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Polymères dirigés et équation KPZ

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Titre : Polymères dirigés et équation KPZ

Résumé : Cette thèse est consacrée à l'étude des liens entre les polymères dirigés en milieu aléatoire, l'équation de la chaleur stochastique avec bruit multiplicatif (SHE) et l'équation Kardar-Parisi- Zhang (KPZ). En dimension d'espace $d = 1$, l'équation KPZ et l'équation SHE font partie d'une classe de modèles appelée la classe d'universalité KPZ. Il est conjecturé que le modèle de polymère dirigé en environnement aléatoire en fasse également partie, mais en dehors de choix d'environnement spécifiques, une preuve de cette conjecteure semble pour le moment hors d'atteinte. Toutefois, on peut montrer dans un cadre général que la fonction de partition du polymère approche la solution de l'équation SHE, à condition de se placer dans un régime de paramètres particulier. En dimension supérieure, définir des solutions des équations KPZ et SHE est un sujet encore mal compris. Un plan d'approche pour étudier ce problème est de considérer une régularisation de ces équations. Il se trouve que les solutions régularisées sont reliées aux fonctions de partition d'un modèle de polymère dirigé, et on peut mettre à profit les techniques de la littérature sur les polymères pour étudier ces solutions.

Mots clefs : Polymères dirigés, environnement aléatoire, équation KPZ, équation de la chaleur stochastique, fonction de partition, régime intermédiaire, faible désordre, limite d'échelle proche du point critique, théorème centrale limite, martingales.

Title: Directed polymers and the KPZ equation

Abstract: This thesis is dedicated to the study of the links between directed polymers in random environment, the stochastic heat equation with multiplicative noise (SHE) and the Kardar-Parisi-Zhang equation (KPZ). In space dimension $d = 1$, the KPZ equation and the SHE equation belong to a class of models which is called the KPZ universality class. The model of directed polymer in random environment is conjectured to also belong to this class, but except for specific choices of environments, a general proof of this conjecture seems for now out of reach. Nevertheless, one can prove that under general environments and under a specific regime of parameters, the point-to-point partition function of the polymer converges towards the solution of the SHE equation. In higher space dimension, it is not clear whether the KPZ and SHE equations should be well-posed. A plan to study this problem is to consider a regularization of these equations. It turns out that the solutions of the regularized equations are linked to the partition functions of a directed polymer model, and one can use standard polymer techniques to study these solutions.

Keywords: Directed polymers, random environment, KPZ equation, stochastic heat equation, partition function, intermediate disorder, weak disorder, near-critical scaling limits, central limit theorem, martingales.

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Table des Matières

Part I

Introduction

Quelques sigles usuels

- **KPZ** : Kardar-Parisi-Zhang ; utilisé pour désigner l'équation KPZ (28) (ou (44) dans sa forme régularisée) ou la classe d'universalité KPZ,
- **SHE** : *Stochastic Heat Equation* ; désigne l'équation de la chaleur stochastique avec bruit multiplicatif (29) (ou (42) dans sa forme régularisée),
- **EW** : Edwards-Wilkinson ; désigne l'équation EW (25) ou la classe d'universalité EW,
- **P2P** : point-à-point ; désigne les fonctions de partition point-à-point du polymère (équation (20)),
- **P2L** : point-à-ligne ; désigne les fonctions de partition point-à-ligne du polymère (équations (10) ou (3)),
- La **Région $\mathbf{L^2}$** : voir Section 1.3.

Généralités

Cette thèse est consacrée à l'étude des liens entre les polymères dirigés en milieu aléatoire, l'équation de la chaleur stochastique avec bruit multiplicatif (SHE) et l'équation Kardar-Parisi-Zhang (KPZ), en différentes dimensions d'espace. L'exploitation des liens entre ces modèles a connu, ces dix dernières années, un essor important avec la publication d'articles importants [3, 1, 86]. En dimension $d = 1$, l'équation KPZ et l'équation SHE font partie d'une classe de modèle possédant des coefficients d'échelle et des limites d'échelle non-standards, appelée la *classe d'universalité KPZ*. Il est possible de montrer que certains modèles particuliers de polymères, dits exactement solubles, font partie de la même classe, mais un des problèmes ouverts du domaine est de montrer l'universalité de ce résultat pour des modèles généraux de polymères dirigés. Toutefois, sous un changement d'échelle diffusif et dans un régime particulier de paramètres, on peut montrer que la fonction de partition point-à-point des polymères généraux converge vers la solution de l'équation de la chaleur stochastique, montrant un résultat d'universalité KPZ *faible* des modèles de polymères.

En dimension supérieure, il n'est pas encore certain que les équations KPZ et SHE soient bien posées. Pour étudier ces équations en dimension supérieure, l'approche que l'on considérera ici suit celle introduite dans [14, 15], consistant à régulariser les équations KPZ et SHE avec un bruit moyenné en espace, puis de regarder la limite des solutions lorsque la régularisation est dissipée. Il se trouve que dans un certain régime de paramètres, les solutions de ces équations régularisées sont reliées aux fonctions de partition d'un modèle de polymère dirigé, et l'on peut mettre à profit les techniques et résultats de la littérature des polymères pour les étudier.

1 Les polymères dirigés en milieu aléatoire

1.1 Origine et histoire du modèle

Le modèle de polymère dirigé en milieu aléatoire a été introduit dans la littérature physique par Huse et Henly [98], puis traduit dans le langage mathématique par Imbrie et Spencer [102] et développé dans l'article de synthèse de Krug et Spohn [109].

Dans ce modèle, un *polymère*, c'est à dire une longue chaîne de monomères, est placé dans un milieu dans lequel il se déploie et où il est perturbé par la présence d'impuretés, aussi appelées l'*environnement*. Mathématiquement, le polymère est décrit soit par la trajectoire d'une marche aléatoire, soit par celle d'un processus aléatoire continu (par exemple, de type mouvement brownien). L'environnement est quant à lui modélisé par un champ de variables aléatoires (discret ou continu), défini sur l'espace dans lequel le polymère se meut.

Tous les modèles de polymères que l'on va considérer seront *dirigés*, au sens où les trajectoires du polymère seront forcées de s'étendre dans une direction donnée de l'espace. En particulier, bien que les trajectoires seront plongées dans un espace de dimension $1 + d$, où $d \geq 1$, le polymère n'aura que d degrés de liberté. La motivation à diriger le polymère, est qu'il ne peut alors pas intersecter sa propre trajectoire, ce qui simplifie la structure de corrélation du modèle.

1.2 Les modèles de polymères

Dans ce manuscrit, nous considérerons alternativement trois modèles de polymères dirigés différents :

- Le polymère brownien en environnement poissonien,
- Le polymère brownien en environnement bruit blanc régularisé en espace,
- Le polymère discret en environnement i.i.d.

La dénomination "polymère brownien" désigne le fait que les trajectoires des polymères sont issues du mouvement brownien. Quant au troisième modèle, ses trajectoires sont des marches aléatoires.

Dans la suite, on posera $d \geq 1$ la *dimension* du modèle, et β le paramètre du modèle représentant l'*inverse de la température* du système. Selon les cas, on considérera $\beta \in [0, \infty)$ ou $\beta \in \mathbb{R}$.

Pour les polymères browniens (cf. Section 1.2 pour le polymère discret), on définit :

- Ω_{traj} l'ensemble des trajectoires continues de $[0, \infty)$ à valeurs dans $\mathbb{R}^d$, c'est à dire $\Omega_{\mathrm{traj}} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$,
- $(\Omega_{\mathrm{traj}}, \mathcal{F}, (\mathrm{P}_x)_{x\in\mathbb{R}^d})$ un espace de probabilité, où P_x est la mesure de Wiener du mouvement brownien partant de $x \in \mathbb{R}^d$,
- $(B_s)_{s\geq 0}$ la trajectoire canonique de Ω_{traj}. En particulier, sous P_x, $(B_s)_{s\geq 0}$ est un mouvement brownien à valeurs dans $\mathbb{R}^d$, tel que $\mathrm{P}_x(B_0 = x) = 1$.

Pour les trois modèles de polymères, on définira :

- $(\Omega, \mathcal{G}, \mathbb{P})$ l'espace de probabilité sur lequel est construit l'*environnement* aléatoire,
- $\mathrm{P}_{\beta,t}$ la *mesure de probabilité du polymère* sur les trajectoires de Ω_{traj}, à température inverse β et horizon de temps t.

Autres notations :

- L'espérance par rapport à P_x est notée E_x et l'espérance par rapport à $\mathbb{P}$ est notée $\mathbb{E}$.
- On pose également $\mathrm{P} = \mathrm{P}_0$ et $\mathrm{E} = \mathrm{E}_0$.

Le polymère brownien en environnement poissonien

Les principaux objets du modèle de polymère brownien en environnement poissonien sont :

- L'*environnement* aléatoire ω : c'est un processus ponctuel de Poisson défini sur $\mathbb{R}_+ \times \mathbb{R}^d$:

  $$\omega = \sum_i \delta_{(T_i, X_i)},$$

  de mesure d'intensité $\nu \mathrm{d}t\mathrm{d}x$, où $\nu > 0$.

- Le tube $V_t(B)$: pour $r > 0$, soit $U(x)$ la boule fermée de $\mathbb{R}^d$, de centre $x \in \mathbb{R}^d$ et de **volume** r^d. On définit ensuite $V_t(B)$ comme le tube de longueur t, qui entoure la trajectoire $(B_s)_{s\leq t}$,

  $$V_t(B) = \{(s, x) : s \in (0, t], x \in U(B_s)\} \subset (0, t] \times \mathbb{R}^d. \tag{1}$$

- La variable $H_t(B) = \omega(V_t(B))$: c'est l'*énergie d'une trajectoire* $(B_s)_{s\leq t}$. Elle indique le nombre de points du nuage de Poisson qui sont vus par la trajectoire avant le temps t.

Pour une réalisation l'environnement ω fixée et un horizon de temps $t > 0$, on définit la mesure de Gibbs du polymère $\mathrm{P}_{\beta,t}$ sur Ω_{traj} par la densité :

$$\mathrm{dP}_{\beta,t}(B) = \frac{1}{Z_t(\omega,\beta,r)} \exp\{\beta H_t(B)\}\, \mathrm{dP}(B), \tag{2}$$

où

$$Z_t(\omega,\beta,r) = \mathrm{P}\left[\exp\left\{\beta\omega(V_t)\right\}\right], \tag{3}$$

est la constante de renormalisation, aussi appelée *fonction de partition* ou *fonction de partition point à ligne (P2L)*, qui assure que $\mathrm{P}_{\beta,t}$ soit une mesure de probabilité. Sous la mesure du polymère, la trajectoire est attirée par les points de l'environnement lorsque $\beta > 0$; pour $\beta < 0$, elle est répoussée. On notera que la mesure du polymère est définie $\mathbb{P}$-presque sûrement : elle dépend du tirage particulier de l'environnement.

Remarque 1.1. *Défini ainsi, le polymère cherche à maximiser son énergie H_t. On notera que dans la littérature physique, c'est plutôt $-H_t$ que l'on appelle énergie, afin que le polymère cherche à minimiser son énergie.*

L'énergie libre *annealed* (i.e. moyennée sur l'environnement) du système est donnée par

$$\lambda(\beta) = t^{-1} \log \mathbb{E}[\exp\{\beta\omega(V_t)\}] = \nu r^d(e^\beta - 1),$$

où la dernière égalité provient du fait que $\omega(V_t)$ suit une loi de Poisson de paramètre $\nu t r^d$.

Ce modèle a été introduit par Nobuo Yoshida en tant que modèle de polymères et sa première apparition dans la littérature est dans [49]. L'environnement, décrit par les points du processus de Poisson, représente des impuretés disséminées aléatoirement dans l'espace. Pour $\beta < 0$ le modèle est lié au mouvement brownien en obstacles poissoniens [65, 152]. La limite $\beta \to -\infty$ est étudiée dans [73]. Lorsque $\beta \to +\infty$, le modèle est relié à la percolation de premier passage euclidienne [95, 94] avec pour exposant $\alpha = 2$.

Le chapitre II du manuscrit, co-écrit avec Francis Comets, est constitué de notes formant un état de l'art des résultats connus sur le polymère dirigé poissonien.

Le polymère brownien en environnement bruit blanc régularisé

Le second modèle de polymère brownien, que l'on peut déjà trouver dans [142], et plus récemment considéré dans [130] dans l'étude de l'équation de la chaleur stochastique en dimension $d \geq 3$ (cf. Section 4.3), est également à trajectoires de type brownien, mais son environnement est cette fois un bruit blanc régularisé en espace $\xi^{(\phi)}$, défini par :

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad \xi^{(\phi)}(t,x) := \int_{\mathbb{R}^d} \phi(x-y)\xi(t,\mathrm{d}y), \tag{4}$$

où $\phi \in \mathcal{C}_c^\infty$ est à support compact et symétrique, et où ξ est un *bruit-blanc* espace-temps, c'est à dire un processus gaussien centré vérifiant formellement

$$\mathbb{E}[\xi(s,y)\xi(t,x)] = \delta(t-s)\delta(x-y).$$

Remarque 1.2. *On peut voir le bruit-blanc comme une distribution aléatoire sur $\mathbb{R} \times \mathbb{R}^d$, telle que pour toute fonction test $f(t,x)$, on a que*[1] $\int_{\mathbb{R}}\int_{\mathbb{R}^d} f(t,x)\xi(\mathrm{d}t,\mathrm{d}x) := \langle f, \xi\rangle$ *suit une variable gaussienne centrée, de variance* $\iint f^2(t,x)\mathrm{d}t\mathrm{d}x$.

De façon similaire au polymère poissonien, l'énergie d'une trajectoire $(B_s)_{s\leq t}$ est donnée par

$$\mathscr{H}_t(B) = \int_0^t \xi^{(\phi)}(\mathrm{d}s, B_s) = \int_0^t \int_{\mathbb{R}^d} \phi(B_s - y)\xi(\mathrm{d}s,\mathrm{d}y),$$

et correspond cette fois à la "somme" des valeurs moyennées (en espace) du bruit blanc, vues depuis la trajectoire.

[1] On utilisera alternativement les notations $\xi(\mathrm{d}t,\mathrm{d}x)$, $\xi(t,\mathrm{d}x)\mathrm{d}t$ ou $\xi(t,x)\mathrm{d}t\mathrm{d}x$ dans l'intégrale contre le bruit blanc.

On définit alors la mesure du polymère par :

$$\mathrm{dP}_{\beta,t}^{(\phi)} = \frac{1}{\mathscr{Z}_t(\xi,\beta)} \exp\{\beta \mathscr{H}_t(B)\}\, \mathrm{dP}, \tag{5}$$

où

$$\mathscr{Z}_t = \mathscr{Z}_t(\xi,\beta) = \mathrm{E}\left[\exp\left\{\beta \mathscr{H}_t(B)\right\}\right] \tag{6}$$

est la fonction de partition du polymère. L'énergie libre *annealed* du système est donnée par

$$\lambda(\beta) = t^{-1} \log \mathbb{E}[\exp\{\beta \mathscr{H}_t\}] = \frac{V(0)\beta^2}{2}, \tag{7}$$

avec

$$V(x) = \int_{\mathbb{R}^d} \phi(x+y)\phi(y)\, \mathrm{d}y = \phi \star \phi(x),$$

où la deuxième égalité provient de la symétrie de ϕ.

Par symétrie du modèle, on considère seulement le cas $\beta \geq 0$. Lorsque $\beta > 0$, les trajectoires du polymère sont attirées par grandes valeurs du bruit blanc.

Le polymère discret en environnement i.i.d.

Ici, on considère comme espace d'états l'ensemble des marches aléatoires sur $\mathbb{Z}^d$, c'est à dire $\Omega_{\mathrm{traj}} = \{(S_k)_{k\geq 0}, S_k \in \mathbb{Z}^d\}$. On prendra $(S_k)_{k\geq 0}$ la marche canonique et P_x une mesure de probabilité, telle que sous P_x, $(S_k)_{k\geq 0}$ est une marche aléatoire simple partant de $x \in \mathbb{Z}^d$, i.e. sous P_x, $S_1 - S_0, \dots, S_{k+1} - S_k$ sont des variables indépendantes et

$$\mathrm{P}_x(S_0 = x) = 1, \quad \mathrm{P}_x(S_{k+1} - S_k = \pm \mathbf{e}) = \frac{1}{2d},$$

où $\mathbf{e}$ est un quelconque vecteur de la base canonique de $\mathbb{R}^d$. On notera à nouveau $\mathrm{P} = \mathrm{P}_0$.

L'environnement est une famille i.i.d. de variables aléatoires non-constantes $\eta(i,x)$, avec $i \in \mathbb{N}, x \in \mathbb{Z}^d$. On se placera toujours sous l'hypothèse de moments exponentiels :

$$\forall \beta \in \mathbb{R}, \quad \mathbb{E}[\exp(\beta \eta(i,x))] < \infty. \tag{8}$$

L'énergie d'une trajectoire est donnée par

$$H_n(S) = \sum_{i=1}^{n} \eta(i, S_i),$$

et la mesure de Gibbs du polymère est définie par :

$$\mathrm{dP}_{\beta,n}(S) = \frac{\exp\left\{\beta H_n(S)\right\}}{Z_n(\eta,\beta)} \mathrm{dP}(S), \tag{9}$$

où

$$Z_n = Z_n(\eta,\beta) = \mathrm{E}\left[\exp\left\{\beta H_n(S)\right\}\right]$$

est la fonction de partition du polymère. L'énergie libre *annealed* du système est donnée par

$$\lambda(\beta) = n^{-1} \log \mathbb{E}[\exp\{\beta H_n\}] = \log \mathbb{E}[\exp\{\beta \eta(i,x)\}].$$

On trouvera de nombreux résultats et commentaires dans le livre récent dédié au modèle discret [41].

1.3 Résultats généraux sur les polymères

Hors mention contraire, les propriétés décrites dans cette section concernent les trois modèles de polymères introduits. Pour fixer les esprits, on choisira Z_t comme représentant la fonction de partition d'un des polymère (continu ou discret), et on se restreindra au cas où β est positif.

Régions de fort et faible désordre

La *fonction de partition* normée ou *fonction de partition point-à-ligne (P2L) normée* :

$$W_t = W_t(\beta) := Z_t\, e^{-t\lambda(\beta)}, \tag{10}$$

est une *martingale* positive d'espérance 1. Elle vérifie la dichotomie suivante :

Théorème 1.3 (Transition de phase faible désordre/fort désordre)**.** *Il existe un paramètre $\beta_c \in [0,\infty]$, tel que , lorsque $t \to \infty$,*

(i) Pour tout $\beta \in [0,\beta_c)$, $W_t \to W_\infty$, $\mathbb{P}$-p.s., avec $\mathbb{P}(W_\infty > 0) = 1$,

(ii) Pour tout $\beta \in \mathbb{R}_+ \setminus [0,\beta_c^+]$, $W_t \to 0$, $\mathbb{P}$-p.s.

De plus, lorsque $d \geq 3$, on a $\beta_c > 0$.

Remarque 1.4. *En dimensions $d = 1$ ou $d = 2$, il a été montré que $\beta_c = 0$ pour les polymères poissonnien* [49] *et discret* [46].

L'étude de la fonction de partition normée remonte à [21]. On trouvera une preuve de ce théorème dans [47] pour le modèle discret, à la Partie II Chapitre 3 pour le polymère poissonnien et dans [130] pour le modèle en bruit blanc régularisé.

Lorsque (i) est vérifié, on dit que le polymère est dans la région de *faible désordre.* Lorsque c'est (ii) qui est vérifié, on dit que le polymère est dans la région de *fort désordre.* Pour $d \geq 3$ et dans la région de faible désordre, l'environnement a une influence faible sur le polymère, qui se comporte alors de façon similaire au mouvement brownien ($\beta = 0$). On peut par exemple montrer que sa trajectoire est *diffusive* : cela a d'abord été prouvé [21, 102] pour le polymère discret dans une région plus restreinte (la région L^2, cf. Section 1.3), puis dans toute la région de faible désordre, via un théorème central limite fonctionnel, dans [50]. Pour le polymère poissonnien, cela a été montré dans la région L^2 dans [48, Th. 2.1.1], puis dans toute la région de faible désordre (cf. Partie II, Chapitre 5), tandis qu'une preuve pour la région L^2 du polymère en bruit blanc régularisé est donnée dans [25].

Dans toute la région de fort désordre et en toutes dimensions, on s'attend en revanche à ce que la trajectoire du polymère soit *superdiffusive*, c'est à dire que $B_t \approx t^a$ lorsque $t \to \infty$, avec $a > 1/2$ (en dimension $d = 1$, il est de plus conjecturé que $a = 2/3$, cf. Section 3.1). Des arguments de superdiffusivité existent pour les trois modèles [48, 17, 33, 123, 134], mais les résultats restent pour l'instant loin d'être optimaux. En fort désordre, il est également supposé que la trajectoire du polymère soit *localisée*, c'est à dire qu'elle devrait être presque sûrement concentrée dans certaines parties de l'espace. La propriété de localisation n'a pour l'instant été montrée que pour β assez grand : on pourra consulter la Partie II, Chapitre 7 pour le polymère poissonnien, [46, 8, 37] pour le polymère discret et [26] pour le polymère en environnement bruit blanc régularisé.

Énergie libre

Une autre quantité caractéristique du modèle est l'énergie libre *quenched* (i.e. non intégrée sur l'environnement). C'est une quantité *déterministe*, définie par :

$$p(\beta) = \lim_{t\to\infty} \frac{1}{t} \ln Z_t(\beta), \quad \mathbb{P}\text{-p.s.} \tag{11}$$

où $t^{-1} \ln Z_t(\omega,\beta,r)$ représente l'énergie libre *quenched* à horizon t. On pourra trouver une preuve de la convergence de cette quantité pour le polymère discret dans [41] et pour le polymère poissonnien à la Partie II, Chapitre 2.

Comme conséquence de l'inégalité de Jensen, il y a toujours inégalité entre les énergies libres *quenched* et *annealed* : $p(\beta) \leq \lambda(\beta)$. De plus, il y a une transition de phase (en β) entre égalité et stricte inégalité. On voit directement que le désorde faible entraîne l'égalité entre les énergies :

$$\beta < \beta_c \implies p(\beta) = \lambda(\beta). \tag{12}$$

Lorsqu'en revanche

$$p(\beta) < \lambda(\beta), \tag{13}$$

on dit alors que le système est dans la région de *désordre très fort* (impliquant donc le désordre fort). Dans ce cas, la différence entre les deux énergies – aussi appelée l'*exposant de Lyapunov quenched* – décrit la décroissance exponentielle de W_t. Le problème de savoir si les régions de fort et très fort désordre coïncident est toujours ouvert.

La région L^2 $(d \geq 3)$

On définit la région L^2 comme la région on la martingale $(W_t)_{t\geq 0}$ est bornée dans L^2. Elle consiste en la région $[0, \beta_{L^2})$, où

$$\beta_{L^2} = \sup\left\{\beta \geq 0 : \mathbb{E}\left[W_\infty^2\right] < \infty\right\}. \tag{14}$$

Dans cette région, $W_t \xrightarrow{L^2} W_\infty$, ce qui implique $\beta_{L^2} \leq \beta_c$, c'est à dire que la région L^2 est une sous-région du faible désordre. Lorsque $d \geq 3$ on peut vérifier que $\beta_{L^2} > 0$ – ce qui fournit en particulier une preuve de $\beta_c > 0$. L'argument peut être trouvé, par exemple, dans la Partie II Chapitre 3 pour le polymère poissonnien et dans [47] pour le polymère discret, ou enfin dans [130] pour le polymère en bruit blanc régularisé. À noter que dans le cas du polymère discret, on peut calculer la valeur de β_{L^2}.

La région L^2 est une partie de la région de faible désordre, dans laquelle on peut mettre à profit l'intégrabilité pour calculer certaines quantités qui ne sont pas définies en dehors de la région. Un exemple de son utilisation est le théorème suivant (écrit dans sa version polymère discret) :

Théorème 1.5 (Théorème locale limite pour les polymères dans la région L^2)**.** *Soit $\beta \in [0, \beta_2)$ et $\alpha > 0$. Pour toute suite $(l_t)_{t\geq 0}$, vérifiant que $l_n \to \infty$ et $l_n = o(n^a)$ pour un certain $a < 1/2$, on a, pour tout $x \in \mathbb{Z}^d$:*

$$\mathrm{E}\left[\exp(\beta H_n - \lambda(\beta)n) | S_n = x\right] = W_{l_n} \overleftarrow{W}^x_{n,l_n} + \delta^x_n, \tag{15}$$

où $\overleftarrow{W}^x_{n,l} = \mathrm{E}_x\left[\exp\left\{\sum_0^l \beta\eta(n-i, S_i) - (l+1)\lambda(\beta)\right\}\right]$ est la fonction de partition normée à ***environnement retourné en temps****, et où, lorsque $n \to \infty$,*

$$\sup_{|x|\leq\alpha\sqrt{n}} \mathbb{E}\left[|\delta^x_n|^2\right] \to 0. \tag{16}$$

Le théorème a été premièrement montré dans le cas du polymère discret dans [149], puis transféré au cas du polymère poissonnien dans [159]. Pour le polymère en environnement bruit blanc régularisé, le théorème est prouvé par le Lemme 2.3 de la Partie IV.

Remarque 1.6. *De manière approximative, le théorème locale limite indique que dans la région L^2,*

$$\mathrm{P}_{\beta,n}(S_k = x) \approx \overleftarrow{W}^x_{n,l_n} \mathrm{P}(S_k = x),$$

avec $\mathrm{P}_{\beta,n}$ la mesure de Gibbs du polymère définie en (9)*. Ainsi, la probabilité de transition du polymère ne devrait différer de celle de la marche aléatoire simple, uniquement par une influence locale de l'environnement au voisinage du point d'arrivée.*

1.4 Fluctuations de la queue de la fonction de partition normée dans la région L^2 entière, en dimension $d \geq 3$

Dans cette section, on travaillera avec le polymère discret.

Si l'on considère le modèle de polymère sur un arbre plutôt que sur la grille $\mathbb{Z}^d$, les auteurs de [62] ont montré que l'on peut alors répondre à des questions difficiles d'accès sur la grille. Le fait que deux chemins deviennent indépendants après branchement, simplifie en particulier la structure de corrélation. En grande dimension, on s'attend à ce que le polymère sur la grille satisfasse des propriétés similaires : après un certain temps, deux chemins ne devraient plus se rencontrer. Ainsi, outre son propre intérêt, le modèle sur l'arbre sert également de modèle jouet du polymère sur la grille. Dans cette optique, le polymère sur le **m-tree**, introduit dans [52], permet de comparer certaines grandeurs du polymère sur l'arbre à celles du polymère sur la grille (cf. [41]). Toutefois, les caractéristiques des deux modèles divergent en de nombreux points (cf. [41]) et le Théorème 1.7 en illustre certains aspects.

Le modèle de polymère sur un arbre est directement relié au modèle de marche aléatoire branchante (voir par ex. [41]). Pour ce modèle, un des sujets de recherche actifs est l'étude des fluctuations de la queue de la martingale de Biggins $W_\infty - W_n$ [100, 141, 121, 101]. Dans la littérature des processus de branchement, l'étude des fluctuations de la queue des martingales caractéristiques remonte à [91, 92] pour le modèle de Galton-Watson. Dans le cadre des polymères sur la grille, on verra en Section 4.4 que l'on peut également relier ce problème à l'étude de la convergence de l'équation KPZ régularisée en dimension $d \geq 3$.

Pour le polymère discret, on a le résultat suivant :

Théorème 1.7. *Pour tout $\beta \in (0, \beta_{L^2})$, lorsque $n \to \infty$,*

$$n^{\frac{d-2}{4}} (W_\infty - W_n) \xrightarrow{(d)} \sigma W_\infty G, \tag{17}$$

et

$$n^{\frac{d-2}{4}} \frac{W_\infty - W_n}{W_n} \xrightarrow{(d)} \sigma G, \tag{18}$$

où G est une variable aléatoire gaussienne standard qui est indépendante de W_∞, et où $\sigma = \sigma(\beta)$ est défini en Partie VI, à l'équation (24).

Remarque 1.8. *On peut de plus vérifier que la première convergence est* ***stable*** *et la seconde* ***mixing*** *(cf. Partie VI).*

Ce théorème a premièrement été montré pour le polymère discret dans [45] dans une partie restreinte de la région L^2, puis pour le polymère en bruit blanc régularisé (cf. Partie V), également dans une partie restreinte de la région L^2.

On verra dans la Partie VI que comme énoncé au Théorème 1.7, on peut effectivement étendre le résultat dans toute la région L^2 du polymère discret. Pour prouver cela, la technique employée consiste à se ramener au théorème locale limite (Théorème 1.5), dont l'application à la méthode développée dans [45] est naturelle et permet d'éviter le recours au moment d'ordre 4. L'idée générale de la preuve est détaillée en Section 2.1 de la partie VI.

Puisque $\sigma(\beta) \to \infty$ lorsque $\beta \to \beta_{L^2}$, il est attendu que le résultat soit optimal, dans le sens où, en dehors de la région L^2, une autre vitesse de convergence de $W_\infty - W_n$ ou d'autres lois limites devraient apparaître[2]. Pour le modèle de marche aléatoire branchante, la question de déterminer les vitesses de convergence et les lois limites, selon les différentes régions de température, est étudiée dans les références citées plus haut. En particulier, au voisinage du point critique du modèle, des lois alpha-stable sont attendues pour les fluctuations. Elles ont été montrées au point critique [121].

Dans les cas réguliers de la marche aléatoire branchante (ou de manière équivalente pour le modèle associé de polymère sur un arbre), la vitesse de convergence de la martingale de Biggins est exponentielle dans la région de très haute température. Dans le cas du polymère sur la grille, les corrélations provenant d'intersections entre des chemins à temps long ralentissent la convergence, qui devient alors polynomiale (Théorème 1.7).

Dans la Partie VI, on montre également le corollaire suivant, concernant l'énergie libre :

Corollary 1.9. *Pour tout $\beta \in (0, \beta_{L^2})$, lorsque $n \to \infty$,*

$$n^{\frac{d-2}{4}} (\log W_n - \log W_\infty) \xrightarrow{(d)} \sigma G, \tag{19}$$

où G et σ sont définis comme précédemment.

On notera que le résultat similaire, pour le polymère en bruit blanc régularisé, est déduit en Partie V, dans une partie restreinte de la région L^2.

[2]Pour $\beta = \beta_{L^2}$, on s'attend à ce que les fluctuations soient toujours gaussiennes mais que la vitesse de convergence soit différente

1.5 Polymères et équation de la chaleur stochastique en dimension $d \geq 1$

Dans cette section, sauf mention contraire, W représente la fonction de partition normée d'un des trois modèles de polymères introduits en Section 1.2 avec $d \geq 1$.

Cas de la fonction de partition P2P : on définit la *fonction de partition point-à-point normée* (P2P) par :

$$W(t,x) = W(\beta,t,x) := \rho(t,x)\mathrm{E}_0\left[\exp\{\beta H_t - \lambda(\beta)t\} | B_t = x\right], \tag{20}$$

où

$$\rho(t,x) = (2\pi)^{d/2} e^{-|x|^2/2t} \tag{21}$$

est le noyau de la chaleur (pour le polymère discret, on aurait en facteur $p(n,x) = \mathrm{P}_0(S_n = x)$).

Alors, $W(t,x)$ vérifie une équation de la chaleur stochastique, avec un *bruit multiplicatif* dépendant de l'environnement, avec la condition initiale Dirac $W(0,x) = \delta_0(x)$. Par exemple, pour le polymère poissonnien, on trouve :

$$\begin{cases} \partial_t W(t,x) = \frac{1}{2}\Delta W(t,x) + (e^\beta - 1)W(t-,x)\bar{\omega}(\mathrm{d}t \times U(x)), \\ W(0,x) = \delta_0(x), \end{cases} \tag{22}$$

où $\bar{\omega}(\mathrm{d}t,\mathrm{d}x) = \omega(\mathrm{d}t,\mathrm{d}x) - \nu\mathrm{d}t\mathrm{d}x$ est le processus de Poisson compensé et $U(x) \subset \mathbb{R}^d$ est la boule de volume r^d centrée en $x \in \mathbb{R}^d$. Une formulation faible de cette l'équation est prouvée à la Partie II Section 6.2. La version correspondante de l'équation, pour le polymère en bruit blanc régularisé, est étudiée en Partie IV. Pour la version discrète, on pourra consulter [1].

Cas de la fonction de partition retournée en temps : Soit $\overleftarrow{W}(t,x)$ la fonction de partition P2P, partant du point (t,x), d'horizon t et d'*environnement retourné en temps*. Par exemple, pour le polymère en bruit blanc régularisé, on définit

$$\overleftarrow{\mathscr{W}}(\beta,t,x) = \mathrm{E}_x\left[\exp\left\{\beta\int_0^t\int_{\mathbb{R}^d}\phi(B_s - y)\xi(t-s,y)\mathrm{d}s\mathrm{d}y - \frac{V(0)\beta^2 t}{2}\right\}\right], \tag{23}$$

où $V(0)$ a été définie en (7). Pour le polymère discret, on a donné son expression plus haut (Théorème 1.5). Alors $\overleftarrow{W}(t,x)$ vérifie la même équation stochastique de la chaleur que $W(t,x)$, avec cette fois la *condition initiale plate* $W(0,\cdot) \equiv 1$. Pour le polymère bruit blanc, on trouve par exemple :

$$\begin{cases} \partial_t \overleftarrow{\mathscr{W}}(t,x) = \frac{1}{2}\Delta\overleftarrow{\mathscr{W}}(t,x) + \beta\overleftarrow{\mathscr{W}}(t,x)\xi^{(\phi)}(t,x), \\ \overleftarrow{\mathscr{W}}(0,\cdot) \equiv 1, \end{cases} \tag{24}$$

où $\xi^{(\phi)}$ est le bruit blanc régularisé défini en (4).

Remarque 1.10. *Pour le polymère en bruit blanc régularisé, on peut passer de* (24) *à* (23) *grâce à la formule de Feynman-Kac (cf.* [130]*). On notera la présence du terme correctif* $\frac{V(0)\beta^2 t}{2}$ *dans l'exponentielle, puisqu'ici l'équation est stochastique et doit être interprétée au sens d'Itô.*

Remarque 1.11. *La boule $U(x)$ qui apparaît dans* (22)*, joue un rôle similaire à la régularisation locale du bruit en espace via ϕ dans* (24)*.*

2 L'équation KPZ et l'équation SHE en dimension $d = 1$

2.1 Un modèle d'interface aléatoire à l'équilibre

Considérons l'équation

$$\frac{\partial\mathscr{U}}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2\mathscr{U}}{\partial x^2}(t,x) + \beta\xi(t,x), \tag{25}$$

où $t \geq 0$, $x \in \mathbb{R}$ et $\beta > 0$ est un paramètre constant. Cette équation, introduite par Edwards et Wilkinson dans [71], est appelée indifféremment équation Edwards-Wilkinson (EW) ou équation de la chaleur stochastique avec *bruit additif* (noter que contrairement à (24), (22), ou plus loin pour SHE (29), le bruit dans (25) n'est *pas* multiplié par $\mathscr{U}$).

La quantité $\mathscr{U}(t,x)$ représente la fonction de hauteur, au temps t et au point x, d'une surface aléatoire dynamique de dimension 1, dont l'évolution est caractérisée par :

- une relaxation locale venant du terme laplacien,
- un bruit $\beta\xi(t,x)$ non corrélé en temps et en espace.

Il est possible d'exprimer simplement la solution de l'équation linéaire (25) via la formule :

$$\mathscr{U}(t,x) = \int_{\mathbb{R}} \rho(t,x-y)\mathscr{U}(0,y)\,\mathrm{d}y + \beta \int_0^t \int_{\mathbb{R}} \rho(t-s,x-y)\xi(\,\mathrm{d}s,\,\mathrm{d}y). \tag{26}$$

Pour un profil initialement plat $\mathscr{U}(0,\cdot) \equiv 0$, $\mathscr{U}(t,x)$ est une processus *gaussien* centrée, de variance $C\sqrt{t}$ avec C constante. En outre, dans ce cas, on peut vérifier que

$$\mathscr{U}(\varepsilon^{-2}t, \varepsilon^{-1}x) \overset{\text{loi}}{=} \varepsilon^{-1/2}\mathscr{U}(t,x),$$

où l'on observe un rapport 1 : 2 : 4 entre les exposants de l'ordre des fluctuations de $\mathscr{U}$, de l'espace et du temps.

D'un point de vue physique, l'équation modélise le comportement d'une interface séparant deux phases à l'équilibre : il n'y a pas de tension de surface entre les deux phases et la surface cherche à tout moment à minimiser son énergie. Il est attendu que l'équation soit un objet universel pour décrire différentes interfaces séparant deux phases en situation d'équilibre [7].

2.2 L'équation KPZ en dimension 1

Dans le but d'étudier des modèles de croissance d'interface séparant deux phases physiques en situation *hors équilibre*, Kardar, Parisi et Zhang [107] ont proposé d'ajouter un terme non-linéaire à l'équation Edwards-Wilkinson :

$$\frac{\partial h}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + \frac{1}{2}\left(\frac{\partial h}{\partial x}(t,x)\right)^2 + \beta\xi(t,x), \tag{27}$$

introduisant ainsi *l'équation KPZ*. La quantité $h(t,x)$ représente la fonction de hauteur, au temps t et au point x, d'une surface aléatoire dynamique de dimension 1, dont l'évolution est caractérisée par :

- une relaxation locale venant du terme laplacien,
- un bruit $\beta\xi(t,x)$ non corrélé en temps et en espace,
- un terme quadratique favorisant une croissance locale dans une direction.

À cause du terme non-linéaire, la solution n'est pas invariante par symétrie $h \to -h$ et h croît à une certaine vitesse $h(t,x) \sim vt$ lorsque $t \to \infty$ – c'est à dire que l'une des deux phases séparées par l'interface (dans ce cas, celle du dessous) est favorisée par rapport à l'autre. Une des prédictions des auteurs était que l'équation devrait permettre de décrire les fluctuations de certaines grandeurs, dans une large classe de modèles de physique statistique. Cela a depuis été mathématiquement confirmé (voir Section 3.1). Une seconde prédiction était que des exposant de changement d'échelle non-standards devraient apparaître lorsque l'on étudie l'équation. Il a fallu attendre plus de vingt ans avant que cela ne soit montré avec rigueur (cf. Section 3.1).

Il est difficile de donner un sens à l'équation même, en partie à cause du terme quadratique qui rend l'équation non linéaire et implique la multiplication de deux distributions (cf. Remarque 4.1). En particulier, c'est en fait l'équation

$$\frac{\partial h}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + \left(\frac{1}{2}\left(\frac{\partial h}{\partial x}(t,x)\right)^2 - \infty\right) + \beta\xi(t,x), \tag{28}$$

auquel on peut donner un sens, où le terme en $-\infty$ apparaît dans l'équation pour compenser la divergence du terme quadratique et résulte d'un passage à la limite non trivial. Une première méthode [14], introduite dans les années 90, consiste à définir la solution de l'équation KPZ via la transformation Hopf-Cole de l'équation de la chaleur stochastique (cf. Section 2.3). C'est seulement une vingtaine d'années plus tard, avec l'important travail de Hairer [86] basé sur la théorie des chemins rugueux, que l'on a pu donner un sens intrinsèque à l'équation. Depuis lors, d'autres outils pour étudier certaines EDP stochastiques singulières (dont l'équation KPZ) ont été développés, notamment la théorie des distributions paracontrolées [83], l'approche du groupe de renormalisation [111], l'approche par martingales [85, 82, 77] ou la théorie des structures de régularité [87]. Pour une introduction générale à l'équation KPZ, on pourra consulter [74] ainsi que les références données en Section 3.1 sur la classe d'universalité KPZ.

Certains types de conditions initiales jouent des rôles particuliers (voir [53, Section 1.2.5] ou [67]). Comme exemples fondamentaux, on peut citer : la condition initiale $h(0, x) = B(x)$ où B est le mouvement brownien étendu sur $\mathbb{R}$, qui donne lieu à une solution *stationnaire* [75] ; les conditions initiales plates ($h(0, \cdot) = 0$) et *narrow-wedge* (cf. Remarque 2.1), qui permettent notamment de faire un lien entre l'équation KPZ et les modèles de polymères.

2.3 L'équation de la chaleur stochastique avec bruit multiplicatif en dimension $d = 1$

L'équation de la chaleur stochastique *avec bruit multiplicatif* (SHE pour *stochastic heat equation*) :

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, x) + \beta u(t, x)\xi(t, x), \tag{29}$$

où $t \geq 0$, $x \in \mathbb{R}$, est une equation aux dérivées partielles stochastique linéaire, dont le bruit est multiplié par $u(t, x)$. On notera que pour cette équation, il n'est pas nécessaire de soustraire une quantité infinie. La solution de l'équation KPZ vérifie la *transformation Hopf-Cole* :

$$h(t, x) = \log u(t, x). \tag{30}$$

Pour définir l'équation SHE et justifier la transformation (30), l'approche développée dans [14] consiste à donner sens à (29) en passant dans un premier temps par une régularisation du bruit (voir Section 4.1). De cette manière, l'équation de la chaleur *régularisée* est alors bien posée grâce au calcul d'Itô. Ensuite, les auteurs montrent que lorsque l'on dissipe la régularisation du bruit, l'équation régularisée converge vers un processus non trivial, lequel définit alors la solution de l'équation SHE. L'équation KPZ s'obtient enfin par la transformation Hopf-Cole. Le terme "$-\infty$" dans l'équation KPZ correspond alors à la limite du terme correctif d'Itô, qui apparaît via la relation (30) appliquée aux solutions régularisées. Cette procédure, consistant à régulariser une équation mal-posée, puis d'étudier la limite en relâchant la régularisation est classique en théorie constructive des champs aléatoires.

Les deux types de conditions initiales $u(0, x) = \delta_0(x)$ (condition initiale dite *narrow-wedge*) et $u(0, \cdot) \equiv 1$ (condition initiale dite *plate*) permettent de relier l'équation de la chaleur stochastique aux modèles de polymères dirigés (cf. Section 1.5).

Remarque 2.1. *La condition initiale narrow-wedge pour l'équation KPZ peut être définie via celle de SHE et la transformation Hopf-Cole.*

Pour des conditions initiales positives, bornées et à support compact, il a été d'abord montré par [128], que $u(t, x)$ est presque-sûrement strictement positive dès lors que $t > 0$. Plus récemment, le résultat a été étendu dans [126], à la condition initiale *narrow-wedge* (et ainsi toute condition initiale positive). La preuve repose sur l'interprétation de $u(t, x)$ comme fonction de partition d'un polymère et se base sur le résultat du régime intermédiaire (voir Section 3.2).

2.4 Le polymère continu ($d = 1$) [1]

Pour cette section, on se restreint à la dimension $d = 1$.

En Section 1.5, on a vu que les fonctions de partition point-à-point $W(t, x)$ suivent l'équation de la chaleur stochastique, avec différents bruits multiplicatifs, dépendant de l'environnement du modèle de polymère. En particulier, la fonction de partition du polymère brownien, en environnement bruit blanc régularisé, vérifie l'équation (24). Lorsque $d = 1$, l'équation (24) est également bien définie si l'on remplace le bruit régularisé $\xi^{(\phi)}$ par le bruit non régularisé ξ, puisque l'on trouve alors l'équation SHE (29). Pour $d = 1$, on peut montrer que la solution $u(t, x)$ de SHE (29) correspond, elle aussi, à la fonction de partition P2P d'un polymère, dont l'environnement est cette fois le bruit blanc *non régularisé*. Ce polymère est appelé le *polymère continu*.

Pour tous les modèles de polymères dirigés, il est immédiat de vérifier que sous la mesure du polymère $\mathrm{P}_{\beta,t}$ (et à environnement fixé), la trajectoire $(B_s)_{s\leq t}$ est un processus de Markov, dont le noyau de transition s'écrit :

$$\mathrm{P}_{\beta,t}\left(B_{s_i} \in \mathrm{d}y_i, i = 1 \dots n\right) = \frac{W(0,0;s_1,y_1)W(s_n,y_n;t,\star)}{W(0,0,t,\star)} \prod_{i=1}^{n} W(s_{i-1},y_{i-1};s_i,y_i), \tag{31}$$

où la quantité $W(s, y; t, x)$ désigne la fonction de partition point-à-point d'un polymère partant de (s, y) et arrivant en (t, x) et $W(s, y; t, \star)$ désigne la fonction de partition point-à-ligne partant de (s, y) et d'horizon t, données par

$$\begin{aligned} W(\beta, s_{i-1}, y_i; s_i, y_i) &= W(\beta, s_i - s_{i-1}, y_i - y_{i-1}) \circ \theta_{s_{i-1},y_{i-1}}, \\ W(\beta, s, y; t, \star) &= W_{t-s}(\beta) \circ \theta_{s,y}, \end{aligned}$$

où $\theta_{s,y}$ est le shift de vecteur $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ sur l'environnement. Ainsi, à environnement fixé, la donnée des fonctions de partition $W(s_{i-1}, y_i; s_i, y_i)$ permet de définir le processus de Markov associé au polymère.

Réciproquement, pour définir un polymère de fonction de partition point-à-point donnée par la solution $u(t, x)$ de SHE (29), les auteurs de [1] partent de l'équation (31) et introduisent une mesure de polymère $\mathrm{Q}_{\beta,t}$ sur $\mathcal{C}([0, 1], \mathbb{R})$, vérifiant :

$$\mathrm{Q}_{\beta,t}\left(B_{s_i} \in \mathrm{d}y_i, i = 1 \dots n\right) = \frac{u(0,0;s_1,y_1)u(s_n,y_n;t,\star)}{u(0,0,t,\star)} \prod_{i=1}^{n} u(s_{i-1},y_{i-1};s_i,y_i)\mathrm{d}y_i, \tag{32}$$

avec, pour θ le shift sur le bruit blanc,

$$\begin{aligned} u(s, y; t, x) &= u(t - s, x - y) \circ \theta_{s,y}, \\ u(s, y; t, \star) &= \int_{\mathbb{R}} u(s, y; t, x)\mathrm{d}x. \end{aligned}$$

Ainsi, on peut interpréter que :

- la mesure $Q_{\beta,t}$ correspond à la mesure d'un polymère brownien en environnement bruit blanc (non régularisé), aussi appelé le *polymère continu*,
- la variable $u(0, 0; t, \star)$ réprésente la fonction de partition point-à-ligne du polymère continu,
- la solution $u(t, x)$ de l'équation de la chaleur stochastique représente la fonction de partition point-à-point.

Remarque 2.2. *Il existe aussi une écriture informelle de $u(t, x)$ sous forme d'intégrale sur l'espace des trajectoires, qui fait écho aux propos de la Section 1.5 :*

$$u(t, x) = \rho(t, x)\ \mathrm{E}_0 \left[: \exp : \left(\beta \int_0^t \xi(u, B_u)\mathrm{d}u \right) \middle| B_t = x \right]. \tag{33}$$

où $: \exp :$ *représente l'exponentielle de Wick* [105]. *Voir Partie III, Section 10.3.1.*

Remarque 2.3. *L'écriture* (33) *est informelle car l'énergie d'une trajectoire* $\int_0^t \xi(u, B_u)\mathrm{d}u$ *n'est pas bien définie. C'est aussi pour cette raison que l'on ne peut pas directement introduire la mesure* $\mathrm{Q}_{\beta,t}$ *comme une mesure de Gibbs, comme c'est le cas pour les trois modèles de polymères considérés auparavant. En fait, on peut même montrer que la mesure* $\mathrm{Q}_{\beta,t}$ *du polymère continu n'est* $\mathbb{P}$*-presque sûrement pas continue par rapport à la mesure de Wiener* [1].

On va voir en Section 3.1, qu'en dimension $d = 1$, le polymère continu, l'équation KPZ et SHE sont des objets universels dans la classe des modèles de polymères dirigés.

3 La classe d'universalité KPZ en dimension $d = 1$

Dans cette Section, on se restreint encore à la dimension $d = 1$.

3.1 Propriétés de la classe KPZ

L'exemple du modèle de polymère dirigé

On considère dans cette section un des modèles de polymère introduits en Section 1.2, avec $d = 1$ et $\beta > 0$ (désordre fort). Un des défis de longue date est de prouver la *superdiffusivité* (avec exposant $2/3 > 1/2$) du modèle sous la mesure du polymère :

$$\sup_{0\leq s\leq t} |B_s| \approx t^{2/3} \quad \text{lorsque } t \to \infty. \tag{34}$$

Un second problème est l'étude de l'ordre des fluctuations de l'énergie libre $t^{-1} \log Z_t$ autour de sa limite :

$$\ln Z_t - \mathbb{P}[\ln Z_t] \approx t^{1/3} \quad \text{lorsque } t \to \infty. \tag{35}$$

Il est attendu les exposants qui apparaissent dans les deux problèmes soient en fait reliés (voir Partie II Chapitre 6). On s'attend de plus à ce que les fluctuations fassent intervenir des lois non standards (en particulier non gaussiennes) :

$$\frac{\ln Z_t - p(\beta)t}{\sigma(\beta)t^{1/3}} \xrightarrow{(d)} TW_1 \quad \text{lorsque } t \to \infty, \tag{36}$$

où TW_1 est la loi de Tracy-Widom associé au GOE, qui apparaît dans l'étude du bord du spectre du GOE [155, 156] et $\sigma(\beta)$ est une constante.

Pour l'instant, les fluctuations (36) n'ont été montrées que pour des modèles exactement solubles (c'est à dire où l'on peut calculer de nombreuses quantités du modèle – par ex. les fonctions de partition). Pour le modèle soluble de polymère dit *log-gamma*, introduit et étudié dans [145], il a été prouvé que l'exposant des fluctuations (35) est au plus 1/3, via des techniques uniquement probabilistes. Puis, pour ce même modèle, le résultat plus fin (36) a été montré à l'aide de techniques combinatoires [55, 23].

La classe d'universalité KPZ en dimension $d = 1$

Les propriétés décrites au-dessus sont en fait des propriétés universelles, caractéristiques de la *classe d'universalité KPZ*. De manière plus ou moins approximative, cette classe est généralement décrite comme étant une famille de modèles de croissance aléatoire, qui partagent :

1. Une relation de type 1:2:3 entre l'échelle des fluctuations, l'échelle de l'espace et l'échelle du temps (comme pour les exposants $t^{1/3}$, $t^{2/3}$ et $t^{3/3}$ en Section 3.1),

2. Des lois limites universelles non standards sous ce changement d'échelle, dépendant uniquement des conditions initiales du système.

On pourra se référer aux différentes notes sur la classe d'universalité KPZ [54, 53, 136, 168, 24], ainsi qu'à [89] qui comprend de nombreuses références dans la littérature physique. On y trouvera dans ces notes de nombreuses références sur les modèles de cette classe, qui comprend notamment : certains systèmes de particules en interactions (*asymmetric simple exclusion process* ASEP, le modèle de déposition balistique) des modèles de trajectoires en environnement aléatoire (polymères dirigés, percolation de premier et dernier passage), des EDP stochastiques (l'équation KPZ, l'équation de la chaleur stochastique avec bruit multiplicatif).

On notera que les modèles, pour lesquels il est pour l'instant possible de montrer les propriétés 1 et 2, sont toujours reliés à des modèles exactement solubles. C'est en particulier le cas de l'équation KPZ, dont l'appartenance à la classe d'universalité KPZ a été montrée au début des années 2010, suite à une série de travaux [3, 144, 66, 27]. Dans [3], les auteurs réussissent à donner rigoureusement une expression à la distribution de $h(t, x)$, en se basant sur deux résultats : celui de Bertini et Giacomin en 1997 [16], où il est montré que l'équation KPZ apparaît comme limite de la fonction de hauteur de l'ASEP ; et sur la série d'articles de Tracy et Widom [154, 157, 158], dans lesquels des propriétés de solvabilité sont utilisées afin de calculer la distribution de ladite fonction de hauteur. Mentionnons également, entre autres, le calcul de la distribution de transition à mi-parcours du polymère continu (membre de la classe KPZ), dans le régime stationnaire, mené dans [119].

De façon plus précise, la classe d'universalité KPZ peut être définie comme l'ensemble des modèles possédant un champ $h(t,x)$, tel que sous le changement d'échelle KPZ 1:2:3 et pour une constante $C_\varepsilon \to \infty$, la quantité

$$\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - C_\varepsilon t$$

converge en loi vers un champ universel $\mathbf{h}(t,x)$, indépendant du modèle, mais dépendant de ses conditions initiales, appelé le *point fixe KPZ*. En particulier, le point fixe KPZ doit être invariant sous le changement d'échelle KPZ 1:2:3. Pour l'instant, à la connaissance de l'auteur, il n'existe pas de modèle où l'on puisse prouver la convergence vers le point fixe en temps et en espace. On pourra consulter [122] pour une référence récente sur le point fixe KPZ.

Universalité KPZ faible

Le fait est qu'en dehors des modèles solubles, il n'existe pas de techniques générales permettant de prouver qu'un modèle est dans la classe d'universalité KPZ. En revanche, on peut se tourner vers une question plus abordable, qui est celle de l'universalité faible.

La *conjecture d'universalité KPZ faible* stipule que l'équation KPZ est un objet universel de la classe d'universalité KPZ. L'idée générale est que l'équation KPZ devrait apparaître en tant que limite d'échelle, au paramètre critique, pour les modèles vérifiant une transition de phase entre un comportement de type KPZ et un comportement différent de la classe KPZ. Lorsque c'est vérifié, on dit que le modèle appartient à la classe d'universalité faible. Via la transformation Hopf-Cole, on peut également se contenter, lorsque c'est possible, de montrer la convergence du modèle vers l'équation SHE.

Remarque 3.1. *Il est à noter que l'équation KPZ n'est pas invariante sous le changement d'échelle KPZ 1:2:3. En particulier, ce n'est* ***pas*** *un point fixe KPZ.*

La première preuve d'universalité faible remonte à [16] pour le modèle ASEP, pour lequel les fluctuations renormalisées de la fonction de hauteur convergent en distribution vers la solution de l'équation KPZ, lorque le coefficient d'asymétrie est amené vers 0. Plus récemment, l'universalité faible a été montrée pour les polymères dirigés discret [2] et poissonnien [57] (cf. Section 3.2). On renvoie au dernier paragraphe de la Partie II Section 1.5 pour y trouver d'autres références d'universalité faible et une discussion sur les différentes méthodes de preuve.

3.2 Le régime de désordre intermédiaire des polymères en dimension $d = 1$.

Pour cette section, on se place en dimension $d = 1$ et on considère le modèle discret ou poissonnien. On se place dans le régime particulier, où le paramètre de température inverse dépend du temps de cette manière :

$$\beta_t = \beta t^{-1/4}, \quad \text{où } \beta > 0,$$

satisfaisant $\beta_t \to 0$ lorsque $t \to \infty$.

Remarque 3.2. *Dans le cas du polymère poissonnien, on peut également introduire une dépendance en temps les paramètres ν et r. En particulier, on peut garder β fixe. Voir en Partie III.*

On rappelle que les notations $W_t(\beta)$, $W(\beta,t,x)$, $W(\beta,s,y,t,x)$, $u(0,0;1,\star)$ et $u\,(S,Y;T,X)$ désignent respectivement les fonctions de partition P2L et P2P définies en (10), (20) et en Section 2.4.

Le théorème qui suit montre que sous un changement d'échelle diffusif et pour β_t comme au dessus, les fonctions de partition convergent vers les fonctions de partition du polymère *continu* (cf. Section 2.4) :

Theorem 3.3. *Lorsque $t \to \infty$, pour $S,Y,T,X \in \mathbb{R}$ fixés :*

$$W_t(\beta_t) \xrightarrow{(d)} \int_{\mathbb{R}} u(1,x)\mathrm{d}x = u(0,0;1,\star), \tag{37}$$

$$\sqrt{t}\,W\left(\beta_t; tS, \sqrt{t}Y; tT, \sqrt{t}X\right) \xrightarrow{(d)} u\left(S, Y; T, X\right), \tag{38}$$

De plus, on a la convergence en terme de processus en (T, X)*:*

$$\sqrt{t}\,W\left(\beta_t; tT, \sqrt{t}X\right) \xrightarrow{(d)} u\left(T, X\right). \tag{39}$$

Ce résultat a été d'abord prouvé dans [2] dans le cadre du modèle de polymère discret, puis dans le cadre du polymère poissonnien (cf. Partie III). On pourra également consulter la partie II, Chapitre 10 du manuscrit, pour y trouver un résumé du résultat et de la preuve dans le cas poissonnien.

On notera qu'en dimension $d = 1$, $\beta_c = 0$ est le paramètre critique séparant un régime conjecturé KPZ ($\beta > 0$) et le mouvement brownien standard ($\beta = 0$). Ainsi, la convergence (39) implique que la solution de l'équation de la chaleur stochastique apparaît comme limite, sous un changement d'échelle diffusif et au paramètre critique ($\beta_t \to 0$), montrant par là l'universalité KPZ faible des modèles de polymères. On mettra l'accent sur le fait qu'aucune propriété de solvabilité n'est nécessaire pour montrer ces résultats.

En termes de polymères, le théorème montre que le *polymère continu* est un objet limite universel (de la classe KPZ) des polymères en dimension $d = 1$. De plus, le régime considéré est un entre-deux entre le régime de fort désordre pour les polymères ($\beta > \beta_c$) et un régime de type faible désordre ($\beta < \beta_c$). En effet, on observe d'une part un rescaling diffusif, un moment d'ordre deux pour la fonctions de partition et un théorème local limite (cf. [2]), qui sont typiques du faible désordre, d'autre part des objets limites qui font partie de la classe d'universalité KPZ, typiques du désordre fort. C'est pourquoi ce régime est aussi appelé *régime intermédiaire*.

Pour le polymère discret, les auteurs de [2] ont conjecturé que le résultat devrait s'étendre à l'hypothèse plus faible que (8) d'un 6-ième moment sur l'environnement $\eta(i, x)$. Cela a été par la suite prouvé dans [63]. Dans ce même article, les auteurs ont émis des conjectures sur le comportement du polymère à environnement à queue lourde lorsque $\beta_n \to 0$. Ces conjectures, ainsi qu'une compréhension de cinq différents régimes sous cette limite ont été montrées dans [11]. Enfin, mentionnons que dans [28], le régime de désordre intermédiaire $\beta_n \to 0$ a été également étudié pour les modèles d'accrochage, de polymère à longs sauts et d'Ising en champ magnétique aléatoire.

4 Les équations KPZ et SHE en dimension supérieure

En dimension quelconque, l'équation KPZ est formellemement donnée par

$$\frac{\partial h}{\partial t}(t, x) = \frac{1}{2}\Delta h(t, x) + \left(\frac{1}{2}\left|\nabla h(t, x)\right|^2 - \infty\right) + \beta\xi(t, x), \tag{40}$$

où $x \in \mathbb{R}^d$ pour $d \geq 1$ et où ∇ représente le gradient. Par la transformation Hopf-Cole (30), on peut se ramener à nouveau à l'équation SHE en dimension $d \geq 1$:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\Delta u(t, x) + \beta u(t, x)\xi(t, x). \tag{41}$$

En dimension $d \geq 2$, on ne sait pas pour l'instant pas donner à sens à ces deux équations.

Si l'on considère (40) sans terme quadratique (c'est à dire l'équation Edwards-Wilkinson (25) en dimension $d \geq 1$), l'irrégularité du bruit blanc se transmet à la solution, et dès que $d \geq 2$, celle-ci n'est définie qu'en terme de distribution. Plus précisément, en dimension $d \geq 1$, le bruit blanc a une régularité *parabolique* de Hölder négative $(-d/2 - 1 - \delta)$ pour tout $\delta > 0$, où le terme "parabolique" signifie que la régularité en temps compte pour le double de la régularité en espace (voir Appendix de la Partie IV). Dans la formule similaire à (26) en dimension $d \geq 1$, la deuxième intégrale a alors pour régularité parabolique $(-d/2 + 1 - \delta)$, l'idée générale étant que la convolution avec le noyau de la chaleur augmente la régularité parabolique de 2.

Ainsi, on s'attend également à ce que des solutions de (40) ou (41) ne puissent être définies qu'en termes de distributions lorsque $d \geq 2$. En particulier, un des problèmes qui intervient est le fait de devoir multiplier deux distributions entre elles (soit ∇h avec elle-même, soit u et ξ).

Remarque 4.1. *En dimension $d = 1$, on peut vérifier que les solutions ont effectivement une régularité parabolique $(1/2 - \delta)$-hölderienne pour tout $\delta > 0$. Dans ce cas, l'équation KPZ implique également la multiplication de deux distributions via le terme quadratique, mais pas l'équation SHE.*

4.1 Procédure de régularisation

L'approche pour étudier l'équation (40) que l'on va considérer ici, suit celle développée dans [14] pour la dimension $d = 1$ et [15] pour la dimension $d = 2$. Elle consiste à d'abord utiliser la transformation Hopf-Cole pour se ramener à l'équation SHE (41), puis à remplacer le bruit blanc par une approximation régularisée en espace :

$$\xi_\epsilon(t,x) = \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\xi(t,\mathrm{d}y),$$

où

$$\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon),$$

avec ϕ lisse et supportée sur un compact, est une approximation de la Dirac en 0, i.e. $\phi_\varepsilon \to \delta_0$ au sens des distributions, impliquant que $\xi_\varepsilon \to \xi$, lorsque $\varepsilon \to 0$. On considère alors l'équation de la chaleur stochastique régularisée :

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{1}{2}\Delta u_\varepsilon + \beta_\varepsilon u_\varepsilon \xi_\varepsilon. \tag{42}$$

Après avoir régularisé le bruit, il est en fait possible de définir (42) au sens classique d'Itô. Ensuite, pour un bon choix de β_ε, on cherchera à obtenir une limite non-triviale lorsque $\varepsilon \to 0$.

Par la transformation Hopf-Cole, on peut également étudier

$$h_\varepsilon = \log u_\varepsilon, \tag{43}$$

qui vérifie alors l'équation KPZ régularisée :

$$\frac{\partial h_\varepsilon}{\partial t}(t,x) = \frac{1}{2}\Delta h_e(t,x) + \left(\frac{1}{2}|\nabla h(t,x)|^2 - C_\varepsilon\right) + \beta_\varepsilon \xi_\varepsilon(t,x), \tag{44}$$

où $C_\epsilon \to \infty$ est un terme de compensation pour l'intégrale d'Itô.

En dimension $d = 1$, il a été prouvé dans [14] que pour $\beta_\varepsilon = \beta$, où β est constant, la solution régularisée u_ε converge vers la solution de l'équation SHE (29) et h_ε vers la solution l'équation KPZ (28). Pour la dimension $d \geq 3$, le choix de $\beta_\varepsilon = \beta\varepsilon^{(d-2)/2}$, pour β *petit* donne lieu à des limites non triviales (cf. Section 4.3), qui ne permettent a priori pas de définir une solution pour SHE ou KPZ. En dimension $d = 2$, on peut prendre $\beta_\varepsilon = \hat{\beta}\sqrt{|\log\varepsilon|}^{-1}$ (voir Section 4.5).

4.2 Réécriture du problème en termes de polymères

Pour tout $d \geq 1$, il est possible de relier la solution u_ε de (42) à la fonction de partition d'un polymère retournée en temps grâce à la formule de Feynman-Kac (comme en Remarque 1.10). En appliquant la formule avec $u_\epsilon(0,\cdot) \equiv 1$, on obtient que la solution de (42) s'écrit

$$u_\varepsilon(t,x) = \mathrm{E}_x\left[\exp\left\{\beta_\varepsilon \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(B_u - z)\xi(t-u,z)\mathrm{d}u\mathrm{d}z - \frac{\beta_\varepsilon^2 \varepsilon^{-d} V(0)\, t}{2}\right\}\right], \tag{45}$$

où l'on rappelle que $V = \phi \star \phi$.

Puis, par le changement d'échelle diffusif $u = \varepsilon^2 s$, $z = \varepsilon y$ et la propriété $B_u \overset{\text{loi}}{=} \varepsilon B_s$, on peut écrire que

$$u_\varepsilon(t,x) = \mathrm{E}_0\left[\exp\left\{\beta_\varepsilon \varepsilon^{-(d-2)/2} \int_0^{\varepsilon^{-2}t} \int_{\mathbb{R}^d} \phi(B_s - y)\xi^{(\varepsilon,t,x)}(\mathrm{d}s,\mathrm{d}y) - \frac{\beta_\varepsilon^2 \varepsilon^{-d} V(0)\, t}{2}\right\}\right] \tag{46}$$

où
$$\xi^{(\varepsilon,t,x)}(\mathrm{d}s,\mathrm{d}y) := \epsilon^{-(d+2)/2}\xi(t-\varepsilon^2 s, x-\varepsilon y)\mathrm{d}(\varepsilon^2 s)\mathrm{d}(\varepsilon y),$$

est le bruit blanc sous le changement d'échelle diffusif, *retourné en temps*, translaté par le vecteur $(\varepsilon^{-2}t, \varepsilon^{-1}x)$. On notera que par les propriétés d'échelle et d'invariance par translation du bruit blanc, on retrouve que

$$\forall(\varepsilon,t,x), \quad \xi^{(\varepsilon,t,x)} \overset{\text{loi}}{=} \xi.$$

Remarque 4.2. *Autrement dit, $u_\varepsilon(t,x)$ s'écrit comme la fonction de partition retournée en temps définie en (23), sous un changement d'échelle diffusif conservant la loi de l'environnement, à la température inverse $\beta_\varepsilon \varepsilon^{-(d-2)/2}$. En particulier, pour tout $\varepsilon > 0$, on a*

$$u_\varepsilon(t,x) \overset{\text{loi}}{=} \overleftarrow{\mathscr{W}}(\beta_\varepsilon\, \varepsilon^{-(d-2)/2}, \varepsilon^{-2}t, \varepsilon^{-1}x),$$

où l'égalité en loi est vraie en tant que processus en temps et en espace.

4.3 Cas de la dimension $d \geq 3$

Limites ponctuelles

Lorsque $d \geq 3$ et pour

$$\beta_\varepsilon = \beta \varepsilon^{(d-2)/2}, \quad \beta \geq 0, \tag{47}$$

on obtient directement de (46) que

$$u_\varepsilon(t,x) = \mathscr{W}_{\varepsilon^{-2}t}\left(\xi^{(\varepsilon,t,x)}\right), \tag{48}$$

où $\mathscr{W}_T$ est la fonction de partition *normée* du polymère brownien en environnement bruit blanc régularisé par ϕ, à température inverse *fixée* β, donnée par :

$$\mathscr{W}_T(\xi) = \mathrm{E}_0\left[\exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(B_s - y)\xi(\mathrm{d}s, \mathrm{d}y) - \frac{V(0)\beta^2 T}{2}\right\}\right].$$

Grâce à cette identification, les auteurs de [130] ont montré que, pour le paramètre critique β_c séparant le faible désordre du fort désordre du polymère (voir 1.3), on a lorsque $\beta < \beta_c$, pour $\varepsilon \to 0$,

$$\forall t > 0, x \in \mathbb{R}^d, \quad u_\varepsilon(t,x) \xrightarrow{(d)} \mathscr{W}_\infty \quad \text{et} \quad h_\varepsilon(t,x) \xrightarrow{(d)} \log \mathscr{W}_\infty, \tag{49}$$

avec $\mathscr{W}_\infty > 0$ p.s. et, lorsque $\beta > \beta_c$,

$$\forall t > 0, x \in \mathbb{R}^d, \quad u_\varepsilon(t,x) \xrightarrow{\mathbb{P}} 0.$$

Pour montrer ce résultat, l'idée est de combiner les propriétés :

$$\forall (t,x,\varepsilon),\ u_\varepsilon(t,x) \overset{\text{loi}}{=} \mathscr{W}_{\varepsilon^{-2}t} \quad \text{et} \quad \mathscr{W}_T \to \mathscr{W}_\infty \ \ \mathbb{P}\text{-p.s.}, \tag{50}$$

où la première identité découle de $\xi^{(\varepsilon,t,x)} \overset{\text{loi}}{=} \xi$ et (48).

Remarque 4.3. *Le fait que l'on observe cette transition de phase en fonction de β, confirme l'intérêt de considérer l'échelle β_ε définie en (47). Voir également la Section 4.5. Dans la suite, on va se concentrer sur le cas $\beta < \beta_c$.*

Enfin, on peut également déduire de (46), que le terme d'Itô qui apparaît dans (44), provient du terme compensateur dans l'intégrale

$$C_\varepsilon = \frac{\beta^2 V(0)\, \varepsilon^{-2}}{2}. \tag{51}$$

Approximation par un champ stationnaire. Moments

Pour obtenir la convergence ponctuelle (49), on est passé d'une d'une convergence p.s. pour la fonction de partition du polymère à une convergence en loi ; pour conserver une convergence plus forte, on peut simplement observer via (48) et la propriété $\mathscr{W}_T \to \mathscr{W}_\infty$, que lorsque $\varepsilon \to 0$,

$$\forall t > 0, x \in \mathbb{R}^d, \quad u_\varepsilon(t,x) - \mathfrak{u}\left(\xi^{(\varepsilon,t,x)}\right) \xrightarrow{\mathbb{P}} 0, \tag{52}$$

où, si $\mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d)$ désigne un espace de Besov adapté au bruit blanc (cf. Appendix de la Partie IV), on a défini

$$\mathfrak{u} = \mathfrak{u}_{\beta,\phi} : \mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d) \to (0,\infty), \tag{53}$$

comme étant une représentation de $\mathscr{W}_\infty$, c'est à dire que $\mathfrak{u}$ est mesurable et telle que $\mathfrak{u}(\xi) = \mathscr{W}_\infty$.

Jusqu'à présent, on s'est uniquement penché sur la condition initiale $u_\epsilon(0,\cdot)\equiv 1$. Si l'on s'intéresse à des conditions initiales plus générales, la formule de Feynman-Kac donne

$$u_\varepsilon(t,x) = \mathrm{E}_0\left[u_\varepsilon(0,x+\varepsilon B_{\varepsilon^{-2}t})\exp\left\{\beta\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d}\phi(B_s-y)\xi^{(\varepsilon,t,x)}(\mathrm{d}s,\mathrm{d}y) - \frac{V(0)\beta^2\varepsilon^{-2}t}{2}\right\}\right].$$

En notant que $\overline{u}(t,x) = \mathrm{E}_0\left[u_\varepsilon\left(0,x+\varepsilon B_{\varepsilon^{-2}t}\right)\right] = \mathrm{E}_x\left[u_\varepsilon(0,B_t)\right]$ vérifie

$$\partial_t\overline{u} = \frac{1}{2}\Delta\overline{u}, \qquad \overline{u}(0,x) = u_\varepsilon(0,x),$$

l'idée derrière le théorème qui suit est que pour β suffisamment petit, il y a indépendence asymptotique des conditions initiales :

$$u_\varepsilon(t,x) \approx \overline{u}(t,x)\mathrm{E}_0\left[\exp\left\{\beta\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d}\phi(B_s-y)\xi^{(\varepsilon,t,x)}(\mathrm{d}s,\mathrm{d}y) - \frac{V(0)\beta^2\varepsilon^{-2}t}{2}\right\}\right],$$

puis de faire d'approcher l'espérance du terme de droite par $\mathfrak{u}(\xi^{(\varepsilon,t,x)})$, comme en (52).

