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Quasi-stationnarité avec frontières mobiles

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[1] Ouin !

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Introduction

La présente thèse se propose d'étudier des processus de Markov généraux absorbés par une zone de l'espace d'état qui bouge au cours du temps. Nous nous intéresserons plus particulièrement aux comportements asymptotiques de tels processus. Une notion naturelle à considérer alors est celle de *quasi-stationnarité*. Cette introduction est donc décomposée en trois parties, la première d'entre elles étant une présentation de la théorie de la quasi-stationnarité au sens classique, c'est-à-dire lorsque le bord absorbant est immobile, la seconde introduit le cas à frontière mobile, la troisième présente plus en détails le contenu de la thèse.

0.1 Introduction à la quasi-stationnarité

0.1.1 Distributions quasi-stationnaires

Considérons $(\Omega, (\mathcal{F}_t)_{t\in I}, (X_t)_{t\in I}, (\mathbb{P}_x)_{x\in E\cup\{\partial\}})$ un processus de Markov homogène défini sur l'espace d'état $E \cup \{\partial\}$ où $(E, \mathcal{E})$ est un espace mesurable et $\partial \notin E$ est un élément absorbant de X, c'est-à-dire

$$X_t = \partial, \quad \forall t \geq \tau_\partial,$$

où τ_∂ est le temps d'atteinte de $\{\partial\}$:

$$\tau_\partial := \inf\{t \in I : X_t = \partial\}.$$

I représente l'espace de temps sur lequel X est indexé tel que $0 \in I$ (typiquement $\mathbb{N}$ ou $\mathbb{R}_+$). Pour toute mesure de probabilité μ sur $E \cup \{\partial\}$, on définit $\mathbb{P}_\mu := \int_{E\cup\{\partial\}} \mu(dx)\mathbb{P}_x$ de telle manière que, pour toute loi μ, $\mathbb{P}_\mu(X_0 \in \cdot) = \mu$.

Ces processus absorbés suscitent un grand intérêt dans de nombreux domaines. Par exemple, ils apparaissent naturellement lorsqu'on s'intéresse aux dynamiques de populations. On peut penser alors au *processus de naissance et de mort* qui représente l'évolution d'une taille de population au cours du temps (un processus à valeurs entières qui croît ou décroît en fonction des naissances et des morts se réalisant de façon aléatoire), ou encore au *processus de Galton-Watson* qui décrit l'évolution du nombre d'individus par génération. Pour ces deux processus, $E = \mathbb{N}^*$ et $\partial = 0$ puisque $X_t = 0$ signifie que, au temps t, il n'y a plus d'individus pour procréer. Un autre exemple concret sera traité plus loin dans l'introduction (sous-section 0.2.2).

De manière générale, lorsque l'on veut étudier un processus de Markov, il est assez habituel d'étudier son *ergodicité*, c'est-à-dire son comportement asymptotique. Il est bien connu que si le processus $(X_t)_{t\in I}$ converge en loi vers une mesure de probabilité π, alors π est une *mesure stationnaire* (ou invariante), c'est-à-dire une mesure vérifiant

$$\mathbb{P}_\pi(X_t \in \cdot) = \pi, \quad \forall t \in I. \tag{1}$$

Ainsi, une manière d'accéder au comportement asymptotique d'un processus de Markov est de travailler sur les mesures stationnaires qui, contrairement aux mesures limites, ne sont pas définies à partir d'une propriété asymptotique du processus. L'ensemble des mesures stationnaires d'un processus sera alors l'ensemble des limites possibles. La raison de cette propriété est la *propriété de Markov* : en effet, notant $\mu_t = \mathbb{P}_\mu(X_t \in \cdot)$, la propriété de Markov implique que, pour tout $s, t \in I$,

$$\mu_{t+s} = \mathbb{P}_{\mu_s}(X_t \in \cdot). \tag{2}$$

Ainsi, on constate naturellement que, en faisant tendre s vers l'infini et en invoquant la continuité de $\mu \to \mathbb{P}_\mu(X_t \in \cdot)$ pour la convergence étroite, la limite de $(\mu_t)_{t\in I}$ (si elle existe) doit satisfaire la relation de point fixe (1).

Dans notre cas, nous allons faire l'hypothèse que, quel que soit le point de départ $x \in E$, le processus X atteint $\mathbb{P}_x$-presque sûrement ∂ en un temps fini:

$$\mathbb{P}_x(\tau_\partial < \infty) = 1, \quad \forall x \in E. \tag{3}$$

En l'occurence, la limite de la loi de X_t est vite trouvée puisqu'il s'agit de la mesure de Dirac δ_∂. C'est par ailleurs l'unique mesure de probabilité stationnaire de X, car on rappelle qu'une mesure stationnaire ne charge que les points récurrents.

Cette mesure de Dirac limite ne présente, pour ainsi dire, que peu d'intérêt en pratique. Si on reprend l'exemple d'un processus de naissance et de mort satisfaisant la condition d'absorption (3), il existe d'autres phénomènes asymptotiques beaucoup plus pertinents à étudier lorsqu'on s'intéresse à l'extinction d'une population. C'est le cas, par exemple, du *phénomène de plateau de mortalité*: il s'agit d'étudier la probabilité que la population s'éteigne dans une tranche de temps $[t, t + h]$, sachant qu'elle n'était pas éteinte jusque-là. Le phénomène de plateau se réalise alors lorsque cette probabilité conditionnelle se stabilise autour d'une valeur limite quand la tranche de temps se rapproche de l'infini. Nous renvoyons le lecteur à l'introduction de thèse de D. Villemonais [55] où plusieurs références sont données concernant l'étude du taux de mortalité d'une population.

Pour rendre compte mathématiquement de ces phénomènes de plateau, l'étude des lois marginales du processus ne nous est d'aucune utilité. L'idée est alors de travailler, non pas sur la loi de X_t, mais sur la loi de X_t conditionné à ne pas avoir été absorbé au temps t, puis de travailler sur le comportement asymptotique de cette loi. Notamment, si la loi conditionnelle converge, la limite sera appelée *distribution quasi-limite (QLD)* :

Definition 1. a) Une *distribution quasi-limite* est une distribution α à support dans E telle qu'il existe une loi initiale μ telle que

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t) = \alpha. \tag{4}$$

où la convergence est à comprendre au sens de la convergence étroite des mesures.

b) *Le domaine d'attraction* d'une QLD α, note $\mathcal{D}(\alpha)$, est l'ensemble des lois initiales pour lesquelles on a la convergence (4).

c) On dit que la QLD α est *universelle* lorsque son domaine d'attraction est l'espace des mesures de probabilité définies sur E.

Lorsque la convergence (4) a lieu pour $\mu = \delta_x$ pour tout $x \in E$, on parle alors de *limite de Yaglom*[2] :

Definition 2. α est une *limite de Yaglom* si, pour tout $x \in E$,

$$\mathbb{P}_x(X_t \in \cdot | \tau_\partial > t) \xrightarrow[t\to\infty]{\mathcal{L}} \alpha.$$

Autrement dit, α est une limite de Yaglom si

$$\{\delta_x, x \in E\} \subset \mathcal{D}(\alpha).$$

L'existence d'une distribution quasi-limite semble, de manière générale, plus difficile à démontrer que la convergence en loi du processus $(X_t)_{t\in I}$ à cause du conditionnement, d'autant plus que l'événement par lequel on conditionne est lui même dépendant du temps. Néanmoins, on peut observer que ces probabilités conditionnelles vérifient aussi une propriété de type (2). En effet, notant cette fois-ci $\mu_t := \mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t)$, on a, pour tout $s, t \in I$,

$$\begin{aligned}
\mu_{t+s} &= \frac{\mathbb{P}_\mu(X_{t+s} \in \cdot, \tau_\partial > t+s)}{\mathbb{P}_\mu(\tau_\partial > t+s)} \\
&= \frac{\mathbb{E}_\mu(\mathbb{1}_{\tau_\partial > s} \mathbb{P}_{X_s}(X_t \in \cdot, \tau_\partial > t))}{\mathbb{E}_\mu(\mathbb{1}_{\tau_\partial > s} \mathbb{P}_{X_s}(\tau_\partial > t))} \\
&= \frac{\mathbb{E}_\mu(\mathbb{P}_{X_s}(X_t \in \cdot, \tau_\partial > t) | \tau_\partial > s)}{\mathbb{E}_\mu(\mathbb{P}_{X_s}(\tau_\partial > t) | \tau_\partial > s)} \\
&= \frac{\mathbb{P}_{\mu_s}(X_t \in \cdot, \tau_\partial > t)}{\mathbb{P}_{\mu_s}(\tau_\partial > t)} \\
&= \mathbb{P}_{\mu_s}(X_t \in \cdot | \tau_\partial > t).
\end{aligned}$$

Donc, de la même manière que pour les mesures stationnaires, on s'attend à ce que ces distributions quasi-limites soient un point fixe des applications $\mu \mapsto \mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t)$, pour tout $t \in I$. Ces points fixes sont alors appelés *distribution quasi-stationnaire*.

[2]Cette appellation provient du mathématicien A.M. Yaglom, qui a commencé à étudier ces types d'objets dans le cadre d'un processus de Galton-Watson sous-critique dans [57]

Definition 3. On dit que la mesure de probabilité α est une *mesure quasi-stationaire (QSD)* si pour tout $t \in I$,

$$\mathbb{P}_\alpha(X_t \in \cdot | \tau_\partial > t) = \alpha. \tag{5}$$

De par l'égalité $\mu_{t+s} = \mathbb{P}_{\mu_s}(X_t \in \cdot | \tau_\partial > t)$ établie ci-dessus pour tout $s, t \in I$, on remarque que les notions de distribution quasi-limite et quasi-stationnaire sont équivalentes, si bien que nous obtenons la chaine d'implication suivante:

$$\text{Limite de Yaglom} \Rightarrow QSD \Leftrightarrow QLD.$$

Une autre manière de voir les distributions quasi-stationnaires est de considérer, pour une mesure de probabilité μ sur E, le processus de Markov $(X_t^\mu)_{t\in I}$ défini comme suit :

- $(X_t^\mu)_{t\in I}$ se comporte comme $(X_t)_{t\in I}$ dans E
- Lorsqu'il atteint ∂, il est réinjecté dans E suivant la loi μ.

Alors α est une mesure quasi-stationnaire pour $(X_t)_{t\in I}$ si et seulement si α est une mesure invariante pour le processus $(X_t^\alpha)_{t\in I}$. Cette interprétation est notamment utilisée dans les articles de M. Benaïm et B. Cloez [4] et M. Benaïm, B. Cloez et F.Panloup [6], traitant d'algorithmes d'approximation de QSD basés sur ce principe.

Dans les parties suivantes, nous allons donner quelques méthodes permettant de démontrer l'existence, et parfois l'unicité, de distributions quasi-stationnaires. Loin de pouvoir exposer la théorie dans son intégralité, nous renvoyons le lecteur curieux au livre de P. Collet, S. Martinez, J. San Martin [19] et au survey de S. Méléard et D.Villemonais [42]. Il existe aussi une bibliographie rédigée par P.Polett[3] présentant une liste exhaustive de travaux portant sur les distributions quasi-stationnaires.

0.1.2 Approche spectrale

Notant $(Q_t)_{t\in I}$ le semi-groupe associé au processus X et en utilisant la notation

$$\mu Q_t := \mathbb{P}_\mu(X_t \in \cdot)$$

pour toute mesure de probabilité μ et pour tout $t \in I$, la relation (1) est alors équivalente à

$$\pi Q_t = \pi, \quad \forall t \in I.$$

Autrement dit, la mesure invariante π peut être vue comme un vecteur propre "à gauche" commun à tous les Q_t de valeur propre commune égale à 1. De la même manière, nous allons voir qu'une mesure quasi-stationnaire peut être aussi vue comme un vecteur propre "à gauche" commun à tous les P_t, où $(P_t)_{t\in I}$ est le semi-groupe défini par, pour tout $t \in I$,

$$P_t f(x) = \mathbb{E}_x(f(X_t)\mathbb{1}_{\tau_\partial > t}), \quad \forall f \text{ mesurable borné }, \forall x \in E. \tag{6}$$

[3] https://people.smp.uq.edu.au/PhilipPollett/papers/qsds/qsds.pdf

Ce semi-groupe est alors dit sous-markovien, c'est-à-dire que l'on a $P_t\mathbb{1} \leq \mathbb{1}$ pour tout $t \in I$. Ce semi-groupe agit aussi par dualité sur l'espace des mesures de probabilité à support dans E et on adopte la notation

$$\mu P_t := \int_E \mu(dx)\mathbb{P}_x(X_t \in \cdot, \tau_\partial > t).$$

La prochaine proposition énonce que, sous $\mathbb{P}_\alpha$, le temps d'atteinte τ_∂ suit une loi exponentielle d'un certain paramètre $\lambda > 0$. Dans la littérature, la proposition est énoncée en ces termes :

Proposition 1 (Proposition 2, [42])**.** *Si α est une QSD, alors il existe $\lambda > 0$ tel que pour tout $t \in I$,*

$$\mathbb{P}_\alpha(\tau_\partial > t) = e^{-\lambda t}. \tag{7}$$

En combinant (5) et (7), on démontre la proposition suivante:

Proposition 2. *α est une mesure quasi-stationnaire si et seulement s'il existe $\lambda > 0$ tel que*

$$\alpha P_t = e^{-\lambda t}\alpha, \qquad \forall t \geq 0.$$

La constante λ est alors celle définie dans la relation (7)*.*

Autrement dit, pour tout $t \geq 0$, la QSD α est un vecteur propre de P_t (agissant "à gauche") de valeur propre $e^{-\lambda t}$. On comprend alors que l'utilisation de la théorie spectrale de ce type d'opérateurs peut être un outil pertinent pour démontrer l'existence de distributions quasi-stationnaires.

Temps discret : $I = \mathbb{N}$

Dans le cas $I = \mathbb{N}$, la proposition 2 devient : α est une distribution quasi-stationnaire si et seulement s'il existe $\rho \in (0, 1)$ tel que

$$\alpha P_1 = \rho\alpha.$$

P_1 est alors l'opérateur de transition sous-Markovien de $(X_n)_{n\in\mathbb{N}}$.

Dans le cas où E est un espace d'état fini, P_1 est alors est une matrice sous-stochastique de taille finie, que l'on appelle *sous-matrice de transition*. Dans ce cas, l'existence et l'unicité de la QSD est une conséquence directe du théorème de Perron-Frobenius appliqué à cette sous-matrice P_1. Si de plus on suppose que P_1 est apériodique, alors la distribution quasi-stationnaire est universelle (cf. Définition 1. c)). Ce résultat a été démontré par J.N. Darroch et E. Seneta dans [20].

Dans le cas où E est un espace dénombrable infini, E. Seneta et D. Vere-Jones démontrent dans [50] que, si P_1 est irréductible et apériodique, l'existence d'une distribution quasi-limite est équivalente à la R-positivité du processus X au sens défini dans [54] : pour tout $(i, j) \in E^2$, la suite $(R^n\mathbb{P}_i(X_n = j, \tau_\partial > n))_{n\in\mathbb{N}}$ ne converge pas vers 0, c'est-à-dire que $\mathbb{P}(X_n = j, \tau_\partial > n)$ ne va pas plus vite vers 0 que toutes les suites géométriques.

Temps continu : $I = \mathbb{R}_+$

Considérons maintenant le cas où $I = \mathbb{R}_+$. Il nous est possible de caractériser la distribution quasi-stationnaire via le générateur du semi-groupe $(P_t)_{t\geq 0}$.

Notons $\mathcal{D}(L) := \{f \text{ mesurable bornée} : \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} \text{ existe pour tout } x\}$. Le générateur L est alors l'opérateur linéaire défini par, pour tout $f \in \mathcal{D}(L)$,

$$Lf = \lim_{t\to 0} \frac{P_t f - f}{t}$$

au sens de la convergence simple. Il est alors démontré dans [42] la proposition suivante :

Proposition 3 (Proposition 4, [42])**.** *Supposons qu'il existe $D \subset \mathcal{D}(L)$ tel que, pour tout $A \subset E$ mesurable, on peut trouver une suite de fonctions $(f_n)_{n\in\mathbb{N}}$ uniformément bornées dans D convergeant simplement vers $\mathbb{1}_A$.*

Alors α est une distribution quasi-stationnaire de X si et seulement s'il existe $\lambda > 0$ tel que, pour tout $f \in D$,

$$\int_E Lf(x)\alpha(dx) = -\lambda \int_E f(x)\alpha(dx). \tag{8}$$

La constante λ est alors celle definie dans la relation (7)*.*

Autrement dit, α est une distribution quasi-stationnaire si et seulement si α est un vecteur propre de l'opérateur

$$\mu \to \left(f \to \int_E Lf(x)\mu(dx) \right)$$

de valeur propre $-\lambda$. Dans le cas où $\alpha(dx) = f(x)\mu(dx)$, la relation (8) est équivalente à dire que la fonction densité f est une fonction propre de L^* de valeur propre $-\lambda$, où L^* l'opérateur adjoint de L pour le produit scalaire $(f, g) \to \int_E f(x)g(x)\mu(dx)$.

Dans le cas où E est un espace d'état fini, on peut aussi introduire l'opérateur L, qui est alors une matrice finie et α est peut être vu comme un vecteur propre à gauche de L associé à la valeur propre λ. Dans ce cadre, J.N. Darroch et E. Seneta ont démontré dans [21] l'existence et l'unicité d'une QSD quand L est irréductible et que cette QSD est une QLD universelle lorsque L est de plus apériodique. Très similaire au cas "temps discret", les résultats reposent sur le théorème de Mandl [40, Théorème 1] et la décomposition spectrale des matrices $P_t = e^{tL}$.

Processus de naissance et de mort

Dans le cas d'un processus de naissance et de mort unidimensionnels vivant sur $E = \mathbb{N}^*$ et absorbé en 0, le générateur L est une matrice de taille infinie. Si on associe à chaque état $i \in \mathbb{N}$ un taux de naissance λ_i et un taux de mort μ_i, alors, en notant $x = (x_n)_{n\in\mathbb{N}^*}$, on a

$$(Lx)_i = \mu_i x_{i-1} - (\lambda_i + \mu_i)x_i + \lambda_i x_{i+1}$$

Il nous est possible d'obtenir une décomposition spectrale par le biais des polynômes orthogonaux de S.Karlin et J.L.McGregor, introduits dans [33], qui sont les polynômes $(Q_n)_{n\in\mathbb{N}}$ définis par le système d'équations suivant: pour tout $x \in \mathbb{R}$,

$$Q_0(x) = 1$$
$$-xQ_0(x) = -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0 Q_1(x)$$
$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \quad \forall n \geq 1$$

Ainsi, utilisant ces polynômes orthogonaux, l'existence de distributions quasi-stationnaires peut être démontrée spectralement. Nous renvoyons le lecteur à la série de travaux initiés par Cavender [12], Kijima & Seneta [34] et Van Doorn [52].

Diffusions unidimensionnelles

Si le processus est une diffusion unidimensionnelle vivant sur $\mathbb{R}_+^*$ et absorbée en 0, le générateur L est alors un opérateur différentiel. Dans le cas particulier où le processus $(X_t)_{t\geq 0}$ suit l'EDS suivante

$$dX_t = dB_t - V(X_t)dt \tag{9}$$

où V est $\mathcal{C}_1$ sur $(0,\infty)$, alors le générateur du processus $(X_t)_{t\geq 0}$ est l'opérateur différentiel

$$Lf(x) = \frac{1}{2}f''(x) - V(x)f'(x), \quad \forall f \in C^2(\mathbb{R}_+), f(0) = 0.$$

En définissant alors

$$\gamma(x) := 2\int_1^x V(y)dy \quad \text{et} \quad \mu(dx) := e^{-\gamma(x)}dx,$$

il est bien connu que μ est une mesure réversible de L, c'est-à-dire que l'opérateur L est symétrique pour le produit scalaire $(f,g) \to \int_0^\infty f(x)g(x)\mu(dx)$. Il est démontré dans [11] que, sous l'hypothèse que

$$-\inf_{x\in(0,\infty)} V^2(x) - V'(x) < \infty \quad \text{et} \quad \lim_{x\to\infty} V^2(x) - V'(x) = +\infty,$$

le générateur L admet un spectre discret dont toutes les valeurs propres sont strictement négatives et il existe une fonction propre η_1 associée à la plus grande valeur propre qui est strictement positive sur $(0,\infty)$. Cette propriété permet alors de démontrer que, si de plus $\eta_1 \in \mathbb{L}^1(\mu)$ et $\mathbb{P}_x(\tau_0 < \tau_\infty) = 1$ pour tout $x > 0$, alors la mesure

$$\alpha(dx) := \frac{\eta_1(x)\mu(dx)}{\int_0^\infty \eta_1(y)\mu(dy)}$$

est une distribution quasi-stationnaire pour $(X_t)_{t\geq 0}$ et

$$\{\text{Mesure de proba. à support compact dans } (0,\infty)\} \subset \mathcal{D}(\alpha). \tag{10}$$

De manière générale, la quasi-stationnarité de diffusions unidimensionnelles a suscité un grand intérêt. Nous renvoyons par exemple aux articles [38, 36, 51, 41], qui utilisent tous des arguments spectraux.

0.1.3 Conditions de Champagnat-Villemonais

Malgré la grande efficacité des méthodes spectrales pour démontrer l'existence de distributions quasi-stationnaires et la convergence des processus conditionnés vers celle-ci, on ne peut l'utiliser que pour des opérateurs de transition P_1 (dans le cas $I = \mathbb{N}$) ou des générateurs L (dans le cas $I = \mathbb{R}_+$) dont la décomposition spectrale est connue. Or, pour des processus très intéressants, la décomposition de l'opérateur en question peut s'avérer relativement difficile à obtenir.

Dans cette partie, nous allons donner d'autres méthodes permettant de démontrer l'existence d'une distribution quasi-stationnaire pour les processus de Markov absorbés. Ces méthodes ont été introduites et développées par N. Champagnat et D. Villemonais et sont inspirées de méthodes de type Lyapunov (ou condition de drift) exposées par exemple dans le livre [43] de S.P. Meyn et R.L. Tweedie. Une de ces méthodes repose sur la *condition de Doeblin* pour le processus non conditionnée, qui est explicitée ci-dessous:

Condition de Doeblin. *Il existe $t_0 \in I$, $c > 0$ et une mesure de probabilite ν tels que*

$$\mathbb{P}_x(X_{t_0} \in \cdot) \geq c\nu, \quad \forall x \in E.$$

Cette condition assure alors *une ergodicité exponentielle uniforme en variation totale*, c'est-a-dire qu'il existe une mesure stationnaire π, ainsi que deux constantes $C > 0$ et $\gamma > 0$ telles que, pour toute loi initiale μ,

$$||\mathbb{P}_\mu(X_t \in \cdot) - \pi||_{TV} \leq Ce^{-\gamma t} \tag{11}$$

où l'on définit *la distance en variation totale* entre deux mesures de probabilité μ et ν comme suit:

$$||\mu - \nu||_{TV} := \sup_{||f||_\infty \leq 1} \left| \int_E f(x)\mu(dx) - \int_E f(x)\nu(dx) \right|$$

En particulier, la décroissance (11) implique qu'il y a *unicité de la mesure stationnaire* et que la famille $(\mathbb{P}_\mu(X_t \in \cdot))_{t \in I}$ converge étroitement vers celle-ci, et ce quelque soit la mesure initiale μ.

Dans [14], N. Champagnat et D. Villemonais ont établi des conditions directement inspirées de la condition de Doeblin impliquant aussi une décroissance exponentielle de la distance en variation totale entre le processus conditionné et l'unique distribution quasi-stationnaire du processus. Les conditions sont exprimées en ces termes :

Conditions fortes de Champagnat-Villemonais. *Il existe une mesure de probabilité ν à support dans E, $t_0 \in I$ et deux constantes $c_1, c_2 > 0$ telles que*

(A1)

$$\mathbb{P}_x(X_{t_0} \in \cdot | \tau_\partial > t_0) \geq c_1\nu, \quad \forall x \in E$$

(A2)

$$\mathbb{P}_\nu(\tau_\partial > t) \geq c_2\mathbb{P}_x(\tau_\partial > t), \quad \forall x \in E, \forall t \in I$$

Si $(A1)$ est la version conditionnelle de la condition de Doeblin, il est nécessaire de supposer en plus une inégalité de type Harnack $(A2)$. On a alors le théorème suivant :

Théorème 1 (Champagnat-Villemonais, [14])**.** $(A1)$ *et* $(A2) \Leftrightarrow$ *Le processus* $(X_t)_{t\in I}$ *admet une distribution quasi-stationnaire* α *et il existe* $C, \gamma > 0$ *tels que, pour toute mesure initiale* μ *et* $t \in I$*,*

$$||\mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t) - \alpha||_{TV} \leq Ce^{-\gamma t} \tag{12}$$

De même que pour la condition de Doeblin, les conditions $(A1) - (A2)$ impliquent l'unicité et l'universalité de la QSD.

Il est intéressant de rapprocher ces conditions de celles fournies par R. Knobloch et L. Partzsch dans [35] puisque l'on retrouve aussi dans cet article la condition de Doeblin conditionnelle $(A1)$, mais l'inégalité d'Harnack est remplacée par une condition portant sur l'existence d'une fonction propre "à droite" du semi-groupe sous-Markovien associé au processus bornée sur E (voir condition (C) p.112 dans [35]).

Les processus qui satisfont ces conditions de Champagnat-Villemonais sont "moralement" les processus apériodiques (à noter qu'une T-périodicité serait contradictoire avec $(A1)$), vivant sur des compacts ou ayant la propriété de revenir rapidement sur les compacts. Cette dernière propriété est notamment vérifiée par *les processus descendant de l'infini* dans le cas unidimensionnel, c'est-à-dire des processus définis sur $\mathbb{R}$ vérifiant la propriété suivante

$$\exists y \in \mathbb{R}, \exists t \geq 0, \lim_{x\to\infty} \mathbb{P}_x(\tau_y < t) > 0$$

ou, de manière équivalente (voir [3]),

$$\sup_{x>0} \mathbb{E}_x(\tau_0) < \infty$$

où τ_y est le temps d'atteinte de y par le processus. Par exemple, si $(X_t)_{t\geq 0}$ est la diffusion en (9), il faut "moralement" que le drift V soit sur-linéaire. Notamment, le processus d'Ornstein-Uhlenbeck $(V(x) = \lambda x)$ ne satisfait pas cette propriété de descente de l'infini. Du reste, M. Lladser et J.San Martin ont démontré dans [39] qu'un processus d'Ornstein-Uhlenbeck admet une infinité de QSD.

Il est alors intéressant d'affaiblir ces conditions $(A1) - (A2)$ afin de traiter des cas plus généraux. Dans [53], A. Velleret a donné des critères, directement inspirés de ces conditions, qui permettent d'obtenir une borne dépendant de la mesure initiale. Plus précisement la constante C dans (12) dépend de μ, c'est-à-dire que l'on a

$$||\mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t) - \alpha||_{TV} \leq C(\mu)e^{-\gamma t} \tag{13}$$

avec un γ ne dépendant pas de μ. Plus précisément encore, les critères de Velleret entrainent aussi l'unicité et l'universalité de la distribution quasi-stationnaire α, puisqu'en fait l'application $\mu \to C(\mu)$ ne prend pas de valeurs infinies. Par ailleurs, contrairement au conditions $(A1) - (A2)$, ces critères permettent des phénomènes du type : il existe une suite $(\mu_t)_{t\in\mathbb{R}_+}$ telle que

$$||\mathbb{P}_{\mu_t}(X_t \in \cdot | \tau_\partial > t) - \mathbb{P}_\mu(X_t \in \cdot | \tau_\partial > t)||_{TV} \geq \epsilon$$

où μ est une certaine mesure de référence et $\epsilon > 0$ un certain seuil.

De même, N. Champagnat et D. Villemonais ont introduit d'autres critères dans [15], faisant appel cette fois-ci à des fonctions de Lyapunov, permettant d'obtenir un contrôle du type (13) avec cette fois-ci une fonction $\mu \to C(\mu)$ qui peut prendre des valeurs infinies pour certaines mesures initiales :

Conditions faibles de Champagnat-Villemonais. *Il existe des réels strictement positifs $\gamma_1, \gamma_2, c_1, c_2$ et c_3, $t_1, t_2 \in I$, une fonction mesurable $\psi_1 : E \to [1, +\infty)$, et une mesure de probabilite ν sur un sous-ensemble mesurable $L \subset E$ tel que*

(F0) (Propriété de Markov forte). Définissant

$$\tau_L := \inf\{t \geq 0 : X_t \in L\}, \tag{14}$$

on suppose que pour tout $x \in E$, $X_{\tau_L} \in L$, $\mathbb{P}_x$-presque sûrement sur l'événement $\{\tau_L < \infty\}$ et pour tout $t > 0$ et pour tout $f : E \cup \{\partial\} \to \mathbb{R}_+$ mesurable,

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{\tau_L \leq t < \tau_\partial}\right] = \mathbb{E}_x\left[\mathbb{1}_{\tau_L \leq t \wedge \tau_\partial}\mathbb{E}_{X_{\tau_L}}\left[f(X_{t-u})\mathbb{1}_{t-u<\tau_\partial}\right]|_{u=\tau_L}\right].$$

(F1) (Condition de Doeblin locale). $\forall x \in L$,

$$\mathbb{P}_x(X_{t_1} \in \cdot) \geq c_1 \nu(\cdot \cap L).$$

(F2) (Fonction de Lyapunov). On a $\gamma_1 < \gamma_2 < 1$ et

$$\begin{aligned}
&\mathbb{E}_x(\psi_1(X_{t_2})\mathbb{1}_{t_2<\tau_L\wedge\tau_\partial}) \leq \gamma_1^{t_2}\psi_1(x),\ \forall x \in E\\
&\mathbb{E}_x(\psi_1(X_t)\mathbb{1}_{t<\tau_\partial}) \leq c_2,\ \forall x \in L,\ \forall t \in [0, t_2],\\
&\gamma_2^{-t}\mathbb{P}_x(X_t \in L) \xrightarrow[t\to+\infty]{} +\infty,\ \forall x \in L.
\end{aligned}$$

(F3) (Inégalité d'Harnack locale). On a

$$\sup_{t\geq 0} \frac{\sup_{y\in L}\mathbb{P}_y(t < \tau_\partial)}{\inf_{y\in L}\mathbb{P}_y(t < \tau_\partial)} \leq c_3$$

et ils démontrent le théorème suivant.

Théorème 2 (Champagnat-Villemonais, [15])**.** *Sous les hypothèses (F), $(X_t)_{t\in I}$ admet une QSD α. De plus, il existe $\gamma > 0$ et $C > 0$ tels que, pour toute mesure de probabilité μ sur E satisfaisant $\int_E \psi_1(x)\mu(dx) < \infty$ et $\mu(L) > 0$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \alpha\|_{TV} \leq C\, e^{-\gamma t}\, \frac{\int_E \psi_1(x)\mu(dx)}{\int_E \psi_2(x)\mu(dx)}, \qquad \forall t \in I, \tag{15}$$

où $\psi_2(x) = \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0 t_2}-1}\sum_{k=0}^{n_0}\gamma_2^{-kt_2}\mathbb{P}_x(X_{kt_2} \in L)$ pour $n_0 \geq 1$ suffisamment grand.

En d'autres termes, la fonction $\mu \to C(\mu)$ s'exprime explicitement comme

$$C : \mu \to C \frac{\int_E \psi_1(x)\mu(dx)}{\int_E \psi_2(x)\mu(dx)}$$

et notamment cette fonction prend des valeurs infinies lorsque $\int_E \psi_1(x)\mu(dx) = \infty$ et/ou $\int_E \psi_2(x)\mu(dx) = 0$. De plus, (15) implique que α est l'unique distribution quasi-stationnaire satisfaisant $\frac{\int_E \psi_1(x)\alpha(dx)}{\int_E \psi_2(x)\alpha(dx)} < \infty$ et

$$\left\{ \mu \text{ mesure de proba.} : \frac{\int_E \psi_1(x)\mu(dx)}{\int_E \psi_2(x)\mu(dx)} < \infty \right\} \subset \mathcal{D}(\alpha)$$

Par conséquent, il n'est pas obligatoire dans l'absolu d'avoir unicité de la distribution quasi-stationnaire. Notamment, le processus d'Ornstein-Uhlenbeck (dont on rappelle qu'il ne satisfait pas $(A1) - (A2)$) satisfait les conditions (F).

Avant de définir la notion de Q-processus, remarquons que cette théorie de la quasi-stationnarité peut être développée pour des semi-groupes non-conservatifs, au sens où l'on a $P_t \mathbb{1}_E \neq 1$ pour tout temps $t \geq 0$. En particulier, les méthodes de N. Champagnat et D. Villemonais se généralisent bien pour ce type de semi-groupes. Concernant les conditions globales (A1)-(A2), la quasi-ergodicité de semi-groupes non-conservatifs a été étudié par V. Bansaye, B. Cloez et P. Gabriel dans [1]. Par ailleurs, ces mêmes auteurs et A. Marguet ont étendu très récemment leurs résultats aux conditions locales (F) dans l'article [2]. Enfin, mentionnons l'article [24] de G. Ferré, M. Rousset et G. Stoltz, dans lequel les auteurs étudient la quasi-ergodicité de semi-groupes de Feynman-Kac satisfaisant des critères de type Hairer-Mattingly (voir [26]).

0.1.4 Q-processus

Nous allons maintenant introduire le *Q-processus* d'une chaîne de Markov : Le Q-processus d'une chaîne $(X_t)_{t\in I}$ est tout simplement la loi du processus $(X_t)_{t\in I}$ sous la loi $(\mathbb{Q}_x)_{x\in E}$, où $\mathbb{Q}_x$ est la mesure de probabilité définie par :

$$\mathbb{Q}_x(\Gamma) = \lim_{T\to\infty} \mathbb{P}_x(\Gamma | \tau_\partial > T), \qquad \forall t \geq 0, \forall \Gamma \in \mathcal{F}_t$$

En l'occurence, la notion de Q-processus n'est pas bien défini si la famille $T \to \mathbb{P}_x(\cdot|\tau_\partial > T)$ ne converge pas étroitement.

Le Q-processus peut alors être interprété comme la loi de $(X_t)_{t\in I}$ conditionné à ne jamais être absorbé par ∂. Il est alors assez naturel d'étudier un tel objet dans le cadre de la quasi-stationnarité, mais le Q-processus d'un processus de Markov peut très bien être défini alors que le processus n'admet pas de distribution quasi-stationnaire. L'exemple le plus connu sans doute est le mouvement brownien unidimensionnel absorbé en 0 : ce processus n'admet pas de mesure quasi-stationnaire, mais admet pourtant un Q-processus, qui est le processus de Bessel 3, défini comme étant le processus $(Y_t)_{t\geq 0}$ suivant l'équation différentielle stochastique

$$dY_t = dB_t + \frac{1}{Y_t} dt$$

où $(B_t)_{t\geq 0}$ désigne un brownien unidimensionnel.

