existe E_{-n}. Soit $(\overline{M}, D_M, g^{TM})$ est une surface aux pointes. On note par $\|\cdot\|_M$ la métrique induite sur $\omega_M(D)$ au-dessus de M comme dans la Construction 1.2.8. On note par $\|\cdot\|^W$ la métrique de Wolpert $\otimes_{P\in D_M} \omega_{\overline{M}}|_P$ induit par g^{TM}.*

Soit $g^{T\overline{M}}$ une métrique de Kähler au-dessus de $\overline{M}$, et soit $\|\cdot\|_{\overline{M}}$ une métrique hermitienne sur $\omega_M(D)$ au-dessus de $\overline{M}$. On note par $\|\cdot\|_{\overline{M}}^{D_M}$ la métrique au-dessus $\otimes_{P\in D_M} \omega_{\overline{M}}|_P$ induit par $g^{T\overline{M}}$. Soit ξ un fibré vectoriel holomorphe au-dessus de $\overline{M}$, et soit h^ξ et h_0^ξ sont deux métriques hermitiennes sur ξ au-dessus de $\overline{M}$.

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)/\|\cdot\|_Q\left(g^{T\overline{M}}, h_0^\xi \otimes \|\cdot\|_{\overline{M}}^{2n}\right)\Big) \\
&= \int_M^{\mathbf{r}} \Big[\widetilde{\mathrm{Td}}\big(\omega_{\overline{M}}^{-1}, g^{T\overline{M}}, g^{TM}\big)\mathrm{ch}\big(\xi, h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big) \\
&\qquad + \mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_0^\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}\big) \\
&\qquad + \mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\mathrm{ch}\big(\xi, h^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, \|\cdot\|_{\overline{M}}^{2n}, \|\cdot\|_M^{2n}\big)\Big]^{[2]} \\
&+ \frac{\mathrm{rk}(\xi)}{6}\ln\Big(\|\cdot\|^W/\|\cdot\|_{\overline{M}}^{D_M}\Big) - \frac{1}{2}\sum_{P\in D_M}\ln\Big(\det(h^\xi/h_0^\xi)|_P\Big) \\
&+ \Big(\#(D_M)\cdot \mathrm{rk}(\xi)\cdot E_{-n}\Big).
\end{aligned} \tag{1.2.23}$$

Remarque 1.2.7. *On pourra même calcluler E_{-n} explicitement. Ce calcul est très lié à la preuve de la théorème principale de la troisième partie de cette thèse et elle est une objet d'une travaille en cours.*

1.2.2 Théorème de courbure et théorème de la régularité

Dans cette partie de la thèse on va appliquer les résultats de la section précédente à l'étude de la métrique de Quillen sur des familles dégénérescentes de courbes épointées.

On fixe une application holomorphe, propre, surjective $\pi : X \to S$ entre deux variétés complexes X, S, telle que pour chaque $t \in S$, l'espace $X_t := \pi^{-1}(t)$ est une courbe complexe qui a au pire des singularités ordinaires, c'est-à-dire des singularités de type $\{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$. Dans la terminologie de [20], $\pi : X \to S$ est une f.s.o. - famille holomorphe de surfaces de Riemann à singularités ordinaires. On note $\Sigma_{X/S} \subset X$ le sous-variété de points singulières dans les fibres de π. On note $\Delta = \pi_*(\Sigma_{X/S})$ le diviseur de fibres singulières. Soit $D_{X/S} \subset X$ un diviseur induit par une sous-variété $|D_{X/S}|$ qui intersecte $\pi^{-1}(|\Delta|)$ transversalement et telle que $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ est localement un isomorphisme. En d'autres termes, on suppose que pour tout $s \in S$, il y a un voisinage U de s, et des sections holomorphes disjointes $\sigma_1, \dots, \sigma_m : U \to X$ de π, qui ne passent pas par des points singuliers et tels que l'identité suivante est valable

$$D_{X/S}|_{\pi^{-1}(U)} := \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m). \tag{1.2.24}$$

Ces sections modéliseraient les positions des pointes dans notre famille.

On fixe la norme $\|\cdot\|^{\omega}_{X/S}$ sur le fibré relatif canonique $\omega_{X/S}$ au-dessus de $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$. On suppose que la restriction de cette norme sur les fibres non-singulières $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ de π induit la métrique de Kähler g^{TX_t} sur $X_t \setminus \{\sigma_1(t), \dots, \sigma_m(t)\}$ pour laquelle le triplet $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ est une surface aux pointes.

Définition 1.2.8. *Pour une variété complexe Y et un diviseur $D_0 \subset Y$, on note $\|\cdot\|^{\mathrm{div}}_{D_0}$ la norme canonique singulière sur $\mathscr{O}_Y(D_0)$, définie par l'identité suivante*

$$\|s_{D_0}\|^{\mathrm{div}}_{D_0}(x) = 1, \tag{1.2.25}$$

où s_{D_0}, $\mathrm{div}(s_{D_0}) = D_0$, est une section canonique de diviseur D_0, et $x \in Y \setminus |D_0|$.

On munit le fibré en droites canonique tordu

$$\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S}) \tag{1.2.26}$$

de la norme $\|\cdot\|_{X/S}$ sur $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, induite par $\|\cdot\|^{\omega}_{X/S}$ et $\|\cdot\|^{\mathrm{div}}_{D_{X/S}}$.

Soit ξ un fibré vectoriel holomorphe au-dessus de X et h^{ξ} une métrique hermitien sur ξ au-dessus de $X \setminus \pi^{-1}(|\Delta|)$. On considère les droites complexes $\lambda\big(\xi|_{X_t} \otimes \omega_{X/S}(D)^n|_{X_t}\big)$, $t \in S \setminus |\Delta|$ définies comme dans (1.2.6). La construction de Grothendick-Knudsen-Mumford [76] (cf. aussi [23, §3]) munit ces droites complexes avec la structure du fibré en droites holomoprhe sur S, qu'on note $\lambda\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big) := \big(\det R^{\bullet}\pi_*(\xi \otimes \omega_{X/S}(D)^n)\big)^{-1}$.

Les normes de Quillen (1.2.7) (resp. de Wolpert, voir Définition 1.2.3), définit point par point, induisent la norme de Quillen sur le fibré en droites $\|\cdot\|_Q\,(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S})$ sur $\lambda\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big)$ (resp. de Wolpert $\|\cdot\|^{W}_{X/S}$ sur $\det(\pi_*(\omega_{X/S}|_{|D_{X/S}|})))$ au-dessus de $S \setminus |\Delta|$. On note

$\det\xi := \Lambda^{\max}\xi$ le fibré en droites sur X et par $h^{\det\xi}$ la métrique induite par h^{ξ} sur $\det\xi$. On considère la norme

$$\|\cdot\|_{\mathscr{L}_n} := \left(\|\cdot\|_Q\left(g^{TX_t}, h^{\xi}\otimes\|\cdot\|^{2n}_{X/S}\right)\right)^{12}\otimes\left(\|\cdot\|^{W}_{X/S}\right)^{-\mathrm{rk}(\xi)} \otimes\left(\|\cdot\|^{\mathrm{div}}_{\Delta}\right)^{\mathrm{rk}(\xi)}\otimes\left(\det(\pi_*(h^{\det\xi}|_{|D_{X/S}|})\right)^3 \quad (1.2.27)$$

sur le fibré en droites

$$\mathscr{L}_n := \det\left(R^{\bullet}\pi_*(\xi\otimes\omega_{X/S}(D)^n)\right)^{-12}\otimes\left(\det(\pi_*(\omega_{X/S}|_{|D_{X/S}|})\right)^{-\mathrm{rk}(\xi)} \otimes\mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)}\otimes\left(\det(\pi_*(\det\xi|_{|D_{X/S}|})\right)^6. \quad (1.2.28)$$

Notre première but dans cette partie de la thèse est d'étudier la régularité de $\|\cdot\|_{\mathscr{L}_n}$ sur $S\setminus|\Delta|$ et ses singularités près de Δ. On va démontrer que les singularités de $\|\cdot\|_{\mathscr{L}_n}$ sont suffisamment raisonnables pour qu'on puisse définir la forme de Chern de $(\mathscr{L}_n, (\|\cdot\|_{\mathscr{L}_n})^2)$ comme un courant sur S. On calcule ce courant explicitement, et ça nous donne le raffinement du théorème de Riemann-Roch-Grothendieck au niveau des courants.

Pour préciser les hypothèses qu'on mets sur les données, on va utiliser deux types de singularités des métriques sur le fibrés vectoriells le long d'un diviseur a croisements normaux: la condition "*good*" au sens de Mumford [95] et la condition plus faible *pre-log-log* de Burgos Gil-Kramer-Kühn [36]. Sans rentrer dans les détails, contentons-nous de dire qu'une métrique "*good*" ou *pre-log-log* est une métrique lisse au complément du diviseur à croisements normaux, dont le comportement au voisinage de celui-ci est singulier et de type logarithmique en valeur absolue d'une équation locale du diviseur. Citons les hypothèses, qu'on considère dans cette thèse.

Hypothèse S1. La métrique hermitienne h^{ξ} est lisse sur X; la norme hermitienne $\|\cdot\|_{X/S}$ est lisse sur $X\setminus|D_{X/S}|$ et pre-log-log d'ordre infini avec des singularités le long de $D_{X/S}$.

Hypothèse S2. Le diviseur Δ a des croisements normaux. La métrique hermitienne h^{ξ} est pre-log-log avec des singularités le long de $\pi^{-1}(\Delta)$; la norme hermitienne $\|\cdot\|_{X/S}$ est pre-log-log avec des singularités le long de $\pi^{-1}(\Delta)\cup D_{X/S}$.

Hypothèse S3. Le diviseur Δ a des croisements normaux. La métrique hermitienne h^{ξ} est lisse sur X; la norme hermitienne $\|\cdot\|_{X/S}$ est continue sur $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$, et son comportement au voisinage de $\Sigma_{X/S}\cup|D_{X/S}|$ est singulier de type double logarithme en valeur absolue du maximum d'équations locales, qui engendrent $\Sigma_{X/S}\cup|D_{X/S}|$. On suppose que $\|\cdot\|_{X/S}$ est "good" sur $X\setminus|D_{X/S}|$ avec des singularités le long $\pi^{-1}(\Delta)$, et l'accouplement de $c_1(\omega_{X/S}(D),\|\cdot\|^2_{X/S})$ avec des champs de vecteurs lisses verticaux sur $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$ est continue sur $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$ et singulier de type double logarithme en valeur absolue d'une équation locale, qui engendre $|D_{X/S}|$.

Remarque 1.2.9. *Si Δ a des croisements normaux, alors **S1** implique **S2** et **S3**.*

Disons quelques mots de motivation à propos de ces hypothèses. Hypothèse **S1** est une généralisation des hypothèses de Bismut-Bost [20] pour $m\neq 0$. Les hypothèses **S2** et **S3** sont intéressantes comme à partir des travaux de Wolpert [116] et Freixas [58], on voit, que des surfaces hyperboliques dégénérescentes les satisfait.

Disons quelques mots pourquoi on considère des hypothèses “pre-log-log” et pas les hypothèses “good”. La raison principale vient du fait que par le résultat de Burgos Gil-Kramer-Kühn [36], des formes de Bott-Chern associes aux métriques pre-log-log sont de type pre-log-log. Comme des formes de Bott-Chern apparaissent naturellement dans la formule d'anomalie (voir Théorème 1.2.2) et dans la définition de normes de Deligne [42, (6.3.1)], cette propriété est fondamentale dans les questions qu'on étudie. On note aussi que les métriques “good” ne satisfont pas cette propriété.

Introduisons encore quelques notions des métriques hermitiennes singulières.

Définition 1.2.10. *Soit Y une variété complexe et D_0 un diviseur dans Y.*

a) On suppose que D_0 a des croisements normaux et que la fonction $f : Y \setminus |D_0| \to \mathbb{R}$ est continue avec la croissance de type double logarithme le long de D_0. On note $[f]_{L^1}$ le courant sur Y, donné par l'extension L^1 de f sur Y. On dit que f est “nice” avec des singularités le long D_0 si les courants $\partial[f]_{L^1}$, $\overline{\partial}[f]_{L^1}$, $\partial\overline{\partial}[f]_{L^1}$ sont données par l'intégration des formes continues sur $Y \setminus |D_0|$ avec la croissance de type double logarithme le long de D_0.

b) Pour $x \in D_0$, on fixe un ouvert $U \subset Y$, $x \in U$. Soient $h_1, \ldots, h_k$, $k \in \mathbb{N}$ des fonctions holomorphes $dh_i(x) \neq 0$, $i = 1, \ldots, k$ et $n_1, \ldots, n_k \in \mathbb{N}$ sont telles que D_0 est défini sur U par l'équation $\{h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k} = 0\}$. On dit que la fonction lisse $f : Y \setminus |D_0| \to \mathbb{R}$ est “very nice” avec des singularités le long D_0 si pour chaque $x \in D_0$ il existe des fonctions lisses $f_0, \ldots, f_k : U \to \mathbb{C}$, telles que

$$f = f_0 + \sum_1^k f_i |h_i|^2 \ln |h_i|. \tag{1.2.29}$$

c) Soit L un fibré en droites holomorphe Y, et soit h^L une métrique hermitienne continue sur L au dessous de $Y \setminus |D_0|$. Pour $x \in Y$, fixons une section holomorphe υ non-nulle de L au voisinage U de x. On dit que h^L est “very nice” (resp. “nice”) avec des singularités le long de D_0 si pour chaque x et υ, la fonction $\ln h^L(\upsilon, \upsilon)$ est “very nice” (resp. “nice”) avec des singularités le long de D_0.

Remarque 1.2.11. *a) Pour une métrique hermitienne h^L, qui est soit “nice” soit “very nice”, on définit la première forme de Chern comme un courant sur Y, donné par l'identité suivante*

$$c_1(L, h^L) := \frac{\partial\overline{\partial}[\ln h^L(\upsilon, \upsilon)]_{L^1}}{2\pi\sqrt{-1}}. \tag{1.2.30}$$

Le courant $c_1(L, h^L)$ est fermé. De plus, par la théorie de Chern-Weil il représente la classe $c_1(L)$ dans la cohomologie de Y.

b) Directement de la définition de la métrique pre-log-log, on voit que si h^L est lisse sur $Y \setminus D_0$ et “nice” avec des singularités le long de D_0, alors h^L est pre-log-log.

Théorème 1.2.3 (Théorème de continuité, [55, Theorem C])**.** *Soit $\pi : X \to S$ une famille de courbes complexes à singularités ordinaires. Soit $\Sigma_{X/S}$ la sous-variété de points singuliers sur les fibres, et soit $\Delta := \pi_*(\Sigma_{X/S})$ le diviseur de courbes singulières. Soit ξ un fibré vectoriel holomorphe au-dessus de X et h^ξ une métrique hermitien sur ξ au-dessus de $X \setminus \pi^{-1}(|\Delta|)$.*

Soit $D_{X/S} \subset X$ un diviseur induit par une sous-variété $|D_{X/S}|$ qui intersecte $\pi^{-1}(|\Delta|)$ transversalement et telle que $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ est localement un isomorphisme. Soit $\|\cdot\|^{\omega}_{X/S}$ la norme sur le fibré en droites canonique $\omega_{X/S}$ au-dessus de $X \setminus (\pi^{-1}(\Delta) \cup |D_{X/S}|)$. On suppose que la restriction de $\|\cdot\|^{\omega}_{X/S}$ sur chaque fibre non singulière $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ de π induit la métrique de Kähler avec pointes à $|D_{X/S}| \cap X_t$.

On utilise la même notation pour le fibré en droites $\mathscr{L}_n$ (voir (1.2.28)), et la norme $\|\cdot\|_{\mathscr{L}_n}$ (voir (1.2.27)).

*1) Sous l'hypothèse **S1**, la norme $\|\cdot\|_{\mathscr{L}_n}$ est "very nice" (donc, lisse au-dessus de $S \setminus |\Delta|$) avec des singularités le long de Δ.*

*2) Sous l'hypothèse **S2**, la norme $\|\cdot\|_{\mathscr{L}_n}$ est "nice" avec des singularités le long de Δ.*

*3) Sous l'hypothèse **S3**, la norme $\|\cdot\|_{\mathscr{L}_n}$ s'éteint continument sur S.*

Remarque 1.2.12. *a) Pour $m = 0$, Théorème 1.2.31 donne un résultat de Bismut-Bost [20, Théorème 2.2]. Cependant, on note que notre preuve utilise [20, Théorème 2.2].*

*b) On peut se demander si pour n'importe lesquelles $\pi : X \to S$, $D_{X/S}$, ... il existe une métrique $\|\cdot\|^{\omega}_{X/S}$ satisfaisant les hypothèses **S1**. On le démontre dans la section §3.4.6.*

Théorème 1.2.4 (Théorème de courbure, [55, Theorem D])**.** *On utilise les notations du Théorème 1.2.3. Sous l'hypothèse **S1** (resp. **S2**), le courant*

$$\pi_*\Big[\mathrm{Td}(\omega_{X/S}(D)^{-1}, \|\cdot\|^{-2}_{X/S})\mathrm{ch}(\xi, h^{\xi})\mathrm{ch}(\omega_{X/S}(D)^{n}, \|\cdot\|^{2n}_{X/S})\Big]^{(2,2)} \tag{1.2.31}$$

est $L^1_{loc}(S)$. On note par le même symbole son extension L^1 sur S. Cette extension est fermé. De plus, au sens de courants sur S, on a l'identité suivante

$$\begin{aligned} &c_1\big(\mathscr{L}_n, \|\cdot\|^2_{\mathscr{L}_n}\big) \\ &\qquad = -12\pi_*\Big[\mathrm{Td}\big(\omega_{X/S}(D)^{-1}, \|\cdot\|^{-2}_{X/S}\big)\mathrm{ch}\big(\xi, h^{\xi}\big)\mathrm{ch}\big(\omega_{X/S}(D)^{n}, \|\cdot\|^{2n}_{X/S}\big)\Big]^{(2,2)}. \end{aligned} \tag{1.2.32}$$

Remarque 1.2.13. *Sous l'hypothèse **S1** et $m = 0$, Théorème 1.2.4 est exactement Bismut-Bost [20, Théorème 2.1]. Cependant, on note que notre preuve de Théorème 1.2.4 sous l'hypothèse **S1** utilise [20, Théorème 2.1]. Sous l'hypothèse **S2**, par contre, on n'utilise que la théorème de courbure de Bismut-Gillet-Soulé [23].*

Décrivons ci-dessous quelques applications des ces résultats à la géométrie de l'espace de modules $\mathscr{M}_{g,m}$ de courbes stables m-épointées du genre $g \in \mathbb{N}$, $2g - 2 + m > 0$. On note $\overline{\mathscr{M}}_{g,m}$ *la compactification de Deligne-Mumford* de $\mathscr{M}_{g,m}$, par $\partial\mathscr{M}_{g,m} := \overline{\mathscr{M}}_{g,m} \setminus \mathscr{M}_{g,m}$ le *diviseur compactifiant*, par $\mathscr{C}_{g,m}$ et $\overline{\mathscr{C}}_{g,m}$ *les courbes universelles* sur $\mathscr{M}_{g,m}$ et $\overline{\mathscr{M}}_{g,m}$ respectivement. On note

$$\Pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m} \tag{1.2.33}$$

la projection universelle. Soit $D_{g,m}$ le diviseur sur $\overline{\mathscr{C}}_{g,m}$, induit par les points épointées. On note $\omega_{g,m}$ le fibré relatif canonique de Π, et par $\omega_{g,m}(D)$ le fibré en droites relatif canonique tordu, donné par la formule

$$\omega_{g,m}(D) := \omega_{g,m} \otimes \mathscr{O}_{\overline{\mathscr{C}}_{g,m}}(D_{g,m}). \tag{1.2.34}$$

Par le théorème d'uniformization, (cf. [49, Chapter IV], [9, Lemma 6.2], [10]), on munit $\omega_{g,m}(D)$ de la norme hermitienne $\|\cdot\|_{g,m}^{\mathrm{hyp}}$ telle que sa restriction sur chaque fibre de π induit par Construction 1.2.8 la métrique dont la courbure scalaire est -1. On considère le fibré determinant $\lambda(j^*(\omega_{g,m}(D)^n))$, $n \leq 0$ sur S, qui est aussi souvent appelé *le fibré de Hodge*, et on la munit de la métrique de Quillen $\|\cdot\|_{g,m}^{Q,n}$ induit par $\|\cdot\|_{g,m}^{\mathrm{hyp}}$. On munit le fibré en droites $\det(\Pi_*(\omega_{g,m}|_{|D_{g,m}|}))$ de la métrique de Wolpert $\|\cdot\|_{g,m}^{W}$ induite. On note ω_{WP} la forme de Weil-Petersson sur $\mathscr{M}_{g,m}$.

Corollaire 1.2.14. *La norme*

$$\|\cdot\|_{g,m}^{H,n} := (\|\cdot\|_{g,m}^{Q,n})^{12} \otimes (\|\cdot\|_{g,m}^{W})^{-1} \otimes \|\cdot\|_{\partial\mathscr{M}_{g,m}}^{\mathrm{div}} \tag{1.2.35}$$

sur le fibré en droites

$$\lambda_{g,m}^{H,n} := \lambda(j^*(\omega_{g,m}(D)^n))^{12} \otimes (\det(\Pi_*(\omega_{g,m}|_{|D_{g,m}|})))^{-1} \otimes \mathscr{O}_{\overline{\mathscr{M}}_{g,m}}(\partial\mathscr{M}_{g,m}) \tag{1.2.36}$$

est "good" au sens de Mumford avec des singularités le long de $\partial\mathscr{M}_{g,m}$. De plus, elle s'éteint continument sur $\overline{\mathscr{M}}_{g,m}$ et elle est lisse sur $\mathscr{M}_{g,m}$.

Remarque 1.2.15. *C'est possible de déduire le résultat du Corollaire 1.2.14 à partir de l'isomorphisme de Deligne [42, Théorème 11.4], du théorème de Riemann-Roch arithmetique pour des surfaces épointées, prouvé dans cette forme par Gillet-Soulé dans [66] (cf. [65, Proposition 1.5.2]) poir $m = 0$, $n \leq 0$, par Freixas dans [59, Theorem 6.2] pour $n = 0$, $m \in \mathbb{N}$ et dans [60, Theorem 6.2] pour $n < 0$, $m \in \mathbb{N}$, et en utilisant "goodness" de la métrique de Deligne associé, prouvé par Freixas dans [58, Theorem 5.2.1 and Remark 5.2.4]. Notre démonstration est différente, car on obtient Corollaire 1.2.14 directement par Théorème 1.2.32.*

Par Corollaire 1.2.14 et Remarque 1.2.11, on voit que la première forme de Chern de $(\lambda_{g,m}^{H,n}, (\|\cdot\|_{g,m}^{H,n})^2)$ est bien-définie comme un courant sur $\overline{\mathscr{M}}_{g,m}$. Comme conséquence triviale du Théorème 1.2.4, on a

Corollaire 1.2.16. *On conserve les notations du Corollaire 1.2.14. La forme ω_{WP} a la croissance de type log-log le long de $\partial\mathscr{M}_{g,m}$. On note $[\omega_{WP}]_{L^1}$ son extension L^1 sur $\overline{\mathscr{M}}_{g,m}$. Cette extension est fermé. De plus, au sens de courants sur $\overline{\mathscr{M}}_{g,m}$, on a l'identité suivante*

$$c_1\Big(\lambda_{g,m}^{H,n}, (\|\cdot\|_{g,m}^{H,n})^2\Big) = -\pi^{-2}\Big(6n^2 - 6n + 1\Big)\big[\omega_{WP}\big]_{L^1}. \tag{1.2.37}$$

Remarque 1.2.17. *a) Par le résultat de Wolpert [117, Theorem 5], [114, Corollary 5.11], et Théorème 1.2.6, Corollaire 1.2.16 est une extension sur $\overline{\mathscr{M}}_{g,m}$ du théorème de courbure de Takhatajan-Zograf [107, Theorem 1] sur $\mathscr{M}_{g,m}$. Nos méthodes sont très différents de méthodes de Takhatajan-Zograf.*

b) Le fait que la métrique de Weil-Petersson a la croissance log-log près de $\partial\mathscr{M}_{g,m}$ suit déjà d'un résultat de Masur [89, Theorem 1]. Le fait que l'extension L^1 de ω_{WP} est fermé a été prouvé par Wolpert dans [113, Theorem 2.3] en utilisant le résultat de Masur.

c) C'est possible de déduire le résultat du Corollaire 1.2.16 à partir de l'isomorphisme de

Deligne et du théorème de Riemann-Roch arithmetique pour des courbes epointées, prouvé par Gillet-Soulé et Freixas (voir Remarque 1.2.15), et du quelques propriétés des métriques "good", voir Mumford [95, Proposition 1.2]. Notre démonstration du Corollaire 1.2.16 est très différent. En fait, Corollaire 1.2.16 découle directement des Théorèmes 1.2.32, 1.2.4. Notons aussi que dans ce cas plusieurs difficultés techniques disparaissent à la démonstration du Théorème 1.2.4 à cause du fait que la métrique de Weil-Petersson est lisse sur $\mathscr{M}_{g,m}$. En particulier, la théorie de potentielles pour des courantes de type log-log, qu'on a developpé, n'est pas nécessaire, car on pourrait simplement utiliser les résultat de Mumford [95, Proposition 1.2].

Comme conséquence triviale de Théorème 1.2.4 et Remarque 1.2.11, on a

Corollaire 1.2.18. *On conserve les notations des Corollaires 1.2.14, 1.2.16. La classe cohomologique de $\pi^{-2}\cdot\left[\omega_{WP}\right]_{L^1}$ dans $H^2(\overline{\mathscr{M}}_{g,m},\mathbb{R})$ coïncide avec $c_1(\lambda_{g,m}^{H,n})$.*

Remarque 1.2.19. *Corollaire 1.2.18 suit aussi des résultats de Wolpert, voir [111, Lemma 5.4], [113, Theorem 1.3, §2, Theorem 4.1], mais sa preuve est très différent. En particulier, il a utilisé l'expression de ω_{WP} en coordonnées Fenchel-Nielsen, il a étudié la régularité de l'application $i:\overline{\mathscr{M}}_{g,0}^{FN}\to\overline{\mathscr{M}}_{g,0}$, où $\overline{\mathscr{M}}_{g,0}^{FN}$ est une variété homéomorphe à $\overline{\mathscr{M}}_{g,0}$ muni de la structure différentielle venant des coordonnées de Fenchel-Nielsen. Après Wolpert a fait un calcul explicit de l'accouplement de forme Weil-Petersson et certaines 2-cycles analytiques en $\overline{\mathscr{M}}_{g,0}$.*

Comme conséquence triviale de Théorèmes 1.2.33, 1.2.4, on a

Corollaire 1.2.20. *La forme de Weil-Petersson ω_{WP} a un potentiel local continue sur $\overline{\mathscr{M}}_{g,m}$.*

Remarque 1.2.21. *Corollaire 1.2.20 a été prouvé par Wolpert dans [112, §2]. Il a utilisé cette corollaire pour donner une preuve analytique de l'amplitude de la classe de la forme de Weil-Petersson, et donc donner une preuve analytique de la projectivité de $\overline{\mathscr{M}}_{g,m}$ indépendant de la preuve algébrique par Knudsen-Mumford, [76], [74], [75]. Nos méthodes sont constructives, donc ils n'utilisent pas le lemme de $\partial\overline{\partial}$, et ils sont très différentes de méthodes non-constructives de [112, §2].*

Comme conséquence triviale de Théorèmes 1.2.32, 1.2.4 et (1.2.30), on a

Corollaire 1.2.22. *On peut décomposer la forme de Weil-Petersson ω_{WP} en somme*

$$\omega_{WP}=-\pi^2\alpha+d\beta, \tag{1.2.38}$$

où les formes α,β sont lisses sur $\mathscr{M}_{g,m}$, et β, $d\beta$ ont la croissance de type double logarithme le long de $\partial\overline{\mathscr{M}}_{g,m}$, et il existe une métrique hermitienne lisse h_{sm} sur $\lambda_{g,m}^{H,0}$ au-dessus de $\overline{\mathscr{M}}_{g,m}$ telle que

$$\alpha=c_1(\lambda_{g,m}^{H,0},h_{sm}). \tag{1.2.39}$$

En particulier, on a l'identité suivante

$$\int_{\mathscr{M}_{g,m}}\omega_{WP}^{\wedge(3g-3+m)}=(-\pi^2)^{3g-3+m}\int_{\overline{\mathscr{M}}_{g,m}}c_1(\lambda_{g,m}^{H,0})^{\wedge(3g-3+m)}. \tag{1.2.40}$$

Le côté gauche de (1.2.40) est donc une multiple rationnelle de la puissance de π.

Remarque 1.2.23. *L'identité (1.2.40) a été prouvé par Wolpert dans [112, Corollary 5.3, Lemma 5.4], mais nos méthodes sont très différentes, voir Remarque 1.2.19.*

1.2.3 Théorème de restriction et théorème de compatibilité

Dans le Théorème 1.2.33, on a vu que sous l'hypothèse S3, la norme (1.2.27) sur le fibré en droites (1.2.28) s'étende continument sur S. Le but principale de cette partie est de donner l'expression géométrique de cette extension. Ou, autrement dit, on cherche à comprendre le comportement de la métrique de Quillen lorsque des aux pointes apparaissent en dégénérescence.

Plus précisément, on fixe une application holomorphe, propre, surjective $\pi : X \to S$ entre deux variétés complexes X, S, telle que pour chaque $t \in S$, l'espace $X_t := \pi^{-1}(t)$ est une courbe complexe qui a au pire des singularités ordinaires. On note $\Sigma_{X/S} \subset X$ le sous-variété de points singulières dans les fibres de π. On note $\Delta = \pi_*(\Sigma_{X/S})$ le diviseur de fibres singulières. On suppose que Δ a des croisement normaux. Soit $D_{X/S} \subset X$ un diviseur induit par une sous-variété $|D_{X/S}|$ qui intersecte $\pi^{-1}(|\Delta|)$ transversalement et telle que $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ est localement un isomorphisme.

En rétrécissant la base S, on peut toujours supposer qu'il existent des sections holomorphes disjointes $\sigma_1, \dots, \sigma_m : U \to X$ de π, qui ne passent pas par des points singuliers et tels que l'identité suivante est valable

$$D_{X/S} := \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m). \tag{1.2.41}$$

Comme on a supposé que Δ a des croisements normaux, en rétrécissant la base S, on peut toujours supposer que pour un certain $l \in \mathbb{N}$, le diviseur Δ se décompose comme

$$\Delta = k \cdot \Delta_0 + k_1 \cdot \Delta_1 + \cdots + k_l \cdot \Delta_l, \tag{1.2.42}$$

où Δ_i, $i = 0, \dots, l$ sont des diviseurs induits par les sous-variétés $|\Delta_i|$ et $k, k_j \in \mathbb{N}^*$, $j = 1, \dots, l$. On note $\Delta_j^0 := \Delta_j \cap \Delta_0$ le diviseur induit sur $S' := |\Delta_0|$, et par Δ' le diviseur sur S' donné par

$$\Delta' := k_1 \cdot \Delta_1^0 + \cdots + k_l \cdot \Delta_l^0. \tag{1.2.43}$$

On note $\iota : S' \to S$ l'inclusion évidente. On note $Z := \pi^{-1}(S')$, $Z_t := \pi^{-1}(t)$, $t \in S'$, et par $\rho : Y \to Z$ la normalisation de Z. On note $\pi' : Y \to S'$ la famille des surfaces, induite par le diagramme commutatif suivant

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\iota} & S \end{array} \tag{1.2.44}$$

La restriction des sections holomorphes $\sigma_1, \dots, \sigma_m$ sur S' induit les sections holomorphes, qu'on note $\sigma_1', \dots, \sigma_m' : S' \to Y$.

Soit $\Sigma_{Z/S'}$ le lieu des points, qui se normalisent par ρ. La sous-variété $\Sigma_{Z/S'}$ est une union de certaines composantes connexes de $\Sigma_{X/S}$. On note

$$\kappa : \Sigma_{Z/S'} \hookrightarrow X \tag{1.2.45}$$

l'inclusion évidente. La restriction de π' sur $\rho^{-1}(\kappa(\Sigma_{Z/S'}))$ est le revêtement de degré $2k$, voir (1.2.42). En rétrécissant la base, on peut supposer que c'est un revêtement trivial, il existe donc

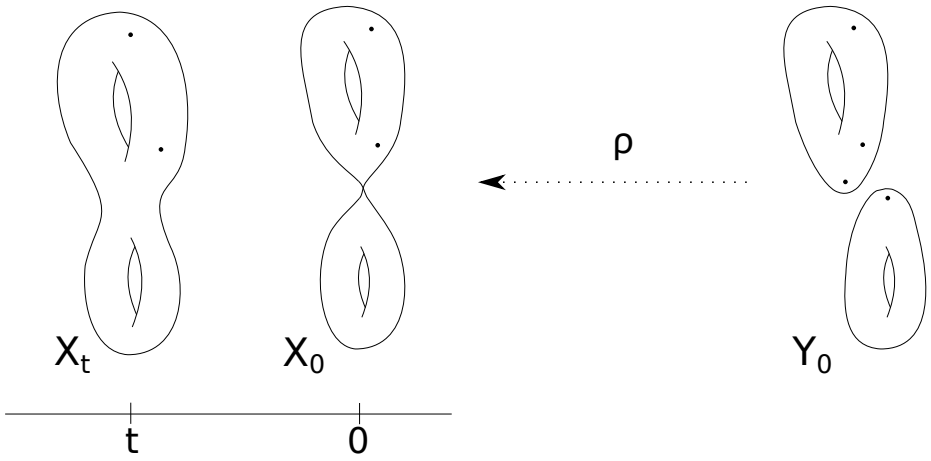

Figure 1.4: Une famille dégénérescente. Notre objectif est de relier la restriction du norme en fibres singulières avec une norme sur la normalisation. De gauche à droite, les points représentent les éléments dans $D_{X/S}|_{X_t}$, $D_{X/S}|_{X_0}$ et $D_{Y/S'}|_{Y_0}$.

des sections holomorphes $\sigma'_{m+1}, \dots, \sigma'_{m+2k} : S' \to Y$ telles que $\rho^{-1}(\Sigma_{Z/S'}) = \cup_{i=1}^{2k} \mathrm{Im}(\sigma'_{m+i})$ et $\rho \circ \sigma'_{m+2i-1} = \rho \circ \sigma'_{m+2i}$, $i = 1, \dots, k$. On définit le diviseur $D_{Y/S'}$ sur Y par

$$D_{Y/S'} := \mathrm{Im}(\sigma'_1) + \cdots + \mathrm{Im}(\sigma'_{m+2k}). \tag{1.2.46}$$

On définit le *fibré en droites canonique tordu* par

$$\omega_{Y/S'}(D) := \omega_{Y/S'} \otimes \mathscr{O}_Y(D_{Y/S'}). \tag{1.2.47}$$

On a l'isomorphisme canonique

$$\rho^*(\omega_{X/S}(D)) \simeq \omega_{Y/S'}(D). \tag{1.2.48}$$

Sous l'hypothèse S3, l'isomorphisme (1.2.48) induit la norme hermitienne $\|\cdot\|_{Y/S'}$ sur $\omega_{Y/S'}(D)$ au-dessus de $Y \setminus |D_{Y/S'}|$ par

$$\|\cdot\|_{Y/S'} := \rho^*(\|\cdot\|_{X/S}). \tag{1.2.49}$$

On note $\|\cdot\|^{\omega}_{Y/S'}$ la norme hermitienne sur $\omega_{Y/S'}$ induite par $\|\cdot\|_{Y/S'}$ comme dans la Construction 1.2.8. Les normes $\|\cdot\|_{Y/S'}, \|\cdot\|^{\omega}_{Y/S'}$ sont défini au-dessus de $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$.

On suppose que la norme hermitienne $\|\cdot\|^{\omega}_{Y/S'}$ sur $Y \setminus (\pi^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$ est telle que sa restriction sur chaque fibre non singulière $Y_t := (\pi')^{-1}(t)$, $t \in S' \setminus |\Delta'|$ de π' induit la métrique kählérienne g^{TY_t}, pour laquel le triplet $(Y_t, \{\sigma'_1(t), \dots, \sigma'_{m+2k}(t)\}, g^{TY_t})$ est une surface aux pointes. (1.2.50)

On note $\|\cdot\|^{W}_{Y/S'}$ la norme de Wolpert sur $\otimes_{i=1}^{m+2k}(\sigma'_i)^*\omega_{Y/S'}$, induite par $\|\cdot\|^{\omega}_{Y/S'}$. Maintenant, par une analogie avec (1.2.27), (1.2.28), on définit la norme hermitienne

$$\|\cdot\|_{\mathscr{L}'_n} := \big(\|\cdot\|_Q\,(g^{TY_t}, \rho^*(h^\xi)\otimes\|\cdot\|^{2n}_{Y/S'})\big)^{12}\otimes\big(\|\cdot\|^W_{Y/S'}\big)^{-\mathrm{rk}(\xi)} \otimes\big(\|\cdot\|^{\mathrm{div}}_{\Delta'}\big)^{\mathrm{rk}(\xi)}\otimes\big(\otimes_{i=1}^{m+2k}\,(\sigma'_i\circ\rho)^*h^{\det\xi}\big)^3 \quad (1.2.51)$$

sur le fibré en droites

$$\mathscr{L}'_n := \lambda\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big)^{12}\otimes(\otimes_{i=1}^{m+2k}(\sigma'_i)^*\omega_{Y/S'})^{-\mathrm{rk}(\xi)} \otimes\mathscr{O}_{S'}(\Delta')^{\mathrm{rk}(\xi)}\otimes\big(\otimes_{i=1}^{m+2k}\,(\sigma'_i\circ\rho)^*\det\xi\big)^6. \quad (1.2.52)$$

On note $N_{\Sigma_{Z/S'}/X}$ (resp. $N_{S'/S}$) le fibré vectoriel normal de $\Sigma_{Z/S'}$ en X (resp. de S' en S). Comme les fibres de la famille π n'ont que des singularités ordinaires, la projection π induit l'isomorphisme canonique

$$d\pi^2 : \wedge^2 N_{\Sigma_{Z/S'}/X}\to\kappa^*\pi^*N_{S'/S}. \quad (1.2.53)$$

Pour chaque $i=1,\dots,k$, l'application de normalisation ρ induit l'isomorphisme canonique

$$(\sigma'_{m+2i-1})^*(TY/S')\otimes(\sigma'_{m+2i})^*(TY/S')\to\wedge^2 N_{\Sigma_{Z/S'}/X}. \quad (1.2.54)$$

On note ω_S et $\omega_{S'}$ les fibrés en droites canoniques sur S et S'. En combinant les duales des isomorphismes (1.2.53), (1.2.54), on obtient l'isomorphisme canonique

$$(\omega_S\otimes\omega_{S'}^{-1})|_{S'}\to(\sigma'_{m+2i-1})^*(\omega_{Y/S'})\otimes(\sigma'_{m+2i})^*(\omega_{Y/S'}). \quad (1.2.55)$$

Le morphisme de résidu de Poincaré donne l'isomorphisme canonique

$$(\omega_S^k\otimes\mathscr{O}_S(k\Delta_0))|_{S'}\to\omega_{S'}^k. \quad (1.2.56)$$

En combinant l'isomorphisme (1.2.55), appliqué pour chaque $i=1,\dots,k$, l'isomorphisme (1.2.56) et en multipliant par $(\otimes_{i=1}^m\sigma_i^*\omega_{X/S})^{-1}\otimes\mathscr{O}_S(\sum k_i\Delta_i)$, on obtient l'isomorphisme canonique

$$\big(\big(\otimes_{i=1}^m\,\sigma_i^*\omega_{X/S}\big)^{-1}\otimes\mathscr{O}_S(\Delta)\big)\big|_{S'}\to\big(\otimes_{i=1}^{m+2k}\,(\sigma'_i)^*\omega_{Y/S'}\big)^{-1}\otimes\mathscr{O}_{S'}(\Delta'). \quad (1.2.57)$$

Pour $t\in S'$, on a la suite exacte de faisceaux (cf. [18, (5.53)])

$$0\to\mathscr{O}_{Z_t}\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big)\to\rho_*\mathscr{O}_{Y_t}\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big) \to\mathscr{O}_{\Sigma_{Z/S'}}\big(\kappa^*\xi\otimes\det(\rho_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})\big)\to 0, \quad (1.2.58)$$

où la première flèche est induite par le pull-back, et la deuxième flèche est la différence de résidus évaluées en points de $\rho^{-1}(\Sigma_{Z/S'})$. La suite exacte courte (1.2.58) induite l'isomorphisme canonique (cf. [18, (5.55)])

$$\lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big)|_{S'}\to\lambda\big(j^*(\rho^*(\xi)\otimes\omega_{Y/S'}(D)^n)\big) \otimes\det\big(\pi_*(\kappa^*(\xi))\big)\otimes\det\big((\pi\circ\rho)_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}}\big)^{\mathrm{rk}(\xi)}. \quad (1.2.59)$$

On note que $\det((\pi \circ \rho)_* \mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})$ est un fibré en droites avec un carré canoniquement trivial. À partir de maintenant, on ne mentionne pas explicitement ses puissances. Trivialement, on a un isomorphisme

$$\det\left(\pi_*(\kappa^*(\xi))\right)^2 \to \left(\otimes_{i=1}^{2k} (\sigma'_{m+i} \circ \rho)^* \det \xi\right) \otimes \left(\det \pi_* \mathscr{O}_{\Sigma_{Z/S'}}\right)^{2\cdot \mathrm{rk}(\xi)}. \tag{1.2.60}$$

La composition des isomorphismes (1.2.57), (1.2.59) et (1.2.60) induit l'isomorphisme canonique

$$\mathscr{L}_n|_{S'} \to \mathscr{L}'_n \otimes \left(\det \pi_* \mathscr{O}_{\Sigma_{Z/S'}}\right)^{12\cdot \mathrm{rk}(\xi)}, \tag{1.2.61}$$

qui est le protagoniste de cette partie de la thèse.

Pour $k \in \mathbb{N}^*$, on note

$$C_0 = -6\log(\pi), \qquad C_k = -6(1+k)\log(2) - 6(1+2k)\log(\pi) - 6\log((2k)!). \tag{1.2.62}$$

Le résultat suivant est le résultat principal de cette partie de la thèse, et il décrit l'extension continue de la norme (1.2.27) en termes des mêmes objets qu'on a utilisés dans la définition de (1.2.27).

Théorème 1.2.5 (Théorème de restriction, [56, Theorem 1.2])**.** *Soit $\pi : X \to S$ une famille de courbes complexes à singularités ordinaires. On suppose que le diviseur de courbes singuliers Δ se décompose comme (1.2.42). Soit $k \in \mathbb{N}$ et S' sont comme dans (1.2.43).*

Soit $\sigma_1, \ldots, \sigma_m : S \to X$ des sections holomorphes disjointes de π, qui ne passent pas par des points singuliers des fibres. On note par $D_{X/S}$ le diviseur (1.2.41).

Soit $\|\cdot\|^{\omega}_{X/S}$ le norme hermitienne sur le fibré en droites canonique $\omega_{X/S}$ au-dessus de $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ telle que sa restriction sur chaque fibre $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ induit la métrique kählerienne g^{TX_t} as-dessus de $X_t \setminus \{\sigma_1(t), \ldots, \sigma_m(t)\}$ telle que le triplet $(X_t, \{\sigma_1(t), \ldots, \sigma_m(t)\}, g^{TX_t})$ est une surface aux pointes dans le sens de §1.2.1.

*Soit (ξ, h^{ξ}) fibré hermitienne au-dessus de X. On utilise la notation $\|\cdot\|_{\mathscr{L}_n}$ et $\mathscr{L}_n$ comme dans (1.2.27), (1.2.28). On définit une famille de courbes complexes à singularités ordinaires $\pi' : Y \to S'$ comme dans (1.2.44). On suppose que les hypothèses **S3** du Théorème 1.2.3 et les hypothèses (1.2.50) sont vérifiées. On utilise la notation $\|\cdot\|_{\mathscr{L}'_n}$ et $\mathscr{L}'_n$ comme dans (1.2.51), (1.2.52).*

La norme $\|\cdot\|_{\mathscr{L}_n}$ s'étend continument S et sous l'isomorphisme (1.2.61), l'identité suivante est vraie

$$\|\cdot\|_{\mathscr{L}_n}|_{S'} = \exp(m \cdot \mathrm{rk}(\xi) \cdot C_{-n}) \cdot \|\cdot\|_{\mathscr{L}'_n}. \tag{1.2.63}$$

Remarque 1.2.24. *On note qu'un théorème similaire a été prouvé par Bismut dans [18, Theorems 0.2, 0.3], et malgré le fait qu'on utilise [18, Theorems 0.3] dans notre preuve de Théorème 1.2.5, les situations géométriques considérées ici et dans [18] sont très différentes. Contrairement à [18], la situation décrit dans cet article est adapté pour des familles des surfaces de Riemann dégénérescentes. On note que même dans le cas $m = 0$, la fibre singulière dans notre situation a toujours au moins deux pointes. En particulier, on ne peut pas obtenir Théorème 1.2.5 directement à partir de [18, Theorems 0.2, 0.3] et de la formule d'anomalie.*

Décrivons le deuxième résultat de cette partie de la thèse. On fixe le surface aux pointes $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ à courbure scalaire constante. Comme $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ est une surface aux pointes, la torsion analytique $T(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n})$ est bien-défini.

Alternativement, soit $Z_{(\overline{M},D_M)}(s), s \in \mathbb{C}$ la fonction zêta de Selberg, donnée pour $\mathrm{Re}(s) > 1$ par la formule suivante

$$Z_{(\overline{M},D_M)}(s) = \prod_{\gamma}\prod_{k=0}^{\infty}(1 - e^{-(s+k)l(\gamma)})^2, \tag{1.2.64}$$

où γ parcours l'ensemble de toutes les géodésiques fermées simples non orientées sur $(M, g^{TM}_{\mathrm{hyp}})$, et $l(\gamma)$ est la longueur de γ. La fonction $Z_{(\overline{M},D_M)}(s)$ admet une extension méromorphe sur $\mathbb{C}$ avec une zéro simple en $s = 1$ (voir par exemple [44, (5.3)]).

Soit $\zeta(s) := \sum_{i=1}^{\infty} i^{-s}$ la fonction zêta de Riemann. Pour $k \in \mathbb{N}^*$, on met

$$\begin{aligned} c_0 &= 4\zeta'(-1) - \tfrac{1}{2} + \log(2\pi), \\ c_k &= \textstyle\sum_{l=0}^{k-1}(2k-2l-1)\big(\log(2k+2kl-l^2-l) - \log(2)\big) + (\tfrac{1}{3} + k + k^2)\log(2) \\ &\quad + (2k+1)\log(2\pi) + 4\zeta'(-1) - 2(k+\tfrac{1}{2})^2 - 4\textstyle\sum_{l=1}^{k-1}\log(l!) - 2\log(k!). \end{aligned} \tag{1.2.65}$$

Pour $k \in \mathbb{N}$, on note $C_k : \mathbb{N}^2 \to \mathbb{R}$, $E : \mathbb{N}^2 \to \mathbb{R}$ les fonctions suivantes

$$\begin{aligned} B_k(g,m) &= \exp\Big((2 - 2g(\overline{M}) - m)\frac{c_k}{2}\Big), \\ E(g,m) &= \exp\Big((g(\overline{M}) + 2 - m)\frac{\log(2)}{3}\Big). \end{aligned} \tag{1.2.66}$$

En particulier, on voit que pour tout $k \in \mathbb{N}$, $(g,m) \in \mathbb{N}^2$, on a

$$B_k(g+m, 0) = B_k(g,m) \cdot B_k(1,1)^m. \tag{1.2.67}$$

Alors pour $l \in \mathbb{Z}$, $l < 0$, Takhtajan-Zograf dans [107, (6)] ont proposé[2] l'analogue de défini à l'aide de la fonction zêta de Selberg par

$$\begin{aligned} T_{TZ}(g^{TM}_{\mathrm{hyp}}, 1) &= E(g(\overline{M}), m) \cdot B_0(g(\overline{M}), m) \cdot Z'_{(\overline{M},D_M)}(1), \\ T_{TZ}(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2l}) &= B_{-l}(g(\overline{M}), m) \cdot Z_{(\overline{M},D_M)}(-l+1). \end{aligned} \tag{1.2.68}$$

Théorème 1.2.6 (Théorème de compatibilité, [56, Theorem 1.4])**.** *Notre définition de la torsion analytique est compatible avec la définition de Takhtajan-Zograf. C'est-à-dire pour n'importe laquelle surface aux pointes $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ muni de la métrique de la courbure scalaire constant, on a*

$$T(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n}) = T_{TZ}(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n}). \tag{1.2.69}$$

[2] La constante devant la fonction zêta de Selberg n'a pas apparu dans [107], car le résultat de Takhtajan-Zograf n'en dépend pas. La constante de la normalisation de $T_{TZ}(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2l})$ coïncide avec Freixas [59], [60].

Remarque 1.2.25. *Pour $m = 0$, Théorème 1.2.6 a été prouvé par Phong-D'Hoker [44, (7.30)], [45, (3.6)] (voir [104], [29, (50)] et [99, (9)]). Notre preuve est basée sur leur résultat. On note aussi que Albin-Rochon dans [4] ont prouvé (1.2.69) jusqu'à une multiplication par une constante universelle, mais nos approches sont très différentes.*

Décrivons maintenant les applications des Théorèmes 1.2.5, 1.2.6 dans l'étude de l'espace des modules $\mathscr{M}_{g,m}$ de surfaces de Riemann m-épointées de genre $g \in \mathbb{N}$, $2g-2+m>0$.

Pour la définition des morphismes de clutching

$$\begin{aligned} &\alpha_{ij} : \overline{\mathscr{M}}_{g-1,m+2} \to \overline{\mathscr{M}}_{g,m}, \\ &\beta^P_{(g_1,m_1),(g_2,m_2)} : \overline{\mathscr{M}}_{g_1,m_1+1} \times \overline{\mathscr{M}}_{g_2,m_2+1} \to \overline{\mathscr{M}}_{g,m}, \end{aligned} \tag{1.2.70}$$

où $i<j$, $i,j=1,\dots,m+2$; $m_1,m_2 \in \mathbb{N}$, $g_1,g_2 \in \mathbb{N}$, $m_1+m_2=m$, $g_1+g_2=g$, $2g_1+m_1-2>0$, $2g_2+m_2-2>0$ et $P \in \{I,J \subset \{1,2,\dots,m\} : I \cap J = \emptyset, I \cup J = \{1,2,\dots,m\}, |I|=m_1, |J|=m_2\}$, voir Knudsen [74]. On rappelle que le diviseur compactifiant $\partial\mathscr{M}_{g,m}$ peut s'écrire en termes de (1.2.70) par (voir [8, p.262])

$$\left|\partial\mathscr{M}_{g,m}\right| = \Big(\cup \operatorname{Im}(\alpha_{ij})\Big) \cup \Big(\cup \operatorname{Im}\left(\beta^P_{(g_1,m_1),(g_2,m_2)}\right)\Big). \tag{1.2.71}$$

À partir de maintenant et jusqu'à la fin, par souci de brièveté, on supprime les indices de α, β.

Après une application de la formule d'adjonction, qui affirme la trivialité canonique du fibré en droites $\Pi_*(\omega_{g,m}(D)|_{|D_{g,m}|})$, l'isomorphisme (1.2.61) spécifie dans ce cas aux isomorphismes

$$\alpha^*\lambda^{H,n}_{g,m} \simeq \lambda^{H,n}_{g-1,m_2}, \tag{1.2.72}$$

$$\beta^*\lambda^{H,n}_{g,m} \simeq \lambda^{H,n}_{g_1,m_1+1} \boxtimes \lambda^{H,n}_{g_2,m_2+1}, \tag{1.2.73}$$

qui respectent la structure $\mathbb{Z}$ naturelle de fibrés en droites (1.2.36), voir Knudsen [75, Theorem 4.2] (cf. [59]). Ici on a utilisé la notation $L_X \boxtimes L_Y$ pour le fibrés en droites sur $X \times Y$, qui est donné par $\pi_X^* L_X \otimes \pi_Y^* L_Y$ pour certains fibrés en droites L_X, L_Y sur les variétés complexes X et Y respectivement, et projections naturelles $\pi_X : X \times Y \to X$ et $\pi_Y : X \times Y \to Y$.

Théorème 1.2.26 (Théorème de restriction sur $\overline{\mathscr{M}}_{g,m}$)**.** *a) L'isomorphisme (1.2.72) est une isométrie si le côté gauche est muni de $\|\cdot\|^{H,n}_{g,m}$, et le côté droite est muni de $\exp(mC_{-n}) \cdot \|\cdot\|^{H,n}_{g-1,m+2}$.*

b) De même, l'isomorphisme (1.2.73) est une isométrie si le côté gauche est muni de la norme $\|\cdot\|^{H,n}_{g,m}$, le côté droite est muni de la norme $\exp(mC_{-n}) \cdot (\|\cdot\|^{H,n}_{g_1,m_1+1} \boxtimes \|\cdot\|^{H,n}_{g_2,m_2+1})$.

Remarque 1.2.27. *Dans [59, Corollary 6.5], Freixas a prouvé Théorème 1.2.26b) pour $n=0$ avec la norme Quillen, défini comme un produit de la torsion analytique de Takhtajan-Zograf et de la norme L^2. Puis dans [60, Theorem 5.3] il a généralisé ce résultat pour $n \leq 0$. Par Théorème 1.2.6, ses résultats découle du Théorème 1.2.26. Cependant, on note que notre preuve du Théorème 1.2.5, qui est une généralisation du Théorème 1.2.26 est basé sur le calcul effectué par Freixas dans ces articles.*

1.2.4 Conclusion

Tout d'abord, on note que Théorème 1.2.5 suggère que la renormalisation

$$T^{\mathrm{ren}}(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}) := \exp(m \cdot \mathrm{rk}(\xi) C_{-n}/12) \cdot T(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}) \tag{1.2.74}$$

est plus naturel du point de vue du théorème de restriction. La même normalisation a été fait par Freixas dans [59, Definition 2.2], [60, Definition 4.2] pour (ξ, h^{ξ}) trivial et M surface hyperbolique stable, muni de la métrique à courbure scalaire constante -1.

Proposons maintenant des directions dans lesquelles nos résultats pourront être utiles.

En combinant la définition de la torsion analytique pour des surfaces avec des pointes hyperboliques, qu'on a introduit, et la torsion analytique d'orbifold de Ma [83], on peut définir la torsion analytique $T(g^{TM}, h^{\xi} \otimes \|\cdot\|^{2n}_{M})$ pour une orbisurface (M, g^{TM}) avec des pointes $D_M \subset \overline{M}$ et singularités $D'_M \subset \overline{M}$, où $n \leq 0$ et $\|\cdot\|_M$ est la norme induite sur le fibré en droites orbifold canonique tordu $\omega_{\overline{M}}(D)$. Cette définition devrait généraliser à la fois la torsion analytique de Takhtajan-Zograf [109], qui est fait pour des orbisurfaces hyperboliques stables et (ξ, h^{ξ}) trivial, et de Freixas-von Pippich [61], qui est fait pour des orbisurfaces hyperboliques stables, (ξ, h^{ξ}) trivial et $n = 0$.

Comme nos méthodes dans la preuve du Théorème 1.2.1 sont locales, l'analogue du Théorème 1.2.1 devra toujours être vraie. Précisons aussi que on a obtenu Théorème 1.2.2 en combinant Théorème 1.2.1 et la formule d'anomalie de Bismut-Gillet-Soulé [21, Theorem 1.23]. Si on remplace la dernière référence par son analogue orbifold de Ma [83, Theorem 0.1], il serait possible d'obtenir un analogue du Théorème 1.2.2 pour des orbisurfaces. Ça nous donne un espoir qu'on peut déduire des analogues de théorèmes de courbure et restriction pour des orbisurfaces par des mêmes méthodes qu'on a utilisé dans le cas de surfaces de Riemann.

Ça sera très interessant de comprendre la relation entre cette approche avec des résultates des articles de Takhtajan-Zograf [109] et Freixas-Pippich, [61].

Part I

Asymptotic of the analytic torsion

Chapter 2

On the full asymptotic of the analytic torsion

Abstract. The purpose of this article is to study the asymptotic expansion of Ray-Singer analytic torsion associated with powers p of a given positive line bundle over a compact n-dimensional complex manifold, as $p \to \infty$. Here we prove that the asymptotic expansion contains only the terms of the form $p^{n-i} \log p, p^{n-i}$ for $i \in \mathbb{N}$. For the first two leading terms it was proved by Bismut-Vasserot. We calculate the coefficients of the terms $p^{n-1} \log p, p^{n-1}$ in the Kähler case and thus answer the question posed in the recent work of Klevtsov-Ma-Marinescu-Wiegmann about quantum Hall effect. Our second result concerns the general asymptotic expansion of the analytic torsion for a compact complex orbifold.

2.1 Introduction

The holomorphic analytic torsion was introduced by Ray-Singer in [103]. It is a number $T(g^{TM}, h^E)$ defined for a holomorphic Hermitian vector bundle (E, h^E) over a compact Hermitian manifold (M, g^{TM}, Θ) as the regularized determinant of the Kodaira Laplacian $\square^E = \overline{\partial}^E \overline{\partial}^{E*} + \overline{\partial}^{E*}\overline{\partial}^E$, acting on the vector space of sections of the vector bundle $\Lambda^\bullet(T^{*(0,1)}M) \otimes E$.

Let L be a positive Hermitian line bundle over M, $\dim_{\mathbb{C}} M = n$. In [27], Bismut-Vasserot obtained the asymptotics of $\log T(g^{TM}, h^{L^p \otimes E})$ as $p \to +\infty$, (here $L^p := L^{\otimes p}$), and they gave an explicit formula for the coefficients of the leading terms $p^n \log p, p^n$ of the expansion. This asymptotic expansion played an important role in a result of arithmetic ampleness (see Gillet-Soulé [66], [106, Chapter VIII]). In this article we obtain a general formula for it in the orbifold's setting. The general strategy of the proof is the same as in the article [27]: we study this asymptotic expansion by studying the heat kernel of the rescaled Kodaira Laplacian $\square^{L^p \otimes E}/p$. We use functional analysis approach inspired by Bismut-Lebeau [25] and realized in Ma-Marinescu [85, §5.5]. Certainly, one expects that the probability approach of [27] could also be applied. In Theorem 2.1.3 we also give an explicit formula for the coefficients of the subsequent terms $p^{n-1} \log p, p^{n-1}$.

Now let's describe our results more precisely. Let (M, g^{TM}, Θ) be a compact Hermitian manifold of complex dimension n. Let (E, h^E) be a holomorphic Hermitian vector bundle over M with first Chern class $c_1(E)$ and rank $\mathrm{rk}(E)$. Let (L, h^L) be a Hermitian positive line bundle over M. Let's denote by ω the 2-form defined by

$$\omega := c_1(L, h^L) := \frac{\sqrt{-1}}{2\pi} R^L, \tag{2.1.1}$$

where R^L is the curvature of the Chern connection on (L, h^L). We define $\mathring{R}^L \in \mathrm{End}(T^{(1,0)}M)$ by

$$g^{TM}(\mathring{R}^L U, \overline{V}) = R^L(U, \overline{V}), \quad U, V \in T^{(1,0)}M. \tag{2.1.2}$$

We denote by $T(g^{TM}, h^{L^p \otimes E})$ the analytic torsion of $L^p \otimes E$ associated with g^{TM}, h^L, h^E (see Definition 2.2.3). From now on, "a local coefficient" means that it can be expressed as an integral of a density defined locally over M. Our first result (cf. Theorem 2.2.7) is

Theorem 2.1.1. *There are local coefficients $\alpha_i, \beta_i \in \mathbb{R}, i \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, as $p \to +\infty$*

$$-2 \log T(g^{TM}, h^{L^p \otimes E}) = \textstyle\sum_{i=0}^k p^{n-i}(\alpha_i \log p + \beta_i) + o(p^{n-k}), \tag{2.1.3}$$

Moreover, the coefficients α_i do not depend on g^{TM}, h^L, h^E.

Remark 2.1.2. Moreover, in the case if M is the fiber of a proper holomorphic submersion, we prove in Section 2.3.3 that α_i, β_i are smooth over the base of the family, and derivatives over the base commute with the asymptotics (2.1.3).

We note that in [27, Theorem 8], Bismut-Vasserot proved Theorem 2.1.1 for $k = 0$. They computed

$$\alpha_0 = \frac{n\,\mathrm{rk}(E)}{2} \int_M \frac{\omega^n}{n!}, \quad \beta_0 = \frac{\mathrm{rk}(E)}{2} \int_M \log\Big(\det \frac{\mathring{R}^L}{2\pi}\Big) \frac{\omega^n}{n!}. \tag{2.1.4}$$

Theorem 2.1.3. *If $\Theta = \omega$, we have*

$$\alpha_1 = \frac{(3n+1)\mathrm{rk}(E)}{12}\int_M c_1(TM)\frac{\omega^{n-1}}{(n-1)!} + \frac{n}{2}\int_M c_1(E)\frac{\omega^{n-1}}{(n-1)!}, \tag{2.1.5}$$

$$\beta_1 = \frac{\mathrm{rk}(E)}{24}(24\zeta'(-1) + 2\log(2\pi) + 7)\int_M c_1(TM)\frac{\omega^{n-1}}{(n-1)!} + \frac{1}{2}\int_M c_1(E)\frac{\omega^{n-1}}{(n-1)!}. \tag{2.1.6}$$

Remark 2.1.4. In the special case when M is a Riemann surface Theorem 2.1.3 gives a precise version of some results concerning quantum Hall effect in physics, see [73, p. 839], [52, §5].

In Section 2.4.3 we verify this result for the case $M = \mathbb{CP}^1, L = \mathcal{O}(1)$ by calculating the coefficients of the asymptotic expansion of $T(g^{FS}, h^{\mathcal{O}(p)})$ as $p \to +\infty$, for the Fubini-Study metric g^{FS}. In Section 2.4.3 we also discuss the informal relation of our result with the arithmetic Riemann-Roch theorem of Gillet-Soulé [66]. We also make a connection with [73], where Klevtsov-Ma-Marinescu-Wiegmann conjectured [73, p.839] the coefficient of the term $\log p$ and the constant term for Riemann surfaces. As it turns out, their conjecture is true for $\log p$, but not for the constant term, see Section 2.4.3.

Our last result (cf. Theorem 2.5.12 for a precise statement) is a generalization of Theorem 2.1.1 to the orbifold's case. Let $(\mathcal{M}, g^{T\mathcal{M}}, \Theta)$ be a compact effective Hermitian orbifold with strata $\Sigma\mathcal{M}$ (see Definition 2.5.5). We denote by $\Sigma\mathcal{M}^{[j]}$ for $j \in J$ the connected components of $\Sigma\mathcal{M}$, by n_j it's dimension. Let $(\mathcal{E}, h^{\mathcal{E}})$ be a proper holomorphic Hermitian orbifold vector bundle (see Definition 2.5.3) on $\mathcal{M}$ and let $(\mathcal{L}, h^{\mathcal{L}})$ be a proper Hermitian positive orbifold line bundle on $\mathcal{M}$.

Theorem 2.1.5. *There are local coefficients $\widetilde{\alpha}_i, \widetilde{\beta}_i \in \mathbb{R}$ and $m_j \in \mathbb{N}, \gamma_{j,i}, \kappa_{j,i} \in \mathbb{R}, j \in J, i \in \mathbb{N}$ such that we have the following asymptotic expansion for any $k \in \mathbb{N}$, as $p \to +\infty$*

$$\begin{aligned} -2\log T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}}) = &\sum_{i=0}^{k} p^{n-i}\big(\widetilde{\alpha}_i \log p + \widetilde{\beta}_i\big) \\ &+ \sum_{i=0}^{k+n_j-n}\sum_{j\in J}\frac{p^{n_j-i}}{m_j}e^{\sqrt{-1}\theta_j p}\big(\gamma_{j,i}\log p + \kappa_{j,i}\big) + o(p^{n-k}). \end{aligned} \tag{2.1.7}$$

The values $\theta_j, \gamma_{j,i}, \kappa_{j,i}, m_j$ depend only on the local geometry around the singular set of $\mathcal{M}$, and

$$\widetilde{\alpha}_0 = \frac{n\,\mathrm{rk}(\mathcal{E})}{2}\int_{\mathcal{M}}\frac{\omega^n}{n!}, \quad \widetilde{\beta}_0 = \frac{\mathrm{rk}(\mathcal{E})}{2}\int_{\mathcal{M}}\log\Big(\det\frac{\mathring{R}^{\mathcal{L}}}{2\pi}\Big)\frac{\omega^n}{n!}, \tag{2.1.8}$$

where $\tilde{\omega}$ and $\mathring{R}^{\mathcal{L}}$ are the orbifold analogues of (2.1.1) and (2.1.2). There are $c_j \neq 0$ such that

$$\gamma_{j,0} = \begin{cases} c_j\int_{\Sigma\mathcal{M}^{[j]}}\frac{\tilde{\omega}^{n-1}}{(n-1)!}, & \text{if } \operatorname{codim}\Sigma\mathcal{M}^{[j]} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.9}$$

Similarly to the manifold's case, the constants $\widetilde{\alpha}_i$, $\gamma_{j,i}$ do not depend on $g^{T\mathcal{M}}$,$h^{\mathcal{L}^p}$, $h^{\mathcal{E}}$. When $\Theta = \omega$, $\widetilde{\alpha}_1, \widetilde{\beta}_1$ are given by (2.1.5) and (2.1.6) after replacing M by $\mathcal{M}$.

Corollary 2.1.6. The set $\{T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}}) : p \in \mathbb{N}\}$ detects the singularities of codimension 1.

Remark 2.1.7. 1. Since for the stabilizers G_x of $x \in \mathcal{M}$, there are only finitely many possible values of $\{|G_x|, x \in M\}$, one can take $q \in \mathbb{N}$, such that G_x acts as identity on $\mathcal{L}_x^q$ for any $x \in M$; thus, $q\theta_j \in 2\pi\mathbb{N}$ (see (2.5.5)) and the asymptotic expansion for $p = qk, k \in \mathbb{N}$ has only terms $p^{n-i}\log p, p^{n-i}$ for $i \in \mathbb{N}$ in (2.1.7).

2. We see that Theorem 2.1.5 is a generalization of Theorem 2.1.1, but to facilitate, we present firstly a proof of Theorem 2.1.1 and then explain the necessary modifications to get Theorem 2.1.5.

3. The coefficients $\widetilde{\alpha}_i, \widetilde{\beta}_i$ are the orbifold's versions of α_i, β_i. Rigorously, this means that each α_i, β_i is an integral of a local quantity and $\widetilde{\alpha}_i, \widetilde{\beta}_i$ are just the integrals of the same quantities defined in an orbifold chart. Thus, by Theorem 2.1.5 we see that if the singularities of $\mathcal{M}$ appear in codimension at least 2, the coefficients of $p^{n-1}\log p, p^{n-1}$ of the expansion of $\log T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}})$ are given by the same formulas as in Theorem 2.1.3. In general, we may express the coefficient $\kappa_{j,0}$ with the help of Mellin transform (see Theorem 2.5.12, (2.5.50)), but we don't pursue the simplification of this formula.

When an orbifold $\mathcal{M}$ is obtained as a quotient of a transversal locally free CR S^1-action on a smooth CR manifold, Theorem 2.1.5 gives a refinement of the main result of Hsiao-Huang [68], see Section 2.5.3 for detailed explanation.

Now we describe some history of related problems and propose some directions in which our results might be useful. In the article [28], Bismut-Vasserot generalized [27] by computing the asymptotic expansion of $\log T(g^{TM}, h^{E\otimes \mathrm{Sym}^p\zeta})$, as $p \to +\infty$, where (ζ, g^ζ) is a Hermitian Griffiths-positive vector bundle and (E, h^E) is a holomorphic Hermitian vector bundle. Recently, Puchol [101] obtained a generalization of this result to the family case. Let's describe his result more precisely.

Let $\pi : X \to B$ be a proper holomorphic Kähler fibration with a compact fiber M in the sense of [22, Definition 1.4], i.e. there exists a closed (1,1)-form ω_{fam} such that its restriction on the fibers of π gives a Kähler form. Let (E, h^E) be a holomorphic Hermitian vector bundle over X. We suppose that the direct image sheaf $R^\bullet\pi_*E$ is locally free, i.e. the Dolbeaut cohomology of E along the fibers is a holomorphic bundle. In [24], Bismut-Köhler introduced the torsion form $\mathcal{T}(\omega_{\text{fam}}, h^E)$, which is a smooth differential form on B, satisfying

$$\mathcal{T}(\omega_{\text{fam}}, h^E)^{[0]} = -2\log T(g^{TM}, h^E), \text{ where } [0] \text{ denotes 0-degree component,} \tag{2.1.10}$$

$$\frac{\overline{\partial}\partial}{2\pi\sqrt{-1}}\mathcal{T}(\omega_{\text{fam}}, h^E) = \sum_i (-1)^i \mathrm{ch}(H^i(M, E|_M), h^{H^i(M,E|_M)}) - \int_M \mathrm{Td}(TM, h^{TM})\mathrm{ch}(E, h^E),$$

where $h^{H^\bullet(M,E|_M)}$ is L^2-metric, and $\mathrm{ch}(\cdot,\cdot)$, $\mathrm{Td}(\cdot,\cdot)$ are the corresponding Chern and Todd forms. In particular, we see that the second identity gives a refinement of Grothendieck-Riemann-Roch theorem on the level of differential forms. In [35] Freixas-Burgos-Liţcanu gave an axiomatic definition of those torsion forms and later used this result in [34] to generalize the arithmetic Grothendieck-Riemann-Roch theorem. See [88] and [66], [63] for another interesting applications of torsion forms in Arakelov geometry.

Puchol in [101] obtained the first term of the asymptotic expansion of $\mathcal{T}(\omega_{\text{fam}}, F_p)$ when F_p is the direct image of the sheaf associated to the increasing powers p of a line bundle, which is positive

along fibers. The main result of Bismut-Vasserot in [28] follows from considering the direct image of the canonical line bundle on the projective fibration associated to the vector bundle F_p on a "family" of manifolds over a point. In [19, §3], Bismut generalized the definition of torsion forms to the case of a holomorphic fibration (which is not necessarily Kähler). It is natural to expect that one can combine our result with [19, §3] and [101] to get a general asymptotic expansion of the torsion forms for a holomorphic fibration. However in this paper we only work with the analytic torsion under the assumptions of Bismut-Vasserot in [27]. We hope, in this way we can present clearly the ideas and avoid to introduce the sophisticated techniques as Toeplitz operators (cf. [85, §7]), Bismut superconnection [15], etc. We hope to come back to the general case very soon.

A similar question in realms of the real analytic torsion was considered in [94], [26]. See also [11], [33] for related topics. For the analytic torsion on orbiolds, see [83], [61]. See [118], [119] for the application of the analytic torsion to the moduli space of K3 surfaces and [87] for the application in Calabi-Yau theefolds. There are many applications of the analytic torsion in Arakelov geometry, see [78] and later works of these authors, where they proved Lefschetz fixed point formula in Arakelov geometry. The results on the equivariant analytic torsion play an important role in their proof.

This article is organized as follows. In Section 2 we recall some properties of the Mellin transform and the definition of the holomorphic analytic torsion. We give a proof of Theorem 2.1.1, relying on some technical tools, which we prove later in Section 3. In Section 3 we also explain some facts about diagonal and off-diagonal expansion of the heat kernel of the operator $\square^{L^p\otimes E}/p$. In Section 4 we prove Theorem 2.1.3, we compare it with [73] and we give a relation to the arithmetic Riemann-Roch theorem. In Section 5 we recall the basics of the orbifolds, we prove Theorem 2.1.5 and we describe a connection between Theorem 2.1.5 and [68].

Notation. In this article denote by $\mathbb{N}^*$ the set $\mathbb{N}\setminus\{0\}$, by $T^{(1,0)}M$ the holomorphic tangent bundle of M (see §2.2) and by $T^{(0,1)}M := \overline{T^{(1,0)}M}$ the antiholomorphic tangent bundle,

$$T^{*(0,1)}M = (T^{(0,1)}M)^*, \qquad \Omega^{(0,j)}(M,E) = \mathscr{C}^\infty\big(M, \Lambda^j(T^{*(0,1)}M)\otimes E\big),$$
$$\Omega^{(0,\bullet)}(M,E) = \oplus\Omega^{(0,j)}(M,E), \qquad \Omega^{(0,>0)}(M,E) = \oplus_{j>0}\Omega^{(0,j)}(M,E).$$

Let N be the number operator on the $\mathbb{Z}$-graded vector space $\Omega^{(0,\bullet)}(M,E)$, i.e.

$$N\cdot\alpha = j\alpha, \qquad \alpha\in\Omega^{(0,j)}(M,E). \tag{2.1.11}$$

This induces a $\mathbb{Z}_2$-grading $\epsilon = (-1)^N$ on $\Omega^{(0,\bullet)}(M,E)$. In general, let A be an operator which acts on $\mathbb{Z}_2$-graded vector space (V,ϵ), its supertrace is defined as $\mathrm{Tr_s}\big[A\big] = \mathrm{Tr}\big[\epsilon A\big]$. Sometimes, to make things more precise, we denote its trace/supertrace by $\mathrm{Tr}^V[A], \mathrm{Tr_s}^V[A]$.

Note. This chapter has been published in the same form as it appears here in [53].

2.2 Asymptotics of heat kernels, Theorem 2.1.1

This is an introductory section. In Section 2.1 we recall the definition of the holomorphic analytic torsion. In Section 2.2 we recall some machinery for studying it and we give a proof of Theorem

2.1.1. Compared to [27] and [85, §5.4], the major contribution of this section is Proposition 2.2.10.

2.2.1 Holomorphic analytic torsion

Before explaining our geometric situation, let's recall the Mellin transform:

Definition 2.2.1 (The Mellin transform)**.** Let $f \in \mathscr{C}^\infty(]0,+\infty[)$ satisfies the following assumptions

1. There exists $m \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, there is an asymptotic expansion as $t \to +0$

$$f(t) = \sum_{i=-m}^{k} f_i t^i + o(t^k), \tag{2.2.1}$$

2. There are $\lambda, C > 0$ such that for $t \gg 1$

$$|f(t)| \leq Ce^{-t\lambda}. \tag{2.2.2}$$

The Mellin transform of f is the function $\mathrm{M}\big[f\big]$, defined on the complex half-plane $\mathrm{Re}\, z > m$ by

$$\mathrm{M}\big[f\big](z) := \tfrac{1}{\Gamma(z)} \int_0^{+\infty} f(t) t^{z-1}\, dt. \tag{2.2.3}$$

It is well-known that $\mathrm{M}\big[f\big]$ extends holomorphically around 0, and we have (cf. [12, Lemma 9.35])

$$\begin{aligned} &\mathrm{M}\big[f\big](0) = f_0, \\ &\mathrm{M}\big[f\big]'(0) = \int_0^1 \big(f(t) - \textstyle\sum_{i=-m}^{0} f_i t^i\big) \tfrac{dt}{t} + \int_1^{+\infty} f(t) \tfrac{dt}{t} + \textstyle\sum_{i=-m}^{-1} \tfrac{1}{i} f_i - \Gamma'(1) f_0. \end{aligned} \tag{2.2.4}$$

Notation 2.2.2. Let's suppose that a function $f :]0,+\infty[\to \mathbb{R}$ satisfies (2.2.1). We denote f_i by $f^{[i]}$.

Now let's recall the main object of this article: the analytic torsion. Let (M, J) be a complex manifold with complex structure J. Let g^{TM} be a Riemannian metric on TM compatible with J, and let $\Theta = g^{TM}(J\cdot, \cdot)$ be the associated $(1,1)$-form. We call (M, g^{TM}, Θ) a Hermitian manifold.

Let (M, g^{TM}, Θ) be a compact Hermitian manifold of complex dimension n. The Riemann volume form dv_M is given by

$$dv_M := \tfrac{1}{n!} \Theta^n. \tag{2.2.5}$$

Let's denote by r^M the scalar curvature of g^{TM} and by $\langle \cdot, \cdot \rangle$ the $\mathbb{C}$-linear extension of g^{TM} to $TM \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $T^{(1,0)}M$ the i-eigenspace of $J \in \mathrm{End}(TM \otimes_{\mathbb{R}} \mathbb{C})$ and by $T^{(0,1)}M$ the $-i$-eigenspace. Then g^{TM} induces a Hermitian metric $h^{T^{(1,0)}M}$ on $T^{(1,0)}M$ by the isomorphism $X \mapsto (X - iJX)/\sqrt{2}, X \in TM$. Let's denote by $R^{\det}$ the curvature of the Chern (Hermitian holomorphic) connection over $(\det T^{(1,0)}M, h^{\det})$, where $h^{\det}$ is the Hermitian metric on $\det T^{(1,0)}M$ induced by h^{TM}. In other words,

$$R^{T^{(1,0)}M} = (\nabla^{T^{(1,0)}M})^2, \quad R^{\det} = \mathrm{Tr}\big[R^{T^{(1,0)}M}\big], \tag{2.2.6}$$

where $\nabla^{T^{(1,0)}M}$ is the Chern connection on $(T^{(1,0)}M, h^{T^{(1,0)}M})$.

Now, let E be a holomorphic vector bundle on M with a Hermitian metric h^E. We call (E, h^E) a holomorphic Hermitian vector bundle. We denote by ∇^E its Chern connection and by $R^E = (\nabla^E)^2$ its curvature.

Let's denote by $\langle \cdot, \cdot \rangle_{L^2}$ the L^2-scalar product on $\Omega^{(0,\bullet)}(M, E)$, defined by

$$\langle \alpha, \alpha' \rangle_{L^2} := \int_M \langle \alpha, \alpha' \rangle_h(x)\, dv_M(x), \quad \text{for any} \quad \alpha, \alpha' \in \Omega^{(0,\bullet)}(M, E), \tag{2.2.7}$$

where $\langle \cdot, \cdot \rangle_h$ is the pointwise Hermitian product on $\Lambda(T^{*(0,1)}M) \otimes E$, induced by $h^{T^{(1,0)}M}$ and h^E.

Let $\overline{\partial}^E$ be the Dolbeaut operator acting on the Dolbeaut complex $\Omega^{(0,\bullet)}(M, E)$. We denote by $\overline{\partial}^{E*}$ the formal adjoint of $\overline{\partial}^E$ with respect to $\langle \cdot, \cdot \rangle_{L^2}$. The Kodaira Laplacian is given by

$$\square^E := \overline{\partial}^E \overline{\partial}^{E*} + \overline{\partial}^{E*} \overline{\partial}^E. \tag{2.2.8}$$

The operator $\square^E$ preserves the $\mathbb{Z}$-grading on $\Omega^{(0,\bullet)}(M, E)$. We also define

$$D^E := \sqrt{2}(\overline{\partial}^E + \overline{\partial}^{E*}), \quad \text{then} \quad (D^E)^2 = 2\square^E. \tag{2.2.9}$$

By Hodge theory, the operator $\square^E$ has finite dimensional kernel. We denote by P the orthogonal projection onto this kernel and by $P^\perp = \mathrm{Id} - P$ the orthogonal projection onto its orthogonal complement. By the standard facts on heat kernels (see [12, Theorem 2.30, Proposition 2.37]), we can define the zeta-function: for $z \in \mathbb{C}, \operatorname{Re} z > n$ we set

$$\zeta_E(z) := -\mathrm{M}\big[\mathrm{Tr_s}\big[N \exp(-u\square^E) P^\perp\big]\big]. \tag{2.2.10}$$

Definition 2.2.3. The analytic torsion of Ray-Singer of (E, h^E) is defined as

$$T(g^{TM}, h^E) := \exp\left(-\tfrac{1}{2}\zeta_E'(0)\right). \tag{2.2.11}$$

Remark 2.2.4. Let $\det(\square^E|_{\Omega^i})$ be the regularized determinant of $\square^E|_{\Omega^{(0,i)}(M)}$, then

$$T(g^{TM}, h^E) = \textstyle\prod_i \det\left(\square^E|_{\Omega^i}\right)^{-(-1)^i i/2}. \tag{2.2.12}$$

2.2.2 Asymptotics of the analytic torsion on manifolds

In this section we present a proof of Theorem 2.1.1. We follow closely the strategy of the proof of the main theorem in [27] and we defer the proof of some technical details to Section 2.3.3.

Let (M, g^{TM}, Θ) be a compact Hermitian manifold and let (E, h^E), (L, h^L) be holomorphic Hermitian vector bundles over M. We suppose that (L, h^L) is a positive line bundle, i.e.

$$R^L(U, \overline{U}) > 0, \quad \text{for any} \quad U \in T^{(1,0)}M. \tag{2.2.13}$$

We denote by $\square_p$ the Laplacian associated to $L^p \otimes E$ and by $\zeta_p, p \in \mathbb{N}$ the zeta-function $\zeta_{L^p \otimes E}$. For $x, y \in M$, we denote by $\exp(-u\square_p/p)(x, y)$ the smooth kernel with respect to the volume form dv_M of the heat operator $\exp(-u\square_p/p)$.

Theorem 2.2.5 ([27, Theorem 4], [40, Theorem 1.2]). *There are smooth sections $a_{i,u}(x)$, $i \in \mathbb{N}$ of $\oplus_{l\geq 0}\mathrm{End}(\Lambda^l(T^{*(0,1)}M)\otimes E)$ over M such that for every $u>0$, we have*

$$\exp(-u\Box_p/p)(x,x) = \sum_{i=0}^{k} a_{i,u}(x)p^{n-i} + O(p^{n-k-1}), \quad \textit{as} \quad p\to+\infty, \tag{2.2.14}$$

and the estimate is uniform in $x\in M$ and u, as u varies in a compact subspace of $]0,+\infty[$.

For the proof of the following proposition see Section 2.3.3.

Proposition 2.2.6. There are smooth sections $a_i^{[j]}(x)$ of $\oplus_{l\geq 0}\mathrm{End}(\Lambda^l(T^{*(0,1)}M)\otimes E)$ such that

$$a_{i,u}(x) = \sum_{j=-n}^{k} a_i^{[j]}(x)u^j + o(u^k), \tag{2.2.15}$$

as $u\to 0$, for any $k\in\mathbb{N}$. Moreover, there are $c_i, d_i>0$ such that for any $u\gg 1, x\in M$

$$\left|a_{i,u}^{[>0]}(x)\right| \leq c_i\exp(-d_i u), \tag{2.2.16}$$

where $[>0]$ means the projection onto positive degree terms.

The estimation (2.2.16) was proved in [40, Theorem 1.2]. Now we can restate Theorem 2.1.1 in a precise way

Theorem 2.2.7. *There are local coefficients $\alpha_i,\beta_i\in\mathbb{R}, i\in\mathbb{N}$ such that for any $k\in\mathbb{N}$, as $p\to+\infty$*

$$\zeta_p'(0) = \sum_{i=0}^{k} p^{n-i}\left(\alpha_i\log p+\beta_i\right) + o(p^{n-k}), \tag{2.2.17}$$

as $p\to\infty$, where

$$\alpha_i = \int_M \mathrm{Tr_s}\big[Na_i^{[0]}(x)\big]\,dv_M(x), \quad \beta_i \;= -\mathrm{M}_u\big[\int_M \mathrm{Tr_s}\big[Na_{i,u}(x)\big]\,dv_M(x)\big]'(0). \tag{2.2.18}$$

To prove Theorem 2.2.7, we need to introduce the constants $b_{p,i}\in\mathbb{R}$ for $i\geq -n, p\in\mathbb{N}^*$, which satisfy the following asymptotic expansion for any $k\in\mathbb{N}$ (cf. [12, Theorem 2.30])

$$p^{-n}\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] = \sum_{i=-n}^{k} b_{p,i}u^i + o(u^{k+1}), \quad \text{as} \quad u\to+0. \tag{2.2.19}$$

We also need the next three propositions, for their proof see Section 2.3.3.

Proposition 2.2.8. As $p\to\infty$, the following expansion holds for any $k\in\mathbb{N}$

$$b_{p,i} = \sum_{j=0}^{k} b_i^{[j]}p^{-j} + o(p^{-k}), \quad \text{with} \quad b_i^{[j]} = \int_M \mathrm{Tr_s}\big[Na_j^{[i]}(x)\big]\,dv_M(x). \tag{2.2.20}$$

The following propositions are essential extensions of [27, Theorem 2] (cf. [85, §5.5]). They form the core of the proof.

Proposition 2.2.9. For any $k\in\mathbb{N}, u_0>0$ there exist $C>0$ such that for any $u\in]0,u_0[, p\in\mathbb{N}^*$:

$$\begin{aligned} p^k\Big|\Big(p^{-n}\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] - \sum_{j=-n}^{0} u^j b_{p,j}\Big) \\ - \sum_{i=0}^{k-1} p^{-i}\Big(\int_M \mathrm{Tr_s}\big[Na_{i,u}(x)\big]\,dv_M(x) - \sum_{j=-n}^{0} u^j b_j^{[i]}\Big)\Big| \leq Cu. \end{aligned} \tag{2.2.21}$$

Proposition 2.2.10. For any $k \in \mathbb{N}, u_0 > 0$ there are $c, C > 0$ such that for $u > u_0, p \in \mathbb{N}^*$:

$$p^k \Big| p^{-n} \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - \sum_{j=0}^{k-1} p^{-j} \int_M \mathrm{Tr_s}\big[N a_{j,u}(x)\big]\, dv_M(x) \Big| \leq C \exp(-cu). \quad (2.2.22)$$

We point out that both of those Propositions are obtained for $k = 0$ in [27, Theorem 2]. The proof of Proposition 2.2.9 for any k is more-or-less parallel to the case $k = 0$. However, in Proposition 2.2.10, the original spectral gap approach works only for $k = 0$.

Proof of Theorem 2.2.7. We introduce the function

$$\tilde{\zeta}_p(z) = p^{z-n} \zeta_p(z). \quad (2.2.23)$$

It satisfies the following

$$p^{-n} \zeta_p'(0) = -\log(p) \tilde{\zeta}_p(0) + \tilde{\zeta}_p{}'(0), \quad (2.2.24)$$

$$\tilde{\zeta}_p(z) = -p^{-n} \mathrm{M}_u \big[\mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big]\big](z). \quad (2.2.25)$$

We remark that Theorem 2.2.7 "follows" formally from Theorem 2.2.5, (2.2.4), (2.2.24) and (2.2.25). Now we are going to make this reasoning precise.

Using (2.2.4) and (2.2.25), we obtain

$$\begin{aligned} \tilde{\zeta}_p{}'(0) = &- \int_0^1 \Big(p^{-n} \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - \sum_{j=-n}^{0} b_{p,j} u^j \Big) \frac{du}{u} \\ &- \int_1^{+\infty} p^{-n} \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] \frac{du}{u} - \sum_{j=-n}^{-1} \frac{b_{p,j}}{j} + \Gamma'(1) b_{p,0}, \quad (2.2.26) \\ \tilde{\zeta}_p(0) = &- b_{p,0}. \quad (2.2.27) \end{aligned}$$

The following notation makes sense due to Proposition 2.2.6:

$$\nu^{[i]} = -\mathrm{M}_u \big[\textstyle\int_M \mathrm{Tr_s}\big[N a_{i,u}(x)\big]\, dv_M(x) \big]'(0). \quad (2.2.28)$$

By (2.2.4) and (2.2.20), we have

$$\begin{aligned} \nu^{[i]} = &- \int_0^1 \Big(\int_M \mathrm{Tr_s}\big[N a_{i,u}(x)\big]\, dv_M(x) - \sum_{j=-n}^{0} u^j b_j^{[i]} \Big) \frac{du}{u} \\ &- \int_1^{+\infty} \int_M \mathrm{Tr_s}\big[N a_{i,u}(x)\big]\, dv_M(x) \frac{du}{u} - \sum_{j=-n}^{-1} \frac{1}{j} b_j^{[i]} + \Gamma'(1) b_0^{[i]}. \quad (2.2.29) \end{aligned}$$

Suppose that the following limit holds for any $k \in \mathbb{N}$

$$\lim_{p \to +\infty} p^k \big(\tilde{\zeta}_p{}'(0) - \textstyle\sum_{i=0}^{k-1} \nu^{[i]} p^{-i} \big) = \nu^{[k]}. \quad (2.2.30)$$

Then from (2.2.24), (2.2.26), (2.2.27), (2.2.28), (2.2.30) and Proposition 2.2.8, we obtain Theorem 2.2.7.

Now let's prove (2.2.30). By (2.2.26) and (2.2.29) it suffices to prove that for $k \in \mathbb{N}$, as $p \to \infty$,

$$\begin{aligned}
1) \int_0^1 p^k \Big(\Big(p^{-n} \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - \sum_{j=-n}^{0} u^j b_{p,j} \Big) \\
- \sum_{i=0}^{k-1} p^{-i} \Big(\int_M \mathrm{Tr_s}\big[N a_{i,u}(x)\big]\, dv_M(x) - \sum_{j=-n}^{0} u^j b_j^{[i]} \Big) \Big) \frac{du}{u} \\
\to \int_0^1 \Big(\int_M \mathrm{Tr_s}\big[N a_{k,u}(x)\big]\, dv_M(x) - \sum_{j=-n}^{0} u^j b_j^{[k]} \Big) \frac{du}{u}, \qquad (2.2.31)
\end{aligned}$$

$$\begin{aligned}
2) \int_1^{+\infty} p^k \Big(p^{-n} \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - \sum_{j=0}^{k-1} p^{-j} \int_M \mathrm{Tr_s}\big[N a_{j,u}(x)\big]\, dv_M(x) \Big) \\
\to \int_1^{+\infty} \int_M \mathrm{Tr_s}\big[N a_{k,u}(x)\big]\, dv_M(x) \frac{du}{u}, \qquad (2.2.32)
\end{aligned}$$

$$3) p^k \Big(b_{p,j} - \sum_{i=0}^{k-1} b_j^{[i]} p^{-i} \Big) \to b_j^{[k]}. \qquad (2.2.33)$$

The first and second limits are consequences of Lebesgue dominated convergence theorem and Propositions 2.2.9, 2.2.10 correspondingly. The third one is a consequence of Proposition 2.2.8.

Now, we will prove that $\alpha_i, i \in \mathbb{N}$ do not depend on g^{TM}, h^L, h^E. Let $c \in \mathbb{R} \to g_c^{TM}, h_c^L, h_c^E$ be some variations of the metrics on TM, L, E. We suppose that g_c^{TM} is compatible with the complex structure J of M. We denote by $*_c$ the Hodge-star operator associated to g_c^{TM} and by $\Box_{p,c}$ the Kodaira Laplacian, associated to g_c^{TM}, h_c^L, h_c^ξ. From [23, Theorems 1.18], there are constants $M_{j,c}^p$, $j \geq -1$, $p \in \mathbb{N}^*$ such that for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
- \mathrm{Tr_s}\Big[\Big((*_c)^{-1} \frac{\partial *_c}{\partial c} + p (h_c^L)^{-1} \frac{\partial h_c^L}{\partial c} + (h_c^E)^{-1} \frac{\partial h_c^E}{\partial c} \Big) \exp(-u\Box_{p,c}/2) \Big] \\
= \sum_{j=-1}^{k} M_{j,c}^p u^j + o(u^k). \qquad (2.2.34)
\end{aligned}$$

Now, from [23, (1.117)], we have

$$-2 \frac{\partial}{\partial c} \log T(g_c^{TM}, h_c^{L^p \otimes E}) = -M_{0,c}^p + \mathrm{Tr_s}\Big[(*_c)^{-1} \frac{\partial *_c}{\partial c} P_c \Big], \qquad (2.2.35)$$

where P_c is the orthogonal projection onto $\ker(\Box_{p,c})$ with respect to g_c^{TM}, h_c^L, h_c^E. We remark that

$$- \mathrm{Tr_s}\Big[\Big((*_c)^{-1} \frac{\partial *_c}{\partial c} + p (h_c^L)^{-1} \frac{\partial h_c^L}{\partial c} + (h_c^E)^{-1} \frac{\partial h_c^E}{\partial c} \Big) \exp(-u\Box_{p,c}/2p) \Big]$$

$$= \sum_{j=-1}^{k} M_{j,c}^{p} p^{-j} u^{j} + o(u^{k}). \quad (2.2.36)$$

Now, from (2.2.36) we see that, similarly to Proposition 2.2.8, $M_{0,c}^{p}$ has an asymptotic expansion of the form (2.2.20), as $p \to \infty$. From [85, Theorem 4.1.1] we see that the asymptotics of $\mathrm{Tr_s}\big[(*_c)^{-1}\frac{\partial *_c}{\partial c}P_c\big]$ contains only powers of p. Thus, the change of the metric doesn't affect α_i, since only the powers of p appear in the asymptotics of (2.2.35). $\square$

2.3 Heat kernel of the high power of positive line bundle

Here we recall some fundamental results about the asymptotic expansion of the heat kernel of $\square_p/p$. For this we use the localization procedure of [46, §2], [86, §3 .4]. In our context this procedure is more natural than the one from [85], [40] since it respects the degrees of differential forms. This property permits us to give simple proofs of long-time estimates on the heat kernel (see Theorems 2.3.9, 2.3.10). Certainly, the original localization procedure from [85], [40] also gives the final result, but then one has to inevitably use some results on the Bergman kernel.

This section is organized as follows. In Section 3.1 we recall how to localize the calculation of the asymptotic expansion and how to tackle this localisation. Almost all the results of Section 3.1 appeared in [85] and were inspired by [25]. In Section 3.2 we recall the off-diagonal expansion of the heat kernel of the local version of the operator $\square_p/p$. Finally, in Section 3.3 we prove Propositions 2.2.6, 2.2.8, 2.2.9, 2.2.10; thus, completing the proof of Theorem 2.2.7.

2.3.1 Localization of the asymptotic expansion of the heat kernel

In this section we recall a localization procedure from [46] of the asymptotic expansion of $\exp(-u\square_p/p)(x,x), x \in M$ as $p \to +\infty$. We conserve the notation from Section 2.2.

To work with non Kähler metrics we recall the definition of *Bismut connection*. Let (X, g^{TX}, Θ_X) be a Hermitian manifold. Let S^B be a 1-form with values in the antisymmetric elements of $\mathrm{End}(T^{(1,0)}X)$, which satisfies (see [16, Definition 1.4])

$$\langle S^B(U)V, W\rangle = \tfrac{1}{2}\sqrt{-1}\big((\partial - \overline{\partial})\Theta_X\big)(U,V,W). \quad (2.3.1)$$

Definition 2.3.1 ([16, (1.15)], cf. also [85, Definition 1.2.9])**.** The Bismut connection ∇^B on TX is defined by $\nabla^B = \nabla^{TX} + S^B$, where ∇^{TX} is the Levi-Civita connection on (TX, g^{TX}).

The connection ∇^B preserves the complex structure of TX. Its family version was also defined by Bismut in [19, §3.6 and Theorem 3.8.1].

Theorem 2.3.2 (Bismut-Vasserot [27, Theorem 1])**.** *There exists $c > 0$ such that*

$$\mathrm{Spec}(\square_p) \subset \{0\} \cup [cp, +\infty[, \qquad \ker(\square_p) \subset \Omega^{(0,0)}(M, L^p \otimes E), \quad \textit{for } p \gg 1.$$

For $e = v^{(1,0)} + v^{(0,1)} \in T^{(1,0)}M \oplus T^{(0,1)}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ we denote by $c(e)$ the operator on $\Omega^{(0,\bullet)}(M)$, defined by

$$c(e) = \sqrt{2}(\overline{v}^{(1,0),*} \wedge -i_{v^{(0,1)}}), \tag{2.3.2}$$

where $\wedge$ and i are the exterior and interior product respectively. Let $e_1, \ldots, e_{2n}$ be an orthonormal frame of (TM, g^{TM}) and $e^1, \ldots, e^{2n}$ its dual frame. We define

$$^c(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_j}) = c(e_{i_1})c(e_{i_2}) \cdots c(e_{i_j}), \tag{2.3.3}$$

for $0 < i_1 < \ldots < i_j \leq n$. We extend this operation $\mathbb{C}$-linearly for any $B \in \Lambda^\bullet(T^*M \otimes_{\mathbb{R}} \mathbb{C})$.

We take $x \in M$. Let $\psi : M \supset U \to V \subset \mathbb{C}^n$ be a holomorphic local chart such that $B(0, 4\epsilon) \subset V$ and the restriction of E over U is trivial, where

$$0 < \epsilon < r_{\text{inj}}/4, \text{ where } r_{\text{inj}} \text{ is the injectivity radii of M .} \tag{2.3.4}$$

We denote in the sequel $X_0 := T_xM \simeq U$. We denote by $\rho : \mathbb{R} \to [0, 1]$ a smooth positive function such that

$$\rho(u) = \begin{cases} 0, & \text{for } |u| > 4, \\ 1, & \text{for } |u| < 2. \end{cases} \tag{2.3.5}$$

We define a Riemannian metric $g^{TX_0}(Z) = g^{TM}(\rho(|Z|/\epsilon)Z)$ over X_0. We choose a holomorphic frame of E over U and we introduce the Hermitian product h^{E_0} on $E_0 = X_0 \times E_x$ over X_0 by $h^{E_0}(Z) = h^E(\rho(|Z|/\epsilon)Z)$, where h^E is the matrix of the Hermitian product in the chosen holomorphic frame. Then g^{TX_0}, h^{E_0} coincide with g^{TM} and h^E over $B(0, 2\epsilon)$ and with trivial structures g^{TM}_x, h^E_x away from $B(0, 4\epsilon)$. We denote by R^{E_0} the Chern curvature of (E_0, h^{E_0}) and by Θ_0 the Hermitian form associated to g^{TX_0}.

Let σ be a holomorphic frame of L over U. It defines a trivialisation $\psi : L|_U \to U \times \mathbb{C}$. We define a function $\phi(Z), Z \in X_0$ by $e^{-2\phi(Z)} = |\sigma|^2_{h^L}(Z)$. Let's denote by $\phi^{[1]}$ and $\phi^{[2]}$ the first and second order Taylor expansions of ϕ at x, i.e.

$$\phi^{[1]}(Z) = \sum_{j=1}^{n} \Big(\frac{\partial \phi}{\partial z_j}(x) z_j + \frac{\partial \phi}{\partial \overline{z}_j}(x) \overline{z}_j \Big), \tag{2.3.6}$$

$$\phi^{[2]}(Z) = \mathrm{Re}\Big(\sum_{j,k=1}^{n} \Big(\frac{\partial^2 \phi}{\partial z_j \partial z_k}(x) z_j z_k + \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_j}(x) z_j \overline{z}_k \Big) \Big), \tag{2.3.7}$$

where $(z_1, \ldots, z_n)$ are the complex coordinates of Z. We define a function $\phi_\epsilon(Z)$ over X_0 by

$$\phi_\epsilon(Z) = \rho(|Z|/\epsilon)\phi(Z) + (1 - \rho(|Z|/\epsilon))\big(\phi(x) + \phi^{[1]}(Z) + \phi^{[2]}(Z)\big). \tag{2.3.8}$$

Let $h^{L_0}_\epsilon$ be the metric on $L_0 := X_0 \times \mathbb{C}$ defined by $|1|^2_{h^{L_0}_\epsilon} = e^{-2\phi_\epsilon(Z)}$. Let ∇^{L_0} be the Chern connection on $(L_0, h^{L_0}_\epsilon)$ and let $R^{L_0}_\epsilon$ be the curvature of it. Then by [46, (2.28)]

$$R^{L_0}_\epsilon \text{ is positive for } \epsilon > 0 \text{ small enough.} \tag{2.3.9}$$

From now on, we fix $\epsilon > 0$ which satisfies (2.3.4) and (2.3.9). We trivialize L_0 by a unitary section S_x of $(L_0, h^{L_0}_\epsilon)$, which we write as

$$S_x = e^\tau 1 \quad \text{with} \quad \tau(0) = \phi(x), \tag{2.3.10}$$

where the function τ is given by $\tau(Z) = \phi(x) - 2\int_0^1 (i_Z \partial\phi_\epsilon)_{tZ} dt$, so that S_x satisfies $\nabla^{L_0}_Z S_x = 0$. By abuse of notation, we drop ϵ from the notation introduced before.