Plus précisément, pour β_{L^2} défini comme en (14), i.e.

$$\beta_{L^2} = \sup\left\{\beta>0 : \mathbb{E}\left[\mathscr{W}_\infty^2\right] = E_0\left[e^{\beta^2\int_0^\infty V(\sqrt{2}B_s)\,\mathrm{d}s}\right] < \infty\right\},$$

et, suivant (43) et (53), en posant

$$\mathfrak{h} = \log\mathfrak{u},$$

on a :

Théorème 4.4 (Théorème 1.1 Partie IV)**.** *Pour tout $\beta\in(0,\beta_{L^2})$,*

- *(Condition initiale continue majorée) Soit $h_\varepsilon(0,\cdot) = h_0$ où $h_0:\mathbb{R}^d\to\mathbb{R}$ est continue et majorée. Alors, pour tout $t>0, x\in\mathbb{R}^d$, on a lorsque $\varepsilon\to 0$,*

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \log\overline{u}(t,x) \xrightarrow{\mathbb{P}} 0\ , \tag{54}$$

où $\overline{u}$ satisfait:

$$\partial_t\overline{u} = \frac{1}{2}\Delta\overline{u}, \qquad \overline{u}(0,\cdot) = \exp h_0.$$

- *(Condition initale "narrow-wedge".) On suppose que*

$$\lim_{t\searrow 0}\exp h_\varepsilon(t,\cdot) = \delta_{x_0}(\cdot),$$

pour $x_0\in\mathbb{R}^d$. Alors, pour tout $t>0, x\in\mathbb{R}^d$, on a lorsque $\varepsilon\to 0$,

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \mathfrak{h}(\xi_{(\varepsilon,x_0)}) - \log\rho(t,x-x_0) \xrightarrow{\mathbb{P}} 0\ , \tag{55}$$

où ρ est le noyau de la chaleur d-dimensionnel défini en (21) et

$$\xi_{(\varepsilon,x_0)}(s,x) = \varepsilon^{\frac{d+2}{2}}\xi(\varepsilon^2 s, x_0+\varepsilon x)$$

a la loi d'un bruit blanc standard.

Ce théorème appelle à plusieurs remarques importantes :

- Ce théorème montre qu'à un terme déterministe près, $\mathfrak{h}\left(\xi^{(\varepsilon,t,x)}\right)$ est une approximation en probabilités de $h_\varepsilon(t,x)$ qui est indépendante de la condition initiale h_0,
- La loi de $\mathfrak{h}\left(\xi^{(\varepsilon,t,x)}\right)$ est constante en t,x et ε, égale à la loi de $\log \mathscr{W}_\infty$ qui est la loi limite ponctuelle qui apparaît dans (49),
- Pour tout $\varepsilon > 0$, $\mathfrak{h}\left(\xi^{(\varepsilon,t,x)}\right)$ est une solution stationnaire (i.e. sa loi ne dépend pas de t,x) de l'équation KPZ régularisée (44).[3]

On pourra rapprocher ces résultats de ceux de [69], où les auteurs étudient l'approximation de la solution l'équation de la chaleur stochastique (42) (avec un bruit régularisé en temps et en espace), par des solutions indépendantes de la condition initiale et approchant une solution stationnaire, toujours dans le régime de faible désordre.

Grâce à l'interprétation de $\mathfrak{h}$ en terme d'énergie libre d'un polymère, il est possible d'adapter la preuve de [32] du résultat analogue, dans le cadre du polymère discret, au théorème suivant :

Théorème 4.5 (Théorème 1.1, Partie 1.3)**.** *Soit $\beta \in (0,\beta_{L^2})$. Alors, pour tout $\theta > 0$, il existe une constante $C \in (0,\infty)$, telle que*

$$\mathbb{P}[\mathfrak{h} \leq -\theta] \leq C\mathrm{e}^{-\theta^2/2}.$$

En particulier, $\mathfrak{h} \in L^p(\mathbb{P})$ pour tout $p \in \mathbb{R}$.

Remarque 4.6. *Comme conséquence directe du théorème Théorème 4.5 et de sa preuve, on peut voir que dans la région L^2 entière, $(u_\varepsilon(t,x))_\varepsilon$ et $(\mathscr{W}_T)_T$ sont bornées dans L^p pour tout p négatif. En particulier, $\mathfrak{u} \in L^p$ pour tout p négatif.*

Remarque 4.7. *Le résultat du Théorème 4.5 a été montré indépendamment dans [68], dans une région plus restreinte de la région L^2. Elle constitue un ingrédient important de la preuve du résultat montré dans cet article*

Comme conséquence des théorèmes 4.4 et 4.5, on a le corollaire suivant, qui indique que les limites $\varepsilon \to 0$ et $u_0 \to \delta_{x_0}$ ne commutent pas :

Corollaire 4.8. *Supposons $\beta \in (0,\beta_{L^2})$ et $h_\varepsilon(0,\cdot) = h_0$, où h_0 est continue et majorée. Alors, pour tout $x_0 \in \mathbb{R}^d$, on a*

$$\lim_{e^{h_0}\to\delta_{x_0}} \lim_{\varepsilon\to 0} \mathbb{E}h_\varepsilon^{(h_0)} - \lim_{\varepsilon\to 0} \lim_{e^{h_0}\to\delta_{x_0}} \mathbb{E}h_\varepsilon^{(h_0)} = -\mathbb{E}\mathfrak{h} > 0\ .$$

4.4 Fluctuations en dimension $d \geq 3$

Fluctuations ponctuelles

On suppose pour l'instant que $h_\varepsilon(0,\cdot) \equiv 0$ (ou de manière équivalente $(u_\varepsilon(0,\cdot) \equiv 1)$. Dans ce cas, on a par (54),

$$\forall t > 0, x \in \mathbb{R}^d, \quad h_\varepsilon(t,x) - \mathfrak{h}\left(\xi^{(\varepsilon,t,x)}\right) \xrightarrow{\mathbb{P}} 0,$$

lorsque $\varepsilon \to 0$, et on peut s'intéresser à la vitesse de convergence et aux fluctuations de cette limite vers 0. En particulier, on recherche des lois limites universelles, permettant de déterminer par exemple la classe d'universalité de notre modèle (voir Section 4.5 pour une discussion autour cette question).

Pour étudier ces fluctuations, on peut d'abord regarder celles de l'équation de la chaleur, qui vérifie

$$\mathfrak{u}\left(\xi^{(\varepsilon,t,x)}\right) - u_\varepsilon(t,x) \stackrel{\text{loi}}{=} \mathscr{W}_\infty - \mathscr{W}_T, \quad \text{avec } T = \varepsilon^{-2}t \to \infty,$$

lorsque $\varepsilon \to 0$. On est ainsi ramené à étudier les fluctuations de la queue de la fonction de partition normalisée d'un polymère, c'est à dire que l'on est ramené au problème de la Section 1.4. On obtient les résultats suivant :

[3] Pour voir cela, on peut par exemple utiliser l'équation d'auto-consistance (Partie V équation (24) avec $S = \infty$) et la formule de Feynman-Kac.

Théorème 4.9 (Théorème 1.2, Partie V)**.** *Il existe $\beta_0 \in (0, \beta_{L^2})$, tel que pour tout $\beta < \beta_0$, $t > 0$ et $x \in \mathbb{R}^d$,*

$$\varepsilon^{-\frac{d-2}{2}} \frac{u_\varepsilon(t,x) - \mathfrak{u}(\xi^{(\varepsilon,t,x)})}{\mathfrak{u}(\xi^{(\varepsilon,t,x)})} \xrightarrow{(d)} N\big(0, \sigma^2(\beta) t^{-\frac{d-2}{2}}\big) . \tag{56}$$

lorsque $\varepsilon \to 0$, où

$$\sigma^2(\beta) = \frac{2\beta^2}{(d-2)(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \; V(\sqrt{2}y) \; E_y \left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}B_s)\, \mathrm{d}s} \right]. \tag{57}$$

Théorème 4.10 (Théorème 1.1, Partie V)**.** *Il existe $\beta_0 \in (0, \beta_{L^2})$, tel que pour tout $\beta < \beta_0$, $t > 0$ et $x \in \mathbb{R}^d$,*

$$\varepsilon^{-\frac{d-2}{2}} \left[h_\varepsilon(t,x) - \mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big) \right] \xrightarrow{(d)} N\big(0, \sigma^2(\beta) t^{-(d-2)/2}\big), \tag{58}$$

où σ^2 est donné par (57).

Remarque 4.11. *Comme pour le polymère discret, on a $\sigma^2(\beta) \to \infty$ lorsque $\beta \to \beta_{L^2}$. En particulier, on s'attend à ce que les deux théorèmes précédents soient valides dans toute la région L^2 et que les lois limites ou que l'ordre des fluctuations soient différents pour $\beta \in (\beta_{L^2}, \beta_c)$.*

Remark 4.12. *Il est en fait assez naturel de trouver les mêmes limites pour les fluctuations considérées de h_ε et u_ε, voir Remarque 2.4 de la Partie V.*

Fluctuations en tant que distribution

On suppose pour l'instant que $u_0 = u_\varepsilon(0, \cdot)$ est une fonction continue et bornée. Dans [130], il a été montré que pour toute fonction test $f \in \mathcal{C}_c^\infty$, on a dans la région L^2:

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) f(x) \mathrm{d}x \xrightarrow{\mathbb{P}} \int_{\mathbb{R}^d} \overline{u}(t,x) f(x) \mathrm{d}x, \tag{59}$$

où $\overline{u}$ satisfait:

$$\partial_t \overline{u} = \frac{1}{2} \Delta \overline{u}, \qquad \overline{u}(0, \cdot) = u_0.$$

Ce résultat contraste avec (49) (et le Théorème 4.4), car ici, la limite après integration contre une fonction test est *déterministe*. Ce phénomène est une conséquence de la décroissance rapide de la corrélation en espace de $u_\varepsilon(t,x)$ (cf. Partie V, Section 2.1) : intégrer f contre $u_\varepsilon(t,x)$ mène ainsi a une loi des grands nombres.

Les fluctuations autour de la limite dans (59) ont été d'abord étudiées dans [81]. On a le théorème suivant (observer que $\mathbb{E}[u_\varepsilon(t,x)] = \overline{u}(t,x)$):

Theorem 4.13 ([81])**.** *Il existe $\beta_0 \in (0, \beta_{L^2})$, tel que pout tout $\beta < \beta_0$, pour $\varepsilon \to 0$,*

$$\varepsilon^{-(d-2)/2} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) f(x) \mathrm{d}x \xrightarrow{(d)} \int_{\mathbb{R}^d} \mathscr{V}(t,x) f(x) \mathrm{d}x, \tag{60}$$

où $\mathscr{V}$ suit l'équation de la chaleur avec bruit additif :

$$\partial_t \mathscr{V}(t,x) = \frac{1}{2} \Delta \mathscr{V}(t,x) + \beta \nu(\beta) \overline{u}(t,x)\, \xi(t,x), \qquad \mathscr{V}(0,x) = 0, \tag{61}$$

et où ν^2 est donné par l'intégrale qui apparaît dans la définition de σ^2 en (57).

Puis, dans l'article récent [68], les auteurs ont apporté une preuve alternative au résultat, apparu dans la littérature physique dans [120], déterminant la nature des fluctuations de h_ε (avec la condition initiale nulle) au sens des distributions :

Theorem 4.14. *Il existe $\beta_0 \in (0, \beta_{L^2})$, tel que pout tout $\beta < \beta_0$, sous la condition $h_\varepsilon(0,\cdot) \equiv 0$,*

$$\varepsilon^{-(d-2)/2} \int_{\mathbb{R}^d} (h_\varepsilon(t,x) - \mathbb{E}[h_\varepsilon(t,x)])\, f(x)\mathrm{d}x \xrightarrow{(d)} \int_{\mathbb{R}^d} \mathscr{U}(t,x) f(x)\mathrm{d}x, \tag{62}$$

où $\mathscr{U}$ suit l'équation de la chaleur avec bruit blanc additif (équation Edwards-Wilkinson) :

$$\partial_t \mathscr{U} = \frac{1}{2}\Delta \mathscr{U} + \beta\nu(\beta)\,\xi, \qquad \mathscr{U}(0,x) = 0. \tag{63}$$

On pourra noter que la vitesse de convergence $\varepsilon^{-(d-2)/2}$ dans les théorèmes 4.13 et 4.14 est la même que celle observée aux théorèmes 4.9 et 4.10. En outre, $\mathscr{U}$ est un processus gaussien, et comme pour le théorème 4.9, le membre de droite dans l'équation (62) est une variable aléatoire gaussienne dont la variance diverge également lorsque $\beta \to \beta_{L^2}$. Toutefois, les résultats sont de nature différente (convergence ponctuelle et au sens des distributions). En particulier, on remarquera que contrairement à (56) et (58), les quantités $\varepsilon^{-(d-2)/2}(u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)])$ et $\varepsilon^{-(d-2)/2}(h_\varepsilon(t,x) - \mathbb{E}[h_\varepsilon(t,x)])$ dans (60) et (62) divergent ponctuellement.

4.5 Commentaires

Cas de la dimension $d = 2$

La situation est similaire en dimension $d = 2$. Dans ce cas, $\beta_\varepsilon = \hat{\beta}\sqrt{|\log\varepsilon|}^{-1}$ et il a été montré dans [29], que $h_\varepsilon(t,x)$ converge ponctuellement vers une loi gaussienne pour $\hat{\beta} < \hat{\beta}_c$, et vers 0 en probabilité pour $\hat{\beta} > \hat{\beta}_c$. En outre, on a $\hat{\beta}_{L^2} = \hat{\beta}_c$ que l'on peut déterminer. Enfin, la quantité $\sqrt{|\log\varepsilon|}^{-1}(h_\varepsilon - \mathbb{E}[h_\varepsilon])$ a également été étudiée dans [38, 31]. Il y a été montré qu'elle converge aussi vers l'équation EW dans la région sous-critique.

Liens avec le comportement sous échelle diffusive du polymère et de l'équation SHE régularisée. Correspondance entre les échelles températures.

En dimension $d \geq 3$, pour $\beta_\varepsilon = \beta\varepsilon^{(d-2)/2}$, la solution u_ε de (42) vérifie $u_\varepsilon \overset{\text{loi}}{=} \overleftarrow{\mathscr{W}}(\beta, \varepsilon^{-2}\cdot, \varepsilon^{-1}\cdot)$, où $\overleftarrow{\mathscr{W}}$ est la fonction de partition retournée en temps définie en (23) (cf. Remarque 4.2). Le régime que l'on considère revient ainsi à prendre une limite diffusive du polymère, à température inverse β fixée. De manière équivalente, cela revient à étudier, sous un changement d'échelle diffusif, le comportement asymptotique de la solution de l'équation SHE en bruit blanc régulier (24), pour β fixe.

En dimension $d = 1$, pour $\beta_\varepsilon = \beta$, on trouve que $u_\varepsilon \overset{\text{loi}}{=} \overleftarrow{\mathscr{W}}(\beta\varepsilon^{-1/2}, \varepsilon^{-2}\cdot, \varepsilon^{-1}\cdot)$, où, cette fois, le paramètre β du polymère est *non constant*. On remarquera que l'on est précisément ramené au régime intermédiaire décrit en Section 3.2 (identifier $t \leftrightarrow \varepsilon^{-2}$).

En dimension $d = 2$, pour le choix de $\beta_\varepsilon = \hat{\beta}\sqrt{|\log\varepsilon|}^{-1}$, on a $u_\varepsilon \overset{\text{loi}}{=} \overleftarrow{\mathscr{W}}(\beta_\varepsilon, \varepsilon^{-2}\cdot, \varepsilon^{-1}\cdot)$. Dans ce cas seulement, la température du polymère coïncide avec β_ε.

Remarque 4.15. *Les différentes échelles de la température inverse du polymère correspondent aux ordres de grandeur de $\sqrt{R_{\varepsilon^{-2}}}$, où $R_t := \mathrm{E}^{\otimes 2}[\int_0^t \mathbf{1}(|B_t - \widetilde{B}_t \leq 1)\mathrm{d}t]$, pour B et $\widetilde{B}$ deux browniens indépendants. On a en effet $R_t \approx Ct^{1/2}$ pour $d = 1$, $R_t \approx C\log t$ pour $d = 2$ et $R_t = O(1)$ pour $d \geq 3$. Ces ordres de grandeurs sont liés à la notion d'intensité du désordre, voir* [29, 10].

Approche perturbative

En dimension $d \geq 3$, on a $\beta_\varepsilon \to 0$ lorsque $\varepsilon \to 0$. On peut ainsi voir les équations (42) et (44) comme des perturbations des équations déterministes (i.e. sans bruit) par un faible bruit. Bien que le bruit disparaisse formellement des équations lorsque $\varepsilon \to 0$, il conserve toutefois une influence du point de vue de la convergence ponctuelle (49), puisque les solutions convergent vers des variables aléatoires non triviales. En revanche, du point de vue des distributions, u_ε converge vers la solution de la chaleur sans bruit (59).

Lorsque l'on considère ensuite les fluctuations de h_ε comme plus haut, on peut remarquer que $\widetilde{h}_\varepsilon := \varepsilon^{-(d-2)/2} h_\varepsilon$ vérifie l'équation

$$\frac{\partial \widetilde{h}_\varepsilon}{\partial t}(t,x) = \frac{1}{2}\Delta \widetilde{h}_e(t,x) + \frac{\varepsilon^{(d-2)/2}}{2}\left|\nabla \widetilde{h}_\varepsilon(t,x)\right|^2 - D_\varepsilon + \beta \xi_\varepsilon(t,x), \tag{64}$$

où

$$D_\varepsilon = \varepsilon^{-(d-2)/2} C_\varepsilon = \varepsilon^{-(d+2)/2}\beta^2 V(0)/2.$$

L'équation (64) peut cette fois être interprétée comme une perturbation non linéaire de l'équation de la chaleur avec bruit additif (équation EW). Comme on l'a vu, lorsque β est assez petit, la quantité $\widetilde{h_\varepsilon} - \mathbb{E}[\widetilde{h}_\varepsilon]$ converge en tant que distribution vers la solution de l'équation EW. Ainsi, pour β petit et au sens des distributions, la perturbation n'a asymptotiquement pas d'influence.

On a également que la variable $\varepsilon^{-(d-2)/2}\mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big)$ est une solution stationnaire de (64) (cf. point trois page 22). Le théorème 4.10 indique que ponctuellement, la différence entre la solution avec condition initiale nulle et la solution stationnaire converge vers une variable gaussienne.

Classes d'universalité et problèmes ouverts

De nombreuses questions restent en suspens. Que se passe-t-il pour $\beta = \beta_{L^2}$ ou dans la région $\beta \in (\beta_{L^2}, \beta_c)$? On sait déjà que les variances limites de $\varepsilon^{-(d-2)/2}(h_\varepsilon - \mathbb{E}[h_\varepsilon])$ et $\varepsilon^{-(d-2)/2}(h_\varepsilon - \mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big)$ divergent lorsque $\beta \to \beta_{L^2}$. Ainsi, comme cela a été mentionné pour les fluctuations de la queue de la martingale, on s'attend à ce que d'autres lois limites, ou d'autres changements d'échelle que $\varepsilon^{-(d-2)/2}$ apparaissent.

En dimension $d = 1$, on sait que h_ε converge vers l'équation KPZ, qui fait partie de la classe d'universalité KPZ. En dimension $d \geq 3$ et dans la région L^2, les résultats présentés en Section 4.4 (coefficients d'échelles standards, lois limites gaussiennes, équation de la chaleur stochastique avec bruit additif) sont en fort contraste avec ce qui serait attendu d'une classe d'universalité KPZ en dimensions supérieures. En particulier, en dimension $d = 1$, l'équation EW est une équation typique de la *classe d'universalité Edwards-Wilkinson*, qui est une seconde classe de modèles de croissance aléatoire, distincte de la classe KPZ, dans laquelle les modèles partagent des coefficients d'échelle de type 1-2-4 et des lois limites gaussiennes (voir Section 2.1).

Remarque 4.16. *On notera également que les variances $\sigma^2(\beta)$ et $\nu^2(\beta)$ qui apparaissent dans les limites gaussiennes des fluctuations en Section 4.4 dépendent de la fonction ϕ, c'est à dire de la procédure de régularisation des équations, ce qui n'est pas le cas en dimension $d = 1$.*

Du fait, par exemple, de la diffusivité du polymère ($|B_t| \approx t^{1/2}$) dans la région $(0, \beta_c)$, on ne s'attend pas à retrouver des caractéristiques de la classe KPZ avant le point critique β_c. Pour $\beta \geq \beta_c$, la situation est pour l'instant loin d'être comprise. Pourrait-on y retrouver des comportements de la classe d'universalité KPZ ? En particulier, peut-on espérer retrouver et définir l'équation de la chaleur avec bruit multiplicatif ou l'équation KPZ ? Lorsque $\beta > \beta_c$, la quantité $h_\varepsilon + c_\varepsilon t$, pour une constante $c_\varepsilon \to \infty$, n'est pas conjecturée tendue. Un régime qui pourrait se révéler intéressant est la fenêtre autour du point critique $\beta \to \beta_c$. En dimension $d = 2$, les auteurs de [30] ont étendu les résultats de [15] et sont capables de déterminer, en considérant une limite du second ordre au point critique $\hat{\beta} \to \hat{\beta}_c$, les premiers moments d'une loi limite non triviale. Contrairement au cas $d = 2$ où l'on peut déterminer $\hat{\beta}_c$, on ne connaît en revanche pas β_c pour $d \geq 3$.

5 Résultats de la thèse et guide de lecture des chapitres

Les parties suivantes du manuscrit sont des copies quasiment conformes de pré-publications que l'on pourra trouver aux références indiquées ci-dessous.

Notes sur le polymère poissonnien [43]. *En collaboration avec Francis Comets.* La Partie II du manuscrit, est issue de notes formant un état de l'art du modèle de polymère poissonnien. On y trouvera également de nouveaux résultats, notamment:

- Dans la section 3 des notes, une preuve (absente de la littérature) montrant la dichotomie entre faible désordre et fort désordre, faisant intervenir de nouvelles estimées de continuité en espace de la fonction de partition, obtenues grâce à un couplage miroir de mouvements brownien. On en déduit également l'uniforme intégrabilité de W_t en faible désordre.
- Dans la section 4, une approche originale sur l'énergie libre directionnelle, propre au modèle poissonnien.
- Dans la section 7, un argument montrant la diffusivité du polymère dans la région de faible désordre, basée sur la transformation de Camerón-Martin.

Dans ces notes, on trouvera également au Chapitre 10 une description succincte de la preuve du régime de désordre intermédiaire pour le polymère poissonnien (Partie III).

Le régime de désordre intermédiaire pour le polymère possonnien [57]. La Partie III est dédiée à la preuve que le polymère poissonnien admet une limite d'échelle diffusive lorsque $\beta \to 0$ (cf. Section 3.2 de l'Introduction). En particulier, on y montre que les fonctions de partitons du polymères convergent vers celles du polymère *continu*, lesquelles sont décrites par la solution de l'équation de la chaleur stochastique.

Renormaliser l'équation KPZ en dimension $d \geq 3$ en faible désordre [42]. *En collaboration avec Francis Comets et Chiranjib Mukherjee.* La partie IV est consacrée à l'étude des propriétés de l'équation KPZ régularisée en dimension 3, pour un bruit d'intensité faible, dans le régime où la régularisation est dissipée. On y compare les comportements pour différentes conditions initiales et on prouve que les moments de la loi limite stationnaire sont finis (voir aussi la Section 4.3 de l'Introduction).

Fluctuations Gaussiennes et vitesse de convergence de l'équation KPZ [44]. *En collaboration avec Francis Comets et Chiranjib Mukherjee.* Dans la partie V, on étudie l'équation KPZ régularisée en espace, en dimension $d \geq 3$ et pour un bruit d'intensité faible, dans le régime où la régularisation est dissipée. On montre que les fluctuations de la solution, rencentrée par l'énergie libre du polymère brownien sous l'échelle diffusive, sont gaussiennes. On détermine de plus leur ordre de grandeur. On prouve des résultats similaires pour l'équation de la chaleur stochastique régularisée et pour la fonction de partition à horizon fini du polymère.

Remarque 5.1. *La version présentée dans ce manuscrit diffère de celle de [44] par la présence du résultat énoncé à la Partie V Théorème 1.1.*

Fluctuations gaussiennes pour la queue de la fonction de partition en dimension $d \geq 3$ et dans toute la région L^2 [58]. *En collaboration avec Shuta Nakajima.* En Partie VI, on considère le modèle de polymère discret sous un environnement discret i.i.d. On étudie les fluctuations de la queue de la martingale $W_\infty - W_n$. On étudie leur ordre de grandeur et on prouve qu'elle sont asymptotiquement gaussiennes dans la région L^2 entière.

*
*   *

Pour clore cette introduction, j'aimerais mentionner l'article [9] publié pendant la thèse, en collaboration avec Assaf Shapira, Lucas Benigni et Kay Jörg Wiese. Nous y montrons que la dimension de Hausdorff de l'ensemble des records du mouvement brownien fractionnaire, est égale à son paramètre de Hurst.

Part II

Brownian Polymers in Poissonian Environment: a survey

Chapter 1

Introduction

This survey is a joint work with Francis Comets, which is based on a course presented by the latter at the Research School in Marseille, March 6-10, 2017. The school was organized by the chair holders, Kostya Khanin and Senya Shlosman, of the Jean-Morlet chair 2017 of CIRM,

Random Structures in Statistical Mechanics and Mathematical Physics.

The model is a space-time continuous directed polymer in random environment. In this regard, it is one of the most basic such model and it plays a fundamental role. Directed polymers are described by random paths, which are influenced by randomly located impurities which may be attractive or repellent. Such models have been widely considered in statistical physics, disordered systems and stochastic processes.

As an informal definition we model the polymer by a random path $\mathbf{x} = (\mathbf{x}(t); t \geq 0)$ taking values in $\mathbb{R}^d$ and interacting with time-space Poisson points (t_i, x_i) called environment. The path sees such a point if at time t_i it is located within a fixed distance r from x_i. Denoting by $\#_t(\mathbf{x}) = \sum_{i:t_i \leq t} \mathbf{1}_{|\mathbf{x}(t_i)-x_i| \leq r}$ the number of Poisson points seen by the path $\mathbf{x}$ up to time t, the model with time horizon t at inverse-temperature parameter β is associated to the Hamiltonian

$$-\frac{1}{2}\int_0^t |\dot{\mathbf{x}}(s)|^2 ds + \beta \#_t(\mathbf{x}) \ .$$

In this model where the path is Brownian and the medium is Poissonian, we benefit from nice formulas and strong tools from stochastic calculus for Gaussian or Poisson measure and martingale techniques.

The notes are essentially based on references [49, 48, 51, 57], gathering and unifying the matter scattered in these references, and containing novel contributions and perspectives as emphasized below. It also parallels the book [41] which deals similar models in the discrete framework, and we warn the reader of the existence of many results available for one particular model but not for the others. We do not reproduce all details or computations, but we rather try to give the general picture and the essential arguments.

Let us mention the main highlights in this survey and also the new results:

1. We establish in section 3 a fine continuity estimate under spatial shifts for the limit of the martingale. This is achieved by a smart use of mirror coupling.

2. Section 4 contains a nice original account on directional free energy. We develop a full approach of disorder strength based on directional free energy.

3. In section 7 we develop an original approach to diffusivity at weak disorder, based on Camerón-Martin transformation (see theorem 7.2.2).

4. Section 10 is dedicated to the intermediate disorder regime and KPZ equation. We give a synthetic account with all the central ideas.

Keywords: Directed polymers, random environment; weak disorder, intermediate disorder, strong disorder; free energy; Poisson processes, martingales.

AMS 2010 subject classifications: Primary 60K37. Secondary 60Hxx, 82A51, 82D30.

Chapter 2

Free energy and phase transition

Notations and conventions: all through the notes, we will use the same symbols $P, \mathbb{P}, \dots$ to denote probability measures and mathematical expectations; e.g., $P[X]$ is the P-expectation of the random variable X.

In this section, we introduce the model and two central thermodynamic quantities, the quenched and the annealed free energies.

2.1 Polymer model

The model is defined as a Brownian motion in a random potential.

• *The free measure :* $(B = \{B_t\}_{t\geq 0}, P_x)$ is a Brownian motion on the d-dimensional Euclidean space $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$. We will use short notation $P_0 = P$.

• *The random environment:* $\omega = \sum_i \delta_{(T_i, X_i)}$ is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $\nu dt dx$, where ν is a positive parameter. We suppose that ω is defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and we define $\mathcal{G}_t$ to be the σ-field generated by the environment up to time t:

$$\omega_t = \omega_{|(0,t]\times\mathbb{R}^d}\ , \quad \mathcal{G}_t = \sigma\left(\omega_t(A); A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)\right), \tag{2.1}$$

where $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}_+ \times \mathbb{R}^d$.

From these two basic ingredients, we define the object we consider in the notes. Fix $r > 0$, and let $U(x)$ denote Euclidean (closed) ball in $\mathbb{R}^d$ with radius $\gamma_d^{-1/d} r$,

$$U(x) = \mathrm{B}(x, \gamma_d^{-1/d} r).$$

with γ_d the volume of the unit ball, so $U(x)$ has volume r^d. The tube around path B is the following subset of $(0,t] \times \mathbb{R}^d$:

$$V_t(B) = \{(s,x) : s \in (0,t], x \in U(B_s)\}\ . \tag{2.2}$$

When the indicator function

$$\chi_{s,x} = \mathbf{1}\{x \in U(B_s)\} = \mathbf{1}\{|x - B_s| \leq \gamma_d^{-1/d} r\} \tag{2.3}$$

has value 1 [resp., 0], the path B does see [resp. does not see] the point (s,x). For a fixed path B, the quantity defined by

$$\omega(V_t) = \int_{(0,t]\times\mathbb{R}^d} \chi_{s,x}\, \omega(ds, dx), \tag{2.4}$$

is the number of Poisson points seen by the path B up to time t, playing the role of $\#_t$ in the Introduction. Note that under $\mathbb{P}$, the variable $\omega(V_t)$ is Poisson distributed with mean $\nu t r^d$.

• *The polymer measure:* Fixing a realization ω of the Poisson point process and a value of the time horizon $t > 0$, we define the probability measure $P_t^{\beta,\omega}$ on the path space $\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d)$ equipped with its Borel field by

$$dP_t^{\beta,\omega} = \frac{1}{Z_t(\omega,\beta,r)} \exp\{\beta\omega(V_t)\}\, dP, \tag{2.5}$$

where $\beta \in \mathbb{R}$ is a parameter (the inverse temperature), where

$$Z_t = Z_t(\omega,\beta,r) = P\left[\exp\left(\beta\omega(V_t)\right)\right] \tag{2.6}$$

is the normalizing constant making $P_t^{\beta,\omega}$ a probability measure on the path space.

The model has been introduced by Nobuo Yoshida as a polymer model, and first appeared in [49] in the literature. For $\beta > 0$ the path is attracted by the Poisson points, and repelled otherwise. The Poisson environment represents randomly dispatched impurities. For negative β the model relates to Brownian motion in Poissonian obstacles [65, 152] which can be traced back to works of Smoluchowski [35]. Here we consider a directed version, in contrast to crossings [165, 164, 166, 167] where the path is stretched ballistically. Our model with $\beta \to +\infty$ is related to Euclidean first passage percolation [95, 94] with exponent $\alpha = 2$ therein.

Also, for a branching Brownian motion in random medium [147, 148], Z_t is equal to the mean population size in the medium given by ω.

2.2 Some key formulas and notations

We first recall three basic formulas that we will use repeatedly.

- For all non-negative and all non-positive measurable functions h on $\mathbb{R}_+ \times \mathbb{R}^d$, the Poisson formula for exponential moments (chapter 3. of [115]) writes

$$\mathbb{P}\left[e^{\int h(s,x)\omega_t(dsdx)}\right] = \exp\int_{]0,t]\times\mathbb{R}^d} \nu dsdx \left(e^{h(s,x)} - 1\right) . \tag{2.7}$$

  The formula remains true when h is replaced by ih, for any real integrable function h.

- Introducing the notation

$$\lambda(\beta) = e^{\beta} - 1 , \tag{2.8}$$

  the linearization formula for Bernoulli writes

$$e^{\beta\mathbf{1}_A} - 1 = (e^{\beta} - 1)\mathbf{1}_A = \lambda(\beta)\mathbf{1}_A . \tag{2.9}$$

- For all $s \geq 0$, we have

$$\int_{\mathbb{R}^d} \chi_{s,x}\, \mathrm{d}x = r^d. \tag{2.10}$$

2.3 Quenched free energy

It is defined as the rate of growth of the partition function, and it is a self-averaging property.

Theorem 2.3.1. *The quenched free energy*

$$p(\beta,\nu) = \lim_{t\to\infty} \frac{1}{t} \ln Z_t(\omega,\beta,r)$$

exists a.s. and in L^p-norm for all $p \geq 1$, and is deterministic,

$$p(\beta,\nu) = \sup_{t>0} \frac{1}{t}\mathbb{P}[\ln Z_t] .$$

Remark 2.3.2. *We omit the parameter $r > 0$ from the notation for the free energy. The reason is that, in contrast to β and ν, it is kept fixed most of the time.*

□ Let $\theta_{t,x}$ the space-time shift operator on the environment space,

$$\theta_{t,x}\big(\sum_i \delta_{(T_i,X_i)}\big) = \sum_i \delta_{(T_i-t,X_i-x)}\ .$$

By Markov property of the Brownian motion, we have for $s,t \geq 0$,

$$\begin{aligned} Z_{t+s} &= P\left[e^{\beta\omega(V_t)}e^{\beta\omega(V_{t+s}\setminus V_t)}\right] \\ &= P\left[e^{\beta\omega(V_t)}P\left[e^{\beta\omega(V_{t+s}\setminus V_t)}\big|B_t\right]\right] \\ &= P\left[e^{\beta\omega(V_t)}Z_s\circ\theta_{t,B_t}\right] && (2.11) \\ &= Z_t \times P_t^{\beta,\omega}[Z_s\circ\theta_{t,B_t}]\ , && (2.12) \end{aligned}$$

a remarkable identity expressing the Markov structure of the model. Let $u(t) = \mathbb{P}[\ln Z_t]$. By the independence property of Poisson points, $\omega_{|]s,t]}$ is independent of $\mathcal{G}_s$ for all $0 \leq s \leq t$. Then, denoting by $\mathbb{P}^{\mathcal{G}_t}$ the conditional expectation and conditional probability given $\mathcal{G}_t$, we have

$$\begin{aligned} u(t+s) &= \mathbb{P}\big[\ln P_t^{\beta,\omega}[Z_s\circ\theta_{t,B_t)}]\big] + \mathbb{P}\ln Z_t \\ &\overset{\text{Jensen}}{\geq} \mathbb{P}P_t^{\beta,\omega}[\ln Z_s\circ\theta_{t,B_t)}] + u(t) \\ &= \mathbb{P}\mathbb{P}^{\mathcal{G}_t}P_t^{\beta,\omega}[\ln Z_s\circ\theta_{t,B_t)}] + u(t) \\ &\overset{\text{Fubini}}{=} \mathbb{P}\big[P_t^{\beta,\omega}[\mathbb{P}^{\mathcal{G}_t}[\ln Z_s\circ\theta_{t,B_t)}]]\big] + u(t) \\ &= \mathbb{P}\big[P_t^{\beta,\omega}[u(s)]\big] + u(t) \qquad\qquad (\omega\text{ shift invariant}) \\ &= u(s)+u(t)\ . \end{aligned}$$

Hence the function $u(t)$ is superadditive. By the superadditive lemma, we get the existence of the limit

$$\lim_{t\to\infty}\frac{u(t)}{t} = \sup_{t>0}\frac{u(t)}{t}\ .$$

Now, anticipating the concentration inequality (6.11) and the continuous time bridging (6.14), we derive that

$$\frac{1}{t}\big(\ln Z_t - \mathbb{P}[\ln Z_t]\big) \longrightarrow 0$$

almost surely and in L^p for all $p \geq 1$ finite. ■

2.4 Annealed free energy and hierarchy of moments

We compute the expectation of the partition function over the medium using (2.4) and Fubini,

$$\begin{aligned} \mathbb{P}[Z_t] &= P\mathbb{P}\left[e^{\beta\int\chi_{s,x}\omega_t(ds,dx)}\right] \\ &\overset{(2.7)}{=} P\exp\int_{]0,t]\times\mathbb{R}^d}\left(e^{\beta\int\chi_{s,x}}-1\right)\nu dsdx \\ &\overset{(2.9)}{=} P\left[\exp\lambda(\beta)\int_{]0,t]\times\mathbb{R}^d}\chi_{s,x}\nu dsdx\right] \\ &= \exp\{t\nu\lambda r^d\}\ . && (2.13) \end{aligned}$$

Hence $\mathbb{P}[Z_t]$ grows in time at exponential rate $p^{(1)}(\beta,\nu) = \nu\lambda(\beta)r^d$. More generally, it is natural to consider the rate of growth of the s-th moment of the partition function,

$$p^{(s)}(\beta,\nu) = \lim_{t\to\infty}\frac{1}{st}\ln\mathbb{P}[Z_t^s]\ ,\quad s>0.$$

By Hölder inequality, $\|Z\|_r \leq \|Z\|_s$ for $r \leq s$, these rates are non-decreasing in s, and for integer values, they can be expressed by handy variational formulas using large deviation theory. By Jensen's inequality, we have for all t

$$\frac{1}{t}\mathbb{P}[\ln Z_t] \leq \frac{1}{t}\ \ln \mathbb{P}[Z_t] = p^{(1)}(\nu,\beta)\ , \tag{2.14}$$

yielding the so-called *annealed bound* :

$$p(\beta,\nu) \leq p^{(1)}(\nu,\beta)\ .$$

Summarizing the above, we have a chain of inequalities

$$p(\beta,\nu) \leq p^{(1)}(\beta,\nu) \leq \ldots \leq p^{(k)}(\beta,\nu) \leq p^{(k+1)}(\beta,\nu) \leq \ldots .$$

It is commun folklore that in a large class of models, the first inequalities in the above chain are equalities, while they become strict from $k^* = \inf\{k \geq 0 : p^{(k)}(\beta,\nu) < p^{(k+1)}(\beta,\nu)$ (with the convention $p^{(0)} = p$). Considering the sequence of rates $(p^{(k)}; k \geq 1)$ is classical approach to intermittency [34, 108, 125] and sect. 2.4 of [14].

In the directed case, we focus at $k = 0, 1$ only, since the latter is explicit.

Proposition 2.4.1. *Basic properties of the free energy:*

1. *For $\beta \neq 0, \nu > 0$, we have $\beta\nu r^d < p(\beta,\nu) \leq \nu\lambda(\beta)r^d$.*
2. *$\beta \to p(\beta,\nu)$ is convex.*
3. *The excess free energy*

$$\psi(\beta,\nu) = \nu\lambda(\beta)r^d - p(\beta,\nu) \tag{2.15}$$

is non-decreasing in $|\beta|$ and in ν. It is jointly continuous.

□ The second inequality in item 1 is the annealed bound. The first one follows from an inifinite-dimensional version of Jensen's inequality; this version being curiously overlooked in the literature, we recall the full statement:

Lemma 2.4.2 (Lemma A.1 in [137])**.** *Let g be a bounded measurable function on a product space $\mathcal{X} \times \mathcal{Y}$, μ a probability measure on $\mathcal{X}$ and ρ a probability measure on $\mathcal{Y}$. Then*

$$\ln \int_{\mathcal{X}} e^{\int_{\mathcal{Y}} g(x,y)d\rho(y)} d\mu(x) \leq \int_{\mathcal{Y}} \left[\ln \int_{\mathcal{X}} e^{g(x,y)} d\mu(x)\right] d\rho(y).$$

We apply it with $\rho = \mathbb{P}, \mu = P, g(x,y) = \beta\omega(V_t)$ to get the desired bound[1]. However this bound is not so great here, since the simple one $p(\beta,\nu) \geq t^{-1}\mathbb{P}[\ln Z_t]$ for a fixed t (which comes from superadditivity of $u(t)$) is not linear, but strictly convex in β and then already better.

Item 2 is the standard convexity of free energy,

$$\frac{\partial^2}{\partial\beta^2} \ln Z_t = \mathrm{Var}_{P_t^{\beta,\omega}}\big(\omega(V_t))\big) > 0\ ,$$

where $\mathrm{Var}_{P_t^{\beta,\omega}}$ denotes the variance under the polymer measure in a fixed environment ω.

We now turn towards item 3, in the case $\beta \geq 0$ (the other case being similar). We use specific properties of the medium, infinite divisibility: for $\nu, \Delta > 0$, we note that the superposition $\omega + \hat{\omega}$ of two independent PPP with intensities ν and Δ is a PPP with intensity $\nu + \Delta$. Writing $\mathbb{P}$ the expectation over both variables $\omega, \hat{\omega}$, we compute by conditioning

$$\begin{aligned}
\mathbb{P} \ln Z_t(\omega) &\overset{\beta\geq 0}{\leq} \mathbb{P} \ln Z_t(\omega + \hat{\omega}) \\
&= \mathbb{P}\mathbb{P}\left[\ln Z_t(\omega + \hat{\omega})|\omega\right] \\
&\overset{\text{Jensen}}{\leq} \mathbb{P} \ln \mathbb{P}\left[Z_t(\omega + \hat{\omega})|\omega\right] \\
&= \mathbb{P} \ln Z_t(\omega) + t\Delta\lambda(\beta)r^d.
\end{aligned}$$

[1] We explain in this note why the Lemma is an infinite-dimensional version of Jensen's inequality: the functional $\psi(f) = \ln \int_{\mathcal{X}} e^{f(x)} d\mu(x)$ is convex, and the function $f(\cdot) = g(\cdot, y)$ is randomly chosen with $\rho(dy)$.

This proves monotonicity of ψ in ν. This proves at the same time continuity in ν (locally uniformly in β) and the joint continuity in (β, ν).

The plain identity $\mathbb{P}[Yf(Y)] = \theta\mathbb{P}[f(Y+1)]$ for a r.v. Y distributed as a Poisson law with mean θ has a counterpart for PPP, an integration by parts formula known as Slivnyak-Mecke formula (e.g., p.50 in [151] or th. 4.1 in [115]): Let $\mathcal{M}$ be the space of point measures on $(0,t] \times \mathbb{R}^d$ and $h : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}_+$ measurable, then

$$\mathbb{P}\left[\int h(s,x;\omega_t)\omega_t(ds,dx)\right] = \int_{(0,t]\times\mathbb{R}^d} \mathbb{P}\big[h(s,x;\omega_t+\delta_{s,x})\big]\ \nu dsdx\ . \tag{2.16}$$

With this in hand, we can show monotonicity of ψ in β:

$$\begin{aligned}
\frac{\partial}{\partial\beta}\mathbb{P}\big[\ln Z_t\big] &= \mathbb{P}P_t^{\beta,\omega}[\omega(V_t)] \\
&= \mathbb{P}\int \omega_t(dsdx)\frac{P\left[\chi_{s,x}e^{\beta\omega(V_t)}\right]}{Z_t} \\
&\overset{(2.16)}{=} \mathbb{P}\int_{(0,t]\times\mathbb{R}^d} \nu dsdx \frac{P\left[\chi_{s,x}e^{\beta(\omega(V_t)+\delta_{s,x})}\right]}{P\left[e^{\beta(\omega(V_t)+\delta_{s,s})}\right]} \\
&\overset{(2.9)}{=} \mathbb{P}\int_{(0,t]\times\mathbb{R}^d} \nu dsdx \frac{P\left[e^{\beta}\chi_{s,x}e^{\beta\omega(V_t)}\right]}{P\left[\big(\lambda(\beta)\chi_{s,x}+1\big)e^{\beta\omega(V_t)}\right]} \\
&= \mathbb{P}\int_{(0,t]\times\mathbb{R}^d} \nu dsdx \frac{e^{\beta}P_t^{\beta,\omega}[\chi_{s,x}]}{1+\lambda(\beta)P_t^{\beta,\omega}[\chi_{s,x}]}\ .
\end{aligned} \tag{2.17}$$

Define

$$\psi_t(\beta,\nu) = t^{-1}\mathbb{P}\left[\nu\lambda r^d - \ln Z_t\right]. \tag{2.18}$$

With the identity

$$\frac{\partial}{\partial\beta}\nu\lambda(\beta)r^d t = \nu e^{\beta}r^d t = e^{\beta}\int_{(0,t]\times\mathbb{R}^d} \nu dsdx P_t^{\beta,\omega}[\chi_{s,x}]$$

we obtain

$$\frac{\partial}{\partial\beta}\psi_t(\beta,\nu) = \frac{1}{t}e^{\beta}\lambda(\beta)\nu\int_{(0,t]\times\mathbb{R}^d} dsdx\ \mathbb{P}\frac{P_t^{\beta,\omega}[\chi_{s,x}]^2}{1+\lambda(\beta)P_t^{\beta,\omega}[\chi_{s,x}]}\ , \tag{2.19}$$

which has the sign of β. So the limit ψ of ψ_t is increasing in $|\beta|$. ■

2.5 Phase transition

An important consequence of monotonicity and continuity of ψ in $|\beta|$ in Proposition 2.4.1 is the existence and uniqueness of the critical temperatures introduced in the next statement, which is a direct consequence of the above.

Theorem 2.5.1. *There exist $\beta_c^+(\nu), \beta_c^-(\nu)$ with $-\infty \le \beta_c^- \le 0 \le \beta_c^+ \le +\infty$ such that*

$$\begin{cases} \psi(\beta,\nu) = 0 & \text{if} \quad \beta \in [\beta_c^-, \beta_c^+] \\ \psi(\beta,\nu) > 0 & \text{if} \quad \beta < \beta_c^- \text{ or } \beta > \beta_c^+. \end{cases} \tag{2.20}$$

Moreover, $|\beta_c^\pm(\nu)|$ is non-increasing in ν.

These values $\beta_c^+(\nu), \beta_c^-(\nu)$ are called *critical* (inverse) *temperatures* at density ν (they depend on r as well). The domains in the (β,ν)-half-plane defined by the first and second line in (2.20) are called *high and low temperature region* respectively. The boundary between the two regions is called the *critical line*, and a *phase transition* in the statistical mechanics sense occurs: the quenched free energy $p(\beta,\nu)$ is equal to the annealed free energy $p^{(1)}(\beta,\nu) = \nu\lambda r^d$ – an analytic function – but analyticity of $p(\beta,\nu)$ breaks down when crossing the critical line.

To summarize our finding, we define the high temperature region and the low temperature region

$$\mathcal{D} = \{(\beta,\nu) : \psi(\beta,\nu) = 0\}, \qquad \mathcal{L} = \{(\beta,\nu) : \psi(\beta,\nu) > 0\}.$$

They are are delimited by the critical lines $\beta_c^-(\nu)$ and $\beta_c^+(\nu)$ from Definition 2.5.1. In the next sections we will discuss non-triviality of the critical lines, as well as fine properties. In section 5 we will understand that they correspond to delocalized or localized behavior respectively.

Chapter 3

Weak Disorder, Strong Disorder

3.1 The normalized partition function

In this section, we introduce a natural martingale that will play an important role in many results concerning the asymptotic behavior of the polymer.

For any fixed path of the brownian motion, $\{\omega(V_t)\}_{t\geq 0}$ is a Poisson process of intensity νr^d and has associated exponential martingales $\{\exp\big(\beta\omega(V_t) - \lambda(\beta)\nu r^d t\big)\}_{t\geq 0}$. Hence, for $t \geq 0$, the *normalized partition function*

$$W_t = e^{-\lambda(\beta)\nu r^d t} Z_t, \tag{3.1}$$

defines a positive, mean 1, càdlàg martingale with respect to $\{\mathcal{G}_t\}_{t\geq 0}$.

By Doob's martingale convergence theorem [138, Chapter 2, Corollary 2.11], we get the existence of a random variable W_∞ such that

$$W_\infty = \lim_{t\to\infty} W_t \quad \text{a.s.} \tag{3.2}$$

Theorem 3.1.1. *There is a dichotomy: either the limit W_∞ is almost-surely positive, or it is almost-surely zero. Otherwise stated, we have either*

$$\mathbb{P}\{W_\infty > 0\} = 1, \tag{3.3}$$

or

$$\mathbb{P}\{W_\infty = 0\} = 1. \tag{3.4}$$

Proof. Denote by e_t the renormalized weight

$$e_t = \exp(\beta\omega(V_t(B)) - \lambda(\beta)\nu r^d t) \tag{3.5}$$

By the Markovian property (2.11), we get that for all positive times t and s,

$$W_{s+t} = P[e_t\, W_s \circ \theta_{t,B_t}]. \tag{3.6}$$

In Section 3.2.1, we will justify that one can take the limit as $s \to \infty$ in this equality, in order to get that a.s.

$$W_\infty = P[e_t\, W_\infty \circ \theta_{t,B_t}]. \tag{3.7}$$

Then, notice that (3.7) also writes

$$W_\infty = W_t \int_{\mathbb{R}^d} P_t^{\beta,\omega}(B_t \in \mathrm{d}x)\, W_\infty \circ \theta_{t,x}. \tag{3.8}$$

Since $W_t > 0$ $\mathbb{P}$-a.s and since $P_t^{\beta,\omega}$ has positive density with respect to Lebesgue's measure, we obtain by (3.8) that

$$\forall t > 0, \quad \{W_\infty = 0\} = \{W_\infty \circ \theta_{t,x} = 0,\ x\text{-a.e.}\},$$

or, equivalently,

$$\{W_\infty = 0\} = \left\{\int_{\mathbb{R}^d} P(B_t \in \mathrm{d}x)\, W_\infty \circ \theta_{t,x} = 0\right\}.$$

The event of the right-hand side belong to the σ-field $\mathcal{G}_{[t,\infty)} = \sigma\left(\omega(A); A \in \mathcal{B}([t,\infty)\times\mathbb{R}^d)\right)$ completed by null sets, so

$$\{W_\infty = 0\} \in \bigcap_{t>0}^{\infty} \mathcal{G}_{[t,\infty)}.$$

The theorem now follows from Komogorov's 0-1 law. ■

This dichotomy calls for a definition.

Definition 3.1.2. *We say that the polymer is in the* ***weak disorder*** *phase when $W_\infty > 0$ almost surely. We say it is in the* ***strong disorder*** *phase when $W_\infty = 0$ almost surely.*

The phase diagram is connected in the β-parameter space.

Theorem 3.1.3. *There exist two critical parameters $\bar{\beta}_c^- \in [-\infty, 0]$ and $\bar{\beta}_c^+ \in [0,\infty]$, depending only on ν, r and d, such that*

- *For all $\beta \in (\bar{\beta}_c^-, \bar{\beta}_c^+) \cup \{0\}$, the polymer belongs to the weak disorder phase.*
- *For all $\beta \in \mathbb{R} \setminus [\bar{\beta}_c^-, \bar{\beta}_c^+]$, the polymer belongs to the strong disorder phase.*

Proof. Let θ be a real number in $(0,1)$ and denote $Y_t = W_t^\theta$ for all $t \geq 0$. The family $(Y_t)_{t\geq 0}$ is a collection of positive random variables verifying

$$\sup_{t\geq 0} \mathbb{P}\big[Y_t^{1/\theta}\big] = \sup_{t\geq 0} \mathbb{P}[W_t] = 1 < \infty.$$

As $1/\theta$ is strictly greater than 1, this relation implies the uniform integrablity of $(Y_t)_{t\geq 0}$. Since the process $(W_t^\theta)_{t\geq 0}$ converges almost surely to W_∞^θ, we get from uniform integrability that

$$\lim_{t\to\infty} \mathbb{P}\left[W_t^\theta\right] = \mathbb{P}\left[W_\infty^\theta\right]. \tag{3.9}$$

Now, one can observe that the right hand side term is positive if and only if (3.3) holds and that it is zero if and only if (3.4) holds. To prove the theorem, it is then enough to prove that $\beta \mapsto \mathbb{P}\left[W_\infty^\theta\right]$ is a non-increasing function of $|\beta|$ and choose for example

$$\bar{\beta}_c^+ = \inf\{\beta \geq 0 : \mathbb{P}\left[W_\infty^\theta\right] = 0\}, \tag{3.10}$$

which does not depend on $\theta \in (0,1)$. Using (3.9), we now just have to show that $\beta \mapsto \mathbb{P}\left[W_t^\theta\right]$ is an non-increasing function of $|\beta|$ for all positive t. By standard arguments, we get that

$$\begin{aligned}
\frac{\partial}{\partial\beta}\mathbb{P}\left[W_t^\theta\right] &= \mathbb{P}\left[\theta W_t^{\theta-1}\frac{\partial}{\partial\beta}W_t\right] \\
&= \mathbb{P}\left[\theta W_t^{\theta-1} P\left[\left(\omega(V_t) - \lambda'(\beta)\nu r^d t\right) e^{\beta\omega(V_t)-\lambda(\beta)\nu r^d t}\right]\right] \\
&= \theta\, P\, \mathbb{P}\left[W_t^{\theta-1}\left(\omega(V_t) - \lambda'(\beta)\nu r^d t\right) e^{\beta\omega(V_t)-\lambda(\beta)\nu r^d t}\right].
\end{aligned}$$

Introducing the probability measure $\mathbb{P}^\beta$ on point measures, given by

$$\mathrm{d}\mathbb{P}^\beta(\omega) = e^{\beta\omega(V_t)-\lambda r^d \nu t}\mathrm{d}\mathbb{P}(\omega),$$

the derivative of $\mathbb{P}[W_t^\theta]$ is now given by

$$\frac{\partial}{\partial\beta}\mathbb{P}\left[W_t^\theta\right] = \theta\, P\ \mathbb{P}^\beta\left[W_t^{\theta-1}(\omega(V_t) - r^d\nu\lambda' t)\right]. \tag{3.11}$$

In Proposition 3.1.4 just below, we will see that under the probability measure $\mathbb{P}^\beta$, ω is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$.

We can then use the *Harris-FKG inequality* for Poisson processes [114, th. 11 p. 31] in order to bound the above expectation. Indeed, the variable $\omega(V_t) - r^d\nu\lambda' t$ is an increasing function of the point process and by definition, the process $W_t^{\theta-1}$ is then a decreasing function of ω when $\beta \geq 0$ (resp. increasing when $\beta < 0$). Applying the FKG inequality, we find that for positive β

$$\mathbb{P}^\beta\left[W_t^{\theta-1}(\omega(V_t) - \lambda' r^d\nu t)\right] \leq \mathbb{P}^\beta\left[W_t^{\theta-1}\right]\mathbb{P}^\beta\left[(\omega(V_t) - r^d\nu\lambda' t)\right] = 0, \tag{3.12}$$

where the last equality is a result of the relation

$$\mathbb{P}[\omega(V_t)e^{\beta\omega(V_t)}] = \lambda'(\beta)r^d\nu t.$$

The same result with opposite inequality comes when $\beta < 0$. Thus, we get from (3.11) and (3.12) that $\mathbb{P}[W_t^\theta]$ is a non-increasing function of $|\beta|$. ∎

We recall at this point that Poisson processes with mutually absolutely continuous intensity measures are themselves mutually absolutely continuous.

Proposition 3.1.4. *Let η be a Poisson point process on a measurable space E, of intensity measure μ. Let f be a function such that $e^f - 1 \in L^1(\mu)$. Then, under the probability measure $\mathbb{Q}$ defined by*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_E f(x)\,\eta(\mathrm{d}x) - \int_E (e^{f(x)} - 1)\,\mathrm{d}\mu(x)\right),$$

the process η is a Poisson point process of intensity measure $e^f\,\mathrm{d}\mu$.

Proof. Let g be any non-negative measurable function. As the Laplace functional characterizes Poisson processes (theorem 3.9 in [115]), we compute it for the point process η under the measure $\mathbb{Q}$:

$$\begin{aligned}\mathbb{Q}\exp\left\{-\int_E g(s)\,\eta(\mathrm{d}x)\right\} &= \mathbb{P}\exp\left\{\int_E f(x) - g(x)\,\eta(\mathrm{d}x)\right\}e^{-\int(e^{f(x)}-1)\,\mathrm{d}\mu(x)}\\ &= \exp\left\{\int_E (e^{f(x)-g(x)} - 1)\,\mathrm{d}\mu(x)\right\}e^{-\int(e^{f(x)}-1)\,\mathrm{d}\mu(x)}\\ &= \exp\left\{\int_E (e^{-g(x)} - 1)e^{f(x)}\,\mathrm{d}\mu(x)\right\},\end{aligned}$$

where the second equality is an application of (2.7). The expression we obtain corresponds, as claimed, to a Poisson point process of intensity measure $e^f\,\mathrm{d}\mu$. ∎

3.2 The self-consistency equation and UI properties in the weak disorder

3.2.1 Proof of the self-consistency equation on W_∞

In this section, we prove that one can take the limit in the identity $W_{s+t} = P[e_t\,W_s \circ \theta_{t,B_t}]$ and obtain the equation of self-consistency:

$$\forall t \geq 0, \quad W_\infty = P[e_t\,W_\infty \circ \theta_{t,B_t}] \quad \text{a.s.} \tag{3.13}$$

A part of the problem is that we only have almost sure convergence of the $W_s \circ \theta_{t,x}$ for countable number of x's. To deal with this issue, we show that the quantity

$$W_t(x) := e^{-\lambda(\beta)\nu r^d t}P_x\left[\exp\left(\beta\omega(V_t)\right)\right], \tag{3.14}$$

does not vary too much with x, in the sense of the following lemma:

Lemma 3.2.1. *There exists a constant $C = C(\beta, \nu, r)$, such that, for all $t \in [0, \infty]$ and $x, y \in \mathbb{R}^d$,*

$$\mathbb{P}\left[|W_t(x) - W_t(y)|\right] \leq C|x - y|. \tag{3.15}$$

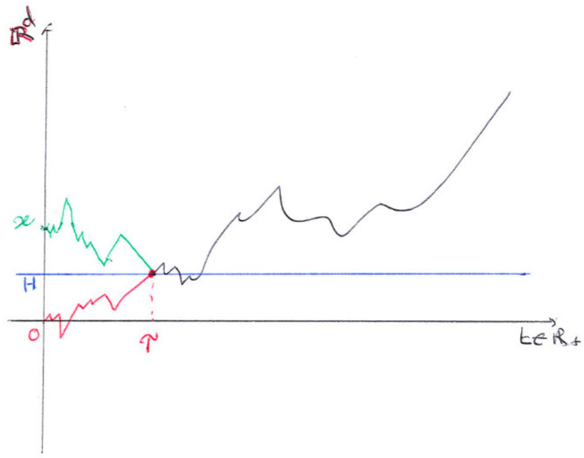

Figure 3.1: Mirror coupling in $d = 1$.

Proof. To simplify the notations, we only consider the case where $y = 0$. To recover the lemma, it is enough to argue that the Poisson environment is invariant in law under a translation in space of vector y.

To prove (3.15) for $y = 0$, we write the difference of the two martingales as expectations over two coupled Brownian motions, in the same environment. The coupling we consider is the mirror coupling, which is defined as follows (see [96] for more details).

At time 0, one of the Brownian motions is starting at 0 and one is starting at $x \in \mathbb{R}^d$. Denote by H be the hyperplane bisecting the segment $[0, x]$, which is the hyperplane passing by $x/2$ and orthogonal to the vector x. Let also

$$\tau = \inf\{t \geq 0 | B_t \in H\},$$

be the first hitting time of H by B. Then, define $\widetilde{B}$ as the path that coincides with the reflection of path of B with respect to H for times before τ, and that coincides with B after τ.

The process $\widetilde{B}$ has the law of a Brownian motion starting from x. Moreover, the time τ is the first time B and $\widetilde{B}$ meet. After τ, the processes coincide. The variable τ has the following cumulative distribution function:

$$P(\tau \geq z) = \phi_z(|x|), \tag{3.16}$$

where, for positive z,

$$\phi_z(|x|) = \frac{2}{\sqrt{2\pi z}} \int_0^{|x|/2} e^{-u^2/2z} \mathrm{d}u,$$

and where $|\cdot|$ is the Euclidean distance. In the litterature, a coupling that satisfies this relation is said to be maximal (see [96]). Let

$$e_t = \exp(\beta\omega(V_t(B)) - \lambda(\beta)\nu r^d t), \quad \tilde{e}_t = \exp(\beta\omega(V_t(\widetilde{B})) - \lambda(\beta)\nu r^d t),$$

which we factorize in the contributions before and after B and $\widetilde{B}$ coalesce, so that, for $t \in [0,\infty)$,

$$\begin{aligned}\mathbb{P}[|W_t(x) - W_t(0)|] &\leq \mathbb{P}P[|\tilde{e}_t - e_t|] \\ &= P\mathbb{P}[|(\tilde{e}_{t\wedge\tau} - e_{t\wedge\tau})e_{(t-t\wedge\tau)^+} \circ \theta_{t\wedge\tau, B_{t\wedge\tau}}|] \\ &= P\mathbb{P}[|\tilde{e}_{t\wedge\tau} - e_{t\wedge\tau}|],\end{aligned}$$

where the last equality is a result of the independance of the Poisson environment before and strictly after time $t \wedge \tau$.

Then, we distinguish the cases where B or $\widetilde{B}$ encounter a point of the environment before $t\wedge\tau$, and the cases where they don't. We get that $P\mathbb{P}[|\tilde{e}_{t\wedge\tau} - e_{t\wedge\tau}|]$ writes:

$$\begin{aligned}&P\mathbb{P}\left[|e_{t\wedge\tau} - \tilde{e}_{t\wedge\tau}|\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B))>1,\, \omega(V_{t\wedge\tau}(\widetilde{B}))>1\}}\right] + P\mathbb{P}\left[(e_{t\wedge\tau} - e^{-\lambda\nu r^d t\wedge\tau})\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B))>1,\, \omega(V_{t\wedge\tau}(\widetilde{B}))=0\}}\right] \\ &\quad + P\mathbb{P}\left[(\tilde{e}_{t\wedge\tau} - e^{-\lambda\nu r^d t\wedge\tau})\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B)=0,\, \omega(V_{t\wedge\tau}(\widetilde{B})>1\}}\right].\end{aligned}$$

We first use the triangle inequality in the first expectation of the sum, and neglect the negative terms in two other expectations. Then, recombining the terms, one obtains that

$$\begin{aligned}\mathbb{P}[|W_t(x) - W_t(0)|] &\leq 2P\mathbb{P}\left[e_{t\wedge\tau}\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B))>1\}}\right] + 2P\mathbb{P}\left[\tilde{e}_{t\wedge\tau}\mathbf{1}_{\{\omega(V_{t\wedge\tau}(\widetilde{B}))>1\}}\right] \\ &= 4P\mathbb{P}\left[e_{t\wedge\tau}\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B))>1\}}\right], \end{aligned} \tag{3.17}$$

where the equality is a consequence of invariance in law of the Poisson environment under the translation by x.

For any fixed s, the variable $\omega(V_s)$ is a Poisson r.v. of parameter $\nu r^d s$, so that

$$\mathbb{P}\left[e_s \mathbf{1}_{\{\omega(V_s(B)>1\}}\right] = \sum_{k=1}^{\infty} \mathbb{P}[e_s \mathbf{1}_{\{\omega(V_s)=k\}}] = \sum_{k=1}^{\infty}\left[e^{\beta k - \lambda\nu r^d s}\frac{(\nu r^d s)^k}{k!}e^{-\nu r^d s}\right],$$

from which we get by standard computations that

$$P\mathbb{P}\left[e_{t\wedge\tau}\mathbf{1}_{\{\omega(V_{t\wedge\tau}(B)>1\}}\right] = P[1-\exp(-e^{\beta}\nu r^d t\wedge\tau)] \leq P[1-\exp(-e^{\beta}\nu r^d \tau)]. \tag{3.18}$$

To control this last expectation, we will first notice that $\tau/|x|^2$ is independent of $|x|$, and we will compute its density. By equation (3.16) and the change of variable $u = |x|\sqrt{(z)}v$, we get that

$$\mathbb{P}(\tau/|x|^2 > z) = \frac{1}{\sqrt{2\pi}}\int_0^{1/2\sqrt{z}} e^{-v^2/2}\mathrm{d}v.$$

This function of z is continuous and everywhere differentiable on $[0,\infty)$. The variable $\tau/|x|^2$ hence admits a density with respect to the Lebesgue measure, given by

$$f(z) = \frac{1}{4\sqrt{2\pi}}\frac{e^{-1/8z}}{z^{3/2}}.$$

Therefore, one gets that the right-hand side of (3.18) can be written as

$$\begin{aligned}&P\left[\left(1-\exp(-e^{\beta}\nu r^d\tau|x|^{-2}|x|^2)\right)\mathbf{1}_{\{\tau/|x|^2\leq 1\}}\right] + P\left[\left(1-\exp(-e^{\beta}\nu r^d\tau|x|^{-2}|x|^2)\right)\mathbf{1}_{\{\tau/|x|^2>1\}}\right] \\ &\qquad \leq e^{\beta}\nu r^d|x|^2 + \int_1^{\infty}\left(1-\exp(-e^{\beta}\nu r^d z|x|^2)\right)f(z)\mathrm{d}z.\end{aligned}$$

As there is some constant $C>0$ such that $f(z) \leq Cz^{-3/2}$, after the change of variables $z|x|^2 = u$, the integral above can be bounded by

$$|x|\int_0^{\infty} C\left(1-\exp\left(-e^{\beta}\nu r^d u\right)\right)v^{-3/2}\mathrm{d}v,$$

where one can check that the integral converges. Since its value only depends on β, ν and r, we finally get that there exists some constant $C' = C(\beta, \nu, r)$, such that

$$\mathbb{P}[|W_t(x) - W_t(0)|] \leq 4P[1 - \exp(-e^{\beta}\nu r^d \tau)] \leq C'(|x|^2 + |x|),$$

where the first inequality comes from combining (3.17) and (3.18). Since the $W_t(x)$ variables have bounded expectations, this proves the lemma in the case where $t < \infty$. The $t = \infty$ case is then a consequence of Fatou's lemma. ■

We can now show that the self-consistency equation holds. Let $\delta > 0$ be a parameter that will go to 0. For all $q \in \mathbb{Z}^d$, define $\Delta(q)$ to be the cube of length δ centered at δq, so that all the cubes form a partition of the space $\mathbb{R}^d$. We get that the right-hand side of (3.6) satisfies

$$\begin{aligned} P[e_t\, W_s \circ \theta_{t,B_t}] &= \sum_{q\in\mathbb{Z}^d} P[e_t\, W_s \circ \theta_{t,B_t}; B_t \in \Delta(q)] \\ &= \sum_{q\in\mathbb{Z}^d} P[e_t\, W_s \circ \theta_{t,\delta q}; B_t \in \Delta(q)] + A_s^\delta, \end{aligned} \tag{3.19}$$

where $A_s^\delta = \sum_{q\in\mathbb{Z}^d} P[e_t\,(W_s \circ \theta_{t,B_t} - W_s \circ \theta_{t,\delta q}); B_t \in \Delta(q)]$.

First observe that $\mathbb{P}$-almost surely, $W_s \circ \theta_{t,\delta q}$ converges to $W_\infty \circ \theta_{t,\delta q}$ for all $q \in \mathbf{Z}^d$, so Fatou's lemma entails

$$\mathbb{P}\text{-a.s.,} \quad \liminf_{s\to\infty} \sum_{q\in\mathbb{Z}^d} P[e_t\, W_s \circ \theta_{t,\delta q}; B_t \in \Delta(q)] \geq \sum_{q\in\mathbb{Z}^d} P[e_t\, W_\infty \circ \theta_{t,\delta q}; B_t \in \Delta(q)],$$

so that, by (3.19) and letting $s \to \infty$ in (3.6),

$$\mathbb{P}\text{-a.s.,} \quad W_\infty \geq \sum_{q\in\mathbb{Z}^d} P[e_t\, W_\infty \circ \theta_{t,\delta q}; B_t \in \Delta(q)] + \liminf_{s\to\infty} A_s^\delta. \tag{3.20}$$

Furthermore, using the fact that W is a martingale, one can check that $s \mapsto A_s^\delta$ is also a martingale with respect to the filtration $\{\mathcal{G}_{t+s}\}_{s\geq 0}$. For any time $S \geq 0$, Lemma 3.2.1 implies that it satisfies

$$\begin{aligned} \mathbb{P}[|A_S^\delta|] &\leq \sum_{q\in\mathbb{Z}^d} P\left[\mathbb{P}[e_t\, |W_S \circ \theta_{t,B_t} - W_S \circ \theta_{t,\delta q}|]\mathbf{1}_{B_t\in\Delta(q)}\right] \\ &\leq C(\delta^2+\delta) \sum_{q\in\mathbb{Z}^d} P\left[\mathbf{1}_{B_t\in\Delta(q)}\right] \\ &= C(\delta^2+\delta), \end{aligned}$$

where, in the second inequality, we have factorized by $\mathbb{P}[e_t] = 1$, using the independence under $\mathbb{P}$ of the environment before and strictly after time t. Thus, by Doob's inequality [106, Th. 3.8 (i), Ch. 1], we have for all $u > 0$,

$$\mathbb{P}\left[\sup_{0\leq s\leq S} |A_s^\delta| \geq u\right] \leq \frac{C(\delta^2+\delta)}{u},$$

where we can let $S \to \infty$ by monotone convergence. This implies that $\sup_{s\geq 0} |A_s^\delta|$ converges in probability to 0, when $\delta \to 0$, which in turn implies that

$$\liminf_{s\to\infty} |A_s^\delta| \xrightarrow{\mathbb{P}} 0.$$

Then, using Lemma 3.2.1 in the case where $t = \infty$, the same computation as above would show that

$$\mathbb{P}\left[\left|\sum_{q\in\mathbb{Z}^d} P\left[e_t\, W_\infty \circ \theta_{t,B_t}; B_t \in \Delta(q)\right] - \sum_{q\in\mathbb{Z}^d} P\left[e_t\, W_\infty \circ \theta_{t,\delta q}; B_t \in \Delta(q)\right]\right|\right] \leq C(\delta^2+\delta),$$

and hence,

$$\sum_{q\in\mathbb{Z}^d} P\left[e_t\, W_\infty \circ \theta_{t,\delta q}; B_t \in \Delta(q)\right] \xrightarrow{L^1} P[e_t\, W_\infty \circ \theta_{t,B_t}].$$

In particular, we have shown that the right-hand side of (3.20) converges in probability to $P[e_t\, W_\infty \circ \theta_{t,B_t}]$, so that almost surely

$$W_\infty \geq P[e_t\, W_\infty \circ \theta_{t,B_t}].$$

Observe that the two quantities have the same expectations to conclude that (3.13) holds.

3.2.2 Uniform integrability in the weak disorder

Proposition 3.2.2. *The martingale W_t is uniformly integrable if and only if the polymer is in the weak disorder phase, i.e. $W_\infty > 0$, $\mathbb{P}$-almost surely.*

Proof. If W_t is UI, then W_t converges in L^1 to W_∞, so that $\mathbb{P}[W_\infty] = 1 > 0$, therefore weak disorder must hold by the dichotomy.

Suppose now that the polymer is in weak disorder and set

$$X_{t,x} = \frac{W_\infty \circ \theta_{t,x}}{\mathbb{P}[W_\infty]}.$$

The self-consistency equation (3.13) writes $X_{0,0} = P[e_t X_{t,B_t}]$, so that, as $X_{t,x}$ is independent of $\mathcal{G}_t$ for all x,

$$\mathbb{P}[X_{0,0}|\mathcal{G}_t] = P\left[e_t \mathbb{P}[X_{t,B_t}]\right] = P[e_t].$$

This shows that for all $t \geq 0$,

$$W_t = \mathbb{P}[X_{0,0}|\mathcal{G}_t] \quad \text{a.s.}$$

Hence, $(W_t)_{t\geq 0}$ is uniformly integrable since the family of the right-hand side is a uniformly integrable martingale. ∎

3.3 The $\mathbf{L^2}$-region

Theorem 3.3.1. *(i) There exist two critical parameters ${\beta_2}^- \in [-\infty, 0]$ and ${\beta_2}^+ \in [0,\infty]$, depending only on ν, r and d, such that, if $\beta \in ({\beta_2}^-, {\beta_2}^+) \cup \{0\}$ then*

$$\sup_{t\in\mathbb{R}} \mathbb{P}[W_t^2] < \infty, \tag{3.21}$$

and such that the supremum is infinite if $\beta \in \mathbb{R} \setminus [\beta_2^-, \beta_2^+]$.

(ii) Furthermore, if $d \geq 3$, there exists a constant $c(d) \in (0,\infty)$, such that (3.21) holds whenever

$$\lambda(\beta)^2 \nu r^{d+2} < c(d), \tag{3.22}$$

(iii) In particular, ${\beta_2}^- < 0$ and ${\beta_2}^+ > 0$ whenever $d \geq 3$.

(iv) Also for $d \geq 3$, when $\nu r^{d+2} < c(d)$ the constant in (3.22), we have ${\beta_2}^- = -\infty$.