En toute généralité, comment démontre-t-on l'existence d'un Q-processus? Prenons $x \in E$, $t \in I$ et $\Gamma \in \mathcal{F}_t$. Alors, pour tout $T \geq t$,

$$\begin{aligned}\mathbb{P}_x(\Gamma|\tau_\partial > T) &= \frac{\mathbb{P}_x(\Gamma, \tau_\partial > T)}{\mathbb{P}_x(\tau_\partial > T)} \\ &= \mathbb{E}_x\left(\mathbb{1}_{\Gamma,\tau_\partial>t}\frac{\mathbb{P}_{X_t}(\tau_\partial > T-t)}{\mathbb{P}_x(\tau_\partial > T)}\right),\end{aligned}$$

où la dernière égalité résulte de la propriété de Markov. Si l'on veut aller plus loin, en réutilisant la notation

$$\mu_t := \mathbb{P}_x(X_t \in \cdot|\tau_\partial > t),$$

on peut même écrire :

$$\mathbb{P}_x(\Gamma|\tau_\partial > T) = \mathbb{E}_x\left(\frac{\mathbb{1}_{\Gamma,\tau_\partial>t}}{\mathbb{P}_x(\tau_\partial > t)}\frac{\mathbb{P}_{X_t}(\tau_\partial > T-t)}{\mathbb{P}_{\mu_t}(\tau_\partial > T-t)}\right) \tag{16}$$

où, une fois de plus, on a utilisé la propriété de Markov. Cette réécriture permet de voir assez facilement que l'existence d'un Q-processus est intimement liée à la convergence d'objets du type

$$\frac{\mathbb{P}_\mu(\tau_\partial > t)}{\mathbb{P}_\nu(\tau_\partial > t)}$$

quand t tend vers l'infini, pour μ et ν des mesures de probabilité sur E. Une telle limite est appelée dans la littérature *"mean ratio limit"*. Lorsque l'approche spectrale le permet, on peut montrer, en utilisant le fait que $\mathbb{P}_\mu(\tau_\partial > t) = \mu P_t \mathbb{1}_E$, que le mean ratio $\frac{\mathbb{P}_\mu(\tau_\partial>t)}{\mathbb{P}_\nu(\tau_\partial>t)}$ converge vers $\frac{\int_E \eta(x)\mu(dx)}{\int_E \eta(x)\nu(dx)}$, où η est un vecteur propre "à droite" du semi-groupe $(P_t)_{t\geq 0}$ défini en (6) et tel que

$$P_t\eta = e^{-\lambda t}\eta, \quad \forall t \in I, \tag{17}$$

où λ est la constante définie dans la formule (7). Notamment, si le passage à la limite sous l'espérance dans la formule (16) est permis, alors nous avons pour tout $\Gamma \in \mathcal{F}_t$ et pour tout $x \in E$,

$$\begin{aligned}\lim_{T\to\infty} \mathbb{P}_x(\Gamma|\tau_\partial > T) &= \mathbb{E}_x\left(\frac{\mathbb{1}_{\Gamma,\tau_\partial>t}}{\mathbb{P}_x(\tau_\partial > t)}\frac{\eta(X_t)}{\int_E \eta(x)\mu_t(dx)}\right) \\ &= \mathbb{E}_x\left(\mathbb{1}_{\Gamma,\tau_\partial>t}\frac{\eta(X_t)}{\mathbb{E}_x(\eta(X_t)\mathbb{1}_{\tau_\partial>t})}\right) \\ &= \mathbb{E}_x\left(\mathbb{1}_{\Gamma,\tau_\partial>t}\frac{e^{\lambda t}\eta(X_t)}{\eta(x)}\right),\end{aligned}$$

puisque, pour tout $t \in I$,

$$\mathbb{E}_x(\eta(X_t)\mathbb{1}_{\tau_\partial>t}) = P_t\eta(x) = e^{-\lambda t}\eta(x).$$

Le Q-processus est alors la h-transformée (ou la transformée de Doob) du processus $(X_t)_{t\in I}$ relatif à la fonction η. Par conséquent, le Q-processus est lui aussi un processus de Markov. De plus, si le processus X admet une distribution quasi-stationnaire α, alors la mesure de probabilité β définie par

$$\beta(dx) = \frac{\eta(x)\alpha(dx)}{\int_E \eta(x)\alpha(dx)}$$

est une mesure stationnaire pour le Q-processus. Autrement dit, l'existence de quasi-stationnaire n'indique rien sur l'existence d'un Q-processus mais nous renseigne sur son ergodicité.

Le Q-processus est bien défini lorsque les conditions de Champagnat-Villemonais (fortes ou faibles) sont satisfaites. En particulier, si un processus vérifie les conditions $(A1) - (A2)$, alors nous avons le résultat suivant :

Proposition 4 (Proposition 2.3, [14])**.** *Sous* $(A1)-(A2)$*, il existe une fonction* η *strictement positive sur* E *telle que*

$$\eta(x) = \lim_{t\to\infty} e^{\lambda t}\mathbb{P}_x(\tau_\partial > t) = \lim_{t\to\infty} \frac{\mathbb{P}_x(\tau_\partial > t)}{\mathbb{P}_\alpha(\tau_\partial > t)}, \tag{18}$$

où la convergence est uniforme sur E*. De plus, la fonction* η *ainsi définie satisfait* (17) *et la relation* $\alpha(\eta) = 1$*.*

De fait, par cette proposition, il est clair que la "mean ratio limit" évoquée plus haut converge bien vers le ratio $\mu(\eta)/\nu(\eta)$. L'existence du Q-processus s'en déduit en utilisant un théorème de pénalisation de B.Roynette, P.Vallois et M.Yor ([47]). De plus, sous les conditions $(A1) - (A2)$, le Q-processus est Harris-récurrent et admet comme unique mesure (de probabilité) invariante la mesure β définie par

$$\beta(dx) = \eta(x)\alpha(dx).$$

Dans le cas où les conditions (F) sont satisfaites, nous obtenons le même genre de résultat que sous les conditions (A), excepté que :

- la convergence (18) n'est plus uniforme sur E, mais a lieu pour la norme $f \to ||f||_{L^\infty(\psi_1)} := \left|\left|\frac{f}{\psi_1}\right|\right|_\infty$.
- le Q-processus n'est défini que pour des états initiaux appartenant à l'ensemble $E' := \{x \in E : \eta(x) \neq 0\}$. En particulier, pour tout $x \in E'$, pour tout $t \in I$,

$$\mathbb{Q}_x(X_t \in \cdot) = \mathbb{E}_x\left(\mathbb{1}_{X_t\in\cdot,\tau_\partial>t} e^{\lambda t}\frac{\eta(X_t)}{\eta(x)}\right) =: \delta_x\tilde{P}_t.$$

  De plus, la mesure définie sur E' par

$$\beta(dx) = \eta(x)\alpha(dx)$$

  est une mesure invariante pour le semi-groupe $(\tilde{P}_t)_{t\in I}$ et, pour tout μ sur E',

$$||\mu\tilde{P}_t - \beta||_{TV} \underset{t\to\infty}{\longrightarrow} 0$$

0.1.5 Distribution quasi-ergodique

Le dernier concept que nous allons définir dans cette première section est la notion de *distribution quasi-ergodique* :

Definition 4. On dit que β est une *distribution quasi-ergodique* s'il existe une loi initiale μ telle que, pour toute fonction f mesurable bornée sur E,

$$\mathbb{E}_\mu\left(\frac{1}{t}\int_0^t f(X_s)ds\middle|\tau_\partial > t\right) \underset{t\to\infty}{\longrightarrow} \int_E f(x)\beta(dx). \tag{19}$$

Autrement dit, la convergence (19) est une version conditionnée du théorème ergodique pour les processus de Markov non absorbés. Dans ce dernier cas, le théorème ergodique stipule que, pour tout f, si le processus est Harris-récurrent, la moyenne empirique $\frac{1}{t}\int_0^t \delta_{X_s}ds$ converge presque sûrement vers sa mesure invariante. De par la nécessité de conditionner par la non-absorption, la convergence vers la distribution quasi-ergodique n'a de sens qu'étroitement.

De plus, contrairement au théorème ergodique non conditionné, la distribution quasi-ergodique diffère de la distribution quasi-stationnaire. Les premiers résultats d'existence de distribution quasi-ergodique dans la littérature concernent les chaînes de Markov à temps discret sur des espaces d'état finis et ont été démontré par J.N.Darroch et E.Seneta dans [20]. De même que pour les mesures quasi-stationnaires, en utilisant le théorème de Perron-Frobenius, ils démontrent la convergence (19) pour les chaînes irréductibles apériodiques avec

$$\beta(dx) = \eta(x)\alpha(dx),$$

où η est le vecteur propre à droite de Perron-Frobenius de la sous-matrice de transition P_1 vérifiant $\int_E \eta(x)\alpha(dx)$. Notons alors que, dans cet exemple précis, la distribution quasi-ergodique coincide avec la mesure stationnaire du Q-processus (voir la sous-section précédente). Le même genre de résultat a été démontré spectralement par G.He et H.Zhang (voir [27]) pour des diffusions unidimensionnelles satisfaisant les conditions de Littin (cf. [38]).

On peut penser alors que cette notion de distribution quasi-ergodique est étroitement liée avec celle de Q-processus. Le premier résultat allant dans ce sens est certainement celui de L.Breyer et G.Roberts ([9]):

Théorème 3 (Breyer-Roberts, [9])**.** *Si le Q-processus est Harris-récurrent et admet pour mesure invariante β, alors, pour tout $x \in E$,*

$$\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot|\tau_\partial > t)ds \underset{t\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \beta \tag{20}$$

Lorsque les conditions (A1)-(A2) sont satisfaites, il est démontré dans [13] que la convergence (20) a lieu en variation totale, et qu'il existe une constante $C > 0$ telle que, pour toute mesure initiale μ,

$$\left\|\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot|\tau_\partial > t)ds - \beta\right\|_{TV} \leq \frac{C}{t}. \tag{21}$$

Un contrôle similaire à (21) peut être établi lorsque le processus satisfait les conditions (F). Nous le verrons dans le chapitre 4.

Remark 1. D'autres critères généraux (voir [58]) ne portant pas sur la notion de Q-processus implique l'existence de distributions quasi-ergodiques.

0.2 Quasi-stationnarité pour des semi-groupes inhomogènes et applications aux frontières mobiles

0.2.1 Processus de Markov inhomogènes

Après avoir défini le concept de distribution quasi-stationnaire dans le cadre homogène, ainsi que les autres concepts tournant autour de cette notion (quasi-limite, quasi-ergodique, Q-processus), il est possible d'étendre ces notions dans le cadre inhomogène.

Considérons alors un processus $(\Omega, (\mathcal{F}_{s,t})_{s\leq t}, (Q_{s,t})_{s\leq t}, (X_t)_{t\in I}, (\mathbb{P}_{x,s})_{x\in E, s\in I})$ inhomogène en temps vivant sur $E\cup\{\partial\}$ où $\partial\notin E$ est un élément absorbant de $(X_t)_{t\in I}$. En particulier, pour tout $s\in I$ et $x\in E$, $\mathbb{P}_{x,s}$ est une mesure de probabilité telle que $\mathbb{P}_{x,s}(X_s=x)=1$. En mimant les définitions de la section précédente, on appelle :

- *mesure quasi-stationnaire* toute mesure α vérifiant

$$\mathbb{P}_{\alpha,s}(X_t\in\cdot|\tau_\partial>t)=\alpha,\quad \forall s\leq t, \tag{22}$$

- *mesure quasi-limite* toute mesure α telle qu'il existe une mesure initiale μ et un temps $s\in I$ telle que

$$\mathbb{P}_{\mu,s}(X_t\in\cdot|\tau_\partial>t)\xrightarrow[t\to\infty]{\mathcal{L}}\alpha, \tag{23}$$

- *Q-processus* la loi de $(X_t)_{t\in I}$ sous $(\mathbb{Q}_{x,s})_{x\in E,s\geq 0}$, où, pour tout $x\in E$ et $s\in I$,

$$\mathbb{Q}_{x,s}(\Gamma)=\lim_{T\to\infty}\mathbb{P}_{x,s}(\Gamma|\tau_\partial>T),\quad \forall\,\Gamma\in\mathcal{F}_{s,t}, \tag{24}$$

- *distribution quasi-ergodique* toute mesure β telle qu'il existe une loi initiale μ et $s\in I$ telle que

$$\frac{1}{t}\int_s^t\mathbb{P}_{\mu,s}(X_u\in\cdot|\tau_\partial>t)du\xrightarrow[t\to\infty]{\mathcal{L}}\beta. \tag{25}$$

A cause de l'inhomogénéité, on s'attend à ce que la notion de distribution quasi-stationnaire soit mal définie, sauf cas exceptionnel (par exemple lorsque le processus en question est un processus de Markov homogène dont on a changé le temps de façon non linéaire). De même, la notion de quasi-limite ne semble être bien définie que lorsque l'on a une stabilisation de l'inhomogénéité à l'infini (le lecteur peut consulter les notes de cours de M. Benaïm [5] traitant de la notion de pseudotrajectoire asymptotique, très liée avec cette propriété de stabilisation). De plus, l'équivalence entre QSD et QLD bien connue dans le cadre homogène n'est plus vérifiée dans le cadre inhomogène : s'il est toujours vrai que l'on a l'implication $QSD\Rightarrow QLD$, nous perdons la réciproque puisque la quasi-limite ne

s'interprète plus comme point fixe d'opérateurs particuliers. En revanche, il est difficile de savoir si les notions de Q-processus ou de distribution quasi-ergodique sont robustes et s'étendent au cas inhomogène.

Une première réponse est apportée par N.Champagnat et D.Villemonais dans l'article [18]. En adaptant les conditions $(A1)-(A2)$ définies plus tôt, ils démontrent que, sous certaines conditions, on peut obtenir une *propriété de mélange* (ou *quasi-ergodicité faible*) pour un processus de Markov inhomogène, ainsi qu'un Q-processus. Ces conditions, adaptées des conditions $(A1)-(A2)$, sont les suivantes : Il existe une famille de mesures de probabilité $(\nu_s)_{s\in I}$ sur E telle que, pour tout $s\in I$,

(A'1) il existe $c_{1,s}>0$ tel que

$$\mathbb{P}_{x,s}(X_{s+1}\in\cdot|\tau_\partial>s+1)\geq c_{1,s}\nu_{s+1},\qquad \forall x\in E,$$

(A'2) et il existe $c_{2,s}>0$ tel que

$$\mathbb{P}_{\nu_s,s}(\tau_\partial>t)\geq c_{2,s}\mathbb{P}_{x,s}(\tau_\partial>t),\qquad \forall x\in E,\forall t\geq 0$$

Sous ces conditions, nous avons le résultat suivant :

Théorème 4 (Champagnat-Villemonais,[18])**.** *Sous les conditions $(A'1)-(A'2)$, pour tout μ_1,μ_2 mesures de probabilité sur E, pour tout $s\leq t$,*

$$||\mathbb{P}_{\mu_1,s}(X_t\in\cdot|\tau_\partial>t)-\mathbb{P}_{\mu_2,s}(X_t\in\cdot|\tau_\partial>t)||_{TV}\leq 2\prod_{k=0}^{\lfloor t-s\rfloor-1}(1-c_{1,t-k}c_{2,t-k})$$

La propriété de mélange est définie par le fait que la distance en variation totale entre $\mathbb{P}_{\mu_1,s}(X_t\in\cdot|\tau_\partial>t)$ et $\mathbb{P}_{\mu_2,s}(X_t\in\cdot|\tau_\partial>t)$ tende vers 0 quand t tend vers l'infini. En particulier, si $c_{1,s}$ et $c_{2,s}$ ne dépendent pas de s, la propriété de mélange est géométrique, au sens où la convergence ci-dessus est géométrique. Le théorème précédent autorise des propriétés de mélange polynômial, pourvu que les coefficients $c_{1,s}$ et $c_{2,s}$ ne tendent pas trop vite vers 1 quand s tend vers l'infini.

Le deuxième résultat important du papier [18] concerne le Q-processus :

Théorème 5. *Sous l'hypothèse de convergence suivante:*

$$\liminf_{t\to\infty}\frac{1}{c_{1,t}c_{2,t}}\prod_{k=0}^{\lfloor t-s\rfloor-1}(1-c_{1,t-k}c_{2,t-k})=0,$$

il existe un Q-processus pour la chaîne $(X_t)_{t\in I}$, c'est-à-dire que la mesure de probabilité $\mathbb{Q}_{x,s}$ (définie comme en (24)) est bien définie pour tout $x\in E$ et $s\in I$.

En revanche, les auteurs ne se sont pas intéressés à l'existence de la distribution quasi-ergodique. De plus, à part cette contribution très générale de N.Champagnat et D.Villemonais, la quasi-stationnarité pour des processus de Markov inhomogènes a suscité peu d'intérêt pour le moment. Citons tout de même le travail de P. Del Moral et

D.Villemonais [22], s'intéressant à la propriété de mélange pour des diffusions inhomogènes; ainsi que l'article [1] de V.Bansaye, B.Cloez et P.Gabriel, qui s'intéresse entre autre à la quasi-ergodicité de semi-groupes inhomogènes non-conservatifs lorsque ces derniers admettent une propriété d'homogénéité asymptotique (définition (H'0) dans [1, Sous-section 3.1]).

0.2.2 Lymphocytes T cytotoxiques et nodule tumoral

Afin d'introduire la problématique de cette thèse, prenons un modèle concret issu du domaine de la biologie. Plus particulièrement, nous allons considérer le modèle de particules introduit par P. Cattiaux, C. Christophe et S. Gadat dans le papier [10] concernant le comportement trajectoriel de lymphocytes T cytotoxiques (CTL) absorbés par un nodule tumoral. En notant N le nombre de lymphocytes, nous numérotons les lymphocytes de 1 à N et nous notons $(Z_t^i)_{t\geq 0}$ la trajectoire du i-ième lymphocyte. Quant à lui, le nodule tumoral sera representé par la boule $A_t := B(0, R_t)$ de $\mathbb{R}^3$, où R_t est le rayon du nodule qui peut bouger au cours du temps.

Maintenant introduisons la dynamique des N lymphocytes :

- A l'instant initial, les N particules bougent de manière indépendante en suivant les équations differentielles stochastiques suivantes

$$dZ_t^j = dB_t^j - aZ_t^j dt, \quad \forall 1 \leq j \leq N,$$

  où $a > 0$ et $(B^j)_{1\leq j\leq N}$ sont N mouvements browniens mutuellement indépendants.

- Lorsqu'une première particule atteint le nodule, elle est alors "absorbée", et les $N-1$ particules restantes suivent indépendamment les unes des autres

$$dZ_t^j = dB_t^j - \left(a + \frac{b}{N}\right) Z_t^j dt,$$

  où $b > 0$. Autrement dit, au moment ou la première particule a été absorbée, on a "intensifié" le paramètre d'Ornstein-Uhlenbeck d'un incrément $\frac{b}{N}$.

- Lorsqu'une deuxième particule est absorbée, on intensifie encore d'un increment $\frac{b}{N}$ le paramètre d'Ornstein-Uhlenbeck des $N-2$ particules restantes. Les EDS deviennent alors

$$dZ_t^j = dB_t^j - \left(a + \frac{2b}{N}\right) Z_t^j dt.$$

- On continue ce procédé d'intensification jusqu'à ce que toutes les particules soient absorbées.

Pour tout $1 \leq j \leq N$, notons T_A^j le temps que met la j-ème particule pour atteindre le nodule. Alors la dynamique avant absorption de la j-ieme particule sera modelisée par l'EDS

$$Z_t^j = Z_0^j + B_t^j - \int_0^t \left(a + \frac{b}{N}\sum_{i=1}^{N} \mathbb{1}_{T^i<s}\right) Z_s^j ds, \quad \forall t < T_A^j,$$

et nous remarquerons au passage que la dynamique présentée ci-dessus est inhomogène en temps.

Puisqu'il y a absorption, la question de la quasi-stationnarité apparait naturellement. Par exemple, nous pouvons nous intéresser à la quasi-stationarité de la première particule $Z := Z^1$ absorbée après le temps $T_A := T_A^1$.

Sans parler du fait que le processus est inhomogène dans cet exemple précis (on peut considérer à la place une dynamique homogène), le fait que le bord absorbant bouge au cours du temps provoque en soi de l'inhomogénéité pour le semi-groupe associé à $(Z_t)_{t\geq 0}$. Par exemple, imaginons maintenant que le processus $(Z_t)_{t\geq 0}$ soit un processus homogène; on peut alors lui associer un triplet $((\mathcal{F}_t)_{t\geq 0}, (Q_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^3})$. La famille d'opérateurs $(P_t)_{t\geq 0}$, où $P_t f(x) = \mathbb{E}_x(f(Z_t)\mathbb{1}_{T_A>t})$, n'est pas un semi-groupe. En revanche, on peut construire un semi-groupe $(P_{s,t})_{s\leq t}$ inhomogène, défini par : pour tout $\mathbb{R}^3$, $s \leq t$, $f : \mathbb{R}^3 \to \mathbb{R}$ bornée borélienne,

$$P_{s,t}f(x) := \mathbb{E}_x(f(Z_{t-s})\mathbb{1}_{T_A\circ\theta_s>t-s}),$$

où θ_s est *l'opérateur de translation* : pour tout h défini sur $\mathbb{R}_+$,

$$h \circ \theta_s : t \to h(t+s).$$

Si la dynamique $(Z_t)_{t\geq 0}$ est inhomogène, définie par le triplet $((\mathcal{F}_{s,t})_{s\leq t}, (Q_{s,t})_{s\leq t}, (\mathbb{P}_{x,s})_{x\in\mathbb{R}^3, s\geq 0})$, on définit alors le semi-groupe $(P_{s,t})_{s\leq t}$ comme suit : pour tout $\mathbb{R}^3$, $s \leq t$, $f : \mathbb{R}^3 \to \mathbb{R}$ bornée borélienne,

$$P_{s,t}f(x) := \mathbb{E}_{x,s}(f(Z_t)\mathbb{1}_{T_A>t}).$$

0.2.3 Quasi-stationarité avec frontières mobiles

Le but de cette thèse est donc d'éclaircir, autant que faire se peut, la notion de quasi-stationnarité pour des processus (homogènes ou non) absorbés par des frontières mobiles. Dans toute cette thèse, nous allons supposer que les frontières bougent de manière déterministe. Considérer des frontières mobiles aléatoires, indépendantes du processus ou non, permet d'imaginer de nombreux modèles plus réalistes que le cas de frontières mobiles déterministes (en particulier, dans l'exemple des CTL, l'évolution du rayon du nodule R_t est solution d'une EDO dépendant de l'espérance du nombre de CTL absorbés). Ce cas n'est pas traité ici et peut être l'objet de développements futurs.

Plus précisément, l'objectif de cette thèse est de comprendre quelles notions définies dans le cadre de la quasi-stationnarité (quasi-stationnaire, quasi-limite, quasi-ergodique, Q-processus) sont robustes pour le cas du conditionnement avec frontières mobiles. La prochaine section a pour but de détailler, chapitre par chapitre, les avancées de compréhension ainsi que les principaux résultats de cette thèse.

0.3 Description détaillée des chapitres de la thèse

Cette thèse se décompose en quatres chapitres. Les trois premiers sont basés sur des articles publiés ou soumis. Le dernier chapitre est un projet en cours, en processus de finalisation.

0.3.1 Chapitre 1 : Quasi-stationarité et quasi-ergodicité pour des chaînes de Markov à temps discret absorbées par une frontière périodique

Ce premier chapitre correspond à l'article [45] publié dans *ALEA Lat. Am. J. Probab. Math. Stat.* en 2018. Il s'intéresse principalement aux chaînes de Markov à temps discret définies sur des espaces d'état finis absorbées par des frontières périodiques.

Tout d'abord, il est démontré que, dans le cas où le mouvement du bord "se fait ressentir" (voir la condition (1.3) du chapitre 1), la notion de distribution quasi-stationnaire, telle que définie dans (22), est mal posée (Proposition 5). Ce résultat est démontré dans le cas de chaîne à temps discret sans hypothèse concernant le mouvement du bord et la nature de l'espace d'état; une version peut être adaptée pour des processus à temps continu, ce qui rend ce résultat en fait très général.

Il est ensuite démontré que, lorsque le bord est périodique, il ne peut exister de distribution quasi-limite (Proposition 6). De même, la nature des espaces d'état et de temps ne rentre pas en ligne de compte.

Ensuite, ce chapitre s'attaque à la notion de distribution quasi-ergodique pour une chaîne de Markov $(X_n)_{n\in\mathbb{N}}$ irréductible définie sur un espace d'état fini E, absorbée par une frontière périodique, de période γ. Il y est démontré (Théorème 3) que, démarrant d'une certaine loi initiale μ, si la chaîne $(X_n, \overline{n})_{n\in\mathbb{N}}$ (où $\bar{n}$ correpond à n modulo γ) définie sur $E \times \mathbb{Z}/\gamma\mathbb{Z}$ vérifie une certaine condition vis-à-vis de cette loi μ, alors il existe une mesure β telle que

$$\mathbb{E}_\mu \left(\frac{1}{n} \sum_{k=0}^{n} \mathbb{1}_{X_k=x} \middle| \tau_A > n \right) \underset{n\to\infty}{\longrightarrow} \beta(x), \qquad \forall x \in E \tag{26}$$

Cette mesure β n'est autre que le projeté sur l'espace E de la distribution quasi-ergodique de $(X_n, \overline{n})_{n\in\mathbb{N}}$ dont l'existence est prouvée au Théorème 2. Cette dernière chaîne est une chaîne absorbée par un sous-ensemble de l'espace d'état $E \times \mathbb{Z}/\gamma\mathbb{Z}$ fixe. Il suffit donc d'appliquer les résultats portant sur l'existence d'une distribution quasi-ergodique dans le cas homogène. Cependant, l'article [20] qui traite des chaînes de Markov définies sur des espaces finis ne s'intéresse qu'au cas où la chaîne est irréductible apériodique. Il a fallu donc dans un premier temps généraliser leurs travaux sur les distributions quasi-ergodiques pour des chaînes réductibles et pouvant présenter une certaine périodicité sur les ilôts d'irréductibilité.

A noter que, dans le cas où la période de la chaîne et la période γ du bord sont des entiers premiers entre eux, la convergence (26) a lieu quelle que soit la mesure initiale μ, ce qui implique notamment l'unicité de la mesure quasi-ergodique (Corollaire 1). Cela vient du fait que cette condition de primalité sur les périodes est équivalente à l'irréductibilité de la chaîne $(X_n, \overline{n})_{n\in\mathbb{N}}$. En dehors de ce cas, nous n'avons pas unicité de la distribution quasi-ergodique et/ou la convergence (26) n'a pas lieu pour toute mesure initiale μ. En particulier, l'exemple traité dans ce chapitre possède deux distributions quasi-ergodiques et la moyenne conditionnelle converge vers l'une ou l'autre suivant que la mesure initiale charge ou non l'ensemble des nombres impairs.

L'existence du Q-processus découle ensuite naturellement des travaux sur les distributions quasi-ergodiques. On démontre, entre autre, que, pour tout état initial x en dehors de l'ensemble absorbant, la convergence

$$\mathbb{Q}_x(X_{t_1} \in \cdot, \ldots, X_{t_n} \in \cdot) = \lim_{T\to\infty} \mathbb{P}_x(X_{t_1} \in \cdot, \ldots, X_{t_n} \in \cdot | \tau_A > T)$$

est bien définie pour tout $t_1 \leq \ldots \leq t_n$. En particulier, le Q-processus de $(X_n)_{n\in\mathbb{N}}$ est une chaîne de Markov inhomogène.

0.3.2 Chapitre 2 : Q-processus et propriétés asymptotiques de processus de Markov conditionnés à ne pas atteindre de frontières mobiles

Le Chapitre 2 est basé sur le preprint [44], soumis en 2018. L'ambition de cet article est de retrouver les résultats obtenus dans le Chapitre 1 pour des espaces d'états et de temps plus généraux. En particulier, en fin de chapitre, nous nous intéressons à la quasi-stationnarité de diffusions unidimensionnelles absorbées par une frontière mobile.

Les démonstrations du chapitre 1 utilisées pour le Q-processus et la distribution quasi-ergodique utilisent fortement la théorie spectrale. Ceci est possible car la chaîne $(X_n)_{n\in\mathbb{N}}$ étant définie sur un espace fini, la chaîne $(X_n, \overline{n})_{n\in\mathbb{N}}$ est également définie sur un espace fini. Si on veut étendre cette méthode au cas d'une diffusion $(X_t)_{t\geq 0}$, le processus markovien $(X_t, \overline{t})_{t\geq 0}$ est homogène, mais est une diffusion dégénérée, pour laquelle la théorie spectrale n'est pas connue et difficile à mettre en place. Ce chapitre 2 va par conséquent s'inspirer des conditions fortes de Champagnat-Villemonais pour établir les hypothèses clés du chapitre.

Plus exactement, les hypothèses de base du chapitre sont les conditions (A'1)-(A'2) (introduites précédemment dans la sous-section 0.2.1) pour lesquelles les coefficients $c_{1,s}$ et $c_{2,s}$ ne dépendent pas de s. Comme on l'a vu précédemment, cette hypothèse entraîne une propriété de mélange géométrique et l'existence du Q-processus, mais il n'y a pas de résultat concernant l'existence de distribution quasi-ergodique.

Pour obtenir l'existence d'une distribution quasi-ergodique, nous allons suivre le même raisonnement que celui de N.Champagnat et D.Villemonais dans [17] dans le cas homogène : On établit tout d'abord une convergence exponentielle en variation totale de $\mathbb{P}_x(X_{[0,s]} \in \cdot | \tau_\partial > t)$ vers la loi du Q-processus $\mathbb{Q}_x(X_{[0,s]} \in \cdot)$ du type

$$||\mathbb{Q}_x(X_{[0,s]} \in \cdot) - \mathbb{P}_x(X_{[0,s]} \in \cdot | \tau_\partial > t)||_{TV} \leq Ce^{-\gamma(t-s)},$$

puis on s'en sert pour établir l'inégalité

$$\begin{aligned}\left|\left|\frac{1}{t}\int_0^t \mathbb{Q}_x(X_s \in \cdot)ds - \frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_\partial > t)ds\right|\right|_{TV} &\leq \frac{C}{t}\int_0^t e^{-\gamma(t-s)}ds \\ &\leq \frac{C}{t}.\end{aligned}$$

Ainsi, l'existence d'une distribution quasi-ergodique résulte du théorème ergodique du Q-processus, qui est Harris récurrent sous les hypothèses (A1)-(A2).

Le premier résultat du chapitre 2, qui traite donc du cas inhomogène, est que, pour tout $s \leq t$ et $x \in E_s := E \setminus A_s$, il existe $\gamma > 0$ tel que pour tout $T > t$

$$||\mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) - \mathbb{P}_{x,s}(X_{[s,t]} \in \cdot | \tau_A > T)||_{TV} \leq C_{s,t,x} e^{-\gamma(T-t)}.$$

où le coefficient $C_{s,t,x} < \infty$ est explicité dans le Théorème 9. De plus si, pour tout $s \in I$ et $x \in E_s$, $C_{s,x} := \sup_{t \geq s} C_{s,t,x} < \infty$, alors

$$\left|\left|\frac{1}{t}\int_s^t \mathbb{Q}_{x,s}(X_u \in \cdot)du - \frac{1}{t}\int_s^t \mathbb{P}_{x,s}(X_u \in \cdot | \tau_A > t)du\right|\right|_{TV} \leq \frac{C_{s,x}}{t}. \tag{27}$$

Par conséquent, s'il existe un théorème ergodique pour le Q-processus inhomogène, on en déduit la forme de la distribution quasi-ergodique pour $(X_t)_{t \in I}$. Cette distribution quasi-ergodique est par ailleurs unique, et la convergence vaut pour toute loi initiale μ.

Au sein de ce chapitre, on étudie plus spécifiquement deux types de comportements de frontières mobiles. Le premier est celui d'une frontière périodique. Dans ce cas, les coefficients $C_{s,t,x}$ sont tous majorés par 1, ce qui entraîne que l'on a (27). Pour conclure à l'existence et l'unicité de la distribution quasi-ergodique, nous utilisons un résultat type théorème ergodique concernant les processus de Markov inhomogènes périodiques dû à R.Höpfner et H.Kutoyants dans [29] (le même genre de résultat se retrouve dans les deux articles [30, 31] écrits par R. Höpfner, E. Löcherbach et M. Thieullen). Ce résultat d'existence et d'unicité de distribution quasi-ergodique pour le cas de frontières périodiques n'est rien d'autre que le Théorème 10 du Chapitre 2.

Le second comportement étudié est la décroissance du bord $(A_t)_{t \in I}$ (au sens de l'inclusion) jusqu'à stabilisation vers un ensemble fixe A_∞. A ça, il faut ajouter quelques hypothèses, notamment que les conditions (A1)-(A2) sont vérifiées pour le processus $(X_t)_{t \in I}$ absorbé en A_∞. Sous ces conditions, nous démontrons l'existence et l'unicité d'une distribution quasi-limite (Théorème 12), ainsi que l'existence et l'unicité d'une distribution quasi-ergodique (Théorème 13). De plus, ces objets sont universels.

0.3.3 Chapitre 3 : Quasi-stationarité pour le mouvement brownien "renormalisé"

Le chapitre 3 est tiré du preprint [46], soumis en 2018. L'idée ici est un peu différent des chapitres 1 et 2 puisqu'il s'agit cette fois-ci d'étudier la quasi-stationarité au sens classique (quand le bord ne bouge pas) d'un processus de Markov inhomogène en temps. L'ambition première de cette étude était néanmoins d'étudier une sorte de quasi-stationarité d'un mouvement brownien unidimensionnel absorbé par les bords fuyants $t \to \{-(t+1)^\kappa, (t+1)^\kappa\}$, où κ est un nombre strictement positif. Au lieu de cela, nous avons préféré travailler sur le processus inhomogène $(X_t)_{t \geq 0}$ défini par

$$X_t := \frac{B_t}{(t+1)^\kappa}, \quad \forall t \geq 0.$$

absorbé en $\{-1, 1\}$, que nous nommons *mouvement brownien "renormalisé"*. Dans ce chapitre, nous démontrons qu'il y a une transition de phase concernant la quasi-stationarité du mouvement brownien renormalisé. Plus précisément (voir Théorèmes 14, 15 et 17),

- Pour $\kappa > \frac{1}{2}$ - *phase sur-critique* - le processus conditionné convergence en loi vers la mesure de Dirac en 0.

- Pour $\kappa = \frac{1}{2}$ - *phase critique* - le processus conditionné converge en loi vers la distribution quasi-stationnaire d'un processus d'Ornstein-Uhlenbeck absorbé en $\{-1, 1\}$ associé au générateur

$$Lf(x) = \frac{1}{2}f''(x) - \frac{1}{2}xf'(x).$$

- Pour $\kappa < \frac{1}{2}$ - *phase sous-critique* - le processus conditionné converge en loi vers la distribution quasi-stationnaire d'un mouvement brownien absorbé en $\{-1, 1\}$.

Dans les trois cas, les distributions quasi-limites sont toutes universelles.

Les résultats du cas $\kappa = \frac{1}{2}$ se déduisent du fait que le processus $X_t = \frac{B_t}{\sqrt{t+1}}$ est, en loi, un processus d'Ornstein-Uhlenbeck changé en temps de façon non-linéaire (le changement est $t \to \log(t+1)$). Ainsi, les résultats standard sur ce processus homogène concernant la quasi-limite et le Q-processus se transportent bien. Cependant, la distribution quasi-ergodique de $(X_t)_{t\geq 0}$ n'est pas égale à la distribution quasi-ergodique du processus d'Ornstein-Uhlenbeck. Plus précisément, en notant $(Z_t)_{t\geq 0}$ le processus d'Ornstein-Uhlenbeck et $\tau_Z := \inf\{t \geq 0 : |Z_t| = 1\}$, la distribution quasi-ergodique de $(X_t)_{t\geq 0}$ est absolument continue par rapport à la mesure quasi-stationaire de $(Z_t)_{t\geq 0}$ et sa densité est la fonction $x \to \mathbb{E}_x(\tau_Z)$.

Pour le cas sous-critique, on démontre la convergence vers la quasi-stationnaire d'un mouvement brownien en établissant une propriété de type "pseudotrajectoire asymptotique" avec le processus $Y_t = \int_0^t \frac{1}{(s+1)^\kappa} dB_s$, qui est égal en loi à un mouvement brownien changé en temps. Cette propriété de pseudotrajectoire asymptotique va nous permettre de transporter les propriétés asymptotiques de $(Y_t)_{t\geq 0}$, en particulier (X_t) et $(Y_t)_{t\geq 0}$ auront la même distribution quasi-limite et la même distribution quasi-ergodique. On démontre aussi l'existence d'un Q-processus inhomogène pour le processus $(X_t)_{t\in\mathbb{R}_+}$, qui sera une pseudotrajectoire asymptotique du Q-processus de $(Y_t)_{t\geq 0}$.