We define a positive function $k : X_0 \to \mathbb{R}$ by the identity

$$dv_{X_0}(Z) = k(Z)\, dv_{T_xM}(Z), \tag{2.3.11}$$

where dv_{T_xM}, $dv_{X_0}(Z)$ are the Riemann volume forms on X_0, induced by g^{TM}_x and g^{TX_0} respectively. We see, in particular, that $k(0) = 1$. We denote by $h^{\det_0}$ the Hermitian metric, induced on $\det T^{(1,0)}X_0$ by g^{TX_0}.

We define a smooth self-adjoint section Φ_{E_0} of $\oplus_{i\geq 0}\mathrm{End}(\Lambda^i(T^{*(0,1)}X_0)\otimes E_0)$ over X_0 by

$$\Phi_{E_0} := \tfrac{1}{4} r^{X_0} + \tfrac{1}{2}{}^c(R^{E_0} + R^{\det_0}) + \tfrac{1}{2}\sqrt{-1}^c(\overline{\partial}\partial\Theta_0) - \tfrac{1}{8}|(\overline{\partial} - \partial)\Theta_0|^2, \tag{2.3.12}$$

where r^{X_0} is the scalar curvature of g^{TX_0} and $R^{\det_0}$ is the Chern curvature of $(\det T^{(1,0)}X_0, h^{\det_0})$.

We denote by $\nabla^{B,\Lambda^{0,\bullet}_0}$ the natural extension of the Bismut connection ∇^B of (X_0, g^{TX_0}) on $\Lambda^\bullet(T^{*(0,1)}X_0)$ (see [85, (1.4.27)]). We set

$$L_{p,x} := \Delta^{B,\Lambda^{0,\bullet}_0\otimes L^p_0\otimes E_0} + \tfrac{1}{2}p^c(R^{L_0}) + \Phi_{E_0}, \tag{2.3.13}$$

where $\Delta^{B,\Lambda^{0,\bullet}_0\otimes L^p_0\otimes E_0}$ is the Bochner Laplacian on $\Lambda^\bullet(T^{*(0,1)}X_0)\otimes L^p_0\otimes E_0$ associated with

$$\nabla^{B,\Lambda^{0,\bullet}_0\otimes L^p_0\otimes E_0} := \nabla^{B,\Lambda^{0,\bullet}_0}\otimes 1\otimes 1 + 1\otimes\nabla^{L^p_0}\otimes 1 + 1\otimes 1\otimes\nabla^{E_0}, \tag{2.3.14}$$

and g^{TX_0}, h^{L_0}, h^{E_0}. By the trivialization as above, we have $\Lambda^\bullet(T^{*(0,1)}X_0)\otimes L^p_0\otimes E_0 \simeq \Lambda^\bullet(T^{*(0,1)}_xM)\otimes L^p_x\otimes E_x$. The operator $L_{p,x}$ preserves $\mathbb{Z}$-grading on $\Omega^{(0,\bullet)}(X_0, L^p_x\otimes E_x)$ and the following formula holds (see [16, Theorem 1.3])

$$L_{p,x} = 2\big(\overline{\partial}^{X_0}_p + \overline{\partial}^{X_0*}_p\big)^2, \tag{2.3.15}$$

where $\overline{\partial}^{X_0}_p$ is the Dolbeaut operator acting on $\Omega^{(0,\bullet)}(X_0, L^p_0\otimes E_0)$, and $\overline{\partial}^{X_0*}_p$ is its adjoint with respect to the L^2-norm induced by g^{TX_0}, h^{L_0} and h^{E_0}. Then $L_{p,x}$ is self-adjoint with respect to this norm.

All the constructions made here could be performed uniformly in a neighbourhood of $x \in M$. For the rest of this article we denote by $\mathscr{C}^m(M)$ the $\mathscr{C}^m$-norm with respect to the parameter x.

By (2.3.15) and the positivity of R^{L_0}, we have

Theorem 2.3.3 ([27, Theorem 1] cf. also the proof of [85, Theorem 1.5.7, 1.5.8])**.** *There is $\mu > 0$ such that*

$$\mathrm{Spec}(L_{p,x}) \subset \{0\}\cup[\mu p, +\infty[, \qquad \ker(L_{p,x_0}) \subset \Omega^{(0,0)}(X_0, L^p_0\otimes E_0), \quad \textit{for} \quad p \gg 1.$$

For $(Z, Z') \in X_0 \times X_0$ we denote by $\exp(-uL_{p,x})(Z, Z')$ the smooth kernel of the heat operator $\exp(-uL_{p,x})$ with respect to the volume form dv_{X_0}. We have

Lemma 2.3.4 ([85, Lemma 1.6.5, (5.5.73)]). There are constants $C, c > 0, l \in \mathbb{N}$ such that uniformly on $p \in \mathbb{N}^*, u \in]0, +\infty[, x \in M$ and $Z, Z' \in T_xM; |Z|, |Z'| < \epsilon$, we have

$$\left| \exp(-u\Box_p/p)(\exp_x(Z), \exp_x(Z')) - \exp(-uL_{p,x}/(2p))(Z, Z') \right| \leq Cp^l \exp(-cp/u). \quad (2.3.16)$$

Remark 2.3.5. Since the proof of Lemma 2.3.4 relies on finite propagation speed of solutions of the hyperbolic equations [85, Theorem D.2.1] and on the fact that $\Box_p$ is essentially self-adjoint operator, and those facts hold for orbifolds (see [83, p.230]), Lemma 2.3.4 itself holds for orbifolds.

More properties of the asymptotics of heat kernel. Now we recall a procedure of replacing a discrete parameter $p \in \mathbb{N}$ in the construction of $L_{p,x}$ to a continuous $t \in [0, 1]$. This permits us to interpret the asymptotic expansion in p as an instance of a Taylor expansion in t.

We recall that we fixed a unitary section S_x of L_x, so that we can say that $L_{p,x}, p \in \mathbb{N}^*$ act on $\Omega^{(0,\bullet)}(X_0, E_0)$, which is independent of p. For $s \in \Omega^{(0,\bullet)}(X_0, E_0), Z \in X_0, t = 1/\sqrt{p}$, we set

$$\begin{aligned} &(S_t s)(Z) := s(Z/t), \\ &\nabla_t := S_t^{-1} t k^{1/2} \nabla^{B, \Lambda_0^{0,\bullet} \otimes L_0^p \otimes E_0} k^{-1/2} S_t, \\ &L_{2,x}^t := S_t^{-1} t^2 k^{1/2} L_{p,x} k^{-1/2} S_t. \end{aligned} \quad (2.3.17)$$

By [85, (1.6.31)] the definition of $\nabla_t, L_{2,x}^t$ extends for $t \in]0, 1]$. Moreover, the operator $L_{2,x}^t$ is self-adjoint with respect to (2.3.18).

In what follows, we will repeatedly use the results from [85]. Their localization procedure is different from ours, but their arguments apply directly when one chooses the Sobolev norms as

$$\|s\|_{t,0}^2 := \int_{X_0} \left\| s(Z) \right\|_{t,h}^2 dv_{T_xM}(Z), \quad (2.3.18)$$

$$\|s\|_{t,m}^2 := \sum_{k=0}^{m} \sum_{i_1, \ldots, l_k = 1}^{2n} \left\| \nabla_{t, e_{i_1}} \ldots \nabla_{t, e_{i_k}} s \right\|_{t,0}^2, \quad (2.3.19)$$

where $s \in \Omega^{(0,\bullet)}(X_0, E_0)$, $\|\cdot\|_{t,h}(Z)$ is the pointwise norm induced by $g^{TX_0}(tZ), h^{E_0}(tZ)$, and $e_1, \ldots, e_{2n}$ are as in (2.3.3).

Let $w_1, \ldots, w_n$ be an orthonormal frame of $(T_x^{(1,0)}M, h_x^{T^{(1,0)}M})$ and let $w^1, \ldots, w^n$ be its dual frame. We denote by $\nabla_{0,\cdot}$ the connection on the vector bundle $\Lambda^\bullet(T^{*(0,1)}X_0) \otimes E_0$ and by $L_{2,x}^0$ the operator on $\Omega^{(0,\bullet)}(X_0, E_0)$, defined by the formulas

$$\begin{aligned} \nabla_{0,\cdot} &:= \nabla_\cdot + \tfrac{1}{2} R_x^L(Z, \cdot), \\ L_{2,x}^0 &:= -\textstyle\sum_i \nabla_{0,e_i}^2 + 2\sum_{i,j} R_x^L(w_i, \overline{w}_j) \overline{w}^j \wedge i_{\overline{w}_i} - \sum_i R_x^L(w_i, \overline{w}_i). \end{aligned} \quad (2.3.20)$$

Lemma 2.3.6 ([85, Lemma 1.6.6]). The family of operators $\nabla_t, L_{2,x}^t$ is smooth in t and

$$\nabla_t \to \nabla_0, \qquad L_{2,x}^t \to L_{2,x}^0, \quad \text{as} \quad t \to 0. \quad (2.3.21)$$

We define the operators $\mathcal{O}_1, \mathcal{O}_2, \ldots$ by the following expansion, as $t \to 0$,

$$L^t_{2,x} = L^0_{2,x} + t\mathcal{O}_1 + t^2\mathcal{O}_2 + \cdots + t^k\mathcal{O}_k + o(t^k), \quad k \in \mathbb{N}. \tag{2.3.22}$$

We also denote by $\exp(uL^t_{2,x})(Z, Z')$ the smooth kernel of the heat operator $\exp(-uL^t_{2,x})$ with respect to dv_{T_xM}. Then for $t = 1/\sqrt{p}$, we have (cf. [85, (1.6.66)])

$$\exp(-uL_{p,x}/p)(Z, Z') = p^n \exp(-uL^t_{2,x})(Z/t, Z'/t)k^{-1/2}(Z)k^{-1/2}(Z'). \tag{2.3.23}$$

Now we recall some properties of the operator $L^t_{2,x}$. The reason why we are interested in it is Lemma 2.3.4 and (2.3.23). By [85, (4.2.31), (4.2.40)], we have

Proposition 2.3.7. The function $t \in]0, 1] \to \exp(-uL^t_{2,x})(0, 0)$ extends smoothly to $[0, 1]$ by taking the value $\exp(-uL^0_{2,x})(0, 0)$ at $t = 0$. All its derivatives are uniformly bounded on $x \in M$ and u, varying in a compact subset of $]0, +\infty[$. Moreover,

$$\frac{\partial^{2i+1}}{\partial t^{2i+1}} \exp(-uL^t_{2,x})(0, 0)|_{t=0} = 0 \tag{2.3.24}$$

By Lemma 2.3.4, Proposition 2.3.7 and (2.3.23), we see that in Theorem 2.2.5 we have

$$a_{k,u}(x) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial t^{2k}} \exp(-uL^t_{2,x}/2)(0, 0)|_{t=0}. \tag{2.3.25}$$

Theorem 2.3.8. *For $t \in [0, 1]$, there are sections $B_{t,r} \in \oplus_{j\geq 0}\mathscr{C}^\infty(M, \mathrm{End}(\Lambda^j(T^{*(0,1)}M) \otimes E))$, $r \in \mathbb{Z}, r \geq -n$, such that for any $k, m \in \mathbb{N}, u_0 > 0$ there is $C > 0$ such that for any $u \in]0, u_0]$*

$$\Big| \exp(-uL^t_{2,x}/2)(0, 0) - \sum_{r=-n}^{k} B_{t,r}(x)u^r \Big|_{\mathscr{C}^m(M\times[0,\, t_0])} \leq Cu^{k+1}, \tag{2.3.26}$$

where the second coordinate of $M \times [0,\, t_0]$ represents t. Moreover,

$$\frac{\partial^{2i+1}}{\partial t^{2i+1}} B_{t,r}(x)|_{t=0} = 0. \tag{2.3.27}$$

Proof. The proof of (2.3.26) is done in [85, (5.5.91)]. By (2.3.24) and (2.3.26), we get (2.3.27). □

By Theorem 2.3.3 and (2.3.17) there are $t_0, \mu > 0$ such that for $t \in [0, t_0]$, we have

$$\mathrm{Spec}(L^t_{2,x}) \subset \{0\} \cup [\mu, +\infty[, \qquad \ker(L^t_{2,x}) \subset \Omega^{(0,0)}(X_0, E_0). \tag{2.3.28}$$

We fix t_0, which satisfies (2.3.28). From now on, we only work with $t < t_0$.

Recall that $L^t_{2,x}$ preserves $\mathbb{Z}$-grading on $\Omega^{(0,\bullet)}(X_0, E_0)$. We denote by $L^{t,>0}_{2,x}$ the restriction of the operator $L^t_{2,x}$ on the positive degree. From (2.3.28),

$$\exp(-uL^{t,>0}_{2,x}) = F_u(L^t_{2,x})|_{\Omega^{(0,>0)}(X_0,E_0)}, \tag{2.3.29}$$

where $F_u(L^t_{2,x})|_{\Omega^{(0,>0)}(X_0,E_0)}$ is the restriction on positive degree terms of the operator $F_u(L^t_{2,x})$ defined in [85, (4.2.21), (4.2.22)] . We denote by $\exp(-uL^{t,>0}_{2,x})(Z, Z'); Z, Z' \in X_0$ the smooth kernel of the heat operator $\exp(-uL^{t,>0}_{2,x})$ with respect to the volume form dv_{T_xM} on X_0. Then $\exp(-uL^{t,>0}_{2,x})(Z, Z')$ is the restriction of $\exp(-uL_{2,x})(Z, Z')$ on positive degree.

Theorem 2.3.9. *For any $u_0 > 0, m \in \mathbb{N}$, there are constants $c, C > 0$, such that for $u > u_0$*

$$\left| \exp(-uL_{2,x}^{t,>0}/2)(0,0) \right|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C\exp(-cu), \tag{2.3.30}$$

where second coordinate of $M \times [0,\, t_0]$ represents t. Moreover, we have

$$a_{k,u}^{[>0]}(x) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial t^{2k}}\exp(-uL_{2,x}^{t,>0}/2)(0,0)|_{t=0}, \qquad \frac{\partial^{2k+1}}{\partial t^{2k+1}}\exp(-uL_{2,x}^{t,>0}/2)(0,0)|_{t=0} = 0, \tag{2.3.31}$$

where $[>0]$ means the projection onto positive degree terms.

Proof. Estimation (2.3.30) is a consequence of [85, Corollary 4.2.6, (4.2.31)] and (2.3.29). The identities (2.3.31) follow directly from (2.3.24), (2.3.25), (2.3.30) and the discussion before Theorem 2.3.9. □

2.3.2 Off-diagonal estimations of the heat kernel and related quantities

In this section we explain some results concerning off-diagonal expansion of the heat kernel of $L_{2,x}^t$. We don't claim originality on those results, as some of them already appeared in [85, §4.2, §5.5] and some were implicit. This section is only used in the proof of Theorem 2.1.5 in Section 2.5. We fix t_0 as in (2.3.28). We have the following off-diagonal version of Theorem 2.3.9.

Theorem 2.3.10. *For any $m \in \mathbb{N}, u_0 > 0$ there are $c, C, C' > 0$ such that for any $x \in M, u \geq u_0, Z, Z' \in T_xM$, we have the following inequality*

$$\left| \exp(-uL_{2,x}^{t,>0})(Z,Z') \right|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C(1+|Z|+|Z'|)^{C'}\exp(-cu - c|Z-Z'|^2/u), \tag{2.3.32}$$

where the second coordinate of $M \times [0,\, t_0]$ represents t.

Proof. By [85, Theorem 4.2.5], we get for some $c, C, C' > 0$

$$\left| \exp(-uL_{2,x}^{t})(Z,Z') \right|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C(1+|Z|+|Z'|)^{C'}\exp(cu - c|Z-Z'|^2/u). \tag{2.3.33}$$

By [85, Corollary 4.2.6, (4.2.31)] and (2.3.29), we get

$$\left| \exp(-uL_{2,x}^{t,>0})(Z,Z') \right|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C(1+|Z|+|Z'|)^{C'}\exp(-cu - c|Z-Z'|), \tag{2.3.34}$$

for some $c, C, C' > 0$. We multiply (2.3.33) and (2.3.34) with suitable powers to get (2.3.32). □

Theorem 2.3.11. *For any $m \in \mathbb{N}, u_0, c_0 > 0$ there are $c, C, C' > 0$ such that for $x \in M, u \in]0, u_0]$ and $Z, Z' \in T_xM, |Z - Z'| \geq c_0$ we have the following inequality*

$$\left| \exp(-uL_{2,x}^{t})(Z,Z') \right|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C(1+|Z|+|Z'|)^{C'}\exp(-c|Z-Z'|^2/u), \tag{2.3.35}$$

where the second coordinate of $M \times [0,\, t_0]$ represents t.

Proof. The proof of this theorem proceeds exactly as in [85, Theorem 4.2.5] with only one modification. One has to change the condition for [85, (4.2.12)] by the following one: for any $m, m' \in \mathbb{N}, c, c_0 > 0$ there are $c', C, C' > 0$ such that for $u \in]0, u_0], h \geq c_0, a \in \mathbb{C}, |\operatorname{Im} a| \leq c'$ the following inequality holds (cf. [85, (4.2.12)])

$$|a|^m |K_{u,h}^{(m')}(a)| \leq C \exp(-c' h^2/u), \tag{2.3.36}$$

where $K_{u,h}$ is defined in [85, (4.2.11)]. We leave the details to the reader. □

We denote $v := \sqrt{u}$. Let Δ^{T_xM} be the Bochner Laplacian on (T_xM, g_x^{TM}). We set

$$L_{3,x}^t := \rho(|Z|/\epsilon) L_{2,x}^t + (1 - \rho(|Z|/\epsilon)) \Delta^{T_xM}, \tag{2.3.37}$$

with ρ as in (2.3.5) and $\epsilon > 0$ satisfying (2.3.4), (2.3.9). We denote

$$L_{4,x}^{t,v} := S_v^{-1} u L_{3,x}^t S_v, \quad \text{with} \quad v = \sqrt{u}, \tag{2.3.38}$$

and S_v is as in (2.3.17). We introduce the Sobolev norms

$$\|s\|_{t,v,0}^2 := \int_{X_0} \left\| s(Z) \right\|_h^2 dv_{T_xM}(Z), \tag{2.3.39}$$

$$\|s\|_{t,v,m}^2 := \sum_{k=0}^{m} \sum_{i_1,\ldots,l_k=1}^{2n} \left\| \nabla_{e_{i_1}} \ldots \nabla_{e_{i_k}} s \right\|_{t,0}^2, \tag{2.3.40}$$

where $s \in \mathscr{C}^\infty(X_0, \Lambda(T_Z^{*(0,1)} X_0) \otimes E_0)$, $\|\cdot\|_h$ is the pointwise norm induced by g_x^{TM}, h_x^E, ∇ is a usual derivative and $e_1, \ldots, e_{2n}$ are as in (2.3.3). We denote by $\mathbf{H}_{t,v}^m, m \in \mathbb{N}$ the Sobolev spaces induced by those norms.

Then, similarly to [25, Theorem 11.26], [85, Theorem 1.6.7], there are $c_1, c_2 > 0$ such that for $t \in]0, 1], v \in]0, 1]$, we have the following estimations

$$\operatorname{Re}\langle L_{4,x}^{t,v} s, s \rangle_{t,v,0} \geq c_1 \|s\|_{t,v,1}^2 - c_2 \|s\|_{t,v,0}^2, \tag{2.3.41}$$

for $s \in \mathscr{C}^\infty(X_0, \Lambda(T_Z^{*(0,1)} X_0) \otimes E_0)$ of compact support.

Then, similarly to [85, Theorem 1.6.8], for any $\lambda \in \mathbb{C}$ as in [85, Figure 1.1], the inverse operator $(\lambda - L_{4,x}^{t,v})^{-1}$ is bounded as an operator operator on $\mathbf{H}_{t,v}^0$. Then one can define the heat operator $\exp(-w L_{4,x}^{t,v}), w > 0$ by the integration over a contour of $(\lambda - L_{4,x}^{t,v})^{-1}$ as it was done in [85, (1.6.48)].

Similarly, we define the heat operator $\exp(-w L_{3,x}^t), w > 0$. Even though the operators $L_{3,x}^t, L_{4,x}^{t,v}$ are not self-adjoint, by [85, (1.6.31)], their adjoints are of the same form as the operators themselves. Thus, all the arguments on the estimation of the kernels of $\exp(-w L_{3,x}^t), \exp(-w L_{4,x}^{t,v})$ can be repeated line in line from [85].

Now, similarly to (2.3.23), we have

$$\exp(-u L_{3,x}^t)(Z, Z') = u^{-n} \exp(-L_{4,x}^{t,v})(Z/v, Z'/v), \tag{2.3.42}$$

where we denote by $\exp(-L_{3,x}^t)(Z, Z'), \exp(-L_{4,x}^{t,v})(Z, Z')$ the smooth kernels of the heat operators $\exp(-L_{3,x}^t), \exp(-L_{4,x}^t)$ corresponding to the volume form dv_{T_xM}. We have the following analogue of Lemma 2.3.4, which follows from [85, (5.5.81)] and (2.3.42)

Proposition 2.3.12. There exists $u_0 > 0$ such that for any $m \in \mathbb{N}$, there are $c, C > 0$ such that for any $u > u_0, Z, Z' \in T_xM$, $|Z|, |Z'| < \epsilon$, we have

$$\big|\exp(-uL^t_{2,x})(Z,Z') - u^{-n}\exp(-L^{t,v}_{4,x})(Z/v, Z'/v)\big|_{\mathscr{C}^m(M\times[0,\,t_0])} \leq C\exp(-c/u), \tag{2.3.43}$$

where the second coordinate of $M \times [0,\, t_0]$ represents t.

Proposition 2.3.13. For any $v_0 > 0, m \in \mathbb{N}$ there are $c, C, C' > 0$ such that for $Z, Z' \in T_xM$, we have

$$\big|\exp(-L^{t,v}_{4,x})(Z,Z')\big|_{\mathscr{C}^m(M\times[0,\,t_0]\times[0,\,v_0])} \leq C(1+|Z|+|Z'|)^{C'}\exp(-c|Z-Z'|^2), \tag{2.3.44}$$

where the second and third coordinates of $M \times [0,\, t_0] \times [0, v_0]$ represents t and v respectively.

Proof. When we fix $v = 1$, this proposition is a special case of [85, Theorem 4.2.5]. In general, since the operator $L^{t,v}_{4,x}$ depends smoothly on (t, v), we may repeat the argument of the proof of [85, Theorem 4.2.5] as if the parameter t in that theorem had two components (t, v). □

2.3.3 Proof of Propositions 2.2.6, 2.2.8, 2.2.9, 2.2.10

Here we finally prove Propositions 2.2.6, 2.2.8, 2.2.9, 2.2.10; thus, completing the proof of Theorem 2.2.7. Then we also explain Remark 2.1.4.

Proof of Proposition 2.2.6. From Theorem 2.3.8 and (2.3.25), we get (2.2.15) with

$$a^{[j]}_k(x) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial t^{2k}}B_{t,j}(x)|_{t=0}. \tag{2.3.45}$$

Now (2.2.16) follows from Theorem 2.3.9. □

Proof of Proposition 2.2.8. Firstly, we make a connection between $b_{p,i}$ and $B_{t,i}, t = \frac{1}{\sqrt{p}}$, defined in Theorem 2.3.8. By Theorem 2.3.8, Lemma 2.3.4 and (2.3.23) we see that there is $l \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, there exist $c, C, C' > 0$ such that for any $p \in \mathbb{N}^*, u \in]0, 1]$, we have

$$\big|p^{-n}\exp(-u\square_p/p)(x,x) - \textstyle\sum_{r=-n}^{k} B_{t,r}(x)u^r\big| \leq Cu^{k+1} + Cp^l\exp(-cp/u) \leq C'u^{k+1}, \tag{2.3.46}$$

thus, by (2.2.19), we have

$$b_{p,i} = \textstyle\int_M \mathrm{Tr_s}\big[NB_{t,i}(x)\big]\, dv_M(x), \quad t = 1/\sqrt{p}. \tag{2.3.47}$$

From (2.3.27), we get the estimation from (2.2.20) with

$$b^{[j]}_i = \frac{1}{(2j)!}\frac{\partial^{2j}}{\partial t^{2j}}\Big(\int_M \mathrm{Tr_s}\big[NB_{t,i}(x)\big]\, dv_M(x)\Big)|_{t=0}. \tag{2.3.48}$$

Finally, (2.2.20) follows from (2.3.45) and (2.3.48). □

Proof of Proposition 2.2.9. By Theorem 2.3.8 and (2.3.24) we see that for any $k \in \mathbb{N}; u_0, t_0 > 0$ there exists $C > 0$ such that for any $u \in]0, u_0], t \in]0, t_0]$, we have

$$\Big|\frac{1}{t^{2k}}\Big[\exp(-uL^t_{2,x}/2)(0,0) - \sum_{r=-n}^{0} u^r B_{t,r}(x) \\ - \sum_{i=0}^{k-1} \frac{t^{2i}}{(2i)!}\frac{\partial^{2i}}{\partial t^{2i}}\Big(\exp(-uL^t_{2,x}/2)(0,0) - \sum_{r=-n}^{0} u^r B_{t,r}(x)\Big)|_{t=0}\Big]\Big| \leq Cu, \quad (2.3.49)$$

for any $x \in M$. We conclude by Lemma 2.3.4, (2.3.23), (2.3.25), (2.3.47) and (2.3.48). □

Proof of Proposition 2.2.10. We distinguish 2 cases:

1. $u > \sqrt{p}$. In this case we proceed similarly to [85, Theorem 5.5.11]. Theorem 2.3.2 implies the first inequality in the following series of estimations and the second one is true by Theorem 2.2.5

$$p^{-n+k}\mathrm{Tr}\big[\exp(-u\square_p^{(>0)}/p)\big] \leq p^{-n}\mathrm{Tr}\big[\exp(-\square_p^{(>0)}/p)\big]p^k \exp(-\tfrac{u-1}{p}cp) \\ \leq C'p^k \exp(-cu/2)\exp(-cu/2) \leq C''\exp(-cu/2). \quad (2.3.50)$$

Now, by Proposition 2.2.6 we obtain the following estimate for some $c_i, d_i, d', d > 0$ and any $x \in M$

$$p^{k-j}\big|\mathrm{Tr_s}\big[Na_{j,u}(x)\big]\big| \leq c_j p^{k-j}\exp(-d_j u) \leq d'\exp(-du/2). \quad (2.3.51)$$

Then (2.2.22) follows from (2.3.50) and (2.3.51).

2. $u \leq \sqrt{p}$. This case is subtler. Let $t = 1/\sqrt{p}$. By Theorem 2.3.9, we have

$$p^k\Big|\int_M \mathrm{Tr_s}\big[N\exp(-uL^t_{2,x}/2)(0,0)\big]\,dv_M(x) \\ - \textstyle\sum_{i=0}^k p^{-i}\int_M \mathrm{Tr_s}\big[Na_{i,u}(x)\big]\,dv_M(x)\Big| \leq C\exp(-cu). \quad (2.3.52)$$

We conclude by Lemma 2.3.4, (2.3.23), (2.3.52) and inequality $e^{-cp/u} \leq e^{-c\sqrt{p}/2}e^{-cu/2}$. □

Proof of Remark 2.1.4. Here we prove that the calculation of the asymptotics of the analytic torsion in Theorem 2.1.1 commutes with derivatives over the base in a family of manifolds.

More precisely, let $\pi : X \to B$ be a proper holomorphic submersion of complex manifolds. We note by $T\pi$ the relative tangent bundle. Let L, E be respectively a holomorphic line and vector bundles over X. We endow L, E with Hermitian metrics h^L, h^E, and suppose that the metric h^L is positive along the fibers. We endow the fibers $M_s := \pi^{-1}(s), s \in B$ with a Kähler metric g^{TM}_s, which is smooth in $s \in B$. Let's denote by $T(g^{TM}_s, h^{L^p\otimes E}|_{M_s})$ for $p \in \mathbb{N}$, the analytic torsion of $L^p \otimes E|_{M_s}$ associated with $g^{TM}_s, h^L|_{M_s}, h^E|_{M_s}$. Then by Theorem 2.1.1, for any $s \in B$, there are local coefficients $\alpha_i(s), \beta_i(s) \in \mathbb{R}, i \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, as $p \to +\infty$, we have

$$-2\log T(g^{TM}_s, h^{L^p\otimes E}|_{M_s}) = \textstyle\sum_{i=0}^k p^{n-i}\big(\alpha_i(s)\log p + \beta_i(s)\big) + o(p^{n-k}). \quad (2.3.53)$$

First of all, from [23, Theorem 1.3], for any compact $K \subset B$, there is p_0 such that for $p \geq p_0$, the function $\log T(g_s^{TM}, h^{L^p \otimes E}|_{M_s})$ is smooth over K, for $p \geq p_0$. We will explain that the functions $\alpha_i(s), \beta_i(s)$ are also smooth in $s \in B$ and for any $l \in \mathbb{N}$, we have

$$\Big\| -2\log T(g_s^{TM}, h^{L^p \otimes E}|_{M_s}) - \sum_{i=0}^{k} p^{n-i}\Big(\alpha_i(s)\log p + \beta_i(s)\Big)\Big\|_{\mathscr{C}^l(K)} \leq cp^{n-k}, \tag{2.3.54}$$

for some $c > 0$. From the proof of Theorem 2.1.1, we see that it is enough to explain why Theorem 2.2.5 and Propositions 2.2.6, 2.2.8, 2.2.9, 2.2.10 hold uniformly in $\mathscr{C}^k(\pi^{-1}(K))$, for any $k \in \mathbb{N}$. For brevity, we prove only the extension of Theorem 2.2.5, as other extensions are done in a similar way. For $s \in B$, we denote by $\square_{p,s}$ the Kodaira Laplacian on M_s associated with $(L^p \otimes E)|_{M_s}$. We need to prove that there are smooth sections $a_{i,u}(x)$, $i \in \mathbb{N}$ of $\oplus_{j\geq 0}\mathrm{End}(\Lambda^j(T^{*(0,1)}\pi) \otimes E)$ over X, such that for any $l \in \mathbb{N}$, $u > 0$, there is $c > 0$ such that for any $p \in \mathbb{N}^*$, we have

$$\Big\| \exp(-u\square_{p,\pi(x)}/p)(x,x) - \sum_{i=0}^{k} a_{i,u}(x)p^{n-i}\Big\|_{\mathscr{C}^l(\pi^{-1}(K))} \leq cp^{n-k-1}. \tag{2.3.55}$$

But to do so, essentially, we have to repeat the proof of [85, Theorem 4.2.5] with practically no change, since x, varying in the fiber, is already treated as a parameter in it. We need only to replace the words "uniformly on $x \in M_s$" by "uniformly on $x \in \pi^{-1}(K)$". $\square$

2.4 Proof of Theorem 2.1.3

In this section we calculate the coefficients α_1, β_1 from Theorem 2.1.1. More precisely, in Section 4.1 we fix the notation and we derive the formal expressions for α_1, β_1 in terms of $a_{1,u}$. In Section 4.2 we prove Theorem 2.1.3. For this we express $\mathscr{O}_1, \mathscr{O}_2$ in terms of creation and annihilation operators and we use the Duhamel's formula for the derivative of the heat kernel to calculate explicitly $A(u)$. This is the most technical part of the article. In Section 4.3 we verify Theorem 2.1.3 on the projective line, we describe how Theorem 2.1.3 is related to arithmetic Riemann-Roch theorem [106] and we make a connection between Theorem 2.1.3 and a result from the article [73, §4] by Klevtsov-Ma-Marinescu-Wiegmann.

2.4.1 Formal expressions for α_1, β_1

Recall that (M, g^{TM}, Θ) is a compact Kähler manifold of complex dimension n and $(E, h^E), (L, h^L)$ are holomorphic Hermitian vector bundles over M. We suppose

$$\Theta = \omega = \frac{\sqrt{-1}}{2\pi}R^L. \tag{2.4.1}$$

We take $x \in M$. For the calculation we use the localization procedure from [85, §1.6.2], where authors use the normal coordinates instead of holomorphic. We do so since some part of the

calculation was done in this context before. The formula (2.3.25), which is the only prerequisite we need from Section 2.3, still holds for this localization, since it relies on the wave-propagation technique (see [85, Theorem 4.2.3]). In this section every notation from Section 3 should be thought in the realms of the localization procedure from [85, §1.6.2].

For the sake of convenience, in this section we use the following notation

$$\begin{aligned} A(u) &= \int_M \mathrm{Tr_s}\big[Na_{1,u}(x)\big]\, dv_M(x), \\ R(u) &= C_u \int_M \mathrm{Tr_s}\Big[Ne^{-2\pi uN}\mathrm{Id}_{\Lambda^\bullet(T_x^{*(0,1)}M)\otimes E_x}\Big]\, dv_M(x). \end{aligned} \tag{2.4.2}$$

Proposition 2.4.1. For any $u > 0$, we have

$$\lim_{p\to\infty} p\big(p^{-n}\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] - R(u)\big) = A(u), \tag{2.4.3}$$

and the convergence is uniform as u varies in a compact subset of $]0,+\infty[$.

Proof. It follows from definitions of $A(u), R(u)$ and Theorems 2.2.5, 2.4.3. □

Now by (2.2.18), we have the following identities (see Notation 2.2.2)

$$\alpha_1 = A^{[0]}, \qquad \beta_1 = -M[A]'(0). \tag{2.4.4}$$

2.4.2 Proof of Theorem 2.1.3

In this section we prove Theorem 2.1.3. For this we give an explicit formula for $a_{1,u}(x), A(u)$ and then plug it in (2.4.4).

Let $w_1,\dots,w_n$ be an orthonormal basis of $(T_x^{(1,0)}M, h^{T_x^{(1,0)}M})$ and let $w^1,\dots,w^n$ be its dual basis. For $j = 1,\dots,n$, the vectors $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j), e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$ form an orthonormal basis of T_xM. This basis identifies T_xM and $\mathbb{R}^{2n}$. Let's introduce the complex coordinates $(z_1,\dots,z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ such that $Z = z + \overline{z}$ and $w_j = \sqrt{2}\frac{\partial}{\partial z_j}, \overline{w}_j = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$. We may consider $z, \overline{z}$ as vector fields by identifying z to $\sum_i z_i \frac{\partial}{\partial z_j}$ and $\overline{z}$ to $\sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$,

Now we define *creation* and *annihilation* operators (see (2.3.20), (2.4.1))

$$b_j = -2\nabla_{0,\frac{\partial}{\partial z_j}} = -2\frac{\partial}{\partial z_j} + \pi\overline{z}_j, \qquad b_j^+ = 2\nabla_{0,\frac{\partial}{\partial \overline{z}_j}} = 2\frac{\partial}{\partial \overline{z}_j} + \pi z_j. \tag{2.4.5}$$

We recall that by $\langle\cdot,\cdot\rangle$ we mean the $\mathbb{C}$-bilinear extension of g^{TM}. From now on, we use Einstein summation convention.

Theorem 2.4.2. *The following identities hold*

$$L_{2,x}^0 = \sum_j b_j b_j^+ + 4\pi N, \qquad \mathcal{O}_1 = 0, \tag{2.4.6}$$

$$\begin{aligned}\mathcal{O}_2 = &\frac{1}{3}\Big\langle R_x^{TM}\Big(\overline{z},\frac{\partial}{\partial z_i}\Big)\overline{z},\frac{\partial}{\partial z_j}\Big\rangle b_i^+ b_j^+ + \frac{1}{3}\Big\langle R_x^{TM}\Big(z,\frac{\partial}{\partial \overline{z}_i}\Big)z,\frac{\partial}{\partial \overline{z}_j}\Big\rangle b_i b_j \\ &- \frac{1}{3}\Big\langle R_x^{TM}\Big(z,\frac{\partial}{\partial \overline{z}_i}\Big)\overline{z},\frac{\partial}{\partial z_j}\Big\rangle b_i b_j^+ - \frac{1}{3}\Big\langle R_x^{TM}\Big(\overline{z},\frac{\partial}{\partial z_i}\Big)z,\frac{\partial}{\partial \overline{z}_j}\Big\rangle b_i^+ b_j \\ &- 2R_x^E\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\Big) - \frac{r_x^M}{6} \\ &+ \frac{2}{3}\Big\langle R_x^{TM}\Big(\overline{z},\frac{\partial}{\partial z_i}\Big)\frac{\partial}{\partial \overline{z}_i}\frac{\partial}{\partial z_j}\Big\rangle b_j^+ - \frac{2}{3}\Big\langle R_x^{TM}\Big(z,\frac{\partial}{\partial \overline{z}_i}\Big)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\Big\rangle b_j \\ &+ \frac{\pi}{3}\Big\langle R_x^{TM}(z,\overline{z})\overline{z},\frac{\partial}{\partial z_i}\Big\rangle b_i^+ - \frac{\pi}{3}\Big\langle R_x^{TM}(z,\overline{z})z,\frac{\partial}{\partial \overline{z}_i}\Big\rangle b_i \\ &- R_x^E\Big(\overline{z},\frac{\partial}{\partial z_i}\Big)b_i^+ + R_x^E\Big(z,\frac{\partial}{\partial \overline{z}_i}\Big)b_i \\ &- R_x^{\Lambda^\bullet(T^{*(0,1)}M)}\Big(\overline{z},\frac{\partial}{\partial z_i}\Big)b_i^+ + R_x^{\Lambda^\bullet(T^{*(0,1)}M)}\Big(z,\frac{\partial}{\partial \overline{z}_i}\Big)b_i \\ &+ 2R_x^{\det}\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\Big)\overline{w}^j\wedge i_{\overline{w}_i} + 4R_x^E\Big(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\Big)\overline{w}^j\wedge i_{\overline{w}_i}.\end{aligned} \tag{2.4.7}$$

Proof. In [40, Theorem 5.1] (cf. [85, Theorem 4.1.25]) authors obtained this result in degree $(0,0)$. In $\mathcal{O}_2$, the last 2 lines of its formula is the only contribution of non-zero degree. Theorem 2.4.2 was obtained in [84, Theorem 2.2] for $Spin^c$-Dirac operator. □

From (2.3.20), (2.4.1), (2.4.6) and Mehler formula for harmonic oscillator (see [85, Appendix E 2.2]), we get

Theorem 2.4.3. *We have the following identity*

$$\exp(-uL_{2,x}^0)(Z,0) = e^{-4\pi uN}C_{2u}\exp(-B_{2u}\|Z\|^2)\mathrm{Id}_{\Lambda^\bullet(T_x^{*(0,1)}M)\otimes E_x}, \tag{2.4.8}$$

where the operator N is defined in (2.1.11), $Z=(z_1,\dots,z_n)$,$\|Z\|^2=\sum|z_i|^2$ and

$$B_u = \frac{\pi}{2\tanh(\pi u)}, \qquad C_u = \frac{1}{(1-e^{-2\pi u})^n}. \tag{2.4.9}$$

From Duhamel's formula (see [40, Theorem 4.17]), (2.3.25) and (2.4.6), we get

$$a_{1,u}(x) = -\int_0^{u/2}\int_{X_0} e^{-vL_{2,x}^0}(0,Z)(\mathcal{O}_2 e^{-(u/2-v)L_{2,x}^0})(Z,0)dZ\,dv \tag{2.4.10}$$

From (2.4.10), to calculate $a_{1,u}$ we have to calculate $\mathcal{O}_2 e^{-uL_{2,x}^t}(Z,0)$ for $Z\in T_xM$. To simplify this calculation, we *omit* the terms of the form $P(z_1,\overline{z}_1,z_2,\cdots,\overline{z}_n)\exp(-vL_{2,x}^0/2)(Z,0)$, where P is a monomial with different degrees of z_i and $\overline{z}_i$ for some $i\in\mathbb{N}^*, i\le n$, since from Theorem 2.4.3 those terms disappear after the integration in Z in (2.4.10). We denote by $\sim$ the identification up to such *omission*. We note

$$\begin{aligned} &R_{i\bar{j}k\bar{l}} = \langle R^{TM}_x(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial z_k}, \tfrac{\partial}{\partial \overline{z}_l}\rangle, \quad R^E_{i\bar{j}} = R^E_x\big(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_j}\big), \\ &R^\Lambda_{i\bar{j}} = R^{\Lambda^{0,\bullet}(T^*M)}_x\big(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_j}\big), \qquad R^{\det}_{i\bar{j}} = R^{\det}_x\big(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_j}\big), \end{aligned} \tag{2.4.11}$$

where $R^{\det}$ is the Chern curvature of $(\det T^{(1,0)}M, h^{\det})$ for the induced by $h^{T^{(1,0)}M}$ Hermitian metric $h^{\det}$. We constantly use the following well-known symmetries of the curvature tensor

$$R_{i\bar{i}j\bar{j}} = R_{i\bar{j}j\bar{i}} = R_{j\bar{j}i\bar{i}} = R_{j\bar{i}i\bar{j}}, \qquad R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}. \tag{2.4.12}$$

Lemma 2.4.4. For $u > 0, Z \in T_xM$, we have

$$\begin{aligned} (\mathcal{O}_2 e^{-uL^t_{2,x}})(Z,0) \sim \Big[&\frac{2}{3}R_{i\bar{i}j\bar{j}}|z_i|^2|z_j|^2(2\pi^2 - \delta_{ij}\pi^2) - \frac{4}{3}R_{i\bar{i}j\bar{j}} - 2R^E_{i\bar{i}} \\ &+ 2\pi R^E_{i\bar{i}}|z_i|^2 + 2\pi R^\Lambda_{i\bar{i}}|z_i|^2 + 2R^{\det}_{i\bar{j}}\overline{w}^j \wedge i_{\overline{w}_i} + 4R^E_{i\bar{j}}\overline{w}^j \wedge i_{\overline{w}_i}\Big] e^{-uL^t_{2,x}}(Z,0). \end{aligned} \tag{2.4.13}$$

Proof. From Theorem 2.4.2, we get

$$\begin{aligned} &(\mathcal{O}_2 e^{-uL^t_{2,x}})(Z,0) \sim \\ &\Big[\frac{1}{3}(2-\delta_{ij})R_{i\bar{i}j\bar{j}}\big(\overline{z}_i\overline{z}_j b_i^+ b_j^+ + z_i z_j b_i b_j\big) \\ &+ \frac{1}{3}(1-\delta_{ij})R_{i\bar{i}j\bar{j}}\big(z_i\overline{z}_j b_i b_j^+ + z_j\overline{z}_i b_i^+ b_j\big) \\ &+ \frac{1}{3}R_{i\bar{i}j\bar{j}}\big(z_j\overline{z}_j b_i b_i^+ + z_j\overline{z}_j b_i^+ b_i\big) - 2R^E_{i\bar{i}} - \frac{1}{6}r^M_x \\ &+ \frac{2}{3}R_{i\bar{i}j\bar{j}}\big(\overline{z}_j b_j^+ - z_j b_j\big) - \frac{\pi}{3}(2-\delta_{ij})R_{i\bar{i}j\bar{j}}z_j\overline{z}_j\big(\overline{z}_i b_i^+ + z_i b_i\big) \\ &+ R^E_{i\bar{i}}\big(\overline{z}_i b_i^+ + z_i b_i\big) + R^\Lambda_{i\bar{i}}\big(\overline{z}_i b_i^+ + z_i b_i\big) \\ &+ 2R^{\det}_{i\bar{j}}\overline{w}^j \wedge i_{\overline{w}_i} + 4R^E_{i\bar{j}}\overline{w}^j \wedge i_{\overline{w}_i}\Big] e^{-uL^t_{2,x}}(Z,0), \end{aligned} \tag{2.4.14}$$

where δ_{ij} is the Kronecker delta. We have the following formulas from Theorem 2.4.3

$$\begin{aligned} &(b_i e^{-uL^t_{2,x}})(Z,0) = (\pi + 2B_{2u})\overline{z}_i e^{-uL^t_{2,x}}(Z,0), \\ &(b_i^+ e^{-uL^t_{2,x}})(Z,0) = (\pi - 2B_{2u})z_i e^{-uL^t_{2,x}}(Z,0). \end{aligned} \tag{2.4.15}$$

Let's recall the following identity

$$r^M_x = \textstyle\sum_{i,j}\langle R(e_i,e_j)e_i, e_j\rangle = 2\sum_{i,j}\langle R(w_i,\overline{w}_i)w_j, \overline{w}_j\rangle = 8\sum_{i,j} R_{i\bar{i}j\bar{j}}. \tag{2.4.16}$$

From (2.4.14), (2.4.15) and (2.4.16), we get (2.4.13). □

Lemma 2.4.5. For $u > 0$ and $x \in M$, we have

$$a_{1,u}(x) = \Big[-\frac{4}{3}R_{i\bar{i}j\bar{j}}(1-e^{-2\pi u})^{-2}\Big(\frac{u}{2}(1+4e^{-2\pi u}+e^{-4\pi u}) - \frac{3}{4\pi}(1-e^{-4\pi u})\Big) \tag{2.4.17}$$

$$
\begin{aligned}
&+\frac{4}{6}R_{i\bar{i}j\bar{j}}u+R^E_{i\bar{i}}u-2R^E_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u}-\frac{1}{2\pi}(1-e^{-2\pi u})\Big)\\
&-2R^{\Lambda}_{i\bar{i}}(1-e^{-2\pi u})^{-1}\Big(\frac{u}{2}+\frac{u}{2}e^{-2\pi u}-\frac{1}{2\pi}(1-e^{-2\pi u})\Big)\\
&-\big(R^{\det}_{i\bar{j}}\overline{w}^j\wedge i_{\overline{w}_i}+2R^E_{i\bar{j}}\overline{w}^j\wedge i_{\overline{w}_i}\big)u\Big]\frac{e^{-2\pi uN}}{(1-e^{-2\pi u})^n}.
\end{aligned}
$$

Proof. From Theorem 2.4.3 and the fact that $\exp(-uL^t_{2,x}), u>0$ is a semigroup, we get

$$
\int_0^u dv\int_{X_0}e^{-vL^0_{2,x}}(0,Z)e^{-(u-v)L^0_{2,x}}(Z,0)dZ=u\frac{e^{-4\pi uN}}{(1-e^{-4\pi u})^n}. \tag{2.4.18}
$$

Similarly, we get

$$
\begin{aligned}
&\int_0^u dv\int_{X_0}e^{-vL^0_{2,x}}(0,Z)|z_j|^2e^{-(u-v)L^0_{2,x}}(Z,0)dZ \qquad (2.4.19)\\
&\qquad=\frac{e^{-4\pi uN}}{\pi(1-e^{-4\pi u})^{n+1}}\Big(u+ue^{-4\pi u}-\frac{1}{2\pi}(1-e^{-4\pi u})\Big),\\
&\int_0^u dv\int_{X_0}e^{-vL^0_{2,x}}(0,Z)|z_i|^2|z_j|^2e^{-(u-v)L^0_{2,x}}(Z,0)dZ \qquad (2.4.20)\\
&\qquad=\frac{e^{-4\pi uN}2^{\delta_{ij}}}{\pi^2(1-e^{-4\pi u})^{n+2}}\Big(u(1+4e^{-4\pi u}+e^{-8\pi u})-\frac{3}{4\pi}(1-e^{-8\pi u})\Big).
\end{aligned}
$$

We get (2.4.17) from Lemma 2.4.4, (2.4.10), (2.4.18) and (2.4.19). □

Now, we introduce the functions $g_1,g_2,\tilde{g}_2,\tilde{g}_3:\mathbb{R}\to\mathbb{R}$ by

$$
\begin{aligned}
g_1(u)&=\frac{e^{-2\pi u}}{1-e^{-2\pi u}}, & g_2(u)&=\frac{e^{-2\pi u}}{(1-e^{-2\pi u})^2},\\
\tilde{g}_2(u)&=\frac{ue^{-2\pi u}}{(1-e^{-2\pi u})^2}, & \tilde{g}_3(u)&=\frac{ue^{-2\pi u}}{(1-e^{-2\pi u})^3}.
\end{aligned} \tag{2.4.21}
$$

Lemma 2.4.6. For $u>0$, we have

$$
\begin{aligned}
A(u)=-\mathrm{rk}(E)\int_M c_1(TM)\frac{\omega^{n-1}}{(n-1)!}\big(g_2(u)+\tfrac{n}{2}g_1(u)-2\pi\tilde{g}_3(u)\big)\\
-\int_M c_1(E)\frac{\omega^{n-1}}{(n-1)!}\big(ng_1(u)-2\pi\tilde{g}_2(u)\big).
\end{aligned} \tag{2.4.22}
$$

Proof. Let $a_{i,j}\in\mathrm{End}(E); i,j=1,\dots,n$ then

$$
\mathrm{Tr_s}\big[e^{-2\pi uN}\big]=\mathrm{rk}(E)(1-e^{-2\pi u})^n, \tag{2.4.23}
$$

$$
\mathrm{Tr_s}\big[\textstyle\sum_{k,l=1}^n a_{k,l}\overline{w}^k\wedge i_{\overline{w}_l}e^{-2\pi uN}\big]=\sum_{j=1}^n(-1)^je^{-2\pi ju}\sum_{k=1}^n\mathrm{Tr}^E[a_{k,k}]\binom{n-1}{j-1} \tag{2.4.24}
$$

$$= -\mathrm{Tr}^E\big[\textstyle\sum_{i=1}^n a_{i,i}\big]e^{-2\pi u}(1-e^{-2\pi u})^{n-1}.$$

By taking derivatives of those identities, we get

$$\begin{aligned}&\mathrm{Tr_s}\big[Ne^{-2\pi uN}\big] = -\mathrm{rk}(E)ne^{-2\pi u}(1-e^{-2\pi u})^{n-1},\\ &\mathrm{Tr_s}\big[N\textstyle\sum_{i,j=1}^n a_{i,j}\overline{w}^i\wedge i_{\overline{w}_j}e^{-2\pi uN}\big] = -\mathrm{Tr}^E\big[\textstyle\sum_{i=1}^n a_{i,i}\big]e^{-2\pi u}(1-ne^{-2\pi u})(1-e^{-2\pi u})^{n-2}.\end{aligned} \tag{2.4.25}$$

By (2.4.11), we have

$$\textstyle\sum_i R^{\Lambda}_{i\bar{i}} = \frac{1}{2}\langle R^{T^{(1,0)}M}(w_k,\overline{w}_k)w_i,\overline{w}_j\rangle\overline{w}^j\wedge i_{\overline{w}_i} = \sum_{i,j} R^{\det}_{i\bar{j}}\overline{w}^j\wedge i_{\overline{w}_i}. \tag{2.4.26}$$

For a 2-form α, we define the function $\Lambda_\omega[\alpha]$ by the identity $\Lambda_\omega[\alpha]\frac{\omega^n}{n!} = \alpha\frac{\omega^{n-1}}{(n-1)!}$, then

$$\begin{aligned}&r^M_x = 8\textstyle\sum_{i,j} R_{i\bar{i}j\bar{j}} = 4\sum_i R^{\det}_{i\bar{i}} = 4\pi\Lambda_\omega\big[c_1(T^{(1,0)}M,h^{T^{(1,0)}M})\big],\\ &\textstyle\sum_i \mathrm{Tr}\big[R^E_{i\bar{i}}\big] = \pi\Lambda_\omega\left[c_1(E,h^E)\right].\end{aligned} \tag{2.4.27}$$

By Lemma 2.4.5, (2.4.16), (2.4.25), (2.4.26) and (2.4.27) to get

$$\begin{aligned}\mathrm{Tr_s}\big[Na_{1,u}(x)\big] = -\mathrm{rk}(E)\Lambda_\omega\big[c_1(T^{(1,0)}M,h^{T^{(1,0)}M})\big]\Big(g_2(u)+\tfrac{n}{2}g_1(u)-2\pi\tilde{g}_3(u)\Big)\\ -\Lambda_\omega\big[c_1(E,h^E)\big]\Big(ng_1(u)-2\pi\tilde{g}_2(u)\Big).\end{aligned} \tag{2.4.28}$$

By (2.4.2) and (2.4.28) we deduce (2.4.22). □

Proof of Theorem 2.1.3. We verify that as $u\to 0$,

$$\begin{aligned}&g_1(u) = g_1^{[-1]}u^{-1}-\tfrac{1}{2}+O(u), && g_2(u) = g_2^{[-2]}u^{-2}+g_2^{[-1]}u^{-1}-\tfrac{1}{12}+O(u),\\ &\tilde{g}_2(u) = \tilde{g}_2^{[-1]}u^{-1}+O(u), && \tilde{g}_3(u) = \tilde{g}_3^{[-2]}u^{-2}+\tilde{g}_3^{[-1]}u^{-1}+O(u).\end{aligned} \tag{2.4.29}$$

From Lemma 2.4.6, (2.4.4) and (2.4.29), we get (2.1.5).

Let $\zeta(z)$ be the Riemann zeta function. By (2.4.21), we have

$$\mathrm{M}\big[g_1\big](z) = \tfrac{1}{\Gamma(z)}\textstyle\int_0^{+\infty}\sum_{j\geq 1}e^{-2\pi ju}u^{z-1}\,du = (2\pi)^{-z}\zeta(z), \tag{2.4.30}$$

Similarly, we get

$$\begin{aligned}&\mathrm{M}\big[g_2\big](z) = (2\pi)^{-z}\zeta(z-1), \quad \mathrm{M}\big[\tilde{g}_2\big](z) = z(2\pi)^{-(z+1)}\zeta(z),\\ &\mathrm{M}\big[\tilde{g}_3\big](z) = z(2\pi)^{-(z+1)}\big(\zeta(z-1)+\zeta(z)\big)/2.\end{aligned} \tag{2.4.31}$$

We recall that

$$\zeta'(0) = -\tfrac{1}{2}\log(2\pi), \qquad \zeta(0) = -\tfrac{1}{2}, \qquad \zeta(-1) = -\tfrac{1}{12}. \tag{2.4.32}$$

From Lemma 2.4.6, (2.4.4), (2.4.30), (2.4.31) and (2.4.32), we get (2.1.6). □

2.4.3 Relations to previous works

Verification. The analytic torsion of $\mathbb{CP}^n$ for $n \geq 1$ with trivial line bundle was computed by Gillet-Soulé and Zagier in [65] and it played an important role in the formulation and proof of the arithmetic Riemann-Roch theorem by Gillet-Soulé in [66]. Later, it was reobtained by Bost in [32] as a direct consequence of Bismut-Lebeau immersion theorem [25], [17].

Let's denote now by $M = \mathbb{CP}^1$ and by $L = \mathcal{O}(1)$ the hyperplane line bundle. We endow $\mathcal{O}(-1)$ with the Hermitian metric induced from the inclusion $\mathcal{O}(-1) \hookrightarrow \mathbb{C}^2 : ([z], \lambda z) \mapsto \lambda z, z \in \mathbb{C}^2 \backslash \{0\}$. Let h^L be the dual Hermitian metric on $L = \mathcal{O}(-1)^*$. The Fubiny-Study metric g^{TM} on TM is by definition the metric, associated to the positive 2-form $\omega = c_1(L, h^L) = \frac{1}{2\pi}\sqrt{-1}R^L$. By [77, Theorem 18] or [73, (12), (73), (74)][1], we get for $p \geq 1$

$$\begin{aligned} 2\log T(g^{TM}, h^{L^p}) = 2\textstyle\sum_{j=1}^{p}(p-j)\log(j+1) - (p+1)\log(p+1)! \\ - 4\zeta'(-1) + \tfrac{1}{2}(p+1)^2 - \tfrac{\log(2\pi)}{2}p - \tfrac{2}{3}\log(2\pi). \end{aligned} \tag{2.4.33}$$

By [97, (5.11.1), (5.17.2), (5.17.5)], we have the following asymptotic expansions of Barnes G-function and factorial as $p \to +\infty$

$$\begin{aligned} &\log \textstyle\prod_{i=1}^{p-1} i! = \tfrac{1}{2}p^2\log p - \tfrac{3}{4}p^2 + \tfrac{1}{2}\log(2\pi)p - \tfrac{1}{12}\log p + \zeta'(-1) + O(p^{-1}), \\ &\log p! = p\log p - p + \tfrac{1}{2}\log p + \tfrac{1}{2}\log(2\pi) + \tfrac{1}{12}p^{-1} + O(p^{-2}). \end{aligned} \tag{2.4.34}$$

We note that from [97] we can actually get each coefficient in the expansion of $\log T(g^{TM}, h^{L^p})$. We substitute (2.4.34) into (2.4.33) and we get, as $p \to +\infty$

$$2\log T(g^{TM}, h^{L^p}) = -\tfrac{1}{2}p\log p - \tfrac{2}{3}\log p - \tfrac{1}{6}\log(2\pi) - \tfrac{7}{12} - 2\zeta'(-1) + O(p^{-1}). \tag{2.4.35}$$

Since $\int_M c_1(TM) = 2$, this formula coincides with Theorem 2.1.3 for E is trivial.

To check the coefficients of $c_1(E)$ for general n-dimensional manifold M, the reader may compare the coefficients of $k^{n-1}\log k$ and k^{n-1} from Theorem 2.1.3 applied for $E = L, p = k-1$ and $E = \mathcal{O}_M, p = k$.

Connection with the arithmetic Riemann-Roch theorem. Now let's describe an informal connection between $\zeta'(-1)$, which appears in Theorem 2.1.3, and the one which appears in the R-genus of Gillet-Soulé.

Let X be an arithmetic variety in the sense of the book of Soulé [106, p. 55]. In [64], Gillet-Soulé defined arithmetic Chow groups $\widehat{CH}^k(X)$, for $k \in \mathbb{N}$. Those groups are generated by pairs (Z, g_Z), where Z is a cycle of codimension k and g_Z is a current over $X(\mathbb{C})$ of bi-degree $(k-1, k-1)$, for which $\frac{\bar{\partial}\partial}{2\pi\sqrt{-1}}g_Z + \delta_Z$ is smooth. There is an intersection pairing $\widehat{CH}^r(X) \times \widehat{CH}^q(X) \to \widehat{CH}^{r+q}(X)$ and pushforward operations for morphisms between arithmetic varieties.

[1] Notice the last two terms, which appear because the metric considered in the article [73] differs from our metric by a factor 2π.

Let E be an algebraic vector bundle over X with a Hermitian metric h^E invariant under the complex conjugation over $X(\mathbb{C})$. Then a pair $\overline{E} := (E, h^E)$ is a Hermitian vector bundle over X in the sense of [106, p. 84]. Let $\widehat{\mathrm{ch}}(\overline{E}) \in \oplus_k \widehat{CH}^k(X)_{\mathbb{Q}}$ be the arithmetic Chern character. It satisfies the usual axioms of a Chern character, but it does depend on the choice of the metric. When h^E is replaced by h_0^E, the difference $\widehat{\mathrm{ch}}(E, h^E) - \widehat{\mathrm{ch}}(E, h_0^E)$ is given by $(0, \widetilde{\mathrm{ch}}(h^E, h_0^E))$, where $\widetilde{\mathrm{ch}}(h^E, h_0^E)$ is the Bott-Chern secondary characteristic class (cf. [21, (1.124)]).

Let $f : X \to B$ be a proper morphism between arithmetic varieties, smooth on generic fiber $X(\mathbb{Q})$. Let $\overline{E}$ be a Hermitian vector bundle over X. Grothendieck and Knudsen-Mumford defined an algebraic line bundle $\lambda(E)$ over B, see [76]. The fiber at every point $y \in B$ is the alternated tensor product $\lambda(E)_y = \otimes_{q\geq 0}(\det H^q(f^{-1}(y), E))^{(-1)^q}$. The line bundle $\lambda(E)$ induces a holomorphic line bundle over the set of complex points $B(\mathbb{C})$. Over $X(\mathbb{C})$, the Kähler form ω induces a Hermitian metric h^{Tf} on the relative tangent bundle $Tf_{\mathbb{C}}$. This defines a Hermitian vector bundle $\overline{Tf_{\mathbb{C}}} := (Tf_{\mathbb{C}}, h^{Tf})$. Then one defines Quillen metric on $\lambda(E)$ over $B(\mathbb{C})$ as a product of L^2 metric and analytic torsion of the fiber, see [23, Definition 1.12, Theorem 1.15]. Thus, we get a Hermitian line bundle $\lambda(\overline{E})$ over $B(\mathbb{C})$. The arithmetic Riemann-Roch theorem of Gillet-Soulé [66, Theorem VIII.1'] says

$$\widehat{c}_1(\lambda(\overline{E})) = f_*(\widehat{\mathrm{ch}}(\overline{E})\widehat{\mathrm{Td}}(\overline{Tf_{\mathbb{C}}})) - (0, f_*(\mathrm{ch}(\overline{E}_{\mathbb{C}})\mathrm{Td}(\overline{Tf_{\mathbb{C}}})R(\overline{Tf_{\mathbb{C}}}))), \tag{2.4.36}$$

where $\widehat{\mathrm{Td}}(\overline{Tf_{\mathbb{C}}}) \in \oplus_k \widehat{CH}^k(X)_{\mathbb{Q}}$ is the arithmetic Todd class of $\overline{Tf_{\mathbb{C}}}$ and R is the additive genus of Gillet-Soulé defined by the power series $R(z)$

$$R(z) = \sum_{n\geq 1}^{n \text{ odd}} \Big(2\frac{\zeta'(-n)}{\zeta(-n)} + \sum_{j=1}^{n} \frac{1}{j}\Big)\zeta(-n)\frac{z^n}{n!}. \tag{2.4.37}$$

Now let's suppose $B = \mathrm{Spec}(\mathbb{Z})$. Then $\widehat{CH}^1(B) = \mathbb{R}$, and by [106, Lemma VIII.1.1], we have

$$\widehat{c}_1(\lambda(\overline{E})) = \textstyle\sum_{q=0}^{n}(-1)^q \left(\log \# H^q(X, E)_{tors} - \log \mathrm{Vol}_{L^2}(H^q(X, E))\right) + 2\log T(X, \overline{E}), \tag{2.4.38}$$

where $\mathrm{Vol}_{L^2}(H^q(X, E)_{\mathbb{R}})$ is the L^2 co-volume of the integer lattice $H(X, E)_{\text{free}}$, which is the free part in cohomology $H^q(X, E) \otimes \mathbb{R}$ and $T(X, \overline{E})$ is the analytic torsion associated with $\overline{E}$ and $\overline{Tf_{\mathbb{C}}}$.

Let's consider the simplest case when X is a projective plane from previous paragraph. For the coordinates z_0, z_1 on $\mathbb{C}^2$, we identify the basis of $H^0(\mathbb{CP}^1, \mathcal{O}(p))$ with homogeneous polynomials $x_j = z_0^j z_1^{p-j}, j = 0, \ldots, p$ of degree p. By the fact that $\{x_j\}$ form an orthogonal basis in cohomology with respect to $\|\cdot\|_{L^2}$, by $\left\|x_j\right\|_{L^2} = j!(p-j)!/(p+1)!$, $H^1(\mathbb{CP}^1, \mathcal{O}(p)) = 0$ for $p \geq 1$ and some calculations with characteristic and secondary characteristic classes, we have

$$\begin{aligned}\textstyle\sum_{q=0}^{1}(-1)^q \log \mathrm{Vol}_{L^2}(H^q(\mathbb{CP}^1, \mathcal{O}(p))) &= \textstyle\sum_{j=0}^{p} \log\left\|x_j\right\|_{L^2} \\ &= 2\log \prod_{1}^{p} j! - (p+1)\log(p+1)!,\end{aligned} \tag{2.4.39}$$

$$f_*(\widehat{\mathrm{ch}}(\overline{E})\widehat{\mathrm{Td}}(Tf_{\mathbb{C}})) - (0, f_*(\mathrm{ch}(\overline{E}_{\mathbb{C}})\mathrm{Td}(\overline{Tf_{\mathbb{C}}})R(\overline{Tf_{\mathbb{C}}}))) \\ = \tfrac{1}{2}(p+1)^2 - 4\zeta'(-1) - \tfrac{\log(2\pi)}{2}p - \tfrac{2\log(2\pi)}{3}. \quad (2.4.40)$$

This goes in line with (2.4.33), (2.4.36) and (2.4.38). Now, by (2.4.34) and (2.4.39), we have, as $p \to \infty$

$$\sum_{q=0}^{1}(-1)^q \log \mathrm{Vol}_{L^2}(H^q(\mathbb{CP}^1, \mathcal{O}(p))) = -\frac{1}{2}p^2 - \frac{1}{2}p\log p + \Big(\frac{\log(2\pi)}{2} - 1\Big)p \\ - \frac{2}{3}\log p + 2\zeta'(-1) + \frac{1}{2}\log(2\pi) - \frac{13}{12} + O(p^{-1}). \quad (2.4.41)$$

Thus, the right-hand side of (2.4.41) contains $2\zeta'(-1)$ in the constant term. It is an interesting question if one could understand the appearance of $\zeta'(-1)$ in the asymptotics of the first summand in the right-hand-side of (2.4.38) for a general arithmetic variety without using Theorem 2.1.3.

Relation with the result of Klevtsov-Ma-Marinescu-Wiegmann [73]. Now let's describe a result from [73, §4]. We denote by M a Riemann surface and by g_0^{TM}, g_1^{TM} Riemann metrics on M. Let L be a holomorphic line bundle over M and let h_0^L, h_1^L be Hermitian metrics such that L is positive with respect to any of h_0^L, h_1^L. Let $\mathring{R}_i^L \in \mathrm{End}(T^{(1,0)}M)$ be defined as in (2.1.2) with respect to (h_i^L, g_i^{TM}). We denote by $\Delta_{g_i^{TM}}, dv_{i,M}$ the scalar Laplacian and the volume form associated to g_i^{TM}. Then [73, (73)] says

$$2\log T(g_1^{TM}, h_1^{L^p}) - 2\log T(g_0^{TM}, h_0^{L^p}) = \mathcal{F}_1 - \mathcal{F}_0, \quad (2.4.42)$$

where

$$\mathcal{F}_i = -\frac{1}{2}\int_M p\log\Big(\frac{p\mathring{R}_i^L}{2\pi}\Big)\omega - \frac{1}{3}\int_M \log\Big(\frac{p\mathring{R}_i^L}{2\pi}\Big)c_1(TM) \\ - \frac{1}{48\pi}\int_M \log\big(\mathring{R}_i^L\big)\Delta_{g_i^{TM}}\big(\log \mathring{R}_i^L\big)\, dv_{i,M} + O(p^{-1}). \quad (2.4.43)$$

The authors observed the equality between the first two terms of the expansion of $\mathcal{F}_1$ and the first two terms of the expansion of $2\log T(g_1^{TM}, h_1^{L^p})$ (see (2.1.4)), so they conjectured that the third and forth terms will also coincide. We see by Theorem 2.1.3 that the third term of the asymptotic expansion of $2\log T(g_1^{TM}, h_1^{L^p})$ is $-\frac{1}{3}\int_M c_1(TM)$, which coincides with the third term of $\mathcal{F}_1$. The forth term of the asymptotic expansion of $2\log T(g_1^{TM}, h_1^{L^p})$ is

$$-\tfrac{1}{24}\mathrm{rk}(E)\big(24\zeta'(-1) + 2\log(2\pi) + 7\big)\int_M c_1(TM), \quad (2.4.44)$$

and in $\mathcal{F}_1$ it is 0. So the conjecture is valid for the third term, but not for the forth.

2.5 General asymptotic expansion for orbifolds, Theorem 2.1.5

In this section we prove Theorem 2.1.5. The general framework of Section 2.2.2 stays the same. We are still able to do the localization in the calculation of the asymptotic expansion of the analytic torsion. Once we localize the problem, the analysis differs from the manifold's case only in the neighbourhood of singular points, where the problem reduces to the $G-$manifold case and the results of Section 2.3.2 could be applied.

This section is organized as follows. In Section 5.1 we recall the definition of an orbifold and fix some notation. In Section 5.2 we prove some technical lemmas which facilitate further exposition. In Section 5.3 we establish Theorem 2.5.12, which is the full statement of Theorem 2.1.5. We also explain how this theorem implies the main result of Hsiao-Huang [68].

2.5.1 Orbifold preliminaries

In this section we recall some definitions from orbifolds theory. The content here is taken almost verbatim from the article [83, §1.1] and the book [85, §5.4].

We define a category $\mathcal{M}_s$ as follows: the objects of $\mathcal{M}_s$ are the class of pairs (G, M) where M is a connected smooth manifold and G is a finite group acting effectively on M. Let (G, M) and (G', M') be two objects, then a morphism $\Phi : (G, M) \to (G', M')$ is a family of open embeddings $\phi : M \to M'$ satisfying:

1. For each $\phi \in \Phi$, there is an injective group homomorphism $\lambda_\phi : G \to G'$ such that ϕ is λ_ϕ-equivariant.

2. For $g \in G', \phi \in \Phi$, we define $g\phi : M \to M'$ by $(g\phi)(x) = g(\phi(x))$ for $x \in M$. If $(g\phi)(M) \cap \phi(M) \neq \emptyset$, then $g \in \lambda_\phi(G)$.

3. For $\phi \in \Phi$, we have $\Phi = \{g\phi, g \in G'\}$.

Definition 2.5.1 (Definition of an orbifold)**.** Let $\mathcal{M}$ be a paracompact Hausdorff space and let $\mathcal{U}$ be a covering of $\mathcal{M}$ consisting of connected open subsets. We assume $\mathcal{U}$ is *dense*, i.e.

For any $x \in U \cap U', U, U' \in \mathcal{U}$, there is $U'' \in \mathcal{U}$ such that $x \in U'' \subset U \cap U'$.

Then an orbifold structure $\mathcal{V}$ on $\mathcal{M}$ is the following:

1. For $U \in \mathcal{U}, \mathcal{V}(U) = ((G_U, \tilde{U}) \to U)$ is a ramified covering, giving an isomorphism $U \simeq \tilde{U}/G_U$.

2. For $U, V \in \mathcal{U}, U \subset V$, there is a morphism $\phi_{VU} : (G_U, \tilde{U}) \to (G_V, \tilde{V})$ that covers the inclusion $U \subset V$.

3. For $U, V, W \in \mathcal{U}, U \subset V \subset W$, we have $\phi_{WU} = \phi_{WV} \circ \phi_{VU}$. If $\mathcal{U}'$ is a dense refinement of $\mathcal{U}$ we say that the restriction $\mathcal{V}'$ of the orbifold structure $\mathcal{V}$ to $\mathcal{U}'$ is equivalent to $\mathcal{V}$. A pair of $\mathcal{M}$ and an equivalence class $[\mathcal{V}]$ is called an orbifold.

Remark 2.5.2. This definition corresponds to "an effective orbifold" in the standard terminology.

In Definition 2.5.1, we can replace $\mathcal{M}_s$ by a category with manifolds with additional structure (orientation, Hermitian or Riemannian structure) as objects and maps, which preserve this structure, as morphisms. So we can define oriented, Hermitian or Riemannian orbifolds.

Let $(\mathcal{M}, [\mathcal{V}])$ be an orbifold. For each $x \in \mathcal{M}$, we can choose a small neighbourhood $(G_x, \tilde{U}_x) \to U_x$ such that $x \in \tilde{U}_x$ is a fixed point of G_x. The isomorphism class of G_x doesn't depend on the choice of a chart. Let's define $\mathcal{M}^{sing} = \{x \in \mathcal{M} : |G_x| \neq 1\}$.

Definition 2.5.3. An orbifold vector bundle $\mathcal{E}$ over an orbifold $(\mathcal{M}, \mathcal{V})$ is defined as follows: $\mathcal{E}$ is an orbifold, for $U \in \mathcal{U}, (G_U^{\mathcal{E}}, \tilde{p}_U : \tilde{\mathcal{E}}_U \to \tilde{U})$ is a $G_U^{\mathcal{E}}$-equivariant vector bundle and $(G_U^{\mathcal{E}}, \tilde{\mathcal{E}}_U)$ is an orbifold structure of $\mathcal{E}$ such that the transition maps in this structure are given by equivariant maps of those vector bundles. Moreover, $(G_U = G_U^{\mathcal{E}}/K_U^{\mathcal{E}}, \tilde{U}), K_U^{\mathcal{E}} = \ker(G_U^{\mathcal{E}} \to \mathrm{Diffeo}(\tilde{U}))$ is an orbifold structure of $\mathcal{M}$. If $K_U^{\mathcal{E}} = \{1\}$, we call $\mathcal{E}$ a proper orbifold vector bundle.

For example, the *orbifold tangent bundle* $T\mathcal{M}$ of an orbifold $\mathcal{M}$ is defined by $(G_U, T\tilde{U} \to \tilde{U})$, for $U \in \mathcal{U}$. It is a proper orbifold vector bundle. Let $\mathcal{E} \to \mathcal{M}$ be an orbifold vector bundle. A section $s : \mathcal{M} \to \mathcal{E}$ is smooth (or holomorphic if $\mathcal{M}$ is a complex orbifold), if for each $U \in \mathcal{U}$, $s|_U$ is covered by a $G_U^{\mathcal{E}}$-invariant smooth (or holomorphic) section $\tilde{s}_U : \tilde{U} \to \tilde{\mathcal{E}}_U$.

For an oriented orbifold $\mathcal{M}$ and a form α over $\mathcal{M}$ (i.e., a section of $\Lambda^\bullet(T^*\mathcal{M})$) we define

$$\int_{\mathcal{M}} \alpha := \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\alpha}_U, \text{ where supp } \alpha \subset U \in \mathcal{U} \tag{2.5.1}$$

We can extend this definition by $\mathbb{R}$-linearity to any differential form with compact support.

Lemma 2.5.4 (cf. [85, Lemma 5.4.3])**.** We can choose local coordinates $\tilde{U}_x \subset \mathbb{R}^n$ (or $\mathbb{C}^n$ if orbifold is complex) such that the finite group G_x acts linearly (or $\mathbb{C}$-linearly) on $\tilde{U}_x$.

Let $(1), (h_x^1), \cdots, (h_x^{\rho_x})$ be all the conjugacy classes in G_x. Let $Z_{G_x}(h_x^j)$ be the centralizer of h_x^j in G_x. We also denote by $\tilde{U}_x^{h_x^j}$ the fixed point set of h_x^j in $\tilde{U}_x$. Then we have a natural bijection

$$\{(y, (h_y^j)) | y \in U_x, j = 1, \cdots, \rho_y\} \simeq \coprod_{j=1}^{\rho_x} \tilde{U}_x^{h_x^j}/Z_{G_x}(h_x^j).$$

Definition 2.5.5 (Strata of an orbifold)**.** We can globally define

$$\Sigma\mathcal{M} = \{(x, (h_x^j)) | x \in \mathcal{M}, G_x \neq 1, j = 1, \cdots, \rho_x\}$$

and endow $\Sigma\mathcal{M}$ with a natural orbifold structure defined by

$$\left\{(Z_{G_x}(h_x^j)/K_x^j, \tilde{U}_x^{h_x^j}) \to \tilde{U}_x^{h_x^j}/Z_{G_x}(h_x^j)\right\}_{(x, U_x, j)},$$

where K_x^j is the kernel of the representation $Z_{G_x}(h_x^j) \to \mathrm{Diff}(\tilde{U}_x^{h_x^j})$ and $\mathrm{Diff}(\tilde{U}_x^{h_x^j})$ is the set of diffeomorphisms of $\tilde{U}_x^{h_x^j}$.

Till the end of this section we denote by

$$\Sigma\mathcal{M}^{[j]}, j \in J \quad \text{the connected components of } \Sigma\mathcal{M}, \qquad n_j = \dim_{\mathbb{C}} \Sigma\mathcal{M}^{[j]}, \tag{2.5.2}$$

$$m_j = |K_j| \quad \text{the multiplicity of } \Sigma\mathcal{M}^{[j]}, \tag{2.5.3}$$

where K_j was defined in Definition 2.5.5. We have a natural map $\pi : \Sigma\mathcal{M} \to \mathcal{M}, (x, (h_x^j)) \mapsto x$. Then $\pi|_{\Sigma\mathcal{M}^{[j]}}$ is an embedding.

2.5.2 General setup and some auxiliary lemmas

Let's fix a compact Hermitian orbifold $(\mathcal{M}, g^{T\mathcal{M}}, \Theta)$ of complex dimension n. Then its strata $\Sigma\mathcal{M}$ is naturally a Hermitian orbifold. We fix $x \in \Sigma\mathcal{M}^{[j]}$ and we denote by

$$g_j \in G_x \text{ some element such that } (x, g_j) \in \Sigma\mathcal{M}^{[j]}.$$

Let's denote by $\tilde{\mathcal{N}}_j$ the normal vector bundle to $\tilde{U}_x^{g_j}$ in $\tilde{U}_x$. We introduce the projection $\pi_{(j)} : \tilde{\mathcal{N}}_j \to \Sigma\mathcal{M}^{[j]}$. We see that $\tilde{\mathcal{N}}_j$ is naturally endowed with the Hermitian metric. We denote by $dv_{\mathcal{N}}$ its Riemannian volume form. Exponential mapping gives a map ϕ from the neighbourhood of the zero section of $\tilde{\mathcal{N}}_j$ to the neighbourhood of $\pi(\Sigma\mathcal{M}^{[j]})$ in $\mathcal{M}$. We define a function k_j in this neighbourhood of $\pi(\Sigma\mathcal{M}^{[j]})$ by

$$\phi^* dv_{\mathcal{M}}(x, Z) = k_j(x, Z)((\phi\pi)^* dv_{\Sigma\mathcal{M}^{[j]}}(x)) \wedge dv_{\mathcal{N}}(x, Z), \tag{2.5.4}$$

where $dv_{\mathcal{M}}, dv_{\Sigma\mathcal{M}}$ are the Riemannian volume forms of $\mathcal{M}$ and $\Sigma\mathcal{M}$ respectively. We extend the function k_j to the whole $\mathcal{N}_j$ in such a way that all its derivatives are bounded.

Let $(\mathcal{E}, h^{\mathcal{E}})$ be a Hermitian proper orbifold vector bundle on $\mathcal{M}$. By Lemma 2.5.4, we can define the operator $\overline{\partial}^{\mathcal{E}}$ locally on each local chart $\tilde{U}$ and patch it globally. As usually, we define the operators $\overline{\partial}^{\mathcal{E}*}, \Box^{\mathcal{E}}$. By [83], the heat operator $\exp(-t\Box^{\mathcal{E}})$ has a smooth kernel $\exp(-t\Box^{\mathcal{E}})(x, y), x, y \in \mathcal{M}$ with respect to $dv_{\mathcal{M}}$.

Let $(\mathcal{L}, h^{\mathcal{L}})$ be a holomoprhic Hermitian proper positive orbifold line bundle on $\mathcal{M}$. We denote by $\theta_j \in 2\pi\mathbb{Q}, j \in J$ the number such that for any $x \in \pi(\Sigma\mathcal{M}^{[j]})$

$$\text{the action of } g_j \in G_x \text{ on } \mathcal{L}_x \text{ is given by } e^{\sqrt{-1}\theta_j}. \tag{2.5.5}$$

This number is independent of the choice of x and g_j. We denote by $\Box_p$ the Laplacian $\Box^{\mathcal{L}^p \otimes \mathcal{E}}$ and define the analytic torsion $T(g^{T\mathcal{M}}, h^{\mathcal{L}^p \otimes \mathcal{E}})$ as in Definition 2.2.3. We have

Theorem 2.5.6 ([85, Theorem 5.4.9]). *There exists $\mu > 0$ such that for any $p \gg 1$, we have*

$$\mathrm{Spec}(\Box_p) \subset \{0\} \cup [\mu p, +\infty[, \qquad \ker(\Box_p) \subset \Omega^{(0,0)}(M, L^p \otimes E).$$

Locally, over an orbifold chart, we define the function k (see (2.3.11)) and the operators $\widetilde{L_{p,x}}, \widetilde{L_{2,x}^t}, \widetilde{L_{4,x}^{t,v}}$ (see (2.3.13), (2.3.17) and (2.3.38)) as we did in the manifolds case. Those objects are G_x-invariant. For brevity, we note for $w > 0$

$$e^{-wL_{2,x}^t} = \exp(-w\widetilde{L_{2,x}^t}), \qquad e^{-wL_{4,x}^{t,v}} = \exp(-w\widetilde{L_{4,x}^{t,v}}). \tag{2.5.6}$$

Now we write down some simple corollaries of Section 2.3.2, which simplify largely the proof of Theorem 2.1.5. Here and after, let $(g_1, g_2) \in G_x \times G_x$ acts on

$$(\xi_1, \xi_2) \in (\Lambda^\bullet(T_y^{*(0,1)}\mathcal{M}) \otimes \mathcal{L}_y^p \otimes \mathcal{E}_y) \otimes (\Lambda^\bullet(T_z^{*(0,1)}\mathcal{M}) \otimes \mathcal{L}_z^p \otimes \mathcal{E}_z)^*, y, z \in \mathcal{M}, \text{ by}$$

$$(g_1, g_2)(\xi_1, \xi_2) = (g_1\xi_1, g_2\xi_2) \in (\Lambda^\bullet(T_{g_1y}^{*(0,1)}\mathcal{M}) \otimes \mathcal{L}_{g_1y}^p \otimes \mathcal{E}_{g_1y}) \otimes (\Lambda^\bullet(T_{g_2z}^{*(0,1)}\mathcal{M}) \otimes \mathcal{L}_{g_2z}^p \otimes \mathcal{E}_{g_2z})^*.$$

Lemma 2.5.7. For any $j \in J, u > 0$ fixed, the function

$$\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tZ)\,dv_{\mathcal{N}_j}(Z) \tag{2.5.7}$$

is differentiable in $(x,t) \in \mathcal{M} \times [0,1]$ and it's derivatives $\frac{\partial^{a+b}}{\partial x^a\,\partial t^b}|_{t=0}$ vanish for b odd.

Proof. First of all, the integral makes sense due to Theorem 2.3.10 and to the fact that the action of g_j on $\tilde{\mathcal{N}}_{j,x}$ has no fixed points. Due to the G_x-invariance of $\widetilde{L^t_{2,x}}$ and the fact that g_j acts by isometries, the integral doesn't depend on the choice of g_j.

The first part is a consequence of the Lebesgue dominated convergence theorem and Theorem 2.3.11. The vanishing result follows from the Duhamel's formula (2.4.10) (cf. [40, Theorem 4.17]), [85, Theorem 4.1.7] and the fact that $\exp(-u\widetilde{L^0_{2,x}})(g_j^{-1}Z,Z)$ is an even function in Z (cf. [85, Appendix D]). □

Lemma 2.5.8. For any $j \in J$, the function

$$\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-L^{t,v}_{4,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tvZ)\,dv_{\mathcal{N}_j}(Z) \tag{2.5.8}$$

is differentiable in $(x,t,v) \in \mathcal{M} \times [0,1] \times [0,+\infty[$ and it's derivatives $\frac{\partial^{a+b+c}}{\partial x^a\,\partial t^b\,\partial v^c}|_{t=0,v=0}$ vanish whenever b or c is odd.

Proof. By Proposition 2.3.13, the proof is the same as in Lemma 2.5.7. □

We fix t_0 as in (2.3.28).

Lemma 2.5.9. For any $m \in \mathbb{N}^*, u_0 > 0$ there are $c, C > 0$ such that for any $u \in]0,u_0]; j \in J$

$$\begin{aligned}\Big|\int_{Z\in\tilde{\mathcal{N}}_{j,x}} &\mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tZ)\,dv_{\mathcal{N}_j}(Z)\\ &- u^{-n_j}\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-L^{t,v}_{4,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tvZ)\,dv_{\mathcal{N}_j}(Z)\Big|_{\mathscr{C}^m(\mathcal{M}\times[0,t_0])}\\ &\le C\exp(-c/u).\end{aligned} \tag{2.5.9}$$

Proof. Let's fix $\epsilon > 0$ small enough. We break up the integral

$$\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tZ)\,dv_{\mathcal{N}_j}(Z) \tag{2.5.10}$$

into two parts $I_1 = \int_{|Z|\le\epsilon}$ and $I_2 = \int_{|Z|>\epsilon}$. Similarly, we break the integral

$$\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-L^{t,v}_{4,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tvZ)\,dv_{\mathcal{N}_j}(Z) \tag{2.5.11}$$

into two parts $J_1 = \int'_{|Z|\le\epsilon/v}$ and $J_2 = \int'_{|Z|>\epsilon/v}$. By Theorem 2.3.11 and Proposition 2.3.13, there are constants $c, C > 0$ such that $|I_2|_{\mathscr{C}^{m'}(\mathcal{M})}, |J_2|_{\mathscr{C}^{m'}(\mathcal{M})} \le C\exp(-c/u)$. By Proposition 2.3.12, we get the estimate $|I_1 - J_1|_{\mathscr{C}^{m'}(\mathcal{M})} \le C\exp(-c/u)$. □

Lemma 2.5.10. For any $u_0 > 0, m \in \mathbb{N}$ there exist $c, C > 0$ such that for any $u > u_0, j \in J$

$$\big| \int_{Z \in \tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tZ)\, dv_{\mathcal{N}_j}(Z)\big|_{\mathscr{C}^m(\mathcal{M}\times[0,t_0])} \leq C\exp(-cu). \quad (2.5.12)$$

Proof. It follows from Lebesgue dominated convergence theorem and Theorem 2.3.10. □

Lemma 2.5.11. For any $u_0 > 0; m, k' \in \mathbb{N}$, there exists $C > 0$ such that for any $u \in]0, u_0], v = \sqrt{u}, j \in J$, we have

$$\Big| \int_{Z \in \tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}}(g_j^{-1}Z,Z)\big](k^{-1}k_j)(x,tZ)\, dv_{\mathcal{N}_j}(Z) - \sum_{h=0}^{k'+n_j} \frac{u^h}{(2h)!}\frac{\partial^{2h}}{\partial v^{2h}}\Big(\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-L^{t,v}_{4,x}}(g_j^{-1}Z,Z)\big] \cdot (k^{-1}k_j)(x,tvZ)\, dv_{\mathcal{N}_j}(Z)\Big)|_{v=0}\Big|_{\mathscr{C}^m(\mathcal{M}\times[0,\,t_0])} \leq Cu^{k'+1}. \quad (2.5.13)$$

Proof. This follows immediately from Lemmas 2.5.8, 2.5.9. □

2.5.3 Proof of Theorem 2.1.5

In this section we prove Theorem 2.1.5. Then we will show that Theorem 2.1.5 gives a refinement of the main result of Hsiao-Huang [68]. One of the main ingredients here is Lemma 2.5.13, which localizes the calculation of the asymptotic expansion of the heat kernel near the singular locus. Once this lemma is established, the main strategy of the proof is the same as in the manifold's case from Section 2.2.2. The only technical modification will consist in exploiting the results of Section 2.3.2 on off-diagonal expansion of the heat kernel. We use the notation from Section 2.5.2.

We begin by giving the definition of the sections $\widetilde{a_{i,u}}$ of the vector bundle $\mathrm{End}(\Lambda^\bullet(T^{*(0,1)}\mathcal{M})\otimes\mathcal{E})$, which are the orbifold's counterparts of $a_{i,u}$, defined in Theorem 2.2.5. If $x \in \mathcal{M}$ is nonsingular, we define $\widetilde{a_{i,u}}(x)$ by (2.3.25). If x is singular, we define the following local section (see (2.5.6))

$$a'_{k,u}(x) = \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial t^{2k}} e^{-uL^t_{2,x}/2}(0,0)|_{t=0}. \quad (2.5.14)$$

It is G_U-invariant over an orbifold neighbourhood $\tilde{U}$, so it gives a section of $\mathrm{End}(\Lambda^\bullet(T^{*(0,1)}\mathcal{M})\otimes\mathcal{E})$ over $\tilde{U}/G_U$, which we denote by $\widetilde{a_{i,u}}$.

We prove in Proposition 2.5.15 that $\widetilde{a_{i,u}}$ has an expansion of the form (2.2.1) as $u \to 0$ (so Notation 2.2.2 makes sense), and we can perform the Mellin transform for the trace of $\widetilde{a_{i,u}}$. Now let's state Theorem 2.1.5 precisely.

Theorem 2.5.12. *There are $\widetilde{\alpha}_i, \widetilde{\beta}_i \in \mathbb{R}, i \in \mathbb{N}$ and $\gamma_{j,i}, \kappa_{j,i} \in \mathbb{R}, j \in J, i \in \mathbb{N}$ such that the asymptotic expansion (2.1.7) holds, as $p \to \infty$. Moreover,*

$$\widetilde{\alpha}_i = \int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a}_i^{[0]}(x)\big]\, dv_{\mathcal{M}}(x), \qquad \widetilde{\beta}_i = -\mathrm{M}_u\big[\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{i,u}}(x)\big]\, dv_{\mathcal{M}}(x)\big]'(0). \quad (2.5.15)$$

Also there are functions $c_{j,u,i}, j \in J, u \in]0,+\infty[, i \in \mathbb{N}$ on $\Sigma\mathcal{M}^{[j]}$, given by (2.5.27), such that

1. For $x \in \mathcal{M}$ the value $c_{j,u,i}(x)$ depends only on the local geometry of $\mathcal{M}$ in x and on the action of $g_j \in G_x$ on the normal bundle $\tilde{\mathcal{N}}_{j,x}$,

2. The equations (2.2.1), (2.2.2) hold for the functions $u \mapsto c_{j,u,i}(x)$, $u > 0$, so we can apply the Mellin transform, and Notation 2.2.2 makes sense. We have the following identities:

$$\gamma_{j,i} = \int_{\Sigma\mathcal{M}^{[j]}} c_{j,i}^{[0]}(x)\, dv_{\Sigma\mathcal{M}^{[j]}}(x), \qquad \kappa_{j,i} = -\mathrm{M}_u \left[\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,i}(x)\, dv_{\Sigma\mathcal{M}^{[j]}}(x)\right]'(0). \tag{2.5.16}$$

Finally, the identities (2.1.8), (2.1.9) hold and the proportion $\kappa_{j,0}/\mathrm{Vol}(\Sigma\mathcal{M}^{[j]})$ depends only on the action of $g_j \in G_x, (x, g_j) \in \Sigma\mathcal{M}^{[j]}$ on the normal bundle $\tilde{\mathcal{N}}_{j,x}$ of a fixed point $x \in \Sigma\mathcal{M}^{[j]}$, and for c_j from (2.1.9), we have a precise formula

$$c_j = -n\mathrm{rk}(E)\big(\det(\mathrm{Id} - g_j|_{\tilde{\mathcal{N}}_j})\big)^{-1/2}. \tag{2.5.17}$$

Now we give a proof of Theorem 2.5.12. Let $\epsilon > 0$ be small enough, we introduce

$$\begin{aligned} A(p,u) &= \int_{\mathcal{M}\setminus B(\mathcal{M}^{sing},\epsilon)} \mathrm{Tr_s}\big[N\exp(-u\Box_p/p)(x,x)\big]\, dv_{\mathcal{M}}(x) \\ &\qquad + p^n \int_{B(\mathcal{M}^{sing},\epsilon)} \mathrm{Tr_s}\big[Ne^{-uL_{2,x}^t/2}(0,0)\big]\, dv_{\mathcal{M}}(x). \\ B(p,u) &= \sum_{j\in J} \frac{1}{m_j} p^{n_j} e^{\sqrt{-1}\theta_j p} \int_{\Sigma\mathcal{M}^{[j]}} \int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL_{2,x}^t/2}(g_j^{-1}Z,Z)\big] \\ &\qquad \cdot (k^{-1}k_j)(x,tZ)\, dv_{\mathcal{N}_j}(Z)\, dv_{\Sigma\mathcal{M}^{[j]}}(x). \end{aligned} \tag{2.5.18}$$

The following Lemma which explains the difference of the manifold's case and the orbifold's case, and why the results from Section 2.3.2 are necessary for the orbifold's case.

Lemma 2.5.13. For $\epsilon > 0$ small enough, there are $c, C > 0$ such that for any $p \in \mathbb{N}^*, u > 0$:

$$\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] = A(p,u) + B(p,u) + O(p^C \exp(-cp/u)). \tag{2.5.19}$$

Proof. We suppose ϵ satisfies (2.3.4) and (2.3.9). Let $f : \mathbb{R} \to [0,1]$ be a smooth even function, which satisfies

$$f(v) = \begin{cases} 1, & \text{for } |v| \le \epsilon/2, \\ 0, & \text{for } |v| \ge \epsilon. \end{cases} \tag{2.5.20}$$

For $u > 0, a \in \mathbb{C}$ we denote holomorphic even functions F_u, G_u on $\mathbb{C}$ by (cf. [85, (1.6.13)])

$$\begin{aligned} F_u(a) &= \int_{-\infty}^{+\infty} e^{iva} \exp(-v^2/2) f(\sqrt{u}v) \frac{dv}{\sqrt{2\pi}}, \\ G_u(a) &= \int_{-\infty}^{+\infty} e^{iva} \exp(-v^2/2)(1 - f(\sqrt{u}v)) \frac{dv}{\sqrt{2\pi}}. \end{aligned} \tag{2.5.21}$$

Then we have (cf. [85, (1.6.14)])

$$F_u(vD_p) + G_u(vD_p) = \exp(-v^2 D_p^2/2). \tag{2.5.22}$$

By [85, Proposition 1.6.4, (5.5.72)], there are $c, C > 0, k \in \mathbb{N}$ such that for any $x, x' \in M, p \in \mathbb{N}^*, u > 0$

$$\left|G_{u/p}(\sqrt{u/p}D_p)(x,x')\right| \leq Cp^k \exp(-cp/u). \tag{2.5.23}$$

We construct an open cover of $B(\mathcal{M}^{sing}, \epsilon)$ by a finite number of balls $B_i := B(x_i, \epsilon_i), i \in I$ for $\epsilon_i < 2\epsilon$ and $x_i \in \mathcal{M}^{sing}$. We require that $x \in B(\pi(\Sigma\mathcal{M}^{[j]}), \epsilon) \cap B_i$ implies $x_i \in \pi(\Sigma\mathcal{M}^{[j]})$, for a natural embedding $\pi : \Sigma\mathcal{M} \to \mathcal{M}$. We construct a partition of unity ρ_i subordinate to $B_i, i \in I$. We implicitly identify a neighbourhood of $0 \in T_{x_i}\mathcal{M}$, parametrized by variable Z, with a neighbourhood of x_i.

By the identity $gF_{u/p}(\sqrt{u/p}D_p) = F_{u/p}(\sqrt{u/p}D_p)g$ and finite propagation speed of solutions of the hyperbolic equations, we have the identity (cf. [85, (5.4.18)])

$$\begin{aligned} &F_{u/p}(\sqrt{u/p}D_p)(\exp_{x_i}(Z), \exp_{x_i}(Z)) \\ &\qquad = \textstyle\sum_{g\in G_{x_i}} (g,1) F_{u/p}(\sqrt{u/p}\tilde{D}_p)(\exp_{x_i}(g^{-1}\tilde{Z}), \exp_{x_i}(\tilde{Z})), \end{aligned} \tag{2.5.24}$$

where $Z \in T_{x_i}\mathcal{M}, |Z| \leq \epsilon, \tilde{Z} \in T_{x_i}\tilde{U}_{x_i}$ represents Z in the orbifold chart $\tilde{U}_{x_i}$ of x_i.

In the following series of identities we use Remark 2.3.5, (2.3.23), (2.5.3), (2.5.23), (2.5.24) and the fact that k is G_x-invariant.

$$\begin{aligned} &\textstyle\int_{B(\mathcal{M}^{sing},\epsilon)} \mathrm{Tr_s}\big[N\exp(-u\Box_p/p)(x,x)\big]\, dv_{\mathcal{M}}(x) \\ &= \textstyle\sum_{j\in I} p^n \int_{B(x_j,\epsilon_i)} \rho_j(x) \mathrm{Tr_s}\big[N e^{-uL^t_{2,x}/2}, 0)\big]\, dv_{\mathcal{M}}(x) \\ &\quad + \textstyle\sum_{j\in I} \int_{B(x_j,\epsilon_j)} \rho_j(Z) \mathrm{Tr_s}\big[N \sum_{g\in G_{x_j}\setminus\{1\}} (g,1)\exp(-u\widetilde{L_{p,\pi_{(j)}(Z)}}/(2p))(g^{-1}Z, Z)\big]\, dv_{\mathcal{M}}(Z) \\ &\quad + O(p^{C'}\exp(-cp/u)) \\ &= p^n \textstyle\int_{B(\mathcal{M}^{sing},\epsilon)} \mathrm{Tr_s}\big[N e^{-uL^t_{2,x}/2}(0,0)\big]\, dv_{\mathcal{M}}(x) \\ &\quad + p^n \textstyle\sum_{j\in J} \frac{1}{m_j} e^{\sqrt{-1}\theta_j p} \int_{\Sigma\mathcal{M}^{[j]}} \int_{Z\in\tilde{\mathcal{N}}_{j,x}, |Z|\leq\epsilon} \mathrm{Tr_s}\big[N(g_j,1)e^{-uL^t_{2,x}/2}(g_j^{-1}Z/t, Z/t)\big] \\ &\qquad\qquad \cdot (k^{-1}k_j)(x, tZ)\, dv_{\mathcal{N}_j}(Z) dv_{\Sigma\mathcal{M}^{[j]}}(x) + O(p^C \exp(-cp/u)), \end{aligned} \tag{2.5.25}$$

After a change of variables $Z \to Z/t$ and an application of Theorems 2.3.10, 2.3.11 we conclude. □

The terms $A(p,u), B(p,u)$ appear in each proof till the end of this section. Due to the finite propagation speed of solutions of the hyperbolic equations, the analysis of $A(p,u)$ is always the same as in Section 2.2.2. The main contribution here is the analysis of $B(p,u)$. The following proposition is an orbifold's version of Theorem 2.2.5.

Proposition 2.5.14. For any $k \in \mathbb{N}$ and $u > 0$ fixed, we have as $p \to +\infty$

$$\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] = \sum_{i=0}^{k} p^{n-i}\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{i,u}}(x)\big]\,dv_{\mathcal{M}}(x) + \sum_{i=0}^{k+n_j-n}\sum_{j\in J}\frac{p^{n_j-i}}{m_j}e^{\sqrt{-1}\theta_j p}\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,i}(x)\,dv_{\Sigma\mathcal{M}^{[j]}}(x) + o(p^{n-k}), \quad (2.5.26)$$

where

$$c_{j,u,i}(x) = \frac{1}{(2i)!}\int_{\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\Big[N(g_j,1)\frac{\partial^{2i}}{\partial t^{2i}}\Big(e^{-uL^t_{2,x}/2}(g_j^{-1}Z,Z)(k^{-1}k_j)(x,tZ)\Big)|_{t=0}\Big]dZ, \quad (2.5.27)$$

and the term $o(p^{n-k})$ is uniform as u varies in compact subsets of $]0,+\infty[$.