Definition 3.3.2. *We call the **$\mathbf{L^2}$-region** the set of parameters β, ν, r for which (3.21) holds.*

Proof of Theorem 3.3.1. We introduce the product measure $P^{\otimes 2}$ of two independent Brownian motions B_t and $\tilde{B}_t$ starting from 0 with respective tubes V_t and $\tilde{V}_t$. The main idea is to write

$$W_t^2 = P^{\otimes 2}[e^{\beta\omega(V_t)} e^{\beta\omega(\tilde{V}_t)}] e^{-2\lambda\nu r^d t},$$

so that, using Fubini's theorem,

$$\mathbb{P}[W_t^2] = P^{\otimes 2}\, \mathbb{P}[e^{\beta(\omega(V_t)+\omega(\tilde{V}_t))}] e^{-2\lambda\nu r^d t}. \tag{3.23}$$

One can see that

$$\omega(V_t) + \omega(\tilde{V}_t) = 2\omega(V_t \cap \tilde{V}_t) + \omega(V_t \Delta \tilde{V} t),$$

which is the sum of two independent Poisson random variables; computing their Laplace transforms leads us to

$$\begin{aligned} \mathbb{P}[W_t^2] &= \exp\left(\lambda(2\beta)\nu|V_t \cap \tilde{V}_t| + \lambda\nu|V_t \Delta \tilde{V}_t| - 2\lambda\nu r^d t\right) \\ &= \exp\left(\lambda^2 \nu |V_t \cap \tilde{V}_t|\right) \\ &\underset{t\to\infty}{\longrightarrow} \exp\left(\lambda^2 \nu |V_\infty \cap \tilde{V}_\infty|\right), \end{aligned} \tag{3.24}$$

where the second equality is obtained using $|V_t| = |\tilde{V}_t| = tr^d$ and $\lambda(\beta)^2 = \lambda(2\beta) - 2\lambda(\beta)$, while the limit is justified by monotone convergence. Now,

$$\begin{aligned} |V_\infty \cap \tilde{V}_\infty| &= \int_0^\infty |U(B_t) \cap U(\tilde{B}_t)| \mathrm{d}t \\ &= \int_0^\infty |U(0) \cap U(\tilde{B}_t - B_t)| \mathrm{d}t \\ &\overset{\text{law}}{=} \int_0^\infty |U(0) \cap U(B_{2t})| \mathrm{d}t, \end{aligned}$$

since $(B_t + \tilde{B}_t)_{t\geq 0} \overset{\text{law}}{=} (B_{2t})_{t\geq 0}$. Hence, using monotone convergence and (3.24), we get that

$$\sup_{t\in\mathbb{R}} \mathbb{P}[W_t^2] = \lim_{t\to\infty} \mathbb{P}[W_t^2] = P\left[\exp\left(\frac{\lambda(\beta)^2}{2}\nu \int_0^\infty |U(0) \cap U(B_t)| \mathrm{d}t\right)\right], \tag{3.25}$$

where the first equality is a consequence of $(W_t^2)_{t\geq 0}$ being a submartingale. Equation (3.25) shows that $\sup_t \mathbb{P}[W_t^2]$ is an increasing function of $|\beta|$, which proves part (i) of the theorem.

To prove the second part, we will bound the right hand side of (3.25). First observe that

$$|U(0) \cap U(B_t)| \leq |U(0)|\, \mathbf{1}_{|B_t| \leq 2\gamma_d^{-1/d} r} \overset{\text{law}}{=} r^d \mathbf{1}\left\{\left|B\left(\frac{t\gamma_d^{2/d}}{4r^2}\right)\right| \leq 1\right\},$$

so that, by a change of variables,

$$\sup_{t\in\mathbb{R}} \mathbb{P}[W_t^2] \leq P\left[\exp\left(2\lambda(\beta)^2 \nu r^{2+d} \gamma_d^{-2/d} \int_0^\infty \mathbf{1}_{|B_t|\leq 1}\, \mathrm{d}t\right)\right]. \tag{3.26}$$

When $d \geq 3$, the Brownian motion is transient and has the following property:

$$\alpha(d) = \sup_{x\in\mathbb{R}^d} P_x \left[\int_0^\infty \mathbf{1}_{|B_t|\leq 1}\, \mathrm{d}t\right] < \infty.$$

By Khas'minskii's lemma [152, p. 8, Lemma 2.1], this implies that

$$P\left[\exp\left(u\int_0^\infty \mathbf{1}_{|B_t|\leq 1}\, \mathrm{d}t\right)\right] < (1 - u\alpha)^{-1},$$

whenever $u\alpha < 1$. Looking back at (3.26), this condition finally leads to (3.22).

Part (iii) is obtained by observing that $\lambda(\beta) \to 0$ as $\beta \to 0$, so that condition (3.22) is fulfilled for small enough β.

To prove (iv), just note that the right-hand side of (3.22) tends to νr^{d+2} as $\beta \to -\infty$. ■

3.4 Relations between the different critical temperatures

The critical values $\beta_2^{\pm}, \bar{\beta}_c^{\pm}$ and $\beta_c^{\pm}$ defined in Theorems 2.5.1, 3.1.3 and 3.3.1 are ordered.

Proposition 3.4.1. *The following properties hold:*

(i) For all $d \geq 1$,

$$\beta_2^+ \leq \bar{\beta}_c^+ \leq \beta_c^+ < \infty \quad \textit{and} \quad \beta_c^- \leq \bar{\beta}_c^- \leq \beta_2^-. \tag{3.27}$$

(ii) When $d \geq 3$*, these parameters are all non-zero.*

(iii) When $d \geq 3$ *and* νr^{d+2} *is small enough,* $\beta_c^- = \bar{\beta}_c^- = \beta_2^- = -\infty$.

Remark 3.4.2. *Point (iii) tells us that if the intensity of the Poisson point process or the radius are sufficiently small, the polymer will not really be impacted by the environment.*

Remark 3.4.3. *For* $d = 1, 2$*, it holds that that* $\beta_2^{\pm} = \bar{\beta}_c^{\pm} = \beta_c^{\pm} = 0$*. The reader is referred to the proofs in [10, 13, 49, 113] for other similar models, and can be convinced that the arguments go through in our case [12].*

Remark 3.4.4. *A long-standing conjecture is that* $\beta_c^{\pm} = \bar{\beta}_c^{\pm}$*, i.e., that the weak/strong disorder transition coincide with the high/low temperature one.*

Proof. We first show that $\beta_2^+ \leq \bar{\beta}_c^+$. Suppose $\beta_2^+ > 0$ and let $0 \leq \beta < \beta_2^+$. By definition, we have

$$\sup_{t \geq 0} \mathbb{P}[W_t^2] < \infty,$$

so that $(W_t)_{t\geq 0}$ is a martingale bounded in L^2. Thus, $(W_t)_{t\geq 0}$ converges in L^2 norm, which implies L^1 convergence.

Since $\mathbb{P}[W_t] = 1$ for all t, we get that $\mathbb{P}[W_\infty] = 1$, so (3.3) must hold and hence $\beta \leq \bar{\beta}_c^+$. As it is true for all $\beta < \beta_2^+$, the desired inequality follows directly.

We now turn to the proof of $\bar{\beta}_c^+ \leq \beta_c^+$. Again, suppose that $\bar{\beta}_c^+ > 0$ and let $0 \leq \beta < \bar{\beta}_c^+$. We have $W_\infty > 0$ almost surely, so $\ln W_t \to \ln W_\infty$ almost surely, that is to say

$$\ln Z_t - \lambda \nu r^d t \underset{t\to\infty}{\longrightarrow} \ln W_\infty.$$

Dividing by t, we get that

$$\frac{1}{t} \ln Z_t - \lambda \nu r^d \underset{t\to\infty}{\longrightarrow} 0,$$

thus $p(\beta, \nu, r) = \lambda \nu r^d$, i.e. $\beta \leq \beta_c$. The same argument goes for the negative critical values. This ends the proof of (i).

Points (ii) and (iii) are repeated from Theorem 3.3.1. ∎

Chapter 4

Directional free energy

In this section we make use of the Brownian nature of the polymer and the invariance of the medium under shear transformations, which induces a lot of symmetries in the model, culminating with quadratic shape function and the equality (4.11).

4.1 Point-to-point partition function

With $P_{s,x}^{t,y}$ the Brownian bridge in $\mathbb{R}^d$ joining (s,x) to (t,y), we introduce the point-to-point partition (P2P) function

$$Z_t(\omega,\beta;x) = P_{0,0}^{t,x}\left[\exp\{\beta\omega(V_t)\}\right] \ , \tag{4.1}$$

from which we can recover the point-to-level (P2L) partition function

$$Z_t(\omega,\beta) = \int_{\mathbb{R}^d} Z_t(\omega,\beta;x)\rho(t,x)dx \tag{4.2}$$

by conditioning on B_t. We use the standard notation $\rho(t,x) = (2\pi t)^{-d/2}\exp -|x|^2/2t$ for the heat kernel in $\mathbb{R}^d$. For $\xi \in \mathbb{R}^d$, define the shear transformation $\tau_\xi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+ \times \mathbb{R}^d$ by

$$\tau_\xi(s,x) = (s, x+s\xi) \ ,$$

which is one to one with $\tau_\xi^{-1} = \tau_{-\xi}$. Since τ_ξ acts on the graph of functions $f : \mathbb{R}_+ \to \mathbb{R}^d$, we denote its action on functions by

$$\tau_\xi f : s \mapsto f(s) + s\xi \ , \tag{4.3}$$

so that $\tau_\xi(s, f(s)) = (s, \tau_\xi(f)(s)$. The pushed forward of a point measure by τ_ξ is defined by

$$\tau_\xi \circ \big(\sum_i \delta_{(t_i,x_i)}\big) = \sum_i \delta_{(t_i,x_i+\xi t_i)} = \sum_i \delta_{\tau_\xi(t_i,x_i)} \ , \tag{4.4}$$

where it is clear that $\tau_\xi \circ \omega$ is again a Poisson point process with intensity $\nu dsdx$, i.e., $\tau_\xi \circ \omega = \omega$ in law. With B the canonical process, under the measure $P_{0,0}^{t,t\xi}$ the process $W = \tau_{-\xi}(B)$ is a Brownian bridge $(0,0) \to (0,0)$. Therefore, for all ω,

$$\begin{aligned} Z_t(\omega,\beta;t\xi) &= P_{0,0}^{t,t\xi}\left[\exp\{\beta\omega(V_t(B))\}\right] \\ &= P_{0,0}^{t,t\xi}\left[\exp\{\beta\omega(V_t(\tau_\xi(W)))\}\right] \\ &= P_{0,0}^{0,0}\left[\exp\{\beta\omega(\tau_\xi(V_t(B)))\}\right] \\ &= P_{0,0}^{0,0}\left[\exp\{\beta(\tau_\xi \circ \omega)(V_t(B))\}\right] \\ &= Z_t(\tau_\xi \circ \omega, \beta; 0) \ . \end{aligned} \tag{4.5}$$

This implies that $Z_t(\omega,\beta;x)$ has same law as $Z_t(\omega,\beta;0)$. We can prove that the *directional free energy*, in the direction $\xi \in \mathbb{R}^d$,

$$p^{\mathrm{dir}}(\beta,\nu;\xi) = \lim_{t\to\infty} \frac{1}{t} \ln Z_t(\omega,\beta;t\xi)$$

exists a.s. and in L^p-norm for all $p \geq 1$, and is equal to $\lim_{t\to\infty} t^{-1}\mathbb{P}[\ln Z_t(\omega,\beta;t\xi)]$. The route is quite different from Theorem 2.3.1, it follows the lines of chapter 5 in [152] for the undirected case, that we briefly sketch now: define the tube around the Brownian path between times $s \leq t$, $V_{s,t} = V_{s,t}(B) = \cup_{u\in[s,t]}\{u\} \times U(B_u)$, and also

$$\mathfrak{e}_{s,t}(x,y;\omega) := P\left[e^{\beta\omega(V_{s,t})}\mathbf{1}_{B_t\in U(y)}\big|B_s = x\right] = P_x\left[e^{\beta\theta_{s,x}\circ\omega(V_{t-s})}\mathbf{1}_{B_{t-s}\in U(y)}\right],$$

which is the integral of the P2P partition function over a ball of radius r. Then, the quantity

$$a_{s,t}(x,y;\omega) = \inf_{z\in U(x)} \ln \mathfrak{e}_{s,t}(z,y;\omega)$$

is superadditive, in particular we have

$$a_{0,s+t}(0,(s+t)\xi;\omega) \geq a_{0,s}(0,s\xi;\omega) + a_{s,s+t}(s\xi,(s+t)\xi;\omega),$$

and the subadditive ergodic theorem shows the existence of the limit $t^{-1}a_{0,t}(0,t\xi;\omega)$ as $t\to\infty$, say $p^{\mathrm{dir}}(\beta,\nu;\xi)$, a.s. and in L^1. (L^p-convergence will follow from the concentration inequality, which remains unchanged). Then, one can show that the infimum over $z \in U(x)$ in the definition of $a_{0,t}(0,y;\omega)$ can be dropped in the limit $t\to\infty$, as well as the integration in the definition of $e_{0,t}(0,y;\omega)$ on the fixed domain $U(y)$. Proving these claims requires some work with quite a few technical estimates; we do not write the details here, the reader is referred to section 5.1 in [152].

By (4.5), $Z_t(\omega,\beta;x) \overset{\text{law}}{=} Z_t(\omega,\beta;x')$ and thus, for all $\xi,\xi' \in \mathbb{R}^d$,

$$p^{\mathrm{dir}}(\beta,\nu;\xi) = p^{\mathrm{dir}}(\beta,\nu;\xi')\ . \tag{4.6}$$

4.2 Free energy does not depend on direction

Let P^h be the Wiener measure with drift $h\in\mathbb{R}^d$, i.e., the probability measure on the path space $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ such that for all t,

$$\left(\frac{dP^h}{dP}\right)_{|\mathcal{F}_t} = \exp\left\{h\cdot B_t - \frac{t|h|^2}{2}\right\}\ .$$

By Cameron-Martin formula, under P^h, the canonical process B is a Brownian motion with drift h, i.e., $W = \tau_{-h}(B)$ is a standard Brownian motion under P^h and has the same law as B under P. Thus, the partition function for the drifted Brownian polymer

$$\begin{aligned} Z_t^h(\omega,\beta) &\overset{\text{def.}}{=} P^h\left[\exp\{\beta\omega(V_t(B))\}\right] \\ &= P^h\left[\exp\{\beta\omega(V_t(\tau_h(W)))\}\right] \\ &= P\left[\exp\{\beta\omega(V_t(\tau_h(B)))\}\right] \\ &= Z_t(\tau_{-h}\circ\omega,\beta) \end{aligned} \tag{4.7}$$

as in (4.5). It would be routine, and this time exactly as in the proof of Theorem 2.3.1, to show the existence of free energy for the drifted Brownian polymer

$$p^h(\beta,\nu) := \lim_{t\to\infty} t^{-1}\ln Z_t^h(\omega,\beta)$$

a.s. and in L^p; But in fact, this is even unnecessary since (4.7) yields the existence of the limit. Now, the previous display together with invariance of ω under shear shifts imply that

$$p^h(\beta,\nu) = p(\beta,\nu)\ . \tag{4.8}$$

On the other hand, similar to (4.2) we have

$$Z_t^h(\omega,\beta) = \int_{\mathbb{R}^d} e^{h\cdot x - t|h|^2/2} Z_t(\omega,\beta;x)\rho(t,x)dx\ . \tag{4.9}$$

By Laplace method, it follows from standard work that

$$\begin{aligned} p^h(\beta,\nu) &= \sup_{\xi\in\mathbb{R}^d} \left\{ h\cdot\xi - |h|^2/2 - |\xi|^2/2 + p^{\mathrm{dir}}(\beta,\nu;\xi) \right\} \\ &\stackrel{(4.6)}{=} p^{\mathrm{dir}}(\beta,\nu;0) - |h|^2/2 + \sup_{\xi\in\mathbb{R}^d} \left\{ h\cdot\xi - |\xi|^2/2 \right\} \\ &= p^{\mathrm{dir}}(\beta,\nu;0). \end{aligned} \tag{4.10}$$

Finally, all the above notions of free energy coincide:

$$\boxed{p(\beta,\nu) = p^{\mathrm{dir}}(\beta,\nu;\xi) = p^h(\beta,\nu)\ .} \tag{4.11}$$

Conclusion: The critical values $\beta_c^\pm$ for equality of quenched and annealed free energy are the same for all free energies (P2P in all directions, P2L with all drifts).

4.3 Local limit theorem

In the L^2-region, a local limit theorem was discovered by Sinaï [149] in the discrete case, and extended to our continuous model by Vargas [159].

Define the time-space reversal operator on the environment $\theta^{\leftarrow}_{t,x}$, acting on point measures as

$$\theta^{\leftarrow}_{t,x}\Big(\sum_i \delta_{(t_i,x_i)}\Big) = \sum_i \delta_{(t-t_i,x_i-x)} \tag{4.12}$$

Theorem 4.3.1 (Local limit theorem; [159], Th. 2.9)**.** *Assume $\beta \in (-\beta_2^-, \beta_2^+)$. Then, for any constant $A > 0$ and any positive function ℓ_t tending to ∞ with $\ell_t = o(t^a)$ for some $a < 1/2$,*

$$Z_t(\omega,\beta;x) = W_\infty \times W_\infty \circ \theta^{\leftarrow}_{t,x} + \varepsilon_t(x)\ , \tag{4.13}$$

and

$$Z_t(\omega,\beta;x) = W_{\ell_t} \times W_{\ell_t} \circ \theta^{\leftarrow}_{t,x} + \delta_t(x)\ ,$$

with error terms vanishing as $t\to\infty$,

$$\sup_{|x|\le A\sqrt{t}} \mathbb{P}[|\varepsilon_t(x)|] \to 0\ , \qquad \sup_{|x|\le A\sqrt{t}} \mathbb{P}[|\delta_t(x)|^2] \to 0\ .$$

Intuitively, the local limit theorem states that, the polymer ending at x at time t only "feels" the environment at times s close to 0 and locations close to 0 or close t at "large" times s close to t and locations close to x. (See Figure 4.1.) In between, it behaves like a Brownian bridge.

Conjecture 4.3.2. *We formulate two conjectures:*

- *It is natural to define another pair of critical inverse temperature, analogue to the weak/strong disorder transition:*

$$\begin{aligned} \overline{\beta}_c^{+,dir} &= \sup\{\beta \ge 0 : \lim_t \mathbb{P}[(W_t^{dir,\xi})^{1/2}] > 0\}, \\ \overline{\beta}_c^{-,dir} &= \inf\{\beta \le 0 : \lim_t \mathbb{P}[(W_t^{dir,\xi})^{1/2}] > 0\}. \end{aligned} \tag{4.14}$$

Using Jensen inequality in (4.2)*, it is not difficult to get $\overline{\beta}_c^{+,dir} \in [\beta_2^+, \beta_c^+]$, $\overline{\beta}_c^{-,dir} \in [\beta_c^-, \beta_2^-]$. We conjecture that the equality holds, i.e.,*

$$\overline{\beta}_c^{\pm} = \overline{\beta}_c^{\pm,dir}.$$

- *A long standing conjecture is that the local limit theorem* (4.13) *holds the way all through the weak disorder region. Note that the latter conjecture would imply that the former one holds.*

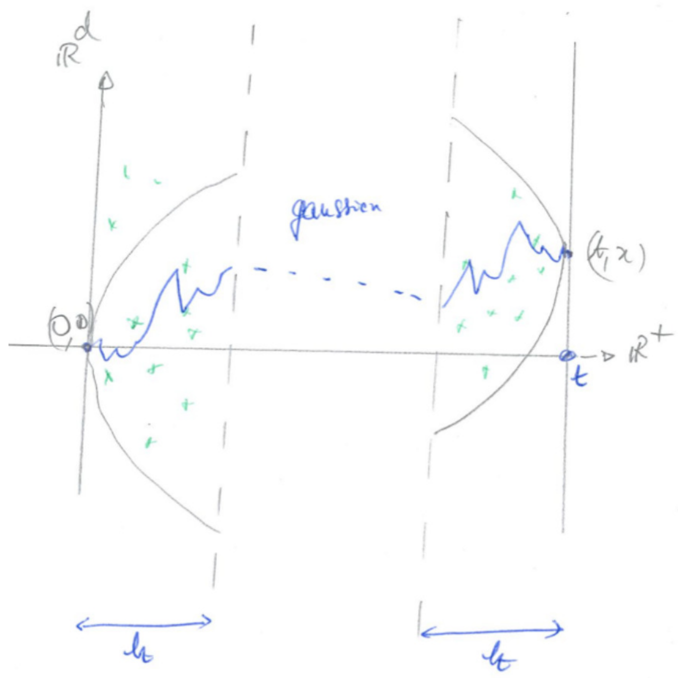

Figure 4.1: The local limit theorem. The P2P partition function only feels details of the environment close to the space-time endpoints $(0,0)$ and (t,x). In between, it behaves like the Gaussian propagator.

Chapter 5

The replica overlap and localization

The utlimate goal of this section is to show that the thermodynamic phase transition of section 2 is a localization transition for the polymer.

To do that, we need fine tools from stochastic analysis. The starting point is Doob-Meyer decomposition, a natural and strong tool to study stochastic processes in which the process is written as the sum of a local martingale (the impredictable part) and a bounded variation predictable process (the tamed part). We start by recalling some martingale properties of the Poisson environment that will prove usefull throughout the following chapters.

5.1 The compensated Poisson measure and some associated martingales

Given the Poisson point process ω, we introduce the compensated measure $\bar{\omega}$,

$$\bar{\omega}(\mathrm{d}s\mathrm{d}x) = \omega(\mathrm{d}s\mathrm{d}x) - \nu\,\mathrm{d}s\mathrm{d}x, \tag{5.1}$$

and we abreviate its restriction to $(0,t] \times \mathbb{R}^d$ by $\bar{\omega}_t$. By definition, for all function $f(s,x,\omega)$ that verifies

$$\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}[|f(s,x,\cdot)|]\mathrm{d}s\mathrm{d}x < \infty, \tag{5.2}$$

the compensated integral of f is given by

$$\int f(s,x,\omega)\bar{\omega}_t(\mathrm{d}s\mathrm{d}x) = \int f(s,x,\omega)\omega_t(\mathrm{d}s\mathrm{d}x) - \int_{[0,t]\times\mathbb{R}^d} f(s,x,\omega)\nu\,\mathrm{d}s\mathrm{d}x. \tag{5.3}$$

Furthermore, we say that a function $f(t,x,\omega)$ is *predictable*, if it belongs to the sigma-field generated by all the functions $g(t,x,\omega)$ that satisfy the following properties:

(i) for all $t > 0$, $(x,\omega) \to g(t,x,\omega)$ is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{G}_t$-measurable;

(ii) for all (x,ω), $t \to g(t,x,\omega)$ is left continuous.

Then, if the function $f(t,x,\omega)$ is predictable, provided that (5.2) holds and that

$$\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}[f(s,x,\cdot)^2]\mathrm{d}s\mathrm{d}x < \infty,$$

the process $t \to \int f\,\mathrm{d}\bar{\omega}_t$ is a square-integrable martingale associated to $(\mathcal{G}_t)_{t\geq 0}$, of previsible bracket [99, Section II.3.]

$$\left\langle \int f\mathrm{d}\bar{\omega} \right\rangle_t = \int_{[0,t]\times\mathbb{R}^d} f^2\,\nu\,\mathrm{d}s\mathrm{d}x. \tag{5.4}$$

The previsible bracket has the property that $(\int f\,\mathrm{d}\bar{\omega}_t)^2 - \langle \int f\,\mathrm{d}\bar{\omega}\rangle_t$ is a martingale. In particular,

$$\mathrm{Var}\left(\int f(s,x,\cdot)\omega_t(\mathrm{d}s\mathrm{d}x)\right) = \int_{[0,t]\times\mathbb{R}^d} \mathbb{P}[f(s,x,\cdot)^2]\nu\,\mathrm{d}s\mathrm{d}x. \tag{5.5}$$

5.2 The Doob-Meyer decomposition of $\ln Z_t$

Since W_t is a martingale and ln is concave, $-\ln(W_t)$ is a submartingale, for which we want to get a Doob-Meyer decomposition [140, Ch.VI].

In the following, we will use the notation $\Delta_s X := X_s - X_{s-}$ for any càdlàg process X. Let $\zeta_t := e^{\beta\omega(V_t)}$. As $\omega(V_t)$ can be expressed as a sum over ω_t, we get by telescopic sum that $\zeta_t = 1 + \int \omega_t(\mathrm{d}s\mathrm{d}x)\Delta_s\zeta$, an integral over $\mathbb{R}_+ \times \mathbb{R}^d$. Averaging over the Brownian path, we also get that Z_t can be expressed as a sum over the process. Therefore, by telescopic sum,

$$\ln Z_t = \int \omega_t(\mathrm{d}s\mathrm{d}x)\Delta_s \ln Z.$$

Now, let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ be any point of the point process ω. As (t,x) is almost surely the only point of ω at time t, we can write that

$$\begin{aligned} Z_t &= P\left[e^{\beta} e^{\beta\omega(V_{t-})}\chi_{t,x}(B)\right] + P\left[e^{\beta\omega(V_{t-})}\left(1-\chi_{t,x}(B)\right)\right] \\ &= P\left[(e^{\beta}-1)e^{\beta\omega(V_{t-})}\chi_{t,x}(B)\right] + P\left[e^{\beta\omega(V_{t-})}\right] \\ &= \left(\lambda(\beta)P_{t-}^{\beta,\omega}[\chi_{t,x}] + 1\right) Z_{t-}\ . \end{aligned}$$

Hence,

$$\ln Z_t = \int \omega_t(\mathrm{d}s\mathrm{d}x) \ln\left(1 + \lambda P_{s-}^{\beta,\omega}[\chi_{s,x}]\right). \tag{5.6}$$

Let g be the function $g(u) = u - \ln(1+u)$, which is positive on $(-1,\infty)$. Then, recalling that $\int_{\mathbb{R}^d} \chi_{s,x} dx = r^d$, we obtain Doob's decomposition

$$-\ln W_t = \lambda\nu r^d t - \ln Z_t = M_t + A_t, \tag{5.7}$$

where the martingale M and the increasing process A are given by

$$M_t = -\int \bar{\omega}_t(\mathrm{d}s\mathrm{d}x) \ln\left(1 + \lambda P_{s-}^{\beta,\omega}[\chi_{s,x}]\right), \tag{5.8}$$

$$\begin{aligned} A_t &= \lambda\nu r^d t - \int_{[0,t]\times\mathbb{R}^d} \ln\left(1 + \lambda P_{s-}^{\beta,\omega}[\chi_{s,x}]\right) \nu\, \mathrm{d}s\mathrm{d}x \\ &= \int_{[0,t]\times\mathbb{R}^d} g\left(\lambda P_{s-}^{\beta,\omega}[\chi_{s,x}]\right) \nu\, \mathrm{d}s\mathrm{d}x. \end{aligned} \tag{5.9}$$

Moreover, the martingale M is square-integrable, and its bracket is given by

$$\langle M \rangle_t = \int_{[0,t]\times\mathbb{R}^d} \left[\ln\left(1 + \lambda P_{s-}^{\beta,\omega}[\chi_{s,x}]\right)\right]^2 \nu\, \mathrm{d}s\mathrm{d}x.$$

5.3 The replica overlap and quenched overlaps

Things being clear from the context, we will use the same notation $|A|$ to denote the Lebesgue measure of Borel subset A of $\mathbb{R}^d$ or $\mathbb{R}_+ \times \mathbb{R}^d$.

Definition 5.3.1. *For any two paths B and $\tilde{B}$, we define the* ***replica overlap*** *$R_t(B,\tilde{B})$ as the mean volume overlap of the two tubes around B and $\tilde{B}$ in time $[0,t]$:*

$$R_t(B,\tilde{B}) = \frac{1}{tr^d}|V_t(B) \cap V_t(\tilde{B})|. \tag{5.10}$$

In a similar way, we define the ***quenched overlaps*** *I_t and J_t as*[1]

$$I_t = \frac{1}{r^d} P_t^{\beta,\omega\otimes 2}\left[|U(B_t) \cap U(\tilde{B}_t)|\right], \tag{5.11}$$

$$J_t = \frac{1}{tr^d} P_t^{\beta,\omega\otimes 2}\left[|V_t(B) \cap V_t(\tilde{B})|\right], \tag{5.12}$$

[1] The product measure $P_t^{\beta,\omega\otimes 2} = P_t^{\beta,\omega} \otimes P_t^{\beta,\omega}$ makes the 2 replicas $B, \tilde{B}$ independent polymer paths sharing the same environment ω.

The variable I_t stands for the expected volume of overlap around the endpoints of two independent polymer paths, while J_t is the expected volume of overlap during the time interval $[0,t]$. Note that both $\int_0^t I_s \mathrm{d}s$ and tJ_t represent an expected volume of overlap in time $[0,t]$, but they will emerge from different circumstances. Similar to $\chi_{s,x} = \mathbf{1}\{x \in U(B_s)\}$ from definition (2.3), we write for short

$$\tilde{\chi}_{s,x} = \mathbf{1}\{x \in U(\tilde{B}_s)\}.$$

Writing

$$|U(B_t) \cap U(\tilde{B}_t)| = \int_{\mathbb{R}^d} \chi_{t,x} \tilde{\chi}_{t,x} dx,$$

we derive two useful formulas:

$$I_t = \frac{1}{r^d} \int_{\mathbb{R}^d} P_t^{\beta,\omega}(\chi_{t,x})^2 dx \ , \qquad J_t = \frac{1}{r^d} \int_{\mathbb{R}^d} dx \frac{1}{t} \int_0^t P_t^{\beta,\omega}(\chi_{s,x})^2 ds \ . \tag{5.13}$$

For better comparisons, we have normalized all quantities in (5.10), (5.11) and (5.12) in such a way that

$$0 \le R_t, I_t, J_t \le 1 \ ,$$

so that we can – and we will – view each of them as a *localization index*:

- R_t close to 1 means that the two fixed paths $B, \tilde{B}$ are close on the interval $[0,t]$;
- I_t close to 1 means that the endpoints of two independent samples of the polymer measure are typically close one from the other;
- J_t close to 1 means that the paths of two independent polymers are close all along the time interval.

The second case corresponds to *endpoint localization* whereas the third one is *path localization.* Mathematically, the quantity I_t appears via Itô's calculus (stochastic differentiation) and J_t via Malliavin calculus (integration by parts). On the contrary, small values of these indices correspond to absence of localization: it means that the polymer spreads more or less uniformly in space without particular preference.

Remark 5.3.2. *When $\beta = 0$, the Gibbs measure $P_t^{\beta,\omega}$ reduces to Wiener measure P, so that $J_t = \int_0^t I_s \mathrm{d}s = tP^{\otimes 2}[R_t(B,\tilde{B})]$, and*

$$P^{\otimes 2}[R_t(B,\tilde{B})] = \frac{1}{t}\int_0^t P^{\otimes 2}\big[|U(B_s) \cap U(\tilde{B}_s)|\big] \mathrm{d}s \le \frac{r^d}{t}\int_0^t P\left(B_{2s} \in U(0)\right) \mathrm{d}s \underset{t\to 0}{\longrightarrow} 0.$$

Thus,

$$\lim_{t\to\infty} J_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t I_s \mathrm{d}s = 0,$$

This indicates that, in absence of interaction with the inhomogeneous medium, there is no localization of the Brownian path.

We now come to the core of the section: localization results.

5.4 Endpoint localization

A naive prediction is that, for small β, the polymer is a small perturbation of Brownian motion for small β, with a comparable behavior, whereas for large β localization takes place and the limits are nonzero. A first theorem shows that this is the case for the quantity I_t:

Theorem 5.4.1. *The following equivalence holds for $\beta \neq 0$:*

$$W_\infty = 0 \iff \int_0^\infty I_s \,\mathrm{d}s = \infty, \quad \mathbb{P}\text{-}a.s. \tag{5.14}$$

In particular, the above integral is a.s. finite for $\beta \in (\bar\beta_c^-, \bar\beta_c^+)$, and a.s. infinite for $\beta < \bar\beta_c^-$ or $\beta > \bar\beta_c^+$. Moreover, we have:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I_s \,\mathrm{d}s = 0 \quad \text{if } \beta \in [\beta_c^-, \beta_c^+] \cap \mathbb{R}, \tag{5.15}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I_s \,\mathrm{d}s > 0 \quad \text{if } \beta \in \mathbb{R} \setminus [\beta_c^-, \beta_c^+]. \tag{5.16}$$

Remark 5.4.2. *Note that $\int_0^\infty I_s \mathrm{d}s$ is a.s. finite or a.s. infinite, by the dichotomy between strong and weak disorder. Similarly, strict positivity of $\liminf_{t\to\infty} t^{-1}\int_0^t I_s ds$ is equivalent to low temperature region $\psi(\beta,\nu) > 0$ in* (2.20). *In fact, we will prove in* (5.19)–(5.23) *that under strong disorder, there exist $c_1, c_2 \in (0,\infty)$, such that*

$$c_1 \int_0^t I_s \,\mathrm{d}s \le -\ln W_t \le c_2 \int_0^t I_s \,\mathrm{d}s, \quad \textit{for large } t, \quad \mathbb{P}-a.s. \tag{5.17}$$

Proof. (Theorem 5.4.1) First observe that one can easily derive (5.15) and (5.16) from (5.14) and (5.17).

To show (5.17), we will relate $\int_0^t I_s \mathrm{d}s$ to the variables M_t and A_t of the Doob decomposition (5.7) as follows. From (5.13), we have

$$\int_{[0,t]\times\mathbb{R}^d} \left(P_s^{\beta,\omega}[\chi_{s,x}]\right)^2 \mathrm{d}s\mathrm{d}x = r^d \int_0^t I_s \,\mathrm{d}s. \tag{5.18}$$

Then, looking at the behavior of $g(u)$ in (5.9) and $\ln(1+u)$ around 0, it is clear that there are two constants $c_1, c_2 > 0$, depending only on β and ν, such that

$$c_1 u^2 \le \nu g(u) \le c_2 u^2, \qquad \nu \ln(1+u)^2 \le c_2 u^2,$$

for all u in $[0,\lambda]$ when $\beta \ge 0$, (resp. $[\lambda, 0]$ when $\beta \le 0$). Together with (5.18), this implies that

$$c_1 \int_0^t I_s \mathrm{d}s \le A_t \le c_2 \int_0^t I_s \mathrm{d}s, \tag{5.19}$$

$$\langle M\rangle_t \le c_2 \int_0^t I_s \mathrm{d}s. \tag{5.20}$$

We then recall two results about martingales. (These facts for the discrete martingales are standard (e.g. [70, p. 255, (4.9),(4.10)]. It is not difficult to adapt the proof for the discrete setting to our case.) Let $\varepsilon > 0$, then

$$\langle M\rangle_\infty < \infty \implies (M_t)_{t\ge 0} \text{ converges a.s.}, \tag{5.21}$$

$$\langle M\rangle_\infty = \infty \implies \lim_{t\to\infty} M_t/\langle M\rangle_t^{\frac{1+\varepsilon}{2}} = 0 \quad \text{a.s..} \tag{5.22}$$

Consequently, we get that $\mathbb{P}$-almost surely,

$$\begin{aligned} \int_0^\infty I_s \mathrm{d}s < \infty &\iff A_\infty < \infty, \langle M\rangle_\infty < \infty \\ &\implies A_\infty < \infty, \lim_{t\to\infty} M_t \text{ exists and is finite} \\ &\implies W_\infty > 0, \end{aligned}$$

where the last implication comes from the Doob decomposition (5.7). By contraposition, this proves the first implication of (5.14). The reverse implication will follow from the arguments below.

The next step is to show (5.17), so we now suppose that we are in the strong disorder setting. Using (5.19), we see that it is enough to show that

$$\lim_{t\to\infty} \frac{-\ln W_t}{A_t} = 1, \quad \mathbb{P}\text{-a.s.} \tag{5.23}$$

or equivalently by the Doob decomposition, that

$$\lim_{t\to\infty} \frac{M_t}{A_t} = 0. \tag{5.24}$$

As we just proved, the strong disorder implies that $\int_0^\infty I_s \mathrm{d}s = \infty$, which in turn implies that $A_\infty = \infty$ by (5.19). Thus, in the case that $\langle M\rangle_\infty < \infty$, the martingale M_t converges so that the condition (5.24) directly holds. When $\langle M\rangle_\infty = \infty$, this is still true as

$$\frac{M_t}{A_t} = \frac{M_t}{\langle M\rangle_t}\frac{\langle M\rangle_t}{A_t} \underset{t\to\infty}{\longrightarrow} 0,$$

by (5.19, 5.20, 5.22).

Finally, what is left to demonstrate is that $\mathbb{P}$-almost surely $W_\infty = 0$ on the event $\int_0^\infty I_s \mathrm{d}s = \infty$. In fact, we showed that this event implies both the limit (5.23) and $A_\infty = \infty$, so that $W_t \to 0$. ■

Endpoint localization: As in the discrete case, we can interpret the results from the present subsection, in terms of localization for the path. Indeed, it is proven in Sect. 8 of [49], that for some constant c_1,

$$c_1 \sup_{y\in\mathbb{R}^d} P_t^{\beta,\omega}[B_s \in U(y)]^2 \le P_t^{\beta,\omega^{\otimes 2}}\left[\left|U(B_s)\bigcap U(\tilde{B}_s)\right|\right] \le \sup_{y\in\mathbb{R}^d} P_t^{\beta,\omega}[B_s \in U(y)] \tag{5.25}$$

(in fact, the inequality on the right is trivial, and the one on the left is the combination of (5.42) and (5.43)).

The maximum appearing in the above bounds should be viewed as the probability of the favorite location for B_s, under the polymer measure $P_t^{\beta,\omega}$; for $s = t$, the supremum is called the probability of the *favorite endpoint*, and the maximizing y is the *location of the favorite endpoint*. Both Theorem 5.4.1 and Theorem 5.5.2 are precise statements that the polymer localizes in the strong disorder regime in a few specific corridors of width $\mathcal{O}(1)$, but spreads out in a diffuse way in the weak disorder regime. If $\psi(\beta,\nu) > 0$, the Cesaro-limit of probability of the favorite endpoint is strictly positive.

5.5 Favorite path and path localization

Recall that the excess free energy ψ from (2.15) is the difference of a smooth function and a convex function. Hence its right-derivative, resp. left-derivative,

$$\left(\frac{\partial\psi}{\partial\beta}\right)_+(\beta,\nu) = \lim_{\beta'\searrow\beta} \frac{\psi(\beta',\nu)-\psi(\beta,\nu)}{\beta'-\beta}, \qquad \left(\frac{\partial\psi}{\partial\beta}\right)_-(\beta,\nu) = \lim_{\beta'\nearrow\beta} \frac{\psi(\beta',\nu)-\psi(\beta,\nu)}{\beta'-\beta},$$

exists for all β and all ν, and satisfy $\left(\frac{\partial\psi}{\partial\beta}\right)_+(\beta,\nu) \le \left(\frac{\partial\psi}{\partial\beta}\right)_-(\beta,\nu)$. For the same reason as above, $\psi(\cdot,\nu)$ is differentiable except on a set which is at most countable, and we can write

$$\left(\frac{\partial\psi}{\partial\beta}\right)_+(\beta,\nu) = \lim_{\beta'\ge\beta} \frac{\partial\psi}{\partial\beta}(\beta',\nu),$$

where the limit is over differentiability points β' tending to β by larger values. A similar statement holds for the left-derivative. For further use, we note that for all fixed ν, $\psi(\cdot,\nu)$ is absolutely continuous again for the same reason as above.

Now, we turn to the properties of the replica overlap J_t. The key fact is the following proposition:

Proposition 5.5.1. *There exist two constants $c_1, c_2 \in (0,\infty)$, depending only on β and ν, such that*

$$\forall\beta > 0, \quad c_1\left(\frac{\partial\psi}{\partial\beta}\right)_+ \le \liminf_{t\to\infty} \mathbb{P}[J_t] \le \limsup_{t\to\infty} \mathbb{P}[J_t] \le c_2\left(\frac{\partial\psi}{\partial\beta}\right)_-, \tag{5.26}$$

$$\forall\beta < 0, \quad -c_1\left(\frac{\partial\psi}{\partial\beta}\right)_- \le \liminf_{t\to\infty} \mathbb{P}[J_t] \le \limsup_{t\to\infty} \mathbb{P}[J_t] \le -c_2\left(\frac{\partial\psi}{\partial\beta}\right)_+. \tag{5.27}$$

Proof. It is not difficult to see from the definition that $tJ_t = \iint_{[0,t]\times\mathbb{R}^d} P_t^{\beta,\omega}[\chi_{s,x}]^2 \mathrm{d}s\mathrm{d}x$. Hence, using equation (2.19) and the fact that $e^{-|\beta|} \leq 1 + \lambda P_t^{\beta,\omega}[\chi_{s,x}] \leq e^{|\beta|}$, we obtain that

$$\lambda\nu e^{\beta-|\beta|} t\mathbb{P}[J_t] \leq t\frac{\partial}{\partial\beta}\psi_t(\beta,\nu) \leq \nu\lambda e^{\beta+|\beta|} t\mathbb{P}[J_t]. \tag{5.28}$$

Moreover, the excess free energy writes $\psi(\beta,\nu) = \nu\lambda(\beta)r^d - p(\beta,\nu)$, where $p(\beta,\nu)$ is a convex function, defined as the limit, for $t \to \infty$, of the convex functions $p_t(\beta,\nu) = \frac{1}{t}\mathbb{P}[\ln Z_t]$. By convexity properties, we know that

$$\left(\frac{\partial p}{\partial\beta}\right)_- \leq \liminf_{t\to\infty} \frac{\partial p_t}{\partial\beta} \leq \limsup_{t\to\infty} \frac{\partial p_t}{\partial\beta} \leq \left(\frac{\partial p}{\partial\beta}\right)_+,$$

which in turns implies that

$$\left(\frac{\partial \psi}{\partial\beta}\right)_+ \leq \liminf_{t\to\infty} \frac{\partial \psi_t}{\partial\beta} \leq \limsup_{t\to\infty} \frac{\partial \psi_t}{\partial\beta} \leq \left(\frac{\partial \psi}{\partial\beta}\right)_-. \tag{5.29}$$

The proposition is then a consequence of (5.28) and these last inequalities. ∎

With this proposition, we can give a characterization of the critical values $\beta_c^\pm$, in terms of the asymptotics of the overlap:

Theorem 5.5.2. *For all $\beta \in [\beta_c^-, \beta_c^+] \cap \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}[J_t] = 0. \tag{5.30}$$

Furthermore,

$$\beta_c^+ = \sup\{\beta' \geq 0 : \forall\beta \in [0,\beta'],\ \lim_{t\to\infty}\mathbb{P}[J_t] = 0\} = \inf\{\beta > 0 : \liminf_{t\to\infty}\mathbb{P}[J_t] > 0\}, \tag{5.31}$$

$$\beta_c^- = \inf\{\beta' \leq 0 : \forall\beta \in [\beta',0],\ \lim_{t\to\infty}\mathbb{P}[J_t] = 0\} = \sup\{\beta < 0 : \liminf_{t\to\infty}\mathbb{P}[J_t] > 0\}. \tag{5.32}$$

Proof. We will focus on the $\beta \geq 0$ case, but the same arguments can be applied to the $\beta \leq 0$ case. Define

$$\delta_c^+ = \sup\{\beta' \geq 0 : \forall\beta \in [0,\beta'],\ \lim_{t\to\infty}\mathbb{P}[J_t] = 0\},$$

so what we need to show in particular is $\beta_c^+ = \delta_c^+$.

To prove the first claim of the theorem, from which $\beta_c^+ \leq \delta_c^+$ follows directly, it is enough, using (5.26), to verify that

$$\forall\beta \leq \beta_c^+,\quad \left(\frac{\partial\psi}{\partial\beta}\right)_-(\beta,\nu) = 0. \tag{5.33}$$

This property is true when $\beta \in [0,\beta_c^+)$, as ψ is constant and set to 0 in this interval. To prove that it extends to β_c^+ if $\beta_c^+ < \infty$, observe that ψ is minimal at β_c^+, so that

$$\left(\frac{\partial\psi}{\partial\beta}\right)_-(\beta_c^+,\nu) \leq 0 \leq \left(\frac{\partial\psi}{\partial\beta}\right)_+(\beta_c^+,\nu).$$

As we saw earlier that $\frac{\partial\psi}{\partial\beta_+} \leq \frac{\partial\psi}{\partial\beta_-}$ always holds, we finally get that $\frac{\partial\psi}{\partial\beta_-}(\beta_c^+) = \frac{\partial\psi}{\partial\beta_+}(\beta_c^+) = 0$.

We now prove $\beta_c^+ \geq \delta_c^+$. Let $\beta > \beta_c^+$ be finite, so that, by definition, $\psi(\beta,\nu) > 0$. As ψ is absolutely continuous with $\psi(0,\nu) = 0$, one can write

$$\psi(\beta,\nu) = \int_0^\beta \frac{\partial\psi}{\partial\beta_+}(\beta',\nu)\mathrm{d}\beta' > 0,$$

which implies that there exists some $\beta' \leq \beta$ such that $\frac{\partial\psi}{\partial\beta+}(\beta',\nu) > 0$. By equation (5.26), we get that $\liminf_{t\to\infty}\frac{1}{t}\mathbb{P}[J_t(\beta)] > 0$, hence $\beta \geq \delta_c^+$. As it is true for all $\beta > \beta_c^+$, we obtain that $\beta_c^+ \geq \delta_c^+$. ∎

The favorite path. Let $\mathcal{M}$ be the set of integer-valued Radon measures on $\mathbb{R}_+ \times \mathbb{R}^d$, equipped with the sigma-field $\mathcal{G}$ generated by the variables $\omega(A)$, $A \in \mathbb{R}_+ \times \mathbb{R}^d$, so that we will consider ω as a process of the probability space $(\mathcal{M}, \mathcal{G}, \mathbb{P})$. It is possible to define, for all fixed time horizon $t > 0$, a measurable function

$$\begin{array}{rccc} \mathsf{Y}^{(t)}: & [0,t] \times \mathcal{M} & \to & \mathbb{R}^d \\ & (s,\omega) & \mapsto & \mathsf{Y}_s^{(t)}, \end{array} \tag{5.34}$$

which satisfies the property that, $\mathbb{P}$-almost surely,

$$\forall s \in [0,t], \quad P_t^{\beta,\omega}\left(B_s \in U\left(\mathsf{Y}_s^{(t)}\right)\right) = \max_{x \in \mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in U(x)\big). \tag{5.35}$$

The reader may refer to [51] for a proper (and rather technical) definition of $\mathsf{Y}^{(t)}$.

Here, the path $s \to \mathsf{Y}_s^{(t)}$ stands for the "optimal path" or the "favorite path" of the polymer, although this path is neither necessarily continuous, nor necessarily unique. Similarly to what we have done previously, we define the overlap with the favorite path R_t^* as the fraction of time any path B stays next to the favorite path:

$$R_t^* = R_t^*(B,\omega) = \frac{1}{t}\int_0^t \mathbf{1}_{B_s \in U(\mathsf{Y}_s^{(t)})} \mathrm{d}s. \tag{5.36}$$

As discussed before, a question of interest is the asymptotic behavior, as $t \to \infty$, of R_t and of R_t^*, and we will see in Theorem 5.5.6 that they are related. In particular, we are interested in determining the regions were one can prove positivity in the limit of these quantities, which can be seen as localization properties of the polymer.

Recall the notations

$$\mathcal{D} = \{(\beta,\nu) : \psi(\beta,\nu) = 0\}, \quad \mathcal{L} = \{(\beta,\nu) : \psi(\beta,\nu) > 0\},$$

of the high and low temperature regions, which are delimited by the critical lines $\beta_c^-(\nu)$ and $\beta_c^+(\nu)$ (cf. Definition 2.5.1). We saw in theorem 5.5.2 that in the $\mathcal{D}$ region, $\lim_{t\to\infty} \mathbb{P}[J_t] = 0$. On the other hand, Proposition 5.5.1 tells us that the limit inferior of $\mathbb{P}[J_t]$ is always positive in the region $\mathcal{L}'$, where

$$\mathcal{L}' = \left\{\beta > 0, \nu > 0 : \left(\frac{\partial \psi}{\partial \beta}\right)_+ > 0\right\} \cup \left\{\beta < 0, \nu > 0 : \left(\frac{\partial \psi}{\partial \beta}\right)_- < 0\right\}. \tag{5.37}$$

From the preceeding considerations, we know that

$$\mathcal{L}' \subset \mathcal{L}\,, \tag{5.38}$$

and a still open question is whether $\mathcal{L}' = \mathcal{L}$ or not.

Remark 5.5.3. *It is a direct consequence of the monotonicity of ψ (point 3 of proposition 2.4.1), that the inequalities on the derivatives of ψ appearing in* (5.37)*, once replaced by large inequalities, are always verified:*

$$\beta < 0 \implies \left(\frac{\partial \psi}{\partial \beta}\right)_- \le 0, \quad \beta > 0 \implies \left(\frac{\partial \psi}{\partial \beta}\right)_+ \ge 0.$$

We first state some results about the localized region:

Proposition 5.5.4. *(i) For any fixed $\nu > 0$ and for large enough positive β, $(\beta,\nu) \in \mathcal{L}'$.*

(ii) For all $(\beta,\nu) \in \mathcal{L}'$, $\liminf_{t\to\infty} \mathbb{P}[J_t] > 0$.

Proof. To prove (i), we use a result of [49, Th. 2.2.2.(b)] where it is shown that there exists a positive constant $C_1 = C_1(r,\nu,d)$, such that, for fixed ν, r and β large enough,

$$p(\beta,\nu) \le C_1 \lambda^{1/2}.$$

By convexity of p in β, we get that for large enough β,

$$\left(\frac{\partial p}{\partial \beta}\right)_+ \leq p(\beta+1,\nu) - p(\beta,\nu) < \nu r^d e^\beta,$$

so that $\left(\frac{\partial \psi}{\partial \beta}\right)_+$ is indeed positive when β is big enough.

The property (ii) is given by proposition 5.5.1. ∎

Remark 5.5.5. *We stress on how strong is the above claim (ii). For $(\beta,\nu) \in \mathcal{L}'$, it implies that there exist $C > 0, \delta > 0$ such that*

$$\liminf_{t\to\infty} \mathbb{P}P_t^{\beta,\omega\otimes 2}\big[R_t(B,\tilde{B}) \geq \delta\big] \geq C\ .$$

In contrast, if $\beta = 0$, there is some $C' > 0$ such that

$$P^{\otimes 2}\big[R_t(B,\tilde{B}) \geq \delta\big] \leq e^{-C't}$$

for all large enough t.

Observe that the properties of R_t and R_t^* are comparable in the following sense:

Theorem 5.5.6. *There exists a constant $c = c(d,r)$ in $(0,1)$, such that*

$$c\left(\mathbb{P}P_t^{\beta,\omega}\big[R_t^*\big]\right)^2 \leq \mathbb{P}[J_t] \leq r^d\, \mathbb{P}P_t^{\beta,\omega}\big[R_t^*\big]. \tag{5.39}$$

In particular, we get that for all $\beta \in \mathcal{L}'$,

$$\liminf_{t\to\infty} \mathbb{P}P_t^{\beta,\omega}\big[R_t^*\big] \geq r^{-d} \liminf_{t\to\infty} \mathbb{P}[J_t] > 0. \tag{5.40}$$

Remark 5.5.7. *Note that equation* (5.40) *gives another feature of path localization of the polymer in the $\mathcal{L}'$ region: we can find a "path" depending only on the environment (here, we found that $\mathsf{Y}^{(t)}$ does the job) such that the* expected proportion of time the random polymer spends in the neighborhood of that "path" is bounded away from 0 *as $t \to \infty$. Under the Gibbs measure, the random polymer sticks to that particular "path". Even though that "path" is not smooth – in fact, it has long jumps – it is an interesting object which sumarizes the attractive effect of the medium.*

Proof. To prove the first point, it is enough to show that there is a $c \in (0,1)$, such that

$$cP_t^{\beta,\omega}\big[R_t^*\big]^2 \leq J_t \leq r^d P_t^{\beta,\omega}\big[R_t^*\big], \tag{5.41}$$

the proposition being then a simple consequence of Jensen's inequality. For the right-hand side inequality of (5.41), observe that by Fubini's theorem,

$$\begin{aligned}
P_t^{\beta,\omega} \otimes P_t^{\beta,\omega}\big[R_t\big] &= \frac{1}{t}\int_0^t \int P_t^{\beta,\omega}[\chi_{s,x}]^2 \mathrm{d}s\mathrm{d}x. \\
&\leq \frac{1}{t}\int_0^t \max_{x\in\mathbb{R}^d} P_t^{\beta,\omega}\left(B_s \in U(x)\right)\mathrm{d}s \times \int_{\mathbb{R}^d} P_t^{\beta,\omega}\big[\chi_{s,x}\big]\mathrm{d}x \\
&= r^d P_t^{\beta,\omega}\left[R_t^*\right].
\end{aligned}$$

In order to obtain the left-hand side inequality, let r_d denote the radius of the ball $U(x)$ and y be any point of $\mathbb{R}^d$. By Cauchy-Schwarz's inequality,

$$\left(\int_{\mathrm{B}(y,r_d/2)} P_t^{\beta,\omega}\big(B_s \in U(z)\big)\mathrm{d}z\right)^2 \leq \big|\mathrm{B}(y,r_d/2)\big| \int_{\mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in U(z)\big)^2,$$

and since for every z in $\mathrm{B}(y, r_d/2)$, the ball $\mathrm{B}(y, r_d/2)$ is included in $U(z)$, this inequality leads to the following:

$$\begin{aligned}\int_{\mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in U(z)\big)^2 \mathrm{d}z &\geq \frac{2^d}{r^d}\left(\int_{\mathrm{B}(y,r_d/2)} P_t^{\beta,\omega}\big(B_s \in \mathrm{B}(y, r_d/2)\big)\mathrm{d}z\right)^2 \\ &= \frac{r^d}{2^d} P_t^{\beta,\omega}\big(B_s \in \mathrm{B}(y, r_d/2)\big)^2. \qquad (5.42)\end{aligned}$$

Now, let $c' = c'(d)$ be the minimal number of copies of $\mathrm{B}(y, r_d/2)$ necessary to cover $U(y)$. Then, by additivity of $P_t^{\beta,\omega}$,

$$\max_{y\in\mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in U(y)\big) \leq c' \max_{y\in\mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in \mathrm{B}(y, r_d/2)\big). \qquad (5.43)$$

Putting things together and integrating on $[0, t]$, we finally get that

$$c\,\frac{1}{t}\int_0^t \max_{y\in\mathbb{R}^d} P_t^{\beta,\omega}\big(B_s \in U(y)\big)^2 \mathrm{d}s \leq \frac{1}{t}\int_0^t \int P_t^{\beta,\omega}[\chi_{s,x}]^2 \mathrm{d}s\mathrm{d}x,$$

where $c = (c')^{-2} r^d/2^d$, from which the left-hand side inequality of (5.41) can be obtained by applying Jensen's inequality with probability measure $\mathrm{d}s/t$ on [0,t].

The second point of the theorem is then a consequence of the first point and proposition 5.5.1. ■

Remark 5.5.8. *Formulas like* (5.39) *and* (5.41) *can be called 2-to-1 formulas since they relate quenched expectations for two independent polymers to expectations for only one polymer (and involving the optimal path). As we have seen from the computations, stochastic analysis brings in second moments, involving 2 replicas of the polymer path. Then, using such formulas, the information is reduced to one polymer path interacting with the favorite path.*

Chapter 6

Formulas for variance and concentration

In this chapter, we introduce the critical exponents of the model and relations between them. The starting points are precise formulas for fluctuations of the partition function (variance and large deviations).

6.1 The critical exponents

There are different ways of defining the critical exponents, see for example [36, 118]. We will not enter the finest details, and we stay at an intuitive level. Although it is not clear that these definitions are all equivalent, the main idea is that the critical exponents are two reals $\xi^{\perp}$ and $\xi^{\|}$ such that

$$\sup_{0\le s\le t}|B_s| \approx t^{\xi^{\perp}(d)} \quad \text{and} \quad \ln Z_t - \mathbb{P}[\ln Z_t] \approx t^{\xi^{\|}(d)} \quad \text{as } t\to\infty. \tag{6.1}$$

The "wandering exponent" $\xi^{\perp}$ is the exponent for the asympotic transversal (or "perpendicular") fluctuations of the path, with respect to the time axis. The polymer is said to be *diffusive* when $\xi^{\perp} = 1/2$ (as for the brownian motion), and it is said to be *super-diffusive* when $\xi^{\perp} > 1/2$. One of the conjectures in polymers is that diffusivity should occure in weak disorder, while super-diffusivity should take place in the strong disorder setting. The number $\xi^{\|}$ denotes the critical exponent for the longitudinal fluctuation of the free energy.

The study of these exponents goes beyond the polymer framework. The reason is that they are expected to take the same value in many different statistical physics models describing growth phenomena. In dimension $d = 1$ this family is called the KPZ universality class (see Section 10). It is conjectured in the physics literature [109] that the two exponents should depend on one other, in the way that

$$\xi^{\|}(d) = 2\xi^{\perp}(d) - 1, \forall d \ge 1 \tag{6.2}$$

Under a certain definition of the exponents, the relation was proved by Chaterjee [36] for first-passage percolation. Auffinger and Damron were able to simplify Chatterjee's proof and extend the result to directed polymers [5, 4].

In dimension $d = 1$, it is conjectured that $\xi^{\perp} = 2/3$ and $\xi^{\|} = 1/3$ for any positive β. For now, this has only been proven for solvable models of polymers: Seppäläinen's discrete log-gamma polymer [145], O'Connell-Yor semi-discrete polymer [127, 146], and also for the the KPZ polymer [6].

In dimension $d \ge 2$, essentially nothing is known. Let's simply mention the (rough) bounds

$$0 \le \xi^{\|} \le 1/2\,, \quad 1/2 \le \xi^{\perp} \le 3/4\,,$$

where the last one will be proved in section 7.

A way to approach $\xi^{\|}$ is to consider the variance of $\ln Z_t$. In what follows, we give a formula to express the variance of $\ln Z_t$ in terms of a stochastic integral, which is obtained through a Clark-Ocone type martingale representation.

6.2 The Clark-Ocone representation

It is a consequence of Itô's work on iterated stochastic integrals [103] any that square-integrable functionals of the Brownian motion can be written as the sum of a constant and an Itô integral. In [39], Clark extended this result to a wider range of functionals, and showed that any martingale that is measurable with respect to the Brownian motion filtration, could be represented as a stochastic integral martingale. Clark was also able to compute the integrand of the representation, for a special class of functionals. Ocone then showed [131] that this computation was linked to Malliavin's calculus, and generalised this idea to a larger class of functionals.

Such representations - called Clark-Ocone representations - also exist in the framework of functionals of a Poisson processes. Denote by ω_{s-} the restriction of ω on $[0,s)\times\mathbb{R}^d$, and consider the derivative operator

$$D_{(s,x)}F(\omega) := F(\omega+\delta_{s,x}) - F(\omega). \tag{6.3}$$

We have:

Theorem 6.2.1. [116, theorem 3.1] *Let $F = F(\omega)$ be a functional of the Poisson process, such that $\mathbb{P}[F^2] < \infty$. Then,*

$$\mathbb{P}\int \mathbb{P}[D_{(s,x)}F(\omega)|\omega_{s-}]^2 \mathrm{d}s\mathrm{d}x < \infty, \tag{6.4}$$

and we have for all $u \geq 0$, that $\mathbb{P}$-a.s.

$$\mathbb{P}[F(\omega)|\omega_u] = \mathbb{P}[F(\omega)] + \int_{[0,u]\times\mathbb{R}^d} \mathbb{P}[D_{(s,x)}F(\omega)|\omega_{s-}]\bar{\omega}(\mathrm{d}s\mathrm{d}x). \tag{6.5}$$

This proves that the square integrable martingale $(\mathbb{P}[F(\omega)|\omega_u])_{u\geq 0}$ admits a stochastic integral martingale representation, with predictable integrand $\mathbb{P}[D_{(s,x)}F(\omega)|\omega_{s-}]$.

6.3 The variance formula

To lighten the writing, we will denote by $\mathcal{G}_{s-}$ the sigma-field generated by ω_{s-} and $\mathbb{P}^{\mathcal{G}_{s-}}$ will stand for the expectation knowing $\mathcal{G}_{s-}$.

Using Jensen's inequality and Tonelli's theorem, it is easy to check that $\ln Z_t$ is a square integrable function of ω. Hence, the process $\left(\mathbb{P}[\ln Z_t|\omega_u]\right)_{u\in[0,t]}$ is a martingale which admits a Clark-Ocone type representation:

$$\mathbb{P}[\ln Z_t|\omega_u] = \mathbb{P}\left[\ln Z_t\right] + \int_{[0,u]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[D_{(s,x)}\ln Z_t\right]\bar{\omega}(\mathrm{d}s\mathrm{d}x), \tag{6.6}$$

where

$$D_{(s,x)}F(\omega) = \ln\frac{P\left[e^{\beta\omega(V_t)}e^{\beta\chi_{s,x}}\right]}{Z_t} = \ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right).$$

As a consequence, one can express the variance of $\ln Z_t$ via (6.6), using the formula for the variance of a Poisson integral (5.5), and find that

$$\mathrm{Var}(\ln Z_t) = \mathbb{P}\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[\ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right)\right]^2 \nu\,\mathrm{d}s\mathrm{d}x. \tag{6.7}$$

This variance formula leads us to the following theorem:

Theorem 6.3.1. *(i) The following lower and upper bounds on the variance hold:*

$$\mathrm{Var}(\ln Z_t) \geq c_-^2\,\mathbb{P}\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[P_t^{\beta,\omega}[\chi_{s,x}]\right]^2 \nu\,\mathrm{d}s\mathrm{d}x, \tag{6.8}$$

$$\mathrm{Var}(\ln Z_t) \leq c_+^2\,\mathbb{P}\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[P_t^{\beta,\omega}[\chi_{s,x}]\right]^2 \nu\,\mathrm{d}s\mathrm{d}x, \tag{6.9}$$

where $c_- = 1-e^{-|\beta|}$ and $c_+ = e^{|\beta|}-1$.

In particular,

$$\mathrm{Var}(\ln Z_t) \leq c_+^2\, t\nu\,\mathbb{P}[J_t]. \tag{6.10}$$

(ii) Letting $c = \nu c_+^2 \exp(c_+)$, the following concentration estimate holds:

$$\mathbb{P}\left(\big|\ln Z_t - \mathbb{P}[\ln Z_t]\big| > u\right) \leq 2\exp\left(-\frac{1}{2}(u \wedge \frac{u^2}{ct})\right). \tag{6.11}$$

Remark 6.3.2. *Recalling Theorem 5.5.2, the inequality* (6.10) *suggests that the variance should be smaller in weak disorder than in strong disorder. It also shows that for all $d \geq 1$, we have $\xi^{\parallel}(d) \leq 1/2$.*

Proof. The two first bounds on the variance are a consequence of the fact that, for all $u \in [0,1]$, we have

$$c_- u \leq |\ln(1+\lambda u)| \leq c_+ u. \tag{6.12}$$

Then, apply Jensen's inequality to the conditional expectation in the right-hand side of (6.9) and use Fubini's theorem such that

$$\begin{aligned}
\mathrm{Var}(\ln Z_t) &\leq c_+^2 \mathbb{P} \int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[P_t^{\beta,\omega}[\chi_{s,x}]\right]^2 \nu\, \mathrm{d}s\mathrm{d}x \\
&\leq c_+^2 \mathbb{P} \int_{[0,t]\times\mathbb{R}^d} P_t^{\beta,\omega}[\chi_{s,x}]^2\, \nu\, \mathrm{d}s\mathrm{d}x \\
&= c_+^2 t\nu\, \mathbb{P}[J_t],
\end{aligned}$$

by definition of J_t. This completes the proof of (i). To prove (6.11), we first denote by $Y_{t,u}$ the mean-zero martingale part appearing in (6.6), i.e.

$$Y_{t,u} := \int_{[0,u]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}} \ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right) \bar{\omega}(\mathrm{d}s\mathrm{d}x).$$

Then, letting $\varphi(v) = e^v - v - 1$ and $a \in [-1,1]$, we define $(M_{t,u})_{u\in[0,t]}$ as the exponential martingale associated to $(Y_{t,u})_{u\in[0,t]}$:

$$M_{t,u} = \exp\left(aY_{t,u} - \int_{[0,u]\times\mathbb{R}^d} \varphi\left(a\cdot \mathbb{P}^{\mathcal{G}_{s-}} \ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right)\right) \nu\, \mathrm{d}s\mathrm{d}x\right).$$

By (6.12) and the observations that χ is less than 1 and that $|\varphi(v)| \leq e^{|v|}v^2/2$ for all v, we have for $a \in [-1,1]$

$$\begin{aligned}
\left|\int_{[0,t]\times\mathbb{R}^d} \varphi\left(a\cdot \mathbb{P}^{\mathcal{G}_{s-}} \ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right)\right) \nu\, \mathrm{d}s\mathrm{d}x\right| &\leq e^{c_+}\frac{c_+^2 a^2}{2}\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[P_t^{\beta,\omega}[\chi_{s,x}]\right]^2 \nu\, \mathrm{d}s\mathrm{d}x \\
&\leq c\frac{a^2}{2}\int_{[0,t]\times\mathbb{R}^d} \mathbb{P}^{\mathcal{G}_{s-}}\left[P_t^{\beta,\omega}[\chi_{s,x}]\right] \mathrm{d}s\mathrm{d}x \\
&= c\frac{a^2}{2}t,
\end{aligned}$$

where $c = \nu c_+^2 e^{c_+}$ and where the second inequality was obtained using Jensen's inequality.

If one denotes by $b_{t,u}$ the integral term in the definition of $M_{t,u}$, we just showed that $b_{t,t} \leq ca^2t/2$, so by Markov's inequality and the martingale property, we obtain that

$$\mathbb{P}\big(\ln Z_t - \mathbb{P}[Z_t] > u\big) = \mathbb{P}\big(M_{t,t} > \exp(au - b_{t,t})\big) \leq \exp(ca^2t/2 - au). \tag{6.13}$$

This implies (6.11) after minimizing the bound for $a \in [-1,1]$, and repeating the same procedure for the lower deviation. ■

From the concentration estimate, one can derive the following almost sure behavior:

Corollary 6.3.3. *For all $\varepsilon > 0$ and as $t \to \infty$,*

$$\ln Z_t - \mathbb{P}[\ln Z_t] = \mathcal{O}\left(t^{\frac{1+\varepsilon}{2}}\right), \quad \mathbb{P}\text{-}a.s. \tag{6.14}$$

Proof. Equation (6.11) implies that for large enough $k \in \mathbb{N}$,

$$\mathbb{P}\left[|\ln Z_k - \mathbb{P}\ln Z_k| > k^{\frac{1+\varepsilon}{2}}\right] \leq 2\exp\big(-\frac{t^\varepsilon}{2c}\big), \tag{6.15}$$

which is summable. By Borel-Cantelli lemma, we obtain that, $\mathbb{P}$-almost surely,

$$|\ln Z_k - \mathbb{P}[\ln Z_k]| \leq k^{\frac{1+\varepsilon}{2}} \qquad \text{for } k \text{ large enough.}$$

To extend this to any $t \geq 0$ and prove (6.14), it suffices to apply the next lemma. ∎

Lemma 6.3.4. *Let $h > 0$. For all $0 \leq s \leq h$,*

$$-c_+\delta_t(h) \leq \ln Z_{t+s} - \ln Z_t \leq c_+\delta_t(h), \tag{6.16}$$

where

$$\delta_t(h) = \int_{[t,t+h]\times\mathbb{R}^d} P^{\beta,\omega}_{s-}[\chi_{s,x}]\omega(\mathrm{d}s\mathrm{d}x),$$

is such that for any $\varepsilon > 0$,

$$\delta_t(h) = \mathcal{O}\big(t^{\frac{1+\varepsilon}{2}}\big), \quad \mathbb{P}\text{-}a.s. \tag{6.17}$$

Proof. We get from the integral writing of $\ln Z_t$ (5.6) that

$$\ln Z_{t+s} - \ln Z_t = \int_{[t,t+h]\times\mathbb{R}^d} \omega(\mathrm{d}s\mathrm{d}x)\ln\big(1+\lambda P^{\beta,\omega}_{s-}[\chi_{s,x}]\big).$$

Hence, (6.16) is simply obtained with (6.12).

Now, introduce the martingale

$$M_t = \int_0^t \omega(\mathrm{d}s\mathrm{d}x)P^{\beta,\omega}_{s-}[\chi_{s,x}] - \nu r^d t,$$

which has bracket $\langle M\rangle_t = \nu\int_0^t I_s \mathrm{d}s \leq \nu r^d t$. Note that $\delta_t(h) = M_{t+h} - M_t + h$, so that (6.17) is thus a consequence of the martingale properties (5.21) and (5.22), since in the case where $\langle M\rangle_\infty$ is infinite, then $|M_t| = o\big(\langle M\rangle_t^{\frac{1+\varepsilon}{2}}\big)$ as $t \to \infty$. ∎

In order to illustrate the general strategy, we now mention a consequence of theorem 6.3.1 (i).

Corollary 6.3.5. *Let $\beta \neq 0$, ξ and $C > 0$. There exists a constant $c_1 = c_1(d, C) \in (0,\infty)$, such that*

$$\begin{aligned}\liminf_{t\to\infty} t^{-(1-d\xi)}\mathrm{Var}(\ln Z_t) &\geq c_1 \liminf_{t\to\infty} \inf_{0\leq s\leq t}\Big(\mathbb{P}P^{\beta,\omega}_t\left(|B_s| \leq C + Ct^\xi\right)\Big)^2 \\ &\geq c_1 \liminf_{t\to\infty}\left(\mathbb{P}P^{\beta,\omega}_t\left(\sup_{0\leq s\leq t}|B_s| \leq C + Ct^\xi\right)\right)^2.\end{aligned}$$

The result suggests that

$$\chi(d) \geq \frac{1-d\xi(d)}{2}.$$

For more details on the result and the proof, we refer the reader to Corollary 2.4.3 in [49].

Chapter 7

Cameron-Martin transform and applications

In this section we extensively use the property that the a-priori measure for the polymer path is Wiener measure. A tilt on the polymer path reflects into a shift on the environment.

7.1 Tilting the polymer

We extend the shear transformation (4.3 – 4.4) to non-linear shifts $\varphi : \mathbb{R}_+ \to \mathbb{R}^d$ by defining

$$\hat{\tau}_\varphi f : s \mapsto f(s) + \varphi(s) , \quad \hat{\tau}_\varphi \circ \Big(\sum_i \delta_{(t_i,x_i)} \Big) = \sum_i \delta_{(t_i, x_i + \varphi(t_i))} , \tag{7.1}$$

so that $\hat{\tau}_\varphi = \tau_\xi$ when $\varphi(t) = t\xi$ and that $\hat{\tau}_\varphi \circ \omega$ has same law as ω.