0.3.4 Chapitre 4 : Quasi-stationnarité pour des semi-groupes sous-Markovien inhomogènes et fonctions de Lyapunov

Ce dernier chapitre correspond à un travail en cours en collaboration avec N. Champagnat et D. Villemonais.

L'idée de ce chapitre est de reprendre les conditions faibles de Champagnat-Villemonais pour des semi-groupes inhomogènes du type

$$P_{s,t}f(x) := \mathbb{E}_{x,s}(f(X_t)\mathbb{1}_{\tau_A > t})$$

où $(X_t)_{t\geq 0}$ est un processus inhomogène et $(A_t)_{t\geq 0}$ est une frontière mobile. Dans un premier temps, nous considérons des conditions (F') pour lesquelles la seule donnée dépendant du temps de départ s est la famille de fonctions de Lyapunov $\psi_{1,s}$. Cette

dépendance est nécessaire lorsque l'on veut passer du cas "temps discret" au cas "temps quelconque".

La première propriété démontrée sous les hypothèses (F') est la propriété de mélange en variation totale pour les semi-groupes inhomogènes. Plus exactement, prenant deux mesures μ_1 et μ_2 au temps s, la distance en variation totale entre $\frac{\mu_1 P_{s,t}}{\mu_1 P_{s,t}\mathbb{1}}$ et $\frac{\mu_2 P_{s,t}}{\mu_2 P_{s,t}\mathbb{1}}$ est majorée par une borne du type

$$Ce^{-\gamma t}\left[\frac{\int_{E_s}\psi_{1,s}(x)\mu_1(dx)}{\int_{E_s}\psi_{2,s}(x)\mu_1(dx)}+\frac{\int_{E_s}\psi_{1,s}(x)\mu_2(dx)}{\int_{E_s}\psi_{2,s}(x)\mu_2(dx)}\right]$$

où C, γ ne dépendent pas de μ_1, μ_2, s et $\psi_{2,s}$ est construit explicitement dans le chapitre 4.

Le deuxième résultat important concerne l'existence du Q-processus. Pour ce faire, nous démontrons que le ratio

$$\frac{\mathbb{P}_{x,s}(\tau_A > t)}{\mathbb{P}_{\nu_s,s}(\tau_A > t)}$$

converge, pour tout $s \in I$ et $x \in E_s$, quand $t \to \infty$ vers $\eta_s(x)$. Puis nous utilisons ce résultat pour démontrer l'existence d'un Q-processus partant d'un point de $E'_s := \{x \in E_s : \eta_s(x) \neq 0\}$. Plus exactement, on démontre qu'il existe C, γ (pas nécessairement les mêmes constantes évoquées lors de la propriété de mélange) tels que

$$||\mathbb{P}_{x,s}[X_{[s,t]} \in \cdot | \tau_A > T] - \mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot)||_{TV} \leq Ce^{-\gamma(T-t)}\left(1+\frac{\psi_{1,s}(x)}{\eta_s(x)}\right)\frac{\psi_{1,s}(x)}{\eta_s(x)}$$

pour tout $s \leq t \leq T$ et $x \in E'_s$. Ce contrôle exponentiel nous permet ensuite de contrôler la distance en variation totale entre $\frac{1}{t}\int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | \tau_A > t)du$ et $\frac{1}{t}\int_s^t \mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)du$ et d'obtenir une vitesse de convergence en $1/t$.

Chapter 1

Quasi-stationarity and quasi-ergodicity for discrete-time Markov chains with absorbing boundaries moving periodically

This chapter is based on the paper [45], published in ALEA in 2018.

Abstract

We are interested in quasi-stationarity and quasi-ergodicity when the absorbing boundary is moving. First we show that, in the moving boundary case, the quasi-stationary distribution and the quasi-limiting distribution are not well-defined when the boundary is oscillating periodically. Then we show the existence of a quasi-ergodic distribution for any discrete-time irreducible Markov chain defined on a finite state space in the fixed boundary case. Finally we use this last result to show the quasi-ergodicity in the moving boundary case.

1.1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $X = (X_n)_{n\in\mathbb{N}}$ be a Markov chain with a finite state space $(E, \mathcal{E})$, $\mathcal{E}$ being the σ-algebra containing all the subset of E. Let $\mathbb{P}_x$ be the probability measure such that $\mathbb{P}_x(X_0 = x) = 1$ and, for any measure μ on E, define $\mathbb{P}_\mu = \int \mathbb{P}_x d\mu(x)$. Denote by $\mathcal{M}_1(E)$ the set of probability measures defined on E.
We define, for each time $n \geq 0$, a subset $A_n \subset E$ called *killing subset at time* n and we denote by E_n the complement of A_n called *survival subset at time* n. We will call $(A_n)_{n\in\mathbb{N}}$ *the moving killing subset* or *the moving killing boundary*. We denote by τ the random variable defined as follows

$$\tau := \inf\{n \geq 0 : X_n \in A_n\} \tag{1.1}$$

For any subset $B \subset E$, we define τ_B as follows

$$\tau_B := \inf\{n \geq 0 : X_n \in B\}$$

and, to make the notation easier, for any $m \in \mathbb{N}$, we denote by τ_m the random variable defined by

$$\tau_m := \tau_{A_m} = \inf\{n \geq 0 : X_n \in A_m\} \tag{1.2}$$

This chapter will deal with quasi-stationary, quasi-limiting and quasi-ergodic distributions that we define as follows.

Definition 5. ν is a *quasi-stationary distribution* if for any $n \geq 0$

$$\mathbb{P}_\nu(X_n \in \cdot | \tau > n) = \nu(\cdot)$$

Definition 6. ν is a *quasi-limiting distribution* if there exist some $\mu \in \mathcal{M}_1(E)$ such that

$$\mathbb{P}_\mu(X_n \in \cdot | \tau > n) \underset{n\to\infty}{\longrightarrow} \nu(\cdot)$$

Definition 7. ν is a *quasi-ergodic distribution* or a *mean-ratio quasi-stationary distribution* if for any $\mu \in \mathcal{M}_1(E)$ and any bounded measurable function f

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) | \tau > n\right) \underset{n\to\infty}{\longrightarrow} \int f d\nu$$

We will also be interested in the existence of a Q-process, which can be interpreted as the process X conditioned never to be absorbed by $(A_n)_{n\in\mathbb{N}}$.

In the case where the sequence $(A_n)_{n\in\mathbb{N}}$ does not depend on the time, the existence of these probability measures was established under several assumptions. See for example [19, 42] for a general review on the theory of quasi-stationary distributions. For modelling purpose, some recent works (see [10]) introduce some Markov processes absorbed by moving boundaries and the classical theory on quasi-stationary distributions does not allow anymore to describe the asymptotic behavior of the process conditioned not to be absorbed. Our purpose is therefore to check whether these probability measures are still well-defined when $(A_n)_{n\geq 0}$ depends on the time or not.

In all what follows, we will assume that for any $x \in E_0$,

$$\mathbb{P}_x(\tau < \infty) = 1$$

and will also assume that the following hypothesis of irreducibility holds

$$\forall n \in \mathbb{N}, \forall x, y \in E_n, \exists m \in \mathbb{N}, \mathbb{P}_x(X_{m\wedge\tau_n} = y) > 0 \tag{1.3}$$

Quasi-stationary distribution will be studied for general moving killing boundaries. However, in a significant part of this chapter we will deal with moving killing boundaries

$(A_n)_{n\in\mathbb{N}}$ which are γ-periodic with $\gamma \geq 2$.
In this chapter, we will actually show that there are no quasi-stationary distributions and quasi-limiting distributions in the sense of Definitions 5 and 6 when the boundaries are moving periodically. However, we will show that the notion of quasi-ergodic distribution and Q-process still makes sense even when the boundary is moving. In particular, we will show the following statement.

Theorem 1. *Assume that $(A_n)_{n\in\mathbb{N}}$ is γ-periodic. Then, for some initial law $\mu \in \mathcal{P}(E)$ and under assumptions which will be spelled out later, there exists a probability measure η such that, for any bounded measurable function f,*

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)\middle|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \int f d\eta$$

The proof is divided to several steps. First we reduce the problem to the study of quasi-stationary distribution in a non moving domain, but for a periodic Markov chain. Then we extend the result proved by Darroch and Senata (see [20]) in the aperiodic case to the periodic situation ($\gamma \in \mathbb{N}^*$).
This chapter ends with an application of this theorem to random walks absorbed by 2-periodic moving boundaries.

1.2 Quasi-stationary distribution with moving killing subset

The following proposition shows that the notion of quasi-stationary distribution as defined in Definition 5 is not relevant when the killing boundary is moving.

Proposition 5. *Assume there exist $l, m \in \mathbb{N}$ such that $A_l \neq A_m$. Then there is no measure $\nu \in \mathcal{M}_1(E)$ satisfying the following property:*

$$\forall n \in \mathbb{N}, \quad \mathbb{P}_\nu(X_n \in \cdot|\tau > n) = \nu(\cdot) \tag{1.4}$$

Proof. For any $n \in \mathbb{N}$, denote by $f_n : \mathcal{M}_1(E) \to \mathcal{M}_1(E)$ the function defined by

$$f_n : \mu \longrightarrow \mathbb{P}_\mu(X_1 \in \cdot|\tau_n > 1) \tag{1.5}$$

where τ_n is defined in (1.2) and denote by μ_n the probability measure defined by

$$\mu_n = \mathbb{P}_\mu(X_n \in \cdot|\tau > n) \tag{1.6}$$

By the Markov property, we have for any $n \in \mathbb{N}^*$

$$\mu_n = \mathbb{P}_{\mu_{n-1}}(X_1 \in \cdot|\tau_n > 1) = f_n(\mu_{n-1})$$

Thus, by induction, we obtain for any $n \in \mathbb{N}$

$$\mathbb{P}_\mu(X_n \in \cdot|\tau > n) = f_n \circ \ldots \circ f_1(\mu)$$

We deduce from this equality that

$$\forall n \in \mathbb{N},\ \mathbb{P}_\nu(X_n \in \cdot | \tau > n) = \nu(\cdot) \Longleftrightarrow \forall n \in \mathbb{N},\ f_n(\nu) = \nu$$
$$\Longleftrightarrow \forall n \in \mathbb{N},\ \mathbb{P}_\nu(X_1 \in \cdot | \tau_n > 1) = \nu(\cdot)$$

In other words, ν is a quasi-stationary distribution in the moving sense if and only if it is a quasi-stationary distribution in the non-moving sense for all the subsets A_n. In particular, if ν satisfies (1.4),

$$\nu(\cdot) = \mathbb{P}_\nu(X_1 \in \cdot | \tau_l > 1) \text{ and } \nu(\cdot) = \mathbb{P}_\nu(X_1 \in \cdot | \tau_m > 1)$$

where l and m have been mentioned in the statement of the proposition. However, since the assumption of irreducibility (1.3) holds, the previous statement is impossible since the support of the quasi-stationary distributions are different. □

Remark 2. Proposition 5 can be extended to any general Markov process defined on any space state. In particular, for continuous-time Markov processes defined on a metric space (E, d), we may replace the assumption of irreducibility (1.3) by the following assumption

$$\forall t \in \mathbb{R}_+, \forall x, y \in E_t, \forall \epsilon > 0, \exists t_0 \in \mathbb{R}_+, \mathbb{P}_x(X_{t_0 \wedge \tau_t} \in B(y, \epsilon)) > 0$$

where $B(y, \epsilon) := \{z \in E : d(y, z) < \epsilon\}$.

Notice moreover that we did not need any assumption about the behavior of $(A_n)_{n\in\mathbb{N}}$. In all what follows, we consider that $(A_n)_{n\in\mathbb{N}}$ is γ-periodic with $\gamma \geq 2$.

1.3 Quasi-limiting distribution when the killing subset is moving periodically

We are now interested in knowing whether the definition of quasi-limiting distribution given in definition 6 is meaningful when the killing subset is moving or not. In the usual case, it is well known (see [42] p. 345) that quasi-stationary distribution and quasi-limiting distribution are equivalent notions. This implies that the non-existence of a quasi-stationary distribution implies the non-existence of any quasi-limiting distribution. However, this equivalence does not hold anymore in the moving case. Consider for example $(A_n)_{n\geq 0}$ such that there exists n_0 such that for any $n \geq n_0$, $A_n = A_{n_0}$ and assume that there exists a quasi-stationary distribution ν_{n_0} (in the non-moving sense) such that for any probability measure μ on E_{n_0},

$$\mathbb{P}_\mu(X_n \in \cdot | \tau_{n_0} > n) \underset{n\to\infty}{\longrightarrow} \nu_{n_0}$$

Thus, by the Markov property, for any $\mu \in \mathcal{M}_1(E)$ such that $\mathbb{P}_\mu(\tau > k) > 0$ for all $k \in \mathbb{N}$ and any $n \geq 0$,

$$\mathbb{P}_\mu(X_{n+n_0} \in \cdot | \tau > n + n_0) = \mathbb{P}_{\mu_{n_0}}(X_n \in \cdot | \tau_{A_{n_0}} > n) \underset{n\to\infty}{\longrightarrow} \nu_{n_0}$$

where μ_n is defined in (1.6) for any $n \in \mathbb{N}$.
From now on, we will assume that $(A_n)_{n\in\mathbb{N}}$ is periodic and will denote by γ its period. We will show that quasi-limiting distribution is not well defined when the killing subset is moving periodically.

Proposition 6. *Assume $(A_n)_{n\in\mathbb{N}}$ is γ-periodic and there exist $0 \leq l, m \leq \gamma - 1$ such that $A_l \neq A_m$.*
Then there is no $\nu \in \mathcal{M}_1(E)$ satisfying the following property:

$$\exists \mu \in \mathcal{M}_1(E), \quad \mathbb{P}_\mu(X_n \in \cdot | \tau > n) \underset{n\to\infty}{\longrightarrow} \nu(\cdot)$$

Proof. Consider again the functions f_m defined in (1.5):

$$f_m : \mu \longrightarrow \mathbb{P}_\mu(X_1 \in \cdot | \tau_m > 1)$$

Then using the periodicity of $(A_n)_{n\in\mathbb{N}}$ and by the Markov property, for any $k \in \{1, \ldots, \gamma\}$, $m \in \mathbb{N}^*$ and $\mu \in \mathcal{M}_1(E)$

$$\mathbb{P}_\mu(X_{k+m\gamma} \in \cdot | \tau > k + m\gamma) = g_k \circ f^m(\mu) \tag{1.7}$$

with

$$g_k = f_k \circ \ldots \circ f_1$$
$$f = f_\gamma \circ \ldots \circ f_1$$

Assume that there exists $\mu \in \mathcal{M}_1(E)$ such that the sequence $(\mathbb{P}_\mu(X_m \in \cdot | \tau > m))_{m\in\mathbb{N}}$ converges to its limit ν. Then

$$\begin{aligned} \nu &= \lim_{m\to\infty} \mathbb{P}_\mu(X_{m\gamma} \in \cdot | \tau > m\gamma) \\ &= \lim_{m\to\infty} f^m(\mu) \end{aligned}$$

So for any $k \in \{1, \ldots, \gamma\}$

$$\nu = g_k(\nu) = \mathbb{P}_\nu(X_k \in \cdot | \tau > k)$$

In other words, for any $k \in \{1, \ldots, \gamma\}$,

$$\nu = f_k(\nu)$$

We thus may conclude in the proof of proposition 5. □

The previous statement implies therefore that the quasi-limiting distribution as defined in Definition 6 is not well-defined when the moving killing subset is periodic. However, according to the proof of the previous proposition, it seems that the sequence of these conditioned probabilities could have some limit points.

The following proposition allows us to pass from a moving problem to a non-moving problem. The existence of limit points will be therefore a consequence of the existence of classical quasi-stationary distributions for the transformed Markov chain.

Proposition 7. *For any $0 \leq m \leq \gamma - 1$ and $\mu \in \mathcal{M}_1(E)$, there is a Markov chain $(X_n^{(m)})_{n\in\mathbb{N}}$ such that*

$$\mathbb{P}_\mu((X_m, X_{m+\gamma}, \dots, X_{m+n\gamma}) \in \cdot | \tau > m+n\gamma) = \mathbb{P}_{\mu_m}((X_0^{(m)}, X_1^{(m)}, \dots, X_n^{(m)}) \in \cdot | \tau_m^{(m)} > n) \tag{1.8}$$

where μ_m is defined in (1.6) *and $\tau_m^{(m)} = \inf\{n \in \mathbb{N} : X_n^{(m)} \in A_m\}$.*

Proof. According to the Markov property, it is enough to show that for any γ-periodic sequence of subsets $B = (B_n)_{n\in\mathbb{N}}$ and any measure $\mu \in \mathcal{M}_1(E)$, there exists a Markov chain $(Z_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{P}_\mu((X_\gamma, \dots, X_{n\gamma}) \in \cdot | \tau(B) > n\gamma) = \mathbb{P}_\mu((Z_1, \dots, Z_n) \in \cdot | \tilde{\tau}_{B_0} > n) \tag{1.9}$$

where $\tau(B) = \inf\{m \geq 0 : X_m \in B_m\}$ and $\tilde{\tau}_{B_0} = \inf\{n \in \mathbb{N} : Z_n \in B_0\}$. Denote F_0 the complement of B_0. For any $x \in F_0$ define $p(x, \cdot)$ by

$$\begin{aligned} p(x, A) &= \mathbb{P}_x(X_\gamma \in A, \tau_B > \gamma), \quad \forall A \subset F_0 \\ p(x, B_0) &= 1 - p(x, F_0) \end{aligned}$$

and we denote by $(Z_n)_{n\in\mathbb{N}}$ the Markov chain for which the transition kernel is p. We will show by induction that, for any $\phi_1, \dots, \phi_n$ bounded measurable functions,

$$\mathbb{E}_\mu(\phi_1(X_\gamma) \dots \phi_n(X_{n\gamma}) \mathbb{1}_{\tau(B)>n\gamma}) = \mathbb{E}_\mu(\phi_1(Z_1) \dots \phi_n(Z_n) \mathbb{1}_{\tilde{\tau}_{B_0}>n})$$

By definition of $(Z_n)_{n\in\mathbb{N}}$, for any probability measure μ and any bounded measurable function ϕ,

$$\mathbb{E}_\mu(\phi(Z_1) \mathbb{1}_{\tilde{\tau}_{B_0}>1}) = \mathbb{E}_\mu(\phi(X_\gamma) \mathbb{1}_{\tau(B)>\gamma})$$

which entails the base case. Now assume that the equality for $n-1$ is satisfied. Let $\phi_1, \dots, \phi_n$ be some bounded measurable functions. Then

$$\begin{aligned} \mathbb{E}_\mu(\phi_1(X_\gamma) \dots \phi_n(X_{n\gamma}) \mathbb{1}_{\tau(B)>n\gamma}) &= \mathbb{E}_\mu(\phi_1(X_\gamma) \mathbb{1}_{\tau(B)>\gamma} \mathbb{E}_{X_\gamma}(\phi_2(X_\gamma) \dots \phi_n(X_{(n-1)\gamma}) \mathbb{1}_{\tau(B)>(n-1)\gamma})) \\ &= \mathbb{E}_\mu(\phi_1(Z_1) \mathbb{1}_{\tilde{\tau}_{B_0}>1} \mathbb{E}_{Z_1}(f_2(Z_1) \dots \phi_n(Z_{n-1}) \mathbb{1}_{\tilde{\tau}_{B_0}>(n-1)})) \\ &= \mathbb{E}_\mu(\phi_1(Z_1) \dots \phi_n(Z_n) \mathbb{1}_{\tilde{\tau}_{B_0}>n}) \end{aligned}$$

This concludes the proof. □

1.4 Existence of quasi-ergodic distribution with periodic moving killing subsets

In this section, our aim is to show the existence of a quasi-ergodic distribution as defined in Definition 7 when the boundary is moving periodically. This section will be split into three parts :

1. We will first study quasi-ergodicity in the non-moving case (when $A_n = A_0, \forall n \in \mathbb{N}$) for irreducible Markov chains.

2. Then we will use the results obtained in the first part to deduce quasi-ergodicity for general Markov chains (irreducible or not), but still considering non-moving boundaries

3. Finally we will show the existence of quasi-ergodic distribution when $(A_n)_{n\in\mathbb{N}}$ is moving periodically.

1.4.1 Quasi-ergodic distribution in the classical non-moving sense for irreducible periodic Markov chains

In this subsection we will study the quasi-ergodicity of one irreducible Markov chain $Y = (Y_n)_{n\in\mathbb{N}}$ in the classical non-moving sense, that is when the killing edge does not move. Without loss of generality, assume Y is defined in the state space $E_0 = \{0, \dots, K\}$ and that the cemetery is $\{0\}$. In this subsection and the following, τ will be defined as (1.1) but refering to Y, that is

$$\tau = \inf\{n \geq 0 : Y_n = 0\}$$

Denote by P the transition matrix of Y. Since 0 is an absorbing state for Y, P has the following form

$$P = \begin{pmatrix} 1 & 0 \\ v & Q \end{pmatrix}$$

where Q is *the sub-transition matrix*. We will assume that Q is irreducible (i.e. $\forall x, y \in E_0, \exists n \in \mathbb{N}, Q^n(x, y) > 0$). As a result we can define T_x the period of x as

$$T_x := gcd\{n \in \mathbb{N} : \mathbb{P}_x(Y_n = x, \tau > n) > 0\}$$

where *gcd* refers to the greatest common divisor. By irreducibility of Q, all the x have the same period and we denote by T this common period.
The existence of quasi-ergodic distributions has already been proved by Darroch and Seneta in [20] when $T = 1$. However we will see that this result is not enough for our purpose and we need to extend it to periodic Markov chains.
Due to the periodicity of Q, there exist $(C_i)_{0\leq i\leq T-1}$ a partition of E_0 such that if the support of the initial distribution μ is included in C_0, then for any $n \in \mathbb{N}$ and $0 \leq k \leq T-1$,

$$\mathbb{P}_\mu(Y_{k+nT} \in C_k, \tau > k + nT) = 1$$

Without loss of generality, we construct $(C_i)_{0\leq i\leq T-1}$ such that $1 \in C_0$. Formally $(C_i)_{0\leq i\leq T-1}$ are defined by

$$C_0 := \{y \in E_0 : \exists n \in \mathbb{N}^*, \mathbb{P}_1(Y_{nT} = y, \tau > nT) > 0\} \tag{1.10}$$

$$\forall 1 \leq i \leq T-1, \quad C_i := \{y \in E_0 : \exists x \in C_{i-1}, \mathbb{P}_x(Y_1 = y) > 0\} \tag{1.11}$$

For each $j \in \{0, \dots, T-1\}$ and any $v \in \mathbb{C}^K$, we will denote by $v^{(j)}$ the sub-vector of v restricted to C_j.
It is well known by the Perron-Frobenius theorem that the spectral radius

$$\rho := \max\{|\lambda| : \lambda \in Sp(Q)\}$$

is a simple eigenvalue of Q and that one can find a left-eigenvector $\nu = (\nu(j))_{1\leq j\leq K}$ and a right-eigenvector $\xi = (\xi(j))_{1\leq j\leq K}$ for ρ (i.e. $\nu Q = \rho\nu$ and $Q\xi = \rho\xi$) such that $\nu(j) > 0$ and $\xi(j) > 0$ for all $j \in \{1, \dots, K\}$. We may choose ν and ξ such that

$$< \nu, \mathbb{1} >=< \nu, \xi >= 1$$

where $< \cdot, \cdot >$ is the usual Hermitian product on $\mathbb{C}^K$. Moreover, since Q is T-periodic,

$$\{\lambda_k := \rho e^{\frac{2ik\pi}{T}} : 0 \leq k \leq T-1\} \subset Sp(Q)$$

and each λ_k is simple. For each λ_k we can obtain a left eigenvector v_k and a right-eigenvector w_k from ν and ξ with the following transformation

$$\forall j \in \{0, \dots, T-1\}, v_k^{(j)} = e^{-i\frac{2\pi jk}{T}}\nu^{(j)} \text{ and } w_k^{(j)} = e^{i\frac{2\pi jk}{T}}\xi^{(j)} \tag{1.12}$$

See Theorem 1.7 in [[49],p.23-24] for more details.
The vectors $(v_i)_{0\leq i\leq T-1}$ are linearly independent. We can complete this family into a basis $\mathcal{V} = (v_i)_{0\leq i\leq K-1}$ such that $v_i \in Span^{\perp}(v_0, \dots, v_{T-1})$ for all $T \leq i \leq K-1$ where

$$Span^{\perp}(v_0, \dots, v_{T-1}) = \{v \in \mathbb{C}^K :< v, v_i >= 0,\ \forall i \in \{0, \dots, T-1\}\}$$

Let us denote by R the matrix representing the change of basis from the canonical basis to $\mathcal{V}$. Then we have the following decomposition

$$Q = R \begin{pmatrix} \lambda_0 & & & \\ & \ddots & & 0 \\ & & \lambda_{T-1} & \\ & 0 & & Q_0 \end{pmatrix} R^{-1}$$

where Q_0 is a $(K-T)\times(K-T)$ matrix. We define the matrix Q' by

$$Q' = R \begin{pmatrix} 0 & 0 \\ 0 & Q_0 \end{pmatrix} R^{-1}$$

Proposition 8. *Let $f : \{1, \dots, K\} \to \mathbb{R}$ be a bounded measurable function. Then for any $x \in \{1, \dots, K\}$ and $n \in \mathbb{N}^*$,*

$$\mathbb{E}_x\left(\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)\mathbb{1}_{\tau>n}\right) = \rho^n\varphi(f)\sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T}} < w_l, \delta_x >< v_l, \mathbb{1} > +o(\rho^n)$$

where

$$\varphi(f) = \sum_{i=1}^{K} f(i)\nu(i)\xi(i)$$

.

Proof. Let $f : \{1, \dots, K\} \to \mathbb{R}$ be a bounded measurable function. In this proof we will consider probability measures on $\{1, \dots, K\}$ and functions defined on $\{1, \dots, K\}$ as K-vectors. Thus for any $x \in \{1, \dots, K\}$ we can say

$$\mathbb{E}_x \left(f(Y_n) \mathbb{1}_{\tau > n} \right) =< \delta_x Q^n, f > \tag{1.13}$$

where δ_x is the Dirac measure on x. For any $x \in \{1, \dots, K\}$, define $(\alpha_k(x))_{0 \leq k \leq T-1}$ the unique family of $\mathbb{C}^K$ such that there is $\mu_x \in Span^{\perp}(v_0, \dots, v_{T-1})$ such that

$$\delta_x = \sum_{k=0}^{T-1} \alpha_k(x) v_k + \mu_x$$

We will use the following lemma whose proof is postponed after the proof of the theorem.

Lemma 1. *For any* $0 \leq k \leq T-1$*,*

$$(\alpha_k(x))_{x \in E_0} = w_k$$

where w_k *is defined in* (1.12)

Thus we can write

$$\delta_x = \sum_{k=0}^{T-1} w_k(x) v_k + \mu_x \tag{1.14}$$

So, using (1.13) and (1.14), for any $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}_x \left(f(X_n) \mathbb{1}_{\tau > n} \right) &=< \sum_{k=0}^{T-1} w_k(x) v_k Q^n, f > + < \mu_x Q^n, f > \\ &= \sum_{k=0}^{T-1} \overline{\lambda_k^n w_k(x)} < v_k, f > + < \mu_x (Q')^n, f > \end{aligned}$$

Now, using the Markov property, for any $k \leq n$,

$$\begin{aligned} \mathbb{E}_\mu (f(Y_k) \mathbb{1}_{\tau > n}) &= \mathbb{E}_\mu (\mathbb{1}_{\tau > k} f(Y_k) \mathbb{P}_{Y_k} (\tau > n - k)) &(1.15) \\ &= \mathbb{E}_\mu (\mathbb{1}_{\tau > k} g_{k,n}(f)(Y_k)) &(1.16) \end{aligned}$$

where, for any $y \in E_0$,

$$g_{k,n}(f)(y) = f(y) \mathbb{P}_y (\tau > n - k)$$

Then,

$$\begin{aligned} g_{k,n}(f)(y) &= f(y) < \delta_y Q^{n-k}, \mathbb{1} > \\ &= \sum_{m=0}^{T-1} \overline{\lambda_m^{n-k}} f(y) \overline{w_m(y)} < v_m, \mathbb{1} > + f(y) < \mu_y (Q')^{n-k}, \mathbb{1} > \end{aligned}$$

Define, for any $k \in \{0, \dots, T-1\}$ and $n \in \mathbb{N}$,

$$g_k(f) : y \to f(y)\overline{w_l(y)}$$
$$w_n(f) : y \to f(y) < \mu_y(Q')^n, \mathbb{1} >$$

Then, using (1.16), for any $k \leq n$,

$$\begin{aligned}\mathbb{E}_x(f(Y_k)\mathbb{1}_{\tau>n}) &=< \delta_x Q^k, g_{k,n}(f) > \\ &= \sum_{l=0}^{T-1} \overline{\lambda_l^k w_l(x)} < v_l, g_{k,n}(f) > + < \mu_x(Q')^k, g_{k,n}(f) > \\ &= A_{k,n} + B_{k,n} + C_{k,n} + D_{k,n}\end{aligned}$$

where

$$\begin{aligned}A_{k,n} &= \sum_{l=0}^{T-1}\sum_{m=0}^{T-1} \overline{\lambda_l^k \lambda_m^{n-k} w_l(x)} < v_l, g_m(f) >< v_m, \mathbb{1} > \\ B_{k,n} &= \sum_{l=0}^{T-1} \overline{\lambda_l^k w_l(x)} < v_l, w_{n-k}(f) > \\ C_{k,n} &= \sum_{m=0}^{T-1} \overline{\lambda_m^{n-k}} < v_m, \mathbb{1} >< \mu_x(Q')^k, g_m(f) > \\ D_{k,n} &=< \mu_x(Q')^k, w_{n-k}(f) >\end{aligned}$$

Hence for any $n \in \mathbb{N}^*$

$$\sum_{k=0}^{n-1} \mathbb{E}_x(f(Y_k)\mathbb{1}_{\tau>n}) = \sum_{k=0}^{n-1} A_{k,n} + \sum_{k=0}^{n-1} B_{k,n} + \sum_{k=0}^{n-1} C_{k,n} + \sum_{k=0}^{n-1} D_{k,n} \tag{1.17}$$

i) Study of $\sum_{k=0}^{n-1} A_{k,n}$

For any $n \in \mathbb{N}^*$,

$$\begin{aligned}\sum_{k=0}^{n-1} A_{k,n} &= \sum_{l=0}^{T-1}\sum_{m=0}^{T-1} \left(\sum_{k=0}^{n-1} \overline{\lambda_l^k \lambda_m^{n-k}} \right) \overline{w_l(x)} < v_l, g_m(f) >< v_m, \mathbb{1} > \\ &= \sum_{l=0}^{T-1} n\overline{\lambda_l^n w_l(x)} < v_l, g_l(f) >< v_l, \mathbb{1} > + \sum_{l \neq m} \overline{\lambda_m} \left(\frac{\overline{\lambda_l^n} - \overline{\lambda_m^n}}{\overline{\lambda_l} - \overline{\lambda_m}} \right) \overline{w_l(x)} < v_l, g_m(f) >< v_m, \mathbb{1} >\end{aligned}$$

On the one side,

$$\sum_{l=0}^{T-1} n\overline{\lambda_l^n w_l(x)} < v_l, g_l(f) >< v_l, \mathbb{1} >= n\rho^n \sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T}} \overline{w_l(x)} < v_l, g_l(f) >< v_l, \mathbb{1} >$$

On the other side, for any $0 \leq l \neq m \leq T-1$,

$$\overline{\lambda_m}\left(\frac{\overline{\lambda_l^n}-\overline{\lambda_m^n}}{\overline{\lambda_l}-\overline{\lambda_m}}\right) = \rho e^{-\frac{2im\pi}{T}}\left(\frac{\rho^n e^{-\frac{2inl\pi}{T}}-\rho^n e^{-\frac{2inm\pi}{T}}}{\rho e^{-\frac{2il\pi}{T}}-\rho e^{-\frac{2im\pi}{T}}}\right)$$
$$= \rho^n e^{-\frac{2im\pi}{T}}\left(\frac{e^{-\frac{2inl\pi}{T}}-e^{-\frac{2inm\pi}{T}}}{e^{-\frac{2il\pi}{T}}-e^{-\frac{2im\pi}{T}}}\right)$$

$\left(e^{-\frac{2im\pi}{T}}\left(\frac{e^{-\frac{2inl\pi}{T}}-e^{-\frac{2inm\pi}{T}}}{e^{-\frac{2il\pi}{T}}-e^{-\frac{2im\pi}{T}}}\right)\right)_{n\in\mathbb{N}}$ is bounded, hence

$$\frac{1}{n}\times e^{-\frac{2im\pi}{T}}\left(\frac{e^{-\frac{2inl\pi}{T}}-e^{-\frac{2inm\pi}{T}}}{e^{-\frac{2il\pi}{T}}-e^{-\frac{2im\pi}{T}}}\right) \underset{n\to\infty}{\longrightarrow} 0$$

We deduce that, for any $0 \leq l \neq m \leq T-1$,

$$\rho^n e^{-\frac{2im\pi}{T}}\left(\frac{e^{-\frac{2inl\pi}{T}}-e^{-\frac{2inm\pi}{T}}}{e^{-\frac{2il\pi}{T}}-e^{-\frac{2im\pi}{T}}}\right) = o(n\rho^n)$$

and therefore

$$\sum_{l\neq m}\overline{\lambda_m}\left(\frac{\overline{\lambda_l^n}-\overline{\lambda_m^n}}{\overline{\lambda_l}-\overline{\lambda_m}}\right)\overline{w_l(x)} < v_l, g_m(f) >< v_m, \mathbb{1} >= o(n\rho^n)$$

since this is a finite sum. Hence

$$\sum_{k=0}^{n-1} A_{k,n} = n\rho^n \sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T}}\overline{w_l(x)} < v_l, g_l(f) >< v_l, \mathbb{1} > +o(n\rho^n)$$

ii) Study of $\sum_{k=0}^{n-1} B_{k,n}$

For any $y \in E$, $n \in \mathbb{N}$ and $0 \leq l \leq T-1$

$$\sum_{k=1}^{n-1}\overline{\lambda_l^k} w_{n-k}(f)(y) = f(y) < \mu_y\left(\sum_{k=0}^{n-1}\lambda_l^k (Q')^{n-k}\right), \mathbb{1} >$$
$$= f(y) < \mu_y Q'(\lambda_l I_K - Q')^{-1}(\lambda_l^n I_K - (Q')^n), \mathbb{1} >$$

where I_K is the $K\times K$-identity matrix. For any $0 \leq l \leq T-1$ and $n \in \mathbb{N}$,

$$\lambda_l^n I_k - (Q')^n = \rho^n(e^{\frac{2i\pi nl}{T}} I_k - \rho^{-n}(Q')^n)$$

and $(e^{\frac{2i\pi nl}{T}} I_k - \rho^{-n}(Q')^n)_{n\in\mathbb{N}}$ is bounded since the spectral radius of Q' is smaller than ρ. Hence

$$\frac{1}{n}\left(e^{\frac{2i\pi nl}{T}} I_k - \rho^{-n}(Q')^n\right) \underset{n\to\infty}{\longrightarrow} 0$$

where 0 is understood as the zero matrix, and we deduce that

$$< \mu_y Q'(\lambda_l I_K - Q')^{-1}(\lambda_l^n I_K - (Q')^n), \mathbb{1} >= o(n\rho^n)$$

As a result, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n-1} \overline{\lambda_l^k} w_{n-k}(f)(y) = o(n\rho^n)$$

Hence for any $n \in \mathbb{N}$

$$\sum_{k=0}^{n-1} B_{k,n} = \sum_{l=0}^{T-1} \overline{w_l(x)} < v_l, \sum_{k=0}^{n-1} \overline{\lambda_l^k} w_{n-k}(f) >= o(n\rho^n)$$

iii) Study of $\sum_{k=0}^{n-1} C_{k,n}$

In the same way as $\sum_{k=0}^{n-1} B_{k,n}$,

$$\begin{aligned}\sum_{k=0}^{n-1} C_{k,n} &= \sum_{k=0}^{n-1} \sum_{m=0}^{T-1} \overline{\lambda_m^{n-k}} < v_m, \mathbb{1} >< \mu_x (Q')^k, g_m(f) > \\ &= \sum_{m=0}^{T-1} < v_m, \mathbb{1} >< \mu_x \left(\sum_{k=0}^{n-1} \lambda_m^{n-k} (Q')^k \right), g_m(f) >\end{aligned}$$

For any $0 \leq m \leq T-1$ and $n \geq 1$,

$$\sum_{k=0}^{n-1} \lambda_m^{n-k} (Q')^k = \lambda_m \times (\lambda_m I_K - Q')^{-1}(\lambda_m^n I_K - (Q')^n)$$

We already showed that for any $0 \leq m \leq T-1$ and $n \geq 1$

$$(\lambda_m I_K - Q')^{-1}(\lambda_m^n I_K - (Q')^n) = o(n\rho^n)$$

Finally,

$$\sum_{k=0}^{n-1} C_{k,n} = o(n\rho^n)$$

iv) Study of $\sum_{k=0}^{n-1} D_{k,n}$

Finally, let us denote by $(q')_{i,j}^{(n)}$, for $i, j \in \{1, \ldots, K-T\}$ and $n \in \mathbb{N}$, the coefficient of $(Q')^n$ located at the ith row and the jth column. Then for any $n \in \mathbb{N}$

$$\sum_{k=0}^{n-1} D_{k,n} = \sum_{i,j,l,m} \mu_x(j) f(i) \mu_i(m) \left(\sum_{k=0}^{n-1} (q')_{m,l}^{(n-k)} (q')_{i,j}^{(k)} \right) \tag{1.18}$$

Let $i,j,l,m \in \{1,\dots,K\}$. By definition of the matrix Q', the spectral radius of Q' is strictly smaller than ρ. We deduce from this

$$(q')_{i,j}^{(n)} = o(\rho^n), \quad (q')_{m,l}^{(n)} = o(\rho^n) \tag{1.19}$$

In particular there is a positive number C such that for any $n \in \mathbb{N}$ and $m,l \in \{1,\dots,K\}$,

$$(q')_{m,l}^{(n-k)} \leq C\rho^{n-k}$$

Hence,

$$\sum_{k=0}^{n-1}(q')_{m,l}^{(n-k)}(q')_{i,j}^{(k)} \leq C\sum_{k=0}^{n-1}\rho^{n-k}(q')_{i,j}^{(k)} \tag{1.20}$$

$$= Cn\rho^n\left(\frac{1}{n}\sum_{k=1}^{n-1}\rho^{-k}(q')_{i,j}^{(k)}\right) \tag{1.21}$$

However, by (1.19), $\rho^{-n}q_{i,j}^{(n)} \to 0$ when n tends to infinity and using Cesaro's lemma,

$$\frac{1}{n}\sum_{k=0}^{n-1}\rho^{-k}(q')_{i,j}^{(k)} \underset{n\to\infty}{\longrightarrow} 0$$

Thus using (1.18) and (1.21), we deduce that

$$\sum_{k=0}^{n-1}D_{k,n} = o(n\rho^n)$$

Hence, gathering all these results and using (1.17),

$$\sum_{k=0}^{n-1}\mathbb{E}_x(f(Y_k)\mathbb{1}_{\tau>n}) = n\rho^n\sum_{l=0}^{T-1}e^{-\frac{2inl\pi}{T}}\overline{w_l(x)} < v_l, g_l(f) >< v_l, \mathbb{1} > +o(n\rho^n) \tag{1.22}$$

However we have for any $l \in \{0,\dots,T-1\}$

$$\begin{aligned} < v_l, g_l(f) > &= \sum_{j=1}^{K} f(j)\overline{v_l(j)w_l(j)} \\ &= \sum_{j=0}^{T-1}\sum_{x\in C_j} f(x)\overline{v_l(x)w_l(x)} \\ &= \sum_{j=0}^{T-1}\sum_{x\in C_j} f(x)\overline{e^{-i\frac{2\pi jl}{T}}\nu(x)e^{i\frac{2\pi jl}{T}}\xi(x)} \\ &=< v_0, g_0(f) > \end{aligned}$$

As a result,

$$\mathbb{E}_x\left(\sum_{k=0}^{n-1}f(Y_k)\mathbb{1}_{\tau>n}\right) = n\rho^n < v_0, g_0(f) > \sum_{l=0}^{T-1}e^{-\frac{2inl\pi}{T}}\overline{w_l(x)} < v_l, \mathbb{1} > +o(n\rho^n)$$

□

Now we prove Lemma 1 quoted in the previous proof.