Proof. By Lemma 2.3.4, Proposition 2.3.7 and (2.5.18), we get

$$A(p,u) = \textstyle\sum_{i=0}^{k} p^{n-i}\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{i,u}}(x)\big]\,dv_{\mathcal{M}}(x) + o(p^{n-k}). \quad (2.5.28)$$

By Lemma 2.5.10, (2.5.18) and (2.5.27), we get

$$B(p,u) = \textstyle\sum_{i=0}^{k+n_j-n}\sum_{j\in J}\frac{1}{m_j}p^{n_j-i}e^{\sqrt{-1}\theta_j p}\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,i}(x)\,dv_{\Sigma\mathcal{M}^{[j]}}(x) + o(p^{n-k}). \quad (2.5.29)$$

Now, Lemma 2.5.13, (2.5.28) and (2.5.29) imply the proposition. □

The next proposition is an analogue of Proposition 2.2.6. It implies that we can do the Mellin transform in u for $\widetilde{a_{i,u}}(x), c_{j,u,i}(x)$; thus, the statement of Theorem 2.5.12 makes sense.

Proposition 2.5.15. There are smooth sections $\widetilde{a}_i^{[l]}(x), i\in\mathbb{N}, l\in\mathbb{Z}, l\geq -n$ of the vector bundle $\mathrm{End}(\Lambda^\bullet(T^{*(0,1)}\mathcal{M})\otimes\mathcal{E})$ such that the following asymptotic expansion holds for any $k\in\mathbb{N}$:

$$\widetilde{a_{i,u}}(x) = \textstyle\sum_{l=-n}^{k}\widetilde{a}_i^{[l]}(x)u^l + o(u^k), \quad \text{as} \quad u\to 0. \quad (2.5.30)$$

Moreover, there are $c_i, d_i > 0$ such that we have the following estimation

$$|N\widetilde{a_{i,u}}(x)| \leq c_i\exp(-d_i u). \quad (2.5.31)$$

Similarly for $i\in\mathbb{N}, j\in J, h\in\mathbb{Z}, h\geq -n_j$ there are functions $c_{ji}^{[h]}(x)$ on $\Sigma\mathcal{M}^{[j]}$ such that for any $k\in\mathbb{N}$, we have the following asymptotic expansion, as $u\to 0$:

$$c_{j,u,i}(x) = \textstyle\sum_{h=-n_j}^{k} c_{ji}^{[h]}(x)u^h + o(u^k). \quad (2.5.32)$$

Moreover, there are $c_i, d_i > 0$ such that we have the following estimation

$$|c_{j,u,i}(x)| \leq c_i\exp(-d_i u). \quad (2.5.33)$$

Proof. The statements about $\widetilde{a_{i,u}}(x)$ are proved in the same way as Proposition 2.2.6. Estimation (2.5.32) follows from Lemma 2.5.11 and (2.5.27). Moreover, it proves that

$$c_{jk}^{[h]}(x) = \frac{1}{(2k)!(2(h+n_j))!}\int_{Z\in\tilde{\mathcal{N}}_{j,x}} \mathrm{Tr_s}\Big[N(g_j,1) \cdot \frac{\partial^{2(k+h+n_j)}}{\partial t^{2k}\partial v^{2(h+n_j)}}\Big(e^{-L_{4,x}^{t,v}/2}(g_j^{-1}Z,Z)(k^{-1}k_j)(x,tvZ)\Big)|_{t=0,v=0}\Big]\,dZ. \quad (2.5.34)$$

Estimation (2.5.33) follows from Lemma 2.5.10 and (2.5.27). □

We also have the following version of Proposition 2.2.8 and (2.2.19).

Proposition 2.5.16. For $p\in\mathbb{N}^*$ there are $\widetilde{b_{p,i}}\in\mathbb{C}, i\in\mathbb{Z}, i\geq -n$ and $\widetilde{b_{j,p,i}}\in\mathbb{C}, j\in J, i\in\mathbb{Z}, i\geq -n_j$ such that for any $k\in\mathbb{N}$, we have the following asymptotic expansion, as $u\to 0$:

$$\mathrm{Tr_s}\big[N\exp(-u\Box_p/p)\big] = p^n\sum_{i=-n}^{k}\widetilde{b_{p,i}}u^i + \sum_{j\in J}\frac{p^{n_j}}{m_j}e^{\sqrt{-1}\theta_j p}\sum_{i=-n_j}^{k}\widetilde{b_{j,p,i}}u^i + o(u^k), \quad (2.5.35)$$

We also have the following expansions as $p\to+\infty$

$$\widetilde{b_{p,i}} = \textstyle\sum_{l=0}^{k}\widetilde{b}_i^{[l]}p^{-l} + o(p^{-k}), \qquad \text{with} \qquad \widetilde{b}_i^{[l]} = \int_{\mathcal{M}}\mathrm{Tr_s}\big[N\tilde{a}_l^{[i]}(x)\big]\,dv_{\mathcal{M}}(x), \quad (2.5.36)$$

$$\widetilde{b_{j,p,i}} = \textstyle\sum_{h=0}^{k}\widetilde{b_{ji}}^{[h]}p^{-h} + o(p^{-k}), \qquad \text{with} \qquad \widetilde{b_{ji}}^{[h]} = \int_{\Sigma\mathcal{M}^{[j]}}c_{jh}^{[i]}(x)\,dv_{\Sigma\mathcal{M}^{[j]}}(x). \quad (2.5.37)$$

Moreover, for $x\in\Sigma\mathcal{M}^{[j]}$, $\widetilde{b_{j,p,i}}(x)$ depends only on the geometry of $\mathcal{M}^{[j]}$, $\mathcal{N}_j$ at x and on the action of g_j on $\tilde{\mathcal{N}}_{j,x}$.

Proof. Similarly to Proposition 2.2.8, by (2.5.18), we get

$$A(p,u) = p^n\textstyle\sum_{i=-n}^{k}\widetilde{b_{p,i}}u^i + o(u^k). \quad (2.5.38)$$

This proves (2.5.36). Now let's denote for $j\in J, i\in\mathbb{N}, i\geq -n_j$

$$\widetilde{b_{j,p,i}} = \frac{1}{(2(i+n_j))!}\int_{\Sigma\mathcal{M}^{[j]}}\int_{Z\in\tilde{\mathcal{N}}_{j,x}}\frac{\partial^{2(i+n_j)}}{\partial v^{2(i+n_j)}}\Big(\mathrm{Tr_s}\big[N(g_j,1)e^{-L_{4,x}^{t,v}/2}(g_j^{-1}Z,Z)\big] \cdot (k^{-1}k_j)(x,tvZ)\Big)|_{v=0}\,dv_{\mathcal{N}_j}(Z)\,dv_{\Sigma\mathcal{M}^{[j]}}(x). \quad (2.5.39)$$

By Lemma 2.5.11 and (2.5.18), we get

$$B(p,u) = \textstyle\sum_{j\in J}\frac{1}{m_j}p^{n_j}e^{\sqrt{-1}\theta_j p}\sum_{i=-n_j}^{k}\widetilde{b_{j,p,i}}u^i + o(u^k). \quad (2.5.40)$$

By Lemma 2.5.13, (2.5.38) and (2.5.40), we get (2.5.35). Now (2.5.37) follows from Lemma 2.5.8 and (2.5.34). □

Now we prove the orbifold's analogue of Proposition 2.2.9.

Theorem 2.5.17. *For any $k \in \mathbb{N}, u_0 > 0$ there exists $C > 0$ such that for $u \in]0, u_0], p \in \mathbb{N}^*$, we get*

$$\begin{aligned} p^k \Big| & \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - p^n \sum_{i=-n}^{0} \widetilde{b_{p,i}} u^i - \sum_{j\in J} \frac{p^{n_j}}{m_j} e^{\sqrt{-1}\theta_j p} \sum_{i=-n_j}^{0} \widetilde{b_{j,p,i}} u^i \\ & - \sum_{h=0}^{n+k} p^{n-h} \Big(\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{h,u}}(x)\big]\, dv_{\mathcal{M}}(x) - \sum_{i=-n}^{0} \widetilde{b}_i^{[h]} u^i \Big) \\ & - \sum_{h=0}^{k+n_j} \sum_{j\in J} \frac{p^{n_j-h}}{m_j} e^{\sqrt{-1}\theta_j p} \Big(\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,h}(x)\, dv_{\Sigma\mathcal{M}^{[j]}}(x) - \sum_{i=-n_j}^{0} \widetilde{b_{ji}}^{[h]} u^i \Big) \Big| \le Cu. \end{aligned} \tag{2.5.41}$$

Proof. We apply (2.5.18) and the same techniques as in Proposition 2.2.9 to get that there exists $C > 0$ such that for any $u \in]0, u_0], p \in \mathbb{N}^*$,

$$\begin{aligned} p^k \Big| & \big(A(p,u) - p^n \textstyle\sum_{i=-n}^{0} \widetilde{b_{p,i}} u^i\big) \\ & - \textstyle\sum_{h=0}^{n+k} p^{n-h} \big(\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{h,u}}(x)\big]\, dv_{\mathcal{M}}(x) - \sum_{i=-n}^{0} \widetilde{b}_i^{[h]} u^i \big) \Big| \le Cu. \end{aligned} \tag{2.5.42}$$

By Lemma 2.5.11, (2.5.18), (2.5.27), (2.5.37) and (2.5.39) we have

$$\begin{aligned} p^k \Big| & \Big(B(p,u) - \textstyle\sum_{j\in J} \frac{1}{m_j} p^{n_j} e^{\sqrt{-1}\theta_j p} \sum_{i=-n_j}^{0} \widetilde{b_{j,p,i}} u^i \Big) - \sum_{h=0}^{k+n_j} \sum_{j\in J} \frac{1}{m_j} p^{n_j-h} \\ & \cdot e^{\sqrt{-1}\theta_j p} \Big(\textstyle\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,h}(x)\, dv_{\Sigma\mathcal{M}^{[j]}}(x) - \sum_{i=-n_j}^{0} \widetilde{b_{ji}}^{[h]} u^i \Big) \Big| \le Cu. \end{aligned} \tag{2.5.43}$$

Now, by Lemma 2.5.13, (2.5.42) and (2.5.43), we get (2.5.41). □

Now we prove the orbifold's analogue of Proposition 2.2.10.

Theorem 2.5.18. *For any $k \in \mathbb{N}, u_0 > 0$ there exists $c, C > 0$ such that for $u > u_0, p \in \mathbb{N}^*$*

$$\begin{aligned} p^k \Big| & \mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] - \sum_{i=0}^{n+k} p^{n-i} \int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{i,u}}(x)\big]\, dv_{\mathcal{M}}(x) \\ & - \sum_{i=0}^{k+n_j} \sum_{j\in J} \frac{1}{m_j} p^{n_j-i} e^{\sqrt{-1}\theta_j p} \int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,i}(x)\, dv_{\Sigma\mathcal{M}^{[j]}}(x) \Big| \le C \exp(-cu). \end{aligned} \tag{2.5.44}$$

Proof. We have to distinguish two cases:
1. $u > \sqrt{p}$. Similarly to Proposition 2.2.10, we get by Theorem 2.5.6 and Proposition 2.5.14.

$$\mathrm{Tr_s}\big[N \exp(-u\Box_p/p)\big] \le \exp(-cu), \qquad \widetilde{a_{i,u}}(x) \le \exp(-cu). \tag{2.5.45}$$

We conclude by (2.5.33), (2.5.45) and inequality $\exp(-cu) \leq \exp(-c\sqrt{p}/2)\exp(-cu/2)$.
2. $u \leq \sqrt{p}$. Similarly to Proposition 2.2.10, we get

$$p^k\Big|A(p,u) - \sum_{i=0}^{n+k} p^{n-i}\int_{\mathcal{M}} \mathrm{Tr_s}\big[N\widetilde{a_{i,u}}(x)\big]\,dv_{\mathcal{M}}(x)\Big| \leq C\exp(-cu). \tag{2.5.46}$$

By Lemma 2.5.10, (2.5.18) and (2.5.27), we have

$$p^k\Big|B(p,u) - \sum_{i=0}^{k+n_j}\sum_{j\in J}\frac{1}{m_j}p^{n_j-i}e^{\sqrt{-1}\theta_j p}\int_{\Sigma\mathcal{M}^{[j]}} c_{j,u,i}(x)\,dv_{\Sigma\mathcal{M}^{[j]}}(x)\Big| \leq C\exp(-cu). \tag{2.5.47}$$

We conclude by Lemma 2.5.13, (2.5.46), (2.5.47) and inequality $e^{-cp/u} \leq e^{-c\sqrt{p}/2}e^{-cu/2}$. □

Now, we can repeat the argument of Theorem 2.2.7 to get (2.1.7). The identities (2.1.8) follow from (2.1.4) and (2.5.14). The identities (2.5.16) are proved in the same way as (2.2.18).

Now let's explain why the constants $\widetilde{\alpha}_i$, $\gamma_{j,i}$ do not depend on $g^{T\mathcal{M}}$,$h^{\mathcal{L}^p}$, $h^{\mathcal{E}}$. In [83, Theorem 0.1], Ma proved an analogue of the anomaly formula for orbifolds. Due to this formula, we have an analogous formula to (2.2.35). By [41, Theorems 1,2] the asymptotic expansion of the Bergman kernel has only terms of the form p^i, $p^i e^{\sqrt{-1}\theta_j p}$. Similarly to Proposition 2.5.16, we conclude that the orbifolds analogue of the term $M^p_{0,c}$ has only terms of the form p^i, $p^i e^{\sqrt{-1}\theta_j p}$ in its asymptotic expansion. Thus, under the change of the metric, only the coefficients of the terms p^i, $p^i e^{\sqrt{-1}\theta_j p}$ change, so the constants $\widetilde{\alpha}_i$, $\gamma_{j,i}$ do not depend on $g^{T\mathcal{M}}$,$h^{\mathcal{L}^p}$, $h^{\mathcal{E}}$

Now let's prove (2.1.9). To simplify our calculation, we work under the assumption $\Theta = \omega$. We have the following formula [85, Appendix E 2.2]

$$\begin{aligned}(g_j,1)&e^{-uL^0_{2,x}}(g_j^{-1}Z,Z)\\ &= e^{-4\pi uN}C_{2u}\exp\Big(-\frac{\pi}{\tanh(2\pi u)}\|Z\|^2 + \frac{\pi}{\sinh(2\pi u)}\langle g_j^{-1}Z,Z\rangle\Big).\end{aligned} \tag{2.5.48}$$

By (2.4.25), we get

$$\begin{aligned}\mathrm{Tr_s}\big[N(g_j,1)e^{-uL^0_{2,x}}(g_j^{-1}Z,Z)\big] &= -\mathrm{rk}(E)ne^{-4\pi u}\frac{1}{1-e^{-4\pi u}}\\ &\cdot\exp\Big(-\frac{\pi}{\tanh(2\pi u)}\|Z\|^2 + \frac{\pi}{\sinh(2\pi u)}\langle g_j^{-1}Z,Z\rangle\Big).\end{aligned} \tag{2.5.49}$$

We denote by $\phi_{j,k}, k = 1,\ldots,n-n_j$ the angles of the rotation of the map $g_j|_{\widetilde{\mathcal{N}}_{j,x}}$. From (2.5.27), (2.5.49), we see

$$c_{j,u,0}(x) = -\frac{\mathrm{rk}(E)ne^{-\pi u}\sinh(\pi u)^{n-n_j-1}}{2\prod_{k=1}^{n-n_j}\sqrt{\cosh(\pi u)^2 - 2\cos(\phi_{j,k})\cosh(\pi u)+1}}. \tag{2.5.50}$$

Since $\phi_{j,k} \notin 2\pi\mathbb{Z}$ for $k \in 1,\ldots,m_j$, the function $c_{j,u,0}(x)$ is continuous at $u = 0$. We get (2.1.9), (2.5.17) from (2.5.16) and (2.5.50).

Relation between Theorem 2.1.5 and the result of Hsiao-Huang in [68]. In a recent article [68], Hsiao-Huang considered a compact connected strongly pseudoconvex CR manifold X with a S^1-transversal locally-free CR action. They considered a *rigid* CR vector bundle E over X (being *rigid* is equivalent to being a pull-back of the holomorphic orbifold vector bundle $\mathcal{E}$ over the quotient $\mathcal{M} = X/S^1$, see [68, Definition 2.4]). They decomposed the space $\Omega^{(0,\bullet)}(X, E)$ into Fourier components

$$\Omega_p^{(0,\bullet)}(X, E) = \{u \in \Omega^{(0,\bullet)}(X, E) | (e^{i\theta})^* u = e^{ip\theta} u, \text{ for all } \theta \in [0, 2\pi[\}, \quad p \in \mathbb{Z}. \tag{2.5.51}$$

Now, the restriction $\overline{\partial}_{b,p}$ of the tangential Cauchy-Riemann operator $\overline{\partial}_b$ endows $\Omega_p^{(0,\bullet)}(X, E)$ with a structure of a differential complex. We endow X with S^1-invariant Riemannian metric g^{TX}, compatible with CR-structure and S^1-action, i.e. g^{TX} is the orthogonal sum of a pull-back of J-invariant metric over the complex orbifold $(X/S^1, J)$ and the trivial metric induced on the S^1-directions, see [68, p.4, last paragraph]. We endow E with S^1-invariant Hermitian metric h^E satisfying *rigidity assumption*, i.e. it is a pull-back of a Hermitian metric on X/S^1, see [68, Definition 2.5]. Then we can endow $\Omega_p^{(0,\bullet)}(X, E)$ with the Hermitian metric induced by the L^2-scalar product. We denote by $\square_{b,p}$ the *Kohn Laplacian*, defined by $\square_{b,p} = \overline{\partial}_{b,p}\overline{\partial}_{b,p}^* + \overline{\partial}_{b,p}^*\overline{\partial}_{b,p}$. We associate the *analytic torsion* $\widetilde{T}_p(g^{TX}, h^E)$ to $\square_{b,p}$, as in the case of Kodaira Laplacian (see Definition 2.2.3, (2.2.10)). The main result in the article [68, Theorem 1.1] is the calculation of the first term of the asymptotic expansion of $\log \widetilde{T}_p(g^{TX}, h^E)$, as $p \to +\infty$.

Let X be a circle bundle associated to the dual of a positive line bundle L over a compact Hermitian complex manifold M. Let $\mathcal{E}$ be a vector bundle over a quotient manifold $M = X/S^1$ and E is a vector bundle $\pi^*\mathcal{E}$ for $\pi : X \to M$. The authors constructed a chain isometry [68, p.2, operator A_m] between differential complexes $(\Omega_p^{(0,\bullet)}(X, E), \overline{\partial}_{p,b})$ and $(\Omega^{(0,\bullet)}(M, L^p \otimes \mathcal{E}), \overline{\partial}^{L^p\otimes\mathcal{E}})$. Thus, $\operatorname{Spec} \square_{b,p} = \operatorname{Spec} \square^{L^p\otimes\mathcal{E}}$ and $\widetilde{T}_p(g^{TX}, h^E) = T(g^{TM}, h^{L^p\otimes\mathcal{E}})$. By [68, (1.4), (1.5)] in this case their result is equivalent to the original result of Bismut and Vasserot [27].

Now, in general it has been proven by Ornea and Verbitsky [98, Theorems 1.11, 5.1] that a compact connected strongly pseudoconvex CR manifold X with a S^1-transversal locally-free CR action is a circle bundle associated to the dual of a positive line bundle $\mathcal{L}$ over a compact Hermitian orbifold $\mathcal{M} = X/S^1$. Similarly to the case when $\mathcal{M}$ is a manifold, there is a chain isometry between differential complexes $(\Omega_p^{(0,\bullet)}(X, E), \overline{\partial}_{p,b})$ and $(\Omega^{(0,\bullet)}(\mathcal{M}, \mathcal{L}^p \otimes \mathcal{E}), \overline{\partial}^{\mathcal{L}^p\otimes\mathcal{E}})$. Then Theorem 2.1.5 gives the asymptotic expansion of $\log T(g^{T\mathcal{M}}, h^{\mathcal{L}^p\otimes\mathcal{E}}), p \to +\infty$. After reformulating this in terms of geometric objects on X, as it was done in the case where $\mathcal{M}$ is a manifold in [68, (1.4), (1.5)], Theorem 2.1.5 implies the main theorem of the article [68, Theorem 1.1]. We also point out that the fractional powers on p in fact do not indeed appear in the asymptotic expansion of $\log \widetilde{T}_p(g^{TX}, h^E)$.

Part II

Analytic torsion for surfaces with cusps

Chapter 3

Relative compact perturbation theorem and anomaly formula.

Abstract. We define and study the analytic torsion associated with a Riemann surface with cusps and a Hermitian vector bundle having at most logarithmic singularities around cusps.

More precisely, we fix a compact Riemann surface and a finite set of points, which we call cusps. We fix a Kähler metric defined away from from those points such that it can be expressed as the Poincaré metric over a punctured disk in some local holomorphic coordinates around the cusps. We fix a holomorphic vector bundle over the total space of the compact Riemann surface and endow it with a Hermitian metric defined away from the cusps. We suppose that this metric has at most logarithmic singularities, coming from the induced metric on the negative power of the canonical line bundle twisted by the divisor line bundle associated with the divisor of cusps.

Then we define the analytic torsion associated with this data. We provide a relation between this analytic torsion and the analytic torsion of a surface with flattened Kähler and Hermitian metrics. Then we establish the anomaly formula, which generalizes the Polyakov formula and describes how the analytic torsion changes under the change of the metric and the Hermitian structure. The results of this paper will be used in the sequel to study the Quillen metric in families.

3.1 Introduction

The goal of this article is to define and study the analytic torsion associated with a Riemann surface with hyperbolic cusps and a holomorphic Hermitian vector bundle with at most logarithmic singularities around the cusps. To define the analytic torsion, we use the regularization of the heat trace, obtained by subtracting a universal contribution coming from the model case of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

We provide a relation between this analytic torsion the analytic torsion of the compactified surface. Then we prove the anomaly formula, which describes how this analytic torsion changes with the change of metric and Hermitian structure on the vector bundle.

We stress out that in our definition we do not require the metric to be of constant scalar curvature everywhere, so the metric of the surface can be as "bad" as we wish over a compact part of the surface. Also we do not put any restriction neither on the holomorphic vector bundle, neither on the Hermitian metric over it. In particular, we do not suppose that it comes from some representation of the associated Fuchsian group.

More precisely, let $\overline{M}$ be a compact Riemann surface, $D_M = \{P_1^M, \ldots, P_m^M\}$ be a finite set of distinct points in $\overline{M}$. Let g^{TM} be a Kähler metric on the punctured Riemann surface $M := \overline{M} \setminus D_M$.

For $\epsilon \in]0, 1]$, $i = 1, \ldots, m$, let $z_i^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon) := \{z \in \mathbb{C} : |z| \leq \epsilon\}$ be a local holomorphic coordinate around P_i^M. We denote

$$V_i^M(\epsilon) := \{x \in M : |z_i^M(x)| < \epsilon\}. \tag{3.1.1}$$

We say that g^{TM} is *Poincaré-compatible* with coordinates $z_1^M, \ldots, z_m^M$ if for any $i = 1, \ldots, m$, there is $\eta > 0$ such that $g^{TM}|_{V_i^M(\eta)}$ is induced by the Hermitian form

$$\frac{\sqrt{-1} dz_i^M d\overline{z}_i^M}{\left|z_i^M \ln |z_i^M|\right|^2}. \tag{3.1.2}$$

We say that g^{TM} is a *metric with cusps* if it is Poincaré-compatible with some holomorphic coordinates near D_M. A triple $(\overline{M}, D_M, g^{TM})$ of a Riemann surface $\overline{M}$, a set of punctures D_M and a metric with cusps g^{TM} is called a *surface with cusps* (cf. [93]).

For example, if a pointed surface $(\overline{M}, D_M)$ is stable, i.e. the genus $g(\overline{M})$ of $\overline{M}$ satisfies

$$2g(\overline{M}) - 2 + m > 0, \tag{3.1.3}$$

then, by the uniformization theorem (cf. [49, Chapter IV], [9, Lemma 6.2]), there is the *canonical hyperbolic metric* g_{hyp}^{TM} of constant scalar curvature -1 on M. Once again, by the uniformization theorem, there are local holomorphic coordinates z_i^M of P_i^M, $i = 1, \ldots, m$, such that g_{hyp}^{TM} is induced by (3.1.2) in the neighbourhood of D_M. Thus, $(\overline{M}, D_M, g_{\mathrm{hyp}}^{TM})$ is a surface with cusps.

Let ξ be a holomorphic vector bundle over a complex manifold X with a Hermitian metric h^ξ over X. A pair (ξ, h^ξ) is called a *Hermitian vector bundle* over X.

From now on, we fix a surface with cusps $(\overline{M}, D_M, g^{TM})$ and a Hermitian vector bundle (ξ, h^ξ) over it. We denote by $\omega_{\overline{M}} := T^{*(1,0)}\overline{M}$ the *canonical line bundle* over $\overline{M}$. Let $\mathscr{O}_{\overline{M}}(D_M)$ be the line bundle associated to the divisor D_M. The *twisted canonical line bundle* on $\overline{M}$ is defined as

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{3.1.4}$$

The metric g^{TM} endows the line bundle ω_M (resp. $\omega_M(D)$) with the induced Hermitian metric $\|\cdot\|_M^\omega$ (resp. with $\|\cdot\|_M$ via the canonical isomorphism $\omega_M(D) \simeq \omega_M$) over M. In other worlds, there is $\epsilon > 0$, such that for the canonical section s_{D_M} of $\mathscr{O}_{\overline{M}}(D_M)$, over $V_i^M(\epsilon)$, we have

$$\left\|dz_i^M\right\|_M^\omega = \left|z_i^M \ln|z_i^M|\right|, \qquad \left\|dz_i^M \otimes s_{D_M}/z_i^M\right\|_M = \left|\ln|z_i^M|\right|. \tag{3.1.5}$$

We denote by $\square^{\xi\otimes\omega_M(D)^n}$ the Kodaira Laplacian associated with (M, g^{TM}) and $(\xi \otimes \omega_M(D)^n, h^\xi \otimes \|\cdot\|_M^{2n})$.

In this article, apart from the discussion of the L^2-norm, we only consider the restriction of $\square^{\xi\otimes\omega_M(D)^n}$ on the sections of degree 0.

Assume first $m = 0$, then the analytic torsion was defined by Ray-Singer [103, Definition 1.2] as the regularized determinant of $\square^{\xi\otimes\omega_M(D)^n}$. More precisely, let $\lambda_i, i \in \mathbb{N}$ be the non-zero eigenvalues of $\square^{\xi\otimes\omega_M(D)^n}$. By Weyl's law, for $\mathrm{Re}(s) > 1$, the associated zeta-function

$$\zeta_M(s) := \sum \lambda_i^s, \tag{3.1.6}$$

is well-defined and it is holomorphic in that region. Moreover, as it can be seen by the small-time expansion of the heat kernel and the classical properties of the Mellin transform, it extends meromorphically to $\mathbb{C}$. This extension is holomorphic at 0, and the *analytic torsion* is defined by

$$T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)). \tag{3.1.7}$$

By (3.1.6) and (3.1.7), we may interpret the analytic torsion as

$$T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) := \prod_{i=0}^{\infty} \lambda_i. \tag{3.1.8}$$

Now, assume $m > 0$. Then M is non-compact, and the heat operator associated to $\square^{\xi\otimes\omega_M(D)^n}$ is no longer of trace class. Also the spectrum of $\square^{\xi\otimes\omega_M(D)^n}$ is not discrete in general. Thus, neither the definition (3.1.7), nor the interpretation (3.1.8) are applicable, and another approach should be used.

Suppose for the moment that $(\overline{M}, D_M)$ satisfies (3.1.3). Let g_{hyp}^{TM} be the canonical hyperbolic metric of constant scalar curvature -1. We denote by $Z_{(\overline{M},D_M)}(s), s \in \mathbb{C}$ the Selberg zeta-function, which is given for $\mathrm{Re}(s) > 1$ by the absolutely converging product:

$$Z_{(\overline{M},D_M)}(s) = \prod_\gamma \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})^2, \tag{3.1.9}$$

where γ runs over the set of all simple closed non-oriented geodesics on M with respect to $g^{TM}_{\rm hyp}$, and $l(\gamma)$ is the length of γ. The function $Z_{(\overline{M},D_M)}(s)$ admits a meromorphic extension to the whole complex s-plane with a simple zero at $s=1$ (see for example [44, (5.3)]). We denote by $\|\cdot\|^{\rm hyp}_M$ the norm induced by $g^{TM}_{\rm hyp}$ on $\omega_M(D)$ over M.

In this situation, for $l\in\mathbb{Z}$, $l<0$, Takhtajan-Zograf in [107, (6)] proposed the analogue of the analytic torsion defined via Selbrerg zeta function as

$$T_{TZ}(g^{TM}_{\rm hyp},(\|\cdot\|^{\rm hyp}_M)^{2n}):=\begin{cases}\exp\Big((g(\overline{M})+2-m)\frac{\log(2)}{3}-\chi(M)\frac{c_0}{2}\Big)\cdot Z'_{(\overline{M},D_M)}(1), & \text{for } n=0,\\ \exp(-c_{-n}\chi(M)/2)\cdot Z_{(\overline{M},D_M)}(-n+1), & \text{for } n<0,\end{cases}\tag{3.1.10}$$

where $\chi(M)=2-2g(\overline{M})-m$ is the Euler characteristics of M, and for $k\in\mathbb{N}^*$, we put

$$\begin{aligned}c_0&=4\zeta'(-1)-\tfrac12+\ln(2\pi),\\ c_k&=\textstyle\sum_{l=0}^{k-1}(2k-2l-1)\big(\ln(2k+2kl-l^2-l)-\ln(2)\big)+(\frac13+k+k^2)\ln(2)\\ &\quad+(2k+1)\ln(2\pi)+4\zeta'(-1)-2(k+\tfrac12)^2-4\textstyle\sum_{l=1}^{k-1}\ln(l!)-2\ln(k!).\end{aligned}\tag{3.1.11}$$

Remark 3.1.1. To explain the values c_k, $k\in\mathbb{N}$, it was shown by Phong-D'Hoker [44, (7.30)], [45, (3.6)] (see also [104], [29, (50)] and [99, (9)]), that the definition (3.1.10) coincides with (3.1.7)[1]. In other words, the two definitions are compatible for M stable, $m=0$, $g^{TM}=g^{TM}_{\rm hyp}$ and $n\leq 0$.

The advantage of the definition (3.1.10) is an explicit formula in terms of "simple" geometric objects and, thus, suitability for the variational-type arguments (see [107], [51]). However, it only works for the complete hyperbolic metric $g^{TM}_{\rm hyp}$ of constant scalar curvature -1 on M and trivial Hermitian vector bundle (ξ,h^ξ).

Our first goal of this article is to give a definition (see Definition 3.2.16) of the analytic torsion $T(g^{TM},h^\xi\otimes\|\cdot\|^{2n}_M)$ for $n\leq 0$,[2] which generalizes both (3.1.7) and (3.1.10). Our definition is done using formula (3.1.7), where in place of a trace we use a regularized version of it, obtained as subtracting a universal spectral contribution of $\mathbb{CP}^1\setminus\{0,1,\infty\}$. Later [56] we show that our definition actually coincides with (3.1.10) for hyperbolic surfaces of constant scalar curvature and (ξ,h^ξ) trivial (thus, extending the results of Phong-D'Hoker [44, (7.30)], [45, (3.6)]).

In this article, after giving a formal definition of the determinant of the Laplacian, we provide two results for computing it. The first one, Theorem A, which we also call the *relative compact perturbation theorem*, expresses the quotient of two Quillen norms associated with surfaces with the same number of cusps through a quotient of two Quillen norms associated with surfaces without cusps. The second one, Theorem B, which we also call the *anomaly formula*, explains how the Quillen norm changes under the change of g^{TM}, h^ξ. It shows that although the Quillen norm is a global invariant, the variation of it, induced by the change of the metric and the Hermitian

[1] It's easy to see that $T(g^{TM}_{\rm hyp},(\|\cdot\|^{\rm hyp}_M)^{2n})$ corresponds to $\det'(\frac12\Delta^-_n)$ in the notation of [45, (1.1)], [29, (3)]. Since for $c>0$, by [45, §3], we have $\det'(c\Delta^-_n)=\det'(c\Delta^+_{-n})$, coefficients (3.1.11) for $k\in\mathbb{N}^*$ can be read of from [29, (50)] for $c=1/2$ (cf. [60, Definition 4.2]) and for $k=0$ from [44, (7.23), (7.30)], [104, Corollary 1] (cf. [59, (6.3)]).

[2] By Serre duality, if one prefers to work with positive line bundles, we can interpret it as the analytic torsion of the vector bundle $\xi^*\otimes\omega_M^{-n+1}(D_M)^{-n}$ associated to $(g^{TM},(h^\xi)^*\otimes\|\cdot\|^{-2n}_M\otimes(\|\cdot\|^{\omega}_M)^2)$, for $n\leq 0$.

structure, can be expressed as an integral of a local quantity. We see that this local quantity has an explicit contribution localized near the cusps. This contribution does not have analogues for compact surfaces and it describes the variation of the Poincaré-compatible coordinates induced by the variation of Kähler metric. The study of the heat kernel associated to $h^{\xi} \otimes \|\cdot\|_M^{2n}$ on a surface with cusps $(\overline{M}, D_M, g^{TM})$ plays the foremost role in our approach.

Now let's describe our results more precisely. For $n \leq 0$, we explain in the end of Section 3.2.1 how to endow the complex line

$$\big(\det H^{\bullet}(\overline{M}, \xi \otimes \omega_M(D)^n)\big)^{-1} := \big(\Lambda^{\max} H^0(\overline{M}, \xi \otimes \omega_M(D)^n)\big)^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, \xi \otimes \omega_M(D)^n), \quad (3.1.12)$$

with the L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n})$. In the compact case it coincides with the L^2-norm induced by the harmonic forms associated with g^{TM} and $h^{\xi} \otimes \|\cdot\|_M^{2n}$. Then we define the Quillen norm on the complex line (3.1.12) by

$$\|\cdot\|_Q(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) = T(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n})^{1/2} \cdot \|\cdot\|_{L^2}(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}). \quad (3.1.13)$$

To motivate, when $m = 0$, this coincides with the usual definition of the Quillen norm from [21, 1.64] and [23, Definition 1.5].

Now let's give some definitions, which are essential for our first theorem.

Definition 3.1.2 (Flattening of a metric)**.** Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps. We say that a (smooth) metric g_{f}^{TM} over $\overline{M}$ is a *flattening* of g^{TM} if there is $\nu > 0$ such that g^{TM} is induced by (3.1.2) over $V_i^M(\nu)$, and

$$g_{\mathrm{f}}^{TM}|_{M \setminus (\cup_i V_i^M(\nu))} = g^{TM}|_{M \setminus (\cup_i V_i^M(\nu))}. \quad (3.1.14)$$

The supremum of all $\nu > 0$, satisfying (3.1.14) is called the *tightness* of the flattening.

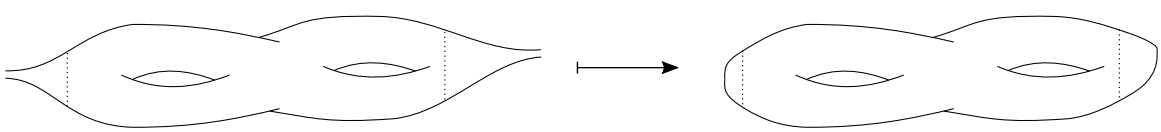

Figure 3.1: An example of a flattening. The regions between the dashed lines are isometric.

Let $(\overline{N}, D_N, g^{TN})$ be another surface with cusps and let g_{f}^{TN} be a *flattening* of g^{TN}. We say that the flattenings g_{f}^{TM} and g_{f}^{TN} are *compatible*, if for any $i = 1, \ldots, m$, we have

$$((z_i^N)^{-1} \circ z_i^M)^*(g_{\mathrm{f}}^{TM}|_{V_i^M(\nu)}) = g_{\mathrm{f}}^{TN}|_{V_i^N(\nu)}, \quad (3.1.15)$$

for some $\nu > 0$, satisfying (3.1.14) and

$$g_{\mathrm{f}}^{TN}|_{N \setminus (\cup_i V_i^N(\nu))} = g^{TN}|_{N \setminus (\cup_i V_i^N(\nu))}. \quad (3.1.16)$$

Similarly, we define the notion of *flattenings* $\|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ for Hermitian norms $\|\cdot\|_M, \|\cdot\|_N$. We say that the flattenings $\|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ are *compatible* if they satisfy similar conditions to (3.1.14), (3.1.16), and for any $i = 1, \ldots, m$, we have

$$((z_i^N)^{-1} \circ z_i^M)^* \big(\|\cdot\|_M / \|\cdot\|_M^{\mathrm{f}}\big)|_{V_i^M(\nu)} = \big(\|\cdot\|_N / \|\cdot\|_N^{\mathrm{f}}\big)|_{V_i^N(\nu)}. \tag{3.1.17}$$

Remark 3.1.3. The definitions of flattenings g_{f}^{TM} of g^{TM} and $\|\cdot\|_M^{\mathrm{f}}$ of $\|\cdot\|_M$ are independent, and there is no relation between them as in (3.1.5).

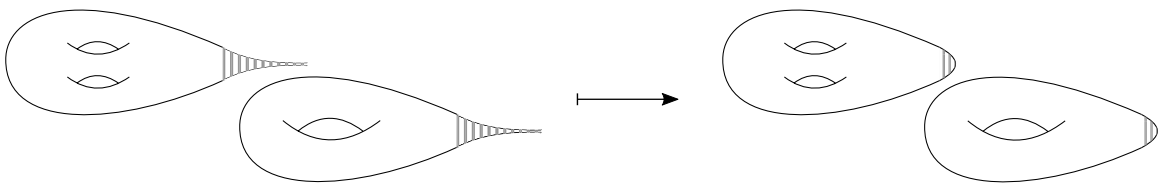

Figure 3.2: An example of compatible flattenings. The striped regions are isometric.

Theorem A (Relative compact perturbation)**.** *Let $(\overline{M}, D_M, g^{TM})$, $(\overline{N}, D_N, g^{TN})$ be two surfaces with the same number of cusps. Let (ξ, h^ξ) be a Hermitian vector bundle over $\overline{M}$ of rank $\mathrm{rk}(\xi)$. We denote by $\|\cdot\|_M$, $\|\cdot\|_N$ the norms induced by g^{TM}, g^{TN} as in (3.1.5) on $\omega_M(D)$ and $\omega_N(D)$ over M and N respectively. Let $g_{\mathrm{f}}^{TM}, g_{\mathrm{f}}^{TN}, \|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ be compatible flattenings of $g^{TM}, g^{TN}, \|\cdot\|_M \|\cdot\|_N$ respectively. Then for any $n \in \mathbb{Z}, n \leq 0$, we have*

$$\begin{aligned} 2\ln\Big(\|\cdot\|_Q\big(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f}}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\big)\Big) \\ - 2\mathrm{rk}(\xi)\ln\Big(\|\cdot\|_Q\big(g^{TN}, \|\cdot\|_N^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\big)\Big) \\ = \int_M c_1(\xi, h^\xi)\Big(2n\ln(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M) + \ln(g_{\mathrm{f}}^{TM}/g^{TM})\Big). \end{aligned} \tag{3.1.18}$$

In other words, the relative Quillen norm can be computed through a compact perturbation.

Remark 3.1.4. a) Philosophically, Theorem A should be interpreted as the anomaly formula, which permits "erasing" the cusps. To make this analogy even more apparent, we rewrite (3.1.18) in the following form (cf. (3.1.23))

$$\begin{aligned} 2\ln\Big(\|\cdot\|_Q\big(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f}}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\big)\Big) \\ - 2\mathrm{rk}(\xi)\ln\Big(\|\cdot\|_Q\big(g^{TN}, \|\cdot\|_N^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\big)\Big) \\ = \int_M \widetilde{\mathrm{Td}}\big(\omega_M^{-1}, g_{\mathrm{f}}^{TM}, g^{TM}\big)c_1\big(\xi, h^\xi\big) + \int_M c_1\big(\xi, h^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f}})^{2n}, \|\cdot\|_M^{2n}\big). \end{aligned} \tag{3.1.19}$$

b) Suppose that (ξ, h^{ξ}) is trivial in the ν-neighbourhood of the cusps, where $\nu > 0$ is the tightness of the flattenings g_{f}^{TM} and $\|\cdot\|_M^{\mathrm{f}}$. Then we simplify Theorem A to

$$\frac{\|\cdot\|_Q\left(g^{TM}, h^{\xi}\otimes\|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q\left(g_{\mathrm{f}}^{TM}, h^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f}})^{2n}\right)} = \frac{\|\cdot\|_Q\left(g^{TN}, \|\cdot\|_N^{2n}\right)^{\mathrm{rk}(\xi)}}{\|\cdot\|_Q\left(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\right)^{\mathrm{rk}(\xi)}}. \tag{3.1.20}$$

In fact, in our proof of Theorem A, we reduce the main statement to (3.1.20).

c) It is possible to restate Theorem A in the way, which doesn't use the language of compatible flattenings. It says that the quantity

$$\begin{aligned}2\mathrm{rk}(\xi)^{-1}\ln\Big(\|\cdot\|_Q\left(g^{TM}, h^{\xi}\otimes\|\cdot\|_M^{2n}\right)/\|\cdot\|_Q\left(g_{\mathrm{f}}^{TM}, h^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f}})^{2n}\right)\Big)\\ -\mathrm{rk}(\xi)^{-1}\int_M c_1(\xi,h^{\xi})\Big(2n\ln(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)+\ln(g_{\mathrm{f}}^{TM}/g^{TM})\Big)\end{aligned} \tag{3.1.21}$$

depends only on the integer $n\in\mathbb{Z}$, $n\leq 0$ and the functions $(g_{\mathrm{f}}^{TM}/g^{TM})|_{V_i^M(1)}\circ(z_i^M)^{-1}:\mathbb{D}^*\to\mathbb{R}$, $(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)|_{V_i^M(1)}\circ(z_i^M)^{-1}:\mathbb{D}^*\to\mathbb{R}$, for $i=1,\ldots,m$. This reformulation is particularly useful when one studies the variation of the Quillen norm in a family setting.

d) For $n=0$ and (ξ,h^{ξ}) trivial, Theorem A was proved by Jorgenson-Lundelius in [71, Theorem 7.3] and Albin-Aldana-Rochon in [3, Theorem 5.2]. The fact that the geometry near the cusps of (M,g^{TM}) and (N,g^{TN}) coincides is used extensively in their proofs. This doesn't hold in our case due to the presence of (ξ,h^{ξ}), and the techniques we use are different even in the case when (ξ,h^{ξ}) is trivial. We note that in [3, Definition 2.2], authors also consider funnel singularities.

The main feature of our techniques is that they are implicit, and unlike [71], we avoid studying the precise contribution of the continuous spectrum to the heat kernel.

Our next result explains how the Quillen norm changes under the conformal change of the metric with cusps. Let's recall that by [21, Theorem 1.27], the Bott-Chern classes of a vector bundle ξ with Hermitian metrics h_1^{ξ}, h_2^{ξ} are natural differential forms (strictly speaking, those are classes of differential forms, see Remark 3.1.7b)) defined so that they satisfy

$$\begin{aligned}\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{Td}}(\xi,h_1^{\xi},h_2^{\xi}) &= \mathrm{Td}(\xi,h_1^{\xi})-\mathrm{Td}(\xi,h_2^{\xi}),\\ \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{ch}}(\xi,h_1^{\xi},h_2^{\xi}) &= \mathrm{ch}(\xi,h_1^{\xi})-\mathrm{ch}(\xi,h_2^{\xi}),\end{aligned} \tag{3.1.22}$$

where Td, ch are Todd and Chern forms. By [21, Theorem 1.27], we have the following identities

$$\widetilde{\mathrm{ch}}(\xi,h_1^{\xi},h_2^{\xi})^{[0]} = 2\widetilde{\mathrm{Td}}(\xi,h_1^{\xi},h_2^{\xi})^{[0]} = \ln\big(\det(h_1^{\xi}/h_2^{\xi})\big). \tag{3.1.23}$$

If, moreover, $\xi := L$ is a line bundle, we have

$$\widetilde{\mathrm{ch}}(L,h_1^L,h_2^L)^{[2]} = 6\widetilde{\mathrm{Td}}(L,h_1^L,h_2^L)^{[2]} = \ln(h_1^L/h_2^L)\Big(c_1(L,h_1^L)+c_1(L,h_2^L)\Big)/2, \tag{3.1.24}$$

where c_1 is the first Chern form.

Definition 3.1.5. For a surface with cusps $(\overline{M}, D_M, g^{TM})$, the *Wolpert norms* $\|\cdot\|_i^W$ on the complex lines $\omega_{\overline{M}}|_{P_i^M}$, $i = 1, \ldots, m$, are defined by $\|dz_i^M\|_i^W = 1$. It induces the Wolpert norm $\|\cdot\|^W$ on the complex line $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i^M}$.

Remark 3.1.6. Since the Poincaré-compatible coordinates are uniquely defined up to a multiplication by a unimodular complex number, the norms $\|\cdot\|_i^W$ are well-defined. They were originally defined by Wolpert in [117, Definition 1] for hyperbolic surfaces of constant scalar curvature -1.

Theorem B (Anomaly formula for metrics with cusps)**.** *Let g^{TM}, g_0^{TM} be two metrics on M such that both triples $(\overline{M}, D_M, g^{TM})$, $(\overline{M}, D_M, g_0^{TM})$ are surfaces with cusps. We denote by $\|\cdot\|_M$, $\|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$, and by $\|\cdot\|^W$, $\|\cdot\|_0^W$ the associated Wolpert norms. Let h^ξ, h_0^ξ be two Hermitian metrics on ξ over $\overline{M}$. Then the right-hand side of the following equation is finite, and*

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\left(g_0^{TM}, h_0^\xi \otimes (\|\cdot\|_M^0)^{2n}\right)/\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)\Big)\\
&\quad= \int_M \Big[\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\
&\qquad\quad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h^\xi, h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\
&\qquad\quad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h_0^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}, (\|\cdot\|_M^0)^{2n}\big)\Big]^{[2]}\\
&\quad- \frac{\mathrm{rk}(\xi)}{6}\ln\Big(\|\cdot\|^W/\|\cdot\|_0^W\Big) + \frac{1}{2}\sum \ln\Big(\det(h^\xi/h_0^\xi)|_{P_i^M}\Big).
\end{aligned} \tag{3.1.25}$$

Remark 3.1.7. a) The anomaly formula was firstly proved by Polyakov in [100] for $m = 0$, $n = 0$ and (ξ, h^ξ) trivial, who used it to compute some integrals over random surfaces which arise in mathematical physics. It was generalized by Bismut-Gillet-Soulé [23, Theorem 1.23] for $m = 0$, but in any arbitrary dimension. For $m = 0$, in [50], Fay gave an alternative proof of (3.1.25), which doesn't use the formalism of heat kernels. Our proof relies on the anomaly formula for $m = 0$.

b) Strictly speaking, the integral in (3.1.25) is not well-defined, since $\widetilde{\mathrm{ch}}$, $\widetilde{\mathrm{Td}}$ are only well-defined as *classes* up to an element of the form $\partial\alpha + \overline{\partial}\beta$. Since a priori nothing is known about the growth of α, β near D_M, the integrals of $\partial\alpha$ and $\overline{\partial}\beta$ over M might not converge (leave alone being equal to 0 by "Stokes theorem"). For the purposes of this article, however, it is enough to think of $\widetilde{\mathrm{ch}}$, $\widetilde{\mathrm{Td}}$ as *forms*, defined by (3.1.23) and (3.1.24). An alternative way to interpret those classes is through the Bott-Chern theory for pre-log-log Hermitian vector bundles, introduced by Burgos Gil-Kramer-Kühn in [36] (cf. [55, §2.4]).

c) Experts will notice the difference between the terms under the integral in the right-hand side of (3.1.25) and the terms, which appear in the right-hand side of the anomaly formula of Bismut-Gillet-Soulé [23, Theorem 1.23] (see (3.3.3)), where in the arguments of Todd class and secondary Todd class we have ω_M in place of $\omega_M(D)$. However, this difference is not a real issue, since for

the current of integration δ_{D_M} along D_M, we have the following identities over $\overline{M}$:

$$\begin{aligned}
&\widetilde{\mathrm{Td}}\big(\omega_M^{-1},(\|\cdot\|_M^{\omega})^{-2},(\|\cdot\|_M^{\omega,0})^{-2}\big)=\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1},\|\cdot\|_M^{-2},(\|\cdot\|_M^{0})^{-2}\big),\\
&\big[\mathrm{Td}\big(\omega_M^{-1},(\|\cdot\|_M^{\omega,0})^{-2}\big)\big]^{[2]}=\big[\mathrm{Td}\big(\omega_M(D)^{-1},(\|\cdot\|_M^{0})^{-2}\big)\big]^{[2]}+\frac{1}{2}\delta_{D_M},\\
&\big[\widetilde{\mathrm{ch}}\big(\omega_M(D)^{n},\|\cdot\|_M^{2n},(\|\cdot\|_M^{0})^{2n}\big)\big]^{[0]}|_{D_M}=0,
\end{aligned}\tag{3.1.26}$$

where $[0]$, $[2]$ stand for the part of degree 0 and 2, and in the second identity we used Poincaré–Lelong equation. Nevertheless, we prefer to state Theorem B in the given form, since in the sequel we will use that the Hermitian line bundles $(\omega_M(D),\|\cdot\|_M)$, $(\omega_M(D),\|\cdot\|_M^0)$ are pre-log-log with singularities along D_M in the terminology of Burgos Gil-Kramer-Kühn [36], and the Hermitian line bundles $(\omega_M,\|\cdot\|_M^{\omega})$, $(\omega_M,\|\cdot\|_M^{\omega,0})$ do not satisfy those properties.

d) Let $\phi : M \to \mathbb{R}$ be a smooth function such that

$$g_0^{TM}=e^{2\phi}g^{TM},\tag{3.1.27}$$

In the case when ϕ has compact support in M, Theorem B follows from the anomaly formula of Bismut-Gillet-Soulé (see Theorem 3.3.1), Theorem A and (3.1.26).

The difference between Theorem B and Theorem 3.3.1 is in the last two terms of (3.1.25):

$$-\frac{\mathrm{rk}(\xi)}{6}\ln\Big(\|\cdot\|^{W}/\|\cdot\|_0^{W}\Big)+\frac{1}{2}\sum\ln\Big(\det(h^{\xi}/h_0^{\xi})|_{P_i^M}\Big).\tag{3.1.28}$$

For $n=0$ and (ξ,h^{ξ}) trivial, Albin-Aldana-Rochon in [2, Theorem 2.9] got a version of Theorem B. Here authors do not require ϕ to be of compact support but have some extra decay at cusps (see [2, (2.11)]). The conformal transformations for ϕ with this type of decay assumptions do not alter the Wolpert norm, and, thus, the terms (3.1.28) do not appear. We note that in [2], authors also consider funnel singularities. The anomaly formula for surfaces with only funnel singularities was proved before by Borthwick-Judge-Perry, [31].

In our applications [55, Theorems C, D], [56, Theorem 1.2], we use substantially that by applying anomaly formula, we can trivialize the Poincaré-compatible coordinates horizontally in the family of Riemann surfaces with hyperbolic cusps. Thus, the appearance of the terms (3.1.28) is of fundamental importance in what follows.

e) Similar theorem appeared in the paper of Lundelius [81, Theorem 1.1] for $n=0$ and (ξ,h^{ξ}) trivial. However, we disagree with his result, as it differs from ours in the last two terms of (3.1.25). From [81, p. 226, line 4], his proof should only work for ϕ of compact support in M.

To motivate this paper, we discuss several applications of Theorems A, B, which are proved in the sequel [55], [56]. All those results are done in a family setting, i.e. we fix a holomorphic, proper map $\pi : X \to S$ of complex analytic manifolds such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a complex curve with at most double point singularities. We also fix disjoint sections $\sigma_1,\ldots,\sigma_m : S \to X$, which avoid singular points of the fibers, and we denote by $D_{X/S}$ the divisor, given by $\mathrm{Im}(\sigma_1)+\ldots+\mathrm{Im}(\sigma_m)$.

1. Regularity and asymptotics of the Quillen norm in a degenerating family of surfaces, [55, Theorem C]. We consider the determinant line bundle $\lambda(j^*(\xi\otimes\omega_{X/S}(D)^n)) := (\det R^{\bullet}\pi_*(\xi\otimes$

$\omega_{X/S}(D)^n))^{-1}$, $n \leq 0$, where ξ is a holomorphic vector bundle over X and $\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S})$ is the twisted relative canonical line bundle. We endow the vector bundles ξ, $\omega_{X/S}(D)$ with Hermitian metrics h^ξ,$\|\cdot\|_{X/S}$ satisfying some mild hypotheses. Let $|\Delta|$ be the locus of singular curves of π. We define the Quillen norm $\|\cdot\|_Q$ on $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$ over $S \setminus |\Delta|$, as a family version of (3.1.13). Then we study the regularity and singularities of $\|\cdot\|_Q$ near $|\Delta|$. We also explicit some conditions under which the renormalized Quillen norm is continuous at the singular fibers.

The hypotheses, which we put on $\|\cdot\|_{X/S}$ are mild enough to include the case of hyperbolic surfaces. In this particular case, the asymptotics of the associated analytic torsion was studied before by Wolpert [115], Lundelius [81], Jorgenson-Lundelius [72], and many others. The paper of Bismut-Bost [20] neither considers the case of hyperbolic cusps nor the singularities of the metric near the singular fibers. However, their result plays a fundamental role in our study.

2. *Curvature theorem for surfaces with cusps, [55, Theorem D].* We show that the metric $\|\cdot\|_Q$ from previous paragraph is good enough, so that one can define its Chern form as a current. Then we give an explicit formula for this current, which refines the Riemann-Roch-Grothendieck theorem on the level of currents.

In particular, if we consider the family of hyperbolic surfaces, this extends the curvature theorem of Takhtajan-Zograf [107, Theorem 1] over the moduli space of curves to its Deligne-Mumford compactification. If we consider the case when there is no cusps, we get a generalization of Bismut-Bost [20, Théorème 2.1] to the case of degenerating metrics.

3. *Restriction and compatibility theorems, [56].* We relate the restriction of the renormalized Quillen norm $\|\cdot\|_Q$ at the locus of singular fibers $|\Delta|$ with the Quillen norm of the normalization of singular fibers. By a combination of this result with the analogical statement for Takhtajan-Zograf analytic torsion (see (3.1.10)), we deduce the compatibility between our definition of the analytic torsion and the one of Takhtajan-Zograf. This generalizes the result of Jorgenson-Lundelius [72, Corollary 4.3], where authors did it for hyperbolic surfaces, (ξ, h^ξ) trivial and $n = 0$.

Let's describe how the present article is related to mathematical physics. Indeed, in [73], Klevtsov-Ma-Marinescu-Wiegmann related the asymptotics of the generating functional for the integer quantum Hall effect when the flux of the magnetic field through a Riemann surface tends to infinity, and the asymptotics of the analytic torsion associated to an increasing power of a positive line bundle. As the anomaly formula for Riemann surfaces played an essential role in their study (see [73, Theorem 2]), the present article lays a foundation to extend their result to the case of surfaces with hyperbolic cusps.

Finally, let's discuss how the theory developed here can be adapted to the orbifold Riemann surfaces due to recent interest in orbifold setting (see [61], [109]). By combining the definition of the analytic torsion here and of the orbifold analytic torsion due to Ma [83], for an orbisurface (M, g^{TM}) with cusps $D_M \subset \overline{M}$, we may define the analytic torsion $T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$, where $n \leq 0$ and $\|\cdot\|_M$ is the induced norm on the orbifold twisted line bundle $\omega_M(D)$. Similarly to the manifolds case, this definition should generalize the definitions of the analytic torsion for stable hyperbolic orbisurfaces and (ξ, h^ξ) trivial due to Takhtajan-Zograf [109], Freixas-von Pippich [61].

Since our methods in the proof of Theorem A are purely local, the analogue of Theorem A would still hold for orbisurfaces. Since we got Theorem B by combining Theorem A and the anomaly formula of Bismut-Gillet-Soulé [21, Theorem 1.23], by replacing the last reference by its

orbifold analogue of Ma [83, Theorem 0.1], it is possible to get an analogue of Theorem B. Then by combining the calclucation of the norm for the Mumford isomorphism in the orbifold setting due to Freixas-von Pippich [61] and the anomaly formula, it should be possible to get the orbifold analogue of Mumford isometry for any orbisurface with metric with cusps and a Hermitian vector bundle over it. We hope to return to this question very soon.

We note that our definition of the analytic torsion is related to the definition of the relative analytic torsion due to Lundelius and Jorgenson-Lundelius, which was given for (ξ, h^{ξ}) trivial and $n = 0$ in [81], [70], [71] (see Remarks 3.1.7e), 3.2.17c)), and the definition of Albin-Rochon (see Remark 3.2.17d)), which was given for (ξ, h^{ξ}) trivial and $n = 0$ in [4, §7.1].

The b-trace of Melrose [90], used in the definition of Albin-Rochon, should also give the definition of the analytic torsion in our case, however we decided to work in a relative setting, and b-trace does not appear here explicitly. This gives us more flexibility to establish some estimates on the heat kernel which are used extensively in the proof of Theorem A.

Now, let's describe the structure of this paper. In Section 2, we develop spectral theory for surfaces with cusps. We introduce the notion of the analytic torsion and Quillen norm, which are used throughout the article. In Section 3, we prove Theorem A. For this, we study the families of metrics which "converge" to the metric with cusps in a special way. In Section 4, we prove Theorem B. The main idea is to use Theorem A and to obtain Theorem B by the anomaly formula of Bismut-Gillet-Soulé [21, Theorem 1.23].

Notation. For $\epsilon > 0$ and $(\overline{M}, D_M), (\overline{N}, D_N), \xi$ as in the statement of Theorem A, we denote

$$\begin{aligned}
&D(\epsilon) = \{u \in \mathbb{C} : |u| < \epsilon\}, \quad && D^*(\epsilon) = \{u \in \mathbb{C} : 0 < |u| < \epsilon\},\\
&\mathbb{D} := D(1), && \mathbb{D}^* = D^*(1),\\
&\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M), &&\\
&E_M^{\xi,n} := \xi \otimes \omega_M(D)^n, && E_N^n := \omega_N(D)^n.
\end{aligned} \tag{3.1.29}$$

By $g^{T\mathbb{D}^*}$ we denote the metric on $\mathbb{D}^*$, induced by (3.1.2), and by $dv_{\mathbb{D}^*}$ the associated Riemannian volume form. By $\mathrm{Spec}(A)$ we denote the spectrum of a self-adjoint operator A, acting on some Hilbert space. We denote by $B^M(x, r)$ the geodesic ball of radius $r > 0$ around $x \in M$ in a Riemannian surface M with Riemannian metric g^{TM}.

We denote by $L_X \boxtimes L_Y$ the holomorphic line bundle over $X \times Y$, which is given by $\pi_X^* L_X \otimes \pi_Y^* L_Y$ for some line bundles L_X, L_Y over the complex manifolds X and Y respectively and natural projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.

Note. This part of the thesis can be found on the ArXiv, see [54].

3.2 Spectral theory of surfaces with cusps

In this section we study spectral properties of surfaces with cusps and define the analytic torsion.

More precisely, in Section 2.1 we set up the notation and state the spectral gap theorem. In Section 2.2 we state several estimations of the heat kernel associated with a hyperbolic surface, we define the regularized heat trace and the analytic torsion. Section 2.3 is the most technical one.

Here we prove the estimations on the heat kernel of the hyperbolic punctured disc. Finally, in Section 2.4 we prove the statements from Sections 2.1, 2.2.

3.2.1 The setting of the problem and the spectral gap theorem

Let $(\overline{M}, D_M, g^{TM})$ be a Riemann surface with cusps and let (ξ, h^ξ) be a Hermitian vector bundle over $\overline{M}$. We denote by $\|\cdot\|_M$ the Hermitian norm induced by g^{TM} on $\omega_M(D)$ (see (3.1.4)) over M.

Let $\alpha, \alpha' \in \mathscr{C}_c^\infty(M, E_M^{\xi,n})$. The L^2-scalar product is defined by

$$\langle \alpha, \alpha' \rangle_{L^2} := \int_M \langle \alpha(x), \alpha'(x) \rangle_h dv_M(x), \tag{3.2.1}$$

where dv_M is the Riemannian volume form on (M, g^{TM}), and $\langle \cdot, \cdot \rangle_h$ is the pointwise Hermitian product induced by $h^\xi, \|\cdot\|_M$. By (3.1.2), the right-hand side of (3.2.1) is finite for $n \leq 0$.

We define the Hilbert space $(L^2(E_M^{\xi,n}), \langle \cdot, \cdot \rangle_{L^2})$, as the L^2-completion of the space $\mathscr{C}_c^\infty(M, E_M^{\xi,n})$ with respect to $\langle \cdot, \cdot \rangle_{L^2}$. Sometimes when we want to insist on the choice of g^{TM}, h^ξ and $\|\cdot\|_M$, we denote this space by $L^2(g^{TM}, h^\xi \otimes (\|\cdot\|_M)^{2n})$.

We denote by $\square^{E_M^{\xi,n}}$ the Kodaira Laplacian on $\mathscr{C}_c^\infty(M, E_M^{\xi,n})$, given by

$$\square^{E_M^{\xi,n}} := (\overline{\partial}^{E_M^{\xi,n}})^* \overline{\partial}^{E_M^{\xi,n}}, \tag{3.2.2}$$

where $(\overline{\partial}^{E_M^{\xi,n}})^*$ is the formal adjoint of $\overline{\partial}^{E_M^{\xi,n}}$ with respect to $\langle \cdot, \cdot \rangle_{L^2}$. Since (M, g^{TM}) is complete, the operator $\square^{E_M^{\xi,n}}$ is essentially self-adjoint on $L^2(E_M^{\xi,n})$ (cf. [85, Corollary 3.3.4]). We denote its closure by the same symbol.

In this article we are mostly interested in the heat operator $\exp(-t\square^{E_M^{\xi,n}}), t > 0$. We denote

$$\exp^\perp(-t\square^{E_M^{\xi,n}}) := \exp(-t\square^{E_M^{\xi,n}}) - P_M, \tag{3.2.3}$$

where P_M is the orthogonal projection onto $\ker(\square^{E_M^{\xi,n}})$. We denote by

$$\exp(-t\square^{E_M^{\xi,n}})(x,y), \exp^\perp(-t\square^{E_M^{\xi,n}})(x,y) \in (E_M^{\xi,n})_x \boxtimes (E_M^{\xi,n})_y^*, \quad \text{for} \quad x, y \in M, \tag{3.2.4}$$

the smooth kernels of $\exp(-t\square^{E_M^{\xi,n}}), \exp^\perp(-t\square^{E_M^{\xi,n}})$ with respect to dv_M. Then

$$\exp(-t\square^{E_M^{\xi,n}})(x,x), \exp^\perp(-t\square^{E_M^{\xi,n}})(x,x) \in \mathrm{End}(\xi)_x, \quad \text{for} \quad x \in M. \tag{3.2.5}$$

In Section 3.2, we fix g^{TM}, $h^\xi, \|\cdot\|_M$ and remove them from some notation: by $|\cdot|_{h\times h}$ we mean the pointwise norm on $(\omega_{\overline{M}}^k \otimes E_M^{\xi,n})^* \boxtimes (\omega_{\overline{M}}^l \otimes E_M^{\xi,n})$, $k, l \in \mathbb{Z}$ induced by $h^\xi, \|\cdot\|_M$, g^{TM}; by $|\cdot|$ we mean either the modulus of a complex number, or the pointwise norm on the vector bundle $\mathrm{End}(\xi)$ induced by h^ξ. We defer the proof of the next theorem until Section 3.2.4.

Theorem 3.2.1. *For $n \leq 0$, the operator $\square^{E_M^{\xi,n}}$ has a spectral gap near 0. More precisely, we have*

$$H^0(\overline{M}, E_M^{\xi,n}) = \ker(\square^{E_M^{\xi,n}}), \tag{3.2.6}$$

and there is $\mu > 0$ such that

$$\mathrm{Spec}\big(\square^{E_M^{\xi,n}}\big) \cap]0, \mu] = \emptyset. \tag{3.2.7}$$

Remark 3.2.2. As it would follow from our proof, there are $c_1, c_1 > 0$ such that the set

$$\mathrm{Spec}\big(\square^{E_M^{\xi,n}}\big) \cap [0, c_1\sqrt{-n} + c_2] \qquad \text{is discrete} \tag{3.2.8}$$

for any (M, g^{TM}), (ξ, h^ξ), $\|\cdot\|_M$ and $n \leq 0$. We leave the verification of the details to the interested reader. For $n = 0$, (ξ, h^ξ) trivial, and $c_2 = 1/4$, this was proved by Müller in [93, §6].

For $n = 0$, our proof of Theorem 3.2.1 relies on the result of Müller [93, §6, Proposition 6.9], who proves Theorem 3.2.1 for $n = 0$ and (ξ, h^ξ) trivial. In case of $n < 0$, we obtain Theorem 3.2.1 by gluing the estimates in the neighbourhood of cusp, coming from Nakano's inequality (cf. [85, Theorem 1.4.14]), and the estimates away from the cusps coming from the spectral gap for the Dirichlet Laplacian of a surface with boundary.

Finally, let's discuss the construction of the L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ on the line bundle (3.1.12). By the isomorphism (3.2.6), we may endow $H^0(\overline{M}, E_M^{\xi,n})$ with the L^2-scalar product induced by (3.2.1). Similarly to the analysis in the proof of (3.2.6), we have a natural isomorphism

$$\ker(\square_1^{E_M^{\xi,n}}) = \begin{cases} H^1(\overline{M}, E_M^{\xi,n}), & \text{for } n = 0, \\ H^1(\overline{M}, E_M^{\xi,n} \otimes \mathscr{O}_{\overline{M}}(D_M)), & \text{for } n \leq -1, \end{cases} \tag{3.2.9}$$

where $\square_1^{E_M^{\xi,n}} = \overline{\partial}^{E_M^{\xi,n}}(\overline{\partial}^{E_M^{\xi,n}})^*$ is the Kodaira Laplacian associated with 1-forms with values in $E_M^{\xi,n}$. We induce the L^2-scalar product on $H^1(\overline{M}, E_M^{\xi,n})$ by the natural inclusion

$$H^1(\overline{M}, E_M^{\xi,n}) \hookrightarrow H^1(\overline{M}, E_M^{\xi,n} \otimes \mathscr{O}_{\overline{M}}(D_M)), \qquad \alpha \mapsto \alpha \otimes s_{D_M}, \tag{3.2.10}$$

where s_{D_M} is the canonical holomorphic section of $\mathscr{O}_{\overline{M}}(D_M)$. Those scalar products induce the natural L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ on the line bundle (3.1.12).

3.2.2 Relative spectral theory for surfaces with cusps

The main goal of this section is to define the *analytic torsion* for any (ξ, h^ξ), $n \leq 0$, $m \in \mathbb{N}$. This extends the relative definition due to Jorgenson-Lundelius [71, Definition 1.9], which they gave in the case $n = 0$ and (ξ, h^ξ) trivial. The challenge here is that unlike in [71], the precise contribution of the continuous spectrum to the heat kernel is unknown, moreover the local geometry near the cusp depends on (ξ, h^ξ). We circumvent this difficulty by the analytic localization techniques of Bismut-Lebeau [25, §11] and by the parametrix construction for the heat kernel (cf. [12, §2.4, 2.5]). The parametrix construction is particularly useful when we would estimate the effect of non-triviality of (ξ, h^ξ) (see Theorem 3.2.6 and (3.2.20)).

We fix $n \in \mathbb{Z}$. Let the function $\rho_M : M \to [1, +\infty[$ be given by

$$\rho_M(x) = \begin{cases} 1 & \text{for } x \in M \setminus (\cup_i V_i^M(1/2)), \\ \sqrt{|\ln|z_i(x)||} & \text{for } x \in V_i^M(1/2), i = 1, \ldots, m. \end{cases} \tag{3.2.11}$$

Remark 3.2.3. The function $(\rho_M(x))^{-2}$ is proportional to the injectivity radius at point x of (M, g^{TM}).

We denote by $\mathrm{d}(\cdot,\cdot)$ the distance function on (M, g^{TM}). Now we can state the main theorems of this section. Their proofs are delayed until Section 3.2.4.

Theorem 3.2.4. *For any $l, l' \in \mathbb{N}$, there are $c, c', C > 0$ such that for any $t > 0$, $x, x' \in M$, we have*

$$\left|(\nabla_x)^l(\nabla_{x'})^{l'}\exp(-t\square^{E_M^{\xi,n}})(x,x')\right|_{h\times h} \leq C\rho_M(x)\rho_M(x')t^{-1-(l+l')/2}\cdot \exp(ct - c'\cdot \mathrm{d}(x,x')^2/t), \quad (3.2.12)$$

where ∇ is induced by the Levi-Civita connection and the Chern connections of (ξ, h^ξ) and $(\omega_M(D), \|\cdot\|_M)$. Also, if $n \leq 0$, then there are $c, C > 0$ such that for any $t > 0$, we have

$$\left|(\nabla_x)^l(\nabla_{x'})^{l'}\exp^\perp(-t\square^{E_M^{\xi,n}})(x,x')\right|_{h\times h} \leq C\rho_M(x)\rho_M(x')t^{-4-l-l'}\exp(-ct). \quad (3.2.13)$$

Remark 3.2.5. a) By Remark 3.2.3, we see that for $n = 0$, (ξ, h^ξ) trivial and $k, l = 0$, (3.2.12) is exactly the Moser's estimate [92, p. 115-117] (cf. [38, Theorem VIII.8]) for a hyperbolic surface. The proof of (3.2.12) is different from [92, p. 115-117] and it uses an explicit construction of the parametrix of the heat kernel.

b) By using the same techniques as in the proof of (3.2.13), we may deduce that for any $l, l' \in \mathbb{N}$, there is $C > 0$ such that

$$\left|(\nabla_x)^l(\nabla_{x'})^{l'}\exp(-t\square^{E_M^{\xi,n}})(x,x')\right|_{h\times h} \leq C\rho_M(x)\rho_M(x')t^{-4-l-l'}. \quad (3.2.14)$$

The estimate (3.2.14) is unfortunately not enough for our needs, since we use (3.2.12) in the proof of (3.2.15). However, by Remark 3.2.18, if one considers only (ξ, h^ξ) which are trivial around the cusps, then the estimate (3.2.14) is enough to prove (3.2.17), and all the analysis associated with the parametrix construction is not necessary.

Now, let M, N and all related notions be as in the statement of Theorem A.

Theorem 3.2.6. *For any $k \in \mathbb{N}$, there are $\epsilon, c, c', C > 0$ such that for any $t > 0$, $u \in \mathbb{C}$, $|u| \leq \epsilon$:*

$$\left|\exp(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big) - \mathrm{Id}_\xi\cdot\exp(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\right|$$
$$\leq C|\ln|u||\exp(ct)\cdot\min\Big\{|\ln|u||^{-k} + \exp(-c'(\ln|\ln|u||)^2/t); \quad (3.2.15)$$
$$|u|^{1/3} + \exp(-c'/t)\Big\}. \quad (3.2.16)$$

Moreover, if $n \leq 0$, then there are $\varsigma < 1$ and $c, C > 0$ such that

$$\left|\exp^\perp(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big) - \mathrm{Id}_\xi\cdot\exp^\perp(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\right|$$
$$\leq C|\ln|u||^\varsigma t^{-4}\exp(-ct). \quad (3.2.17)$$

Remark 3.2.7. As we explain in the course of the proof of Theorem 3.2.6, if (ξ, h^{ξ}) is trivial around the cusps, then the estimates (3.2.15), (3.2.16) could be easily improved to

$$\Big|\exp(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_{\xi}\cdot\exp(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big| \\ \leq C|\ln|u||\exp(-c'(\ln|\ln|u||)^2/t). \quad (3.2.18)$$

To prove (3.2.15), (3.2.16) in full generality, we use Duhamel's formula and (3.2.12).

Theorem 3.2.8. *There are smooth bounded functions $a_{\xi,j}^{M,n}: M\to \mathrm{End}(\xi)$, $j\geq -1$ such that for any $x\in M$, $t_0>0$, $k\in\mathbb{N}$, there is $C>0$ such that for any $t\in]0,t_0]$, we have*

$$\Big|\exp(-t\square^{E_M^{\xi,n}})\big(x,x\big)-\sum_{j=-1}^{k}a_{\xi,j}^{M,n}(x)t^j\Big|\leq Ct^k. \quad (3.2.19)$$

Moreover, if $x\in M\setminus(\cup_i V_i^M(e^{-t^{-1/3}}))$, then C can be chosen independently of $t\in]0,t_0]$ and x.

Also, there is $\epsilon>0$, such that for any $l\in\mathbb{N}$, $j\geq -1$, there is $C>0$ such that for any $u\in\mathbb{C}$, $0<|u|\leq\epsilon$, $i=1,\ldots,m$, we have

$$\Big|(\nabla_u)^l\Big(a_{\xi,j}^{M,n}\big((z_i^M)^{-1}(u)\big)-\mathrm{Id}_{\xi}a_j^{N,n}\big((z_i^N)^{-1}(u)\big)\Big)\Big|_h\leq C|u|^{1/3}, \quad (3.2.20)$$

where ∇ is induced by the Levi-Civita connection and Chern connections associated with (ξ,h^{ξ}) and $(\omega_{\mathbb{D}}(0),\|\cdot\|_{\mathbb{D}})$.

From now on till the end of this section, we denote by

$$P:=\mathbb{C}\mathrm{P}^1\setminus\{0,1,\infty\}, \quad (3.2.21)$$

and by g^{TP} the unique hyperbolic metric of constant scalar curvature -1 over P with cusps at $D_P=\{0,1,\infty\}$. We use the notations $\|\cdot\|_P$, $V_i^P(\epsilon)$, E_P^n, ... and denote by z^P the Poincaré-compatible coordinate of $0\in\mathbb{C}\mathrm{P}^1$ of (P,g^{TP}).

Definition 3.2.9. We define the *regularized heat trace* by

$$\begin{aligned}\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big]:=&\int_{M\setminus(\cup_i V_i^M(\eta))}\mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})(x,x)\big]dv_M(x)\\ &-\frac{m\cdot\mathrm{rk}(\xi)}{3}\cdot\int_{P\setminus(\cup_i V_i^P(\eta))}\mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_P^n})(x,x)\big]dv_P(x)\\ &+\sum_i\int_{D^*(\eta)}\Big(\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)\Big]\\ &\qquad-\mathrm{rk}(\xi)\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_P^n})\big((z^P)^{-1}(u),(z^P)^{-1}(u)\big)\Big]\Big)dv_{\mathbb{D}^*}(u),\end{aligned} \quad (3.2.22)$$

where $\eta>0$ is such that Theorem 3.2.6 and (3.1.2) hold.

Remark 3.2.10. a) From the fact that there is a holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ permuting D_P and inducing the isometry on (P, g^{TP}), the coordinate z^P in (3.2.22) can be changed to a Poincaré-compatible coordinate associated with 1 or ∞, and this would result in the same definition.

b) Essentially, in our definition of the regularized heat trace, we take out the diverging part of the usual heat trace. This idea is very similar to the famous b-trace, defined by Melrose in [90, Lemma 4.62], which was used in the context of Riemann surfaces with cusps by Albin-Rochon [4].

Proposition 3.2.11. The Definition 3.2.9 makes sense and it is independent of $\epsilon > 0$. We also have

$$\begin{aligned}\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big] := \lim_{r\to 0}\Bigg(&\int_{M\setminus(\cup_i V_i^M(r))}\mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})(x,x)\big]dv_M(x)\\ &-\frac{\mathrm{rk}(\xi)}{3}\int_{P\setminus(\cup_i V_i^P(r))}\mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_P^n})(x,x)\big]dv_P(x)\Bigg).\end{aligned} \tag{3.2.23}$$

Proof. The first two integrals in the right-hand side of (3.2.22) are bounded by (3.2.13). The last one is bounded by (3.2.17) and the fact that for any $\varsigma < 1$, we have

$$\int_{D(\epsilon)}\frac{\sqrt{-1}du d\overline{u}}{|u|^2|\ln|u||^{2-\varsigma}} \leq +\infty. \tag{3.2.24}$$

The independence on $\epsilon > 0$ is trivial. The formula (3.2.23) follows trivially from (3.2.17). □

A similar quantity $\mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\square^{E_M^{\xi,n}})\big]$ (see also [71, Definition 1.1] for the relative version) is defined similarly to (3.2.22), where we put $\exp$ in place of $\exp^{\perp}$. It is well-defined by (3.2.15), (3.2.24) and the fact that for any $c' > 0$ and $\epsilon > 0$ small enough, there is $C > 0$ such that for any $t > 0$:

$$\int_{D(\epsilon)}\exp\big(-c'(\ln|\ln|u||)^2/t\big)\frac{\sqrt{-1}du d\overline{u}}{|u|^2|\ln|u||} \leq Ct^{1/2}\exp\big(-(c'/2)(\ln|\ln\epsilon|)^2/t\big). \tag{3.2.25}$$

By (3.2.6), the relation between Definition 3.2.9 and $\mathrm{Tr}\big[\exp(-t\square^{E_M^{\xi,n}})\big]$ is given by

$$\begin{aligned}&\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big]\\ &\qquad = \mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\square^{E_M^{\xi,n}})\big] - \dim H^0(\overline{M}, E_M^{\xi,n}) + \frac{\mathrm{rk}(\xi)}{3}\dim H^0(\overline{P}, E_P^n).\end{aligned} \tag{3.2.26}$$

Remark 3.2.12. In [71, §3], Jorgenson-Lundelius defined the relative heat trace

$$\mathrm{Tr}^{\mathrm{rel}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}}); \exp^{\perp}(-t\square^{E_N^n})\big] \tag{3.2.27}$$

for (ξ, h^{ξ}) trivial and $n = 0$. Directly from the definition, in this case we have

$$\begin{aligned}&\mathrm{Tr}^{\mathrm{rel}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}}); \exp^{\perp}(-t\square^{E_N^n})\big] = \mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big] - \mathrm{rk}(\xi)\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_N^n})\big],\\ &\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big] = \frac{1}{3}\mathrm{Tr}^{\mathrm{rel}}\big[3\exp^{\perp}(-t\square^{E_M^{\xi,n}}); m\exp^{\perp}(-t\square^{E_P^n})\big],\end{aligned} \tag{3.2.28}$$

where $3\exp^{\perp}(-t\square^{E_M^{\xi,n}})$ (resp. $m\exp^{\perp}(-t\square^{E_P^n})$) means the heat operator on $M \sqcup M \sqcup M$ (resp. on $P \sqcup \cdots \sqcup P$) with the induced geometry.

By Theorem 3.2.8, the functions $\mathrm{Tr}\big[a^{M,n}_{\xi,j}(x)\big]$, $a^{P,n}_j(x)$ are integrable over M and P respectively. For $j \geq -1$, we denote

$$\begin{aligned} A^{M,n}_{\xi,j,0} &:= \int_M \mathrm{Tr}\big[a^{M,n}_{\xi,j}(x)\big] dv_M(x) - \frac{\mathrm{rk}(\xi)}{3} \int_P a^{P,n}_j(x) dv_P(x), \\ A^{M,n}_{\xi,j} &= A^{M,n}_{\xi,j,0} - \dim H^0(\overline{M}, E^{\xi,n}_M) + \frac{\mathrm{rk}(\xi)}{3} \dim H^0(\overline{P}, E^n_P). \end{aligned} \tag{3.2.29}$$

Proposition 3.2.13. For any $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, we have

$$\Big| \mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box^{E^{\xi,n}_M})\big] - \sum_{j=-1}^{k} A^{M,n}_{\xi,j} t^j \Big| \leq C t^k. \tag{3.2.30}$$

Proof. First of all, by (3.2.26), it is enough to prove that for any $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, we have

$$\Big| \mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\Box^{E^{\xi,n}_M})\big] - \sum_{j=-1}^{k} A^{M,n}_{\xi,j,0} t^j \Big| \leq C t^k. \tag{3.2.31}$$

By Theorem 3.2.8, for any $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, we have

$$\begin{aligned} &\Big| \int_{M \setminus (\cup_i V^M_i(e^{-t^{-1/3}}))} \Big[\mathrm{Tr}\big[\exp(-t\Box^{E^{\xi,n}_M})(x,x)\big] - \sum_{j=-1}^{k} \mathrm{Tr}\big[a^{M,n}_{\xi,j}(x)\big] t^j \Big] dv_M(x) \Big| \leq C t^k, \\ &\Big| \int_{P \setminus (\cup_i V^P_i(e^{-t^{-1/3}}))} \Big[\exp(-t\Box^{E^n_P})(x,x) - \sum_{j=-1}^{k} a^{P,n}_j(x) t^j \Big] dv_P(x) \Big| \leq C t^k. \end{aligned} \tag{3.2.32}$$

Since for $u \in \mathbb{C}$, $0 < |u| \leq e^{-t^{-1/3}}$, we have $t^{-1/3} \leq |\ln |u||$, by (3.2.15), (3.2.24) and (3.2.25), for any $k \in \mathbb{N}$ there are $c, C > 0$ such that for any $t \in]0, t_0]$, $i = 1, \ldots, m$, we have

$$\begin{aligned} \int_{D(e^{-t^{-1/3}})} \Big| &\mathrm{Tr}\Big[\exp(-t\Box^{E^{\xi,n}_M})\big((z^M_i)^{-1}(u), (z^M_i)^{-1}(u)\big) \Big] \\ &- \mathrm{rk}(\xi) \exp(-t\Box^{E^n_P})\big((z^P)^{-1}(u), (z^P)^{-1}(u)\big) \Big| dv_{\mathbb{D}^*}(u) \leq C t^k + C \exp(-c t^{-1/2}). \end{aligned} \tag{3.2.33}$$

Also, by (3.2.20), for any $j \in \mathbb{N}$, $i = 1, \ldots, m$ there are $c, C > 0$, such that we have

$$\begin{aligned} \int_{D(e^{-t^{-1/3}})} \Big| &\mathrm{Tr}\Big[a^{M,n}_{\xi,j}\big((z^M_i)^{-1}(u)\big) \Big] \\ &- \mathrm{rk}(\xi) a^{P,n}_j\big((z^P)^{-1}(u)\big) \Big| dv_{\mathbb{D}^*}(u) \leq C \exp(-c t^{-1/3}). \end{aligned} \tag{3.2.34}$$

We see that (3.2.31) holds by (3.2.32), (3.2.33) and (3.2.34). □

Proposition 3.2.14. For any $t_0 > 0$, there are $c, C > 0$ such that for any $t \geq t_0$, we have

$$\left|\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big]\right| \leq C\exp(-ct). \tag{3.2.35}$$

Proof. By Theorem 3.2.4 since ρ_M is bounded over $M \setminus (\cup_i V_i^M(\eta))$, $\eta > 0$ for some $c, C > 0$ and for any $t \geq t_0$, we get

$$\begin{aligned} \left|\int_{M\setminus(\cup_i V_i^M(\eta))} \mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})(x,x)\big]dv_M(x)\right| &\leq C\exp(-ct), \\ \left|\int_{P\setminus(\cup_i V_i^P(\eta))} \mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_P^{n}})(x,x)\big]dv_P(x)\right| &\leq C\exp(-ct). \end{aligned} \tag{3.2.36}$$

By (3.1.2), (3.2.17) and (3.2.24), we deduce that there are $c, C > 0$ such that for any $t \geq t_0$, we have

$$\begin{aligned} \Big|\int_{D(\eta)} \Big(&\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)\Big] \\ &- \mathrm{rk}(\xi)\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_P^{n}})\big((z^P)^{-1}(u),(z^P)^{-1}(u)\big)\Big]\Big)dv_{\mathbb{D}^*}(u)\Big| \leq C\exp(-ct). \end{aligned} \tag{3.2.37}$$

We conclude by (3.2.36) and (3.2.37). □

Definition 3.2.15. We define the *regularized spectral zeta function* $\zeta_M(s)$ for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ by

$$\zeta_M(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} \mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big]t^s\frac{dt}{t}. \tag{3.2.38}$$

By Propositions 3.2.13 and 3.2.14, the function $\zeta_M(s)$ is holomorphic for $\mathrm{Re}(s) > 1$ and has a meromorphic extension to the entire s-plane. Classically, this extension, which we also denote by $\zeta_M(s)$, is holomorphic at $s = 0$.

Definition 3.2.16. We define the *analytic torsion* by

$$T(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)) \cdot T_{TZ}(g^{TP}, \|\cdot\|_P^{2n})^{m\cdot \mathrm{rk}(\xi)/3}. \tag{3.2.39}$$

Remark 3.2.17. a) In the forthcoming paper we show that under the conditions (3.1.3), we have

$$T(g_{\mathrm{hyp}}^{TM}, (\|\cdot\|_M^{\mathrm{hyp}})^{2n}) = T_{TZ}(g_{\mathrm{hyp}}^{TM}, (\|\cdot\|_M^{\mathrm{hyp}})^{2n}). \tag{3.2.40}$$

For the moment, we content ourselves by noting that (3.2.40) holds for $M = P$ by the choice of the last multiplicand in (3.2.39).

b) Explicitly, we have the following identity (see Proposition 3.2.13 for the definition of a_{-1}^M):

$$\zeta_M'(0) = \int_0^1 \Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big] - \frac{A_{\xi,-1}^{M,n}}{t} - A_{\xi,0}^{M,n}\Big)\frac{dt}{t}$$

$$+\int_1^{+\infty}\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big]\frac{dt}{t}+A_{\xi,-1}^{M,n}-\Gamma'(-1)A_{\xi,0}^{M,n}. \quad (3.2.41)$$

c) By (3.2.38), the relation between the relative analytic torsion, defined by Jorgenson-Lundelius [71] for (ξ,h^{ξ}) trivial and $n=0$, and our definition is

$$T^{\mathrm{rel}}(g^{TM},1;g^{TN},1)=\frac{T(g^{TM},1)}{T(g^{TN},1)}. \quad (3.2.42)$$

d) In [4], Albin-Rochon , for $n=0$, gave an alternative definition of the analytic torsion $T_{AR}(g^{TM})$. By [5, (1.24)], [4, §7] and (3.2.15), the relation between their definition and ours is

$$\frac{T_{AR}(g^{TM})}{T_{AR}(g^{TN})}=\frac{T(g^{TM},1)}{T(g^{TN},1)}, \quad (3.2.43)$$

for M,N as in the statement of Theorem A. Their definition is based on b-trace of Melrose [90], see Remark 3.2.10b).

e) In his thesis [58, Corollary 8.2.2], Freixas explicitly evaluated (see (3.1.9))

$$\log Z'_{(\overline{P},D_P)}(1)=4\zeta'(-1)+\log 2\pi+\frac{10}{9}\log 2. \quad (3.2.44)$$

By combining (3.1.10), (3.2.44), we may give an explicit formula for $T_{TZ}(g^{TP},1)$ in (3.2.39).

3.2.3 Heat kernel on the punctured hyperbolic disc and elliptic estimates

In this section we recall the well-known construction [12, §2.4, 2.5] of the parametrix, applied for the heat kernel on the punctured hyperbolic disc, endowed with a Hermitian vector bundle. We also prove the elliptic estimates for Kodaira Laplacian on a punctured hyperbolic disc.

Let's explain the setting in this section. Let (ξ,h^{ξ}) be Hermitian vector bundle over $\mathbb{D}$. Let

$$\omega_{\mathbb{D}}(0):=\omega_{\mathbb{D}}\otimes\mathscr{O}_{\mathbb{D}}(0) \quad (3.2.45)$$

be the twisted canonical line bundle as in (3.1.4), and let $\|\cdot\|_{\mathbb{D}}$ be the norm on $\omega_{\mathbb{D}}(0)$ over $\mathbb{D}^*$, induced by $g^{T\mathbb{D}^*}$ as in (3.1.5). We denote by the same symbol restriction of h^{ξ} to $\mathbb{D}^*$. By Cartan's Theorem A, we fix a holomorphic trivialization $e_1,\ldots,e_{\mathrm{rk}(\xi)}$ of ξ over $\mathbb{D}$. We may always chose it in such a way that it becomes a normal trivialization (cf. [43, Proposition V.12.10]), i.e. we have

$$h^{\xi}(e_i,e_j)(u)=\delta_{ij}+O(|u|^2). \quad (3.2.46)$$

Let $\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n}$, $n\in\mathbb{Z}$ be the Kodaira Laplacian associated with $h^{\xi}\otimes\|\cdot\|_{\mathbb{D}}^{2n}$ on $(\mathbb{D}^*,g^{T\mathbb{D}^*})$. Let

$$\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})(z_1,z_1)\in(\xi\otimes\omega_{\mathbb{D}}(0)^n)^*_{z_1}\boxtimes(\xi\otimes\omega_{\mathbb{D}}(0)^n)_{z_2},\quad\text{for}\quad z_1,z_1\in\mathbb{D}^*, \quad (3.2.47)$$

be the smooth kernel of the heat operator $\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})$ with respect to the volume form $dv_{\mathbb{D}^*}$.

We consider the covering

$$\rho:\mathbb{H}\to\mathbb{D}^*,\qquad z\mapsto e^{\sqrt{-1}z}. \quad (3.2.48)$$

The metric $g^{T\mathbb{H}} := \rho^*(g^{T\mathbb{D}^*})$ is equal to the standard hyperbolic metric on the upper half-plane. The Deck transformations of ρ are generated by the isometry

$$U : \mathbb{H} \to \mathbb{H}, \qquad z \mapsto z + 2\pi. \tag{3.2.49}$$

Let $\|\cdot\|_{\mathbb{H}}$ be the norm on $\omega_{\mathbb{H}}$, given by $\rho^*(\|\cdot\|_{\mathbb{D}})$. For $z = (x, y) := x + \sqrt{-1}y$, we have

$$g_z^{T\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}, \qquad \|dz\|_{\mathbb{H}}(z) = y. \tag{3.2.50}$$

Let $\square^{\xi\otimes\omega_{\mathbb{H}}^n}$ be the Kodaira Laplacian associated with $g^{T\mathbb{H}}$, $\rho^*(h^\xi) \otimes \|\cdot\|_{\mathbb{H}}^{2n}$ on $(\mathbb{H}, g^{T\mathbb{H}})$, and let

$$\exp(-t\square^{\xi\otimes\omega_{\mathbb{H}}^n})(z_1, z_2) \in (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n)^*_{z_1} \boxtimes (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n)_{z_2}, \quad \text{for} \quad z_1, z_2 \in \mathbb{H}, \tag{3.2.51}$$

be the smooth kernel of the heat operator $\exp(-t\square^{\xi\otimes\omega_{\mathbb{H}}^n})$ with respect to the Riemannian volume form $dv_{\mathbb{H}}$ on $\mathbb{H}$, induced by $g^{T\mathbb{H}}$. For $z_1, z_2 \in \mathbb{D}$, the relation between (3.2.47) and (3.2.51) is given by

$$\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})(z_1, z_2) = \sum_{i\in\mathbb{Z}} \exp(-t\square^{\xi\otimes\omega_{\mathbb{H}}^n})(\tilde{z}_1, U^i\tilde{z}_2), \tag{3.2.52}$$

where $\tilde{z}_i \in \mathbb{H}$, $\rho(\tilde{z}_i) = z_i$ for $i = 1, 2$.

Since $(\mathbb{H}, g^{T\mathbb{H}})$ is a compete manifold, we may use the framework of [12, §2.4, 2.5] to construct the parametrix of $\exp(-t\square^{\xi\otimes\omega_{\mathbb{H}}^n})$. Let us briefly recall the main steps of this construction. By doing so, we also provide some uniform estimates on the heat kernels.

We denote by $\mathrm{d}(z_1, z_2)$, $z_1, z_2 \in \mathbb{H}$ the Riemannian distance associated with $g^{T\mathbb{H}}$, we have

$$\mathrm{d}\big((x_1, y_1), (x_2, y_2)\big) = 2\ln\left(\frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2}}{2\sqrt{y_1 y_2}}\right). \tag{3.2.53}$$

Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\psi(u) = \begin{cases} 1 & \text{for } |u| < 1/2, \\ 0 & \text{for } |u| > 1. \end{cases} \tag{3.2.54}$$

For $k \in \mathbb{N}$, $z_1, z_2 \in \mathbb{H}$, $t > 0$, let $k_{t,k}^{\xi\otimes\omega_{\mathbb{H}}^n} \in \mathscr{C}^\infty(\mathbb{H} \times \mathbb{H}, (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n) \boxtimes (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n)^*)$ be given by (cf. [12, (2.7)])

$$k_{t,k}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1, z_2) := \frac{\psi(\mathrm{d}(z_1, z_2)^2)}{t} \exp\Big(-\frac{\mathrm{d}(z_1, z_2)^2}{4t}\Big)\Big(\sum_{i=0}^{k} t^i \Phi_{i,n}^\xi(z_1, z_2)\Big), \tag{3.2.55}$$

where $\Phi_{i,n}^\xi \in \mathscr{C}^\infty(\mathbb{H} \times \mathbb{H}, (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n) \boxtimes (\rho^*(\xi) \otimes \omega_{\mathbb{H}}^n)^*)$, $i \geq 0$ are symmetric (i.e. $\Phi_{i,n}^\xi(z_1, z_2) = (\Phi_{i,n}^\xi(z_2, z_1))^*$) and given by the procedure, described in [12, Theorem 2.26]. We denote by $\Phi_{i,n}$, $i \geq 0$ those sections associated to (ξ, h^ξ) trivial. Now let's state the main result of this section.

Theorem 3.2.18. *The sections $\Phi^{\xi}_{i,n}$ are uniformly $\mathscr{C}^{\infty}$-bounded in the following sense: for any $i, l, l' \in \mathbb{N}$, there is $C > 0$ such that for any $z_1, z_2 \in \mathbb{H}$, we have*

$$\left|(\nabla_{z_1})^l (\nabla_{z_2})^{l'} \Phi^{\xi}_{i,n}(z_1, z_2)\right|_{h\times h} \leq C, \tag{3.2.56}$$

where ∇ is induced by the Levi-Civita connection and Chern connections associated with (ξ, h^{ξ}), $(\omega_{\mathbb{H}}(0), \|\cdot\|_{\mathbb{H}})$, and $|\cdot|_{h\times h}$ is the associated pointwise norm.

Moreover, for any $i, l, l' \in \mathbb{N}$, there is $C > 0$ such that for any $z_1, z_2 \in \mathbb{H}$, we have

$$\left|(\nabla_{z_1})^l (\nabla_{z_2})^{l'} \big(\Phi^{\xi}_{i,n} - \mathrm{Id}_{\xi} \cdot \Phi_{i,n}\big)(z_1, z_2)\right|_{h\times h} \leq C \exp(-(\operatorname{Im} z_1 + \operatorname{Im} z_2)/6). \tag{3.2.57}$$

Proof. Let's fix $z_0 \in \mathbb{H}$, $z_0 = (x_0, y_0)$. For $z \in \mathbb{H}$, $r > 0$ we denote by $B^{\mathbb{H}}(z, r) \subset \mathbb{H}$ the hyperbolic disc of radius r around z. We consider the isometry

$$g_{z_0} : (\mathbb{H}, g^{T\mathbb{H}}) \to (\mathbb{H}, g^{T\mathbb{H}}), \qquad (x, y) \mapsto ((x - x_0)/y_0, y/y_0). \tag{3.2.58}$$

As $g_{z_0}(z_0) = (0, 1) := \sqrt{-1}$, we have $g_{z_0}(B^{\mathbb{H}}(z_0, 1)) = B^{\mathbb{H}}(\sqrt{-1}, 1)$. We recall that by the procedure, described in [12, Theorem 2.26], the sections $\Phi^{\xi}_{i,n}(z, \cdot)$ are defined locally, i.e. they depend only on the restriction of $(\mathbb{H}, g^{T\mathbb{H}})$, (ξ, h^{ξ}) over $B^{\mathbb{H}}(z, 1)$, and if $\mathrm{d}(z, z_2) > 1$, then $\Phi^{\xi}_{i,n}(z, z_2) = 0$. Moreover, if one changes "smoothly" the parameters $g^{T\mathbb{H}}$, h^{ξ}, then the sections $\Phi^{\xi}_{i,n}(z, \cdot)$ change smoothly "at the same rate". Let's make the last point precise and adapt it for our situation.

Let h^{ξ}_z, $h^{\xi,0}_z$, $z \in \mathbb{H}$ be two families of Hermitian metrics on $(g_z^{-1}\rho)^*\xi$ over $B^{\mathbb{H}}(\sqrt{-1}, 1)$, and let $\Phi^{\xi}_{i,n,z}(\sqrt{-1}, \cdot)$, $\Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, \cdot)$ be the corresponding sections from (3.2.55). Suppose that there is $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $l \in \mathbb{N}$, there is $C > 0$ such that for any $z_2 \in B^{\mathbb{H}}(\sqrt{-1}, 1)$:

$$\begin{aligned} &\left|\nabla^l (h^{\xi}_z)(z_2)\right|_h && \leq C, \\ &\left|\nabla^l (h^{\xi}_z - h^{\xi,0}_z)(z_2)\right|_h && \leq C f(\operatorname{Im} z). \end{aligned} \tag{3.2.59}$$

From the procedure, described in [12, Theorem 2.26], the sections $\Phi^{\xi}_{i,n,z}(\sqrt{-1}, \cdot)$, $\Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, \cdot)$ are obtained iteratively by applying the Laplacian associated with h^{ξ}_z and $h^{\xi,0}_z$ to $\Phi^{\xi}_{i-1,n,z}(\sqrt{-1}, \cdot)$ and $\Phi^{\xi,0}_{i-1,n,z}(\sqrt{-1}, \cdot)$ respectively and integrating over the geodesics of length ≤ 1, emanating from $\sqrt{-1}$. Thus, for any $l \in \mathbb{N}$ there is $C > 0$ such that for any $z_2 \in B^{\mathbb{H}}(\sqrt{-1}, 1)$, we have

$$\begin{aligned} &\left|(\nabla_{z_2})^l \Phi^{\xi}_{i,n,z}(\sqrt{-1}, z_2)\right|_h && \leq C, \\ &\left|(\nabla_{z_2})^l \big(\Phi^{\xi}_{i,n,z} - \Phi^{\xi,0}_{i,n,z}\big)(\sqrt{-1}, z_2)\right|_h && \leq C f(\operatorname{Im} z), \end{aligned} \tag{3.2.60}$$

or, as we stated before, the sections $\Phi^{\xi}_{i,n,z}(\sqrt{-1}, \cdot)$, $\Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, \cdot)$, $i \geq 0$ change "at the same rate".