Let $\varphi \in H^1_{0,\mathrm{loc}} := \big\{ \varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d); \varphi(0) = 0, \dot{\varphi} \in L^2_{\mathrm{loc}} \big\}$ where the dot denotes time derivative. Introduce the probability measure on the path space P^φ which restriction to $\mathcal{F}_t$ has density relative to P given by

$$\Big(\frac{dP^\varphi}{dP}\Big)_{|\mathcal{F}_t} = \exp\left\{ \int_0^t \dot{\varphi}(s) dB(s) - \frac{1}{2} \int_0^t |\dot{\varphi}(s)|^2 ds \right\}$$

for all $t > 0$, with B the canonical process. Then, by Cameron-Martin theorem, under the measure P^φ the process $W(t) = B(t) - \varphi(t)$ is a standard Brownian motion, and as in (4.7) we write

$$\begin{aligned} P\left[\exp\{\beta\omega(V_t(B))\} e^{\int_0^t \dot{\varphi}(s)dB(s) - \frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds} \right] &= P^\varphi \left[\exp\{\beta\omega(V_t(\hat{\tau}_\varphi(W))\}\right] \\ &= P\left[\exp\{\beta\omega(V_t(\hat{\tau}_\varphi(B)))\}\right] \\ &= P\left[\exp\{\beta\omega(\hat{\tau}_\varphi(V_t(B)))\}\right] \\ &= P\left[\exp\{\beta(\hat{\tau}_{-\varphi} \circ \omega)(V_t(B))\}\right] \\ &= Z_t(\hat{\tau}_{-\varphi} \circ \omega, \beta) , \end{aligned}$$

yielding

$$P_t^{\beta,\omega} \left[\exp\{ \int_0^t \dot{\varphi}(s) dB(s)\} \right] = \exp\{ \frac{1}{2} \int_0^t |\dot{\varphi}(s)|^2 ds\} \times \frac{W_t(\hat{\tau}_{-\varphi} \circ \omega, \beta)}{W_t(\omega, \beta)} . \tag{7.2}$$

7.2 Consequences for weak disorder regime

In this section we assume that $\beta \in (\bar{\beta}_c^-, \bar{\beta}_c^+)$, more precisely that $W_\infty(\omega,\beta) = \lim_t W_t(\omega,\beta) > 0$ a.s. In particular, for a fixed $\varphi \in H^1_0$ (i.e., $\dot{\varphi} \in L^2$), we derive from (7.2) that

$$P_t^{\beta,\omega} \left[\exp\{ \int_0^t \dot{\varphi}(s) dB(s)\} \right] \longrightarrow \exp\{ \frac{1}{2} \|\dot{\varphi}\|_2^2\} \times \frac{W_\infty(\hat{\tau}_{-\varphi} \circ \omega, \beta)}{W_\infty(\omega, \beta)} \tag{7.3}$$

a.s. as $t \to \infty$.

In view of (7.2), a natural question is continuity of W_∞ in the ω variable: how does the limit depend on the environment ?

Lemma 7.2.1. *Assume that $W_\infty(\omega, \beta) > 0$. Let $\varphi_T \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ be a family indexed by $T > 0$ with $\varphi_T(0) = 0$ which vanishes locally uniformly, i.e.,*

$$\forall t > 0, \quad \|\varphi_T\|_{\infty,t} := \sup\{|\varphi_T(s)|; s \in [0,t]\} \to 0 \quad \text{as } T \to \infty.$$

Then we have, as $T \to \infty$,

$$W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta) \longrightarrow W_\infty(\omega, \beta) \quad \text{in } L^1\text{–norm}, \tag{7.4}$$

$$W_T(\hat{\tau}_{\varphi_T} \circ \omega, \beta) \longrightarrow W_\infty(\omega, \beta) \quad \text{in } L^1\text{–norm}. \tag{7.5}$$

The result can be compared to lemma 3.2.1, where we have already considered the effect of a shift on the environment. In the notation of (7.1), that lemma deals with constant shifts $\varphi = \varphi^{(x)}$ such that $\varphi^{(x)}(t) \equiv x$ for all t, and implies that

$$x \mapsto W_\infty(\hat{\tau}_{\varphi^{(x)}} \circ \omega) \quad \text{is Lipschitz continuous from } \mathbb{R}^d \text{ to } L^1.$$

In the above lemma 7.2.1, the shift is not anymore constant.

Proof. Fix $t > 0$ and decompose the difference $W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_\infty(\omega, \beta)$ as

$$\{W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_t(\hat{\tau}_{\varphi_T} \circ \omega, \beta)\} + \{W_t(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_t(\omega, \beta)\} + \{W_t(\omega, \beta) - W_\infty(\omega, \beta)\}.$$

Now, using triangular inequality and invariance in law of ω under the shear transformation, we get

$$\begin{aligned} \|W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_\infty(\omega, \beta)\|_1 &\leq 2\|W_\infty(\omega, \beta) - W_t(\omega, \beta)\|_1 + \|W_t(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_t(\omega, \beta)\|_1 \\ &=: 2\varepsilon(t) + \varepsilon_t(\varphi_T) \end{aligned} \tag{7.6}$$

with $\varepsilon(t) = \|W_\infty - W_t\|_\infty$ and $\varepsilon_t(\cdot)$ defined by the above formula. By assumption on β and proposition 3.2.2, we have $\lim_{t\to\infty} \varepsilon(t) = 0$. On the other hand, for fixed t, $W_t(\hat{\tau}_{\varphi_T} \circ \omega, \beta) \to W_t(\omega, \beta)$ a.s. as $T \to \infty$, and the variables $(W_t(\hat{\tau}_{\varphi_T} \circ \omega, \beta); T > 0)$, $W_t(\omega, \beta)$ are uniformly integrable. Thus, the above convergence holds in L^1, which, combined with (7.6), completes the proof of (7.4).

The proof of (7.5) is quite similar, writing this time the difference

$$W_T(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_\infty(\omega, \beta) = \{W_T(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta)\} + \{W_\infty(\hat{\tau}_{\varphi_T} \circ \omega, \beta) - W_\infty(\omega, \beta)\},$$

observing that the first term in the right-hand side has L^1-norm equal to $\|W_T - W_\infty\|_1$, that the second one vanishes a.s. and is uniformly integrable. This completes the proof. ■

For $a \in \mathbb{R}^d, t > 0$ define the function

$$\varphi_{a,t}(s) = \frac{s \wedge t}{\sqrt{t}} a\ .$$

From (7.2) with $\varphi = \varphi_{a,t}$, we have

$$P_t^{\beta,\omega}\left[e^{a \cdot \frac{B(t)}{\sqrt{t}}}\right] = e^{|a|^2/2} \times \frac{W_t(\hat{\tau}_{-\varphi_{a,t}} \circ \omega, \beta)}{W_t(\omega, \beta)}\ . \tag{7.7}$$

Theorem 7.2.2. *Assume weak disorder, i.e., that $W_\infty > 0$. Then, as $t \to \infty$,*

$$P_t^{\beta,\omega}\left[e^{a \cdot \frac{B(t)}{\sqrt{t}}}\right] \xrightarrow{\mathbb{P}} e^{|a|^2/2}.$$

Proof. Note that the family $(-\varphi_{a,t}, t > 0)$ satisfies the assumptions of lemma 7.2.1. Writing formula (7.7) as

$$P_t^{\beta,\omega}\left[e^{a \cdot \frac{B(t)}{\sqrt{t}}}\right] - e^{|a|^2/2} = e^{|a|^2/2} \times \frac{W_t(\hat{\tau}_{-\varphi_{a,t}} \circ \omega, \beta) - W_t(\omega, \beta)}{W_t(\omega, \beta)}\ ,$$

we see from Slutsky's lemma that this quantity vanishes as $t \to \infty$. ■

Remark 7.2.3. *In a suitable sense, this result shows that the polymer is diffusive if $W_\infty > 0$. Its interest is that it covers the* full *weak disorder region, in contrast to [48, Th. 2.1.1] which only applies to the L^2 region. It can be viewed as a step to prove diffusivity at weak disorder region.*

7.3 Moderate and large deviations at all temperature

In this section, $\beta \in \mathbb{R}$ is arbitrary.

Theorem 7.3.1. *For $\mathbb{P}$-a.e. realization of the environment ω, the following holds.*

(i) For all Borel $A \subset \mathbb{R}^d$,

$$\begin{aligned} -\inf\{\frac{|\xi|^2}{2}; \xi \in \mathring{A}\} \quad &\leq \liminf_{t\to\infty} \tfrac{1}{t} \ln P_t^{\beta,\omega}\left[\tfrac{B(t)}{t} \in A\right] \\ &\leq \limsup_{t\to\infty} \tfrac{1}{t} \ln P_t^{\beta,\omega}\left[\tfrac{B(t)}{t} \in A\right] \quad \leq -\inf\{\frac{|\xi|^2}{2}; \xi \in \overline{A}\}. \end{aligned}$$

(ii) Let t_n be a positive sequence increasing to ∞, let $\chi > 0$ such that

$$\sum_{n\geq 1} \mathbb{P}(|\ln Z_{t_n}(\omega,\beta) - \mathbb{P}[\ln Z_{t_n}(\omega,\beta)]| > t_n^{\chi}) < \infty, \tag{7.8}$$

and let $\xi > (1+\chi)/2$. Then, for all Borel $A \subset \mathbb{R}^d$,

$$\begin{aligned} -\inf\{\frac{|a|^2}{2}; a \in \mathring{A}\} \quad &\leq \liminf_{t\to\infty} t_n^{-(2\xi-1)} \ln P_{t_n}^{\beta,\omega}\left[\tfrac{B(t_n)}{t_n^{\xi}} \in A\right] \\ &\leq \limsup_{t\to\infty} t_n^{-(2\xi-1)} \ln P_{t_n}^{\beta,\omega}\left[\tfrac{B(t_n)}{t_n^{\xi}} \in A\right] \quad \leq -\inf\{\frac{|a|^2}{2}; a \in \overline{A}\}. \end{aligned}$$

Part (i) is the almost sure large deviation principle for the polymer endpoint. The rate function $a \mapsto |a|^2/2$ is called the shape function in the polymer framework. In view of (4.10–4.11) it is no surprise.

Part (ii) is an almost sure moderate deviation principle. The rate function is the same as before. Since (7.8) is expected to hold for all $\chi > \xi^{\|}$, we derive that the polymer endpoint lies at time t_n within a distance $t_n^{\xi^{\|}+\varepsilon}$ with overwhelming probability for all positive ε. Hence we get a relation between characteristic exponents

$$\xi^{\perp} \leq \frac{1+\xi^{\|}}{2}\ . \tag{7.9}$$

Since $\xi^{\|} \leq 1/2$ from theorem 6.3.1 and corollary 6.3.3, we derive a bound for all values of d:

$$\xi^{\perp} \leq 3/4\ . \tag{7.10}$$

Proof. We start with (i). Using (7.2) with $\varphi = \varphi_t$ given by $\varphi_t(s) = (s \wedge t)a$, we get

$$\begin{aligned} \ln P_t^{\beta,\omega}[e^{a\cdot B(t)}] &= \frac{t|a|^2}{2} + \ln W_t(\tau_a \circ \omega, \beta) - \ln W_t(\omega,\beta) \\ &= \frac{t|a|^2}{2} + \big(\ln W_t(\tau_a\circ\omega,\beta) - \mathbb{P}[\ln W_t(\tau_a\circ\omega,\beta)]\big) + \big(\ln W_t(\omega,\beta) - \mathbb{P}[\ln W_t(\omega,\beta)]\big), \end{aligned}$$

and by theorem 2.3.1 the set Ω_a of environments such that

$$\lim_{t\to\infty} \frac{1}{t} \ln P_t^{\beta,\omega}[e^{a\cdot B(t)}] = \frac{|a|^2}{2} \tag{7.11}$$

has full $\mathbb{P}$-measure. We now claim that on the event $\overline{\Omega} = \bigcap_{a\in\mathbb{Q}^d} \Omega_a$ the above limit holds for all $a \in \mathbb{R}^d$. Indeed, the maps $a \mapsto \frac{1}{t} \ln P_t^{\beta,\omega}[e^{a\cdot B(t)}]$ are convex, and the convergence is locally uniform on the closure $\mathbb{R}^d$ of $\mathbb{Q}^d$, see Th. 10.8 in [139]. Then, the large deviation principle (i) follows from the Gärtner-Ellis-Baldi theorem ([60], p.44, Th.2.3.6), with the rate function

$$|\xi|^2/2 = \sup\{|a|^2/2; a \in \mathbb{R}^d\}.$$

For the proof of (ii) we proceed as above. By (7.2) with $\varphi_t(s) = (s \wedge t)t^{\xi-1}a$, we first write

$$\begin{aligned} t^{-(2\xi-1)} \ln P_t^{\beta,\omega}[e^{t^{\xi-1}a\cdot B(t)}] &= \frac{|a|^2}{2} + t^{-(2\xi-1)}\big(\ln W_t(\hat{\tau}_{-\varphi_t} \circ \omega, \beta) - \mathbb{P}[\ln W_t(\omega,\beta)]\big) \\ &\quad + t^{-(2\xi-1)}\big(\ln W_t(\omega,\beta) - \mathbb{P}[\ln W_t(\omega,\beta)]\big). \end{aligned}$$

Now, in order to take the limit we restrict to the sequence $t = t_n$: using Borel-Cantelli lemma, we indeed get the a.s. limit (7.11) along the sequence t_n. From then on, the other arguments go through without any change. ∎

Chapter 8

Phase diagram in the (β, ν)-plane, $d \geq 3$

In this section, $r > 0$ is kept fixed, and we discuss the phase diagram in the remaining parameters β, ν. Recall ψ form (2.15) and $\psi_t(\beta, \nu) = \nu\lambda(\beta)r^d - t^{-1}\mathbb{P}[\ln Z_t(\omega, \beta)]$ from (2.18). Recall the notations

$$\mathcal{D} = \{(\beta, \nu) : \psi(\beta, \nu) = 0\}, \quad \mathcal{L} = \{(\beta, \nu) : \psi(\beta, \nu) > 0\}, \quad \texttt{Crit} = \mathcal{D} \bigcap \overline{\mathcal{L}}.$$

$\mathcal{D}$ is is the delocalized phase, $\mathcal{L}$ the localized phase.They are also high temperature/low density and low temperature/high density phases respectively. They are separated by the critical curve $\texttt{Crit} = \{(\beta_c^+(\nu), \nu); \nu > 0\} \cup \{(\beta_c^-(\nu), \nu); \nu > \nu_c\}$. We have seen that, in dimension $d = 1$ or 2, $\mathcal{D}$ reduces to the axis $\beta = 0$, so we assume $d \geq 3$ in this section.

We introduce

$$\nu_c = \sup\{\nu > 0 : \beta_c^- = -\infty\} . \tag{8.1}$$

Then, $\nu_c \in (0, \infty)$, and

$$\nu > \nu_c \iff \beta_c^- > -\infty .$$

A central question in polymer models is to estimate the critical curve [61, 76].

8.1 Strategy for critical curve estimates

We follow the idea of [51], that is to

find curves $\nu(\beta)$ along which ψ is monotone.

We now sketch the strategy. By computing the derivative with the chain rule

$$\frac{d}{d\beta}\psi_t(\beta, \nu(\beta)) = \nu' \frac{\partial}{\partial \nu}\psi_t + \frac{\partial}{\partial \beta}\psi_t,$$

one sees that, along the smooth curve $\mathcal{C}_a^\alpha$

$$\nu(\beta) = a|\lambda(\beta)|^{-\alpha} \tag{8.2}$$

for positive constants a, α, it takes the amenable form

$$t\frac{\partial}{\partial \beta}\psi_t(\beta, \nu(\beta)) = \nu' \times \mathbb{P}\left[\int_{(0,t]\times\mathbb{R}^d} h_\alpha(\lambda P_t^{\beta,\omega}[\chi_{s,x}])dsdx\right] , \tag{8.3}$$

where

$$h_\alpha(u) = u - \frac{u^2}{\alpha(1+u)} - \ln(1+u) \tag{8.4}$$

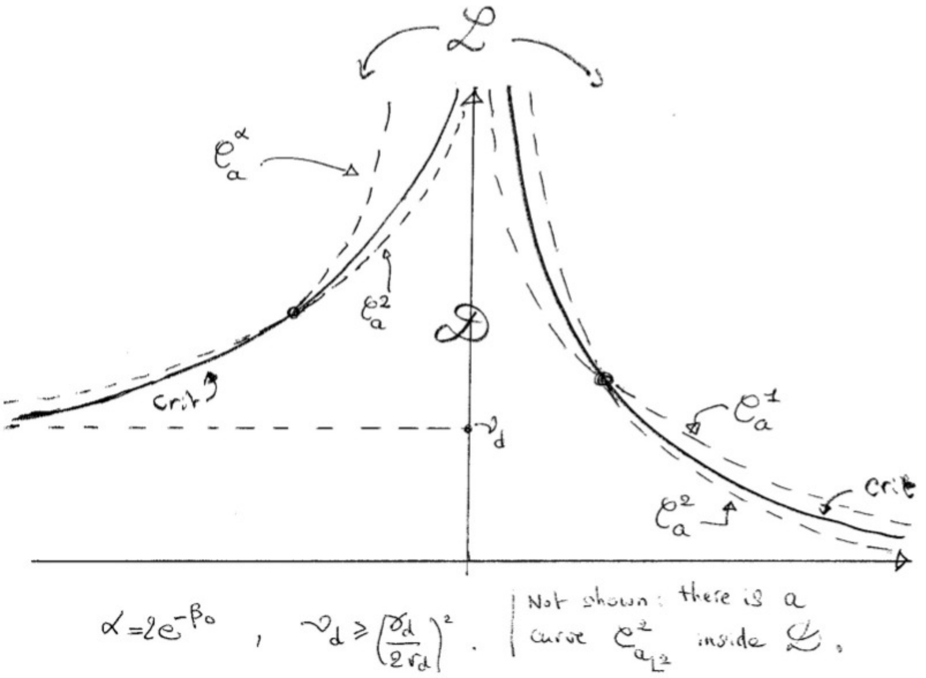

Figure 8.1: Estimating the critical curve. $\mathcal{C}_a^\alpha$ denotes the curve (8.2).

for $u > -1$. Now, the questions boils down to controling the sign of the function on the relevant interval with endpoints 0 and $\lambda(\beta)$:

$$h_\alpha(u) \quad \begin{cases} \geq 0 & \text{if } \alpha = 2 \text{ and} & u \geq 0, \\ \leq 0 & \text{if } \alpha = 2 \text{ and} & u \in (-1, 0], \\ \leq 0 & \text{if } \beta > 0, \alpha \leq \alpha(\beta) \text{ and} & u \in [0, \lambda], \\ \geq 0 & \text{if } \beta < 0, \alpha \geq \alpha(\beta) \text{ and} & u \in [\lambda, 0], \end{cases}$$

with $\alpha(\beta) = \frac{(e^\beta - 1)^2}{e^\beta(e^\beta - 1 - \beta)}$, $\alpha(0) = 2$. With this at hand, we can bound the critical curve from above and below with curves of the form (8.2) and specific α's, as indicated on Figure 8.1.

8.2 Main results

We start with an estimate of the free energy. In [51, Th. 5.3.1], the asymptotics of the free energy is determined as $\nu\beta^2$ diverges and β remains bounded. We state it below, it is key for our results.

Lemma 8.2.1. *Let $\beta_0 \in (0, \infty)$ arbitrary. Then, as $\nu\beta^2 \to \infty$ and $|\beta| \leq \beta_0$, we have*

$$p(\nu, \beta) = \beta\nu r^d + \mathcal{O}((\nu\beta^2 r^d)^{5/6}) .$$

We refer to the above paper for the involved, technical proof.

Among all the results, we mention:

Theorem 8.2.2. *Let $d \geq 3$.*

(i) the functions $\beta_c^\pm(\nu)$ are locally Lipshitz and strictly monotone;

(ii) we have[1]

$$\beta_c^+(d, \nu) \asymp \ln(1/\nu) \quad \textit{as } \nu \searrow 0; \tag{8.5}$$

(ii) we have

$$|\beta_c^\pm(d, \nu)| \asymp 1/\sqrt{\nu} \quad \text{as} \quad \nu \nearrow \infty. \tag{8.6}$$

[1] For positive functions we write $f(x) \asymp g(x)$ as $x \to x_0$ if the ratio remains bounded from 0 and from ∞ as $x \to x_0$.

8.3 Main steps

The derivative of ψ_t in β has been obtained in (2.17). We explain how to obtain the derivative in ν, and for clarity, in the sequel of the section we write $\mathbb{P} = \mathbb{P}_\nu$ to make the dependence in ν explicit.

Lemma 8.3.1. *We have*

$$\frac{\partial}{\partial \nu}\mathbb{P}_\nu[\ln Z_t(\beta,\omega)] = \int_{[0,t]\times\mathbb{R}^d} dsdx\ \mathbb{P}_\nu\ \ln[1+\ln P_t^{\beta,\omega}(\chi_{s,x})]\ . \tag{8.7}$$

Proof. : For $k \geq 1$, let

$$Z_{t,k} = P[e^{\beta\omega(V_t)}; A_k]\ , \quad \text{with } A_k = \{B_s \in [-k,k]^d,\ \forall s \leq t\},$$

and $p_{t,k}(\beta,\nu) = t^{-1}\mathbb{P}_\nu \ln Z_{t,k}$. Let $K^r = [-k-r, k+r]^d$ and $K_t = (0,t]\times K^r$. By Proposition 3.1.4,

$$\mathbb{P}_\nu[\ln Z_{t,k}] = \mathbb{P}_1\left[\rho_{t,\nu}\ln Z_{t,k}\right] \quad \text{with} \quad \rho_{t,\nu} = \exp\left(\omega_t(K_t)\ln\nu - (\nu-1)t|K^r|\right).$$

Thus, $tp_{t,k}(\beta,\nu)$ is differentiable in ν, with derivative

$$\begin{aligned}
\frac{1}{\nu}\mathbb{P}_\nu\left[\int_{K_t}\overline{\omega}(dsdx)\ln Z_{t,k}\right] &\overset{(2.16)}{=} \int_{K_t} dsdx\ \mathbb{P}_\nu \ln\frac{Z_{t,k}(\omega+\delta_{s,x})}{Z_{t,k}(\omega)} \\
&= \int_{[0,t]\times\mathbb{R}^d} dsdx\ \mathbb{P}_\nu \ln\Big(1+\ln P_t^{\beta,\omega}[\chi_{s,x}|A_k]\Big)\ .
\end{aligned}$$

Now, we write

$$tp_{t,k}(\beta,\nu) - tp_{t,k}(\beta,1) = \int_1^\nu d\nu' \int_{[0,t]\times\mathbb{R}^d} dsdx\ \mathbb{P}_{\nu'}\ln\Big(1+\ln P_t^{\beta,\omega}[\chi_{s,x}|A_k]\Big)\ .$$

By dominated convergence theorem (see details in [51], Lemma 7.2.1), we can take the limit $k\to\infty$, and obtain the desired statement. ∎

We come to the core of the proof. With the derivatives of ψ_t in both variables, from Lemma 8.3.1 and (2.17), we obtain

$$\begin{aligned}
t\frac{d}{d\beta}\psi_t(\beta,\nu(\beta)) &= \nu' t\frac{\partial}{\partial\nu}\psi_t + t\frac{\partial}{\partial\beta}\psi_t \\
&= \nu'\times\mathbb{P}\int_{[0,t]\times\mathbb{R}^d} dsdx\left\{\lambda P_t^{\beta,\omega}[\chi_{s,x}] + \frac{\nu}{\nu'}e^\beta\lambda\frac{P_t^{\beta,\omega}[\chi_{s,x}]^2}{1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]} - \ln\left(1+\lambda P_t^{\beta,\omega}[\chi_{s,x}]\right)\right\}.
\end{aligned}$$

Recall that $e^\beta = \lambda'$; we recover the simpler formula (8.3) along the curves of equation

$$\frac{\lambda'\nu}{\lambda\nu'} = -\frac{1}{\alpha},$$

that is, the curves h_α from (8.4). From then on, the rest of the proof is a tedious but elementary exercise in calculus, performed in [51]. We will not dive any further in the details of the proof, that the reader can find in this reference together with many fine estimates. We summarize the section by giving a qualitative picture of the phase diagram in Figure 8.2.

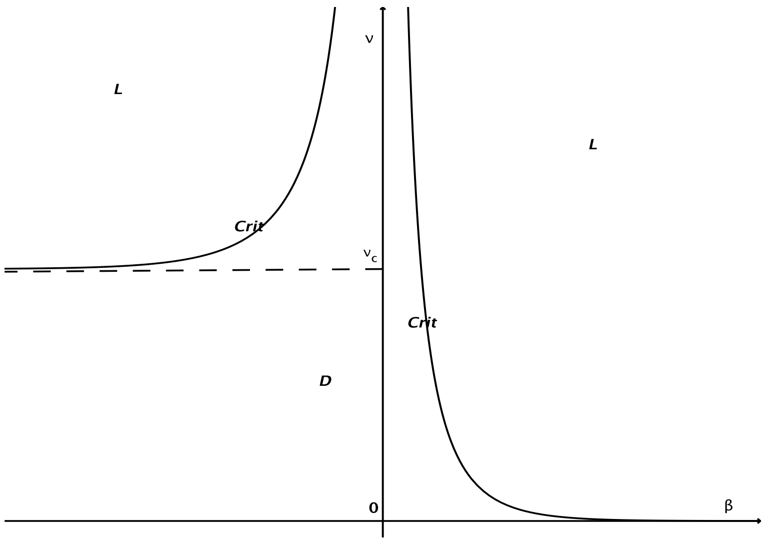

Figure 8.2: Shape of the phase diagram, $d \geq 3$.

Chapter 9

Complete localization

As we send some parameters to 0 or ∞, the present model converges to other related polymer models. A first instance is the intermediate disorder regime of section 10, where parameters β, ν, r are scaled with the polymer size, see (10.29).

9.1 A mean field limit

Another instance is the mean field limit: independently of the polymer length, we let

$$\nu \to \infty \text{ and } \beta \to 0 \text{ in such a way that } \nu\beta^2 \to b^2 \in (0, \infty).$$

Then, the rewards given by the Poisson medium get denser and weaker in this asymptotics so that they turn into a Gaussian environment, given by generalized Gaussian process $g(t, x)$ with mean 0 and covariance

$$\mathbf{E}[g(t,x)g(s,y)] = b^2\delta(t-s)|U(x) \cap U(y)|,$$

where $|\ \cdot\ |$ above denotes the Lebesgue measure. In other words, the environment is gaussian, which is correlated in space but not in time – it is Brownian-like. Here, the limit of our model is a Brownian directed polymer in a Gaussian environment, introduced in [142], with partition function

$$\mathfrak{Z}_t = P\left[\exp\{\int_0^t g(s, B(s))ds\}\right].$$

We do not elaborate this asymptotics, but instead we focus on the case $b = \infty$.

9.2 The regime of complete localization

This corresponds to letting, independently of the polymer length,

$$\nu \to \infty,\ |\beta| \le \beta_0, \quad \text{such that} \quad \nu\beta^2 \to \infty. \tag{9.1}$$

(The parameter r is kept fixed.) Precisely, we first let $t \to \infty$ and then take the limit (9.1).

Theorem 9.2.1. *Under the assumption* (9.1)*,*

$$1 - \mathcal{O}\left((\nu\beta^2)^{-1/6}\right) \le \liminf_{t\to\infty} \mathbb{P}\big[P_t^{\beta,\omega}(R_t^*)\big]$$
$$\le \limsup_{t\to\infty} \mathbb{P}\big[P_t^{\beta,\omega}(R_t^*)\big] \le 1 - \mathcal{O}\left((\nu\beta^2)^{-1/6}\right).$$

This statement describes the strong localization properties of the polymer path. The time-average $\frac{1}{t}\int_0^t \mathbf{1}_{B_s \in U(\mathsf{Y}^{(t)}(s))} ds$ is the *time fraction* the polymer spends together with the favourite path. We know that when $(\partial\psi)/(\partial\beta) \neq 0$ the time fraction is positive. The claim here is that it is almost the maximal value 1, in the limit (9.1). For a benchmark, we recall that, for the free measure P, for all smooth path Y and all $\delta > 0$, there exists a positive C such that for large t,

$$P\left(\frac{1}{t}\int_0^t \mathbf{1}_{B_s \in U(\mathsf{Y}(s))} ds \geq \delta\right) \leq e^{-Ct} \tag{9.2}$$

(In fact, it is not difficult to see (9.2) for $Y \equiv 0$ by applying Donsker-Varadhan's large deviations [65] for the occupation measure of Brownian motion. Then, one can use Girsanov transformation to extend (9.2) to the case of smooth path Y.)

Proof. Let $\hat{\phi}_t(\beta) = t^{-1}(\mathbb{P}\ln Z_t - t\nu\beta r^d)$. We assume $\beta \geq 0$, the other case being similar. By convexity of $\hat{\phi}_t$,

$$\begin{aligned}
\hat{\phi}_t(2\beta) - \hat{\phi}_t(\beta) &\overset{\text{convex}}{\geq} \beta \frac{\partial \hat{\phi}_t}{\partial \beta}(\beta) \\
&\overset{(2.17)}{=} \frac{\beta\nu\lambda(\beta)}{t}\int_{(0,t]\times\mathbb{R}^d} dsdx\, \mathbb{P}\, \frac{P_t^{\beta,\omega}[\chi_{s,x}] - P_t^{\beta,\omega}[\chi_{s,x}]^2}{1+\lambda(\beta)P_t^{\beta,\omega}[\chi_{s,x}]} .
\end{aligned}$$

Bounding from above the denominator in the integral by e^{β_0}, we get

$$1 - \mathbb{P}P_t^{\beta,\omega^{\otimes 2}}[R_t] \leq e^{\beta_0}\frac{\hat{\phi}_t(2\beta) - \hat{\phi}_t(\beta)}{\beta\lambda\nu r^d} . \tag{9.3}$$

Now, using the bound in Lemma 8.2.1, we derive

$$\text{R.H.S.}(9.3) = \mathcal{O}((\nu\beta^2)^{-1/6} .$$

Similarly,

$$\begin{aligned}
\hat{\phi}_t(\beta) - \hat{\phi}_t(\beta/2) &\overset{\text{convex}}{\leq} (\beta/2) \frac{\partial \hat{\phi}_t}{\partial \beta}(\beta) \\
&= \frac{\beta\nu\lambda(\beta)}{2t}\int_{(0,t]\times\mathbb{R}^d} dsdx\, \mathbb{P}\, \frac{P_t^{\beta,\omega}[\chi_{s,x}] - P_t^{\beta,\omega}[\chi_{s,x}]^2}{1+\lambda(\beta)P_t^{\beta,\omega}[\chi_{s,x}]} ,
\end{aligned}$$

leading to

$$1 - \mathbb{P}P_t^{\beta,\omega^{\otimes 2}}[R_t] \geq \frac{\hat{\phi}_t(\beta) - \hat{\phi}_t(\beta/2)}{\beta\lambda\nu r^d} = \mathcal{O}((\nu\beta^2)^{-1/6} ,$$

and to the desired result. ∎

We can extract fine additional information and geometric properties of the Gibbs measure. For $\delta \in (0, 1/2)$ define the (δ, t)-negligible set as

$$\mathcal{N}_{\delta,t}^{\eta} = \left\{(s,x) \in [0,t]\times\mathbb{R}^d : P_t^{\beta,\omega}(\chi_{s,x}) \leq \delta\right\},$$

and the (δ, t)-predominant set as

$$\mathcal{P}_{\delta,t}^{\eta} = \left\{(s,x) \in [0,t]\times\mathbb{R}^d : P_t^{\beta,\omega}(\chi_{s,x}) \geq 1-\delta\right\}.$$

As suggested by the names, $\mathcal{N}_{\delta,t}^{\eta}$ is the set of space-time locations the polymer wants to stay away from, and $\mathcal{P}_{\delta,t}^{\eta}$ is the set of locations the polymer likes to visit. Both sets depend on the environment.

Corollary 9.2.2. *For all* $0 < \delta < 1/2$*, we have, under the assumption* (9.1)*,*

$$\limsup_{t\to\infty} \mathbb{P}\left[\frac{1}{t}\left|(\mathcal{N}_{\delta,t}^{\eta} \cup \mathcal{P}_{\delta,t}^{\eta})^{\complement}\right|\right] = \mathcal{O}\left((\nu\beta^2)^{-1/6}\right), \tag{9.4}$$

$$\limsup_{t\to\infty} \mathbb{P}P_t^{\beta,\omega}\left[\frac{1}{t}\left|V_t(B) \bigcap \mathcal{N}_{\delta,t}^{\eta}\right|\right] = \mathcal{O}\left((\nu\beta^2)^{-1/6}\right), \tag{9.5}$$

$$\limsup_{t\to\infty} \mathbb{P}P_t^{\beta,\omega}\left[\frac{1}{t}\left|V_t(B)^{\complement} \bigcap \mathcal{P}_{\delta,t}^{\eta}\right|\right] = \mathcal{O}\left((\nu\beta^2)^{-1/6}\right). \tag{9.6}$$

Recall that $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$, and note that $|\mathcal{N}^{\eta}_{\delta,t}| = |V_t(B)^{\complement}| = \infty$.

The limits (9.4), (9.5), (9.6), bring information on how is the corridor around the favourite path where the measure concentrates for large $\nu\beta^2$. We depict the main features for large $\nu\beta^2$:

- most (in Lebesgue measure) time-space locations become negligible or predominant,
- most (in Lebesgue and Gibbs measures) negligible locations are outside the tube around the polymer path,
- most (in Lebesgue and Gibbs measures) predominant locations are inside the tube around the polymer path.

The trace $\{x \in \mathbb{R}^d : P_t^{\beta,\omega}(\chi_{s,x}) \geq 1-\delta\}$ at time t of the (δ,t)-predominant set is reminiscent of the ϵ-atoms discovered in [160], with $\epsilon = 1-\delta$, and discussed in [8]. These references study the time and space discrete setting, and restrict to the end point of the polymer.

Proof. It suffices to prove, for all $\delta \in (0,1/2]$,

$$\frac{1}{tr^d}\Big|\Big\{(s,x)\in[0,t]\times\mathbb{R}^d : P_t^{\beta,\omega}(\chi_{s,x})\in[\delta,1-\delta]\Big\}\Big| \leq \frac{1}{\delta(1-\delta)}P_t^{\beta,\omega^{\otimes 2}}\Big(1-R_t\Big) \tag{9.7}$$

$$P_t^{\beta,\omega}\left[\frac{1}{tr^d}\Big|V_t(B)\bigcap\Big\{(s,x) : P_t^{\beta,\omega}(\chi_{s,x}) \leq \delta\Big\}\Big|\right] \leq \frac{1}{1-\delta}\,P_t^{\beta,\omega^{\otimes 2}}\Big(1-R_t\Big) \tag{9.8}$$

$$P_t^{\beta,\omega}\left[\frac{1}{tr^d}\Big|V_t(B)^{\complement}\bigcap\Big\{(s,x) : P_t^{\beta,\omega}(\chi_{s,x}) \geq 1-\delta\Big\}\Big|\right] \leq \frac{1}{1-\delta}\,P_t^{\beta,\omega^{\otimes 2}}\Big(1-R_t\Big). \tag{9.9}$$

Note that

$$u(1-u) \geq (1-\delta)u\mathbf{1}_{u<\delta} + \delta(1-\delta)\mathbf{1}_{u\in[\delta,1-\delta]} + (1-\delta)(1-u)\mathbf{1}_{u>1-\delta}.$$

Setting $A_s = \{x : P_t^{\beta,\omega}(\chi_{s,x}) \in [\delta,1-\delta]\}$ and writing

$$\begin{aligned}
P_t^{\beta,\omega^{\otimes 2}}(1-R_t) &= \frac{1}{tr^d}\int_0^t\int_{\mathbb{R}^d}\Big[P_t^{\beta,\omega}(\chi_{s,x}) - P_t^{\beta,\omega}(\chi_{s,x})^2\Big]\,dxds \\
&\geq \frac{1}{tr^d}\int_0^t\int_{A_s}\Big[P_t^{\beta,\omega}(\chi_{s,x}) - P_t^{\beta,\omega}(\chi_{s,x})^2\Big]\,dxds \\
&\geq \delta(1-\delta)\frac{1}{tr^d}\int_0^t\Big|\{x : P_t^{\beta,\omega}(\chi_{s,x}) \in [\delta,1-\delta]\}\Big|\,ds,
\end{aligned}$$

which yields (9.7). For the next one, we write

$$\begin{aligned}
P_t^{\beta,\omega^{\otimes 2}}(1-R_t) &= \frac{1}{tr^d}\int_0^t\int_{\mathbb{R}^d}\Big[P_t^{\beta,\omega}(\chi_{s,x}) - P_t^{\beta,\omega}(\chi_{s,x})^2\Big]\,dxds \\
&\geq (1-\delta)\frac{1}{tr^d}\int_0^t\int_{\mathbb{R}^d}P_t^{\beta,\omega}(\chi_{s,x})\mathbf{1}_{P_t^{\beta,\omega}\chi_{s,x})<\delta}dsdx \\
&= (1-\delta)P_t^{\beta,\omega}\left[\frac{1}{tr^d}\int_0^t\int_{\mathbb{R}^d}\mathbf{1}_{P_t^{\beta,\omega}(\chi_{s,x})<\delta, B_s\in U(x)}\right]dsdx,
\end{aligned}$$

which is (9.8). The last claim can be proved similarly. ∎

Chapter 10

The Intermediate Regime ($d = 1$)

In this chapter, we focus on dimension $d = 1$, where the polymer is in the strong disorder phase as soon as $\beta \neq 0$ is kept fixed (Remark 3.4.3).

10.1 Introduction

Although it is believed that the model satisfies the following non-standard critical exponents:

$$\sup_{0 \le s \le t} |B_s| \approx t^{2/3} \quad \text{and} \quad \ln Z_t - \mathbb{P}[\ln Z_t] \approx t^{1/3} \quad \text{as } t \to \infty, \tag{10.1}$$

proofs are missing at this moment. It is also expected that the fluctuations of the free energy around its mean are of Tracy-Widom type:

Conjecture 10.1.1. *For all non-zero β, ν and r, there exists some constant $\sigma(\beta, \nu)$ such that, as $t \to \infty$,*

$$\frac{\ln Z_t - p(\beta, \nu) t}{\sigma(\beta, \nu) t^{1/3}} \xrightarrow{(d)} F_{\mathrm{GOE}} \tag{10.2}$$

where the F_{GOE} is the Tracy-Widom GOE distribution [155].

These properties are characteristics of the KPZ universality class. They are in sharp contrast to the weak disorder regime, where one knows to a large extent that $B_t \approx t^{1/2}$ (Theorem 7.2.2), and where the free energy $\ln Z_t$ has order one fluctuations around its mean (3.3), which are features of the Edward-Wilkinson universality class.

The KPZ universality class is a family of models of random surfaces dynamics that share non-gaussian statistics, non-standard critical exponents and scaling relations (3-2-1 in time, space and fluctuations, as in (10.1)). Members of this class include some interacting particles systems (asymmetric simple exclusion prosses (ASEP), interacting Brownian motions), paths in random environment (directed polymers, first and last passage percolation), stochastic PDEs (KPZ equation, stochastic Burgers equation, stochastic reaction-diffusion equations). The reader may refer to [54] for a non-technical review on the KPZ universality class.

The Kardar-Parisi-Zhang (KPZ) equation is the non-linear stochastic partial differential equation:

$$\frac{\partial \mathcal{H}}{\partial T}(T, X) = \frac{1}{2} \frac{\partial^2 \mathcal{H}}{\partial X^2}(T, X) + \frac{1}{2} \left(\frac{\partial \mathcal{H}}{\partial X}(T, X) \right)^2 + \beta \eta(T, X), \tag{10.3}$$

where $\beta \in \mathbb{R}$ and η is a random measure on $[0,1] \times \mathbb{R}$ called the space-time *Gaussian white noise*, which verifies that:

(i) For all measurable sets $A_1, \ldots, A_k$ of $[0,1] \times \mathbb{R}$, $\big(\eta(A_1), \ldots, \eta(A_k)\big)$ is a centered Gaussian vector.

(ii) For all measurable sets A, B of $[0,1] \times \mathbb{R}$, then $\mathbb{P}[\eta(A)\eta(B)] = |A \cap B|$.

The KPZ equation models the behavior of a random interface growth and was introduced by Kardar, Parisi and Zhang [107] in 1986. It is difficult to make sense of this equation and Bertini-Cancrini [14] argued that a possible definition of $\mathcal{H}_\beta$ could be given by the so-called *Hopf-Cole transformation*:

$$\mathcal{H}_\beta(T,X) = \ln \mathcal{Z}_\beta(T,X), \tag{10.4}$$

where $\mathcal{Z}_\beta$ is the solution of the stochastic heat equation (SHE):

$$\frac{\partial \mathcal{Z}_\beta}{\partial T}(T,X) = \frac{1}{2}\frac{\partial^2 \mathcal{Z}_\beta}{\partial X^2}(T,X) + \beta \mathcal{Z}_\beta(T,X)\eta(T,X). \tag{10.5}$$

In a breakthrough paper [3], Amir, Corwin and Quastel were able to describe the pointwise distribution of $\mathcal{H}_\beta(T,X)$ by exploiting the weak universality of the ASEP model. It results from this that the KPZ equation lies in the KPZ universality class.

The *weak KPZ universality conjecture* states that the KPZ equation is a universal object of the KPZ class. As a general idea, the KPZ equation should appear as a scaling limit at critical parameters for models that feature a phase transition between the Edward-Wilkinson class (4-2-1 scaling) and the KPZ class. This was first verified for the model of ASEP [16], and more recently for the discrete and Brownian directed polymers [2, 57]. The proofs rely on the Hopf-Cole transformation, which enables one to switch between the KPZ equation and the stochastic heat equation. In this chapter, we essentially summarize the arguments of [57] to explain why the Brownian polymer in Poisson environment model verifies the weak KPZ universality.

10.2 Connections between stochastic heat equation(s) and directed polymers

10.2.1 The continuum case

A special case of interest for the SHE, where $\mathcal{Z}_\beta(T,X)$ can be seen as the point-to-point partition function of a directed polymer, placed at $X=0$ at time $T=0$, is when

$$\mathcal{Z}_\beta(0,X) = \delta_0(X). \tag{10.6}$$

In this case, $\mathcal{Z}_\beta(T,X)$ can be expressed through the following shortcut (cf. Section 10.3.1):

$$\mathcal{Z}_\beta(T,X) = \rho(T,X)\ P_{0,0}^{T,X}\left[:\exp: \left(\beta \int_0^T \eta(u,B_u)\mathrm{d}u\right)\right], \tag{10.7}$$

where $\rho(t,x) = e^{-x^2/2t}/\sqrt{2\pi t}$.

This equation is similar to the definition of the point-to-point partition function a polymer with Brownian path and white noise environment. Alberts, Khanin and Quastel [1] were in fact able to construct a polymer measure with P2P partition function given by $\mathcal{Z}_\beta(T,X)$. As both the environment and the paths of the polymer are continuous, it was named *the continuum directed random polymer*.

Similarly to the Poisson polymer, the P2P free energy $\mathcal{F}_\beta(T,X)$ can be defined as

$$\mathcal{F}_\beta(T,X) = \ln \frac{\mathcal{Z}_\beta(T,X)}{\rho(T,X)}, \tag{10.8}$$

so that the free energy of the polymer and the solution of the KPZ equation follow the relation:

$$\mathcal{F}_\beta(T,X) = \mathcal{H}_\beta(T,X) + X^2/2T + \ln\sqrt{2\pi T}. \tag{10.9}$$

10.2.2 The Poisson case

Introduce the renormalized point-to-point partition function:

$$W(t,x;\omega,\beta,r) = \rho(t,x)P_{0,0}^{t,x}\left[\exp\{\beta\omega(V_t) - \lambda(\beta)\nu r^d t\}\right]. \tag{10.10}$$

We will often shorten the notation $W(t,x;\omega,\beta,r) = W(t,x)$ when no confusion can arise. Compared to $Z_t(\omega,\beta;x)$ of (4.1), a major difference is that it encorporates the Gaussian kernel as a factor. In the next theorem, we state that the renormalized P2P partition function verifies a weak formulation of the following *stochastic heat equation with multiplicative Poisson noise*:

$$\partial_t W(t,x) = \frac{1}{2}\Delta W(t,x) + \lambda W(t-,x)\bar{\omega}(\mathrm{d}t \times U(x)). \tag{10.11}$$

When $\beta = 0$, it reduces to the usual heat equation.

Theorem 10.2.1 (Weak solution). *For all $\varphi \in \mathcal{D}(\mathbb{R})$ and $t \geq 0$, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x = \varphi(0) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} W(s,x)\Delta\varphi(x)\mathrm{d}x \\ + \lambda \int_{\mathbb{R}} \mathrm{d}x\varphi(x) \int_{(0,t]\times\mathbb{R}} \bar{\omega}(\mathrm{d}s,\mathrm{d}y)W(s-,x)\mathbf{1}_{|y-x|\leq r/2}\,.\end{aligned} \tag{10.12}$$

Proof. Let $\xi_t = \exp(\beta\omega(V_t(B)) - \lambda(\beta)\nu r^d t)$ and observe that

$$\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x = P[\xi_t\varphi(B_t)].$$

Then, recalling that $\omega(V_t(B)) = \int \chi_{s,x}\,\omega_t(\mathrm{d}s\mathrm{d}x)$, we use Itô's formula [99, Section II.5] for fixed B to get that

$$\begin{aligned}\xi_t &= 1 - \lambda\nu r^d \int_0^t \xi_s \mathrm{d}s + \lambda \int_{(0,t]\times\mathbb{R}} \xi_{s-}\chi_{s,x}\,\omega(\mathrm{d}s\mathrm{d}x) \\ &= 1 + \lambda \int_{(0,t]\times\mathbb{R}} \xi_{s-}\chi_{s,x}\,\bar{\omega}(\mathrm{d}s\mathrm{d}x),\end{aligned} \tag{10.13}$$

as almost surely, $\mathbb{P}$-a.s. $\xi_s = \xi_{s-}$ a.e.

As a difference of two increasing processes, ξ is of finite variation over all bounded time intervals. Also note that one can get an expression to the measure associated to ξ from the last equation. By the integration by part formula [104, p.52],

$$\xi_t\varphi(B_t) = \xi_0\varphi(B_0) + \int_0^t \xi_{s-}\mathrm{d}\varphi(B_s) + \int_0^t \varphi(B_s)\mathrm{d}\xi_s + [\xi, \varphi(B)]_t,$$

where $[\xi, \varphi(B)]_t = 0$ since $\varphi(B)$ is continuous. Applying Itô 's formula on $\mathrm{d}\varphi(B)$ and then taking P-expectation (which cancels the martingale term in the Itô formula), one obtains by (10.13) that $\mathbb{P}$-a.s.

$$\begin{aligned}&\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x \\ &= \varphi(0) + \frac{1}{2}\int_0^t P[\xi_{s-}\Delta\varphi(B_s)]\mathrm{d}s + \lambda \int_{(0,t]\times\mathbb{R}} P[\varphi(B_s)\xi_{s-}\chi_{s,y}]\bar{\omega}(\mathrm{d}s\mathrm{d}y) \\ &= \varphi(0) + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \Delta\varphi(x)W(s-,x)\mathrm{d}x\mathrm{d}s + \lambda \int_{(0,t]\times\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x)\mathbf{1}_{|y-x|\leq r/2}W(s-,x)\mathrm{d}x\right)\bar{\omega}(\mathrm{d}s\mathrm{d}y).\end{aligned}$$

To conclude the proof, observe that we can apply Fubini's theorem to the last integral since for all $t > 0$,

$$\begin{aligned}\mathbb{P}\int_{(0,t]\times\mathbb{R}} P[|\varphi(B_s)|\xi_{s-}\chi_{s,y}]\omega(\mathrm{d}s\mathrm{d}y) &= \nu e^{\beta} \int_{(0,t]\times\mathbb{R}} \mathbb{P}[\xi_{s-}]P[|\varphi(B_s)|\chi_{s,y}]\mathrm{d}s\mathrm{d}y \\ &= \nu e^{\beta} r \int_0^t P[|\varphi(B_s)|]\mathrm{d}s < \infty,\end{aligned}$$

where we have used the Mecke equation, cf. (2.16) or [115, 4.1] in the first equality. ■

10.3 Chaos expansions

Let us first introduce some notations. For any $k \geq 1$, $s_1, \dots, s_k \in \mathbb{R}_+$ and $x_1, \dots, x_k \in \mathbb{R}$, write $\mathbf{s} = (s_1, \dots, s_k)$ and $\mathbf{x} = (x_1, \dots, x_k)$. Let

$$\Delta_k(0,t) = \{\mathbf{s} \in [u,t]^k \,|\, 0 < s_1 < \cdots < s_k \leq t\}, \tag{10.14}$$

be the k-dimensional simplex and $\Delta_k = \Delta_k(0,1)$.

10.3.1 The continuum case

We give here the definition of a *mild* solution to the stochastic heat equation, and we will see how this leads to an expression of the solution as a Wiener chaos expansion. We first mention that it is possible (cf. [105]) to extend the integral over the space time white noise to any square integrable function:

Proposition 10.3.1. *There exists an isometry $I_1 : L^2\big([0,1] \times \mathbb{R}\big) \mapsto L^2(\Omega, \mathcal{G}, \mathbb{P})$ verifying:*

(i) For all measurable set A of $[0,1] \times \mathbb{R}$, we have $I_1(A) = \eta(A)$.

(ii) For all $g \in L^2$, the variable $I_1(g)$ is a centered Gaussian variable of variance $\|g\|^2_{L^2([0,1]\times\mathbb{R})}$.

We call $I_1(g)$ the *Wiener integral* which also writes $I_1(g) = \int_{[0,1]} \int_{\mathbb{R}} g(s,x)\, \eta(\mathrm{d}s, \mathrm{d}x)$.

It is said that $\mathcal{Z}$ is a *mild* solution to the stochastic heat equation (10.5) if, for all $0 \leq S < T \leq 1$,

$$\begin{aligned}\mathcal{Z}(T,X) = &\int_{\mathbb{R}} \rho(T-S, X-Y)\mathcal{Z}(S,Y)\mathrm{d}Y \\ &+ \beta \int_S^T \int_{\mathbb{R}} \rho(T-U, X-Y)\mathcal{Z}(U,Y)\eta(U,Y)\mathrm{d}U\mathrm{d}Y,\end{aligned} \tag{10.15}$$

and if for all $T \geq 0$, $\mathcal{Z}(T,X)$ is measurable with respect to the white noise on $[0,T] \times \mathbb{R}$.

Remark 10.3.2. *As a motivation to look at this form of the equation, one can check that if $\mathcal{Z}(T,X)$ satisfies* (10.15) *with a smooth deterministic function $\eta(U,Y)$, then $\mathcal{Z}(T,X)$ is a solution to the SHE* (10.5) *with smooth noise.*

Remark 10.3.3. *Under some integrability condition, it can be shown that there is a unique mild solution - up to indistinguishability - to the SHE with Dirac initial condition* [14]. *This solution is continuous in time and space for $(T,X) \in (0,1] \times \mathbb{R}$, and it is continuous in $T = 0$ in the space of distributions. Furthermore, $\mathcal{Z}_\beta(T,X)$ can be shown to be positive for all $T > 0$* [126, 128].

Using the initial condition $\mathcal{Z}(0,X) = \delta_X$, we get by iterating equation (10.15) that

$$\begin{aligned}\mathcal{Z}(T,X) = \rho(T,X) + \beta &\int_0^T \int \rho(T-U, X-Y)\rho(U,Y)\eta(U,Y)\mathrm{d}U\mathrm{d}Y \\ + \beta^2 \iint_{0<R<U\leq T} &\iint_{\mathbb{R}^2} \rho(T-U, X-Y)\rho(U-R, Y-Z)\mathcal{Z}(R,Z) \\ &\times \eta(U,Y)\eta(R,Z)\mathrm{d}U\mathrm{d}Y\mathrm{d}R\mathrm{d}Z.\end{aligned}$$

It is possible to give a proper definition of these iterated integrals, and one can find the details of such a procedure in [105, Chapter 7]. We give a few properties of these integrals:

Property 10.3.4. *For all $k > 0$, there exists a map $I_k : L^2(\Delta_k \times \mathbb{R}^k) \mapsto L^2(\Omega, \mathcal{G}, \mathbb{P})$, which has the following properties:*

(i) For all $g \in L^2(\Delta_k \times \mathbb{R}^k)$ and $h \in L^2(\Delta_j \times \mathbb{R}^j)$, the variable $I_k(g)$ is centered and

$$\mathbb{P}\left[I_k(g)I_j(h)\right] = \ \delta_{k,j}\ < g,h >_{L^2(\Delta_k\times\mathbb{R}^k)}\ . \tag{10.16}$$

(ii) The map I_k is linear, in the sense that for all square-integrable f, g and reals λ, μ,

$$\mathbb{P}\text{-a.s.} \quad I_k(\lambda f + \mu g) = \lambda\, I_k(f) + \mu\, I_k(g).$$

The operator I_k is called the ***multiple Wiener integral****, and for $g \in L^2(\Delta_k \times \mathbb{R}^k)$, we also write*

$$I_k(g) = \int_{\Delta_k} \int_{\mathbb{R}^k} g(\mathbf{t}, \mathbf{x}) \eta^{\otimes k}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}).$$

Remark 10.3.5. *As a justification of the "iterated integral" property, it can be shown that the map I_k extends to $L^2([0,1]^k \times \mathbb{R}^k)$, where it verifies that for all orthogonal family $(g_1, \dots, g_k)$ of functions in $L^2([0,1] \times \mathbb{R})$:*

$$I_k\left(\bigotimes_{j=1}^{k} g_j\right) = \prod_{j=1}^{k} I_1(g_j), \tag{10.17}$$

where $\bigotimes$ denotes the tensor product: $(\bigotimes_{j=1}^k g_j)(\mathbf{s}, \mathbf{x}) = \prod_{j=1}^k g_j(s_j, x_j)$.

By repeating the above iteration procedure, one gets that:

$$\mathcal{Z}(T, X) = \rho(T, X) + \sum_{k=1}^{\infty} \beta^k \int_{\Delta_k} \int_{\mathbb{R}^k} \rho^k(\mathbf{S}, \mathbf{Y}; T, X) \eta^{\otimes k}(\mathrm{d}\mathbf{S}, \mathrm{d}\mathbf{Y}), \tag{10.18}$$

where we have used the notation, for $\mathbf{s} \in \Delta_k(s,t)$ and $\mathbf{y} \in \mathbb{R}^d$,

$$\rho^k(\mathbf{s}, \mathbf{y}\,; t, x) = \rho(s_1, y_1) \left(\prod_{j=1}^{k-1} \rho(s_{j+1} - s_j, y_{j+1} - y_j)\right) \rho(t - s_k, x - y_k).$$

The infinite sum (10.18) is called a *Wiener chaos expansion*. By the covariance structure of the Wiener integrals (10.16), all the integrals in the sum are orthogonal and to prove that (10.18) converges in L^2, it suffices to check that (see [1]):

$$\sum_{k=0}^{\infty} \|\rho^k(\cdot, \cdot; T, X)\|^2_{L^2(\Delta_k \times \mathbb{R}^k)} < \infty.$$

The ratio $\frac{\rho^k(\mathbf{s}, \mathbf{y}\,; t, x)}{\rho(t,x)}$ is the k-steps transition function of a Brownian bridge, starting from $(0,0)$ and ending at (t,x). From this observation, it is possible to introduce an alternative expression of the mild solution $\mathcal{Z}(T,X)$ of SHE equation, via a Feynman-Kac formula:

$$\mathcal{Z}(T, X) = \rho(T, X)\ P_{0,0}^{T,X}\left[: \exp : \left(\beta \int_0^T \eta(u, B_u) \mathrm{d}u\right)\right], \tag{10.19}$$

The Wick exponential $: \exp :$ of a Gaussian random variable ξ is defined by

$$: \exp(\xi) := \sum_{k=0}^{\infty} \frac{1}{k!} : \xi^k :$$

where the $: \xi^k :$ notation stands for the Wick power of a random variable (cf.[105]). The integral $\int_0^T \eta(u, B_u)\mathrm{d}u$, on the other hand, is not well defined, and to understand how to go from (10.19) to (10.18), one should use the following identification:

$$P_{0,0}^{T,X}\left[: \left(\beta \int_0^T \eta(u, B_u) \mathrm{d}u\right)^k :\right] = \beta^k k! \int_{\Delta_k} \int_{\mathbb{R}^k} \frac{\rho^k(\mathbf{S}, \mathbf{Y}\,; T, X)}{\rho(T, X)} \eta^{\otimes k}(\mathrm{d}\mathbf{S}\mathrm{d}\mathbf{Y}).$$

From now on, we suppose that $\mathcal{Z}_\beta(T,X)$ is defined through equation (10.18). Integrating over X this equation leads to the definition of the partition function of the continuum polymer:

$$\mathcal{Z}_\beta = \sum_{k=0}^{\infty} \beta^k I_k(\rho^k), \tag{10.20}$$

where ρ^k is the k-th dimensional Brownian transition function, defined for $(\mathbf{s},\mathbf{x}) \in \Delta_k \times \mathbb{R}^k$ by:

$$\begin{aligned}\rho^k(\mathbf{s},\mathbf{x}) &= \rho(s_1,x_1)\left(\prod_{j=1}^{k-1}\rho(s_{j+1}-s_j, x_{j+1}-x_j)\right)\\ &= P\left(B_{s_1}\in \mathrm{d}x_1,\dots,B_{s_k}\in \mathrm{d}x_k\right),\end{aligned} \tag{10.21}$$

with the convention that $\rho^0 = 1$. The motivation for writing the partition function as in (10.20) is that (10.18) writes $\mathcal{Z}_\beta(T,X) = \sum_{k=0}^{\infty}\beta^k I_k(\rho^k(\cdot;T,X))$.

10.3.2 The Poisson case

We want to express W_t in a similar way as (10.20), this time with Poisson iterated integrals. We give here the basic definitions of these integrals and one can refer to [115] for more details[1].

Definition 10.3.6. *For any positive integer k, define the* ***k-th factorial measure*** $\omega_t^{(k)}$ *to be the point process on $\Delta_k(0,t)\times\mathbb{R}^k$, such that, for any measurable set $\mathbf{A}\subset\Delta_k(0,t)\times\mathbb{R}^k$,*

$$\omega_t^{(k)}(\mathbf{A}) = \sum_{\substack{(s_1,x_1),\dots,(s_k,x_k)\in\,\omega_t\\ s_1<\dots<s_k}} \mathbf{1}_{((s_1,x_1),\dots,(s_k,x_k))\in\mathbf{A}}. \tag{10.22}$$

Otherwise stated,

$$\omega_t^{(k)} = \sum_{\substack{(s_1,x_1),\dots,(s_k,x_k)\in\,\omega_t\\ s_1<\dots<s_k}} \delta_{((s_1,x_1),\dots,(s_k,x_k))}\,. \tag{10.23}$$

These factorial measures define naturaly a multiple integral for the point process ω_t. Contrary to the Wiener integrals, these integrals are not centered, so what we really want is to define a multiple integral for the compensated process $\bar\omega_t$. This is done as follows:

Definition 10.3.7. *For $k\geq 1$ and $g\in L^1(\Delta_k(0,t)\times\mathbb{R}^k)$, denote the* ***multiple Wiener-Itô integral*** *of g as*

$$\bar\omega_t^{(k)}(g) := \sum_{J\subset[k]}(-1)^{k-|J|}\int_{\Delta_k\times\mathbb{R}^k} g(\mathbf{s},\mathbf{x})\,\omega_t^{(|J|)}(\mathrm{d}\mathbf{s}_J,\mathrm{d}\mathbf{x}_J)\,\nu^{k-|J|}\,\mathrm{d}\mathbf{s}_{J^c}\,\mathrm{d}\mathbf{x}_{J^c}. \tag{10.24}$$

When $k=0$, define $\bar\omega_t^{(0)}$ to be the identity on $\mathbb{R}$.

The two following results can be found in [115]:

Proposition 10.3.8. *For $k\geq 1$, the map $\bar\omega_t{}^{(k)}$ can be extended to a map*

$$\begin{array}{rcccl}\bar\omega_t^{(k)} & : & L^2(\Delta_k(0,t)\times\mathbb{R}^k) & \to & L^2(\Omega,\mathcal{G},\mathbb{P})\\ & & g & \mapsto & \bar\omega_t^{(k)}(g),\end{array}$$

which coincides with the above definition of $\bar\omega_t^{(k)}$ on the functions of $L^1\cap L^2(\Delta_k(0,t)\times\mathbb{R}^k)$.

Property 10.3.9. *(i) For any $k\geq 1$ and $g\in L^2(\Delta_k(0,t)\times\mathbb{R}^k)$, we have $\mathbb{P}\big[\bar\omega_t^{(k)}(g)\big]=0$.*

[1]Note that for simplicity, we choosed to define here the integrals for functions of the simplex, so that some normalizing $k!$ terms and symmetrisation of some objects should be added to match the definitions in [115].

(ii) For any $k \geq 1$ and $l \geq 1$, $g \in L^2(\Delta_k(0,t) \times \mathbb{R}^k)$ and $h \in L^2(\Delta_l(0,t) \times \mathbb{R}^l)$, the following covariance structure holds:

$$\mathbb{P}\left[\bar{\omega}_t^{(k)}(g)\,\bar{\omega}_t^{(l)}(h)\right] = \delta_{k,l}\ \nu^k < g, h >_{L^2(\Delta_k(0,t)\times\mathbb{R}^k)} . \tag{10.25}$$

(iii) The map $\bar{\omega}_t^{(k)}$ is linear, in the sense that for all square-integrable f, g and reals λ, μ,

$$\mathbb{P}\text{-}a.s.\quad \bar{\omega}_t^{(k)}(\lambda f + \mu g) = \lambda\,\bar{\omega}_t^{(k)}(f) + \mu\,\bar{\omega}_t^{(k)}(g).$$

Proposition 10.3.10. [57] *The renormalized partition function admits the following Wiener-Itô chaos expansion:*

$$W_t = \sum_{k=0}^{\infty} \bar{\omega}_t^{(k)}(\Psi^k), \tag{10.26}$$

where the sum converges in L^2 and where, for all $\mathbf{s} \in \Delta_k$, $\mathbf{x} \in \mathbb{R}^k$ and $k \geq 0$, we have set:

$$\Psi^k(\mathbf{s}, \mathbf{x}) = \lambda(\beta)^k\, P\left[\prod_{i=1}^{k} \chi_{s_i,x_i}(B)\right], \tag{10.27}$$

with the convention that an empty product equals 1.

Sketch of proof. We follow the proof of Lemma 18.9 in [115]. By definition we have that $W_t = P\left[e^{\beta\omega(V_t(B)-t\lambda r\nu}\right]$. Hence, assuming that Fubini's theorem applies to the RHS of (10.26), it is enough to show that $\mathbb{P} \times P$-almost surely:

$$e^{\beta\omega(V_t(B))-t\lambda r\nu} = \sum_{k=0}^{\infty} \bar{\omega}_t^{(k)}\left((\lambda\chi)^{\otimes k}\right), \tag{10.28}$$

where, for all $\mathbf{s} \in \Delta_k(0,t)$, $\mathbf{x} \in \mathbb{R}^k$, we have defined $(\lambda\chi)^{\otimes k}(\mathbf{s}, \mathbf{x}) = \prod_{j=1}^{k} \lambda(\beta)\chi_{s_j,x_j}(B)$. Then, observe that:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \bar{\omega}_t^{(k)}\left((\lambda\chi)^{\otimes k}\right) \\
&= \sum_{k=0}^{\infty} \sum_{J\subset[k]} (-1)^{k-|J|} \int_{\Delta_k(0,t)\times\mathbb{R}^k} \prod_{i=1}^{k} \lambda\chi_{s_i,x_i}\, \omega_t^{(|J|)}(\mathrm{d}\mathbf{s}_J, \mathrm{d}\mathbf{x}_J)\, \nu^{k-|J|}\, \mathrm{d}\mathbf{s}_{J^c}\, \mathrm{d}\mathbf{x}_{J^c} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{1}{k!} \int_{[0,t]^k\times\mathbb{R}^k} \prod_{i=1}^{k} \lambda\chi_{s_i,x_i}\, \omega_t^{(j)}(\mathrm{d}\mathbf{s}_{[j]}, \mathrm{d}\mathbf{x}_{[j]})\, \nu^{k-j}\, \mathrm{d}\mathbf{s}_{[j]^c}\, \mathrm{d}\mathbf{x}_{[j]^c} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!}\, j!\, \omega^{(j)}\left((\lambda\chi)^{\otimes j}\right) \sum_{k=j}^{\infty} \frac{1}{(k-j)!} (-t\lambda\nu r)^{k-j} \\
&= e^{-t\lambda r\nu} \sum_{j=0}^{\infty} \omega^{(j)}\left((\lambda\chi)^{\otimes j}\right).
\end{aligned}$$

Then, if we let $(s_1, x_1), \ldots, (s_N, x_N)$ with $s_1 < \cdots < s_N$ be the points of ω that lie in the tube $V_t(B)$, we get by definition of the $\omega_t^{(j)}$'s that

$$\sum_{j=0}^{\infty} \omega_t^{(j)}\left((\lambda\chi)^{\otimes j}\right) = \sum_{J\subset[N]} \prod_{i\in J} \left(e^{\beta\chi_{s_i,x_i}} - 1\right) = \prod_{i=1}^{N} e^{\beta\chi_{s_i,x_i}} = e^{\beta\omega(V_t(B))},$$

where the last equality comes from a telescopic sum (and the convention that an empty product is 1). This implies (10.28).

To prove convergence in L^2 of the sum in the RHS of (10.26), notice that the terms are pairwise orthogonal and verify:

$$\begin{aligned}\mathbb{P}\left[\bar{\omega}_t^{(k)}\left(T_k\, W_t\right)^2\right] &= \lambda^{2k}\int_{\Delta_k(0,t)\times\mathbb{R}^k} P\left[\prod_{i=1}^k \chi_{s_i,x_i}(B)\right]^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}\\ &\leq \frac{\lambda^{2k}}{k!}\int_{[0,t]^k\times\mathbb{R}^k} P\left[\prod_{i=1}^k \chi_{s_i,x_i}(B)\right]\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}\\ &= \frac{(\lambda^2 t\nu r)^k}{k!},\end{aligned}$$

whose sum converges. ■

10.4 The intermediate regime

We now consider parameters $\beta_t \in \mathbb{R}$, $\nu_t > 0$ and $r_t > 0$ that depend on time t, and we fix a parameter $\beta^* \in \mathbb{R}^*$. We assume that they verify the following asymptotic relations, as $t \to \infty$:

$$\begin{gathered}\text{(a) } \nu_t r_t^2 \lambda(\beta_t)^2 \sim (\beta^*)^2 t^{-1/2}, \quad \text{(b) } \nu_t r_t^3 \lambda(\beta_t)^3 \to 0,\\ \text{(c) } r_t/\sqrt{t} \to 0.\end{gathered} \tag{10.29}$$

Suppose for example that $r_t = \nu_t = 1$. Then, the scaling conditions are equivalent to $\beta_t = \beta^* t^{-1/4}$, so that we can see the scaling as a limit from strong disorder ($\beta > 0$) to weak disorder ($\beta = 0$).

We also note that similar computation to what led to (3.26), one gets that

$$\mathbb{P}\left[W_t^2\right] \leq P\left[\exp\left(2\lambda(\beta_t)^2 \nu_t r_t^2 \int_0^t \mathbf{1}_{|B_s|\leq 1}\,\mathrm{d}s\right)\right],$$

with $P\left[\int_0^t \mathbf{1}_{|B_s|\leq 1}\,\mathrm{d}s\right] \sim Ct^{1/2}$, so, again by Khas'minskii's lemma, one gets that W_t is bounded in L^2 as soon as

$$\lambda(\beta_t)^2 \nu_t r_t^2 t^{1/2} = O(1),$$

which is condition (a). We emphasize the fact that for fixed parameters $\beta, \nu, r > 0$, the martingale is not bounded in L^2 since $\beta_c = 0$. Hence, the regime we are looking at should be interpreted as a crossover between strong disorder and weak disorder. This explains the name *intermediate disorder regime.*

Remark 10.4.1. *The asymptotics of the parameters are in contrast to the regime of complete localization* (9.1)*, where, for fixed* r*, one let* $\nu\beta^2 \to \infty$*.*

The following theorem states that under the above scaling, the P2L and P2P partition functions of the Poisson polymer converge to the one of the continuum polymer:

Theorem 10.4.2. *Suppose conditions (a), (b) and (c) hold. Then, as* $t \to \infty$*:*

$$W_t(\omega^{\nu_t}, \beta_t, r_t) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}, \tag{10.30}$$

where ω^{ν_t} *is the Poisson point process with intensity measure* $\nu_t \mathrm{d}s\mathrm{d}x$*. Moreover, for all* $S, Y, T, X \in [0,1]$*, we have*

$$\sqrt{t}W\left(tS, \sqrt{t}Y; tT, \sqrt{t}X; \omega^{\nu_t}, \beta_t, r_t\right) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}\left(S, Y; T, X\right), \tag{10.31}$$

where the renormalized P2P partition function from (S,Y) *to* (T,X) *is defined by*

$$W(s, y; t, x; \omega, \beta, r) = W(t-s, x-y; \omega, \beta, r) \circ \theta_{s,y}, \tag{10.32}$$

and similarly for $\mathcal{Z}_{\beta^*}\left(S, Y; T, X\right)$*.*

Remark 10.4.3. *The $\sqrt{t}$ term appears here as a renormalization in the scaling of the heat kernel: $\sqrt{t}\rho\left(tT, \sqrt{t}X\right) = \rho(T, X)$.*

Sketch of proof. We focus on showing (10.30), as the result for the P2P partition function follows from the same technique and remark 10.4.3. Let γ_t be proportional to the vanishing parameter appearing in scaling relation (b):

$$\gamma_t := (\beta^*)^{-3}\nu_t r_t^3 \lambda(\beta_t)^3 \to 0. \tag{10.33}$$

and we now specify the radius for the indicator $\chi^\delta_{s,x}(B) = \mathbf{1}_{|B_s - y| \le \delta/2}$. Introduce the following time-depending functions of $\Delta_k(0,t) \times \mathbb{R}^k$:

$$\phi_t^k(\mathbf{s}, \mathbf{x}) = \gamma_t^{-k}\, \lambda(\beta_t)^k P\left[\prod_{i=1}^k \chi^{r_t/\sqrt{t}}_{s_i, x_i}(B)\right]. \tag{10.34}$$

Note that for all (s,x), the diffusive scaling property of the Brownian motion implies that

$$\chi^{r_t/\sqrt{t}}_{s/t, x/\sqrt{t}} = \mathbf{1}_{|B_{s/t} - x/\sqrt{t}| \le r_t/2\sqrt{t}} \overset{\text{law}}{=} \chi^{r_t}_{s,x}.$$

Therefore, using notation $\widetilde{\phi}_t^k = \phi_t^k(\cdot/t, \cdot/\sqrt{t})$, we see that after simple rescaling, equation (10.27) becomes

$$\gamma_t^k\, \widetilde{\phi}_t^k(\mathbf{s}, \mathbf{x}) = \lambda(\beta)^k\, P\left[\prod_{i=1}^k \chi_{s_i, x_i}(B)\right]. \tag{10.35}$$

Hence, Proposition (10.3.10) and equation (10.35) lead to the following expression of W_t:[2]

$$W_t = \sum_{k=0}^\infty \gamma_t^k\, \bar{\omega}_t^{(k)}\left(\widetilde{\phi}_t^k\right). \tag{10.36}$$

Now, we also define the rescaled functions $\tilde{\rho}_t^k = \rho^k(\cdot/t, \cdot/\sqrt{t})$ and we make two claims:

- **Claim 1:** For all $k \ge 0$ and as $t \to \infty$, $\phi_t^k \xrightarrow{L^2} (\beta^*)^k \rho^k$,
- **Claim 2:** As $t \to \infty$, $\sum_{k=0}^\infty (\beta^*)^k \gamma_t^k\, \bar{\omega}_t^{(k)}(\tilde{\rho}_t^k) \xrightarrow{(d)} \sum_{k=0}^\infty (\beta^*)^k I_k(\rho^k) = \mathcal{Z}_{\beta^*}$.