Proof of Lemma 1. Let us start by proving that α_l is a right-eigenvector associated to λ_l. Since Q is a real matrix, it is equivalent to show that $\overline{\alpha_l}$ is a right-eigenvector associated to $\overline{\lambda_l}$.
First remind that α_l is defined by the relations

$$\delta_k = \sum_{l=0}^{T-1} \alpha_l(k) v_l + \delta'_k$$

for any $k \in E_0$ and with $\delta'_k \in Span^{\perp}(v_0, \dots, v_{T-1})$. This implies for any k

$$< \delta_k, v_m >= \sum_{l=0}^{T-1} \overline{\alpha_l(k)} < v_l, v_m >$$

or, in other words,

$$\begin{pmatrix} < \delta_k, v_0 > \\ \vdots \\ < \delta_k, v_{T-1} > \end{pmatrix} = \begin{pmatrix} < v_0, v_0 > & \dots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \dots & < v_{T-1}, v_{T-1} > \end{pmatrix} \begin{pmatrix} \overline{\alpha_0(k)} \\ \vdots \\ \overline{\alpha_{T-1}(k)} \end{pmatrix}$$

Denote by A the matrix

$$A = \begin{pmatrix} < v_0, v_0 > & \dots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \dots & < v_{T-1}, v_{T-1} > \end{pmatrix}$$

A is simply the Gram's matrix of the basis $(v_i)_{0 \le i \le T-1}$. Thus the determinant $\det(A)$ is positive and for any $x \in E_0$

$$\overline{\alpha_l(x)} = \frac{1}{\det(A)} \begin{vmatrix} < v_0, v_0 > & \dots & < \delta_x, v_0 > & \dots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \dots & < \delta_x, v_{T-1} > & \dots & < v_{T-1}, v_{T-1} > \end{vmatrix}$$

where the column $(< \delta_x, v_0 >, \dots, < \delta_x, v_{T-1} >)^T$ is the l-th columns of the matrix. We want to show now that α_l is a right-eigenvector associated to λ_l, that is

$$\forall v \in \mathbb{C}^K, < v, Q\overline{\alpha_l} >= \overline{\lambda_l} < v, \overline{\alpha_l} > \tag{1.23}$$

In fact it is enough to show (1.23) when v is one of left-eigenvectors and when $v \in$

$Span^{\perp}(v_0, \ldots, v_{T-1})$. In the case where $v = v_k$ for $k \in \{0, \ldots, T-1\}$

$$
\begin{aligned}
< v_k, \overline{\alpha_l} > &= \sum_{j=1}^{K} \overline{v_k(j)} \frac{1}{\det(A)} \begin{vmatrix} < v_0, v_0 > & \ldots & < \delta_j, v_0 > & \ldots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \ldots & < \delta_j, v_{T-1} > & \ldots & < v_{T-1}, v_{T-1} > \end{vmatrix} \\
&= \frac{1}{\det(A)} \begin{vmatrix} < v_0, v_0 > & \ldots & < v_k, v_0 > & \ldots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \ldots & < v_k, v_{T-1} > & \ldots & < v_{T-1}, v_{T-1} > \end{vmatrix} \\
&= \begin{cases} 1 & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

We deduce from this

$$< v_k, Q\overline{\alpha_l} >= \overline{\lambda_l} < v_k, \overline{\alpha_l} >, \quad \forall\ 0 \leq k \leq T-1$$

Finally, for any $v \in Span(v_0, \ldots, v_{T-1})^{\perp}$,

$$< v, \alpha_l >= \frac{1}{\det(A)} \begin{vmatrix} < v_0, v_0 > & \ldots & 0 & \ldots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \ldots & 0 & \ldots & < v_{T-1}, v_{T-1} > \end{vmatrix} = 0$$

Thus we have

$$< v, Q\overline{\alpha_l} >= 0 = \overline{\lambda_l} < v_k, \overline{\alpha_l} >$$

because $\overline{{}^tQ}v \in Span(v_0, \ldots, v_{T-1})^{\perp}$.
Hence for each $k \in \{0, \ldots, T-1\}$, there is $\beta_k \in \mathbb{C}$ such that $\alpha_k = \beta_k w_k$ (where w_k is defined at the beginning of the subsection). We will show that $\beta_k = \beta_0 = 1$ for any k.
Remark that A can be written as $\sum_{i=1}^{T} a_{i-1} P_{\sigma_i}$ where P_{σ_i} is the permutation matrix of σ_i where $\sigma_i = (i\ i+1\ \ldots\ i-2\ i-1)$ and $a_0 > 0$ and $a_1, \ldots, a_{T-1} \in \mathbb{C}$. In other words, A is of the following shape

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{T-1} \\ a_{T-1} & a_0 & a_1 & \ldots & a_{T-2} \\ \vdots & \vdots & \ddots & \vdots & \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}$$

with $a_0 > 0$ and $a_1, \ldots, a_{T-1} \in \mathbb{C}^{T-1}$. Moreover, since $1 \in C_0$, $< \delta_1, v_l >=< \delta_1, v_0 >= \nu_1$

for any $l \in \{0, \dots, T-1\}$. As a result, for any $l \in \{0, \dots, T-1\}$,

$$\det(A)\alpha_l(1) = \begin{vmatrix} a_0 & \dots & \nu_1 & \dots & a_{T-1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_1 & \dots & \nu_1 & \dots & a_1 \end{vmatrix} \tag{1.24}$$

$$= \begin{vmatrix} \nu(1) & a_{l+1} & \dots & \dots & a_{l-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \nu(1) & a_{l+2} & \dots & \dots & a_l \end{vmatrix} \tag{1.25}$$

$$= \begin{vmatrix} \nu(1) & a_1 & \dots & \dots & a_{T-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \nu(1) & a_2 & \dots & \dots & a_0 \end{vmatrix} \tag{1.26}$$

$$= \det(A)\alpha_0(1) \tag{1.27}$$

Indeed, from (1.24) to (1.25), we applied a circular permutation for the columns in order to put the vector ${}^t(\nu(1), \dots, \nu(1))$ at the first column, and the determinant stays the same after this transformation. From (1.25) to (1.26), we did a circular permutation on the rows, which does not affect either the determinant.
We deduce from this equality that $\beta_k = \beta_0$ for any $k \in \{0, \dots, T-1\}$ because $w_k(1) = w_0(1)$. Concerning the fact that $\beta_0 = 1$, remark that

$$\begin{aligned} \sum_{i=1}^{K} \nu(i)\overline{\alpha_0(i)} &= \sum_{i=1}^{K} \overline{v_0(i)\alpha_0(i)} \\ &= \frac{1}{\det(A)} \begin{vmatrix} < v_0, v_0 > & \dots & < v_{T-1}, v_0 > \\ \vdots & \ddots & \vdots \\ < v_0, v_{T-1} > & \dots & < v_{T-1}, v_{T-1} > \end{vmatrix} \\ &= 1 \end{aligned}$$

And

$$\sum_{i=1}^{K} \nu(i)\overline{\alpha_0(i)} = \beta_0 \sum_{i=1}^{K} \nu(i)\xi(i) = 1$$

□

The statement of Theorem 1 is meaningful provided the coefficient of the leading term ρ^n is not equal to 0. In the following proposition we prove that this coefficient is actually not 0.

Proposition 9. *For any $n \in \mathbb{N}$ and any x*

$$\sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T}} < w_l, \delta_x >< v_l, \mathbb{1} > \neq 0$$

Proof. Let $x \in E_0$. Then there exists $k \in \{0, \dots, T-1\}$ such that $x \in C_k$. Thus, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T}} \overline{w_l(x)} < v_l, \mathbb{1} > &= \sum_{l=0}^{T-1} e^{-\frac{2i(n+k)l\pi}{T}} \xi(x) \left(\sum_{j=0}^{T-1} \sum_{y \in C_j} e^{\frac{2i\pi lj}{T}} \nu(y) \right) \\
&= \sum_{j=0}^{T-1} \sum_{y \in C_j} \xi(x)\nu(y) \left(\sum_{l=0}^{T-1} e^{-\frac{2i\pi(n+k-j)l}{T}} \right) \\
&= T \sum_{T | n+k-j} \sum_{y \in C_j} \xi(x)\nu(y) \\
&\quad + \underbrace{\sum_{T \nmid n+k-j} \sum_{y \in C_j} \xi(x)\nu(y) e^{\frac{i\pi(n+k-j)(T-1)}{T}} \frac{\sin(\pi(n+k-j))}{\sin(\frac{\pi(n+k-j)}{T})}}_{=0} \\
&= T \sum_{T | n+k-j} \sum_{y \in C_j} \xi(x)\nu(y) > 0
\end{aligned}$$

□

1.4.2 Quasi-ergodic distribution for the classical non-moving sense in the general case

Now assume that the sub-transition matrix Q is not necessarily irreducible. For each $x \in \{1, \dots, K\}$, we denote by D_x the subset of $\{1, \dots, K\}$ defined by

$$D_x := \{y \in \{1, \dots, K\} : \exists n, m \in \mathbb{N}, \mathbb{P}_x(Y_n = y) > 0 \text{ and } \mathbb{P}_y(Y_m = x) > 0\}$$

It is well-known that $(D_x)_{x \in \{1,\dots,K\}}$ are equivalence classes. Note that, for each x, the restriction of Y on D_x is irreducible. Thus we can associate, for each D_x, a period T_x. We can also associate to D_x a spectral radius ρ_x and some left and right-eigenvectors $(v_{x,l})_{0 \le l \le T_x - 1}$ and $(w_{x,l})_{0 \le l \le T_x - 1}$ constructed in the same way as in the subsection 1.4.1. Particularly, for every $x \in \{1, \dots, K\}$, $\nu_x := v_{x,0}$ and $\xi_x := w_{x,0}$ are vectors whose all the components are positive and such that $< \nu_x, \mathbb{1} > = < \nu_x, \xi_x > = 1$. We define also, for any x,

$$\varphi_x : f \to \sum_{j=1}^{|D_x|} f(j)\nu_x(j)\xi_x(j)$$

where $|D_x|$ is the number of elements in D_x. Now fix $\mu \in \mathcal{M}_1(\{1, \dots, K\})$. Denote by $Supp(\mu)$ the support of μ. Then we can define

$$\mathcal{B} = \{x \in \{1, \dots, K\} : Supp(\mu) \cap D_x \neq \emptyset\}$$

$$\rho_{max} = \max_{x \in \mathcal{B}} \rho_x$$

and we define $\mathcal{B}_{max}$ as follows

$$\mathcal{B}_{max} = \{x \in \mathcal{B} : \rho_x = \rho_{max}\}$$

We set the following hypothesis

Hypothesis 1. *There exists $x_{max} \in \{1, \dots, K\}$ such that*

$$\mathcal{B}_{max} = D_{x_{max}}$$

Under this hypothesis, the following notation will be used

$$\nu_{max} = \nu_{x_{max}} \tag{1.28}$$

$$\xi_{max} = \xi_{x_{max}} \tag{1.29}$$

$$\varphi_{max} = \varphi_{x_{max}} \tag{1.30}$$

In all what follows, we have to keep in mind that the definition of $\mathcal{B}_{max}$ implicitly depends on the initial distribution μ (more precisely on the support of μ).
The following statement explains therefore that the quasi-ergodic distribution exists if the Hypothesis 1 holds.

Theorem 2. *Let $\mu \in \mathcal{M}_1(\{1, \dots, K\})$. Then, if the Hypothesis 1 holds, the following convergence holds for any measurable bounded function $f : \{1, \dots, K\} \to \mathbb{R}$,*

$$\mathbb{E}_\mu \left(\frac{1}{n} \sum_{k=0}^{n-1} f(Y_k) | \tau > n \right) \underset{n \to \infty}{\longrightarrow} \varphi_{max}(f)$$

Proof. According to Proposition 8, giving the fact that Y is irreducible on each D_x, we have for any $x \in \{1, \dots, K\}$

$$\mathbb{E}_x \left(\frac{1}{n} \sum_{k=0}^{n-1} f(Y_k) \mathbb{1}_{\tau > n} \right) = \rho_x^n \varphi_x(f) \sum_{l=0}^{T-1} e^{-\frac{2inl\pi}{T_x}} < w_{x,l}, \delta_x >< v_{x,l}, \mathbb{1} > + o(\rho_x^n) \tag{1.31}$$

Thus, for any $\mu \in \mathcal{M}_1(E)$

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)|\tau > n\right) = \frac{\sum_{j=1}^{K}\mu(j)\mathbb{E}_j\left(\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)\mathbb{1}_{\tau>n}\right)}{\sum_{j=1}^{K}\mu(j)\mathbb{P}_j(\tau > n)}$$

$$= \frac{\sum_{j=1}^{K}\mu(j)\rho_j^n\varphi_j(f)\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(\rho_j^n)}{\sum_{j=1}^{K}\mu(j)\rho_j^n\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(\rho_j^n)}$$

$$= \frac{\sum_{j\in\mathcal{B}_{max}}\varphi_j(f)\mu(j)\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(1)}{\sum_{j\in\mathcal{B}_{max}}\mu(j)\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(1)}$$

$$= \frac{\varphi_{max}(f)\sum_{j\in\mathcal{B}_{max}}\mu(j)\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(1)}{\sum_{j\in\mathcal{B}_{max}}\mu(j)\sum_{l=0}^{T_j-1} e^{-\frac{2inl\pi}{T_j}} < w_{j,l}, \delta_x >< v_{j,l}, \mathbb{1} > +o(1)}$$

where the Hypothesis 1 was used for the last equality, implying that $\varphi_j(f) = \varphi_{max}(f)$ for all $j \in \mathcal{B}_{max}$. Note moreover that this hypothesis is useful only to make this equality right.

Then using Proposition 9, we can conclude

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \varphi_{max}(f)$$

□

1.4.3 Quasi-ergodic distribution with periodic moving killing subset

In this subsection we are interested in the quasi-ergodicity of the chain X defined in the Introduction considering that the boundaries are moving γ-periodically. We denote by $Y = (Y_n)_{n\in\mathbb{N}}$ the Markov chain defined on $E \times \mathbb{Z}/\gamma\mathbb{Z}$ by

$$Y_n = (X_n, \overline{n}) \tag{1.32}$$

where $\bar{n}$ is the residue of n, modulo γ. Y is therefore a Markov chain defined on a finite space state, which is irreducible if and only if $gcd(T(X), \gamma) = 1$, where $T(X)$ is the period of $(X_n)_{n\in\mathbb{N}}$. If the chain Y is actually irreducible, the associated period is

$$T = LCM(T(X), \gamma)$$

where $LCM(\cdot,\cdot)$ refers to the least common multiple.
Moreover we have

$$\tau = \inf\{n \geq 0 : X_n \in A_n\} = \inf\{n \geq 0 : Y_n \in \partial\}$$

with

$$\partial := \{(x, \overline{k}) : x \in A_k\}$$

Remark that ∂ is a non moving killing subset for the chain Y. Thus we can apply Theorem 2 to the process Y which yields the following theorem

Theorem 3. *Let $\mu \in \mathcal{M}_1(E_0)$. Assume that $(A_n)_{n\in\mathbb{N}}$ is periodic and Y defined in* (1.32) *satisfies the Hypothesis 1. Then, for any measurable bounded function f,*

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \sum_{(x,y)\in E\times\mathbb{Z}/\gamma\mathbb{Z}-\partial} f(x)\nu_{max}(x,y)\xi_{max}(x,y)$$

where ν_{max} and ξ_{max} are the probability measures defined in (1.28) *and* (1.29) *relatively to Y.*

We can also give the following corollary which requires assumptions on X and $(A_n)_{n\in\mathbb{N}}$.

Corollary 1. *Assume that $(A_n)_{n\in\mathbb{N}}$ is γ-periodic and that $gcd(T,\gamma) = 1$ (where T is the period of X). Then there exists $\eta \in \mathcal{M}_1(E)$ such that, for any $\mu \in \mathcal{M}_1(E_0)$ and any f bounded measurable,*

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \int f d\eta$$

Proof of Theorem 3. It is enough to apply Theorem 2 to the chain Y defined on (1.32) and to deduce the results on X thanks to the following equality

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)|\tau > n\right) = \mathbb{E}_{\mu\otimes\delta_{\overline{0}}}\left(\frac{1}{n}\sum_{k=0}^{n-1} \tilde{f}(Y_k)|\tau > n\right), \quad \forall n \geq 1$$

where $\tilde{f}$ is the projection on the first component. □

1.5 Existence of Q-process with boundaries moving periodically

In this section, we are interested in the Q-process, which can be interpreted as the law of the process X conditioned never to be killed by the moving boundary. As before, we still consider that the boundary moves periodically period γ.
To show the existence of the Q-process, we will consider again the Markov chain Y defined in (1.32), that is defined by

$$Y_n = (X_n, \overline{n}), \quad \forall n \in \mathbb{N}$$

and we take back the notation introduced in subsection 1.4.2 associated to Y.
The following statement ensures the existence of a Q-process even when the boundary moves. However, it is interesting to observe that we lose the homogeneity of the Q-process because of the movement of the killing boundary.

Theorem 4. *For any $n \in \mathbb{N}$ and any $x \in E_0$, the probability measure $\mathbb{Q}_x$ defined by*

$$\mathbb{Q}_x(X_1 \in \cdot, \dots, X_n \in \cdot) = \lim_{m\to\infty} \mathbb{P}_x(X_1 \in \cdot, \dots, X_n \in \cdot | \tau > m)$$

is well-defined and, under the probability $\mathbb{Q}_x$, $(X_n)_{n\in\mathbb{N}}$ is a time-inhomogeneous Markov chain such that for any $n \in \mathbb{N}$, for any $(y, z) \in E_{n-1} \times E_n$

$$\mathbb{Q}_x(X_n = z | X_{n-1} = y) = \frac{\xi_x(z, \overline{n})}{\rho_x \xi_x(y, \overline{n-1})} \mathbb{P}_y(X_1 = z, \tau_{A_n} > 1)$$

Proof. For any $m, n \in \mathbb{N}$, for any $f_1, \dots, f_n$ measurable bounded functions and for any $x \in E_0$,

$$\mathbb{E}_x(f_1(Y_1) \dots f_n(Y_n) | \tau > n+m) = \frac{\mathbb{E}_x(f_1(Y_1) \dots f_n(Y_n) \mathbb{1}_{\tau>n+m})}{\mathbb{P}_x(\tau > n+m)} \tag{1.33}$$

$$= \mathbb{E}_x \left(f_1(Y_1) \dots f_n(Y_n) \frac{\mathbb{1}_{\tau>n} \mathbb{P}_{Y_n}(\tau > m)}{\mathbb{P}_x(\tau > n+m)} \right) \tag{1.34}$$

According to the equality (1.31) applied to the function equal to 1, for any $y \in E \times Z/\gamma\mathbb{Z} - \partial$ and $n \in \mathbb{N}$,

$$\mathbb{P}_y(\tau > n) = \rho_y^n \sum_{l=0}^{T_y - 1} e^{-\frac{2inl\pi}{T_y}} < w_{y,l}, \delta_y >< v_{y,l}, \mathbb{1} > + o(\rho_y^n)$$

Thus, using this in (1.34),

$$\mathbb{E}_x(f_1(Y_1) \dots f_n(Y_n) | \tau > m+n) \tag{1.35}$$

$$= \mathbb{E}_x \left(f_1(Y_1) \dots f_n(Y_n) \frac{\mathbb{1}_{\tau>n} \rho_{Y_n}^m \sum_{l=0}^{T_{Y_n}-1} e^{-\frac{2iml\pi}{T_{Y_n}}} < w_{Y_n,l}, \delta_{Y_n} >< v_{Y_n,l}, \mathbb{1} > + o(\rho_{Y_n}^m)}{\rho_x^{n+m} \sum_{l=0}^{T_x-1} e^{-\frac{2i(n+m)l\pi}{T_x}} < w_{x,l}, \delta_x >< v_{x,l}, \mathbb{1} > + o(\rho_x^{n+m})} \right) \tag{1.36}$$

$$= \mathbb{E}_x \left(f_1(Y_1) \dots f_n(Y_n) \frac{\mathbb{1}_{\tau>n} \rho_x^m \sum_{l=0}^{T_x-1} e^{-\frac{2iml\pi}{T_x}} < w_{x,l}, \delta_{Y_n} >< v_{x,l}, \mathbb{1} > + o(\rho_x^m)}{\rho_x^{n+m} \sum_{l=0}^{T_x-1} e^{-\frac{2i(n+m)l\pi}{T_x}} < w_{x,l}, \delta_x >< v_{x,l}, \mathbb{1} > + o(\rho_x^{n+m})} \right) \tag{1.37}$$

$$= \mathbb{E}_x \left(f_1(Y_1) \dots f_n(Y_n) \frac{\mathbb{1}_{\tau>n} \sum_{l=0}^{T_x-1} e^{-\frac{2iml\pi}{T_x}} < w_{x,l}, \delta_{Y_n} >< v_{x,l}, \mathbb{1} > + o(1)}{\rho_x^n \sum_{l=0}^{T_x-1} e^{-\frac{2i(n+m)l\pi}{T_x}} < w_{x,l}, \delta_x >< v_{x,l}, \mathbb{1} > + o(\rho_x^n)} \right) \tag{1.38}$$

The passage from (1.36) to (1.37) is due to the fact that, for any $n \in \mathbb{N}$, $Y_n \in D_x$ almost surely and the quantities T_x, ρ_x, $w_{x,l}$ and $v_{x,l}$ depends only on D_x.
Since the restriction of the chain Y on D_x is irreducible, we can construct as in the subsection 1.4.1 some clusters $(C_j)_{0 \leq j \leq T_x - 1}$ such that $x \in C_0$ and

$$\mathbb{P}_x(Y_{k+nT_x} \in C_k, \tau > k + nT_x) = 1, \quad \forall k \in \{0, \ldots, T_x - 1\}, \forall n \in \mathbb{N}$$

For any $y \in D_x$, denote by $j(y)$ the integer such that $y \in C_{j(y)}$. Then we deduce from the equality (1.12) in the subsection 1.4.1 that for any $y \in E \times \mathbb{Z}/\gamma\mathbb{Z} - \partial$ and $n \in \mathbb{N}$,

$$e^{-\frac{2inl\pi}{T_x}} < w_{x,l}, \delta_y >= e^{-\frac{2i\pi(n+j(y))l}{T_x}} \xi_x(y)$$

Thus, according to (1.38) and the previous equality,

$$\begin{aligned}
&\mathbb{E}_x(f_1(Y_1) \ldots f_n(Y_n) | \tau > m + n) \\
&= \mathbb{E}_x \left(f_1(Y_1) \ldots f_n(Y_n) \frac{\mathbb{1}_{\tau > n} \xi_x(Y_n) \left(\sum_{l=0}^{T_x - 1} e^{-\frac{2i\pi(m+j(Y_n))l}{T_x}} < v_{x,l}, \mathbb{1} > + o(1) \right)}{\rho_x^n \xi_x(x) \left(\sum_{l=0}^{T_x - 1} e^{-\frac{2i\pi(m+n+j(x))l}{T_x}} < v_{x,l}, \mathbb{1} > + o(1) \right)} \right)
\end{aligned}$$

However, for any $n \in \mathbb{N}$,

$$j(Y_n) = j(x) + n \mod T_x, \quad a.s.$$

and for any $m, n \in \mathbb{N}$,

$$\sum_{l=0}^{T_x - 1} e^{-\frac{2i\pi(m+n+j(x))l}{T_x}} < v_{x,l}, \mathbb{1} > \neq 0$$

Since the state space $E \times \mathbb{Z}/\gamma\mathbb{Z}$ is finite, we may first consider function $f_i(y) = \mathbb{1}_{y = x_i}$, so that quantities in the ratio except $\mathbb{1}_{\tau > n}$ are fixed. This justifies that we can exchange the expectation and the limit as $n \to \infty$ in the previous expression. We deduce that,

$$\mathbb{E}_x(f_1(Y_1) \ldots f_n(Y_n) | \tau > m + n) \underset{m \to \infty}{\longrightarrow} \mathbb{E}_x \left(f_1(Y_1) \ldots f_n(Y_n) \frac{\mathbb{1}_{\tau > n} \xi_x(Y_n)}{\rho_x^n \xi_x(x, \overline{0})} \right)$$

The statement on X is obtained using projection functions and we can deduce from it the transition kernel of the Q-process.

□

1.6 Example : discrete-time random walk

We shall illustrate the previous results by looking at a discrete-time random walk. Let $p \in]0, 1[$. We denote by $(M_n^p)_{n \in \mathbb{N}}$ the Markov chain defined on $\mathbb{Z}$ such that

$$\begin{aligned}
\mathbb{P}(M_{n+1}^p = M_n^p + 1 | M_n^p) &= 1 - p \\
\mathbb{P}(M_{n+1}^p = M_n^p - 1 | M_n^p) &= p
\end{aligned}$$

Before dealing with the quasi-ergodicity with moving boundaries, let us recall some properties about quasi-stationarity concerning random walks. For any $K \geq 1$ we define

$$T_K = \inf\{n \geq 0 : M_n^p \in (-\infty, 0] \cup [K+1, \infty)\}$$

The sub-Markovian transition matrix associate to $(M^p_{n\wedge T_K})_{n\in\mathbb{N}}$ is the matrix $Q_K \in M_K(\mathbb{R})$ defined by :

$$Q_K = \begin{pmatrix} 0 & 1-p & 0 & \dots & 0 & 0 \\ p & 0 & 1-p & \dots & 0 & 0 \\ 0 & p & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1-p \\ 0 & 0 & 0 & \dots & p & 0 \end{pmatrix}$$

For any $K \geq 1$, denote by $P_K(X)$ the characteristic polynomial of Q_K. Using standard algebraic manipulations, one can show that for any $K \geq 1$, the following recurrence relation is satisfied

$$P_{K+2}(X) = -XP_{K+1}(X) - p(1-p)P_K(X)$$

with $P_1(X) = -X$ and $P_2(X) = X^2 - p(1-p)$. We set $P_0(X) = 1$.
For any $K \geq 0$, define

$$U_K(X) = \left(-\frac{1}{\sqrt{p(1-p)}}\right)^K P_K\left(2\sqrt{p(1-p)}X\right)$$

Then the following equation is satisfied

$$U_{K+2}(X) = 2XU_{K+1}(X) - U_K(X)$$

for which $U_0(X) = 1$ and $U_1(X) = 2X$. In other words, the sequence $(U_K)_{K\geq 0}$ are the Chebyshev's polynomials of the second kind and we have for any $\theta \in \mathbb{R}$

$$U_K(\cos(\theta)) = \frac{\sin((K+1)\theta)}{\sin(\theta)}$$

The set of roots of U_K, hence of P_K, is thus well-known. It follows

$$Sp(Q_K) = \left\{\lambda_j := 2\sqrt{p(1-p)}\cos\left(\frac{j\pi}{K+1}\right) : j \in \{1, \dots, K\}\right\}$$

We are interested now in the eigenvectors of Q_K.

Proposition 10. *Let $K \geq 1$. Then, for any $j \in \{1, \dots, K\}$, $Ker(Q_K - \lambda_j I_k) = Span(x_j)$ where*

$$x_j(i) = \left(-\frac{1}{1-p}\right)^{i-1} P_{i-1}(\lambda_j) = \left(\sqrt{\frac{p}{1-p}}\right)^{i-1} \frac{\sin\left(\frac{ij\pi}{K+1}\right)}{\sin\left(\frac{j\pi}{K+1}\right)}, \quad \forall i \in \{1, \dots, K\}$$

Proof. Let $\lambda \in Sp(Q_K)$. We want to find all the eigenvectors $x = (x(i))_{1\leq i\leq K}$ associated to λ such that $x(1) = 1$. We will prove the proposition by double induction.

Base case: According to the relation $Q_K x = \lambda x$, we have

$$\lambda x(1) = (1-p)x(2) \tag{1.39}$$

Having $x(1) = 1$, we will have therefore $x(2) = \frac{\lambda}{1-p} = -\frac{1}{1-p}P_1(\lambda)$, which conclude the base case

Inductive step: Let $i \in \{3, \dots, K-1\}$. We assume that the equality is satisfied for $i-1$ and $i-2$, so we have

$$x(i-2) = \left(-\frac{1}{1-p}\right)^{i-3} P_{i-3}(\lambda)$$
$$x(i-1) = \left(-\frac{1}{1-p}\right)^{i-2} P_{i-2}(\lambda)$$

Using $\lambda x = Q_K x$,

$$\lambda x(i-1) = px(i-2) + (1-p)x(i)$$

So

$$\begin{aligned} x(i) &= \frac{1}{1-p}\left(\lambda x(i-1) - px(i-2)\right) \\ &= \frac{1}{1-p}\left(\lambda\left(-\frac{1}{1-p}\right)^{i-2} P_{i-2}(\lambda) - px(i-2)\right) \\ &= \left(-\frac{1}{1-p}\right)^{i-1}\left(-\lambda P_{i-2}(\lambda) - p(1-p)P_{i-3}(\lambda)\right) \\ &= \left(-\frac{1}{1-p}\right)^{i-1} P_{i-1}(\lambda) \end{aligned}$$

which concludes the proof. □

The previous proposition gives us left and right eigenvectors of Q_K : if we denote by $(v_i)_{1\leq i\leq K}$ (respectively $(w_i)_{1\leq i\leq K}$) the left (respectively right) eigenvectors satisfying $v_i Q_K = \lambda_i v_i$ (respectively $Q_K w_i = \lambda_i w_i$), then

$$v_i(j) = \left(\sqrt{\frac{1-p}{p}}\right)^{j-1} \frac{\sin\left(\frac{ij\pi}{K+1}\right)}{\sin\left(\frac{i\pi}{K+1}\right)}$$
$$w_i(j) = \left(\sqrt{\frac{p}{1-p}}\right)^{j-1} \frac{\sin\left(\frac{ij\pi}{K+1}\right)}{\sin\left(\frac{i\pi}{K+1}\right)}$$

In particular, considering the spectral radius λ_1, the quasi-stationary distribution ν and the right-eigenvector ξ associated to λ_1 satisfying $< \nu, \xi >= 1$ are as follows:

$$\nu(j) = \frac{\left(\sqrt{\frac{1-p}{p}}\right)^{j-1} \sin\left(\frac{j\pi}{K+1}\right)}{\sum_{k=1}^{K} \left(\sqrt{\frac{1-p}{p}}\right)^{k-1} \sin\left(\frac{k\pi}{K+1}\right)}$$

$$\xi(j) = \frac{\sum_{k=1}^{K} \left(\sqrt{\frac{1-p}{p}}\right)^{k-1} \sin\left(\frac{k\pi}{K+1}\right)}{\sum_{k=1}^{K} \sin^2\left(\frac{k\pi}{K+1}\right)} \left(\sqrt{\frac{p}{1-p}}\right)^{j-1} \sin\left(\frac{j\pi}{K+1}\right)$$

We are interested now in moving boundaries. Let $N \geq 1$ and consider the simplest case where $(A_n)_{n\in\mathbb{N}}$ is defined by

$$A_n = \begin{cases} (-\infty, 0] \cup [2N, \infty) & \textit{if } n \textit{ is even} \\ (-\infty, 1] \cup [2N-1, \infty) & \textit{if } n \textit{ is odd} \end{cases} \tag{1.40}$$

Recall the previous notation

$$Y_n^p = (M^p_{n\wedge\tau_0}, \overline{n}) \tag{1.41}$$

with $\overline{n} \in \mathbb{Z}/2\mathbb{Z}$. The chain is not irreducible (if M_0^p is even, then for any n, M_n^p have the same parity as n). It admits exactly two irreducible subsets:

1. $\mathcal{P} = \{(x, y) \in E : x + y \text{ is even}\}$
2. $\mathcal{I} = \{(x, y) \in E : x + y \text{ is odd}\}$

But, as we can see in Figure 1.1, the chain Y^p behaves as a random walk on each irreducible subsets:

1. On $\mathcal{P}$, Y^p has the same behavior as a random walk on $\mathbb{Z}$ starting from $[2, 2N-2]$ absorbed by $\{1, 2N-1\}$.
2. On $\mathcal{I}$, Y^p has the same behavior as a random walk on $\mathbb{Z}$ starting from $[1, 2N-1]$ absorbed by $\{0, 2N\}$.