Now, let h^{ξ}_z, $z \in \mathbb{H}$ be defined by

$$h^{\xi}_z := ((g_z^{-1}\rho)^* h^{\xi})|_{B^{\mathbb{H}}(\sqrt{-1},1)}. \tag{3.2.61}$$

Let the frame $e_1, \dots, e_{\mathrm{rk}(\xi)}$ be as in (3.2.46). Then for $z \in \mathbb{H}$, $z_2 = (x, y) \in B^{\mathbb{H}}(\sqrt{-1}, 1)$, we have

$$h^{\xi}_z((g_z^{-1}\rho)^* e_i, (g_z^{-1}\rho)^* e_j)(z_2) = h^{\xi}(e_i, e_j)(e^{-yy_0+\sqrt{-1}(xy_0+x_0)}). \tag{3.2.62}$$

Let $h^{\xi,0}_z$, $z \in \mathbb{H}$ be defined by

$$h^{\xi,0}_z((g_z^{-1}\rho)^* e_i, (g_z^{-1}\rho)^* e_j)(z_2) = \delta_{ij}, \tag{3.2.63}$$

where δ_{ij} is the Kronecker delta symbol. Let $\Phi^{\xi}_{i,n,z}(\sqrt{-1}, \cdot)$, $\Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, \cdot)$ be the sections from (3.2.55), associated with $\|\cdot\|_{\mathbb{H}}|_{B^{\mathbb{H}}(\sqrt{-1},1)}$, $g^{T\mathbb{H}}|_{B^{\mathbb{H}}(\sqrt{-1},1)}$ and h^{ξ}_z, $h^{\xi,0}_z$ respectively. Then by the locality of $\Phi^{\xi}_{i,n,z}(\sqrt{-1}, \cdot)$, $\Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, \cdot)$, for any $z_2 \in B^{\mathbb{H}}(\sqrt{-1}, 1)$, we have

$$\Phi^{\xi}_{i,n,z}(\sqrt{-1}, z_2) = \Phi^{\xi}_{i,n}(z, g_z^{-1}(z_2)), \qquad \Phi^{\xi,0}_{i,n,z}(\sqrt{-1}, z_2) = \mathrm{Id}_{\xi} \cdot \Phi_{i,n}(z, g_z^{-1}(z_2)). \tag{3.2.64}$$

By the symmetry of $\Phi^{\xi}_{i,n,z}$ and (3.2.64), to complete the proof of Theorem 3.2.18, it is enough to prove the analogue of (3.2.59) for $f(x) = \exp(-x/3)$.

Now, by the formula (3.2.53), we have

$$\min\left\{ \operatorname{Im} z : z \in B^{\mathbb{H}}(\sqrt{-1}, 1)\right\} \geq 1/6. \tag{3.2.65}$$

By (3.2.46), (3.2.62) and (3.2.65), we have (3.2.59) for $f(x) = \exp(-x/3)$, which finishes the proof. □

To compare $k^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(x, y)$ with the heat kernel, we recall the definition of the "defect":

$$r^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2) := \left(\partial_t + \square^{\xi\otimes\omega_{\mathbb{D}}(0)^n}_{\mathbb{D},x}\right) k^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2). \tag{3.2.66}$$

The following theorem says, in particular, that as one increases $k \in \mathbb{N}$, the kernel $k^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2)$ more and more accurately "satisfies" the properties defined by the heat kernel.

Theorem 3.2.19. *For any $t_0 > 0$, the family of kernels $k^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2)$, $t \in]0, t_0]$, $z_1, z_2 \in \mathbb{H}$ defines a uniformly bounded family of operators $K^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}$ on $\mathscr{C}^{\infty}_c(\mathbb{H}, \rho^*(\xi) \otimes \omega^n_{\mathbb{H}})$ such that for any $s \in \mathscr{C}^{\infty}_c(\mathbb{H}, \rho^*(\xi) \otimes \omega^n_{\mathbb{H}})$, the sections $K^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(s)$ converge, as $t \to 0$, to s over any compact subset of $\mathbb{H}$ with all it's derivatives.*

Moreover, for any $l, l', l'' \in \mathbb{N}$, there are $c', C > 0$ such that for any $t \in]0, t_0]$, $z_1, z_2 \in \mathbb{H}$:

$$\left|(\nabla_{z_1})^l (\nabla_{z_2})^{l'} (\partial_t)^{l''} k^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2)\right|_{h\times h} \leq C t^{-1-(l+l')/2-l''} \cdot \psi(\mathrm{d}(z_1, z_2)^2/2) \cdot \exp(-c' \cdot \mathrm{d}(z_1, z_2)^2/t), \tag{3.2.67}$$

$$\left|(\nabla_{z_1})^l (\nabla_{z_2})^{l'} (\partial_t)^{l''} r^{\xi\otimes\omega^n_{\mathbb{H}}}_{t,k}(z_1, z_2)\right|_{h\times h} \leq C t^{k-(l+l')/2-l''} \cdot \psi(\mathrm{d}(z_1, z_2)^2/2) \cdot \exp(-c' \cdot \mathrm{d}(z_1, z_2)^2/t). \tag{3.2.68}$$

Proof. The first statement is done as in [12, Theorem 2.29]. The estimate (3.2.67) follows directly from (3.2.55) and (3.2.56). The proof of (3.2.68) uses (3.2.56), but otherwise it is done in the same way as [12, Theorem 2.29]. □

This theorem means that $k_{t,k}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1,z_2)$ is the parametrix of the heat equation in the sense of [12, p.77]. Thus, we may construct the heat kernel by the procedure, which follows. For $k,k'\in\mathbb{N}$, $z,z'\in\mathbb{H}$, we denote

$$q_{t,k,k'}^{\xi\otimes\omega_{\mathbb{H}}^n}(z,z') := \int_{t\Delta_{k'}}\int_{\mathbb{H}^{k'}} k_{t-t_{k'},k}^{\xi\otimes\omega_{\mathbb{H}}^n}(z,z_{k'})r_{t_{k'}-t_{k'-1},k}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_{k'},z_{k'-1})\cdots \\ \cdots r_{t_1,k}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1,z')\,dv_{\mathbb{H}}(z_{k'})\otimes\cdots\otimes dv_{\mathbb{H}}(z_1)dv_{t\Delta_{k'}}(t_1,\ldots,t_{k'}), \quad (3.2.69)$$

where $\Delta_{k'}$ is the standard k'-simplex, and $dv_{t\Delta_{k'}}(t_1,\ldots,t_{k'})$ is the standard volume form over $t\Delta_{k'}$. Now let's explain why (3.2.69) is well-defined. The integration over $\mathbb{H}^{k'}$ in (3.2.69) is well-defined since by (3.2.54), (3.2.55) and (3.2.66), the functions under the integral vanish if the arguments are too distant, so all the integrations are done in a compact subset. The integration over $t\Delta_{k'}$ is well-defined for $k\geq 1$ by (3.2.55) and (3.2.68). By the same reasons, it is easy to see that if $k\geq (l+l')/2+l''+1$, then the partial derivatives $(\partial_{z_1})^l(\partial_{z_2})^{l'}(\partial_t)^{l''}q_{t,k,k'}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1,z_2)$ exists.

Theorem 3.2.20. *For any $t_0>0$, $k\in\mathbb{N}^*$ and $l,l',l''\in\mathbb{N}$, there are $c',C>0$ such that for any $t\in]0,t_0]$, $z_1,z_2\in\mathbb{H}$ and $k'\in\mathbb{N}$ satisfying $k\geq (l+l')/2+l''+1$, we have*

$$\Big|(\nabla_{z_1})^l(\nabla_{z_2})^{l'}(\partial_t)^{l''}q_{t,k,k'}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1,z_2)\Big|_{h\times h}\leq \frac{C^{k'}t^{kk'-l-l'-2l''}}{(k'-1)!}\exp(-c'\cdot \mathrm{d}(z_1,z_2)^2/t). \quad (3.2.70)$$

Moreover, for any $t\in]0,t_0]$, $z_1,z_2\in\mathbb{H}$, $k\in\mathbb{N}^*$, the series

$$\sum_{k'=0}^{\infty}(-1)^{k'}q_{t,k,k'}^{\xi\otimes\omega_{\mathbb{H}}^n}(z_1,z_2), \quad (3.2.71)$$

converges to $\exp(-t\square_{\mathbb{H}}^{\xi\otimes\omega_{\mathbb{H}}^n})(z_1,z_2)$ in $\mathscr{C}^{2k-2}(\mathbb{H}\times\mathbb{H})$, and for any $l,l',l''\in\mathbb{N}$, satisfying $k\geq (l+l')/2+l''+1$, there is $C>0$ such for any $t\in]0,t_0]$, $z_1,z_2\in\mathbb{H}$, we have

$$\Big|(\nabla_{z_1})^l(\nabla_{z_2})^{l'}(\partial_t)^{l''}\Big(\exp(-t\square^{\xi\otimes\omega_{\mathbb{H}}^n})-k_{t,k}^{\xi\otimes\omega_{\mathbb{H}}^n}\Big)(z_1,z_2)\Big|_{h\times h} \\ \leq Ct^{k-l-l'-2l''}\exp(-c'\cdot \mathrm{d}(z_1,z_2)^2/t). \quad (3.2.72)$$

Proof. First of all, we note that by the weighted mean inequality and the triangle inequality, for $k'\in\mathbb{N}$, $t>t_{k'}>\ldots>t_1>0$, and $z,z',z_1,\ldots,z_{k'}\in\mathbb{H}$, we have

$$\exp\Big(-c'\frac{\mathrm{d}(z,z_{k'})^2}{t-t_{k'}}\Big)\exp\Big(-c'\frac{\mathrm{d}(z_{k'},z_{k'-1})^2}{t_{k'}-t_{k'-1}}\Big)\cdots \\ \cdots\exp\Big(-c'\frac{\mathrm{d}(z_1,z')^2}{t_1}\Big)\leq \exp\Big(-\frac{c'}{t}\mathrm{d}(z,z')^2\Big). \quad (3.2.73)$$

We also note that the integration over each variable $z_1,\ldots,z_{k'}$ is done over a hyperbolic ball of radius 1, which has a constant volume, independently of the choice of its center. From now on, the proof remains verbatim with [12, Lemma 2.22, Theorem 2.23], where one has to replace the appropriate estimates by (3.2.67), (3.2.68) and use (3.2.73) to bound the exponentials. □

Now let's apply all this theory to the study of the heat kernel on the punctured hyperbolic disc. We summarize all the important results, which will be used in Section 3.2.4, in the following theorem, which is a local analogue of (3.2.12) and Theorem 3.2.8.

Theorem 3.2.21. *For any $l, l', l'' \in \mathbb{N}$, there are $t_0 > 0$ $c, c', C > 0$ such that for any $t \in]0, t_0]$, $u, v \in \mathbb{D}^*$, we have*

$$\Big|(\nabla_u)^l(\nabla_v)^{l'}(\partial_t)^{l''}\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})(u,v)\Big|_{h\times h} \leq Ct^{-1-(l+l')/2-l''} \cdot \big(1+|\ln|u||\big)^{1/2}\big(1+|\ln|v||\big)^{1/2}\exp\big(-c'\cdot \mathrm{d}(u,v)^2/t\big). \quad (3.2.74)$$

Moreover, there are bounded sections $a^{\mathbb{D}^*,n}_{\xi,j} \in \mathscr{C}^\infty(\mathbb{D}^*, \mathrm{End}(\xi))$, $j \geq -1$ such that there are $c', C > 0$ such that for any $u \in \mathbb{D}^*$, $k \in \mathbb{N}$ and $t \in]0, t_0]$, we have

$$\Big|\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})(u,u) - \sum_{j=-1}^{k} a^{\mathbb{D}^*,n}_{\xi,j}(u)t^j\Big| \leq \Big(1+|\ln|u||\Big)\Big(Ct^k + \frac{C}{t}\exp\Big(-\frac{c'}{t|\ln|u||^2}\Big)\Big). \quad (3.2.75)$$

Moreover, for any $j \geq -1$, there is $C > 0$ such that for any $u \in \mathbb{D}^*$, we have

$$\Big|(\nabla_u)^l\Big(a^{\mathbb{D}^*,n}_{\xi,j} - \mathrm{Id}_{\mathrm{rk}(\xi)}a^{\mathbb{D}^*,n}_j\Big)(u)\Big|_{h\times h} \leq C|u|^{1/3}, \quad (3.2.76)$$

where we trivialized ξ as in the beginning of this section.

Before proving this theorem, let's prove the following technical

Lemma 3.2.22. There is $t_0 > 0$ such that for any $z_1, z_2 \in \mathbb{H}$, $t \in]0, t_0]$, satisfying $\mathrm{d}(z_1, U^i z_2) \leq \mathrm{d}(z_1, z_2)$ for any $i \in \mathbb{Z}$, we have

$$\sum \exp\big(-\mathrm{d}(z_1, U^i z_2)^2/t\big) \leq C\Big(\big(\mathrm{Im}(z_1)+1\big)\big(\mathrm{Im}(z_2)+1\big)\Big)^{1/2}\exp\Big(-\mathrm{d}(z_1,z_2)^2/(2t)\Big). \quad (3.2.77)$$

Proof. We decompose the sum in (3.2.77) into two parts: for $i^2 \leq 4\,\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)$ and the complementary. Trivially, by the assumption, the first part is bounded by

$$4\Big(\big(\mathrm{Im}(z_1)+1\big)\big(\mathrm{Im}(z_2)+1\big)\Big)^{1/2}\exp\Big(-\mathrm{d}(z_1,z_2)^2/(2t)\Big). \quad (3.2.78)$$

Now, by choosing t_0 small enough, we see that

$$\exp\Big(-\Big(\ln\frac{i^2}{\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)}\Big)^2/t\Big) \leq \frac{\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)}{i^2}. \quad (3.2.79)$$

By (3.2.89) and (3.2.79), we see that

$$\begin{aligned}\sum_{i^2>4\,\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)} &\exp(-\mathrm{d}(z_1,U^iz_2)^2/t)\\ &\leq \big(\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)\big)\cdot\exp\big(-\mathrm{d}(z_1,z_2)^2/(2t)\big)\sum_{i^2>4\,\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)} i^{-2}\\ &\leq \big(\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)\big)^{1/2}\cdot\exp\big(-\mathrm{d}(z_1,z_2)^2/(2t)\big).\end{aligned}\tag{3.2.80}$$

Thus, we conclude by (3.2.78) and (3.2.80). □

Proof of Theorem 3.2.21. Let $u,v\in\mathbb{D}^*$, and let $\tilde{u},\tilde{v}\in\mathbb{H}$ be such that $\rho(\tilde{u})=u$, $\rho(\tilde{v})=v$. Then $\mathrm{Im}(\tilde{u})=|\log|u||$, $\mathrm{Im}(\tilde{v})=|\log|v||$. By (3.2.52), (3.2.72) and Lemma 3.2.22, we have

$$\begin{aligned}&\Big|(\nabla_u)^l(\nabla_v)^{l'}(\partial_t)^{l''}\Big(\exp(-t\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})(u,v)-\sum_{i\in\mathbb{Z}}k_{t,k}^{\xi\otimes\omega_{\mathbb{H}}^n}(\tilde{u},U^i\tilde{v})\Big)\Big|_{h\times h}\\ &\qquad\leq Ct^{k-(l+l')/2-l''}\big(1+|\ln|u||\big)^{1/2}\big(1+|\ln|v||\big)^{1/2}\exp\big(-c'\cdot\mathrm{d}(u,v)^2/t\big).\end{aligned}\tag{3.2.81}$$

Now, by (3.2.89), there is $C>0$ such that for any $z_1,z_2\in\mathbb{H}$, we have

$$\#\Big\{i\in\mathbb{Z}:\mathrm{d}(z_1,U^iz_2)<2\Big\}\leq C\big((\mathrm{Im}(z_1)+1)(\mathrm{Im}(z_2)+1)\big)^{1/2}.\tag{3.2.82}$$

Thus, the number of non-zero terms in the sum under the module in (3.2.81) is bounded by the right-hand side of (3.2.82). So, by (3.2.67) and (3.2.81), we get (3.2.74).

Now, by (3.2.53), there is $C>0$ such that for any $z\in\mathbb{D}^*$ and $i\in\mathbb{Z}^*$, we have

$$\mathrm{d}(z,U^iz)\geq\frac{C}{|\ln|z||}.\tag{3.2.83}$$

Thus, from Lemma 3.2.22, (3.2.81) and (3.2.83), we get (3.2.75) for

$$a_{\xi,j}^{\mathbb{D}^*,n}(z):=\Phi_{j+1}^{\xi\otimes\omega_{\mathbb{H}}^n}(\tilde{z},\tilde{z}),\quad\text{where}\quad\tilde{z}\in\mathbb{H},\rho(\tilde{z})=z,\quad\text{and}\quad j\geq-1.\tag{3.2.84}$$

Now, (3.2.76) follows from (3.2.57) and (3.2.84). □

Finally, as an application of the ideas from the proof of Theorem 3.2.18, let's establish the following elliptic estimates.

Lemma 3.2.23. For any $\alpha>0$, $k\in\mathbb{N}$, there is $C>0$, such that for any $n\in\mathbb{Z}$, $\sigma\in\mathscr{C}^\infty(\mathbb{D}^*,\xi\otimes\omega_{\mathbb{D}}(0)^n)$, $x\in\mathbb{D}^*$, we have

$$\big|\nabla^k\sigma(x)\big|_h\leq C|\log|x||^{1/2}\textstyle\sum_{i=0}^{2+k}(n^{4(2+2k-i)}+1)\big\|(\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})^i\sigma\big\|_{L^2(B^{\mathbb{D}^*}(x,\alpha))}.\tag{3.2.85}$$

Remark 3.2.24. Similar results have appeared in a recent article of Auvray-Ma-Marinescu [9, §4]. Our methods of proof are, however, fundamentally different.

Proof. We conserve the notations from the proof of Theorem 3.2.18.

We denote by $\square_z^{\xi\otimes\omega_{\mathbb{H}}^n}$ the Kodaira Laplacian on $B^{\mathbb{H}}(\sqrt{-1},1)$ associated to $g^{T\mathbb{H}}$, $h_z^{\xi}\otimes\|\cdot\|_{\mathbb{H}}^{2n}$. Let ∇_z be the connection on $B^{\mathbb{H}}(\sqrt{-1},1)$ induced by the Chern connection associated to $h_z^{\xi}, \|\cdot\|_{\mathbb{H}}$ and the Levi-Civita connection on $(\mathbb{H}, g^{T\mathbb{H}})$.

The family of metrics h_z^{ξ} over $B^{\mathbb{H}}(\sqrt{-1},1)$ has bounded geometry by (3.2.59). From this, the fact that $g_z\in\mathrm{Aut}(\mathbb{H})$ preserves $g^{T\mathbb{H}}$ and [85, Lemma 1.6.2], we deduce that for any $0<\alpha<1$, $k\in\mathbb{N}$, there is $C>0$, such that for any $z\in\mathbb{H}$, $n\in\mathbb{Z}$, $\sigma_1\in\mathscr{C}^{\infty}(B^{\mathbb{H}}(\sqrt{-1},1),\rho^*(\xi)\otimes\omega_{\mathbb{H}}^n)$:

$$\big|(\nabla_z^k\sigma_1)(\sqrt{-1})\big|_h\leq C\sum_{i=0}^{2+k}(n^{4(2+2k-i)}+1)\big\|(\square_z^{\xi\otimes\omega_{\mathbb{H}}^n})^i\sigma_1\big\|_{L^2(B^{\mathbb{H}}(\sqrt{-1},\alpha))}. \tag{3.2.86}$$

Now, for $\tilde{\sigma}\in\mathscr{C}^{\infty}(\mathbb{H},\rho^*(\xi)\otimes\omega_{\mathbb{H}}^n)$, we denote $\sigma_1:=((g_z)^{-1})^*\tilde{\sigma}$. Then by the fact that $g_z\in\mathrm{Aut}(\mathbb{H})$ preserves $g^{T\mathbb{H}}$, we trivially have

$$\begin{aligned}&\big|(\nabla_z^k\sigma_1)(\sqrt{-1})\big|_h=\big|(\nabla^k\tilde{\sigma})(z)\big|_h,\\&\big\|(\square_z^{\xi\otimes\omega_{\mathbb{H}}^n})^i\sigma_1\big\|_{L^2(B^{\mathbb{H}}(\sqrt{-1},\alpha))}=\big\|(\square^{\xi\otimes\omega_{\mathbb{H}}^n})^i\tilde{\sigma}\big\|_{L^2(B^{\mathbb{H}}(z,\alpha))}.\end{aligned} \tag{3.2.87}$$

From (3.2.86) and (3.2.87), we deduce the following elliptic estimate on $\mathbb{H}$:

$$\big|(\nabla^k\tilde{\sigma})(\tilde{x})\big|_h\leq C\sum_{i=0}^{2+k}(n^{4(2+2k-i)}+1)\big\|(\square^{\xi\otimes\omega_{\mathbb{H}}^n})^i\tilde{\sigma}\big\|_{L^2(B^{\mathbb{H}}(\tilde{x},\alpha))}. \tag{3.2.88}$$

Now, by (3.2.53), for any $i\neq 0$, we have

$$\mathrm{d}(z_1,U^iz_2)\geq\ln\big(i^2/(\mathrm{Im}(z_1)\,\mathrm{Im}(z_2))\big). \tag{3.2.89}$$

By (3.2.89), we deduce that for any α, there is $C'>0$ such that for any $x\in D^*(1/2)$, $\tilde{x}\in\mathbb{H}$, such that $\rho(\tilde{x})=x$, we have

$$\#\Big\{\tilde{y}\in B^{\mathbb{H}}(\tilde{x},\alpha):\rho(\tilde{y})=y\Big\}\leq C'\cdot|\log|x||. \tag{3.2.90}$$

Thus, by (3.2.90) and the fact that the restriction $\rho|_{B^{\mathbb{H}}(\tilde{x},\alpha)}:B^{\mathbb{H}}(\tilde{x},\alpha)\to B^{\mathbb{D}}(x,\alpha)$ is a surjection, we deduce that for any $\sigma\in\mathscr{C}^{\infty}(\mathbb{D}^*,\xi\otimes\omega_{\mathbb{D}}(0)^n)$, $x\in\mathbb{D}^*$ and $\tilde{x}\in\mathbb{H}$ such that $\rho(\tilde{x})=x$, we have

$$\big\|(\square^{\xi\otimes\omega_{\mathbb{H}}^n})^i(\sigma\circ\rho)\big\|_{L^2(B^{\mathbb{H}}(\tilde{x},\alpha))}\leq(C')^{1/2}|\log|x||^{1/2}\cdot\big\|(\square^{\xi\otimes\omega_{\mathbb{D}}^n})^i\sigma\big\|_{L^2(B^{\mathbb{D}^*}(x,\alpha))} \tag{3.2.91}$$

By (3.2.91) and (3.2.88) applied for $\tilde{\sigma}:=\rho^*\sigma$, we deduce (3.2.85) for $C:=C(C')^{1/2}$. □

Lemma 3.2.25. For any $\beta>1$, $k\in\mathbb{N}$, $n\in\mathbb{Z}$, there is $C>0$, such that for any $\sigma\in\mathscr{C}^{\infty}(\mathbb{D}^*,\xi\otimes\omega_{\mathbb{D}}(0)^n)$, $x\in D(x/(2\beta))\setminus\{0\}$, we have

$$\big|\nabla^k\sigma(x)\big|_h\leq C|\log|x||^{3+k}\sum_{i=0}^{2+k}\big\|(\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})^i\sigma\big\|_{L^2(D(\beta|x|)\setminus D(|x|/\beta))}. \tag{3.2.92}$$

Proof. We use the notation from the proof of Lemma 3.2.23.

First, let's prove that for any $\gamma>1$, $k\in\mathbb{N}$, $n\in\mathbb{Z}$, there is $C>0$, such that for any $z\in\mathbb{H}$, $\tilde{\sigma}\in\mathscr{C}^{\infty}(\mathbb{H},\rho^*(\xi)\otimes\omega_{\mathbb{H}}^n)$, we have the following elliptic estimate on $\mathbb{H}$:

$$\big|(\nabla^k\tilde{\sigma})(z)\big|_h\leq C|\,\mathrm{Im}\,z|^{2+k}\sum_{i=0}^{2+k}\big\|(\square^{\xi\otimes\omega_{\mathbb{H}}^n})^i\tilde{\sigma}\big\|_{L^2(B^{\mathbb{H}}(z,\gamma/\,\mathrm{Im}\,z))}. \tag{3.2.93}$$

Similarly to (3.2.87), we see that in the notations (3.2.86), to prove (3.2.93), it is enough to prove that for any $1 > \delta > 0$, there exists $C > 0$ such that the following estimate holds

$$\big|(\nabla^k_z \sigma_1)(\sqrt{-1})\big|_h \leq C\delta^{-(2+k)} \sum_{i=0}^{2+k} \big\|(\square_z^{\xi\otimes\omega^n_{\mathbb{H}}})^i \sigma_1\big\|_{L^2(B^{\mathbb{H}}(\sqrt{-1},\delta))}. \tag{3.2.94}$$

However, as the family of metrics h^ξ_z, $z \in \mathbb{H}$ has bounded geometry and $g^{T\mathbb{H}}$ differs from the standard Euclidean metric over $B^{\mathbb{H}}(\sqrt{-1}, 1) \subset \mathbb{C}$ by a smooth function, we deduce that it is enough to prove that for a standard Kodaira Laplacian $\square$ on $\mathbb{C}$, for any $1 > \delta > 0$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, $\sigma' \in \mathscr{C}^\infty(D(\delta))$, we have

$$\big|(\nabla^k \sigma')(0)\big|_h \leq C\delta^{-(2+k)} \sum_{i=0}^{2+k} \big\|\square^i \sigma'\big\|_{L^2(D(\delta))}. \tag{3.2.95}$$

But (3.2.95) follows from a standard elliptic estimate on a disc (cf. [85, Lemma 1.6.2]) by using the transformation $\sigma''(x) := \sigma'(\delta x)$.

Directly from (3.2.91), there is $C' > 0$ such that for any $\sigma \in \mathscr{C}^\infty(\mathbb{D}, \xi \otimes \omega_{\mathbb{D}}(0)^n)$, $x \in D(1/2) \setminus \{0\}$ and $\tilde{x} \in \mathbb{H}$ such that $\rho(\tilde{x}) = x$, we have

$$\big\|(\square^{\xi\otimes\omega^n_{\mathbb{H}}})^i(\sigma \circ \rho)\big\|_{L^2(B^{\mathbb{H}}(\tilde{x},\gamma/\operatorname{Im}(\tilde{x})))} \leq C'|\log|x||^{1/2} \big\|(\square^{\xi\otimes\omega^n_{\mathbb{D}}})^i \sigma\big\|_{L^2(B^{\mathbb{D}}(x,\gamma/|\log|x||))} \tag{3.2.96}$$

From (3.2.96) and (3.2.93) applied for $\tilde{\sigma} = \sigma \circ \rho$, we get

$$\big|\nabla^k \sigma(x)\big|_h \leq C|\log|x||^{5/2+k} \sum_{i=0}^{2+k} \big\|(\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n})^i \sigma\big\|_{L^2(B^{\mathbb{D}}(x,\gamma/|\log|x||))}. \tag{3.2.97}$$

However, by (3.2.53), we see that for any $\beta > 1$, there is $\gamma > 0$ such that for any $z \in D(1/(2\beta))$, we have $B^{\mathbb{D}}(z, \gamma/|\log|z||) \subset D(\beta z) \setminus D(z/\beta)$. From this and (3.2.97), we deduce (3.2.92). $\square$

Remark 3.2.26. Let's choose a family of Hermitian metrics h^ξ_η, $\eta \in]0, 1]$ instead of h^ξ, for

$$h^\xi_\eta(e_i, e_j)(u) := \big(1 - \psi(|u|^2/\eta)\big) h^\xi(e_i, e_j)(u) + \psi(|u|^2/\eta)\delta_{ij}, \tag{3.2.98}$$

where ψ is defined in (3.2.54), e_i, $i = 1, \ldots, \operatorname{rk}(\xi)$ is as in (3.2.46). Then all the estimates of this chapter would continue to hold uniformly over $\eta \in]0, 1]$.

Let's briefly explain this point. First of all, as all the results of this section rely on Theorem 3.2.18, it is enough to explain why the uniform analogue of (3.2.59) holds, as it is the main step in the proof of Theorem 3.2.18. But this is due to the fact that for the Hermitian metrics $h^\xi_{z,\eta}$, $z = (x_0, y_0) \in \mathbb{H}$, defined in the notation of Theorem 3.2.18 by (compare with (3.2.61))

$$h^\xi_{z,\eta} := ((g_z^{-1}\rho)^* h^\xi_\eta)|_{B^{\mathbb{H}}(\sqrt{-1},1)}, \tag{3.2.99}$$

we have (compare with (3.2.62))

$$h^\xi_{z,\eta}((g_z^{-1}\rho)^* e_i, (g_z^{-1}\rho)^* e_j)(z_2) = h^\xi_\eta(e_i, e_j)(e^{-yy_0+\sqrt{-1}(xy_0+x_0)}). \tag{3.2.100}$$

Now, if a smooth function $f(y)$, $y > 0$ satisfies $f(y) = 0$, $y > 1$, then the function $f(e^{-y}/\eta)$, $y \in \mathbb{R}$ has bounded derivatives uniformly on $\eta > 0$. From this observation, (3.2.100) implies that for $h^{\xi,0}_z$ defined as in (3.2.63), the following uniform analogue of (3.2.59) holds:

$$\begin{aligned} &\big|(\partial_x)^l(\partial_y)^{l'}(h^\xi_{z,\eta})(z_2)\big|_h &&\leq C, \\ &\big|(\partial_x)^l(\partial_y)^{l'}(h^\xi_{z,\eta} - h^{\xi,0}_z)(z_2)\big|_h &&\leq Ce^{-\operatorname{Im} z/3}. \end{aligned} \tag{3.2.101}$$

Finally, let's mention one consequence of Remark 3.2.26. We add a subscript η to all the objects which depend on h^{ξ}_{η} instead of h^{ξ}.

Lemma 3.2.27. For any $\alpha > 0$, $k \in \mathbb{N}$, there is $C > 0$, such that for any $n \in \mathbb{Z}$, $\sigma \in \mathscr{C}^{\infty}(\mathbb{D}, \xi \otimes \omega_{\mathbb{D}}(0)^n)$, $x \in \mathbb{D}$, we have

$$\left|\nabla^k_{\eta}\sigma(x)\right|_{h,\eta} \leq C|\log|x||^{1/2}\sum_{i=0}^{2+k}(n^{4(2+2k-i)}+1)\left\|(\square^{\xi\otimes\omega_{\mathbb{D}}(0)^n}_{\eta})^i\sigma\right\|_{L^2_{\eta}(B^{\mathbb{D}}(x,\alpha))}. \tag{3.2.102}$$

Proof. Same as the proof of Lemma 3.2.23, as by (3.2.101), the family $h^{\xi}_{z,\eta}$ is bounded. □

3.2.4 Proofs of Theorems 3.2.1, 3.2.4, 3.2.6, 3.2.8

In this section we finally present the proofs of Theorems 3.2.1, 3.2.4, 3.2.6 and 3.2.8.

Proof of Theorem 3.2.1. First of all, for $n \leq 0$, there is $C > 0$ such that for any $z \in D^*(1/2)$:

$$C < \left(|z|^2(\ln|z|)^{2-2n}\right)^{-1}. \tag{3.2.103}$$

Let g^{TM}_{sm} be some smooth metric over $\overline{M}$, and let $\|\cdot\|^{\mathrm{sm}}_M$ be some smooth Hermitian norm on $\omega_M(D)$ over $\overline{M}$. By (3.2.103), there is $C > 0$ such that $g^{TM}_{\mathrm{sm}} \otimes (\|\cdot\|^{\mathrm{sm}}_M)^{2n} \leq Cg^{TM} \otimes \|\cdot\|^{2n}_M$. Thus, we have

$$\ker(\square^{E^{\xi,n}_M}) \subset L^2\left(g^{TM}_{\mathrm{sm}}, h^{\xi} \otimes (\|\cdot\|^{\mathrm{sm}}_M)^{2n}\right). \tag{3.2.104}$$

Let $s \in \ker(\square^{E^{\xi,n}_M})$. By (3.2.104) and the classical L^2-extension theorem (cf. [85, Lemma 2.3.22]), s extends holomorphically to $V^M_i(\epsilon)$. In other words

$$\ker(\square^{E^{\xi,n}_M}) \subset H^0(\overline{M}, E^{\xi,n}_M). \tag{3.2.105}$$

By the finiteness of the volume of (M, g^{TM}), see (3.2.24), we see that that each holomorphic section lies in $L^2(g^{TM}, h^{\xi} \otimes \|\cdot\|^{2n}_M)$, i.e.

$$H^0(\overline{M}, E^{\xi,n}_M) \subset \ker(\square^{E^{\xi,n}_M}). \tag{3.2.106}$$

We deduce (3.2.6) by (3.2.105) and (3.2.106).

For $n = 0$, our proof of Theorem 3.2.1 relies on the result of Müller [93, §6, Proposition 6.9], who proves Theorem 3.2.1 for $n = 0$ and (ξ, h^{ξ}) trivial. In case of $n < 0$, we obtain Theorem 3.2.1 by gluing the estimates in the neighbourhood of cusp, coming from Nakano's inequality (cf. [85, Theorem 1.4.14]), and the estimates away from the cusps coming from the spectral gap for the Dirichlet Laplacian of a surface with boundary.

Let's show that (3.2.7) holds for $n < 0$ and any (ξ, h^{ξ}). In [93, §6], Müller proved (3.2.8) for (ξ, h^{ξ}) trivial, $n = 0$ and $c_2 = 1/4$, see Remark 3.2.2. This implies, in particular, that (3.2.7) holds for (ξ, h^{ξ}) trivial and $n = 0$ (see [93, Proposition 6.9]). He proved (3.2.8) in this case by studying explicitly the spectrum of Kodaira Laplacian of the von Neumann problem in the cusp and using the scattering matrix to relate the continuous spectrum of the manifolds. If the vector

bundle (ξ, h^{ξ}) is trivial around the cusps, the presence of it doesn't change the Hermitian structure around D_M. Thus, the result of Müller extends line by line to the case $n = 0$ and (ξ, h^{ξ}) trivial around the cusps, which we summarize in

$$\mathrm{Spec}(\square^{E_M^{\xi,n}}) \cap [0, 1/4] \qquad \text{is discrete.} \tag{3.2.107}$$

Now, let h^{ξ} be any Hermitian metric on ξ. We will prove that there is $k \in \mathbb{N}$ and $F \subset L^2(E_M^{\xi,n})$, $\mathrm{codim} F = k$, such that we have

$$\inf_{s \in F} \left\{ \frac{\langle \square^{E_M^{\xi,n}} s, \square^{E_M^{\xi,n}} s \rangle_{L^2}}{\langle s, s \rangle_{L^2}} \right\} > 0. \tag{3.2.108}$$

Then, by the min-max theorem (cf. [85, (C.3.3)]), (3.2.6) and (3.2.108), we get (3.2.7).

We choose $\eta \in]0, 1/2]$ small enough, so that (3.1.2) is satisfied for any $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we fix a normal trivialization of ξ over $V_i^M(\eta)$, i.e. a local holomorphic frame $e_1, \ldots, e_{\mathrm{rk}(\xi)}$ of ξ over $V_i^M(\eta)$ as in (3.2.46). Let h_{η}^{ξ} be a Hermitian metric on ξ such that it coincides with h^{ξ} over $M \setminus (\cup_i V_i^M(\eta))$ and over $V_i^M(\eta)$ it is given by (compare with (3.2.98))

$$h_{\eta}^{\xi}((z_i^M)^{-1}(u))(e_i, e_j) = (1 - \psi(|u|^2/\eta)) h^{\xi}((z_i^M)^{-1}(u))(e_i, e_j) + \psi(|u|^2/\eta)\delta_{ij}, \tag{3.2.109}$$

where ψ is defined in (3.2.54), e_i, $i = 1, \ldots, \mathrm{rk}(\xi)$ is as in (3.2.46), and δ_{ij} is the Kronecker delta symbol. Then (ξ, h_{η}^{ξ}) is trivial around the cusps, and there is $C > 0$ such that for any $\eta \in]0, 1/2]$, we have

$$(h_{\eta}^{\xi})^{-1} \frac{\partial h_{\eta}^{\xi}}{\partial z_i^M} \Big((z_i^M)^{-1}(u) \Big) \leq C|u|. \tag{3.2.110}$$

We denote by $\square_{\eta}^{E_M^{\xi,n}}$ the Kodaira Laplacian on (M, g^{TM}), associated with h_{η}^{ξ}. Then over $V_i^M(\eta)$, we have

$$(\overline{\partial}^{\xi})^* = \Big(\|dz_i^M\|_M^{\omega} \Big)^2 \Big(\frac{\partial}{\partial z_i^M} + (h_{\eta}^{\xi})^{-1} \frac{\partial h_{\eta}^{\xi}}{\partial z_i^M} \Big) \cdot \iota_{\partial/\partial \overline{z}_i^M}, \tag{3.2.111}$$

where ι is the contraction and $*$ is the adjoint with respect to the L^2-scalar product induced by h_{η}^{ξ}. By (3.2.2) and (3.2.111), we deduce

$$\square_{\eta}^{E_M^{\xi,n}} - \square^{E_M^{\xi,n}} = \sum_i |z_i^M|^2 (\ln |z_i^M|)^2 \Big((h_{\eta}^{\xi})^{-1} \frac{\partial h_{\eta}^{\xi}}{\partial z_i^M} - (h^{\xi})^{-1} \frac{\partial h^{\xi}}{\partial z_i^M} \Big) \frac{\partial}{\partial \overline{z}_i^M}. \tag{3.2.112}$$

We denote by $\langle \cdot, \cdot \rangle_{L_{\eta}^2}$ the L^2-scalar product induced by g^{TM}, h_{η}^{ξ}. We fix $\eta > 0$ small enough so that $2h^{\xi} > h_{\eta}^{\xi} > h^{\xi}/2$. Then we have $2\langle \cdot, \cdot \rangle_{L^2} > \langle \cdot, \cdot \rangle_{L_{\eta}^2} > \langle \cdot, \cdot \rangle_{L^2}/2$. Now, by (3.2.112) and Cauchy inequality, for $s \in \mathscr{C}_c^{\infty}(M, E_M^{\xi,n})$, as the support of (3.2.112) lies in $\cup V_i^M(\eta^{1/2}/2)$, by (3.2.110):

$$\langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2} \geq \tfrac{1}{2} \langle \square_{\eta}^{E_M^{\xi,n}} s, s \rangle_{L_{\eta}^2} - 2Cm|\eta^2 \ln |\eta|| \big(\langle s, s \rangle_{L_{\eta}^2} \cdot \langle \square_{\eta}^{E_M^{\xi,n}} s, s \rangle_{L_{\eta}^2} \big)^{1/2}. \tag{3.2.113}$$

We fix $\eta > 0$ small enough so that $4Cm|\eta^2 \ln|\eta|| \leq 1/16$, and put

$$F := \Big\langle \big\{ s \in \mathrm{dom}\big(\square_\eta^{E_M^{\xi,n}}\big) : \square_\eta^{E_M^{\xi,n}} s = \lambda s, \quad \text{for} \quad \lambda < 1/4 \big\} \Big\rangle^\perp, \tag{3.2.114}$$

where the orthogonal complement is taken with respect to $\langle \cdot, \cdot \rangle_{L^2_\eta}$. Since (ξ, h^ξ_η) is trivial around the cusps, by (3.2.107), the space F is of finite codimension. By (3.2.113) and (3.2.114), for $s \in F$:

$$\frac{\langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2}}{\langle s, s \rangle_{L^2}} \geq \frac{1}{4} \bigg(\frac{\langle \square_\eta^{E_M^{\xi,n}} s, s \rangle_{L^2_\eta}}{\langle s, s \rangle_{L^2_\eta}} \bigg)^{1/2} \bigg(\bigg(\frac{\langle \square_\eta^{E_M^{\xi,n}} s, s \rangle_{L^2_\eta}}{\langle s, s \rangle_{L^2_\eta}} \bigg)^{1/2} - \frac{1}{4} \bigg) \geq \frac{1}{32}. \tag{3.2.115}$$

Also, by Cauchy inequality, we have

$$\bigg(\frac{\langle \square^{E_M^{\xi,n}} s, \square^{E_M^{\xi,n}} s \rangle_{L^2}}{\langle s, s \rangle_{L^2}} \bigg)^{1/2} \geq \frac{\langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2}}{\langle s, s \rangle_{L^2}} \tag{3.2.116}$$

Then (3.2.115) and (3.2.116) imply (3.2.108), and thus (3.2.7) holds for $n = 0$ and any (ξ, h^ξ).

We remark that similarly to (3.2.113), we have

$$\langle \square_\eta^{E_M^{\xi,n}} s, s \rangle_{L^2_\eta} \geq \tfrac{1}{2} \langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2} - 2m|\eta^2 \ln|\eta|| \big(\langle s, s \rangle_{L^2} \cdot \langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2} \big)^{1/2}. \tag{3.2.117}$$

From (3.2.108) and (3.2.117), in a similar fashion as we got (3.2.108), we deduce that there exists $\mu > 0$ such that for any η small enough, we have

$$\mathrm{Spec}\Big(\square_\eta^{E_M^{\xi,n}} \cap]0, \mu] \Big) = \emptyset. \tag{3.2.118}$$

Now let's show that (3.2.7) holds for $n < 0$ and any (ξ, h^ξ). Similarly, we prove that there are $k \in \mathbb{N}$, $F \subset L^2(E_M^{\xi,n})$, $\mathrm{codim} \overline{F} = k$ satisfying (3.2.108). Then, as before, we would get (3.2.7).

Let $\eta_0 > 0$ be chosen such that g^{TM} is induced by (3.1.2) over $\cup_i V_i^M(\eta_0)$, and

$$\big| [\sqrt{-1} R^\xi, \Lambda^{TM}] \big| \leq 1/4, \qquad \text{over } \cup_i V_i^M(\epsilon_0), \tag{3.2.119}$$

where R^ξ is the curvature of the Chern connection on (ξ, h^ξ), and Λ^{TM} is the contraction with the Hermitian norm induced by g^{TM}. Such ϵ_0 exists since as (ξ, h^ξ) is a Hermitian vector bundle over $\overline{M}$ and $\Lambda^{TM} = O(|z_i^M \ln|z_i^M||^2) dz_i^M d\overline{z}_i^M$ over $V_i^M(\epsilon_0)$, which can be made arbitrarily small by replacing ϵ_0 by a smaller number.

Let $\rho : \overline{M} \to [0, 1]$ be a smooth cut-off function satisfying

$$\rho(x) = \begin{cases} 1 & \text{for } x \in \cup_i V_i^M(\epsilon_0/2), \\ 0 & \text{for } x \in M \setminus (\cup_i V_i^M(\epsilon_0)). \end{cases} \tag{3.2.120}$$

For $s \in \mathscr{C}_c^\infty(M, E_M^{\xi,n})$, we have

$$\langle \square^{E_M^{\xi,n}} s, s \rangle_{L^2} = \langle \square^{E_M^{\xi,n}} (\rho s), \rho s \rangle_{L^2}$$

$$+\langle \Box^{E_M^{\xi,n}}((1-\rho)s),(1-\rho)s\rangle_{L^2}+2\langle \Box^{E_M^{\xi,n}}(\rho s),(1-\rho)s\rangle_{L^2}. \quad (3.2.121)$$

By Cauchy inequality, we see that there is $c_1>0$ such that for any $\epsilon>0$, we have

$$\left|\langle \Box^{E_M^{\xi,n}}(\rho s),(1-\rho)s\rangle_{L^2}\right| \leq \left|\langle \rho(\Box^{E_M^{\xi,n}}s),(1-\rho)s\rangle_{L^2}\right| + \left|\langle [\Box^{E_M^{\xi,n}},\rho]s,(1-\rho)s\rangle_{L^2}\right|, \quad (3.2.122)$$

Since $[\Box^{E_M^{\xi,n}},\rho]$ is a differential operator of order 1 with support in a compact subspace of M, there is $C>0$ such that for any $s\in\mathscr{C}_c^\infty(M,E_M^{\xi,n})$, we have

$$\left\|[\Box^{E_M^{\xi,n}},\rho]s\right\|_{L^2}^2 \leq C\Big(\left\|\Box^{E_M^{\xi,n}}s\right\|_{L^2}^2+\|s\|_{L^2}^2\Big). \quad (3.2.123)$$

By (3.2.123) and Cauchy inequality, there is $c_2>0$ such that for any $\epsilon>0$, we have

$$\begin{aligned}&\left|\langle [\Box^{E_M^{\xi,n}},\rho]s,(1-\rho)s\rangle_{L^2}\right| \leq \epsilon\Big(\left\|\Box^{E_M^{\xi,n}}s\right\|_{L^2}^2+\|s\|_{L^2}^2\Big)+(c_2/\epsilon)\|(1-\rho)s\|_{L^2}^2,\\ &\left|\langle \rho(\Box^{E_M^{\xi,n}}s),(1-\rho)s\rangle_{L^2}\right| \leq \|\Box^{E_M^{\xi,n}}s\|_{L^2}^2+\|(1-\rho)s\|_{L^2}^2.\end{aligned} \quad (3.2.124)$$

Thus, by (3.2.121), (3.2.122) and (3.2.124), we see that

$$\begin{aligned}\langle \Box^{E_M^{\xi,n}}s,s\rangle_{L^2}+(3+2c_1/\epsilon)\|\Box^{E_M^{\xi,n}}s\|_{L^2}^2 &\geq \langle \Box^{E_M^{\xi,n}}(\rho s),\rho s\rangle_{L^2}\\ &+\langle \Box^{E_M^{\xi,n}}((1-\rho)s),(1-\rho)s\rangle_{L^2}-4\epsilon\|s\|_{L^2}^2-(2+2c_2/\epsilon)\|(1-\rho)s\|_{L^2}^2.\end{aligned} \quad (3.2.125)$$

Recall that by Nakano's inequality (cf. [85, Theorem 1.4.14]), we have

$$\langle \Box^{E_M^{\xi,n}}(\rho s),\rho s\rangle_{L^2} \geq \langle [\sqrt{-1}R^{E_M^{\xi,n}},\Lambda^{TM}](\rho s),\rho s\rangle_{L^2}, \quad (3.2.126)$$

where $R^{E_M^{\xi,n}}$ is the curvature of the Chern connection on $E_M^{\xi,n}$. We decompose

$$R^{E_M^{\xi,n}}=R^{\xi}+n\mathrm{Id}_{\xi}\cdot R^{\omega_M(D)}, \quad (3.2.127)$$

where $R^{\omega_M(D)}$ is the curvature of the Chern connection on $(\omega_M(D),\|\cdot\|_M)$. Now, by (3.1.2), over $V_i^M(\eta_0)$, we have

$$[\sqrt{-1}R^{\omega_M(D)},\Lambda^{TM}]=-1/2. \quad (3.2.128)$$

We conclude by (3.2.119), (3.2.126), (3.2.127) and (3.2.128) that for $d:=-n/2-1/4>0$, we have

$$\langle \Box^{E_M^{\xi,n}}(\rho s),\rho s\rangle_{L^2} \geq d\|\rho s\|_{L^2}^2. \quad (3.2.129)$$

As the closure of $M\setminus(\cup_i V_i^M(\epsilon))$ is a compact manifold with boundary, the Dirichlet problem for $\Box^{E_M^{\xi,n}}$ on $M\setminus(\cup_i V_i^M(\epsilon))$ has a discrete set of eigenvalues. Let $\phi_1,\phi_2,\ldots$ be the eigenvectors corresponding to the eigenvalues in the increasing order. There exists $k\in\mathbb{N}$ such that for any s, satisfying $s\perp(1-\rho)\phi_i$, $i=1,\ldots,k$, we have

$$\langle \Box^{E_M^{\xi,n}}((1-\rho)s),(1-\rho)s\rangle_{L^2} \geq (2+d+2c_2/\epsilon)\left\|(1-\rho)s\right\|_{L^2}^2. \quad (3.2.130)$$

Thus, we conclude from (3.2.125), (3.2.129) and (3.2.130) that for some $c_1, c_2 > 0, k \in \mathbb{N}$ and for any $\epsilon > 0$ and s satisfying $s \perp (1-\rho)\phi_i$, $i = 1, \ldots, k$, we have

$$\langle \square^{E_M^{\xi,n}} s, s\rangle_{L^2} + (2 + 2c_1/\epsilon)\|\square^{E_M^{\xi,n}} s\|_{L^2}^2 \geq (d/2 - 4\epsilon)\|s\|_{L^2}^2 . \tag{3.2.131}$$

We set $F = \langle (1-\rho)\phi_1, \ldots, (1-\rho)\phi_k\rangle^{\perp}$, where the orthogonal complement is taken with respect to the L^2-scalar product. Then we take $\epsilon = d/16$ and deduce (3.2.108) from (3.2.116) and (3.2.131). □

We recall that the function $\rho_M : M \to [1, +\infty[$ was defined in (3.2.11). To prove Theorems 3.2.4 and 3.2.6, we need the following technical

Lemma 3.2.28. For any $\alpha > 0$, $k \in \mathbb{N}$, there is $C > 0$, such that for any $n \in \mathbb{Z}$, $\sigma \in \mathscr{C}^{\infty}(M, E_M^{\xi,n})$, $x \in M$, we have

$$\left|\nabla^k \sigma(x)\right|_h \leq C\rho_M(x) \textstyle\sum_{i=0}^{2+k} (n^{4(2+2k-i)} + 1)\left\|(\square^{E_M^{\xi,n}})^i \sigma\right\|_{L^2(B^M(x,\alpha))}. \tag{3.2.132}$$

Proof. Let $\epsilon > 0$. For $x \in M \setminus (\cup_i V_i^M(\epsilon))$, the estimate (3.2.132) follows from [85, Lemma 1.6.2]. For $x \in V_i^M(\epsilon)$, the estimate (3.2.132) follows from Lemma 3.2.23. □

To prove Theorem 3.2.4, we need the following

Lemma 3.2.29. Let $f(t)$, $t > 0$ be a semigroup of operators acting on $L^2(E_M^{\xi,n})$ with smooth kernels $f(t, x, y)$, $x, y \in M$ associated with $dv_M(y)$. Suppose that for any $l, l', l'' \in \mathbb{N}$, there are some $t_0 > 0$, $c', C_1 > 0$, such that for any $t \in]0, t_0]$, $x, y \in M$, we have

$$\left|(\nabla_x)^l (\nabla_y)^{l'} (\partial_t)^{l''} f(t, x, y)\right|_{h\times h} \leq C_1 t^{-1-(l+l')/2-l''} \rho_M(x)\rho_M(y) \exp(-c'\mathrm{d}(x, y)^2/t). \tag{3.2.133}$$

Then there are $c, C > 0$ such that for any $t > 0$, $x, y \in M$, we have

$$\left|(\nabla_x)^l (\nabla_y)^{l'} (\partial_t)^{l''} f(t, x, y)\right|_{h\times h} \leq Ct^{-1-(l+l')/2-l''} \cdot \rho_M(x)\rho_M(y) \exp(ct - c'\mathrm{d}(x, y)^2/t). \tag{3.2.134}$$

Proof. There are essentially three different cases to consider $x, y \in M \setminus (\cup_i V_i^M(1/2))$, $x \in V_i^M(1/2), y \in V_j^M(1/2)$ for some $i \neq j$ and $x, y \in V_i^M(1/2)$ for some $i = 1, \ldots, m$. We only treat the last one, which is the most difficult one, and we leave the rest to the reader.

We denote $u = z_i^M(x)$, $v = z_i^M(y)$. Let's prove by induction that there exists $c, C > 0$ such that for any $k \in \mathbb{N}$, $t < 2^k t_0$, we have

$$\left|(\nabla_u)^l (\nabla_v)^{l'} (\partial_t)^{l''} f(t, u, v)\right|_{h\times h} \leq Ct^{-1-(l+l')/2-l''} (1 + |\ln|u||)^{1/2} (1 + |\ln|v||)^{1/2} \cdot \exp\left(c(2^n - n) - \frac{c'}{t} \cdot \mathrm{d}(u, v)^2\right). \tag{3.2.135}$$

Now, for $k = 0$, (3.2.135) is simply (3.2.133). Once the induction step is done, (3.2.135) would imply (3.2.134). For simplicity, we treat the case $l = l' = l'' = 0$, as the generalization is straightforward.

Let $k \in \mathbb{N}$ and $2^{k-1}t_0 \leq t < 2^k t_0$, then by the semigroup property, we have

$$\big|f(2t,u,v)\big|_{h\times h} \leq \int_M \big|f(t,u,z)\big|_{h\times h} \cdot \big|f(t,z,v)\big|_{h\times h} dv_M(z) \tag{3.2.136}$$

Without losing the generality, suppose $|u| \leq |v|$. We decompose the integration over M into four parts: over $V_i^M(|u|)$, over $V_i^M(|v|) \setminus V_i^M(|u|)$, over $V_i^M(1/2) \setminus V_i^M(|v|)$ for $i = 1, \ldots, m$, and over $M \setminus (\cup V_i^M(1/2))$. We will suppose that $|u|$ is small enough, as if it is not, then the treatment of all those cases reduces to the last one, which is the easiest one. Before treating those cases, let's recall some facts about the geometry of $(\mathbb{D}^*, g^{T\mathbb{D}^*})$ and the induced S^1-action by rotations. First of all, by (3.1.2), for any $u_0 \in V_i^M(1/2)$, the length of the S^1-orbit of u_0 is given by $2\pi/|\ln(|u_0|)|$. Thus, by triangle inequality and S^1-symmetry, for any $u_1 \in V_i^M(1/2)$, we have

$$\mathrm{d}(|u_0|,|u_1|) \leq \mathrm{d}(u_0,u_1) \leq \mathrm{d}(|u_0|,|u_1|) + \min\left\{\frac{2\pi}{|\ln|u_0||}; \frac{2\pi}{|\ln|u_1||}\right\}. \tag{3.2.137}$$

Also, by a trivial calculation, we have

$$\mathrm{d}(|u_0|,|u_1|) = \Big|\ln|\ln|u_0|| - \ln|\ln|u_1||\Big|. \tag{3.2.138}$$

Let $z \in V_i^M(1/2)$, by abuse of notation, we denote $z := z_i^M(z)$.

Let's treat the integration over $|z| < |u|$. By (3.2.137), we have

$$\mathrm{d}(z,v) \geq \mathrm{d}(|u|,|v|) \geq \mathrm{d}(u,v) - \frac{2\pi}{|\ln|u||}. \tag{3.2.139}$$

By (3.2.138) and (3.2.139), since u is small enough, we deduce

$$\mathrm{d}(z,v)^2 \geq \mathrm{d}(|u|,|v|)^2 \geq \mathrm{d}(u,v)^2 - 4\pi. \tag{3.2.140}$$

From the induction hypothesis (3.2.135), (3.2.139) and (3.2.140), we deduce

$$\begin{aligned}\int_{|z|<|u|} \big|f(t,u,z)\big|_{h\times h} \cdot \big|f(t,z,v)\big|_{h\times h} dv_M(z) &\leq 2C^2 t^{-2}(1+|\ln|u||)^{1/2} \\ &\cdot (1+|\ln|v||)^{1/2} \exp\Big(c(2\cdot 2^{n-1} - 2(n-1)) + \frac{4\pi c'}{t} - \frac{c'}{t}\mathrm{d}(u,v)^2\Big) \\ &\cdot \int_{|z|<|u|} \exp\Big(-\frac{c'}{t}\mathrm{d}(|u|,|z|)^2\Big)\frac{\sqrt{-1}dz d\overline{z}}{|z|^2\ln|z|}.\end{aligned} \tag{3.2.141}$$

Now, by (3.2.138), there exists $C_2 > 0$ such that for any $t > 0$, we have

$$\int_{|z|<|u|} \exp\Big(-\frac{c'}{t}\mathrm{d}(|u|,|z|)^2\Big)\frac{\sqrt{-1}dz d\overline{z}}{|z|^2\ln|z|} = 4\pi\int_0^\infty \exp\Big(-\frac{c'}{t}r^2\Big)dr \leq C_2\sqrt{t}. \tag{3.2.142}$$

From (3.2.141) and (3.2.142), we deduce

$$\int_{|z|<|u|} \big|f(t,u,z)\big|_{h\times h}\cdot\big|f(t,z,v)\big|_{h\times h}dv_M(z)$$
$$\leq 2C^2C_2\exp(4\pi^2c'/t)t^{-3/2}\big(1+|\ln|u||\big)^{1/2}\big(1+|\ln|v||\big)^{1/2}$$
$$\cdot\exp\Big(c\big(2\cdot 2^{n-1}-2(n-1)\big)-\frac{c'}{t}\mathrm{d}(u,v)^2\Big) \quad (3.2.143)$$

Thus, by choosing c, C appropriately, by using the bounds on t, we bound the contribution from the integral over $\{|z|<|u|\}$ by the right-hand side of (3.2.135).

Now let's treat the integral over $|u|<|z|<|v|$**.** From (3.2.137), (3.2.138) and the boundness of the Gaussian integral, for some $C>0$, we deduce

$$\int_{|u|<|z|<|v|}\exp\Big(-\frac{c'}{t}\big(\mathrm{d}(u,z)^2+\mathrm{d}(v,z)^2\big)\Big)\frac{\sqrt{-1}dzd\overline{z}}{|z|^2\ln|z|}$$
$$\leq 2\pi\int_{\ln|\ln|u||}^{\ln|\ln|v||}\exp\Big(-\frac{c'}{t}\Big(\big(y-\ln|\ln|u||\big)^2+\big(\ln|\ln|v||-y\big)^2\Big)\Big)dy$$
$$=2\pi\int_0^{\mathrm{d}(|u|,|v|)}\exp\Big(-\frac{c'r^2}{t}-\frac{c'}{t}\big(\mathrm{d}(|u|,|v|)-r\big)^2\Big)dr=2\pi\exp\Big(-\frac{c'}{2t}\mathrm{d}(|u|,|v|)^2\Big)$$
$$\cdot\int_{-\mathrm{d}(|u|,|v|)/2}^{\mathrm{d}(|u|,|v|)/2}\exp\Big(-\frac{c'r^2}{2t}\Big)dr\leq C\sqrt{t}\exp\Big(-\frac{c'}{2t}\mathrm{d}(|u|,|v|)^2\Big). \quad (3.2.144)$$

From the induction hypothesis (3.2.135), (3.2.140) and the bounds on t, we bound the contribution of the integration over $|u|<|z|<|v|$ by the right-hand side of the induction step (3.2.135).

The integral over $|v|<|z|<1/2$ is treated similarly to the integral over $|z|<|u|$.

The integral over $z\in M\setminus\cup_i V_i^M(1/2)$ is the easiest one and it follows from (3.2.73). □

Proof of Theorem 3.2.4. Let's prove (3.2.13) first. From Lemma 3.2.28, there is $C>0$, such that for any $x,x'\in M$, we have

$$\big|(\nabla_x)^l(\nabla_{x'})^{l'}\exp^{\perp}(-t\square^{E_M^{\xi,n}})(x,x')\big|_{h\times h}\leq C\rho_M(x)\rho_M(x')\cdot$$
$$\cdot\sum_{i=0}^{2+l}\sum_{j=0}^{2+l'}\Big\|(\square^{E_M^{\xi,n}})^i\exp^{\perp}(-t\square^{E_M^{\xi,n}})(\square^{E_M^{\xi,n}})^j\Big\|^{0,0}, \quad (3.2.145)$$

where $\|\cdot\|^{0,0}$ is the operator norm between the corresponding L^2 spaces. For any $l\in\mathbb{N}$, $c>0$, there is $C>0$ such that for any $t>0$, we have

$$\sup_{u\geq c}u^l\exp(-tu)\leq Ct^{-l}\exp(-ct/2). \quad (3.2.146)$$

By Theorem 3.2.1, for any $i,j\in\mathbb{N}$, there are $c,C>0$ such that for any $t\geq 0$, we have

$$\Big\|(\square^{E_M^{\xi,n}})^i\exp^{\perp}(-t\square^{E_M^{\xi,n}})(\square^{E_M^{\xi,n}})^j\Big\|^{0,0}\leq Ct^{-(i+j)}\exp(-ct). \quad (3.2.147)$$

From (3.2.145) and (3.2.147), we get (3.2.13).

Let's proceed with a proof of (3.2.12). By Lemma 3.2.29, it's enough to prove it for $t < t_0$ for some $t_0 > 0$. We fix $\epsilon > 0$ small enough, and consider several cases.

Case 1: $x, x' \in M \setminus (\cup_i V_i^M(\epsilon))$. The estimate (3.2.12) for small t is classical and it is proved by using finite propagation speed of solutions of hyperbolic equations (cf. [85, Theorems D.2.1, 4.2.8]) and the parametrix estimates of the heat kernel similar to [12, §2.4, 2.5].

Case 2: $x \in V_i^M(\epsilon)$, $x' \notin V_i^M(2\epsilon)$, for some $i = 1, \ldots, m$. In this case, we prove the estimate (3.2.12) for $t < t_0$ by using finite propagation speed of solutions of hyperbolic equations.

More precisely, for $r > 0$, we introduce smooth even functions (cf. [85, (4.2.11)])

$$\begin{aligned} K_{t,r}(a) &= \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v\sqrt{2t}a) \exp\Big(-\frac{v^2}{2}\Big)\Big(1 - \psi\Big(\frac{\sqrt{2t}v}{r}\Big)\Big)\frac{dv}{\sqrt{2\pi}}, \\ G_{t,r}(a) &= \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v\sqrt{2t}a) \exp\Big(-\frac{v^2}{2}\Big)\psi\Big(\frac{\sqrt{2t}v}{r}\Big)\frac{dv}{\sqrt{2\pi}}, \end{aligned} \tag{3.2.148}$$

where $\psi : \mathbb{R} \to [0, 1]$ was defined in (3.2.54). Let $\widetilde{K}_{t,r}, \widetilde{G}_{t,r} : \mathbb{R}_+ \to \mathbb{R}$ be the smooth functions given by $\widetilde{K}_{t,r}(a^2) = K_{t,r}(a), \widetilde{G}_{t,r}(a^2) = G_{t,r}(a)$. Then the following identities hold

$$\exp(-t\square^{E_M^{\xi,n}}) = \widetilde{G}_{t,r}(\square^{E_M^{\xi,n}}) + \widetilde{K}_{t,r}(\square^{E_M^{\xi,n}}). \tag{3.2.149}$$

By the finite propagation speed of solutions of hyperbolic equations (cf. [85, Theorems D.2.1, 4.2.8]), the section $\widetilde{G}_{t,r}(\square^{E_M^{\xi,n}})\big(y, \cdot\big)$, $y \in M$, depends only on the restriction of $\square^{E_M^{\xi,n}}$ onto $B^M(y, r)$. Moreover, we have

$$\operatorname{supp} \widetilde{G}_{t,r}(\square^{E_M^{\xi,n}})\big(y, \cdot\big) \subset B^M(y, r). \tag{3.2.150}$$

From (3.2.149) and (3.2.150), we get

$$\exp(-t\square^{E_M^{\xi,n}})(y, z) = \widetilde{K}_{t,r}(\square^{E_N^n})(y, z) \quad \text{if} \quad \mathrm{d}(y, z) > r. \tag{3.2.151}$$

From (3.2.148), for any $r_0 > 0$ fixed, there exists $c' > 0$ such that for any $m \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, 1], r > r_0, a \in \mathbb{R}$, the following inequality holds (cf. [85, (4.2.12)])

$$|a|^m |K_{t,r}(a)| \leq C \exp(-c'r^2/t). \tag{3.2.152}$$

Thus, by (3.2.152), for $t \in]0, 1], r > r_0, a \in \mathbb{R}_+$, we have

$$|a|^m |\widetilde{K}_{t,r}(a)| \leq C \exp(-c'r^2/t). \tag{3.2.153}$$

Now, by (3.2.153), there exists $c' > 0$ such that for any $k, k' \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, 1]$ and $r > r_0$, we have

$$\Big\| (\square^{E_M^{\xi,n}})^k \widetilde{K}_{t,r}(\square^{E_M^{\xi,n}})(\square^{E_M^{\xi,n}})^{k'} \Big\|^{0,0} \leq C \exp(-c'r^2/t), \tag{3.2.154}$$

where $\|\cdot\|^{0,0}$ is the operator norm between the corresponding L^2-spaces. Thus, by Lemma 3.2.28, for any $l, l' \in \mathbb{N}$, there are $c', C > 0$ such that for any $x, x' \in M$, $r > r_0$, we have

$$\left|(\nabla_x)^l(\nabla_{x'})^{l'}\widetilde{K}_{t,r}(\square^{E_M^{\xi,n}})(x,x')\right|_{h\times h} \leq C\rho_M(x)\rho_M(x')\exp(-c'r^2/t), \tag{3.2.155}$$

We get (3.2.12) from (3.2.151) and (3.2.155) by taking $r_0 = \frac{1}{4}\mathrm{d}(V_i^M(\epsilon), M \setminus V_i^M(2\epsilon))$ and $r = \frac{1}{2}\mathrm{d}(x,y)$.

Case 3: $x, x' \in V_i^M(2\epsilon)$ for some $i = 1, \ldots, m$. In this case, we prove the estimate (3.2.12) for $t < t_0$ by (3.2.74) and by finite propagation speed of solutions of hyperbolic equations.

We choose a holomorphic trivialization $e_1, \ldots, e_{\mathrm{rk}(\xi)}$ of ξ over $V_i^M(\epsilon)$. By the map $\mathbb{C}^{\mathrm{rk}(\xi)} \to \xi$, given by $(z_1, \ldots, z_{\mathrm{rk}(\xi)}) \mapsto z_1e_1 + \cdots + z_{\mathrm{rk}(\xi)}e_{\mathrm{rk}(\xi)}$, we induce the Hermitian metric h_0^ξ on the trivial vector bundle $\xi_0 := \mathbb{C}^{\mathrm{rk}(\xi)}$ over $D(2\epsilon)$. By using a bump function, we extend h_0^ξ to a Hermitian metric on $\mathbb{C}^{\mathrm{rk}(\xi)}$ over $\mathbb{D}$, which is trivial away from a compact set, and by abuse of notation, we denote the resulting Hermitian metric by h_0^ξ. Let $\square^{E_{\mathbb{D}}^{\xi_0,n}}$ be the Kodaira Laplacian on $(\mathbb{D}, g^{T\mathbb{D}^*})$ associated with $(\xi_0 \otimes \omega_{\mathbb{D}}(0), h_0^\xi \otimes (\|\cdot\|_{\mathbb{D}})^{2n})$.

We denote $u := z_i^M(x)$, $u' := z_i^M(x')$, $r := \mathrm{d}_{\mathbb{D}^*}(u, 2\epsilon)$. Without losing the generality, we suppose $r < \mathrm{d}_{\mathbb{D}^*}(u', 2\epsilon)$. By (3.2.137) and (3.2.138), for some $c > 0$, we have

$$\mathrm{d}(u, 2\epsilon) \geq \ln|\ln|u|| - c. \tag{3.2.156}$$

From the fact that the restriction of $\square^{E_M^{\xi,n}}$ onto $B^M(x,r)$ coincides with the restriction of $\square^{E_{\mathbb{D}}^{\xi_0,n}}$ onto $B^{\mathbb{D}}(u,r)$, by the finite propagation speed of solutions of hyperbolic equations, we have

$$\widetilde{G}_{t,r}(\square^{E_M^{\xi,n}})\big(x,x'\big) = \widetilde{G}_{t,r}(\square^{E_{\mathbb{D}}^{\xi_0,n}})\big(u,u'\big), \tag{3.2.157}$$

for $E_{\mathbb{D}}^n := \xi_0 \otimes \omega_{\mathbb{D}}(D)^n$. Now, from (3.2.149) and (3.2.157), we get

$$\exp(-t\square^{E_M^{\xi,n}})(x,x') - \exp(-t\square^{E_{\mathbb{D}}^{\xi_0,n}})(u,u') = \widetilde{K}_{t,r}(\square^{E_M^{\xi,n}})(x,x') - \widetilde{K}_{t,r}(\square^{E_{\mathbb{D}}^{\xi_0,n}})(u,u'). \tag{3.2.158}$$

Now, we conclude by (3.2.74), (3.2.155), (3.2.156) and (3.2.158). □

Proof of Theorem 3.2.6. First of all, in the case when (ξ, h^ξ) is trivial around the cusps, by choosing ϵ small enough in Case 3 of the proof of Theorem 3.2.4, we see that the Hermitian vector bundle (ξ_0, h_0^ξ) becomes trivial. Thus, (3.2.18) follows from (3.2.155), (3.2.158).

Now let's prove the estimates (3.2.15), (3.2.16). Consider a family of Hermitian metrics h_ϵ^ξ, $\epsilon \in [0,1]$ on ξ such that they coincide with h^ξ over $M \setminus (\cup_i V_i^M(1/2))$ and over $V_i^M(1/2)$, we have

$$h_\epsilon^\xi((z_i^M)^{-1}(u))(e_i,e_j) := (1-\epsilon\psi(4|u|^2))h^\xi((z_i^M)^{-1}(u))(e_i,e_j) + \epsilon\psi(4|u|^2)\delta_{ij}, \tag{3.2.159}$$

where ψ is defined in (3.2.54), e_i, $i = 1, \ldots, \mathrm{rk}(\xi)$ is as in (3.2.46), and δ_{ij} is the Kronecker delta symbol. We denote by $\square_\epsilon^{E_M^{\xi,n}}$ the Kodaira Laplacian on (M, g^{TM}), associated with $h_\epsilon^\xi \otimes \|\cdot\|_M^{2n}$. Then we have (3.2.112) for $\eta := \epsilon$. Moreover, (3.2.110) still holds uniformly on $\eta := \epsilon \in [0,1]$. By

Duhamel's formula (cf. [12, Theorem 2.48]), there exists $\epsilon_0 > 0$ such for any $u \in V_i^M(\epsilon_0)$, we have

$$\partial_\epsilon \exp(-t\square_\epsilon^{E_M^{\xi,n}})(u,u) = -\int_0^t \int_{v\in M} \exp(-(t-s)\square_\epsilon^{E_M^{\xi,n}})(u,v)\cdot \\ \cdot \Big(\partial_\epsilon(\square_\epsilon^{E_M^{\xi,n}})_v \exp(-s\square_\epsilon^{E_M^{\xi,n}})(v,u)\Big) dv_M(v)ds. \quad (3.2.160)$$

Now, the operator (3.2.112) has support over $V_i^M(1/2)$, thus, the integration in (3.2.160) is done only over $V_i^M(1/2)$. Using the coordinate function z_i^M, we identify $V_i^M(1/2)$ with $D^*(1/2) \subset \mathbb{D}$. Now, since the family of Hermitian metrics (3.2.159) is smooth, the estimate (3.2.12) holds uniformly in ϵ, and by (3.2.12), (3.2.110), (3.2.112), there is $C > 0$ such that

$$\Big|\partial_\epsilon \exp(-t\square_\epsilon^{E_M^{\xi,n}})(u,u)\Big| \leq C(1+|\ln|u||)\exp(ct)\int_0^t \int_{v\in D^*(\frac{1}{2})} |v|(1+|\ln|v||)\frac{1}{t-s}\cdot \\ \cdot \frac{1}{s^{3/2}}\cdot \exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big) dv_{\mathbb{D}^*}(v)ds. \quad (3.2.161)$$

For $r \in \mathbb{R}_+$, we decompose

$$\int_{v\in D^*(\frac{1}{2})} |v|(1+|\ln|v||)\exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big) dv_{\mathbb{D}^*}(v) \\ = \int_{v\in B^{\mathbb{D}}(u,r)\cap D^*(\frac{1}{2})} + \int_{v\in D^*(\frac{1}{2})\setminus B^{\mathbb{D}}(u,r)}. \quad (3.2.162)$$

Since for $\tilde{u} \in \mathbb{H}$, $\rho(\tilde{u}) = u$, the restriction $\rho_{B^{\mathbb{H}}(\tilde{U},r)} : B^{\mathbb{H}}(\tilde{u},r) \to B^{\mathbb{D}}(u,r)$ of the covering ρ from Section 3.2.3 is a surjection, which reduces the distances, we have

$$\int_{v\in B^{\mathbb{D}}(u,r)} \exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big) dv_{\mathbb{D}^*}(v) \\ \leq \int_{\tilde{v}\in B^{\mathbb{H}}(\tilde{u},r)} \exp\Big(-\frac{\mathrm{d}(\tilde{u},\tilde{v})^2}{4}(s^{-1}+(t-s)^{-1})\Big) dv_{\mathbb{H}}(\tilde{v}). \quad (3.2.163)$$

However, since $(\mathbb{H}, g^{T\mathbb{H}})$ is isometrically transitive, the right-hand side of (3.2.163) doesn't depend on $\tilde{u}$, i.e. it is a function of $r > 0$. Thus, in further estimation of the right-hand side of (3.2.163), we may suppose that $\tilde{u} = \sqrt{-1}$.

Now let's take $r = 1$. Over $B^{\mathbb{H}}(\sqrt{-1}, r)$, the metric $g^{T\mathbb{H}}$ is equivalent to the standard Euclidean metric. Thus, by the Gaussian integral on $\mathbb{C}$, for some $C > 0$, we have

$$\int_{\tilde{v}\in B^{\mathbb{H}}(\sqrt{-1},r)} \exp\Big(-\frac{\mathrm{d}(\sqrt{-1},\tilde{v})^2}{4}(s^{-1}+(t-s)^{-1})\Big) dv_{\mathbb{H}}(\tilde{v}) \leq \frac{C}{s^{-1}+(t-s)^{-1}}. \quad (3.2.164)$$

Now, there is $C > 0$ such that

$$\int_{v\in D^*(\frac{1}{2})\setminus B^{\mathbb{D}}(u,r)} \exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big)dv_{\mathbb{D}^*}(v)$$
$$\leq \int_{v\in D^*(\frac{1}{2})} \exp\Big(-(s^{-1}+(t-s)^{-1})/4\Big)dv_{\mathbb{D}^*}(v)$$
$$\leq C\exp\Big(-(s^{-1}+(t-s)^{-1})/4\Big), \quad (3.2.165)$$

where in the last line we used the fact that the volume of $D^*(1/2)$ is finite. By (3.2.162), (3.2.163), (3.2.164), (3.2.165), and by the fact that from (3.1.2) and (3.2.138), for $v \in B^{\mathbb{D}}(u,1)$, we have $|v| \leq |u|^{1/e}$, we deduce that there are $c, C > 0$ such that

$$\int_{v\in D^*(\frac{1}{2})} |v|(1+|\ln|v||)\exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big)dv_{\mathbb{D}^*}(v)$$
$$\leq \frac{C|u|^{1/e}|\ln|u||}{s^{-1}+(t-s)^{-1}} + C\exp\Big(-c(s^{-1}+(t-s)^{-1})\Big). \quad (3.2.166)$$

From (3.2.161) and (3.2.166), we get (3.2.16).

Now let's prove (3.2.15). Now let's fix $k \in \mathbb{N}$ and take $r = \mathrm{d}(|u|, |\ln|u||^{-k})$. By (3.2.138):

$$r = -\int_{|u|}^{|\ln|u||^{-k}} \frac{dr}{r|\ln r|} \approx \ln|\ln|u||. \quad (3.2.167)$$

Then by (3.2.164) and (3.2.165), as $r \geq 1$, for some $c, C > 0$, we have

$$\int_{v\in B^{\mathbb{D}}(u,r)\cap D^*(\frac{1}{2})} \exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big)dv_{\mathbb{D}^*}(v)$$
$$= \int_{v\in B^{\mathbb{D}}(u,1)\cap D^*(\frac{1}{2})} + \int_{v\in (B^{\mathbb{D}}(u,r)\cap D^*(\frac{1}{2}))\setminus B^{\mathbb{D}}(u,1)} \leq \frac{C}{s^{-1}+(t-s)^{-1}}$$
$$+ C\exp\Big(-c(s^{-1}+(t-s)^{-1})\Big). \quad (3.2.168)$$

Also, by (3.2.167) and the fact that the volume of $D^*(\frac{1}{2})$ is finite, there are $c, C > 0$, such that

$$\int_{v\in D^*(\frac{1}{2})\setminus B^{\mathbb{D}}(u,r)} \exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big)dv_{\mathbb{D}^*}(v)$$
$$\leq C\exp\Big(-c(\ln|\ln|u||)^2(s^{-1}+(t-s)^{-1})\Big). \quad (3.2.169)$$

By (3.2.162), (3.2.168) and (3.2.169), we have

$$\int_{v\in D^*(\frac{1}{2})} |v|(1+|\ln|v||)\exp\Big(-\frac{\mathrm{d}(u,v)^2}{4}(s^{-1}+(t-s)^{-1})\Big)dv_{\mathbb{D}^*}(v)$$

$$\leq C\Big(\frac{1}{s^{-1}+(t-s)^{-1}}+\exp\Big(-c(s^{-1}+(t-s)^{-1})\Big)\Big)\Big(1+\ln|\ln|u||^k\Big)$$
$$\cdot|\ln|u||^{-k}+C\exp\Big(-c(\ln|\ln|u||)^2(s^{-1}+(t-s)^{-1})\Big). \quad (3.2.170)$$

By (3.2.161) and (3.2.170), we get (3.2.15).

Now let's prove the estimate (3.2.17). We have the identity

$$\exp(-t\square^{E_M^{\xi,n}})(x,x')=\exp^{\perp}(-t\square^{E_M^{\xi,n}})(x,x')+\sum s_i(x)s_i(x')^*, \quad (3.2.171)$$

where s_i is an orthonormal basis of $H^0(\overline{M},E_M^{\xi,n})$ with respect to $\langle\cdot,\cdot\rangle_{L^2}$, see (3.2.1). From (3.2.6), (3.2.15) and (3.2.171), we conclude that there are $c',C>0$, such that for any $t>0$, $u\in D^*(1/2)$:

$$\Big|\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_\xi\cdot\exp^{\perp}(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big|$$
$$\leq C\exp(ct)\Big(|\ln|u||\exp(-c'(\ln|\ln|u||)^2/t)+1\Big). \quad (3.2.172)$$

Also, from (3.2.13), there are $c',C>0$, such that for any $t>0$, $u\in D^*(1/2)$, we have

$$\Big|\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_\xi\cdot\exp^{\perp}(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big|$$
$$\leq C|\ln|u||t^{-4}\exp(-ct). \quad (3.2.173)$$

By Cauchy inequality, we have

$$\exp(-ct-c'(\ln|\ln|u||)^2/t)\leq|\ln|u||^{-2\sqrt{cc'}}. \quad (3.2.174)$$

We get (3.2.17) by multiplying appropriate powers of (3.2.172) with (3.2.173) and using (3.2.174). □

Proof of Theorem 3.2.8. By finite propagation speed of solutions of hyperbolic equations and small-time asymptotics of the heat kernel in a compact manifold, we get (3.2.19). Moreover, the constant C from (3.2.19) could be chosen independently of $x\in M\setminus(\cup_i V_i^M(\epsilon))$, for some $\epsilon>0$.

Now let's suppose $x\in V_i^M(\epsilon)$, for some $i=1,\ldots,m$. We note $u=z_i^M(x)$, and we use (3.2.158) for $h=\mathrm{d}_{\mathbb{D}^*}(u,2\epsilon)$. Then by (3.2.75), (3.2.155) and (3.2.158), we see that there are smooth sections $a_{\xi,j}^{M,n}:M\to\mathrm{End}(\xi)$, as described, and there is $C>0$ such that for any $x\in M$, $t\in]0,t_0]$:

$$\Big|\exp(-t\square^{E_M^{\xi,n}})(x,x)-\sum_{j=-1}^{k}a_{\xi,j}^{M,n}(x)t^j\Big|\leq C\rho_M(x)\Big(t^k+\frac{1}{t}\exp\Big(-\frac{c'}{t|\ln|z_i^M(x)||^2}\Big)$$
$$+\exp\Big(-c(\ln|\ln|z_i^M(x)||)^2/t\Big)\Big), \quad (3.2.175)$$

and for $a_{\xi,j}^{\mathbb{D}^*,n}$, defined as in Theorem 3.2.21, we have

$$a_{\xi,j}^{M,n}(x) = a_{\xi,j}^{\mathbb{D}^*,n}(z_i^M(x)). \tag{3.2.176}$$

From (3.2.175), we conclude that if $x \in M \setminus (\cup_i V_i^M(e^{-t^{-1/3}}))$, then C in (3.2.75) can be chosen independently of $t \in]0, t_0]$ and x.

The statement (3.2.20) and the boundness of $a_{\xi,j}^{M,n}(x)$ follows from (3.2.76) and (3.2.176). □

3.3 Compact perturbation of the cusp: a proof of Theorem A

In this section we prove Theorem A. The proof consists of two steps. In the first step, Section 3.3.2, we prove that by successive "flattenings" of the Hermitian metric h^ξ, the associated Quillen norm converges to the Quillen norm associated with h^ξ. For this, essentially, we use the estimations developed in Section 3.2.3 along with analytic localization techniques of Bismut-Lebeau [25, §11]. In the second step, Section 3.3.3, we restrict ourselves to the case when (ξ, h^ξ) is trivial near the cusps, and we construct a family of flattenings which "approach" the cusp metric in such a way that the associated analytic torsion converges. In this step we use the analytic localization techniques of Bismut-Lebeau [25, §11] along with the maximal principle and some comparison results. Finally, as we explain in Section 3.3.1, those two results are enough to give a complete proof of Theorem A. Moreover, as we will see along the way, we actually prove Theorem B for $g_0^{TM} = g^{TM}$, i.e. for the variation of h^ξ.

3.3.1 General strategy of a proof of Theorem A

Let's recall the setting of the problem and describe the main idea of the proof more precisely. We fix surfaces with cusps $(\overline{M}, D_M, g^{TM}), (\overline{N}, D_N, g^{TN})$, a Hermitian vector bundle (ξ, h^ξ) over $\overline{M}$ and $n \in \mathbb{Z}$ as in the statement of Theorem A. We consider a family of Hermitian metrics h_η^ξ, $\eta \in]0, 1/2]$ on ξ constructed in (3.2.109). The main goal of Section 3.3.2 is to prove the following formula

$$\lim_{\eta\to 0} \|\cdot\|_Q \left(g^{TM}, h_\eta^\xi \otimes \|\cdot\|_M^{2n}\right) = \|\cdot\|_Q \left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right). \tag{3.3.1}$$

As $h_\eta^\xi|_{D_M} = h^\xi|_{D_M}$, we see that (3.3.1) is compatible with Theorem B.

In Section 3.3.3 we construct specific families of flattenings $g_{\mathrm{f},\theta}^{TM}, \|\cdot\|_M^{\mathrm{f},\theta}$, $\theta \in]0, 1]$ such that the corresponding ν from (3.1.14) tends to 0, as $\theta \to 0$. We consider the flattenings $g_{\mathrm{f},\theta}^{TN}, \|\cdot\|_N^{\mathrm{f},\theta}$, which are compatible to $g_{\mathrm{f},\theta}^{TM}, \|\cdot\|_M^{\mathrm{f},\theta}$, see (3.1.15), (3.1.17). Then we prove that for any Hermitian metric h_2^ξ on ξ over $\overline{M}$, for which (ξ, h_2^ξ) is trivial around the cusps, we have

$$\lim_{\theta\to 0} \frac{\|\cdot\|_Q \left(g_{\mathrm{f},\theta}^{TM}, h_2^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n}\right)^{\mathrm{rk}(\xi)}} = \frac{\|\cdot\|_Q \left(g^{TM}, h_2^\xi \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q \left(g^{TN}, \|\cdot\|_N^{2n}\right)^{\mathrm{rk}(\xi)}}. \tag{3.3.2}$$

This is the most technical and challenging part of this section.

Now let's explain how (3.3.1) and (3.3.2) imply Theorem A. Recall that $\widetilde{\mathrm{Td}}$ and $\widetilde{\mathrm{ch}}$ are given by (3.1.23) and (3.1.24). Let's recall the following theorem of Bismut-Gillet-Soulé [23, Theorem 1.23]:

Theorem 3.3.1 (Anomaly formula)**.** *Let $\overline{M}$ be endowed with two (smooth) metrics $g_1^{T\overline{M}}, g_2^{T\overline{M}}$ over $\overline{M}$. We denote by $\|\cdot\|_1^{\omega}, \|\cdot\|_2^{\omega}$ the Hermitian norms on $\omega_{\overline{M}}$ induced by $g_1^{T\overline{M}}, g_2^{T\overline{M}}$ over $\overline{M}$. Let ξ be a holomorphic vector bundle with Hermitian metrics h_1^{ξ}, h_2^{ξ} over $\overline{M}$. We have the following identity*

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\,(g_2^{T\overline{M}}, h_2^{\xi})/\|\cdot\|_Q\,(g_1^{T\overline{M}}, h_1^{\xi})\Big)\\
&\quad=\int_{\overline{M}}\Big[\widetilde{\mathrm{Td}}(\omega_{\overline{M}}^{-1}, (\|\cdot\|_1^{\omega})^{-2}, (\|\cdot\|_2^{\omega})^{-2})\mathrm{ch}(\xi, h_1^{\xi}) + \mathrm{Td}(\omega_{\overline{M}}^{-1}, (\|\cdot\|_2^{\omega})^{-2})\widetilde{\mathrm{ch}}(\xi, h_1^{\xi}, h_2^{\xi})\Big].
\end{aligned}\tag{3.3.3}$$

Now, by Theorem 3.3.1, by the fact that the flattenings $g_{\mathrm{f},\theta}^{TM}, \|\cdot\|_M^{\mathrm{f},\theta}$ and $g_{\mathrm{f},\theta}^{TN}, \|\cdot\|_N^{\mathrm{f},\theta}$ are compatible, and by the fact that (ξ, h^{ξ}) is trivial around the cusps, we see that the term inside of limit in left-hand side of (3.3.2) doesn't depend on the choice of the flattenings for θ small enough. Thus, for any $\theta > 0$ such that (ξ, h_{η}^{ξ}) is trivial over $\cup_i V_i^M(\theta)$ (for example, for $\theta^2 < \eta$), by (3.3.2), we have

$$\frac{\|\cdot\|_Q\,(g_{\mathrm{f},\theta}^{TM}, h_{\eta}^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f},\theta})^{2n})}{\|\cdot\|_Q\,(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n})^{\mathrm{rk}(\xi)}} = \frac{\|\cdot\|_Q\,(g^{TM}, h_{\eta}^{\xi}\otimes\|\cdot\|_M^{2n})}{\|\cdot\|_Q\,(g^{TN}, \|\cdot\|_N^{2n})^{\mathrm{rk}(\xi)}}.\tag{3.3.4}$$

Now, by Theorem 3.3.1, for any $\theta\in]0,1]$, we have

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\,\big(g_{\mathrm{f},\theta}^{TM}, h_{\eta}^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)/\|\cdot\|_Q\,\big(g_{\mathrm{f},\theta}^{TM}, h_2^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\Big)\\
&\qquad=\int_{\overline{M}}\mathrm{Td}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big).
\end{aligned}\tag{3.3.5}$$

From (3.3.4) and (3.3.5), for any $\theta^2 < \eta$, we have

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\,\big(g^{TM}, h_{\eta}^{\xi}\otimes\|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\,\big(g_{\mathrm{f},\theta}^{TM}, h_2^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\Big)\\
&\qquad-2\mathrm{rk}(\xi)\ln\Big(\|\cdot\|_Q\,\big(g^{TN}, \|\cdot\|_N^{2n}\big)/\|\cdot\|_Q\,\big(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n}\big)\Big)\\
&\qquad=\int_{\overline{M}}\mathrm{Td}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big).
\end{aligned}\tag{3.3.6}$$

Trivially, the following identity holds

$$\begin{aligned}
&\int_{\overline{M}}\mathrm{Td}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\\
&\quad=\int_{\overline{M}}\mathrm{Td}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big)+\int_{\overline{M}}\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\\
&\qquad-\int_{\overline{M}}\widetilde{\mathrm{ch}}\big(\xi, h_2^{\xi}, h_{\eta}^{\xi}\big).
\end{aligned}\tag{3.3.7}$$

Now, by (3.1.22), (3.1.23) and (3.1.26), we have

$$\int_M \mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big) = \int_M \mathrm{Td}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big). \tag{3.3.8}$$

Now, by (3.1.22) and Green identities, we have

$$\begin{aligned}\int_M &\Big(\mathrm{Td}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}\big) - \mathrm{Td}\big(\omega_M^{-1}, g^{TM}\big)\Big)\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big)\\ &= \int_M \widetilde{\mathrm{Td}}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}, g^{TM}\big)\big(c_1\big(\xi, h_2^\xi\big) - c_1\big(\xi, h_\eta^\xi\big)\big) + \frac{1}{2}\sum \ln\Big(\det(h_2^\xi/h_\eta^\xi)|_{P_i^M}\Big).\end{aligned} \tag{3.3.9}$$

Similarly, by (3.1.22), we have

$$\begin{aligned}\int_M &\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big)\Big(\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big) - \mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\Big)\\ &= \int_M \big(c_1\big(\xi, h_2^\xi\big) - c_1\big(\xi, h_\eta^\xi\big)\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}, \|\cdot\|_M^{2n}\big).\end{aligned} \tag{3.3.10}$$

By (3.3.6), (3.3.7), (3.3.8), (3.3.9) and (3.3.10), we get

$$\begin{aligned}&2\ln\Big(\|\cdot\|_Q\big(g^{TM}, h_\eta^\xi \otimes \|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f},\theta}^{TM}, h_2^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\Big)\\ &\qquad - 2\mathrm{rk}(\xi)\ln\Big(\|\cdot\|_Q\big(g^{TN}, \|\cdot\|_N^{2n}\big)/\|\cdot\|_Q\big(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n}\big)\Big)\\ &= \int_M \Big(\widetilde{\mathrm{Td}}\big(\omega_M^{-1}, g_{\mathrm{f},\theta}^{TM}, g^{TM}\big) + \widetilde{\mathrm{ch}}\big(\omega_M(D)^n, (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}, \|\cdot\|_M^{2n}\big)\Big)\big(c_1\big(\xi, h_2^\xi\big) - c_1\big(\xi, h_\eta^\xi\big)\big)\\ &\quad + \int_M \mathrm{Td}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big) + \int_M \widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\ &\qquad + \frac{1}{2}\sum \ln\Big(\det(h_2^\xi/h_\eta^\xi)|_{P_i^M}\Big).\end{aligned} \tag{3.3.11}$$

We make $\theta \to 0$ in (3.3.11). By (3.3.2), the uniform bounds on $g_{\mathrm{f},\theta}^{TM}$ and $\|\cdot\|_M^{\mathrm{f},\theta}$ from (3.3.43), Lebesgue dominated convergence theorem and the fact that the Bott-Chern representatives of Chern and Todd classes appear only in degree 0 in the first term of the right hand side of (3.3.11), we deduce

$$\begin{aligned}&2\ln\Big(\|\cdot\|_Q\big(g^{TM}, h_\eta^\xi \otimes \|\cdot\|_M^{2n}\big)/\|\cdot\|_Q\big(g^{TM}, h_2^\xi \otimes (\|\cdot\|_M)^{2n}\big)\Big)\\ &= \int_M \mathrm{Td}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big) + \int_M \widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big)\\ &\qquad - \int_M \widetilde{\mathrm{ch}}\big(\xi, h_2^\xi, h_\eta^\xi\big) + \frac{1}{2}\sum \ln\Big(\det(h_2^\xi/h_\eta^\xi)|_{P_i^M}\Big).\end{aligned} \tag{3.3.12}$$

Now we let $\eta \to 0$. Then by (3.3.1), the fact that the first Chern forms of (ξ, h_η^ξ), $\eta \in]0,1]$ are uniformly bounded and by Lebesgue dominated convergence theorem, we get Theorem B for $g_0^{TM} = g^{TM}$ and $h_0^\xi := h_2^\xi$, i.e trivial around the cusps. By applying this result twice for $h^\xi := h^\xi$, $h_0^\xi := h_2^\xi$ and $h^\xi := h_0^\xi$, $h_0^\xi := h_2^\xi$, and by taking the difference, we get Theorem B for $g_0^{TM} = g^{TM}$ and any h_0^ξ. By this, Theorem 3.3.1, (3.3.4), (3.3.9) and (3.3.10) we deduce Theorem A.

3.3.2 Flattening the Hermitian metric: a proof of (3.3.1)

In this section, we reduce Theorem A to the case (ξ, h^{ξ}) trivial near the cusps. For this, we prove (3.3.1). As we explained in Section 3.3, we consider a family of Hermitian metrics h^{ξ}_{η}, $\eta \in]0, 1/2]$ on ξ constructed in (3.2.109). We denote by $\square^{E^{\xi,n}_M}_{\eta}$ the Kodaira Laplacian on (M, g^{TM}), associated with $(\xi \otimes \omega_M(D)^n, h^{\xi}_{\eta} \otimes \|\cdot\|^{2n}_M)$. Similarly, for all the geometric objects we considered before, the subscript η would mean that instead of h^{ξ}, we use h^{ξ}_{η}.

Theorem 3.3.2. *For $n \leq 0$, there is $\eta_0 > 0$ such that the operators $\square^{E^{\xi,n}_M}_{\eta}$, $\eta \in]0, \eta_0]$ have a uniform spectral gap near 0, i.e. there is $\mu > 0$ such that for any $\eta \in]0, \eta_0]$, we have*

$$H^0(\overline{M}, E^{\xi,n}_M) = \ker(\square^{E^{\xi,n}_M}_{\eta}), \tag{3.3.13}$$

$$\mathrm{Spec}\big(\square^{E^{\xi,n}_M}_{\eta}\big) \cap]0, \mu] = \emptyset. \tag{3.3.14}$$

Proof. For $n = 0$, the statement of Theorem 3.3.2 is exactly (3.2.118). For $n < 0$, the proof of Theorem 3.2.1 remains unchanged, since the first Chern form of (ξ, h^{ξ}_{η}) is bounded, and thus the inequality (3.2.119) continues to hold. □

In this section, we denote by ∇ the connection, induced by the Levi-Civita connection and the Chern connections associated with (ξ, h^{ξ}_{η}) and $(\omega_M(D), \|\cdot\|_M)$. We denote by $\mathrm{d}(\cdot, \cdot)$ the distance function on (M, g^{TM}).

Lemma 3.3.3. For any $l, l' \in \mathbb{N}$, $n \in \mathbb{Z}$, there are $\eta_0, C > 0$, such that for any $\sigma \in \mathscr{C}^{\infty}\big(\overline{M} \times \overline{M}, (E^{\xi,n}_M) \boxtimes (E^{\xi,n}_M)^*\big)$, $x, x' \in M$ and any $\eta \in]0, \eta_0]$, we have

$$\big|(\nabla_x)^l (\nabla_{x'})^{l'} \sigma(x, x')\big|_{h \times h} \leq C \rho_M(x) \rho_M(x') \sum_{i=0}^{2+l} \sum_{i=0}^{2+l'} \Big\| (\square^{E^{\xi,n}_M}_{\eta,z})^i (\square^{E^{\xi,n}_M}_{\eta,z'})^j \sigma(z, z') \Big\|_{L^2,\eta}. \tag{3.3.15}$$

Proof. Let $\epsilon > 0$. For $x \in M \setminus (\cup_i V^M_i(\epsilon))$, the estimate (3.2.132) follows from [85, Lemma 1.6.2]. From $x \in V^M_i(\epsilon)$, the estimate (3.2.132) follows from Lemma 3.2.27. □

Theorem 3.3.4. *For any $l, l' \in \mathbb{N}$, there are $\eta_0, c, c', C > 0$ such that for any $t > 0$, $x, x' \in M$, $\eta \in]0, \eta_0]$, we have*

$$\Big|(\nabla_x)^l (\nabla_{x'})^{l'} \exp(-t \square^{E^{\xi,n}_M}_{\eta})(x, x')\Big|_{h \times h} \leq C \rho_M(x) \rho_M(x') t^{-1-(l+l')/2} \cdot \exp\big(ct - c' \cdot \mathrm{d}(x, x')^2/t\big). \tag{3.3.16}$$

Also, if $n \leq 0$, then there are $c, C > 0$ such that for any $t > 0$, $\eta \in]0, \eta_0]$, we have

$$\Big|(\nabla_x)^l (\nabla_{x'})^{l'} \exp^{\perp}(-t \square^{E^{\xi,n}_M}_{\eta})(x, x')\Big|_{h \times h} \leq C \rho_M(x) \rho_M(x') t^{-4-l-l'} \exp(-ct). \tag{3.3.17}$$

Proof. By Remark 3.2.26, the proof of (3.2.12) works uniformly on η, thus, we get (3.3.16). Now, (3.3.17) follows from Theorem 3.3.2 and Lemma 3.3.3. □

Theorem 3.3.5. *For any $k \in \mathbb{N}$, there are $\eta_0, \epsilon_1, c, c', C > 0$ such that for any $t > 0$, $u \in \mathbb{C}, |u| \leq \epsilon_1$, $\eta \in]0, \eta_0]$, $i = 1, \ldots, m$, we have*

$$\Big|\Big(\exp(-t\Box_\eta^{E_M^{\xi,n}}) - \exp(-t\Box^{E_M^{\xi,n}})\Big)\big((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)\big)\Big| \\ \leq C|\ln|u||\exp(ct)\Big(|\ln|u||^{-k} + \exp(-c'(\ln|\ln|u||)^2/t)\Big). \quad (3.3.18)$$

Moreover, if $n \leq 0$, then there are $\varsigma < 1$ and $c, C > 0$ such that

$$\Big|\Big(\exp^\perp(-t\Box_\eta^{E_M^{\xi,n}}) - \exp^\perp(-t\Box^{E_M^{\xi,n}})\Big)\big((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)\big)\Big| \\ \leq C|\ln|u||^\varsigma t^{-4}\exp(-ct). \quad (3.3.19)$$

Proof. As the proof of Theorem 3.2.6 is based on (3.2.12), which works uniformly on $\eta \in]0, 1/2]$, the analogue of (3.2.15) also works uniformly on η, which implies (3.3.18). The proof of (3.3.19) remains identical to the proof of (3.2.17), one only has to use (3.3.17) instead of (3.2.17). □

Theorem 3.3.6. *There are smooth bounded functions $a_{\xi,\eta,j}^{M,n} : M \to \mathrm{End}(\xi)$, $j \geq -1$ such that for any $x \in M$, $t_0 > 0$, $k \in \mathbb{N}$ there is $C > 0$ such that for any $t \in]0, t_0]$, $\eta \in]0, 1/2]$, we have*

$$\Big|\exp(-t\Box_\eta^{E_M^{\xi,n}})(x, x) - \sum_{j=-1}^{k} a_{\xi,\eta,j}^{M,n}(x)t^j\Big| \leq Ct^k. \quad (3.3.20)$$

Moreover, if $x \in M \setminus (\cup_i V_i^M(e^{-t^{-1/3}}))$, then C can be chosen independently of $t \in]0, t_0]$ and x.