Claim 1 follows from the scaling relations and the fact that $\varepsilon^{-k} P[\prod_{i=1}^k \chi^\varepsilon_{s_i,x_i}(B)] \to \rho^k(\mathbf{s}, \mathbf{x})$ as $\varepsilon \to 0$. For claim 2, we only present the argument for the convergence in law of the $k = 1$ term of the sum. The complete argument relies on this case to extend the convergence to all $k \ge 1$ terms (see [57]). As $\tilde{\omega}^{(1)} = \tilde{\omega}$ and $\rho^1 = \rho$, we can apply the complex exponential formula (see equation (2.7)) to compute the characteristic function of $\gamma_t\, \bar{\omega}_t(\tilde{\rho}_t)$. For $u \in \mathbb{R}$, we obtain:

$$\begin{aligned}
&\mathbb{P}\left[e^{iu\gamma_t \bar{\omega}_t(\tilde{\rho}_t)}\right] \\
&= \exp\left(\int_{[0,t]}\int_{\mathbb{R}} \left(e^{iu\gamma_t \rho(s/t, x/\sqrt{t})} - 1 - iu\gamma_t\, \rho(s/t, x/\sqrt{t})\right)\nu_t\, \mathrm{d}s\mathrm{d}x\right) \\
&= \exp\left(\int_{[0,1]}\int_{\mathbb{R}} \nu_t t^{3/2}\left(e^{iu\gamma_t\rho(s,x)} - 1 - iu\gamma_t\, \rho(s,x)\right)\mathrm{d}s\mathrm{d}x\right).
\end{aligned}$$

By Taylor-Lagrange formula, we get that

$$\forall (s,x) \in [0,1] \times \mathbb{R},\ \nu_t t^{3/2}\left|e^{iu\gamma_t\rho(s,x)} - 1 - iu\gamma_t\, \rho(s,x)\right| \le \nu_t t^{3/2}\gamma_t^2 \frac{u^2}{2}\rho(s,x)^2,$$

[2] Note that from now on, we will always assume that $\omega \overset{\text{law}}{=} \omega^{\nu_t}$, although we will drop the superscript notation.

This gives L^1 domination since since $\rho \in L^2([0,1]\times\mathbb{R})$ and since relations (a) and (b) imply that $\nu_t\gamma_t^2 \sim t^{-3/2}$. Moreover, as $\gamma_t \to 0$, the integrand of the above integral converges pointwise to the function $(s,x) \mapsto -\frac{u^2}{2}\rho^2(s,x)$. Therefore, by dominated convergence, we obtain that as $t\to\infty$,

$$\mathbb{P}\left[e^{iu\gamma_t\tilde{\omega}_t(\tilde{\rho}_t)}\right] \to \exp\left(-\frac{u^2}{2}\|\rho\|_2^2\right).$$

This is the Fourier transform of a centered Gaussian random variable of variance $\|\rho\|_2^2$, which has the same law as $I_1(\rho)$, so that indeed $\bar{\omega}_t^{(1)}(\tilde{\rho}_t^1) \xrightarrow{(d)} I_1(\rho^1)$.

Now, if X_n and Y_n are random variables such that $Y_n \xrightarrow{(d)} Y$ and $\|Y_n - X_n\|_2 \longrightarrow 0$, then $X_n \xrightarrow{(d)} Y$. Therefore, to prove that $W_t \xrightarrow{(d)} \mathcal{Z}_{\beta^*}$, it is enough by claim 2 to show that

$$\left\|\sum_{k=0}^{\infty}\gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{\phi}_t^k) - \sum_{k=0}^{\infty}(\beta^*)^k\gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{\rho}_t^k)\right\|_2^2 \underset{t\to\infty}{\longrightarrow} 0. \tag{10.37}$$

By Pythagoras' identity and linearity of $\tilde{\omega}_t^{(k)}$, we obtain that the above norm writes:

$$\sum_{k=0}^{\infty}\gamma_t^{2k}\,\|\bar{\omega}_t^{(k)}\big(\phi_t^k(\cdot/t,\cdot/\sqrt{t}) - (\beta^*)^k\rho^k(\cdot/t,\cdot/\sqrt{t})\big)\|_2^2.$$

For all $g\in L^2(\Delta_k\times\mathbb{R}^k)$, we get from a substitution of variables that

$$\|\bar{\omega}_t^{(k)}\big(g(\cdot/t,\cdot/\sqrt{t})\|_2^2 = \nu_t^k\|g(\cdot/t,\cdot/\sqrt{t})\|^2_{L^2(\Delta_k(0,t)\times\mathbb{R}^k)} = \nu_t^k t^{3k/2}\|g\|^2_{L^2(\Delta_k\times\mathbb{R}^k)},$$

so that the above sum is given by

$$\sum_{k=0}^{\infty}\gamma_t^{2k}\nu_t^k t^{3k/2}\,\|\phi_t^k - (\beta^*)^k\rho^k\|^2_{L^2(\Delta_k\times\mathbb{R}^k)}.$$

Conditions (a) and (b) imply that $\gamma_t^2\nu_t^k t^{3/2}\sim 1$, so the proof can be concluded by claim 1 and by showing that $\|\phi_t^k\|_2^2$ is dominated by the summable sequence $C^{2k}\|\rho^k\|_2^2$, where $C = C(\beta^*)$ is some positive constant, so that the dominated convergence theorem applies. ∎

10.5 Convergence in terms of processes of the P2P partition function

Let $\mathcal{D}'(\mathcal{R})$ denote the space of distributions on $\mathbb{R}$, and $D\big([0,1],\mathcal{D}'(\mathcal{R})\big)$ the space of càdlàg function with values in the space of distributions, equipped with the topology defined in [124]. We also define the rescaling of the renormalized P2P partition function (10.10):

$$\mathcal{Y}_t(T,X) = \rho(T,X)W\left(tT,\sqrt{t}X;\omega^{\nu_t},\beta_t,r_t\right). \tag{10.38}$$

The two variables function $\mathcal{Y}_t$ can be seen as an element of $D\big([0,1],\mathcal{D}'(\mathcal{R})\big)$, through the mapping $\mathcal{Y}_t : T\mapsto \big(\varphi\mapsto\int\mathcal{Y}_t(T,X)\varphi(X)\mathrm{d}X\big)$. The next theorem states that the rescaled partition function $\mathcal{Y}_t$ converges, in terms of processes, to the solution of the stochastic heat equation:

Theorem 10.5.1. [57] *Suppose that $(\beta_t)_{t\geq 0}$ is bounded by above. As $t\to\infty$, the following convergence of processes holds:*

$$\mathcal{Y}_t \xrightarrow{(d)} \big(T\mapsto\mathcal{Z}_{\beta^*}(T,\cdot)\big), \tag{10.39}$$

where the convergence in distribution holds in $D\big([0,1],\mathcal{D}'(\mathcal{R})\big)$.

For any function $F\in D\left([0,1],\mathcal{D}'(\mathbb{R})\right)$ and $\varphi\in\mathcal{D}(\mathbb{R})$, set

$$F(T,\varphi) := \int F(T,X)\varphi(X)\mathrm{d}X. \tag{10.40}$$

In order to show tightness of $\mathcal{Y}_t$, the tool used in [57] is Mitoma's criterion [124, 161]:

Proposition 10.5.2. *Let $(F_t)_{t\geq 0}$ be a family of processes in $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$. If, for all $\varphi \in \mathcal{D}(\mathbb{R})$, the family $T \to F_t(T,\varphi), t \geq 0$ is tight in the real cadlàg functions space $D([0,1],\mathbb{R})$, then $(F_t)_{t\geq 0}$ is tight in $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$.*

Then, to prove uniqueness of the limit, one can rely on the following proposition:

Proposition 10.5.3 ([124])**.** *Let $(F_t)_{t\geq 0}$ be a tight family of processes in the space $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$. If there exists a process $F \in D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$ such that, for all $n \geq 1$, $T_1, \dots, T_n \in [0,1]$ and $\varphi_1, \dots, \varphi_n \in \mathcal{D}(\mathbb{R})$, we have as $t \to \infty$:*

$$(F_t(T_1,\varphi_1), \dots, F_t(T_n,\varphi_n)) \xrightarrow{(d)} (F_t(T_1,\varphi_1), \dots, F_t(T_n,\varphi_n)) \,,$$

then $F_t \xrightarrow{(d)} F$.

Part III

The intermediate disorder regime for Brownian polymers in Poisson environment

Abstract. We consider the Brownian directed polymer in Poissonian environment in dimension 1+1, under the so-called *intermediate disorder regime* [2], which is a crossover regime between the strong and weak disorder regions. We show that, under a diffusive scaling involving different parameters of the system, the normalized point-to-point partition function of the polymer converges in law to the solution of the stochastic heat equation with Gaussian multiplicative noise. The Poissonian environment provides a natural setting and strong tools, such as the Wiener-Itô chaos expansion [115], which, applied to the partition function, is the basic ingredient of the proof.[1]

1 Introduction

The model of directed polymers in random environment is a simple description of stretched chains which are sensitive to external random impurities. It was first introduced in the physics literature in [98] and several variations of the model have been studied ever since [102, 49, 1, 132]. Although the model has attracted significant attention, many expected results are still far from being proved with mathematical rigor. One can refer to [41] for a recent review in the discrete setting. In the present paper, we consider a Brownian path directed polymer, where the external impurities are represented by a Poisson point process.

1.1 The model and its context

Let $((B_t)_{t\geq 0}, \mathrm{P}_x)$ denote the Brownian motion starting from $x \in \mathbb{R}^d$ and set $\mathrm{P} = \mathrm{P}_0$. The *environment* is a Poisson point process $\omega = \sum_i \delta_{s_i,x_i}$ on $[0,\infty) \times \mathbb{R}^d$ of intensity measure $\nu \mathrm{d}s\mathrm{d}x$. We assume that ω is defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and we will denote by $\omega_t = \omega_{|[0,t]\times\mathbb{R}^d}$ the restriction of the environment to times before $t \geq 0$. Fix $r > 0$ and let $U(x)$ be the ball of volume r^d, centered at $x \in \mathbb{R}^d$. Define $V_t(B)$ as the tube around path B:

$$V_t(B) = \{(s,x) : s \in (0,t], x \in U(B_s)\}. \tag{1}$$

For any fixed path B, the variable $\omega(V_t)$ stands for the energy of the path until time t and corresponds to the number of points that the path encounters; it has the law of a Poisson variable of mean $\nu r^d t$.

The *polymer measure* $\mathrm{P}_t^{\beta,\omega}$ is the Gibbsian probability measure on the space $\mathcal{C}([0,\infty)\times\mathbb{R})$ of continuous paths, defined by its density with respect to the Wiener measure:

$$\mathrm{dP}_t^{\beta,\omega} = \frac{1}{Z_t} \exp\left(\beta\omega(V_t)\right) \mathrm{dP}, \tag{2}$$

where $\beta \in \mathbb{R}^d$ is the inverse temperature parameter, and where Z_t is the *partition function* of the polymer[2]:

$$Z_t(\omega,\beta,r) = \mathrm{P}\left[\exp\left(\beta\omega(V_t)\right)\right]. \tag{3}$$

This model was first introduced by Comets and Yoshida [49]. Under the polymer measure, the path is attracted by the Poisson points when $\beta > 0$, and repelled when $\beta < 0$.

The partition function Z_t has mean $\exp(\lambda(\beta)\nu r^d t)$, where

$$\lambda(\beta) = e^\beta - 1. \tag{4}$$

The *normalized partition function*:

$$W_t(\omega,\beta,r) = e^{-\lambda\nu r^d t} Z_t(\omega,\beta,r), \tag{5}$$

is a mean one, positive martingale such that $W_t \to W_\infty$. In [49], it is shown that there is a dichotomy:

$$\text{either} \quad W_\infty = 0 \ \text{ a.s.} \quad \text{or} \quad W_\infty > 0 \ \text{ a.s.} \tag{6}$$

[1] *AMS 2010 subject classifications.* 60F05, 82D60. **Key words and phrases**. Directed polymers, Near-critical scaling limits, Chaos expansions, KPZ equation.

[2] For every probability measure $\mathbb{P}$ and variable X, we will use the convention that $\mathbb{P}[X]$ denotes the expectation of X under $\mathbb{P}$.

The first case corresponds to the *strong disorder regime* and the second to the *weak disorder regime.* Furthermore, it is proved that there exist critical values $\beta_c^-(\nu, r) \in [-\infty, 0]$ and $\beta_c^+(\nu, r) \in [0, \infty]$, such that weak disorder holds for $\beta_c^- < \beta < \beta_c^+$ and strong disorder holds for $\beta \notin [\beta_c^-, \beta_c^+]$.

A classical quantity in statistical physics is the deterministic *quenched free energy*:

$$p(\beta, \nu, r) = \lim_{t\to\infty} \frac{1}{t} \ln Z_t(\omega, \beta, r), \tag{7}$$

which can be compared to the quenched free energy, that is defined as the limit as $t \to \infty$ of $t^{-1} \ln \mathbb{P}[Z_t]$ and equals $\lambda(\beta)\nu r^d$. The fact that the two energies agree or not also separates two regions, with some critical β thresholds:

$$\text{either} \quad p(\beta, \nu, r) < \lambda(\beta)\nu r^d \quad \text{or} \quad p(\beta, \nu, r) = \lambda(\beta)\nu r^d. \tag{8}$$

As strict inequality in (8) implies strong disorder, this regime is sometimes called *very strong disorder.* When this holds, the difference of the two energies - also called the *quenched Lyapunov exponent* - describes the exponential decay of W_t. Whether or not very strong disorder is equivalent to strong disorder is still an open question.

Path localization properties of the polymer were studied by Comets and Yoshida in [51]. For all time $t \geq 0$, they indentified a *favorite path* $Y_s^{(t)}$, depending only on the Poisson environment, such that for all $s \leq t$,

$$\mathrm{P}_t^{\beta,\omega}(B_s \in U(Y_s^{(t)})) = \max_{x\in\mathbb{R}^d} \mathrm{P}_t^{\beta,\omega}(B_s \in U(x)).$$

Denoting by $R_t^*(B) = \int_0^t \mathbf{1}\{B_s \in U(Y_s^{(t)})\}\mathrm{d}s$ the time fraction any path B spends next to the favorite path, they showed that under strict inequality between the derivatives in β of the two free energies, the following localization property holds:

$$\liminf_{t\to\infty} \mathbb{P}\mathrm{P}_t^{\beta,\omega}[R_t^*] > 0. \tag{9}$$

When this is verified, the polymer spends $\mathrm{P}_t^{\beta,\omega}$-a.s. a positive fraction of time close to a particular path in the environment, which is in contrast with the Brownian motion behavior.

As a preliminary step, one can define a favorite endpoint for the polymer on $[0, t]$ in a similar way (as in [49, Remark 2.3.1]). A complete understanding of the endpoint distribution has been recently achieved [8] in the discrete case.

Under weak disorder, the environment is supposed to have less influence over the polymer measure, so the polymer should behave similarly to Brownian motion ($\beta = 0$) and paths should not be localized. In the case of the discrete polymer, there is a functional central limit theorem on the polymer path, which holds in the whole weak disorder region [50]. For the Brownian polymer, a central limit theorem for the endpoint distribution was only proved in the smaller L^2 region [48]. In this region, a local limit theorem for the discrete and continuous polymers was proved by Vargas [159]. One can observe that in general, continuous models have received less attention and results are still incomplete compared to the discrete models. In contrast with weak disorder, the strong disorder regime should be characterized by localized paths and super-diffusivity ($B_t \approx t^\xi$ as $t \to \infty$ with $\xi > 1/2$), but integrable models put aside, rigorous proofs of this facts seem still out of reach.

We end this section by mentioning two related models:

(i) The branching Brownian motion in Poisson environment [147, 148], for which the mean population size at a given time equals the partition function (3).

(ii) The Brownian polymer in a white noise, mollified in space environment, introduced in [142] and studied in [130] for its relation to the stochastic heat equation in dimension $d \geq 3$.

1.2 KPZ universality for polymers and the intermediate disorder regime

From now on, we focus on dimension $d = 1$. In this case, the polymer is in the strong disorder phase as soon as $\beta \neq 0$. It is expected that under the polymer measure,

$$\sup_{0\leq s\leq t} |B_s| \approx t^{2/3} \quad \text{and} \quad \ln Z_t - \mathbb{P}[\ln Z_t] \approx t^{1/3} \quad \text{as } t \to \infty. \tag{10}$$

Moreover, it is conjectured that:

Conjecture 1.1. *For all non-zero β, ν and r, there exists a constant $\sigma(\beta,\nu,r)$ such that, as $t \to \infty$,*

$$\frac{\ln Z_t - p(\beta,\nu,r)t}{\sigma(\beta,\nu,r)t^{1/3}} \xrightarrow{(d)} F_{\text{GOE}} \,, \tag{11}$$

where the F_{GOE} is the Tracy-Widom GOE distribution [155].

These properties are characteristics of the KPZ universality class (cf. Section 1.5). They are in sharp contrast to the weak disorder regime, where one knows to a large extent that $B_t \approx t^{1/2}$, and where the free energy $\ln Z_t$ has order one fluctuations around its mean [49].

For general models, only non-sharp bounds for the fluctuations in (11) have been obtained yet [49, 123, 134, 163, 165]. For a specific, integrable and discrete model of polymer involving log-gamma distributed weights and some boundary conditions, Seppäläinen [145] was able to obtain sharp bounds with probabilistic methods, while the Tracy-Widom fluctuations were obtained later via Fredholm determinant identities [55, 23]. Similar results were obtained for the O'Connell-Yor polymer [22].

A more accessible question is to look at the so-called *intermediate disorder regime*, in the transition between $\beta > 0$ (strong disorder) and $\beta = 0$ (weak disorder). In the seminal paper [2], Alberts, Khanin and Quastel considered a time-space diffusive rescaling of the discrete directed polymer in i.i.d. environment, where they also rescaled the inverse temperature as $\beta_n = \beta n^{-1/4}$. They proved that under this scaling, the point-to-point and point-to-line partition functions of the polymer converge, in distribution, toward the partition functions of the *continuum directed polymer*, which is a directed polymer of Brownian path and white noise environment.

We now outline the **main results in the paper** (Theorems 2.4 - 2.7). We prove that the intermediate disorder regime also appears as a scaling limit of the Brownian directed polymer in Poisson environment. Here, thanks to the Poissonian environment, the model has a more general scaling (compared to the general discrete model) that involves parameters β, ν and r. In particular, at the price of tuning the other parameters, we show that the intermediate disorder regime can occur while keeping the temperature fixed. Similarly to [2, 28], the result is obtained via chaos expansion of the partition functions. In our case, W_t admits an infinite Wiener-Itô chaos expansion [115] which arises from the nice algebraic Poisson structure.

The paper is structured as follows: In the rest of this introduction, we discuss the link between the point-to-point partition function of the polymer and the stochastic heat equation. We will also say a few words about the KPZ equation, the KPZ universality class and the intermediate disorder regime in the discrete setting. The main results are presented in Section 2. Sections 3 and 4 are devoted to introducing the chaos expansions in the Poisson and white noise environment. In Section 5, we study the asymptotics of Poisson Wiener-Itô integrals. Proofs of the results will finally be given through Section 6.

1.3 The KPZ equation and the stochastic heat equation

The Kardar-Parisi-Zhang equation is the non-linear stochastic partial differential equation:[3]

$$\frac{\partial \mathcal{H}_\beta}{\partial T}(T,X) = \frac{1}{2}\frac{\partial^2 \mathcal{H}_\beta}{\partial X^2}(T,X) + \frac{1}{2}\left(\frac{\partial \mathcal{H}_\beta}{\partial X}(T,X)\right)^2 + \beta\eta(T,X), \tag{12}$$

where $\beta \in \mathbb{R}$ and η stands for the space-time Gaussian white noise (for a definition of this object, see section 4.1). The equation was first introduced in 1986 by Kardar, Parisi and Zhang [107], in the study of scaling behaviors of random interface growth.

Due to the non-linear term, it is difficult to give a proper definition of a solution of the KPZ equation. In [14, 16], it was argued that $\mathcal{H}_\beta$ could be defined by the so-called *Hopf-Cole transformation*:

$$\mathcal{H}_\beta(T,X) = \ln \mathcal{Z}_\beta(T,X), \tag{13}$$

[3] We will try to reserve capital letters $(T, X, ...)$ for the macroscopic scale (KPZ, SHE and continuum polymer) and lower case $(t, x, ...)$ for the microscopic scale (Poisson polymer and associated quantities).

where $\mathcal{Z}_\beta$ is the solution of the stochastic heat equation (SHE):

$$\frac{\partial \mathcal{Z}_\beta}{\partial T}(T,X) = \frac{1}{2}\frac{\partial^2 \mathcal{Z}_\beta}{\partial X^2}(T,X) + \beta \mathcal{Z}_\beta(T,X)\eta(T,X). \tag{14}$$

As a first-approach justification, one can check that the relation (13) defines a solution to (12), whenever $\mathcal{Z}$ is a solution of (14), where the white noise η is replaced with a smooth function.

Developing new tools to make sense of ill-posed stochastic PDEs, Hairer [86] later constructed a method giving a direct notion of solution to the KPZ equation. Hairer further showed that the solution coincided with the solution defined by the Hopf-Cole transformation.

1.4 Connections between the stochastic heat equation(s) and the directed polymers in random environments

The Poisson case

Introduce the normalized *point-to-point (P2P) partition function* :

$$W(t,x;\omega,\beta,r) = \rho(t,x)\mathrm{P}_{0,0}^{t,x}\left[\exp\{\beta\omega(V_t) - \lambda(\beta)\nu r^d t\}\right], \tag{15}$$

where $\mathrm{P}_{0,0}^{t,x}$ is the Brownian bridge $(0,0)\to(t,x)$ and $\rho(t,x) = e^{-x^2/2t}/\sqrt{2\pi t}$ is the heat kernel/Brownian motion transition function. In the next theorem, we state that the normalized P2P partition function verifies a weak formulation of the following stochastic heat equation, with multiplicative Poisson noise:

$$\partial_t W(t,x) = \frac{1}{2}\Delta W(t,x) + \lambda W(t-,x)\bar{\omega}(\mathrm{d}t \times U(x)). \tag{16}$$

Let $\mathcal{D}(\mathbb{R})$ denote the set of infinitely differentiable functions of compact support.

Theorem 1.2. *For all $\varphi \in \mathcal{D}(\mathbb{R})$ and $t \geq 0$, we have $\mathbb{P}$-almost surely*

$$\begin{aligned}\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x = \varphi(0) &+ \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} W(s,x)\Delta\varphi(x)\mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}} \mathrm{d}x\varphi(x) \int_{(0,t]\times\mathbb{R}} \bar{\omega}(\mathrm{d}s,\mathrm{d}y)W(s-,x)\mathbf{1}_{|y-x|\leq r/2}\,.\end{aligned} \tag{17}$$

The proof of the theorem can be found in Section 6.2.

The continuous case

A special case of interest for the SHE, where $\mathcal{Z}_\beta(T,X)$ can be interpreted as the point-to-point partition function of a directed polymer, started at $X=0$ at time $T=0$, is when

$$\mathcal{Z}_\beta(0,X) = \delta_0(X). \tag{18}$$

In this case, $\mathcal{Z}_\beta(T,X)$ can be expressed through the following shortcut (cf. Section 4.4):

$$\mathcal{Z}_\beta(T,X) = \rho(T,X)\ \mathrm{P}_{0,0}^{T,X}\left[:\exp:\left(\beta\int_0^T \eta(u,B_u)\mathrm{d}u\right)\right], \tag{19}$$

where $:\exp:$ denotes the Wick exponential.

Normalization put aside, this equation is similar to the definition of the point-to-point partition of our polymer model (15). Alberts, Khanin and Quastel [1] were indeed able to construct a polymer measure, with P2P partition function given by $\mathcal{Z}_\beta(T,X)$. As both the environment (white noise) and the path (Brownian motion) of the polymer are continuous, it was named *the continuum directed random polymer*.

Some special care has to be taken to construct this measure, as it can be shown that contrary to the Poissonian medium polymer, the continuum polymer measure is almost surely (with respect to the noise) singular with respect to the Wiener measure (See [1]). A way to circumvent this issue is to define directly the transition functions of the polymer through the equation that they should satisfy.

At time horizon $T = 1$, the transition functions of the polymer path X are defined by

$$\mathbb{P}^{\eta}_{1,\beta}(X_{t_1} \in \mathrm{d}x_1, \dots, X_{t_k} \in \mathrm{d}x_k) = \frac{\mathcal{Z}_\beta(t_k, x_k\,; 1, *)}{\mathcal{Z}_\beta(0, 0\,; 1, *)} \prod_{j=0}^{k-1} \mathcal{Z}_\beta(t_j, x_j\,; t_{j+1}, x_{j+1})\, \mathrm{d}x_1 \dots \mathrm{d}x_k$$

where $\mathcal{Z}_\beta(S, Y; T, X)$ is the P2P partition function from (S, Y) to (T, X):

$$\mathcal{Z}_\beta(S, Y; T, X) = \mathcal{Z}_\beta(T - S, X - Y) \circ \theta_{S,Y},$$

letting $\theta_{S,Y}$ be the shift by (S, Y) in the white noise environment, and where

$$\mathcal{Z}_\beta(S, Y\,; 1, *) = \int_{\mathbb{R}} \mathcal{Z}_\beta(S, Y\,; 1, X)\mathrm{d}X.$$

The *point-to-line partition function* of the continuum polymer at time horizon $T = 1$ is given by

$$\mathcal{Z}_\beta = \mathcal{Z}_\beta(0, 0\,; 1, *). \tag{20}$$

Remark 1.3. *One can check that the transition functions of the Brownian polymer in Poisson environment also satisfy the above equation.*

Similarly to the Poisson polymer, the P2P free energy $\mathcal{F}_\beta(T, X)$ can be defined as

$$\mathcal{F}_\beta(T, X) = \ln \frac{\mathcal{Z}_\beta(T, X)}{\rho(T, X)}, \tag{21}$$

so that the free energy of the polymer and the solution of the KPZ equation are related by the equation:

$$\mathcal{F}_\beta(T, X) = \mathcal{H}_\beta(T, X) + X^2/2T + \ln \sqrt{2\pi T}. \tag{22}$$

1.5 The KPZ universality class and the KPZ equation

The KPZ equation belongs to a wide class of mathematical and physical models, called the KPZ universality class, which gathers models that share similar statistical behaviors under long time or scaling limits and particular scaling exponents (3-2-1 in time, space and fluctuations, as in (10)). The reader may refer to [54, 53, 136] for reviews on the KPZ universality class and the KPZ equation, and [168, 24] for reviews on the algebraic structures that lie behind solvable models of the class. Notable models, that are proven or conjectured to belong to this class, include paths in a random environment (directed polymers in random environment, first and last passage percolation), random growing interfaces (corner growth model), interacting particle systems (asymmetric simple exclusion process (ASEP)), stochastic PDEs and random matrices.

In the begining of the 2010s, the computation [3, 144, 66, 27] of the pointwise distribution of $\mathcal{H}_\beta(T, X)$ was a breakthrough and provided the proof that the KPZ equation lied in the KPZ universality class. The rigorous result [3] relied on two main results: the work of Bertini-Giacomin [16], who were able to show that the solution of the KPZ equation appeared as the weak asymmetry limit of the ASEP height function; and on the papers of Tracy-Widom [154, 157, 158], in which the authors obtained formulas to express this height function distribution.

As a consequence of the ASEP weak asymmetry limit of Bertini-Giacomin, the KPZ equation can be seen as a crossover between the positive asymmetry ASEP (which belongs to the KPZ universality class) and the symmetric simple exclusion process (which belongs to the Edwards-Wilkinson universality class with 4-2-1 scaling). The *weak KPZ universality conjecture* states that the KPZ equation is a universal object of the KPZ universality class. As a general idea, the KPZ equation should appear as a scaling limit at critical parameters for models that feature a phase transition between the Edwards-Wilkinson class (4-2-1 scaling) and the KPZ class. In recent years, several techniques have been used to prove convergence of different models to the KPZ equation in weakly asymmetric regimes. The Hopf-Cole transformation is one of them and provides the opportunity to deal with the linear stochastic heat equation, instead of the non-linear KPZ equation; we will rely on it in this paper. This transformation may be applied to models which can be controlled after exponentiation, as for [16, 2, 59, 56, 112]. When one cannot apply the transformation, another tool is the pathwise approach introduced by Hairer [86, 87], along with [83] for the theory of paracontrolled distributions, and considered in [88, 84, 93]. Although very robust, the pathwise analysis requires quantitative estimates that may be hard to obtain. An alternative, relying on stationarity of the models, is the martingale approach developed in [85, 82, 77] and applied to [64, 77, 78, 79, 72]. Finally, we also mention [111] for a renormalisation group approach.

1.6 The intermediate disorder regime for the discrete polymer with i.i.d. weights

The fact that the KPZ equation can emerge as a crossover regime appears in the result of Alberts, Khanin and Quastel [2], who showed that in dimension $d = 1$, the rescaled logarithm of the point-to-point partition function of the discrete directed polymer (see below) converges to $\mathcal{H}_\beta(T, X) = \ln \mathcal{Z}_\beta(T, X)$. The P2P partition function of the polymer is defined as

$$Z_\beta(n, x) = P(S_n = x) \times \mathrm{P}[e^{\beta \sum_{k=0}^{n} w(i,S_i)} | S_n = x], \tag{23}$$

where S is the simple symmetric random walk and $w(i, x)$ are i.i.d. random variables with finite exponential moments. They showed that, as $n \to \infty$,

$$\frac{\sqrt{n}}{2} Z_{\beta n^{-1/4}}(nT, \sqrt{n}X) e^{-\mu(\beta n^{-1/4})nT} \xrightarrow{(d)} \mathcal{Z}_\beta(T, X), \tag{24}$$

where $\mu(\beta) = \ln \mathrm{P}[e^{\beta w(i,x)}]$, and where the limit in distribution is proven in terms of convergence of processes.

When $\beta = 0$ (weak disorder), the polymer measure reduces to the simple symmetric random walk measure. When $\beta > 0$ (strong disorder), a long-standing conjecture is to prove that the discrete polymer model lies in the KPZ universality class. In particular, what is expected is the following, where the 1/3 coefficient and the limiting Tracy-Widom distribution are characteristics of the KPZ universality class:

Conjecture 1.4. [20] *Suppose that the $w(i, x)$ are i.i.d. of finite fifth moment. Then, there exists some constants $c(\beta)$ et $\sigma(\beta)$ such that, as $n \to \infty$,*

$$\frac{\ln Z_\beta(n, 0) - c(\beta)n}{\sigma(\beta)n^{1/3}} \xrightarrow{(d)} F_{\mathrm{GUE}}.$$

Because in the limit (24), $\beta n^{-1/4} \to 0$, the KPZ equation - or equivalently the continuum directed random polymer - can be interpreted as a crossover regime between the weak disorder polymer regime and the strong disorder regime, so that it was named in [2] the *intermediate disorder regime.* Moreover, the intermediate disorder regime features both characteristics of the strong disorder (a limiting universal law that does not depend on the law of the initial environment, a limiting polymer model that in the KPZ universality class) and the weak disorder (a diffusive scaling and a random local limit theorem for the endpoint density [2]).

For the discrete polymer, the authors in [2] conjectured that the result should still hold under the weaker asumption of a sixth-moment on the $w(i, x)$ viariables. This was later proved in [63]. In the same article, the authors gave conjectures on the behaviour of the heavy-tailed polymer under the regime $\beta_n \to 0$. These conjectures as well as the understanding of five different regimes were shown under this regime in [11]. Finally, we also mention the work of [28], where the intermediate disorder regime $\beta_n \to 0$ was also studied for the pinning model, the long-range jumps discrete polymer and the random field Ising model.

2 Main results

We will show that a similar result to convergence (24) holds in our model. In this respect, we consider parameters $\beta_t \in \mathbb{R}$, $\nu_t > 0$ and $r_t > 0$ that depend on time t, and we fix a parameter $\beta^* \in \mathbb{R}^*$. We introduce three asymptotic relations, when $t \to \infty$:

$$\begin{gathered}\text{(a) } \nu_t r_t^2 \lambda(\beta_t)^2 \sim (\beta^*)^2 t^{-1/2}, \quad \text{(b) } \nu_t r_t^3 \lambda(\beta_t)^3 \to 0,\\ \text{(c) } r_t/\sqrt{t} \to 0.\end{gathered} \tag{25}$$

Remark 2.1. *Suppose the radius r_t and the intensity ν_t are constants. Then, the relations imply that β_t scales like $t^{-1/4}$, as in equation* (24).

Remark 2.2 (Interpretation of (25) as disorder intensity)**.** *We note that similar computation to what led to the inequality in Part II Equation* (3.26)*,*

$$\mathbb{P}\left[W_t^2\right] \leq P\left[\exp\left(2\lambda(\beta_t)^2 \nu_t r_t^2 \int_0^t \mathbf{1}_{|B_s|\leq 1}\,\mathrm{d}s\right)\right],$$

with $P\left[\int_0^t \mathbf{1}_{|B_s|\leq 1}\,\mathrm{d}s\right] \sim Ct^{1/2}$, so that, again by Khas'minskii's lemma, one obtains that W_t is bounded in L^2 as soon as

$$\lambda(\beta_t)^2 \nu_t r_t^2 t^{1/2} < \alpha,$$

for some constant $\alpha > 0$. By relation (a), this condition is verified when $|\beta^*|$ is small enough[4]. We emphasize the fact that for fixed parameters $\beta, \nu, r > 0$, the martingale is not bounded in L^2 since we know that $\beta_c^{\pm} = 0$ for $d = 1$. The regime (25) *should thus be interpreted as a crossover between strong disorder $(\beta, \nu, r > 0)$ and weak disorder (L^2-region), which explains the denomination of **intermediate disorder regime**. We say more about the scaling relations in Section 5.1.*

Remark 2.3. *The relations can be compared to the regime of* complete localization *[51], corresponding to the extremal parameters regime (r is fixed):*

$$\nu \to \infty,\ |\beta| \leq \beta_0, \quad \text{such that} \quad \nu\beta^2 \to \infty. \tag{26}$$

In the complete localization *regime, the polymer measure is highly concentrated around a favorite path, and the rest of the environment is neglected.*

Theorem 2.4. *Under conditions (a), (b), (c) and as $t \to \infty$:*

$$W_t(\omega^{\nu_t}, \beta_t, r_t) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}, \tag{27}$$

where ω^{ν_t} is the Poisson point process with intensity measure $\nu_t \mathrm{d}s\mathrm{d}x$.

We will also show that the result extends to the normalized point-to-point partition function:

$$W(s, y; t, x; \omega, \beta, r) = W(t - s, x - y; \omega, \beta, r) \circ \theta_{s,y}, \tag{28}$$

where $\theta_{s,y}$ denotes the shift of vector (s, y) in the Poisson environment: $\theta_{s,y}(\sum_i \delta_{(s_i,y_i)}) = \sum_i \delta_{(s_i-s,y_i-y)}$.

Theorem 2.5. *Let $S, T \geq 0$ and $Y, X \in \mathbb{R}$. Under conditions (a), (b), (c) and as $t \to \infty$:*

$$\sqrt{t} W\left(tS, \sqrt{t}Y; tT, \sqrt{t}X; \omega^{\nu_t}, \beta_t, r_t\right) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}(S, Y; T, X). \tag{29}$$

Remark 2.6. *The $\sqrt{t}$ term appears here as a normalization in the heat kernel scaling: $\sqrt{t}\rho\left(tT, \sqrt{t}X\right) = \rho(T, X)$.*

Let $\mathcal{D}'(\mathcal{R})$ denote the space of distributions on $\mathbb{R}$, and $D\big([0,1], \mathcal{D}'(\mathcal{R})\big)$ the space of càdlàg function with values in the space of distributions, equipped with the topology defined in [124]. We also define the rescaled and normalized P2P partition function (15):

$$\mathcal{Y}_t(T, X) = \rho(T, X) W\left(tT, \sqrt{t}X; \omega^{\nu_t}, \beta_t, r_t\right). \tag{30}$$

The two variables function $\mathcal{Y}_t$ can be seen as an element of $D\left([0,1], \mathcal{D}'(\mathcal{R})\right)$ through the mapping $\mathcal{Y}_t : T \mapsto \left(\varphi \mapsto \int \mathcal{Y}_t(T, X)\varphi(X)\mathrm{d}X\right)$. We have:

[4]In fact, we will see in Lemma 6.6 that under conditions (a)-(c), the normalized point-to-point partition function has finite L^p moments for all $p > 0$ whenever $(\beta_t)_{t\geq 0}$ is bounded by above

Theorem 2.7. *Suppose that $(\beta_t)_{t\geq 0}$ is bounded by above. Then, as $t \to \infty$:*

$$\mathcal{Y}_t \xrightarrow{(d)} \big(T \mapsto \mathcal{Z}_{\beta^*}(T,\cdot)\big), \tag{31}$$

where the convergence in distribution holds in $D\big([0,1], \mathcal{D}'(\mathcal{R})\big)$.

3 The Wiener-Itô integrals with respect to Poisson process

In this section, we expose the basic theory of multiple integration over Poisson processes. We rely on the reviews of Günter Last and Mathew Penrose [115, 114].

Let us first introduce some notations that will prove useful throughout the paper. For any $k \geq 1$, $s_1, \dots, s_k \in \mathbb{R}_+$ and $x_1, \dots, x_k \in \mathbb{R}$, write $\mathbf{s} = (s_1, \dots, s_k)$ and $\mathbf{x} = (x_1, \dots, x_k)$. Let

$$\Delta_k(u,t) = \{\mathbf{s} \in [u,t]^k \,|\, u < s_1 < \cdots < s_k \leq t\}, \tag{32}$$

be the k-dimensional simplex and $\Delta_k = \Delta_k(0,1)$. In addition, for any given function g of $\mathbb{R}_+^k \times \mathbb{R}^k$, define the symmetrized version of g

$$\operatorname{Sym} g\,(\mathbf{s},\mathbf{x}) = \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} g(\pi\mathbf{s}, \pi\mathbf{x}), \tag{33}$$

where $\mathfrak{S}_k$ denotes the set of permutation of $\{1,\dots,k\}$, and where any $\pi \in \mathfrak{S}_k$ acts on $\mathbb{R}^k$ by permutation of indices. We say that a function is *symmetric* whenever $\operatorname{Sym} g = g$. We see from the definition that the function $\operatorname{Sym} g$ is indeed symmetric.

3.1 The factorial measures

For all function f on $\mathbb{R}_+ \times \mathbb{R}$, define its sum over the Poisson process as $\omega(f) := \iint f(s,x)\omega(\mathrm{d}s\mathrm{d}x)$. Suppose for a moment that one wants to evaluate $\mathbb{P}[\omega(f)^2]$. A solution is to decompose the square of the sum as

$$\omega(f)^2 = \sum_{(s,x)\in\omega} f(s,x)^2 + \sum_{(s,x)\neq(s',x')\in\,\omega} f(s,x)f(s',x'), \tag{34}$$

in order to apply the multivariate Mecke equation [115, Th. 4.4] to evaluate the expectation of the second term on the right-hand side. One can observe that this particular term is an integral over a certain point process measure, which depends on ω and defines a measure on $\mathbb{R}_+^2 \times \mathbb{R}^2$, called the second *factorial measure* of ω. The first factorial measure is simply ω.

As an example, we saw that the simple functional $\omega(f)^2$ can be written as a sum of two integrals over factorial measures. More generally, we will see in Theorem 3.7 that any square-integrable functionals of ω can be expressed through an infinite sum of integrals over similar measures; in particular, it will be the case for the partition function W_t.

To give a proper definition of the factorial measures, let $t > 0$ and let $\mathcal{B}_t$ denote the product Borel sets of $[0,t] \times \mathbb{R}$, $\mathcal{B}_t^{\otimes k}$ the product Borel sets of $[0,t]^k \times \mathbb{R}^k$.

Definition 3.1. *For any positive integer k, define the* ***k-th factorial measure*** *$\omega_t^{(k)}$ to be the point process on $[0,t]^k \times \mathbb{R}^k$, such that, for any measurable set $\mathbf{A} \in \mathcal{B}_t^{\otimes k}$,*

$$\omega_t^{(k)}(\mathbf{A}) = \sum\nolimits^{\neq}_{(s_1,x_1),\dots,(s_k,x_k)\in\,\omega_t} \mathbf{1}_{((s_1,x_1),\dots,(s_k,x_k))\in\mathbf{A}}, \tag{35}$$

where the sign $\neq$ indicates that the summation is made over pairwise different (s_i,x_i). Otherwise stated,

$$\omega_t^{(k)} = \sum\nolimits^{\neq}_{(s_1,x_1),\dots,(s_k,x_k)\in\,\omega_t} \delta_{((s_1,x_1),\dots,(s_k,x_k))}\ . \tag{36}$$

Remark 3.2. *If A is a Borel set of $\mathcal{B}_t$ and $A^k = A \times \ldots \times A$, then $\omega_t^{(k)}(A^k)$ is the number of k-tuples of distinct points of ω_t that belong to A, that is*

$$\omega_t^{(k)}(A^k) = \omega_t(A)(\omega_t(A) - 1) \ldots (\omega_t(A) - k + 1),$$

which is the reason why it is called a "factorial" measure. For $A_1, \ldots, A_k$ a collection of pairwise disjoint sets of $\mathcal{B}_t$, the situation is substantially different as we have

$$\omega_t^{(k)}(A_1 \times \cdots \times A_k) = \prod_{i=1}^{k} \omega_t(A_i). \tag{37}$$

Since the sum is over all distinct k-tuples, symmetry plays an important role in factorial measures, and one should keep in mind that symmetric functions are the natural functions to integrate. As an example, the integral of a function g is in fact the integral of its symmetrized function:

$$\begin{aligned}
\omega_t^{(k)}(\operatorname{Sym} g) &= \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \sum_{(s_i, x_i) \in \omega_t}^{\neq} g(\pi \mathbf{s}, \pi \mathbf{x}) \\
&= \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \sum_{(\tilde{s}_i, \tilde{x}_i) \in \omega_t}^{\neq} g(\tilde{\mathbf{s}}, \tilde{\mathbf{x}}) \\
&= \omega_t^{(k)}(g).
\end{aligned}$$

3.2 Multiple stochastic integral over a Poissonian medium

Now that we have defined the factorial measures of ω_t, we seek to do the same for the compensated measure $\bar{\omega}_t$ - also called the *Wiener-Itô integral.* In particular, we still want to avoid points belonging to the diagonal in the integration process.

Let $A_1, \ldots A_k$ be a collection of pairwise disjoint, finite sets of $\mathcal{B}_t$. Then, observe that

$$\begin{aligned}
\prod_{i=1}^{k} \bar{\omega}_t(A_i) &= \prod_{i=1}^{k} \big(\omega_t(A_i) - \nu |A_i|\big) \\
&= \sum_{J \subset [k]} \prod_{i \in J} \left(\int \mathbf{1}_{A_i}(s, x)\, \omega_t(\mathrm{d}s\mathrm{d}x) \right) \prod_{i \in J^c} \left(- \int \mathbf{1}_{A_i}(s, x)\, \nu \mathrm{d}s\mathrm{d}x \right),
\end{aligned}$$

where $[k] = \{1, 2, \ldots, k\}$ will not be confused with the integer part. Using the fact that the A_i are disjoint, by (37) the above product over J can be written as an integral with respect to the measure $\omega_t^{(|J|)}$:

$$\begin{aligned}
&\prod_{i=1}^{k} \bar{\omega}_t(A_i) \\
&= \sum_{J \subset [k]} (-1)^{k-|J|} \int_{[0,t]^k \times \mathbb{R}^k} \left(\prod_{i=1}^{k} \mathbf{1}_{A_i} \right) \omega_t^{(|J|)}(\mathrm{d}\mathbf{s}_J, \mathrm{d}\mathbf{x}_J)\, \nu^{k-|J|}\, \mathrm{d}\mathbf{s}_{J^c}\, \mathrm{d}\mathbf{x}_{J^c},
\end{aligned} \tag{38}$$

where the notations $\mathrm{d}\mathbf{s}_J$ and $\mathrm{d}\mathbf{x}_J$ mean that the integration is done with respect to the variables $(s_i)_{i \in J}$ and $(x_i)_{i \in J}$. This leads to the following definition of the *multiple Wiener-Itô integral*:

Definition 3.3. *For $k \geq 1$ and $g \in L^1([0,t]^k \times \mathbb{R}^k)$, denote the* ***multiple Wiener-Itô integral*** *of g as*

$$\begin{aligned}
&\bar{\omega}_t^{(k)}(g) \\
&:= \sum_{J \subset [k]} (-1)^{k-|J|} \int_{[0,t]^k \times \mathbb{R}^k} g(\mathbf{s}, \mathbf{x})\, \omega_t^{(|J|)}(\mathrm{d}\mathbf{s}_J, \mathrm{d}\mathbf{x}_J)\, \nu^{k-|J|}\, \mathrm{d}\mathbf{s}_{J^c}\, \mathrm{d}\mathbf{x}_{J^c}.
\end{aligned} \tag{39}$$

When $k = 0$, define $\bar{\omega}_t^{(0)}$ to be the identity on $\mathbb{R}$.

The two following results can be found in [115, 114]:

Proposition 3.4. *For $k \geq 1$, the map $\bar{\omega_t}^{(k)}$ can be extended to a map*

$$\begin{array}{rcccc} \bar{\omega}_t^{(k)} & : & L^2([0,t]^k \times \mathbb{R}^k) & \to & L^2(\Omega, \mathcal{G}, \mathbb{P}) \\ & & g & \mapsto & \bar{\omega}_t^{(k)}(g), \end{array}$$

which coincides with the above definition of $\bar{\omega}_t^{(k)}$ on the functions of $L^1 \cap L^2([0,t]^k \times \mathbb{R}^k)$.

Property 3.5. *(i) For $k \geq 1$ and $g \in L^2([0,t]^k \times \mathbb{R}^k)$, then $\mathbb{P}\big[\bar{\omega}_t^{(k)}(g)\big] = 0$.*

(ii) For every square-integrable function g,

$$\bar{\omega_t}^{(k)}(\mathrm{Sym}\, g) = \bar{\omega_t}^{(k)}(g). \tag{40}$$

(iii) For any $k \geq 1$ and $l \geq 1$, $g \in L^2([0,t]^k \times \mathbb{R}^k)$ and $h \in L^2([0,t]^l \times \mathbb{R}^l)$, the following covariance structure holds:

$$\mathbb{P}\left[\bar{\omega}_t^{(k)}(g)\,\bar{\omega}_t^{(l)}(h)\right] = \delta_{k,l}\; k!\,\nu^k < \mathrm{Sym}\, g, \mathrm{Sym}\, h >_{L^2([0,t]^k \times \mathbb{R}^k)} . \tag{41}$$

(iv) The map $\bar{\omega}_t^{(k)}$ is linear, in the sense that for all square-integrable f, g and reals λ, μ,

$$\mathbb{P}\text{-}a.s. \quad \bar{\omega}_t^{(k)}(\lambda f + \mu g) = \lambda\,\bar{\omega}_t^{(k)}(f) + \mu\,\bar{\omega}_t^{(k)}(g).$$

Multiple Wiener-Itô integral of a function on the simplex: When g is a function of $L^2(\Delta_k(0,t) \times \mathbb{R}^k)$, denote by $\widehat{g}$ its extension set to zero outside of the simplex, and define

$$\bar{\omega}_t^{(k)}(g) := \bar{\omega}_t^{(k)}(\widehat{g}). \tag{42}$$

Observing that $[0,t]^k \times \mathbb{R}^k$ is made of $k!$ copies of $\Delta_k(0,t) \times \mathbb{R}^k$, one gets that

$$\mathbb{P}\left[\bar{\omega}_t^{(k)}(g)^2\right] = \nu^k \|g\|^2_{L^2(\Delta_k(0,t) \times \mathbb{R}^k)}. \tag{43}$$

Remark 3.6. *This tells us that $\bar{\omega}_t^{(k)}$ is an isometry on $L^2(\Delta_k(0,t) \times \mathbb{R}^k)$ with measure $\nu^k \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}$. This is one of the reasons why we will mainly consider functions of the simplex.*

3.3 A Wiener-Itô Chaos Expansion of the normalized partition function

The next theorem, proven in [115, §18.4], states that every square-integrable function $F(\omega)$ admits a Wiener-Itô chaos expansion, i.e., it can be written as an infinite sum of orthogonal multiple Wiener-Itô integrals. In order to be able to state the theorem, we first need to introduce a new operator.

For any function $F(\omega_t)$ of the point-process up to time t, define the derivative operator $D_{(s,x)}(F) = F(\omega_t + \delta_{(s,x)}) - F(\omega_t)$, for all $(s,x) \in [0,t] \times \mathbb{R}$. Now, for any given $(s_i, x_i) \in [0,t] \times \mathbb{R}$, $i \leq k$, define the iterated operator:

$$D_{(s_1,x_1),\ldots,(s_k,x_k)} \quad = \quad D_{(s_1,x_1)} \circ D_{(s_2,x_2)} \circ \cdots \circ D_{(s_k,x_k)}\,, \tag{44}$$

and the function $T_k F : [0,t]^k \times \mathbb{R}^k \to \mathbb{R}$, by letting

$$T_k F(s_1, \ldots, s_k, x_1, \ldots, x_k) = \mathbb{P}\big[D_{(s_1,x_1),\ldots,(s_k,x_k)} F(\omega_t)\big]. \tag{45}$$

We also set $T_0 F = \mathbb{P}[F(\omega_t)]$. By induction, we see that

$$D_{(s_1,x_1),\ldots,(s_k,x_k)} F(\omega_t) = \sum_{I \subset [k]} (-1)^{k-|I|} F\left(\omega_t + \sum_{i \in I} \delta_{(s_i,x_i)}\right) .$$

Theorem 3.7. *Let $F(\omega_t)$ be any measurable function of the point process up to time t, verifying $\mathbb{P}\big[F(\omega_t)^2\big] < \infty$. Then, for all k, $T_k F$ is a symmetric, square-integrable function, and we have $\mathbb{P}$-almost surely:*

$$F(\omega_t) = \sum_{k=0}^{\infty} \frac{1}{k!} \bar{\omega}_t^{(k)} \big(T_k F\big). \tag{46}$$

The orthogonal series (46) converges in $L^2(\Omega, \mathcal{G}, \mathbb{P})$, and it is called the ***Wiener-Itô chaos expansion*** *of $F(\omega_t)$.*

Remark 3.8. *Note that the terms in the sum are pairwise orthogonal. As a possible interpretation of the theorem, one can view T_k as a k-th derivative, and the Wiener-Itô expansion as a Taylor expansion.*

We can now apply the theorem to normalized partition function W_t, which is square-integrable. For this purpose, define for all path B and all $\delta > 0$,

$$\chi^{\delta}_{s,x}(B) = \mathbf{1}\{|x - B_s| \le \delta/2\}. \tag{47}$$

Note that when $\delta = r$, we have $\chi^r_{s,x}(B) = \mathbf{1}\{x \in U(B_s)\}$.

Proposition 3.9. *The normalized partition function admits the following Wiener-Itô chaos expansion:*

$$W_t = \sum_{k=0}^{\infty} \frac{1}{k!}\, \bar{\omega}_t^{(k)}\, (T_k\, W_t)\,, \tag{48}$$

where, for all $\mathbf{s} \in [0,t]^k$, $\mathbf{x} \in \mathbb{R}^k$ and $k \ge 0$:

$$T_k\, W_t\, (\mathbf{s}, \mathbf{x}) = \lambda(\beta)^k\, \mathrm{P}\left[\prod_{i=1}^{k} \chi^r_{s_i, x_i}(B)\right], \tag{49}$$

with the convention that an empty product equals 1.

Proof. Fix a path B and observe that

$$\omega\,(V_t(B)) = \int_{[0,t]\times\mathbb{R}} \chi^r_{s,y}(B)\, \omega(\mathrm{d}s\mathrm{d}y).$$

For all $(s,x) \in [0,t] \times \mathbb{R}$, we get that $D_{s,x}\, e^{\beta\omega(V_t)} = \lambda(\beta)\, \chi^r_{s,x}\, e^{\beta\omega(V_t)}$. Hence,

$$T_k\, W_t\, (\mathbf{s}, \mathbf{x}) = \mathbb{P}\mathrm{P}\left[e^{\beta\omega(V_t)} \prod_{i=1}^{k} \lambda\, \chi^r_{s_i,x_i}\right] e^{-t\nu\lambda r^d} = \lambda^k\, \mathrm{P}\left[\prod_{i=1}^{k} \chi^r_{s_i,x_i}\right].$$

∎

4 The Wiener integrals

We repeat the construction of Section 3 for white noise instead of Poisson noise. In this section, we consider another probability space $(\Lambda, \mathcal{F}_\eta, \mathbb{Q})$.

4.1 Stochastic integral over the white noise

Definition 4.1. *A time-space* ***Gaussian white noise*** *environment η, is a random measure on $[0,1] \times \mathbb{R}$, which satisfies the two following properties:*

(i) For all measurable sets $A_1, \dots, A_k$ of $[0,1] \times \mathbb{R}$, $\big(\eta(A_1), \dots, \eta(A_k)\big)$ is a centered Gaussian vector.

(ii) For all measurable sets A, B of $[0,1] \times \mathbb{R}$,

$$\mathbb{Q}[\eta(A)\eta(B)] = |A \cap B|. \tag{50}$$

In the following, we suppose that a white noise process is defined on the probability space $(\Lambda, \mathcal{F}_\eta, \mathbb{Q})$. It is then possible to construct a stochastic integral over the white noise measure, which has the following properties:

Proposition 4.2. *There exists an isometry $I_1 : L^2\big([0,1]\times\mathbb{R}\big) \mapsto L^2(\Lambda, \mathcal{F}_\eta, \mathbb{Q})$ verifying that:*

(i) For all measurable set A of $[0,1]\times\mathbb{R}$, we have $I_1(A) = \eta(A)$.

(ii) For all $g \in L^2$, the variable $I_1(g)$ is a centered Gaussian variable of variance $\|g\|^2_{L^2([0,1]\times\mathbb{R})}$.

We call $I_1(g)$ the *stochastic integral* of g *over the white noise.* Note that the integral will sometimes also be written as

$$I_1(g) = \int_{[0,1]}\int_{\mathbb{R}} g(s,x)\,\eta(\mathrm{d}s,\mathrm{d}x). \tag{51}$$

4.2 Multiple stochastic integral

It is again possible to extend I_1 to a multiple stochastic integral. One can find the details of such a procedure in Janson's book [105, Chapter 7]. This integral has very similar properties to the Wiener-Itô integral:

Theorem 4.3. *For all $k > 0$, there exists a map $I_k : L^2([0,1]^k \times \mathbb{R}^k) \mapsto L^2(\Lambda, \mathcal{F}_\eta, \mathbb{Q})$, which has the following properties:*

(i) If g is any square-integrable function, then $I_k(\operatorname{Sym} g) = I_k(g)$.

(ii) For all $g \in L^2([0,1]^k \times \mathbb{R}^k)$ and $h \in L^2([0,1]^j \times \mathbb{R}^j)$, the variable $I_k(g)$ is centered and

$$\mathbb{Q}\left[I_k(g)I_j(h)\right] = \delta_{k,j}\, k!\, < \operatorname{Sym} g, \operatorname{Sym} h >_{L^2([0,1]^k\times\mathbb{R}^k)}. \tag{52}$$

(iii) For all orthogonal family $(g_1, \dots, g_k)$ of functions in $L^2([0,1]\times\mathbb{R})$, we have

$$I_k\left(\prod_{j=1}^k g_j\right) = \prod_{j=1}^k I_1(g_j). \tag{53}$$

(iv) The map I_k is linear.

Remark 4.4. *Similar to (42), we define multiple Wiener integral of a function defined on the simplex. If g is a function of $L^2(\Delta_k \times \mathbb{R})$, and if $\widehat{g}$ is the extension of g set to zero outside of the simplex, we define $I_k(g) := I_k(\widehat{g})$ and have*

$$\mathbb{Q}[I_k(g)^2] = \|g\|^2_{L^2(\Delta_k\times\mathbb{R}^k)}. \tag{54}$$

Remark 4.5. *We will sometimes use the notation*

$$I_k(g) = \int_{[0,1]^k}\int_{\mathbb{R}^k} g(\mathbf{t},\mathbf{x})\eta^{\otimes k}(\mathrm{d}\mathbf{t},\mathrm{d}\mathbf{x}), \tag{55}$$

where $[0,1]^k$ can be replaced with Δ_k when dealing with functions of the simplex. See [105] for a justification of the tensor product notation.

4.3 Wiener chaos decomposition

Definition 4.6. *For any family $G = (g^k)_{k\geq 0}$ such that for all $k \geq 0$, $g^k \in L^2(\Delta_k \times \mathbb{R}^k)$, and that*

$$\|G\|_2^2 := \sum_{k=0}^{\infty} \|g^k\|^2_{L^2(\Delta_k\times\mathbb{R}^k)} < \infty,$$

*we say that G is an element of the **Fock space** $\bigoplus_{k=0}^{\infty} L^2(\Delta_k \times \mathbb{R}^k)$, which is a normed vector space with norm $\|G\|_2$, also called **Fock norm**.*

The next proposition is a consequence of Remark 4.4:

Proposition 4.7. *The linear map*

$$\begin{array}{rcccc} I & : & \bigoplus_{k=0}^{\infty} L^2(\Delta_k \times \mathbb{R}^k) & \to & L^2(\Omega, \mathcal{F}_\eta, \mathbb{Q}) \\ & & G = (g^k)_{k\geq 0} & \mapsto & \sum_{k=0}^{\infty} I_k(g^k) =: I(G), \end{array} \tag{56}$$

is an isometry. Note that the sum is well defined as an L^2-limit.

Remark 4.8. *If we were dealing with functions of $[0,1]^k \times \mathbb{R}^k$, we would have defined $I(G)$ as $\sum_{k=0}^{\infty} \frac{1}{k!} I_k(g^k)$ and the Fock norm as $\sum_{k=0}^{\infty} \frac{1}{k!} \|g^k\|_2^2$.*

We now consider a key example. Let $k > 0$, and introduce the k-th dimensional Brownian transition function, for $(\mathbf{s}, \mathbf{x}) \in \Delta_k \times \mathbb{R}^k$:

$$\begin{aligned} \rho^k(\mathbf{s}, \mathbf{x}) &= \mathrm{P}\left(B_{s_1} \in \mathrm{d}x_1, \ldots, B_{s_k} \in \mathrm{d}x_k\right) \\ &= \rho(s_1, x_1) \left(\prod_{j=1}^{k-1} \rho(s_{j+1} - s_j, x_{j+1} - x_j) \right), \end{aligned} \tag{57}$$

with the convention that $\rho^0 = 1$.

Proposition 4.9. *The family $R(\beta) = (\beta^k \rho^k)_{k\geq 0}$ is in $\bigoplus_{k=0}^{\infty} L^2(\Delta_k \times \mathbb{R}^k)$ for all $\beta \in \mathbb{R}$. In particular, the variable*

$$\mathcal{Z}_\beta := I\big(R(\beta)\big), \tag{58}$$

is well defined and square-integrable.

Remark 4.10. *This quantity is in fact the continuum polymer partition function (cf. Section 4.4).*

Proof. We will rely on the observation that

$$\rho(s, x)^2 = \frac{1}{2\sqrt{\pi s}} \rho(s/2, x).$$

Expressing ρ^k in terms of product of ρ function, we have, with the convention that $s_0 = x_0 = 0$,

$$\rho^k(\mathbf{s}, \mathbf{x})^2 = \rho^k(\mathbf{s}/2, \mathbf{x}) \prod_{j=1}^{k} \frac{1}{2\sqrt{\pi(s_j - s_{j-1})}},$$

so,

$$\int_{\Delta_k} \int_{\mathbb{R}^k} \rho^k(\mathbf{s}, \mathbf{x})^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x} = 2^{-k} \pi^{-k/2} \int_{\Delta_k} \prod_{j=1}^{k} \frac{1}{\sqrt{s_j - s_{j-1}}} \mathrm{d}\mathbf{s}.$$

The last integral is the normalizing constant of the order $k+1$ Dirichlet distribution, taken with parameter $\alpha = (\frac{1}{2}, \ldots, \frac{1}{2}, 1)$. As this constant is known to be the multivariate Beta function $\prod_{i=1}^{k+1} \Gamma(\alpha_i) \Gamma\big(\sum_{i=1}^{k+1} \alpha_i\big)^{-1}$, we obtain, using the identity $\Gamma(1/2) = \sqrt{\pi}$, that

$$\|R(\beta)\|_2^2 = \sum_{k=0}^{\infty} \frac{(\beta^k)^2}{2^k \Gamma(k/2 + 1)} < \infty.$$

$\blacksquare$

4.4 Construction of the P2P and P2L functions of the continuum polymer

It is said that $\mathcal{Z}$ is a *mild* solution to the stochastic heat equation (14) if, for all fixed $0 \le S < T \le 1$,

$$\begin{aligned}\mathcal{Z}(T,X) = &\int_{\mathbb{R}} \rho(T-S,X-Y)\mathcal{Z}(S,Y)\mathrm{d}Y \\ &+ \beta \int_S^T \int_{\mathbb{R}} \rho(T-U,X-Y)\mathcal{Z}(U,Y)\eta(U,Y)\mathrm{d}U\mathrm{d}Y,\end{aligned} \tag{59}$$

and if for all $T \ge 0$, $\mathcal{Z}(T,X)$ is measurable with respect to the white noise on $[0,T]\times\mathbb{R}$.

Remark 4.11. *As a motivation to look at this form of the equation, one can check that if $\mathcal{Z}(T,X)$ satisfies* (59) *with a smooth deterministic function $\eta(U,Y)$, then $\mathcal{Z}(T,X)$ is a solution to the SHE* (14) *with smooth noise.*

Remark 4.12. *Under some integrability condition, it can be shown that there is a unique mild solution - up to indistinguishability - to the SHE with Dirac initial condition* [14]. *This solution is continuous in time and space for $(T,X) \in (0,1]\times\mathbb{R}$, and it is continuous in $T=0$ in the space of distributions. Furthermore, $\mathcal{Z}_\beta(T,X)$ can be shown to be positive for all $T>0$* [126, 128].

Using the initial condition $\mathcal{Z}(0,X)=\delta_X$, we get by iterating equation (59) for $S=0$, that

$$\begin{aligned}\mathcal{Z}(T,X) = \rho(T,X) &+ \beta \int_0^T \int_{\mathbb{R}} \rho(T-U,X-Y)\rho(U,Y)\eta(U,Y)\,\mathrm{d}U\mathrm{d}Y \\ + \beta^2 \iint_{0<R<U\le T} &\iint_{\mathbb{R}^2} \rho(T-U,X-Y)\rho(U-R,Y-Z)\mathcal{Z}(R,Z) \\ &\qquad \times \eta(U,Y)\eta(R,Z)\,\mathrm{d}U\mathrm{d}Y\,\mathrm{d}R\,\mathrm{d}Z.\end{aligned}$$

By repeating this iteration procedure, the following expansion arises:

$$\mathcal{Z}(T,X) = \sum_{k=0}^{\infty} \beta^k I_k\big(\rho^k(\cdot;0,0;T,X)\big), \tag{60}$$

where we have used the notation, for $\mathbf{s}\in\Delta_k(s,t)$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}&\rho^k(\mathbf{s},\mathbf{x}\,;\,s,y\,;t,x) \\ &= \rho(s_1-s,x_1-y)\left(\prod_{j=1}^{k-1}\rho(s_{j+1}-s_j,x_{j+1}-x_j)\right)\rho(t-s_k,x-x_k),\end{aligned} \tag{61}$$

with the convention that $\rho^0(\cdot,\cdot;s,y;t,x)=\rho(t-s,x-y)$.

The ratio $\frac{\rho^k(\mathbf{s},\mathbf{x}\,;\,s,y\,;t,x)}{\rho(t-s,x-y)}$ is the k-steps transition function of a Brownian bridge, starting from (s,y) and ending at (t,x). From this observation, it is possible to introduce an alternative expression of $\mathcal{Z}(T,X)$, via a Feynman-Kac formula:

$$\mathcal{Z}(T,X) = \rho(T,X)\ \mathrm{P}_{0,0}^{T,X}\left[:\exp:\left(\beta\int_0^T \eta(u,B_u)\mathrm{d}u\right)\right], \tag{62}$$

with $\mathrm{P}_{s,y}^{t,x}$ the Brownian bridge $(s,y)\to(t,x)$. The Wick exponential $:\exp:$ of a Gaussian random variable ξ is defined by

$$:\exp(\xi) := \sum_{k=0}^{\infty}\frac{1}{k!} :\xi^k:$$

where the $:\xi^k:$ notation stands for the Wick power of a random variable (cf.[105]). The integral $\int_0^T \eta(u,B_u)\mathrm{d}u$, on the other hand, is not well defined, and to understand how to go from (62) to (60), one should use the following shortcut:

$$\mathrm{P}_{0,0}^{T,X}\left[:\left(\beta\int_0^T \eta(u,B_u)\mathrm{d}u\right)^k:\right] = \beta^k k! \int_{\Delta_k}\int_{\mathbb{R}^k}\frac{\rho^k(\mathbf{t},\mathbf{x};\,0,0\,;T,X)}{\rho(T,X)}\eta^{\otimes k}(\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x})$$

We will from now on suppose that $\mathcal{Z}_\beta(T,X)$ is defined through equation (60). Again, it is possible to check that the $\rho^k(\cdot\,;\,S,Y;T,X)$ have finite Fock space norm [1]. One can further define the shifted P2P functions:

$$\mathcal{Z}_\beta(S,Y;T,X) = \sum_{k=0}^{\infty} \beta^k I_k\big(\rho^k(\cdot\,;\,S,Y;T,X)\big). \tag{63}$$

Moreover, integrating over X equation (60), we recover the previous definition of the P2L partition function (58):

$$\mathcal{Z}_\beta = \mathcal{Z}_\beta(0,0;1,*) = \sum_{k=0}^{\infty} \beta^k I_k(\rho^k) = I(R(\beta)).$$

We also get that for any test function $\varphi \in \mathcal{C}_c^\infty$:

$$\int_{\mathbb{R}} \mathcal{Z}_\beta(T,X)\varphi(X)\mathrm{d}X = \sum_{k=0}^{\infty} \beta^k I_k\left(\int \rho^k(\cdot\,;0,0;T,X)\varphi(X)\mathrm{d}X\right). \tag{64}$$

5 Asymptotic study of Wiener-Itô integrals

5.1 The scaling relations

From now on, we will suppose that, as $t\to\infty$,

$$\begin{gathered}\text{(a) } \nu_t r_t^2 \lambda(\beta_t)^2 \sim (\beta^*)^2 t^{-1/2}, \quad \text{(b) } \nu_t r_t^3 \lambda(\beta_t)^3 \to 0,\\ \text{(c) } r_t/\sqrt{t} \to 0.\end{gathered} \tag{65}$$

There are two main reasons why we chose these relations. First, as $t\to\infty$, conditions (a) and (b) assure that under a scaling of t in time and $\sqrt{t}$ in space, the Poisson environment over time $[0,t]\times\mathbb{R}$ becomes a white noise environment on $[0,1]\times\mathbb{R}$. This fact is properly stated in Theorem 5.1, and the $k=1$ case of the proof gives good insights about how the parameters relate to one another. The addition of condition (c) ensures that the properly rescaled and normalized $T_k W_t$ functions converge to the Brownian transition functions.

5.2 Gaussian limits of Wiener-Itô integrals

We are interested in the limit of rescaled Wiener-Itô integrals, and more generally at sums of these integrals. We show that we can adapt the techniques developed for the study of U-statistics made out of an i.i.d. sequence of random variables, in chapter 11 of [105].

For any function g defined on $[0,1]^k\times\mathbb{R}^k$ (resp. $\Delta_k\times\mathbb{R}^k$), denote by $\tilde{g}_t$ the rescaled function of g, defined on $[0,t]^k\times\mathbb{R}^k$ (resp. $\Delta_k(0,t)\times\mathbb{R}^k$) and such that

$$\tilde{g}_t(\mathbf{s},\mathbf{x}) = g\big(\mathbf{s}/t, \mathbf{x}/\sqrt{t}\big), \tag{66}$$

and let γ_t be proportional to the vanishing parameter appearing in (b):

$$\gamma_t := (\beta^*)^{-3}\nu_t r_t^3 \lambda(\beta_t)^3 \to 0. \tag{67}$$

Theorem 5.1. *Let $g\in L^2([0,1]^k\times\mathbb{R}^k)$ for $k\geq 0$. The following convergence holds, as $t\to\infty$,*

$$\gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t) \xrightarrow{(d)} \int_{[0,1]^k}\int_{\mathbb{R}^k} g(\mathbf{t},\mathbf{x})\eta^{\otimes k}(\mathrm{d}\mathbf{t},\mathrm{d}\mathbf{x}). \tag{68}$$

The convergence can be extended for any finite collection $k_1,\dots,k_m\in\mathbb{N}$ and $g^1,\dots,g^m$, which satisfy $g^i\in L^2([0,1]^{k_i}\times\mathbb{R}^{k_i})$,

$$\big(\gamma_t^{k_1}\,\bar{\omega}_t^{(k_1)}(\tilde{g}_t^1),\dots,\gamma_t^{k_m}\,\bar{\omega}_t^{(k_m)}(\tilde{g}_t^m)\big) \xrightarrow{(d)} \big(I_{k_1}(g^1),\dots,I_{k_m}(g^m)\big). \tag{69}$$

Corollary 5.2. *Let $G = (g^k)_{k\geq 0}$ belong to the Fock space $\bigoplus_{k\geq 0} L^2(\Delta_k \times \mathbb{R}^k)$. Then, the sum $\bar{\omega}_t(G) := \sum_{k=0}^{\infty} \gamma_t^k\, \bar{\omega}_t^{(k)}(\tilde{g}_t^k)$ is well defined, and when $t \to \infty$,*

$$\bar{\omega}_t(G) = \sum_{k=0}^{\infty} \gamma_t^k\, \bar{\omega}_t^{(k)}(\tilde{g}_t^k) \xrightarrow{(d)} \sum_{k=0}^{\infty} \int_{\Delta_k} \int_{\mathbb{R}^k} g^k(\mathbf{t}, \mathbf{x}) \eta^{\otimes k}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = I(G). \tag{70}$$

The convergence can be extended to a joint convergence for any finite collection $G_1, \dots, G_m \in \bigoplus_{k\geq 0} L^2(\Delta_k \times \mathbb{R}^k)$:

$$\big(\bar{\omega}_t(G_1), \dots, \bar{\omega}_t(G_m)\big) \xrightarrow{(d)} \big(I(G_1), \dots, I(G_m)\big). \tag{71}$$

Remark 5.3. *Note that all functions in the corollary are defined on simplexes, as we have defined the Fock space for such functions.*

We first state a lemma that we will use several times. For a proof of the result, see [19, Ch. 1. Th. 3.2].

Lemma 5.4. *Let $(S, \mathcal{S})$ be a metric space with his Borel σ-field. Suppose that (X_t^n, Y_t) for $t \geq 0$, $n \in \mathbb{N}$ are random variables on S^2 and assume that the following diagram holds :*

$$\begin{array}{ccc} X_t^n & \xrightarrow[t\to\infty]{(d)} & Y^n \\ {\scriptstyle \mathbb{P},\ unif\ in\ t}\Big\downarrow{\scriptstyle n\to\infty} & & {\scriptstyle (d)}\Big\downarrow{\scriptstyle n\to\infty} \\ Y_t & & Y \end{array}$$

then $Y_t \xrightarrow{(d)} Y$.

Proof of Theorem 5.1. We follow [105]. In particular, we will focus in the first place on $k = 1$ and $g \in L^1 \cap L^2([0,1] \times \mathbb{R})$.

$\mathbf{k = 1}$ case. Let $g \in L^1 \cap L^2([0,1] \times \mathbb{R})$. When $k = 1$, we have $\bar{\omega}^{(1)} = \bar{\omega}$, so we can use the complex form of the exponential formula for Poisson point processes (see equation (74) below) to compute the characteristic function of $\gamma_t\, \bar{\omega}_t(\tilde{g}_t)$. Let $u \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{P}\left[e^{iu\gamma_t \bar{\omega}_t(\tilde{g}_t)}\right] \\ &= \exp\left(\int_{[0,t]} \int_{\mathbb{R}} \big(e^{iu\gamma_t g(s/t, x/\sqrt{t})} - 1 - iu\gamma_t\, g(s/t, x/\sqrt{t})\,\big)\nu_t\, \mathrm{d}s\mathrm{d}x \right) \\ &= \exp\left(\int_{[0,1]} \int_{\mathbb{R}} \nu_t t^{3/2} \big(e^{iu\gamma_t g(s,x)} - 1 - iu\gamma_t\, g(s,x)\big)\mathrm{d}s\mathrm{d}x \right), \end{aligned}$$

where the last equality comes from a change of variables.

By Taylor-Lagrange formula, we obtain

$$\nu_t t^{3/2} \left| e^{iu\gamma_t g(s,x)} - 1 - iu\gamma_t\, g(s,x) \right| \leq \nu_t t^{3/2} \gamma_t^2 \frac{u^2}{2} g(s,x)^2,$$

which gives L^1 domination, since g is square-integrable and since conditions (a) and (b) imply that $\nu_t \gamma_t^2 \sim t^{-3/2}$.