Denote by $Y^p_{\mathcal{P}}$ (respectively $Y^p_{\mathcal{I}}$) the Markov chain such that for any $\mu \in \mathcal{M}_1(\mathcal{P})$ (respectively $\mathcal{M}_1(\mathcal{I})$)

$$\mathbb{P}_\mu(Y_1^p \in \cdot) = \mathbb{P}_\mu((Y^p_{\mathcal{P}})_1 \in \cdot) \text{ (respectively } \mathbb{P}_\mu(Y_1^p \in \cdot) = \mathbb{P}_\mu((Y^p_{\mathcal{I}})_1 \in \cdot))$$

Let $\mu \in \mathcal{M}_1(E \times \mathbb{Z}/2\mathbb{Z})$. Then there are $\lambda \in [0, 1]$ and $\mu_{\mathcal{P}}, \mu_{\mathcal{I}} \in \mathcal{M}_1(\mathcal{P}) \times \mathcal{M}_1(\mathcal{I})$ such that

$$\mu = \lambda\mu_{\mathcal{P}} + (1 - \lambda)\mu_{\mathcal{I}}$$

Hence we see that two cases are possible

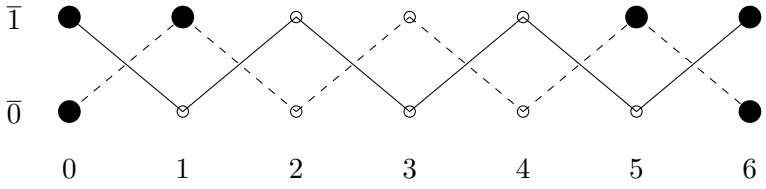

Figure 1.1: The black dots represent the states in ∂. The irreducible subsets $\mathcal{P}$ and $\mathcal{I}$ are represented respectively by the dashed path and the filled path. On each path, we see that Y^p behaves as a random walk.

Proposition 11. *1. if $\lambda = 1$, $\mathcal{B}_{max} = \mathcal{P}$. Then $\rho_{max} = 2\sqrt{p(1-p)}\cos\left(\frac{\pi}{2(N-1)}\right)$, and*

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(M_k^p)|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \sum_{j=2}^{2N-2} f(j)\frac{\sin^2\left(\frac{(j-1)\pi}{2(N-1)}\right)}{\sum_{k=1}^{2N-3}\sin^2\left(\frac{k\pi}{2(N-1)}\right)}$$

2. if $\lambda \neq 1$, $\mathcal{B}_{max} = \mathcal{I}$. Then $\rho_{max} = 2\sqrt{p(1-p)}\cos\left(\frac{\pi}{2N}\right)$, and

$$\mathbb{E}_\mu\left(\frac{1}{n}\sum_{k=0}^{n-1} f(M_k^p)|\tau > n\right) \underset{n\to\infty}{\longrightarrow} \sum_{j=1}^{2N-1} f(j)\frac{\sin^2\left(\frac{j\pi}{2N}\right)}{\sum_{k=1}^{2N-1}\sin^2\left(\frac{k\pi}{2N}\right)}$$

When $(A_n)_{n\in\mathbb{N}}$ is moving as (1.40), the quasi-ergodic distribution is the same as the non-moving quasi-ergodic distribution for one random walk absorbed at $\{0, 2N\}$ except when the support of the initial distribution is included in the set of even numbers. As a matter of fact, if the chain starts from the set of even numbers, it can be absorbed only by $\{1, 2N-1\}$. Remark also that the quasi-ergodic distribution of one random walk does not depend on p anymore.

We have also the existence of a Q-process according to Theorem 4 which is the time-inhomogeneous Markov chain $(Z_n^p)_{n\in\mathbb{N}}$ defined by

$$\mathbb{P}_x(Z_n^p = y \pm 1 | Z_{n-1}^p = y) = \frac{\sin\left(\frac{(y\pm1)\pi}{K(y,n)}\right)}{2\sin\left(\frac{y\pi}{K(y,n)}\right)\cos\left(\frac{\pi}{K(y,n)}\right)}$$

with $K(y,n) = 2N - 1 + (-1)^{n+y}$.

Chapter 2

Q-process and asymptotic properties for Markov processes conditioned not to hit the moving boundaries

This chapter is based on the paper [44], submitted in March 2018.

Abstract

We investigate some asymptotic properties of general Markov processes conditioned not to be absorbed by the moving boundaries. We first give general criteria involving an exponential convergence towards the Q-process, that is the law of the considered Markov process conditioned never to reach the moving boundaries. This exponential convergence allows us to state the existence and uniqueness of quasi-ergodic distribution considering either boundaries moving periodically or stabilizing boundaries. We also state the existence and uniqueness of quasi-limiting distribution when absorbing boundaries stabilize. We finally deal with some examples such as diffusions which are coming down from infinity.

2.1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(X_t)_{t\in I}$ be a Markov process (where $I = \mathbb{Z}_+$ or $\mathbb{R}_+$) defined on a state space E. We associate to E a σ-algebra $\mathcal{E}$. For any $t \in I$, denoted by $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t \in I)$ the σ-field generated by $(X_s)_{0\leq s\leq t\in I}$. Let $(\mathbb{P}_x)_{x\in E}$ be the probability distribution of X such that, for any $x \in E$, $\mathbb{P}_x(X_0 = x) = 1$ and, for any measure μ on E, define $\mathbb{P}_\mu = \int \mathbb{P}_x d\mu(x)$. We naturally denote by $\mathbb{E}_x$ and $\mathbb{E}_\mu$ the corresponding expectations. For any subset $F \subset E$, denote by $\mathcal{M}_1(F)$ the set of probability measures defined on F and $\mathcal{B}(F)$ the set of the bounded measurable function

$f : F \to \mathbb{R}$.
We define, for each time $t \in I$, a subset $A_t \in \mathcal{E}$ called *absorbing subset at time t* and we denote by E_t the complement set of A_t called *survival subset at time t*. We will call $t \to A_t$ *the moving absorbing subset* or *the moving absorbing boundary*. We denote by τ_A the random variable defined as follows

$$\tau_A := \inf\{t \in I : X_t \in A_t\}$$

Moreover, for any $s \in I$, we denote θ_s the shift operator, that is the function from I to I defined by

$$\theta_s : t \to s + t$$

Hence, for any $s \in I$, we define

$$\tau_{A\circ\theta_s} := \inf\{t \in I : X_t \in A_{t+s}\}$$

In all what follows, assume that, for any $s \in I$, $\tau_{A\circ\theta_s}$ is a stopping time for the filtration $(\mathcal{F}_t)_{t\in I}$. This assumption holds when, for example, the Markov process $(X_t)_{t\in I}$, defined on a metric space, is continuous and all the sets $(A_t)_{t\in I}$ are closed.

In this chapter, we will deal with the so-called *Q-process*, *quasi-limiting distribution* and *quasi-ergodic distribution* of X:

Definition 8. Definition of Q-process, quasi-limiting distribution and quasi-ergodic distribution

(i) We say that there is a *Q-process* if there exists a family of probability measures $(\mathbb{Q}_{s,x})_{s\in I, x\in E_s}$ such that for any $s, t \in I$, $x \in E_s$,

$$\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t + T) \xrightarrow[T\in I, T\to\infty]{(d)} \mathbb{Q}_{s,x}(X_{[s,s+t]} \in \cdot)$$

where, for any $u, v \in I$, $X_{[u,v]}$ is the trajectory of $(X_t)_{t\in I}$ between times u and v and where (d) refers to the weak convergence of probability measures. Then the Q-process is the law of $(X_t)_{t\in I}$ under $(\mathbb{Q}_{s,x})_{s\geq 0, x\in E_s}$.

(ii) We say that $\alpha \in \mathcal{M}_1(E)$ is a *quasi-limiting distribution* if, for some $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t\in I, t\to\infty]{(d)} \alpha$$

(iii) We say that $\beta \in \mathcal{M}_1(E)$ is a *quasi-ergodic distribution* if there exists some $\mu \in \mathcal{M}_1(E_0)$ such that,

- $$\frac{1}{n}\sum_{k=0}^{n} \mathbb{P}_\mu(X_k \in \cdot | \tau_A > n) \xrightarrow[n\to\infty]{(d)} \beta$$

  if $I = \mathbb{Z}_+$

- $$\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t\to\infty]{(d)} \beta$$

  if $I = \mathbb{R}_+$.

In the case where $(A_t)_{t\in I}$ does not depend on the time t, Q-process, quasi-limiting and quasi-ergodic distribution are strongly related to the theory of quasi-stationary distributions, i.e. probability measures α such that

$$\mathbb{P}_\alpha(X_t \in \cdot | \tau_A > t) = \alpha, \quad \forall t \in I \tag{2.1}$$

The aim of this chapter is to provide some results about existence and uniqueness of Q-process, quasi-limiting and quasi-ergodic distribution when $(A_t)_{t\in I}$ does depend on time. It has been already shown in [45] that the quasi-stationary distributions defined as in (2.1) do not exist when A is moving under the following assumption of irreducibility:

$$\forall t \in I, \forall x, y \in E_t, \forall \epsilon > 0, \exists t_0 \in I, \mathbb{P}_x(X_{t_0 \wedge \tau_{A_t}} \in B(y, \epsilon)) > 0 \tag{2.2}$$

where $\tau_{A_t} := \inf\{s \in I : X_s \in A_t\}$ and $B(y, \epsilon) := \{z \in E : d(z, y) < \epsilon\}$.

When $(A_t)_{t\in I}$ does not depend on t, quasi-stationary distributions and quasi-limiting distributions are equivalent. When A depends on time, these two notions are not equivalent anymore and quasi-limiting distributions could exist even if quasi-stationary distributions do not. It is shown in [45] that quasi-limiting distributions do not exist if we assume that A is periodic (still assuming the assumption of irreducibility in (2.2)), but a characterization of the existence of quasi-limiting distributions according to the behavior of the moving boundary is still an open question. In what follows, we will deal with quasi-limiting distributions considering converging moving boundaries.

In the non-moving case, existence of quasi-limiting distributions and Q-processes have been shown for several processes : Markov chains on finite state space and countable space, birth and death processes, diffusion processes and others. See [42, 19] for an overview. In the same way, existence of quasi-ergodic distributions has been also shown for such processes. The reader can see [28, 58, 9] for the study on quasi-ergodic distribution in a very general framework.

In this chapter, the study of Q-processes, quasi-limiting and quasi-ergodic distributions will be based on Champagnat-Villemonais type condition : when A does not depend on t, Champagnat and Villemonais show in [14] that if there exists $\nu \in \mathcal{M}_1(E)$ such that

(A1) there exist $t_0 \geq 0$ and $c_1 > 0$ such that

$$\forall x \in E_0, \quad \mathbb{P}_x(X_{t_0} \in \cdot | \tau_A > t_0) \geq c_1 \nu$$

(A2) there exists $c_2 > 0$ such that : $\forall x \in E_0$, $\forall t \geq 0$,

$$\mathbb{P}_\nu(\tau_A > t) \geq c_2 \mathbb{P}_x(\tau_A > t)$$

then there is an exponential convergence to a unique quasi-stationary distribution and there exists a Q-process. In the recent paper [17], they improve significantly their previous results showing an exponential convergence of $\mathbb{P}_x(X_{[0,s]} \in \cdot | \tau_A > t+s)$ towards the Q-process, which entails a uniform convergence of the conditioned mean ratio (starting from any initial law) towards the unique quasi-ergodic distribution. We will use this reasoning to study the notions defined in Definition 8 when $(A_t)_{t\in I}$ is moving.

2.2 Assumptions and general results

From now we assume that $(A_t)_{t\in I}$ could depend on time and that, for any $s,t \in I$ and $x \in E_s$,

$$\mathbb{P}_x(\tau_{A\circ\theta_s} < \infty) = 1, \qquad \text{and} \quad \mathbb{P}_x(\tau_{A\circ\theta_s} > t) > 0.$$

We introduce now the main assumption adapted from the Champagnat-Villemonais conditions introduced in [14]:

Assumption 1. *There exist $(\nu_s)_{s\in I}$ a sequence of probability measures ($\nu_s \in \mathcal{M}_1(E_s)$ for each $s \in I$), and $t_0, c_1, c_2 > 0$ such that*

1. $\forall s \in I, \forall x \in E_s$,
$$\mathbb{P}_x(X_{t_0} \in \cdot | \tau_{A\circ\theta_s} > t_0) \geq c_1 \nu_{s+t_0}$$
2. $\forall s,t \in I, \forall x \in E_s$,
$$\mathbb{P}_{\nu_s}(\tau_{A\circ\theta_s} > t) \geq c_2 \mathbb{P}_x(\tau_{A\circ\theta_s} > t)$$

In this section, the main results and contributions in this chapter are presented. For a general behavior of $(A_t)_{t\in I}$, we will show the following theorem

Theorem 5. *If Assumption 1 holds, then there exists a Q-process (Definition 8 (i)). Furthermore, for any $s,t,T \in I$ and $x \in E_s$*

$$\begin{aligned}&||\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t+T) - \mathbb{Q}_{s,x}(X_{[s,t+s]} \in \cdot)||_{TV} \\ &\qquad \leq \frac{1}{(c_1c_2)^3 \mathbb{P}_x(\tau_{A\circ\theta_s} > t_0)} \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)}{\sup_{y\in E_{s+t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{s+t_0}} > t)} (1-c_1c_2)^{\left\lfloor \frac{T}{t_0} \right\rfloor}\end{aligned}$$

where the total variation norm $||\cdot||_{TV}$ is defined as

$$||\mu_1 - \mu_2||_{TV} = \sup_{B\in\mathcal{E}} |\mu_1(B) - \mu_2(B)|$$

In what follows, we will specifically study two types of behavior for the boundary. In particular, we will consider $(A_t)_{t\in I}$ as a non-increasing nested sequence of subsets (i.e. for any $s \leq t \in I, A_t \subset A_s$). In this case, define

$$A_\infty = \bigcap_{t\in I} A_t \tag{2.3}$$

Then the notion of convergence for non-increasing sequence of subsets is defined as follows.

Definition 9. We say that the non-increasing sequence $(A_t)_{t\in I}$ *converges* if $A_\infty \neq \emptyset$. Then A_∞ is *the limit of* $(A_t)_{t\in I}$

Denote by E_∞ the complement of A_∞. Then,

$$E_\infty = \bigcup_{t\in I} E_t$$

Let us now set the following assumptions :

Assumption 2. *1. Strong Markov property: For any τ stopping time of $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ and for any $x \in E$,*

$$\mathbb{P}_x((X_{\tau+t})_{t\in I} \in \cdot, \tau < \infty | \mathcal{F}_\tau) = \mathbb{1}_{\tau<\infty}\mathbb{P}_{X_\tau}((X_t)_{t\in I} \in \cdot)$$

2. Convergence in law for the hitting times : For any $x \in E_0$, $(\tau_{A\circ\theta_s})_{s\geq 0}$ converges in law towards τ_{A_∞} under $\mathbb{P}_x$, where $\tau_{A_\infty} := \inf\{t \geq 0 : X_t \in A_\infty\}$.

3. Continuity at time: For any $s \geq 0$ and $x \in E_0$, the functions $t \to \mathbb{P}_x(\tau_{A\circ\theta_s} > t)$ and $t \to \mathbb{P}_x(\tau_{A_\infty} > t)$ are continuous.

4. Continuity at state: For any $t \in I$, the function $x \to \mathbb{P}_x(\tau_{A_\infty} > t)$ is continuous.

Then the existence anf the uniqueness of the quasi-limiting distribution for converging boundaries is stated in the following theorem.

Theorem 6. *Assume that $(A_t)_{t\in I}$ is a non-increasing nested sequence of subsets converging towards $A_\infty \neq \emptyset$. Assume moreover that Assumption 1 and 2 hold, and that the Champagnat-Villemonais criteria $(A1) - (A2)$ hold for the non-moving boundary A_∞ (we recall that the assumptions $(A1)$ and $(A2)$ are introduced in the section 2.1).*
Then there exists a unique probability measure α such that, for any $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t\to\infty]{(d)} \alpha$$

The next results concern quasi-ergodic distribution. We show the existence and the uniqueness of the quasi-ergodic distribution for $(X_t)_{t\in I}$ when $(A_t)_{t\in I}$ is non-increasing and converges, but also when $(A_t)_{t\in I}$ is periodic. The statement is the following :

Theorem 7. *Assume that Assumption 1 holds.*

a) If $(A_t)_{t\in I}$ is γ-periodic with $\gamma > 0$,

b) or if the assumptions of Theorem 6 are satisfied

Then there exists a unique probability measure β such that, for any $\mu \in \mathcal{M}_1(E_0)$,

- $\frac{1}{n}\sum_{k=0}^{n} \mathbb{P}_\mu(X_k \in \cdot | \tau_A > n) \xrightarrow[n\to\infty]{(d)} \beta$, *if* $I = \mathbb{Z}_+$
- $\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t)ds \xrightarrow[t\to\infty]{(d)} \beta$, *if* $I = \mathbb{R}_+$

In the last part of this chapter, we show that the Assumption 1 are satisfied for some one-dimensional diffusion processes which are coming down from infinity. In particular, Q-process and quasi-ergodic distribution exist for the two behaviors of moving boundaries described above.

2.3 Exponential convergence towards Q-process and quasi-ergodic distribution

First, we recall Proposition 3.1. and Theorem 3.3. of [18]. In their paper, N.Champagnat and D.Villemonais took a time-inhomogeneous Markov process and $(Z_{s,t})_{s\leq t}$ a collection of multiplicative nonnegative random variables (i.e. satisfying $Z_{s,r}Z_{r,t} = Z_{s,t},\ \ \forall s \leq r \leq t$) such that, for any $s \leq t \in I$ and $x \in E_s$, $\mathbb{E}_x(Z_{s,t}) > 0$ and $\sup_{y\in E_s} \mathbb{E}_y(Z_{s,t}) < \infty$. In our case, $(X_t)_{t\in I}$ is time-homogeneous, however the penalization $(Z_{s,t})_{s\leq t}$ we shall use is given by

$$Z_{s,t} = \mathbb{1}_{\tau_{A\circ\theta_s}>t-s}, \qquad \forall s \leq t$$

and is time-inhomogeneous because $(A_t)_{t\in I}$ depends on t. Adapting their notation, we define $(\phi_{s,t})_{s\leq t}$ the non-linear semi-group defined by : $\forall s \leq t, \mu \in \mathcal{M}_1(E_s), f \in \mathcal{B}(E_t)$

$$\phi_{s,t}(\mu)(f) := \mathbb{E}_\mu(f(X_{t-s})|\tau_{A\circ\theta_s} > t-s)$$

Let $t_0 \in I$. For any $s \geq t_0$ and $x_1, x_2 \in E_{s-t_0}$, define v_{s,x_1,x_2} and v_s as follows

$$v_{s,x_1,x_2} = \min_{j=1,2} \phi_{s-t_0,s}(\delta_{x_j}) \tag{2.4}$$

$$v_s = \min_{x\in E_{s-t_0}} \phi_{s-t_0,s}(\delta_x) \tag{2.5}$$

where the minimum of several measures is understood as the largest measure smaller than all the considered measures. Finally, for any $s \geq t_0$, define

$$d_s = \inf_{t\geq 0, x_1,x_2\in E_{s-t_0}} \frac{\mathbb{P}_{v_{s,x_1,x_2}}(\tau_{A\circ\theta_s} > t)}{\sup_{x\in E_s} \mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \tag{2.6}$$

$$d'_s = \inf_{t\geq 0} \frac{\mathbb{P}_{v_s}(\tau_{A\circ\theta_s} > t)}{\sup_{x\in E_s} \mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \tag{2.7}$$

In particular, $v_s \leq v_{s,x_1,x_2}$ and $d'_s \leq d_s$. We can now state Proposition 3.1. and Theorem 3.3. of [18] in our situation (see [18] for a more general framework) :

Proposition 12 (Proposition 3.1. ([18]))**.** *For any $s \in I$ such that $d'_s > 0$ and $y \in E_s$, there exists a finite constant $C_{s,y}$ only depending on s and y such that, for all $x \in E_s$ and $t, u \geq s + t_0$ with $t \leq u$,*

$$\left|\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t-s)}{\mathbb{P}_y(\tau_{A\circ\theta_s} > t-s)} - \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > u-s)}{\mathbb{P}_y(\tau_{A\circ\theta_s} > u-s)}\right| \leq C_{s,y} \inf_{v\in[s+t_0,t]} \frac{1}{d'_v} \prod_{k=0}^{\left\lfloor \frac{v-s}{t_0}\right\rfloor -1} (1-d_{v-k}) \tag{2.8}$$

In particular, if

$$\liminf_{t\in I, t\to\infty} \frac{1}{d'_t} \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0}\right\rfloor -1} (1-d_{t-k}) = 0 \tag{2.9}$$

for all $s \geq 0$, there exists a positive bounded function $\eta_s : E_s \to (0, \infty)$ such that

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t - s)}{\mathbb{P}_y(\tau_{A\circ\theta_s} > t - s)} = \frac{\eta_s(x)}{\eta_s(y)}, \qquad \forall x, y \in E_s$$

where, for any fixed y, the convergence holds uniformly in x. η_s satisfies for all $x \in E_s$ and $s \leq t \in I$,

$$\mathbb{E}_x(\mathbb{1}_{\tau_{A\circ\theta_s}>t-s}\eta_t(X_{t-s})) = \eta_s(x)$$

In addition, the function $s \to ||\eta_s||_\infty$ is locally bounded on $[0, \infty)$.

Theorem 8 (Theorem 3.3 ([18]))**.** *Assume that*

$$\liminf_{t\in I, t\to\infty} \frac{1}{d'_t} \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0} \right\rfloor - 1} (1 - d_{t-k}) = 0.$$

Then there exists $(\mathbb{Q}_{s,x})_{s\in I, x\in E_s}$ such that

$$\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t + T) \xrightarrow[T\in I, T\to\infty]{(d)} \mathbb{Q}_{s,x}(X_{[s,s+t]} \in \cdot), \quad \forall s, t \in I, x \in E_s$$

and $\mathbb{Q}_{s,x}$ is given by

$$\mathbb{Q}_{s,x}(X_{[s,t]} \in \cdot) = \mathbb{E}_x \left(\mathbb{1}_{X_{[0,t-s]}\in\cdot} \frac{\mathbb{1}_{\tau_{A\circ\theta_s}>t-s}\eta_t(X_{t-s})}{\mathbb{E}_x(\mathbb{1}_{\tau_{A\circ\theta_s}>t-s}\eta_t(X_{t-s}))} \right), \qquad \forall s \leq t \in I, x \in E_s \quad (2.10)$$

Furthermore, under $(\mathbb{Q}_{s,x})_{s\in I,x\in E_s}$, $(X_t)_{t\in I}$ is a time-inhomogeneous Markov process. Finally, this process is asymptotically mixing in the sense that, for any $s \leq t \in I$ and $x \in E_s$,

$$||\mathbb{Q}_{s,x}(X_t \in \cdot) - \mathbb{Q}_{s,y}(X_t \in \cdot)||_{TV} \leq 2 \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0} \right\rfloor - 1} (1 - d_{t-k})$$

We proceed now with the proof of Theorem 1, hence we assume that Assumption 1 holds for the process $(X_t)_{t\in I}$. Let $t_0 \in I$ as defined in Assumption 1 and, considering such a choice of t_0, define v_{s,x_1,x_2}, v_s, d_s and d'_s as in (2.4), (2.5), (2.6) and (2.7) respectively. As a result, by Assumption 1, for any $s \in I$

$$d_s \geq d'_s \geq c_1 c_2 > 0 \quad (2.11)$$

Hence, by Proposition 12, (2.8) is satisfied and we have for any $s < s + t_0 \leq t \leq u$ and $x, y \in E_s$,

$$\left| \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t - s)}{\mathbb{P}_y(\tau_{A\circ\theta_s} > t - s)} - \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > u - s)}{\mathbb{P}_y(\tau_{A\circ\theta_s} > u - s)} \right| \leq C_{s,y} \times \frac{1}{c_1 c_2} (1 - c_1 c_2)^{\left\lfloor \frac{t-s}{t_0} \right\rfloor} \quad (2.12)$$

From this last equation, we can expect an exponential convergence of the family of probability measures $(\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta} > T + t))_{T\geq 0}$ towards the Q-process. In the following theorem, we show this exponential convergence and we provide some general assumptions for the existence of quasi-ergodic theorem.

Theorem 9. *Let $(X_t)_{t\geq 0}$ be a Markov process satisfying Assumption 1.*

1. *Then, for any $s,t,T \in I$ and $x \in E_s$,*

$$||\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t+T) - \mathbb{Q}_{s,x}(X_{[s,t+s]} \in \cdot)||_{TV} \leq \frac{1}{(c_1c_2)^3 \mathbb{P}_x(\tau_{A\circ\theta_s} > t_0)} \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)}{\sup_{y\in E_{s+t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{s+t_0}} > t)} (1-c_1c_2)^{\left\lfloor \frac{T}{t_0} \right\rfloor}$$

   where $\mathbb{Q}_{s,x}$ is defined by (2.10) *in Theorem 8*

2. *If moreover*

$$\forall x \in E_0 \qquad \sup_{t\geq 0} \frac{\mathbb{P}_x(\tau_A > t)}{\sup_{y\in E_{t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{t_0}} > t)} < \infty \tag{2.13}$$

   and if the Q-process satisfies a mean ergodic theorem, i.e. there exists a probability measure β such that for any $x \in E_0$,

$$\frac{1}{t}\int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot) ds \xrightarrow[t\to\infty]{(d)} \beta \tag{2.14}$$

   Then for any $x \in E_0$,

$$\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t\to\infty]{(d)} \beta$$

The statement of this theorem is implicitly written for $I = \mathbb{R}_+$. Obviously, the statement holds when $I = \mathbb{Z}_+$ and, from now, we will confuse integral and sum to deal with quasi-ergodic distributions when the time space I will not be specify.

Proof of Theorem 9. First we will show the exponential convergence towards the Q-process essentially thanks to (2.12). In the second step, we will show the existence and uniqueness of quasi-ergodic distribution using a method similar to that used in [17].

Step 1 : Exponential convergence towards the Q-process

We may extend (2.12) to general initial law μ and π : putting moreover $1/c_1c_2$ inside the constant, there exists $C_{s,\pi} > 0$ only depending on s and π such that, for any $s \geq 0$ and any $0 \leq t \leq u$,

$$\left| \frac{\mathbb{P}_\mu(\tau_{A\circ\theta_s} > u)}{\mathbb{P}_\pi(\tau_{A\circ\theta_s} > u)} - \frac{\mathbb{P}_\mu(\tau_{A\circ\theta_s} > t)}{\mathbb{P}_\pi(\tau_{A\circ\theta_s} > t)} \right| \leq C_{s,\pi}(1-c_1c_2)^{\left\lfloor \frac{t}{t_0} \right\rfloor}$$

Thus, by Theorem 8 and letting $u \to \infty$,

$$\left| \frac{\mu(\eta_s)}{\pi(\eta_s)} - \frac{\mathbb{P}_\mu(\tau_{A\circ\theta_s} > t)}{\mathbb{P}_\pi(\tau_{A\circ\theta_s} > t)} \right| \leq C_{s,\pi}(1-c_1c_2)^{\left\lfloor \frac{t}{t_0} \right\rfloor} \tag{2.15}$$

where, for a general $\mu \in \mathcal{M}_1(E)$, and for any $f \in \mathcal{B}(E)$,

$$\mu(f) := \int_E f d\mu$$

Using twice the Markov property, for any $s, t, T \in I$ and for any $x \in E_s$,

$$\begin{aligned}
\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > T + t) &= \mathbb{E}_x \left(\mathbb{1}_{X_{[0,t]} \in \cdot} \frac{\mathbb{1}_{\tau_{A\circ\theta_s} > t} \mathbb{P}_{X_t}(\tau_{A\circ\theta_{s+t}} > T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > T + t)} \right) \\
&= \mathbb{E}_x \left(\mathbb{1}_{X_{[0,t]} \in \cdot} \frac{\mathbb{1}_{\tau_{A\circ\theta_s} > t} \mathbb{P}_{X_t}(\tau_{A\circ\theta_{s+t}} > T)}{\mathbb{E}_x(\mathbb{1}_{\tau_{A\circ\theta_s} > t} \mathbb{P}_{X_t}(\tau_{A\circ\theta_{s+t}} > T))} \right) \\
&= \mathbb{E}_x \left(\mathbb{1}_{X_{[0,t]} \in \cdot} \mathbb{1}_{\tau_{A\circ\theta_s} > t} \frac{\mathbb{P}_{X_t}(\tau_{A\circ\theta_{s+t}} > T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t) \mathbb{P}_{\phi_{s,t,x}}(\tau_{A\circ\theta_{s+t}} > T))} \right)
\end{aligned}$$

where

$$\phi_{s,t,x} := \mathbb{P}_x(X_t \in \cdot | \tau_{A\circ\theta_s} > t) \tag{2.16}$$

Using this last equality and (2.10), for any $s, t, T \in I$, for any $x \in E_s$ and any $B \in \mathcal{E}$,

$$\begin{aligned}
&\left| \mathbb{P}_x(X_{[0,t]} \in B | \tau_{A\circ\theta_s} > T + t) - \mathbb{Q}_{s,x}(X_{[s,t+s]} \in B) \right| \\
&\qquad = \left| \mathbb{E}_x \left(\frac{\mathbb{1}_{X_{[0,t]} \in B} \mathbb{1}_{\tau_{A\circ\theta_s} > t}}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \left(\frac{\mathbb{P}_{X_t}(\tau_{A\circ\theta_{s+t}} > T)}{\mathbb{P}_{\phi_{s,t,x}}(\tau_{A\circ\theta_{s+t}} > T))} - \frac{\eta_{s+t}(X_t)}{\phi_{s,t,x}(\eta_{s+t})} \right) \right) \right| \\
&\qquad \leq C_{s+t,\phi_{s,t,x}} (1 - c_1 c_2)^{\left\lfloor \frac{T}{t_0} \right\rfloor} \mathbb{E}_x \left(\frac{\mathbb{1}_{X_{[0,t]} \in B} \mathbb{1}_{\tau_{A\circ\theta_s} > t}}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \right)
\end{aligned}$$

where the last inequality follows from (2.15). Moreover, for any $s, t \in I$,

$$\mathbb{E}_x \left(\frac{\mathbb{1}_{X_{[0,t]} \in B} \mathbb{1}_{\tau_{A\circ\theta_s} > t}}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \right) = \mathbb{P}_x \left(X_{[0,t]} \in B | \tau_{A\circ\theta_s} > t \right) \leq 1, \quad \forall B \in \mathcal{E}$$

Hence, for any $s, t \in I$, $x \in E_s$ and $B \in \mathcal{E}$,

$$\left| \mathbb{P}_x(X_{[0,t]} \in B | \tau_{A\circ\theta_s} > T + t) - \mathbb{Q}_{s,x}(X_{[s,t+s]} \in B) \right| \leq C_{s+t,\phi_{s,t,x}} (1 - c_1 c_2)^{\left\lfloor \frac{T}{t_0} \right\rfloor}$$

Note that [18] provides an explicit formula of $C_{s,y}$ in the proof of Proposition 3.1. for s and y fixed. Adapting this formula for a general probability measure π and recalling that we put the term $1/c_1c_2$ inside $C_{s,\pi}$, one explicit formula of $C_{s,\pi}$ for $s \in I$ can be

$$C_{s,\pi} = \frac{1}{c_1 c_2} \frac{\sup_{z \in E_s} \mathbb{P}_z(\tau_{A\circ\theta_s} > v_s - s)}{d'_{v_s} \mathbb{P}_\pi(\tau_{A\circ\theta_s} > v_s - s)} \tag{2.17}$$

where $v_s \in I$ is the smaller time $v \geq s+t_0$ such that $d'_v > 0$ (with d'_v as defined in (2.11)). Since $\sup_{z\in E_s} \mathbb{P}_z(\tau_{A\circ\theta_s} > v_s - s) \leq 1$ and $d'_v > c_1c_2 > 0$ by Assumption 1 for any $v \in I$, then, for any $s \in I$ and $\pi \in \mathcal{M}_1(E_s)$,

$$C_{s,\pi} \leq \frac{1}{(c_1c_2)^2\mathbb{P}_\pi(\tau_{A\circ\theta_s} > v_s - s)}$$

In our case, by Assumption 1, $d'_v > c_1c_2 > 0$ for any time v, which implies that $v_s = s + t_0$ for any $s \in I$. As a result, for any $s \in I$ and $\pi \in \mathcal{M}_1(E_s)$,

$$C_{s,\pi} \leq \frac{1}{(c_1c_2)^2\mathbb{P}_\pi(\tau_{A\circ\theta_s} > t_0)}$$

Thus, for any $s, t \in I$ and $x \in E_s$

$$C_{s+t,\phi_{s,t,x}} \leq \frac{1}{(c_1c_2)^2\mathbb{P}_{\phi_{s,t,x}}(\tau_{A\circ\theta_{s+t}} > t_0)} \tag{2.18}$$

According to Markov property, for any $s, t \in I$ and $x \in E_s$,

$$\mathbb{P}_{\phi_{s,t,x}}(\tau_{A\circ\theta_{s+t}} > t_0) = \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t + t_0)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)} \tag{2.19}$$

Moreover, by Assumption 1, for any $s, t \in I$ and $x \in E_s$,

$$\begin{aligned}
\mathbb{P}_x(\tau_{A\circ\theta_s} > t + t_0) &= \mathbb{P}_x(\tau_{A\circ\theta_s} > t_0)\mathbb{P}_{\phi_{s,t_0,x}}(\tau_{A\circ\theta_{s+t_0}} > t) && (2.20)\\
&\geq c_1\mathbb{P}_x(\tau_{A\circ\theta_s} > t_0)\mathbb{P}_{\nu_{s+t_0}}(\tau_{A\circ\theta_{s+t_0}} > t) && (2.21)\\
&\geq c_1c_2\mathbb{P}_x(\tau_{A\circ\theta_s} > t_0) \sup_{y\in E_{s+t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{s+t_0}} > t) && (2.22)
\end{aligned}$$

Hence, combining equations (2.18), (2.19) and (2.22), for any $s, t \in I$ and $x \in E_s$,

$$\begin{aligned}
C_{s+t,\phi_{s,t,x}} &\leq \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)}{(c_1c_2)^2\mathbb{P}_x(\tau_{A\circ\theta_s} > t + t_0)}\\
&\leq \frac{1}{(c_1c_2)^3\mathbb{P}_x(\tau_{A\circ\theta_s} > t_0)} \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)}{\sup_{y\in E_{s+t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{s+t_0}} > t)}
\end{aligned}$$

This conclude the first step

Step 2 : Convergence towards the quasi-ergodic distribution

We just proved that for any $0 \leq s \leq t$ and $x \in E_0$,

$$\begin{aligned}
&||\mathbb{P}_x(X_s \in \cdot|\tau_A > t) - \mathbb{Q}_{0,x}(X_s \in \cdot)||_{TV} && (2.23)\\
&\qquad \leq \frac{1}{(c_1c_2)^3\mathbb{P}_x(\tau_A > t_0)} \frac{\mathbb{P}_x(\tau_A > s)}{\sup_{y\in E_{t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{t_0}} > s)} (1 - c_1c_2)^{\left\lfloor \frac{t-s}{t_0} \right\rfloor} && (2.24)
\end{aligned}$$

By the assumption (2.13), for any $x \in E_0$, there exists $C_x < \infty$ such that

$$\frac{\mathbb{P}_x(\tau_A > s)}{\sup_{y \in E_{t_0}} \mathbb{P}_y(\tau_{A \circ \theta_{t_0}} > s)} \leq C_x, \quad \forall s \geq 0$$

As a result for any $0 \leq s \leq t$,

$$\begin{aligned}
&\left|\left|\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds - \frac{1}{t}\int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot) ds\right|\right|_{TV} \\
&\qquad \leq \frac{C_x}{(c_1c_2)^3 \mathbb{P}_x(\tau_A > t_0) t} \int_0^t (1 - c_1c_2)^{\left\lfloor \frac{t-s}{t_0} \right\rfloor} ds \\
&\qquad \leq \frac{C_x}{(c_1c_2)^3 \mathbb{P}_x(\tau_A > t_0) t} \int_0^t (1 - c_1c_2)^{\frac{t-s}{t_0} - 1} ds \\
&\qquad = \left(-\frac{C_x t_0}{(c_1c_2)^3 (1 - c_1c_2) \log(1 - c_1c_2) \mathbb{P}_x(\tau_A > t_0)}\right) \times \frac{1 - (1 - c_1c_2)^{\frac{t}{t_0}}}{t}
\end{aligned}$$

Let β as defined in (2.14). Then for any $x \in E_0$ and $f \in \mathcal{B}(E)$,

$$\begin{aligned}
&\left|\frac{1}{t}\int_0^t \mathbb{E}_x(f(X_s) | \tau_A > t) ds - \beta(f)\right| \\
&\leq \left|\left|\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds - \frac{1}{t}\int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot) ds\right|\right|_{TV} + \left|\frac{1}{t}\int_0^t \mathbb{E}^{\mathbb{Q}}_{0,x}(f(X_s)) ds - \beta(f)\right| \\
&\leq \left(-\frac{C_x t_0}{(c_1c_2)^3 (1 - c_1c_2) \log(1 - c_1c_2) \mathbb{P}_x(\tau_A > t_0)}\right) \times \frac{1 - (1 - c_1c_2)^{\frac{t}{t_0}}}{t} \\
&\qquad + \left|\frac{1}{t}\int_0^t \mathbb{E}^{\mathbb{Q}}_{0,x}(f(X_s)) ds - \beta(f)\right|
\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}}_{0,x}$ is the expectation with respect to $\mathbb{Q}_{0,x}$. Then, using the ergodic theorem for the Q-process,

$$\left|\frac{1}{t}\int_0^t \mathbb{E}_x(f(X_s) | \tau_A > t) ds - \beta(f)\right| \underset{t \to \infty}{\longrightarrow} 0$$

□

2.4 Some behaviors of moving boundaries and quasi-ergodicity

In this section, we will focus on two types of behavior for the moving boundaries

1. when A is γ-periodic with $\gamma > 0$
2. when A is non-increasing and converges at infinity towards $A_\infty \neq \emptyset$

Under Assumption 1, the existence of the Q-process is provided by Theorem 8 (Theorem 3.3, [18]) and we get moreover an exponential convergence towards the Q-process provided by Theorem 9. Now we want to investigate on the existence of quasi-ergodic distribution in the two cases described above.