Also, there is $\epsilon_1 > 0$, such that for any $l \in \mathbb{N}$, $j \geq -1$, there is $C > 0$ such that for any $u \in \mathbb{C}$, $0 < |u| \leq \epsilon_1$, $i = 1, \ldots, m$, $\eta \in]0, 1/2]$, we have

$$\Big|(\nabla_u)^l\Big(a_{\xi,\eta,j}^{M,n} - a_{\xi,j}^{M,n}\Big)\big((z_i^M)^{-1}(u)\big)\Big|_h \leq C|u|^{1/3}, \quad (3.3.21)$$

Moreover, for any $x \in M \setminus (\cup_i V_i^M(\eta^{1/2}))$, we have

$$a_{\xi,\eta,j}^{M,n}(x) = a_{\xi,j}^{M,n}(x). \quad (3.3.22)$$

Proof. By Remark 3.2.26, the proof of Theorem 3.2.8 works uniformly on η. Thus, only the last statement (3.3.22) needs to be justified, as we don't have its analogue in Theorem 3.2.8. But it simply follows from the fact that the coefficients of the small-time expansion of the heat kernel are local and the fact that h_η^ξ coincides with h^ξ over $M \setminus (\cup_i V_i^M(\eta^{1/2}))$. □

Theorem 3.3.7. *There are $\eta_0, c', C > 0$ such that for any $t > 0, \eta \in]0, \eta_0]$ and $x \in M \setminus (\cup_i V_i^M(|\ln\eta|^{-1}))$, we have*

$$\Big|\big(\exp(-t\Box_\eta^{E_M^{\xi,n}}) - \exp(-t\Box^{E_M^{\xi,n}})\big)(x, x)\Big| \leq C\rho_M(x)^2\exp(-c'(\ln|\ln\eta|)^2/t). \quad (3.3.23)$$

Proof. We put $r = \mathrm{d}(V_i^M(\eta^{1/2}), M \setminus V_i^M(|\ln\eta|^{-1}))$. Then by (3.2.138), we have $r \simeq \ln|\ln\eta|$. Similarly to (3.2.157), using (3.2.138) and the fact that h_η^ξ coincides with h^ξ over $M \setminus (\cup_i V_i^M(\eta))$, by the finite propagation speed of solutions of hyperbolic equations, there is $c > 0$ such that for any $x \in M \setminus (\cup V_i^M(|\ln\eta|^{-1}))$, we have

$$\widetilde{G}_{t,r}(\square_\eta^{E_M^{\xi,n}})(x,\cdot) = \widetilde{G}_{t,r}(\square^{E_M^{\xi,n}})(x,\cdot). \tag{3.3.24}$$

Then, similarly to (3.2.158), we have (see (3.2.148))

$$\big(\exp(-t\square_\eta^{E_M^{\xi,n}}) - \exp(-t\square^{E_M^{\xi,n}})\big)(x,x) = \big(\widetilde{K}_{t,r}(\square_\eta^{E_M^{\xi,n}}) - \widetilde{K}_{t,r}(\square^{E_M^{\xi,n}})\big)(x,x). \tag{3.3.25}$$

Now, similarly to (3.2.154), for any $k, k' \in \mathbb{N}$, there are $c', C > 0$ such that for any $t > 0$, we have

$$\Big\|(\square_\eta^{E_M^{\xi,n}})^k\big(\widetilde{K}_{t,r}(\square_\eta^{E_M^{\xi,n}})\big)(\square_\eta^{E_M^{\xi,n}})^{k'}\Big\|^{(0,0)} \leq C\exp(-c'r^2/t). \tag{3.3.26}$$

From (3.3.15) and (3.3.26), similarly to (3.2.155), for some $c', C > 0$ and for any $x \in M$, we get

$$\Big|\widetilde{K}_{t,r}(\square_\eta^{E_M^{\xi,n}})(x,x)\Big| \leq C\rho_M(x)^2\exp(-c'r^2/t). \tag{3.3.27}$$

Now, from (3.2.155), (3.3.25) and (3.3.27), we get (3.3.23). □

Now we can relate the regularized heat traces associated with h_η^ξ and h^ξ.

Theorem 3.3.8. *There are $c, C > 0$, $\varsigma > 0$, $t_0 > 0$, such that for any $t > t_0, \eta \in]0, e^{-3}]$, we have*

$$\Big|\mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square_\eta^{E_M^{\xi,n}})\big] - \mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square^{E_M^{\xi,n}})\big]\Big| \leq C\big(\ln|\ln\eta|\big)^{-\varsigma}\exp(-ct). \tag{3.3.28}$$

Proof. First of all, by (3.3.17) and (3.3.19), in the same way as in Proposition 3.2.14, we get

$$\Big|\mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square_\eta^{E_M^{\xi,n}})\big] - \mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square^{E_M^{\xi,n}})\big]\Big| \leq C\exp(-ct). \tag{3.3.29}$$

Now, by (3.2.26), we have

$$\begin{aligned}\mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square_\eta^{E_M^{\xi,n}})\big] &- \mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square^{E_M^{\xi,n}})\big] \\ &= \mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\square_\eta^{E_M^{\xi,n}})\big] - \mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\square^{E_M^{\xi,n}})\big].\end{aligned} \tag{3.3.30}$$

Trivially, there is $C > 0$ such that for any $\eta \in]0, e^{-3}]$, we have

$$\int_{D(1/2)\setminus D(|\ln\eta|^{-1})} \frac{\sqrt{-1}dzd\overline{z}}{|z|^2|\ln|z||} \leq C\ln\ln|\ln\eta|. \tag{3.3.31}$$

We decompose the integration in the definition of $\mathrm{Tr}\big[\exp(-t\square_\eta^{E_M^{\xi,n}})\big]$, analogical to Definition 3.2.9, into two parts: over $\cup_i V_i^M(|\ln\eta|^{-1})$ and over $M \setminus (\cup_i V_i^M(|\ln\eta|^{-1}))$. By bounding the first

part of the integral corresponding to the right-hand side of (3.3.30) by (3.2.25), (3.3.18) and second part by (3.3.23) and (3.3.31), we see that there are $c, c', C > 0$ such that for any $t > 0$, $\eta \in]0, e^{-3}]$, we have

$$\mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\Box_{\eta}^{E_M^{\xi,n}})\big] - \mathrm{Tr}^{\mathbf{r}}\big[\exp(-t\Box^{E_M^{\xi,n}})\big] \leq \frac{C\exp(ct)}{\ln|\ln\eta|} + C(1+t)\ln\ln|\ln\eta|\exp\Big(ct - \frac{c'}{t}(\ln\ln|\ln\eta|)^2\Big). \quad (3.3.32)$$

By multiplying (3.3.29) and (3.3.32) with suitable powers, and using (3.2.174), (3.3.30), we get (3.3.28). □

Now, for $j \geq -1$, we denote (compare with (3.2.29))

$$A_{\xi,\eta,j}^{M,n} = \int_M \mathrm{Tr}\big[a_{\xi,\eta,j}^{M,n}(x)\big]dv_M(x) - \frac{\mathrm{rk}(\xi)}{3}\int_P a_j^{P,n}(x)dv_N(x) - \dim H^0(\overline{M}, E_M^{\xi,n}) + \frac{\mathrm{rk}(\xi)}{3}\dim H^0(\overline{P}, E_P^n). \quad (3.3.33)$$

The integrals in (3.3.33) converge by Theorem 3.2.8.

Proposition 3.3.9. For any $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, we have

$$\Big|\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box_{\eta}^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{k} A_{\xi,\eta,j}^{M,n}t^j\Big| \leq Ct^k. \quad (3.3.34)$$

Proof. It is proved in the same way as Proposition 3.2.13 with one modification: instead of using Theorems 3.2.6, 3.2.8, we use Theorems 3.3.5, 3.3.6. □

Theorem 3.3.10. *For any $t_0 > 0$, there is $C > 0$, such that for any $t \in]0, t_0]$, $\eta \in]0, e^{-3}]$:*

$$\Big|\int_0^1 \Big(\Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box_{\eta}^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,\eta,j}^{M,n}t^j\Big) - \Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,j}^{M,n}t^j\Big)\Big)\frac{dt}{t}\Big| \leq C(\ln\ln|\ln\eta|)^{-1/3}. \quad (3.3.35)$$

Proof. First of all, by Propositions 3.2.13, 3.3.9, we get

$$\Big|\Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box_{\eta}^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,\eta,j}^{M,n}t^j\Big) - \Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,j}^{M,n}t^j\Big)\Big| \leq Ct. \quad (3.3.36)$$

Now, by (3.3.21) and (3.3.22), there are $\eta_0, C > 0$ such that for any $\eta \in]0, \eta_0]$, $j = -1, 0$, we have

$$\big|A_{\xi,\eta,j}^{M,n} - A_{\xi,j}^{M,n}\big| \leq C\eta^{1/6}. \quad (3.3.37)$$

Also, by Theorem 3.3.7 and (3.3.31), there are $\eta_0, c', C > 0$ such that for any $t \in]0, t_0], \eta \in]0, \eta_0]$:

$$\int_{M\setminus(\cup_i V_i^M(|\ln\eta|^{-1}))} \Big|\mathrm{Tr}\Big[\big(\exp(-t\square_\eta^{E_M^{\xi,n}}) - \exp(-t\square^{E_M^{\xi,n}})\big)(x,x)\Big]\Big| \\ \leq C\ln\ln|\ln\eta|\exp\Big(-\frac{c'}{t}(\ln|\ln\eta|)^2\Big). \quad (3.3.38)$$

Also, by (3.2.25) and (3.3.18), there are $\eta_0, c', C > 0$ such that for any $t \in]0, t_0], \eta \in]0, \eta_0]$, we have

$$\int_{V_i^M(|\ln\eta|^{-1})} \Big|\mathrm{Tr}\Big[\big(\exp(-t\square_\eta^{E_M^{\xi,n}}) - \exp(-t\square^{E_M^{\xi,n}})\big)(x,x)\Big]\Big| \\ \leq \frac{C}{\ln|\ln\eta|} + C\exp\Big(-\frac{c'}{t}(\ln\ln|\ln\eta|)^2\Big). \quad (3.3.39)$$

Thus, by (3.3.30), (3.3.37), (3.3.38) and (3.3.39), there are $c', C > 0$ such that for any $t \in]0, t_0]$, we have

$$\Big|\Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square_\eta^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,\eta,j}^{M,n} t^j\Big) - \Big(\mathrm{Tr}^{\mathbf{r}}\big[\exp^\perp(-t\square^{E_M^{\xi,n}})\big] - \sum_{j=-1}^{0} A_{\xi,j}^{M,n} t^j\Big)\Big| \\ \leq \frac{C\eta^{1/6}}{t} + \frac{C}{\ln|\ln\eta|} + C\ln\ln|\ln\eta|\exp\Big(-\frac{c'}{t}(\ln\ln|\ln\eta|)^2\Big). \quad (3.3.40)$$

Now, by multiplying (3.3.36) and (3.3.40) with appropriate powers, and integrating on t from 0 to 1, we deduce Theorem 3.3.10. □

Proof of (3.3.1). By Theorems 3.3.8, 3.3.10, (3.2.41), (3.3.37) and Lebesgue dominated convergence theorem, we have

$$\lim_{\eta\to 0} T(g^{TM}, h_\eta^\xi \otimes \|\cdot\|_M^{2n}) = T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}). \quad (3.3.41)$$

However, trivially from (3.2.1), we have

$$\lim_{\eta\to 0} \|\cdot\|_{L^2}(g^{TM}, h_\eta^\xi \otimes \|\cdot\|_M^{2n}) = \|\cdot\|_{L^2}(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}). \quad (3.3.42)$$

By (3.3.41) and (3.3.42), we get (3.3.1). □

3.3.3 Flattening the Riemannian metric: a proof of (3.3.2)

In this section we introduce the notion of a *tight family of flattenings*, which "approach" the cusped metric, and study some of its properties. We study how the relative heat trace converges as this family of flattenings "converges" to the cusped metric, and from this study we deduce (3.3.2).

From now and till the end of Section 3.3, we suppose that (ξ, h^ξ) is trivial near the cusps.

Definition 3.3.11. We say that the flattenings $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$, $\theta \in]0,1]$ (cf. Definition 3.1.2) of g^{TM}, $\|\cdot\|_M$ are *n-tight*, $n \in \mathbb{Z}$ if they satisfy the following requirements:

1. We have $g^{TM}_{\mathrm{f},\theta}|_{M\setminus(\cup_i V_i^M(\theta))} = g^{TM}|_{M\setminus(\cup_i V_i^M(\theta))}$ and similarly for $\|\cdot\|^{\mathrm{f},\theta}_M$.

2. For all $i = 1,\ldots,m$, the following identity holds over $V_i^M(\theta^2)$: $\left\|dz_i^M \otimes s_{D_M}/z_i^M\right\|^{\mathrm{f},\theta}_M = |\ln\theta|$, where s_{D_M} is the canonical section of $\mathscr{O}_M(D_M)$.

3. There are flattenings $g^{TM}_{\mathrm{sm}}, \|\cdot\|^{\mathrm{sm}}_M$ of $g^{TM}, \|\cdot\|_M$ and $\tau > 0$ such that for any $\theta \in]0, e^{-3}]$:

$$\begin{aligned} &g^{TM}_{\mathrm{sm}} \otimes (\|\cdot\|^{\mathrm{sm}}_M)^{2n} \leq g^{TM}_{\mathrm{f},\theta} \otimes (\|\cdot\|^{\mathrm{f},\theta}_M)^{2n} \leq \tau \cdot g^{TM} \otimes \|\cdot\|^{2n}_M\,, \\ &\|\cdot\|^{\mathrm{sm}}_M \leq \|\cdot\|^{\mathrm{f},\theta}_M \leq \|\cdot\|_M\,. \end{aligned} \tag{3.3.43}$$

4. We have the following analogue of Lemma 3.2.28: *for any $m \in \{-n, n\}$, there is $C > 0$ such that for any $\sigma \in \mathscr{C}^\infty\big(M, E_M^{\xi,m}\big)$, $\theta \in]0, e^{-3}]$, and for any $x \in M$, we have*

$$\big|\sigma(x)\big|_{h,\theta} \leq C\rho_{M,\theta}(x) \textstyle\sum_{i=0}^{2} \big\|(\square^{E_M^{\xi,m}}_{\mathrm{f},\theta})^i\sigma\big\|_{L^2,\theta}, \tag{3.3.44}$$

where $\square^{E_M^{\xi,m}}_{\mathrm{f},\theta}$ is the Laplacian associated with $g^{TM}_{\mathrm{f},\theta}$ and $h^\xi \otimes (\|\cdot\|^{\mathrm{f},\theta}_M)^{2m}$; $|\cdot|_{h,\theta}$ is the pointwise norm induced by h^ξ and $\|\cdot\|^{\mathrm{f},\theta}_M$; $\|\cdot\|_{L^2,\theta}$ is the L^2 norm induced by $g^{TM}_{\mathrm{f},\theta}, h^\xi, \|\cdot\|^{\mathrm{f},\theta}_M$; and the function $\rho_{M,\theta} : \overline{M} \to [1,\infty[$ is given by

$$\rho_{M,\theta}(x) = \begin{cases} 1 & \textit{for } x \in M \setminus (\cup_i V_i(1/2)), \\ \sqrt{|\ln|z_i^M(x)||} & \textit{for } x \in V_i^M(1/2) \setminus V_i^M(\theta^3), \\ (\ln\theta)^6 & \textit{for } x \in V_i^M(\theta^3). \end{cases} \tag{3.3.45}$$

In Section 3.3.5, we show that for any $n \in \mathbb{Z}$, $n \leq 0$, n-tight families of flattenings exist.

We fix $n \in \mathbb{Z}, n \leq 0$ and n-tight families of flattenings $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$, $\theta \in]0,1]$. From (3.3.43):

$$g^{TM}_{\mathrm{f},\theta} \leq \tau \cdot g^{TM}. \tag{3.3.46}$$

Recall that $\square^{E_M^{\xi,n}}$ is the Kodaira Laplacian associated to g^{TM}, h^ξ and $\|\cdot\|_M$. We set $\mu > 0$ as in (3.2.7), and let τ be as in (3.3.43). We defer the proof of the following theorem until Section 3.3.4.

Theorem 3.3.12. *The operator $\square^{E_M^{\xi,n}}_{\mathrm{f},\theta}$ has a uniform spectral gap near 0, i.e. for any $\theta \in]0, e^{-3}]$:*

$$\ker(\square^{E_M^{\xi,n}}_{\mathrm{f},\theta}) \simeq H^0(\overline{M}, E_M^{\xi,n}), \tag{3.3.47}$$

$$\mathrm{Spec}(\square^{E_M^{\xi,n}}_{\mathrm{f},\theta}) \cap]0, \mu/\tau[= \emptyset. \tag{3.3.48}$$

In what follows, we denote the smooth kernels of $\exp(-t\square^{E_M^{\xi,n}}_{\mathrm{f},\theta}), \exp^\perp(-t\square^{E_M^{\xi,n}}_{\mathrm{f},\theta})$ with respect to the Riemannian volume form $dv_{M,\theta}$ induced by $g^{TM}_{\mathrm{f},\theta}$ by

$$\exp(-t\square^{E_M^{\xi,n}}_{\mathrm{f},\theta})(x,y), \exp^\perp(-t\square^{E_M^{\xi,n}}_{\mathrm{f},\theta})(x,y) \in (E_M^{\xi,n})^*_x \boxtimes (E_M^{\xi,n})_y, \quad \text{for} \quad x, y \in M. \tag{3.3.49}$$

Theorem 3.3.13. *There are $c, C > 0$ such that for any $t > 0$, $x \in M$, $\theta \in]0, e^{-3}]$, we have*

$$\Big| \exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})(x,x) \Big| \leq C\rho_{M,\theta}(x)^2 t^{-4} \exp(-ct). \tag{3.3.50}$$

Proof. The proof is the same as the proof of Theorem 3.2.4, one only has to change the use of Lemma 3.2.28 by (3.3.44) and of Theorem 3.2.1 by Theorem 3.3.12. □

Theorem 3.3.14. *There are $c', C > 0$ such that for any $t > 0, \theta \in]0, e^{-3}]$ and $x \in M \setminus (\cup_i V_i^M(|\ln\theta|^{-1}))$, we have*

$$\Big| \big(\exp(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}}) - \exp(-t\Box^{E_M^{\xi,n}}) \big)(x,x) \Big| \leq C\rho_{M,\theta}(x)^2 \exp(-c'(\ln|\ln\theta|)^2/t). \tag{3.3.51}$$

Proof. The proof is the same as the proof of Theorem 3.3.7. □

We construct the flattenings $g_{\mathrm{f},\theta}^{TN}, \|\cdot\|_N^{\mathrm{f},\theta}$, $\theta \in]0,1]$ of $g^{TN}, \|\cdot\|_N$, which are compatible with $g_{\mathrm{f},\theta}^{TM}$, $\|\cdot\|_M^{\mathrm{f},\theta}$. Trivially, the flattenings $g_{\mathrm{f},\theta}^{TN}$, $\|\cdot\|_N^{\mathrm{f},\theta}$, $\theta \in]0,1]$ are n-tight. The following theorem is an analogue of Theorem 3.2.6, and it forms the core of this section. Its proof is defered to Section 3.3.4.

Theorem 3.3.15. *There are $c, c', C > 0$, $\varsigma < 1$ such that for any $t > 0, \theta \in]0, e^{-3}], i = 1, \ldots, m$, and $u \in \mathbb{C}, |u| \leq |\ln\theta|^{-1}$, we have*

$$\begin{aligned} &\Big| \exp(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)\big) - \mathrm{Id}_\xi \exp(-t\Box_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u), (z_i^N)^{-1}(u)\big) \Big| \\ &\qquad \leq C|\ln\max(\theta,|u|)| \cdot \exp(-c'(\ln|\ln\max(\theta,|u|)|)^2/t). \end{aligned} \tag{3.3.52}$$

$$\begin{aligned} &\Big| \exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)\big) - \mathrm{Id}_\xi \exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u), (z_i^N)^{-1}(u)\big) \Big| \\ &\qquad \leq C|\ln\max(\theta,|u|)|^{\varsigma} \cdot t^{-4}\exp(-ct). \end{aligned} \tag{3.3.53}$$

Now, for brevity, we denote

$$\begin{aligned} X_\theta(t) := &\ \mathrm{Tr}\big[\exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big] - \mathrm{rk}(\xi)\mathrm{Tr}\big[\exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_N^n})\big] \\ &- \mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box^{E_M^{\xi,n}})\big] + \mathrm{rk}(\xi)\mathrm{Tr}^{\mathbf{r}}\big[\exp^{\perp}(-t\Box^{E_N^n})\big] \end{aligned} \tag{3.3.54}$$

Let's use those theorems to study the convergence of heat traces. The main theorem here is

Theorem 3.3.16. *There are $c, c', C > 0$ such that for any $t > 0, \theta \in]0, e^{-3}]$, we have*

$$\big|X_\theta(t)\big| \leq C\exp(-ct - c'(\ln|\ln\theta|)^2/t). \tag{3.3.55}$$

Proof. Let's denote

$$A_M^{\perp}(t) = \int_{M\setminus(\cup_i V_i^M(|\ln\theta|^{-1}))} \Big(\mathrm{Tr}\big[\exp^{\perp}(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})(x,x)\big] - \mathrm{Tr}\big[\exp^{\perp}(-t\Box^{E_M^{\xi,n}})(x,x)\big] \Big) dv_{M,\theta}(x),$$

$$
\begin{aligned}
A_N^{\perp}(t) &= \int_{N\setminus(\cup_i V_i^N(|\ln\theta|^{-1}))} \Big(\mathrm{Tr}\big[\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_N^n})(x,x)\big] - \mathrm{Tr}\big[\exp^{\perp}(-t\square^{E_N^n})(x,x)\big]\Big) dv_{N,\theta}(x),\\
B_\theta^{\perp}(t) &= \sum_i \int_{D(|\ln\theta|^{-1})} \Big(\mathrm{Tr}\Big[\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)\Big] \\
&\qquad - \mathrm{rk}(\xi)\mathrm{Tr}\Big[\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big]\Big) dv_\theta(u),\\
B^{\perp}(t) &= \sum_i \int_{D(|\ln\theta|^{-1})} \Big(\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)\Big] \\
&\qquad - \mathrm{rk}(\xi)\mathrm{Tr}\Big[\exp^{\perp}(-t\square^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big]\Big) dv_{\mathbb{D}^*}(u),
\end{aligned}
\tag{3.3.56}
$$

and $dv_{N,\theta}, dv_\theta$ are the Riemannian volume forms induced by $g_{\mathrm{f},\theta}^{TN}$ and $((z_i^M)^{-1})^* g_{\mathrm{f},\theta}^{TM}$ correspondingly. Then we have

$$
X_\theta(t) = A_M^{\perp}(t) + A_N^{\perp}(t) + B_\theta^{\perp}(t) + B^{\perp}(t). \tag{3.3.57}
$$

By (3.1.2), (3.3.31), (3.3.45) and (3.3.46), there is $C>0$ such that for any $\theta \in]0, e^{-3}]$, we have

$$
\int_{M\setminus(\cup_i V_i^M(|\ln\theta|^{-1}))} \rho_{M,\theta}(x)^2 dv_{M,\theta}(x) \leq C(\ln\ln|\ln\theta|). \tag{3.3.58}
$$

By Theorem 3.3.13, (3.2.13) and (3.3.58), there are $c, C>0$ such that

$$
|A_M^{\perp}(t)|, |A_N^{\perp}(t)| \leq C(\ln\ln|\ln\theta|)t^{-4}\exp(-ct). \tag{3.3.59}
$$

By (3.1.2), (3.2.24) and (3.3.46), for any $\varsigma<1$, there is $C>0$ such that for any $\theta \in]0, e^{-3}]$, we have

$$
\int_{V_i^M(|\ln\theta|^{-1})} \big|\ln\max(\theta, |z_i^M(x)|)\big|^{\varsigma} dv_{M,\theta}(x) \leq C. \tag{3.3.60}
$$

By (3.3.53) and (3.3.60), there are $c, C>0$ such that

$$
|B_\theta^{\perp}(t)| \leq Ct^{-4}\exp(-ct). \tag{3.3.61}
$$

By (3.2.17) and (3.2.24), there are $c, C>0$ such that

$$
|B^{\perp}(t)| \leq Ct^{-4}\exp(-ct). \tag{3.3.62}
$$

By (3.3.59), (3.3.61) and (3.3.62), for some $c, C>0$, and for any $t>0$, $\theta \in]0, e^{-3}]$, we have

$$
\big|X_\theta(t)\big| \leq C(1+t^{-4})(\ln\ln|\ln\theta|)\exp(-ct). \tag{3.3.63}
$$

Now, alternatively, we may also write

$$
X_\theta(t) = A_M(t) + A_N(t) + B_\theta(t) + B(t), \tag{3.3.64}
$$

where $A_M(t)$, $A_N(t)$, $B_\theta(t)$, $B(t)$ are as in (3.3.56), but we put $\exp$ in place of $\exp^{\perp}$.

By Theorem 3.3.14 and (3.3.58), there are $c', C > 0$ such that for any $\theta \in]0, e^{-3}]$, we have

$$|A_M(t)|, |A_N(t)| \leq C(\ln\ln|\ln\theta|)\exp(-c'(\ln|\ln\theta|)^2/t). \tag{3.3.65}$$

By (3.3.46), we have

$$\int_{V_i^M(|\ln\theta|^{-1})} |\ln\max(\theta, |u|)| dv_{M,\theta}(x) \leq C\ln|\ln\theta|. \tag{3.3.66}$$

By (3.3.52) and (3.3.66), for some $c', C > 0$, we have

$$|B_\theta(t)| \leq C\ln|\ln\theta|\exp(-c'(\ln\ln|\ln\theta|)^2/t). \tag{3.3.67}$$

By (3.2.18) and (3.2.25), for some $c', C > 0$, we have

$$|B(t)| \leq C(1+t)\exp(-c'(\ln\ln|\ln\theta|)^2/t). \tag{3.3.68}$$

By (3.3.65), (3.3.67) and (3.3.68), we conclude that

$$\big|X_\theta(t)\big| \leq C(1+t)(\ln|\ln\theta|)\exp(-c'(\ln\ln|\ln\theta|)^2/t). \tag{3.3.69}$$

By multiplying (3.3.63) with power $1-\mu \in]1/2, 1]$ and (3.3.69) with power μ, for some $c, c', C > 0$:

$$\big|X_\theta(t)\big| \leq C(1+t)(1+t^{-4})(\ln|\ln\theta|)^\mu\big(\ln\ln|\ln\theta|\big)\exp(-ct - c'\mu(\ln\ln|\ln\theta|)^2/t). \tag{3.3.70}$$

By (3.2.174) and (3.3.70), we deduce (3.3.55) by taking μ small enough. □

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, let's denote the approximated regularized zeta-function by

$$\zeta_M^\theta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} \mathrm{Tr}\Big[\exp^\perp(-t\Box_{\mathrm{f},\theta}^{E_M^{\xi,n}})\Big]t^s\frac{dt}{t}. \tag{3.3.71}$$

As usually, $\zeta_M^\theta(s)$ has a meromorphic extension to the entire s-plane, and this extension is holomorphic at 0. We recall that the zeta-function ζ_M was defined in Definition 3.2.15.

Proposition 3.3.17. For any $\theta \in]0, e^{-3}]$, the difference $\zeta_M^\theta(s) - \mathrm{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \mathrm{rk}(\xi)\zeta_N(s)$ is a holomorphic function on $\mathbb{C}$. Moreover, as $\theta \to 0$, we have

$$\zeta_M^\theta(s) - \mathrm{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \mathrm{rk}(\xi)\zeta_N(s) \to 0, \tag{3.3.72}$$

uniformly for s varying in a compact subset of $\mathbb{C}$. In particular, as $\theta \to 0$, we have

$$\frac{T(g_{\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n})}{T(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n})^{\mathrm{rk}(\xi)}} \to \frac{T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})}{T(g^{TN}, \|\cdot\|_N^{2n})^{\mathrm{rk}(\xi)}}. \tag{3.3.73}$$

Proof. First of all, by Definition 3.2.15, (3.3.54) and (3.3.71), we have

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} X_\theta(t)t^s\frac{dt}{t} = \zeta_M^\theta(s) - \mathrm{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \mathrm{rk}(\xi)\zeta_N(s), \tag{3.3.74}$$

Now, by Theorem 3.3.16, the function $X_\theta(t)$ has subexponential growth near 0 and ∞, thus, by (3.3.74), the left-hand side of (3.3.72) is a holomorphic function over $\mathbb{C}$ for any $\theta \in]0, e^{-3}]$.

Also, by Theorem 3.3.16, there are $c, c', C > 0$, $\varsigma > 0$ such that for any $t > 0$, $\theta \in]0, e^{-3}]$:

$$X_\theta(t) \leq C|\log\theta|^{-\varsigma}\exp(-ct - c'/t). \tag{3.3.75}$$

In particular, by (3.3.75), as $\theta \to 0$, we have

$$X_\theta(t) \to 0. \tag{3.3.76}$$

By (3.3.74), (3.3.75), (3.3.76) and Lebesgue dominated convergence theorem, we deduce (3.3.72). Now, (3.3.73) follows from Definitions 3.2.15, 3.2.16, (3.3.71) and (3.3.72). □

We denote by $\|\cdot\|_{L^2}\left(g_{\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\right)$ the L^2-norm over the line bundle (3.1.12) induced by $g_{\mathrm{f},\theta}^{TM}$, h^ξ, $\|\cdot\|_M^{\mathrm{f},\theta}$. By *properties 1,3* of tight families and Lebesgue dominated convergence, we have

$$\begin{aligned}\lim_{\theta\to 0}\|\cdot\|_{L^2}\left(g_{\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\right) &= \|\cdot\|_{L^2}\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right),\\ \lim_{\theta\to 0}\|\cdot\|_{L^2}\left(g_{\mathrm{f},\theta}^{TN}, (\|\cdot\|_N^{\mathrm{f},\theta})^{2n}\right) &= \|\cdot\|_{L^2}\left(g^{TN}, \|\cdot\|_N^{2n}\right).\end{aligned} \tag{3.3.77}$$

From (3.3.73) and (3.3.77), as $\theta \to 0$, we get (3.3.2) for $h_2^\xi := h^\xi$.

3.3.4 Proofs of Theorems 3.3.12, 3.3.15

In this section we prove Theorems 3.3.12, 3.3.15, which were announced in Section 3.3.3. In the proof of Theorem 3.3.12 we use the homogeneity of the Laplacian. In the proof of Theorem 3.3.15, we use the analytic localization techniques, the maximal principle and sup-characterization of the Bergman kernel. We recall that we suppose that (ξ, h^ξ) is trivial around the cusps.

Proof of Theorem 3.3.12. First of all, (3.3.47) is a consequence of Hodge theory for compact manifolds. To prove (3.3.48), by (3.3.43), it is enough to prove the following: *let $\tau > 0$, g_0^{TM} be a Kähler metric on $\overline{M}$ and let $\|\cdot\|_M^0$ be a Hermitian norm on $\omega_M(D)$ over $\overline{M}$ such that over M, we have*

$$g_0^{TM} \otimes (\|\cdot\|_M^0)^{2n} \leq \tau \cdot g^{TM} \otimes \|\cdot\|_M^{2n}, \tag{3.3.78}$$

$$\|\cdot\|_M^0 \leq \|\cdot\|_M. \tag{3.3.79}$$

Let $\langle\cdot,\cdot\rangle_{L_0^2}$ be the L^2-scalar product associated with g_0^{TM}, h^ξ, $\|\cdot\|_M^0$, and let $\square_0^{E_M^{\xi,n}}$ be the associated Kodaira Laplacian. Then for $n \leq 0$, we have

$$\inf\left\{\mathrm{Spec}\left(\square^{E_M^{\xi,n}}\right)\setminus\{0\}\right\} \leq \tau\cdot\inf\left\{\mathrm{Spec}\left(\square_0^{E_M^{\xi,n}}\right)\setminus\{0\}\right\}. \tag{3.3.80}$$

Let's prove this statement. By (3.2.24), we deduce

$$\mathscr{C}^\infty(\overline{M}, E_M^{\xi,n}) \subset \mathrm{Dom}(\square^{E_M^{\xi,n}}). \tag{3.3.81}$$

In the following series of transformations, we use (3.3.79) and $n \leq 0$ to get the inequality. For $s \in \mathscr{C}^\infty(\overline{M}, E_M^{\xi,n})$ we have

$$\langle \square_0^{E_M^{\xi,n}} s, s\rangle_{L_0^2} = \langle \overline{\partial}^{E_M^{\xi,n}} s, \overline{\partial}^{E_M^{\xi,n}} s\rangle_{L_0^2} \geq \langle \overline{\partial}^{E_M^{\xi,n}} s, \overline{\partial}^{E_M^{\xi,n}} s\rangle_{L^2} = \langle \square^{E_M^{\xi,n}} s, s\rangle_{L^2}. \tag{3.3.82}$$

Also, from (3.3.78), we have

$$\langle s, s\rangle_{L_0^2} \leq \tau \cdot \langle s, s\rangle_{L^2}. \tag{3.3.83}$$

From (3.3.82) and (3.3.83), we deduce

$$\frac{\langle \square_0^{E_M^{\xi,n}} s, s\rangle_{L_0^2}}{\langle s, s\rangle_{L_0^2}} \geq \frac{\langle \square^{E_M^{\xi,n}} s, s\rangle_{L^2}}{\tau \cdot \langle s, s\rangle_{L^2}}. \tag{3.3.84}$$

We denote $k = \dim H^0(\overline{M}, E_M^{\xi,n})$. By the min-max theorem (cf. [85, (C.3.3)]) and (3.3.81), we have

$$\inf\left\{\mathrm{Spec}\left(\square^{E_M^{\xi,n}}\right) \setminus \{0\}\right\} = \inf_{F \subset \mathscr{C}^\infty(\overline{M}, E_M^{\xi,n})} \left\{ \sup_{s \in F} \left\{ \frac{\langle \square^{E_M^{\xi,n}} s, s\rangle_{L^2}}{\langle s, s\rangle_{L^2}} \right\} : \dim F = k+1 \right\}. \tag{3.3.85}$$

Then (3.3.80) follows from (3.3.84) and (3.3.85). □

Proof of Theorem 3.3.15. This proof uses all the properties of tight families. The presence of the line bundle $\omega_M(D)$ makes analysis more difficult, and we have to consider 2 cases: $\theta^3 < |u| < |\log\theta|^{-1}$ and $|u| \leq \theta^3$. The main feature exploited in the first case is that we have elliptic estimate with the needed power of logarithm (3.3.44), (3.3.45). The main feature exploited in the second case is the *property 2* of tight families along with the maximal principle (cf. [38, p. 180]).

Let's prove (3.3.52) for $\theta^3 \leq |u| \leq |\log\theta|^{-1}$. We put $r = \mathrm{d}(u, 1/2)$, then by (3.2.138), $r \approx \ln|\ln|u||$. In this case, similarly to (3.2.157), by the fact that our flattenings are compatible, (ξ, h^ξ) is trivial near the cusps, and by the finite propagation speed of solutions of hyperbolic equations, we have

$$\widetilde{G}_{t,r}(\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u), \cdot\big) = \mathrm{Id}_\xi \cdot \widetilde{G}_{t,r}(\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u), \cdot\big), \tag{3.3.86}$$

where $\widetilde{G}_{t,r}$ is as in (3.2.148). Then, similarly to (3.2.158), by (3.3.86), we have

$$\begin{aligned}\exp(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u), \cdot\big) - \mathrm{Id}_\xi \cdot \exp(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u), \cdot\big)\\ = \widetilde{K}_{t,r}(\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u), \cdot\big) - \mathrm{Id}_\xi \cdot \widetilde{K}_{t,r}(\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u), \cdot\big).\end{aligned} \tag{3.3.87}$$

Now, similarly to (3.3.27), from (3.2.153), (3.3.44) and (3.3.87), for any $\theta^3 \leq |u|, |v| \leq |\log\theta|^{-1}$, we get

$$\Big|\exp(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(v)\big) - \mathrm{Id}_\xi \exp(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(v)\big)\Big|_{h\times h,\theta} \leq C|\ln|u||\exp(-c'(\ln|\ln|u||)^2/t). \quad (3.3.88)$$

In particular, (3.3.88) implies (3.3.52) for $\theta^3 \leq |u| \leq |\log\theta|^{-1}$.

Let's prove (3.3.52) for $|u| \leq \theta^3$. We trivialize $(\omega_M(D),\|\cdot\|_M)$, $(\omega_N(D),\|\cdot\|_N)$ as in *property 2* of tight families. Then, since (ξ, h^ξ) is trivialized around the cusps, for $v, w \in D(\theta^3)$, we may look at $\exp^\perp(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})((z_i^M)^{-1}(v),(z_i^M)^{-1}(w))$ and $\mathrm{Id}_\xi \exp^\perp(-t\square_{\mathrm{f},\theta}^{E_N^n})((z_i^N)^{-1}(v),(z_i^N)^{-1}(w))$ as at the functions over $D(\theta^3)\times D(\theta^3)$ with values in $\mathrm{End}(\xi|_{P_i^M})$.

For $v, w \in D(\theta^3)$, we denote

$$F(v,w,t) := \exp(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(v),(z_i^M)^{-1}(w)\big) - \mathrm{Id}_\xi \exp(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(v),(z_i^N)^{-1}(w)\big). \quad (3.3.89)$$

We write $F(v,w,t) = (F_{kl}(v,w,t))_{k,l=1}^{\dim\xi}$ for the components of the matrix from $\mathrm{End}(\xi|_{P_i^M})$. We notice that the functions $F_{kl}(v,w,t)$ satisfy the heat equation with zero initial data in $D(\theta^3)\times D(\theta^3)\times]0,+\infty[$, i.e. for any $k,l = 1,\ldots,\dim\xi$, we have

$$\Big(\frac{\partial}{\partial t} + \square_{\mathrm{f},\theta}\Big)F_{kl}(u,v,t) = 0 \quad \text{and} \quad \lim_{t\to 0} F_{kl}(u,v,t) = 0, \quad (3.3.90)$$

where $\square_{\mathrm{f},\theta}$ is the Laplace–Beltrami operator induced by $((z_i^M)^{-1})^* g_{\mathrm{f},\theta}^{TM}$ on $D(\theta^3)$. Thus, by the maximal principle (cf. [38, p. 180]), for $|u| \leq \theta^3$, we get

$$|F_{kl}(u,u,t)| \leq \sup_{\tau'\in[0,t]}\sup_{|w|=\theta^3}|F_{kl}(u,w,\tau')|. \quad (3.3.91)$$

By applying the maximal principle again, we get

$$|F_{kl}(u,w,\tau')| \leq \sup_{\tau\in[0,\tau']}\sup_{|v|=\theta^3}|F_{kl}(v,w,\tau)|. \quad (3.3.92)$$

By (3.3.88), there are $c', C > 0$ such that for any $\theta \in]0,e^{-3}]$, and $|v|, |w| = \theta^3$, we have

$$|F_{kl}(v,w,\tau)| \leq |\ln\theta|\exp(-c'(\ln|\ln\theta|)^2/\tau). \quad (3.3.93)$$

By (3.3.91), (3.3.92) and (3.3.93), we get (3.3.52) for $|u| \leq \theta^3$. Thus, (3.3.52) is completely proved.

Now let's prove (3.3.53). By Theorem 3.3.13, there are $c, C > 0$ such that for any $|u| \leq |\ln\theta|^{-1}$, $\theta \in]0,e^{-3}]$, $t > 0$, we have

$$\Big|\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_{\xi}\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big| \\ \leq C(\ln\max(\theta,|u|))^{12}t^{-4}\exp(-ct). \quad (3.3.94)$$

Now, for any $x,x'\in M$, we have

$$\exp(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})(x,x')=\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})(x,x')+B_{\theta}^{E_M^{\xi,n}}(x,x'), \quad (3.3.95)$$

where $B_{\theta}^{E_M^{\xi,n}}(x,x')$ is the Bergman kernel, defined by

$$B_{\theta}^{E_M^{\xi,n}}(x,x')=\sum s_i(x)(s_i(x'))_{\theta}^{*}, \quad (3.3.96)$$

for an orthonormal base $\{s_i\}$ of $H^0(\overline{M},E_M^{\xi,n})$ with respect to the L^2-scalar product induced by $g_{\mathrm{f},\theta}^{TM}$, h^{ξ}, $\|\cdot\|_M^{\mathrm{f},\theta}$, and $(\cdot)_{\theta}^{*}$ is the dual with respect to $|\cdot|_{h,\theta}$. By [39, Lemma 3.1], we have

$$B_{\theta}^{E_M^{\xi,n}}(x,x)=\max\left\{\frac{|s(x)|_{h,\theta}^2}{\|s\|_{L^2,\theta}^2}:s\in H^0(\overline{M},E_M^{\xi,n})\setminus\{0\}\right\}. \quad (3.3.97)$$

By (3.3.43) and the fact that $n\leq 0$, we see that for any $s\in\mathscr{C}^{\infty}(\overline{M},E_M^{\xi,n})$, we have

$$|s(x)|_{h,\theta}\leq|s(x)|_{h,\mathrm{sm}},\qquad \|s\|_{L^2,\theta}\geq\|s\|_{L^2,\mathrm{sm}}, \quad (3.3.98)$$

where $|\cdot|_{h,\mathrm{sm}}$ is the pointwise norm induced by h^{ξ}, $\|\cdot\|_{\mathrm{sm}}$, and $\|\cdot\|_{L^2,\mathrm{sm}}$ is the L^2-norm induced by h^{ξ}, $\|\cdot\|_{\mathrm{sm}}$, g_{sm}^{TM}. From (3.3.97) and (3.3.98), we deduce

$$B_{\theta}^{E_M^{\xi,n}}(x,x)\leq B_{\mathrm{sm}}^{E_M^{\xi,n}}(x,x), \quad (3.3.99)$$

where $B_{\mathrm{sm}}^{E_M^{\xi,n}}(x,x')$ is the Bergman kernel associated with h^{ξ}, $\|\cdot\|_{\mathrm{sm}}$, g_{sm}^{TM}. Thus, from (3.3.52), (3.3.95) and (3.3.99), there is $C>0$ such that for any $\theta\in]0,1/2]$, $|u|<\theta^3$, we have

$$\Big|\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_{\xi}\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big| \\ \leq C\Big(1+|\ln\max(\theta,|u|)|\exp\big(-c'(\ln|\ln\max(\theta,|u|)|)^2/t\big)\Big). \quad (3.3.100)$$

By multiplying (3.3.94) with power $\mu\in]0,1/2[$ and (3.3.100) with power $1-\mu$, we have

$$\Big|\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_M^{\xi,n}})\big((z_i^M)^{-1}(u),(z_i^M)^{-1}(u)\big)-\mathrm{Id}_{\xi}\exp^{\perp}(-t\square_{\mathrm{f},\theta}^{E_N^n})\big((z_i^N)^{-1}(u),(z_i^N)^{-1}(u)\big)\Big| \\ \leq C|\ln\max(\theta,|u|)|^{1+11\mu}t^{-4}\exp(-c\mu t-c'(\ln|\ln\max(\theta,|u|)|)^2/t) \\ +C|\ln\max(\theta,|u|)|^{12\mu}t^{-4}\exp(-c\mu t). \quad (3.3.101)$$

By (3.2.174) and (3.3.101), we finally get (3.3.53) by taking μ small enough. □

3.3.5 Existence of tight families of flattenings

Here we prove by an explicit construction that for any $n \in \mathbb{Z}$, $n \leq 0$, there are n-tight families of flattenings $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$ (see Definition 3.3.11). To simplify the notation, for $0 < a < b$, $i = 1, \ldots, m$, we denote (see (3.1.1))

$$C^M_i(a,b) := V^M_i(b) \setminus V^M_i(a). \tag{3.3.102}$$

As in Section 3.3.3, we suppose that (ξ, h^ξ) is trivial near the cusps.

Before giving the details, let's describe in words our construction. The metrics $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$ are equal to $g^{TM}, \|\cdot\|_M$ over $M \setminus (\cup_i V^M_i(\theta))$. The metric $\|\cdot\|^{\mathrm{f},\theta}_M$ gets "flattened" over the set $C^M_i(\theta^2, \theta)$ so that it differs from $\|\cdot\|_M$ by a multiplication by a function, which is bounded by a constant independent of θ. Over $V^M_i(\theta^2), \|\cdot\|^{\mathrm{f},\theta}_M$ is flat with a normalization as in *property 2* of tight families. The metric $g^{TM}_{\mathrm{f},\theta}$ coincide with g^{TM} over $M \setminus (\cup_i V^M_i(\theta^4))$. It gets "flattened" over the set $C^M_i(\theta^4/4, \theta^4)$, so that it differs from g^{TM} by a bounded function. Finally, over $V^M_i(\theta^4/4)$ it is flat such that the Riemannian manifolds $(V^M_i(\theta^4/4), g^{TM}_{\mathrm{f},\theta})$ and $(D(2), (dx^2 + dy^2)/(\ln\theta)^2)$ are isometric up to a multiplication by a constant independent of θ.

Now let's make this description more precise by giving explicit formulas.

Let $\phi : [0, +\infty[\to [0,1]$ be some smooth decreasing function satisfying

$$\phi(u) = \begin{cases} 1 & \text{for } u \in [0,1], \\ 0 & \text{for } u \in [2, +\infty[. \end{cases} \tag{3.3.103}$$

Fix $\theta \in]0, 1/2]$. We denote by $\|\cdot\|^{\mathrm{f},\theta}_M$ the Hermitian norm on $\omega_M(D)$ such that $\|\cdot\|^{\mathrm{f},\theta}_M$ coincides with $\|\cdot\|_M$ over $M \setminus (\cup_i V^M_i(\theta))$, and over $V^M_i(\theta)$, it satisfies

$$\left\| dz^M_i \otimes s_{D_M}/z^M_i \right\|^{\mathrm{f},\theta}_M (x) = |\ln\theta| \cdot \left| \frac{\ln|z^M_i(x)|}{\ln\theta} \right|^{\phi(\ln|z^M_i(x)|/\ln\theta)}. \tag{3.3.104}$$

Let the metric $g^{TM}_{\mathrm{f},\theta}$ coincide with g^{TM} over $M \setminus (\cup_i V^M_i(\theta^4))$, and over $V^M_i(\theta^4)$ be induced by

$$\left(\frac{|z^M_i \ln|z^M_i||^2}{|\theta^4 \ln\theta^4|^2} \right)^{\phi(2|z^M_i|^2/\theta^8)} \cdot \frac{\sqrt{-1} dz^M_i d\overline{z}^M_i}{|z^M_i \ln|z^M_i||^2}. \tag{3.3.105}$$

Then we see that the metrics $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$ verify the description given in the beginning of the section. Let $\square^{E^{\xi,n}_M}_{\mathrm{f},\theta}, \|\cdot\|_{L^2,\theta}$ be the Kodaira Laplacian and the L^2-norm induced by $g^{TM}_{\mathrm{f},\theta}$, $h^\xi \otimes (\|\cdot\|^{\mathrm{f},\theta}_M)^{2n}$.

Theorem 3.3.18. *The flattenings $g^{TM}_{\mathrm{f},\theta}, \|\cdot\|^{\mathrm{f},\theta}_M$, $\theta \in]0, 1/2]$ are n-tight.*

Proof. We see directly from (3.3.104) and (3.3.105) that all the requirements for tightness are trivially satisfied with only one exception - the estimate (3.3.44). As $g^{TM}_{\mathrm{f},\theta}$ and $\|\cdot\|^{\mathrm{f},\theta}_M$ coincide with g^{TM} and $\|\cdot\|_M$ over $M \setminus (\cup_i V^M_i(\theta))$, by Lemma 3.2.28, (3.2.138), it is enough to prove that for any $n \in \mathbb{Z}$, there is $C > 0$ such that for any $\sigma \in \mathscr{C}^\infty(\overline{M}, E^{\xi,n}_M)$, $x \in V^M_i(\theta^{1/2})$, the estimate (3.3.44) holds.

Let's prove (3.3.44) for $x \in C_i^M(\theta^3, \theta^{1/2})$. Let's denote by $\|\cdot\|_{\mathbb{D}}^{\mathrm{f},\theta}$ the Hermitian norm on $\omega_{\mathbb{D}}(0)$ (see (3.2.45)), given by the formula (compare with (3.3.104))

$$\big\|dz \otimes s_0/z\big\|_{\mathbb{D}}^{\mathrm{f},\theta}(z) = |\ln\theta| \cdot \left|\frac{\ln|z|}{\ln\theta}\right|^{\phi(\ln|z|/\ln\theta)}. \tag{3.3.106}$$

We denote by $\square_{\mathrm{f},\theta}^{\omega_{\mathbb{D}}^n}$ the Kodaira Laplacian on $(\mathbb{D}^*, g^{T\mathbb{D}^*})$ associated to $(\omega_{\mathbb{D}}(0)^n, (\|\cdot\|_{\mathbb{D}}^{\mathrm{f},\theta})^{2n})$.

By (3.2.138), for $x \in M \setminus (\cup_i V_i^M(\theta^3))$, we have $B(x, \ln(4/3)) \subset M \setminus (\cup_i V_i^M(\theta^4))$. By this, the fact that g^{TM} coincides with $g_{\mathrm{f},\theta}^{TM}$ over $M \setminus (\cup_i V_i^M(\theta^4))$ and the fact that (ξ, h^ξ) is trivial around the cusps, we see that to prove (3.3.44) for $x \in C_i^M(\theta^3, \theta^{1/2})$, it is enough to prove that for any $n \in \mathbb{Z}$, there is $C > 0$ such that for any $\sigma' \in \mathscr{C}^\infty(\mathbb{D}, \omega_{\mathbb{D}}(0)^n)$, $z \in D(\theta^{1/2}) \setminus D(\theta^3)$, we have

$$\big|\sigma'(z)\big|_{h,\theta} \leq C|\log|z||^{1/2} \sum_{i=0}^{2} \big\|(\square_{\mathrm{f},\theta}^{\omega_{\mathbb{D}}^n})^i \sigma'\big\|_{L^2(B^{\mathbb{D}}(z,\ln(4/3))),\theta}. \tag{3.3.107}$$

Let's prove (3.3.107). Recall that for $z_0 \in \mathbb{H}$, in (3.2.58), we have defined $g_{z_0} \in \mathrm{Aut}(\mathbb{H})$, and in (3.2.48), we have defined the covering map $\rho : \mathbb{H} \to \mathbb{D}^*$.

By proceeding in the same way as in the proof of Lemma 3.2.23, we see that it is enough to prove that for any $z_0 \in \mathbb{H}$, the family of metrics

$$\|\cdot\|_{z_0,\mathbb{H}}^{\mathrm{f},\theta} := \big((g_{z_0}^{-1}\rho)^* \|\cdot\|_{\mathbb{D}}^{\mathrm{f},\theta}\big)\big|_{B^{\mathbb{H}}(\sqrt{-1},1)} \tag{3.3.108}$$

is uniformly $\mathscr{C}^\infty$-bounded for $\theta \in]0, 1/2]$ and $z_0 \in \mathbb{H}$ such that $\rho(z_0) \in D(\theta^{1/2}) \setminus D(\theta^3)$. However, similarly to (3.2.62), for any $z_0 = (x_0, y_0) \in \mathbb{H}$, $z_2 = (x, y) \in B^{\mathbb{H}}(\sqrt{-1}, 1)$, by (3.3.104), we have

$$\begin{aligned}\left\|\frac{(g_{z_0}^{-1}\rho)^*(dz \otimes s_0/z)}{|\log\theta|}\right\|_{z_0,\mathbb{H}}^{\mathrm{f},\theta}(z_2) &= \left\|\frac{dz \otimes s_0/z}{|\log\theta|}\right\|_{\mathbb{D}}^{\mathrm{f},\theta}(e^{-yy_0+\sqrt{-1}(xy_0+x_0)}) \\ &= \big(yy_0/|\log\theta|\big)^{\phi(yy_0/|\log\theta|)},\end{aligned} \tag{3.3.109}$$

which is uniformly bounded for $(x, y) \in B^{\mathbb{H}}(\sqrt{-1}, 1)$ and $|\log\theta|/2 \leq y_0 \leq 3|\log\theta|$. But since $\rho(z_0) \in D(\theta^{1/2}) \setminus D(\theta^3)$ if and only if $|\log\theta|/2 \leq y_0 \leq 3|\log\theta|$, we conclude that (3.3.107) holds for $z \in D(\theta^{1/2}) \setminus D(\theta^3)$. Thus, (3.3.44) holds for $x \in V_i^M(\theta^{1/2}) \setminus V_i^M(\theta^3)$.

Let's prove (3.3.44) for $x \in C_i^M(\theta^4/4, \theta^3)$. Since the Hermitian line bundle $(\omega_M(D), \|\cdot\|_M^{\mathrm{f},\theta})$ is trivial over $V_i^M(\theta^2)$, without loss of generality we may and we will suppose $n = 0$. Since $C_i^M(|x|/2, 2|x|) \subset C_i^M(\theta^4/8, 2\theta^3)$, by Lemma 3.2.25, we have

$$\big|\sigma(x)\big| \leq C(\ln\theta)^3 \sum_{j=0}^{2} \big\|(\square^{E_M^{\xi,n}})^j \sigma\big\|_{L^2(C_i^M(\theta^4/2, 2\theta^3))}. \tag{3.3.110}$$

By a trivial calculation, we have

$$\Big(\frac{|z\ln|z||^2}{\theta^8|\ln\theta^4|^2}\Big)^{-\psi(|z|^2/\theta^8)} \frac{\partial}{\partial z}\Big(\frac{|z\ln|z||^2}{\theta^8|\ln\theta^4|^2}\Big)^{\psi(|z|^2/\theta^8)}$$

$$= \Big(\frac{\psi(|z|^2/\theta^8)(\ln|z|+1/2)}{z\ln|z|} + \ln\Big(\frac{|z\ln|z||^2}{\theta^8|\ln\theta^4|^2}\Big)\psi'(|z|^2/\theta^2))\overline{z}\theta^{-8}\Big), \quad (3.3.111)$$

By (3.3.105) and (3.3.111), there is $C>0$ such that for any $\theta\in]0,1/2]$, we have

$$|\square(g^{TM}/g^{TM}_{\mathrm{f},\theta})| < C(\ln\theta)^2, \qquad |\partial(g^{TM}/g^{TM}_{\mathrm{f},\theta})|_{h,\theta} < C|\ln\theta|, \quad (3.3.112)$$

over $C_i^M(\theta^4/8, 2\theta^3)$. As $(\omega_M(D), \|\cdot\|^{\mathrm{f},\theta}_M)$ is trivial over $V_i^M(2\theta^3)$, the following identity holds

$$\square^{E^{\xi,n}_M}_{\mathrm{f},\theta} = (g^{TM}/g^{TM}_{\mathrm{f},\theta})\cdot\square^{E^{\xi,n}_M}. \quad (3.3.113)$$

By (3.3.110), (3.3.112) and (3.3.113), we get (3.3.44) for $x\in C_i^M(\theta^4,\theta^3)$.

Let's prove (3.3.44) for $x\in V_i^M(\theta^4)$. First of all, we recall that by Sobolev inequality and standard elliptic estimates, we have for some $C>0$ and any $h\in\mathscr{C}^\infty(D(2))$, $x\in D(1)$:

$$\big|h(x)\big| \le C\sum_{i=0}^2 \big\|\square^i h\big\|_{L^2_{\mathrm{st}}}, \quad (3.3.114)$$

where $\|\cdot\|_{L^2_{\mathrm{st}}}$ is the L^2-norm induced by the standard Euclidean metric g_{st} over $D(2)$, and $\square$ is the Kodaira Laplacian induced by g_{st}. We denote by $g_{\mathrm{st},\theta}$ the rescaled Euclidean metric given by

$$g_{\mathrm{st},\theta} := \frac{dx^2+dy^2}{(\ln\theta)^2}. \quad (3.3.115)$$

Let $\|\cdot\|_{L^2_{\mathrm{st}},\theta}$ be the L^2-norm induced by $g_{\mathrm{st},\theta}$, let $\square_\theta$ be the Kodaira Laplacian induced by $g_{\mathrm{st},\theta}$. Analogically to (3.3.113), the estimation (3.3.114) implies that

$$\big|h(x)\big| \le C(\ln\theta)^4\sum_{i=0}^2 \big\|\square_\theta^i h\big\|_{L^2_{\mathrm{st}},\theta}, \quad (3.3.116)$$

By (3.3.105), the spaces $(D(2), g_{\mathrm{st},\theta})$ and $(V_i^M(\theta^4), g^{TM}_{\mathrm{f},\theta})$ are isometric up to a constant independent of θ. Thus, by (3.3.116), we deduce (3.3.44) for $x\in V_i^M(\theta^4)$.

Now, all the cases have been considered, thus, the proof of Theorem 3.3.18 is finished. □

3.4 The anomaly formula: a proof of Theorem B

In this section we prove Theorem B. First of all, we recall that in Section 3.3 we proved Theorem B for $g_0^{TM}=g^{TM}$, i.e. when we have only the variation of h^ξ. Thus, it's left to prove Theorem B for $h_0^\xi = h^\xi$ and under the supposition that (ξ, h^ξ) is trivial around the cusps. Let's describe the idea of the proof. We construct a family of flattenings which "approach" the cusp metric and we use Theorem A to relate the corresponding relative Quillen norms. Then we apply the anomaly formula Bismut-Gillet-Soulé [23, Theorem 1.23] (see Theorem 3.3.1) and calculate the limit of the right-hand side of (3.3.3), as the family of flattenings "approach" the cusp metric.

Before giving a proof of Theorem B, let's fix some notation. By suppositions of Theorem B, for $\epsilon>0$, there are holomorphic functions $h_i^\phi : D(\epsilon)\to D(1)$, $i=1,\dots,m$, such that g_0^{TM} is Poincaré-compatible with coordinates $h_i^\phi(z_i^M)$ around $P_i^M\in D_M$. We note

$$z_i^{0,M} := h_i^\phi(z_i^M). \quad (3.4.1)$$

By Definition 3.1.5 of the Wolpert norm, we have the following identity

$$\ln\Big(\|\cdot\|^W/\|\cdot\|_0^W\Big)=\sum\ln\big|(h_i^{\phi})'(0)\big|. \tag{3.4.2}$$

First of all, let's describe why the right-hand side of (3.1.25) is finite. For $\epsilon>0$, in $V_i^M(\epsilon)$:

$$c_1\big(\omega_M(D),(\|\cdot\|_M)^2\big)|_M=\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\ln\big(\|s\|_M^2\big)=O\big(|z_i^M\ln|z_i^M||^{-2}\big), \tag{3.4.3}$$

where s is a local holomorphic frame of $\omega(D_M)$. Similar estimation holds for the norm $\|\cdot\|_M^0$. The identity (3.1.27) says

$$\frac{e^{2\phi}dz_i^Md\overline{z}_i^M}{\big|z_i^M\ln|z_i^M|\big|^2}=\frac{dz_i^{0,M}d\overline{z}_i^{0,M}}{\big|z_i^{0,M}\ln|z_i^{0,M}|\big|^2}. \tag{3.4.4}$$

By (3.1.5) and (3.4.4), we see that over $V_i^M(\epsilon)$, we have

$$\ln\big(\|\cdot\|_M^0/\|\cdot\|_M\big)=O\big(|\ln|z_i^M||^{-1}\big). \tag{3.4.5}$$

By (3.1.23), (3.1.24), (3.2.24), (3.4.3) and (3.4.5), we conclude that the right-hand side of (3.1.25) is finite.

Now let's describe the precise family of flattenings we choose. Recall that the function $\psi:\mathbb{R}\to[0,1]$ was defined in (3.2.54). Let $g_{\mathrm{f},\theta}^{TM}$ be a metric over $\overline{M}$ such that it coincides with g^{TM} away from $\cup_iV_i^M(\theta)$, and over $V_i^M(\theta)$ it is induced by

$$\Big|z_i^M\ln|z_i^M|\Big|^{-2\psi(\ln|z_i^M|/\ln\theta)}\sqrt{-1}dz_i^Md\overline{z}_i^M, \tag{3.4.6}$$

for all $i=1,\ldots,m$. Similarly, let $\|\cdot\|_M^{\mathrm{f},\theta}$ be the smooth metric on $\omega_M(D)$ over $\overline{M}$ such that it coincides with $\|\cdot\|_M$ away from $\cup_iV_i^M(\theta)$, and over $V_i^M(\theta)$, $i=1,\ldots,m$, we have

$$\Big\|dz_i^M\otimes s_{D_M}/z_i^M\Big\|_M^{\mathrm{f},\theta}=\Big|\ln|z_i^M|\Big|^{\psi(\ln|z_i^M|/\ln\theta)}, \tag{3.4.7}$$

where s_{D_M} is the canonical section of $\mathscr{O}_{\overline{M}}(D_M)$, $\mathrm{div}(s_{D_M})=D_M$.

For $\epsilon>0$, $i=1,\ldots,m$, we denote

$$V_i^{0,M}(\epsilon):=\{x\in M:|z_i^{0,M}(x)|\leq\epsilon\}. \tag{3.4.8}$$

Let $g_{0,\mathrm{f},\theta}^{TM},\|\cdot\|_{0,M}^{\mathrm{f},\theta}$ be the flattenings of $g_0^{TM},\|\cdot\|_M^0$, compatible with the flattenings $g_{\mathrm{f},\theta}^{TM},\|\cdot\|_M^{\mathrm{f},\theta}$ (cf. (3.1.15), (3.1.16)). More precisely, the metrics $g_{0,\mathrm{f},\theta}^{TM},\|\cdot\|_{0,M}^{\mathrm{f},\theta}$ coincide with $g_0^{TM},\|\cdot\|_M^0$ away from $\cup_iV_i^{0,M}(\theta)$, and over $V_i^{0,M}(\theta)$ the metric $g_{0,\mathrm{f},\theta}^{TM}$ is induced by

$$\Big|z_i^{0,M}\ln|z_i^{0,M}|\Big|^{-2\psi(\ln|z_i^{0,M}|/\ln\theta)}\sqrt{-1}dz_i^{0,M}d\overline{z}_i^{0,M}. \tag{3.4.9}$$

Also, for s_{D_M} as in (3.4.7), we have

$$\left\| dz_i^{0,M} \otimes s_{D_M}/z_i^{0,M} \right\|_{0,M}^{\mathrm{f},\theta} = \left| \ln |z_i^{0,M}| \right|^{\psi(\ln |z_i^{0,M}|/\ln\theta)}. \tag{3.4.10}$$

Let's denote by $\|\cdot\|_{\mathrm{f},\theta,\mathrm{M}}^{\omega,0}, \|\cdot\|_{\mathrm{f},\theta,\mathrm{M}}^{\omega}$ the norms on $\omega_{\overline{M}}$ over $\overline{M}$ induced by $g_{0,\mathrm{f},\theta}^{TM}$ and $g_{\mathrm{f},\theta}^{TM}$ respectively.

Proof of (3.1.25). By Theorem A, for any $\theta \in]0,1]$, we have

$$2\ln\left(\frac{\|\cdot\|_Q\left(g_0^{TM}, h^\xi \otimes \|\cdot\|_{0,M}^{2n}\right)}{\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)}\right) = 2\ln\left(\frac{\|\cdot\|_Q\left(g_{0,\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_{0,M}^{\mathrm{f},\theta})^{2n}\right)}{\|\cdot\|_Q\left(g_{\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\right)}\right). \tag{3.4.11}$$

We show that the limit of the right-hand side of (3.4.11), as $\theta \to 0$ is exactly the right-hand side of (3.1.25). Then Theorem B would follow from (3.4.11).

We denote by $\|\cdot\|_{\mathrm{f},\theta,M}^{\omega}, \|\cdot\|_{\mathrm{f},\theta,M}^{\omega,0}$ the norms on ω_M induced by $g_{\mathrm{f},\theta}^{TM}$ and $g_{0,\mathrm{f},\theta}^{TM}$. Set

$$\begin{aligned}\Phi(\theta) := \Big[&\widetilde{\mathrm{Td}}\big(\omega_M^{-1}, (\|\cdot\|_{\mathrm{f},\theta,M}^{\omega})^{-2}, (\|\cdot\|_{\mathrm{f},\theta,M}^{\omega,0})^{-2}\big)\mathrm{ch}\big(\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D), (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\big)\\ &+ \mathrm{Td}\big(\omega_M^{-1}, (\|\cdot\|_{\mathrm{f},\theta,\mathrm{M}}^{\omega,0})^{-2}\big)\mathrm{ch}\big(\xi, h^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D), (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}, (\|\cdot\|_{0,M}^{\mathrm{f},\theta})^{2n}\big)\Big]^{[2]}.\end{aligned} \tag{3.4.12}$$

Then, by Theorem 3.3.1, we have

$$2\ln\left(\frac{\|\cdot\|_Q\left(g_{0,\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_{0,M}^{\mathrm{f},\theta})^{2n}\right)}{\|\cdot\|_Q\left(g_{\mathrm{f},\theta}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f},\theta})^{2n}\right)}\right) = \int_M \Phi(\theta). \tag{3.4.13}$$

where $\widetilde{\mathrm{Td}}$ and $\widetilde{\mathrm{ch}}$ are given by (3.1.23) and (3.1.24). We decompose the right-hand side of (3.4.13) into integrals over $M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0,M}(\theta)))$ and over $V_i^M(\theta) \cup V_i^{0,M}(\theta)$, $i = 1, \ldots, m$. Since the flattenings $g_{0,\mathrm{f},\theta}^{TM}, g_{\mathrm{f},\theta}^{TM}$ and $\|\cdot\|_{0,M}^{\mathrm{f},\theta}, \|\cdot\|_M^{\mathrm{f},\theta}$ coincide with g_0^{TM}, g^{TM} and $\|\cdot\|_M^0, \|\cdot\|_M$ over $M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0,M}(\theta)))$, and the quantities under the integration in the anomaly formula are local, we see by Lebesgue dominated convergence theorem, by the finiteness of the right-hand side of (3.1.25) and by (3.1.26), that the integral of $\Phi(\theta)$ over $M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0,M}(\theta)))$ converges to the integral part in the right-hand side of (3.1.25), as $\theta \to 0$.

Now let's study the contribution over $\cup_i(V_i^M(\theta) \cup V_i^{0,M}(\theta))$ of the integral in (3.4.13). We note that in the case when ϕ from (3.1.27) has compact support in M, this integral is actually zero for θ sufficiently small (which is consistent with the statement of Theorem B).

From the discussion above, (3.4.2), (3.4.13), and the fact that we restrict ourselves to the case (ξ, h^ξ) trivial around the cusps, Theorem B would follow from the following

Lemma 3.4.1. As $\theta \to 0$, we have

$$\int_{V_i^M(\theta) \cup V_i^{0,M}(\theta)} \Phi(\theta) \to -\frac{\mathrm{rk}(\xi)}{6} \ln |(h_i^\phi)'(0)|. \tag{3.4.14}$$

Proof. All the subsequent formulas should be regarded as being valid over $V_i^M(\theta)\cup V_i^{0,M}(\theta)$. By (3.4.6) and (3.4.7), we have

$$\begin{aligned} c_1\big(\omega_M,(\|\cdot\|^{\omega}_{\mathrm{f},\theta,M})^2\big) &= -\frac{\sqrt{-1}\partial\overline{\partial}}{2\pi}\Big(\psi\big(\ln|z_i^M|/\ln\theta\big)\cdot\big(2\ln|z_i^M|+2\ln|\ln|z_i^M||\big)\Big)\\ &= \Big[\frac{\ln|z_i^M|\psi''\big(\ln|z_i^M|/\ln\theta\big)}{2|z_i^M\ln\theta|^2}+\frac{\psi'\big(\ln|z_i^M|/\ln\theta\big)}{|z_i^M|^2\ln\theta}\\ &\qquad + O\Big(\frac{\ln|\ln|z_i^M||}{|z_i^M\ln|z_i^M||^2}\Big)\Big]\frac{-\sqrt{-1}dz_i^M d\overline{z}_i^M}{2\pi}, \end{aligned} \tag{3.4.15}$$

$$\begin{aligned} c_1(\omega_M(D),(\|\cdot\|^{\mathrm{f},\theta}_M)^2) &= -\frac{\sqrt{-1}\partial\overline{\partial}}{2\pi}\Big(\psi\big(\ln|z_i^M|/\ln\theta\big)\cdot\big(2\ln|\ln|z_i||\big)\Big)\\ &= O\Big(\frac{\ln|\ln|z_i^M||}{|z_i^M\ln|z_i^M||^2}\Big)dz_i^M d\overline{z}_i^M. \end{aligned} \tag{3.4.16}$$

By $dz_i^{0,M}=(h_i^{\phi})'(z_i^M)\cdot dz_i^M$ and $\ln|\ln|z_i^{0,M}||=\ln|\ln|z_i^M||+O(1/|\ln|z_i^{0,M}||)$, we deduce

$$\begin{aligned} \ln\big(\|\cdot\|^{\omega,0}_{\mathrm{f},\theta,M}/\|\cdot\|^{\omega}_{\mathrm{f},\theta,M}\big) &= \psi\big(\ln|z_i^{0,M}|/\ln\theta\big)\Big(\ln|z_i^{0,M}|+\ln|\ln|z_i^{0,M}||\Big)\\ &\quad -\psi\big(\ln|z_i^M|/\ln\theta\big)\Big(\ln|z_i^M|+\ln|\ln|z_i^M||\Big)-\ln|(h_i^{\phi})'(z_i^M)|\\ &= \ln|(h_i^{\phi})'(0)|\Big(-1+\psi\big(\ln|z_i^M|/\ln\theta\big)+\psi'\big(\ln|z_i^M|/\ln\theta\big)\frac{\ln|z_i^M|}{\ln\theta}\Big)\\ &\qquad + O\Big(\frac{\ln|\ln|z_i^M||}{|\ln|z_i^M||}\Big), \end{aligned} \tag{3.4.17}$$

$$\begin{aligned} \ln\big(\|\cdot\|^{\mathrm{f},\theta}_{0,M}/\|\cdot\|^{\mathrm{f},\theta}_M\big) &= \psi\big(\ln|z_i^{0,M}|/\ln\theta\big)\ln|\ln|z_i^{0,M}||\\ &\quad -\psi\big(\ln|z_i^M|/\ln\theta\big)\ln|\ln|z_i^M||=O\Big(\frac{\ln|\ln|z_i^M||}{|\ln|z_i^M||}\Big). \end{aligned} \tag{3.4.18}$$

Finally, from (3.4.15) and the analogical statement for $\|\cdot\|^{\omega,0}_{\mathrm{f},\theta,\mathrm{M}}$, we easily get

$$\partial\overline{\partial}\ln\big(\|\cdot\|^{\omega,0}_{\mathrm{f},\theta,M}/\|\cdot\|^{\omega}_{\mathrm{f},\theta,M}\big)=O\Big(\frac{\ln|\ln|z_i^M||}{|z_i^M\ln|z_i^M||^2}\Big)dz_i^M d\overline{z}_i^M. \tag{3.4.19}$$

From Theorem 3.3.1, (3.1.23), (3.1.24) and (3.4.15) - (3.4.19), we get

$$\begin{aligned} \int_{V_i^M(\theta)\cup V_i^{0,M}(\theta)}\Phi(\theta) &= -\frac{\mathrm{rk}(\xi)}{3}\int_{V_i^M(\theta)\cup V_i^{0,M}(\theta)}\Big[c_1\big(\omega_M,(\|\cdot\|^{\omega}_{\mathrm{f},\theta,M})^2\big)\ln\Big(\frac{\|\cdot\|^{\omega,0}_{\mathrm{f},\theta,M}}{\|\cdot\|^{\omega}_{\mathrm{f},\theta,M}}\Big)\\ &\qquad + O\Big(\frac{\ln|\ln|z_i^M||}{|z_i^M\ln|z_i^M||^2}\Big)dz_i^M d\overline{z}_i^M\Big]. \end{aligned} \tag{3.4.20}$$

From (3.4.15), (3.4.17) and (3.4.20), we get

$$\begin{aligned}
&\lim_{\theta\to 0}\int_{V_i^M(\theta)\cup V_i^{0,M}(\theta)}\Phi(\theta)=\frac{2\ln|(h_i^\phi)'(0)|}{3}\cdot \mathrm{rk}(\xi)\\
&\quad\cdot\lim_{\theta\to 0}\int_{\theta}^{\theta^{1/2}}\frac{1}{r}\Big(\psi''\Big(\frac{\ln r}{\ln\theta}\Big)\frac{\ln r}{2(\ln\theta)^2}+\psi'\Big(\frac{\ln r}{\ln\theta}\Big)\frac{1}{\ln\theta}\Big)\Big(-1+\psi\Big(\frac{\ln r}{\ln\theta}\Big)+\psi'\Big(\frac{\ln r}{\ln\theta}\Big)\frac{\ln r}{\ln\theta}\Big)dr\\
&\quad=-\frac{2\ln|(h_i^\phi)'(0)|}{3}\cdot \mathrm{rk}(\xi)\cdot\int_{1/2}^{1}\Big(-\psi'(u)+\psi'(u)\psi(u)+u\psi'(u)^2\\
&\qquad\qquad -u\psi''(u)/2+u\psi''(u)\psi(u)/2+u^2\psi'(u)\psi''(u)/2\Big)du,
\end{aligned}\tag{3.4.21}$$

where in the last identity we used the change of variables $u:=\ln r/\ln\theta$. By the integration by parts and (3.2.54), we have

$$\begin{aligned}
&\int_{1/2}^{1}\psi'(u)du=-1, && \int_{1/2}^{1}u\psi''(u)\psi(u)du=\frac{1}{2}-\int_{1/2}^{1}u\psi'(u)^2du,\\
&\int_{1/2}^{1}u\psi''(u)du=1, && \int_{1/2}^{1}u^2\psi'(u)\psi''(u)du=-\int_{1/2}^{1}u\psi'(u)^2du,\\
&\int_{1/2}^{1}\psi'(u)\psi(u)du=-\frac{1}{2}.
\end{aligned}\tag{3.4.22}$$

We get (3.4.14) from (3.4.21), (3.4.22). □

Chapter 4

Regularity, asymptotics and curvature theorem.

Abstract. We study the Quillen metric on the determinant line bundle associated with a family of complex curves with cusps, which admit singular fibers.

More precisely, we fix a family of complex curves, which admit at most double-point singularities. We endow the fibers of this family with Kähler metrics, defined away from a finite set of points. We suppose that the metrics on the fibers have Poincaré-type singularities near the fixed points, and that those points form a divisor on the total space of the family, which is disjoint from the singularities of the fibers.

We fix a holomorphic vector bundle over the total space of the family and endow it with a Hermitian metric defined away from the fixed points. We suppose that this metric has at most logarithmic singularities, coming from the induced norm on the negative power of the relative canonical line bundle twisted by the divisor line bundle associated with the divisor of the fixed points.

The determinant line bundle associated with the holomorphic line bundle is naturally endowed with the Quillen metric defined using the analytic torsion from the first paper of this series [54]. We study the regularity of this Quillen metric and its asymptotics near the locus of singular curves. The singularities of the asymptotics turn out to be reasonable enough, so that the curvature of the Chern connection of the determinant line bundle endowed with the Quillen norm is well-defined as a current over the base. We derive the explicit formula for this current, which gives a refinement of Riemann-Roch-Grothendieck theorem at the level of currents. This generalizes the curvature formulas of Takhtajan-Zograf and Bismut-Bost.

Our assumptions on the degeneration of the metric are very mild, and our study applies to the family of degenerating pointed hyperbolic surfaces. As a consequence, we get some regularity results on the Weil-Petersson form over the moduli space of pointed curves. Those regularity results are enough to conclude a well-known fact, originally due to Wolpert, that the Weil-Petersson volume of the moduli space of pointed curves is a rational multiple of a power of π.

4.1 Introduction

We study the Quillen metric on the determinant line bundle associated with a family of complex curves with cusps, which admit singular fibers.

To make things slightly more formal, we recall that for complex manifolds X and S, a proper holomorphic map $\pi : X \to S$ and a holomorphic vector bundle ξ over X, the construction of Grothendick-Knudsen-Mumford [76] (cf. also [23, §3]) associates the "determinant of the direct image of ξ" - the holomorphic line bundle over S, which we denote by $\det(R^\bullet \pi_* \xi)$. In the special case when the cohomology groups $H^\bullet(\pi^{-1}(\cdot), \xi)$ have constant dimension, they form holomorphic vector bundles, and we have a natural isomorphism $\det(R^\bullet \pi_* \xi) = \otimes_i (\Lambda^{\max} H^i(\pi^{-1}(\cdot), \xi))^{(-1)^i}$. If X and S are quasiprojective and π is projective, by a theorem of Riemann-Roch-Grothendieck (cf. [30, §7]), the first Chern class of $\det(R^\bullet \pi_* \xi)$ is expressed as a push-forward by π of characteristic classes of ξ and the relative tangent bundle TX/S.

Assume temporarily that π is a Kähler fibration, i.e. it is a submersion such that there exists a closed $(1,1)$-form on X such that its restriction to the fibers of π gives a Kähler form. In this case, Bismut-Gillet-Soulé in [23] gave an analytic construction of the line bundle $\lambda(j^*\xi)$, which they proved in [23, Theorem 3.14] to coincide with $(\det(R^\bullet \pi_* \xi))^{-1}$. Now, endow the vector bundles $\xi, TX/S$ with Hermitian metrics h^ξ and $h^{TX/S}$ over X. As the dimension of the cohomology of the fibers might change from point to point, the L^2-metric of the fibers on the line bundle $\lambda(j^*\xi)$ is not necessarily continuous. Nevertheless, Bismut-Gillet-Soulé in [23, Theorems 1.3, 1.6] showed that the Quillen norm $\|\cdot\|_Q$ on $\lambda(j^*\xi)$, defined as a product of the holomorphic analytic torsion and the L^2-metric of the fiber, is smooth. When π is trivial of relative dimension 1, this metric was previously defined by Quillen in [102]. The curvature theorem of Bismut-Gillet-Soulé [23, Theorem 1.9] expresses the curvature of the Chern connection on $(\lambda(j^*\xi), \|\cdot\|_Q^2)$ as an integral over the fibers of π of the differential form associated by Chern-Weil theory with a cohomological

class appearing on the right-hand-side of Riemann-Roch-Grothendieck theorem. Thus, curvature formula of Bismut-Gillet-Soulé gives a refinement of Riemann-Roch-Grothendieck theorem.

One of the main goals of this article is to prove a generalisation of the curvature formula of Bismut-Gillet-Soulé in relative dimension 1. In this generalisation our fibers are allowed to have hyperbolic cusps and double-point singularities. Moreover, the Hermitian metric h^{ξ} is also allowed to have logarithmic singularities at cusps.

More precisely, let $\pi : X \to S$ be a family of complex curves with ordinary singularities. We denote by $\Sigma_{X/S} \subset X$ the submanifold of singular points of the fibers (see Corollary 4.2.3), and by $\Delta := \pi_*(\Sigma_{X/S})$ the divisor of singular fibers. Let $D_{X/S} \subset X$ be a divisor induced by a submanifold $|D_{X/S}|$ intersecting $\pi^{-1}(|\Delta|)$ transversally and such that $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ is locally an isomorphism. In other words, we suppose that for any $s \in S$, there is a neighbourhood U of s, and disjoint holomorphic sections $\sigma_1, \dots, \sigma_m : U \to X$ of π, which do not pass through singular points such that over U the following identity holds

$$D_{X/S}|_{\pi^{-1}(U)} := \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m). \tag{4.1.1}$$

Those sections would model the positions of cusps in our family.

Let the norm $\|\cdot\|^{\omega}_{X/S}$ on the canonical line bundle $\omega_{X/S}$ (see Section 4.2.2) over $X \setminus (\pi^{-1}(\Delta) \cup |D_{X/S}|)$ be such that its restriction over each nonsingular fiber $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ of π induces the Kähler metric with cusps at $|D_{X/S}| \cap X_t$ (see Definition 4.2.5). The goal of this article is to study the Quillen norm associated with the family of cusped curves $(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$. The Quillen norm here uses the analytic torsion, defined in the first article of this series [54].

We denote by $\|\cdot\|^{W}_{X/S}$ the Wolpert norm induced by $\|\cdot\|^{\omega}_{X/S}$ (see Definition 4.2.6) on the line bundle $\det(\pi_*(\omega_{X/S}|_{|D_{X/S}|}))$. This norm measures how the local structure of the metric $\|\cdot\|^{\omega}_{X/S}$ changes in the neighborhood of $|D_{X/S}|$.

Construction 4.1.1. For a complex manifold Y and a divisor $D_0 \subset Y$, let $\|\cdot\|^{\mathrm{div}}_{D_0}$ be the singular norm on $\mathscr{O}_Y(D_0)$, defined by

$$\|s_{D_0}\|^{\mathrm{div}}_{D_0}(x) = 1, \tag{4.1.2}$$

where s_{D_0}, $\mathrm{div}(s_{D_0}) = D_0$, is the canonical section of the divisor D_0, and $x \in Y \setminus |D_0|$.

We endow the twisted canonical line bundle

$$\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S}) \tag{4.1.3}$$

with the canonical Hermitian norm $\|\cdot\|_{X/S}$ over $\pi^{-1}(S \setminus |\Delta|) \setminus |D_{X/S}|$, induced by $\|\cdot\|^{\omega}_{X/S}, \|\cdot\|^{\mathrm{div}}_{D_{X/S}}$.

Let ξ be a holomorphic vector bundle over X, and let h^{ξ} be a Hermitian metric over $\pi^{-1}(S \setminus |\Delta|)$. For $n \leq 0$, we endow the line bundle $\det(R^{\bullet}\pi_*(\xi \otimes \omega_{X/S}(D)^n))^{-1}$ with the Quillen norm

$$\|\cdot\|_Q\,(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}), \quad t \in S \setminus |\Delta|, \tag{4.1.4}$$

over $S \setminus |\Delta|$, defined as the product of the L^2-norm and the analytic torsion of the fiber (see Definition 4.2.8). We denote $\det \xi := \Lambda^{\max}\xi$, and by $h^{\det \xi}$ the Hermitian metric on $\det \xi$ induced by h^{ξ}. Now, we denote the norm

$$\|\cdot\|_{\mathscr{L}_n} := \left(\|\cdot\|_Q\left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right)\right)^{12} \otimes \left(\|\cdot\|^W_{X/S}\right)^{-\mathrm{rk}(\xi)} \otimes \left(\|\cdot\|^{\mathrm{div}}_\Delta\right)^{\mathrm{rk}(\xi)} \otimes \left(\det(\pi_*(h^{\det\xi}|_{|D_{X/S}|})\right)^3 \quad (4.1.5)$$

on the line bundle

$$\mathscr{L}_n := \det\left(R^\bullet\pi_*(\xi\otimes\omega_{X/S}(D)^n)\right)^{-12} \otimes \left(\det(\pi_*(\omega_{X/S}|_{|D_{X/S}|})\right)^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes \left(\det(\pi_*(\det\xi|_{|D_{X/S}|})\right)^6. \quad (4.1.6)$$

Our first goal is to study - under various assumptions on the data - the regularity of $\|\cdot\|_{\mathscr{L}_n}$ over $S\setminus|\Delta|$ and its singularities near Δ. We show that the singularities of $\|\cdot\|_{\mathscr{L}_n}$ are reasonable enough to define the curvature of the Chern connection of $(\mathscr{L}_n, (\|\cdot\|_{\mathscr{L}_n})^2)$ as a current over S. Then we compute this current explicitly, which would give us a refinement of Riemann-Roch-Grothendieck theorem on the level of currents. In particular, when $\pi : X \to S$ is a family of hyperbolic surfaces without singular fibers and (ξ, h^ξ) is trivial, we get Takhtajan-Zograf formula[1] [107, Theorem 1]. When the metrics $h^\xi, \|\cdot\|_{X/S}$ are smooth, i.e. there are no cusps and no degeneration of the metric near singular fibers, we get a formula of Bismut-Bost [20, Théorème 2.2]. Thus, our formula unifies those two curvature theorems.

Now let's state precisely the different assumptions on the data, which we consider in this article.

Assumption S1. The Hermitian metric h^ξ extends smoothly over X; the Hermitian norm $\|\cdot\|_{X/S}$ extends smoothly over $X\setminus|D_{X/S}|$ and it is pre-log-log of infinite order, with singularities along $D_{X/S}$ (cf. Definition 4.2.14).

Assumption S2. The divisor Δ has normal crossings. The Hermitian metric h^ξ is pre-log-log with singularities along $\pi^{-1}(\Delta)$ (cf. Definition 4.2.14); the Hermitian norm $\|\cdot\|_{X/S}$ is pre-log-log with singularities along $\pi^{-1}(\Delta)\cup D_{X/S}$.

Assumption S3. The divisor Δ has normal crossings. The Hermitian metric h^ξ extends smoothly over X; the Hermitian norm $\|\cdot\|_{X/S}$ is continuous over $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$, has log-log growth with singularities along $\Sigma_{X/S}\cup|D_{X/S}|$ (cf. Definitions 4.2.13), is good in the sense of Mumford on $X\setminus|D_{X/S}|$ with singularities along $\pi^{-1}(\Delta)$ (cf. Definition 4.2.14), and the coupling of $c_1(\omega_{X/S}(D), \|\cdot\|^2_{X/S})$ with two smooth vertical vector fields over $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$ is continuous over $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$ and has log-log growth with singularities along $|D_{X/S}|$ (cf. Definition 4.2.11b)).

Remark 4.1.2. a) If Δ has normal crossings, then, trivially, **S1** implies both **S2** and **S3**.

b) In Proposition 4.2.23, we prove that for any $\pi : X \to S$, $D_{X/S}$ as before, there is a Hermitian norm $\|\cdot\|_{X/S}$ on $\omega_{X/S}$ over $X\setminus|D_{X/S}|$ satisfying Assumption **S1**.

Let's motivate those assumptions. Assumption **S1** is more of a laboratory example to show the strongest regularity result for $\|\cdot\|_{\mathscr{L}_n}$ we could achieve. Also it generalizes to the non-compact case the hypothesises from Bismut-Bost [20] (cf. Bismut [18]). Assumption **S2** and **S3** are interesting because the main example of degenerating hyperbolic surfaces (see Construction 4.5.2) satisfies

[1] In fact, Takhtajan-Zograf used a version of analytic torsion defined through lengths of closed geodesics. In [56], we show that their definition is compatible with ours.

them (see Proposition 4.5.6). Assumption **S2** is well-adapted to the curvature theorem, Theorem D, and Assumption **S3** - to the continuity theorem, Theorem C.

Now let's argue why instead of more well-known notion of good vector bundles due to Mumford [95] (see Definition 4.2.14), we use the notion of pre-log-log vector bundles due to Burgos Gil-Kramer-Kühn [36] (see Definition 4.2.14). By Proposition 4.5.6, for our main example of hyperbolic surfaces from Construction 4.5.2, the metric $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ is good. This is stronger than being pre-log-log (see for example [36, Lemma 4.26]). As it easily follows from formulas for Bott-Chern classes of degree 0, the Bott-Chern classes associated with good metrics is not necessarily of Poincaré growth (see Definition 4.2.11d)), cf. [36, §4.5]. Nevertheless, as it was proved by Burgos Gil-Kramer-Kühn [36, Theorem 4.55] (cf. Theorem 4.2.19) in its equivalence class one can choose a representative which is pre-log-log. Now, the Bott-Chern classes enter naturally in the anomaly formula and the definition of Deligne norms [42, (6.3.1)]. Thus, Deligne norm associated with good Hermitian vector bundles is not good in general. Pre-log-log condition is a natural condition to form a class of metrics, such that the associated Deligne norms (see Freixas [58, Theorem 5.1.3]) and renormalized Quillen norms (see Theorem C2) almost[2] stay in this class. We will make this point more clear and use it extensively in our forthcoming paper [57] on Deligne-Mumford isometry.

To describe the singularities of the norm (4.1.5) near $|\Delta|$, we need the following notions.

Definition 4.1.3. Let Y be a complex manifold and let D_0 be a divisor in Y.

a) Suppose D_0 has normal crossings and the function $f : Y \setminus |D_0| \to \mathbb{R}$ is continuous and has log-log growth along D_0 (see Definition 4.2.11a)). We denote by $[f]_{L^1}$ the current over Y, given by the L^1-extension of f over Y. We say that f is *nice with singularities along* D_0 if the currents $\partial[f]_{L^1}$, $\overline{\partial}[f]_{L^1}$, $\partial\overline{\partial}[f]_{L^1}$ are defined by the integration against continuous forms over $Y \setminus |D_0|$, which have log-log growth along D_0 (see Definition 4.2.11b)).

b) Let $x \in D_0$ and let $U \subset Y$, $x \in U$ be an open subset. Let $h_1, \dots, h_k$, $k \in \mathbb{N}$ be local holomorphic functions such that $dh_i(x) \neq 0$, $i = 1, \dots, k$ and $n_1, \dots, n_k \in \mathbb{N}$ are such that D_0 is defined over U by $\{h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k} = 0\}$. We say that a smooth function $f : Y \setminus |D_0| \to \mathbb{R}$ is *very nice with singularities along* D_0 if for any $x \in D_0$ there are smooth functions $f_0, \dots, f_k : U \to \mathbb{C}$:

$$f = f_0 + \sum_1^k f_i |h_i|^2 \ln |h_i|. \tag{4.1.7}$$

c) Let L be a holomorphic line bundle over Y, and let h^L be a continuous Hermitian metric on L over $Y \setminus |D_0|$. For $x \in Y$, fix a local holomorphic frame υ of L in a neighbourhood U of x. We say that h^L is *very nice* (resp. *nice*) *with singularities along* D_0 if for any x and υ, the function $\ln h^L(\upsilon, \upsilon)$ is *very nice* (resp. *nice*) *with singularities along* D_0.

Remark 4.1.4. a) Trivially, for D_0 with normal crossings, every *very nice* Hermitian metric with singularities along D_0 is *nice* Hermitian metric with singularities along D_0.

[2] As we say in Theorem C2, the renormalized Quillen norm associated with a pre-log-log Hermitian vector bundle is nice in the sense of Definition 4.1.3. The notion of niceness is slightly less stronger than the notion of pre-log-log Hermitian vector bundle. Hovewer, as it follows from elliptic regularity (see also the proof of Corollary 4.1.8), a nice vector bundle is pre-log-log if and only if its curvature is smooth over $S \setminus |\Delta|$.

b) For a Hermitian metric h^L, which is either nice or very nice, we define the first Chern class as a current over Y by

$$c_1(L, h^L) := \frac{\partial\overline{\partial}[\ln h^L(v, v)]_{L^1}}{2\pi\sqrt{-1}}. \tag{4.1.8}$$

We note that with such a definition it is trivial that $c_1(L, h^L)$ is a closed current and the associated cohomological class coincides with $c_1(L)$.

Our first result describes the singularities of the norm (4.1.4) near $|\Delta|$.

Theorem C (Continuity theorem)**.** *Let $\pi : X \to S$ be a family of complex curves with at most double point singularities. Let $\Sigma_{X/S}$ be the submanifold of singular points on the fibers, and let $\Delta := \pi_*(\Sigma_{X/S})$ be the divisor of singular curves. Let ξ be a holomorphic vector bundle over X, and let h^ξ be a Hermitian metric on ξ over $\pi^{-1}(S \setminus |\Delta|)$.*

Let $D_{X/S} \subset X$ be a divisor induced by a submanifold $|D_{X/S}|$ intersecting $\pi^{-1}(|\Delta|)$ transversally and such that $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ is locally an isomorphism. Let the norm $\|\cdot\|^{\omega}_{X/S}$ on the canonical line bundle $\omega_{X/S}$ over $X \setminus (\pi^{-1}(\Delta) \cup |D_{X/S}|)$ be such that its restriction over each non-singular fiber $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ of π induces the Kähler metric with cusps at $|D_{X/S}| \cap X_t$ (see Definition 4.2.5).

We use the same notation for the line bundle $\mathscr{L}_n$ (see (4.1.6)), and the norm $\|\cdot\|_{\mathscr{L}_n}$ (see (4.1.5)).

*1) Under Assumption **S1**, the norm $\|\cdot\|_{\mathscr{L}_n}$ is very nice with singularities along Δ (hence, smooth over $S \setminus |\Delta|$).*

*2) Under Assumption **S2**, the norm $\|\cdot\|_{\mathscr{L}_n}$ is nice with singularities along Δ.*

*3) Under Assumption **S3**, the norm $\|\cdot\|_{\mathscr{L}_n}$ is continuous over S.*

Remark 4.1.5. a) When $m = 0$, Theorem C1 gives the result of Bismut-Bost [20, Théorème 2.2]. However, we note that the proof presented here relies on [20, Théorème 2.2]. In [57], we give another proof of Theorem C, which relies on the extension of Deligne-Mumford isomorphism and regularity results for Deligne metrics. This would give us, in particular, an alternative proof of [20, Théorème 2.2], which depends only on the results of Bismut-Gillet-Soulé [23], and not on the results of Bismut-Bost [20].

b) In the forthcoming paper [56], under Assumption **S3**, we exhibit a relation between the restriction of $\mathscr{L}_n$ over the singular locus and the Quillen norm of the normalisation of the singular fibers. In other words, this theorem describes the behaviour of a Quillen norm under adjunction of cusps obtained by degeneration. In particular, it gives a geometric interpretation of the continuous extension of $\|\cdot\|_{\mathscr{L}_n}$ onto the singular locus.

We note that by Theorems C.1, C.2 and Remark 4.1.4.b), the current $c_1(\mathscr{L}_n, \|\cdot\|^2_{\mathscr{L}_n})$ is well-defined over S. The next theorem gives an explicit expression for this current.

We recall that the Chern and Todd forms of the Hermitian vector bundle (ξ, h^ξ) are defined as

$$\begin{aligned} \mathrm{ch}(\xi, h^\xi) &= \mathrm{rk}(\xi) + c_1(\xi, h^\xi) + \tfrac{1}{2}c_1(\xi, h^\xi)^2 - c_2(\xi, h^\xi) + \cdots, \\ \mathrm{Td}(\xi, h^\xi) &= \mathrm{rk}(\xi) + \tfrac{1}{2}c_1(\xi, h^\xi) + \tfrac{1}{12}\big(c_1(\xi, h^\xi)^2 + c_2(\xi, h^\xi)\big) + \cdots, \end{aligned} \tag{4.1.9}$$

where the dots mean higher degree terms.

Theorem D (Curvature theorem)**.** *We use the notations from Theorem C. Under Assumption **S1** (resp. **S2**), the current*

$$\pi_*\Big[\mathrm{Td}(\omega_{X/S}(D)^{-1},\|\cdot\|_{X/S}^{-2})\mathrm{ch}(\xi,h^{\xi})\mathrm{ch}(\omega_{X/S}(D)^{n},\|\cdot\|_{X/S}^{2n})\Big]^{(2,2)} \tag{4.1.10}$$

is $L^1_{loc}(S)$ (resp. has log-log growth along Δ in the sense of Definition 4.2.11b), so in particular, it is also $L^1_{loc}(S)$). We denote by the same symbol the trivial L^1-extension of this current over S. This extension is closed. Moreover, the following identity of currents over S holds

$$c_1\big(\mathscr{L}_n,\|\cdot\|^2_{\mathscr{L}_n}\big)=-12\pi_*\Big[\mathrm{Td}\big(\omega_{X/S}(D)^{-1},\|\cdot\|_{X/S}^{-2}\big)\mathrm{ch}\big(\xi,h^{\xi}\big)\mathrm{ch}\big(\omega_{X/S}(D)^{n},\|\cdot\|_{X/S}^{2n}\big)\Big]^{(2,2)}. \tag{4.1.11}$$

Remark 4.1.6. a) Assume **S1** holds and $m=0$, then Theorem D is exactly the curvature formula of Bismut-Bost [20, Théorème 2.1]. We note, however, that the proof of Theorem D in this case relies on [20, Théorème 2.1].

b) Assume **S2** holds, then our proof relies on the curvature theorem of Bismut-Gillet-Soulé [23, Theorem 1.9], on the proof of Theorem C and on potential theory of log-log currents, which we develop in Section 4.4.1. In [57], we give an alternative proof, which uses the extension of Deligne-Mumford isomorphism.

Let's say a word about the way we prove this theorem. For simplicity, we suppose that there are no singular fibers, i.e. $|\Delta|=\emptyset$. Then our previous result, [54, Theorem A], permits us to compare the Quillen metric associated with the cusped metric and the Quillen metric associated with the compact metric. Once we apply [54, Theorem A] to a family of cusped Riemann surfaces, we get a family of compact Riemann surfaces. Then, after some easy calculations, we see that Theorem D reduces to the curvature theorem of Bismut-Gillet-Soulé [23, Theorem 1.9], which finishes the proof.

For a complete proof of the following two corollaries, see Section 4.4.

Corollary 4.1.7. We use the notation of Theorem C. Assume **S2** and **S3** hold. Then the trivial L^1-extension of the current (4.1.10) from Theorem D has a local continuous potential over S, which can be written explicitly through a product of the $\|\cdot\|_{\mathscr{L}_n}$-norm of a holomorphic frame of $\mathscr{L}_n$.

Corollary 4.1.8. We use the notation of Theorem C. Assume **S2** holds, and assume that the current (4.1.10) is smooth over S. Then the Hermitian norm $\|\cdot\|_{\mathscr{L}_n}$ on the line bundle $\mathscr{L}_n$ is smooth over S.

Remark 4.1.9. If h^{ξ} and $\|\cdot\|_{X/S}$ satisfy **S1**, and the forms $c_1(\xi,h^{\xi})$, $c_2(\xi,h^{\xi})$, $c_1(\omega_{X/S}(D),\|\cdot\|_{X/S})$ vanish in the neighbourhood of $\Sigma_{X/S}$, then the current (4.1.10) is smooth over S.

Now let's describe some of the applications of those results to the moduli space $\mathscr{M}_{g,m}$ of m-pointed stable curves of genus g, $2g-2+m>0$. We denote by $\overline{\mathscr{M}}_{g,m}$ the *Deligne-Mumford compactification* of $\mathscr{M}_{g,m}$, by $\partial\mathscr{M}_{g,m}:=\overline{\mathscr{M}}_{g,m}\setminus\mathscr{M}_{g,m}$ the *compactifying divisor*, by $\mathscr{C}_{g,m}$ and $\overline{\mathscr{C}}_{g,m}$ the universal curves over $\mathscr{M}_{g,m}$ and $\overline{\mathscr{M}}_{g,m}$ respectively. We denote by $\Pi:\overline{\mathscr{C}}_{g,m}\to\overline{\mathscr{M}}_{g,m}$ the universal projection. We denote by $D_{g,m}$ the divisor on $\overline{\mathscr{C}}_{g,m}$, formed by m fixed points. We denote

by $\omega_{g,m}$ the relative canonical line bundle of Π and by $\omega_{g,m}(D)$ the twisted relative canonical line bundle:

$$\omega_{g,m}(D) := \omega_{g,m} \otimes \mathscr{O}_{\overline{\mathscr{C}}_{g,m}}(D_{g,m}). \tag{4.1.12}$$

By the uniformization theorem (cf. [49, Chapter IV], [9, Lemma 6.2], [10]), we endow $\omega_{g,m}(D)$ with the Hermitian norm $\|\cdot\|_{g,m}^{\mathrm{hyp}}$, such that its restriction over each fiber of π induces by Construction 4.1.1 the canonical hyperbolic metric of constant scalar curvature -1. This endows the determinant line bundle $\lambda(j^*(\omega_{g,m}(D)^n))$, $n \leq 0$, which is also sometimes called the *Hodge line bundle*, with the induced Quillen metric $\|\cdot\|_{g,m}^{Q,n}$. Also the line bundle $\det(\Pi_*(\omega_{g,m}|_{|D_{X/S}|}))$ is endowed with the associated Wolpert norm $\|\cdot\|_{g,m}^{W}$. We denote by ω_{WP} the Weil-Petersson form over $\mathscr{M}_{g,m}$ (cf. Section 4.5.1). The following corollaries will be proved in Section 4.5.2.