Using again this asymptotic equivalence and the fact that $\gamma_t \to 0$, we get that the integrand converges pointwise to the function $(s,x) \mapsto -\frac{u^2}{2} g^2(s,x)$. Therefore, dominated convergence proves that, as $t \to \infty$,

$$\mathbb{P}\left[e^{iu\gamma_t \bar{\omega}_t(\tilde{g}_t)}\right] \to \exp\left(-\frac{u^2}{2} \|g\|_2^2\right).$$

The limiting term is the Fourier transform of a centered Gaussian random variable of variance $\|g\|_2^2$, which has the same law as $I_1(g)$. This proves the first part of the theorem in the $k = 1$ and $L^1 \cap L^2$ case.

To prove the second part, we use the Cramér-Wold device which tells us that for a collection of real random variables, it is equivalent to show convergence in distribution of all finite linear combinations or to show joint convergence.

Thus, let $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and $g^1, \dots, g^m \in L^1 \cap L^2([0,1] \times \mathbb{R})$. By linearity of the stochastic integrals

$$\sum_{i=1}^m \alpha_i \, \gamma_t \, \bar{\omega}_t(\tilde{g}_t^i) = \gamma_t \, \bar{\omega}_t \left(\sum_{i=1}^m \alpha_i \, \tilde{g}_t^i \right) \xrightarrow{(d)} I_1 \left(\sum_{i=1}^m \alpha_i \, g^i \right) = \sum_{i=1}^m \alpha_i \, I_1(g^i),$$

where the convergence as $t \to \infty$ is ensured by the foregoing, since the combination $\sum_{i=1}^m \alpha_i g^i$ is square-integrable. By the Cramér-Wold device and as $t \to \infty$,

$$\big(\gamma_t \, \bar{\omega}_t(\tilde{g}_t^1), \dots, \gamma_t \, \bar{\omega}_t(\tilde{g}_t^m)\big) \xrightarrow{(d)} \big(I_1(g^1), \dots, I_1(g^m)\big).$$

$\mathbf{k \geq 1}$ case. Let $k \geq 1$ and let $g^1, \dots, g^k$ be the indicator functions of disjoint, finite and measurable sets $A_1, \dots, A_k \subset [0,1] \times \mathbb{R}$, and consider

$$g(\mathbf{s}, \mathbf{x}) = g^1(s_1, x_1) \dots g^k(s_k, x_k). \tag{72}$$

Equation (38) writes

$$\gamma_t^k \, \bar{\omega}_t^{(k)}(\tilde{g}_t) = \prod_{i=1}^k \gamma_t \, \bar{\omega}_t(\tilde{g}_t^i),$$

so joint convergence of the $\gamma_t \, \bar{\omega}_t(\tilde{g}_t^i)$, $i \leq k$, from the $k=1$ case, implies that

$$\bar{\omega}_t^{(k)}(\tilde{g}_t) \xrightarrow{(d)} \prod_{i=1}^k I_1(g^i) = I_k(g),$$

where the equality comes from property (53) and the fact that the g^i's are orthogonal in L^2.

In fact, if one takes $g^1, \dots, g^m$ of the form (72), so that they write $g^i(\mathbf{s}, \mathbf{x}) = \prod_{j=1}^{k_i} g^{i,j}(s_j, x_j)$, the same argument, combined with the joint convergence of the $\gamma_t \, \bar{\omega}_t(\tilde{g}_t^{i,j})$ for $1 \leq i \leq m$ and $1 \leq j \leq k_i$, proves that

$$\left(\gamma_t^{k_1} \, \bar{\omega}_t^{(k_1)}(\tilde{g}_t^1), \dots, \gamma_t^{k_m} \, \bar{\omega}_t^{(k_m)}(\tilde{g}_t^m)\right) \xrightarrow{(d)} \big(I_{k_1}(g^1), \dots, I_{k_m}(g^m)\big). \tag{73}$$

Now, denote by V_k the linear subspace of $L^2([0,1]^k \times \mathbb{R})$ spanned by the functions of the form (72), with fixed dimension k. By linear combinations, (73) can be extended for any collection $\big(g^i \in V_{k_i}\big)_{1 \leq i \leq m}$, so this proves the whole theorem for functions of V_k, $k \geq 1$.

It is a standard result that V_k is dense in $L^2([0,1]^k \times \mathbb{R}^k)$ for all $k \geq 1$. Let then g be any function of $L^2([0,1]^k \times \mathbb{R}^k)$ and $(g^n)_{n \geq 1}$ be a sequence of functions of V_k that converges to g in L^2 norm. Conditions (a) and (b) imply that $\nu_t \, t^{3/2} \gamma_t^2 \sim 1$. Hence, by the covariance structures and the linearity of $\bar{\omega}_t^{(k)}$, we obtain for large enough t:

$$\begin{aligned}
\mathbb{P}\left[\left(\gamma_t^k \bar{\omega}_t^{(k)}(\tilde{g}_t^n) - \gamma_t^k \bar{\omega}_t^{(k)}(\tilde{g}_t)\right)^2\right] &= \nu_t^k \gamma_t^{2k} k! \|(g^n - g)(\cdot/t, \cdot/\sqrt{t})\|^2_{L^2([0,t]^k \times \mathbb{R}^k)} \\
&= t^{3k/2} \nu_t^k \gamma_t^{2k} k! \|g^n - g\|^2_{L^2([0,1]^k \times \mathbb{R}^k)} \\
&\leq 2k! \|g^n - g\|^2_{L^2([0,1]^k \times \mathbb{R}^k)} \to 0,
\end{aligned}$$

as $n \to \infty$. Similarly for I_k:

$$\mathbb{Q}\left[\big(I_k(g^n) - I_k(g)\big)^2\right] = k! \|g^n - g\|^2_{L^2([0,1]^k \times \mathbb{R}^k)} \to 0$$

We get the following diagram:

$$\begin{tikzcd}
\gamma_t \, \bar{\omega}_t^{(k)}(\tilde{g}_t^n) \arrow[r, "(d)", "t \to \infty"'] \arrow[d, "L^2\text{, unif in } t"', "n \to \infty"] & I_k(g^n) \arrow[d, "(d)"', "n \to \infty"] \\
\gamma_t \, \bar{\omega}_t^{(k)}(\tilde{g}_t) & I_k(g)
\end{tikzcd}$$

so that $\gamma_t\,\bar{\omega}_t^{(k)}(\tilde{g}_t) \to I_k(g)$ by Lemma 5.4. This proves the first part of the theorem. The joint convergence can be shown by the same argument, using again the Cramér-Wold device and approaching any linear combinations of Wiener-Itô integrals of L^2 functions with linear combinations of integrals of V_k functions. ■

Proof of Corollary 5.2. We focus on the first part of the corollary, since the joint convergence follows from the Cramér-Wold device and linearity of $\bar{\omega}_t(G)$ and $I(G)$.

First and by definition, we know that $\sum_{k=0}^M I_k(g^k) \xrightarrow{L^2} \sum_{k=0}^\infty I_k(g^k)$ as $M\to\infty$. Moreover, since we are now dealing with functions on the simplex, equation (43) leads to

$$\|\bar{\omega}_t^{(k)}(\tilde{g}_t^k)\|_2^2 = \nu_t^k \|g^k(\cdot/t,\cdot/\sqrt{t})\|^2_{L^2(\Delta_k(0,t)\times\mathbb{R}^k)} = \nu_t^k\, t^{3k/2}\|g^k\|^2_{L^2(\Delta_k\times\mathbb{R}^k)}.$$

Conditions (a) and (b) imply that $\nu_t\, t^{3/2}\sim\gamma_t^{-2}$. Hence, as $\|g^k\|_2^2$ is summable, we obtain by absolute convergence that, uniformly in t and as $M\to\infty$,

$$\sum_{k=0}^M \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k) \xrightarrow{L^2} \sum_{k=0}^\infty \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k).$$

Moreover, it is a consequence the joint convergence part of the theorem that, for all M and when $t\to\infty$,

$$\sum_{k=0}^M \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k) \xrightarrow{(d)} \sum_{k=0}^M I_k(g^k).$$

Putting things together, we get the following diagram:

$$\begin{array}{ccc}
\sum_{k=0}^M \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k) & \xrightarrow[t\to\infty]{(d)} & \sum_{k=0}^M I_k(g^k) \\
{\scriptstyle L^2,\ \text{unif in } t}\Big\downarrow{\scriptstyle M\to\infty} & & {\scriptstyle (d)}\Big\downarrow{\scriptstyle M\to\infty} \\
\sum_{k=0}^\infty \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k) & & \sum_{k=0}^\infty I_k(g^k),
\end{array}$$

so by Lemma 5.4,

$$\bar{\omega}_t(G) = \sum_{k=0}^\infty \gamma_t^k\,\bar{\omega}_t^{(k)}(\tilde{g}_t^k) \xrightarrow{(d)} \sum_{k=0}^\infty I_k(g^k).$$

■

6 Proofs

6.1 Some useful formulas

- For all non-negative and all non-positive measurable functions h, defined on $\mathbb{R}_+\times\mathbb{R}^d$, the Poisson formula for exponential moments (chapter 3. of [115]) writes

$$\mathbb{P}\left[e^{\int h(s,x)\omega_t(dsdx)}\right] = \exp\int_{]0,t]\times\mathbb{R}} \nu \mathrm{d}s\mathrm{d}x\left(e^{h(s,x)}-1\right). \tag{74}$$

  The formula remains true when h is replaced by ih, for any real integrable function h.

- For all $s\geq 0$, we have

$$\int_{\mathbb{R}} \chi^r_{s,x}\,\mathrm{d}x = r. \tag{75}$$

6.2 Proof of Theorem 1.2 : SHE in the Poisson setting

Proof. Let $\xi_t = \exp(\beta\omega(V_t(B)) - \lambda(\beta)\nu r^d t)$ and observe that

$$\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x = \mathrm{P}[\xi_t\varphi(B_t)].$$

Then, recalling that $\omega(V_t(B)) = \int \chi_{s,x}\,\omega_t(\mathrm{d}s\mathrm{d}x)$, we use Itô's formula [99, Section II.5.] to get that

$$\begin{aligned}\xi_t &= 1 - \lambda\nu r^d\int_0^t \xi_s\mathrm{d}s + \lambda\int_{(0,t]\times\mathbb{R}} \xi_{s-}\chi_{s,x}\,\omega(\mathrm{d}s\mathrm{d}x)\\ &= 1 + \lambda\int_{(0,t]\times\mathbb{R}} \xi_{s-}\chi_{s,x}\,\bar\omega(\mathrm{d}s\mathrm{d}x),\end{aligned} \tag{76}$$

as almost surely, $\mathbb{P}$-a.s. $\xi_s = \xi_{s-}$ a.e.

As a difference of two increasing processes, ξ is of finite variation over all bounded time intervals. Also note that one can get an expression to the measure associated to ξ from the last equation. By the integration by part formula [104, p.52],

$$\xi_t\varphi(B_t) = \xi_0\varphi(B_0) + \int_0^t \xi_{s-}\mathrm{d}\varphi(B_s) + \int_0^t \varphi(B_s)\mathrm{d}\xi_s + [\xi,\varphi(B)]_t,$$

where $[\xi,\varphi(B)]_t = 0$ since $\varphi(B)$ is continuous. Applying Itô 's formula on $\mathrm{d}\varphi(B)$ and then taking P-expectation (which cancels the martingale term in the Itô formula), one obtains that $\mathbb{P}$-a.s.

$$\begin{aligned}&\int_{\mathbb{R}} W(t,x)\varphi(x)\mathrm{d}x\\ &= \varphi(0) + \frac{1}{2}\int_0^t \mathrm{P}[\xi_{s-}\Delta\varphi(B_s)]\mathrm{d}s + \lambda\int_{(0,t]\times\mathbb{R}} \mathrm{P}[\varphi(B_s)\xi_{s-}\chi_{s,y}]\bar\omega(\mathrm{d}s\mathrm{d}y)\\ &= \varphi(0) + \frac{1}{2}\int_0^t\int_{\mathbb{R}} \Delta\varphi(x)W(s-,x)\mathrm{d}x\mathrm{d}s\\ &\qquad + \lambda\int_{(0,t]\times\mathbb{R}}\left(\int_{\mathbb{R}}\varphi(x)\mathbf{1}_{|y-x|\le r/2}W(s-,x)\mathrm{d}x\right)\bar\omega(\mathrm{d}s\mathrm{d}y).\end{aligned}$$

To conclude the proof, observe that we can apply Fubini's theorem to the last integral since for all $t>0$,

$$\begin{aligned}\mathbb{P}\int_{(0,t]\times\mathbb{R}} \mathrm{P}[|\varphi(B_s)|\xi_{s-}\chi_{s,y}]\omega(\mathrm{d}s\mathrm{d}y) &= \nu e^{\beta}\int_{(0,t]\times\mathbb{R}} \mathbb{P}[\xi_{s-}]\mathrm{P}[|\varphi(B_s)|\chi_{s,y}]\mathrm{d}s\mathrm{d}y\\ &= \nu e^{\beta} r\int_0^t \mathrm{P}[|\varphi(B_s)|]\mathrm{d}s < \infty,\end{aligned}$$

where we have used the Mecke equation [115, 4.1] in the first equality. ■

6.3 Proof of Theorem 2.4 : convergence of the P2L partition function

Introduce the following time-depending functions of $[0,1]^k\times\mathbb{R}^k$:

$$\phi_t^k(\mathbf{s},\mathbf{x}) = \gamma_t^{-k}\,\lambda(\beta_t)^k\mathrm{P}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{r_t/\sqrt{t}}(B)\right]\mathbf{1}_{\Delta_k}(\mathbf{s},\mathbf{x}). \tag{77}$$

Note that for all (s,x), the diffusive scaling property of the Brownian motion implies that

$$\chi_{s/t,x/\sqrt{t}}^{r_t/\sqrt{t}} = \mathbf{1}_{|B_{s/t}-x/\sqrt{t}|\le r_t/2\sqrt{t}} \overset{\text{law}}{=} \chi_{s,x}^{r_t}.$$

Therefore, using notation $\widetilde{\phi}_t^k := (\widetilde{\phi_t^k})_t = \phi_t^k(\cdot/t, \cdot/\sqrt{t})$, we see that after simple rescaling, equation (49) gives

$$\gamma_t^k \, \widetilde{\phi}_t^k = T_k W_t \mathbf{1}_{\Delta_k(0,t)}. \tag{78}$$

Besides, observe that by the symmetric property of $T_k W_t$ and the invariance of the Wiener-Itô integrals under symmetrization (40), we obtain[5]:

$$\omega_t^{(k)}\big(T_k W_t\big) = k!\, \omega_t^{(k)}\big(T_k W_t \mathbf{1}_{\Delta_k(0,t)}\big).$$

Hence, Proposition (3.9) and equation (78) lead to the following expression of W_t:

$$W_t = \sum_{k=0}^{\infty} \gamma_t^k \, \bar{\omega}_t^{(k)} \left(\widetilde{\phi}_t^k\right). \tag{79}$$

Considering from now on ϕ_t^k as a function of the simplex, this writing of W_t is of the type $\bar{\omega}_t(G)$ (cf. Corollary 5.2), although there is a time dependence in the ϕ_t^k functions. The purpose of the two following lemmas is to study the asymptotic behavior of these functions, as $t \to \infty$.

Approximations in L^2-norm

Lemma 6.1. *Let k be a positive integer. We have the following properties:*

(i) For all $\mathbf{s} \in \Delta_k$, there exists a non-negative function $h_{\mathbf{s}} \in L^2(\mathbb{R}^k)$, such that

$$\forall \varepsilon \in (0,1],\ \mathbf{x} \in \mathbb{R}^k, \quad \varepsilon^{-k}\, \mathrm{P}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{\varepsilon}(B)\right] \le h_{\mathbf{s}}(\mathbf{x}).$$

(ii) There exists a non-negative function $H \in L^2(\Delta_k)$, such that

$$\forall \varepsilon > 0,\ \forall \mathbf{s} \in \Delta_k, \quad \int_{\mathbb{R}^k} \left(\varepsilon^{-k}\, \mathrm{P}\Big[\prod_{i=1}^k \chi_{s_i,x_i}^{\varepsilon}(B)\Big]\right)^2 \mathrm{d}\mathbf{x} \le H(\mathbf{s}).$$

(iii) We have the pointwise convergence, as $\varepsilon \to 0$,

$$\forall \mathbf{s} \in \Delta_k, \forall \mathbf{x} \in \mathbb{R}^k, \quad \varepsilon^{-k}\, \mathrm{P}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{\varepsilon}(B)\right] \to \rho^k(\mathbf{s},\mathbf{x}).$$

Proof. We use the convention $s_0 = y_0 = x_0 = u_0$.

(i) By Markov property of the Brownian Motion,

$$\begin{aligned}
&\varepsilon^{-k}\, \mathrm{P}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{\varepsilon}(B)\right] \\
&= \varepsilon^{-k} \int\limits_{\mathbb{R}^k} \prod_{i=1}^k \mathbf{1}_{|x_i - y_i| \le \varepsilon/2}\, \rho(s_i - s_{i-1}, y_i - y_{i-1}) \mathrm{d}\mathbf{y} && (80) \\
&= \int\limits_{[-\frac{1}{2},\frac{1}{2}]^k} \prod_{i=1}^k \rho\big(s_i - s_{i-1}, x_i - x_{i-1} + \varepsilon(u_i - u_{i-1})\big) \mathrm{d}\mathbf{u}, && (81)
\end{aligned}$$

where we have taken $y_i = x_i + \varepsilon u_i$.

[5] Note that from now on, we will always assume that $\omega \overset{\text{law}}{=} \omega^{\nu_t}$, even if we drop the superscript notation.

Observe that, for $0 < \varepsilon \leq 1$ and $u \in [-1,1]$,

$$\frac{e^{-\frac{(x+\varepsilon u)^2}{2s}}}{\sqrt{2\pi s}} = \rho(s,x) e^{-\varepsilon \frac{2xu}{2s}} e^{-\varepsilon^2 \frac{u^2}{2s}} \leq \rho(s,x) e^{|x|/s},$$

which leads to the following domination :

$$\varepsilon^{-k}\,\mathrm{P}\left[\prod_{i=1}^k \chi^{\varepsilon}_{s_i,x_i}(B)\right] \leq \prod_{i=1}^k \rho(s_i - s_{i-1}, x_i - x_{i-1}) e^{|x_i - x_{i-1}|/(s_i - s_{i-1})}. \tag{82}$$

Define $h_{\mathbf{s}}(\mathbf{x})$ to be the right-hand side of (82), so that what is left to prove is that $h_{\mathbf{s}} \in L^2(\mathbb{R}^k)$ for $\mathbf{s} \in \Delta_k$. With the change of variables $z_i = x_i - x_{i-1}$ of Jacobian $J = 1$ and by Tonelli's theorem

$$\begin{aligned}\int_{\mathbb{R}^k} h_{\mathbf{s}}(\mathbf{x})^2 \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^k} \prod_{i=1}^k \frac{e^{-(x_i - x_{i-1})^2/(s_i - s_{i-1})}}{2\pi(s_i - s_{i-1})} e^{2|x_i - x_{i-1}|/(s_i - s_{i-1})} \mathrm{d}\mathbf{x} \\ &= \prod_{i=1}^k \int_{\mathbb{R}} \frac{e^{-z_i^2/(s_i - s_{i-1})}}{2\pi(s_i - s_{i-1})} e^{2|z_i|/(s_i - s_{i-1})} \mathrm{d}z_i,\end{aligned}$$

which is finite as each integral converges.

(ii) One can first note that for all $s > 0$, $\rho(s,x) \leq 1/\sqrt{2\pi s}$. This combined with equation (80) gives us, for all $\mathbf{s} \in \Delta_k$,

$$\begin{aligned}\mathrm{P}\left[\prod_{i=1}^k \chi^{\varepsilon}_{s_i,x_i}(B)\right] &\leq (2\pi)^{-k/2} \prod_{i=1}^k \frac{1}{\sqrt{s_i - s_{i-1}}} \int_{\mathbb{R}^k} \prod_{i=1}^k \mathbf{1}_{|x_i - y_i| \leq \varepsilon/2}\, \mathrm{d}\mathbf{y} \\ &\leq \varepsilon^k \prod_{i=1}^k \frac{1}{\sqrt{s_i - s_{i-1}}}.\end{aligned} \tag{83}$$

Let $H(\mathbf{s}) = \prod_{i=1}^k (s_i - s_{i-1})^{-1/2}$ be the product appearing in the last inequality. We saw in the proof of Proposition 4.9 that H is an element of $L^1(\Delta_k)$. Furthermore:

$$\begin{aligned}\int_{\mathbb{R}^k} \left(\varepsilon^{-k}\,\mathrm{P}\Big[\prod_{i=1}^k \chi^{\varepsilon}_{s_i,x_i}(B)\Big]\right)^2 \mathrm{d}\mathbf{x} &\leq H(\mathbf{s}) \int_{\mathbb{R}^k} \varepsilon^{-k}\,\mathrm{P}\Big[\prod_{i=1}^k \chi^{\varepsilon}_{s_i,x_i}(B)\Big] \mathrm{d}\mathbf{x} \\ &= H(\mathbf{s}),\end{aligned} \tag{84}$$

where we have used Tonelli's theorem in the equality.

(iii) This result can be derived from equation (81), using continuity in x of $\rho(s,x)$ for a fixed $s > 0$. ∎

From the last lemma, we can derive L^2 properties of ϕ_t^k:

Lemma 6.2. *Let k be a positive integer. We have:*

(i) The following convergence holds:

$$\|\phi_t^k - (\beta^*)^k \rho^k\|_{L^2(\Delta_k \times \mathbb{R}^k)} \underset{t\to\infty}{\longrightarrow} 0.$$

(ii) There exists a positive constant $C = C(\beta^*)$, such that

$$\sup_{t\in[0,1]} \|\phi_t^k\|_{L^2([0,1]^k \times \mathbb{R}^k)} \leq C^k \|\rho^k\|_{L^2(\Delta_k \times \mathbb{R}^k)}.$$

Proof. (i) Recall that on $\Delta_k(0,t) \times \mathbb{R}^k$:

$$\phi_t^k(\mathbf{s},\mathbf{x}) = \gamma_t^{-k}\,\lambda^k \mathrm{P}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{r_t/\sqrt{t}}(B)\right].$$

Conditions (a) and (b) imply that, as $t \to \infty$,

$$\gamma_t^{-1}\lambda(\beta_t) \sim \beta^* \frac{\sqrt{t}}{r_t}, \tag{85}$$

which leads to the existence of a constant $c > 1$, such that, for t large enough and all $k \geq 1$:

$$\gamma_t^{-k}\lambda^k \leq c^k |\beta^*|^k \left(\frac{r_t}{\sqrt{t}}\right)^{-k}. \tag{86}$$

Observe that $r_t/\sqrt{t} \to 0$ by condition (c). Then, Lemma 6.1 implies the existence of two non-negative dominating functions $h_{\mathbf{s}} \in L^2(\mathbb{R}^k)$ and $H \in L^1(\Delta_k)$, such that, for large enough t and $\mathbf{s} \in \Delta_k$,

$$\forall \mathbf{x} \in \mathbb{R}^k \quad \phi_t^k(\mathbf{s},\mathbf{x}) \leq h_{\mathbf{s}}(\mathbf{x}), \tag{87}$$

and

$$\int_{\mathbb{R}^k} \phi_t^k(\mathbf{s},\mathbf{x})^2 \mathrm{d}\mathbf{x} \leq H(\mathbf{s}). \tag{88}$$

Furthermore, because of the equivalence (85), point (iii) of the same lemma shows that we have the pointwise convergence:

$$\forall \mathbf{s} \in [0,1],\ \forall \mathbf{x} \in \mathbb{R}^k, \quad \phi_k^t(\mathbf{s},\mathbf{x}) \underset{t\to\infty}{\longrightarrow} (\beta^*)^k \rho^k(\mathbf{s},\mathbf{x}).$$

From (87), we get by the dominated convergence theorem that

$$\forall \mathbf{s} \in \Delta_k, \quad \int_{\mathbb{R}^k} \left(\phi_t^k(\mathbf{s},\mathbf{x}) - (\beta^*)^k \rho^k(\mathbf{s},\mathbf{x})\right)^2 \mathrm{d}\mathbf{x} \underset{t\to\infty}{\longrightarrow} 0,$$

and as we have

$$\int_{\mathbb{R}^k} \left(\phi_t^k(\mathbf{s},\mathbf{x}) - (\beta^*)^k \rho^k(\mathbf{s},\mathbf{x})\right)^2 \mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^k} 2\phi_t^k(\mathbf{s},\mathbf{x})^2 + 2(\beta^*)^{2k}\rho^k(\mathbf{s},\mathbf{x})^2 \mathrm{d}\mathbf{x},$$

where the right hand side is dominated in $L^1(\Delta_k)$ using equation (88), we can use the dominated convergence theorem and obtain the convergence

$$\|\phi_k^t - (\beta^*)^k \rho^k\|_{L^2(\Delta_k \times \mathbb{R}^k)}^2 = \int_{\Delta_k} \left(\int_{\mathbb{R}^k} \left(\phi_k^t(\mathbf{s},\mathbf{x}) - (\beta^*)^k \rho^k(\mathbf{s},\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}\right) \mathrm{d}\mathbf{s} \underset{t\to\infty}{\longrightarrow} 0.$$

(ii) Using inequalities (86) and (84), we get

$$\|\phi_k^t\|_{L^2([0,1]^k \times \mathbb{R}^k)}^2 \leq c^k |\beta^*|^k \int_{\Delta_k} \prod_{i=1}^k \frac{1}{\sqrt{s_i - s_{i-1}}} \mathrm{d}\mathbf{s} = C^k \|\rho^k\|_{L^2(\Delta_k \times \mathbb{R}^k)}^2,$$

with $C = 2\sqrt{\pi} c |\beta^*|$. ■

Proof of the theorem

We are now ready to prove Theorem 2.4. From Proposition 4.9, we know that $R(\beta^*) = ((\beta^*)^k \rho^k)_{k\geq 0} \in \bigoplus_{k\geq 0} L^2(\Delta_k \times \mathbb{R}^k)$, so we get from Corollary 5.2 that when $t \to \infty$,

$$\sum_{k=0}^{\infty} (\beta^*)^k \gamma_t^k\, \bar{\omega}_t^{(k)}(\tilde{\rho}_t^k) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}. \tag{89}$$

In addition to this, we saw at equation (79) that the normalized partition function writes $W_t = \sum_{k=0}^{\infty} \gamma_t^k \,\bar{\omega}_t^{(k)} \left(\tilde{\phi}_t^k\right)$, where $\phi_t^k \xrightarrow{L^2} (\beta^*)^k \rho^k$ by Lemma 6.2. It is a standard result that if X_n and Y_n are real random variables such that $Y_n \xrightarrow{(d)} Y$ and $\|Y_n - X_n\|_2 \longrightarrow 0$, then $X_n \xrightarrow{(d)} Y$. Hence, in order to prove that $W_t \xrightarrow{(d)} \mathcal{Z}_{\beta^*}$, it suffices to show that

$$\left\| \sum_{k=0}^{\infty} \gamma_t^k \,\bar{\omega}_t^{(k)}(\tilde{\phi}_t^k) - \sum_{k=0}^{\infty} (\beta^*)^k \gamma_t^k \,\bar{\omega}_t^{(k)}(\tilde{\rho}_t^k) \right\|_2^2 \underset{t\to\infty}{\longrightarrow} 0. \tag{90}$$

By linearity of $\tilde{\omega}_t^{(k)}$ and orthogonality of the terms for two different k, we get from Pythagoras' identity that the norm can be written as

$$\sum_{k=0}^{\infty} \gamma_t^{2k} \,\|\bar{\omega}_t^{(k)}\big(\phi_t^k(\cdot/t, \cdot/\sqrt{t}) - (\beta^*)^k \rho^k(\cdot/t, \cdot/\sqrt{t})\big)\|_2^2.$$

For all $g \in L^2(\Delta_k \times \mathbb{R}^k)$, equation (43) and a substitution of variables lead to

$$\|\bar{\omega}_t^{(k)}\big(g(\cdot/t, \cdot/\sqrt{t})\|_2^2 = \nu_t^k \|g(\cdot/t, \cdot/\sqrt{t})\|^2_{L^2(\Delta_k(0,t)\times\mathbb{R}^k)} = \nu_t^k t^{3k/2} \|g\|^2_{L^2(\Delta_k\times\mathbb{R}^k)},$$

so that the above sum is given by

$$\sum_{k=0}^{\infty} \gamma_t^{2k} \nu_t^k t^{3k/2} \,\|\phi_t^k - (\beta^*)^k \rho^k\|^2_{L^2(\Delta_k\times\mathbb{R}^k)}.$$

Conditions (a) and (b) imply that $\gamma_t^2 \nu_t^k t^{3/2} \sim 1$, so by lemma 6.2, the summand tends to zero, as $t \to \infty$, and it is dominated by $C^{2k} \|\rho^k\|_2^2$, where $C = C(\beta^*)$ is some positive constant. As this dominating sequence is summable (Proposition 4.9), the dominated convergence theorem concludes the proof.

6.4 Proof of Theorem 2.5 : convergence of the point-to-point partition function

Using again Theorem 3.7, and after similar normalization to what was done in the beginning of Section 6.3, we find that

$$\sqrt{t} W(\beta_t, tS, \sqrt{t}Y; tT, \sqrt{t}X) = \sum_{k=0}^{\infty} \gamma_t^k \,\bar{\omega}_t^{(k)} \left(\widetilde{\psi}_t^k(S, Y; T, X)\right), \tag{91}$$

where

$$\begin{aligned} &\psi_t^k(S, Y; T, X)(\mathbf{s}, \mathbf{x}) \\ &= \gamma_t^{-k} \lambda(\beta_t)^k \rho(T - S, X - Y) \mathrm{P}_{S,Y}^{T,X} \left[\prod_{i=1}^{k} \chi_{s_i, x_i}^{r_t/\sqrt{t}}(B) \right] \mathbf{1}_{\Delta_k(S,T)}. \end{aligned}$$

Analogous calculations to those of Section 6.3 will show that for all $k \geq 0$, as $t \to \infty$,

$$\psi_t^k(S, Y; T, X) \xrightarrow{L^2} \beta^{*k} \rho^k(\cdot\,;\, S, Y; T, X),$$

where, by Corollary 5.2, the limiting functions have the property that

$$\sum_{k=0}^{\infty} \gamma_t^k \beta^{*k} \,\bar{\omega}^{(k)} \left(\tilde{\rho}_t^k(\cdot\,;\, S, Y; T, X)\right) \xrightarrow{(d)} \mathcal{Z}_{\beta^*}(S, Y; T, X). \tag{92}$$

The theorem then follows from similar arguments to those of the proof of the convergence of the point-to-line partition function, that is by showing that the right-hand side of (91) and the left-hand side of (92) are close in L^2 norm.

6.5 Proof of Theorem 2.7 : convergence in terms of processes

In order to show tightness of $\mathcal{Y}_t$, we will rely on Mitoma's criterion [124, 161]. It will help us reduce the problem of showing tightness of a two variables process, to the problem of showing tightness of a set of one variable processes.

In what follows, for any function $F \in D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$ and $\varphi \in \mathcal{D}(\mathbb{R})$, we set

$$F(T,\varphi) := \int F(T,X)\varphi(X)\mathrm{d}X. \tag{93}$$

Proposition 6.3 ([124])**.** *Let $(F_t)_{t\geq 0}$ be a family of processes in $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$. If, for all $\varphi \in \mathcal{D}(\mathbb{R})$, the family $T \to F_t(T,\varphi), t \geq 0$ is tight in the real cadlàg functions space $D([0,1],\mathbb{R})$, then $(F_t)_{t\geq 0}$ is tight in $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$.*

Then, in order to prove uniqueness of the limit, we use the following proposition:

Proposition 6.4 ([124])**.** *Let $(F_t)_{t\geq 0}$ be a tight family of processes in the space $D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$. If there exists a process $F \in D\left([0,1], \mathcal{D}'(\mathbb{R})\right)$ such that, for all $n \geq 1$, $T_1, \dots, T_n \in [0,1]$ and $\varphi_1, \dots, \varphi_n \in \mathcal{D}(\mathbb{R})$, we have as $t \to \infty$:*

$$\left(F_t(T_1,\varphi_1), \dots, F_t(T_n,\varphi_n)\right) \xrightarrow{(d)} \left(F_t(T_1,\varphi_1), \dots, F_t(T_n,\varphi_n)\right),$$

then $F_t \xrightarrow{(d)} F$.

Identification of the limit

Proposition 6.5. *Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then, for all $T \geq 0$ and as $t \to \infty$,*

$$\mathcal{Y}_t(T,\varphi) := \int \mathcal{Y}_t(T,X)\varphi(X)\mathrm{d}X \xrightarrow{(d)} \int \mathcal{Z}_{\beta^*}(T,X)\varphi(X)\mathrm{d}X. \tag{94}$$

Moreover, the convergence extends to a joint convergence as in Proposition 6.4.

Proof. Once again, we rely on Theorem 3.7 and similar normalization to the beginning of Section 6.3 to get that

$$\mathcal{Y}_t(T,\varphi) = \sum_{k=0}^{\infty} \gamma_t^k \,\bar{\omega}^{(k)}\left(\widetilde{\psi}_t^k(T,\varphi)\right),$$

where

$$\begin{aligned}&\psi_t^k(T,\varphi)(\mathbf{s},\mathbf{x}) \\ &= \gamma_t^{-k}\lambda(\beta_t)^k \int_{\mathbb{R}} \rho(T,X)\mathrm{P}_{0,0}^{T,X}\left[\prod_{i=1}^k \chi_{s_i,x_i}^{r_t/\sqrt{t}}(B)\right]\varphi(X)\mathrm{d}X\,\mathbf{1}_{\Delta_k}(\mathbf{s},\mathbf{x}).\end{aligned}$$

Then, for all $k \geq 0$ and as $t \to \infty$, we have:

$$\psi_t^k(T,\varphi) \xrightarrow{L^2} g^k := \beta^{*k} \int_{\mathbb{R}} \rho^k(\cdot\,;0,0;T,X)\varphi(X)\mathrm{d}X.$$

To see this, apply Cauchy-Schwarz's inequality to obtain that

$$\begin{aligned}
&\|\psi_t^k(T,\varphi) - g^k\|_2^2 \\
&= \int_{\Delta_k\times\mathbb{R}^k} \left(\int_{\mathbb{R}} \varphi(X) \left(\gamma_t^{-k} \lambda(\beta_t)^k \rho(T,X) \mathrm{P}_{0,0}^{T,X} \left[\prod_{i=1}^k \chi_{s_i,x_i}^{r_t/\sqrt{t}} \right] \right.\right. \\
&\qquad\qquad \left.\left. - \beta^{*k} \rho^k(\mathbf{s},\mathbf{x},0,0;T,X) \right) \mathrm{d}X \right)^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x} \\
&\le \|\varphi\|_2^2 \int_{\Delta_k\times\mathbb{R}^{k+1}} \left(\gamma_t^{-k} \lambda(\beta_t)^k \rho(T,X) \mathrm{P}_{0,0}^{T,X} \left[\prod_{i=1}^k \chi_{s_i,x_i}^{r_t/\sqrt{t}} \right] \right. \\
&\qquad\qquad \left. - \beta^{*k} \rho^k(\mathbf{s},\mathbf{x};0,0;T,X) \right)^2 \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{x}\mathrm{d}X.
\end{aligned}$$

By similar estimates to those obtained in Section 6.3, we get that this last integral goes to 0.

Equation (64) and Corollary 5.2 imply that

$$\sum_{k=0}^{\infty} \gamma_t^k \, \bar{\omega}^{(k)} \left(\tilde{g}_t^k \right) \xrightarrow{(d)} \int_{\mathbb{R}} \mathcal{Z}_{\beta^*}(S,X)\varphi(X)\mathrm{d}X,$$

so that convergence (94) follows from the same arguments we used for the convergence of the P2L functions. Finally, the joint convergence can be obtained using (71) in Corollary 5.2.

■

Tightness

The process Λ^t is a Poisson point process of intensity measure $t^{3/2}\nu_t \, \mathrm{d}S\mathrm{d}X$. By simple rescaling of the Poisson stochastic heat equation (17), one can write that

$$\mathcal{Y}_t(T,\varphi) = \varphi(0) + A_T^t(\varphi) + M_T^t(\varphi), \tag{95}$$

where

$$\begin{aligned}
A_T^t(\varphi) &= \frac{1}{2}\int_0^T \int_{\mathbb{R}} \Delta\varphi(X)\mathcal{Y}_t(S,X)\mathrm{d}S\mathrm{d}X, \\
M_T^t(\varphi) &= \lambda(\beta_t) \int_{\mathbb{R}} \varphi(X) \left(\int_{(0,T]\times\mathbb{R}} \mathcal{Y}_t(S-,X) \mathbf{1}_{|X-Y|\le r_t/2\sqrt{t}} \, \overline{\Lambda^t}(\mathrm{d}S\mathrm{d}Y) \right) \mathrm{d}X.
\end{aligned}$$

We will show that both $A^t(\varphi)$ and $M^t(\varphi)$ are tight in $\mathcal{D}([0,1],\mathbb{R})$. If this is proven, then $\left(A^t(\varphi), M^t(\varphi)\right)$ is tight, hence $\mathcal{Y}_t(\cdot,\varphi)$ is tight.

Tightness of $A^t(\varphi)$: To prove tightness, we will use Kolmogorov's criterion [18, Theorem 12.3]. For this we need estimates on the moments on the variations of $A^t(\varphi)$, which will be derived through the next lemma:

Lemma 6.6. *Suppose that $(\beta_t)_{t\ge0}$ is bounded by above. Then, there exists a constant $C = C(\beta^*,p)$ verifying $0 < C < \infty$ and that for t large enough and all $p > 1$, $T > 0$ and $X \in \mathbb{R}$,*

$$\mathbb{P}[\mathcal{Y}_t(T,X)^p] \le C\rho(T,X)^p. \tag{96}$$

Suppose for a moment that the lemma is proven, and let $p \ge 2$ be an integer and $U \le T$ in $[0,1]$. We have:

$$\begin{aligned}
\mathbb{P}\left[\left|A_T^t(\varphi) - A_U^t(\varphi)\right|^p\right] &\le 2^{-p}\mathbb{P}\left[\left(\int_{[U,T]\times\mathbb{R}} |\Delta\varphi(X)| \, \mathcal{Y}_t(S,X)\mathrm{d}S\mathrm{d}X\right)^p\right] \\
&\le 2^{-p}\|\Delta\varphi\|_\infty^p \int_{[U,T]^p\times\mathbb{R}^p} \mathbb{P}\left[\prod_{i=1}^p \mathcal{Y}_t(S_i,X_i)\right] \mathrm{d}\mathbf{S}\mathrm{d}\mathbf{X}.
\end{aligned}$$

By Lemma 6.6, the functions $\mathcal{Y}_t(S_i, X_i)/\rho(S_i, X_i)$ are bounded in L^p, so we can use the generalized Hölder inequality to bound the expectation of the product in the right-hand side. We get that there is constant $C = C(p) > 0$ such that

$$\begin{aligned}\mathbb{P}\left[\left|\left\langle A^t(\varphi)\right\rangle_T - \left\langle A^t(\varphi)\right\rangle_U\right|^p\right] &\leq C\|\Delta\varphi\|_\infty^p \int_{[U,T]^p\times\mathbb{R}^p} \prod_{i=1}^p \rho(S_i, X_i)\mathrm{d}\mathbf{S}\mathrm{d}\mathbf{X} \\ &= C\|\Delta\varphi\|_\infty^p \left(\int_{[U,T]\times\mathbb{R}} \rho(S,X)\mathrm{d}S\mathrm{d}X\right)^p \\ &= C\|\Delta\varphi\|_\infty^p\, |T-U|^p.\end{aligned}$$

This shows that $A^t(\varphi)$ verifies the assumptions of Kolmogorov's criterion.

Proof of Lemma 6.6. Let $p > 1$, $T > 0$ and $X \in \mathbb{R}$. We have:

$$\mathcal{Y}_t(T,X)^p = \rho(T,X)^p \mathrm{P}^{\otimes p}\left[\exp\int_{[0,tT]\times\mathbb{R}} \sum_{i=1}^p \beta_t \chi_{s,x}^{r_t}(X^i)\,\omega(\mathrm{d}s\mathrm{d}x)\right] e^{-ptT\lambda\nu_t r_t},$$

where $X^1, \ldots, X^p$ are independent Brownian bridges from $(0,0)$ to $(tT, \sqrt{t}X)$. By the exponential formula (74), we get

$$\frac{\mathbb{P}[\mathcal{Y}_t(T,X)^p]}{\rho(T,X)^p} = \mathrm{P}^{\otimes p}\left[\exp\int_{[0,tT]\times\mathbb{R}}\left(e^{\sum_{i=1}^p \beta_t\chi_{s,x}^{r_t}(X^i)} - 1\right)\nu_t\mathrm{d}s\mathrm{d}x\right] e^{-ptT\lambda\nu_t r_t}.$$

Then, observe that

$$\begin{aligned}e^{\sum_{i=1}^p \beta_t\chi_{s,x}^{r_t}(X^i)} &= \prod_{i=1}^p\left(1+\lambda\chi_{s,x}^{r_t}(X^i)\right) \\ &= 1 + \sum_{i=1}^p \lambda\chi_{s,x}^{r_t}(X^i) + \sum_{k=2}^p \lambda^k \sum_{p_1<\cdots<p_k\leq p}\prod_{i=1}^k \chi_{s,x}^{r_t}(X^{p_i}).\end{aligned}$$

Using equation (75), we are left with:

$$\begin{aligned}&\frac{\mathbb{P}[\mathcal{Y}_t(T,X)^p]}{\rho(T,X)^p} = \\ &\mathrm{P}^{\otimes p}\left[\prod_{k=2}^p \prod_{p_1<\cdots<p_k\leq p} \exp\left(\nu_t\lambda(\beta_t)^k\int_{[0,tT]\times\mathbb{R}} \prod_{i=1}^k \chi_{s,x}^{r_t}(X^{p_i})\,\mathrm{d}s\mathrm{d}x\right)\right].\end{aligned} \tag{97}$$

We claim that for each $q > 0$ and $k \geq 2$, there exists a constant $C = C(q, \beta^*) > 0$, such that for all t large enough and all T, X,

$$\mathrm{P}^{\otimes p}\left[\exp\left(q\nu_t|\lambda(\beta_t)|^k\int_{[0,tT]\times\mathbb{R}}\prod_{i=1}^k \chi_{s,x}^{r_t}(X^{p_i})\,\mathrm{d}s\mathrm{d}x\right)\right] \leq C. \tag{98}$$

If this is proven, then the generalized Hölder inequality implies that the right-hand side of (97) is bounded, which is the claim of the lemma.

First, notice that for all $k \geq 2$,

$$\begin{aligned}\int_0^{tT}\int_{\mathbb{R}}\prod_{i=1}^k \chi_{s,x}^{r_t}(X^{p_i})\mathrm{d}x\mathrm{d}s &\leq \int_0^{tT}\int_{\mathbb{R}} \chi_{s,x}^{r_t}(X^{p_1})\chi_{s,x}^{r_t}(X^{p_2})\mathrm{d}x\mathrm{d}s \\ &\leq \int_0^{tT}\int_{\mathbb{R}} \mathbf{1}_{|X_s^{p_1}-X_s^{p_2}|\leq r_t/2}\,\chi_{s,x}^{r_t}(X^{p_1})\mathrm{d}x\mathrm{d}s \\ &= tr_t\int_0^T \mathbf{1}_{|X_{tS}^{p_1}-X_{tS}^{p_2}|\leq r_t/2}\,\mathrm{d}S\end{aligned}$$

As $X^{p_1}_{tS} - X^{p_2}_{tS} \overset{\text{law}}{=} \sqrt{2t}\tilde{X}_S$, where $\tilde{X}$ is a Brownian bridge $(0,0) \to (T,0)$, we can bound the left hand side of (98) by

$$\sum_{m=0}^{\infty} \frac{1}{m!}(qtr_t\nu_t|\lambda(\beta_t)|^k)^m \mathrm{P}\left[\left(\int_0^T \mathbf{1}_{|\tilde{X}_S| \le r_t/\sqrt{8t}}\,\mathrm{d}S\right)^m\right].$$

By symmetry, we have:

$$\begin{aligned}
&\frac{1}{m!}\mathrm{P}\left[\left(\int_0^T \mathbf{1}_{|\tilde{X}_S| \le r_t/\sqrt{8t}}\,\mathrm{d}S\right)^m\right] \\
&= \int_{\Delta_m(0,T)} \mathrm{P}\left[\prod_{i=1}^m \mathbf{1}_{|\tilde{X}_{S_i}| \le r_t/\sqrt{8t}}\right]\mathrm{d}\mathbf{S} \\
&= \int_{\Delta_m(0,T)} \int_{[-\frac{r_t}{2\sqrt{2t}},\frac{r_t}{2\sqrt{2t}}]} \frac{\prod_{i=1}^{m+1}\rho(S_i - S_{i-1}, X_{i-1} - X_i)}{\rho(T,0)}\mathrm{d}\mathbf{S}\mathrm{d}\mathbf{X},
\end{aligned}$$

where $S_0 = X_0 = X_{m+1} = 0$ and $S_{m+1} = T$. Using that $\rho(S,X) \le \sqrt{2\pi S}^{-1}$, we get that

$$\begin{aligned}
&\frac{1}{m!}\mathrm{P}\left[\left(\int_0^T \mathbf{1}_{|\tilde{X}_S| \le r_t/\sqrt{8t}}\,\mathrm{d}S\right)^m\right] \\
&\le (\sqrt{2\pi})^{-m}\sqrt{T}\left(\frac{r_t}{\sqrt{2t}}\right)^m \int_{\Delta_m(0,T)} \prod_{i=1}^{m+1} \sqrt{S_i - S_{i-1}}^{-1}\mathrm{d}\mathbf{S} \\
&\le C^m r_t^m t^{-m/2} T^{m/2} \int_{\Delta_m(0,1)} \sqrt{U_i - U_{i-1}}^{-1}\mathrm{d}\mathbf{U} \\
&= C^m r_t^m t^{-m/2} T^{m/2} \frac{\sqrt{\pi}^m}{\Gamma(m/2)},
\end{aligned}$$

where $C > 0$ is some constant and where the value of the integral in the third equation was identified via the Dirichlet distribution.

Since β_t is assumed to be bounded by above, $\lambda(\beta_t)$ is bounded. Then, as $k \ge 2$, the scaling relation (a) implies that there exists a constant $C_1 = C_1(\beta^*) > 0$, such that $t^{1/2}r_t^2\nu_t\lambda(\beta_t)^k \le C_1$. Added to the fact that $T \le 1$, we get that there exists a finite constant $C_2 > 0$, depending only on β^* and q, such that for t large enough:

$$\sum_{m=0}^{\infty} \frac{1}{m!}(qtr_t\nu_t|\lambda|^k)^m \mathrm{P}\left[\left(\int_0^T \mathbf{1}_{|\tilde{X}_S| \le r_t/\sqrt{8t}}\,\mathrm{d}S\right)^m\right] \le \sum_{m=0}^{\infty} \frac{C_2^m}{\Gamma(m/2)} < \infty.$$

This proves (98), which ends the proof of the lemma.

■

Tightness of $M^t(\varphi)$: The process $T \mapsto M^t_T(\varphi)$ is a martingale (see equation (99) below) with respect to the filtration induced by $(\omega_{tT})_{T\in[0,1]}$. It is therefore possible to rely on Aldous' criterion to show tightness:

Theorem 6.7 (Aldous' criterion for martingales [104, Chap. VI Theorem 4.13])**.** *Let $(N^t)_{t\ge 0}$ be a family of martingales in $D([0,1],\mathbb{R})$. Assume that:*

(i) The family $(N^t_0)_{t\ge0}$ is tight.

(ii) The family of previsible brackets $(\langle N^t\rangle)_{t\ge0}$ is tight in $\mathcal{C}([0,1],\mathbb{R})$.

Then, the $(N^t)_{t\ge0}$ are tight in $D([0,1],\mathbb{R})$.

In our case, point (i) is immediately verified, as $M_0^t(\varphi) = 0$. To show that point (ii) holds, we use Kolmogorov's criteria. We have:

$$M_T^t(\varphi) = \int_{(0,T]\times\mathbb{R}} f(S,Y,\omega)\overline{\Lambda^t}(\mathrm{d}S\mathrm{d}Y), \tag{99}$$
$$\text{where}\quad f(S,Y,\omega) = \lambda(\beta_t)\int_{\mathbb{R}} \mathcal{Y}_t(S-,X)\varphi(X)\mathbf{1}_{|X-Y|\leq r_t/2\sqrt{t}}\,\mathrm{d}X.$$

Since f is predictable and Λ^t has intensity $t^{3/2}\nu_t\,\mathrm{d}S\mathrm{d}X$, the bracket can be expressed [99, Section II.3.] by

$$\left\langle M^t(\varphi)\right\rangle_T = t^{3/2}\nu_t\int_{[0,T]\times\mathbb{R}} f(S,Y,\omega)^2\,\mathrm{d}S\mathrm{d}Y.$$

Now, by Cauchy-Schwarz inequality, we have:

$$f(S,Y,\omega)^2 \leq \frac{r_t\lambda^2}{2\sqrt{t}}\int_{\mathbb{R}}\mathcal{Y}_t(S-,X)^2\varphi(X)^2\mathbf{1}_{|X-Y|\leq r_t/2\sqrt{t}}\,\mathrm{d}X,$$

so for all $U\leq T$ in $[0,1]$ and integer $p\geq 2$, we get by Tonelli's theorem that

$$\begin{aligned}
&\mathbb{P}\left[\left|\left\langle M^t(\varphi)\right\rangle_T - \left\langle M^t(\varphi)\right\rangle_U\right|^p\right]\\
&\leq \frac{t^{3p/2}\nu_t^p r_t^p\lambda^{2p}}{2^p t^{p/2}}\mathbb{P}\left[\int_{[U,T]\times\mathbb{R}}\mathcal{Y}_t(S-,X)^2\varphi(X)^2\left(\int_{\mathbb{R}}\mathbf{1}_{|X-Y|\leq r_t/2\sqrt{t}}\,\mathrm{d}Y\right)\mathrm{d}S\mathrm{d}X\right]^p\\
&= 2^{-p}t^{p/2}\nu_t^p r_t^{2p}\lambda^{2p}\,\mathbb{P}\left[\int_{[U,T]\times\mathbb{R}}\mathcal{Y}_t(S-,X)^2\varphi(X)^2\,\mathrm{d}S\mathrm{d}X\right]^p.
\end{aligned}$$

Then, observe that by our scaling relations (25), the quantity $t^{1/2}\nu_t r_t\lambda(\beta_t)^2$ is bounded by some constant $C>0$. Expanding the power of the integral, we get:

$$\mathbb{P}\left[\left|\left\langle M^t(\varphi)\right\rangle_T - \left\langle M^t(\varphi)\right\rangle_U\right|^p\right] \leq C\|\varphi\|_\infty^{2p}\int_{[U,T]^p\times\mathbb{R}^p}\mathbb{P}\left[\prod_{i=1}^p\mathcal{Y}_t(S_i-,X_i)^2\right]\mathrm{d}\mathbf{S}\mathrm{d}\mathbf{X}$$

As we know by Lemma 6.6 that $\mathcal{Y}_t(S,X)/\rho(S,X)$ is bounded in L^{2p}, we can again use the generalized Hölder inequality to bound the expectation of the product, and obtain that there is constant $C'>0$ such that

$$\begin{aligned}
\mathbb{P}\left[\left|\left\langle M^t(\varphi)\right\rangle_T - \left\langle M^t(\varphi)\right\rangle_U\right|^p\right] &\leq C'\|\varphi\|_\infty^{2p}\int_{[U,T]^p\times\mathbb{R}^p}\prod_{i=1}^p\rho(S_i,X_i)^2\mathrm{d}\mathbf{S}\mathrm{d}\mathbf{X}\\
&= C'\|\varphi\|_\infty^{2p}\left(\int_{[U,T]\times\mathbb{R}}\rho(S,X)^2\mathrm{d}S\mathrm{d}X\right)^p\\
&= C'\|\varphi\|_\infty^{2p}\,\pi^{-p/2}\,|T^{1/2}-U^{1/2}|^p.
\end{aligned}$$

Thus, Kolmogorov's tightness criterion [18, Theorem 12.3] applies, so the bracket $\langle M^t(\varphi)\rangle$ is tight. Hence Aldous' criterion applies, which concludes the proof of tightness of $M^t(\varphi)$.

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Part IV

Renormalizing the Kardar-Parisi-Zhang equation in $d \geq 3$ in weak disorder

Abstract (Joint work with Francis Comets and Chiranjib Mukherjee) We study the Kardar-Parisi-Zhang equation in spatial dimension 3 or larger driven by a Gaussian space-time white noise with a small convolution in space. When the noise intensity is small, it is known that the solutions converge to a random limit as the smoothing parameter is turned off. We identify this limit, in the case of general initial conditions ranging from flat to droplet. We provide strong approximations of the solution which obey exactly the limit law. We prove that this limit has sub-Gaussian lower tails, implying existence of all negative (and positive) moments.[1]

1 Introduction and main results.

1.1 KPZ equation and its regularization.

We consider the *Kardar-Parisi-Zhang* (KPZ) equation written informally as

$$\frac{\partial}{\partial t} h = \frac{1}{2}\Delta h + \left[\frac{1}{2}|\nabla h|^2 - \infty\right] + \xi \qquad u \colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \tag{1}$$

and driven by a totally uncorrelated Gaussian space-time white noise ξ. More precisely, ξ on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\{\xi(\varphi)\}_{\varphi \in \mathcal{S}(\mathbb{R}_+\times\mathbb{R}^d)}$ of Gaussian random variables

$$\xi(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{d}t \; \mathrm{d}x \; \xi(t,x)\, \varphi(t,x)$$

with mean 0 and covariance

$$\mathbb{E}\big[\xi(\varphi_1)\; \xi(\varphi_2)\big] = \int_0^\infty \int_{\mathbb{R}^d} \varphi_1(t,x)\varphi_2(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

The equation (1) describes the evolution of a growing interface in $d+1$ dimension [107, 143] and also appears as the scaling limit for $d=1$ of front propagation of the certain exclusion processes ([16, 53]) as well as that of the free energy of the discrete directed polymer ([2]). It should be noted that, on a rigorous level, only distribution-valued solutions are expected for (1), and thus it is already ill-posed in $d=1$ stemming from the inherent non-linearity of the equation and the fundamental problem of squaring or multiplying random distributions. For $d=1$, studies related to the above equation have enjoyed a huge resurgence of interest in the last decade starting with the important work [86] which gave an intrinsic precise notion of a *solution* to (1).

We now fix a spatial dimension $d \geq 3$. As remarked earlier, since (1) is a-priori ill-posed, we will study its regularized version

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon \;, \qquad h_\varepsilon(0,x) = 0, \tag{2}$$

which is driven by the spatially mollified noise

$$\xi_\varepsilon(t,x) = (\xi(t,\cdot) \star \phi_\varepsilon)(x) = \int \phi_\varepsilon(x-y)\xi(t,y)\,\mathrm{d}y.$$

with $\phi_\varepsilon(\cdot) = \varepsilon^{-d}\phi(\cdot/\varepsilon)$ being a suitable approximation of the Dirac measure δ_0 and C_ε being a suitable divergent (renormalization) constant. We will work with a fixed mollifier $\phi : \mathbb{R}^d \to \mathbb{R}_+$ which is smooth and spherically symmetric, with $\mathrm{supp}(\phi) \subset B(0,\frac{1}{2})$ and $\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1$. Then, $\{\xi_\varepsilon(t,x)\}$ is a centered Gaussian field with covariance

$$\mathbb{E}[\xi_\varepsilon(t,x)\xi_\varepsilon(s,y)] = \delta(t-s)\,\varepsilon^{-d} V\big((x-y)/\varepsilon\big),$$

where $V = \phi \star \phi$ is a smooth function supported in $B(0,1)$. We also remark that in (2), the multiplicative parameter β can be taken to be positive without loss of generality, while by rescaling, no multiplicative

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parameter is needed in (1), see [135]. Also in spatial dimensions $d \geq 3$, the factor $\varepsilon^{\frac{d-2}{2}}$ is the correct scaling – a small enough $\beta > 0$ guarantees a non-trivial random limit of h_ε as $\varepsilon \to 0$, see the discussion in Section 1.3.

The goal of the present article is to consider *general solutions* of (2), namely the solutions of (2) with various initial conditions and prove that as the mollification parameter ε is turned off, the renormalized solution of (2) converges to a meaningful random limit as long as β remains small enough. We use Feynman-Kac representation of the solution of stochastic heat equation and results from directed polymers. Not only do we identify the distributional limit of h_ε, but we also provide a sequence (indexed by ε) of functions of the noise such that

- it is a strong approximation of h_ε, i.e. the difference tends to 0 in norm,
- all terms in the sequence obey the limit law.

The above functions for the flat initial condition are defined from the martingale limit of a random polymer model taken at some rescaled, shifted and time-reversed version of the noise. The similar functions for other intial conditions can be derived from the martingale limit taken at various version of the noise and the heat kernel. We finally show that it has sub-Gaussian lower tails in this regime, which implies existence of all negative and positive moments of this object. Besides new contributions, we gather and reformulate results which are atomized in the literature, often stated in a primitive form and hidden by necessary technicalities. We end up the introduction with a rather complete account on the state-of-the-art. We now turn to a more precise description of our main results.

1.2 Main results.

In order to state our main results, we will introduce the following notation which will be consistently used throughout the sequel. Recall the definition of the space-time white noise $\xi \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ which is a random tempered distribution (defined in all times, including negative ones), and for any $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, $\varepsilon > 0$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\xi^{(\varepsilon,t,x)}(\varphi) \overset{\text{(def)}}{=} \varepsilon^{-\frac{d+2}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \varphi\big(\varepsilon^{-2}(t-s), \varepsilon^{-1}(y-x)\big)\xi(s,y)ds\,dy.$$

Equivalently,

$$\xi^{(\varepsilon,t,x)}(s,y) = \varepsilon^{\frac{d+2}{2}} \xi\bigg(\varepsilon^2\Big(\frac{t}{\varepsilon^2} - s\Big), \varepsilon\Big(y - \frac{x}{\varepsilon}\Big)\bigg) \tag{3}$$

so that by invariance under space-time diffusive rescaling, time-reversal and spatially translation, $\xi^{(\varepsilon,t,x)}$ is itself a Gaussian white noise and possesses the same law as ξ. This is also the reason why we define the noise above also for negative times. To abbreviate notation, we will also write

$$\xi^{(\varepsilon,t)} = \xi^{(\varepsilon,t,0)}. \tag{4}$$

We also need specify the definition(s) of the *critical disorder parameter.* Note that (2) is inherently non-linear. The Hopf-Cole transformation suggests that

$$u_\varepsilon = \exp h_\varepsilon \tag{5}$$

solves the linear multiplicative noise stochastic heat equation (SHE)

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon\, \xi_\varepsilon\,, \qquad u_\varepsilon(0,x) = 1, \tag{6}$$

provided that the stochastic integral in (6) is interpreted in the classical Itô-Skorohod sense and that we choose

$$C_\varepsilon = \beta^2(\phi \star \phi)(0)\varepsilon^{-2}/2 = \beta^2 V(0)\varepsilon^{-2}/2 \tag{7}$$

equal to the Itô correction below. Then, the generalized Feynman-Kac formula ([110, Theorem 6.2.5]) provides a solution to (6)

$$u_\varepsilon(t,x) = E_x\bigg[\exp\bigg\{\beta\varepsilon^{\frac{d-2}{2}} \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(W_{t-s} - y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\, t\, \varepsilon^{-2}}{2}\, V(0)\bigg\}\bigg]\,.$$

with E_x denoting expectation with respect to the law P_x of a d-dimensional Brownian path $W = (W_s)_{s\geq 0}$ starting at $x \in \mathbb{R}^d$, which is independent of the noise ξ. Plugging (3) in the previous formula, using Brownian scaling and time-reversal, we get the a.s. equality

$$u_\varepsilon(t,x) = \mathscr{Z}_{\frac{t}{\varepsilon^2}}\left(\xi^{(\varepsilon,t)}; \frac{x}{\varepsilon}\right) \tag{8}$$

where

$$\mathscr{Z}_T(x) = \mathscr{Z}_T(\xi;x) = E_x\left[\exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s,y)\, \mathrm{d}s\, \mathrm{d}y - \frac{\beta^2 T}{2}\, V(0)\right\}\right], \tag{9}$$

is the *normalized partition function* of the *Brownian directed polymer* is a white noise environment ξ, or equivalently, the total-mass of a *Gaussian multiplicative chaos* in the Wiener space ([130, Section 4]).

It follows that there exists $\beta_c \in (0,\infty)$ and a strictly positive non-degenerate random variable $\mathscr{Z}_\infty(x)$ so that, a.s. as $T \to \infty$,

$$\mathscr{Z}_T(x) \to \begin{cases} \mathscr{Z}_\infty(x) & \text{if } \beta \in (0,\beta_c), \\ 0 & \text{if } \beta \in (\beta_c,\infty). \end{cases} \tag{10}$$

See [130], or [41] for a general reference. Moreover, $(\mathscr{Z}_T)_{T\geq 0}$ is uniformly integrable for $\beta < \beta_c$, which we will always assume from now on. Now, let $\mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d)$ denote the path-space of the white noise (see Appendix for a precise definition) and

$$\mathfrak{u} = \mathfrak{u}_{\beta,\phi} : \mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d) \to (0,\infty),$$

be any arbitrary representative of the random limit $\mathscr{Z}_\infty = \mathscr{Z}_\infty(0)$; in particular $\mathfrak{u}(\xi) = \mathscr{Z}_\infty$. Then, $\mathbb{E}[\mathfrak{u}] = 1$, and throughout the sequel we will write (recall (5) and (8))

$$\mathfrak{h} = \log \mathfrak{u}\,. \tag{11}$$

Since $\mathfrak{u}$ is non constant with $\mathbb{E}\mathfrak{u} = 1$, we have $\mathbb{E}\mathfrak{h} < 0$.

Finally, we also define another critical disorder parameter:

$$\beta_{L^2} = \sup\left\{\beta > 0 : E_0\left[e^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)\, \mathrm{d}s}\right] < \infty\right\}$$

which corresponds to the *L^2-region of the polymer model* (see (14)). In $d \geq 3$, it is easy to see that for β small enough, $\sup_{x\in\mathbb{R}^d} E_x[\beta\int_0^\infty V(W_s)\, \mathrm{d}s] < 1$, so that by Khas'minskii's lemma, $E_0\big[\exp\big\{\beta\int_0^\infty V(W_s)\, \mathrm{d}s\big\}\big] < \infty$, so this implies that $\beta_{L^2} > 0$. Furthermore, for $\beta < \beta_{L^2}$, convergence (10) becomes an L^2-convergence, hence $0 < \beta_{L^2} < \beta_c < \infty$.

We are now ready to state our main results.

Theorem 1.1. *Assume $d \geq 3$ and recall that $\mathfrak{h}$ is defined in* (11)*.*

- *(Flat initial condition.) Fix $\beta \in (0,\beta_c)$ and consider the solution h_ε to* (2) *with $h_\varepsilon(0,\cdot) = 0$. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,*

$$h_\varepsilon(t,x) - \mathfrak{h}\big(\xi^{(\varepsilon,t,x)}\big) \xrightarrow{\mathbb{P}} 0\,.$$

- *(General initial condition.) Fix $\beta \in (0,\beta_{L^2})$ and consider the solution h_ε to* (2) *with $h_\varepsilon(0,\cdot) = h_0$ for some $h_0 : \mathbb{R}^d \to \mathbb{R}$ which is continuous and bounded from above. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,*

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \log \overline{u}(t,x) \xrightarrow{\mathbb{P}} 0\,, \tag{12}$$

where

$$\partial_t \overline{u} = \frac{1}{2}\Delta\overline{u}, \qquad \overline{u}(0,x) = \exp h_0(x).$$

- *(Droplet or narrow-wedge initial condition.) Fix $\beta \in (0, \beta_{L^2})$ and consider the solution h_ε to (2) such that*

$$\lim_{t \searrow 0} \exp h_\varepsilon(t, \cdot) = \delta_{x_0}(\cdot)$$

for some $x_0 \in \mathbb{R}^d$. Then, for all $t > 0, x \in \mathbb{R}^d$, we have as $\varepsilon \to 0$,

$$h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) - \mathfrak{h}(\xi_{(\varepsilon,x_0)}) - \log \rho(t, x - x_0) \xrightarrow{\mathbb{P}} 0 \,, \tag{13}$$

where ρ is the d-dimensional Gaussian kernel, and

$$\xi_{(\varepsilon,x_0)}(s,x) = \varepsilon^{\frac{d+2}{2}} \xi(\varepsilon^2 s, x_0 + \varepsilon x)$$

is a space-time Gaussian white noise.

We obtain an immediate corollary to Theorem 1.1.

Corollary 1.2. *Fix $\beta \in (0, \beta_{L^2})$ and denote by $h_\varepsilon^{(h_0)}$ the solution of (2) with initial condition $h_\varepsilon(0, \cdot) = h_0$, where h_0 is continuous and bounded from above. Then for any $x_0 \in \mathbb{R}^d$,*

$$\lim_{e^{h_0} \to \delta_{x_0}} \lim_{\varepsilon \to 0} h_\varepsilon^{(h_0)} \neq \lim_{\varepsilon \to 0} \lim_{e^{h_0} \to \delta_{x_0}} h_\varepsilon^{(h_0)}$$

Our next main result is the following which provides a sub-Gaussian upper tail estimate on the limit $\mathfrak{h}$ defined in (11).

Theorem 1.3. *Let $d \geq 3$ and $\beta \in (0, \beta_{L^2})$. Then for any $\theta > 0$, there exists a constant $C \in (0, \infty)$ so that*

$$\mathbb{P}[\mathfrak{h} \leq -\theta] \leq C e^{-\theta^2/2}.$$

In particular, $\mathfrak{h} \in L^p(\mathbb{P})$ for any $p \in \mathbb{R}$.

From Theorems 1.1 and 1.3, we derive

Corollary 1.4. *In the hypothesis of Corollary 1.2, we have for any $x_0 \in \mathbb{R}^d$,*

$$\lim_{e^{h_0} \to \delta_{x_0}} \lim_{\varepsilon \to 0} \mathbb{E} h_\varepsilon^{(h_0)} - \lim_{\varepsilon \to 0} \lim_{e^{h_0} \to \delta_{x_0}} \mathbb{E} h_\varepsilon^{(h_0)} = -\mathbb{E}\mathfrak{h} > 0 \,.$$

1.3 Literature remarks and discussion.

In the present set up, by finding a non-trivial limit when letting the regularization parameter vanish we have obtained a non-trivial renormalization of KPZ equation (1). Let us stress the main specificity of Theorem 1.1. The approximating sequence $(\mathfrak{h}(\xi^{(\varepsilon,t,x)}); \varepsilon > 0)$ in the case of flat initial condition combines three interesting properties:

- it is constant in law for all (ε, t, x), with law given by the one of $\log \mathscr{Z}_\infty$;
- for all $\varepsilon > 0$, it is a stationary solution of the regularized KPZ equation in (2) (with non-zero initial condition),
- it approximates $h_\varepsilon(t, x)$ in probability.

(Similar properties hold for the other initial conditions). Since it depends on ε, it is not a (strong) limit, but it can be used similarly. In particular, fluctuations can be studied as shown in Part V of the manuscript. This is quite different from using a deterministic centering, e.g., $\tilde{h}_\varepsilon(t,x) = h_\varepsilon(t,x) - \mathbb{E} h_\varepsilon(t,x)$. As mentioned in [68], $\tilde{h}_\varepsilon$ does not converge to 0 pointwise, but it does as a distribution. Integrating $\tilde{h}_\varepsilon$ in space against test functions cause oscillations to cancel. On the contrary, in our result $h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)}) \to 0$ pointwise, and *we do not need any averaging in space.*

We also emphasize that our results concern studying the asymptotic behavior of the solution to the non-linear equation (2), and are not restricted to the linear multiplicative noise stochastic heat equation (see (6)). Furthermore, the statements of the results concern the solution itself, without need of integrating spatially against test functions. However, note that the limit obtained in Theorem 1.1 *does* depend on the smoothing procedure ϕ as well as on the disorder parameter β and it is not universal (in particular, for $\beta < \beta_{L^2}$, the variance of $\exp(\mathfrak{h})$ can be computed from the RHS of (14) for $x = 0$, and it depends on the mollification). Thus the present scenario lies in total contrast with the 1-dimensional spatial case where the limit can be defined by a chaos expansion [2] (with the parameter β absorbed by scaling) or via the theory of regularity structures ([87]) which also produces a renormalized limit which does not depend on the mollification scheme.

In Part V of the manuscript, we also investigate the rate of the convergence of $h_\varepsilon \to \mathfrak{h}$ for small enough β and show that $\varepsilon^{\frac{d-2}{2}}[h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)})] \xrightarrow{\text{law}} N(0, \sigma^2(\beta))$, for each fixed $x \in \mathbb{R}^d$ and $t > 0$. For larger β, the so-called *KPZ regime* is expected to take place with different limits, different scaling exponents and non-Gaussian limiting distributions. In particular, the variance in the above Gaussian distribution is given by

$$\sigma^2(\beta) = C_d \int_{\mathbb{R}^d} \mathrm{d}y\, V(\sqrt{2}y) E_y[\exp\{\beta^2 \int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s\}]$$

which already diverges for $\beta > \beta_{L^2}$ indicating that the amplitude of the fluctuations, or at least their distributional nature, changes at this point. However the KPZ regime is not expected before the critical value β_c. Hence this region $\beta \in (\beta_{L^2}, \beta_c)$ remains mysterious.

The correlation structure of the limit $\mathfrak{u}$ which were computed in [44]. It was shown that, for β small enough,

$$\mathrm{Cov}\big(\mathscr{Z}_\infty(0), \mathscr{Z}_\infty(x)\big) = \begin{cases} E_{x/\sqrt{2}}\Big[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds} - 1\Big] & \forall x \in \mathbb{R}^d, \\ \mathfrak{C}_1 \Big(\frac{1}{|x|}\Big)^{d-2} & \forall\, |x| \geq 1, \end{cases} \tag{14}$$

with $\mathfrak{C}_1 = E_{\mathbf{e_1}/\sqrt{2}}[\exp\{\beta^2 \int_0^\infty V(\sqrt{2}W_s)ds\} - 1]$. The above correlation structure underlines that $u_\varepsilon(t,x)$ and $u_\varepsilon(t,y)$ become asymptotically independent, so that the spatial averages $\int_{\mathbb{R}^d} f(x)\, u_\varepsilon(t,x)\,\mathrm{d}x \to \int f(x)\overline{u}(t,x)\,\mathrm{d}x$ become deterministic and $\overline{u}$ solves the unperturbed heat equation $\partial_t \overline{u} = \frac{1}{2}\Delta\overline{u}$. As remarked earlier, the spatially averaged fluctuations $\varepsilon^{1-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)[u_\varepsilon(t,x) - \overline{u}(t,x)]\,\mathrm{d}x$ were shown to converge ([120, 81, 68]) to the averages of the heat equation with additive space-time white noise with variance given by (a constant multiple of) $\sigma^2(\beta)$, which also underlines the Edwards-Wilkinson regime in weak disorder. For averaged fluctuations of similar nature in $d = 2$ we refer to [38, 31, 80].

Finally, we mention that the similar question of finding an approximating solution being independent of the initial condition, for the regularized (in time and space) SHE in dimension $d \geq 3$, has been investigated in [69] under the weak disorder regime.

2 Proof of Theorem 1.1

We now consider the regularized KPZ equation (6) as before, but with different initial data and identify the limit of the solution up to leading order. For notational brevity, we will write

$$\Phi_T = \Phi_T(\xi; W) = \exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2 T}{2} V(0)\right\} \tag{15}$$

where $V = \phi \star \phi$ so that $\mathscr{Z}_T(x) = \mathscr{Z}_T(\xi; x) = E_x[\Phi_T]$ and $\mathbb{E}[\mathscr{Z}_T] = 1$.