2.4.1 Quasi-ergodic distribution when A is γ-periodic

In this subsection, we will work on periodic moving boundaries and we will assume that the Markov process $(X_t)_{t\in I}$ satisfies the Assumption 1. In particular, considering Assumption 1 for $s = 0$, for any $x \in E_0$ and $t \in I$,

1. $\mathbb{P}_x(X_{t_0} \in \cdot | \tau_A > t_0) \geq c_1\nu_{t_0}$
2. $\mathbb{P}_{\nu_0}(\tau_A > t) \geq c_2\mathbb{P}_x(\tau_A > t)$

It is always possible to choose t_0 in Assumption 1 such that $t_0 = \gamma\mathbb{N}$, since every time t_1 greater than t_0 suits. As a matter of fact, for a given t_0 for which Assumption 1 holds and $t_1 \geq t_0$, by Markov property, for any $x \in E_s$, one has,

$$\mathbb{P}_x(X_{t_1} \in \cdot | \tau_{A\circ\theta_s} > t_1) = \mathbb{P}_{\phi_{s,s+t_1-t_0}(\delta_x)}(X_{t_0} \in \cdot | \tau_{A\circ\theta_{s+t_1-t_0}} > t_0), \tag{2.25}$$

where we recall that, for any $s \leq t$ and $\mu \in \mathcal{M}_1(E_s)$, $\phi_{s,t}(\mu) := \mathbb{P}_\mu(X_{t-s} \in \cdot | \tau_{A\circ\theta_s} > t-s)$. Then, writing the first condition of Assumption 1 at time $s+t_1-t_0$ and $y \in E_{s+t_1-t_0}$ as

$$\mathbb{P}_y(X_{t_0} \in \cdot, \tau_{A\circ\theta_{s+t_1-t_0}} > t_0) \geq c_1\nu_{s+t_1}\mathbb{P}_y(\tau_{A\circ\theta_{s+t_1-t_0}} > t_0),$$

integrating over $\phi_{s,s+t_1-t_0}(\delta_x)(dy)$ and dividing by $\mathbb{P}_{\phi_{s,s+t_1-t_0}(\delta_x)}(\tau_{A\circ\theta_{s+t_1-t_0}} > t_0)$, it follows from (2.25) that

$$\begin{aligned}\mathbb{P}_x(X_{t_1} \in \cdot | \tau_{A\circ\theta_s} > t_1) &= \mathbb{P}_{\phi_{s,s+t_1-t_0}(\delta_x)}(X_{t_0} \in \cdot | \tau_{A\circ\theta_{s+t_1-t_0}} > t_0) \\ &\geq c_1\nu_{s+t_1}.\end{aligned}$$

From now, t_0 will be taken such that $t_0 = n_0\gamma$ with $n_0 \in \mathbb{N}$. Moreover, by periodicity of A, it is easy to see that $(\nu_s)_{s\geq 0}$ can be chosen as a γ-periodic sequence. As a result, one has

$$\nu_{t_0} = \nu_{n_0\gamma} = \nu_0$$

In all what follows, we will consider such a choice of $(\nu_s)_{s\geq 0}$. The aim is to obtain the convergence of the conditioned mean ratio towards a quasi-ergodic distribution which will be unique. Let us state the result.

Theorem 10. *Assume A is γ-periodic with $\gamma > 0$, and assume that Assumption 1 is satisfied. Then for any $x \in E_0$,*

$$\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t)ds \xrightarrow[t\to\infty]{(d)} \frac{1}{\gamma}\int_0^\gamma \mathbb{Q}_{0,\beta_\gamma}(X_s \in \cdot)ds$$

where β_γ is the invariant measure of $(X_{n\gamma})_{n\in\mathbb{N}}$ under $\mathbb{Q}_{0,\cdot}$, i.e.

$$\forall n \in \mathbb{N}, \quad \beta_\gamma = \mathbb{Q}_{0,\beta_\gamma}(X_{n\gamma} \in \cdot) := \int_{E_0} \beta_\gamma(dx)\mathbb{Q}_{0,x}(X_t \in \cdot).$$

Proof. Since A is γ-periodic and $t_0 \in \gamma\mathbb{N}$, one has $E_0 = E_{t_0}$ and for any $x \in E_0$ and $t \in I$,

$$\frac{\mathbb{P}_x(\tau_A > t)}{\sup_{y \in E_{t_0}} \mathbb{P}_y(\tau_{A \circ \theta_{t_0}} > t)} \leq \frac{\mathbb{P}_x(\tau_A > t)}{\mathbb{P}_x(\tau_{A \circ \theta_{t_0}} > t)} = 1$$

Hence the condition (2.13) is satisfied. Now we want to show an ergodic theorem for the time-inhomogeneous Markov process $(X_t)_{t \geq 0}$ under $(\mathbb{Q}_{s,x})_{s \geq 0, x \in E_s}$. Since $(A_t)_{t \geq 0}$ is γ-periodic, for any $0 \leq s \leq t$, for any $x \in E_s$,

$$\mathbb{Q}_{s+k\gamma,x}(X_{t+k\gamma} \in \cdot) = \mathbb{Q}_{s,x}(X_t \in \cdot), \quad \forall k \in \mathbb{Z}_+ \tag{2.26}$$

Moreover, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathbb{Q}_{0,x}(X_{n\gamma} \in \cdot) &= \lim_{t \to \infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > t) \\
&= \lim_{m \in \mathbb{Z}_+, m \to \infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > m\gamma) \\
&= \lim_{m \in \mathbb{Z}_+, m \to \infty} \mathbb{P}_x(Y_n \in \cdot | \tau_\partial > m)
\end{aligned}$$

where τ_∂ is defined by

$$\tau_\partial = \begin{cases} \inf\{n \geq 1 : \exists t \in ((n-1)\gamma, n\gamma], X_t \in A_t\} & \text{if } Y_0 \in E_0 \\ 0 & \text{if } Y_0 \in A_0 \end{cases}$$

and $(Y_n)_{n \in \mathbb{Z}_+}$ is the time-homogeneous Markov chain defined by

$$Y_n = \begin{cases} X_{n\gamma} \text{ for } n < \tau_\partial \\ \partial \quad \text{otherwise} \end{cases}$$

where ∂ plays the role of an absorbing state for $(Y_n)_{n \in \mathbb{Z}_+}$. In other words, τ_∂ is an absorbing time for $(Y_n)_{n \in \mathbb{Z}_+}$ and, under $(\mathbb{Q}_{0,x})_{x \in E_0}$, the chain $(X_{n\gamma})_{n \in \mathbb{Z}_+}$ is the Q-process of $(Y_n)_{n \in \mathbb{Z}_+}$.
By Assumption 1 and recalling that we chose $(\nu_s)_{s \geq 0}$ as γ-periodic, $(Y_n)_{n \in \mathbb{Z}_+}$ satisfies the following Champagnat-Villemonais type condition :

1.
$$\forall x \in E_0, \quad \mathbb{P}_x(Y_{n_0} \in \cdot | \tau_\partial > n_0) \geq c_1 \nu_0$$

2.
$$\forall x \in E_0, \forall n \in \mathbb{Z}_+, \quad \mathbb{P}_{\nu_0}(\tau_\partial > n) \geq c_2 \mathbb{P}_x(\tau_\partial > n)$$

where we recall that $n_0 = \frac{t_0}{\gamma}$. Hence, by Theorem 3.1 in [14], there exists $\beta_\gamma \in \mathcal{M}_1(E_0)$, $C > 0$ and $\rho \in (0,1)$ such that for any $n \in \mathbb{Z}_+$,

$$||\mathbb{Q}_{0,x}(X_{n\gamma} \in \cdot) - \beta_\gamma||_{TV} \leq C\rho^n, \quad \forall x \in E_0$$

This implies that, under $\mathbb{Q}_{0,\cdot}$, $(X_{n\gamma})_{n\in\mathbb{N}}$ is Harris recurrent. We can therefore apply Theorem 2.1 in [29] and deduce that, for any nonnegative function f,

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}_{0,\beta_\gamma}\left(\frac{1}{\gamma}\int_0^\gamma f(X_s)ds\right), \quad \mathbb{Q}_{0,x}\text{-almost surely, } \forall x\in E_0$$

where $\mathbb{E}^{\mathbb{Q}}_{0,\mu}(G)=\int G d\mathbb{Q}_{0,\mu}$ for any measurable nonnegative function G and $\mu\in\mathcal{M}_1(E_0)$. It extends to $f\in\mathcal{B}(E)$ using $f=f_+-f_-$ with f_+, f_- non negative functions. Thus, by bounded Lebesgue's convergence theorem, for any $x\in E_0$ and for any $f\in\mathcal{B}(E)$,

$$\frac{1}{t}\int_0^t \mathbb{E}^{\mathbb{Q}}_{0,x}(f(X_s))ds \underset{t\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}_{0,\beta_\gamma}\left(\frac{1}{\gamma}\int_0^\gamma f(X_s)ds\right)$$

Hence the condition (2.14) is satisfied. We conclude the proof using the second part of Theorem 9.

□

Remark 3. In [29], Höpfner and Kutoyants claimed their results for Markov processes with continuous paths. It is easy to see using their arguments that the statement in Theorem 2.1. can be generalized to any time-inhomogeneous Markov processes $(X_t)_{t\in I}$ such that the condition of periodicity (2.26) is satisfied and the chain $(X_{n\gamma})_{n\in\mathbb{Z}_+}$ is Harris recurrent. Other statements are made in Proposition 5 of [31]. See also the formula (2.2) in [30].

2.4.2 Quasi-ergodic distribution when A converges at infinity

In this subsection, we assume that A is non-increasing and let A_∞ as defined in (2.3). We assume that $A_\infty\neq\emptyset$.

We recall that E_∞ is the complement of A_∞. Define

$$\tau_{A_\infty} := \inf\{t\geq 0 : X_t\in A_\infty\}$$

and set the following assumption :

Assumption 3. *there exists $\nu_\infty\in\mathcal{M}_1(E_\infty)$ such that*

1. *there exist $t_\infty\geq 0$ and $c_{1,\infty}>0$ such that*

$$\mathbb{P}_x(X_{t_\infty}\in\cdot|\tau_{A_\infty}>t_\infty)\geq c_{1,\infty}\nu_\infty, \qquad \forall x\in E_\infty$$

2. *there exists $c_{2,\infty}>0$ such that*

$$\mathbb{P}_{\nu_\infty}(\tau_{A_\infty}>t)\geq c_{2,\infty}\mathbb{P}_x(\tau_{A_\infty}>t), \qquad \forall x\in E_\infty, \forall t\geq 0$$

In what follows, we will first state the existence and uniqueness of a quasi-limiting distribution under Assumptions 1, 2 and 3. Then we will deal with quasi-ergodic distribution.

Quasi-limiting distribution

First we state the following proposition which will be useful to prove the theorem on the existence and the uniqueness of the quasi-limiting distribution.

Proposition 13. *Under Assumptions 1 and 3, for any $B \in \mathcal{E}$, the quantities*

$$\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t) \text{ and } \liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t)$$

do not depend on any couple (s,μ) such that $\mu \in \mathcal{M}_1(E_s)$.

Proof. We recall the part of Theorem 2.1 in [18] adapted to our case:

Theorem 11 (Theorem 2.1., [18])**.** *For any $s \in I$, for any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$, for any $t \geq s + t_0$,*

$$||\mathbb{P}_{\mu_1}(X_{t-s} \in \cdot|\tau_{A\circ\theta_s} > t-s) - \mathbb{P}_{\mu_2}(X_{t-s} \in \cdot|\tau_{A\circ\theta_s} > t-s)||_{TV} \leq 2 \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0} \right\rfloor - 1} (1 - d_{t-k}) \tag{2.27}$$

Let $B \in \mathcal{E}$. First we remark that, for s fixed, $\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t)$ does not depend on $\mu \in \mathcal{M}_1(E_s)$. This is straightforward since, thanks to (2.27), for any $s \geq 0$ and any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$||\mathbb{P}_{\mu_1}(X_t \in \cdot|\tau_{A\circ\theta_s} > t) - \mathbb{P}_{\mu_2}(X_t \in \cdot|\tau_{A\circ\theta_s} > t)||_{TV} \underset{t\to 0}{\longrightarrow} 0$$

which implies that for any $s \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$

$$\limsup_{t\to\infty} \mathbb{P}_{\mu_1}(X_t \in B|\tau_{A\circ\theta_s} > t) = \limsup_{t\to\infty} \mathbb{P}_{\mu_2}(X_t \in B|\tau_{A\circ\theta_s} > t) \tag{2.28}$$

Now for any $u \geq 0$, denoting $\mu_u := \mathbb{P}_\mu(X_u \in \cdot|\tau_{A\circ\theta_s} > u)$ for any $\mu \in \mathcal{M}_1(E_s)$,

$$\begin{aligned}
\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t) &= \limsup_{t\to\infty} \mathbb{P}_\mu(X_{t+u} \in B|\tau_{A\circ\theta_s} > t+u) &(2.29)\\
&= \limsup_{t\to\infty} \mathbb{P}_{\mu_u}(X_t \in B|\tau_{A\circ\theta_{s+u}} > t) &(2.30)\\
&= \limsup_{t\to\infty} \mathbb{P}_\nu(X_t \in B|\tau_{A\circ\theta_{s+u}} > t) &(2.31)
\end{aligned}$$

where we used first Markov property, and then (2.28) with a given probability measure $\nu \in \mathcal{M}_1(E_{s+u})$.
Hence (2.28) and (2.29),(2.30), (2.31) show that $\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t)$ does not depend on any couple (s,μ) satisfying $s \in I$ and $\mu \in \mathcal{M}_1(E_s)$. A similar reasoning shows that $\liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B|\tau_{A\circ\theta_s} > t)$ does not depend on s and μ either.

□

Before showing the existence of quasi-limiting and quasi-ergodic distribution, let us state the following proposition providing a uniform-in-time convergence of the time-inhomogeneous conditioned semi-group towards the time-homogeneous limit semi-group .

Proposition 14. *Assume Assumptions 2 and 3. Then for any $x \in E_0$,*

$$\lim_{s\to\infty} \sup_{t,T\in I} ||\mathbb{P}_x(X_t \in \cdot | \tau_{A\circ\theta_s} > t+T) - \mathbb{P}_x(X_t \in \cdot | \tau_{A_\infty} > t+T)||_{TV} = 0 \tag{2.32}$$

Proof. Let $x \in E_0$. Then, for any $s, t, T \in I$ and $B \in \mathcal{E}$,

$$\begin{aligned}
&|\mathbb{P}_x(X_t \in B | \tau_{A\circ\theta_s} > t+T) - \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > t+T)| \\
&= \left| \frac{\mathbb{P}_x(\tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t+T)} \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} - \frac{\mathbb{P}_x(X_t \in B, \tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} \right| \\
&\leq \left| \frac{\mathbb{P}_x(\tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t+T)} \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} - \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} \right| \\
&\quad + \left| \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} - \frac{\mathbb{P}_x(X_t \in B, \tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} \right| \\
&\leq \left| \frac{\mathbb{P}_x(\tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t+T)} - 1 \right| \times \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} \\
&\quad + \left| \frac{\mathbb{P}_x(X_t \in B, \tau_{A\circ\theta_s} > t+T) - \mathbb{P}_x(X_t \in B, \tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A_\infty} > t+T)} \right| \\
&\leq \left| \frac{\mathbb{P}_x(\tau_{A_\infty} > t+T)}{\mathbb{P}_x(\tau_{A\circ\theta_s} > t+T)} - 1 \right| + \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t+T < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t+T)}
\end{aligned}$$

where we used several times the fact that $A_\infty \subset A_t$ for any t (in particular to say that $\mathbb{P}_x(\tau_{A\circ\theta_s} > u) \leq \mathbb{P}_x(\tau_{A_\infty} > u)$ for any $u \in I$). Hence it is enough to prove that

$$\sup_{t\in I} \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \underset{s\to\infty}{\longrightarrow} 0 \tag{2.33}$$

As a matter of fact, (2.33) is equivalent to

$$\sup_{t\in I} \left| \frac{\mathbb{P}_x(\tau_{A\circ\theta_s} > t)}{\mathbb{P}_x(\tau_{A_\infty} > t)} - 1 \right| \underset{s\to\infty}{\longrightarrow} 0$$

and it is easy to check that, for general functions $(s,t) \to f(s,t)$, $(f(s,\cdot))_{s\in I}$ converges uniformly towards the constant function equal to 1 if and only if $\left(\frac{1}{f(s,\cdot)}\right)_{s\in I}$ also converges uniformly towards 1.
Since A is non-increasing, for any $s \leq s'$ and $t \in I$,

$$\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \geq \frac{\mathbb{P}_x(\tau_{A\circ\theta_{s'}} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)}$$

Moreover, using *the convergence in law for the hitting times* of Assumption 2, one has, for any $t \in I$,

$$\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \underset{s\to\infty}{\longrightarrow} 0$$

Finally, by *the strong Markov property* of Assumption 2, for any $t \in I$,

$$\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty}) = \mathbb{E}_x(\mathbb{1}_{\tau_{A\circ\theta_s}\leq t}\phi(X_{\tau_{A\circ\theta_s}}, \tau_{A\circ\theta_s}, t))$$

where $\phi(\cdot,\cdot,\cdot)$ is defined as follows

$$\forall z \in E_\infty, \forall 0 \leq u \leq t, \quad \phi(z,u,t) = \mathbb{P}_z(\tau_{A_\infty} > t-u)$$

By Assumption 3, there exists a unique quasi-stationary distribution α_∞ for the process $(X_t)_{t\in I}$ absorbed at A_∞ (Theorem 2.1., [14]). Then we recall (see [42, 19]) that there exists $\lambda_\infty > 0$ such that

$$\mathbb{P}_{\alpha_\infty}(\tau_{A_\infty} > t) = e^{-\lambda_\infty t}, \quad \forall t \in I$$

In [14] it is shown (Proposition 2.3.) that there exists a positive bounded function η_∞ defined on E_∞ such that

$$\eta_\infty(x) = \lim_{t\to\infty} e^{\lambda_\infty t}\mathbb{P}_x(\tau_{A_\infty} > t), \quad \forall x \in E_\infty \tag{2.34}$$

where the convergence holds for the uniform norm on E_∞. Thus, by [Proposition 2.3, [14]], $\mathbb{1}_{\tau_{A\circ\theta_s}\leq t}\frac{\phi(X_{\tau_{A\circ\theta_s}},\tau_{A\circ\theta_s},t)}{\mathbb{P}_x(\tau_{A_\infty}>t)}$ is uniformly bounded and converges almost surely towards $\mathbb{1}_{\tau_{A\circ\theta_s}<\infty}e^{\lambda_\infty\tau_{A\circ\theta_s}}\frac{\eta_\infty(X_{\tau_{A\circ\theta_s}})}{\eta_\infty(x)}$. Then, by the bounded Lebesgue's convergence theorem, for any $s \in I$,

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} &= \lim_{t\to\infty}\mathbb{E}_x\left(\mathbb{1}_{\tau_{A\circ\theta_s}\leq t}\frac{\phi(X_{\tau_{A\circ\theta_s}},\tau_{A\circ\theta_s},t)}{\mathbb{P}_x(\tau_{A_\infty} > t)}\right)\\ &= \mathbb{E}_x\left(\mathbb{1}_{\tau_{A\circ\theta_s}<\infty}e^{\lambda_\infty\tau_{A\circ\theta_s}}\frac{\eta_\infty(X_{\tau_{A\circ\theta_s}})}{\eta_\infty(x)}\right)\end{aligned}$$

For any $s \in I$, we can therefore define $t \to \frac{\mathbb{P}_x(\tau_{A\circ\theta_s}\leq t<\tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty}>t)}$ on the Alexandroff extension $I \cup \{\infty\}$ setting

$$\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq \infty < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > \infty)} := \lim_{t\to\infty}\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} = \mathbb{E}_x\left(\mathbb{1}_{\tau_{A\circ\theta_s}<\infty}e^{\lambda_\infty\tau_{A\circ\theta_s}}\frac{\eta_\infty(X_{\tau_{A\circ\theta_s}})}{\eta_\infty(x)}\right)$$

Then, like any $t \in \mathbb{R}$, $\left(\frac{\mathbb{P}_x(\tau_{A\circ\theta_s}\leq\infty<\tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty}>\infty)}\right)_{s\geq 0}$ is non-increasing and since η_∞ vanishes on A_∞ and η_∞ is continuous (this is due to *the continuity at state* of Assumption 2 and the uniform convergence (2.34)),

$$\lim_{s\to\infty}\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq \infty < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > \infty)} = \lim_{s\to\infty}\lim_{t\to\infty}\frac{\mathbb{P}_x(\tau_{A\circ\theta_s} \leq t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} = 0$$

We conclude to the uniform convergence (2.33) using Dini's theorem for a decreasing sequence of functions. □

Let us state the theorem on existence and uniqueness of quasi-limiting distribution.

Theorem 12. *Assume that $(A_t)_{t\geq 0}$ is a non-increasing nested sequence of subsets converging towards $A_\infty \neq 0$ and assume Assumptions 1, 2 and 3.*

Then there exists a unique $\alpha \in \mathcal{M}_1(E_\infty)$ such that, for any $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t\to\infty]{(d)} \alpha$$

Proof. Fix $B \in \mathcal{E}$ and note that, by Assumption 3, for any $\mu \in \mathcal{M}_1(E_\infty)$

$$\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A_\infty} > t) = \liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A_\infty} > t) = \alpha_\infty(B)$$

where we recall that α_∞ is the quasi-stationary distribution of $(X_t)_{t\in I}$ absorbed at A_∞. By Proposition 13, for a given $s \in I$, $\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t)$ and $\liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t)$ do not depend on $\mu \in \mathcal{M}_1(E_s)$. Denote therefore by F_{sup} and F_{inf} the functions defined by, for any $s \in I$ and any $\mu \in \mathcal{M}_1(E_s)$,

$$F_{sup}(s) = \limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t) = \limsup_{t\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A\circ\theta_s} > t)$$

$$F_{inf}(s) = \liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t) = \liminf_{t\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A\circ\theta_s} > t)$$

for a given $x \in E_0$. Then F_{sup} and F_{inf} do not depend on s either (by Proposition 13), hence for any $s \in I$,

$$F_{sup}(s) = \lim_{u\to\infty} F_{sup}(u)$$

$$F_{inf}(s) = \lim_{u\to\infty} F_{inf}(u)$$

Moreover, by the uniform convergence (2.32) of Proposition 14,

$$\begin{aligned}\lim_{u\to\infty} F_{sup}(u) &= \lim_{u\to\infty} \limsup_{t\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A\circ\theta_u} > t) \\ &= \limsup_{t\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > t) \\ &= \alpha_\infty(B)\end{aligned}$$

Similarly,

$$\lim_{u\to\infty} F_{inf}(u) = \alpha_\infty(B)$$

Hence, for any $s \in I$ and $\mu \in \mathcal{M}_1(E_s)$,

$$\limsup_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t) = \liminf_{t\to\infty} \mathbb{P}_\mu(X_t \in B | \tau_{A\circ\theta_s} > t) = \alpha_\infty(B)$$

□

Quasi-ergodic distribution

Now we can state the existence and uniqueness of the quasi-ergodic distribution :

Theorem 13. *Under the assumptions of Theorem 12, for any $x \in E_0$,*

$$\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t\to\infty]{(d)} \beta_\infty$$

where β_∞ is the unique invariant measure of the Q-process of $(X_t)_{t\geq 0}$ absorbed by A_∞.

Proof. Since A is non-increasing, for any $x \in E_0$ and $t \in I$,

$$\frac{\mathbb{P}_x(\tau_A > t)}{\sup_{y\in E_{t_0}} \mathbb{P}_y(\tau_{A\circ\theta_{t_0}} > t)} \leq \frac{\mathbb{P}_x(\tau_A > t)}{\mathbb{P}_x(\tau_{A\circ\theta_{t_0}} > t)} \leq 1$$

Hence the assumption (2.13) in Theorem 9 holds.

Now, we will show that the Q-process converges weakly towards a probability measure. Fix $B \in \mathcal{E}$. Since we have the following inequality shown in Theorem 3.3 of [18]

$$||\mathbb{Q}_{s,x}(X_t \in \cdot) - \mathbb{Q}_{s,y}(X_t \in \cdot)||_{TV} \leq 2 \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0} \right\rfloor - 1} (1 - d_{t-k})$$

where we recall that d_s is defined in (2.6) and we define, for any $s \in I$, and $\mu \in \mathcal{M}_1(E_s)$, $\mathbb{Q}_{s,\mu}(X_t \in \cdot) := \int_{E_s} \mu(dx)\mathbb{Q}_{s,x}(X_t \in \cdot)$. We get therefore that for any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$

$$\limsup_{t\to\infty} \mathbb{Q}_{s,\mu_1}(X_t \in B) = \limsup_{t\to\infty} \mathbb{Q}_{s,\mu_2}(X_t \in B)$$

and we can therefore use the reasoning of the proof of Proposition 13 to show that, for any $s, u \in I$, for any $\mu, \nu \in \mathcal{M}_1(E_s) \times \mathcal{M}_1(E_{s+u})$,

$$\limsup_{t\to\infty} \mathbb{Q}_{s,\mu}(X_t \in B) = \limsup_{t\to\infty} \mathbb{Q}_{s+u,\nu}(X_t \in B)$$

In particular, for any $s \in I$, $\mu \in \mathcal{M}_1(E_s)$ and $x \in E_0$,

$$\limsup_{t\to\infty} \mathbb{Q}_{s,\mu}(X_t \in B) = \lim_{u\to\infty} \limsup_{t\to\infty} \mathbb{Q}_{u,x}(X_t \in B)$$

By the uniform convergence (2.32) of Proposition 14, for any $s \in I$, $\mu \in \mathcal{M}_1(E_s)$ and $x \in E_0$,

$$\begin{aligned}
\limsup_{t\to\infty} \mathbb{Q}_{s,\mu}(X_t \in B) &= \lim_{u\to\infty} \limsup_{t\to\infty} \mathbb{Q}_{u,x}(X_t \in B) \\
&= \lim_{u\to\infty} \limsup_{t\to\infty} \lim_{T\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A\circ\theta_u} > t + T) \\
&= \limsup_{t\to\infty} \lim_{T\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > t + T) \\
&= \limsup_{t\to\infty} \mathbb{Q}_x^\infty(X_t \in B)
\end{aligned}$$

where for any $x \in E_\infty$

$$\mathbb{Q}_x^\infty(X_t \in B) = \lim_{T\to\infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > t+T)$$

is well-defined by [Theorem 3.1, [14]] under Assumption 3. This theorem states moreover that $(X_t)_{t\in I}$ admits under $(\mathbb{Q}_x^\infty)_{x\in E_\infty}$ a unique invariant measure β_∞ and for any $x \in E_\infty$

$$\lim_{t\to\infty} \mathbb{Q}_x^\infty(X_t \in \cdot) = \beta_\infty$$

Thus, for any $B \in \mathcal{E}$, $s \in I$ and $x \in E_s$

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{Q}_{s,x}(X_t \in B) &= \beta_\infty(B) \\ &= \liminf_{t\to\infty} \mathbb{Q}_{s,x}(X_t \in B)\end{aligned}$$

Finally, thanks to the convergence in law of the Q-process we just prove, we can deduce the weak ergodic theorem using Cesaro's rule

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot)ds = \beta_\infty$$

Hence the condition (2.14) holds. As a result we can apply the second part of Theorem 9 and conclude the proof. □

2.5 Example : Diffusion on $\mathbb{R}$

Let $(X_t)_{t\geq 0}$ be a diffusion on $\mathbb{R}$ satisfying the following stochastic differential equation

$$dX_t = dW_t - V(X_t)dt \tag{2.35}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is Brownian motion on $\mathbb{R}$ and $V \in \mathcal{C}^1(\mathbb{R})$ non-negative on $\mathbb{R}_+$. We assume that, under $\mathbb{P}_x$, there exists a strongly unique non explosive solution of (2.35) such that $X_0 = x$ almost surely.
Let h be a positive bounded $\mathcal{C}^1$-function. We define τ_h the random time defined by

$$\tau_h = \inf\{t \geq 0 : X_t = h(t)\}$$

and more generally, for any $s \geq 0$, we define $\tau_{h\circ\theta_s}$ by

$$\tau_{h\circ\theta_s} = \inf\{t \geq 0 : X_t = h(t+s)\}$$

2.5.1 Assumptions and preliminaries

We assume that $(X_t)_{t\in\mathbb{R}_+}$ comes down from infinity (in the sense given in [11]), that is, there exists $y > h_{max} := \sup_{s\geq 0} h(s)$ and $t > 0$ such that

$$\lim_{x\to\infty} \mathbb{P}_x(\tau_y < t) > 0 \tag{2.36}$$

where, for any $z \in \mathbb{R}$,

$$\tau_z := \inf\{t \geq 0 : X_t = z\}$$

In this case, as remarked in the subsection 4.5.2. of [13], one has

$$\int_0^1 \frac{1}{x} \left(\sup_{y \in (z, \Lambda_z^{-1}(x)]} \frac{1}{\Lambda_z(y)} \int_z^y \Lambda_z(\xi)^2 m(d\xi) \right) dx < \infty$$

where, for any $z \geq 0$, Λ_z is the scale function of X satisfying $\Lambda_z(z) = 0$ and defined by

$$\Lambda_z(x) = \int_z^x e^{2\int_0^y V(\xi)d\xi} dy, \quad \forall x \geq z \tag{2.37}$$

and m is the speed measure of $(X_t)_{t\geq 0}$ defined by

$$m(d\xi) = 2e^{-2\int_0^\xi V(\xi')d\xi'} d\xi$$

In particular, for any $z \geq 0$, the process $Y^z := (\Lambda_z(X_t))_{t\geq 0}$ is a local martingale and, since X is solution of (2.35), by Itô's formula, for any $t \geq 0$,

$$Y_t^z = Y_0^z + \int_0^t \Lambda_z'(\Lambda_z^{-1}(Y_s^z)) dW_s.$$

Note that $\Lambda_z' = \Lambda_0' = e^{2\int_0^\cdot V(\xi)d\xi}$ for any z. So denoting for any $x, z \geq 0$ $\sigma_z(x) := \Lambda_z'(\Lambda_z^{-1}) = \Lambda_0'(\Lambda_z^{-1})$, one has

$$dY_t^z = \sigma_z(Y_t^z) dW_t$$

Adapting [Theorem 4.6.,[13]] for general diffusions, we deduce that for any $t > 0$, there exists $A_t^z < \infty$ such that

$$\mathbb{P}_x(t < \tau_z) \leq A_t^z \Lambda_z(x), \quad \forall x \geq z$$

So let $u_1 \geq 0$ arbitrarily chosen. One has for any $z \geq 0$,

$$\mathbb{P}_x(u_1 < \tau_z) \leq A_{u_1}^z \Lambda_z(x), \quad \forall x \geq z \tag{2.38}$$

or, equivalently,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_z) \leq A_{u_1}^z x, \quad \forall x \geq 0$$

Denoting, for any $r \geq 0$ and for any process $(R_t)_{t\geq 0}$, $\tau_r(R) := \inf\{t \geq 0 : R_t = r\}$, one has for any $z \geq 0$ and $x \geq r$,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_{\Lambda_z^{-1}(r)}) = \mathbb{P}\left(\tau_r(Y^z) > u_1 | Y_0^z = x\right)$$

Since $z \to \Lambda_z^{-1}(x)$ is increasing for any $x > 0$ and Λ_0 is non-decreasing (because V is non-negative on $\mathbb{R}_+$), then, for any $x > 0$ and for any $z \geq z'$,

$$\sigma_z(x) \geq \sigma_{z'}(x) \tag{2.39}$$

Thus, using the same reasoning as in the proof of Lemma 4.2. in the paper [18], it is possible to show that (2.39) implies that, for any $z \geq z'$ and $x \geq r$

$$\mathbb{P}\left(\tau_r(Y^{z'}) > u_1 \middle| Y_0^{z'} = x\right) \geq \mathbb{P}\left(\tau_r(Y^z) > u_1 \middle| \tilde{Y}_0^z = x\right)$$

or, equivalently,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_{\Lambda_z^{-1}(r)}) \leq \mathbb{P}_{\Lambda_{z'}^{-1}(x)}(u_1 < \tau_{\Lambda_{z'}^{-1}(r)}) \tag{2.40}$$

Taking $z' = r = 0$, for any $x \geq 0$,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_z) \leq \mathbb{P}_{\Lambda_0^{-1}(x)}(u_1 < \tau_0) \leq A_{u_1}^0 x, \quad \forall x \geq 0$$

In conclusion, one has, for any $z \geq 0$,

$$\mathbb{P}_x(u_1 < \tau_z) \leq A_{u_1}^0 \Lambda_z(x), \quad \forall x \geq z \tag{2.41}$$

One set $A := A_{u_1}^0$. Let us now state and prove the following lemma.