Corollary 4.1.10. The norm

$$\|\cdot\|_{g,m}^{H,n} := (\|\cdot\|_{g,m}^{Q,n})^{12} \otimes (\|\cdot\|_{g,m}^{W})^{-1} \otimes \|\cdot\|_{\partial\mathscr{M}_{g,m}}^{\mathrm{div}} \tag{4.1.13}$$

on the line bundle

$$\lambda_{g,m}^{H,n} := \lambda(j^*(\omega_{g,m}(D)^n))^{12} \otimes (\det(\Pi_*(\omega_{g,m}|_{|D_{X/S}|})))^{-1} \otimes \mathscr{O}_{\overline{\mathscr{M}}_{g,m}}(\partial\mathscr{M}_{g,m}) \tag{4.1.14}$$

is good in the sense of Mumford with singularities along $\partial\mathscr{M}_{g,m}$. Moreover, it extends continuously over $\overline{\mathscr{M}}_{g,m}$ and it is smooth over $\mathscr{M}_{g,m}$.

Remark 4.1.11. It is possible to deduce the result of Corollary 1.2.14 from Deligne's isomorphism [42, Théorème 11.4], from Riemann-Roch arithmetic theorem for pointed stable surfaces, proved in this form by Gillet-Soulé in [66] (cf. [65, Proposition 1.5.2]) for $m = 0$, $n \leq 0$, by Freixas in [59, Theorem 6.2] for $m \in \mathbb{N}$, $n = 0$ and in [60, Theorem 6.2] for $n < 0$, $m \in \mathbb{N}$, and by "goodness" of the associated Deligne metric, proved by Freixas in [58, Theorem 5.2.1 and Remark 5.2.4]. Our proof is different because we get Corollary 4.1.10 directly from Theorem C2.

By Corollary 4.1.10 and Remark 4.1.4, we see that the first Chern form of $(\lambda_{g,m}^{H,n}, (\|\cdot\|_{g,m}^{H,n})^2)$ is well-defined as a current over $\overline{\mathscr{M}}_{g,m}$. Let's state the curvature theorem in this context.

Corollary 4.1.12. We use the notation from Corollary 4.1.10. The form ω_{WP} has log-log growth along the boundary $\partial\mathscr{M}_{g,m}$ of the moduli space of curves. We denote by $[\omega_{WP}]_{L^1}$ its L^1-extension to $\overline{\mathscr{M}}_{g,m}$. This extension is closed. Moreover, the following identity of currents over $\overline{\mathscr{M}}_{g,m}$ holds

$$c_1\big(\lambda_{g,m}^{H,n}, (\|\cdot\|_{g,m}^{H,n})^2\big) = -\pi^{-2}\big(6n^2 - 6n + 1\big)\big[\omega_{WP}\big]_{L^1}. \tag{4.1.15}$$

Remark 4.1.13. a) Recall that in [117, Theorem 5], Wolpert proved that $c_1(\det(\Pi_*(\omega_{g,m}|_{|D_{X/S}|})), (\|\cdot\|_{g,m}^{W})^2)$ is equal up to an explicit constant to a Kähler form on $\mathscr{M}_{g,m}$, which was defined previously by Takhtajan-Zograf in [107, (8)]. In [114, Corollary 5.11], Wolpert expressed ω_{WP} as an integral $\int_\pi c_1(\omega_{g,m}(D), (\|\cdot\|_{g,m}^{\mathrm{hyp}})^2)^2$. By those results, we see that Theorem D extends the curvature theorem of Takhtajan-Zograf [107, Theorem 1] from $\mathscr{M}_{g,m}$ to $\overline{\mathscr{M}}_{g,m}$. Our methods are very different from the methods of Takhtajan-Zograf, as we don't use the variational approach with Beltrami differentials.

b) The fact that the Weil-Petersson form has log-log growth along the boundary $\partial\mathscr{M}_{g,m}$ also follows from an old result of Masur [89, Theorem 1]. The fact that the current $[\omega_{WP}]_{L^1}$ is closed was previously proved by Wolpert in [113, Theorem 2.3]. See also the recent article of Melrose-Zhu [91] for related results.

c) It is possible to deduce the result of Corollary 4.1.12 from the Deligne isomorphism and the arithmetic Riemann-Roch theorem for pointed stable surfaces, proved by Gillet-Soulé and Freixas (see Remark 4.1.11), and some properties of "good" metrics, see Mumford [95, Proposition 1.2]. Our proof of Corollary 4.1.12 is very different. In fact, Corollary 4.1.12 follows directly from Theorems C2, D. Note also that in this case several technical difficulties disappear in the proof of Theorem D because the Weil-Petersson metric is smooth on $\mathscr{M}_{g,m}$. In particular, the potential theory for log-log type currents, which we develop in §4.4.1, is not necessary, because we can simply use the results of Mumford [95, Proposition 1.2].

As a direct consequence of Corollary 4.1.12 and Remark 4.1.4b), we get

Corollary 4.1.14. We use the notation from Corollary 4.1.12. The cohomological class of $\pi^{-2}\cdot\big[\omega_{WP}\big]_{L^1}$ in $H^2(\overline{\mathscr{M}}_{g,m},\mathbb{R})$ coincides with $c_1(\lambda_{g,m}^{H,n})$.

Remark 4.1.15. Corollary 4.1.14 is originally due to Wolpert, let's see how it follows from his results.

In [113, Theorem 1.3], Wolpert has shown that ω_{WP} can be written nicely in Fenchel-Nielsen coordinates. From this formula, he deduced in [113, §2] that the form ω_{WP} extends smoothly to the differential form ω_{WP}^{FN} over the space $\overline{\mathscr{M}}_{g,0}^{FN}$, which is homeomorphic to $\overline{\mathscr{M}}_{g,0}$, but with differential structure coming from Fenchel-Nielsen coordinates. Then, by studying the regularity of the application $i:\overline{\mathscr{M}}_{g,0}^{FN}\to\overline{\mathscr{M}}_{g,0}$, he concluded in [113, Theorem 4.1] that the cohomology class $[\omega_{WP}]$, induced by the current $[\omega_{WP}]_{L^1}$, and the cohomology class $[\omega_{WP}^{FN}]$, induced by the differential form ω_{WP}^{FN}, coincide.

Then in [111, Lemma 5.4], for $m=0$, Wolpert proved the identity $\pi^{-2}\cdot[\omega_{WP}^{FN}]=c_1(\lambda_{g,m}^{H,n})$. He did it by constructing explicitly $2+[g/2]$ analytic 2-cycles in $\overline{\mathscr{M}}_{g,0}$, which generate the cohomology group $H^{6g-8}(\overline{\mathscr{M}}_{g,0},\mathbb{R})$, and then he explicitly evaluated the intersection pairing of $[\omega_{WP}^{FN}]$ and $c_1(\lambda_{g,m}^{H,n})$ with those 2-cycles to show that $[\omega_{WP}]$ and $[\omega_{WP}^{FN}]$ coincide. The expression of ω_{WP} in terms of Fenchel-Nielsen coordinates plays a fundamental role in his calculation. As it was later remarked by Arbarello-Cornalba [7, end of §1], the reasoning of Wolpert works for any $m\in\mathbb{N}$. By combining all those results, we get Corollary 4.1.14 from the results of Wolpert.

We note that our proof is very different, and it doesn't appeals neither to Fenchel-Nielsen coordinates, neither to an explicit construction of the generators of $H^{6g-8}(\overline{\mathscr{M}}_{g,0},\mathbb{R})$.

Let's state the following corollaries, regarding the Weil-Petersson form.

Corollary 4.1.16. The Weil-Petersson form ω_{WP} has a local continuous potential.

Remark 4.1.17. Corollary 4.1.16 was originally proved by Wolpert in [112, §2]. He later used it to give a complex-analytic proof of the ampleness of Weil-Petersson form, which was originally proved by Knudsen-Mumford [76], [74], [75]. Our method of the proof is constructive, and doesn't use $\partial\overline{\partial}$-lemma, thus, it is very different from the non-constructive proof in [112, §2].

Corollary 4.1.18. We can decompose the Weil-Petersson form ω_{WP} as

$$\omega_{WP} = -\pi^2\alpha + d\beta, \tag{4.1.16}$$

for some smooth forms α, β over $\mathscr{M}_{g,m}$. Moreover, there is a smooth Hermitian metric h_{sm} on $\lambda_{g,m}^{H,0}$ over $\overline{\mathscr{M}}_{g,m}$ such that

$$\alpha = c_1(\lambda_{g,m}^{H,0}, h_{sm}), \tag{4.1.17}$$

and β, $d\beta$ have log-log growth along $\partial\overline{\mathscr{M}}_{g,m}$. In particular, we have

$$\int_{\mathscr{M}_{g,m}} \omega_{WP}^{\wedge(3g-3+m)} = (-\pi^2)^{3g-3+m} \int_{\overline{\mathscr{M}}_{g,m}} c_1(\lambda_{g,m}^{H,0})^{\wedge(3g-3+m)}. \tag{4.1.18}$$

As a consequence, we see that the left-hand side of (4.1.18) is a rational multiple of a power of π.

Remark 4.1.19. a) The decomposition (4.1.16) can be also deduced from studying the singularities of the Deligne metric on the Deligne-Weil product, as it was done implicitly by Freixas in his PhD thesis [58, Theorem 5.1.3].

b) The identity (4.1.18) was originally proved by Wolpert in [113, Corollary 5.3, Lemma 5.4], see also Remark 4.1.15. Our proof of (4.1.18) is very different.

c) Similarly, local index formulas have been used by Takhtajan-Zograf in [108, §5.3, §5.4] to give an alternative approach of computing symplectic volumes of the moduli spaces of parabolic bundles, which were originally calculated by Witten in [110, Formula 3.18].

Finally, let's describe our last application. For this, let's recall that Deligne in [42, §7] defined a holomorphic line bundle $\langle\omega_{g,m}(D), \omega_{g,m}(D)\rangle$ over $\overline{\mathscr{M}}_{g,m}$, which is now called the Deligne-Weil product. For $m = 0$, in [42, §8], he endowed it with the Hermitian norm $\|\cdot\|_{g,m}^{\mathrm{Del}}$, which is now called the Deligne norm. Later, Freixas in his PhD thesis [58, Theorem 5.1.3] generalized the construction of this norm for $m \in \mathbb{N}$. The natural isomorphism

$$\lambda_{g,m}^{H,0} \to \big\langle\omega_{g,m}(D), \omega_{g,m}(D)\big\rangle^{-1} \tag{4.1.19}$$

was constructed by Mumford in [96, p. 102] for $m = 0$. Then Freixas in [59, Theorem 3.10] extended it for $m \in \mathbb{N}$. Those isomorphisms are canonical and can be characterized uniquely up to a multiplication by -1 as the morphisms which respect $\mathbb{Z}$-structure of the corresponding line bundles, coming from the fact that $\overline{\mathscr{M}}_{g,m}$ and $\overline{\mathscr{C}}_{g,m}$ can be equipped with a structure of arithmetic varieties, for which Π is an arithmetic morphism in the sense of the book of Soulé [106]. see [76], [74], [65, §1.5], [59], [42].

When one applies the general isomorphism of Deligne [42, Construction 7.5] to the universal family of punctured stable curves, one gets an isomorphism

$$\lambda_{g,m}^{H,0} \to \lambda_{g,m}^{H,n} \otimes \big\langle\omega_{g,m}(D), \omega_{g,m}(D)\big\rangle^{6n^2-6n}. \tag{4.1.20}$$

For $m = 0$, the isomorphism (4.1.20) was also constructed by Mumford [96, Theorem 5.10 and p. 102]. The isomorphism (4.1.20) respects the induced $\mathbb{Z}$-structure. This is due to the fact that by [42], the construction of Deligne works for schemes over any ring and his construction is compatible with the base change.

Corollary 4.1.20. The isomorphisms (4.1.19), (4.1.20) over $\overline{\mathscr{M}}_{g,m}$ are isometries up to a constant when the line bundles are induced by $\|\cdot\|_{g,m}^{\mathrm{Del}}$ and $\|\cdot\|_{g,m}^{H,n}$.

Remark 4.1.21. a) For isomorphism (4.1.19), this fact was firstly proved for $m = 0$ up to an undetermined constant by Deligne in [42]. He relied extensively on the theory of Bismut-Gillet-Soulé [21], [22], [23]. Later, the constant was computed explicitly by Gillet-Soulé in [65, Proposition 1.5.2], [66] for $m = 0$, where authors relied heavily on arithmetic Riemann-Roch theorem. By using those results, Freixas in [59, Theorem 6.1] developed the arithmetic Riemann-Roch theorem for hyperbolic surfaces with cusps and proved that (4.1.19) is an isometry up to an explicit constant for any $m \in \mathbb{N}$. In his theory he used the version of the analytic torsion defined through the lengths of closed geodesics as in Takhtajan-Zograf [107]. He relied heavily on the results of the asymptotics of the Selberg Zeta Function due to Wolpert [115].

For isomorphism (4.1.20), this was firstly proved by Deligne [42, Théorème 11.4] in the case $m = 0$. He also proved that the constant up to which this isometry holds is 1. In [60, Theorem 6.1], Freixas extended the result of Deligne for any $m \in \mathbb{N}$ by using the definition of the analytic torsion through the lengths of closed geodesics. As we prove in our forthcoming paper [56], this definition of the analytic torsion coincides with our definition.

b) Although our methods do not determine precisely the norm of the isomorphism, they have an advantage of being a 1-paragraph consequences of the curvature theorem, Theorem D. In fact, from Theorem D we deduce that the norm of the isomorphisms (4.1.19), (4.1.20) is pluriharmonic over $\overline{\mathscr{M}}_{g,m}$. Thus, it is a constant. For more details, see §4.5.2.

Finally, let's mention that in the related work of Albin-Rochon [4], authors obtain local family index formula for the direct image $R^\bullet\pi_*(\omega_{g,m}(D)^n)$ over $\mathscr{M}_{g,m}$. They are able to do to because the dimension of the cohomology of the fiber does not vary in a family, and thus $R^\bullet\pi_*(\omega_{g,m}(D)^n)$ forms a holomorphic vector bundle over $\mathscr{M}_{g,m}$. In our situation, the cohomology of the fiber may vary, and similarly to [21], [22], [23], we work only with the first Chern form.

Now let's describe the plan of this article. In Section 2, we recall the basic definitions of the subject, we recall the definition of the Quillen metric, different notions of singularities of vector bundles and Bott-Chern forms. In Section 3, we prove an analytic proposition, which studies the singularities of a push-forward of a differential form in f.s.o. Then we use it to prove Theorem C. In Section 4, we develop potential theory for currents of log-log growth and then we prove Theorem D and Corollaries 4.1.7, 4.1.8. In Section 5, we recall the necessary prerequisites related to the moduli space of pointed stable curves and prove Corollaries 4.1.10, 4.1.12, 4.1.16, 4.1.18, 4.1.20.

Notation. For $\epsilon > 0$, we denote

$$D(\epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}, \qquad D^*(\epsilon) = \{z \in \mathbb{C} : 0 < |z| < \epsilon\}. \tag{4.1.21}$$

For a vector space E, we denote $\det E := \Lambda^{\max} E$.

For a holomorphic vector bundle ξ over a complex manifold X with a Hermitian metric h^ξ over X, the pair (ξ, h^ξ) is called a *Hermitian vector bundle* over X.

For a divisor $D_0 \subset Y$ in a complex manifold Y, we denote by s_{D_0} the canonical holomorphic section of $\mathscr{O}_Y(D_0)$, $\mathrm{div}(s_{D_0}) = D_0$, and by δ_{D_0} the current of integration along D_0. For a current T over $Y \setminus D_0$, which is in $L^1_{loc}(Y)$, we denote by $[T]_{L^1}$ the L^1 extension of T over Y.

Let $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, $q \in \mathbb{N}$ be a multi-index. We denote by $|\alpha| = \sum \alpha_i$.
Note. This part of the thesis can be found on the ArXiv, see [55].

4.2 Families of nodal curves and related notions

In this section we recall the relevant notations. More precisely, in Section 2.1, we recall the notion of the analytic torsion from the first paper of these series, [54] (for related notions of analytic torsions, see Lundelius [81], Jorgenson-Lundelius [70], [71]; Albin-Rochon [5], [4]; see also [54, §2.2] for a brief summary about the connections between those definitions). In Section 2.2, we recall the definition of holomorphic families of Riemann surfaces with ordinary singularities, we define the notion of a family of surfaces with cusps and the Wolpert norm on the restriction of the relative canonical line bundle to the cusps. In Section 2.3, we recall the properties of the determinant line bundle due to Grothendick-Knudsen-Mumford [76] and Bismut-Gillet-Soulé [23]. We also recall the definition of the Quillen norm for non-compact surfaces, which was done in [54]. In Section 2.4, we recall several notions of singularities of Hermitian metrics on holomorphic line bundles. In Section 2.5, we recall the theory of Bott-Chern currents for singular Hermitian metrics and prove some useful properties of those currents. Finally, in Section 2.6, we prove that the class of pre-log-log metrics of infinite order is not empty.

4.2.1 The analytic torsion

Let $\overline{M}$ be a compact Riemann surface, and let $D_M = \{P_1^M, \dots, P_m^M\}$ be a finite set of distinct points in $\overline{M}$. Let g^{TM} be a Kähler metric on the punctured Riemann surface $M = \overline{M} \setminus D_M$.

For $\epsilon \in]0, 1[$, $i = 1, \dots, m$, let $z_i^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon) = \{z \in \mathbb{C} : |z| \leq \epsilon\}$ be a local holomorphic coordinate around P_i^M, and

$$V_i^M(\epsilon) := \{x \in M : |z_i^M(x)| < \epsilon\}. \tag{4.2.1}$$

We say that g^{TM} is *Poincaré-compatible* with coordinates $z_1^M, \dots, z_m^M$ if for any $i = 1, \dots, m$, there is $\epsilon > 0$ such that $g^{TM}|_{V_i^M(\epsilon)}$ is induced by the Hermitain form

$$\frac{\sqrt{-1} dz_i^M d\overline{z}_i^M}{\left|z_i^M \ln |z_i^M|\right|^2}. \tag{4.2.2}$$

We say that g^{TM} is a *metric with cusps* if it is Poincaré-compatible with some holomorphic coordinates of D_M. A triple $(\overline{M}, D_M, g^{TM})$ of a Riemann surface $\overline{M}$, a set of punctures D_M and a metric with cusps g^{TM} is called a *surface with cusps* (cf. [93]).

From now on, we fix a surface with cusps $(\overline{M}, D_M, g^{TM})$ and a Hermitian vector bundle (ξ, h^ξ) over it. We denote by $\omega_M := T^{*(1,0)}\overline{M}$ the *canonical line bundle* over $\overline{M}$. We denote by $\|\cdot\|_M^\omega$ the norm induced on $\omega_{\overline{M}}$ by g^{TM} over M. Let $\mathscr{O}_M(D_M)$ be the line bundle associated with the divisor D_M. The *twisted canonical line bundle* is defined as

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{4.2.3}$$

The metric g^{TM} endows by Construction 4.1.1 the line bundle $\omega_M(D)$ with the induced Hermitian metric $\|\cdot\|_M$ over M.

We denote by $\square^{\xi\otimes\omega_M(D)^n}$ the Kodaira Laplacian associated with g^{TM} and $(\xi\otimes\omega_M(D)^n, h^\xi\otimes\|\cdot\|_M^{2n})$. In this article we only consider the action of $\square^{\xi\otimes\omega_M(D)^n}$ on the sections of degree 0.

We recall that for $m=0$, the analytic torsion was defined by Ray-Singer [103] as the regularized determinant of $\square^{\xi\otimes\omega_M(D)^n}$. More precisely, let $\lambda_i, i\in\mathbb{N}$ be the non-decreasing sequence of non-zero eigenvalues of $\square^{\xi\otimes\omega_M(D)^n}$. Classically, the associated zeta-function

$$\zeta_M(s) := \sum \lambda_i^s \tag{4.2.4}$$

is defined for $\mathrm{Re}(s)>1$, $s\in\mathbb{C}$ and it extends meromorphically to the entire s-plane. This extension is holomorphic at 0, and the *analytic torsion* is defined by

$$T(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)). \tag{4.2.5}$$

However, for $m>0$, the heat operator associated with $\square^{\xi\otimes\omega_M(D)^n}$ is no longer of trace class. Thus, the definition (4.2.5) is no longer applicable, but, as it was shown in the previous paper of this series [54, §2.2], by taking out the diverging part in the definition of the heat trace, we can extend the definition of zeta-function $\zeta_M(s)$ to define the *analytic torsion* $T(g^{TM}, h^\xi\otimes\|\cdot\|_M^{2n})$ for $m\geq 0$ by the same formula (4.2.5).

4.2.2 Families of nodal curves

In this section we recall the definition of a holomorphic family of Riemann surfaces with ordinary singularities (cf. Bismut-Bost [20]) and some of its properties. We introduce its cusped version, which we call a family of surfaces with cusps, and we also define the Wolpert norm.

Definition 4.2.1. A *holomorphic family of Riemann surfaces with ordinary singularities* is a holomorphic, proper, surjective map $\pi: X\to S$ of complex manifolds, such that for every $t\in S$, the space $X_t := \pi^{-1}(t)$ is a complex curve whose singularities are at most ordinary double points. As a shortcut, we will call π a f.s.o. (from french "famille à singularités ordinaires").

Proposition 4.2.2 ([20, Proposition 3.1])**.** Let $\pi: X\to S$ be a f.s.o. Then for every $x\in X$, there are local holomorphic coordinates $(z_0,\dots,z_q)$ of $x\in X$ and $(w_1,\dots,w_q)$ of $\pi(x)\in S$, such that π is locally defined by one of the following identities

$$w_i = z_i, \qquad \text{for} \quad i=1,\dots,q, \tag{4.2.6}$$

$$w_1 = z_0 z_1; \quad w_i = z_i, \qquad \text{for} \quad i=2,\dots,q. \tag{4.2.7}$$

Corollary 4.2.3 ([20, §3(a)])**.** Let $\pi: X\to S$ be a f.s.o., and let $\Sigma_{X/S}\subset X$ be the locus of double points of the fibers of π. Then the following holds

a) $\Sigma_{X/S}$ is a submanifold of X of codimension 2;

b) the map $\pi|_{\Sigma_{X/S}}: \Sigma_{X/S}\to S$ is a closed immersion;

c) the map $\pi|_{X\setminus\Sigma_{X/S}}: X\setminus\Sigma_{X/S}\to S$ is a submersion.

In particular, the direct image $\Delta=\pi_*(\Sigma_{X/S})$ is a divisor in S.

Notation 4.2.4. We use the notation Δ, $\Sigma_{X/S}$ for the divisor and the submanifold from Corrolary 4.2.3.

For a complex manifold X, we denote by Π_X the sheaf of holomorphic sections of the vector bundle $T^{*(1,0)}X$, and by ω_X the line bundle $\det(T^{*(1,0)}X)$.

Let's recall the construction of the *relative canonical* line bundle $\omega_{X/S}$ of a f.s.o. $\pi : X \to S$. Define the sheaf $\Pi_{X/S}$ by the exact sequence:

$$\pi^*\Pi_S \to \Pi_X \to \Pi_{X/S} \to 0. \tag{4.2.8}$$

By Corollary 4.2.3, the exact sequence (4.2.8) becomes exact to the left when restricted to $X \setminus \Sigma_{X/S}$:

$$0 \to \pi^*\Pi_S|_{X\setminus\Sigma_{X/S}} \to \Pi_X|_{X\setminus\Sigma_{X/S}} \to \Pi_{X/S}|_{X\setminus\Sigma_{X/S}} \to 0. \tag{4.2.9}$$

By taking determinants of (4.2.9), we deduce the isomorphism

$$\Pi_{X/S}|_{X\setminus\Sigma_{X/S}} = (\omega_X \otimes \pi^*\omega_S^{-1})|_{X\setminus\Sigma_{X/S}}. \tag{4.2.10}$$

We define

$$\omega_{X/S} := \omega_X \otimes \pi^*\omega_S^{-1}. \tag{4.2.11}$$

Then $\omega_{X/S}$ is the unique extension of $\Pi_{X/S}|_{X\setminus\Sigma_{X/S}}$ over X. This line bundle is called the *relative canonical* line bundle of $\pi : X \to S$.

Let $x \in \Sigma_{X/S}$. Take local coordinates $(z_0, \dots, z_q)$ on an open neighbourhood V of $x \in X$ and local coordinates $(w_1, \dots, w_q)$ of $\pi(x) \in S$, as in (4.2.7). Then the manifold $\Sigma_{X/S} \cap V$ is given by

$$\{z_0 = 0 \text{ and } z_1 = 0\}. \tag{4.2.12}$$

Consider the sections dz_0/z_0 and dz_1/z_1 of Π_X, defined over the sets $\{z_0 \neq 0\}$ and $\{z_1 \neq 0\}$ respectively. The images of dz_0/z_0 and $-dz_1/z_1$ in $\omega_{X/S}$ coincide over $\{z_0 z_1 \neq 0\}$, since

$$\frac{dz_0}{z_0} + \frac{dz_1}{z_1} = \pi^*\frac{dw_1}{w_1}. \tag{4.2.13}$$

Thus, they define a nowhere vanishing section σ of $\omega_{X/S}$ over $V \setminus \Sigma_{X/S}$. Since $\Sigma_{X/S}$ is of codimension 2, the section σ extends to a nowhere vanishing section over V of the line bundle $\omega_{X/S}$.

Definition 4.2.5 (Family of surfaces with cusps)**.** A *holomorphic family of Riemann surfaces with ordinary singularities and cusps* is a f.s.o. $\pi : X \to S$, disjoint sections $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ and a Hermitian metric $\|\cdot\|^{\omega}_{X/S}$ on $\omega_{X/S}$ over $\pi^{-1}(S \setminus |\Delta|) \setminus (\cup_i \operatorname{Im}(\sigma_i))$, such that for any $t \in S \setminus |\Delta|$, the restriction of $\|\cdot\|^{\omega}_{X/S}$ over $\pi^{-1}(t) \setminus (\cup_{i=1}^m \sigma_i(t))$ induces the Kähler metric g^{TX_t} over $X_t \setminus (\cup_{i=1}^m \sigma_i(t))$ such that the associated triple $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ becomes a surface with cusps. As a short-cut, we call $(\pi; \{\sigma_1, \dots, \sigma_m\}; \|\cdot\|^{\omega}_{X/S})$ a f.s.c.

In fact, all the results of this article (except for Proposition 4.2.23) are local over the base. Thus, we will always assume that one can write the divisor of cusps $D_{X/S}$ as in (4.1.1). We also denote by D_{X_t} the restriction of $D_{X/S}$ on $X_t := \pi^{-1}(t)$, $t \in S$.

Definition 4.2.6 (Wolpert norm)**.** Let $(\pi;\sigma_1,\dots,\sigma_m;\|\cdot\|^{\omega}_{X/S})$ be a f.s.c. Let $t\in S\setminus|\Delta|$, and let z_i be a holomorphic coordinate of $\sigma_i(t)\subset X_t$ such that (see (4.2.2))

$$\|dz_i\|^{\omega}_{X/S}=|z_i\ln|z_i||. \tag{4.2.14}$$

By the uniformization theorem, such a holomorphic coordinate is uniquely defined up to a multiplication by a unitary complex constant. We define the norm $\|\cdot\|^{W}_{i}$ on $\sigma_i^*(\omega_{X/S})$ pointwise by

$$\|\sigma_i^*dz_i\|^{W}_{i}(t)=1. \tag{4.2.15}$$

The *Wolpert norm* $\|\cdot\|^{W}_{X/S}$ is defined as the product norm on $\otimes_i\sigma_i^*(\omega_{X/S})$, induced by $\|\cdot\|^{W}_{i}$.

We stress out that in general, we make no claims about the continuity of $\|\cdot\|^{W}_{X/S}$.

Remark 4.2.7. Let $(\pi;\{\sigma_1,\dots,\sigma_m\};\|\cdot\|^{\omega,\mathrm{hyp}}_{X/S})$ be from Construction 4.5.2. Wolpert in [117, Definition 1] defined the norm $\|\cdot\|^{W,\mathrm{hyp}}_{X/S}$, which coincides with the norm from Definition 4.2.6 in this particular case. In [117, Theorem 5] he showed that $\|\cdot\|^{W,\mathrm{hyp}}_{X/S}$ is smooth over $S\setminus|\Delta|$.

4.2.3 Determinant line bundles and Quillen norms

In this section we recall the notion of the determinant line bundle due to Grothendick-Knudsen-Mumford [76] and then, similarly to Bismut-Gillet-Soulé [23], but basing on the definition of the analytic torsion from [54, Definition 2.17], we introduce the notion of the Quillen norm on the determinant line bundle.

Let $\pi:X\to S$ be a f.s.o., and let ξ be a holomorphic vector bundle over X. We denote

$$\det(R^{\bullet}\pi_*\xi)_t:=\det H^0(X_t,\xi)\otimes(\det H^1(X_t,\xi))^{-1},\quad t\in S, \tag{4.2.16}$$

where we identified ξ with its sheaf of holomorphic sections. By Grothendick-Knudsen-Mumford [76] (cf. [20, Proposition 4.1]) the family of complex lines $(\det(R^{\bullet}\pi_*\xi)_t)_{t\in S}$ is endowed with a natural structure of holomorphic line bundle $\det(R^{\bullet}\pi_*\xi)$ over S.

Now, suppose $(\pi;\{\sigma_1,\dots,\sigma_m\};\|\cdot\|^{\omega}_{X/S})$ is a f.s.c. For $t\in S\setminus|\Delta|$, we denote by dv_{X_t} the Riemannian volume form on $X_t\setminus(\cup_{i=1}^m\sigma_i(t))$, induced by $\|\cdot\|^{\omega}_{X/S}$ on the fiber X_t. Endow the twisted canonical line bundle $\omega_{X/S}(D)$ with the norm $\|\cdot\|_{X/S}$ from Construction 4.1.1. Let ξ be endowed with a Hermitian metric h^{ξ} over $\pi^{-1}(S\setminus|\Delta|)$. For $n\in\mathbb{Z},n\le 0$, we define the L^2-scalar product $\langle\cdot,\cdot\rangle_{L^2}$ on $\mathscr{C}^{\infty}(X_t,\xi\otimes\omega_{X/S}(D)^n)$ and $\mathscr{C}^{\infty}(X_t,\xi\otimes\omega_{X/S}(D)^n\otimes\overline{\omega}_{X/S})$ by

$$\langle s_1,s_2\rangle_{L^2}=\int_{X_t}\langle s_1(x),s_2(x)\rangle_h dv_{X_t}(x), \tag{4.2.17}$$

where s_1,s_2 are either in $\mathscr{C}^{\infty}(X_t,\xi\otimes\omega_{X/S}(D)^n)$ or in $\mathscr{C}^{\infty}(X_t,(\xi\otimes\omega_{X/S}(D)^n)^*\otimes\overline{\omega}_{X/S})$, and $\langle\cdot,\cdot\rangle_h$ is the pointwise Hermitian product induced by h^{ξ},$\|\cdot\|^{\omega}_{X/S}$ and $\|\cdot\|_{X/S}$. As we explained in [54, Section 2.1], the right-hand side of (4.2.17) is finite, and (4.2.17) defines the L^2-scalar product on the vector spaces $H^0(X_t,\xi\otimes\omega_{X/S}(D)^n)$, $H^1(X_t,\xi\otimes\omega_{X/S}(D)^n)$. We denote by $\|\cdot\|_{L^2}(g^{TX_t},h^{\xi}\otimes\|\cdot\|^{2n}_{X/S})$ the induced L^2-norm on the complex line $(\det(R^{\bullet}\pi_*(\xi\otimes\omega_{X/S}(D)^n))_t)^{-1}$.

Definition 4.2.8. The Quillen norm on the line bundle $(\det(R^\bullet\pi_*(\xi\otimes\omega_{X/S}(D)^n)))^{-1}$ over $S\setminus|\Delta|$ is defined for $t\in S\setminus|\Delta|$ by

$$\|\cdot\|_Q\left(g^{TX_t},h^\xi\otimes\|\cdot\|^{2n}_{X/S}\right):=T\left(g^{TX_t},h^\xi\otimes\|\cdot\|^{2n}_{X/S}\right)^{1/2}\cdot\|\cdot\|_{L^2}\left(g^{TX_t},h^\xi\otimes\|\cdot\|^{2n}_{X/S}\right). \tag{4.2.18}$$

4.2.4 Singular Hermitian vector bundles

In this section we recall several notions of singularities for Hermitian vector bundles. Then we show that one can actually characterize the Grothendick-Knudsen-Mumford determinant line bundle $\det(R^\bullet\pi_*(\xi\otimes\omega_{X/S}(D)^n))^{-1}$ as an extension of Bismut-Gillet-Soulé determinant line bundle $\lambda(j^*(\xi\otimes\omega_{X/S}(D)^n))$ through the singularities of the Quillen metric.

We work with a complex manifold Y of dimension $q+1$, a *normal crossing divisor* $D_0\subset Y$ and a submanifold $F\subset Y$.

Definition 4.2.9. A triple $(U;z_0,\dots,z_q;l)$ of an open set $U\subset Y$, coordinates $z_0,\dots,z_q:U\to\mathbb{C}$ and $l\in\mathbb{N}$ is called an *adapted chart* for D_0 (resp. F) at $x\in|D_0|$ (resp. $x\in F$) if $U=\{(z_0,\dots,z_q)\in\mathbb{C}^{q+1}:|z_i|<1,\text{ for all } i=0,\dots,q\}$ and $|D_0|\cap U$ (resp. $F\cap U$) is defined by $\{z_0\cdots z_l=0\}$ (resp. $\{z_0=0,\dots,z_l=0\}$).

Notation 4.2.10. Let $(U;z_0,\dots,z_q;l)$ be an *adapted chart* for D_0. We denote

$$d\zeta_k=\begin{cases}dz_k/(z_k\ln|z_k|), & \text{if}\quad 0\le k\le l,\\ dz_k, & \text{if}\quad l+1\le k\le q.\end{cases} \tag{4.2.19}$$

Definition 4.2.11. a) [36, Definition 2.17] A differential form over $Y\setminus|D_0|$ (resp. a locally bounded section of the wedge algebra on cotangent space) has *log-log growth of order* $k\in\mathbb{N}$ *(resp. weakly log-log growth) on* Y*, with singularities along* D_0, if it can be expressed as a linear combination of monomials constructed using $d\zeta_k,\overline{d\zeta_k}$, $k=0,\dots,q$ with coefficients $f\in\mathscr{C}^\infty(Y\setminus|D_0|)$ (resp. $f\in L^\infty(Y\setminus|D_0|)$) such that for any $\mathbf{k},\mathbf{k}'\in\mathbb{N}^{q+1}$, $|\mathbf{k}|+|\mathbf{k}'|\le K$ (resp. for $\mathbf{k},\mathbf{k}'=0$), for some adapted chart $(U;z_0,\dots,z_q;l)$ of D_0 at $x\in|D_0|$, and for some $C>0$, $p\in\mathbb{N}$, we have

$$\left|\frac{\partial^{|\mathbf{k}|}}{\partial z^{\mathbf{k}}}\frac{\partial^{|\mathbf{k}'|}}{\partial\overline{z}^{\mathbf{k}'}}f(z_0,\dots,z_q)\right|\le C|z|^{-|\mathbf{k}_0|-|\mathbf{k}'_0|}\prod_{k=0}^{l}\left(\ln|\ln|z_k||\right)^p+C, \tag{4.2.20}$$

where $\mathbf{k}_0,\mathbf{k}'_0\in\mathbb{N}^{l+1}$ are the projections of $\mathbf{k},\mathbf{k}'$ onto first $l+1$ components, and $\partial^{|\mathbf{k}|}/\partial z^{\mathbf{k}}$, $\partial^{|\mathbf{k}'|}/\partial\overline{z}^{\mathbf{k}'}$ are the multinomial notations for the differentiations. When we don't precise $k\in\mathbb{N}$, by our convention, this means $k=0$.

b) [58, Definition 2.1] A function $f:Y\setminus F\to\mathbb{C}$ has *log-log growth on* Y*, with singularities along* F (resp. *has logarithmic singularities along* F), if for any $x\in Y$, for some adapted chart $(U;z_0,\dots,z_q;l)$ of F at x, and for some $C>0$, $p\in\mathbb{N}$, we have

$$\begin{aligned}&|f(z_0,\dots,z_q)|\le C\Big(\ln\big|\ln\big(\max\nolimits_{k=0}^{l}\{|z_k|\}\big)\big|\Big)^p+C,\\ &\Big(\text{resp. }|f(z_0,\dots,z_q)|\le C\big|\ln\big(\max\nolimits_{k=0}^{l}\{|z_k|\}\big)\big|^p+C\Big).\end{aligned} \tag{4.2.21}$$

c) A function $f : Y \setminus |D_0| \to \mathbb{C}$ has *logarithmic singularities of order* $k \in \mathbb{N}$ *along* D_0, if $f \in \mathscr{C}^\infty(Y/|D_0|, \mathbb{C})$, and for any $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^{q+1}$, for some adapted chart $(U; z_0, \ldots, z_q; l)$ of D_0 at $x \in |D_0|$, and some $C > 0$, $p \in \mathbb{N}$, we have

$$\Big|\frac{\partial^{|\mathbf{k}|}}{\partial z^{\mathbf{k}}}\frac{\partial^{|\mathbf{k'}|}}{\partial \overline{z}^{\mathbf{k}'}} f(z_0, \ldots, z_q)\Big| \leq C|z|^{-|\mathbf{k}_0|-|\mathbf{k}'_0|}\prod_{k=0}^{l}|\ln|z_k||^p + C, \tag{4.2.22}$$

where $\mathbf{k}_0, \mathbf{k}'_0 \in \mathbb{N}^{l+1}$ are the projections of $\mathbf{k}, \mathbf{k}'$ onto first $l+1$ components.

d) [95, p. 240] A differential form over $Y \setminus |D_0|$ has *Poincaré growth on* Y*, with singularities along* D_0, if it can be expressed as a linear combination of monomials constructed using $d\zeta_k, \overline{d\zeta_k}$, $k = 0, \ldots, q$ with coefficients $f \in \mathscr{C}^\infty(Y \setminus |D_0|) \cap L^\infty(Y \setminus |D_0|)$.

e) A current T over $Y \setminus |D_0|$ has *log-log growth (resp. Poincaré growth) on* Y *with singularities along* D_0 if it is represented by integration of a L^1_{loc}-form and there is a form α with log-log growth (resp. Poincaré growth) on Y, with singularities along D_0, such that $-\alpha \leq T \leq \alpha$.

f) [37, Definition 7.1] A differential form α is *pre-log-log of order* $k \in \mathbb{N}$ *on* Y*, with singularities along* D_0, if α, $\partial\alpha$, $\overline{\partial}\alpha$, $\partial\overline{\partial}\alpha$ have log-log growth of order k on Y, with singularities along D_0. Again, when the order is not precised, by our convention, we suppose $k = 0$.

g) [58, Definition 2.14] A smooth function $f : Y \setminus |D_0| \to \mathbb{C}$ is *P-singular, with singularities along* D_0, if ∂f, $\overline{\partial} f$, $\partial\overline{\partial} f$ have Poincaré growth on Y, with singularities along D_0.

Proposition 4.2.12 (Burgos Gil-Kramer-Kühn [37, Proposition 7.6])**.** a) Any differential form $Y \setminus |D_0|$ with log-log growth with singularities along D_0 is locally integrable.

b) If α is pre-log-log form on Y with singularities along D_0, then

$$[d\alpha]_{L^1} = d[\alpha]_{L_1}. \tag{4.2.23}$$

Definition 4.2.13. Let L be a holomorphic line bundle over Y and let h^L be a smooth Hermitian metric on L over $Y \setminus (F \cup |D_0|)$. We say that the metric h^L *has log-log growth with singularities along* $F \cup |D_0|$ if for any local holomorphic frame υ of L, the function $\ln h(\upsilon, \upsilon)$, has log-log growth on Y, with singularities along $F \cup |D_0|$.

Definition 4.2.14 ([95, p. 242], [36, Definition 4.29] (cf. [58, Definition 3.1]))**.** Let ξ be a holomorphic vector bundle over Y and let h^ξ be a Hermitian metric on ξ over $Y \setminus |D_0|$. We say that the metric h^ξ *is pre-log-log of order* $k \in \mathbb{N}$ *(resp. good) with singularities along* D_0 if for any local holomorphic frame $e_1, \ldots, e_{\mathrm{rk}(\xi)}$ of ξ, the functions $h^\xi(e_i, e_j)$, $i, j \in 1, \ldots, \mathrm{rk}(\xi)$, $(\det(H))^{-1}$, for $H = (h^\xi(e_i, e_j))_{i,j=1}^{\mathrm{rk}(\xi)}$ have logarithmic singularities of order $k \in \mathbb{N}$ (resp. order 0), and the entries of the matrix $(\partial H)H^{-1}$, are pre-log-log of order $k \in \mathbb{N}$ (resp. P-singular), with singularities along D_0. Again, when the order is not precised, by our convention, we suppose $k = 0$.

For line bundles, we have the following easy criteria of pre-log-log and good conditions:

Proposition 4.2.15 ([58, Proposition 3.2])**.** Let L be a holomorphic line bundle over Y and let h^L be a smooth Hermitian metric on L over $Y \setminus |D_0|$. Then h^L is pre-log-log (resp. good) with singularities along D_0 if and only if for every local holomorphic frame υ of L over $U \subset Y$, the function $\ln h^L(\upsilon, \upsilon)$ is pre-log-log (resp. P-singular), with singularities along D_0.

The following proposition explains how nice vector bundles can be used to precise the extension of a given line bundle.

Proposition 4.2.16. Let $(L', h^{L'})$ be a continuous Hermitian line bundle over $Y \setminus |D_0|$. Then there is at most one holomorphic line bundle L over Y which extends L' in such way that the Hermitian metric $h^{L'}$ becomes nice with singularities along D_0 in the sense of Definition 4.1.3b).

Proof. The proof is similar to the proof of the statement about the *good vector bundles* proved by Mumford in [95, Proposition 1.3] with only one change: for an open $U \subset Y$, the frames of L are given by

$$\Big\{ s \in \mathscr{C}^\infty(U \setminus |D_0|, L') : \ln h^{L'}(s,s) \text{ has log-log singularities along } D_0 \Big\}. \tag{4.2.24}$$

From now on, the proof repeats [95, Proposition 1.3], and we leave it to the interested reader. □

Now we use the notation from Theorem C. Theorem C and Proposition 4.2.16 gives us a possibility to deduce the characterization through the Quillen metric of Grothendick-Knudsen-Mumford determinant line bundle $\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))^{-1}$ from the as an extension of Bismut-Gillet-Soulé determinant line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$.

Corollary 4.2.17. Suppose Assumption **S2** (resp. **S3**) hold. Then the line bundle $\mathscr{L}_n$ over S, is the only extension of the line bundle

$$\lambda\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big)^{12} \otimes \big(\det(\pi_*(\omega_{X/S}|_{|D_{X/S}|})\big)^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes \big(\det(\pi_*(\det \xi|_{|D_{X/S}|})\big)^6, \tag{4.2.25}$$

over $S \setminus |\Delta|$, for which the norm $\|\cdot\|_{\mathscr{L}_n}$ over $S \setminus |\Delta|$ is *nice* with singularities along Δ (resp. *continuous* over S).

Proof of Corollary 4.2.17 modulo Theorem C. By Theorems C2, C3 we see that the extension $\mathscr{L}_n$ satisfies the required properties. Now, Proposition 4.2.16 shows that there is at most one extension, which finishes the proof. □

4.2.5 Bott-Chern currents and pre-log-log Hermitian vector bundles

In this section we recall the theory of Bott-Chern currents associated with pre-log-log Hermitian vector bundles, which was implicit in the paper [36] and which extends previous work of Bismut-Gillet-Soulé [21, Theorem 1.29]. We work with a complex manifold Y, and a normal crossing divisor $D_0 \subset Y$.

Now, let's define the vector space

$$P_{\mathrm{pll}}(Y, D_0) := \oplus_i P_{\mathrm{pll}}^{(i,i)}(Y, D_0), \tag{4.2.26}$$

where $P^{(i,j)}_{\mathrm{pll}}(Y, D_0)$ are the vector spaces of pre-log-log differential forms on Y of degree (i, j) with singularities along D_0. We denote

$$\begin{aligned} &P'_{\mathrm{pll}}{}^{(i,i)}(Y, D_0) = \partial(P^{(i-1,i)}_{\mathrm{pll}}(Y, D_0)) + \overline{\partial}(P^{(i,i-1)}_{\mathrm{pll}}(Y, D_0)), \\ &P'_{\mathrm{pll}}(Y, D_0) := \oplus_i P'_{\mathrm{pll}}{}^{(i,i)}(Y, D_0). \end{aligned} \tag{4.2.27}$$

Then, by Definition 4.2.11f), we have $P'_{\mathrm{pll}}(Y, D_0) \subset P_{\mathrm{pll}}(Y, D_0)$.

Let $\phi : \mathrm{M}_k(\mathbb{C}) \to \mathbb{C}$ be a polynomial map, which is invariant under conjugation. For a Hermitian vector bundle (ξ, h^ξ) over Y, we denote by $\phi(\xi, h^\xi)$ the differential form, constructed by $\phi(\xi, h^\xi) = \phi(-\frac{1}{2\pi\sqrt{-1}} \cdot R^{h^\xi})$, where R^{h^ξ} is the curvature of the Chern connection on (ξ, h^ξ).

Proposition 4.2.18. Let h^ξ be a Hermitian metric on ξ, which is pre-log-log of order $k \in \mathbb{N}$ (resp. good) over $Y \setminus D_0$ with singularities along D_0. Then the form $\phi(\xi, h^\xi)$ is pre-log-log of order k (resp. good) on Y with singularities along D_0, and the induced current $[\phi(\xi, h^\xi)]_{L^1}$ represents the cohomology class of $\phi(\xi) \in H^{2k}(Y, \mathbb{C})$.

Proof. This was proved for good Hermitian metrics by Mumford in [95, Theorem 1.4]. The proof for pre-log-log Hermitian metrics remains identical, see [37, Proposition 7.25]. □

We consider a short exact sequence of vector bundles over Y, endowed with Hermitian metrics over $Y \setminus D_0$, which are pre-log-log with singularities along D_0.

$$0 \longrightarrow (E_0, h^{E_0}) \longrightarrow (E_1, h^{E_1}) \longrightarrow (E_2, h^{E_2}) \longrightarrow 0. \tag{4.2.28}$$

Theorem 4.2.19. *For ϕ as above, there is a unique way to attach to every exact sequence $(E_\bullet, h^{E_\bullet})$ as in (4.2.28) a class $\tilde{\phi}(E_\bullet, h^{E_\bullet}) \in P_{\mathrm{pll}}(Y, D_0)/P'_{\mathrm{pll}}(Y, D_0)$ such that*

a) $\frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}\tilde{\phi}(E_\bullet, h^{E_\bullet}) = \phi(E_0 \oplus E_2, h^{E_0} \oplus h^{E_2}) - \phi(E_1, h^{E_1})$.

b) If (4.2.28) induces an isometry $(E_1, h^{E_1}) = (E_0 \oplus E_2, h^{E_0} \oplus h^{E_2})$, then $\tilde{\phi}(E_\bullet, h^{E_\bullet}) = 0$.

c) If Y' is another complex manifold, D'_0 is a normal crossing divisor in Y', and $f : Y' \to Y$ is a holomorphic map such that $f^{-1}(D_0) \subset D'_0$, then $\tilde{\phi}(f^*E_\bullet, f^*h^{E_\bullet}) = f^*\tilde{\phi}(E_\bullet, h^{E_\bullet})$.

d) For a Hermitian exact square

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E^0_0, h^{E_0}) & \longrightarrow & (E^0_1, h^{E^0_1}) & \longrightarrow & (E^0_2, h^{E^0_2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E^1_0, h^{E^1_0}) & \longrightarrow & (E^1_1, h^{E^1_1}) & \longrightarrow & (E^1_2, h^{E^1_2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E^2_0, h^{E^2_0}) & \longrightarrow & (E^2_1, h^{E^2_1}) & \longrightarrow & (E^2_2, h^{E^2_2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.2.29}$$

we have in $P_{\mathrm{pll}}(Y, D_0)/P'_{\mathrm{pll}}(Y, D_0)$*:*

$$\tilde{\phi}(E^1_\bullet, h^{E^1_\bullet}) - \tilde{\phi}(E^0_\bullet \oplus E^2_\bullet, h^{E^0_\bullet} \oplus h^{E^2_\bullet}) - \tilde{\phi}(E^\bullet_1, h^{E^\bullet_1}) + \tilde{\phi}(E^\bullet_0 \oplus E^\bullet_2, h^{E^\bullet_0} \oplus h^{E^\bullet_2}) = 0. \quad (4.2.30)$$

Proof. This theorem was proved in [36, Theorem 4.64] for the pre-log-log vector bundles of infinite order, but their proof remains valid for pre-log-log vector bundles. This is due to the fact that in [36, Theorem 4.64], the estimates of the *higher* derivatives of the underlying metric are only used in the estimates of *higher* derivatives of the Bott-Chern form. □

Remark 4.2.20. a) By [21, Theorem 1.29], we see that if h^{E_i} are smooth, then $\tilde{\phi}(E_\bullet, h^{E_\bullet})$ coincides with the Bott-Chern form constructed in [21].

b) When the exact sequence $(\xi_\bullet, h^{\xi_\bullet})$ consists of two elements

$$(\xi_\bullet, h^{\xi_\bullet}) : 0 \longrightarrow (\xi, h^\xi) \longrightarrow (\xi, h^\xi_0) \longrightarrow 0, \quad (4.2.31)$$

we denote for simplicity

$$\tilde{\phi}(\xi, h^\xi, h^\xi_0) := \tilde{\phi}(\xi_\bullet, h^{\xi_\bullet}). \quad (4.2.32)$$

By Theorem 4.2.19a), we have

$$\frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\tilde{\phi}(\xi, h^\xi, h^\xi_0) = \phi(\xi, h^\xi) - \phi(\xi, h^\xi_0). \quad (4.2.33)$$

In this paper we will only need to consider the short exact sequences consisting of 2 terms. Nevertheless, we state the theorem for general short exact sequences, since in the forthcoming paper [57] about Deligne-Mumford isometry, we use it in full generality.

c) It's disputable if one needs the last axiom. In the original set of the axioms [21, Theorem 1.29] for smooth metrics, the last axiom is shown to be a consequence of the first three [21, Theorem 1.20, Corollary 1.30].

Proposition 4.2.21. Let $(E_\bullet, h^{E_\bullet})$ be the exact sequence from (4.2.28). Taking into account the isomorphism $\det E_1 \simeq \det E_0 \otimes \det E_2$, we have

$$\widetilde{\mathrm{ch}}(E_\bullet, h^{E_\bullet})^{[0]} = \ln\det\big((h^{E_0}h^{E_2})/h^{E_1}\big). \quad (4.2.34)$$

Proof. First of all, we have

$$\widetilde{\mathrm{ch}}(E_\bullet, h^{E_\bullet})^{[0]} = \widetilde{c}_1(E_\bullet, h^{E_\bullet}). \quad (4.2.35)$$

By the uniqueness of the Bott-Chern classes from Theorem 4.2.19, it is enough to see that (4.2.34) satisfies the requirements of Theorem 4.2.19 for $\phi = \mathrm{Tr}$, i.e. representing the first Chern form. □

Proposition 4.2.22. Let ξ be a holomorphic vector bundle over Y, and let h^ξ, h^ξ_0 be pre-log-log Hermitian metrics on ξ with singularities along D_0. Then

$$\widetilde{\mathrm{ch}}(\xi, h^\xi, h^\xi_0)^{[0]} = \ln\det(h^\xi/h^\xi_0). \quad (4.2.36)$$

Moreover, if ξ is of rank 1, then we have

$$\widetilde{\mathrm{ch}}(\xi, h^\xi, h^\xi_0)^{[2]} = \frac{1}{2}\ln\det(h^\xi/h^\xi_0)\cdot\Big(c_1(\xi, h^\xi_0) + c_1(\xi, h^\xi)\Big). \quad (4.2.37)$$

Proof. First of all, the identity (4.2.36) follows directly from Proposition 4.2.21.

Now, we note that for smooth metrics (4.2.37) follows from [21, Theorems 1.27, 1.30]. To prove (4.2.37) in full generality, we point out that two pre-log-log Hermitian metrics h_0^ξ, h_1^ξ on holomorphic *line* bundles can be joined by a family of uniformly pre-log-log Hermitian line bundles h_t^ξ such that $(h_t^\xi)^{-1}\partial_t h_t^\xi$ has uniform log-log growth. Take for example $h_t^\xi := (h^\xi)^t (h_{\mathrm{sm}}^\xi)^{1-t}$ for some smooth Hermitian metric h_{sm}^ξ. Thus, the construction from [21, §e)] works perfectly well for pre-log-log Hermitian *line* bundles. Thus, the reasoning of [21, Theorems 1.27, 1.30] still holds in the pre-log-log case, and this implies (4.2.37) for pre-log-log Hermitian metrics. □

4.2.6 Existence of pre-log-log metrics of infinite order

In this section we prove that the class of pre-log-log metrics of infinite order is not empty. More formally, we prove

Proposition 4.2.23. For any f.s.o. $\pi : X \to S$, a divisor $D_{X/S}$ in X induced by a submanifold $|D_{X/S}|$ intersecting singular fibers transversally and such that $\pi|_{|D_{X/S}|} : |D_{X/S}| \to S$ is a local isomorphism, there is a metric $\|\cdot\|_{X/S}$ on $\omega_{X/S}(D)$ over $X \setminus |D_{X/S}|$ which is pre-log-log of infinite order with singularities along $D_{X/S}$.

The idea of the proof is as follows. We would like to construct a *smooth* function $h : U \to \mathbb{C}$, defined in a neighbourhood U around $|D_{X/S}|$ such that its restriction over each fiber becomes a local *holomorphic coordinate*. If such a function exists, we can construct the needed metric by gluing a smooth metric away from $|D_{X/S}|$ and a metric over the fibers obtained by the restriction induced by the 2-form

$$\frac{\sqrt{-1}dhd\overline{h}}{|h\log|h||^2} \tag{4.2.38}$$

However, it's quite easy to see that the existence of such a function would imply that the normal vector bundle to $|D_{X/S}|$ in X is trivial. Thus, in general, such h does not exist. Nevertheless, we prove that such a function h exists, if we allow it to be multivalued with respect to a multiplication by a unimodular function over the base.

To make it formal, let's introduce some notation. For any open $U \subset X$, we define the set

$$\mathscr{C}^\infty_{\pi,\mathrm{hol}}(U,\mathbb{C}) = \Big\{ f \in \mathscr{C}^\infty(U,\mathbb{C}) : f \text{ is holomorphic along the fibers of } \pi \Big\}. \tag{4.2.39}$$

Trivially, the multiplicative group $\mathscr{C}^\infty(\pi(U), S^1)$ acts on $\mathscr{C}^\infty_{\pi,\mathrm{hol}}(U,\mathbb{C})$ by multiplication. Here we view S^1 injected in $\mathbb{C}$ as a unit circle.

For any open subset $U \subset X$, we define the set

$$\mathscr{A}_\pi(U) = \mathscr{C}^\infty_{\pi,\mathrm{hol}}(U,\mathbb{C})/\mathscr{C}^\infty(\pi(U), S^1). \tag{4.2.40}$$

By standard partition of unity argument, $\mathscr{A}_\pi(U)$ form a sheaf over X, which we denote $\mathscr{A}_\pi$.

Lemma 4.2.24. There is a neighborhood V of $|D_{X/S}|$ and $h \in \mathscr{A}_\pi(V)$ such that $\{h = 0\} = D_{X/S}$.

Before giving a proof let's see how it helps us to prove Proposition 4.2.23.

Proof of Proposition 4.2.23. Let V and h be as in Lemma 4.2.24. Then although the 2-form (4.2.38) is not well-defined, its restriction over the fibers is perfectly well-defined. We denote by $\|\cdot\|^{\mathrm{sing}}_{X/S}$ the norm on $\omega_{X/S}(D)$ over $V \setminus |D_{X/S}|$ induced by the restriction of this form over each fiber. Trivially, it is pre-log-log of infinite order with singularities along $D_{X/S}$.

Let $\|\cdot\|^{\mathrm{sm}}_{X/S}$ be a smooth metric on $\omega_{X/S}(D)$ over $X \setminus U$ for some $U \Subset V$, $|D_{X/S}| \subset U$. By using partition of unity, we glue the norms $\|\cdot\|^{\mathrm{sm}}_{X/S}$ and $\|\cdot\|^{\mathrm{sing}}_{X/S}$ to get a norm $\|\cdot\|_{X/S}$ on $\omega_{X/S}(D)$ over $X \setminus |D_{X/S}|$. As $\|\cdot\|^{\mathrm{sing}}_{X/S}$ is pre-log-log of infinite order with singularities along $D_{X/S}$, the norm $\|\cdot\|_{X/S}$ is pre-log-log of infinite order with singularities along $D_{X/S}$ as well. Moreover, as h is holomorphic over the fibers, the norm $\|\cdot\|_{X/S}$ induces Kähler metric along the fibers with cusps at $D_{X/S}$. Thus, the norm $\|\cdot\|_{X/S}$ satisfies the hypothesizes of Proposition 4.2.23. □

Proof of Lemma 4.2.24. We will construct such neighborhood V and such section h explicitly. We choose some covering $\{U_i, i \in I\}$ of S. Let V_i be a neighborhood of $D_{X/S} \cap \pi^{-1}(U_i)$ such that $\pi(V_i) = U_i$ and there are local coordinates $(z_0^i, z_1^i, \dots, z_q^i)$ of V_i such that

$$\pi(z_0^i, z_1^i, \dots, z_q^i) = (z_1^i, \dots, z_q^i). \tag{4.2.41}$$

We suppose that V_i are small enough so that for any $i, j \in I$, the image of the functions z_0^i/z_0^j over $V_i \cap V_j$ is contained in some proper sector of $\mathbb{C}$.

We denote $f_i := z_0^i$, and consider it as an element in $\mathscr{A}_\pi(V_i)$. Let ρ_i be a partition of unity in S subordinate to $\{U_i, i \in I\}$. We define $h_i \in \mathscr{A}_\pi(V_i)$ by

$$h_i = f_i \cdot \prod_{\substack{j \in I \\ V_j \cap V_i \neq \emptyset}} \left(\frac{f_j}{f_i}\right)^{\rho_j \circ \pi}. \tag{4.2.42}$$

Here, due to our supposition on the image of z_0^j/z_0^i, the function $(f_j/f_i)^{\rho_j \circ \pi}$ is well-defined over $V_i \cap V_j$ up to a multiplication by a function, which is a pull-back of a smooth function over S. By this and the definition of the partition of unity, we see that $h_i \in \mathscr{A}_\pi(V_i)$ glue into an element $h \in \mathscr{A}_\pi(\cup_{i \in I} V_i)$, which satisfies the assumptions of the lemma. □

4.3 Regularity and singularities: a proof of Theorem C

In this section we prove Theorem C. More precisely, in Section 3.1, we prove a technical proposition about the regularity of a push-forward of a differential form in f.s.o. In Section 3.2, we recall the necessary prerequisites for the proof of Theorem C: the *compactification theorem* [54, Theorem A], the *anomaly formula for surfaces with cusps* [54, Theorem B], and the result of Bismut-Bost [20, Théorème 2.2], describing the asymptotics of the Quillen norm associated with a smooth metric over the total space near the singular fibers. In Section 3.3, we use it to prove Theorem C.

4.3.1 Pushforward of differential forms in f.s.o.

In this section we will study the singularities of a pushforward of a differential form in f.s.o. This study will be used extensively in the proof of Theorem C. The main result of this section is

Proposition 4.3.1. Let $\pi : X \to S$ be a f.s.o., and let $D \subset X$ be a divisor intersecting $\pi^{-1}(\Delta)$ transversally and such that $\pi|_D : D \to S$ is locally an isomorphism.

a) Let α be a smooth 2-form over $X \setminus |D|$ with log-log growth of infinite order along D. Then the function $\pi_*[\alpha]$ over $S \setminus |\Delta|$ is *very nice* with singularities along Δ (cf. Definition 4.1.3).

b) Suppose that Δ has normal crossings. Let α be a pre-log-log differential 2-form over $X \setminus (\pi^{-1}(|\Delta|) \cup |D|)$, with singularities along $\pi^{-1}(\Delta) \cup D$. Then the function $\pi_*[\alpha]$ over $S \setminus |\Delta|$ is *nice* with singularities along Δ (cf. Definition 4.1.3).

c) Suppose that Δ has normal crossings. Let α be a differential $(1,1)$-form over $X \setminus (\Sigma_{X/S} \cup |D|)$, such that it has Poincaré growth on $X \setminus |D|$ with singularities along $\pi^{-1}(\Delta)$, and the coupling of α with smooth vertical vector fields over $X \setminus (\Sigma_{X/S} \cup |D|)$ is continuous and has log-log growth with singularities along $|D|$ (cf. Definition 4.2.11b)). Let $f : X \setminus (\Sigma_{X/S} \cup |D|) \to \mathbb{R}$ be a continuous function, with log-log growth along $\Sigma_{X/S} \cup |D|$. Then the function $\pi_*[f\alpha]$ extends continuously over S.

Proof. **The Proposition 4.3.1a)** was proved by Igusa [69] in the case $D = \emptyset$ (for precisely this version, see Bismut-Bost [20, Théorèmes 12.2, 12.3]). Now let's describe the proof for $D \neq \emptyset$.

We take $t_0 \in S$, and let $U \subset S$ be a small neighbourhood of t_0 such that $\pi|_D : D \to S$ is an isomorphism on each connected component over $\pi^{-1}(U)$. For simplicity, we suppose that $D|_{\pi^{-1}(U)}$ has only one connected component. We choose coordinates $(z_0, \ldots, z_q)$ of $(\pi|_D)^{-1}(t_0) \in \pi^{-1}(U)$ and $(w_1, \ldots, w_q)$ of U as in (4.2.6), such that D is given by the equation $\{z_0 = 0\}$ in $\pi^{-1}(U)$.

For $c > 0$ small enough, we denote $V_{1,c} = \{x \in U : |z_0(x)| < c\}$, and decompose the integration over the fiber in $\pi_*[\alpha]$ into two parts: $(\pi|_{V_{1,c}})_*[\alpha]$ and $(\pi|_{\pi^{-1}(U)\setminus V_{1,c}})_*[\alpha]$. The function $(\pi|_{\pi^{-1}(U)\setminus V_{1,c}})_*[\alpha]$ induces *very nice* Hermitian metric on $S \setminus |\Delta|$ with singularities along Δ by the mentioned result of Igusa.

Let's treat the first part. Trivially, for any $p \in \mathbb{Z}$, we have

$$\int_{|z_1|<c} \frac{(\ln|\ln|z_1||)^p \sqrt{-1} dz_1 d\bar{z}_1}{|z_1 \ln|z_1||^2} = 4\pi \int_0^c \frac{(\ln|\ln(r)|)^p dr}{r(\ln(r))^2}$$
$$= 4\pi \int_0^{-2/\ln(c)} \ln(y)^p dy < +\infty. \quad (4.3.1)$$

Since α is pre-log-log over $\pi^{-1}(U) \setminus |D|$, with singularities along D, by Lebesgue dominated convergence theorem and (4.3.1), we deduce that

$$(\pi|_{V_{1,c}})_*[\alpha] \quad \text{is continuous.} \quad (4.3.2)$$

By taking horizontal derivatives with respect to the coordinates entering the definition of log-log growth of *infinite* order of α, we deduce in the same way that the form $(\pi|_{V_{1,c}})_*[\alpha]$ is smooth, which concludes the proof.

Now let's prove 4.3.1b). It is essentially the repetition of the proof of [58, Theorem 5.1.3] from the thesis of Freixas.

We take $t_0 \in S \setminus |\Delta|$, and let $U \subset S \setminus |\Delta|$ be a small neighbourhood of t_0 as in the previous case. As before, we suppose for simplicity that $D|_{\pi^{-1}(U)}$ has only one connected component. We choose coordinates $(z_0, \ldots, z_q)$ of $(\pi|_D)^{-1}(t_0) \in \pi^{-1}(U)$ and $(w_1, \ldots, w_q)$ of U as before, and let $V_{1,c}$ be defined as above. We decompose the integration over the fiber in $\pi_*[\alpha]$ into two parts: $(\pi|_{V_{1,c}})_*[\alpha]$ and $(\pi|_{\pi^{-1}(U)\setminus V_{1,c}})_*[\alpha]$.

Trivially, as $\pi|_{\pi^{-1}(U)\setminus V_{1,c}}$ is a submersion, and the form α is continuous over $\pi^{-1}(U) \setminus V_{1,c}$ for any $c > 0$, the form $(\pi|_{\pi^{-1}(U)\setminus V_{1,c}})_*[\alpha]$ is continuous over U for any $c > 0$. By this and the identity

$$\lim_{c\to 0}(\pi|_{V_{1,c}})_*[\alpha] = 0, \tag{4.3.3}$$

which follows from (4.3.1), we conclude that $\pi_*[\alpha]$ is continuous over $S \setminus |\Delta|$. Let's prove that $\pi_*[\alpha]$ has log-log growth along Δ.

We fix $t_0 \in |\Delta|$. For simplicity, we suppose that the curve $X_{t_0} = \pi^{-1}(t_0)$ has only one double point singularity at $x_0 \in X_{t_0}$. We choose coordinates $(z_0, \ldots, z_q)$ at $x_0 \in X$ and $(w_1, \ldots, w_q)$ at $t_0 \in S$ as in (4.2.7). For $c > 0$ small enough, we denote $U = \{t \in S : |w_1| < c\}$ and

$$V_{2,c} = \{x \in \pi^{-1}(U) : |z_0(x)|, |z_1(x)| < c\}. \tag{4.3.4}$$

Let's prove that $(\pi|_{V_{2,c}})_*[\alpha]$ has log-log growth along Δ. The divisor $\pi^{-1}(\Delta)$ is given over $V_{2,c}$ by equations $\{z_0 = 0\} + \{z_1 = 0\}$. Let $c < 1/2$. We note that since $z_0 z_1 = w_1$, the estimates

$$\ln|\ln|z_0||, \ln|\ln|z_1|| \leq \ln|\ln|w_1||, \tag{4.3.5}$$

are valid in $V_{2,c}$. By (4.3.5), there is a function $f : S \setminus |\Delta| \to \mathbb{R}$ with log-log growth along Δ such that function $(\pi|_{V_{2,c}})_*[\alpha]$ is bounded by

$$f \cdot \int_{H_{w_1,c}} \left(\frac{\sqrt{-1}dz_0 d\overline{z}_0}{|z_0 \ln|z_0||^2} + \frac{|dz_0 d\overline{z}_1|}{|z_0 \ln|z_0|||z_1 \ln|z_1||} + \frac{\sqrt{-1}dz_1 d\overline{z}_1}{|z_1 \ln|z_1||^2} \right), \tag{4.3.6}$$

for $H_{w_1,c} = \{(z_0, z_1) : z_0 z_1 = w_1; |z_0|, |z_1| < c\}$. Trivially, there is $C > 0$ such that for any $|\omega_1| < c^2$, we have

$$\int_{H_{w_1,c}} \frac{|dz_0 d\overline{z}_1|}{|z_0 \ln|z_0|||z_1 \ln|z_1||} \leq \int_{H_{w_1,c}} \frac{\sqrt{-1}dz_0 d\overline{z}_0}{|z_0 \ln|z_0||^2} < C. \tag{4.3.7}$$

By (4.3.6) and (4.3.7), we conclude that $(\pi|_{V_{2,c}})_*[\alpha]$ has log-log growth along Δ.

Now, as before, for simplicity, we suppose that $D|_{\pi^{-1}(U)}$ has only one connected component. We choose coordinates $(z_0, \ldots, z_q)$ of $\pi^{-1}(U)$ and $(w_1, \ldots, w_q)$ of U as in (4.2.6), and we conserve the notation $V_{1,c}$ from the previous step. Moreover, we suppose that $\pi^{-1}(\Delta)$ is given by the equation $\{z_1 = 0\}$ over $V_{1,c}$. From (4.3.1), the fact that α has log-log growth along $\pi^{-1}(\Delta) \cup D$, which is given by $\{z_0 z_1 = 0\}$ in U, and the fact that $\pi|_{V_{1,c}}$ is a submersion, we prove that $(\pi|_{V_{1,c}})_*[\alpha]$ has log-log growth along Δ.

Finally, as $\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})}$ is a submersion, and the form α has log-log growth along $\pi^{-1}(\Delta)$, the form $(\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})})_*[\alpha]$ has log-log growth along Δ. Thus, we deduce that $\pi_*[\alpha]$ has log-log growth along Δ.

Now, to prove that $\pi_*[\alpha]$ is nice, we have to study the distributional derivatives $\partial[\pi_*[\alpha]]_{L_1}$, $\overline{\partial}[\pi_*[\alpha]]_{L_1}$, $\partial\overline{\partial}[\pi_*[\alpha]]_{L_1}$. Let's concentrate on the study of $\partial[\pi_*[\alpha]]_{L_1}$, as the others are similar. First of all, by Fubini theorem, since $\pi^{-1}(\Delta)$ is Lebesgue negligible, we have

$$\big[\pi_*[\alpha]\big]_{L_1} = \pi_*\big[[\alpha]_{L_1}\big]. \tag{4.3.8}$$

By Stokes theorem, we have

$$\partial\pi_*\big[[\alpha]_{L_1}\big] = \pi_*\big[\partial[\alpha]_{L_1}\big]. \tag{4.3.9}$$

By the fact that α is pre-log-log, and by Proposition 4.2.12, we have

$$\partial[\alpha]_{L_1} = [\partial\alpha]_{L_1}. \tag{4.3.10}$$

Thus, by (4.3.8), (4.3.9) and (4.3.10), we see that it is enough to prove that the differential form $\pi_*[\partial\alpha]$ over $S\setminus|\Delta|$ is continuous and has log-log growth along Δ. The continuity over $S\setminus|\Delta|$ is proved as before. Let's prove that it has log-log growth along Δ. As before, we decompose the integration $\pi_*[\partial\alpha]$ into three parts: $(\pi|_{V_{1,c}})_*[\partial\alpha]$, $(\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})})_*[\partial\alpha]$ and $(\pi|_{V_{2,c}})_*[\partial\alpha]$. Since $\pi|_{V_{1,c}}$ and $\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})}$ are submersions, the first two parts are treated in the same way as before. Let's concentrate on the last part $(\pi|_{V_{2,c}})_*[\partial\alpha]$.

By (4.3.5) and (4.3.7), similarly to (4.3.6), there is a function $f : S\setminus|\Delta|\to\mathbb{R}$ with log-log growth along Δ such that the form $(\pi|_{V_{2,c}})_*[\partial\alpha]$ is bounded by

$$f\cdot\int_{H_{w_1,c}}\frac{\sqrt{-1}dz_0d\overline{z}_0}{|z_0\ln|z_0||^2}\bigg(\frac{dz_1}{|z_1\ln|z_1||}+\frac{d\overline{z}_1}{|z_1\ln|z_1||}+\beta\bigg), \tag{4.3.11}$$

where β is some bounded differential form in variables $z_2,\dots,z_q$. Now, by the identity $z_0z_1=w_0$ and $|z_0|,|z_1|\leq c$, there is a constant $C>0$ such that we have

$$\begin{aligned}\bigg|\int_{H_{w_1,c}}&\frac{\sqrt{-1}dz_0d\overline{z}_0}{|z_0\ln|z_0||^2}\frac{dz_1}{z_1|\ln|z_1||}\bigg|\\ &=\bigg|\frac{dw_0}{w_0}\bigg|\int_{H_{w_1,c}}\frac{\sqrt{-1}dz_0d\overline{z}_0}{|z_0\ln|z_0||^2\cdot|\ln|w_0/z_0||}\leq C\bigg|\frac{dw_0}{w_0|\ln|w_0||}\bigg|.\end{aligned} \tag{4.3.12}$$

By (4.3.11) and (4.3.12), we deduce that the form $(\pi|_{V_{2,c}})_*[\partial\alpha]$ has log-log growth along Δ.

Now let's prove 4.3.1c). By the proof of Proposition 4.3.1b), we see that $\pi_*[f\alpha]$ is continuous over $S\setminus|\Delta|$. Now, let $t_0\in|\Delta|$, $U\in S$ and $V_{i,c}$, $i=1,2$, be as before. Trivially, since $f\alpha$ is continuous over $\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})$, and $\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})}$ is a submersion, we see that $(\pi|_{\pi^{-1}(U)\setminus(V_{1,c}\cup V_{2,c})})_*[f\alpha]$ is continuous for any $c>0$. Let's prove that for $i=1,2$, we have

$$\lim_{c\to0}(\pi|_{V_{i,c}})_*[f\alpha]=0, \tag{4.3.13}$$

If (4.3.13) holds, we would immediately conclude that $\pi_*[f\alpha]$ is continuous.

By the fact that $(\pi|_{V_{i,c}})_*[f\alpha]$ depends only on the coupling of $f\alpha$ with two vertical vector fields, and the fact that those couplings have log-log growth on $X \setminus \Sigma_{X/S}$ with singularities along D, we deduce (4.3.13) for $i = 1$ from (4.3.1).

Since α has Poincaré growth on $X \setminus |D|$ with singularities along $\pi^{-1}(\Delta)$, and f has log-log growth along $\Sigma_{X/S}$, we deduce that there are $C > 0$, $p \in \mathbb{N}$ such that

$$(\pi|_{V_{2,c}})_*[f\alpha] \leq C \int_{H_{w_1,c}} \Bigg(\frac{(\ln|\ln|z_0||)^p \sqrt{-1} dz_0 d\overline{z}_0}{|z_0 \ln|z_0||^2} + \frac{(\ln|\ln|z_0||)^p (\ln|\ln|z_1||)^p |dz_0 d\overline{z}_1|}{|z_0 \ln|z_0|||z_1 \ln|z_1||} + \frac{(\ln|\ln|z_1||)^p \sqrt{-1} dz_1 d\overline{z}_1}{|z_1 \ln|z_1||^2} \Bigg), \quad (4.3.14)$$

where $H_{w_1,c}$ is as in (4.3.6). By (4.3.1), (4.3.14) and Cauchy inequality, we deduce (4.3.13) for $i = 2$, which finally proves that $\pi_*[f\alpha]$ is continuous over S. □

The next proposition explains why Proposition 4.3.1 is well-suited to our Assumptions **S1**, **S2**, **S3**. Let L be a holomorphic line bundle over X and let h_i^L, $i = 1, 2$ be smooth Hermitian metrics on L over $X \setminus (\pi^{-1}(|\Delta|) \cup |D|)$.

Proposition 4.3.2. a) Suppose that h_i^L, $i = 1, 2$ extend smoothly over $X \setminus |D|$, and they are pre-log-log of infinite order, with singularities along D. Then there is a differential form α in the class $[\widetilde{\mathrm{ch}}(L, h_1^L, h_2^L)]^{[2]} \in P_{\mathrm{pll}}(X \setminus \pi^{-1}(|\Delta|), D)$, which satisfies the hypothesis of Proposition 4.3.1a).

b) Suppose that h_i^L, $i = 1, 2$ are pre-log-log, with singularities along $\pi^{-1}(|\Delta|) \cup |D|$. Then there is a differential form α in the class $[\widetilde{\mathrm{ch}}(L, h_1^L, h_2^L)]^{[2]} \in P_{\mathrm{pll}}(X \setminus \pi^{-1}(|\Delta|), D)$, which satisfies the hypothesis of Proposition 4.3.1b).

c) Suppose that h_i^L, $i = 1, 2$ extend continuously over $X \setminus (\Sigma_{X/S} \cup |D|)$, have log-log growth with singularities along $\Sigma_{X/S} \cup |D|$, are good in the sense of Mumford on $X \setminus |D|$ with singularities along $\pi^{-1}(\Delta)$, and the coupling with two vertical vector fields of $c_1(L, h_i^L)$, $i = 1, 2$ are continuous over $X \setminus (\Sigma_{X/S} \cup |D|)$ and has log-log growth on $X \setminus \Sigma_{X/S}$ with singularities along D.

Then there is a function f and a differential form α such that $f\alpha$ is in the class $[\widetilde{\mathrm{ch}}(L, h_1^L, h_2^L)]^{[2]} \in P_{\mathrm{pll}}(X \setminus \pi^{-1}(|\Delta|), D)$, and they satisfy the hypothesis of Proposition 4.3.1c).

Proof. It follows directly from Propositions 4.2.15, 4.2.22. For Proposition 4.3.2c), take $f = [\widetilde{\mathrm{ch}}(L, h_1^L, h_2^L)]^{[0]}$ and $\alpha = (c_1(L, h_1^L) + c_1(L, h_2^L))/2$.

□

4.3.2 Some properties of the Quillen metric

Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps. Let's recall some notions from [54].

Definition 4.3.3 (Flattening of a metric, [54, Definition 1.2])**.** We say that a (smooth) metric g_{f}^{TM} over $\overline{M}$ is a *flattening* of g^{TM} if there is $\epsilon > 0$ such that g^{TM} is induced by (4.2.2) over $V_i^M(\epsilon)$, and

$$g_{\mathrm{f}}^{TM}|_{M \setminus (\cup_i V_i^M(\epsilon))} = g^{TM}|_{M \setminus (\cup_i V_i^M(\epsilon))}. \quad (4.3.15)$$

Similarly, we defined the notion of *flattening* $\|\cdot\|_M^{\mathrm{f}}$ for Hermitian norm $\|\cdot\|_M$. For brevity, we state a version of [54, Theorem A, Remark 1.4.d)], which doesn't use the language of *compatible flattenings* from [54].

Theorem A (Compact perturbation)**.** *Let g_{f}^{TM}, $\|\cdot\|_M^{\mathrm{f}}$ be some flattenings of g^{TM} and $\|\cdot\|_M$ respectively, then the quantity*

$$\begin{aligned}2\mathrm{rk}(\xi)^{-1}\ln\Big(\|\cdot\|_Q\left(g^{TM},h^{\xi}\otimes\|\cdot\|_M^{2n}\right)/\|\cdot\|_Q\left(g_{\mathrm{f}}^{TM},h^{\xi}\otimes(\|\cdot\|_M^{\mathrm{f}})^{2n}\right)\Big)\\ -\mathrm{rk}(\xi)^{-1}\int_M c_1(\xi,h^{\xi})\Big(2n\ln(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)+\ln(g_{\mathrm{f}}^{TM}/g^{TM})\Big)\end{aligned}\tag{4.3.16}$$

depends only on the number $n\in\mathbb{Z}$, $n\leq 0$ and the functions $(g_{\mathrm{f}}^{TM}/g^{TM})|_{V_i^M(1)}\circ(z_i^M)^{-1}:\mathbb{D}^*\to\mathbb{R}$ and $(\|\cdot\|_M^{\mathrm{f}}/\|\cdot\|_M)|_{V_i^M(1)}\circ(z_i^M)^{-1}:\mathbb{D}^*\to\mathbb{R}$, for $i=1,\ldots,m$.

Now let's recall the anomaly formula for surfaces with cusps, which explains how the Quillen norm changes under the conformal change of the metric with cusps.

Let's recall that by [21, Theorem 1.27] (cf. Theorem 4.2.19) and (4.1.9), the Bott-Chern forms of a vector bundle ξ with (smooth) Hermitian metrics h_1^{ξ}, h_2^{ξ} over $\overline{M}$ satisfy (see also (4.2.36), (4.2.37))

$$\widetilde{\mathrm{Td}}(\xi,h_1^{\xi},h_2^{\xi})^{[0]}=\widetilde{\mathrm{ch}}(\xi,h_1^{\xi},h_2^{\xi})^{[0]}/2,\tag{4.3.17}$$

If, moreover, $\xi:=L$ is a line bundle, we have

$$\widetilde{\mathrm{Td}}(L,h_1^{L},h_2^{L})^{[2]}=\widetilde{\mathrm{ch}}(L,h_1^{L},h_2^{L})^{[2]}/6.\tag{4.3.18}$$

Theorem B (Anomaly formula for metrics with cusps)**.** *Let g^{TM}, g_0^{TM} be two metrics on M such that both triples $(\overline{M},D_M,g^{TM})$, $(\overline{M},D_M,g_0^{TM})$ are surfaces with cusps. We denote by $\|\cdot\|_M$, $\|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$, and by $\|\cdot\|^W$, $\|\cdot\|_0^W$ the associated Wolpert norms. Let h^{ξ}, h_0^{ξ} be two Hermitian metrics on ξ over $\overline{M}$. Then the right-hand side of the following equation is finite, and*

$$\begin{aligned}&2\ln\Big(\|\cdot\|_Q\left(g_0^{TM},h_0^{\xi}\otimes(\|\cdot\|_M^0)^{2n}\right)/\|\cdot\|_Q\left(g^{TM},h^{\xi}\otimes\|\cdot\|_M^{2n}\right)\Big)\\ &=\int_M\Big[\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1},\|\cdot\|_M^{-2},(\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi,h^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n,\|\cdot\|_M^{2n}\big)\\ &\qquad+\mathrm{Td}\big(\omega_M(D)^{-1},(\|\cdot\|_M^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi,h^{\xi},h_0^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n,\|\cdot\|_M^{2n}\big)\\ &\qquad+\mathrm{Td}\big(\omega_M(D)^{-1},(\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi,h_0^{\xi}\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n,\|\cdot\|_M^{2n},(\|\cdot\|_M^0)^{2n}\big)\Big]^{[2]}\\ &-\frac{\mathrm{rk}(\xi)}{6}\ln\Big(\|\cdot\|^W/\|\cdot\|_0^W\Big)+\frac{1}{2}\sum_{P\in D_M}\ln\Big(\det(h^{\xi}/h_0^{\xi})|_P\Big).\end{aligned}\tag{4.3.19}$$

Now, let's recall the result of Bismut-Bost [20, Théorème 2.2] on the asymptotics of the Quillen norm (see also Bismut [18] for its generalization to higher dimension and Ma [82] for the family

version of [18]). For this, we fix a f.s.o. $\pi : X \to S$ and smooth Hermitian vector bundles $(\omega_{X/S}, \|\cdot\|^{\omega,\mathrm{sm}}_{X/S})$, $(\xi, h^{\xi}_{\mathrm{sm}})$ over X. We denote by $g^{TX_t}_{\mathrm{sm}}$ the metric on X_t, $t \in S \setminus |\Delta|$, induced by $(\omega_{X/S}, \|\cdot\|^{\omega,\mathrm{sm}}_{X/S})$, and by $\|\cdot\|_Q\,(g^{TX_t}_{\mathrm{sm}}, h^{\xi}_{\mathrm{sm}})$ the Quillen norm on $\det(R^\bullet\pi_*\xi)^{-1}$.

Theorem 4.3.4 (Continuity theorem of Bismut-Bost)**.** *The norm* $\|\cdot\|_Q\,(g^{TX_t}_{\mathrm{sm}}, h^{\xi}_{\mathrm{sm}})^{12} \otimes (\|\cdot\|^{\mathrm{div}}_{\Delta})^{\mathrm{rk}(\xi)}$ *on the line bundle* $\det(R^\bullet\pi_*\xi)^{-12} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)}$ *over* $S \setminus |\Delta|$ *is very nice on* S *with singularities along* Δ *in the sense of Definition 4.1.3.*

4.3.3 Proof of Theorem C

We use the notation from Theorem C and (4.1.1). Since all the statements are local, it suffices to prove them in a neighbourhood U of $t_0 \in S$. We prove them all at the same time in three steps: in *Step 1* we see that by Theorem B, we can trivialize the Poincaré-compatible coordinates associated to g^{TX_t}. In *Step 2*, by Theorem A, we reduce the problem to the problem without cusps. Finally, in *Step 3*, by the anomaly formula of Bismut-Gillet-Soulé (cf. Theorem B), we reduce the problem to the problem with smooth metrics, which is exactly Theorem 4.3.4. For the proof of Theorem C1, this step is unnecessary since the metrics, which are obtained after Step 2 are already smooth. In the first two steps the reduction is done by modifying norms $\|\cdot\|^{\omega}_{X/S}, \|\cdot\|_{X/S}$ only in the neighbourhood of $|D_{X/S}|$.

Step 1. Let $V_{i,c}$, $i = 1, \dots, m$, $c > 0$ (resp. U) be a neighbourhood of $\sigma_i(t_0)$ (resp. t_0) such that for some local coordinates $(z_0, \dots, z_q)$ of $\sigma_i(t_0)$ and $(w_1, \dots, w_q)$ of $t_0 \in S$, satisfying (4.2.6), we have $V_{i,c} = \{x \in \pi^{-1}(U) : |z_0| < c\}$ and $\{z_0(x) = 0\} = \{\sigma_i(t) : t \in U\}$. For simplicity, we note $V_i := V_{i,1}$. Let $\nu : \mathbb{R}_+ \to [0,1]$ be a smooth function satisfying

$$\nu(u) = \begin{cases} 1, & \text{if} \quad u < 1/2, \\ 0, & \text{if} \quad u > 3/4. \end{cases} \tag{4.3.20}$$

We denote by $\|\cdot\|^{\omega,0}_{X/S}$ the norm on $\omega_{X/S}$ over $\pi^{-1}(U \setminus |\Delta|) \setminus |D_{X/S}|$ such that $\|\cdot\|^{\omega,0}_{X/S}$ coincides with $\|\cdot\|^{\omega}_{X/S}$ away from $\cup_i V_i$, and over $(\cup_i V_i) \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, we have

$$\|dz_0\|^{\omega,0}_{X/S} = \big|z_0 \ln |z_0|\big|^{\nu(|z_0|)} \cdot \big(\|dz_0\|^{\omega}_{X/S}\big)^{1-\nu(|z_0|)}. \tag{4.3.21}$$

Let $\|\cdot\|^{0}_{X/S}$ be the induced norm on $\omega_{X/S}(D)$ as in Construction 4.1.1, and let $g^{TX_t}_0$, $t \in S$ be the induced metric with cusps on X_t. Then by Construction 4.1.1 and (4.3.21), we see that if h^{ξ}, $\|\cdot\|_{X/S}$ satisfy Assumptions **S1** or **S2** or **S3**, then h^{ξ}, $\|\cdot\|^{0}_{X/S}$ satisfy Assumptions **S1** or **S2** or **S3** correspondingly. In fact, this property along with the fact that $\|\cdot\|^{\omega,0}_{X/S}$ doesn't vary in the horizontal direction around the cusps are the only facts we need from the construction (4.3.21).

We denote by $\|\cdot\|^{W,0}_{X/S}$ the Wolpert norm (see Definition 4.2.6) on $\otimes_{i=1}^m \sigma_i^*\omega_{X/S}$ induced by $g^{TX_t}_0$. By Theorem B, applied pointwise for the line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))^{12} \otimes (\otimes_{i=1}^m \sigma_i^*\omega_{X/S})^{-\mathrm{rk}(\xi)}$,

for any $t \in U \setminus |\Delta|$, we deduce

$$\begin{aligned}
&\frac{1}{6}\ln\Big(\|\cdot\|_Q\left(g_0^{TX_t}, h^\xi\otimes(\|\cdot\|^0_{X/S})^{2n}\right)^{12}\otimes\left(\|\cdot\|^{W,0}_{X/S}\right)^{-\mathrm{rk}(\xi)}\Big)\\
&\qquad\qquad -\frac{1}{6}\ln\Big(\|\cdot\|_Q\left(g^{TX_t}, h^\xi\otimes(\|\cdot\|_{X/S})^{2n}\right)^{12}\otimes\left(\|\cdot\|^{W}_{X/S}\right)^{-\mathrm{rk}(\xi)}\Big)\\
&=\pi_*\Big[\widetilde{\mathrm{Td}}\big(\omega_{X/S}(D)^{-1},\|\cdot\|^{-2}_{X/S},(\|\cdot\|^0_{X/S})^{-2}\big)\mathrm{ch}\big(\xi,h^\xi\big)\mathrm{ch}\big(\omega_{X/S}(D)^n,\|\cdot\|^{2n}_{X/S}\big)\\
&\qquad+\mathrm{Td}\big(\omega_{X/S}(D)^{-1},(\|\cdot\|^0_{X/S})^{-2}\big)\mathrm{ch}\big(\xi,h^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_{X/S}(D)^n,\|\cdot\|^{2n}_{X/S},(\|\cdot\|^0_{X/S})^{2n}\big)\Big]^{[2]}.
\end{aligned}\tag{4.3.22}$$

We note that the conformal factor corresponding to the change of the metric from $\|\cdot\|^{\omega}_{X/S}$ to $\|\cdot\|^{\omega,0}_{X/S}$ is non-trivial in the neighborhood of the cusp. Thus, we use Theorem B with the conformal factor which doesn't have compact support in the punctured surface.

By Propositions 4.3.1, 4.3.2, we see that the right-hand-side of (4.3.22) is very nice on $U \setminus |\Delta|$ with singularities along Δ under Assumption **S1**, it is nice on $U \setminus |\Delta|$ with singularities along Δ under Assumption **S2**, and it is continuous on S under Assumption **S3**. By this and (4.3.22), we see that it is enough to prove Theorem C for the metrics $\|\cdot\|^0_{X/S}, \|\cdot\|^{\omega,0}_{X/S}, \|\cdot\|^{W,0}_{X/S}$ instead $\|\cdot\|_{X/S}, \|\cdot\|^{\omega}_{X/S}$, $\|\cdot\|^{W}_{X/S}$.

We note, however, that the norm $\|\cdot\|^{W,0}_{X/S}$ is trivial over $U\setminus|\Delta|$, thus, it's enough to prove Theorem C for the norm $\|\cdot\|_Q\left(g_0^{TX_t}, h^\xi\otimes(\|\cdot\|^0_{X/S})^{2n}\right)^{12}\otimes(\|\cdot\|^{\mathrm{div}}_{\Delta})^{\mathrm{rk}(\xi)}$ on the line bundle $\det(R^\bullet\pi_*(\xi\otimes\omega_{X/S}(D)^n))^{-12}\otimes\mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)}$ in place of the norm $\|\cdot\|_{\mathscr{L}_n}$ on the line bundle $\mathscr{L}_n$.

Step 2. We denote $V_i' = V_{i,1/2} \subset V_i$, and by $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ the norm on $\omega_{X/S}$ over $\pi^{-1}(U \setminus |\Delta|)$ such that $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ coincides with $\|\cdot\|^{\omega,0}_{X/S}$ away from $\cup_i V_i'$, and over V_i', we have

$$\|dz_0\|^{\omega,\mathrm{cmp}}_{X/S} = |z_0\ln|z_0||^{1-\nu(2|z_0|)},\tag{4.3.23}$$

where $\nu:\mathbb{R}\to[0,1]$ is as in (4.3.20). We denote by $g^{TX_t}_{\mathrm{cmp}}$ the induced metric on X_t. We denote by $\|\cdot\|^{\mathrm{cmp}}_{X/S}$ the norm on $\omega_{X/S}(D)$ over $\pi^{-1}(U \setminus |\Delta|)$, such that $\|\cdot\|^{\mathrm{cmp}}_{X/S}$ coincides with $\|\cdot\|^0_{X/S}$ away from $\cup_i V_i'$, and over V_i' we have

$$\|dz_0\otimes s_{D_{X/S}}/z_0\|^{\mathrm{cmp}}_{X/S} = |\ln|z_0||^{1-\nu(2|z_0|)}.\tag{4.3.24}$$

By (4.3.23), (4.3.24), we see that if $h^\xi, \|\cdot\|^0_{X/S}$ satisfy Assumptions **S1** or **S2** or **S3**, then $h^\xi\otimes(\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}, \|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ satisfy Assumptions **S1** or **S2** or **S3** for $D_{X/S}=\emptyset$ correspondingly. In fact, this property along with the fact that $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ doesn't vary in the horizontal direction around the cusps are the only facts we need from the construction (4.3.23).

Now, as the norms $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}, \|\cdot\|^{\omega,0}_{X/S}, \|\cdot\|^{\mathrm{cmp}}_{X/S}, \|\cdot\|^0_{X/S}$ do not vary in the horizontal direction around the cusps, by Theorem A, we see that the function

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\left(g^{TX_t}_{\mathrm{cmp}}, h^\xi\otimes(\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}\right)/\|\cdot\|_Q\left(g_0^{TX_t}, h^\xi\otimes(\|\cdot\|^0_{X/S})^{2n}\right)\Big)\\
&\qquad\qquad-\pi_*\Big[c_1(\xi,h^\xi)\Big(n\widetilde{c}_1(\omega_{X/S}(D),(\|\cdot\|^0_{X/S})^2,(\|\cdot\|^{\mathrm{cmp}}_{X/S})^2)
\end{aligned}$$

$$- \widetilde{c}_1(\omega_{X/S}, (\|\cdot\|^{\omega,0}_{X/S})^2, (\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S})^2)\Big)\Big]^{[2]} \quad (4.3.25)$$

is constant over $U \setminus |\Delta|$. By Propositions 4.2.18, 4.3.1, 4.3.2, we see that the term under the integration in (4.3.25) is very nice on $U \setminus |\Delta|$ with singularities along Δ under Assumption **S1**, it is nice on $U \setminus |\Delta|$ with singularities along Δ under Assumption **S2**, and it is continuous on S under Assumption **S3**. By this and (4.3.25), it is enough to prove Theorem C for the metrics $h^{\xi}, \|\cdot\|^{\mathrm{cmp}}_{X/S}$, $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ instead of $h^{\xi}, \|\cdot\|^{0}_{X/S}, \|\cdot\|^{\omega,0}_{X/S}$.

Step 3. Let $h^{\xi}_{\mathrm{sm}}, \|\cdot\|^{\mathrm{sm}}_{X/S}, \|\cdot\|^{\omega,\mathrm{sm}}_{X/S}$ be some smooth metrics on ξ, $\omega_{X/S}(D)$ and $\omega_{X/S}$ respectively over X. We denote by $g^{TX_t}_{\mathrm{sm}}$ the Riemannian metric on X_t, induced by $\|\cdot\|^{\omega,\mathrm{sm}}_{X/S}$. By the anomaly formula of Bismut-Gillet-Soulé [23, Theorem 1.27] (cf. Theorem B for $m = 0$), for $t \in U \setminus |\Delta|$:

$$\begin{aligned}
&2\ln\Big(\|\cdot\|_Q\left(g^{TX_t}_{\mathrm{cmp}}, h^{\xi} \otimes (\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}\right)/\|\cdot\|_Q\left(g^{TX_t}_{\mathrm{sm}}, h^{\xi}_{\mathrm{sm}} \otimes (\|\cdot\|^{\mathrm{sm}}_{X/S})^{2n}\right)\Big)\\
&\quad = \pi_*\Big[\widetilde{\mathrm{Td}}\big(\omega_M^{-1}, (\|\cdot\|^{\mathrm{sm}}_M)^{-2}, (\|\cdot\|^{\mathrm{cmp}}_M)^{-2}\big)\mathrm{ch}\big(\xi, h^{\xi}_{\mathrm{sm}}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|^{\mathrm{sm}}_M)^{2n}\big)\\
&\qquad + \mathrm{Td}\big(\omega_M^{-1}, (\|\cdot\|^{\mathrm{cmp}}_M)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h^{\xi}_{\mathrm{sm}}, h^{\xi}\big)\mathrm{ch}\big(\omega_M(D)^n, (\|\cdot\|^{\mathrm{sm}}_M)^{2n}\big)\\
&\qquad + \mathrm{Td}\big(\omega_M^{-1}, (\|\cdot\|^{\mathrm{cmp}}_M)^{-2}\big)\mathrm{ch}\big(\xi, h^{\xi}\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, (\|\cdot\|^{\mathrm{sm}}_M)^{2n}, (\|\cdot\|^{\mathrm{cmp}}_M)^{2n}\big)\Big].
\end{aligned} \quad (4.3.26)$$

By Theorem 4.2.19 and Propositions 4.2.18, 4.3.1, 4.3.2, the right-hand side of (4.3.26) is very nice on $U \setminus |\Delta|$ with singularities along Δ. By this and (4.3.26), we see that it is enough to prove Theorem C for the metrics $h^{\xi}_{\mathrm{sm}}, \|\cdot\|^{\mathrm{sm}}_{X/S}, \|\cdot\|^{\omega,\mathrm{sm}}_{X/S}$ instead of $h^{\xi}, \|\cdot\|^{\mathrm{cmp}}_{X/S}, \|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$. But for $h^{\xi}_{\mathrm{sm}}, \|\cdot\|^{\mathrm{sm}}_{X/S}$, $\|\cdot\|^{\omega,\mathrm{sm}}_{X/S}$, Theorem C follows directly from Theorem 4.3.4 by Remark 4.1.2. Thus, we conclude Theorem C.

Remark 4.3.5. Quite easily, we see that if in hypothesizes **S2** instead of pre-log-log we demand all the metrics to be good, the Hermitian norm $\|\cdot\|_{\mathscr{L}_n}$ would become good.

4.4 Potential theory for log-log currents, a proof of Theorem D

In Section 4.1 we introduce the potential theory for currents with log-log growth and in Section 4.2 we use it to prove Theorem D. Then we deduce Corollaries 4.1.7, 4.1.8 from Theorem D.

4.4.1 Potential theory for currents with log-log growth

In this section we provide the potential theory for currents with log-log growth. We denote $U = \{(z_1, \ldots, z_q) \in \mathbb{C}^q : |z_i| < 1, \text{ for all } i = 1, \ldots, q\}$, and let $D_i \subset U$ be defined by the equation $\{z_i = 0\}$. We denote $D = \cup_{i=1}^{l} D_i$, for some $l \leq q$. Before stating the main result of this section, we need the following lemma.

Lemma 4.4.1. Let T be a closed $(1,1)$-current over $U \setminus D$ with log-log growth along D. Then the trivial L^1-extension $[T]_{L^1}$ of T, is a closed current over U. Also, in a small neighbourhood of

D, the current $[T]_{L^1}$ can be represented as a difference of two positive closed currents with log-log growth along D.

Remark 4.4.2. If we could assume that current T is induced by integration of a smooth differential form, Lemma 4.4.1 would follow from simple integration by parts, similar to [95, Proposition 1.2]. Such a supposition, however, is too strong for our needs, and we will use Skoda-El Mir's theorem to prove Lemma 4.4.1.