We also remind the reader that u_ε solves (6) and $h_\varepsilon = \exp[u_\varepsilon]$ solves (2) with $C_\varepsilon = \frac{\beta^2 \varepsilon^{-2}}{2} V(0)$. Finally, recall that $u_\varepsilon(t,x) = \mathscr{Z}_{\frac{t}{\varepsilon^2}}\big(\xi^{(\varepsilon,t)}; \frac{x}{\varepsilon}\big)$ with $\xi^{(\varepsilon,t)}$ given by (3) and (4).

2.1 General initial condition: Proof of (12).

Fix continuous functions $u_0 : \mathbb{R}^d \to (0, +\infty)$ and $h_0 : \mathbb{R}^d \to \mathbb{R}$ which are bounded from above, consider the solution of SHE

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon\, \xi_\varepsilon\,, \qquad u_\varepsilon(0,x) = u_0(x)\,, \tag{16}$$

or, equivalently by the relations $u_\varepsilon = \exp h_\varepsilon$ and $u_0 = \exp h_0$, the solution of KPZ

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon\,, \qquad h_\varepsilon(0,x) = h_0(x), \tag{17}$$

As before, we have the Feynman-Kac representation

$$u_\varepsilon(t,x) = E_{x/\varepsilon}\big[u_0(\varepsilon W_{\varepsilon^{-2}t})\,\Phi_{\varepsilon^{-2}t}(\xi^{(\varepsilon,t)};W)\big] \tag{18}$$

with $\xi^{(\varepsilon,t)}$ as above.

Lemma 2.1. *For $\beta \in (0, \beta_{L^2})$,*

$$E_{x/\varepsilon}\big[u_0(\varepsilon W_{\varepsilon^{-2}t})\Phi_{\varepsilon^{-2}t}(\xi;W)\big] - \mathfrak{u}(\xi\circ\theta_{x/\varepsilon})\,\overline{u}(t,x) \xrightarrow{L^2} 0,$$

where θ_x denotes the canonical spatial translation in the path space $\mathscr{C}^\alpha$ of the white noise and $\overline{u}$ solves $\partial_t \overline{u} = \frac{1}{2}\Delta\overline{u}$ with $\overline{u}(0,\cdot) = u_0(\cdot)$.

Proof. Note that

$$E_0[u_0(x+\varepsilon W_{\varepsilon^{-2}t})] = E_x[u_0(W_t)] = \overline{u}(t,x)$$

Then

$$\begin{aligned}
&\mathbb{E}\left[\left(E_{x/\varepsilon}\left[u_0(\varepsilon W_{\varepsilon^{-2}t})\Phi_{\varepsilon^{-2}t}(\xi;W)\right] - \overline{u}(t,x)E_{x/\varepsilon}\left[\Phi_{\varepsilon^{-2}t}(\xi;W)\right]\right)^2\right] \qquad (19)\\
&= E_0^{\otimes 2}\left[e^{\beta^2\int_0^{\varepsilon^{-2}t} V\left(W_s^{(1)}-W_s^{(2)}\right)\mathrm{d}s}\left(u_0\left(x+\varepsilon W^{(1)}_{\varepsilon^{-2}t}\right) - \overline{u}(t,x)\right)\left(u_0\left(x+\varepsilon W^{(2)}_{\varepsilon^{-2}t}\right) - \overline{u}(t,x)\right)\right].
\end{aligned}$$

Furthermore,

$$\left(\int_0^{\varepsilon^{-2}t} V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s, \varepsilon W^{(1)}_{\varepsilon^{-2}t}, \varepsilon W^{(2)}_{\varepsilon^{-2}t}\right) \overset{\text{law}}{\longrightarrow} \left(\int_0^\infty V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s, Z_t^{(1)}, Z_t^{(2)}\right),$$

where the right hand side is a triplet of three independent random variables, with $Z_t^{(1)}$ and $Z_t^{(2)}$ distributed as W_t. Hence, expectation (19) vanishes as $\varepsilon \to 0$, provided that u_0 is bounded and continuous, and because of uniform integrability which is implied by

$$E_0^{\otimes 2}\left[\exp\left\{(1+\delta)\beta^2\int_0^\infty V\left(W_s^{(1)} - W_s^{(2)}\right)\,\mathrm{d}s\right\}\right] < \infty, \tag{20}$$

for $\beta < \beta_{L^2}$ and $\delta > 0$ small enough. The proof is concluded by the observation that $E_{x/\varepsilon}\left[\Phi_{\varepsilon^{-2}t}(\xi;W)\right] - \mathfrak{u}(\xi\circ\theta_{x/\varepsilon}) \overset{L^2}{\to} 0$. ∎

We now end the

Proof of (12): For $\beta < \beta_{L^2}$, for all t,x, as $\varepsilon \to 0$, we first show

$$u_\varepsilon(t,x) - \mathfrak{u}\big(\xi^{(\varepsilon,t,x)}\big)\,\overline{u}(t,x) \xrightarrow{L^2} 0\,. \tag{21}$$

Note that (21) follows directly from Lemma 2.1 and (18). Then, since $\mathfrak{u} > 0$, taking logarithm we deduce the convergence in probability (12). ∎

Remark 2.2. *Recall that in [130] it was shown that, for any smooth function f with compact support, $\int_{\mathbb{R}^d} u_\varepsilon(t,x)f(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d}\overline{u}(t,x)f(x)\,\mathrm{d}x$. Note that unlike the latter statement, no smoothing in space is needed in the present context. In fact, we can recover the spatially averaged statement from above. Indeed, fast decorrelation in space of $\xi^{(\varepsilon,t,x)}$ as $\varepsilon \to 0$, ergodicity and smoothness justify the equivalence below:*

$$\begin{aligned}
\int u_\varepsilon(t,x)f(x)dx &\overset{(21)}{=} \int \mathfrak{u}(\xi^{(\varepsilon,t,x)})\overline{u}(t,x)f(x)dx + o(1)\\
&\sim \mathbb{E}[\mathfrak{u}(\xi^{(\varepsilon,t,x)})]\int\overline{u}(t,x)f(x)\,\mathrm{d}x\\
&= \int\overline{u}(t,x)f(x)\,\mathrm{d}x.
\end{aligned}$$

∎

2.2 Narrow-wedge initial condition: Proof of (13).

Fix $x_0 \in \mathbb{R}^d$, and consider the solution of SHE

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon \, \xi_\varepsilon \;, \qquad \lim_{t \searrow 0} u_\varepsilon(t,\cdot) = \delta_{\varepsilon^{-1}x_0}(\cdot) \;, \tag{22}$$

or, equivalently by the relation $u_\varepsilon = \exp h_\varepsilon$, the solution of KPZ

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon \;, \qquad \lim_{t \searrow 0} \exp h_\varepsilon(t,\cdot) = \delta_{\varepsilon^{-1}x_0}(\cdot) \;. \tag{23}$$

By Feynman-Kac formula, the solution of SHE now admits a Brownian bridge representation:

$$\begin{aligned} & u_\varepsilon(t,x) \\ &= \rho(t,x-x_0)\, E_{0,\varepsilon^{-1}x_0}^{\varepsilon^{-2}t,\varepsilon^{-1}x}\left[\exp\left\{\beta \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d} \phi(W_s-y)\xi_{(\varepsilon,0)}(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\, t}{2\,\varepsilon^2}\; V(0)\right\}\right] && (24) \\ &= \rho(t,x-x_0) E_{0,0}^{\varepsilon^{-2}t,\varepsilon^{-1}x}\left[\exp\left\{\beta \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d} \phi(W_s-y)\xi_{(\varepsilon,x_0)}(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\, t}{2\,\varepsilon^2}\; V(0)\right\}\right] && (25) \end{aligned}$$

where $E_{0,x}^{t,y}$ denotes expectation with respect to a Brownian bridge starting at x and conditioned to be found at y at time t, ρ is the d-dimensional Gaussian kernel and

$$\xi_{(\varepsilon,x_0)}(S,Y) = \varepsilon^{(d+2)/2}\xi(\varepsilon^2 S, x_0 + \varepsilon Y) \tag{26}$$

so that we again have $\xi_{(\varepsilon,x_0)} \overset{\text{law}}{=} \xi$.

The following Lemma follows the approach for proving the local limit theorem as in [149, 159].

Lemma 2.3. *For $\beta \in (0,\beta_{L^2})$, for any $A > 0$,*

$$\sup_{|x|\leq A} \left\| E_{0,0}^{\varepsilon^{-2}t,\varepsilon^{-1}x}[\Phi_{\varepsilon^{-2}t}(\xi,\cdot)] - \mathfrak{u}(\xi)\; \mathfrak{u}\big(\xi^{(1,\varepsilon^{-2}t,\varepsilon^{-1}x)}\big)\right\|_{L^1(\mathbb{P})} \to 0. \tag{27}$$

Proof. We will write $X = \varepsilon^{-1}x$, $T = \varepsilon^{-2}t$ and let $m = m_\varepsilon$ be a time parameter, such that $m_\varepsilon \to \infty$ and $m_\varepsilon = o(T)$, as $\varepsilon \to 0$. We use the notation:

$$\Phi_{S,T}(\xi;W) := \exp\left\{\beta \int_S^T \int_{\mathbb{R}^d} \phi(W_s-y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\,(T-S)}{2}\; V(0)\right\}.$$

Step 1: We first want to approximate $E_{0,0}^{T,X}[\Phi_T]$ by $E_{0,0}^{T,X}[\Phi_m \Phi_{T-m,T}]$ in L^2-norm, so we compute the difference:

$$\mathbb{E}\left[\left(E_{0,0}^{T,X}[\Phi_T - \Phi_m\Phi_{T-m,T}]\right)^2\right] = E_{0,0}^{T,0}\left[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_s)ds} - \mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_s)ds}\mathrm{e}^{\beta^2\int_{T-m}^T V(\sqrt{2}W_s)ds}\right].$$

To show that the right hand side which goes to 0 as $\varepsilon \to 0$, it suffices to observe that, for all $a > 0$,

$$\lim_{\varepsilon\to 0} E_{0,0}^{T,0}\left[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\;\; \mathbf{1}\left\{\int_m^{T-m} V(\sqrt{2}W_s)\;\mathrm{d}s > a\right\}\right] = 0.$$

To prove this, we use Hölder's inequality similarly to (20), and apply [44, Lemma 3.5] (alternatively [159, Corollary 3.8]) and transience of Brownian motion for $d \geq 3$, which implies, since $m \to \infty$, that

$$\lim_{\varepsilon\to 0} E_{0,0}^{T,0}\left[1\left\{\int_m^{T-m} V(\sqrt{2}W_s)\;\mathrm{d}s > a\right\}\right] = 0.$$

Step 2: We wish to use Markov property and symmetry of the Brownian bridge to show that $E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right]$ factorizes asymptotically into the product $E_0\left[\Phi_m\right]E_0[\Phi_m(\xi^{(1,T,X)})]$, which satisfies:

$$\sup_{x\in\mathbb{R}}\left\|E_0\left[\Phi_m\right]E_0[\Phi_m(\xi^{(1,T,X)})]-\mathfrak{u}(\xi)\,\mathfrak{u}(\xi^{(1,\varepsilon^{-2}t,\varepsilon^{-1}x)})\right\|_1\to 0,$$

as $\varepsilon\to 0$ for $\beta<\beta_{L^2}$, by Cauchy-Schwarz inequality and invariance in law of the white noise with a shift by X. Hence, we compute

$$E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right]=\int_{\mathbb{R}^d}\frac{\rho_{T/2}(Y)\rho_{T/2}(X-Y)}{\rho_T(X)}E_{0,0}^{T/2,Y}\left[\Phi_m\right]E_{T/2,Y}^{T,X}\left[\Phi_{T-m,T}\right]\mathrm{d}Y. \tag{28}$$

After change of variable by setting $Y=\sqrt{T}y$ in the above integral and since $\rho(Ts,\sqrt{T}z)=T^{-d/2}\rho(s,z)$, observe that by Jensen's inequality and dominated convergence, we can prove that

$$\sup_{|x|\le A}\left\|E_{0,0}^{T,X}\left[\Phi_m\Phi_{T-m,T}\right]-E_0\left[\Phi_m\right]E_0[\Phi_m(\xi^{(1,T,X)})]\right\|_1\to 0,$$

if we can show that for all fixed $y\in\mathbb{R}^d$,

$$\sup_{|x|\le A}\left\|E_{0,0}^{T/2,\sqrt{T}(y-x)}\left[\Phi_m\right]-E_0\left[\Phi_m\right]\right\|_1\to 0.$$

To prove this, we use the density of the Brownian bridge, at truncated time horizon, with respect to the Brownian motion ([44, Lemma 3.4]), to get that:

$$E_{0,0}^{T/2,\sqrt{T}(y-x)}\left[\Phi_m\right]-E_0\left[\Phi_m\right]=E_0\left[\Phi_m\left(\frac{\rho_{T/2-m}\big(\sqrt{T}(y-x)-W_m\big)}{\rho_{T/2}\big(\sqrt{T}(y-x)\big)}-1\right)\right].$$

After rescaling, the difference inside the parenthesis goes almost surely to 0, for y fixed and uniformly in $|x|\le A$; we conclude the proof of the lemma using Hölder's inequality.

■

We can now conclude the
Proof of (13)**:** Let h_ε be the narrow-wedge height function solution of of (23). We need to show that for $\beta<\beta_{L^2}$, for all t,x, as $\varepsilon\to 0$,

$$h_\varepsilon(t,x)-\mathfrak{h}(\xi_{(\varepsilon,x_0)})-\mathfrak{h}(\xi^{(\varepsilon,t,x)})-\log\rho(t,x-x_0)\xrightarrow{\mathbb{P}}0,$$

with $\xi_{(\varepsilon,x_0)}$ in (26). We use the representation (25) and the property $\xi_{(\varepsilon,x_0)}\overset{\text{law}}{=}\xi$, so that we can exchange ξ with $\xi_{(\varepsilon,x_0)}$, in convergence (27) taken with endpoint $\varepsilon^{-1}(x-x_0)$. This leads to the above convergence in probability for the logarithm, proving (13). ■

3 Proof of Theorem 1.3.

We focus on showing that for $\beta<\beta_{L^2}$, $\log\mathscr{Z}_\infty$ admits a sub-Gaussian lower tails estimate, that is, for some $C\in(0,\infty)$ and any $\theta>0$,

$$\mathbb{P}[\log\mathscr{Z}_\infty\le-\theta]\le C\mathrm{e}^{-\theta^2/C}. \tag{29}$$

We invoke a second moment method combined with the Talagrand's concentration inequality as in [32] (see also [126, Section 2.2]). As the Cauchy-Schwarz inequality, which is a central tool in this proof is not directly available in the continuous setting, we choose to introduce a discretization of the white noise to recover it.

Consider $\mathcal{R}_n$ a tiling of $[0,2^n]\times[-2^n,2^n]^d$, composed of cubes of length 2^{-n}, such that every cube of $\mathcal{R}_n$ can be divided in 2^{d+1} cubes of $\mathcal{R}_{n+1}$. We define a discrete version of $\mathscr{Z}_T$ through:

$$\mathscr{Z}_T^{(n)}=\mathrm{E}\left[\exp\left\{\beta\int_0^T\int_{\mathbb{R}^d}\phi_W^{(n)}(s,y)\,\xi(s,y)\mathrm{d}s\mathrm{d}y-\frac{\beta^2}{2}\left\|\phi_W^{(n)}\right\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right\}\right],$$

where,

- $\phi_W(s,y) := \phi(W_s - y)$,
- $\phi_W^{(n)}(s,y) = \inf_R \phi_W$ if (s,y) is in a cube R of $\mathcal{R}_n$, and 0 otherwise.

We stress that $\phi_W^{(n)}$ is non-decreasing with n and converges almost surely to ϕ_W.

Using the Gaussian covariance structure, we have that

$$\begin{aligned}\mathbb{E}\left[\left(\mathscr{Z}_T - \mathscr{Z}_T^{(n)}\right)^2\right] &= \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi_{W^{(1)}}\phi_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right]\\ &\quad - 2\mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right] + \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\frac{\beta^2}{2}\int_{[0,T]\times\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y}\right].\end{aligned}$$

For $\beta < \beta_{L^2}$ and since $\phi_W^{(n)} \leq \phi_W$, we immediatly obtain from the monotone convergence theorem that the right-hand side goes to zero in the limit $T \to \infty$ followed by $n \to \infty$. By Doob's L^2 inequality applied to the martingale $\mathscr{Z} - \mathscr{Z}^{(n)}$, this implies in particular that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\mathscr{Z}_\infty - \mathscr{Z}_\infty^{(n)}\right)^2\right] = 0. \tag{30}$$

Since $\phi_W^{(n)}$ is set to be 0 outside of $[0,2^n]\times[-2^n,2^n]^d$, the set $\mathcal{C}_n$ containing the centers of the cubes of $\mathcal{R}_n$ is finite. Hence, as each cube has volume $2^{-(d+1)n}$, we can write that

$$\int_0^\infty \int_{\mathbb{R}^d} \phi_W^{(n)}(s,y)\,\xi(s,y)\mathrm{d}s\mathrm{d}y = \frac{1}{2^{\frac{(d+1)n}{2}}} \sum_{(i,x)\in\mathcal{C}_n} \phi_W^{(n)}(i,x)\,\xi_n(i,x), \tag{31}$$

where the $\xi_n(i,x)$ are independent centered Gaussian random variables of variance 1. Then, we define the polymer measure $\widehat{P}_{\beta,\xi_n}$ of renormalized partition function $\mathscr{Z}_\infty^{(n)}$:

$$\widehat{P}_{\beta,\xi_n}(\mathrm{d}W) = \frac{1}{\mathscr{Z}_\infty^{(n)}}\,\Phi^{(n)}(W)\mathrm{P}(\mathrm{d}W),$$

where we have set, using (31),

$$\Phi^{(n)}(W) = \exp\left\{\frac{\beta}{2^{\frac{(d+1)n}{2}}}\sum_{(i,x)\in\mathcal{C}_n}\phi_W^{(n)}(i,x)\,\xi_n(i,x) - \frac{\beta^2}{2}\left\|\phi_W^{(n)}\right\|^2_{L^2([0,\infty]\times\mathbb{R}^d)}\right\}.$$

Finally, we let $\widehat{E}_{\beta,\xi_n}$ denote expectation corresponding to $\widehat{P}_{\beta,\xi_n}$.

Now, we can compare the free energies of two realizations of the noise $\xi_n(i,x)$ and $\xi_n'(i,x)$:

$$\begin{aligned}&\log\mathscr{Z}_\infty^{(n)}(\xi_n) - \log\mathscr{Z}_\infty^{(n)}(\xi_n')\\ &= \log\widehat{E}_{\beta,\xi_n'}\left[\exp\left\{\frac{\beta}{2^{\frac{(d+1)n}{2}}}\sum_{(i,x)\in\mathcal{C}_n}\phi_W^{(n)}(i,x)\ (\xi_n(i,x) - \xi_n'(i,x))\right\}\right]\\ &\geq \beta\sum_{(i,x)\in\mathcal{C}_n} 2^{-\frac{(d+1)n}{2}}\,\widehat{\mathrm{E}}_{\beta,\xi_n'}\left[\phi_W^{(n)}(i,x)\right](\xi_n(i,x) - \xi_n'(i,x))\\ &\geq -\beta\sqrt{\widehat{E}^{\otimes 2}_{\beta,\xi_n'}\left[\int_0^\infty\int_{\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\right]}\ \mathrm{d}(\xi_n,\xi_n'),\end{aligned}$$

where $\mathrm{d}(\cdot,\cdot)$ denotes the euclidean distance on $\mathbb{R}^{\mathrm{Card}(\mathcal{C}_n)}$, and where we used Jensen's and Cauchy-Schwarz inequalities for respectively the first and second lower bounds.

Let m and C be two positive constants, and consider the set:

$$\mathcal{A}_n = \left\{\xi_n : \mathscr{Z}_\infty^{(n)}(\xi_n) \geq m,\ \widehat{E}^{\otimes 2}_{\beta,\xi_n}\left[\int_0^\infty\int_{\mathbb{R}^d}\phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\right] \leq C^2\right\}.$$

For $\xi_n' \in \mathcal{A}_n$, the above computation implies that

$$\log \mathscr{Z}_\infty^{(n)}(\xi_n) \geq \log m - \beta C \mathrm{d}\left(\xi_n, \mathcal{A}_n\right), \tag{32}$$

therefore, assuming

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{A}_n) > 0, \tag{33}$$

property (29) results from (32), (30) and the following Gaussian concentration inequality (Lemma 4.1 in [32], extracted from [153]):

$$\mathbb{P}\left(\mathrm{d}(\xi_n, \mathcal{A}_n) > u + \sqrt{2\log\frac{1}{\mathbb{P}(\mathcal{A}_n)}}\right) \leq e^{-\frac{u^2}{2}}. \tag{34}$$

We now prove (33). By convergence (30), and since $\mathscr{Z}_\infty > 0$ a.s., we can find $m > 0$, such that for n large enough,

$$\mathbb{P}\left(\mathscr{Z}_\infty^{(n)} > m\right) \geq \frac{1}{2}.$$

Then, for a large enough C,

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_n) &\geq \mathbb{P}\left(\mathscr{Z}_\infty^{(n)} \geq m,\ \mathrm{E}^{\otimes 2}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\ e^{\beta\Phi^{(n)}(W^{(1)})+\beta\Phi^{(n)}(W^{(2)})}\right] \leq mC^2\right) \\
&\geq \mathbb{P}\left(\mathscr{Z}_\infty^{(n)} \geq m\right) - \mathbb{P}\left(\mathrm{E}^{\otimes 2}\left[\int_0^\infty \int_{\mathbb{R}^d} \phi^{(n)}_{W^{(1)}}\phi^{(n)}_{W^{(2)}}(s,y)\mathrm{d}s\mathrm{d}y\ e^{\beta\Phi^{(n)}(W^{(1)})+\beta\Phi^{(n)}(W^{(2)})}\right] > mC^2\right) \\
&\geq \frac{1}{2} - \frac{1}{mC^2}\mathrm{E}^{\otimes 2}\left[\int_0^\infty V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s\ \mathrm{e}^{\beta^2\int_0^\infty V(W_s^{(1)}-W_s^{(2)})\,\mathrm{d}s}\right] > 0.
\end{aligned}$$

In the above display, the first lower bound follows from the definition of $\mathcal{A}$, while we used $\mathbb{P}[A \cap B] \geq \mathbb{P}[A] - \mathbb{P}[B^c]$ in the second lower bound. The third inequality comes from Markov's inequality and the upper-bound $\phi_W^{(n)} \leq \phi_W$. Positivity of the left hand-side of the third line is assured for C large enough, provided that $\beta < \beta_{L^2}$.

This entails that the KPZ limit $|\mathfrak{h}|$ (recall (11)) has all positive and negative moments for all $\beta < \beta_{L^2}$. Indeed, letting $\log_- = \log \wedge 0$, the sub-Gaussian decay of the left tail of $\log \mathscr{Z}_\infty$ (29) gives $\mathbb{E}[\exp\{\nu \log_- \mathscr{Z}_\infty\}] < \infty$, for all $\nu \in \mathbb{R}$. Moreover, by definition of the L^2 region, we have $\mathbb{E}[\exp\{2\log \mathscr{Z}_\infty\}] < \infty$. Hence, $\log \mathscr{Z}_\infty$ admits all positive and negative moments. ∎

Remark 3.1. *After finishing the writing of the present article, we learned that another proof of the negative moments of $\mathscr{Z}_\infty$ has been recently proposed in [97, 68] using a continuous approximation of the white noise. A proof of the corresponding result for the KPZ equation in dimension 2, which relies on the convexity of the free energy and the Malliavin derivative, can be found in [31].*

4 Appendix.

We will include some elementary facts regarding the regularity properties of space-time white noise ξ. For any $z, z' \in \mathbb{R}\times\mathbb{R}^d$, we will denote by $\|\cdot\|$ the *parabolic distance* given by $\|z-z'\| = |t-t'|^{1/2} + \sum_{i=1}^d |x_j - x_j'|$, where $z = (t,x)$ and $z' = (t', x')$. Recall that the Hölder space of positive exponent $\alpha \in (0,1)$ consists of all functions $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ such that for any compact set $K \subset \mathbb{R}\times\mathbb{R}^d$,

$$\sup_{z,z'\in K, z\neq z'} \frac{|u(z) - u(z')|}{\|z - z'\|^\alpha} < \infty.$$

The corresponding Hölder (Besov) space of *negative regularity* is defined as follows. First for any $k \in \mathbb{N}$, let B_k denote the space of all smooth functions $\varphi\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ which are supported on the unit ball in $(\mathbb{R}\times\mathbb{R}^d, \|\cdot\|)$ such that

$$\|\varphi\|_{B_k} \overset{(\mathrm{def})}{=} \sup_{\beta : |\beta| \leq k} \sup_{z \in \mathbb{R}\times\mathbb{R}^d} |D^\beta \varphi(z)| \leq 1.$$

Then for any fixed $\alpha < 0$, we define the space $\mathscr{C}^\alpha$ to be the space of all tempered distributions $\eta \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ such that for any compact set $K \subset \mathbb{R} \times \mathbb{R}^d$,

$$\|\eta\|_{\mathscr{C}^\alpha(K)} \stackrel{\text{(def)}}{=} \sup_{z \in K} \sup_{\substack{u \in B_k \\ \lambda \in (0,1]}} \left| \frac{\langle \eta, \Theta_z^\lambda u \rangle}{\lambda^\alpha} \right| < \infty,$$

where $k = \lceil -\alpha \rceil$ and

$$(\Theta_z^\lambda u)(s,y) = \lambda^{-(d+2)} u\big(\lambda^{-2}(t-s), \lambda^{-1}(y-x)\big) \qquad z = (t,x).$$

A crucial estimate on the Besov norm $\| \cdot \|_{\mathscr{C}^\alpha(K)}$ is given by

$$\|\eta\|_{\mathscr{C}^\alpha(K)} \leq C \sup_{n \geq 0} \sup_{z \in (2^{-2n}\mathbb{Z} \times 2^{-n}\mathbb{Z}^d) \cap \widetilde{K}} 2^{-n\alpha} \big\langle \eta, \Theta_z^{2^{-n}} u \big\rangle, \tag{35}$$

where $\widetilde{K}$ is also a compact set slightly larger than K and u is a single, well-chosen test function (which can be constructed by wavelets, see [87]). Recall that if ξ is space-time white noise (i.e, $\mathbb{E}[\langle \xi, \varphi_1 \rangle \langle \xi, \varphi_2 \rangle] = \int \varphi_1(t,x)\varphi_2(t,x)\, \mathrm{d}t\, \mathrm{d}x$), then with

$$\langle \xi_\lambda, \varphi \rangle = \langle \xi, \Theta_0^\lambda \varphi \rangle \qquad \text{we have} \qquad \mathbb{E}[\langle \xi_\lambda, \varphi \rangle^2] = \lambda^{-(d+2)} \int_{\mathbb{R}^{d+1}} \varphi^2(t,x)\, \mathrm{d}t\, \mathrm{d}x. \tag{36}$$

The following result, which is a consequence of Kolmogorov's lemma and (35), then implies the desired regularity property of ξ.

Lemma 4.1. *Fix $\alpha < 0$ and $p \geq 1$ and let η be a linear map from $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ to the space of random variables. Suppose there exists $C \in (0, \infty)$ such that for all $z \in \mathbb{R} \times \mathbb{R}^d$ and all $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ with compact support in $\mathbb{R} \times \mathbb{R}^d$ with $\sup_w |u(w)| \leq 1$ one has*

$$\mathbb{E}[|\eta(\Theta_z^\lambda u)|^p] \leq C \lambda^{\alpha p} \qquad \forall \lambda \in (0,1].$$

Then there exists a random distribution $\widetilde{\eta}$ in $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ such that for all $\alpha' < \alpha - \frac{(d+2)}{p}$ and compact set K,

$$\mathbb{E}\big[\|\widetilde{\eta}\|^p_{\mathscr{C}^{\alpha'}(K)}\big] < \infty \qquad \text{and } \eta(u) = \widetilde{\eta}(u) \quad \text{a.s.}$$

■

Since for any $p \geq 1$, $\mathbb{E}[\langle \xi_\lambda, \varphi \rangle^p] \leq C_p \mathbb{E}[\langle \xi_\lambda, \varphi \rangle^2]^{p/2}$, then Lemma 4.1 and (36) imply that ξ has regularity $\mathscr{C}^{-\frac{d}{2}-1-\delta}$ for any $\delta > 0$.

Part V

Gaussian fluctuations and rate of convergence of the Kardar-Parisi-Zhang equation for $d \geq 3$

Abstract (Joint work with Francis Comets and Chiranjib Mukherjee). We consider the smoothened KPZ equation on $\mathbb{R}^d$. For $d \geq 3$ and small noise intensity, the solution can be approximated by the rescaled free energy of the Brownian directed polymer. In this regime, we study the order of the point-wise fluctuations and prove that the normalized fluctuations converge in distribution towards a Gaussian variable. We also prove that the same result holds for the smoothened stochastic heat equation as well as that for the underlying martingale of the Brownian directed polymer.[1]

1 Introduction and the result.

1.1 Introduction and summary.

We consider the *Kardar-Parisi-Zhang* (KPZ) equation written informally as

$$\frac{\partial}{\partial t} h = \frac{1}{2}\Delta h + \left[\frac{1}{2}|\nabla h|^2 - \infty\right] + \xi \qquad h\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \tag{1}$$

and driven by a totally uncorrelated Gaussian space-time white noise ξ with covariance given by $\mathbb{E}[\xi(s,x)\,\xi(t,y)] = \delta_0(t-s)\,\delta_0(x-y)$. More precisely, ξ on $\mathbb{R}_+ \times \mathbb{R}^d$ is a family $\{\xi(\varphi)\}_{\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)}$ of Gaussian random variables

$$\xi(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} dt\, dx\; \xi(t,x)\,\varphi(t,x)$$

with mean 0 and covariance $\mathbb{E}\big[\xi(\varphi_1)\ \xi(\varphi_2)\big] = \int_0^\infty \int_{\mathbb{R}^d} \varphi_1(t,x)\varphi_2(t,x)\,\mathrm{d}t\,\mathrm{d}x$

The equation (1) describes the evolution of a growing interface in $(d+1)$ dimension [107, 143] and also appears as the scaling limit of front propagation of certain exclusion processes ([16]) as well as that of the free energy of the discrete directed polymer ([2]). It should be noted that, on a rigorous level, only distribution-valued solutions are expected for (1), and thus the inherent non-linearity of the equation and the problem of squaring (or multiplying) random distributions make (1) already ill-posed in $d = 1$. For spatial dimension $d = 1$, studies related to the above equation have enjoyed a huge resurgence of interest in the last decade starting with the important work [86] which gave an intrinsic precise notion of a *solution* to (1).

We now fix a spatial dimension $d \geq 3$. As remarked earlier, since (1) is a-priori ill-posed, we will study its regularized version

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon\ , \qquad h_\varepsilon(0,x) = 0, \tag{2}$$

driven by the spatially mollified noise $\xi_\varepsilon = \xi \star \phi_\varepsilon$, i.e.,

$$\xi_\varepsilon(t,x) = \int \phi_\varepsilon(y-x)\xi(t,y)\,\mathrm{d}y,$$

Here $\phi_\varepsilon(\cdot) = \varepsilon^{-d}\phi(\cdot/\varepsilon)$ is a suitable approximation of the Dirac measure δ_0 with $\phi : \mathbb{R}^d \to \mathbb{R}_+$ being a fixed smooth and spherically symmetric function with $\mathrm{supp}(\phi) \subset B(0, \frac{1}{2})$ and $\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1$. Moreover,

$$C_\varepsilon = \frac{\beta^2 \varepsilon^{d-2} (\phi_\varepsilon \star \phi_\varepsilon)(0)}{2} = \frac{\beta^2 \varepsilon^{-2} (\phi \star \phi)(0)}{2} \tag{3}$$

is a divergent constant (as $\varepsilon \to 0$). Note that in this setup $\{\int_0^t \xi_\varepsilon(s,x) ds\}$ is a mean-zero Gaussian process with covariance

$$\mathbb{E}\left[\int_0^t \xi_\varepsilon(r,x) dr \int_0^s \xi_\varepsilon(r,y) dr\right] = (s \wedge t)\,\varepsilon^{-d} V((x-y)/\varepsilon),$$

where $V = \phi \star \phi$ is a smooth function supported in $B(0,1)$.

We remark that the multiplicative parameter β can be taken to be positive without loss of generality, while by rescaling, no multiplicative parameter is needed in (1), see [135]. Also in the dimensions $d \geq 3$,

[1] **Keywords:** SPDE, stochastic heat equation, directed polymers, random environment, weak disorder, Edwards-Wilkinson limit. **AMS 2010 subject classifications:** Primary 60K35. Secondary 35R60, 35Q82, 60H15, 82D60

the factor $\varepsilon^{\frac{d-2}{2}}$ is the correct scaling – a small enough $\beta > 0$ guarantees a non-trivial random limit $\mathfrak{h}$ of h_ε as $\varepsilon \to 0$, while for large β, the scenario is quite different. The goal of the present article is to explicitly characterize the rescaled pointwise fluctuations $\varepsilon^{-\frac{d-2}{2}}[h_\varepsilon(t,x) - \mathfrak{h}(t,x)]$ as $\varepsilon \to 0$, when $d \geq 3$ and β is chosen small enough.

We now remark on the related results obtained for the corresponding multiplicative noise heat equation. Note that (2) is inherently non-linear. The Hopf-Cole transformation suggests that $u_\varepsilon = \exp[h_\varepsilon]$ solves the linear multiplicative noise stochastic heat equation (SHE)

$$\frac{\partial}{\partial t} u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \beta \varepsilon^{\frac{d-2}{2}} u_\varepsilon \xi_\varepsilon \ , \qquad u_\varepsilon(0,x) = 1, \tag{4}$$

with C_ε chosen as in (3) provided that the stochastic integral in (4) is interpreted in the classical Itô sense. Then the Feynman-Kac formula provides a solution to (4)

$$u_\varepsilon(t,x) = E_x\left[\exp\left\{\beta\varepsilon^{\frac{d-2}{2}} \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(W_{t-s} - y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2\, t\, \varepsilon^{-2}}{2}\, V(0)\right\}\right]. \tag{5}$$

with E_x denoting expectation with respect to the law P_x of a d-dimensional Brownian path $W = (W_s)_{s\geq 0}$ starting at $x \in \mathbb{R}^d$, which is independent of the noise ξ. Note that by time reversal, for any fixed $t > 0$ and $\varepsilon > 0$,

$$u_\varepsilon(t,\cdot) \overset{\text{law}}{=} \mathscr{Z}_{\varepsilon^{-2}t}(\varepsilon^{-1}\cdot)\ , \tag{6}$$

where

$$\mathscr{Z}_T(x) = E_x\left[\exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(W_s - y)\xi(s,y)\,\mathrm{d}s\ \mathrm{d}y - \frac{\beta^2 T}{2} V(0)\right\}\right], \tag{7}$$

See (Eq.(2.6) in [130]) for details. Then it was shown [130, Theorem 2.1 and Remark 2.2] that for $\beta > 0$ sufficiently small and any test function $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x)\, f(x)\ \mathrm{d}x \to \int_{\mathbb{R}^d} \overline{u}(t,x)\, f(x)\ \mathrm{d}x \tag{8}$$

as $\varepsilon \to 0$ in probability, with $\overline{u}$ solving the heat equation

$$\partial_t \overline{u} = \frac{1}{2}\Delta\overline{u} \tag{9}$$

with unperturbed diffusion coefficient. Furthermore, it was also shown in [130] that, with β small enough, and for any $t > 0$ and $x \in \mathbb{R}^d$, $u_\varepsilon(t,x)$ converges in law to a non-degenerate random variable $\mathscr{Z}_\infty$ which is almost surely strictly positive, while $u_\varepsilon(t,x)$ converges in probability to zero if β is chosen large. The law of $\mathscr{Z}_\infty$ was not determined in [130].

1.2 Main results.

Consider the random process obtained by space-time rescaling, time-reversing and spatially translating the white noise,

$$\xi^{(\varepsilon,t,x)}(\varphi) \overset{(\text{def})}{=} \varepsilon^{-\frac{d+2}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}^d} \varphi\big(\varepsilon^{-2}(t-s), \varepsilon^{-1}(y-x)\big)\xi(s,y)ds\,dy$$

Equivalently,

$$\xi^{(\varepsilon,t,x)}(S,Y) = \varepsilon^{(d+2)/2}\xi\big(\varepsilon^2(T-S), \varepsilon(Y-X)\big) \qquad \text{with } X = \varepsilon^{-1}x, T = \varepsilon^{-2}t. \tag{10}$$

Then, $\xi^{(\varepsilon,t,x)}$ is itself a Gaussian white noise and possesses the same law as ξ. This is also the reason why we define the noise above also for negative times. To abbreviate notation, we will also sometimes write $\xi^{(\varepsilon,t)} = \xi^{(\varepsilon,t,0)}$. Recall the identity (6). Then if we denote by $\mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d)$ the path space of the white noise ξ (see Appendix of Part IV in the manuscript), then by convergence 10 of Part IV, for $d \geq 3$ and β smaller than some critical value $\beta_c > 0$, there exists a strictly positive measurable function

$$\mathfrak{u} = \mathfrak{u}^{(\beta,\phi)} : \mathcal{B} \to (0,+\infty)$$

such that

$$\mathbb{E}[\mathfrak{u}(\xi)] = 1 \qquad \text{and} \quad u_\varepsilon(t,x) - \mathfrak{u}(\xi^{(\varepsilon,t,x)}) \xrightarrow{\mathbb{P}} 0 \quad \text{as } \varepsilon \to 0. \tag{11}$$

Throughout the rest of the article, we will denote by

$$\mathfrak{h} = \log \mathfrak{u} \tag{12}$$

and we remind the reader that $h_\varepsilon = \log u_\varepsilon$. Here is our first main result.

Theorem 1.1 (Gaussian fluctuations of the KPZ equation)**.** *Assume $d \geq 3$. Consider the solution h_ε to (2) with $h_\varepsilon(0,\cdot) = 0$. There exists $\beta_0 \in (0, \beta_c)$ such that for $\beta < \beta_0$, for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$\varepsilon^{-\frac{d-2}{2}} \big[h_\varepsilon(t,x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)})\big] \xrightarrow{\text{law}} N\big(0, \sigma^2(\beta) t^{-(d-2)/2}\big), \tag{13}$$

as $\varepsilon \to 0$, where

$$\sigma^2(\beta) = \frac{2\beta^2}{(d-2)(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \; V(\sqrt{2}y) \; E_y\Big[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s}\Big]. \tag{14}$$

We remark that the variance $\sigma^2(\beta)$ already blows up for $\beta > \beta_c$. We now turn to the fluctuations of the solution u_ε of the stochastic heat equation (4) and the corresponding martinagle M_T.

Theorem 1.2. *Fix $d \geq 3$ and $\beta < \beta_0$ as in Theorem 1.1. Then for all $x \in \mathbb{R}^d$, and as $T \to \infty$,*

$$T^{\frac{d-2}{4}} \left(\frac{\mathscr{Z}_T(x) - \mathscr{Z}_\infty(x)}{\mathscr{Z}_T(x)}\right) \xrightarrow{\text{law}} N\big(0, \sigma^2(\beta)\big), \tag{15}$$

where $\sigma^2(\beta)$ is defined in (14). Consequently, for all $x \in \mathbb{R}^d$, $t > 0$, as $\varepsilon \to 0$,

$$\varepsilon^{-\frac{d-2}{2}} \left(\frac{\mathfrak{u}(\xi^{(\varepsilon,t,x)})}{u_\varepsilon(t,x)} - 1\right) \xrightarrow{\text{law}} N\big(0, \sigma^2(\beta) t^{-\frac{d-2}{2}}\big)\,. \tag{16}$$

Moreover, $\varepsilon^{-\frac{d-2}{2}} \left(\frac{u_\varepsilon(t,x)}{\mathfrak{u}(\xi^{(\varepsilon,t,x)})} - 1\right)$ converges in law to the same limit.

Remark 1.3. *Following the standard terminology used in the literature on discrete directed polymers, the Feynman-Kac representation (7) relates $\mathscr{Z}_T$ (and thus, u_ε) to the (quenched) polymer partition function, and existence of a strictly positive limit $\mathscr{Z}_\infty$ for small disorder strength β is referred to as the weak-disorder regime, while for β large, a vanishing partition function $\lim_{T\to\infty} \mathscr{Z}_T$ underlines the strong disorder phase ([47]). The polymer model corresponding to (7) is known as Brownian directed polymer in a Gaussian environment, and the reader is refered to [43] for a review of a similar model driven by a Poissonian noise.*

Remark 1.4. *While we do not discuss it in detail, Theorem 1.1 and Theorem 1.2 provide Edwards-Wilkinson type limit for the KPZ and the stochastic heat equation in the weak disorder regime. We mention two recent articles ([81], [120]) where a similar problem has been studied in a different context. It was shown [81, Theorem 1.2] that, if $\beta > 0$ is chosen sufficiently small, then for $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,*

$$\varepsilon^{1-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{d}x \,\big[u_\varepsilon(t,x) - \mathbb{E}(u_\varepsilon(t,x))\big]\, f(x) \Rightarrow \int_{\mathbb{R}^d} \mathrm{d}x\, \mathscr{U}(t,x)\; f(x) \tag{17}$$

where $\mathscr{U}$ solves the heat equation with additive noise, or the Edwards-Wilkinson equation:

$$\partial_t \mathscr{U} = \frac{1}{2}\Delta \mathscr{U} + \beta\sigma(\beta)\,\overline{u}\,\xi, \qquad \mathscr{U}(0,x) = 0, \tag{18}$$

and $\overline{u}$ is the solution of the heat equation (9). We remark that the nature of the results in (17) and in Theorem 1.2, as well as their proofs are different. In particular, in the present case we consider pointwise fluctuations of the form $\mathscr{Z}_T(x) - \mathscr{Z}_\infty(x)$ for $x \in \mathbb{R}^d$ (i.e., we **do not** *study the* **spatially smoothened averages** *of $\mathscr{Z}_T(x) - \mathbb{E}[\mathscr{Z}_T(x)]$).*

The case when the noise ξ is smoothened both in time and space has also recently been considered. If $F(t,x) = \int_{\mathbb{R}^d}\int_0^\infty \phi_1(t-s)\phi_2(x-y)\,\mathrm{d}B(s,y)$ is the mollified noise, and $\hat{u}_\varepsilon(t,x) = u(\varepsilon^{-2}t, \varepsilon^{-1}x)$

with u solving $\partial_t u = \frac{1}{2}\Delta u + \beta F(t,x)u$, then it was shown in [129] that for any $\beta > 0$ and $x \in \mathbb{R}^d$, $\frac{1}{\kappa(\varepsilon,t)}\,\mathbb{E}[\hat{u}_\varepsilon(t,x)] \to \hat{u}(t,x)$ as $\varepsilon \to 0$, where $\kappa(\varepsilon,t)$ is a divergent constant and $\hat{u}(t,x)$ solves the homogenized heat equation

$$\partial_t \hat{u} = \frac{1}{2}\mathrm{div}\big(\mathrm{a}_\beta \nabla \hat{u}\big) \tag{19}$$

with diffusion coefficient $\mathrm{a}_\beta \in \mathbb{R}^{d\times d}$. It was then shown in [81, Theorem 1.1] that, for $\beta > 0$ small enough, a result of the form (17) holds also for the rescaled and spatially averaged fluctuations $\varepsilon^{1-d/2}\int \mathrm{d}x f(x)[\hat{u}_\varepsilon(t,x) - \mathbb{E}(\hat{u}_\varepsilon(t,x)]$, and the limit $\mathscr{U}$ again satisfies the additive noise stochastic heat equation $\partial_t \mathscr{U} = \frac{1}{2}\mathrm{div}\big(\mathrm{a}_\beta \nabla \mathscr{U}\big) + \beta\nu^2(\beta)\,\hat{u}\,\xi$ with diffusivity a_β and variance $\nu^2(\beta)$, and $\hat{u}$ solves (19). Note that, unlike (18), due to the presence of time correlations, in this case both the diffusion matrix and the variance of the noise are homogenized in the limit $\varepsilon \to 0$.

Finally we briefly comment on the strategy for the proof of Theorem 1.2 for which we loosely follow [45] as a guiding philosophy. The first step relies on a technical fact stated in Proposition 2.2 whose proof constitutes Section 3. However, a key step for the proof of Theorem 1.2 is utterly disparate from [45]. In particular, we do not take the approach via central limit theorem for martingales or use stable and mixing convergence (see [90]) as in [45] which can conceivably be adapted to the present case. Instead, we invoke techniques from stochastic calculus as in [40] which are well-suited and efficient in the present scenario. The details can be found in Section 2.

2 Proof of Theorem 1.1 and Theorem 1.2.

2.1 Rate of decorrelation.

In this section we will provide the following elementary result, which provides an estimate on the asymptotic decorrelation of $u_\varepsilon(x)$ and $u_\varepsilon(y)$ as $\varepsilon \to 0$. This estimate also underlines the fact that smoothing $u_\varepsilon(x)$ w.r.t. any $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ makes $\int_{\mathbb{R}^d} \mathrm{d}x\, u_\varepsilon(x)\, f(x)$ deterministic (recall (8)).

Proposition 2.1. *Let $d \geq 3$ and β small enough.*

- *We have:*

$$\mathrm{Cov}\big(\mathscr{Z}_\infty(0), \mathscr{Z}_\infty(x)\big) = \begin{cases} E_{x/\sqrt{2}}\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_s)ds} - 1\Big] & \forall x \in \mathbb{R}^d, \\ \mathfrak{C}_1\Big(\frac{1}{|x|}\Big)^{d-2} & \forall\, |x| \geq 1, \end{cases} \tag{20}$$

  with $\mathfrak{C}_1 = E_{\mathbf{e_1}/\sqrt{2}}\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_s)ds} - 1\Big]$.

- *Finally,*

$$\big\|\mathscr{Z}_\infty - \mathscr{Z}_T\big\|_2^2 \quad\sim\quad \mathfrak{C}_1\mathfrak{C}_2\mathbb{E}\big[\mathscr{Z}_\infty^2\big]T^{-\frac{d-2}{2}} \qquad \text{as } T \to \infty. \tag{21}$$

  with $\mathfrak{C}_2 = E\big[\big(\sqrt{2}/|Z|\big)^{d-2}\big]$, where Z is a centered Gaussian vector with covariance I_d.

Proof. For any Brownian path $W = (W_s)_{s\geq 0}$ we set

$$\Phi_T(W) = \exp\left\{\beta\int_0^T\int_{\mathbb{R}^d}\phi(W_s - y)\xi(s,y)\,\mathrm{d}s\,\mathrm{d}y - \frac{\beta^2 T}{2}V(0)\right\} \tag{22}$$

and see that, for any $n \in \mathbb{N}$,

$$\mathbb{E}\bigg[\prod_{i=1}^n \Phi_T(W^{(i)})\bigg] = \exp\bigg\{\beta^2\int_0^T \sum_{1\leq i<j\leq n} V\big(W_s^{(i)} - W_s^{(j)}\big)ds\bigg\}. \tag{23}$$

Then, $\mathrm{Var}\big(\mathscr{Z}_T(x)\big) = E_0\big[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_s)ds} - 1\big]$ and the first line of equation (20) follows from (23) with $n = 2$ and Brownian scaling. Now, the second line of (20) follows by considering the hitting time of the unit ball for $\sqrt{2}W$ and spherical symmetry of V.

We now show (21) as follows. For two independent paths $W^{(1)}$ and $W^{(2)}$ (which are also independent of the noise ξ), we will denote by $\mathcal{F}_T$ the σ-algebra generated by both paths until time T. Then, by Markov property, for $0 < S \leq \infty$,

$$\mathscr{Z}_{T+S}(x) = E_x\big[\Phi_T(W)\ \mathscr{Z}_S \circ \theta_{T,W_T}\big]\ , \tag{24}$$

where for any $t > 0$ and $x \in \mathbb{R}^d$, $\theta_{t,x}$ denotes the canonical spatio-temporal shift in the white noise environment. Hence, by (24),

$$\begin{aligned}
\|\mathscr{Z}_\infty - \mathscr{Z}_T\|_2^2 &= \mathbb{E}\bigg[E_0^{\otimes 2}\Big\{\Phi_T(W^{(1)})\Phi_T(W^{(2)})\Big(\mathscr{Z}_\infty \circ \theta_{T,W_T^{(1)}} - 1\Big)\Big(\mathscr{Z}_\infty \circ \theta_{T,W_T^{(2)}} - 1\Big)\Big\}\bigg] \\
&= E_0^{\otimes 2}\Big[\mathrm{e}^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s} \times \mathrm{Cov}\big(\mathscr{Z}_\infty(W_T^{(1)}), \mathscr{Z}_\infty(W_T^{(2)})\big)\Big] \\
&= E_0^{\otimes 2}\Big[\mathrm{e}^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s} \times E_0^{\otimes 2}\Big[\mathrm{Cov}\big(\mathscr{Z}_\infty(W_T^{(1)}), \mathscr{Z}_\infty(W_T^{(2)})\big)\Big|\mathcal{F}_T\Big]\Big] \\
&\sim \mathfrak{C}_1 E_0^{\otimes 2}\bigg[\mathrm{e}^{\beta^2 \int_0^T V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s} \times \bigg(\frac{2}{|W_T^{(1)} - W_T^{(2)}|}\bigg)^{d-2}\bigg] \qquad \text{(by (20))} \\
&= \mathfrak{C}_1 E_0\bigg[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\bigg(\frac{\sqrt{2}}{|W_T|}\bigg)^{d-2}\bigg]
\end{aligned}$$

Then (21) is proved once we show

$$E_0\bigg[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\bigg(\frac{1}{|W_T|}\bigg)^{d-2}\bigg] \sim E_0\Big[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s}\Big] E_0\bigg[\bigg(\frac{1}{|W_T|}\bigg)^{d-2}\bigg] \tag{25}$$

But as $T \to \infty$,

$$\bigg(\int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s, T^{-1/2}W_T\bigg) \xrightarrow{\text{law}} \bigg(\int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s, Z\bigg)$$

with $Z \sim N(0, I_d)$ being independent of the Brownian path W, and then (21) follows from the requisite uniform integrability

$$\sup_{T \geq 1} E_0\left[\left(\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\bigg(\frac{T^{1/2}}{|W_T|}\bigg)^{d-2}\right)^{1+\delta}\right] < \infty \tag{26}$$

for $\delta > 0$. By Hölder's inequality and Brownian scaling, for any $p, q \geq 1$ with $1/p + 1/q = 1$,

$$\begin{aligned}
\text{(l. h. s.) of (26)} &\leq E_0\Big[\mathrm{e}^{q(1+\delta)\beta^2 \int_0^T V(\sqrt{2}W_s)\,\mathrm{d}s}\Big]^{1/q}\ E_0\bigg[\frac{1}{|W_1|^{p(1+\delta)(d-2)}}\bigg]^{1/p} \\
&\leq E_0\Big[\mathrm{e}^{q(1+\delta)\beta^2 \int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s}\Big]^{1/q}\ \bigg[\int_{\mathbb{R}^d} \mathrm{d}x\ \frac{1}{|x|^{p(1+\delta)(d-2)}}\mathrm{e}^{-|x|^2/2}\bigg]^{1/p} \\
&\leq C\bigg[\int_0^\infty \mathrm{d}r\ r^{d-1}\ \frac{1}{r^{p(1+\delta)(d-2)}}\mathrm{e}^{-r^2/2}\bigg]^{1/p}
\end{aligned}$$

Then the last integral is seen to be finite provided we choose $\delta > 0$ and $p > 1$ small enough so that $1 < p(1+\delta) \leq \frac{d}{d-2}$. ∎

2.2 Proof of Theorem 1.2.

In this section we will prove Theorem 1.2. We start by computing the stochastic differential and bracket of the martingale $\mathscr{Z}_T$ defined as follows:

$$\begin{aligned}
\mathrm{d}\mathscr{Z}_T &= \beta E_0\bigg[\Phi_T(W)\ \int_{\mathbb{R}^d} \phi(y - W_T)\xi(T, y)\,\mathrm{d}y\bigg]\,\mathrm{d}T = \beta E_0\Big[\Phi_T(W)\ \ \phi \star \xi(T, W_T)\Big]\ dT\ , \\
\mathrm{d}\langle\mathscr{Z}\rangle_T &= \beta^2 E_0^{\otimes 2}\Big[\Phi_T(W^{(1)})\Phi_T(W^{(2)})V\big(W_T^{(1)} - W_T^{(2)}\big)\Big]\,\mathrm{d}T & (27) \\
&= \beta^2 \mathscr{Z}_T^2 \times E_{0,\beta,T}^{\otimes 2}\big[V(W_T^{(1)} - W_T^{(2)})\big]\ \mathrm{d}T\ , & (28)
\end{aligned}$$

where $E^{\otimes 2}_{0,\beta,T}$ is the expectation taken with respect to the product of two independent polymer measures,

$$P_{0,\beta,T}(\mathrm{d}W^{(i)}) = \frac{1}{Z_{\beta,T}} \exp\left\{\beta \int_0^T \int_{\mathbb{R}^d} \phi(W^{(i)}_s - y)\, \xi(y,s)\,\mathrm{d}s\,\mathrm{d}y\right\} P(\mathrm{d}W^{(i)}) \qquad i=1,2,$$

with $Z_{\beta,T} = e^{-\frac{\beta^2}{2}TV(0)}\mathscr{Z}_T$. The proof of Theorem 1.2 splits into two main steps. The first step involves showing the following estimate whose proof constitutes Section 3:

Proposition 2.2. *There exists $\beta_0 \in (0,\infty)$, such that for all $\beta < \beta_0$, as $T \to \infty$,*

$$T^{\frac{d}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_T - \mathfrak{C}_3 \mathscr{Z}_T^2 \xrightarrow{L^2} 0,$$

with $\mathfrak{C}_3 = \mathfrak{C}_3(\beta) = \frac{d-2}{2}\,\sigma^2(\beta)$ and $\sigma^2(\beta)$ from (14).

For the second step, we define a sequence $\{G^{(T)}_\tau\}_{\tau\geq 1}$ of stochastic processes on time interval $[1,\infty)$, with

$$G^{(T)}_\tau = T^{\frac{d-2}{4}}\left(\frac{\mathscr{Z}_{\tau T}}{\mathscr{Z}_T} - 1\right), \qquad \tau \geq 1. \tag{29}$$

Then, for all T, $G^{(T)}$ is a continuous martingale for the filtration $\mathcal{B}^{(T)} = (\mathcal{B}^{(T)}_\tau)_{\tau\geq 1}$, where $\mathcal{B}^{(T)}_\tau$ denotes the σ-field generated by the white noise ξ up to time τT. Then we need the following result, which provides convergence at the process level:

Theorem 2.3. *For $\beta < \beta_0$, as $T \to \infty$, we have convergence*

$$G^{(T)} \xrightarrow{\text{law}} G \tag{30}$$

on the space of continuous functions on $[1,\infty)$, equipped with the topology of uniform convergence on compact intervals, where G is a mean zero Gaussian process with independent increments and variance

$$g(\tau) = \sigma^2(\beta)\,[1 - \tau^{-\frac{d-2}{2}}].$$

Proof of Theorem 1.2 (Assuming Theorem 2.3): Note that (16) in Theorem 1.2 follows immediately from (15). To deduce (15), we write

$$\begin{aligned} T^{\frac{d-2}{4}}\left(\frac{\mathscr{Z}_\infty}{\mathscr{Z}_T} - 1\right) &= G^{(T)}_\infty \\ &= G^{(T)}_\tau + \frac{T^{\frac{d-2}{4}}[\mathscr{Z}_\infty - \mathscr{Z}_{\tau T}]}{\mathscr{Z}_T}, \end{aligned}$$

and we consider the last term. By (21), the numerator has L^2-norm tending to 0 as $\tau \to \infty$ uniformly in $T \geq 1$ whereas the denominator has a positive limit. Then, the last term vanishes in the double limit $T \to \infty, \tau \to \infty$, and therefore

$$\lim_{T\to\infty} T^{\frac{d-2}{4}}\left(\frac{\mathscr{Z}_\infty}{\mathscr{Z}_T} - 1\right) = \lim_{\tau\to\infty}\lim_{T\to\infty} G^{(T)}(\tau),$$

which is the Gaussian law with variance $g(\infty) = \sigma^2(\beta)$ by Theorem 2.3. Theorem 1.2 is proved. ∎

We now complete the
Proof of Theorem 2.3 (Assuming Proposition 2.2): From the definition (29) we compute the bracket of the square-integrable martingale $G^{(T)}$,

$$\begin{aligned} \langle G^{(T)}\rangle_\tau = \frac{T^{\frac{d-2}{2}}}{\mathscr{Z}_T^2}\,\langle \mathscr{Z}\rangle_{\tau T} &= \frac{T^{\frac{d-2}{2}}}{\mathscr{Z}_T^2}\int_1^{\tau T}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_s \mathrm{d}s \\ &= \frac{T^{\frac{d}{2}}}{\mathscr{Z}_T^2}\int_1^{\tau}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_{\sigma T} \mathrm{d}\sigma \end{aligned}$$

by replacing the variables $s = \sigma T$. Then,

$$\begin{aligned}
\langle G^{(T)}\rangle_\tau - g(\tau) &= \int_1^\tau \left[\frac{(\sigma T)^{\frac{d}{2}}}{\mathscr{Z}_T^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_{\sigma T} - \mathfrak{C}_3\right]\sigma^{-d/2}\,\mathrm{d}\sigma \\
&= \int_1^\tau \frac{\mathscr{Z}_{\sigma T}^2}{\mathscr{Z}_T^2}\left[\frac{(\sigma T)^{\frac{d}{2}}}{\mathscr{Z}_{\sigma T}^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_{\sigma T} - \mathfrak{C}_3\right]\sigma^{-d/2}\,\mathrm{d}\sigma + \frac{\mathfrak{C}_3}{\mathscr{Z}_T^2}\int_1^\tau \left[\mathscr{Z}_{\sigma T}^2 - \mathscr{Z}_T^2\right]\sigma^{-d/2}d\sigma \\
&=: I_1 + I_2
\end{aligned}$$

As $T \to \infty$ the last integral vanishes in L^2 and I_2 vanishes in probability. For $\varepsilon \in (0, 1]$, introduce the event

$$A_\varepsilon = \left\{\sup\left\{\mathscr{Z}_t; t \in [0,\infty]\right\} \vee \sup\left\{\mathscr{Z}_t^{-1}; t \in [0,\infty]\right\} \leq \varepsilon^{-1}\right\}$$

and observe that $\lim_{\varepsilon\to 0}\mathbb{P}(A_\varepsilon) = 1$ since $\mathscr{Z}_t$ is continuous, positive with a positive limit. So, we can estimate the expectation of I_1 by

$$\mathbb{E}\big[\mathbf{1}_{A_\varepsilon}|I_1|\big] \leq \frac{\tau}{\varepsilon^6}\sup_{t\geq T}\left\{\left\| t^{\frac{d}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{Z}\rangle\right)_t - \mathfrak{C}_3\mathscr{Z}_t^2\right\|_1\right\},$$

which vanishes by Proposition 2.2. Thus, $\langle G^{(T)}\rangle \to g$ in probability. Since for the sequence of continuous martingales $G^{(T)}$ the brackets converge pointwise to a deterministic limit g, we derive that the sequence $G^{(T)}$ itself converges in law to a Brownian motion with time-change given by g, that is, the process G defined in the statement of Theorem 2.3 (see [104, Theorem 3.11 in Chapter 8]), which is proved now. ■

2.3 Proof of Theorem 1.1.

Proposition 2.2, together with the following computation also provides the proof of Theorem 1.1 (see Remark 2.4 for an alternative proof using directly Theorem 1.2). Indeed, by Itô's formula and equations (27) and (28), we can write

$$\log \mathscr{Z}_T = N_T - \frac{1}{2}\langle N\rangle_T, \tag{31}$$

where

$$N_T = \beta\int_0^T\int_{\mathbb{R}^d} E_{0,\beta,t}[\phi(y - W_t)]\dot{B}(t,y)\,\mathrm{d}y\,\mathrm{d}t, \quad \langle N\rangle_T = \beta^2\int_0^T E_{0,\beta,s}^{\otimes 2}\left[V(W_s^{(1)} - W_s^{(2)})\right]\mathrm{d}s,$$

with N being a martingale. Then, Proposition 2.2 shows that, in probability and as $T \to \infty$,

$$T^{d/2}\frac{\mathrm{d}}{\mathrm{d}T}\langle N\rangle_T \to \frac{d-2}{2}\sigma^2(\beta). \tag{32}$$

Mimicking the proof of Theorem 2.3, this implies that the bracket of the rescaled martingale $N^{(T)}$: $\tau \to T^{(d-2)/4}(N_{\tau T} - N_T)$ converges to the deterministic function $\tau \to \sigma^2(\beta)(1 - \tau^{-\frac{d-2}{2}})$. Hence, by the functional central limit for martingales (Theorem 3.11 in [104]), $N^{(T)}$ converges to a gaussian process, with the above function as bracket. Moreover, by (32), one gets that $T^{(d-2)/4}(\langle N\rangle_\infty - \langle N\rangle_T)$ converges to 0 in probability.

Theorem 1.1 follows by similar arguments to the proof of Theorem 1.2, letting together $\tau \to \infty$ and $T \to \infty$. ■

Remark 2.4. *Theorem 1.1 can also be alternatively deduced from* (16) *in Theorem 1.2 and the following lemma:*

Lemma 2.5. *If $X_\varepsilon \to N(0, \sigma^2)$ in distribution, then for any $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0, \Phi'(0) = 1$ and $a > 0$, we have $\varepsilon^{-a}\Phi(\varepsilon^a X_\varepsilon) \to N(0, \sigma^2)$.*

Thus, choosing $\Phi(x) = \log(1 + x)$ and $a = \frac{d-2}{2}$ in (16) *in Theorem 1.2 we get*

$$\varepsilon^{-\frac{d-2}{2}}\left(h_{\varepsilon,t}(x) - \mathfrak{h}(\xi^{(\varepsilon,t,x)})\right) \to N(0, \sigma^2(\beta)t^{-\frac{d-2}{2}}),$$

proving Theorem 1.1. ■

3 Proof of proposition 2.2

Denote for short by L_T the quantity of interest,

$$\begin{aligned} L_T &:= T^{\frac{d}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathscr{Z} \rangle \right)_T - \mathfrak{C}_3 \mathscr{Z}_T^2 \\ &= E_0^{\otimes 2}\left[\Phi_T(W^{(1)}) \Phi_T(W^{(2)}) \Big(T^{\frac{d}{2}} V\big(W_T^{(1)} - W_T^{(2)}\big) - \mathfrak{C}_3 \Big) \right], \end{aligned}$$

and proceed in two steps.

3.1 The first moment.

We first want to show that

Proposition 3.1. *There exists $\beta_1 \in (0,\infty)$ such that for all $\beta < \beta_1$, if we choose*

$$\mathfrak{C}_3 = (2\pi)^{-d/2} \int E_y \left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2} W_t)\,\mathrm{d}t} \right] V(\sqrt{2} y)\,\mathrm{d}y\,, \tag{33}$$

then $\mathbb{E}(L_T) \to 0$ as $T \to \infty$.

■

The rest of Section 3.1 is devoted to the proof of Proposition 3.1. For any $t > s \geq 0$ and $x, y \in \mathbb{R}^d$, we will denote by $P_{s,x}^{t,y}$ the law (and by $E_{s,x}^{t,y}$ the corresponding expectation) of the Brownian bridge starting at x at time s and conditioned to reach y at time $t > s$. We will also write

$$\rho(t,x) = (2\pi t)^{-d/2} \mathrm{e}^{-|x|^2/2t}$$

to be the standard Gaussian kernel.

We note that

$$\begin{aligned} \mathbb{E}(L_T) &= E_0^{\otimes 2}\left[\mathrm{e}^{\beta^2 \int_0^T V(W_t^{(1)} - W_t^{(2)})\,\mathrm{d}t} \Big(T^{\frac{d}{2}} V\big(W_T^{(1)} - W_T^{(2)}\big) - \mathfrak{C}_3 \Big) \right] \\ &= E_0 \left[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2} W_t)\,\mathrm{d}t} \Big(T^{\frac{d}{2}} V(\sqrt{2} W_T) - \mathfrak{C}_3 \Big) \right] \end{aligned}$$

and

$$E_0 \left[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2} W_t)\,\mathrm{d}t} T^{\frac{d}{2}} V(\sqrt{2} W_T) \right] = \int_{\mathbb{R}^d} V(\sqrt{2} y) E_{0,0}^{T,y} \left[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2} W_t)\,\mathrm{d}t} \right] T^{\frac{d}{2}} \rho(T,y)\,\mathrm{d}y.$$

Now, we fix a sequence $m = m(T)$, such that $m \to \infty$ and $m = o(T)$ as $T \to \infty$, which helps us prove Proposition 3.1 in two steps:

Proposition 3.2. *For small enough β, we have for any $y \in \mathbb{R}^d$ and as $T \to \infty$,*

$$E_{0,0}^{T,y} \left[\mathrm{e}^{\beta^2 \int_0^T V(\sqrt{2} W_t)\,\mathrm{d}t} \right] = \mathcal{T}_1 + o(1),$$

where

$$\mathcal{T}_1 = E_{0,0}^{T,y} \left[\mathrm{e}^{\beta^2 \int_{[0,m] \cup [T-m,T]} V(\sqrt{2} W_t)\,\mathrm{d}t} \right].$$

Proposition 3.3. *For small enough β, we have as $T \to \infty$,*

$$\begin{aligned} \mathcal{T}_1 &\sim E_{0,0}^{T,y} \left[\mathrm{e}^{\beta^2 \int_{[0,m]} V(\sqrt{2} W_t)\,\mathrm{d}t} \right] E_{0,0}^{T,y} \left[\mathrm{e}^{\beta^2 \int_{[T-m,T]} V(\sqrt{2} W_t)\,\mathrm{d}t} \right] \\ &\to E_0 \left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2} W_t)\,\mathrm{d}t} \right] E_y \left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2} W_t)\,\mathrm{d}t} \right]. \end{aligned}$$

We will provide some auxiliary results which will be needed to prove Proposition 3.2 and Proposition 3.3. First, we state a simple consequence of Girsanov's theorem:

Lemma 3.4. *For any $s < t$ and $y, z \in \mathbb{R}^d$, the Brownian bridge $P_{0,y}^{t,z}$ is absolutely continuous w.r.t. $P_{0,y}$ on the σ-field $\mathcal{F}_{[0,s]}$ generated by the Brownian path until time $s < t$, and*

$$\frac{\mathrm{d}P_{0,y}^{t,z}}{\mathrm{d}P_{0,y}}\bigg|_{\mathcal{F}_{[0,s]}} = \frac{\rho(t-s, z-W_s)}{\rho(t, z-y)} \leq \left(\frac{t}{t-s}\right)^{d/2} \exp\left\{\frac{|z-y|^2}{2t}\right\}. \tag{34}$$

■

We will need the following version of Khas'minskii's lemma for the Brownian bridge:

Lemma 3.5. *If $E_0\left[2\beta^2 \int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s\right] < 1$, then*

$$\sup_{z,x\in\mathbb{R}^d, t>0} E_{0,x}^{t,z}\left[\exp\left\{\beta^2 \int_0^t V(\sqrt{2}W_s)\,\mathrm{d}s\right\}\right] < \infty.$$

Proof. By Girsanov's theorem, for any $s < t$, $\alpha \in \mathbb{R}^d$ and $A \in \mathcal{F}_{[0,s]}$,

$$P_{0,x}^{t,z}(A) = E_x^{(\alpha)}\left[\frac{\rho^{(\alpha)}(t-s; z-W_s)}{\rho^{(\alpha)}(t, z-x)} \mathbf{1}_A\right] \tag{35}$$

where $E^{(\alpha)}$ (resp. $P^{(\alpha)}$) refers to the expectation (resp. the probability) with respect to Brownian motion with drift α and transition density

$$\rho^{(\alpha)}(t,z) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{-\frac{|z-t\alpha|^2}{2t}\right\}.$$

With $\alpha = (z-x)/t$ and $s = t/2$, applying (35), we get

$$P_{0,x}^{t,z}(A) \leq 2^{d/2}\; P_x^{(\alpha)}(A).$$

Replacing A by $e^{2\beta^2 \int_0^{t/2} V(\sqrt{2}W_s)\,\mathrm{d}s}$, we have

$$\begin{aligned}\sup_{z,x\in\mathbb{R}^d, t>0} E_{0,x}^{t,z}\left[\exp\left\{2\beta^2 \int_0^{t/2} V(\sqrt{2}W_s)\,\mathrm{d}s\right\}\right] &\leq 2^{d/2} \sup_\alpha E^{(\alpha)}\left[\exp\left\{2\beta^2 \int_0^{t/2} V(\sqrt{2}W_s)\,\mathrm{d}s\right\}\right] \\ &\leq 2^{d/2} \frac{1}{1-a} < \infty,\end{aligned}$$

where the second upper bound follows from Khas'minskii's lemma provided we have

$$2\beta^2 \sup_{x,\alpha} E_x^{(\alpha)}\left[\int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s\right] \leq a < 1.$$

But since the expectation in the above display is equal to $\int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}z V(\sqrt{2}z)\; \rho^{(\alpha)}(s, z-x)$ and is maximal for $x = 0$ and $\alpha = 0$, the requisite condition reduces to

$$2\beta^2 E_0\left[\int_0^\infty V(\sqrt{2}W_s)\,\mathrm{d}s\right] < 1,$$

which is satisfied by our assumption. Finally, the lemma follows from the observation

$$\exp\left\{\beta^2 \int_0^t V(\sqrt{2}W_s)\;\mathrm{d}s\right\} \leq \frac{1}{2}\left[\exp\left\{2\beta^2 \int_0^{t/2} V(\sqrt{2}W_s)\,\mathrm{d}s\right\} + \exp\left\{2\beta^2 \int_{t/2}^t V(\sqrt{2}W_s)\,\mathrm{d}s\right\}\right]$$

combined with time reversibility of Brownian motion. ■

Recall that $V = \phi\star\phi$ is bounded and has support in a ball of radius 1 around the origin, and therefore, for some constant $c, c' > 0$, and any $a > 0$,

$$P_0\Big[\int_m^\infty \mathrm{d}s\, V(\sqrt{2}W_s) > a\Big] \le \frac{c}{a}\int_m^\infty \frac{\mathrm{d}s}{s^{3/2}}\int_{B(0,1)} \mathrm{d}y V(\sqrt{2}y)\exp\Big\{-\frac{|y|^2}{2s}\Big\} \le \frac{c'\|V\|_\infty}{am^{1/2}} \to 0$$

as $m\to\infty$, implying

Lemma 3.6. *For any $a > 0$, $\lim_{T\to\infty}\ P_0\big[\int_m^\infty \mathrm{d}s\, V(\sqrt{2}W_s) > a\big] = 0$.*

By Lemma 3.5, we also have

Lemma 3.7. *For any $a > 0$,*

$$\lim_{T\to\infty}\sup_{z\in\mathbb{R}^d} P_{0,0}^{T,z}\Big[\int_m^{T-m} V(\sqrt{2}W_s)\,\mathrm{d}s > a\Big] = 0.$$

Proof of Proposition 3.2. Note that, for any $a > 0$, we only need to show that

$$\limsup_{T\to\infty}\sup_{y\in\mathbb{R}^d} E_{0,0}^{T,y}\Big[\mathrm{e}^{\beta^2\int_0^T V(\sqrt{2}W_t)\,\mathrm{d}t}\ \mathbf{1}\Big\{\int_m^{T-m} V(\sqrt{2}W_s)\ \mathrm{d}s > a\Big\}\Big] = 0.$$

But the above convergence follows by Hölder's inequality, Lemma 3.5 and Lemma 3.7. ■

We now turn to the proof of

Proof of Proposition 3.3. Condition on the position of the Brownian bridge at time $T/2$, then use reversal property of the Brownian bridge and change of variable $z\to\sqrt{T}z$, to get:

$$\begin{aligned}
\mathcal{T}_1 &= \int_{\mathbb{R}^d} E_{0,0}^{T/2,z}\Big[e^{\beta^2\int_{[0,m]} V(\sqrt{2}W_t)\,\mathrm{d}t}\Big] E_{T/2,z}^{T,y}\Big[\mathrm{e}^{\beta^2\int_{[T-m,T]} V(\sqrt{2}W_t)\,\mathrm{d}t}\Big]\frac{\rho(T/2,z)\rho(T/2,y-z)}{\rho(T,y)}\,\mathrm{d}z\\
&= \int_{\mathbb{R}^d} E_{0,0}^{T/2,z\sqrt{T}}\Big[e^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\Big] E_{0,y}^{T/2,z\sqrt{T}}\Big[\mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\Big]\frac{\rho(1/2,z)\rho(1/2,z-y/\sqrt{T})}{\rho(1,y/\sqrt{T})}\,\mathrm{d}z.
\end{aligned}$$

We now claim that, for fixed z,

$$E_{0,y}^{T/2,z\sqrt{T}}\Big[\mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\Big] \sim E_y\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\Big]. \tag{36}$$

Then, by dominated convergence theorem applied to the above integral, where the expectations in the integrand are bounded thanks to Lemma 3.5, we obtain that:

$$\begin{aligned}
\mathcal{T}_1 &\sim \int_{\mathbb{R}^d} E_0\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\Big] E_y\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\Big]\frac{\rho(1/2,z)\rho(1/2,z)}{\rho(1,0)}\,\mathrm{d}z\\
&= E_0\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\Big] E_y\Big[\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\Big].
\end{aligned}$$

To prove (36), we use Lemma 3.4:

$$\begin{aligned}
&E_{0,y}^{T/2,z\sqrt{T}}\Big[\mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\Big]\\
&= \frac{1}{\rho(T/2,z\sqrt{T}-y)}E_y\Big[\mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\rho(T/2-m,z\sqrt{T}-\sqrt{2}W_m)\Big]\\
&= \frac{1}{\rho(1/2,z-y/\sqrt{T})\big(\pi(1-\frac{2m}{T})\big)^{d/2}}E_y\Big[\mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\mathrm{e}^{-\frac{|z-\sqrt{2/T}\,W_m|^2}{1-2m/T}}\Big].
\end{aligned}$$

By monotone convergence and the fact that $m = o(T)$, we obtain:

$$P\text{-a.s.}\quad \mathrm{e}^{\beta^2\int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\to\mathrm{e}^{\beta^2\int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\quad\text{and}\quad \mathrm{e}^{-\frac{|z-\sqrt{2/T}\,W_m|^2}{1-2m/T}}\to\mathrm{e}^{-2z^2}.$$

Then, we have the following uniform integrability property for small $\delta > 0$ and small β:

$$E_y\left[\left(\mathrm{e}^{\beta^2 \int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\mathrm{e}^{-\frac{|z-\sqrt{2/T}\,W_m|^2}{1-2m/T}}\right)^{1+\delta}\right] \leq E_y\left[\mathrm{e}^{(1+\delta)\beta^2 \int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\right] < \infty.$$

Hence,

$$E_{0,y}^{T/2,z\sqrt{T}}\left[\mathrm{e}^{\beta^2 \int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\right] \to \frac{\mathrm{e}^{-2z^2}}{\rho(1/2,z)\pi^{d/2}} E_y\left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\right] = E_y\left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\right].$$

■

3.2 Second moment.

The goal of this section is to show

Proposition 3.8. *There exists $\beta_0 \in (0,\infty)$, such that for all $\beta < \beta_0$, $\mathbb{E}(L_T^2) \to 0$.*

For the proof of the above result, it is enough to show that $\limsup_{T\to\infty} \mathbb{E}(L_T^2) \leq 0$. Computing second moment, we get an integral over four independent Brownian paths:

$$\begin{aligned}
\mathbb{E}(L_T^2) &= E_0^{\otimes 4}\Big[\prod_{i\in\{1,3\}}\Big(T^{\frac{d}{2}}V(W_T^{(i)}-W_T^{(i+1)}) - \mathfrak{C}_3\Big)\mathrm{e}^{\beta^2 \sum_{1\leq i<j\leq 4}\int_0^T V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t}\Big] \\
&= E_0^{\otimes 4}\Big[\ \prod_{i\in\{1,3\}}\Big\{\mathrm{e}^{\beta^2 \int_0^T V(W_t^{(i)}-W_t^{(i+1)})\,\mathrm{d}t}\Big(T^{\frac{d}{2}}V(W_T^{(i)}-W_T^{(i+1)}) - \mathfrak{C}_3\Big)\Big\} \\
&\qquad\qquad \times \mathrm{e}^{\beta^2 \sum^* \int_0^T V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t}\ \Big]
\end{aligned}$$

where the sum $\sum^*$ is considered for 4 pairs $(i,j), 1 \leq i < j \leq 4$ different from $(1,2)$ and $(3,4)$.