Lemma 2. *There exists $u_0 \geq 0$, $\kappa > 0$ and a family of probability measures $(\psi_z)_{z\in[0,h_{max}]}$ such that, for any $z \in [0, h_{max}]$,*

$$\mathbb{P}_x(X_u \in \cdot | \tau_z > u) \geq \kappa \psi_z, \quad \forall x > z, \forall u > u_0 \tag{2.42}$$

The difference between this lemma and [13, Theorem 4.1] is that the time u_0 and the constant κ do not depend on z. The sketch of the proof is inspired from the proof of the Theorem 4.1 presented in [13, Subsection 5.1].

Proof. The following proof is divided into three steps.

Step 1. : Mimicking the Step 1 in the proof of [13, Theorem 4.1]

The aim of this first step is to prove that there exist $\epsilon, c > 0$ not depending on z such that

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon | \tau_z > u_1) \geq c, \quad \forall x > z \tag{2.43}$$

Since, for any $z \in [0, h_{max}]$, $\Lambda_z(X)$ is a local martingale, one has for any $x \in (z, \Lambda_z^{-1}(1))$,

$$\begin{aligned}
\Lambda_z(x) &= \mathbb{E}_x(\Lambda_z(X_{u_1 \wedge \tau_z \wedge \tau_{\Lambda_z^{-1}(1)}})) \\
&= \mathbb{P}_x(\tau_z > u_1)\mathbb{E}_x(\Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}}) | \tau_z > u_1) + \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z \leq u_1)
\end{aligned}$$

By the Markov property,

$$\begin{aligned}
\mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z \leq u_1) &\leq \mathbb{E}_x(\mathbb{1}_{\tau_{\Lambda_z^{-1}(1)} < \tau_z \wedge u_1} \mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1)) \\
&\leq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z)\mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1) \\
&= \Lambda_z(x)\mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1)
\end{aligned}$$

where the following identity is used

$$\mathbb{P}_x(\tau_a < \tau_b) = \frac{\Lambda_z(x) - \Lambda_z(b)}{\Lambda_z(a) - \Lambda_z(b)}, \qquad \forall x \in [a, b]$$

As a result, using (2.41), one has, for any $x \in (z, \Lambda_z^{-1}(1))$,

$$\mathbb{E}_x(1 - \Lambda_z(X_{1\wedge\tau_{\Lambda_z^{-1}(1)}})|1 < \tau_z) \leq 1 - \frac{1}{A'_z}$$

where $A'_z := A/\mathbb{P}_{\Lambda_z^{-1}(1)}(u_1 < \tau_z)$. But, since $z \in [0, h_{max}]$, the inequality (2.40) applied to $r = 0$ implies that

$$\mathbb{P}_{\Lambda_z^{-1}(1)}(u_1 < \tau_z) \geq \mathbb{P}_{\Lambda_{h_{max}}^{-1}(1)}(u_1 < \tau_{h_{max}})$$

So, defining $A' := A/\mathbb{P}_{\Lambda_{h_{max}}^{-1}(1)}(u_1 < \tau_{h_{max}})$, one has

$$\mathbb{E}_x(1 - \Lambda_z(X_{u_1\wedge\tau_{\Lambda_z^{-1}(1)}})|u_1 < \tau_z) \leq 1 - \frac{1}{A'}, \qquad \forall x \in (z, \Lambda_1^{-1}(1))$$

Thus, using the Markov inequality,

$$\mathbb{P}_x\left(\Lambda_z(X_{u_1\wedge\tau_{\Lambda_z^{-1}(1)}}) \leq \frac{1}{2A'-1}\middle|\tau_z > u_1\right) \leq 1 - \frac{1}{2A'}$$

Then, since $A' > 1$ by (2.41), $1/(2A'-1) < 1$. Thus, for any $\epsilon \in (0, 1/(2A'-1))$ and $x \in (z, \Lambda_z^{-1}(1/(2A'-1)))$,

$$\begin{aligned}
&\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon, \tau_z > u_1) \\
&\geq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1/(2A'-1))} < u_1 \wedge \tau_z, \tau_{\Lambda_z^{-1}(\epsilon)} \circ \theta_{\tau_{\Lambda_z^{-1}(1/(2A'-1))}} > u_1 + \tau_{\Lambda_z^{-1}(1/(2A'-1))}) \\
&= \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1/(2A'-1))} < u_1 \wedge \tau_z)\mathbb{P}_{\Lambda_z^{-1}(1/(2A'-1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\
&\geq \mathbb{P}_x\left(\Lambda_z(X_{u_1\wedge\tau_{\Lambda_z^{-1}(1)}\wedge\tau_z}) \geq 1/(2A'-1)\right)\mathbb{P}_{\Lambda_z^{-1}(1/(2A'-1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\
&\geq \frac{\mathbb{P}_x(\tau_z > u_1)}{2A'}\mathbb{P}_{\Lambda_z^{-1}(1/(2A'-1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\
&\geq \frac{\mathbb{P}_x(\tau_z > u_1)}{2A'}\mathbb{P}_{\Lambda_{h_{max}}^{-1}(1/(2A'-1))}(\tau_{\Lambda_{h_{max}}^{-1}(\epsilon)} > u_1)
\end{aligned}$$

where (2.40) is used again. So, if ϵ is chosen such that $\mathbb{P}_{\Lambda_{h_{max}}^{-1}(1/(2A'-1))}(\tau_{\Lambda_{h_{max}}^{-1}(\epsilon)} > u_1) > 0$ (it is possible since $\mathbb{P}_{\Lambda_{h_{max}}^{-1}(1/(2A'-1))}(\tau_z > u_1) > 0$), then there exist $\epsilon \in (0, 1/(2A'-1))$ and $c > 0$ (not depending on z) such that, for any $x \in (z, \Lambda_z^{-1}(1/(2A'-1)))$,

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon|\tau_z > u_1) \geq c$$

For $x \geq \Lambda_z^{-1}(1/(2A'-1))$,

$$\begin{aligned}\mathbb{P}_x(\Lambda_z(X_{u_1}) > \epsilon | \tau_z > u_1) &\geq \mathbb{P}_x(\Lambda_z(X_{u_1}) > \epsilon, \tau_z > u_1) \\ &\geq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ &\geq \mathbb{P}_{\Lambda_z^{-1}(1/(2A'-1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ &\geq \mathbb{P}_{\Lambda_{h_{max}}^{-1}(1/(2A'-1))}(\tau_{\Lambda_{h_{max}}^{-1}(\epsilon)} > u_1) > 0\end{aligned}$$

Finally, there exist $\epsilon \in (0, 1/(2A'-1))$ and $c > 0$ (not depending on z) such that, for any $x \geq z$,

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon | \tau_z > u_1) \geq c$$

Step 2. Mimicking the steps 2 and 3 in the proof of [13, Theorem 4.1].

Now, taking the exact same reasoning as the one presented in the second step of the proof of Theorem 4.1 [13, Subsection 5.1], one can prove that, for any $z \in [0, h_{max}]$, for all $x \geq \epsilon$,

$$\mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,z}}) \in \cdot, \tau_z > u_{2,z}) \geq c_{1,z}\psi_z$$

where

- $u_{2,z}$ can be any time satisfying

$$\inf_{y>z} \mathbb{P}_y(\tau_z < u_{2,z}) =: c'_{1,z} > 0$$

- $$c_{1,z} := c'_{1,z}\mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\tau_z > u_{2,z})$$

- $$\tilde{\nu}_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,z}}) \in \cdot | \tau_z > u_{2,z})$$

In particular, for $z = 0$, one choose $u_{2,0}$ such that

$$\inf_{y>0} \mathbb{P}_y(\tau_0 < u_{2,0}) > 0 \tag{2.44}$$

Hence, for any $z \in [0, h_{max}]$ and $x > z$,

$$\mathbb{P}_x(\tau_z < u_{2,0}) \geq \mathbb{P}_x(\tau_0 < u_{2,0}) \geq \inf_{y>0} \mathbb{P}_y(\tau_0 < u_{2,0}) = c'_{1,0}$$

Hence, for any $z \in [0, h_{max}]$,

$$c'_{1,z} = \inf_{x>z} \mathbb{P}_x(\tau_z < u_{2,0}) > c'_{1,0}$$

In other words, we can set for any $z \in [0, h_{max}]$

$$u_{2,z} = u_{2,0}$$

Hence, one can define for any $z \in [0, h_{max}]$,

$$c_{1,z} := c'_{1,z}\mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\tau_z > u_{2,0}), \quad \tilde{\nu}_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,0}}) \in \cdot | \tau_z > u_{2,0})$$

As a result, doing the same computation as those presented in Step 3 of the proof of Theorem 4.1 in [13], and defining $u_0 := u_1 + u_{2,0}$, for any $x > z$,

$$\mathbb{P}_x(\Lambda_z(X_{u_0}) \in \cdot | \tau_z > u_0) \geq c_{1,z} c \tilde{\nu}_z \geq c c'_{1,0} \mathbb{P}_{\Lambda_{h_{max}}^{-1}(\epsilon)}(\tau_{h_{max}} > u_{2,0}) \tilde{\nu}_z$$

In conclusion, to get (almost) (2.42), one has to set $\kappa := c c'_{1,0} \mathbb{P}_{\Lambda_{h_{max}}^{-1}(\epsilon)}(\tau_{h_{max}} > u_{2,0})$ and $\psi_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(X_{u_{2,0}} \in \cdot | \tau_z > u_{2,0})$ and one has

$$\mathbb{P}_x(X_{u_0} \in \cdot | \tau_z > u_0) \geq \kappa \psi_z, \quad \forall x > z \tag{2.45}$$

Step 3 : Conclusion

By (2.45) one has for any $x > z$,

$$\mathbb{P}_x(X_{u_0} \in \cdot, \tau_z > u_0) \geq \kappa \psi_z \mathbb{P}_x(\tau_z > u_0)$$

Integrating the above equality over $\mu(dx)$, we deduce that for any $\mu \in \mathcal{M}_1((z, \infty))$,

$$\mathbb{P}_\mu(X_{u_0} \in \cdot, \tau_z > u_0) \geq \kappa \psi_z \mathbb{P}_\mu(\tau_z > u_0)$$

that is to say

$$\mathbb{P}_\mu(X_{u_0} \in \cdot | \tau_z > u_0) \geq \kappa \psi_z, \quad \forall \mu \in \mathcal{M}_1((z, \infty))$$

In particular, using Markov property and denoting $\varphi_{z,u-u_0,x} := \mathbb{P}_x(X_{u-u_0} \in \cdot | \tau_z > u - u_0)$, for any $u \geq u_0$ and $x > z$,

$$\mathbb{P}_x(X_u \in \cdot | \tau_z > u) = \mathbb{P}_{\varphi_{z,u-u_0,x}}(X_{u_0} \in \cdot | \tau_z > u_0) \geq \kappa \psi_z$$

which is (2.42) □

Before showing that the Assumption 1 is satisfied when h is periodic or converging, we will need to give some hypothesis on the function V as defined in (2.35). In the both case we will deal with, the absorbing function h will be Lipschitz, i.e.

$$L := \sup_{s \leq t} \frac{|h(t) - h(s)|}{|t - s|} < \infty$$

Now we state the assumption we need on the function V

Assumption 4 (Hypothesis on V)**.**

- *V is such that the process X satisfying* (2.35) *comes down from infinity*
- *V is positive and increasing on $[-Lu_0, \infty)$ (where u_0 is mentionned in Lemma 2)*
- $\sup_{x \in \mathbb{R}} V'(x) - V^2(x) < \infty$

Note that the functions $V : x \to (x+c)^\alpha$ with $\alpha > 1$ and $c > 0$ are suitable functions.

2.5.2 Periodic absorbing function

In this subsection, we will assume that h is γ-periodic ($\gamma > 0$) and we want to show the existence of Q-process and quasi-ergodic distribution for diffusion processes coming down from infinity.

Proposition 15. *Under the assumptions described in Subsection 2.5.1, Assumption 1 is satisfied. In particular, we obtain the exponential convergence towards the Q-process of Theorem 9.*

Proof. We will show that the four points in Assumption 1 are satisfied.

1. Denote by $\mathcal{T}_{max}$ the set defined by

$$\mathcal{T}_{max} = \{t \geq 0 : h(t) = h_{max}\}$$

where we recall that $h_{max} = \sup_{s\geq 0} h(s)$. The main part of this proof is to show that there exists $C_{max} > 0$ such that, for any $s \in \mathcal{T}_{max}$ and any $u \in [u_0, u_0 + \gamma]$

$$\mathbb{P}_x(X_u \in \cdot | \tau_{h\circ\theta_s} > u) \geq C_{max}\psi_{h_{max}}, \quad \forall x > h_{max} \tag{2.46}$$

where u_0 and $\psi_{h_{max}}$ are defined in Lemma 2. Then we will generalize (2.46) to any $s \geq 0$ using Markov property.

First step : Proof of (2.46)
Let $s \in \mathcal{T}_{max}$. For any $x > h_{max}$, for any $t \geq 0$,

$$\mathbb{P}_x(X_t \in \cdot | \tau_{h\circ\theta_s} > t) \geq \frac{\mathbb{P}_x(\tau_{h_{max}} > t)}{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)} \mathbb{P}_x(X_t \in \cdot | \tau_{h_{max}} > t)$$

Using the Champagnat-Villemonais type condition (2.42) for $z = h_{max}$, for any $u \geq u_0$,

$$\mathbb{P}_x(X_u \in \cdot | \tau_{h_{max}} > u) \geq \kappa\psi_{h_{max}}, \quad \forall x \in (h_{max}, \infty)$$

Then we obtain for any $u_0 \leq u \leq u_0 + \gamma$,

$$\begin{aligned}\mathbb{P}_x(X_u \in \cdot | \tau_h > u) &\geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u)}{\mathbb{P}_x(\tau_{h\circ\theta_s} > u)} \kappa\psi_{h_{max}} \\ &\geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{h\circ\theta_s} > u_0)} \kappa\psi_{h_{max}}\end{aligned}$$

Recalling that h is Lispchitz and that we defined $L = \sup_{s\leq t} \frac{|h(t)-h(s)|}{|t-s|}$, for any $x \in (h_{max}, \infty)$,

$$\frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{h\circ\theta_s} > u_0)} \geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)}$$

where

$$\tau_{u\to h_{max}-Lu} := \inf\{t \geq 0 : X_t = h_{max} - Lt\}$$

To show that

$$\inf_{x\in(h_{max},\infty)} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} > 0$$

using a continuity argument, it is enough to show that

$$\liminf_{x\to h_{max}} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} > 0 \tag{2.47}$$

and

$$\liminf_{x\to \infty} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} > 0 \tag{2.48}$$

(2.48) is obvious since

$$\liminf_{x\to \infty} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} \geq \lim_{x\to\infty} \mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma) > 0$$

Thus let us focus on (2.47). Our strategy will be to reduce the study to the case of a Brownian motion. Denote by $(M_t)_{t\geq 0}$ the exponential local martingale defined by, for any t,

$$\begin{aligned} M_t &:= \exp\left(-\int_0^t V(W_s)dW_s - \frac{1}{2}\int_0^t V^2(W_s)ds\right) \\ &= \exp\left(F(W_0) - F(W_t) + \frac{1}{2}\int_0^t (V'(W_s) - V^2(W_s))ds\right) \end{aligned}$$

where F is a primitive of V that we choose as a positive function on $[-Lu_0,\infty)$ (it is possible since F is necessarily non-decreasing by the assumptions on V). Under $\mathbb{P}_x$ for $x \in (h_{max}, h_{max}+1]$, $W_0 = x$ almost surely. Moreover denote by $\tau^W_{h_{max}}$ and $\tau^W_{u\to h_{max}-Lu}$ the following random times

$$\tau^W_{h_{max}} := \inf\{t \geq 0 : W_t = h_{max}\}$$

$$\tau^W_{u\to h_{max}-Lu} := \inf\{t \geq 0 : W_t = h_{max} - Lt\}$$

Thus, since F is non-decreasing, the stopped local martingale $(M_{t\wedge u_0\wedge \tau^W_{u\to h_{max}-Lu}})_{t\geq 0}$ is almost surely bounded by $\exp\left(F(h_{max}+1) + \frac{u_0}{2}\sup_{y\in\mathbb{R}} V'(y) - V^2(y)\right)$ and is therefore a martingale. Likewise, the stopped local martingale $(M_{t\wedge u_0+\gamma\wedge\tau^W_{h_{max}}})_{t\geq 0}$ is also a martingale. By Girsanov theorem,

$$\begin{aligned} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} &= \frac{\mathbb{E}_x\left(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma\wedge\tau^W_{h_{max}}}\right)}{\mathbb{E}_x\left(\mathbb{1}_{\tau^W_{u\to h_{max}-Lu} > u_0} M_{u_0\wedge\tau^W_{u\to h_{max}-Lu}}\right)} \\ &= \frac{\mathbb{E}_x\left(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma}\right)}{\mathbb{E}_x\left(\mathbb{1}_{\tau^W_{u\to h_{max}-Lu} > u_0} M_{u_0}\right)} \end{aligned}$$

For any $x \in (h_{max}, h_{max}+1]$,

$$\mathbb{E}_x(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma}) \geq \mathbb{E}_x\left(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma} \mathbb{1}_{\sup_{s\in[0,u_0+\gamma]} W_s \leq h_{max}+2}\right)$$

On the event $\{\sup_{s\in[0,u_0+\gamma]} W_s \leq h_{max}+2\}$,

$$M_{u_0+\gamma} \geq \exp\left(-F(h_{max}+2) + \frac{u_0+\gamma}{2} \inf_{s\in[h_{max},h_{max}+2]}(V'(s) - V^2(s))\right) =: \tilde{M}_{u_0+\gamma}$$

As a result,

$$\mathbb{E}_x(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma}) \geq \tilde{M}_{u_0+\gamma} \mathbb{E}_x\left(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} \mathbb{1}_{\sup_{s\in[0,u_0+\gamma]} W_s \leq h_{max}+2}\right)$$
$$\geq \tilde{M}_{u_0+\gamma} \mathbb{P}_x\left(\tau^W_{h_{max}} > u_0+\gamma\right) \inf_{y\in(h_{max},h_{max}+1]} \mathbb{P}_y\left(\sup_{s\in[0,u_0+\gamma]} W_s \leq h_{max}+2 \middle| \tau^W_{h_{max}} > u_0+\gamma\right)$$

Noting that

$$\lim_{y\to h_{max}} \mathbb{P}_y\left(\sup_{s\in[0,u_0+\gamma]} W_s \leq h_{max}+2 \middle| \tau^W_{h_{max}} > u_0+\gamma\right) = \mathbb{P}\left(\sup_{s\in[0,u_0+\gamma]} W^+_s \leq 2\right) > 0$$

where $(W^+_t)_{t\geq 0}$ is a Brownian meander (see [23], Theorem 2.1.), we deduce finally that there exists $c > 0$ such that for any $x \in (h_{max}, h_{max}+1]$

$$\mathbb{E}_x(\mathbb{1}_{\tau^W_{h_{max}} > u_0+\gamma} M_{u_0+\gamma}) \geq c\mathbb{P}_x\left(\tau^W_{h_{max}} > u_0+\gamma\right)$$

On the other side, as we said before, $(M_{t\wedge u_0 \wedge \tau^W_{u\to h_{max}-Lu}})_{t\geq 0}$ is almost surely bounded by $\exp\left(F(h_{max}+1) + \frac{u_0}{2}\sup_{y\in\mathbb{R}} V'(y) - V^2(y)\right)$. Hence there exists $d > 0$ such that, for any $x \in (h_{max}, h_{max}+1]$,

$$\mathbb{E}_x\left(\mathbb{1}_{\tau^W_{u\to h_{max}-Lu} > u_0} M_{u_0}\right) \leq d\mathbb{P}_x(\tau^W_{u\to h_{max}-Lu} > u_0)$$

As a result, for any $(h_{max}, h_{max}+1]$,

$$\frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} \geq \frac{c}{d} \frac{\mathbb{P}_x(\tau^W_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau^W_{u\to h_{max}-Lu} > u_0)}$$

For any $x > h_{max}$, denote by $p^W_{h_{max}}(x,\cdot)$ and $p^W_{u\to h_{max}-Lu}(x,\cdot)$ the density functions of $\tau^W_{h_{max}}$ and $\tau^W_{u\to h_{max}-Lu}$ which are known to be equal to

$$p^W_{h_{max}}(x,t) = \frac{x-h_{max}}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x-h_{max})^2}{2t}\right)$$

$$p^W_{u\to h_{max}-Lu}(x,t) = \frac{x-h_{max}}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(x-h_{max}+Lt)^2\right)$$

Then, for any $x \in (h_{max}, h_{max}+1]$,

$$\frac{\mathbb{P}_x(\tau^W_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau^W_{u\to h_{max}-Lu} > u_0)} = \frac{\int_{u_0+\gamma}^{\infty} p^W_{h_{max}}(x,t)dt}{\int_{u_0}^{\infty} p^W_{u\to h_{max}-Lu}(x,t)dt}$$

By l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to h_{max}} \frac{\mathbb{P}_x(\tau^W_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau^W_{u\to h_{max}-Lu} > u_0)} &= \lim_{x\to h_{max}} \frac{\int_{u_0+\gamma}^{\infty} \partial_x p^W_{h_{max}}(x,t)dt}{\int_{u_0}^{\infty} \partial_x p^W_{u\to h_{max}-Lu}(x,t)dt} \\ &= \frac{\int_{u_0+\gamma}^{\infty} \partial_x p^W_{h_{max}}(h_{max},t)dt}{\int_{u_0}^{\infty} \partial_x p^W_{u\to h_{max}-Lu}(h_{max},t)dt} > 0\end{aligned}$$

As a result,

$$\liminf_{x\to h_{max}} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu} > u_0)} \geq \frac{c}{d} \lim_{x\to h_{max}} \frac{\mathbb{P}_x(\tau^W_{h_{max}} > u_0+\gamma)}{\mathbb{P}_x(\tau^W_{u\to h_{max}-Lu} > u_0)} > 0$$

In conclusion, $\inf_{x\in(h_{max},\infty)} \frac{\mathbb{P}_x(\tau_{h_{max}}>u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu}>u_0)} > 0$ and (2.46) holds with $C_{max} = \kappa \times \inf_{x\in(h_{max},\infty)} \frac{\mathbb{P}_x(\tau_{h_{max}}>u_0+\gamma)}{\mathbb{P}_x(\tau_{u\to h_{max}-Lu}>u_0)}$

Second step : Generalization and conclusion
Now let $s \geq 0$. Then there exists $s' \geq 0$ such that $s+s' \in \mathcal{T}_{max}$. As a result we can construct a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ as follows

$$g(s) = \inf\{s' \geq 0 : s+s' \in \mathcal{T}_{max}\} \tag{2.49}$$

In particular, $g(s) = 0$ if $s \in \mathcal{T}_{max}$. Since h is a continuous function, $s+g(s) \in \mathcal{T}_{max}$ for any $s \geq 0$. Moreover, since h is γ-periodic, then for any $s \geq 0$, $g(s) \leq \gamma$.
Thus for any $x \in E_s$,

$$\mathbb{P}_x(X_{u_0+\gamma} \in \cdot | \tau_{h\circ\theta_s} > u_0+\gamma) = \mathbb{P}_{\phi_{s,g(s),x}}(X_{u_0+\gamma-g(s)} \in \cdot | \tau_{h\circ\theta_{s+g(s)}} > u_0+\gamma-g(s))$$

thanks to Markov property, where we recall (see (2.16)) that

$$\phi_{s,g(s),x} = \mathbb{P}_x(X_{g(s)} \in \cdot | \tau_{h\circ\theta_s} > g(s)) \in \mathcal{M}_1((h_{max},\infty))$$

Now by (2.46), for any $x > h(s)$,

$$\begin{aligned}\mathbb{P}_x(X_{u_0+\gamma} \in \cdot | \tau_{h\circ\theta_s} > u_0+\gamma) &= \mathbb{P}_{\phi_{s,g(s),x}}(X_{u_0+\gamma-g(s)} \in \cdot | \tau_{h\circ\theta_{s+g(s)}} > u_0+\gamma-g(s)) \\ &\geq C_{max}\psi_{h_{max}}\end{aligned}$$

since $u_0+\gamma-g(s) \in [u_0, u_0+\gamma]$. As a result the first condition in Assumption 1 holds denoting for any $s \geq 0$,

$$\begin{aligned}\nu_s &= \psi_{h_{max}} \\ t_0 &= \gamma + u_0 \\ c_1 &= C_{max}\end{aligned} \tag{2.50}$$

2. For the second condition of Assumption 1, we will use some part of the proof of [Theorem 4.1, [13]]. First we recall [Lemma 5.1.,[13]]

Lemma 3 (Lemma 5.1., [13]). *There exists $a > h_{max}$ such that $\psi_{h_{max}}([a,\infty)) > 0$ and, for any $k \in \mathbb{N}$,*

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h_{max}}} \geq a) \geq e^{-\rho k u_0}$$

with $\rho > 0$.

So let a as in the previous lemma. It is shown in [13] that we can choose $b > a$ large enough such that

$$\sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty \tag{2.51}$$

Using Markov property,

$$\mathbb{P}_a(t < \tau_{h \circ \theta_s}) \geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h_{max}}} \geq b) \mathbb{P}_b(t < \tau_{h \circ \theta_s})$$

for any $t \geq 0$ and any $s_0 = k_0 \gamma$ with $k_0 \in \mathbb{N}$. Then, for $s_0 > 0$ fixed, $C := 1/\mathbb{P}_a(X_{s_0 \wedge \tau_{h_{max}}} \geq b) < \infty$, and for any $t \geq 0$ and any $s \geq 0$,

$$\mathbb{P}_b(t < \tau_{h \circ \theta_s}) \leq C \mathbb{P}_a(t < \tau_{h \circ \theta_s})$$

Thanks to Markov property again, for any $u \leq t \in \mathbb{R}_+$

$$\mathbb{P}_a(X_{u \wedge \tau_{h_{max}}} \geq a) \mathbb{P}_a(t - u < \tau_{h \circ \theta_{s+u}}) \leq \mathbb{P}_a(t < \tau_{h \circ \theta_s})$$

According to Markov property, for any $u \in \mathbb{R}_+$,

$$\begin{aligned} \mathbb{P}_a(X_{u \wedge \tau_{h_{max}}} \geq a) &\geq \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h_{max}}} \geq a) \mathbb{P}_a(X_{(u - \lfloor \frac{u}{u_0} \rfloor u_0) \wedge \tau_{h_{max}}} \geq a) \\ &\geq C' \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h_{max}}} \geq a) \end{aligned}$$

where

$$C' := \inf_{v \in [0, u_0]} \mathbb{P}_a(X_{v \wedge \tau_{h_{max}}} \geq a) > 0$$

since $v \to \mathbb{P}_a(X_{v \wedge \tau_{h_{max}}} \geq a)$ is continuous and $\mathbb{P}_a(X_{v \wedge \tau_{h_{max}}} \geq a) > 0$ for any

$v \in [0, u_0]$. Gathering all these inequalities and using also Lemma 3, for any $x \geq b$,

$$\begin{aligned}
\mathbb{P}_x(t < \tau_{h\circ\theta_s}) &\leq \mathbb{P}_x(\tau_b > t) + \int_0^t \mathbb{P}_b(t-u < \tau_{h\circ\theta_{s+u}})\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{-\rho t} + C\int_0^t \mathbb{P}_a(t-u < \tau_{h\circ\theta_{s+u}})\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{-\rho\lfloor t/u_0\rfloor u_0} \\
&\qquad + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t \frac{1}{\mathbb{P}_a(X_{\lfloor u/u_0\rfloor u_0\wedge\tau_{h_{max}}} \geq a)}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}e^{-\rho\left(\left\lfloor \frac{t}{u_0}\right\rfloor+1\right)u_0} + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}\mathbb{P}_a(X_{(\lfloor t/u_0\rfloor+1)u_0\wedge\tau_{h_{max}}} \geq a) \\
&\qquad + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}\mathbb{P}_a(\tau_{h\circ\theta_s} > t) + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du)
\end{aligned}$$

We deduce from (2.51) that, for any $t \geq 0$,

$$\sup_{x\geq b} \mathbb{P}_x(t < \tau_{h\circ\theta_s}) \leq C''\mathbb{P}_a(t < \tau_{h\circ\theta_s})$$

where

$$C'' = \left(e^{\rho u_0} + \frac{C}{C'}\right)\sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b}) < \infty$$

Since $\psi_{h_{max}}([a,\infty)) > 0$, we conclude the point 2. of Assumption 1 setting

$$c_2 := \frac{\psi_{h_{max}}([a,\infty))}{C''}$$

□

2.5.3 When h is decreasing and converges at infinity

Now we consider h as a decreasing $\mathcal{C}^1$-function going to 0 as t goes to infinity. In particular, $h_{max} = h(0)$.

Since this is a diffusion process on $\mathbb{R}_+$, $(X_t)_{t\geq 0}$ satisfies the strong Markov property and the two assumptions of continuity presented in Assumption 2. Moreover, since $t \to X_t$ is continuous almost surely and, for any $s \geq 0$, $\tau_{h(s)}$ is the hitting time of the closed set $[-1, h(s)]$, then $\tau_{h(s)} \underset{s\to\infty}{\longrightarrow} \tau_0$ almost surely, which implies that $\tau_{h\circ\theta_s} \underset{s\to\infty}{\longrightarrow} \tau_0$ almost surely. This entails the convergence in law of the hitting times of Assumption 2.

Now let us state and prove the following proposition.

Proposition 16. *Assumption 1 holds. In particular, we obtain the exponential convergence towards the Q-process of Theorem 9. Moreover, since Assumptions 2 and 3 hold, there exist a unique quasi-limiting distribution and a quasi-ergodic distribution.*

Proof. 1. Adapting exactly the same reasoning as Proposition 15, we can show that for any $s \geq 0$ and any $x > h(s)$,

$$\mathbb{P}_x(X_{u_0} \in \cdot | \tau_{h \circ \theta_s} > u_0) \geq \tilde{d}_s \kappa_0 \psi_{h(s)}$$

where we recall that u_0, κ_0 and ψ_z are such that (2.42) holds, and where d_s is defined by

$$\tilde{d}_s = \frac{\mathbb{P}_x(\tau_{h(s)} > u_0)}{\mathbb{P}_x(\tau_{u \to h(s) - Lu} > u_0)}$$

We have therefore to show that

$$\inf_{s \geq 0} \tilde{d}_s > 0$$

For any $z \in [0, h(0)]$ define $(X_t^{(z)})_{t \geq 0}$ by the solution of

$$dX_t^{(z)} = dW_t - V(X_t^{(z)} + z)dt$$

In particular, $X^{(0)} \overset{(d)}{=} X$. Likewise, for any $y \in \mathbb{R}$ and $z \in [0, h(0)]$, we denote by $\tau_y^{(z)} := \inf\{t \geq 0 : X_t^{(z)} = y\}$ and $\tau_{u \to y - Lu}^{(z)} := \inf\{t \geq 0 : X_t^{(z)} = y - Lt\}$. Since V is positive and increasing on $[-Lu_0, \infty)$, then, using Theorem 1.1 in [[32], Chapter VI, p.437], we can show that for any $x > 0$ and $z \in [0, h(0)]$,

$$\mathbb{P}_x(\tau_0^{(z)} > u_0) \geq \mathbb{P}_x(\tau_0^{(h(0))} > u_0)$$

and that

$$\mathbb{P}_x(\tau_{u \to -Lu}^{(z)} > u_0) \leq \mathbb{P}_x(\tau_{u \to -Lu}^{(0)} > u_0)$$

Then, for any $x > 0$ and $s \geq 0$,

$$\begin{aligned}
\mathbb{P}_{x+h(s)}(\tau_{h(s)} > u_0) &= \mathbb{P}_x(\tau_0^{(h(s))} > u_0) \\
&\geq \mathbb{P}_x(\tau_0^{(h(0))} > u_0) \\
&\geq \frac{\mathbb{P}_x(\tau_0^{(h(0))} > u_0)}{\mathbb{P}_x(\tau_{u \to -Lu}^{(0)} > u_0)} \mathbb{P}_{x+h(s)}(\tau_{u \to h(s) - Lu} > u_0)
\end{aligned}$$

To conclude, it is enough to see that $\inf_{x>0} \frac{\mathbb{P}_x(\tau_0^{(h(0))} > u_0)}{\mathbb{P}_x(\tau_{u \to -Lu}^{(0)} > u_0)} > 0$ using the same techniques as the point 1 of Proposition 15.
As a result the first hypothesis of Assumption 1 holds setting for any $s \geq 0$,

$$\nu_s = \begin{cases} \psi_{h(0)} & \text{if } s \leq u_0 \\ \psi_{h(s-u_0)} & \text{if } s > u_0 \end{cases}$$

$$t_0 = u_0$$

$$c_1 = \kappa_0 \times \inf_{s \geq 0} \tilde{d}_s$$

2. Noting that for any $z \in [0, h(0)]$ and any $y \geq h(0)$, $\psi_z([y, \infty)) > 0$, then, by Lemma 3, there exists $a > h(0)$ such that, for any $z \in [0, h(0)]$, $\psi_z([a, \infty)) > 0$ and for any $k \in \mathbb{N}$

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h(0)}} \geq a) \geq e^{-\rho k u_0}$$

where $\rho > 0$. We deduce that for any $s \geq 0$

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h(s)}} \geq a) \geq e^{-\rho k u_0}$$

As in the proof of Proposition 15, we can choose $b > a$ large enough such that

$$\sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty$$

Since h is non-increasing, for any $s, t \geq 0$ and $s_0 \geq 0$,

$$\mathbb{P}_b(\tau_{h \circ \theta_{s+s_0}} > t) \geq \mathbb{P}_b(\tau_{h \circ \theta_s} > t)$$

Hence according to Markov property,

$$\begin{aligned} \mathbb{P}_a(t < \tau_{h \circ \theta_s}) &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h(s)}} \geq b) \mathbb{P}_b(t < \tau_{h \circ \theta_s}) \\ &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h(0)}} \geq b) \mathbb{P}_b(t < \tau_{h \circ \theta_s}) \end{aligned}$$

for any $t \geq 0$ and any $s_0 \geq 0$. Hence, for s_0 fixed, $C := \frac{1}{\mathbb{P}_a(X_{s_0 \wedge \tau_{h(0)}} \geq b)} < \infty$, and for any $t \geq 0$,

$$\mathbb{P}_b(t < \tau_{h \circ \theta_s}) \leq C \mathbb{P}_a(t < \tau_{h \circ \theta_s})$$

According to Markov property again, for any $u \leq t \in \mathbb{R}_+$

$$\mathbb{P}_a(X_{u \wedge \tau_{h(s)}} \geq a) \mathbb{P}_a(t - u < \tau_{h \circ \theta_{s+u}}) \leq \mathbb{P}_a(t < \tau_{h \circ \theta_s})$$

Thanks to Markov property, for any $u \in \mathbb{R}_+$,

$$\begin{aligned} \mathbb{P}_a(X_{u \wedge \tau_{h(s)}} \geq a) &\geq \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h(s)}} \geq a) \mathbb{P}_a(X_{(u - \lfloor \frac{u}{u_0} \rfloor u_0) \wedge \tau_{h(s)}} \geq a) \\ &\geq \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h(0)}} \geq a) \mathbb{P}_a(X_{(u - \lfloor \frac{u}{u_0} \rfloor u_0) \wedge \tau_{h(s)}} \geq a) \\ &\geq C' \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h(s)}} \geq a) \end{aligned}$$

where

$$C' := \inf_{v \in [0, u_0]} \mathbb{P}_a(X_{v \wedge \tau_{h(0)}} \geq a) > 0$$

since $v \to \mathbb{P}_a(X_{v \wedge \tau_0} \geq a)$ is continuous and $\mathbb{P}_a(X_{v \wedge \tau_0} \geq a) > 0$ for any $v \in [0, u_0]$.