Proof. For $p \in \mathbb{N}$, we denote the functions

$$\begin{aligned} a_p^0(z,w) &= (\ln|\ln|z|^2|)^p|w|^2, \\ a_p(z,w) &= -(\ln|\ln|z|^2|)^p(\ln|\ln|w|^2|)^p, \end{aligned} \tag{4.4.1}$$

over $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$ and $\mathbb{C}^2 \setminus (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C})$ respectively. Then it's easy to see that the dominating terms of differential forms $\sqrt{-1}\partial\overline{\partial}a_p^0(z,w)$, $\sqrt{-1}\partial\overline{\partial}a_p(z,w)$ are given by

$$\begin{aligned} &\frac{-\sqrt{-1}p(\ln|\ln|z|^2|)^{p-1}|w|^2 dz d\overline{z}}{|z\ln|z|^2|^2} + \sqrt{-1}(\ln|\ln|z|^2|)^p dw d\overline{w}, \\ &\sqrt{-1}p(\ln|\ln|z|^2|)^{p-1}(\ln|\ln|w|^2|)^{p-1}\left(\frac{(\ln|\ln|z|^2|)dw d\overline{w}}{|w\ln|w|^2|^2} + \frac{(\ln|\ln|w|^2|)dz d\overline{z}}{|z\ln|z|^2|^2}\right), \end{aligned} \tag{4.4.2}$$

respectively. From this, we see that for some open neighbourhood $V \subset U$ of S, the forms $\sqrt{-1}\partial\overline{\partial}a_p(z,w)$, $\sqrt{-1}\partial\overline{\partial}(a_p(z,w) + a_p^0(z,v))$ are positive on $\mathbb{C}^2 \setminus (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C})$ and $\mathbb{C}^3 \setminus (\mathbb{C}^2 \times \{0\} \cup \mathbb{C} \times \{0\} \times \mathbb{C} \cup \{0\} \times \mathbb{C}^2)$ respectively. Now, any differential form α over $U \setminus D$ with log-log growth along D can be bounded from above and below by a linear combination of

$$\begin{aligned} &\frac{\sqrt{-1}(\ln|\ln|z_i|^2|)^{p-1} dz_j d\overline{z}_j}{|z_j\ln|z_j|^2|^2}, \qquad i,j = 1,\dots,l, \\ &\sqrt{-1}(\ln|\ln|z_i|^2|)^p dz_j d\overline{z}_j, \qquad i = 1,\dots,l; j = 1,\dots,q. \end{aligned} \tag{4.4.3}$$

So, since T has log-log growth along D, by (4.4.2), there are $C > 0, p \in \mathbb{N}$ such that for

$$A_p := (q+1)\sum_{i,j=1}^{l} a_p(z_i,z_j) + \sum_{i=1}^{l}\sum_{j=1+l}^{q} a_p^0(z_i,z_j), \tag{4.4.4}$$

we have the following inequalities over V:

$$-C\sqrt{-1}\partial\overline{\partial}A_p \le T \le C\sqrt{-1}\partial\overline{\partial}A_p. \tag{4.4.5}$$

Thus, the current $T + C\sqrt{-1}\partial\overline{\partial}A_p$ is closed, positive in V, and by (4.3.1), (4.4.4), (4.4.5) it has finite mass. Thus, by Skoda-El Mir's theorem (cf. [43, Theorem III.2.3]), $[T + C\sqrt{-1}\partial\overline{\partial}A_p]_{L^1}$ is a closed positive current over V. Similarly, $[\sqrt{-1}\partial\overline{\partial}A_p]_{L^1}$ is a closed positive current over V. So

$$[T]_{L^1} = [T + C\sqrt{-1}\partial\overline{\partial}A_p]_{L^1} - C[\sqrt{-1}\partial\overline{\partial}A_p]_{L^1} \tag{4.4.6}$$

is a closed current over U. Also (4.4.6) gives the needed decomposition of $[T]_{L^1}$ as a difference of positive currents. □

Remark 4.4.3. Since our current is locally represented as a difference of two positive currents, its Lelong numbers (cf. [43, Definition III.5.4]) are well-defined, see also Remark 4.4.7.

The main goal of this section is to prove the following

Proposition 4.4.4. Let $\phi : U \setminus D \to \mathbb{R}$ be a continuous function with log-log growth along D. Suppose that for the induced current $[\phi]$ over $U \setminus D$, the current

$$T := \frac{\partial\overline{\partial}[\phi]}{2\pi\sqrt{-1}}, \tag{4.4.7}$$

over $U \setminus D$ has log-log growth along D. Then we have the following identity of currents over U

$$\frac{\partial\overline{\partial}[\phi]_{L^1}}{2\pi\sqrt{-1}} = [T]_{L^1}. \tag{4.4.8}$$

Remark 4.4.5. When $l = 1$, this result implies Yoshikawa [118, Proposition 3.11], where he obtained this for T of Poincaré growth. If T extends smoothly over U, Proposition 4.4.4 is a special case of Bismut-Bost [20, Proposition 10.2]. We note, however, that in our applications, the condition of being pre-log-log and not smooth is essential, see Section 4.5.

To prove Proposition 4.4.4, we need the following weak analogue of Poincaré lemma for currents of log-log growth:

Lemma 4.4.6. Let T be a closed $(1,1)$-current over U with log-log growth along D. For any $x \in U$, there is a neighborhood V of x and a function $\psi \in L^1_{loc}(V)$ with log-log growth along D, satisfying

$$\frac{\partial\overline{\partial}[\psi]_{L^1}}{2\pi\sqrt{-1}} = [T]_{L^1}. \tag{4.4.9}$$

Remark 4.4.7. This lemma, implies, that Lelong numbers of T (see Remark 4.4.3) vanish.

Proof. We recall that the functions $A_p : U \setminus D \to \mathbb{R}$, $p \in \mathbb{N}$ were defined in (4.4.4). Let $C > 0$, $p \in \mathbb{N}$ be as in (4.4.5). By Siu [105, Proof of Lemma 5.3], since the current $[T + C\sqrt{-1}\partial\overline{\partial}A_p]_{L^1}$ is closed and positive, there is an open subset $V' \subset U$ and a plurisubharmonic (cf. [85, Definition B.2.16]) function R over V', such that

$$\sqrt{-1}\partial\overline{\partial}R = \left[T + C\sqrt{-1}\partial\overline{\partial}A_p\right]_{L^1}. \tag{4.4.10}$$

Moreover, since

$$0 < [2C\sqrt{-1}\partial\overline{\partial}A_p - T]_{L^1} = \sqrt{-1}\partial\overline{\partial}(3C[A_p]_{L^1} - R). \tag{4.4.11}$$

Thus, by plurisubharmonicity (cf. [47, Proposition A.15]), there is $C_0 > 0$, such that almost everywhere, we have

$$-C_0 + 3CA_p \leq R \leq C_0, \tag{4.4.12}$$

in particular, since A_p has log-log growth along D, we deduce by (4.4.12) that $R - C[A_p]_{L^1}$ has log-log growth along D. By (4.4.10), we get (4.4.9) for $\psi := -2\pi(R - CA_p)$. □

Proof of Proposition 4.4.4. Let ψ be a function on $V \subset\subset U$ as in Lemma 4.4.6, such that

$$\frac{\partial\overline{\partial}[\psi]_{L^1}}{2\pi\sqrt{-1}} = [T]_{L^1}. \tag{4.4.13}$$

We denote

$$\chi = \phi - \psi. \tag{4.4.14}$$

Then χ is pluriharmonic on $V \setminus D$ and has log-log singularities along D. We'll prove that χ is pluriharmonic on V. Once it will be done, Proposition 4.4.4 would follow from (4.4.7), (4.4.13) and (4.4.14). Let $z' = (0, z'_2, \dots, z'_q) \in V$ be such that $z' \notin D_i$ for any $i \geq 2$, i.e. $z_i \neq 0$. Then the function

$$\chi_0(z) := \chi(z, z'_2, \dots, z'_q) \tag{4.4.15}$$

is harmonic over $D^*(\epsilon)$, for some $\epsilon > 0$, and has log-log growth along $0 \in D(\epsilon)$. By [20, p. 71-72], the function χ_0 extends to a harmonic function over $D(\epsilon)$. By repeating this for z' in a small neighbourhood of fixed z' in D_1, we see that χ extends over $V \setminus (\cup_{i=2}^l D_i)$, such that it's restriction on discs $\{(z, z'_2, \dots, z'_n) : |z| \leq \epsilon\}$ are harmonic. By the maximum principle and the fact that χ is smooth over $V \setminus (\cup_{i=1}^l D_i)$, we see that this extension is actually locally bounded in $V \setminus (\cup_{i=2}^l D_i)$, so by [43, Theorem 5.24], the function χ is pluriharmonic over $V \setminus (\cup_{i=2}^l D_i)$. By repeating this argument for $i = 2, \dots, l$, we see that χ is actually pluriharmonic over V. □

4.4.2 Proof of Theorem D and Corollaries 4.1.7, 4.1.8

We use the notation from Theorem D. The main ingredients of the proof are Theorems A, B, Proposition 4.4.4 and the curvature theorem of Bismut-Bost [20, Théorème 2.2], which we now recall.

We borrow the notation from Theorem 4.3.4. By Theorem 4.3.4, the Hermitian norm $\|\cdot\|_Q\,(g^{TX_t}_{\mathrm{sm}}, h^{\xi}_{\mathrm{sm}})^{12} \otimes (\|\cdot\|^{\mathrm{div}}_{\Delta})^{\mathrm{rk}(\xi)}$ is very nice over S with singularities along Δ. In particular, by Remark 4.1.4, its first Chern form is well-defined.

Theorem 4.4.8 ([20, Théorème 2.2]). *The following identity of currents over S holds*

$$c_1\Big(\lambda(j^*\xi)^{12} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)}, \|\cdot\|_Q\,(g^{TX_t}_{\mathrm{sm}}, h^{\xi}_{\mathrm{sm}})^{24} \otimes \big(\|\cdot\|^{\mathrm{div}}_{\Delta}\big)^{2\mathrm{rk}(\xi)}\Big)$$
$$= -12\pi_*\Big[\mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,\mathrm{sm}}_{X/S})^{-2}\big)\mathrm{ch}(\xi, h^{\xi}_{\mathrm{sm}})\Big]^{[4]}. \tag{4.4.16}$$

Proof of Theorem D. **Let's treat Assumption S1 first.** As in Theorem C, the statement is local over the base. So, for any $t_0 \in S$, it is enough to prove (4.1.11) in some neighbourhood $U \subset S$ of t_0. The fact that the current (4.1.10) is $L^1_{loc}(S)$ follows from Lebesgue dominated convergence theorem and [20, Proposition 5.2]. The fact that its closure is a d-closed current follows from the fact that it is obtained as a pushforward of a closed form and (4.3.8).

We denote the analogue of the norm (4.1.5), associated with $\|\cdot\|^0_{X/S}$, where in place by

$$\|\cdot\|^0_{\mathscr{L}_n} := \|\cdot\|_Q\,(g^{TX_t}_0, h^{\xi} \otimes (\|\cdot\|^0_{X/S})^{2n})^{12} \otimes (\|\cdot\|^{W,0}_{X/S})^{-\mathrm{rk}(\xi)} \otimes (\|\cdot\|^{\mathrm{div}}_{\Delta})^{\mathrm{rk}(\xi)} \otimes (\otimes_{i=1}^m \sigma_i^* h^{\det\xi})^3. \tag{4.4.17}$$

By Proposition 4.2.12, (4.2.33), (4.3.21) and (4.3.22), we deduce that

$$c_1\Big(\mathscr{L}_n, \big(\|\cdot\|^0_{\mathscr{L}_n}\big)^2\Big) - c_1\Big(\mathscr{L}_n, \big(\|\cdot\|_{\mathscr{L}_n}\big)^2\Big) = -12\pi_*\Big[\mathrm{ch}(\xi, h^\xi)\Big(\mathrm{Td}\big(\omega_{X/S}(D)^{-1}, (\|\cdot\|^0_{X/S})^{-2}\big)$$
$$\cdot\, \mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^0_{X/S})^{2n}\big) - \mathrm{Td}\big(\omega_{X/S}(D)^{-1}, \|\cdot\|^{-2}_{X/S}\big)\mathrm{ch}\big(\omega_{X/S}(D)^n, \|\cdot\|^{2n}_{X/S}\big)\Big)\Big]^{[4]}. \quad (4.4.18)$$

From (4.4.18), wee see that it is enough to prove (4.1.11) for the norms $\|\cdot\|^{\omega,0}_{X/S}, \|\cdot\|^0_{X/S}$ instead of the norms $\|\cdot\|^{\omega}_{X/S}, \|\cdot\|_{X/S}$.

By Proposition 4.2.12, (4.3.23), (4.3.24), and the fact that (4.3.25) is constant, we deduce

$$\begin{aligned} &c_1\Big(\lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big), \|\cdot\|_Q\,(g^{TX_t}_{\mathrm{cmp}}, h^\xi\otimes(\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n})^2\Big) \\ &\qquad\qquad - c_1\Big(\lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big), \|\cdot\|_Q\,(g^{TX_t}_0, h^\xi\otimes(\|\cdot\|^0_{X/S})^{2n})^2\Big) \\ &= \pi_*\Big[\mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,0}_{X/S})^{-2}\big)c_1(\xi, h^\xi)\mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^0_{X/S})^{2n}\big) \\ &\qquad\qquad - \mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S})^{-2}\big)c_1(\xi, h^\xi)\mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}\big)\Big]^{[4]}. \end{aligned} \quad (4.4.19)$$

Now, since the norms $\|\cdot\|^{\omega,0}_{X/S}, \|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$ and $\|\cdot\|^0_{X/S}, \|\cdot\|^{\mathrm{cmp}}_{X/S}$ coincide away from $\cup V'_i$, and over $\cup V'_i$ they vary only in the horizontal direction, we deduce that

$$\Big[\mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,0}_{X/S})^{-2}\big)\mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^0_{X/S})^{2n}\big)$$
$$- \mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S})^{-2}\big)\mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}\big)\Big]^{[4]} = 0. \quad (4.4.20)$$

Thus, by (4.4.20), we can interpret $c_1(\xi, h^\xi)$ in the right-hand side of (4.4.19) as the Chern form $\mathrm{ch}(\xi, h^\xi)$. By Poincaré-Lelong formula and (4.1.9), we deduce that

$$\mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,0}_{X/S})^{-2}\big)^{[2]} = \mathrm{Td}\big(\omega_{X/S}(D)^{-1}, (\|\cdot\|^0_{X/S})^{-2}\big)^{[2]} + \delta_{D_{X/S}}/2. \quad (4.4.21)$$

Now, by Theorem 4.4.8 applied for $\xi := \xi\otimes\omega_{X/S}(D)^n$, $h^\xi := h^\xi\otimes(\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}$ and the metric $g^{TX_t}_{\mathrm{cmp}}$, $t\in U$ induced by $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$, (4.4.19), (4.4.20) and (4.4.21), we deduce Theorem D in Assumption **S1**.

Let's treat Assumption S2. First of all, from Proposition 4.3.1, the current (4.1.10) has log-log growth along Δ. Also, since (4.1.10) is a pushforward of a closed current, it is a d-closed current over $S\setminus|\Delta|$. By Lemma 4.4.1, the L^1-trivial extension of this current is also d-closed. Thus, by Theorem C2 and Proposition 4.4.4, we see that it is enough to prove Theorem D over $S\setminus|\Delta|$ without the boundary term δ_Δ.

Now, as before, the statement is local over the base. So, for any $t_0\in S\setminus|\Delta|$, it is enough to prove (4.1.11) in some neighbourhood $U\subset S\setminus|\Delta|$ of t_0.

Trivially, (4.4.18) still holds over U under Assumption **S2** over U. Similarly (4.4.19) also continues to hold over U. Thus, by (4.4.18)-(4.4.21), we deduce that it is enough to prove Theorem D

for $\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}, \|\cdot\|^{\mathrm{cmp}}_{X/S}$ in place of $\|\cdot\|^{\omega}_{X/S}, \|\cdot\|_{X/S}$. However, by Theorem 4.4.8 (in the current situation it reduces to the special case of the curvature theorem of Bismut-Gillet-Soulé [23, Theorem 1.9]), we get

$$c_1\Big(\lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big), \|\cdot\|_Q\,(g^{TX_t}_{\mathrm{cmp}}, h^{\xi}\otimes(\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n})^2\Big)$$
$$= -\pi_*\Big[\mathrm{Td}\big(\omega^{-1}_{X/S}, (\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S})^{-2}\big)\mathrm{ch}(\xi, h^{\xi})\mathrm{ch}\big(\omega_{X/S}(D)^n, (\|\cdot\|^{\mathrm{cmp}}_{X/S})^{2n}\big)\Big]^{[4]}, \quad (4.4.22)$$

which finishes the proof of Theorem D under Assumption **S2**. □

Proof of Corollary 4.1.7. Fix a local holomorphic frame v of $\mathscr{L}_n$. Then by Theorems C2, D and (4.1.8), we have the identity

$$\frac{\partial\overline{\partial}\ln(\|\cdot\|^2_{\mathscr{L}_n})}{2\pi\sqrt{-1}} = -12\pi_*\Big[\mathrm{Td}(\omega_{X/S}(D)^{-1}, \|\cdot\|^{-2}_{X/S})\mathrm{ch}(\xi, h^{\xi})\mathrm{ch}(\omega_{X/S}(D)^n, \|\cdot\|^{2n}_{X/S})\Big]. \quad (4.4.23)$$

However, by Theorem C3, the function G is continuous, which finishes the proof by (4.4.23). □

Proof of Corollary 4.1.8. It follows from Theorem C3, (4.4.23) and the regularity theory of elliptic partial differential equations (cf. [62, Corollary 8.11]). □

4.5 Applications to the moduli space of stable pointed curves

In this section we apply the results of Sections 4.3, 4.4 to study the Hodge line bundle on the moduli space of pointed curves. This section is organized as follows: in Section 5.1, we recall the local description of the moduli space $\overline{\mathscr{M}}_{g,m}$ of m-pointed stable curves of genus g and of the universal projection map $\Pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$. Then we recall the definition of the Weil-Petersson metric with Wolpert theorem, expressing it as a push-out of Chern forms under the universal projection map. In Section 5.2 we recall the pinching expansion of the hyperbolic metric. From this, we see that the twisted canonical line bundle $\omega_{g,m}(D)$ over $\overline{\mathscr{C}}_{g,m}$ (see (4.1.12)) satisfies Assumptions **S2**, **S3**. Then we prove Corollaries 4.1.10, 4.1.12, 4.1.16, 4.1.18, 4.1.20.

4.5.1 Orbifold structure of $\overline{\mathscr{M}}_{g,m}$ and $\overline{\mathscr{C}}_{g,m}$

We follow closely the expositions of Wolpert [116], and we use the notation from Section 4.1.

We fix $\mathbf{M} := (\overline{M}, D_M) \in \mathscr{M}_{g,m}$. Let Γ be a Fuchsian group of type (g, m) such that $M := \overline{M} \setminus D_M$ is isomorphic to the quotient $\Gamma \setminus \mathbb{H}$ of the hyperbolic space. Recall that the space of Beltrami differentials $H^1(\overline{M}, T^{1,0}\overline{M}\otimes\mathscr{O}_{\overline{M}}(-D_M))$ with the obvious action by the automorphisms group $\mathrm{Aut}(\mathbf{M})$ gives a local chart for $\mathscr{M}_{g,m}$ in the following way. We take $[\nu_0] \in H^1(\overline{M}, T^{1,0}\overline{M}\otimes\mathscr{O}_{\overline{M}}(-D_M))$. By locally resolving $\overline{\partial}$-equation around the cusps, we may choose a representative $\nu \in \mathscr{C}^\infty(\overline{M}, \overline{\omega}_{\overline{M}}\otimes T^{1,0}\overline{M}\otimes\mathscr{O}_{\overline{M}}(-D_M))$ in the class $[\nu_0]$, which has compact support in $M :=$

$\overline{M} \setminus D_M$. Denote by $\nu_{\mathbb{H}}$ the pull-back of ν on $\mathbb{H}$. By a theorem of Ahlfors [1, Theorem V.5], if $|\nu|_{\mathscr{C}^0} < 1$, then the Beltrami equation

$$\begin{cases} \overline{\partial} f^{\nu}(z) = \nu_{\mathbb{H}}(z)\partial f^{\nu}(z), & \text{for } z \in \mathbb{H}, \\ \overline{\partial} f^{\nu}(z) = \overline{\nu_{\mathbb{H}}}(\overline{z})\partial f^{\nu}(z), & \text{for } z \in \mathbb{C} \setminus \mathbb{H}, \end{cases} \tag{4.5.1}$$

has a unique solution in the class of diffeomorphisms of $\mathbb{C} \cup \{\infty\}$, fixing $0, 1, \infty \in \mathbb{C} \cup \{\infty\}$. We denote

$$\Gamma^{\nu} := f^{\nu}\Gamma(f^{\nu})^{-1}, \tag{4.5.2}$$

then, classically (cf. [1, p. 69]), Γ^{ν} is the Fuchsian group of type (g, m), and f^{ν} defines a diffeomorphism

$$\tilde{f}^{\nu} : \Gamma \setminus \mathbb{H} \to \Gamma^{\nu} \setminus \mathbb{H}, \tag{4.5.3}$$

which is holomorphic if and only if $[\nu_0] = 0$.

Now, we choose $\nu_1, \ldots, \nu_N \in \mathscr{C}_c^{\infty}(M, \overline{\omega}_{\overline{M}} \otimes T^{1,0}\overline{M} \otimes \mathscr{O}_{\overline{M}}(-D_M))$ such that the associated cohomology classes form a basis in $H^1(\overline{M}, T^{1,0}\overline{M} \otimes \mathscr{O}_{\overline{M}}(-D_M))$. Let $c > 0$ be small enough. For $s = (s_1, \ldots, s_N) \in D(c)^N$, we denote

$$\nu(s) := \sum s_i \nu_i. \tag{4.5.4}$$

Now, let (U_α, z_α) be an atlas of $\overline{M}$. Then $W_\alpha := (\tilde{f}^{\nu(s)} \circ z_\alpha, s)$ is a chart mapping $U_\alpha \times D(c)^N \to \mathbb{C}^{N+1}$. This defines a holomorphic atlas on $X_0 := \cup_{s \in S}(\Gamma^{\nu(s)} \setminus \mathbb{H})$, for which the obvious projection $\pi_0 : X_0 \to D(c)^N$ is a holomorphic submersion of codimension 1 (cf. [116, §2.4.C]). Now, since the sections $\nu_1, \ldots, \nu_N$ have compact support in M, by (4.5.1), the local coordinate z_i^M of $\overline{M}$, centered at $P_i^M \in D_M$ extends to a holomorphic function $z_i : U \subset X_0 \to D^*(\epsilon)$, for some $\epsilon > 0$ and open neighbourhood U of P_i^M. Thus, the conformal completion of $D^*(\epsilon)$ induces the compactification X of X_0 such that $X \setminus X_0 = \cup_{i=1}^m \mathrm{Im}(\sigma_i)$ for some non-intersecting holomorphic functions $\sigma_i : D(c)^N \to X$. Also, trivially, the action of $\mathrm{Aut}(\mathbf{M})$ over $H^1(\overline{M}, T^{1,0}\overline{M} \otimes \mathscr{O}_{\overline{M}}(-D_M))$ induces the action on X, which preserves $\sigma_1, \ldots, \sigma_m$.

By Serre duality, for $\mathbf{M} := (\overline{M}, D_M) \in \mathscr{M}_{g,m}$, we have the isomorphism

$$H^1(\overline{M}, T^{1,0}\overline{M} \otimes \mathscr{O}_{\overline{M}}(-D_M)) \simeq H^0(\overline{M}, \omega_{\overline{M}}^2 \otimes \mathscr{O}_{\overline{M}}(D_M)). \tag{4.5.5}$$

By the uniformization theorem, there is the unique hyperbolic metric g_{hyp}^{TM} of constant scalar curvature -1 over M with cusps at D_M. We endow the space $H^0(\overline{M}, \omega_{\overline{M}}^2 \otimes \mathscr{O}_{\overline{M}}(D_M)) \subset \mathscr{C}^{\infty}(\overline{M}, \omega_{\overline{M}}^2 \otimes \mathscr{O}_{\overline{M}}(D_M))$ with the L^2-scalar product from (4.2.17). This defines the Kähler metric on $\overline{\mathscr{M}}_{g,m}$, which is called the *Weil-Petersson metric*. The *Weil-Petersson form*, which we denote by ω_{WP}, is the Kähler form associated with the Weil-Petersson metric.

By the uniformization theorem, the relative canonical line bundle $\omega_{g,m}$ of Π can be endowed with the Hermitian metric $\|\cdot\|_{g,m}^{\omega,\mathrm{hyp}}$ over $\mathscr{C}_{g,m}$ in such a way that the restriction of this metric over each fiber induces the hyperbolic Kähler metric of constant scalar curvature -1 on the fibers. By Teichmüller theory, this metric is smooth over $\mathscr{C}_{g,m}$. Let $D_{g,m}$ be the divisor in $\overline{\mathscr{C}}_{g,m}$, which is

formed by the fixed points of the fibers. We endow the twisted canonical line bundle $\omega_{g,m}(D)$ (cf. (4.1.12)) with the induced norm $\|\cdot\|^{\mathrm{hyp}}_{g,m}$ as in Construction 4.1.1. The following interpretation of ω_{WP} lies in the core of our applications.

Theorem 4.5.1 (Wolpert [114, Corollary 5.11], (cf. [58, Corollary 5.2.2])). *The following identity of smooth forms over $\mathscr{M}_{g,m}$ holds:*

$$\omega_{WP} = \pi^2 \Pi_* \Big[c_1\big(\omega_{g,m}(D), \|\cdot\|^{\mathrm{hyp}}_{g,m}\big)^2 \Big]. \tag{4.5.6}$$

Now, to describe the local structure of $\overline{\mathscr{M}}_{g,m}$, $\overline{\mathscr{C}}_{g,m}$ near the boundary, we describe the deformations of a pointed complex curve

$$(\overline{R}, D_R) \in \partial \mathscr{M}_{g,m}, \quad D_R \subset \overline{R}, \quad \#D_R < \infty \tag{4.5.7}$$

with double-point singularities $\Sigma_R = \{q_1, \dots, q_l\} \subset \overline{R}$, $\Sigma_R \cap D_R = \emptyset$.

We write$\overline{R} \setminus \Sigma_R = \cup_{i=1}^k R_i$, for some open Riemann surfaces R_i. Then D_R induces the marked points $D^0_{R_i}$ on R_i. We compactify each R_i to $\overline{R}_i$ by filling the created punctures, appearing after deletion of the nodes, and denote

$$R^0 = \overline{R} \setminus (\Sigma_R \cup D_R), \quad D_{R_i} := D^0_{R_i} \cup (\overline{R}_i \setminus R_i). \tag{4.5.8}$$

Suppose that for any $i = 1, \dots, k$, the marked surfaces $(\overline{R}_i, D_{R_i})$ are stable. We describe small deformations of $(\overline{R}, D_R)$ in terms of small deformations of $(\overline{R}_i, D_{R_i})$ and so-called *plumbing construction*, which we are going to describe now.

For every $j = 1, \dots, l$, the complex curve $\overline{R} \setminus \{q_j\}$, has a couple of punctures $\{a_j, b_j\}$ at the place of q_j. For the punctures q_j, we consider

1. A neighbourhood A_j of the puncture a_j, biholomorphic to a punctured disc. We denote by U_j the conformal completion of A_j, obtained by formally adding a_j. Then U_j is biholomorphic to an open disc. Let $F_j : U_j \to \mathbb{C}$ be a holomorphic coordinate mapping with $F_j(a_j) = 0$;

2. Similarly, a neighbourhood B_j of the puncture b_j, its conformal completion V_j and a coordinate mapping $G_j : V_j \to \mathbb{C}$ satisfying $G_j(b_j) = 0$;

3. A small complex parameter $t_j \in \mathbb{C}$.

We suppose that the sets A_j and B_j are mutually disjoint for $j = 1, \dots, l$, and they are disjoint from D_R. Let $c > 0$ be such that $D(c) \subset \mathbb{C}$ is contained in $\mathrm{Im}(F_j), \mathrm{Im}(G_j)$, for all j. Assume that $|t_j| < c^2$, for all j. We denote $t = (t_1, \dots, t_l) \in D(c^2)^l$. For $d = (d_1, \dots, d_l) \in D(c)^l$, we note

$$R^{d,*} = R^0 \setminus \cup_{j=1}^l \big(\{|F_j| \le |d_j|\} \cup \{|G_j| \le |d_j|\}\big). \tag{4.5.9}$$

Consider the equivalence relation on points of $R^{t/c,*}$ generated by: $p \sim q$ if there exists $j = 1, \dots, l$, such that $|t_j|/c \le |F_j(p)| \le c$, $|t_j|/c \le |G_j(q)| \le c$ and $F_j(p)G_j(q) = t_j$. Form the identification space $\overline{R}_t = R^{t/c,*}/\sim$. By the construction, D_R induces the set of points D_{R_t} on

$\overline{R}_t$. We say that the compact pointed complex curve $(\overline{R}_t, D_{R_t})$ is the *plumbing construction* for (R, D_R) associated with the *plumbing data* $\{(U_j, V_j, F_j, G_j, t_j)\}_j$. Trivially, we see that a set $X := \cup_{t\in D(c^2)^l} R_t$ can be endowed with a structure of a complex manifold, for which $\pi : X \to D(c^2)^l$ is a proper holomorphic map of codimension 1.

Now let's present a construction which combines the deformations using Beltrami differentials and the plumbing families.

Construction 4.5.2. Let $(\overline{R}, D_R)$, Σ_R, $(\overline{R}_i, D_{R_i})$, $i = 1, \dots, k$ be as in (4.5.7), (4.5.8). Choose a plumbing data $\{(U_j, V_j, F_j, G_j, t_j)\}_j$. Observe that one can take U_j, V_j so small so that there are Beltrami differentials $\nu_1, \dots, \nu_N$ such that each of them is compactly supported in exactly one connected component of $\overline{R} \setminus \cup_j(U_j \cup V_j)$ and the associated cohomology classes $[\nu_1], \dots, [\nu_N]$ form a basis in $\oplus_{i=1}^k H^1(\overline{R}_i, T^{1,0}\overline{R}_i \otimes \mathscr{O}_{\overline{R}_i}(-D_{R_i}))$. To simplify the exposition, we suppose that $k = 1$, i.e. that R^0 is connected. Let $\nu(s)$, $s \in D(c)^N$ be defined as in (4.5.4) for c small enough.

Let Γ be a Fuchsian group such that R^0 is isomorphic to the quotient $\Gamma \setminus \mathbb{H}$. We write $\Gamma^s := f^{\nu(s)}\Gamma(f^{\nu(s)})^{-1}$ for $f^{\nu(s)}$ as in (4.5.1) and define a Riemann surface $R^0_s := \Gamma^s \setminus \mathbb{H}$. Observe that since the support of $\nu(s)$ is contained in $\overline{R} \setminus \cup_j(U_j \cup V_j)$, by (4.5.1),the coordinates F_j, G_j induce holomorphic charts on R^0_s. We can complete R^0_s by adding points representing $\{F_j = 0\}$ and $\{G_j = 0\}$. By identifying those pairs of points, we get a compact complex curve $\overline{R}_s$. The set of singular points of $\overline{R}_s$ is in the obvious bijection with Σ_R. Now, again by the fact that the support of $\nu(s)$ is contained in $\overline{R} \setminus (\cup_j(U_j \cup V_j))$, the plumbing data $\{(U_j, V_j, F_j, G_j, t_j)\}_j$ on $\overline{R}$ induces the plumbing data on $\overline{R}_s$. Thus, for $t \in D(c)^l$, we form a complex curve $\overline{R}_{s,t}$.

Proposition 4.5.3 (Wolpert [116, p. 434])**.** Construction 4.5.2 has the following properties:

a) The complex parameters (s, t) in $S := D(c)^{N+l}$ are local coordinates for local manifold covers of $\overline{\mathscr{M}}_{g,m}$ in a neighbourhood of a point defined by $(\overline{R}, D_R)$. The divisor of singular curves Δ is given by $\{z_{N+1} = 0\} + \cdots + \{z_{N+l} = 0\}$, thus, has normal crossings.

b) The set $X := \cup_{(s,t)\in S}\overline{R}_{s,t}$ can be endowed with a structure of a complex manifold such that the projection $\pi : X \to S$ is a f.s.o.

c) The fixed points D_R induce the holomorphic sections $\sigma_1, \dots, \sigma_m : S \to X$. Then $(\pi, \sigma_1, \dots, \sigma_m)$ provides a description for local manifold covers of $\Pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$.

4.5.2 Pinching expansion and proof of Corollaries 4.1.10, 4.1.12, 4.1.16, 4.1.18, 4.1.20

In this section we will explain why the hyperbolic metric over the universal curve satisfies Assumptions **S2**, **S3**. For this, we recall the *pinching expansion* of the hyperbolic metric. Then we establish Corollaries 4.1.10, 4.1.12, 4.1.16, 4.1.18, 4.1.20.

Pinching expansion describes the behaviour of the hyperbolic metric near the boundary of the universal curve. It compares the hyperbolic metric with so-called *grafted metric*, which is more accessible for analysis. We follow closely the description of Wolpert [116].

Let $(\overline{R}, D_R)$, R^0, Σ_R, $(\overline{R}_i, D_{R_i})$, $i = 1, \dots, k$ be as in Section 4.5.1. Let $\{(U_j, V_j, F_j, G_j, t_j)\}_j$ be a plumbing data for $\overline{R}$. Consider the plumbing construction $R_{s,t}$, $(s, t) \in S := D(c)^{N+l}$, $c > 0$. The grafted metric is built from the hyperbolic metric on R^0 and the *hyperbolic metric on*

a cylinder, see (4.5.16), (4.5.18). Let's describe this construction more precisely.

Let $g_{\mathrm{hyp}}^{TR^0}$ be the hyperbolic metric of constant scalar curvature -1 with cusps on $\cup_{i=1}^k(\overline{R}_i, D_{R_i})$. Let u_j, v_j, $j = 1, \dots, l$ be some Poincaré-compatible coordinates around $\cup_{i=1}^k D_{R_i}$ with respect to g^{TR^0}. We denote

$$\alpha_j = \left|(F_j \circ u_j^{-1})'(0)\right|^{-1}, \qquad \beta_j = \left|(G_j \circ v_j^{-1})'(0)\right|^{-1}. \tag{4.5.10}$$

We renormalize the coordinates

$$f_j = \alpha_j F_j, g_j = \beta_j G_j, \quad \tau_j := \alpha_j \beta_j t_j. \tag{4.5.11}$$

Trivially from Section 4.5.1, the plumbing construction for $(U_j, V_j, F_j, G_j, t_j)$ coincides with the plumbing construction for $(U_j, V_j, f_j, g_j, \tau_j)$.

Let $\nu_{1,0} : R^0 \to [0, 1]$, $\nu_2 : \mathbb{R} \to [0, 1]$ be smooth functions, satisfying

$$\nu_{1,0}(x) = \begin{cases} 1, & \text{for } x \in R^{(1+\delta)c,*}, \\ 0, & \text{for } x \in R^0 \setminus R^{(1-\delta)c,*}. \end{cases} \tag{4.5.12}$$

$$\nu_2(w) \ = \begin{cases} 0, & \text{for } w < 1/2 - 2\delta, \\ 1, & \text{for } w > 1/2 + 2\delta. \end{cases} \tag{4.5.13}$$

Since the function $\nu_{1,0}$ is zero in the pinching collar, it induces the functions $\nu_{1,s,t} : R_{s,t} \to [0, 1]$, $(s, t) \in D(c)^{N+l}$ by zero away from $R^{(1-\delta)c,*}$.

Let $0 < c, \delta < 1$ be some small real constants. Since $f_j'(0), g_j'(0) = 1$, the inner boundary of annuli $\{c < |f_j| < 2c\}$, $\{c < |g_j| < 2c\}$ are approximately $\{u_j = c\}$ and $\{v_j = c\}$ respectively.

We suppose that $c > 0$ is chosen in such a way that the metric g^{TR^0} is induced by

$$\begin{aligned} &\frac{\sqrt{-1}du_j d\overline{u}_j}{|u_j \log |u_j|^2|^2}, \quad \text{over } \big\{|f_j| < 2c\big\}, \\ &\frac{\sqrt{-1}dv_j d\overline{v}_j}{|v_j \log |v_j|^2|^2}, \quad \text{over } \big\{|g_j| < 2c\big\}. \end{aligned} \tag{4.5.14}$$

We denote by $g_{j,1}^{\mathrm{Cyl}}$ the metric over the subset of X_t, given by

$$\big\{|\tau_j|^{1/2+2\delta} < |f_j| < e^{2\delta}c\big\} = \big\{e^{-2\delta}|\tau_j|/c < |g_j| < |\tau_j|^{1/2-2\delta}\big\}, \tag{4.5.15}$$

which is induced by the Kähler form

$$\left(\frac{\pi}{|u_j| \log |\tau_j|}\left(\sin \frac{\pi \log |u_j|}{\log |\tau_j|}\right)^{-1}\right)^2 \sqrt{-1}du_j d\overline{u}_j. \tag{4.5.16}$$

Similarly, we denote by $g_{j,2}^{\mathrm{Cyl}}$ the metric over the subset of X_t, given by

$$\big\{|\tau_j|^{1/2+2\delta} < |g_j| < e^{2\delta}c\big\} = \big\{e^{-2\delta}|\tau_j|/c < |f_j| < |\tau_j|^{1/2-2\delta}\big\}, \tag{4.5.17}$$

which is induced by the Kähler form

$$\left(\frac{\pi}{|v_j|\log|\tau_j|}\left(\sin\frac{\pi\log|v_j|}{\log|\tau_j|}\right)^{-1}\right)^2\sqrt{-1}dv_j d\overline{v}_j. \tag{4.5.18}$$

The *grafted metric* $g_{\text{gft}}^{TR_{s,t}}$ is given by

$$g_{\text{gft}}^{TR_{s,t}} = \left(g_{j,1}^{\text{Cyl}}(g_{j,2}^{\text{Cyl}}/g_{j,1}^{\text{Cyl}})^{\nu_2(\log|f_j|/\log|\tau_j|)}\right)^{1-\nu_{1,s,t}}(g_{\text{hyp}}^{TR_0})^{\nu_{1,s,t}}. \tag{4.5.19}$$

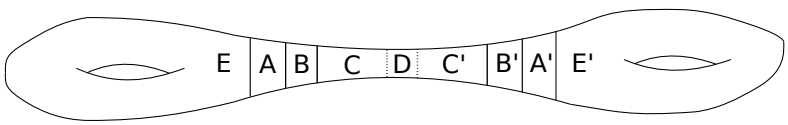

Figure 4.1: The grafted metric. Over the regions E,E', the metric $g_{\text{gft}}^{TR_{s,t}}$ is isometric to $g_{\text{hyp}}^{TR^0}$. Over the regions A, A' the metric $g_{\text{gft}}^{TR_{s,t}}$ is Poincaré-compatible with u_j and v_j respectively (see (4.2.2)). Over the regions B, B' the metric $g_{\text{gft}}^{TR_{s,t}}$ is a geometric interpolation between the Poincaré metric and (4.5.16), (4.5.18) respectively. Over the regions C, C' the metric $g_{\text{gft}}^{TR_{s,t}}$ is given by (4.5.16) and (4.5.18) respectively. Finally, over the region D, the metric $g_{\text{gft}}^{TR_{s,t}}$ is a geometric interpolation between (4.5.16) and (4.5.18). The dashed lines represent that the boundary of the region, given by the conditions $\{|f_j| = \text{const}\}$, $\{|g_j| = \text{const}\}$, and filled lines represent the boundary given by $\{|u_j| = \text{const}\}$, $\{|v_j| = \text{const}\}$.

Remark 4.5.4. If $f_j = u_j$ and $g_j = v_j$, then since $u_j v_j = \tau_j$, the metrics $g_{j,2}^{\text{Cyl}}$, $g_{j,1}^{\text{Cyl}}$ coincide over the set $\{|\tau_j|^{1/2+2\delta} < |f_j| < |\tau_j|^{1/2-2\delta}\}$, and the formula for $g_{\text{gft}}^{TR_{s,t}}$ becomes simpler. This corresponds to the *model grafting* in the terminology of Wolpert [116].

Let's recall the pinching expansion of the hyperbolic metric. Let's denote by $g_{\text{hyp}}^{TR_{s,t}}$ the hyperbolic metric with cusps on $(R_{s,t}, D_{R_{s,t}})$. The following results was proved in the compact case by Wolpert [116, Expansion 4.2] and in the non-compact case by Freixas [58, Theorem 4.3.1]:

Theorem 4.5.5 (The pinching expansion)**.** *For $(s,t) \in S \setminus |\Delta|$, we have*

$$g_{\text{hyp}}^{TR_{s,t}} = g_{\text{gfh}}^{TR_{s,t}}\Big(1 + \sum O\big(|\log|t_j||^{-2}\big)\Big), \tag{4.5.20}$$

where the O-term is for the $\mathscr{C}^\infty$ norm over $R_{s,t} \setminus D_{R_{s,t}}$ with respect to $g_{\text{hyp}}^{TR_{s,t}}$.

The metrics $g_{\text{hyp}}^{TR_{s,t}}$ induce the Hermitian norm $\|\cdot\|_{g,m}^{\omega,\text{hyp}}$ on $\omega_{g,m}$ over $\mathscr{C}_{g,m} \setminus D_{g,m}$ from Section 4.5.1. By Teichmüller theory, the norm $\|\cdot\|_{g,m}^{\omega,\text{hyp}}$ is smooth over $\mathscr{C}_{g,m} \setminus D_{g,m}$. We denote by $\|\cdot\|_{g,m}^{\text{hyp}}$ the induced norm on $\omega_{g,m}(D)$. We denote by $\Sigma_{g,m}$ the set on double points singularities in $\overline{\mathscr{C}}_{g,m}$. Then

Proposition 4.5.6. The hyperbolic metric $\|\cdot\|^{\mathrm{hyp}}_{g,m}$ satisfies Assumptions **S2** and **S3**.

Proof. We recall that the goodness and continuity of the hyperbolic metric was proved for compact surfaces by Wolpert in [116, Theorem 5.8] and for non-compact by Freixas in [58, Theorem 4.0.1]. By [36, Lemma 4.26], any good line bundle is pre-log-log. Thus, Assumption **S2** is satisfied by $\|\cdot\|^{\mathrm{hyp}}_{X/S}$.

By Proposition 4.5.3c), it is enough to prove that f.s.c. $(\pi : X \to S; \sigma_1, \ldots, \sigma_m; \|\cdot\|^{\mathrm{hyp}}_{X/S})$ from Construction 4.5.2 satisfies Assumptions **S3**.

First of all, since the metric $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ has constant scalar curvature -1, we see that the coupling of $c_1(\omega_{X/S}(D), (\|\cdot\|^{\mathrm{hyp}}_{X/S})^2)$ with two vertical vector fields is expressed through the coupling of the fiberwise volume form. By the continuity and goodness of $\|\cdot\|^{\mathrm{hyp}}_{X/S}$, we deduce that the coupling of $c_1(\omega_{X/S}(D), (\|\cdot\|^{\mathrm{hyp}}_{X/S})^2)$ with smooth vertical vector fields is continuous over $X \setminus (\Sigma_{X/S} \cup |D_{X/S}|)$ and has log-log growth on $X \setminus \Sigma_{X/S}$ with singularities along $|D_{X/S}|$.

Now let's prove the fact that $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ has log-log growth with singularities along $\Sigma_{X/S} \cup |D_{X/S}|$. First of all, by Theorem 4.5.5, it is enough to prove so for the Hermitian norm $\|\cdot\|^{\mathrm{gft}}_{X/S}$ induced by $g^{TR_{s,t}}_{\mathrm{gft}}$ on $\omega_{X/S}(D)$. Trivially, we have

$$\frac{2}{\pi}x \leq \sin(x) \leq x, \qquad \text{for } x \in [0, \pi/2]. \tag{4.5.21}$$

We fix $C > 0$ such that over $D(2c)$, we have the inequalities

$$\|f_j \circ u_j^{-1}\|_{\mathscr{C}^1} < C, \qquad \|g_j \circ v_j^{-1}\|_{\mathscr{C}^1} < C, \tag{4.5.22}$$

are satisfied. Then by the identity $f_j g_j = \tau_j$ and (4.5.22), we deduce that there is $C_1 > 0$ such that

$$\Big| \log|\tau_j| - \log|u_j| - \log|v_j| \Big| \leq C_1. \tag{4.5.23}$$

By (4.5.16), (4.5.18), (4.5.21), (4.5.22) and (4.5.23), we deduce that there is $C_2 > 0$ such that

$$\big\|du_j/u_j\big\|^{\mathrm{gft}}_{X/S} \leq \begin{cases} C_2|\log|u_j||, & \text{over} \quad \big\{|\tau_j|^{1/2} < |g_j| < e^{2\delta}c\big\}, \\ C_2|\log|v_j||, & \text{over} \quad \big\{|\tau_j|^{1/2} < |f_j| < e^{2\delta}c\big\}, \end{cases} \tag{4.5.24}$$

which implies by (4.2.13) that $\|\cdot\|^{\mathrm{gft}}_{X/S}$ has log-log growth with singularities along $\Sigma_{X/S} \cup |D_{X/S}|$. □

Now let's explain some applications of Sections 4.3, 4.4. But before we drag the attention of the reader to the fact that $\overline{\mathscr{M}}_{g,m}$ is an orbifold. However, since all the theorems of this article are local, they can be applied in an orbifold chart, and the final statements continues to hold for families of complex curves over an orbifold.

Proof of Corollaries 4.1.10. It is a direct consequence of Theorem C, Corollary 4.1.8, Remark 4.3.5 and Proposition 4.5.6. □

Proof of Corollary 4.1.12. It follows from Theorem D and Proposition 4.5.6. □

Proof of Corollary 4.1.16. It follows from Theorem 4.5.1, Corollary 4.1.7 and Proposition 4.5.6. □

Proof of Corollary 4.1.18. The decomposition (4.1.16) follows directly from Theorems C2, 4.5.1, Proposition 4.5.6 and (4.2.33). Now let's prove the identity (4.1.18). From (4.1.16), it is enough to prove that for $N := 3g - 3 + m$ and any $i = 1, \dots, N$, we have

$$\int_{\mathscr{M}_{g,m}} \alpha^{N-i}(d\beta)^i = 0. \tag{4.5.25}$$

Since $d\beta$ has log-log growth along $\partial\mathscr{M}_{g,m}$ and α is smooth over $\overline{\mathscr{M}}_{g,m}$, we have

$$\int_{\mathscr{M}_{g,m}} \alpha^{N-i}(d\beta)^i = \lim_{\epsilon\to 0}\int_{\mathscr{M}_{g,m}\setminus B(\partial\mathscr{M}_{g,m},\epsilon)} \alpha^{N-i}(d\beta)^i, \tag{4.5.26}$$

where $B(\partial\mathscr{M}_{g,m},\epsilon)$ is an ϵ-tubular neighbourhood of $\partial\mathscr{M}_{g,m}$ in $\overline{\mathscr{M}}_{g,m}$. By Stokes theorem

$$\begin{aligned}\int_{\mathscr{M}_{g,m}\setminus B(\partial\mathscr{M}_{g,m},\epsilon)} &\alpha^{N-i}(d\beta)^i \\ &= \int_{\mathscr{M}_{g,m}\setminus B(\partial\mathscr{M}_{g,m},\epsilon)} d\Big(\beta\alpha^{N-i}(d\beta)^{i-1}\Big) = \int_{\partial B(\partial\mathscr{M}_{g,m},\epsilon)} \beta\alpha^{N-i}(d\beta)^{i-1}.\end{aligned} \tag{4.5.27}$$

Trivially, for any $k \in \mathbb{N}$, $p \in \mathbb{N}$, we have the following identity

$$\lim_{\epsilon\to 0}\int_{\substack{|z_0|=\epsilon\\ |z_1|,\dots,|z_k|=\epsilon}} \frac{\log|\log|z_0||^p|dz_0|}{|z_0\log|z_0||}\prod_{i=1}^{k}\frac{\sqrt{-1}\log|\log|z_i||^p dz_i d\overline{z}_i}{|z_i\log|z_i||^2} = 0 \tag{4.5.28}$$

As β, $d\beta$ have log-log growth and α is smooth, similarly to [95, Proposition 1.2], by (4.5.28):

$$\lim_{\epsilon\to 0}\int_{\partial B(\partial\mathscr{M}_{g,m},\epsilon)} \beta\alpha^{N-i}(d\beta)^{i-1} = 0 \tag{4.5.29}$$

From (4.5.26), (4.5.27) and (4.5.29), we deduce (4.5.25). □

Proof of Corollary 4.1.20. As it was proved by Deligne in [42, Proposition 8.5] for $m = 0$, and Freixas in [58, Theorem 5.1.3, Corollary 5.1.4] for $m \in \mathbb{N}$, the Hermitian norm $\|\cdot\|^{\mathrm{Del}}_{g,m}$ on the Deligne-Weil product $\langle\omega_{g,m}(D), \omega_{g,m}(D)\rangle$ is smooth over $\mathscr{M}_{g,m}$, and over $\mathscr{M}_{g,m}$ we have

$$c_1\Big(\langle\omega_{g,m}(D), \omega_{g,m}(D)\rangle, \big(\|\cdot\|^{\mathrm{Del}}_{g,m}\big)^2\Big) = \pi^{-2}\omega_{WP}. \tag{4.5.30}$$

By another result of Freixas, [58, Theorem 5.1.3], the Hermitian norm $\|\cdot\|^{\mathrm{Del}}_{g,m}$ on the Deligne-Weil product $\langle\omega_{g,m}(D), \omega_{g,m}(D)\rangle$ is nice over $\overline{\mathscr{M}}_{g,m}$, with singularities along $\partial\mathscr{M}_{g,m}$. Thus, by Remark

4.1.4, the first Chern form is well-defined as a current, and from Proposition 4.4.4, (4.5.30), we have the following identity of currents over $\overline{\mathscr{M}}_{g,m}$

$$c_1\Big(\langle\omega_{g,m}(D),\omega_{g,m}(D)\rangle,\big(\|\cdot\|_{g,m}^{\mathrm{Del}}\big)^2\Big)=\pi^{-2}\big[\omega_{WP}\big]_{L^1}. \tag{4.5.31}$$

By Corollary 4.1.12 and (4.5.31), the norm of the isomorphisms (4.1.19) and (4.1.20) is a pluri-harmonic functions over $\overline{\mathscr{M}}_{g,m}$. As $\overline{\mathscr{M}}_{g,m}$ is compact, we deduce that they are constant, which finishes the proof. □

Chapter 5

Quillen metric for singular families of Riemann surfaces with cusps.

Abstract. In this article we study the behaviour of the Quillen metric for the family of Riemann surfaces with cusps when the additional cusps are created by degeneration.

More precisely, by our previous results, we see that the renormalisation of the Quillen metric associated with a family of Riemann surfaces with cusps extends continuously over the locus of singular curves. The main result of this article shows that, modulo some explicit universal constant, this continuous extension coincides with the Quillen metric of the normalisation of singular curves. This result shows that the Quillen metric is compatible with adjunction of cusps.

As an application, we obtain the compatibility between our definition of the analytic torsion and the definition of Takhtajan-Zograf using lengths of closed geodesics. As another application, we obtain the compatibility of the Quillen metric with clutching morphisms in the moduli space of pointed stable curves.

5.1 Introduction

In this article we study the behaviour of the Quillen metric for the family of Riemann surfaces with cusps when the additional cusps are created by degeneration.

Let X and S be complex manifolds, and let $\pi : X \to S$ be a proper holomorphic map. The

construction of Grothendick-Knudsen-Mumford [76] (cf. also [23, §3]) associates for every holomorphic vector bundle ξ over X the "determinant of the direct image of ξ" - the holomorphic line bundle over S, which we denote (cf. (5.2.40))

$$\lambda(j^*\xi)^{-1} := \det(R^\bullet \pi_* \xi). \tag{5.1.1}$$

Let's fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a complex curve whose singularities are at most ordinary double points (in the terminology of [20], [55], a f.s.o.). We denote by $\Sigma_{X/S} \subset X$ the submanifold of singular points of the fibers (see Corollary 5.2.5). We denote by $\Delta = \pi_*(\Sigma_{X/S})$ the divisor formed by the locus of the singular fibers π. In this article we only consider π for which the associated divisor Δ has normal crossings.

Let $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ be disjoint holomorphic sections of π. We denote by $D_{X/S}$ the divisor on X, given by

$$D_{X/S} = \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m). \tag{5.1.2}$$

Let the norm $\|\cdot\|^{\omega}_{X/S}$ on the canonical line bundle $\omega_{X/S}$ (see (5.2.36)) over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ be such that its restriction over each nonsingular fiber $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ of π induces the Kähler metric g^{TX_t} on $X_t \setminus \{\sigma_1(t), \dots, \sigma_m(t)\}$ such that the triple $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ is a surface with cusps in the sense of [54], [93], [59] (see Section 5.2.1).

Construction 5.1.1. For a complex manifold Y and a divisor $D_0 \subset Y$, let $\|\cdot\|^{\mathrm{div}}_{D_0}$ be the singular norm on $\mathscr{O}_Y(D_0)$, defined by

$$\|s_{D_0}\|^{\mathrm{div}}_{D_0}(x) = 1, \quad \text{for any } x \in Y \setminus D_0, \tag{5.1.3}$$

where s_{D_0} is the canonical section of the divisor D_0 with $\mathrm{div}(s_{D_0}) = D_0$.

We endow the *twisted canonical line bundle*

$$\omega_{X/S}(D) := \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S}) \tag{5.1.4}$$

with the canonical norm $\|\cdot\|_{X/S}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, induced by $\|\cdot\|^{\omega}_{X/S}$ and $\|\cdot\|^{\mathrm{div}}_{D_{X/S}}$.

Let (ξ, h^ξ) be a holomorphic Hermitian vector bundle over X. Let $h^{\det \xi}$ be the induced Hermitian metric on $\det \xi := \Lambda^{\max}\xi$. In [54, §2.1], we've seen that for $n \leq 0$, the L^2-scalar product[1]

$$\langle \alpha, \alpha' \rangle_{L^2} := \frac{1}{2\pi} \int_{X_t} \langle \alpha(x), \alpha'(x) \rangle_h dv_{X_t}(x), \quad \alpha, \alpha' \in \mathscr{C}^\infty(X_t, \xi \otimes \omega_{X/S}(D)^n), \tag{5.1.5}$$

where $\langle \cdot, \cdot \rangle_h$ is the pointwise scalar product, dv_{X_t} is the induced Riemannian volume form on (X_t, g^{TX_t}), induces the natural L^2-norm on the determinant line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$, $n \leq 0$ over $S \setminus |\Delta|$. In [54], [55] (cf. Definition 5.2.7), we've defined the Quillen norm $\|\cdot\|_Q$ $(g^{TX_t}, h^\xi \otimes$

[1] Our normalisation is different from the one used in [54], [55] by a factor 2π. The reason for such a normalisation is to make things compatible with [42], [66], [25], [59], in particular, to make Serre duality an isometry, see (5.2.30).

$\|\cdot\|^{2n}_{X/S}$) on the determinant line bundle $\lambda(j^*(\xi\otimes\omega_{X/S}(D)^n))$, $n\leq 0$ over $S\setminus|\Delta|$ as the product of the analytic torsion of the fiber $T(g^{TX_t},h^\xi\otimes\|\cdot\|^{2n}_M)$ from [54], and the L^2-norm of the fiber, see Section 5.2.1. As we explained in [55], this definition gives a non-compact 1-dimensional version of the definition of the Quillen norm of Bismut-Gillet-Soulé [23] and generalizes the definition of Quillen [102], which was given for $n,m=0$ and π, (ξ,h^ξ) trivial.

Let's denote by $\|\cdot\|^W_{X/S}$ the Wolpert norm on $\otimes_{i=1}^m\sigma_i^*(\omega_{X/S})$ induced by $\|\cdot\|^\omega_{X/S}$ (see Definition 5.2.2). The necessary definitions for the following passage are given in Definitions 5.2.11, 5.2.12.

We suppose that the Hermitian norm $\|\cdot\|_{X/S}$ on $\omega_{X/S}(D)$ extends continuously over $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$, has log-log growth with singularities along $\Sigma_{X/S}\cup|D_{X/S}|$, is good in the sense of Mumford on $X\setminus|D_{X/S}|$ with singularities along $\pi^{-1}(\Delta)$, and the coupling of $c_1(\omega_{X/S}(D),\|\cdot\|^2_{X/S})$ with two smooth vertical vector fields over $X\setminus(\Sigma_{X/S}\cup|D_{X/S}|)$ has log-log growth with singularities along $D_{X/S}$. (5.1.6)

Then in [55, Theorem C3] (cf. Theorem 5.2.8) we proved that the norm

$$\|\cdot\|_{\mathscr{L}_n}:=\big(\|\cdot\|_Q\,(g^{TX_t},h^\xi\otimes\|\cdot\|^{2n}_{X/S})\big)^{12}\otimes\big(\|\cdot\|^W_{X/S}\big)^{-\mathrm{rk}(\xi)}\otimes\big(\|\cdot\|^{\mathrm{div}}_\Delta\big)^{\mathrm{rk}(\xi)}\otimes(\otimes_{i=1}^m\sigma_i^*h^{\det\xi})^3 \tag{5.1.7}$$

on the line bundle

$$\mathscr{L}_n:=\lambda\big(j^*(\xi\otimes\omega_{X/S}(D)^n)\big)^{12}\otimes(\otimes_{i=1}^m\sigma_i^*\omega_{X/S})^{-\mathrm{rk}(\xi)}\otimes\mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)}\otimes(\otimes_{i=1}^m\sigma_i^*\det\xi)^6 \tag{5.1.8}$$

extends continuously over S. The main goal of this article is to give the precise value of this extension.

More precisely, as Δ has normal crossings, for any $t\in S$, by shrinking the base S, we may always suppose that for some $l\in\mathbb{N}$, the divisor Δ decomposes in the neighbourhood of t as

$$\Delta=k\cdot\Delta_0+k_1\cdot\Delta_1+\cdots+k_l\cdot\Delta_l, \tag{5.1.9}$$

where Δ_i, $i=0,\dots,l$ are divisors induced by the submanifolds $|\Delta_i|$ and $k,k_j\in\mathbb{N}^*$, $j=1,\dots,l$. Let $\Delta_j^0:=\Delta_j\cap\Delta_0$ be the induced divisor on $S':=|\Delta_0|$, and let Δ' be the divisor on S' given by

$$\Delta':=k_1\cdot\Delta_1^0+\cdots+k_l\cdot\Delta_l^0. \tag{5.1.10}$$

Let $\iota:S'\to S$ be the obvious inclusion. We denote $Z:=\pi^{-1}(S')$, $Z_t:=\pi^{-1}(t)$, $t\in S'$, and let $\rho:Y\to Z$ be the normalization of Z. We denote by $\pi':Y\to S'$ the family of surfaces, induced by the following commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi} \\ S' & \xrightarrow{\iota} & S \end{array} \tag{5.1.11}$$

The restriction of the holomorphic sections $\sigma_1,\dots,\sigma_m$ on S' induce the holomorphic sections, which we denote by $\sigma_1',\dots,\sigma_m':S'\to Y$.

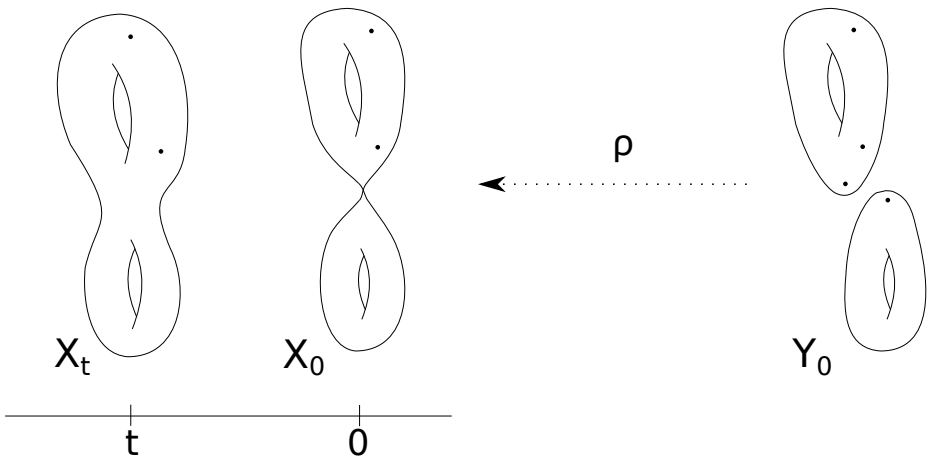

Figure 5.1: A degenerating familiy. Our goal is to relate a norm on the line bundle at the singular fiber with a norm on the line bundle on its normalisation. Points represent the elements in $D_{X/S}|_{X_t}$, $D_{X/S}|_{X_0}$ and $D_{Y/S'}|_{Y_0}$. Notice the added marked points on the normalisation.

Let $\Sigma_{Z/S'}$ be the locus of points, which get normalized in ρ. The manifold $\Sigma_{Z/S'}$ is a union of some connected components of $\Sigma_{X/S}$, thus, it has codimension 2 in X (see Corollary 5.2.5). Let

$$\kappa : \Sigma_{Z/S'} \hookrightarrow X \tag{5.1.12}$$

the obvious inclusion. Then the restriction of π' to $\rho^{-1}(\kappa(\Sigma_{Z/S'}))$ is the covering of degree $2k$, see (5.1.9). By shrinking the base, we may suppose that it is a trivial cover, so there are holomoprhic sections $\sigma'_{m+1}, \dots, \sigma'_{m+2k} : S' \to Y$ such that $\rho^{-1}(\Sigma_{Z/S'}) = \cup_{i=1}^{2k} \operatorname{Im}(\sigma'_{m+i})$ and $\rho \circ \sigma'_{m+2i-1} = \rho \circ \sigma'_{m+2i}$, $i = 1, \dots, k$. We define the divisor $D_{Y/S'}$ over Y by

$$D_{Y/S'} := \operatorname{Im}(\sigma'_1) + \cdots + \operatorname{Im}(\sigma'_{m+2k}). \tag{5.1.13}$$

We also define the *twisted canonical line bundle* of π' as follows

$$\omega_{Y/S'}(D) := \omega_{Y/S'} \otimes \mathscr{O}_Y(D_{Y/S'}). \tag{5.1.14}$$

Then, classically (cf. Section 5.2.1), we have the canonical isomorphism

$$\rho^*(\omega_{X/S}(D)) \simeq \omega_{Y/S'}(D). \tag{5.1.15}$$

Under assumptions (5.1.6), the isomorphism (5.1.15) induces the Hermitian norm$\|\cdot\|_{Y/S'}$ on $\omega_{Y/S'}(D)$ over $Y \setminus |D_{Y/S'}|$ by

$$\|\cdot\|_{Y/S'} := \rho^*(\|\cdot\|_{X/S}). \tag{5.1.16}$$

Let$\|\cdot\|^{\omega}_{Y/S'}$ be the norm on $\omega_{Y/S'}$, which is induced by$\|\cdot\|_{Y/S'}$ as in Construction 5.1.1. Then$\|\cdot\|_{Y/S'}$ and$\|\cdot\|^{\omega}_{Y/S'}$ are Hermitian norms over $Y \setminus ((\pi')^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$.

We suppose that the norm $\|\cdot\|^{\omega}_{Y/S'}$ *over* $Y \setminus (\pi^{-1}(|\Delta'|) \cup |D_{Y/S'}|)$ *is such that its restriction over each nonsingular fiber* $Y_t := \pi^{-1}(t)$, $t \in S' \setminus |\Delta'|$ *of* π' *induces the Kähler metric* g^{TY_t}, *for which the triple* $(Y_t, \{\sigma'_1(t), \ldots, \sigma'_{m+2k}(t)\}, g^{TY_t})$ *is a surface with cusps.* (5.1.17)

We denote by $\|\cdot\|^{W}_{Y/S'}$ the Wolpert norm on $\otimes_{i=1}^{m+2k}(\sigma'_i)^*\omega_{Y/S'}$, induced by $\|\cdot\|^{\omega}_{Y/S'}$. Now, similarly to (5.1.7), (5.1.8), we define the norm

$$\|\cdot\|_{\mathscr{L}'_n} := \big(\|\cdot\|_Q\,(g^{TY_t}, \rho^*(h^{\xi}) \otimes \|\cdot\|^{2n}_{Y/S'})\big)^{12} \otimes \big(\|\cdot\|^{W}_{Y/S'}\big)^{-\mathrm{rk}(\xi)} \otimes \big(\|\cdot\|^{\mathrm{div}}_{\Delta'}\big)^{\mathrm{rk}(\xi)} \otimes \big(\otimes_{i=1}^{m+2k}\,(\sigma'_i \circ \rho)^* h^{\det \xi}\big)^3 \quad (5.1.18)$$

on the line bundle

$$\mathscr{L}'_n := \lambda\big(j^*(\rho^*(\xi) \otimes \omega_{Y/S'}(D)^n)\big)^{12} \otimes (\otimes_{i=1}^{m+2k}(\sigma'_i)^*\omega_{Y/S'})^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_{S'}(\Delta')^{\mathrm{rk}(\xi)} \otimes \big(\otimes_{i=1}^{m+2k}\,(\sigma'_i \circ \rho)^* \det \xi\big)^6. \quad (5.1.19)$$

We denote by $N_{\Sigma_{Z/S'}/X}$ (resp. $N_{S'/S}$) the normal vector bundle of $\Sigma_{Z/S'}$ in X (resp. of S' in S). Since the fibers of X have only double-point singularities, the projection π induces the canonical isomorphism (see (5.2.32), cf. also [18, (2.9)])

$$d\pi^2 : \wedge^2(N_{\Sigma_{Z/S'}/X}) \otimes (\det \rho_*(\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})) \to \kappa^*\pi^* N_{S'/S}. \quad (5.1.20)$$

Also, for the relative tangent bundle TY/S' of π' and for any $i = 1, \ldots, k$, the normalization map ρ induces the canonical isomorphism

$$(\sigma'_{m+2i-1})^*(TY/S') \otimes (\sigma'_{m+2i})^*(TY/S') \to \wedge^2(N_{\Sigma_{Z/S'}/X}). \quad (5.1.21)$$

We denote by ω_S and $\omega_{S'}$ the canonical line bundles over S and S'. By combining the duals of the isomorphisms (5.1.20), (5.1.21), for $i = 1, \ldots, k$, we get the canonical isomorphism

$$(\omega_S \otimes \omega_{S'}^{-1})|_{S'} \to (\sigma'_{m+2i-1})^*(\omega_{Y/S'}) \otimes (\sigma'_{m+2i})^*(\omega_{Y/S'}). \quad (5.1.22)$$

The following isomorphism is given by Poincaré residue morphism (cf. [67, p. 147])

$$(\omega_S^k \otimes \mathscr{O}_S(k\Delta_0))|_{S'} \to \omega_{S'}^k. \quad (5.1.23)$$

By combining the isomoprhism (5.1.22), applied for each $i = 1, \ldots, k$, the isomorphism (5.1.23) and by multiplying by $(\otimes_{i=1}^m \sigma_i^*\omega_{X/S})^{-1} \otimes \mathscr{O}_S(\sum k_i\Delta_i)$, we get the canonical isomorphism

$$\Big(\big(\otimes_{i=1}^m\,\sigma_i^*\omega_{X/S}\big)^{-1} \otimes \mathscr{O}_S(\Delta)\Big)\big|_{S'} \to \big(\otimes_{i=1}^{m+2k}\,(\sigma'_i)^*\omega_{Y/S'}\big)^{-1} \otimes \mathscr{O}_{S'}(\Delta'). \quad (5.1.24)$$

For $t \in S'$, we have the following exact sequence of sheaves (cf. [18, (5.53)])

$$0 \to \mathscr{O}_{Z_t}\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big) \to \rho_*\mathscr{O}_{Y_t}\big(j^*(\rho^*(\xi) \otimes \omega_{Y/S'}(D)^n)\big) \\ \to \mathscr{O}_{\Sigma_{Z/S'}}\big(\kappa^*\xi \otimes \det(\rho_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})\big) \to 0, \quad (5.1.25)$$

where the first map is induced by the pull-back and (5.1.15), and the second map is the difference of residue morphism at $\rho^{-1}(\Sigma_{Z/S'})$. The short exact sequence (5.1.25) of the line bundles over S' induces the canonical isomorphism (cf. [18, (5.55)])

$$\lambda\big(j^*(\xi \otimes \omega_{X/S}(D)^n)\big)|_{S'} \to \lambda\big(j^*(\rho^*(\xi) \otimes \omega_{Y/S'}(D)^n)\big) \\ \otimes \det\big(\pi_*(\kappa^*(\xi))\big) \otimes \det\big((\pi\circ\rho)_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}}\big)^{\mathrm{rk}(\xi)}. \quad (5.1.26)$$

We note that the square of $\det((\pi\circ\rho)_*\mathscr{O}_{\rho^{-1}\Sigma_{Z/S'}})$ is canonically trivialized. From now on, we don't mention those powers explicitly. Trivially, we have an isomorphism

$$\det\big(\pi_*(\kappa^*(\xi))\big)^2 \to \big(\otimes_{i=1}^{2k}(\sigma'_{m+i}\circ\rho)^*\det\xi\big) \otimes \big(\det\pi_*\mathscr{O}_{\Sigma_{Z/S'}}\big)^{2\cdot\mathrm{rk}(\xi)}. \quad (5.1.27)$$

The composition of the isomorphisms (5.1.24), (5.1.26) and (5.1.27) induce the canonical isomorphism

$$\mathscr{L}_n|_{S'} \to \mathscr{L}'_n \otimes \big(\det\pi_*\mathscr{O}_{\Sigma_{Z/S'}}\big)^{12\cdot\mathrm{rk}(\xi)}, \quad (5.1.28)$$

which is the protagonist of this paper.

For $k \in \mathbb{N}^*$, we define

$$C_0 = -6\log(\pi), \qquad C_k = -6(1+k)\log(2) - 6(1+2k)\log(\pi) - 6\log((2k)!). \quad (5.1.29)$$

Now we can state the main result of this article, which describes the continuous extension of the norm (5.1.7) in terms of the same objects that we've used in the definition of (5.1.7).

Theorem 5.1.2 (Restriction theorem)**.** *Let $\pi : X \to S$ be a proper, holomorphic, surjective map of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a complex curve whose singularities are at most ordinary double points. We suppose that the divisor of singular curves Δ decomposes as in (5.1.9). We use the notation for $k \in \mathbb{N}$ and S' as in (5.1.10).*

Let $\sigma_1, \ldots, \sigma_m : S \to X$ be disjoint holomorphic sections of π, which do not pass through singular points of the fibers. We denote by $D_{X/S}$ the divisor (5.1.2).

Let $\|\cdot\|^{\omega}_{X/S}$ be a Hermitian norm on the canonical line bundle $\omega_{X/S}$ (see (5.2.36)) over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ such that its restriction over each $X_t := \pi^{-1}(t)$, $t \in S \setminus |\Delta|$ induces the Kähler metric g^{TX_t} on $X_t \setminus \{\sigma_1(t), \ldots, \sigma_m(t)\}$ such that the triple $(X_t, \{\sigma_1(t), \ldots, \sigma_m(t)\}, g^{TX_t})$ is a surface with cusps in the sense of [54], [93], [59] (see Section 5.2.1).

Let (ξ, h^{ξ}) be a holomorphic Hermitian vector bundle over X. We define $\|\cdot\|_{\mathscr{L}_n}$, $\mathscr{L}_n$ as in (5.1.7) and (5.1.8). Let let family of complex curves $\pi' : Y \to S'$ be constructed as in (5.1.11). We suppose that assumptions (5.1.6) and (5.1.17) hold. We define $\|\cdot\|_{\mathscr{L}'_n}$, $\mathscr{L}'_n$ as in (5.1.18) and (5.1.19).

Then $\|\cdot\|_{\mathscr{L}_n}$ extends continuously over S and under the isomorphism (5.1.28), we have

$$\|\cdot\|_{\mathscr{L}_n}|_{S'} = \exp(m\cdot\mathrm{rk}(\xi)\cdot C_{-n})\cdot\|\cdot\|_{\mathscr{L}'_n}. \quad (5.1.30)$$

Remark 5.1.3. a) We note that a similar theorem was proved by Bismut in [18, Theorems 0.2, 0.3] (cf Theorem 5.3.1). Let's comment on the differences between Theorem 5.1.2 and [18, Theorems 0.2, 0.3].

First of all, Theorem 5.1.2 is for codimension 1 family, where fibers are endowed with metric with cusps singularities, and the result [18, Theorems 0.2] works for families of any codimension but with smooth metric. Also, the way we endow singular fibers with the metric is crucially different. In our case even when the general fiber has no cusps, the metric on the normalization of the singular fiber acquires at least two cusps. This is different from [18, Theorems 0.2, 0.3], where author induce smooth metric on the normalization of the singular fiber. In particular, Theorem 5.1.2 doesn't follow directly from [18, Theorems 0.2, 0.3] and anomaly formula.

b) We firstly prove Theorem 5.1.2 up to some undetermined universal constant. This proof is based on our previous results concerning the Quillen norm for Riemann surfaces with cusps [54, Theorems A, B] and on the mentioned result of Bismut [18, Theorem 0.3]. The reason why we couldn't get the precise constant by this method is that our compact perturbation theorem, [54, Theorem A], is only proved in the relative setting for the moment being. In fact, to prove it in a non-relative setting we use the proof of Theorem 5.1.2.

The evaluation of the constant C_{-n} uses the calculations made by Freixas [59, Corollary 5.8], [59, Theorem 5.3] (cf. Theorem 5.2.15), which were based on the asymptotics of the Selberg zeta function due to Wolpert [115] and on a careful study of the degeneration of the L^2-norm by Freixas.

Now, let's describe our second result. We fix a compact Riemann surface $\overline{M}$ and a set of points $D_M \subset \overline{M}$, $\#D_M = m$, $m < +\infty$. We denote $M := \overline{M} \setminus D_M$. Suppose that a pointed Riemann surface $(\overline{M}, D_M)$ is stable, i.e. the genus $g(\overline{M})$ of $\overline{M}$ satisfies

$$2g(\overline{M}) - 2 + m > 0, \tag{5.1.31}$$

then, by the uniformization theorem (cf. [49, Chapter IV], [9, Lemma 6.2]), there is the *canonical hyperbolic metric* g^{TM}_{hyp} of constant scalar curvature -1 on M with cusps at D_M. We denote by $\|\cdot\|^{\mathrm{hyp}}_M$ the norm induced by g^{TM}_{hyp} on $\omega_M(D)$ over M. Then, as we explain in [54, §2.1], the triple $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ is a surface with cusps (see Section 5.2.1), thus, the analytic torsion $T(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_M)^{2n})$ is well-defined in this case.

Alternatively, we denote by $Z_{(\overline{M},D_M)}(s), s \in \mathbb{C}$ the Selberg zeta-function, which is given for $\mathrm{Re}(s) > 1$ by the absolutely converging product:

$$Z_{(\overline{M},D_M)}(s) = \prod_{\gamma}\prod_{k=0}^{\infty}(1 - e^{-(s+k)l(\gamma)})^2, \tag{5.1.32}$$

where γ runs over the set of all simple non-oriented closed geodesics on $(M, g^{TM}_{\mathrm{hyp}})$, and $l(\gamma)$ is the length of γ. The function $Z_{(\overline{M},D_M)}(s)$ admits a meromorphic extension to the whole complex s-plane with a simple zero at $s = 1$ (see for example [44, (5.3)]).

Let $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$ be the Riemann zeta function. For $k \in \mathbb{N}^*$, we put

$$\begin{aligned} c_0 &= 4\zeta'(-1) - \tfrac{1}{2} + \log(2\pi), \\ c_k &= \textstyle\sum_{l=0}^{k-1}(2k-2l-1)\big(\log(2k+2kl-l^2-l) - \log(2)\big) + (\tfrac{1}{3} + k + k^2)\log(2) \\ &\quad + (2k+1)\log(2\pi) + 4\zeta'(-1) - 2(k+\tfrac{1}{2})^2 - 4\textstyle\sum_{l=1}^{k-1}\log(l!) - 2\log(k!). \end{aligned} \tag{5.1.33}$$

For $k \in \mathbb{N}$, we denote by $B_k : \mathbb{N}^2 \to \mathbb{R}$, $E : \mathbb{N}^2 \to \mathbb{R}$ the following functions

$$\begin{aligned} B_k(g,m) &= \exp\Big((2-2g(\overline{M})-m)\frac{c_k}{2}\Big), \\ E(g,m) &= \exp\Big((g(\overline{M})+2-m)\frac{\log(2)}{3}\Big). \end{aligned} \tag{5.1.34}$$

In particular, we see that for any $k \in \mathbb{N}$, $(g,m) \in \mathbb{N}^2$, we have

$$\begin{aligned} B_k(g+m,0) &= B_k(g,m)\cdot B_k(1,1)^m, \\ E(g+m,0) &= E(g,m)\cdot E(1,1)^m. \end{aligned} \tag{5.1.35}$$

Then, in case for hyperbolic surfaces and (ξ, h^{ξ}) trivial, for $l \in \mathbb{Z}$, $l < 0$, Takhtajan-Zograf in [107, (6)] proposed[2] the analogue of the analytic torsion defined via Selbrerg zeta function as

$$\begin{aligned} T_{TZ}(g^{TM}_{\mathrm{hyp}},1) &= E(g(\overline{M}),m)\cdot B_0(g(\overline{M}),m)\cdot Z'_{(\overline{M},D_M)}(1), \\ T_{TZ}(g^{TM}_{\mathrm{hyp}},(\|\cdot\|^{\mathrm{hyp}}_M)^{2l}) &= B_{-l}(g(\overline{M}),m)\cdot Z_{(\overline{M},D_M)}(-l+1). \end{aligned} \tag{5.1.36}$$

Theorem 5.1.4 (Compatibility theorem)**.** *For any surface with cusps $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$, for which g^{TM}_{hyp} has constant scalar curvature -1, the following identity holds*

$$T(g^{TM}_{\mathrm{hyp}},(\|\cdot\|^{\mathrm{hyp}}_M)^{2n}) = T_{TZ}(g^{TM}_{\mathrm{hyp}},(\|\cdot\|^{\mathrm{hyp}}_M)^{2n}). \tag{5.1.37}$$

Thus, our definition of the analytic torsion is compatible with the definition of Takhtajan-Zograf.

Remark 5.1.5. For $m = 0$, i.e. when surfaces have no cusps, Theorem 5.1.4 was shown by Phong-D'Hoker [44, (7.30)], [45, (3.6)] (see also [104], [29, (50)] and [99, (9)]). Our proof is based on their result. We note that Albin-Rochon in [4] proved (5.1.37) up to a universal constant (see also [55, (2.43)]). Our approach to (5.1.37) is based on degenerating families, which is different from the one of Albin-Rochon.

Now let's describe the applications of Theorems 5.1.2, 5.1.4 in the study of the moduli space $\mathscr{M}_{g,m}$ of m-pointed Riemann surfaces of genus $g \in \mathbb{N}$, $2g-2+m > 0$. We denote by $\overline{\mathscr{M}}_{g,m}$ the *Deligne-Mumford compactification* of $\mathscr{M}_{g,m}$, by $\partial\mathscr{M}_{g,m} := \overline{\mathscr{M}}_{g,m} \setminus \mathscr{M}_{g,m}$ the *compactifying divisor*, by $\mathscr{C}_{g,m}$ and $\overline{\mathscr{C}}_{g,m}$ the universal curves over $\mathscr{M}_{g,m}$ and $\overline{\mathscr{M}}_{g,m}$ respectively. We denote by $\Pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$ the *universal projection*. We denote by $D_{g,m}$ the divisor on $\overline{\mathscr{C}}_{g,m}$, formed by m fixed points. We denote by $\omega_{g,m}$ the relative canonical line bundle of Π, by $\otimes_{i=1}^m \sigma_i^*\omega_{g,m}$ the determinant of the restriction of $\omega_{g,m}$ to the divisor $D_{g,m}$, and by $\omega_{g,m}(D)$ the twisted relative canonical line bundle,

$$\omega_{g,m}(D) := \omega_{g,m} \otimes \mathscr{O}_{\overline{\mathscr{C}}_{g,m}}(D_{g,m}). \tag{5.1.38}$$

By the uniformization theorem (cf. [49, Chapter IV], [9, Lemma 6.2], [10]), we endow $\omega_{g,m}(D)$ with the Hermitian norm $\|\cdot\|^{\mathrm{hyp}}_{g,m}$, such that its restriction over each fiber induces the canonical

[2]The constant in front of Selberg zeta function didn't appear in [107], as their result is independent of it.

hyperbolic metric of constant scalar curvature -1 by Construction 5.1.1. This endows the determinant line bundle $\lambda(j^*(\omega_{g,m}(D)^n))$, $n \leq 0$, (5.2.10), which is usually called the *Hodge line bundle*, with the induced Quillen metric $\|\cdot\|^{Q,n}_{g,m}$. We endow the line bundle $\otimes_{i=1}^m \sigma_i^* \omega_{g,m}$ with the associated Wolpert norm $\|\cdot\|^W_{g,m}$, see Wolpert [117, Definition 1] (cf. [54, Definition 1.5] or Definition 5.2.2).

We recall that we proved in [55, Corollary 1.11] (cf. Theorem 5.2.8) that the norm

$$\|\cdot\|^{H,n}_{g,m} := (\|\cdot\|^{Q,n}_{g,m})^{12} \otimes (\|\cdot\|^W_{g,m})^{-1} \otimes \|\cdot\|^{\mathrm{div}}_{\partial \mathscr{M}_{g,m}} \tag{5.1.39}$$

on the line bundle

$$\lambda^{H,n}_{g,m} := \lambda(j^*(\omega_{g,m}(D)^n))^{12} \otimes (\otimes \sigma_i^* \omega_{g,m})^{-1} \otimes \mathscr{O}_{\overline{\mathscr{M}}_{g,m}}(\partial \mathscr{M}_{g,m}) \tag{5.1.40}$$

extends continuously over $\overline{\mathscr{M}}_{g,m}$. Classically, $\|\cdot\|^{H,n}_{g,m}$ is smooth over $\mathscr{M}_{g,m}$.

For the definition of the clutching morphisms

$$\begin{aligned} &\alpha_{ij} : \overline{\mathscr{M}}_{g-1,m+2} \to \overline{\mathscr{M}}_{g,m}, \\ &\beta^P_{(g_1,m_1),(g_2,m_2)} : \overline{\mathscr{M}}_{g_1,m_1+1} \times \overline{\mathscr{M}}_{g_2,m_2+1} \to \overline{\mathscr{M}}_{g,m}, \end{aligned} \tag{5.1.41}$$

where $i < j$, $i,j = 1,\ldots,m+2$; $m_1, m_2 \in \mathbb{N}$, $g_1, g_2 \in \mathbb{N}$, $m_1 + m_2 = m$, $g_1 + g_2 = g$, $2g_1 + m_1 - 2 > 0$, $2g_2 + m_2 - 2 > 0$ and $P \in \{I, J \subset \{1,2,\ldots,m\} : I \cap J = \emptyset, I \cup J = \{1,2,\ldots,m\}, |I| = m_1, |J| = m_2\}$, see Knudsen [74]. We recall that the compactifying divisor $\partial \mathscr{M}_{g,m}$ can be described in terms of (5.1.41) by (cf. [8, p.262])

$$\left|\partial \mathscr{M}_{g,m}\right| = \Big(\cup \operatorname{Im}(\alpha_{ij}) \Big) \cup \Big(\cup \operatorname{Im}\left(\beta^P_{(g_1,m_1),(g_2,m_2)}\right) \Big). \tag{5.1.42}$$

From now on and till the end of this article, for brevity, we drop the subscripts from α, β.

After an application of adjunction formula, which asserts the canonical triviality of the line bundle $\Pi_*(\omega_{g,m}(D)|_{|D_{g,m}|})$, the isomorphism (5.1.28) specifies in this case to the isomoprhisms

$$\alpha^* \lambda^{H,n}_{g,m} \simeq \lambda^{H,n}_{g-1,m_2}, \tag{5.1.43}$$

$$\beta^* \lambda^{H,n}_{g,m} \simeq \lambda^{H,n}_{g_1,m_1+1} \boxtimes \lambda^{H,n}_{g_2,m_2+1}, \tag{5.1.44}$$

which also remarkably respect the natural $\mathbb{Z}$-structure of the line bundles (5.1.40), as it was proved by Knudsen in [75, Theorem 4.2] (cf. [59]).

Theorem 5.1.6 (Restriction theorem on $\overline{\mathscr{M}}_{g,m}$)**.** *a) The isomorphism (5.1.43) is an isometry if the left-hand side is endowed with* $\|\cdot\|^{H,n}_{g,m}$*, and the right-hand side with* $\exp(m \cdot C_{-n}) \cdot \|\cdot\|^{H,n}_{g-1,m+2}$.

b) Similarly, the isomorphism (5.1.44) is an isometry if the left-hand side is endowed with $\|\cdot\|^{H,n}_{g,m}$*, and the right-hand side is endowed with the norm* $\exp(m \cdot C_{-n}) \cdot (\|\cdot\|^{H,n}_{g_1,m_1+1} \boxtimes \|\cdot\|^{H,n}_{g_2,m_2+1})$.

Remark 5.1.7. For a special family of curves from Section 5.2.3, Freixas proved Theorem 5.1.6 for $n = 0$ in [59, Corollary 5.8] and extended it for $n \leq 0$ in [59, Theorem 5.3].

Instead of the usual definition Quillen norm, Freixas used its version, defined as a product of (5.1.36) and the L^2-norm. By Theorem 5.1.4, his results follows from Theorem 5.1.6. We note, however, that the precise calculation of the constant C_n in Theorem 5.1.2, generalizing 5.1.6, relies on the calculations of Freixas, which he obtained while proving those results.

Finally, we note that Theorem 5.1.2 suggest that the renormalization

$$T^{\mathrm{ren}}(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}) := \exp(m \cdot \mathrm{rk}(\xi) C_{-n}/12) \cdot T(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}) \tag{5.1.45}$$

is more natural from the point of view of restriction theorem. For (ξ, h^{ξ}) trivial and (M, g^{TM}) stable hyperbolic surface, this coincides with the normalization of Freixas in [59, Definition 2.2] and [60, Definition 4.2].

Let's describe the structure of this paper. In Section 2, we recall the definition of the Quillen norm on the family of Riemann surfaces with cusps, the definition of Wolpert norm, and some results from [54], [55], which study those norms. Then we recall an analogue of Theorem 5.1.2 due to Freixas about convergence of the Quillen norm for a special family of hyperbolic Riemann surfaces, where the Quillen norm is defined using (5.1.36). In Section 3 we extend a result of Bismut [18, Theorem 0.3] to Kähler family endowed with non-Kähler metric and give a proof of Theorems 5.1.2, 5.1.4, 5.1.6.

Notation. For a complex manifold X, we denote by Ω_X the sheaf of holomorphic sections of the vector bundle $T^{*(1,0)}X$, and by ω_X the canonical line bundle $\det(T^{*(1,0)}X)$ of X. For a divisor D in X, we denote by s_D the canonical meromorphic section of $\mathscr{O}_X(D)$.

For $\epsilon > 0$, we define

$$D(\epsilon) = \{u \in \mathbb{C} : |u| < \epsilon\}, \quad D^*(\epsilon) = \{u \in \mathbb{C} : 0 < |u| < \epsilon\}. \tag{5.1.46}$$

5.2 Families of nodal curves and related notions

In this section we recall the relevant notations. More precisely, in Section 2.1, we recall the notion of the Quillen norm from [54], [21], [22], [23], the basic notions for families of Riemann surfaces with cusps from [20], [55] and relevant results from [54], [55]. In Section 2.2, we recall several notions of singularities of Hermitian metrics on holomorphic line bundles and a useful regularity result for push-forward of differential forms from [55].

5.2.1 Determinant line bundles, Serre duality and Quillen norms

Let $\overline{M}$ be a compact Riemann surface, and let $D_M = \{P_1^M, \dots, P_m^M\}$ be a finite set of distinct points in $\overline{M}$. Let g^{TM} be a Kähler metric on the punctured Riemann surface $M = \overline{M} \setminus D_M$.

For $\epsilon \in]0, 1[$, let $z_i^M : \overline{M} \supset V_i^M(\epsilon) \to D(\epsilon) = \{z \in \mathbb{C} : |z| \leq \epsilon\}$, $i = 1, \dots, m$, be a local holomorphic coordinate around P_i^M, and

$$V_i^M(\epsilon) := \{x \in M : |z_i^M(x)| < \epsilon\}. \tag{5.2.1}$$

We say that g^{TM} is *Poincaré-compatible* with coordinates $z_1^M, \dots, z_m^M$ if for any $i = 1, \dots, m$, there is $\epsilon > 0$ such that $g^{TM}|_{V_i^M(\epsilon)}$ is induced by the Hermitian form

$$\frac{\sqrt{-1} dz_i^M d\overline{z}_i^M}{\left|z_i^M \log |z_i^M|\right|^2}. \tag{5.2.2}$$

We say that g^{TM} is a *metric with cusps* if it is Poincaré-compatible with some holomorphic coordinates of D_M. A triple $(\overline{M}, D_M, g^{TM})$ of a Riemann surface $\overline{M}$, a set of punctures D_M and a metric with cusps g^{TM} is called a *surface with cusps* (cf. [93]).

From now on, we fix a surface with cusps $(\overline{M}, D_M, g^{TM})$ and a Hermitian vector bundle (ξ, h^ξ) over it. We denote by $\omega_M := T^{*(1,0)}\overline{M}$ the *canonical line bundle* over $\overline{M}$. We denote by $\|\cdot\|_M^\omega$ the norm on $\omega_{\overline{M}}$ induced by g^{TM} over M by the natural identification $TM \ni X \mapsto \frac{1}{2}(X - JX) \in T^{(1,0)}M$, where J is the complex structure of M. Let $\mathscr{O}_{\overline{M}}(D_M)$ be the line bundle associated with the divisor D_M. The *twisted canonical line bundle* is defined as

$$\omega_M(D) := \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M). \tag{5.2.3}$$

The metric g^{TM} endows by Construction 5.1.1 the line bundle $\omega_M(D)$ with the induced Hermitian metric $\|\cdot\|_M$ over M.

We recall here briefly the definition of the *analytic torsion* $T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n})$ for $m \in \mathbb{N}$ from [54, Definition 2.17].

Assume first $m = 0$, then the analytic torsion was defined by Ray-Singer [103, Definition 1.2] as the regularized determinant of the Kodaira Laplacian $\square^{\xi\otimes\omega_M(D)^n}$ associated with (M, g^{TM}) and $(\xi \otimes \omega_M(D)^n, h^\xi \otimes \|\cdot\|_M^{2n})$. More precisely, let $\lambda_i, i \in \mathbb{N}$ be the non-zero eigenvalues of $\square^{\xi\otimes\omega_M(D)^n}$. By Weyl's law, the associated zeta-function

$$\zeta_M(s) := \sum \lambda_i^{-s}, \tag{5.2.4}$$

is defined for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and it is holomorphic in this region. Moreover, we have

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\big[\exp^\perp(-t\square^{\xi\otimes\omega_M(D)^n})\big] t^s \frac{dt}{t}, \tag{5.2.5}$$

where $\exp^\perp(-t\square^{\xi\otimes\omega_M(D)^n})$ is the spectral projection onto the eigenspace corresponding to non-zero eigenvalues. Also, as it can be seen by the small-time expansion of the heat kernel and the usual properties of the Mellin transform, $\zeta_M(s)$ extends meromorphically to the entire s-plane. This extension is holomorphic at 0, and the *analytic torsion* is defined by

$$T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)). \tag{5.2.6}$$

By (5.2.4) and (5.2.6), we may interpret the analytic torsion as

$$T(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}) := \prod_{i=0}^{\infty} \lambda_i. \tag{5.2.7}$$

Now, assume $m > 0$. Then M is non-compact, and the heat operator associated to $\square^{\xi\otimes\omega_M(D)^n}$ is no longer of trace class. Also the spectrum of $\square^{\xi\otimes\omega_M(D)^n}$ is not discrete in general. Thus, neither the definition (5.2.6), nor the interpretation (5.2.7) are applicable.

In [54, Definition 2.10], for $n \leq 0$, we defined the *regularized heat trace* $\mathrm{Tr}^{\mathbf{r}}[\exp^\perp(-t\square^{\xi\otimes\omega_M(D)^n}]$ as a "difference" of the heat trace of $\square^{\xi\otimes\omega_M(D)^n}$ and the heat trace of the

Kodaira Laplacian $\square^{\omega_P(D)^n}$ corresponding to the 3-punctured projective plane $P := \overline{P} \setminus \{0, 1, \infty\}$, $\overline{P} := \mathbb{C}P^1$, endowed with the hyperbolic metric g^{TP} of constant scalar curvature -1 and the induced metric $\|\cdot\|_P$ on $\omega_P(D) := \omega_{\overline{P}} \otimes \mathscr{O}_{\overline{P}}(0+1+\infty)$. Then in [54, Definition 2.16], we defined the *regularized spectral zeta function* $\zeta_M(s)$ for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ by

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\mathbf{r}} \big[\exp^{\perp}(-t\square^{E_M^{\xi,n}}) \big] t^s \frac{dt}{t}. \tag{5.2.8}$$

And we concluded in [54, p. 17], similarly to the case $m = 0$, the function $\zeta_M(s)$ extends meromorphically to $\mathbb{C}$ and 0 is a holomorphic point. Then in [54, Definition 2.17], we defined the regularized analytic torsion as

$$T(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) := \exp(-\zeta_M'(0)/2) \cdot T_{TZ}(g^{TP}, \|\cdot\|_P^{2n})^{m \cdot \mathrm{rk}(\xi)/3}. \tag{5.2.9}$$

Then for $n \leq 0$, in [54, §2.1], we explain how to endow the complex line

$$\begin{aligned} &\big(\det H^{\bullet}(\overline{M}, \xi \otimes \omega_M(D)^n) \big)^{-1} \\ &\qquad := \big(\Lambda^{\max} H^0(\overline{M}, \xi \otimes \omega_M(D)^n) \big)^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, \xi \otimes \omega_M(D)^n), \end{aligned} \tag{5.2.10}$$

with the L^2-norm $\|\cdot\|_{L^2}(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n})$ induced by the L^2-scalar product (5.1.5). In the compact case it coincides with the L^2-norm induced on the harmonic forms.

The Quillen norm on the complex line is defined by

$$\|\cdot\|_Q (g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}) = T(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n})^{1/2} \cdot \|\cdot\|_{L^2} (g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}). \tag{5.2.11}$$

To motivate, when $m = 0$, this coincides with the usual definition of the Quillen norm from Quillen [102], Bismut-Gillet-Soulé [21, (1.64)] and [23, Definition 1.5].

Following [54], we say that a (smooth) metric g_{f}^{TM} over $\overline{M}$ is a *flattening* of g^{TM} if there is $\nu > 0$ such that g^{TM} is induced by (5.2.2) over $V_i^M(\nu)$, and

$$g_{\mathrm{f}}^{TM}|_{M \setminus (\cup_i V_i^M(\nu))} = g^{TM}|_{M \setminus (\cup_i V_i^M(\nu))}. \tag{5.2.12}$$

Similarly, we defined a flattening $\|\cdot\|_M^{\mathrm{f}}$ of the norm $\|\cdot\|_M$.

Theorem 5.2.1 ([54, Theorem A]). *Let $g_{\mathrm{f}}^{TM}, \|\cdot\|_M^{\mathrm{f}}$ be flattenings of $g^{TM}, \|\cdot\|_M$. Then*

$$\begin{aligned} &2\mathrm{rk}(\xi)^{-1} \log \Big(\|\cdot\|_Q \big(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\big) / \|\cdot\|_Q \big(g_{\mathrm{f}}^{TM}, h^{\xi} \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\big) \Big) \\ &\qquad - \mathrm{rk}(\xi)^{-1} \int_M c_1(\xi, h^{\xi}) \Big(2n \log(\|\cdot\|_M^{\mathrm{f}} / \|\cdot\|_M) + \log(g_{\mathrm{f}}^{TM}/g^{TM}) \Big) \end{aligned} \tag{5.2.13}$$

depends only on the integer $n \in \mathbb{Z}$, $n \leq 0$, the functions $(g_{\mathrm{f}}^{TM}/g^{TM})|_{V_i^M(1)} \circ (z_i^M)^{-1} : \mathbb{D}^* \to \mathbb{R}$ and $(\|\cdot\|_M^{\mathrm{f}} / \|\cdot\|_M)|_{V_i^M(1)} \circ (z_i^M)^{-1} : \mathbb{D}^* \to \mathbb{R}$, for $i = 1, \ldots, m$.

Now let's recall another natural norm associated with a surface with cusps

Definition 5.2.2 ([54, Definition 1.5]). For a surface with cusps $(\overline{M}, D_M, g^{TM})$, the *Wolpert norms* $\|\cdot\|_M^{W,i}$ on the complex lines $\omega_{\overline{M}}|_{P_i^M}$, $i = 1, \ldots, m$, is defined by $\|dz_i^M\|_M^{W,i} = 1$. It induces the Wolpert norm $\|\cdot\|_M^W$ on the complex line $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i^M}$.

Let's recall that by [21, Theorem 1.27], the Bott-Chern forms of a vector bundle ξ with Hermitian metrics h_1^ξ, h_2^ξ are natural differential forms (strictly speaking, those are classes of differential forms) defined so that they satisfy

$$\begin{aligned} \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{Td}}(\xi, h_1^\xi, h_2^\xi) &= \mathrm{Td}(\xi, h_1^\xi) - \mathrm{Td}(\xi, h_2^\xi), \\ \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\widetilde{\mathrm{ch}}(\xi, h_1^\xi, h_2^\xi) &= \mathrm{ch}(\xi, h_1^\xi) - \mathrm{ch}(\xi, h_2^\xi), \end{aligned} \tag{5.2.14}$$

where Td, ch are Todd and Chern forms. By [21, Theorem 1.27], we have the following identities

$$\widetilde{\mathrm{ch}}(\xi, h_1^\xi, h_2^\xi)^{[0]} = 2\widetilde{\mathrm{Td}}(\xi, h_1^\xi, h_2^\xi)^{[0]} = \log\big(\det(h_1^\xi/h_2^\xi)\big). \tag{5.2.15}$$

If, moreover, $\xi := L$ is a line bundle, we have

$$\widetilde{\mathrm{ch}}(L, h_1^L, h_2^L)^{[2]} = 6\widetilde{\mathrm{Td}}(L, h_1^L, h_2^L)^{[2]} = \log(h_1^L/h_2^L)\Big(c_1(L, h_1^L) + c_1(L, h_2^L)\Big)/2, \tag{5.2.16}$$

where $c_1(L, h_i^L)$, $i = 1, 2$, is the first Chern form.

Theorem 5.2.3 ([54, Theorem B]). *Let $\phi : M \to \mathbb{R}$ be a smooth function such that for the metric*

$$g_0^{TM} = e^{2\phi} g^{TM}, \tag{5.2.17}$$

the triple $(\overline{M}, D_M, g_0^{TM})$ is a surface with cusps. We denote by $\|\cdot\|_M$,$\|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$, and by $\|\cdot\|_M^W$, $\|\cdot\|_M^{W,0}$ the associated Wolpert norms. Let h_0^ξ be a Hermitian metric on ξ over $\overline{M}$. Then the right-hand side of the following equation is finite, and

$$\begin{aligned} &2\log\Big(\|\cdot\|_Q\left(g_0^{TM}, h_0^\xi \otimes (\|\cdot\|_M^0)^{2n}\right)/\|\cdot\|_Q\left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)\Big) \\ &\quad = \int_M \Big[\widetilde{\mathrm{Td}}\big(\omega_M(D)^{-1}, \|\cdot\|_M^{-2}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big) \\ &\qquad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\xi, h^\xi, h_0^\xi\big)\mathrm{ch}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}\big) \\ &\qquad + \mathrm{Td}\big(\omega_M(D)^{-1}, (\|\cdot\|_M^0)^{-2}\big)\mathrm{ch}\big(\xi, h_0^\xi\big)\widetilde{\mathrm{ch}}\big(\omega_M(D)^n, \|\cdot\|_M^{2n}, (\|\cdot\|_M^0)^{2n}\big)\Big]^{[2]} \\ &\quad - \frac{\mathrm{rk}(\xi)}{6}\log\Big(\|\cdot\|_M^W/\|\cdot\|_M^{W,0}\Big) + \frac{1}{2}\sum \log\Big(\det(h^\xi/h_0^\xi)|_{P_i^M}\Big). \end{aligned} \tag{5.2.18}$$

By a *curve* in this article we mean (cf. [8, Definition of nodal curve on p. 79]) an analytic space such that every one of its points is either smooth or is locally complex-analytically isomorphic to a neighbourhood of the origin in $\{(z_0, z_1) \in \mathbb{C}^2 : z_0 z_1 = 0\}$.