Throughout the rest of the article, for notational convenience, we will write

$$\begin{aligned}
H_m &= \mathrm{e}^{\beta^2 \sum_{1\leq i<j\leq 4}\int_0^m V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t}, \\
H_{T-m,T} &= \prod_{i\in\{1,3\}}\Big\{\mathrm{e}^{\beta^2 \int_{T-m}^T V(W_t^{(i)}-W_t^{(i+1)})\,\mathrm{d}t}\ \Big(T^{d/2}V\Big(W_T^{(i)}-W_T^{(i+1)}\Big) - \mathfrak{C}_4\Big)\Big\}.
\end{aligned} \tag{37}$$

We will now estimate each term in the expectation in (37). Proposition 3.9 stated below enables us to neglect the contributions of $\int_m^{T-m} V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t$ for all i,j and of $\int_{T-m}^T V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t$ for all $(i,j) \neq (1,2),(3,4)$. More precisely, we want to show that

Proposition 3.9. *For $m = m(T)$ as above, there exists a constant $C > 0$ such that, for small enough β,*

$$\limsup_{T\to\infty} \mathbb{E}L_T^2 \leq C \limsup_{T\to\infty} \mathcal{T}_2,$$

where

$$\mathcal{T}_2 = E_0^{\otimes 4}\big[H_m\ H_{T-m,T}\big]. \tag{38}$$

Then, Proposition 3.8 will be a consequence of

Proposition 3.10. *For small enough β, we have as $T \to \infty$:*

$$\begin{aligned}
\mathcal{T}_2 = E_0^{\otimes 4}&\Big[\mathrm{e}^{\beta^2 \sum_{1\leq i<j\leq 4}\int_0^\infty V(W_t^{(i)}-W_t^{(j)})\,\mathrm{d}t}\Big] \\
&\times \Big[E_0^{\otimes 2}\Big(\mathrm{e}^{\beta^2 \int_{T-m}^T V(W_t^{(1)}-W_t^{(2)})\,\mathrm{d}t}\Big[T^{\frac{d}{2}}V(W_T^{(1)}-W_T^{(2)}) - \mathfrak{C}_3\Big]\Big)\Big]^2 + o(1).
\end{aligned} \tag{39}$$

As a result, $\mathcal{T}_2 \to 0$.

3.3 Proof of Proposition 3.9.

We pick up from the first display in (37), and write

$$\mathbb{E}(L_T^2) = \mathcal{T}_2^{(\mathrm{I})} + \mathcal{T}_2^{(\mathrm{II})} \tag{40}$$

where

$$\begin{aligned} \mathcal{T}_2^{(\mathrm{II})} = E_0^{\otimes 4}\Bigg[& \mathrm{e}^{\beta^2 \sum_{1\le i<j\le 4} \int_0^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \prod_{i=\{1,3\}} \Big(T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) - \mathfrak{C}_3 \Big) \\ & \times \mathbf{1}\Big\{ \int_m^{T-m} V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \ge a \text{ for some } 1\le i<j\le 4 \Big\} \Bigg] \end{aligned} \tag{41}$$

and $\mathcal{T}_2^{(\mathrm{I})}$ is defined canonically. We claim that,

$$\limsup_{T\to\infty} \mathcal{T}_2^{(\mathrm{II})} = 0. \tag{42}$$

To prove the above claim, in (41) we first estimate, using that $V, \mathfrak{C}_3 \ge 0$,

$$\prod_{i\in\{1,3\}} \Big[T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) - \mathfrak{C}_3 \Big] \le \prod_{i\in\{1,3\}} \Big[T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) \Big] + \mathfrak{C}_3^2.$$

Note that,

$$\begin{aligned} \mathfrak{C}_3^2 \limsup_{T\to\infty} \; E_0^{\otimes 4}\Bigg[& \mathrm{e}^{\beta^2 \sum_{1\le i<j\le 4} \int_0^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \\ & \times \mathbf{1}\Big\{ \int_m^{T-m} V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \ge a \text{ for some } 1\le i<j\le 4 \Big\} \Bigg] = 0, \end{aligned}$$

by Hölder's inequality combined with Khas'minskii's lemma and Lemma 3.6. Therefore,

$$\begin{aligned} \mathcal{T}_2^{(\mathrm{II})} \le E_0^{\otimes 4}\Bigg[& e^{\beta^2 \sum_{1\le i<j\le 4} \int_0^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \prod_{i=\{1,3\}} \Big(T^{d/2} V(W_T^{(i)} - W_T^{(i+1)}) \Big) \\ & \times \mathbf{1}\Big\{ \int_m^{T-m} V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \ge a \text{ for some } 1\le i<j\le 4 \Big\} \Bigg] + o(1). \end{aligned} \tag{43}$$

Next, we switch from free Brownian motion to the Brownian bridge in the first term on the right hand side above, such that, writing $\mathbf{y} = (y_1, \dots, y_4)$, we get

$$\begin{aligned} \mathcal{T}_2^{(\mathrm{II})} \le & \int_{(\mathbb{R}^d)^4} \mathrm{d}\mathbf{y} \prod_{i\in\{1,3\}} \Big(T^{-d/2} \mathrm{e}^{-\frac{|y_i|^2 + |y_{i+1}|^2}{2T}} V(y_i - y_{i+1}) \Big) \\ & \bigotimes_{i=1}^4 E_{0,0}^{T,y_i} \Bigg[\mathrm{e}^{\beta^2 \sum_{1\le i<j\le 4} \int_0^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \\ & \qquad \times \mathbf{1}\Big\{ \int_m^{T-m} V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \ge a \text{ for some } 1\le i<j\le 4 \Big\} \Bigg] + o(1). \end{aligned}$$

Note that V has support in a ball of radius 1 around 0. We now again use Hölder's inequality which, combined with Lemma 3.5 and Lemma 3.7 finish the proof of (42).

We now turn to estimate $\mathcal{T}_2^{(\mathrm{I})}$, which, by the second display in (37), (40) and (41) is given by

$$\begin{aligned} \mathcal{T}_2^{(\mathrm{I})} = \; & E_0^{\otimes 4}\Bigg[H_m \; H_{T-m,T} \; \Big\{ \mathrm{e}^{\beta^2 \sum^\star \int_{T-m}^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \Big\} \\ & \times \mathrm{e}^{\beta^2 \sum_{1\le i<j\le 4} \int_m^{T-m} V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \mathbf{1}\Big\{ \int_m^{T-m} V\Big(W_t^{(i)} - W_t^{(j)} \Big)\,\mathrm{d}t \le a \text{ for all } 1\le i<j\le 4 \Big\} \Bigg] \\ \le \; & \mathrm{e}^{6\beta^2 a} E_0^{\otimes 4}\Bigg[H_m \; H_{T-m,T} \; \Big\{ \mathrm{e}^{\beta^2 \sum^\star \int_{T-m}^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t} \Big\} \Bigg]. \end{aligned}$$

We again want to ignore the contribution of the last term. But this can be done exactly as in the way we estimated $\mathcal{T}_2^{(\mathrm{II})}$ by splitting interactions for $(i,j) \in \sum^\star$ when $\int_{T-m}^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \geq a$ and $\int_{T-m}^T V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t \leq a$. In order to avoid repetition we omit the details, and conclude the proof of Proposition 3.9. ■

3.4 Proof of Proposition 3.10.

If we denote by $\mathcal{F}_{[0,T/2]}$ the σ-algebra generated by all four Brownian paths until time $T/2$, then, using Markov's property,

$$\begin{aligned}
\mathcal{T}_2 = E_0^{\otimes 4}[H_m\ H_{T-m,T}] &= E_0^{\otimes 4}\left[E_0^{\otimes 4}\left(H_m\ H_{T-m,T}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right)\right] \\
&= E_0^{\otimes 4}\left[E_0^{\otimes 4}\left\{E_0^{\otimes 4}\left(H_m\ H_{T-m,T}\big|\mathcal{F}_{[0,T/2]}\right)\Big|\big(W_{T/2}^{((i)}\big)_{i=1}^4\right\}\right] \\
&= E_0^{\otimes 4}\left[E_0^{\otimes 4}\left\{H_m\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} E_0^{\otimes 4}\left\{H_{T-m,T}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\}\right].
\end{aligned}$$

We will prove that there exists a constant $C < \infty$, such that:

$$\text{(i)} \sup_{T>0} E_0^{\otimes 4}\left\{H_m\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} \leq C \qquad \text{(ii)} \sup_{T>0} E_0^{\otimes 4}\left\{H_{T-m,T}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} \leq C,$$
$$\text{(iii)}\ E_0^{\otimes 4}\left\{H_m\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} \xrightarrow{\text{law}} E_0^{\otimes 4}\left[H_\infty\right],\ \text{as } T \to \infty.$$

where H_∞ is defined as H_m with the time interval $[0,m]$ replaced by $[0,\infty)$, recall (37).

Let us first conclude the proof of Proposition 3.10 assuming the above three assertions. The difference of the two first terms in (39) writes:

$$\begin{aligned}
&\mathcal{T}_2 - E_0^{\otimes 4}\left[H_\infty\right] E_0^{\otimes 4}\left[H_{T-m,T}\right] \\
&= E_0^{\otimes 4}\left[\left(E_0^{\otimes 4}\left\{H_m\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} - E_0^{\otimes 4}\left[H_\infty\right]\right) E_0^{\otimes 4}\left\{H_{T-m,T}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\}\right],
\end{aligned}$$

which goes to 0 as $T \to \infty$ by (i)-(iii), proving (39). Finally, computations of Section 3.1 ensure that:

$$\left[E_0^{\otimes 2}\left\{\mathrm{e}^{\beta^2 \int_{T-m}^T V(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s}\left(T^{d/2} V\left(W_T^{(1)} - W_T^{(2)}\right) - \mathfrak{C}_3\right)\right\}\right] \underset{T\to\infty}{\longrightarrow} 0.$$

We now owe the reader the proofs of (i)-(iii). To prove (i), we use Hölder's inequality to get

$$\begin{aligned}
E_0^{\otimes 4}\left\{H_m\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} &\leq \prod_{1\leq i<j\leq 4} E_0^{\otimes 4}\left[\mathrm{e}^{6\beta^2 \int_0^m V(W_t^{(i)} - W_t^{(j)})\,\mathrm{d}t}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right]^{1/6} \\
&= \prod_{1\leq i<j\leq 4} E_{0,0}^{T/2, W_{T/2}^{(i)} - W_{T/2}^{(j)}}\left[\mathrm{e}^{6\beta^2 \int_0^m V(\sqrt{2}W_t)\,\mathrm{d}t}\right]^{1/6} \\
&\leq \sup_{T,z} E_{0,0}^{T/2,z}\left[\mathrm{e}^{6\beta^2 \int_0^{T/2} V(\sqrt{2}W_t)\,\mathrm{d}t}\right] < \infty,
\end{aligned}$$

by Lemma 3.5. For (ii), we note that by Markov's property,

$$E_0^{\otimes 4}\left\{H_{T-m,T}\Big|\big(W_{T/2}^{(i)}\big)_{i=1}^4\right\} = \prod_{i\in\{1,3\}} E_{W_{T/2}^{(i)} - W_{T/2}^{(i+1)}}\left[\mathrm{e}^{\beta^2 \int_{T/2-m}^{T/2} V(\sqrt{2}W_t)\,\mathrm{d}t}\left(T^{d/2} V\left(\sqrt{2}W_{T/2}\right) - \mathfrak{C}_3\right)\right].$$

We have:

$$\mathfrak{C}_3\, E_{W_{T/2}^{(i)} - W_{T/2}^{(i+1)}}\left[\mathrm{e}^{\beta^2 \int_{T/2-m}^{T/2} V(\sqrt{2}W_t)\,\mathrm{d}t}\right] \leq \mathfrak{C}_3 \sup_z E_z\left[\mathrm{e}^{\beta^2 \int_0^\infty V(\sqrt{2}W_t)\,\mathrm{d}t}\right] < \infty,$$

while, for some constant $C' > 0$,

$$
\begin{aligned}
&E_{W^{(i)}_{T/2}-W^{(i+1)}_{T/2}}\left[e^{\beta^2\int_{T/2-m}^{T/2} V(\sqrt{2}W_t)\,\mathrm{d}t}\; T^{d/2}V\left(\sqrt{2}W_{T/2}\right)\right]\\
&\leq C'\int_{\mathbb{R}^d}\mathrm{d}z\; E^{T/2,z}_{0,W^{(i)}_{T/2}-W^{(i+1)}_{T/2}}\left[\mathrm{e}^{\beta^2\int_{T/2-m}^{T/2} V(\sqrt{2}W_t)\,\mathrm{d}t}\right] V\left(\sqrt{2}z\right)\\
&\leq C'\sup_{T,y,z}\; E^{T/2,z}_{0,y}\left[\mathrm{e}^{\beta^2\int_0^{T/2} V(\sqrt{2}W_t)\;\mathrm{d}t}\right]\int_{B(0,1)}\mathrm{d}z\; V\left(\sqrt{2}z\right)\\
&< \infty,
\end{aligned}
$$

again by Lemma 3.5.

Finally, to prove (iii), we fix any smooth test function $f : \mathbb{R} \to \mathbb{R}$, so that

$$
\begin{aligned}
E_0^{\otimes 4}\left[f\left(E_0^{\otimes 4}\left\{H_m\middle|\left(W^{(i)}_{T/2}\right)_{i=1}^4\right\}\right)\right] &= \int_{(\mathbb{R}^d)^4}\mathrm{d}\mathbf{y}\; f\left(E_{0,0}^{T/2,\mathbf{y}}\left[H_m\right]\right)\prod_{i=1}^4\rho(T/2,y_i)\\
&= \int_{(\mathbb{R}^d)^4}\mathrm{d}\mathbf{z}\; f\left(E_{0,0}^{T/2,\sqrt{T}\mathbf{z}}\left[H_m\right]\right)\prod_{i=1}^4\rho(1/2,z_i).
\end{aligned}
\tag{44}
$$

Now, letting $T \to \infty$, we get similarly to (36) that $E_{0,0}^{T/2,\sqrt{T}\mathbf{z}}\left[H_m\right] \to E_0^{\otimes 4}\left[H_\infty\right]$. By dominated convergence, the RHS of (44) converges to $f\left(E_0^{\otimes 4}\left[H_\infty\right]\right)$, implying (iii). ∎

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Part VI

Fluctuations of the tail of the polymer partition function for $d \geq 3$ in the whole L^2-region.

Abstract (Joint work with Shuta Nakajima). In this paper, we consider the discrete directed polymer model with i.i.d. environment and we study the fluctations of the tail $W_\infty - W_n$ of the normalized partition function. It was proven by Comets and Liu [45] that for sufficiently high temperature, the rescaled fluctuations converge in distribution towards the product of the limiting partition function and an independent Gaussian random variable. We extend the result to the whole L^2-region, which is predicted to be the maximal high-temperature region where the Gaussian fluctuations should occur under the considered scaling.

1 Introduction

1.1 The model

The directed polymer model was first introduced by Huse and Henly in the physics literature [98] and was reformulated in mathematics by Imbrie and Spencer [102]. The model is a description of a long chain of monomers, called a *polymer*, which interacts with impurities that it may encounter on its path. The reader is refered to [41] for a recent review of the model. In the discrete case, the model is defined as follows.

The impurities, also called the *environment*, are modeled by a collection of non-constant, i.i.d. random variables $\omega(i,x)$, $i \in \mathbb{N}, x \in \mathbb{Z}^d$, defined under a probability measure $\mathbb{P}$ of expectation denoted by $\mathbb{E}$. We will assume that $\mathbb{E}[\exp(\beta\omega(i,x))] < \infty$ for all $\beta \in \mathbb{R}$, and define:

$$\lambda(\beta) = \log \mathbb{E}\left[e^{\beta\omega(i,x)}\right].$$

Let $\Omega = \{(S_k)_{k\geq 0}, S_k \in \mathbb{Z}^d\}$ be the state space of the trajectories, and P_x the probability measure on Ω, such that the canonical process $(S_k)_{k\geq 0}$ is the simple random walk on $\mathbb{Z}^d$ starting at position x, i.e. under P_x, $S_1 - S_0, \ldots, S_{k+1} - S_k$ are independent and

$$\mathrm{P}_x(S_0 = x) = 1, \quad \mathrm{P}_x(S_{k+1} - S_k = \pm\mathbf{e}) = \frac{1}{2d},$$

where $\mathbf{e}$ is any vector of the canonical basis of $\mathbb{R}^d$. We denote by E_x the expectation under P_x, and $\mathrm{P} = \mathrm{P}_0, \mathrm{E} = \mathrm{E}_0$.

Then, the *Gibbs measure of the polymer* $\mathrm{P}_{x,\beta,n}$ on Ω is defined as:

$$\mathrm{dP}_{x,\beta,n}(S) = \frac{\exp\left\{\sum_{i=1}^n \beta\omega(i,S_i)\right\}}{Z_n(\beta)} \mathrm{dP}_x(S),$$

where $\beta \geq 0$ stands for the *inverse temperature* of the polymer, and where $Z_n(\beta) = \mathrm{E}\left[\exp\{\sum_{i=1}^n \beta\omega(i,S_i)\}\right]$ is called the *partition function.*

A *polymer path of horizon* n is the realization of $(S_k)_{0\leq k\leq n}$ under the polymer measure $\mathrm{P}_{x,\beta,n}$. The parameter β models the strength of the interaction of the polymer with the environment: the higher β, the more the polymer path is tempted to go through high values of the environment.

The normalized partition function:

$$W_n = Z_n e^{-n\lambda(\beta)} = \mathrm{E}\left[e_n\right], \quad \text{with: } e_n = e^{\sum_{i=1}^n \omega(i,S_i) - n\lambda(\beta)}, \tag{1}$$

is a mean 1, positive martingale with respect to the filtration $\mathcal{F}_n$ generated by the variables $\omega(i,x)$, $i \leq n, x \in \mathbb{Z}^d$. The martingale verifies the following dichotomy [47]: for $d \geq 3$ (which will be assumed from now), there exist some critical parameters $\beta_c^+(d) \in (0,\infty]$ and $\beta_c^- \in [-\infty, 0)$, such that

- For all $\beta_c^- < \beta < \beta_c^+$, $W_n \to W_\infty$ a.s., with $\mathbb{P}(W_\infty > 0) = 1$,
- For all $\beta \in \mathbb{R} \setminus [\beta_c^-, \beta_c^+]$, $W_n \to 0$ a.s.

The region below (β_c^-, β_c^+) is called the *weak disorder regime*, while the region $\mathbb{R} \setminus [\beta_c^-, \beta_c^+]$ is called the *strong disorder regime.* In the weak disorder region, the polymer path is *diffusive* (it was first proved in a more restrained region in [21, 102], then in the whole weak disorder region in [50]), while in the strong disorder regime, it is believed that the polymer path should be *superdiffusive*. Moreover, it was shown that for large enough β, the polymer path localizes [46, 8, 37].

The subregion of the weak disorder, where $W_n \to W_\infty$ in L^2, is called the L^2*-region*. It corresponds to the β-region (see e.g. (9)-(11) in [45]):

$$(\mathbf{L2}) \quad \lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d),$$

where $\pi_d \in (0,1)$ is the probability of return to 0 of the simple random walk:

$$\pi_d = \mathrm{P}(\exists n \geq 1, S_n = 0).$$

Moreover, since $\pi_{d+1} < \pi_d$ for all $d \geq 3$ [133, Lemma 1] and $\pi_3 = 0.3405\ldots$ [150, page 103], condition $(\mathbf{L2})$ is always verified for $|\beta|$ small enough. As the function λ_2 is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$, this implies that

$$(\mathbf{L2}) \Leftrightarrow \beta \in (\beta_2^-, \beta_2^+),$$

where $\beta_2^- = \beta_2^-(d) \in [-\infty, 0)$ and $\beta_2^+ = \beta_2^+(d) \in (0, \infty]$.

Then, again by [21],

$$\mathbb{E}\big[W_\infty^2\big] = \begin{cases} \frac{(1-\pi_d)e^{\lambda_2(\beta)}}{1-\pi_d e^{\lambda_2(\beta)}} & \text{if } \beta \in (\beta_2^-, \beta_2^+), \\ \infty & \text{else.} \end{cases} \tag{2}$$

In the following, we will always assume that $d \geq 3$ and $\beta \in (\beta_2^-, \beta_2^+)$.

1.2 The results

We introduce two additional types of convergences, referring to [45].

Definition 1.1. *Let Y_n be a family of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that Y_n converges to some random variable Y in distribution.*

- *We say that this convergence is stable if for any $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, the law of Y_n under the condition B converges to some probability distribution, which might depend on B.*

- *We say that this convergence is mixing if it is stable and the limit of conditional law is independent of given B. Then this conditional limit is the law of Y itself.*

Theorem 1.2. *For all $\beta \in (\beta_2^-, \beta_2^+)$, as $n \to \infty$,*

$$n^{\frac{d-2}{4}}(W_\infty - W_n) \xrightarrow{\text{law}} \sigma W_\infty G, \tag{3}$$

and

$$n^{\frac{d-2}{4}} \frac{W_\infty - W_n}{W_n} \xrightarrow{\text{law}} \sigma G, \tag{4}$$

where G is a standard centered Gaussian random variable which is independent of W_∞, and $\sigma = \sigma(\beta)$ is defined in (24)*. Moreover, convergence* (3) *is stable and convergence* (4) *is mixing.*

Corollary 1.3. *For all $\beta \in (\beta_2^-, \beta_2^+)$, as $n \to \infty$,*

$$n^{\frac{d-2}{4}}(\log W_\infty - \log W_n) \xrightarrow{\text{law}} \sigma G, \tag{5}$$

where G and σ are as above. Moreover, this convergence is mixing.

The proof of Theorem 4 is given in Section 2.1 and the proof of its corollary can be found in Section 3.6.

1.3 Comments and connections to other models

Our work is an extension of the results of Comets and Liu [45] to the whole L^2-region. Although our proof partially relies on their method, we manage to avoid the 4th-moment computation which is not valid in the whole L^2-region. Instead, we make a natural use of the local limit theorem for polymers [149, 159] and appeal to homogenization via a fine truncation method, rather than brute force moment computation. Moreover, since $\sigma(\beta)$ from (24) blows up at $\beta_2^{\pm}$, we predict that our results are optimal in the sense that another scaling, or other limiting laws should hold for (3)-(5) outside the L^2-region.

In branching process literature, the study of the rate of convergence and the nature of the fluctuations of the tail of caracteristic martingales is a common subject. For the Galton-Watson process, this has been studied in [91, 92]. We note that the rate of convergence is there exponential, while the rate is polynomial in our case. In the model of the branching random walk, the fluctuations of the tail of Biggins' martingale are an active subject of research. In the sub-critical region, it was shown that the fluctuations are of Gaussian nature for small enough inverse temperature parameters [100, 141] and that they become of alpha-stable nature at criticality [121]. What happens close to criticality is still an open question. See also [101] for recent results including complex parameters, for which different types of scaling exponents and both Gaussian and stable laws are exhibited.

In recent works [130, 120, 44, 42], the question of defining the KPZ equation in higher dimension ($d \geq 3$) has been investigated through techniques coming from polymer models. The starting point of these studies is to consider at first the KPZ equation with *mollified* white noise. Then, the goal is to try to find a limit when the mollification is removed (see also [14], where this method was first applied to define the KPZ equation in dimension $d = 1$). Using the intepretation of the mollified solution through the partition function of a polymer, it was shown that for small noise intensity (corresponding to the weak disorder region of the polymer model), the mollified solution converges in law towards the limiting partition function of the polymer. In [44], it was further shown that the difference, between the mollified solution and the rescaled partition function, vanishes at polynomial rate, and that the renormalized difference conveges to a Gaussian in distribution. The result is based on the martingale technique from [45] and is valid in a restrained part of the L^2-region of the polymer. We believe that our method could possibly apply to extend the result to the entire L^2-region of the polymer.

2 Idea of the proof

2.1 A central limit theorem for martingales

As in [45], the main tool to prove Theorem 1.2 is the following theorem:

Theorem 2.1 (Corollary 3.2. in [45])**.** *Let $(M_n)_{n\geq 0}$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with adapted filtration $(\mathcal{F}_n)_{n\geq 0}$, $M_0 = 0$, which is bounded in L^2. Let $D_{k+1} = M_{k+1} - M_k$ for all $k \geq 0$ and let $M_\infty = \lim_{n\to\infty} M_n$ be the a.s. limit of M_n. Also define:*

$$v_n^2 = \mathbb{E}\big[(M_\infty - M_n)^2\big] = \mathbb{E}\sum_{k=n}^{\infty} D_{k+1}^2 \ . \tag{6}$$

Suppose that v_n is always positive and that:

(a) There exists a non-negative and finite random variable V, such that

$$V_n^2 = \frac{1}{v_n^2}\sum_{k=n}^{\infty}\mathbb{E}\left[D_{k+1}^2 \middle| \mathcal{F}_k\right] \xrightarrow{\mathbb{P}} V^2;$$

(b) The following conditional Lindeberg condition holds:

$$\forall \epsilon > 0, \quad \frac{1}{v_n^2}\sum_{k=n}^{\infty}\mathbb{E}\left[D_{k+1}^2 \mathbf{1}_{\{|D_{k+1}|>\epsilon v_n\}} \middle| \mathcal{F}_k\right] \xrightarrow{\mathbb{P}} 0.$$

Then,

$$\frac{M_\infty - M_n}{v_n} \xrightarrow{\text{law}} V\ G, \tag{7}$$

where G is a standard Gaussian random variable which is independent of V. If, additionally, $V \neq 0$ a.s., then

$$\frac{M_\infty - M_n}{V_n} \xrightarrow{\text{law}} G. \tag{8}$$

Moreover, convergence (7) *is stable and convergence* (8) *is mixing.*

To prove Theorem 1.2, we show that condition (a) and (b) hold for $M_n = W_n$ and some suited V. The proof of condition (b) is delayed to Section 3.5. Our main focus will be condition (a): if we let

$$D_{k+1} = W_{k+1} - W_k,$$

then, by estimate (10), condition (a) follows from:

Theorem 2.2. *For all $\beta \in (\beta_2^-, \beta_2^+)$, as $n \to \infty$,*

$$s_n^2 := n^{(d-2)/2} \sum_{k \geq n} \mathbb{E}\big[D_{k+1}^2 | \mathcal{F}_k\big] \xrightarrow{L^1} \sigma^2 W_\infty^2. \tag{9}$$

The structure of the proof for Theorem 2.2 is described in Section 2.2. We now turn to the proof of the main theorem.

Proof of Theorem 1.2. It follows directly from Theorem 2.2 that as $n \to \infty$,

$$v_n^2 := \mathbb{E}\big[(W_\infty - W_n)^2\big] \sim \sigma^2\, \mathbb{E}\big[W_\infty^2\big]\, n^{\frac{d-2}{2}}, \tag{10}$$

where v_n is as in (6) (we also refer the reader to Proposition 2.1 of [45] for a more direct argument). Hence, Theorem 2.2 implies condition (a) with limiting variable V given by

$$V = \mathbb{E}\big[W_\infty^2\big]^{-1/2} W_\infty.$$

Combined with condition (b), Theorem 2.1 implies convergence (7) which in turn gives (3). Then, to get (4) from (8), observe that V_n/V and W_∞/W_n both converge in probability 1, so that by simple multiplication, one can replace V_n by V in (8), and then W_∞ by W_n to obtain (4). ∎

2.2 Structure of the proof of Theorem 2.2

By a standard computation, the summand in (9) satisfies

$$\mathbb{E}\big[D_{k+1}^2 | \mathcal{F}_k\big] = \kappa_2(\beta) \sum_{x \in \mathbb{Z}^d} \mathrm{E}\left[e_k \mathbf{1}_{\{S_{k+1} = x\}}\right]^2, \tag{11}$$

where

$$\kappa_2(\beta) = e^{\lambda_2(\beta)} - 1.$$

In order to study the right-hand side of (11), we appeal to the following theorem:

Theorem 2.3 (Local limit theorem for polymers in the L^2-region [149, 159])**.** *Let $\beta \in (\beta_2^-, \beta_2^+)$ and $\alpha > 0$. For any sequence $(l_k)_{k \geq 0}$, verifying that $l_k \to \infty$ and $l_k = o(k^a)$ for some $a < 1/2$,*

$$\mathrm{E}\left[e_k |\ S_{k+1} = x\right] = W_{l_k} \overleftarrow{W}_{k+1, l_k}^x + \delta_k^x, \tag{12}$$

where $\overleftarrow{W}_{k,l}^y = \mathrm{P}_y \left[\exp\left(\sum_{i=1}^l \beta\omega(k-i, S_i) - l\lambda(\beta)\right)\right]$ is the time-reversed partition function, and where, as $k \to \infty$,

$$\sup_{|x| \leq \alpha\sqrt{k}} \mathbb{E}\left[|\delta_k^x|^2\right] \to 0. \tag{13}$$

Remark 2.4. *Note that we have reformulated the result with endpoint distribution at time $k+1$, for a polymer measure of horizon k, so that the time-reversed partition function $\overleftarrow{W}^x_{k+1,l_k}$ does not take into account the environment at time $k+1$.*

By the local limit theorem for polymers,

$$\begin{aligned} s_n^2 &= \kappa_2(\beta) n^{(d-2)/2} \sum_{k\geq n} \sum_{x\in\mathbb{Z}^d} \mathrm{E}\left[e_k \mathbf{1}_{\{S_{k+1}=x\}}\right]^2 \\ &=: A_n + B_n + C_n + F_n, \end{aligned} \tag{14}$$

where:

$$A_n = \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq \alpha\sqrt{k}} \left(W_{l_k}\, \overleftarrow{W}^x_{k+1,l_k}\right)^2 \mathrm{P}(S_{k+1}=x)^2, \tag{15}$$

and:

$$\begin{aligned} B_n &= 2\kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq \alpha\sqrt{k}} \delta_k^x\ W_{l_k}\, \overleftarrow{W}^x_{k+1,l_k}\, \mathrm{P}(S_{k+1}=x)^2, \\ C_n &= \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq \alpha\sqrt{k}} (\delta_k^x)^2\, \mathrm{P}(S_{k+1}=x)^2, \\ F_n &= \kappa_2(\beta) n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|> \alpha\sqrt{k}} \mathrm{E}\left[e_k \mathbf{1}_{\{S_{k+1}=x\}}\right]^2. \end{aligned}$$

Section 3.2 is dedicated to showing that B_n, C_n and F_n all vanish in L^1 norm. Turning to A_n, we note that $\overleftarrow{W}^x_{k+1,l_k}$ and $\overleftarrow{W}^y_{k+1,l_k}$ are independent whenever $|x-y|_1 > l_k$, so that, by some homogenization argument, we can show that

$$\begin{aligned} A_n &\approx \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq \alpha\sqrt{k}} W^2_{l_k}\, \mathbb{E}\left[\left(\overleftarrow{W}^x_{k+1,l_k}\right)^2\right] \mathrm{P}(S_{k+1}=x)^2 \\ &\to \sigma^2 W^2_\infty, \end{aligned} \tag{16}$$

as $n\to\infty$ and $\alpha\to\infty$ in this order. Approximation (16) is justified in Section 3.3, while, letting $\overline{A}_n$ denote the RHS of (16), convergence in the second line is proved in Proposition 3.11.

3 Proof

Notations

- $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}$ or $\mathbb{R}^d$.
- $|\cdot|_1$ stands for the usual L^1-norm on $\mathbb{R}^d$.
- Let $B(r)$ denote the closed ball of radius r in the Euclidean norm.
- $\mathrm{E}^{\otimes 2}$ and $\mathrm{P}^{\otimes 2}$ will stand for resp. the expectation and the probability measure for two independent simple random walks S and $\widetilde{S}$.
- We write $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_k]$.
- Given two paths S and $\widetilde{S}$, we denote the overlap of S and $\widetilde{S}$, from time m to k, as $N_{m,k} = \sum_{i=m}^k \mathbf{1}_{\{S_i=\tilde{S}_i\}}$. When $m=1$, we simply write N_k. This k can be taken to be infinity.

3.1 Some tools.

Theorem 3.1 (Local central limit theorem for the simple random walk [117])**.** *For all x such that* $\mathrm{P}(S_k = x) \neq 0$*, as* $k \to \infty$*,*

$$\mathrm{P}(S_k = x) = 2\left(\frac{d}{2\pi k}\right)^{d/2} e^{-d\frac{x^2}{2k}} + |x|^{-2}\, O\left(k^{-d/2}\right), \tag{17}$$

$$\mathrm{P}(S_{2k} = 0) \sim 2\left(\frac{d}{4\pi k}\right)^{d/2}, \tag{18}$$

where the big O term is uniform in x.

Proposition 3.2. *There exists $\mathcal{Z}_d > 0$ such that*

$$n^{(d-2)/2}\sum_{k\geq n}\sum_{x\in\mathbb{Z}^d}\mathrm{P}(S_{k+1} = x)^2 \to \mathcal{Z}_d. \tag{19}$$

Proof. For $x \in \mathbb{Z}^d$, let G be the Green function :

$$G(x) = \mathrm{E}_x\left[\sum_{i=0}^{\infty}\mathbf{1}_{S_i=0}\right].$$

Then, by strong Markov property, for any $x \in \mathbb{Z}^d$,

$$\mathrm{P}_x(\exists n \in \mathbb{Z}_{\geq 0},\ S_n = 0) = G(x)/G(0), \tag{20}$$

Since $S_k - \widetilde{S}_k \overset{\text{law}}{=} S_{2k}$ and using again the strong Markov property,

$$\begin{aligned}
&n^{(d-2)/2}\sum_{k\geq n}\sum_{x\in\mathbb{Z}^d}\mathrm{P}(S_{k+1} = x)^2\\
&= n^{(d-2)/2}\mathrm{E}^{\otimes 2}[N_{n+1,\infty}]\\
&= \mathrm{E}^{\otimes 2}\left[n^{(d-2)/2}\mathrm{P}_{S_{n+1}-\tilde{S}_{n+1}}(\exists n \in \mathbb{Z}_{\geq 0},\ S_n = 0)\right]G(0).
\end{aligned}$$

On the other hand, using (20),

$$\mathrm{E}^{\otimes 2}\left[\mathrm{P}_{S_{n+1}-\tilde{S}_{n+1}}(\exists n \in \mathbb{Z}_{\geq 0},\ S_n = 0)\right] = G(0)^{-1}\mathrm{E}^{\otimes 2}[G(S_{n+1} - \tilde{S}_{n+1})].$$

By (23) and (25) in [45], $n^{(d-2)/2}\mathrm{E}^{\otimes 2}[G(S_{n+1} - \tilde{S}_{n+1})]$ converges to a positive constant, which completes the proof. ∎

We will also require the following technical proposition:

Proposition 3.3. *Let v be a non-negative bounded function on $\mathbb{Z}^d$ with $d \geq 3$, such that:*

$$\sup_{x\in\mathbb{Z}^d}\mathrm{E}_x\left[e^{\sum_{k=1}^{\infty}v(S_{2k})}\right] < \infty.$$

Then, there exists $C \in (0,\infty)$, such that for all $n \geq 0$,

$$\mathrm{E}_0\left[e^{\sum_{k=1}^{n}v(S_{2k})}\middle| S_{2(n+1)} = 0\right] \leq C.$$

To prove this proposition, we use an analogue of Lemma 3.3 in [159]:

Lemma 3.4. *Under the assumptions of Proposition 3.3, there exists a constant $C \in (0,\infty)$, such that for all non-negative function f on $\mathbb{Z}^d$,*

$$\sup_{\substack{x\in\mathbb{Z}^d\\ |x|_1:even}}\mathrm{E}_x\left[e^{\sum_{i=1}^{n}v(S_{2i})}f(S_{2n})\right] \leq \frac{C}{n^{d/2}}\sum_{\substack{y\in\mathbb{Z}^d\\ |y|_1:even}}f(y).$$

Proof. We repeat the argument of [159], which is a little simpler in our case. Let $A = \{x \in \mathbb{Z}^d,\ |x|_1 \text{ is even}\}$ be the underlying graph of (S_{2i}). We also let $x \sim y$ denote the fact that x and y are nearest neighbors in A, and $p^{(2)}(x,y)$ be the transition kernel $\mathrm{P}(S_2 = y | S_0 = x)$. For all $x \in A$, define $h(x) = \mathrm{E}_x\left[e^{\sum_{i=1}^{\infty} v(S_{2i})}\right]$; then h satisfies

$$h(x) = \sum_{y \sim x} p^{(2)}(x,y) e^{v(y)} h(y).$$

Hence, similarly to Doob's h-transform, the kernel

$$K(x,y) = \frac{h(y)}{h(x)} e^{v(y)} p^{(2)}(x,y)$$

defines a probability transition kernel of a Markov chain on A, for which $m(x) = h(x)^2 e^{v(x)}$ is a reversible measure.

By assumption, m is uniformly bounded and bounded away from 0, and there exists a constant $c \in (0,\infty)$, such that $K(x,y) \geq c p^{(2)}(x,y)$ for all $x, y \in A$. As by Theorem 4.18 in [162], S_{2n} satisfies the d-isoperimetric inequality (cf. pp. 39-40 therein) on A, this implies that K also satisfies it.

Therefore, we get from Corollary 14.5 in [162] that there exists a finite C, such that

$$\frac{1}{h(x)} \mathrm{E}_x\left[h(S_{2n}) e^{\sum_{i=1}^{n} v(S_{2i})} f(S_{2n})\right] = \sum_{y \in A} K^{(n)}(x,y) f(y) \leq \frac{C}{n^{d/2}} \sum_{y \in A} f(y).$$

This in turn implies the lemma by our assumptions. ■

Proof of Proposition (3.3). Choose $f(y) = \mathbf{1}_{\{y=0\}}$ and $x = 0$ in Lemma 3.4, and conclude using estimate (18) from the local CLT . ■

3.2 Removing the negligeable terms

The following proposition justifies that F_n from Section 2.2 is negligible in L^1-norm.

Proposition 3.5. *We have :*

$$\lim_{\alpha \to \infty} \limsup_{n \to \infty} \mathbb{E}\left[n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \geq \alpha\sqrt{k}} \mathrm{E}\left[e_k \mathbf{1}_{\{S_{k+1}=x\}}\right]^2 \right] = 0.$$

Proof. The finite positive constants C that will arise in the paper may change from line to line, but they will not depend on any varying parameter. We write:

$$\begin{aligned}
&\mathbb{E} n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \geq \alpha\sqrt{k}} \mathrm{E}\left[e_k \mathbf{1}_{\{S_{k+1}=x\}}\right]^2 \\
&= n^{(d-2)/2} \sum_{k \geq n} \mathrm{E}^{\otimes 2}\left[e^{\lambda_2 N_k} \mathbf{1}_{S_{k+1} = \widetilde{S}_{k+1}} \mathbf{1}_{|S_{k+1}| \geq \alpha\sqrt{k}}\right] \\
&= n^{(d-2)/2} \sum_{k \geq n} \mathrm{E}^{\otimes 2}\left[e^{\lambda_2 N_k} \mathbf{1}_{|S_{k+1}| \geq \alpha\sqrt{k}} \,\middle|\, S_{k+1} = \widetilde{S}_{k+1}\right] \mathrm{P}(S_{k+1} = \widetilde{S}_{k+1}).
\end{aligned}$$

As $\mathrm{E}^{\otimes 2}[e^{\lambda_2 N_\infty}] = \mathbb{E}[W_\infty^2]$ is given by the RHS of (2), we can apply Hölder's inequality, for $p^{-1} + q^{-1} = 1$ with the only constraint that $p\lambda_2(\beta) < \log(1/\pi_d)$, and use Lemma 3.4 (note that $S_k - \widetilde{S}_k \overset{\text{law}}{=} S_{2k}$), to get that

$$\mathrm{E}^{\otimes 2}\left[e^{\lambda_2 N_k} \mathbf{1}_{|S_{k+1}| \geq \alpha\sqrt{k}} \middle| S_{k+1} = \widetilde{S}_{k+1}\right] \leq C \mathrm{E}^{\otimes 2}\left[\mathbf{1}_{|S_{k+1}| \geq \alpha\sqrt{k}} \middle| S_{k+1} = \widetilde{S}_{k+1}\right]^{1/q}$$

Then, by the local central limit theorem (Theorem 3.1), there exists some positive constants C, such that for large enough n,

$$\begin{aligned}\mathrm{E}^{\otimes 2}\left[\mathbf{1}_{\{|S_{k+1}|\geq\alpha\sqrt{k}\}}\middle| S_{k+1}=\widetilde{S}_{k+1}\right] &\leq \sum_{|x|\geq\alpha\sqrt{k}}\frac{\mathrm{P}(S_{k+1}=x)^2}{\mathrm{P}^{\otimes 2}(S_{k+1}=\widetilde{S}_{k+1})}\\ &\leq \frac{\max_{|x|\geq\alpha\sqrt{k}}\mathrm{P}(|S_{k+1}|=x)}{\mathrm{P}^{\otimes 2}(S_{k+1}=\widetilde{S}_{k+1})}\\ &\leq C\alpha^{-2}.\end{aligned}$$

It follows from the local CLT that

$$\begin{aligned}&\mathbb{E}n^{(d-2)/2}\sum_{k\geq n}\sum_{|x|\geq\alpha\sqrt{k}}\mathrm{E}\left[e_k\mathbf{1}_{\{S_{k+1}=x\}}\right]^2\\ &\leq C\alpha^{-2/q}\,n^{(d-2)/2}\sum_{k\geq n}\mathrm{P}(S_{k+1}=\widetilde{S}_{k+1})\\ &= C\alpha^{-2/q}\,n^{(d-2)/2}\sum_{k\geq n}\mathrm{P}(S_{2(k+1)}=0)\\ &\leq C\alpha^{-2/q},\end{aligned}$$

which vanishes as $\alpha\to\infty$. ■

Proposition 3.6. *As $n\to\infty$,*

$$B_n = 2\kappa_2(\beta)\,n^{(d-2)/2}\sum_{k\geq n}\sum_{|x|\leq\alpha\sqrt{k}}\delta_k^x\,W_{l_k}\,\overleftarrow{W}_{k+1,l_k}^x\mathrm{P}(S_{k+1}=x)^2\xrightarrow{L^1}0.$$

$$C_n = \kappa_2(\beta)n^{(d-2)/2}\sum_{k\geq n}\sum_{|x|\leq\alpha\sqrt{k}}(\delta_k^x)^2\,\mathrm{P}(S_{k+1}=x)^2\xrightarrow{L^1}0.$$

Proof. From independence of W_{l_k} and $\overleftarrow{W}_{k+1,l_k}^x$, we get from Cauchy-Schwarz inequality:

$$\mathbb{E}\left[\left|\delta_{l_k}^x\,W_{l_k}\,\overleftarrow{W}_{k+1,l_k}^x\right|\right]\leq\mathbb{E}\left[W_{l_k}^2\right]\mathbb{E}\left[(\delta_k^x)^2\right]^{1/2}.$$

The first term of the right-hand side is bounded in the L^2-region, so the convergence for B_n follows simply from (13) and (19). C_n is treated in the same way. ■

3.3 The homogenization result

This section is dedicated to proving the next proposition, which justifies approximation (16):

Proposition 3.7. *Let $\overline{A}_n$ denote the right-hand side of equation* (16). *Then, for any $\alpha>0$,*

$$\lim_{n\to\infty}\mathbb{E}|A_n-\overline{A}_n|=0.$$

To show the result of the proposition, it is enough to prove that, as $k\to\infty$,

$$M_k := k^{d/2}\sum_{|x|\leq\alpha\sqrt{k}}(Y_{k,x}-\mathbb{E}Y_{k,x})\mathrm{P}(S_{k+1}=x)^2\xrightarrow{L^1}0, \tag{21}$$

where $Y_{k,x} = (\overleftarrow{W}^x_{k+1,l_k})^2$. Indeed, for large k, we have $l_k < k/2$ and so W_{l_k} and $\overleftarrow{W}^x_{k+1,l_k}$ are independent. Hence:

$$\begin{aligned}
&\mathbb{E}\left|A_n - \overline{A}_n\right| \\
&\leq \kappa_2\, n^{(d-2)/2} \sum_{k\geq n} \mathbb{E}\left[W_{l_k}^2\right] \mathbb{E}\left[\left|\sum_{|x|\leq\alpha\sqrt{k}} (Y_{k,x} - \mathbb{E}Y_{k,x}) \mathrm{P}(S_{k+1} = x)^2\right|\right] \\
&\leq \kappa_2\, \mathbb{E}[W_\infty^2]\, n^{(d-2)/2} \sum_{k\geq n} k^{-d/2}\, \mathbb{E}[|M_k|],
\end{aligned}$$

where the last term vanishes, as $n \to \infty$, if one assumes convergence (21). To prove this convergence, we rely on a truncation technique:

Lemma 3.8. *Let $\tilde{Y}_{k,x} = Y_{k,x} \wedge (k^{d/2}\, l_k^{-d})$. We have,*

$$\lim_{k\to\infty} \mathbb{E}\left(k^{d/2} \sum_{|x|\leq\alpha\sqrt{k}} \left(\tilde{Y}_{k,x} - \mathbb{E}\tilde{Y}_{k,x}\right) \mathrm{P}(S_{k+1} = x)^2\right)^2 = 0.$$

Proof. Since $Y_{k,x}$ and $Y_{k,y}$ are independent for $|x-y|_1 \geq l_k$,

$$\begin{aligned}
&\mathbb{E}\left(k^{d/2} \sum_{|x|\leq\alpha\sqrt{k}} (\tilde{Y}_{k,x} - \mathbb{E}\tilde{Y}_{k,x}) \mathrm{P}(S_{k+1} = x)^2\right)^2 \\
&= k^d \sum_{|x|,|y|\leq\alpha\sqrt{k}} \mathbb{E}\left(\tilde{Y}_{k,x} - \mathbb{E}\tilde{Y}_{k,x}\right)\left(\tilde{Y}_{k,y} - \mathbb{E}\tilde{Y}_{k,y}\right) \mathrm{P}(S_{k+1} = x)^2 \mathrm{P}(S_{k+1} = y)^2 \\
&\leq Ck^{-d} \sum_{|x-y|_1\leq l_k, |x|\leq\alpha\sqrt{k}} \mathbb{E}\left(\tilde{Y}_{k,x} - \mathbb{E}\tilde{Y}_{k,x}\right)\left(\tilde{Y}_{k,y} - \mathbb{E}\tilde{Y}_{k,y}\right) \\
&\leq Ck^{-d} \sum_{|x-y|_1\leq l_k, |x|\leq\alpha\sqrt{k}} \mathbb{E}\left(\tilde{Y}_{k,0} - \mathbb{E}\tilde{Y}_{k,0}\right)^2,
\end{aligned}$$

where we have used the local central limit theorem in the first inequality; we used the Cauchy-Schwarz inequality and the fact that $\tilde{Y}_{k,x}$ are identically distributed with respect to x in the last one. This is further bounded from above by

$$\begin{aligned}
Ck^{-d/2}\, l_k^d\, \mathbb{E}\left(\tilde{Y}_{k,0} - \mathbb{E}\tilde{Y}_{k,0}\right)^2 &\leq Ck^{-d/2}\, l_k^d\, \mathbb{E}\tilde{Y}_{k,0}^2 \\
&\to 0,
\end{aligned}$$

as $k \to \infty$, where, observing that the family $(Y_{k,0})_k$ is uniformly integrable since W_k^2 converges in L^1, the convergence in the second line is justified by the following lemma. ■

Lemma 3.9. *Let $(X_k)_{k\in\mathbb{N}}$ be a non-negative, uniformly integrable family of random variables. Then, for any sequence $a_k \to \infty$, $a_k^{-1}\mathbb{E}\left[(X_k \wedge a_k)^2\right] \to 0$.*

Proof. By property: $x\mathbb{P}(X_k \geq x) \leq \mathbb{E}[X_k \mathbf{1}_{\{X_k\geq x\}}]$, we have

$$\begin{aligned}
a_k^{-1}\mathbb{E}\left[(X_k \wedge a_k)^2\right] &= a_k^{-1} \int_0^{a_k} 2x\mathbb{P}(X_k \geq x)\, dx \\
&\leq 2a_k^{-1} \int_0^{a_k} \sup_{k\in\mathbb{N}} \mathbb{E}[X_k \mathbf{1}_{\{X_k\geq x\}}] dx \to 0,
\end{aligned}$$

as $k \to \infty$, since $\sup_k \mathbb{E}[X_k \mathbf{1}_{\{X_k\geq x\}}] \to 0$ as $x \to \infty$, by uniform integrability. ■

The next lemma will be used in order to remove the truncation:

Lemma 3.10. *We have:*

$$k^{d/2} \sum_{|x| \leq \alpha\sqrt{k}} \left(Y_{k,x} - \tilde{Y}_{k,x} \right) \mathrm{P}(S_{k+1} = x)^2 \xrightarrow{L^1} 0. \tag{22}$$

Moreover,

$$\lim_{k\to\infty} k^{d/2} \sum_{|x| \leq \alpha\sqrt{k}} \left(\mathbb{E}Y_{k,x} - \mathbb{E}\tilde{Y}_{k,x} \right) \mathrm{P}(S_{k+1} = x)^2 = 0. \tag{23}$$

Proof. Note that

$$\mathbb{E}[|Y_{k,0} - \tilde{Y}_{k,0}|] \leq \mathbb{E}\left[Y_{k,0};\; Y_{k,0} > k^{d/2} l_k^{-d} \right] \to 0.$$

Thus, combining with the local CLT, we safely get (22) and (23). ■

Finally, putting together Lemma 3.8 and Lemma 3.10, we get that $M_k \xrightarrow{L^1} 0$, as desired.

3.4 Proof of Theorem 2.2

Combined to propositions of the two last sections, the following theorem entails Theorem 2.2:

Proposition 3.11. *With σ^2 defined as in* (24)*,*

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \mathbb{E}\left|\overline{A}_n - \sigma^2 W_\infty^2\right| = 0.$$

Proof. Note first that

$$\begin{aligned}
&\mathbb{E}\left| \overline{A}_n - \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq\alpha\sqrt{k}} W_\infty^2\, \mathbb{E}\left[\left(\overleftarrow{W}^x_{k+1,l_k} \right)^2 \right] \mathrm{P}(S_{k+1} = x)^2 \right| \\
&\leq C \sup_{k\geq n} \mathbb{E}|W_\infty^2 - W_{l_k}^2| \to 0,
\end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned}
&\mathbb{E}\left| \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|\leq\alpha\sqrt{k}} W_\infty^2 \left(\mathbb{E}\left[\left(\overleftarrow{W}^x_{k+1,l_k} \right)^2 \right] - \mathbb{E}W_\infty^2 \right) \mathrm{P}(S_{k+1} = x)^2 \right| \\
&\leq C \sup_{k\geq n} \mathbb{E}|W_\infty^2 - W_{l_k}^2| \to 0.
\end{aligned}$$

Moreover, by Proposition 3.5, we have

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \mathbb{E}\left| \kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{|x|>\alpha\sqrt{k}} W_\infty^2\, \mathbb{E}W_\infty^2 \mathrm{P}(S_{k+1} = x)^2 \right| = 0.$$

Therefore, it suffices to show that as $n \to \infty$,

$$\kappa_2(\beta)\, n^{(d-2)/2} \sum_{k\geq n} \sum_{x\in\mathbb{Z}^d} \mathbb{E}W_\infty^2 \mathrm{P}(S_{k+1} = x)^2 \to \sigma^2(\beta),$$

where

$$\sigma^2(\beta) = \frac{(1-\pi_d)(e^{\lambda_2(\beta)} - 1)}{1 - \pi_d e^{\lambda_2(\beta)}} \mathcal{Z}_d. \tag{24}$$

Recalling $\kappa_2(\beta) = e^{\lambda_2(\beta)} - 1$ and $\mathbb{E}W_\infty^2 = \frac{1-\pi_d}{1-\pi_d e^{\lambda_2(\beta)}}$ (cf. (2)), this follows from convergence (19). ■

3.5 Proof of condition (b): the Lindeberg condition

Given the asymptotics of v_n in (10), condition (b) of Theorem 2.1 follows from the following proposition:

Proposition 3.12 (Lindeberg condition)**.** *For any $\epsilon > 0$,*

$$n^{(d-2)/2} \sum_{k \geq n} \mathbb{E}_k \left[D_{k+1}^2 \mathbf{1}_{\{n^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] \xrightarrow{L^1} 0.$$

Proof. We first observe that it is enough to prove that

$$\lim_{k \to \infty} k^{d/2} \mathbb{E} \left[D_{k+1}^2 \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] = 0. \tag{25}$$

Indeed, since $n^{\frac{d-2}{4}} |D_{k+1}| > \epsilon$ implies $k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon$ for $k \geq n$, we would have, assuming (25),

$$\begin{aligned}
&\limsup_{n \to \infty} \mathbb{E} \left[n^{\frac{d-2}{2}} \sum_{k \geq n} \mathbb{E}_k \left(D_{k+1}^2 \mathbf{1}_{\{n^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right) \right] \\
&\leq \limsup_{n \to \infty} n^{\frac{d-2}{2}} \sum_{k \geq n} k^{-d/2} \, \mathbb{E} \left[k^{d/2} D_{k+1}^2 \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] \\
&\leq C \limsup_{k \to \infty} k^{d/2} \, \mathbb{E} \left[D_{k+1}^2 \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] = 0,
\end{aligned}$$

where the third inequality comes from the boundedness of $n^{\frac{d-2}{2}} \sum_{k \geq n} k^{-d/2}$. We now focus on showing (25).

Using $S_k - \widetilde{S}_k \overset{\text{law}}{=} S_{2k}$, we may write

$$\begin{aligned}
\mathbb{E} D_{k+1}^2 &= \kappa_2 \mathrm{E}^{\otimes 2} \left[e^{\lambda_2 N_k} \mathbf{1}_{\{S_{k+1} = \tilde{S}_{k+1}\}} \right] \\
&= \kappa_2 \, \mathrm{E}^{\otimes 2} \left[e^{\lambda_2 N_k} | \; S_{k+1} = \tilde{S}_{k+1} \right] \mathrm{P}^{\otimes 2}(S_{k+1} = \tilde{S}_{k+1}) \\
&= \kappa_2 \, \mathrm{E} \left[e^{\lambda_2 \sum_{i=1}^k \mathbf{1}_{\{S_{2i} = 0\}}} | \; S_{2(k+1)} = 0 \right] \mathrm{P}(S_{2(k+1)} = 0)
\end{aligned}$$

By Proposition 3.3 and the local CLT, we have $k^{d/2} \mathbb{E} D_{k+1}^2 = O(1)$. Thus, applying Markov's inequality, we get

$$\begin{aligned}
\mathbb{P} \left(k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon \right) &= \mathbb{P} \left(k^{\frac{d}{2}} D_{k+1}^2 > \epsilon^2 k \right) \\
&\leq \frac{1}{\epsilon^2 k} k^{\frac{d}{2}} \mathbb{E} D_{k+1}^2 \to 0,
\end{aligned} \tag{26}$$

as $k \to \infty$.

In order to prove (25), we will rely on estimate (26) and uniform integrability properties. We will need the following simple lemma.

Lemma 3.13. *Let $\{X_n\}, \{Y_n\}$ be independent uniformly integrable families of random variables. Then $\{X_n Y_n\}$ is also uniformly integrable.*

Proof. Let us note that $|X_n Y_n| \geq t$, then $|X_n| \geq \sqrt{t}$ or $|Y_n| \geq \sqrt{t}$. Thus,

$$\begin{aligned}
\mathbb{E}[|X_n Y_n|; \; |X_n Y_n| \geq t] &\leq \mathbb{E}[|X_n Y_n|; \; |X_n| \geq \sqrt{t}] + \mathbb{E}[|X_n Y_n|; \; |Y_n| \geq \sqrt{t}] \\
&= \mathbb{E}[|Y_n|] \mathbb{E}[|X_n|; \; |X_n| \geq \sqrt{t}] + \mathbb{E}[|X_n|] \mathbb{E}[|Y_n|; \; |Y_n| \geq \sqrt{t}],
\end{aligned}$$

which uniformly goes to 0 as $t \to \infty$. ■

For all $k \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we write $\eta_k(x) = e^{\beta\omega(k+1,x)-\lambda(\beta)} - 1$. Note that

$$\mathbb{E}\eta_k(x) = 0 \text{ and } \mathbb{E}\eta_k(x)^2 = \kappa_2(\beta).$$

We decompose,

$$\begin{aligned}
D_{k+1} &= W_{k+1} - W_k \\
&= \sum_{|x|\le\alpha\sqrt{k}} \mathrm{E}[e_k \mathbf{1}_{\{S_{k+1}=x\}}]\eta_k(x) \\
&\quad + \sum_{|x|>\alpha\sqrt{k}} \mathrm{E}[e_k \mathbf{1}_{\{S_{k+1}=x\}}]\eta_k(x),
\end{aligned}$$

and first observe that by proposition 3.5, we have

$$\lim_{\alpha\to\infty} \limsup_{k\to\infty} k^{d/2}\mathbb{E}\left[\left(\sum_{|x|>\alpha\sqrt{k}} \mathrm{E}[e_k \mathbf{1}_{\{S_{k+1}=x\}}]\eta_k(x)\right)^2\right] = 0.$$

Then, if we let $(l_k)_k$ be any positive sequence satisfying the conditions of Proposition 2.3, we can write

$$\begin{aligned}
&\sum_{|x|\le\alpha\sqrt{k}} \mathrm{E}[e_k \mathbf{1}_{\{S_{k+1}=x\}}]\eta_k(x) \\
&= \sum_{|x|\le\alpha\sqrt{k}} W_{l_k} \overleftarrow{W}^x_{k+1,l_k} \mathrm{P}(S_{k+1}=x)\eta_k(x) \\
&\quad + \sum_{|x|\le\alpha\sqrt{k}} \delta_{k,x} \mathrm{P}(S_{k+1}=x)\eta_k(x).
\end{aligned}$$

For the second term of the right hand side, it is easy to check as in Proposition 3.6 that

$$\lim_{k\to\infty} k^{d/2}\mathbb{E}\left(\sum_{|x|\le\alpha\sqrt{k}} \delta_{k,x}\mathrm{P}(S_{k+1}=x)\eta_k(x)\right)^2 = 0.$$

For the first term, denoting by $B(r)$ the closed ball of $\mathbb{R}^d$ of radius r, we compute,

$$\begin{aligned}
&k^{d/2}\left(\sum_{|x|\le\alpha\sqrt{k}} W_{l_k}\overleftarrow{W}^x_{k+1,l_k}\mathrm{P}(S_{k+1}=x)\eta_k(x)\right)^2 \\
&= k^{d/2} \sum_{x,y\in B(\alpha\sqrt{k})} W_{l_k}^2 \overleftarrow{W}^x_{k+1,l_k}\overleftarrow{W}^y_{k+1,l_k}\eta_k(x)\eta_k(y)\mathrm{P}(S_{k+1}=x)\mathrm{P}(S_{k+1}=y) \\
&= k^{d/2} \sum_{\substack{x,y\in B(\alpha\sqrt{k}) \\ |x-y|_1\le l_k}} W_{l_k}^2 \overleftarrow{W}^x_{k+1,l_k}\overleftarrow{W}^y_{k+1,l_k}\eta_k(x)\eta_k(y)\mathrm{P}(S_{k+1}=x)\mathrm{P}(S_{k+1}=y) \\
&\quad + W_{l_k}^2 k^{d/2} \sum_{\substack{x,y\in B(\alpha\sqrt{k}) \\ |x-y|_1> l_k}} \overleftarrow{W}^x_{k+1,l_k}\overleftarrow{W}^y_{k+1,l_k}\eta_k(x)\eta_k(y)\mathrm{P}(S_{k+1}=x)\mathrm{P}(S_{k+1}=y) \\
&=: \mathcal{D}_k^{(1)} + W_{l_k}^2\mathcal{D}_k^{(2)}. \qquad (27)
\end{aligned}$$

By Cauchy-Schwarz inequality and Theorem 3.1,

$$\left|\mathcal{D}_k^{(1)}\right| \le Ck^{-d/2} l_k^d \sum_{x\in B(\alpha\sqrt{k})} W_{l_k}^2 \left(\overleftarrow{W}^x_{k+1,l_k}\right)^2 \eta_k(x)^2.$$

By (26), Lemma 3.13 and uniform integrability of W_n^2 (note that W_n^2 converges in L^1), we get that as $k \to \infty$,

$$\begin{aligned} a_k := &\sup_{0 \leq m < k/2} \sup_{x \in \mathbb{Z}^d} \mathbb{E}\left[W_m^2 \left(\overleftarrow{W}_{k+1,m}^x \right)^2 \eta_k(x)^2 \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] \\ &\to 0, \end{aligned}$$

where, in order to use Lemma 3.13, we have restricted the supremum to $m < k/2$, so that W_m and $\overleftarrow{W}_{k+1,m}^x$ are then independent from each other, and, by definition, independent of $\eta_k(x)$. Then, we choose and fix a specific $(l_k)_k$, which satisfies both $l_k^d \, a_k \to 0$ and the conditions of Proposition 2.3 (and hence $l_k < k/2$ for large k). Thereby, as $k \to \infty$,

$$\mathbb{E}\left[k^{-d/2} l_k^d \sum_{x \in B(\alpha\sqrt{k})} W_{l_k}^2 \left(\overleftarrow{W}_{k+1,l_k}^x \right)^2 \eta_k(x)^2 \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] \leq C \, l_k^d \, a_k \to 0.$$

As a consequence, we have

$$\lim_{k \to \infty} \mathbb{E}\left[\mathcal{D}_k^{(1)} \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] = 0.$$

Finally, note that

$$\begin{aligned} &\mathbb{E}\left[\left(\mathcal{D}_k^{(2)} \right)^2 \right] \\ &= k^d \sum_{\substack{x,y \in B(\alpha\sqrt{k}) \\ |x-y|_1 > l_k}} \sum_{\substack{z,w \in B(\alpha\sqrt{k}) \\ |z-w|_1 > l_k}} \mathbb{E}\left[\prod_{u \in \{x,y,z,w\}} \overleftarrow{W}_{k+1,l_k}^u \eta_k(u) \right] \mathrm{P}(S_{k+1} = u), \end{aligned}$$

where, by independence of $\eta_k(u)$ and $\overleftarrow{W}_{k+1,l_k}^u$, each term inside the sum vanishes unless either $x = z$, $y = w$ or $x = w$, $y = z$. Hence, by Theorem 3.1,

$$\begin{aligned} \mathbb{E}\left[\left(\mathcal{D}_k^{(2)} \right)^2 \right] &\leq C k^{-d} \sum_{\substack{x,y \in B(\alpha\sqrt{k}) \\ |x-y|_1 > l_k}} \mathbb{E}\left[\left(\overleftarrow{W}_{k+1,l_k}^x \right)^2 \left(\overleftarrow{W}_{k+1,l_k}^y \right)^2 \eta_k(x)^2 \eta_k(y)^2 \right] \\ &\leq C k^{-d} \sum_{x,y \in B(\alpha\sqrt{k})} \mathbb{E}\left[\left(\overleftarrow{W}_{k+1,l_k}^0 \right)^2 \right]^2 \mathbb{E}\left[\eta_k(0)^2 \right]^2 \\ &= O(1), \end{aligned}$$

where the second inequality comes from the independence of $\overleftarrow{W}_{k+1,l_k}^x$, $\overleftarrow{W}_{k+1,l_k}^y$ whenever $|x - y| > l_k$. Therefore, $\mathcal{D}_k^{(2)}$ is uniformly integrable, so by independence of $W_{l_k}^2$ and $\mathcal{D}_k^{(2)}$ and Lemma 3.13,

$$\lim_{k \to \infty} \mathbb{E}\left[W_{l_k}^2 \mathcal{D}_k^{(2)} \mathbf{1}_{\{k^{\frac{d-2}{4}} |D_{k+1}| > \epsilon\}} \right] = 0.$$

Putting things together, we have shown (25).

■

3.6 Proof of Corollary 1.3

Proof. We write

$$\log W_\infty - \log W_n = \log\left(1 + \frac{W_\infty - W_n}{W_n} \right).$$

By Taylor expansion, there exists a constant $M > 0$, such that for all $|x| < 1/2$, we have

$$|\log(1+x) - x| \le Mx^2. \tag{28}$$

Then, we write $X_n = \frac{W_\infty - W_n}{W_n}$, so that by Theorem 1.2, $n^{\frac{d-2}{4}} X_n \xrightarrow{\text{law}} \sigma G$ and this convergence is mixing. In particular $X_n \xrightarrow{\mathbb{P}} 0$.

By the inequality in (28), we have

$$\mathbb{P}\left(n^{\frac{d-2}{4}} |\log(1+X_n) - X_n| > \epsilon;\ |X_n| < 1/2\right) \le \mathbb{P}\left(Mn^{\frac{d-2}{4}} |X_n|^2 > \epsilon\right),$$

which vanishes as $n \to \infty$. Moreover, $\mathbb{P}(|X_n| \ge 1/2) \to 0$, so that

$$n^{\frac{d-2}{4}} (\log(1+X_n) - X_n) \xrightarrow{\mathbb{P}} 0.$$

Lemma 3.14. *Suppose that $Y_n \xrightarrow{\text{law}} Y$ and $Z_n \xrightarrow{\mathbb{P}} 0$, where Y has a continuous cumulative distribution function. Then*

$$Y_n + Z_n \xrightarrow{\text{law}} Y. \tag{29}$$

Moreover, if, in addition, the convergence $Y_n \xrightarrow{\text{law}} Y$ is mixing, the convergence (29) *is also mixing.*

Proof. Let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space. Recall that the property that $Y_n \xrightarrow{\text{law}} Y$ is mixing is equivalent to that for any $x \in \mathbb{R}$ and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$,

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le x;\ B) = \mathbb{P}(Y \le x)\mathbb{P}(B).$$

We fix $x \in \mathbb{R}$ and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. For any $\epsilon > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}(Y_n + Z_n \le x;\ B) &\le \lim_{n\to\infty} \mathbb{P}(Y_n \le x+\epsilon;\ B) + \lim_{n\to\infty} \mathbb{P}(Z_n < -\epsilon)\\ &= \mathbb{P}(Y \le x+\epsilon)\mathbb{P}(B).\end{aligned}$$

Letting $\epsilon \downarrow 0$, since Y has a continuous cumulative distribution function, we have

$$\limsup_{n\to\infty} \mathbb{P}(Y_n + Z_n \le x;\ B) \le \mathbb{P}(Y \le x)\mathbb{P}(B).$$

Conversely, for any $\epsilon > 0$,

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}(Y_n + Z_n \le x;\ B) &\ge \liminf_{n\to\infty} \mathbb{P}(Y_n + Z_n \le x;\ Z_n \le \epsilon;\ B)\\ &\ge \lim_{n\to\infty} \mathbb{P}(Y_n \le x-\epsilon;\ B) - \lim_{n\to\infty} \mathbb{P}(Z_n > \epsilon)\\ &\ge \mathbb{P}(Y \le x-\epsilon)\mathbb{P}(B).\end{aligned}$$

Similarly, we have

$$\liminf_{n\to\infty} \mathbb{P}(Y_n + Z_n \le x;\ B) \ge \mathbb{P}(Y \le x)\mathbb{P}(B).$$

■

Using this lemma, we get

$$n^{\frac{d-2}{4}} \log\left(1 + \frac{W_\infty - W_n}{W_n}\right) \xrightarrow{\text{law}} \sigma G,$$

and this convergence is mixing. ■

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