Let C be a curve with singularities $\Sigma_C \subset C$. Let $\rho : N \to C$ be the normalization of C. For brevity, we denote the *twisted relative canonical bundle* on N by

$$\omega_N(D) := \omega_N \otimes \mathscr{O}_N(\rho^{-1}\Sigma_C). \tag{5.2.19}$$

We recall that for a point $p \in \rho^{-1}(\Sigma_C)$, and a local holomorphic coordinate z of p, the Poincaré residue morphism $\mathrm{Res}_p : (\omega_N \otimes \mathscr{O}_N(\rho^{-1}\Sigma_C))|_p \to \mathbb{C}$, is defined by

$$\mathrm{Res}_p\Big(\frac{dz \otimes s_{\rho^{-1}\Sigma_C}}{z}\Big) = 1. \tag{5.2.20}$$

Now, recall that the canonical sheaf ω_C of C is defined (cf. [8, p.91]) as an invertible subsheaf

$$\omega_C \subset \rho_*(\omega_N(D)), \tag{5.2.21}$$

defined by the following prescription. A section υ of $\rho_*(\omega_N(D))$, viewed as a section of $\omega_N(D)$, is a section of ω_C if and only if for any $x, y \in N$, $x \neq y$ such that $\rho(x) = \rho(y)$, we have

$$\mathrm{Res}_x(\upsilon) + \mathrm{Res}_y(\upsilon) = 0. \tag{5.2.22}$$

We denote by $\mathscr{C}^\infty_{\mathrm{Res}}(N, \omega_N(D)^n)$ the set of smooth sections υ of $\omega_N(D)^n$ over N such that for any $x, y \in N$, $x \neq y$, $\rho(x) = \rho(y)$, we have

$$\mathrm{Res}_x(\upsilon) = (-1)^n \mathrm{Res}_y(\upsilon). \tag{5.2.23}$$

By definition, we have the short exact sequence of sheaves

$$0 \to \omega_C \to \rho_*(\omega_N(D)) \xrightarrow{\mathrm{Res}} \oplus_{p\in\Sigma_C} \mathscr{O}_p \to 0, \tag{5.2.24}$$

where the last isomorphism is given by the map

$$\upsilon \mapsto \oplus_{p\in\Sigma_C} 1_p \cdot \big(\mathrm{Res}_{x_p}(\upsilon) + \mathrm{Res}_{y_p}(\upsilon)\big), \tag{5.2.25}$$

where $x_p, y_p \in N$ are distinct points satisfying $\rho(x_p) = \rho(y_p) = p$. Then (5.2.24) and the resolution of the sheaf $\omega_N(D)$ by the sheaves of germs of holomorphic forms with values in $\omega_N(D)^n$ induce, for any $n \in \mathbb{Z}$, the natural isomorphisms

$$\begin{aligned} &\rho^* : H^0\big(C, \omega_C^n\big) \to \ker(\overline{\partial}|_{\mathscr{C}^\infty_{\mathrm{Res}}(N,\omega_N(D)^n)}), \\ &\rho^* : H^1\big(C, \omega_C^n\big) \to \mathscr{C}^\infty(N, \overline{\omega}_N \otimes \omega_N(D)^n)/\operatorname{Im}(\overline{\partial}|_{\mathscr{C}^\infty_{\mathrm{Res}}(N,\omega_N(D)^n)}). \end{aligned} \tag{5.2.26}$$

Serre duality (cf. [8, p. 90 - 91]) is the canonical isomorphism

$$H^1(C, \omega_C^n) \to (H^0(C, \omega_C^{1-n}))^*, \tag{5.2.27}$$

given by the following pairing: for $\upsilon \in H^1\big(C, \omega_C^n\big)$ and $\alpha \in H^0(C, \omega_C^{1-n})$, by (5.2.26), we define

$$(\upsilon, \alpha) = \frac{-\sqrt{-1}}{2\pi} \int_N \upsilon \wedge \alpha. \tag{5.2.28}$$

The integration (5.2.28) is well-defined since only the poles of first order appear under the integral. By (5.2.23), Stokes and Residue theorems, (5.2.28) defines a pairing of $H^1(C, \omega_C^n)$ with $H^0(C, \omega_C^{1-n})$.

Now, when X is non-singular and ω_X is endowed with a Hermitian norm $\|\cdot\|_X$, by (5.1.5), the left-hand side and the right-hand side of (5.2.27) are endowed with the induced L^2-norm $\|\cdot\|_{L^2}$. By using the description of Serre duality through the Hodge star operator (cf. [43, p. 310]), we observe that for any $\upsilon \in H^1\big(C, \omega_C^n\big)$, we have

$$\sup\Big\{\big|\int_N \upsilon \wedge \beta\big|^2 : \beta \in H^0\big(C, \omega_C^{1-n}\big) \setminus \{0\}, \quad \|\beta\|_{L^2} = 1\Big\} = 2\pi\|\upsilon\|_{L^2}^2 . \tag{5.2.29}$$

Thus, under the isomorphism (5.2.27), we have

$$(H^1(C, \omega_C^n), \|\cdot\|_{L^2}) = (H^0(C, \omega_C^{1-n})^*, \|\cdot\|_{L^2}^{-1}). \tag{5.2.30}$$

Now let's fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a curve (in the terminology of [20], [55], a f.s.o.).

Proposition 5.2.4 ([20, Proposition 3.1])**.** For every $x \in X$, there are local holomorphic coordinates $(z_0, \ldots, z_q)$ of $x \in X$ and $(w_1, \ldots, w_q)$ of $\pi(x) \in S$, such that π is locally defined either by one of the following identities

$$w_i = z_i, \qquad \text{for} \quad i = 1, \ldots, q, \tag{5.2.31}$$

$$w_1 = z_0 z_1; \quad w_i = z_i, \qquad \text{for} \quad i = 2, \ldots, q. \tag{5.2.32}$$

Corollary 5.2.5 ([20, §3(a)])**.** Let $\Sigma_{X/S} \subset X$ be the locus of double points of the fibers of π. Then:

a) $\Sigma_{X/S}$ is a submanifold of X of codimension 2;

b) the map $\pi|_{\Sigma_{X/S}} : \Sigma_{X/S} \to S$ is a closed immersion;

c) the map $\pi|_{X \setminus \Sigma_{X/S}} : X \setminus \Sigma_{X/S} \to S$ is a submersion.

In particular, the direct image $\Delta = \pi_*(\Sigma_{X/S})$ is a divisor in S.

Notation 5.2.6. We use the notation Δ, $\Sigma_{X/S}$ for the divisor and the submanifold from Corrolary 5.2.5.

Let's recall the construction of the *relative canonical* line bundle $\omega_{X/S}$ of a f.s.o. $\pi : X \to S$. Define the sheaf $\Omega_{X/S}$ by the exact sequence:

$$\pi^*\Omega_S \to \Omega_X \to \Omega_{X/S} \to 0. \tag{5.2.33}$$

By Corollary 5.2.5, the exact sequence (5.2.33) becomes exact to the left when restricted to $X \setminus \Sigma_{X/S}$:

$$0 \to \pi^*\Omega_S|_{X \setminus \Sigma_{X/S}} \to \Omega_X|_{X \setminus \Sigma_{X/S}} \to \Omega_{X/S}|_{X \setminus \Sigma_{X/S}} \to 0. \tag{5.2.34}$$

By taking determinants of (5.2.34), we deduce the isomorphism

$$\Omega_{X/S}|_{X \setminus \Sigma_{X/S}} = (\omega_X \otimes \pi^*\omega_S^{-1})|_{X \setminus \Sigma_{X/S}}. \tag{5.2.35}$$

We define

$$\omega_{X/S} := \omega_X \otimes \pi^* \omega_S^{-1}. \tag{5.2.36}$$

Then $\omega_{X/S}$ is the unique extension of the line bundle $\Omega_{X/S}|_{X\setminus\Sigma_{X/S}}$ over X. This line bundle is called the *relative canonical* line bundle of $\pi : X \to S$.

Let $x \in \Sigma_{X/S}$. Take local coordinates $(z_0, \ldots, z_q)$ on an open neighbourhood V of $x \in X$ and local coordinates $(w_1, \ldots, w_q)$ of $\pi(x) \in S$, as in (5.2.32). Then the manifold $\Sigma_{X/S} \cap V$ is given by

$$\{z_0 = 0 \text{ and } z_1 = 0\}. \tag{5.2.37}$$

Consider the sections dz_0/z_0 and dz_1/z_1 of Π_X, defined over the sets $\{z_0 \neq 0\}$ and $\{z_1 \neq 0\}$ respectively. The images of dz_0/z_0 and $-dz_1/z_1$ in $\omega_{X/S}$ coincide over $\{z_0 z_1 \neq 0\}$, since

$$\frac{dz_0}{z_0} + \frac{dz_1}{z_1} = \pi^* \frac{dw_1}{w_1}. \tag{5.2.38}$$

Thus, they define a nowhere vanishing section σ of $\omega_{X/S}$ over $V \setminus \Sigma_{X/S}$. Since $\Sigma_{X/S}$ is of codimension 2, the section σ extends to a nowhere vanishing section over V of the line bundle $\omega_{X/S}$.

Now, let $s_0 := \pi(x) \in \Delta$, $x \in \Sigma_{X/S}$, and let $\rho : Y_{s_0} \to X_{s_0}$ be the normalization of X_{s_0} at x. Then by the discussion above, there is the canonical isomorphism

$$\rho^* \omega_{X/S} = \omega_{Y_{s_0}} \otimes \mathscr{O}_{Y_{s_0}}(\rho^{-1}(x)), \tag{5.2.39}$$

which induces the isomorphism (5.1.15). The identity (5.2.38) implies that for the natural inclusion $i_{s_0} : X_{s_0} \to X$, the pull-back $(i_{s_0})^* \omega_{X/S}$ is canonically isomorphic to $\omega_{X_{s_0}}$.

Now let's fix disjoint sections $\sigma_1, \ldots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ and a Hermitian metric $\|\cdot\|^{\omega}_{X/S}$ on $\omega_{X/S}$ over $\pi^{-1}(S \setminus |\Delta|) \setminus (\cup_i \operatorname{Im}(\sigma_i))$, such that for any $t \in S \setminus |\Delta|$, the restriction of $\|\cdot\|^{\omega}_{X/S}$ over $\pi^{-1}(t) \setminus (\cup_i \sigma_i(t))$ induces the Kähler metric g^{TX_t} over $X_t \setminus (\cup_i \sigma_i(t))$ such that the associated triple $(X_t, \{\sigma_1(t), \ldots, \sigma_m(t)\}, g^{TX_t})$ becomes a surface with cusps. As a short-cut, we call $(\pi; \sigma_1, \ldots, \sigma_m; \|\cdot\|^{\omega}_{X/S})$ a f.s.c.

Now, let (ξ, h^{ξ}) be a Hermitian vector bundle over X. For $t \in S$, we denote

$$\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))_t := \det H^0(X_t, \xi \otimes \omega_{X/S}(D)^n) \otimes (\det H^1(X_t, \xi \otimes \omega_{X/S}(D)^n))^{-1}. \tag{5.2.40}$$

By Grothendick-Knudsen-Mumford [76] (cf. [20, Proposition 4.1]), the family of complex lines $(\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))_t)_{t \in S}$ is endowed with a natural structure of holomorphic line bundle $\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))$ over S. We denote

$$\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n)) := \big(\det(R^\bullet \pi_*(\xi \otimes \omega_{X/S}(D)^n))\big)^{-1}. \tag{5.2.41}$$

Following [55], the pointwise Quillen norms induce the Quillen norm $\|\cdot\|_Q \left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|^{2n}_{X/S}\right)$ on the line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$. Similarly, the pointwise Wolpert norms glue into the Wolpert norm $\|\cdot\|^{W}_{X/S}$ on $\otimes_{i=1}^m \sigma_i^* \omega_{X/S}$.

Definition 5.2.7. The Quillen norm on the line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))$, $n \leq 0$ over $S \setminus |\Delta|$ is defined for $t \in S \setminus |\Delta|$ by

$$\|\cdot\|_Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right) := T\left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right)^{1/2} \cdot \|\cdot\|_{L^2} \left(g^{TX_t}, h^\xi \otimes \|\cdot\|^{2n}_{X/S}\right). \tag{5.2.42}$$

Now we can recall the special case of continuity theorem

Theorem 5.2.8 ([55, Theorem C3])**.** *Suppose that assumption (5.1.6) holds. Then the Hermitian norm (5.1.7) on the line bundle (5.1.8) extends continuously over S.*

5.2.2 Singular Hermitian vector bundles

In this section we recall several notions of singularities for Hermitian vector bundles.

We work with a complex manifold Y of dimension $q+1$, a *normal crossing divisor* $D_0 \subset Y$ and a submanifold $F \subset Y$.

Definition 5.2.9. A triple $(U; z_0, \ldots, z_q; l)$ of an open set $U \subset Y$, coordinates $z_0, \ldots, z_q : U \to \mathbb{C}$ and $l \in \mathbb{N}$ is called an *adapted chart* for D_0 (resp. F) at $x \in D_0$ (resp. $x \in F$) if $U = \{(z_0, \ldots, z_q) \in \mathbb{C}^{q+1} : |z_i| < 1, \text{ for all } i = 0, \ldots, q\}$ and $D_0 \cap U$ (resp. $F \cap U$) is defined by $\{z_0 \cdots z_l = 0\}$ (resp. $\{z_0 = 0, \ldots, z_l = 0\}$).

Notation 5.2.10. Let $(U; z_0, \ldots, z_q; l)$ be an *adapted chart* for D_0. We denote

$$d\zeta_k = \begin{cases} dz_k/(z_k \log |z_k|^2), & \text{if} \quad 0 \leq k \leq l, \\ dz_k, & \text{if} \quad l+1 \leq k \leq q. \end{cases} \tag{5.2.43}$$

Definition 5.2.11. a) [58, Definition 2.1] A function $f : Y \setminus F \to \mathbb{C}$ has *log-log growth on Y, with singularities along F* if for any $x \in Y$, for some adapted chart $(U; z_0, \ldots, z_q; l)$ of F at x, and for some $C > 0$, $p \in \mathbb{N}$, we have

$$|f(z_0, \ldots, z_q)| \leq C\Big(\log\Big|\log\big(\max_{k=0}^{l}\{|z_k|\}\big)\Big|\Big)^p + C. \tag{5.2.44}$$

b) [95, p. 240] A differential form over $Y \setminus D_0$ has *Poincaré growth on Y, with singularities along D_0*, if it can be expressed as a linear combination of monomials constructed using $d\zeta_k, \overline{d\zeta_k}$, $k = 0, \ldots, q$ with coefficients $f \in \mathscr{C}^\infty(Y \setminus D_0) \cap L^\infty(Y \setminus D_0)$.

c) [58, Definition 2.14] A smooth function $f : Y \setminus D_0 \to \mathbb{C}$ is *P-singular, with singularities along D_0*, if ∂f, $\overline{\partial} f$, $\partial\overline{\partial} f$ have Poincaré growth on Y, with singularities along D_0.

Definition 5.2.12 ([95, p. 242])**.** Let L be a holomorphic line bundle over Y and let h^L be a smooth Hermitian metric on L over $Y \setminus D_0$. Then h^L is good with singularities along D_0 if and only if, for every local holomorphic frame υ of L over $U \subset Y$, the function $\log h^L(\upsilon, \upsilon)$ is P-singular, with singularities along D_0.

Remark 5.2.13. The original definition of Mumford differs from the one presented here. Their equivalence is proved in [58, Proposition 3.2].

Let $\pi : X \to S$ be a f.s.o. such that the corresponding divisor of singular curves Δ has normal crossings. Let D be a divisor on X such that $\pi|_D : D \to S$ is a local isomorphism.

Proposition 5.2.14. Let α be a differential $(1,1)$-form over $X \setminus (\Sigma_{X/S} \cup |D|)$, such that it has Poincaré growth on $X \setminus |D|$ with singularities along $\pi^{-1}(\Delta)$, and the coupling of α with smooth vertical vector fields over $X \setminus (\Sigma_{X/S} \cup |D|)$ is continuous and has log-log growth with singularities along D (cf. Definition 5.2.11.a)). Let $f : X \setminus (\Sigma_{X/S} \cup |D|) \to \mathbb{R}$ be a continuous function, with log-log growth along $\Sigma_{X/S} \cup |D|$.

Then for the normalisation $\rho : Y_t \to X_t$ of X_t, $t \in |\Delta|$, the form $\rho^*(f\alpha)$ is integrable over Y_t. Moreover, the function $\pi_*[f\alpha]$ extends continuously over S, and the value of this extension is

$$\pi_*[f\alpha](t) = \int_{Y_t} \rho^*(f\alpha). \tag{5.2.45}$$

Proof. The first part of the statement was proved in [55, Proposition 3.1c)]. The second statement follows directly from the proof of [55, Proposition 3.1c)]. □

5.2.3 Quillen metric and hyperbolic surfaces

In this section we recall the results of Freixas from [59] and [60], which describe how the Quillen metric, defined using the version of analytic torsion due to Takhtajan-Zograf (see (5.1.36)), behaves in a degenerating family of curves. The family he considers is a special case of plumbing family construction, which is originally due to Wolpert [116, p. 434].

We fix a Riemann surface $\overline{M}$ with m fixed points $D_M = \{P_1^M, \dots, P_m^M\} \subset \overline{M}$ and a Riemann surface $\overline{T}$ of genre 1 with one fixed point $D_T = \{P^T\} \subset \overline{T}$. Take m copies $(\overline{T}_i, P_i^T)$ of $(\overline{T}, P^T)$. Let $g \in \mathbb{N}$ be the genus of $\overline{M}$. Clutching morphisms β (see (5.1.41)), applied to the pairs of points $\{P_1^M, P_1^T\}, \dots, \{P_m^M, P_m^T\}$, realizes the pointed surface $(M, m \cdot T) := (\overline{M}, D_M) \cup (\overline{T}_1, P_1^T) \cup \dots \cup (\overline{T}_m, P_m^T)$ as a point in a compactifying divisor $\partial\mathscr{M}_{g+m,0}$ of $\overline{\mathscr{M}}_{g+m,0}$. The *plumbing family* associated with $(M, m \cdot T)$ is a family of pointed curves representing a transversal direction to $\partial\mathscr{M}_{g+m,0}$ in $\overline{\mathscr{M}}_{g+m,0}$. More precisely, we consider

1. Neighbourhood U_i of $P_i^M \in \overline{M}$, $i = 1, \dots, m$ biholomorphic to an open disc and a holomorphic coordinate mappings $F_i : U_i \to \mathbb{C}$ with $F_i(P_i^M) = 0$;

2. Similarly, a neighbourhood V of $P^T \in \overline{T}$, and a holomorphic coordinate mapping $G : V \to \mathbb{C}$ satisfying $G(P^T) = 0$;

3. A small complex parameter $t \in \mathbb{C}$.

We suppose that U_i are pairwise disjoint. Let $c > 0$ be such that $D(c) \subset \mathbb{C}$ is contained in $\mathrm{Im}(F_i)$, $i = 1, \dots, m$ and $\mathrm{Im}(G)$. We take m copies $G_1, \dots, G_m$ of G, and regard them as functions acting on $\overline{T}_1, \dots, \overline{T}_m$ respectively. Let $|t| < c^2$. For $d \in D(c)$, we note

$$R^{d,*} = \big(\overline{M} \setminus (\cup_{i=1}^m \{|F_i| \le |d|\})\big) \cup \big(\overline{T}_1 \setminus \{|G_1| \le |d|\}\big) \cup \dots \cup \big(\overline{T}_m \setminus \{|G_m| \le |d|\}\big). \tag{5.2.46}$$

Consider the equivalence relation on points of $R^{t/c,*}$ generated by:

$$p \sim q \text{ if } |t|/c \le |F_i(p)| \le c,\ |t|/c \le |G_i(q)| \le c,\ F_i(p)G_i(q) = t. \tag{5.2.47}$$

Form the identification space $X_t = R^{t/c,*}/\sim$. The curve X_t, $t \in D(c^2)$, is called the *plumbing construction* for $(M, m \cdot T)$ associated with the *plumbing data* $(\cup_i U_i, V, \cup_i F_i, G, t)$. Trivially, we see that a set $X := \cup_{t \in D(c^2)} X_t$ can be endowed with a structure of a complex manifold, for which $\pi : X \to S := D(c^2)$ is a proper holomorphic map of codimension 1. This construction is also called the *Bers trick* for $\overline{M}$ (cf. [13], [58, Construction 4.3.2]).

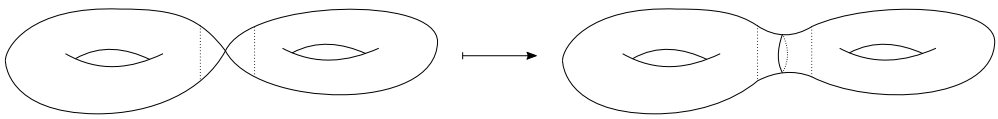

Figure 5.2: Plumbing family. The regions away from the dashed lines are isomorphic.

Now, suppose that the pointed surface $(\overline{M}, D_M)$ is stable, i.e. it satisfies (5.1.31), and let g^{TM}_{hyp} be the canonical hyperbolic metric of constant scalar curvature -1 on $\overline{M} \setminus D_M$ with cusps at D_M. Then one can take the functions F_i, $i = 1, \ldots, m$, from the plumbing construction to be Poincaré-compatible coordinates z^M_i of P^M_i (see (5.2.2)). Similarly, we make the choice for $G_i = z^T$. The plumbing family associated to this plumbing data is called the *canonical plumbing family*.

From now on, we fix the canonical plumbing family $\pi : X \to S := D(c^2)$. Then the divisor of singular curves is given by $\Delta = m \cdot \{0\}$. We denote by

$$\rho : Y_0 := (\overline{M} \cup \overline{T}_1 \cup \cdots \cup \overline{T}_m) \to X_0 \tag{5.2.48}$$

the normalisation of the singular fiber. We denote by

$$\Sigma_{X/S} = \{Q_1, \ldots, Q_m\}, \quad Q_i = \rho(P^M_i), \tag{5.2.49}$$

the set of singular points in X_0. Let $g^{TX_t}_{\mathrm{hyp}}$, $t \neq 0$ be the canonical hyperbolic metric of constant scalar curvature -1 on $X_t \setminus D_{X_t}$ with cusps at D_{X_t}. We denote by $Z_{X_t}(s)$ the Selberg zeta-function associated with X_t, given by the formula (5.1.32). Let $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ be the Hermitian norm on $\omega_{X/S}(D)$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, induced from $g^{TX_t}_{\mathrm{hyp}}$ by Construction 5.1.1.

We consider the determinant line bundle $\lambda(j^*(\omega_{X/S}(D)^n))$, $n \leq 0$, and we endow it over $S \setminus \Delta$ with the Takhtajan-Zograf version of Quillen norm (cf. [58, §6]), given by (compare with (5.2.42))

$$\|\cdot\|^{TZ}_Q \left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right) := T_{TZ}\left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right). \tag{5.2.50}$$

We construct the norm (compare with (5.1.7))

$$\|\cdot\|^{TZ}_{\mathscr{L}_n} := \left(\|\cdot\|^{TZ}_Q \left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right)\right)^{12} \otimes \|\cdot\|^{\mathrm{div}}_{\Delta} \tag{5.2.51}$$

on the line bundle (compare with (5.1.8))

$$\mathscr{L}^{TZ}_n := \lambda\left(j^*(\omega^n_{X/S})\right)^{12} \otimes \mathscr{O}_S(\Delta). \tag{5.2.52}$$

We denote by $\|\cdot\|^{\mathrm{hyp}}_M, \|\cdot\|^{\mathrm{hyp}}_T$ the norms on $\omega_M(D)$, $\omega_T(D)$ induced by the canonical hyperbolic metrics g^{TM}_{hyp}, g^{TT}_{hyp} of constant scalar curvature -1 on $\overline{M} \setminus D_M$, $\overline{T} \setminus D_T$ with cusps at D_M and D_T

respectively. We denote by $\|\cdot\|_M^{W,\mathrm{hyp}}, \|\cdot\|_T^{W,\mathrm{hyp}}$ the associated Wolpert norms on the complex lines $\det(\omega_{\overline{M}}|_{D_M})$ and $\det(\omega_{\overline{T}}|_{D_T})$. Now, we define the norm (compare with (5.1.18))

$$\|\cdot\|_{\mathscr{L}_n'}^{TZ} := \Big(\|\cdot\|_Q^{TZ}\,(g_{\mathrm{hyp}}^{TM}, (\|\cdot\|_M^{\mathrm{hyp}})^{2n}) \otimes \big(\|\cdot\|_Q^{TZ}\,(g_{\mathrm{hyp}}^{TT}, (\|\cdot\|_T^{\mathrm{hyp}})^{2n})\big)^m\Big)^{12} \otimes \big(\|\cdot\|_M^{W,\mathrm{hyp}} \otimes (\|\cdot\|_T^{W,\mathrm{hyp}})^m\big)^{-1} \tag{5.2.53}$$

on the complex line (compare with (5.1.19))

$$\mathscr{L}_n^{TZ\prime} := \Big(\lambda\big(\omega_M(D)^n\big) \otimes \lambda\big(\omega_T(D)^n\big)^m\Big)^{12} \otimes \Big(\det(\omega_{\overline{M}}|_{D_M}) \otimes (\det(\omega_{\overline{T}}|_{D_T}))^m\Big)^{-1}. \tag{5.2.54}$$

Then the isomorphism (5.1.28) gives in our case the canonical isomorphism

$$\mathscr{L}_n^{TZ}|_\Delta \to \mathscr{L}_n^{TZ\prime} \otimes \big(\otimes_{i=1}^m \mathscr{O}_{Q_i}\big)^{12}. \tag{5.2.55}$$

We recall that C_k, $k \in \mathbb{N}$ were defined in (5.1.29). The main theorem of this section is

Theorem 5.2.15 (Freixas, [59, Corollary 5.8], [59, Theorem 5.3])**.** *The norm $\|\cdot\|_{\mathscr{L}_n}^{TZ}$ extends continuously over S, and, under the isomorphism (5.2.55), the following identity holds*

$$\|\cdot\|_{\mathscr{L}_n}^{TZ}|_\Delta = \exp(m \cdot C_{-n}) \cdot \|\cdot\|_{\mathscr{L}_n'}^{TZ}. \tag{5.2.56}$$

5.2.4 Model grafting and pinching expansion

The goal of this section is to recall model grafting construction and the pinching expansion of the hyperbolic metric due to Wolpert [116]. For simplicity, we state his results only for the plumbing family considered in Section 5.2.3. We conserve the notation from Section 5.2.3. To be compatible with further notation, we denote

$$z_0^i := z_i^M, \qquad z_1^i := z_i^T. \tag{5.2.57}$$

By the definition of plumbing family, the coordinates (z_0^i, z_1^i) serve as local holomorphic charts in the neighbourhood $Q_i \in X$. We denote

$$U(Q_i, \epsilon) = \{x \in X : |z_0^i(x)| < \epsilon, |z_1^i(x)| < \epsilon\}. \tag{5.2.58}$$

Again, by the definition of plumbing family, in t-coordinates on S, we have

$$\pi(z_0^i, z_1^i) = z_0^i z_1^i. \tag{5.2.59}$$

The canonical hyperbolic metric on M (resp. T) with cusps at D_M (resp. D_T) induces a metric $g_{\mathrm{hyp}}^{TR^\epsilon}$ on R^ϵ (see (5.2.46) for the definition of R^ϵ). Let ϵ be so small, so that $g_{\mathrm{hyp}}^{TR^\epsilon}$ is induced by

$$\begin{aligned} &\frac{\sqrt{-1}dz_j^M d\overline{z}_j^M}{|z_j^M \log |z_j^M|^2|^2}, \quad \text{over } \big\{|z_j^M| < 2\epsilon\big\}, \\ &\frac{\sqrt{-1}dz_j^T d\overline{z}_j^T}{|z_j^T \log |z_j^T|^2|^2}, \quad \text{over } \big\{|z_j^T| < 2\epsilon\big\}. \end{aligned} \tag{5.2.60}$$

We choose $c = \epsilon^2$ in the plumbing construction from Section 5.2.3. Now, since the manifold $X \setminus (\cup_{i=1}^k U(Q_i, \epsilon))$ is naturally isomorphic to the product $R^\epsilon \times D(\epsilon^2)$, the metric $g_{\text{hyp}}^{TR^\epsilon}$ induces the Kähler metric g^{TX_t} on $X_t \setminus (\cup_{i=1}^k U(Q_i, \epsilon))$.

The *model grafted metric* is built from the metric g^{TX_t} and the *hyperbolic metric on a cylinder*, see (5.2.63). More precisely, let $\nu : X \to [0, 1]$ be smooth function, satisfying

$$\nu(x) = \begin{cases} 0, & \text{for } x \in X \setminus (\cup_{i=1}^k U(Q_i, 2\epsilon)), \\ 1, & \text{for } x \in \cup_{i=1}^k U(Q_i, \epsilon). \end{cases} \tag{5.2.61}$$

For $t \in D(\epsilon^2)$, we denote by $g_{i,t}^{\text{Cyl}}$ the metric over the set

$$\big\{(z_0^i, z_1^i) \in X_t : |t|/(2\epsilon) < |z_0^i| < 2\epsilon\big\}, \tag{5.2.62}$$

induced by the Kähler form

$$\left(\frac{\pi}{|z_0^i| \log |t|}\left(\sin \frac{\pi \log |z_0^i|}{\log |t|}\right)^{-1}\right)^2 \sqrt{-1} dz_0^i d\overline{z}_0^i. \tag{5.2.63}$$

We remark that due to the fact that over X_t, we have $z_0^i z_1^i = t$, the expression (5.2.63) is symmetric with respect to the change of variables $z_0^i \leftrightarrow z_1^i$.

Following Wolpert [116], we define the *model grafted metric* $g_{\text{gft}}^{TX_t}$ as follows: over $X_t \setminus (\cup_{i=1}^m U(Q_i, 2\epsilon))$, $g_{\text{gft}}^{TX_t}$ coincides with g^{TX_t}, and over $U(Q_i, 2\epsilon)$, it is given by

$$g_{\text{gft}}^{TX_t} = \big(g_{i,t}^{\text{Cyl}}\big)^\nu (g^{TX_t})^{1-\nu}. \tag{5.2.64}$$

This metric has the following nice properties

Proposition 5.2.16. The metric $\|\cdot\|_{X/S}^{\text{gft}}$ induced by $g_{\text{gft}}^{TX_t}$ over $X \setminus \pi^{-1}(|\Delta|)$ extends continuously over $X \setminus \Sigma_{X/S}$. Moreover, it is good in the sense of Mumford on $X \setminus \pi^{-1}(|\Delta|)$ with singularities along $\pi^{-1}(\Delta)$

Proof. This is a trivial verification, see for example Wolpert [116, Lemma 1.5]. □

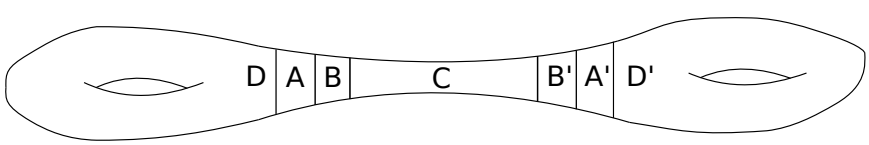

Figure 5.3: The model grafting. Over the regions $D \cup A$, $D' \cup A'$, the metric $g_{\text{gft}}^{TX_t}$ is isometric to g^{TX_t}. Over the regions A, A' it is Poincaré-compatible (see (5.2.2)) with coordinates z_0^i, z_1^i. Over the regions B, B' the metric $g_{\text{gft}}^{TX_t}$ is a geometric interpolation between g^{TX_t} and (5.2.63). Over the region C, the metric $g_{\text{gft}}^{TX_t}$ is given by (5.2.63).

Let's denote by $g^{TX_t}_{\mathrm{hyp}}$ the hyperbolic metric of constant scalar curvature -1 on X_t. As usual, we denote by $\|\cdot\|^{\omega,\mathrm{hyp}}_{X/S}$ the induced Hermitian norm on $\omega_{X/S}$. The pinching expansion describes the behaviour of $g^{TX_t}_{\mathrm{hyp}}$ near the singular fiber, it won't be used explicitly in this article, however, it is very important to understand the motivation behind the the metric $g^{TX_t}_{\varkappa}$ from Section 5.3.2.

Theorem 5.2.17 (The pinching expansion, Wolpert [116, Expansion 4.2])**.** *For $t \in S \setminus |\Delta|$, we have*

$$g^{TX_t}_{\mathrm{hyp}} = g^{TX_t}_{\mathrm{gft}}\Big(1 + O\big(|\log|t||^{-2}\big)\Big), \tag{5.2.65}$$

where the O-term is for the $\mathscr{C}^k$-norm over X_t for any $k \in \mathbb{N}$ with respect to $g^{TX_t}_{\mathrm{hyp}}$.

Remark 5.2.18. We note that in [58, Theorem 4.3.1], Freixas proved Theorem 5.2.17 for degenerating families of hyperbolic surfaces with cusps.

5.3 Quillen metric near singular fibers

The goal of this section is to prove Theorems 5.1.2, 5.1.4, 5.1.6, which are the main statements of this article. It is organized as follows. In Section 4.1 we generalize the result of Bismut [18, Theorem 0.3] about the behaviour of the Quillen norm in a smooth Kähler family of degenerating compact Riemann surfaces. In Section 4.2 we use this generalization with Theorems 5.2.1, 5.2.3, 5.2.8, 5.2.15 to prove Theorems 5.1.2, 5.1.4, 5.1.6.

5.3.1 Quillen metric in a smooth family of Riemann surfaces

In this section we describe a generalisation of the result of Bismut [18, Theorem 0.3] for non-necessarily Kähler manifolds. This theorem describes the behaviour of the Quillen norm in a *smooth* family of degenerating Riemann surfaces endowed with *compact* Riemann metric. It will be used in Section 4.2, but it is also of some independent interest.

Let's fix a holomorphic, proper, surjective map $\pi : X \to S$ of complex manifolds, such that for every $t \in S$, the space $X_t := \pi^{-1}(t)$ is a curve (see Section 5.2.1). Let (ξ, h^ξ) be a Hermitian vector bundle over X. Let g^{TX} be a Riemannian metric over X, which is compatible with the complex structure of X. By h^{TX} we note the Hermitian metric on $T^{(1,0)}X$ induced by g^{TX} by the natural identification $TX \ni Y \mapsto \frac{1}{2}(Y - JY) \in T^{(1,0)}X$, where J is the complex structure of X. We denote by g^{TX_t} the restriction of the metric g^{TX} on X_t, $t \in S \setminus |\Delta|$. Since g^{TX} is compatible with the complex structure, the metric g^{TX_t} is Kähler on X_t. We denote by $\|\cdot\|_Q\,(g^{TX_t}, h^\xi)$ the Quillen norm on the line bundle $\lambda(j^*\xi)$ over $S \setminus |\Delta|$ (see (5.2.11)).

For simplicity, assume that $\dim S = 1$, $S = D(1)$ and $|\Delta| = \{0\}$. We write $\Sigma_{X/S} = \{Q_1, \dots, Q_k\}$. Let $\rho : Y_0 \to X_0$ be the normalisation of X_0. We denote

$$\rho^{-1}(\Sigma_{X/S}) = \{P_1, \dots, P_{2k}\}, \tag{5.3.1}$$

where P_i are enumerated in such a way that $\rho(P_{2j-1}) = \rho(P_{2j}) = Q_j$ for $j = 1, \dots, k$. We denote by $g^{TY_0} := \rho^*(g^{TX})$ the induced Riemannian metric on Y_0 and by $\|\cdot\|^{\omega}_{Y_0}$ the induced Hermitian

norm on ω_{Y_0}. Since g^{TX} is compatible with the complex structure, we see that g^{TY_0} is Kähler on Y_0. We denote by $\|\cdot\|_Q\,(g^{TY_0}, \rho^*(h^\xi))$ the induced Quillen norm on the complex line $\lambda(\rho^*\xi)$.

Let $\|\cdot\|^i_{\Sigma_{X/S}/X}$ be the Hermitian norm induced by the natural isomorphism (5.1.21) on the complex lines $\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}}$, $i = 1, \ldots, k$. More explicitly, let local holomorphic coordinates z_0^i, z_1^i around $Q_i \in X$ and t around $0 \in S$ be as in (5.2.59). We denote

$$a_i = h^{TX}\Big(\frac{\partial}{\partial z_0^i}, \frac{\partial}{\partial z_0^i}\Big), \quad b_i = h^{TX}\Big(\frac{\partial}{\partial z_0^i}, \frac{\partial}{\partial z_1^i}\Big), \quad c_i = h^{TX}\Big(\frac{\partial}{\partial z_1^i}, \frac{\partial}{\partial z_1^i}\Big). \tag{5.3.2}$$

Then, by definition, we have

$$\big\|dz_0^i \otimes dz_1^i\big\|^i_{\Sigma_{X/S}/X} = \big(a_i c_i - |b_i|^2\big)^{-1/2}(Q_i). \tag{5.3.3}$$

We denote by $\|\cdot\|_{\Sigma_{X/S}/X}$ the induced norm on the complex line $\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}})$.

Over S, we introduce the holomorphic line bundle

$$\mathscr{L} := \lambda(j^*\xi)^{12} \otimes \mathscr{O}_S(\Delta)^{2\cdot \mathrm{rk}(\xi)}. \tag{5.3.4}$$

We endow it with a norm

$$\|\cdot\|_{\mathscr{L}} := \|\cdot\|_Q\,(g^{TX_t}, h^\xi)^{12} \otimes (\|\cdot\|^{\mathrm{div}}_\Delta)^{2\cdot \mathrm{rk}(\xi)}. \tag{5.3.5}$$

We bring the attention of the reader to the fact that the power of the divisor line bundle $\mathscr{O}_S(\Delta)$ in $\mathscr{L}$ is different from the construction (5.1.8) (cf. (5.1.7)). This is due to the fact that the geometric setting in this section is different from Section 5.1, as here, for example, the assumption (5.1.17) is not satisfied for the metric induced by g^{TX_t}, thus, the Wolpert norm is not well-defined and the analogue of the norm (5.1.18), doesn't make any sense.

We introduce the complex line

$$\mathscr{L}' := \lambda(\rho^*\xi)^{12} \otimes (\otimes_{i=1}^{2k} \det \rho^*(\xi)|_{P_i})^6 \otimes (\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}}))^{-2\cdot \mathrm{rk}(\xi)}. \tag{5.3.6}$$

We denote by $\|\cdot\|_{\mathscr{L}'}$ the norm on the complex line $\mathscr{L}'$ which is induced by $\|\cdot\|_Q\,(g^{TY_0}, \rho^*(h^\xi))$, h^ξ and $\|\cdot\|_{\Sigma_{X/S}/X}$. Remark that the power of the line bundle $(\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}}))$ in $\mathscr{L}'$ is different from the construction (5.1.19) (cf. (5.1.18)).

Analogically to (5.1.28), one has the following canonical isomorphism

$$\mathscr{L}|_\Delta \to \mathscr{L}' \otimes (\otimes_{i=1}^k \mathscr{O}_{Q_i})^{12\cdot \mathrm{rk}(\xi)}. \tag{5.3.7}$$

Now we can state the main result of this section.

Theorem 5.3.1. *The norm $\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}}$ extends continuously over S. Moreover, under the isomorphism (5.3.7), the following identity holds*

$$\|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}}|_\Delta = \exp\Big(\mathrm{rk}(\xi)\cdot k \cdot \big(24\zeta'(-1) - 6\log(2\pi)\big)\Big) \cdot \|\cdot\|^{\mathrm{cmp}}_{\mathscr{L}'}. \tag{5.3.8}$$

Proof. First of all, let's assume that g^{TX} is Kähler. Then we argue that Theorem 5.3.1 is just a restatement of [18, Theorem 0.3] due to Bismut.

To see this, let's fix a holomorphic coordinate t on S such that $|\Delta| = \{t = 0\}$. We denote by $\|\cdot\|_{\Delta}$ the Hermitian norm on $\mathscr{O}_S(\Delta)$, characterized by

$$\|s_{\Delta}/t^k\|_{\Delta} = 1. \tag{5.3.9}$$

As $\mathrm{div}(s_{\Delta}) = k\{0\}$, we deduce that $\|\cdot\|_{\Delta}$ is smooth over S. By the definition of the singular norm $\|\cdot\|_{\Delta}^{\mathrm{div}}$ from (5.1.3), by (5.3.9), we have

$$\|\cdot\|_{\Delta}^{\mathrm{div}} = |t|^{-k} \cdot \|\cdot\|_{\Delta}. \tag{5.3.10}$$

By (5.2.59), the isomorphism (5.1.24) specifies in our case to

$$\begin{aligned} &\mathscr{O}_S(\Delta)|_{|\Delta|} \to \big(\otimes_{i=1}^{k} (\omega_{Y_0}|_{P_{2i-1}} \otimes \omega_{Y_0}|_{P_{2i}})\big)^{-1}, \\ &\Big(\frac{s_{\Delta}}{t^k}\Big)|_{t=0} \mapsto \big(\otimes_{i=1}^{k} (dz_0^i|_{P_{2i-1}} \otimes dz_1^i|_{P_{2i}})\big)^{-1}. \end{aligned} \tag{5.3.11}$$

We denote by $\|d\pi^2\|$ the norm of the isomorphism (5.3.11). By (5.3.9), we have

$$\big\|d\pi^2\big\| := \Big(\big\| \otimes_{i=1}^{k} (dz_0^i|_{P_{2i-1}} \otimes dz_1^i|_{P_{2i}})\big\|_{\Sigma_{X/S}/X}\Big)^{-1}. \tag{5.3.12}$$

We note that due to our normalisation of the L^2-norm, (5.1.5), the difference between our definition of the Quillen norm, and the one from [22], [23], [18], which we denote by $\|\cdot\|_Q^{BGS}$, is

$$\|\cdot\|_Q\,(g^{TX_t}, h^{\xi}) = \exp\big(\log(2\pi) \cdot \chi(X_t, \xi|_{X_t})/2\big) \cdot \|\cdot\|_Q^{BGS}\,(g^{TX_t}, h^{\xi}), \tag{5.3.13}$$

where $\chi(X_t, \xi|_{X_t})$ is the Euler characteristic, given by

$$\chi(X_t, \xi|_{X_t}) = \dim H^0(X_t, \xi|_{X_t}) - \dim H^1(X_t, \xi|_{X_t}). \tag{5.3.14}$$

By Riemann-Roch theorem, the value $\chi(X_t, \xi|_{X_t})$ is constant over $S \setminus |\Delta|$.

We denote by $\|\cdot\|_Q^{\xi}\,(g^{TY_0}, \rho^*(h^{\xi}))$ the norm on the complex line $\lambda(j^*\xi) \otimes (\otimes_{i=1}^{2k} \det \xi|_{P_i})^6$ induced by $\|\cdot\|_Q\,(g^{TY_0}, \rho^*(h^{\xi}))$ and h^{ξ}. Similarly, due to our normalisation of the L^2-norm, (5.1.5), the difference between our definition of the norm $\|\cdot\|_Q^{\xi}\,(g^{TY_0}, \rho^*(h^{\xi}))$, and the one from [22], [23], [18], which we denote by $\|\cdot\|_Q^{\xi,BGS}\,(g^{TY_0}, \rho^*(h^{\xi}))$, is

$$\|\cdot\|_Q^{\xi}\,(g^{TY_0}, \rho^*(h^{\xi})) = \exp\big(\log(2\pi) \cdot \chi(Y_0, \rho^*(\xi)|_{Y_0})/2\big) \cdot \|\cdot\|_Q^{\xi,BGS}\,(g^{TY_0}, \rho^*(h^{\xi})). \tag{5.3.15}$$

We fix a smooth frame υ of $\lambda(j^*\xi)$ over S. Then by [18, Theorem 0.3, (0.5), the fact that the genus E is additive], under the isomorphisms (5.1.26), (5.1.27), the following identity holds

$$\lim_{t\to 0}\Big(\log\big(\|\upsilon(t)\|_Q^{BGS}(g^{TX_t}, h^{\xi})\big) - \frac{\mathrm{rk}(\xi)}{6}\log\big(\big\|s_{\Delta}(t)\big\|_{\Delta}\big)\Big)$$

$$= \log\left(\|\upsilon(0)\|_Q^{\xi,BGS}(g^{TY_0},\rho^*(h^\xi))\right) + \frac{\mathrm{rk}(\xi)}{6}\log\left\|d\pi^2\right\| + 2\zeta'(-1)\cdot k\cdot \mathrm{rk}(\xi). \quad (5.3.16)$$

Now, by (5.1.25) and the induced long exact sequence, we deduce

$$\chi(X_t,\xi|_{X_t}) = \chi(Y_0,\rho^*(\xi)) - k\cdot\mathrm{rk}(\xi). \quad (5.3.17)$$

However, by (5.3.10), (5.3.12) and (5.3.16), we deduce that

$$\begin{aligned}\lim_{t\to 0}\Big(\log\left(\|\upsilon(t)\|_Q(g^{TX_t},h^\xi)\right) + \frac{\mathrm{rk}(\xi)}{6}\log\Big(\Big\|\frac{s_\Delta(t)}{t^k}\Big\|_\Delta^{\mathrm{div}}\Big)\Big) &= \log\left(\|\upsilon(0)\|_Q^{\xi}(g^{TY_0},\rho^*(h^\xi))\right)\\ &- \frac{\mathrm{rk}(\xi)}{6}\log\Big(\big\|\otimes_{i=1}^k\left(dz_0^i|_{P_{2i-1}}\otimes dz_1^i|_{P_{2i}}\right)\big\|_{\Sigma_{X/S}/X}\Big)\\ &+ \Big(2\zeta'(-1) - \frac{\log(2\pi)}{2}\Big)\cdot k\cdot\mathrm{rk}(\xi), \quad (5.3.18)\end{aligned}$$

which means, in particular, that the norm $\|\cdot\|_{\mathscr{L}}$ extends continuously over S. Moreover, as by (5.3.11), the isomorphism (5.3.7) is given in our situation by

$$\Big(\upsilon^{12}\otimes\Big(\frac{s_\Delta}{t^k}\Big)^{2\cdot\mathrm{rk}(\xi)}\Big)|_{|\Delta|} \mapsto \upsilon(0)^{12}\otimes\Big(\otimes_{i=1}^k\left(dz_0^i|_{P_{2i-1}}\otimes dz_1^i|_{P_{2i}}\right)\Big)^{-2\cdot\mathrm{rk}(\xi)}, \quad (5.3.19)$$

the continuous extension satisfies (5.3.8) by (5.3.18).

Now let's prove (5.3.8) for metric g_0^{TX}, which is not necessarily Kähler. We note that π is locally projective (cf. Bismut-Bost [20, Proposition 3.4]), thus for some small neighbourhood U of $0\in S$, we may find a Kähler metric g^{TX} over $\pi^{-1}(U)$. As the statement of Theorem 5.3.1 is local over the base, without losing the generality, we suppose from now on that g^{TX} is defined over X. We denote by $\|\cdot\|_{\mathscr{L}}^0$ the norm on $\mathscr{L}$, induced by g_0^{TX}. The idea of the proof is to use the anomaly formula of Bismut-Gillet-Soulé [23] (cf. Theorem 5.2.3 for $m=0$) to relate the norms $\|\cdot\|_{\mathscr{L}}^0$ and $\|\cdot\|_{\mathscr{L}}$, and to study the limit of the right-hand side of this formula near the locus of singular curves.

We denote by $\|\cdot\|_{\Sigma_{X/S}/X}^0$ the norm on the line bundle $\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}}\otimes\omega_{Y_0}|_{P_{2i}})$, induced by g_0^{TX} as in (5.3.3). Similarly to (5.3.2), we denote by a_i^0, b_i^0, c_i^0 the corresponding functions associated with g_0^{TX}.

We argue that without losing the generality, we may suppose that $a_i^0, c_i^0 = 1, b_i^0 = 0$. This is true since we could fix a Riemannian metric g_*^{TX} which is compatible with the complex structure satisfying this assumption and then simply apply Theorem 5.3.1 twice for g_*^{TX} and g^{TX} and for g_*^{TX} and g_0^{TX}. By combining the two results, we would get the original statement.

Now, by (5.3.3), we trivially have

$$2\log\Big(\|\cdot\|_{\Sigma_{X/S}/X}/\|\cdot\|_{\Sigma_{X/S}/X}^0\Big) = -\sum_{i=1}^k\log\left(a_ic_i - |b_i|^2\right)(Q_i). \quad (5.3.20)$$

Let the differential form F be given by

$$F = \widetilde{\mathrm{Td}}\big(TX/S, g^{TX/S}, g_0^{TX/S}\big)\mathrm{ch}\big(\xi,h^\xi\big), \quad (5.3.21)$$

where TX/S is the vertical tangent bundle of π, and $g^{TX/S}, g_0^{TX/S}$ are the Hermitian norms on TX/S induced by g^{TX}, g_0^{TX}. By the anomaly formula of Bismut-Gillet-Soulé [23] (cf. Theorem 5.2.3 for $m = 0$), over $X \setminus \Sigma_{X/S}$, we have

$$\log\Big(\|\cdot\|_{\mathscr{L}}/\|\cdot\|_{\mathscr{L}}^0\Big) = 6\int_{X_t} F. \tag{5.3.22}$$

Now, as the map π is a submersion away from $\Sigma_{X/S}$, and the metrics g^{TX}, g_0^{TX} are smooth over X, by (5.3.20) and (5.3.22), to prove Theorem 5.3.1, it is enough to prove that for any $i = 1, \ldots, k$, the following holds

$$\lim_{\epsilon\to 0}\lim_{t\to 0}\int_{X_t\cap U(Q_i,\epsilon)} F = \frac{\mathrm{rk}(\xi)}{6}\log\big(a_ic_i - |b_i|^2\big)(Q_i). \tag{5.3.23}$$

For brevity, we fix $i = 1, \ldots, k$, and denote $z_0 := z_0^i$, $z_1 := z_1^i$. As $z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1}$ is a local holomoprhic frame of TX/S, locally around Q_i, we have

$$g^{TX/S}\Big(z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1}, z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1}\Big) = a_i|z_0|^2 + c_i|z_1|^2 - b_iz_0\overline{z}_1 - \overline{b}_iz_1\overline{z}_0. \tag{5.3.24}$$

By using the fact that $z_0z_1 = t$ over X_t, we deduce that locally around Q_i, we have

$$\begin{aligned} c_1(TX/S, g^{TX})|_{X_t} &= \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\Big(\log\big(a_i|z_0|^2 + c_i|z_1|^2 - b_iz_0\overline{z}_1 - \overline{b}_iz_1\overline{z}_0\big)\Big) \\ &= \frac{4(a_ic_i - |b_i|^2)|z_0|^2|t|^2}{(a_i|z_0|^4 + c_i|t|^2 - b_iz_0^2\overline{t} - \overline{b}_i\overline{z}_0^2t)^2}\frac{dz_0d\overline{z}_0}{2\pi\sqrt{-1}} + o\bigg(\Big(\frac{|t|^2}{|z_0|^6} + \frac{|z_0|^2}{|t|^2}\Big)dz_0d\overline{z}_0\bigg). \end{aligned} \tag{5.3.25}$$

As we are only interested in the limit (5.3.23), the calculation localizes around Q_i, and only the highest order terms matter. Thus, in the calculations, we may suppose that a_i, b_i, c_i are constants equal to $a_i(Q_i), b_i(Q_i), c_i(Q_i)$ respectively, and we may suppress the o-term.

By making the change of variables $y_0 = z_0|t|^{-1/2}$ and using (5.2.15), (5.2.16), we deduce that as $t \to 0$ and $\epsilon \to 0$, we have

$$\begin{aligned} \int_{X_t\cap U(Q_i,\epsilon)} F \sim -\frac{\mathrm{rk}(\xi)}{12}\int_{|t|^{1/2}/\epsilon<|y_0|<\epsilon/|t|^{1/2}}&\bigg(\log\frac{a_i|y_0|^4 + c_i - b_iy_0^2 - \overline{b}_i\overline{y}_0^2}{|y_0|^4+1}\bigg) \\ &\cdot\bigg(\frac{(4a_ic_i - 4|b_i|^2)|y_0|^2}{(a_i|y_0|^4 + c_i - b_iy_0^2 - \overline{b}_i\overline{y}_0^2)^2} + \frac{4|y_0|^2}{(|y_0|^4+1)^2}\bigg)\frac{dy_0d\overline{y}_0}{2\pi\sqrt{-1}}. \end{aligned} \tag{5.3.26}$$

We make the change of variables $x_0 := y_0^2$. The variable x_0 "turns around" $\mathbb{C}$ twice, which kills one of the two additional factors 2 appearing in the denominator in the following identity

$$\int_{X_t\cap U(Q_i,\epsilon)} F \sim -\frac{\mathrm{rk}(\xi)}{24}\int_{|t|^{1/4}/\epsilon^{1/2}<|x_0|<\epsilon^{1/2}/|t|^{1/4}}\bigg(\log\frac{a_i|x_0|^2 + c_i - b_ix_0 - \overline{b}_i\overline{x}_0}{|x_0|^2+1}\bigg)$$

$$\cdot\left(\frac{(4a_ic_i-4|b_i|^2)}{(a_i|x_0|^2+c_i-b_ix_0-\overline{b}_i\overline{x}_0)^2}+\frac{4}{(|x_0|^2+1)^2}\right)\frac{dx_0d\overline{x}_0}{2\pi\sqrt{-1}}. \quad (5.3.27)$$

By making the change of variables $x_0 := x_0 \cdot \exp(\arg(b_i))$, we see that the right-hand side of (5.3.27) depends only on $|b_0| \in \mathbb{R}$. Thus, without losing the generality, we may assume that $b_0 \in \mathbb{R}$. By writing $x_0 = x + \sqrt{-1}y$, from (5.3.27), we see

$$\int_{X_t\cap U(Q_i,\epsilon)} F \sim \frac{\mathrm{rk}(\xi)}{24\pi}\int_{|t|^{1/4}/\epsilon^{1/2}<x^2+y^2<\epsilon^{1/2}/|t|^{1/4}}\left(\log\frac{a_i(x^2+y^2)+c_i-2b_ix}{x^2+y^2+1}\right)$$
$$\cdot\left(\frac{(4a_ic_i-4b_i^2)}{(a_i(x^2+y^2)+c_i-2b_ix)^2}+\frac{4}{(x^2+y^2+1)^2}\right)dxdy. \quad (5.3.28)$$

Now, we remark that by switching to polar coordinates, changing the integration over radius by the integration over its square and applying tedious derivation by parts, for $a, c > 0$, $b \in \mathbb{R}$, $ac-b^2 > 0$, we get the following identity

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\log\frac{a(x^2+y^2)+c-2bx}{x^2+y^2+1}\right)\left(\frac{4ac-4b^2}{(a(x^2+y^2)+c-2bx)^2}+\frac{4}{(x^2+y^2+1)^2}\right)dxdy$$
$$= 4\pi\log(ac-b^2). \quad (5.3.29)$$

By (5.3.28) and (5.3.29), we get (5.3.23), which finishes the proof of Theorem 5.3.1. □

5.3.2 Proofs of Theorems 5.1.2, 5.1.4, 5.1.6

In this section we will finally prove Theorems 5.1.2, 5.1.4, 5.1.6.

Proof of Theorems 5.1.2, 5.1.4. This is the most technical proof of the whole article. It consists of 4 steps, and we start with a short résumé of them. During *Steps 1,2,3*, we will subsequently reduce Theorem 5.1.2 to simpler statements, and in *Step 4* we will completely prove Theorems 5.1.2, 5.1.4. More precisely, in *Step 1* we see that by Theorem 5.2.3, we can trivialize the Poincaré-compatible coordinates associated to g^{TX_t} (thus, the associated Wolpert norm). In *Step 2*, by Theorem 5.2.1, we reduce the problem to the problem without cusps. In *Step 3*, by Theorem 5.3.1, the anomaly formula of Bismut-Gillet-Soulé (cf. Theorem 5.2.3) and Theorem 5.2.1, we get Theorem 5.1.2 but instead of C_n, we have an undetermined constant A_{-n}, which depends only on n, and not on $\pi : X \to S$ or any other geometrical data. Finally, in *Step 4*, by applying Theorem 5.2.15 two times and using the fact that the analytic torsion coincides with (5.1.36) for the 3-punctured hyperbolic sphere (see [54, Remark 2.18a)]), we see that $A_{-n} = C_{-n}$, which would finish the proof of both Theorems 5.1.2 and 5.1.4.

In the first two steps the reduction is done by modifying norms $\|\cdot\|^{\omega}_{X/S}, \|\cdot\|_{X/S}$ only in the neighbourhood of $|D_{X/S}|$ and our technique is the same as in the proofs of [55, Theorems C, D].

We also note that by Theorem 5.2.3, we may suppose that near the singular locus, the Hermitian vector bundle (ξ, h^{ξ}) is trivial over a small neighbourhood of $|D_{X/S}| \cup \Sigma_{X/S}$.

Finally, we note that by Theorem 5.2.8, it is enough to prove Theorem 5.1.2 only in the case $\dim S = 1$, $S = D(1)$, $|\Delta| = \{0\}$. From now on we make those assumptions, and we conserve the relevant notations from Section 5.3.1.

Step 1. Let $V_{i,c}$, $i = 1, \dots, m$, $c > 0$ (resp. U) be a neighbourhood of $\sigma_i(t_0)$ (resp. t_0) such that for some local coordinates $(z_0, \dots, z_q)$ of $\sigma_i(t_0)$ and $(w_1, \dots, w_q)$ of $t_0 \in S$, satisfying (5.2.31), we have $V_{i,c} = \{x \in \pi^{-1}(U) : |z_0(x)| < c\}$ and $\{z_0(x) = 0\} = \{\sigma_i(t) : t \in U\}$. For simplicity, we note $V_i := V_{i,1}$. Let $\nu_0 : \mathbb{R}_+ \to [0, 1]$ be a smooth function satisfying

$$\nu_0(u) = \begin{cases} 0, & \text{if} \quad u < 1/2, \\ 1, & \text{if} \quad u > 3/4. \end{cases} \tag{5.3.30}$$

We denote by $\|\cdot\|^{\omega,0}_{X/S}$ the norm on $\omega_{X/S}$ over $X \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$ such that $\|\cdot\|^{\omega,0}_{X/S}$ coincides with $\|\cdot\|^{\omega}_{X/S}$ away from $\cup_{i=1}^m V_i$, and over $(\cup_{i=1}^m V_i) \setminus (\pi^{-1}(|\Delta|) \cup |D_{X/S}|)$, we have

$$\|dz_0\|^{\omega,0}_{X/S} = \big|z_0 \log |z_0|\big|^{1-\nu_0(|z_0|)} \cdot \big(\|dz_0\|^{\omega}_{X/S}\big)^{\nu_0(|z_0|)}. \tag{5.3.31}$$

Let $\|\cdot\|^0_{X/S}$ be the norm on $\omega_{X/S}(D)$ induced from $\|\cdot\|^{\omega,0}_{X/S}$ as in Construction 5.1.1, and let $g_0^{TX_t}$, $t \in S$ be the induced Kähler metric $X_t \setminus D_{X/S}$ with cusps at $D_{X/S} \cap X_t$. We denote by $\|\cdot\|^{W,0,i}_{X/S}$, $i = 1, \dots, m$, the Wolpert norms (see Definition 5.2.2) on $\sigma_i^* \omega_{X/S}$ induced by $g_0^{TX_t}$, and by $\|\cdot\|^{W,0}_{X/S}$ the induced Wolpert norm on $\otimes_{i=1}^m \sigma_i^* \omega_{X/S}$. Then by Construction 5.1.1 and (5.3.31), we see that if $\|\cdot\|_{X/S}$ satisfies assumptions (5.1.6) and (5.1.17), then $\|\cdot\|^0_{X/S}$ satisfies assumptions (5.1.6) and (5.1.17) as well.

In fact, this property along with the fact that $\|\cdot\|^{\omega,0}_{X/S}$ doesn't vary in the horizontal direction around the cusps are the only facts we need from the construction (5.3.31).

We denote by $g_0^{TY_0}$ the Kähler metric on $Y_0 \setminus D_{Y_0}$, constructed from $\|\cdot\|^0_{Y_0} := \rho^*(\|\cdot\|^0_{X/S})$ as in Construction 5.1.1. We denote by $\|\cdot\|^{W,0}_{Y_0}$ the Wolpert norm on $\otimes_{i=1}^{m+2k} (\sigma_i')^* \omega_{Y_0}$ induced by $g_0^{TY_0}$.

As we supposed that (ξ, h^ξ) is trivial in a neighbourhood of $|D_{X/S}|$, and the metrics $\|\cdot\|_{X/S}$, $\|\cdot\|^0_{X/S}$ differ only in the neighbourhood of $|D_{X/S}|$, by Theorem 5.2.3, applied pointwise for the line bundle $\lambda(j^*(\xi \otimes \omega_{X/S}(D)^n))^{12} \otimes (\otimes_{i=1}^m \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)}$, for any $t \in S \setminus |\Delta|$, we see that we have

$$\begin{aligned} &\frac{1}{6} \log \Big(\|\cdot\|_Q \big(g_0^{TX_t}, h^\xi \otimes (\|\cdot\|^0_{X/S})^{2n}\big)^{12} \otimes \big(\|\cdot\|^{W,0}_{X/S}\big)^{-\mathrm{rk}(\xi)} \Big) \\ &\qquad\qquad - \frac{1}{6} \log \Big(\|\cdot\|_Q \big(g^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S})^{2n}\big)^{12} \otimes \big(\|\cdot\|^{W}_{X/S}\big)^{-\mathrm{rk}(\xi)} \Big) \\ &= \mathrm{rk}(\xi) \cdot \int_{X_t} \Big(\widetilde{\mathrm{Td}}\big(\omega_{X/S}(D)^{-1}, \|\cdot\|^{-2}_{X/S}, (\|\cdot\|^0_{X/S})^{-2}\big) \mathrm{ch}\big(\omega_{X/S}(D)^n, \|\cdot\|^{2n}_{X/S}\big) \\ &\qquad + \mathrm{Td}\big(\omega_{X/S}(D)^{-1}, (\|\cdot\|^0_{X/S})^{-2}\big) \widetilde{\mathrm{ch}}\big(\omega_{X/S}(D)^n, \|\cdot\|^{2n}_{X/S}, (\|\cdot\|^0_{X/S})^{2n}\big) \Big). \end{aligned} \tag{5.3.32}$$

We note that the conformal factor corresponding to the change of the metric from $\|\cdot\|^{\omega}_{X/S}$ to $\|\cdot\|^{\omega,0}_{X/S}$ is non-trivial in the neighborhood of the cusp. Thus, we use Theorem 5.2.3 with the conformal factor which doesn't have compact support in the punctured surface.

By applying Theorem 5.2.3, where instead of flattening, we choose a "partial flattening" applied for the m points, coming from $\sigma_1', \dots, \sigma_m'$, we get

$$\begin{aligned}
&\frac{1}{6}\log\Big(\|\cdot\|_Q\left(g_0^{TY_0}, \rho^*(h^\xi)\otimes(\|\cdot\|_{Y_0}^0)^{2n}\right)^{12}\otimes\left(\|\cdot\|_{Y_0}^{W,0}\right)^{-\mathrm{rk}(\xi)}\Big)\\
&\qquad\qquad -\frac{1}{6}\log\Big(\|\cdot\|_Q\left(g^{TY_0}, \rho^*(h^\xi)\otimes(\|\cdot\|_{Y_0})^{2n}\right)^{12}\otimes\left(\|\cdot\|_{Y_0}^{W}\right)^{-\mathrm{rk}(\xi)}\Big)\\
&=\mathrm{rk}(\xi)\cdot\int_{Y_0}\Big(\widetilde{\mathrm{Td}}\big(\omega_{Y_0}(D)^{-1}, \|\cdot\|_{Y_0}^{-2}, (\|\cdot\|_{Y_0}^0)^{-2}\big)\mathrm{ch}\big(\omega_{Y_0}(D)^n, \|\cdot\|_{Y_0}^{2n}\big)\\
&\qquad\qquad +\mathrm{Td}\big(\omega_{Y_0}(D)^{-1}, (\|\cdot\|_{Y_0}^0)^{-2}\big)\widetilde{\mathrm{ch}}\big(\omega_{Y_0}(D)^n, \|\cdot\|_{Y_0}^{2n}, (\|\cdot\|_{Y_0}^0)^{2n}\big)\Big).
\end{aligned}\tag{5.3.33}$$

By Proposition 5.2.14, we see that the right-hand-side of (5.3.32) extends continuously over S, moreover, as $t\to 0$, by (5.2.45), the right-hand side of (5.3.32) tends to the right-hand side of (5.3.33). Thus, we see that it is enough to prove Theorem 5.1.2 for the metrics $\|\cdot\|_{X/S}^0$, $\|\cdot\|_{X/S}^{\omega,0}$, $\|\cdot\|_{X/S}^{W,0}$ instead of $\|\cdot\|_{X/S}$, $\|\cdot\|_{X/S}^{\omega}$, $\|\cdot\|_{X/S}^{W}$. We also note, that by (5.3.31), for $i=1,\dots,m$, the following identity holds

$$\big\|dz_0|_{\sigma_i(t)}\big\|_{X/S}^{W,0,i}=\big\|dz_0|_{\sigma_i'(0)}\big\|_{Y/S'}^{W,0,i}=1.\tag{5.3.34}$$

Step 2. We denote by $V_i'=V_{i,1/2}\subset V_i$, $i=1,\dots,m$. Let $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ be the Hermitian norm on $\omega_{X/S}$ over $X\setminus\pi^{-1}(|\Delta|)$ such that $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ coincides with $\|\cdot\|_{X/S}^{\omega,0}$ away from $\cup_{i=1}^m V_i'$, and for $\nu_0:\mathbb{R}\to[0,1]$ from (5.3.30), it is given over V_i' by

$$\|dz_0\|_{X/S}^{\omega,\mathrm{cmp}}=|z_0\log|z_0||^{\nu_0(2|z_0|)}.\tag{5.3.35}$$

We denote by $g_{\mathrm{cmp}}^{TX_t}$ the induced Kähler metric on X_t. By (5.3.35), we see that if $\|\cdot\|_{X/S}^{\omega,0}$ satisfies assumptions (5.1.6) and (5.1.17), then $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ satisfies assumptions (5.1.6) and (5.1.17) as well, but for $D_{X/S}=\emptyset$, i.e. without the cusps. In fact, this property along with the fact that $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ doesn't vary in the horizontal direction around the cusps are the only facts we need from the construction (5.3.35).

We denote by $g_{\mathrm{cmp}}^{TY_0}$ the Kähler metric over $Y_0\setminus\rho^{-1}(\Sigma_{X/S})$ induced from $\|\cdot\|_{X/S}^{\omega,\mathrm{cmp}}$ as in Section 1 for $D_{X/S}=\emptyset$. We denote by $\|\cdot\|_{X/S}^{\mathrm{cmp}}$ the norm on $\omega_{X/S}(D)$ over $X\setminus\pi^{-1}(|\Delta|)$, such that $\|\cdot\|_{X/S}^{\mathrm{cmp}}$ coincides with $\|\cdot\|_{X/S}^0$ away from $\cup_{i=1}^m V_i'$, and over V_i' we have

$$\big\|dz_0\otimes s_{D_{X/S}}/z_0\big\|_{X/S}^{\mathrm{cmp}}=|\log|z_0||^{\nu_0(2|z_0|)}.\tag{5.3.36}$$

We denote by $\|\cdot\|_{Y_0}^{\mathrm{cmp}}:=\rho^*(\|\cdot\|_{X/S}^{\mathrm{cmp}})$ the induced Hermitian norm on $\omega_{Y_0}(D)$ over $Y_0\setminus\rho^{-1}(\Sigma_{X/S})$. Now, since in $g_0^{TX_t}$, the Poincaré-compatible coordinates of the cusps are trivialized, by Theorem 5.2.1, we see that for $t\in S\setminus|\Delta|$, the following holds

$$\begin{aligned}
&2\log\Big(\|\cdot\|_Q\left(g_{\mathrm{cmp}}^{TX_t}, h^\xi\otimes(\|\cdot\|_{X/S}^{\mathrm{cmp}})^{2n}\right)/\|\cdot\|_Q\left(g_0^{TX_t}, h^\xi\otimes(\|\cdot\|_{X/S}^0)^{2n}\right)\Big)\\
&\qquad=2\log\Big(\|\cdot\|_Q\left(g_{\mathrm{cmp}}^{TY_0}, \rho^*h^\xi\otimes(\|\cdot\|_{Y_0}^{\mathrm{cmp}})^{2n}\right)/\|\cdot\|_Q\left(g_0^{TY_0}, \rho^*h^\xi\otimes(\|\cdot\|_{Y_0}^0)^{2n}\right)\Big),
\end{aligned}\tag{5.3.37}$$

where we didn't mention the last term of (5.2.13) since (ξ, h^{ξ}) is trivial in the neighbourhood of $|D_{X/S}|$, and the norms $\|\cdot\|^{\mathrm{cmp}}_{X/S}, \|\cdot\|^{0}_{X/S}$ differ only in the neighbourhood of $|D_{X/S}|$. We denote by $\|\cdot\|^{\mathrm{cmp}}_{D_{X/S}}$ the norm on $\mathscr{O}_X(D_{X/S})$, given by $\|\cdot\|^{\mathrm{cmp}}_{X/S}/\|\cdot\|^{\omega,\mathrm{cmp}}_{X/S}$. Then the norm $\|\cdot\|^{\mathrm{cmp}}_{D_{X/S}}$ is trivial away from $\cup_{i=1}^{m} V_i$ and smooth over X.

By (5.1.24), (5.3.34) and (5.3.37), we see that it is enough to prove Theorem 5.1.2 for the Hermitian vector bundles $(\xi \otimes \mathscr{O}_X(D_{X/S})^n, h^{\xi} \otimes (\|\cdot\|^{\mathrm{cmp}}_{D_{X/S}})^{2n})$, $(\omega_{X/S}, \|\cdot\|^{\omega,\mathrm{cmp}}_{X/S})$ and $D_{X/S} = \emptyset$, instead of (ξ, h^{ξ}), $(\omega_{X/S}, \|\cdot\|^{\omega,0}_{X/S})$ and $D_{X/S}$, given by (5.1.2). But as the Hermitian norm $\|\cdot\|^{\mathrm{cmp}}_{D_{X/S}}$ is smooth over X, such a statement would follow from Theorem 5.1.2 for $m = 0$. Thus, we conclude that to prove Theorem 5.1.2 in its full generality, it is enough to prove it only for $m = 0$. From now on, we suppose $m = 0$.

Step 3. The main goal of this step is to show that Theorem 5.1.2 holds, but instead of C_{-n}, we have an undetermined constant A_{-n}, which doesn't depend on $\pi : X \to S$ and any other geometrical data. The idea is to construct a metric $\|\cdot\|^{\varkappa}_{X/S}$ on $\omega_{X/S}$ over $X \setminus |\pi^{-1}(\Delta)|$, for which the assumptions (5.1.6), (5.1.17) hold and to show that Theorem 5.1.2 holds for $\|\cdot\|^{\varkappa}_{X/S}$ but instead of C_{-n}, we have an undetermined constant A_{-n}. Then, by the anomaly formula and Proposition 5.2.14, we deduce that Theorem 5.1.2 holds up to $\exp(k \cdot \mathrm{rk}(\xi) \cdot A_{-n})$ for any norm $\|\cdot\|^{\omega}_{X/S}$, for which the assumptions (5.1.6), (5.1.17) are satisfied.

We proceed in the following way. First, we construct a Riemannian metric $g^{TX}_{\sim}$ preserving the complex structure on X. This metric will be trivial in the neighbourhood of $\Sigma_{X/S}$. By using Theorem 5.3.1, we will study the asymptotics of the norm $\|\cdot\|^{\sim}_{\mathscr{L}_n}$ induced on the line bundle $\mathscr{L}_n$ by $\|\cdot\|^{\sim}_{X/S}$. Then by modifying locally this metric in the neighbourhood of $\Sigma_{X/S}$, we construct a family of metrics $g^{TX_t}_{\varkappa}$ on X_t for $t \in S \setminus |\Delta|$. This family degenerates to the hyperbolic metric at the singular fiber through the family of degenerating hyperbolic cylinders. By applying anomaly theorem and the previous result on the asymptotics of $\|\cdot\|^{\sim}_{\mathscr{L}_n}$, we compute the asymptotics of the norm $\|\cdot\|^{\varkappa}_{\mathscr{L}_n}$ induced on the line bundle $\mathscr{L}_n$ by $g^{TX_t}_{\varkappa}$. As our construction is local around $\Sigma_{X/S}$, the asymptotic we obtain would not depend on any geometry of $\pi : X \to S$. Let's make this reasoning more precise...

Let local coordinates z_0^j, z_1^j (cf. (5.3.1)) around Q_j be as in (5.2.59). We suppose that $D(1) \subset \mathrm{Im}(z_0^j) \cap \mathrm{Im}(z_1^j)$. This is merely a question of normalisation, as for any coordinates z_0^j, z_1^j of X and t of S, and for any $a \in \mathbb{C}$, we may change the coordinates by $a \cdot z_0^j, a \cdot z_1^j$ and $a^2 \cdot t$, and the identity (5.2.59) would be preserved. We specify the function ν from (5.2.61) as follows

$$\nu(x) = \begin{cases} 1, & \text{for } x \in X \setminus (\cup_{i=1}^{k} U(Q_i, 1)), \\ \nu_0(|z_0^i|^2 + |z_1^i|^2), & \text{for } x \in U(Q_i, 1). \end{cases} \tag{5.3.38}$$

By (5.3.30), the function (5.3.38) satisfies the generic conditions (5.2.61) for $\epsilon = 1/2$.

We note that π is locally projective (cf. Bismut-Bost [20, Proposition 3.4]), thus, there is a neighbourhood U of $0 \in S$ such that there is a Kähler metric g_0^{TX} over $\pi^{-1}(U)$. As the statement of Theorem 5.3.1 is local over the base, without losing the generality, we may suppose from now on that g_0^{TX} is defined over X.

We define the Riemannian metric $g^{TX}_{\sim}$ over X so that it coincides with g_0^{TX} over $X \setminus$

$(\cup_{i=1}^k U(Q_i,1))$, and over $U(Q_i,1)$ it is given by

$$g_{\sim}^{TX} = \nu \cdot g_0^{TX} + (1-\nu)\cdot(|dz_0^i|^2 + |dz_1^i|^2). \tag{5.3.39}$$

We note that $g_{\sim}^{TX}$ is not necessarily Kähler, but it is trivially compatible with the complex structure of X. We denote by $g_{\sim}^{TX_t}$ the induced Riemannian metric on X_t, $t \in S \setminus |\Delta|$, and by $g_{\sim}^{TY_0}$ the induced Riemannian metric on Y_0, constructed as in Section 5.3.1, i.e. by $g_{\sim}^{TY_0} = \rho^*(g_{\sim}^{TX})$, where $\rho : Y_0 \to X_0$ is the normalization map. Since $g_{\sim}^{TX}$ is compatible with the complex structure of X, we see that the metrics $g_{\sim}^{TX_t}$, $g_{\sim}^{TY_0}$ are Kähler.

We endow $\omega_{X/S}$ with the Hermitian norm $\|\cdot\|_{X/S}^{\sim,\mathrm{ind}}$ induced by $g_{\sim}^{TX}$ over $X \setminus \Sigma_{X/S}$. We define the Hermitian norm $\|\cdot\|_{X/S}^{\sim}$ on $\omega_{X/S}$ over X as follows. Over $X \setminus (\cup_{i=1}^k U(Q_i,1))$, we demand it to be equal to $\|\cdot\|_{X/S}^{\sim,\mathrm{ind}}$, and over $U(Q_i,1)$, we define it by

$$\left\|dz_0^i/z_0^i\right\|_{X/S}^{\sim} = \nu \cdot \left\|dz_0^i/z_0^i\right\|_{X/S}^{\sim,\mathrm{ind}} + (1-\nu). \tag{5.3.40}$$

Trivially, the Hermitian norm $\|\cdot\|_{X/S}^{\sim}$ on $\omega_{X/S}$ is smooth over X. Moreover, it is trivial on $\cup_{i=1}^k U(Q_i,1/2)$. The norm $\|\cdot\|_{Y_0}^{\sim} := \rho^*(\|\cdot\|_{X/S}^{\sim})$ over $\omega_{Y_0}(D)$ is characterized as follows: over $Y_0 \setminus (\cup_{i=1}^k(\{|z_0^i|<1\}\cup\{|z_1^i|<1\}))$, it is induced by $g_{\sim}^{TY_0}$ as in Construction 5.1.1, over $\{|z_j^i|<1\}$, $j=0,1$, $i=1,\dots,k$, it is given by

$$\left\|dz_j^i \otimes s_{D_{Y_0}}/z_j^i\right\|_{Y_0}^{\sim} = (\nu\circ\rho)\cdot\left\|dz_j^i/z_j^i\right\|_{X/S}^{\sim,\mathrm{ind}} + (1-(\nu\circ\rho)). \tag{5.3.41}$$

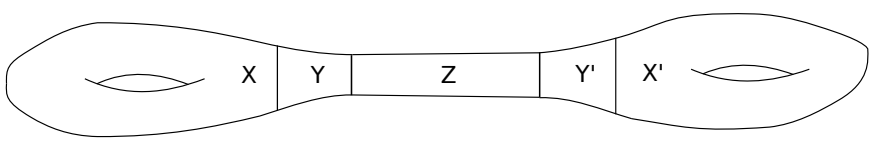

Figure 5.4: The metric $g_{\sim}^{TX_t}$. Over the regions X, X', it is induced by g_0^{TX}. Over the regions Y, Y', it is an interpolation between g_0^{TX} and $|dz_0|^2+|dz_1|^2$, and over the region Z, it is given by $|dz_0|^2+|dz_1|^2$, where $z_0z_1 = t$.

We endow $\mathscr{L}_n$ with the metric $\|\cdot\|_{\mathscr{L}_n}^{\sim}$, induced by the Quillen norm $\|\cdot\|_Q\,(g_{\sim}^{TX_t}, h^\xi \otimes (\|\cdot\|_{X/S}^{\sim})^{2n})$ and the singular norm (5.1.3). We endow $\mathscr{L}_n'$ with the norm $\|\cdot\|_{\mathscr{L}_n'}^{\sim}$, induced by the Quillen norm $\|\cdot\|_Q\,(g_{\sim}^{TY_0}, \rho^*(h^\xi)\otimes(\|\cdot\|_{Y_0}^{\sim})^{2n})$ and the norm $\|\cdot\|_{\Sigma_{X/S}/X}^{\sim}$ (see (5.3.3)) on $\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}}\otimes\omega_{Y_0}|_{P_{2i}})$. By (5.3.39), the norm $\|\cdot\|_{\Sigma_{X/S}/X}^{\sim}$ is characterized by the identity

$$\left\|\otimes_{i=1}^k\,(dz_0^i|_{P_{2i-1}}\otimes dz_1^i|_{P_{2i}})\right\|_{\Sigma_{X/S}/X}^{\sim} = 1. \tag{5.3.42}$$

The metrics $g_{\sim}^{TX_t}, \|\cdot\|_{X/S}^{\sim}$ and $g_{\sim}^{TY_0}, \|\cdot\|_{Y_0}^{\sim}$ satisfy the hypothesis of Theorem 5.3.1. Thus, for

$$A'_{-n} := 24\zeta'(-1) - 6\log(2\pi), \tag{5.3.43}$$

for a smooth local frame υ of $\mathscr{L}_n$, by (5.3.42), under the isomorphism (5.1.28), the following identity holds

$$\lim_{t\to 0}\Big(\log\big(\|\upsilon(t)\|^{\widetilde{}}_{\mathscr{L}_n}\big)-k\cdot\mathrm{rk}(\xi)\cdot\log|t|\Big)=\log\big(\|\upsilon(0)\|^{\widetilde{}}_{\mathscr{L}'_n}\big)+k\cdot\mathrm{rk}(\xi)\cdot A'_{-n}. \tag{5.3.44}$$

Now, let $\widetilde{\nu}:X\to[0,1]$ be defined as ν in (5.3.38), where in place of $\nu_0(\cdot)$, we put $\nu_0(4\cdot)$. Then $\widetilde{\nu}(x)=1$ for $x\in X\setminus(\cup_{i=1}^k U(Q_i,1/2))$. We define the metrics $g_{\varkappa}^{TX_t}$ on X_t, $t\in S\setminus|\Delta|$, as follows: over $X_t\setminus(\cup_{i=1}^k U(Q_i,1/2))$ it coincides with $g_{\varkappa}^{TX_t}$, and over $U(Q_i,1/2)$ it is given by

$$g_{\varkappa}^{TX_t}:=\widetilde{\nu}\cdot g_{\sim}^{TX_t}+(1-\widetilde{\nu})\cdot g_{i,t}^{\mathrm{Cyl}}, \tag{5.3.45}$$

where the metric $g_{j,t}^{\mathrm{Cyl}}$ was defined in (5.2.63). We also define the metric $g_{\varkappa}^{TY_0}$ as follows: over $Y_0\setminus(\cup_{i=1}^k U(Q_i,1/2))$ it coincides with $g_{\sim}^{TY_0}$, and over $U(Q_i,1/2)$ it is given by

$$g_{\varkappa}^{TY_0}:=(\widetilde{\nu}\circ\rho)\cdot g_{\sim}^{TY_0}+\big(1-(\widetilde{\nu}\circ\rho)\big)\cdot\big(g_{i,0}^{\mathrm{Poinc}}+g_{i,1}^{\mathrm{Poinc}}\big), \tag{5.3.46}$$

where the metrics $g_{i,0}^{\mathrm{Poinc}}$, $g_{i,1}^{\mathrm{Poinc}}$ are the metrics induced by the Poincaré metric (5.2.2) with respect to the coordinates z_0^i and z_1^i. We denote by $\|\cdot\|_{X/S}^{\varkappa}$ the Hermitian norm on $\omega_{X/S}$ induced by $g_{\varkappa}^{TX_t}$. By (5.3.45), we see that the Hermitian norm $\|\cdot\|_{X/S}^{\varkappa}$ extends continuously over $X\setminus\Sigma_{X/S}$, and the assumptions (5.1.6) are satisfied. We define the norm $\|\cdot\|_{Y_0}^{\varkappa}$ on $\omega_{Y_0}(D)$ as follows

$$\|\cdot\|_{Y_0}^{\varkappa}=\rho^*(\|\cdot\|_{X/S}^{\varkappa}). \tag{5.3.47}$$

Then we see trivially that $\|\cdot\|_{X/S}^{\varkappa}$ satisfies assumptions (5.1.17), and by (5.3.45), (5.3.46), the associated metric on $Y_0\setminus D_{Y_0}$, constructed as in Section 1, coincides with $g_{\varkappa}^{TY_0}$.

Let's pause and explain this construction. The metrics $g_{\varkappa}^{TX_t}$ degenerate near the singular fibers to a metric with cusps in the similar way as the hyperbolic metrics (see Theorem 5.2.17). The advantage of the metrics $g_{\varkappa}^{TX_t}$ over the hyperbolic one is that over the region $\cup_{i=1}^k U(Q_i,1/2)$, it is independent of any exterior data as $\pi:X\to S$, and over $X_t\setminus(\cup_{i=1}^k U(Q_i,1/2))$, the metric $g_{\varkappa}^{TX_t}$ coincides with a metric $g_{\sim}^{TX_t}$, for which Theorem 5.3.1 holds.

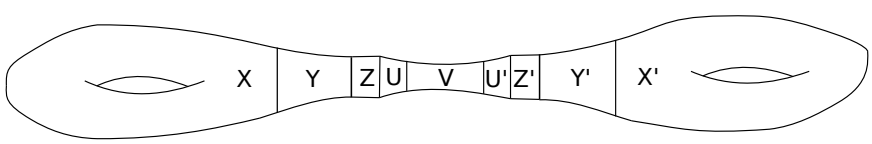

Figure 5.5: The metric $g_{\varkappa}^{TX_t}$. Over the regions X,Y,Z,Z',Y',X' it coincides with $g_{\sim}^{TX_t}$. Over the regions U,U', it is an interpolation between $g_{\sim}^{TX_t}$ and the hyperbolic cylinder metric, (5.2.63), and over the region V, it coincides with the hyperbolic cylinder metric, (5.2.63).

Let $\|\cdot\|_{Y_0}^{W,\varkappa}$ be the Wolpert norm on $\otimes_{i=1}^k(\omega_{Y_0}|_{P_{2i-1}}\otimes\omega_{Y_0}|_{P_{2i}})$. By Definition 5.2.2 and (5.3.46)

$$\big\|\otimes_{i=1}^k\,(dz_0^i|_{P_{2i-1}}\otimes dz_1^i|_{P_{2i}})\big\|_{Y_0}^{W,\varkappa}=1. \tag{5.3.48}$$

We endow the holomorphic line bundle $\mathscr{L}_n$ with the Hermitian norm $\|\cdot\|^{\varkappa}_{\mathscr{L}_n}$, induced by the Quillen norm $\|\cdot\|_Q\,(g^{TX_t}_{\varkappa}, h^{\xi}\otimes(\|\cdot\|^{\varkappa}_{X/S})^{2n})$ and the singular norm (5.1.3). We endow the complex line $\mathscr{L}'_n$ with the Hermitian norm $\|\cdot\|^{\varkappa}_{\mathscr{L}'_n}$, induced by the Quillen norm $\|\cdot\|_Q\,(g^{TY_0}_{\varkappa}, \rho^*(h^{\xi})\otimes(\|\cdot\|^{\varkappa}_{Y_0})^{2n})$ and the Wolpert norm $\|\cdot\|^{W,\varkappa}_{Y_0}$. By the anomaly formula of Bismut-Gillet-Soulé [23] (cf. Theorem 5.2.3 for $m=0$), we deduce that for any $t\in S\setminus|\Delta|$, we have

$$\log\Big(\|\cdot\|^{\varkappa}_{\mathscr{L}_n}/\|\cdot\|^{\sim}_{\mathscr{L}_n}\Big)(t)=6\int_{X_t}G, \tag{5.3.49}$$

where the differential form G is given by

$$\begin{aligned}G=\mathrm{rk}(\xi)\cdot\Big(&\widetilde{\mathrm{Td}}\big(\omega^{-1}_{X/S},(\|\cdot\|^{\sim,\mathrm{ind}}_{X/S})^{-2},(\|\cdot\|^{\varkappa}_{X/S})^{-2}\big)\mathrm{ch}\big(\omega^{n}_{X/S},(\|\cdot\|^{\sim}_{X/S})^{2n}\big)\\&+\mathrm{Td}\big(\omega^{-1}_{X/S},(\|\cdot\|^{\varkappa}_{X/S})^{-2}\big)\widetilde{\mathrm{ch}}\big(\omega^{n}_{X/S},(\|\cdot\|^{\sim}_{X/S})^{2n},(\|\cdot\|^{\varkappa}_{X/S})^{2n}\big)\Big).\end{aligned} \tag{5.3.50}$$

We decompose

$$\int_{X_t}G=\int_{X_t}G_1+\int_{X_t}G_2, \tag{5.3.51}$$

where G_1 (resp. G_2) is given by the same formula as (5.3.50), where in place of $\|\cdot\|^{\varkappa}_{X/S}$ (resp. $\|\cdot\|^{\sim,\mathrm{ind}}_{X/S}$), we put $\|\cdot\|^{\sim}_{X/S}$. By (5.3.50), (5.3.51) and the fact that the norms $\|\cdot\|_{X/S}, \|\cdot\|^{\sim}_{X/S}, \|\cdot\|^{\varkappa}_{X/S}$ coincide over $X\setminus(\cup_{i=1}^{m}U(Q_i,1))$, we conclude that G and G_i, $i=1,2$ have support over $\cup_{i=1}^{k}U(Q_i,1)$. Over $U(Q_i,1)$, by (5.3.39), (5.3.40), (5.3.45), the following identities hold

$$\begin{aligned}&\Big(\Big\|z^i_0\frac{\partial}{\partial z^i_0}-z^i_1\frac{\partial}{\partial z^i_1}\Big\|^{\sim,\mathrm{ind}}_{X/S}\Big)^2\Big|_{X_t}=|z^i_0|^2+|z^i_1|^2,\\&\Big(\Big\|z^i_0\frac{\partial}{\partial z^i_0}-z^i_1\frac{\partial}{\partial z^i_1}\Big\|^{\sim}_{X/S}\Big)^2\Big|_{X_t}=\nu\cdot(|z^i_0|^2+|z^i_1|^2)+(1-\nu),\\&\Big(\Big\|z^i_0\frac{\partial}{\partial z^i_0}-z^i_1\frac{\partial}{\partial z^i_1}\Big\|^{\varkappa}_{X/S}\Big)^2\Big|_{X_t}=\widetilde{\nu}\cdot\big(|z^i_0|^2+|z^i_0|^2\big)+\frac{4\pi(1-\widetilde{\nu})}{\log|t|}\Big(\sin\frac{\pi\log|z^i_0|}{\log|t|}\Big)^{-1}.\end{aligned} \tag{5.3.52}$$

By (5.3.52) and the fact that the norms $\|\cdot\|_{X/S}, \|\cdot\|^{\sim}_{X/S}, \|\cdot\|^{\varkappa}_{X/S}$ coincide over $X\setminus(\cup_{i=1}^{m}U(Q_i,1))$, the functions $\int_{X_t}G_i: S\setminus|\Delta|\to\mathbb{R}$, $i=1,2$, can be written as

$$\int_{X_t}G_i=\mathrm{rk}(\xi)\cdot k\cdot f_i(t), \tag{5.3.53}$$

where $f_i: S\setminus|\Delta|\to\mathbb{R}$ depend only on the choice of the function ν_0. Let's study the functions f_i more precisely.

First, as the norm $\|\cdot\|^{\sim}_{X/S}$ is smooth over X, and the first Chern form of the norm $\|\cdot\|^{\varkappa}_{X/S}$ has Poincaré growth by Proposition 5.2.16, we conclude by (5.2.15), (5.2.16), (5.3.50) and (5.3.52) that the following bound holds.

$$G_2|_{X_t\cap U(Q_i,1)}=O\Big(\log|\log|z^i_0||\frac{dz^i_0d\overline{z}^i_0}{|z^i_0\log|z^i_0||^2}\Big). \tag{5.3.54}$$

Thus, the functions f_2 extends continuously over S. We denote $A^{II}_{-n} := f(0)$.

Now let's treat the function f_1. By (5.3.25) and (5.3.52), for $t \in S \setminus |\Delta|$, we have the following identity

$$c_1(TX/S, g^{TX})|_{X_t \cap U(Q_i,1/2)} = \frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i}{2\pi\sqrt{-1}}. \tag{5.3.55}$$

Now, we note that the norm $\|\cdot\|^{\sim}_{X/S}$ is trivial over $U(Q_i, 1/2)$. By this, (5.2.15), (5.2.16) and (5.3.50) we see that the following identity holds

$$f_1(t) = \frac{1}{12}\int_{2|t|<|z_0^i|<1/2} \log\left(|z_0^i|^2 + |t/z_0^i|^2\right)\frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i}{2\pi\sqrt{-1}} + 2\int_{1/2<|z_0^i|<1} G_1. \tag{5.3.56}$$

By making change of variables $y := z_0^i \cdot |t|^{-1/2}$, we see that

$$\begin{aligned}\int_{2|t|<|z_0^i|<1/2} &\log\left(|z_0^i|^2 + |t/z_0^i|^2\right)\frac{4|z_0^i|^2|t|^2}{(|z_0^i|^4+|t|^2)^2}\frac{dz_0^i d\overline{z}_0^i} \\ &= (\log|t|)\cdot\int_{2|t|^{1/2}<|y|<|t|^{-1/2}/2}\frac{4|y|^2}{(|y|^4+1)^2}\frac{dyd\overline{y}}{2\pi\sqrt{-1}} \\ &\quad + \int_{2|t|^{1/2}<|y|<|t|^{-1/2}/2}\log(|y|^2+|y|^{-2})\frac{4|y|^2}{(|y|^4+1)^2}\frac{dyd\overline{y}}{2\pi\sqrt{-1}}.\end{aligned} \tag{5.3.57}$$

Also, we see that

$$\int_{y\in\mathbb{C}}\frac{4|y|^2}{(|y|^4+1)^2}\frac{dyd\overline{y}}{2\pi\sqrt{-1}} = -\int_0^{+\infty}\frac{8r^3dr}{(r^4+1)^2} = -2. \tag{5.3.58}$$

By a simple calculation, we see that the second summand in the right-hand side of (5.3.57) extends continuously over $\{t = 0\}$. The same holds for the second summand in the right-hand side of (5.3.56). Thus, by (5.3.56), (5.3.57), (5.3.58), we conclude that there is a constant A^I_{-n} such that

$$\lim_{t\to 0}\left(f_1(t) + \frac{\log|t|}{6}\right) = A^I_{-n}. \tag{5.3.59}$$

Thus, by (5.3.49), (5.3.53) and (5.3.59), the norm $\|\cdot\|^{\varkappa}_{\mathscr{L}_n}$ extends continuously over S. Moreover

$$\lim_{t\to 0}\left(\log\left(\|\cdot\|^{\varkappa}_{\mathscr{L}_n}/\|\cdot\|^{\sim}_{\mathscr{L}_n}\right)(t) + k\cdot\mathrm{rk}(\xi)\cdot\log|t|\right) = 6k\cdot\mathrm{rk}(\xi)\cdot(A^I_{-n}+A^{II}_{-n}). \tag{5.3.60}$$

Now, by Theorem 5.2.1, (5.3.42), (5.3.46), (5.3.47) and (5.3.48), we deduce that the quantity

$$(\mathrm{rk}(\xi)\cdot k)^{-1}\log\left(\|\cdot\|^{\varkappa}_{\mathscr{L}'_n}/\|\cdot\|^{\sim}_{\mathscr{L}'_n}\right) := A'''_{-n} \tag{5.3.61}$$

depends only the choice of the function ν_0. Thus, by (5.3.44), (5.3.60) and (5.3.61), we deduce that under the isomorphism (5.1.28), the following holds

$$\|\cdot\|^{\varkappa}_{\mathscr{L}_n}|_{\Delta} = \exp(k\cdot\mathrm{rk}(\xi)\cdot A_{-n})\cdot\|\cdot\|^{\varkappa}_{\mathscr{L}'_n} \quad \text{with} \quad A_{-n} := A'_{-n} + 6(A^I_{-n}+A^{II}_{-n}) - A'''_{-n}. \tag{5.3.62}$$

The essential difference between (5.3.44) and (5.3.62) is that (5.3.44) is a statement in realms of Theorem 5.3.1, and (5.3.62) is a statement in realms of Theorem 5.1.2, which is exactly what we need.

Now, let $\|\cdot\|_{X/S}, \|\cdot\|_{Y_0}$ be any norms which satisfy the assumptions of Theorem 5.1.2. We denote by $\|\cdot\|_{\mathscr{L}_n}, \|\cdot\|_{\mathscr{L}'_n}$ the Hermitian norms on $\mathscr{L}_n, \mathscr{L}'_n$, defined as in Theorem 5.1.2. By Proposition 5.2.14 and the anomaly formula of Bismut-Gillet-Soulé [23] (cf. Theorem 5.2.3 for $m = 0$), applied for the line bundles $\mathscr{L}_n$ and $\mathscr{L}'_n$, we get

$$\lim_{t\to 0} \log\left(\|\cdot\|^{\varkappa}_{\mathscr{L}_n} / \|\cdot\|_{\mathscr{L}_n}\right)(t) = \log\left(\|\cdot\|^{\varkappa}_{\mathscr{L}'_n} / \|\cdot\|_{\mathscr{L}'_n}\right). \tag{5.3.63}$$

From (5.3.62) and (5.3.63), we deduce that the norm $\|\cdot\|_{\mathscr{L}_n}$ extends continuously over S and under the isomorphism (5.1.28), the following holds

$$\|\cdot\|_{\mathscr{L}_n}|_{\Delta} = \exp(k \cdot \mathrm{rk}(\xi) \cdot A_{-n}) \cdot \|\cdot\|_{\mathscr{L}'_n}, \tag{5.3.64}$$

in other words Theorem 5.1.2 holds, but instead of C_n, we have an undetermined constant A_{-n}.

Step 4. The goal of this step is to show that $A_{-n} = C_{-n}$ and to prove Theorem 5.1.4.

For this we consider a stable pointed Riemann surface $(\overline{M}, D_M)$ and the associated canonical plumbing family $\pi : X \to S$ with the canonical hyperbolic norm $\|\cdot\|^{\mathrm{hyp}}_{X/S}$ on $\omega_{X/S}$ from Section 5.2.3.

Then, in the notations of Section 5.2.3, by a theorem of Phong-d'Hooker (cf. Remark 5.1.5), the following identity of norms over $S \setminus |\Delta|$ holds

$$\|\cdot\|_Q\left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right) = \|\cdot\|^{TZ}_Q\left(g^{TX_t}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{X/S})^{2n}\right). \tag{5.3.65}$$

We apply this construction for $(\overline{M}, D_M) := (\overline{T}, D_T)$, where $(\overline{T}, D_T)$ is a 1-pointed torus, considered in Section 5.2.3. Then by Theorem 5.2.15, Step 3 and (5.3.65), we get

$$\exp(A_{-n}/2) \cdot \|\cdot\|_Q\left(g^{TT}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{T})^{2n}\right) = \exp(C_{-n}/2) \cdot \|\cdot\|^{TZ}_Q\left(g^{TT}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{T})^{2n}\right). \tag{5.3.66}$$

By applying (5.3.65) again, but now for any $(\overline{M}, D_{\overline{M}})$, by Theorem 5.2.15, Step 3 and (5.3.66), we see that for any $(\overline{M}, D_M)$, $m := \#D_M$, we have

$$\exp(m \cdot A_{-n}/2) \cdot \|\cdot\|_Q\left(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{M})^{2n}\right) = \exp(C_{-n}/2) \cdot \|\cdot\|^{TZ}_Q\left(g^{TM}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{M})^{2n}\right). \tag{5.3.67}$$

However, by [54, Remark 2.18a)], our definition of the analytic torsion coincides with the definition of Takhtajan-Zograf for the 3-punctured hyperbolic sphere $P := \mathbb{P} \setminus \{0, 1, \infty\}$, i.e.

$$\|\cdot\|_Q\left(g^{TP}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{P})^{2n}\right) = \|\cdot\|^{TZ}_Q\left(g^{TP}_{\mathrm{hyp}}, (\|\cdot\|^{\mathrm{hyp}}_{P})^{2n}\right). \tag{5.3.68}$$

By combining (5.3.67) and (5.3.68), we get $A_{-n} = C_{-n}$, which finishes the proof of Theorem 5.1.2. Also by (5.2.11), (5.2.51) and (5.3.67), we deduce Theorem 5.1.4. □

Proof of Theorem 5.1.6. By [55, Proposition 5.6], the norm $\|\cdot\|^{\mathrm{hyp}}_{g,m}$ satisfies assumptions (5.1.6) and (5.1.17). Thus, Theorem 5.1.6 is a direct consequence of Theorem 5.1.2 and [55, Proposition 5.6]. The fact that the underlying spaces are orbifolds doesn't pose any problem, as our methods are local, and thus, can be applied on an orbifold chart. □

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