Hence, gathering all these inequalities, for any $x \geq b$,

$$\begin{aligned}
\mathbb{P}_x(t < \tau_{h\circ\theta_s}) &\leq \mathbb{P}_x(\tau_b > t) + \int_0^t \mathbb{P}_b(t-u < \tau_{h\circ\theta_{s+u}})\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{-\rho t} + C\int_0^t \mathbb{P}_a(t-u < \tau_{h\circ\theta_{s+u}})\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{-\rho\lfloor \frac{t}{u_0}\rfloor u_0} \\
&\qquad + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t \frac{1}{\mathbb{P}_a(X_{\lfloor \frac{u}{u_0}\rfloor u_0 \wedge \tau_{h(s)}} \geq a)}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}e^{-\rho\left(\left\lfloor \frac{t}{u_0}\right\rfloor +1\right)u_0} + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}\mathbb{P}_a(X_{(\lfloor t/u_0\rfloor +1)u_0\wedge\tau_{h(s)}} \geq a) \\
&\qquad + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du) \\
&\leq \sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b})e^{\rho u_0}\mathbb{P}_a(\tau_{h\circ\theta_s} > t) + \frac{C}{C'}\mathbb{P}_a(t < \tau_{h\circ\theta_s})\int_0^t e^{\rho u}\mathbb{P}_x(\tau_b \in du)
\end{aligned}$$

We deduce from (2.51) that, for any $t \geq 0$,

$$\sup_{x\geq b} \mathbb{P}_x(t < \tau_{h\circ\theta_s}) \leq C''\mathbb{P}_a(t < \tau_{h\circ\theta_s})$$

where

$$C'' := \left(e^{\rho u_0} + \frac{C}{C'}\right)\sup_{x\geq b} \mathbb{E}_x(e^{\rho\tau_b}) < \infty$$

Since $\psi_{h(s)}([a,\infty)) > 0$ for any $s \geq 0$, we conclude the proof of the point 2 setting

$$c_2 := \frac{\inf_{z\in[0,h(0)]}\psi_z([a,\infty))}{C''}$$

□

Chapter 3

Quasi-stationarity for one-dimensional renormalized Brownian motion

This chapter is based on the paper [46], submitted in September 2018.

Abstract

We are interested in the quasi-stationarity of the time-inhomogeneous Markov process

$$X_t = \frac{B_t}{(t+1)^\kappa}$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\kappa \in (0, \infty)$. We first show that the law of X_t conditioned not to go out from $(-1, 1)$ until the time t converges weakly towards the Dirac measure δ_0 when $\kappa > \frac{1}{2}$ as t goes to infinity. Then we show that this conditioned probability converges weakly towards the quasi-stationary distribution of an Ornstein-Uhlenbeck process when $\kappa = \frac{1}{2}$. Finally, when $\kappa < \frac{1}{2}$, it is shown that the conditioned probability converges towards the quasi-stationary distribution of a Brownian motion. We also prove the existence of a Q-process and a quasi-ergodic distribution for $\kappa = \frac{1}{2}$ and $\kappa < \frac{1}{2}$.

3.1 Introduction

3.1.1 Introduction of the problem and quasi-stationarity

In this chapter we are interested in some notions related to quasi-stationarity for a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ killed when crossing the moving boundaries $t \to (-(t+1)^\kappa, (t+1)^\kappa)$, with $\kappa \in (0, \infty)$. Quasi-stationarity with moving boundaries was studied in [44] and [45] for periodic or converging boundaries, but expanding boundaries

were not yet considered. Actually, instead of considering the process B absorbed at $t \to (-(t+1)^\kappa, (t+1)^\kappa)$, we will study the quasi-stationarity of the process $X = (X_t)_{t\geq 0}$ absorbed at $(-1,1)^c$ and defined by

$$X_t := \frac{B_t}{(t+1)^\kappa}, \qquad \forall t < \tau_X$$

where $\tau_X := \inf\{t \geq 0 : |X_t| = 1\}$.

The process X is a time-inhomogeneous Markov process. For any $x \in \mathbb{R}$ and $s \geq 0$, denote by $\mathbb{P}_{x,s}$ the probability measure satisfying $\mathbb{P}_{x,s}(X_s = x) = 1$, and denote by $\mathbb{E}_{x,s}$ its corresponding expectation. Also, for any measure μ, for any $s \geq 0$, one denote by $\mathbb{P}_{\mu,s} := \int_{\mathbb{R}} \mathbb{P}_{x,s}\mu(dx)$ and $\mathbb{E}_{\mu,s} := \int_{\mathbb{R}} \mathbb{E}_{x,s}\mu(dx)$.

A *quasi-stationary distribution of X absorbed at* $(-1,1)^c$ is a probability measure α supported on $(-1,1)$ such that

$$\mathbb{P}_{\alpha,s}(X_t \in \cdot | \tau_X > t) = \alpha, \qquad \forall s \leq t$$

We refer the reader to [19, 42] for more details on the theory. Note however that these references only deal with the time-homogeneous setting and that quasi-stationary distributions for time-inhomogeneous Markov processes do not exist except for particular cases (especially we will see that the existence of one quasi-stationary distribution holds only for $\kappa = \frac{1}{2}$).

Usually, in the literature dealing with quasi-stationarity, mathematicians are interested in showing the weak convergence of marginal laws of Markov processes conditioned not to be absorbed by a cemetery set. The corresponding limit is called *quasi-limiting distribution*. For our purpose, we define a quasi-limiting distribution as follows

Definition 10. α is a quasi-limiting distribution of X if for some initial law μ supported on $(-1,1)$ and for any $s \geq 0$,

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \alpha$$

where the limit refers to the weak convergence of measures. In [42] it is noted that, in the time-homogeneous setting, quasi-stationary distribution and quasi-limit distribution are equivalent notions. In the time-inhomogeneous setting, this equivalence does not hold anymore. More particularly a time-inhomogeneous Markov process could admit a quasi-limit distribution without admitting a quasi-stationary distribution. However every quasi-stationary distribution is necessarily the quasi-limiting distribution.

Quasi-limiting distribution is not the only thing of interest in the theory of quasi-stationarity. Another thing of interest is the *Q-process*, which can be considered as the law of the considered Markov process conditioned not be absorbed. Concerning the process X, we define the Q-process as follows

Definition 11. We say that there is a Q-process for X if there exists a family $(\mathbb{Q}_{x,s})_{x\in(-1,1),s\geq 0}$ of probability measure defined by : for any $x \in (-1,1)$ and for any $s \leq t$

$$\lim_{T\to\infty} \mathbb{P}_{x,s}(X_{[s,t]} \in \cdot | T < \tau_X) = \mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot)$$

where, for any $u \leq v$, $X_{[u,v]}$ is the trajectory of X between times u and v. Then the Q-process is defined as the law of X under $(\mathbb{Q}_{x,s})_{x\in(-1,1),s\geq 0}$.

In general, the Q-process is also a Markov process and the theory of quasi-stationarity allows to get some results of ergodicity for the Q-process.

Finally, a third concept to study is the existence of a *quasi-ergodic distribution* defined as follows

Definition 12. β is a quasi-ergodic distribution of X if for some initial law μ supported on $(-1,1)$ and for any $s \geq 0$,

$$\lim_{t\to\infty} \frac{1}{t} \int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | \tau_X > t) du = \beta$$

In the literature, this notion is also called *mean-ratio quasi-stationary distribution*. The references [19, 42] does not deal with quasi-ergodic distribution. See for example [9, 17] which provide general assumptions implying the existence of quasi-ergodic distribution for time-homogeneous Markov processes, and see [45, 44] for the time-inhomogeneous setting.

Some general results on quasi-stationarity for time-inhomogeneous Markov process are established, particularly in [18], where it is shown that criteria based on Doeblin-type condition implies a mixing property (or *merging* or *weak ergodicity*) and the existence of the Q-process. However it will be difficult to apply these results for our purpose. See also [56, 22, 45, 44] for a few results about quasi-stationarity in the time-inhomogeneous setting, and [1] for ergodic properties for general non-conservative (time-homogeneous and inhomogeneous) semi-group.

Now let us come back to our process X. As we can expect, the existence of quasi-limiting, Q-process and quasi-ergodic distribution will strongly depend on κ. More precisely, three regimes are identified :

- $\kappa > \frac{1}{2}$, we will say that X is *supercritical*
- $\kappa = \frac{1}{2}$, we will say that X is *critical*
- $\kappa < \frac{1}{2}$, we will say that X is *subcritical*

The aim of this chapter is therefore to show the existence of quasi-limiting, Q-process and quasi-ergodic distribution for each regime. More precisely, it will be shown in a first step that, for any probability measure μ on $(-1,1)$ and $s \geq 0$,

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \delta_0$$

in the supercritical regime. This regime is of little interest and the existence of a Q-process and a quasi-ergodic distribution will not be shown. In a second step, the existence of quasi-limiting, Q-process and quasi-ergodic distribution will be stated in the critical regime and these probability measures are connected to the quasi-stationarity of an Ornstein-Uhlenbeck process absorbed at $(-1,1)^c$. Finally, the existence of these three notions will also be established in the subcritical regime where the quasi-stationarity of X is linked with the quasi-stationarity of a Brownian motion on $[-1,1]$.

3.1.2 A few notation

For any $E \subset \mathbb{R}$, one denotes by $\mathcal{M}_1(E)$ the set of the probability measures supported on E and, for any measurable bounded function f on $(-1,1)$ and $\mu \in \mathcal{M}_1((-1,1))$, one denotes by

$$\mu(f) := \int_{(-1,1)} f d\mu$$

For a general Markov process $(\Omega, (\mathcal{F}_t^A)_{t\geq 0}, (A_t)_{t\geq 0}, (\mathbb{P}^A_{x,s})_{x\in\mathbb{R}, s\geq 0})$, one denote, for any probability measure μ on $\mathbb{R}$ and any $s \geq 0$, $\mathbb{P}^A_{\mu,s} = \int_{\mathbb{R}} \mathbb{P}_{x,s}\mu(dx)$. Then the family of probability measures $(\mathbb{P}^A_{\mu,s})_{\mu\in\mathcal{M}_1(\mathbb{R}),s\geq 0}$ satisfies

$$\mathbb{P}^A_{\mu,s}(A_s \in \cdot) = \mu$$

If the process A is time-homogeneous, one define, for any $\mu \in \mathcal{M}_1(\mathbb{R})$, $\mathbb{P}^A_\mu := \mathbb{P}^A_{\mu,0}$ and one has, for any $s \leq t$,

$$\mathbb{P}^A_{\mu,s}(X_{[s,t]} \in \cdot) = \mathbb{P}^A_\mu(X_{[0,t-s]} \in \cdot)$$

For $A = X$, we will keep the notation established at the beginning of the introduction.

3.2 The supercritical regime : $\kappa > \frac{1}{2}$

The following theorem states the existence of a unique quasi-limiting distribution, which is δ_0

Theorem 14. *For any $\mu \in \mathcal{M}_1((-1,1))$ and $s \geq 0$,*

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \delta_0$$

Proof. By Markov's inequality, for any $\epsilon > 0$ and any probability measure μ,

$$\begin{aligned}
\mathbb{P}_{\mu,s}\left(|X_t| \geq \epsilon | \tau_X > t\right) &\leq \frac{\mathbb{E}_{\mu,s}(X_t^2 | \tau_X > t)}{\epsilon^2} \\
&\leq \frac{\mathbb{E}_{\mu,s}(X_t^2)}{\epsilon^2 \mathbb{P}_{\mu,s}(\tau_X > t)} \\
&= \frac{t - s + (s+1)^{2\kappa} \int_{(-1,1)} x^2 d\mu(x)}{\epsilon^2 (t+1)^{2\kappa} \mathbb{P}_{\mu,s}(\tau_X > t)}
\end{aligned}$$

It is well known that $\mathbb{P}_{\mu,s}$-almost surely,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = 1$$

and

$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = -1$$

Thus, since $\kappa > \frac{1}{2}$, $\mathbb{P}_{\mu,s}$-almost surely,

$$X_t = \frac{B_t}{(t+1)^\kappa} \underset{t\to\infty}{\longrightarrow} 0$$

As a result, for any $s \geq 0$ and any probability measure μ on $(-1,1)$,

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(\tau_X > t) = \mathbb{P}_{\mu,s}(\tau_X = \infty) > 0$$

Thus, for any $\epsilon > 0$,

$$\mathbb{P}_{\mu,s}(|X_t| \geq \epsilon | \tau_X > t) \underset{t\to\infty}{\longrightarrow} 0$$

In other words, $(X_t)_{t\geq 0}$ converges in conditional probability towards 0. This conclude the proof. □

3.3 The critical case : $\kappa = \frac{1}{2}$

3.3.1 Existence and uniqueness of a quasi-stationary distribution

We state a first theorem on the existence of the quasi-limiting distribution (and quasi-stationary distribution) in the critical regime.

Theorem 15. *Let α_{OU} be the unique quasi-stationary distribution of the Ornstein-Uhlenbeck process absorbed by $(-1,1)^c$ whose the generator is*

$$L := \frac{1}{2}\Delta - \frac{1}{2}x\nabla \tag{3.1}$$

Then α_{OU} is also the unique quasi-stationary distribution of X and there exist $C_{OU}, \gamma_{OU} > 0$ such that for any probability measure μ on $(-1,1)$ and any $0 \leq s \leq t$,

$$||\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) - \alpha_{OU}||_{TV} \leq C_{OU} \left(\frac{s+1}{t+1}\right)^{\gamma_{OU}} \tag{3.2}$$

In particular, for any $\mu \in \mathcal{M}_1((-1,1))$ and $s \geq 0$, $t \to \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t)$ converges weakly towards α_{OU} when t goes to infinity.

Remark 4. Using the spectral theory of the Ornstein-Uhlenbeck generator, α_{OU} can be computed and

$$\alpha_{OU}(dx) := K \times (1-x^2) e^{-\frac{x^2}{2}} dx$$

where K is the renormalization constant.

Remark 5. It is well-known (see [19, 42]) that there exists $\lambda_{OU} > 0$ such that

$$\mathbb{P}^Z_{\alpha_{OU}}(\tau_Z > t) = e^{-\lambda_{OU} t}, \quad \forall t \geq 0 \tag{3.3}$$

where $\tau_Z := \inf\{t \geq 0 : |Z_t| = 1\}$. Moreover, for any $f \in \{g \in C^2([-1,1]) : g(-1) = g(1) = 0\}$,

$$\alpha_{OU}(Lf) = -\lambda_{OU} \alpha_{OU}(f)$$

where L is defined in (3.1). Using the explicit formula of α_{OU}, it is easy to check that

$$\lambda_{OU} = 1 \tag{3.4}$$

Proof of Theorem 15. Let Z be the Ornstein-Uhlenbeck process whose infinitesimal generator is L. Then, for any probability measure μ on $(-1,1)$ and any $s \geq 0$,

$$\mathbb{P}_{\mu,s}((X_t)_{t\geq s} \in \cdot) = \mathbb{P}^Z_\mu \left(\left(Z_{\log\left(\frac{t+1}{s+1}\right)} \right)_{t\geq s} \in \cdot \right) \tag{3.5}$$

Hence, using (3.5), one has for any $s \leq t$,

$$\begin{aligned} \mathbb{P}_{\alpha_{OU},s}(X_t \in \cdot | \tau_X > t) &= \mathbb{P}^Z_{\alpha_{OU}} \left(Z_{\log\left(\frac{t+1}{s+1}\right)} \in \cdot \middle| \tau_Z > \log\left(\frac{t+1}{s+1}\right) \right) \\ &= \alpha_{OU} \end{aligned}$$

In other words α_{OU} is also the unique quasi-stationary distribution of the time-inhomogeneous Markov process X. Moreover, since Z satisfies the assumptions $(A1)$ and $(A2)$ of [14] (this is actually shown in [13]), then, by Theorem 2.1. in [14], there exist $C_{OU} > 0$ and $\gamma_{OU} > 0$ such that for any $t \geq 0$ and for any probability measure μ,

$$||\mathbb{P}^Z_\mu(Z_t \in \cdot | \tau_Z > t) - \alpha_{OU}||_{TV} \leq C_{OU} e^{-\gamma_{OU} t}$$

Using (3.5) one deduce that, for any $s \leq t$ and for any probability measure μ on $(-1,1)$,

$$||\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) - \alpha_{OU}||_{TV} \leq C_{OU} \left(\frac{s+1}{t+1} \right)^{\gamma_{OU}}$$

This concludes the proof. □

3.3.2 Existence of the Q-process

Before tackling the existence of the Q-process we need the following proposition.

Proposition 17. *There exists a non-negative function $\eta_{OU} : [-1,1] \to \mathbb{R}_+$, positive on $(-1,1)$ and vanishing on $\{-1,1\}$ such that for any $x \in (-1,1)$ and any $s \geq 0$,*

$$\eta_{OU}(x) = \lim_{t\to\infty} \frac{t+1}{s+1} \mathbb{P}_{x,s}(\tau_X > t)$$

where the convergence holds uniformly on $[-1,1]$ and $\alpha_{OU}(\eta_{OU}) = 1$. Moreover the function η_{OU} is bounded, belongs to the domain of L defined in (3.1) *and*

$$L\eta_{OU} = -\lambda_{OU}\eta_{OU} = -\eta_{OU}$$

Remark 6. Actually it is easy to check that

$$\eta_{OU}(x) = K' \times (1 - x^2) \tag{3.6}$$

where K' is the positive constant such that $\alpha_{OU}(\eta_{OU}) = 1$.

An interesting consequence of Proposition 17 can be stated as the following corollary

Corollary 2. *Let B a Brownian motion on $\mathbb{R}$ and denote by*

$$\tau_B^{\sqrt{\cdot}} := \inf\{t \geq 0 : |B_t| \geq \sqrt{t+1}\}$$

Then for any $x \in (-1,1)$,

$$\mathbb{P}_x^B(\tau_B^{\sqrt{\cdot}} > t) \sim_{t\to\infty} K' \frac{1-x^2}{t+1}$$

Proof of Proposition 17 and Corollary 2. Using Proposition 2.3 in [14] applied to the process Z and (3.5), one has for any $x \in (-1,1)$ and $s \geq 0$,

$$\begin{aligned}
\eta_{OU}(x) &= \lim_{t\to\infty} e^{\lambda_{OU}\log(\frac{t+1}{s+1})} \mathbb{P}_x^Z\left(\tau_Z > \log\left(\frac{t+1}{s+1}\right)\right) \\
&= \lim_{t\to\infty} \left(\frac{t+1}{s+1}\right)^{\lambda_{OU}} \mathbb{P}_{x,s}(\tau_X > t) \\
&= \lim_{t\to\infty} \frac{t+1}{s+1} \mathbb{P}_{x,s}(\tau_X > t)
\end{aligned}$$

where we finally used (3.4). This ends the proof of Proposition 17. Now it is easy to see that, for any $x \in (-1,1)$ and $t \geq 0$, $\mathbb{P}_x^B(\tau_B^{\sqrt{\cdot}} > t) = \mathbb{P}_{x,0}(\tau_X > t)$. Thus, using Proposition 17 and (3.6), we conclude the corollary. □

Remark 7. In [8], Breiman shows a similar result for one-dimensional Brownian motion absorbed by a one-sided square boundary. More precisely, denoting $T_c^* := \inf\{t \geq 0 : B_t \geq c\sqrt{t+1}\}$ for any $c > 0$, he shows that $\mathbb{P}_0^B(T_c^* > t) \sim_{t\to\infty} at^{-b(c)}$ for $a > 0$ and b such that $b(1) = 1$. In particular, for $c = 1$, $\mathbb{P}_0^B(T_1^* > t)$ and $\mathbb{P}_0^B(\tau_B^{\sqrt{\cdot}} > t)$ decay as $1/t$. The reader can also see [48] for more general boundaries.

We turn to the existence of the Q-process and its ergodicity.

Proposition 18.

- *There exists a Q-process and the family of probability measure $(\mathbb{Q}_{x,s})_{x\in(-1,1),s\geq 0}$ defined in Definition 11 is given by : for any $x \in (-1,1)$ and $s \leq t$,*

$$\begin{aligned}
\mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) &= \mathbb{E}_{x,s}\left(\mathbb{1}_{X_{[s,t]}\in\cdot,\tau_X>t}\left(\frac{t+1}{s+1}\right)^{\lambda_{OU}} \frac{\eta_{OU}(X_t)}{\eta_{OU}(x)}\right) \\
&= \frac{t+1}{s+1} \times \mathbb{E}_{x,s}\left(\mathbb{1}_{X_{[s,t]}\in\cdot,\tau_X>t} \frac{1-X_t^2}{1-x^2}\right)
\end{aligned}$$

- *The probability measure β_{OU} defined by*

$$\beta_{OU}(dx) := \eta_{OU}(x)\alpha_{OU}(dx) = KK'(1-x^2)^2 e^{-\frac{x^2}{2}} dx$$

is the unique stationary distribution of X under $(\mathbb{Q}_{x,s})_{s\geq 0,x\in(-1,1)}$. Moreover, for any $0 \leq s \leq t$ and any $x \in (-1,1)$,

$$||\mathbb{Q}_{x,s}(X_t \in \cdot) - \beta_{OU}||_{TV} \leq C_{OU}\left(\frac{s+1}{t+1}\right)^{\gamma_{OU}}$$

where C_{OU} and γ_{OU} are the same constant as used in (3.2).

Proof. Straightforward using (3.5) and Proposition 3.1. in [14] applied to the Ornstein-Uhlenbeck process Z □

3.3.3 Quasi-ergodic distribution

Theorem 16. *For any probability measure μ on $(-1,1)$ and any $s \geq 0$, for any measurable set S*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}_{\mu,s}(X_u \in S|\tau_X > t)du = \int_S \mathbb{E}_x^Z\left(\tau_Z\right)\alpha_{OU}(dx)$$

Remark 8. In the time-homogeneous setting, it is usually expected that the quasi-ergodic distribution is the stationary distribution of the Q-process (see [9, 17]). A similar result could even be expected in the time-inhomogeneous case when the Q-process converges weakly at the infinity (see [44]). It is therefore astonishing to see that this is not the case for our process in the critical regime, even though the Q-process admits a stationary measure. In particular, the quasi-ergodic distribution of X is different from the quasi-ergodic distribution of the process Z.

Proof. First, using the variable change $u = s+q(t-s)$, one has, for any $\mu \in \mathcal{M}_1((-1,1))$, $s < t$ and f bounded measurable,

$$\frac{1}{t-s}\int_s^t \mathbb{E}_{\mu,s}(f(X_u)|\tau_X > t)du = \int_0^1 \mathbb{E}_{\mu,s}(f(X_{s+q(t-s)})|\tau_X > t)dq$$

As a result it is enough to show the weak convergence of $\left(\mathbb{P}_{\mu,s}(X_{s+q(t-s)} \in \cdot|\tau_X > t)\right)_{t\geq 0}$ for any $q \in (0,1)$, then to conclude with the Lebesgue's dominated convergence theorem. Let $\mu \in \mathcal{M}_1((-1,1))$, $s \geq 0$, $q \in (0,1)$ and f bounded measurable. Using Markov property and (3.5),for any $t \geq s$,

$$\mathbb{E}_{\mu,s}(f(X_{s+q(t-s)})\mathbb{1}_{\tau_X>t}) = \mathbb{E}_{\mu,s}\left(f(X_{s+q(t-s)})\mathbb{1}_{\tau_X>s+q(t-s)}\mathbb{P}_{X_{s+q(t-s)},s+q(t-s)}(\tau_X > t)\right) \tag{3.7}$$

$$= \mathbb{E}_{\mu,s}\left(f_t\left(X_{s+q(t-s)}\right)\mathbb{1}_{\tau_X>s+q(t-s)}\right) \tag{3.8}$$

where we set for any $y \in (-1,1)$,

$$f_t(y) := f(y)\mathbb{P}_{y,s+q(t-s)}\left[\tau_X > t\right]$$

By (3.5), for any $y \in (-1,1)$ and $t \geq s$,

$$f_t(y) = f(y)\mathbb{P}_y^Z\left[\tau_Z > \log\left(\frac{t+1}{s+q(t-s)+1}\right)\right]$$

Now define for any $y \in (-1,1)$,

$$f_\infty(y) := f(y)\mathbb{P}_y^Z\left[\tau_Z > -\log\left(q\right)\right]$$

It is easy to see that $(f_t)_{t\geq 0}$ converges pointwise towards f_∞. Moreover, a simple curve sketching entails that the function $t \to \frac{t+1}{s+q(t-s)+1}$ is increasing, which implies that the sequence $(f_t)_{t\geq 0}$ is a decreasing sequence of continuous functions defined on $[-1,1]$. Likewise, f_∞ is continuous on $[-1,1]$. As a result, by Dini's theorem for the decreasing sequences of continuous function, the pointwise convergence is equivalent to uniform convergence on $[-1,1]$. Thus,

$$\lim_{t\to\infty} \sup_{y\in(-1,1)} |f_t(y) - f_\infty(y)| = 0 \tag{3.9}$$

Now let us show that

$$\lim_{t\to\infty} \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)})\mathbb{1}_{\tau_X > s+q(t-s)}) = \mu(\eta_{OU})\alpha_{OU}(f_\infty) \tag{3.10}$$

To show this, let us begin with

$$\begin{aligned}
&\frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}\left(f_\infty(X_{s+q(t-s)})\mathbb{1}_{\tau_X > s+q(t-s)}\right) \\
&= \frac{s+q(t-s)+1}{s+1}\mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) \times \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)})|\tau_X > s+q(t-s))
\end{aligned}$$

On the one hand, by Proposition 17,

$$\lim_{t\to\infty} \frac{s+q(t-s)+1}{s+1}\mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) = \mu(\eta_{OU})$$

On the other hand, by (3.2),

$$\lim_{t\to\infty} \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)})|\tau_X > s+q(t-s)) = \alpha_{OU}(f_\infty)$$

(3.10) follows from these two convergences. Now, by (3.10) and (3.9),

$$\begin{aligned}
&\frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}\left(f_t\left(X_{s+q(t-s)}\right)\mathbb{1}_{\tau_X > s+q(t-s)}\right) \\
&\qquad = \frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}\left(f_\infty\left(X_{s+q(t-s)}\right)\mathbb{1}_{\tau_X > s+q(t-s)}\right) \\
&\qquad\quad + \frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}\left(\left[f_\infty\left(X_{s+q(t-s)}\right) - f_t\left(X_{s+q(t-s)}\right)\right]\mathbb{1}_{\tau_X > s+q(t-s)}\right) \\
&\qquad \underset{t\to\infty}{\longrightarrow} \mu(\eta_{OU})\alpha_{OU}(f_\infty)
\end{aligned}$$

because

$$\begin{aligned}
&\left|\frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}\left(\left[f_\infty\left(X_{s+q(t-s)}\right) - f_t\left(X_{s+q(t-s)}\right)\right]\mathbb{1}_{\tau_X > s+q(t-s)}\right)\right| \\
&\qquad \leq \frac{s+q(t-s)+1}{s+1}\mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) \times \sup_{y\in(-1,1)} |f_t(y) - f_\infty(y)| \underset{t\to\infty}{\longrightarrow} 0
\end{aligned}$$

Hence, using (3.7), (3.8),

$$\lim_{t\to\infty} \frac{s+q(t-s)+1}{s+1}\mathbb{E}_{\mu,s}(f(X_{s+q(t-s)})\mathbb{1}_{\tau_X>t}) = \mu(\eta_{OU})\alpha_{OU}(f_\infty)$$
$$= \mu(\eta_{OU})\int_{(-1,1)} f(x)\mathbb{P}_x^Z(\tau_Z > -\log(q))\alpha_{OU}(dx)$$

Moreover, taking $f = \mathbb{1}$, using (3.3) and (3.4),

$$\lim_{t\to\infty} \frac{s+q(t-s)+1}{s+1}\mathbb{P}_{\mu,s}(\tau_X > t) = \mu(\eta_{OU})\mathbb{P}^Z_{\alpha_{OU}}(\tau_Z > -\log(q))$$
$$= \mu(\eta_{OU})q$$

Thus we deduce that

$$\lim_{t\to\infty} \mathbb{E}_{\mu,s}(f(X_{s+q(t-s)})|\tau_X > t) = q^{-1}\int_{(-1,1)} \alpha_{OU}(dx)f(x)\mathbb{P}_x(\tau_Z > -\log(q))$$

Then, by Lebesgue's theorem, for any probability measure μ on $(-1,1)$ and any bounded measurable function f,

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t-s}\int_s^t \mathbb{E}_{\mu,s}(f(X_u)|\tau_X > t)du &= \lim_{t\to\infty}\int_0^1 \mathbb{E}_{\mu,s}(f(X_{s+q(t-s)})|\tau_X > t)dq \\
&= \int_0^1 q^{-1}\int_{(-1,1)} f(x)\mathbb{P}_x^Z(\tau_Z > -\log(q))\alpha_{OU}(dx)dq \\
&= \int_{(-1,1)} \alpha_{OU}(dx)f(x)\int_0^1 q^{-1}\mathbb{P}_x^Z(\tau_Z > -\log(q))dq \\
&= \int_{(-1,1)} \alpha_{OU}(dx)f(x)\int_0^\infty \mathbb{P}_x^Z(\tau_Z > s)ds \\
&= \int_{(-1,1)} \alpha_{OU}(dx)f(x)\mathbb{E}_x^Z(\tau_Z)
\end{aligned}$$

This concludes the proof. □

3.4 The subcritical case : $\kappa < \frac{1}{2}$

In order to show the asymptotic properties of the law of the process X at time t conditioned not be absorbed, we will first show that the process X conditioned not to be absorbed until a further time T is close in total variation norm to the one of the process Y defined by

$$Y_t := \int_0^t \frac{1}{(u+1)^\kappa}dB_u, \quad \forall t \geq 0$$

when the starting time s goes to infinity. Then the existence of quasi-limiting or quasi-ergodic distribution and Q-process will be deduced from the quasi-stationarity of Y.

3.4.1 Approximation by Y through asymptotic pseudotrajectories

Denote by $\tau_Y := \inf\{t \geq 0 : |Y_t| = 1\}$. The aim of this subsection is to show the following proposition :

Proposition 19. *There exists a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $0 \leq s \leq t \leq T$, for any probability measure μ on $(-1, 1)$,*

$$\lim_{s\to\infty} F(s) = 0$$

and such that

$$||\mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_X > T) - \mathbb{P}^Y_{\mu,s}(Y_t \in \cdot|\tau_Y > T)||_{TV} \leq F(s) \tag{3.11}$$

Remark 9. (3.11) provides us with a decay towards 0 uniformly in the initial law, t and T. It can be seen as an analogue of the asymptotic pseudotrajectories introduced by Benaïm and Hirsch in [7]. See also [5] for more details about asymptotic pseudotrajectories in general case.

Proof of Proposition 19. By Itô's formula, one has for any $t \geq 0$,

$$X_t = X_0 + Y_t - < Y, M >_t$$

where

$$M_t := \int_0^t \kappa(u+1)^{\kappa-1} X_u dB_u = \int_0^t \kappa(u+1)^{2\kappa-1} X_u dX_u + \int_0^t (\kappa(u+1)^{\kappa-1} X_u)^2 du$$

For any $s \leq t$, denote by

$$M_{s,t} := M_t - M_s = \int_s^t \kappa(u+1)^{\kappa-1} X_u dB_u$$

and

$$< M, M >_{s,t} := < M, M >_t - < M, M >_s = \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du$$

and denote by $\mathcal{E}(M)_{s,t}$ the exponential local martingale defined by

$$\begin{aligned}
\mathcal{E}(M)_{s,t} &:= \exp\left(M_{s,t} - \frac{1}{2} < M, M >_{s,t}\right) \\
&:= \exp\left(\int_s^t \kappa(u+1)^{2\kappa-1} X_u dX_u + \frac{1}{2}\int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du\right) \\
&= \exp\left(\frac{1}{2} N_{s,t}\right)
\end{aligned}$$

where, for any $s \leq t$,

$$\begin{aligned}
N_{s,t} &:= \kappa(t+1)^{2\kappa-1}X_t^2 - \kappa(s+1)^{2\kappa-1}X_s^2 \\
&\quad - \int_s^t [\kappa(u+1)^{2\kappa-1}]' X_u^2 du - \int_s^t \frac{\kappa}{u+1} du + \int_s^t (\kappa(u+1)^{\kappa-1}X_u)^2 du \\
&= \kappa(t+1)^{2\kappa-1}X_t^2 - \kappa(s+1)^{2\kappa-1}X_s^2 \\
&\quad - \int_s^t [\kappa(u+1)^{2\kappa-1}]' X_u^2 du - \kappa \log\left(\frac{t+1}{s+1}\right) + \int_s^t (\kappa(u+1)^{\kappa-1}X_u)^2 du
\end{aligned}$$

Note that the process $(N_{s,t\wedge\tau_X})_{s\leq t}$ is almost surely uniformly bounded, thus $\mathcal{E}(M)_{s,t\wedge\tau_X}$ is a martingale. For any $t \geq s \geq 0$ and $\mu \in \mathcal{M}_1((-1,1))$, define $\mathbb{G}_{\mu,s}$ the probability measure satisfying

$$\mathbb{G}_{\mu,s}(A) = \mathbb{E}_{\mu,s}(\mathcal{E}(M)_{s,t\wedge\tau_X} \mathbb{1}_A), \qquad \forall A \in \sigma(X_u, s \leq u \leq t)$$

Then, by Girsanov's theorem, the law of $(X_{t\wedge\tau_X})_{t\geq s}$ under $\mathbb{G}_{\mu,s}$ is the law of $(Y_{t\wedge\tau_Y})_{t\geq s}$ under $\mathbb{P}_{\mu,s}$. In particular, for any S measurable set, probability measure μ on $(-1,1)$ and $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\mathbb{P}_{\mu,s}(Y_t \in B, \tau_Y > T) &= \mathbb{G}_{\mu,s}(X_t \in S, \tau_X > T) \\
&= \mathbb{E}_{\mu,s}\left(\mathcal{E}(M)_{s,T\wedge\tau_X} \mathbb{1}_{X_t\in S,\tau_X>T}\right) \\
&= \mathbb{E}_{\mu,s}\left(\mathcal{E}(M)_{s,T} \mathbb{1}_{X_t\in S,\tau_X>T}\right) \\
&= \mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N_{s,T}\right) \mathbb{1}_{X_t\in S,\tau_X>T}\right) \\
&= \left(\frac{s+1}{T+1}\right)^{\frac{\kappa}{2}} \mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right) \mathbb{1}_{X_t\in S,\tau_X>T}\right)
\end{aligned}$$

with $N'_{s,T} = N_{s,T} + \kappa \log\left(\frac{T+1}{s+1}\right)$. Thus,

$$\mathbb{P}_{\mu,s}(Y_t \in B | \tau_Y > T) = \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right) \mathbb{1}_{X_t\in S,\tau_X>T}\right)}{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right) \mathbb{1}_{\tau_X>T}\right)}$$

Thus for any $0 \le s \le t \le T$ and S measurable set,

$$
\begin{aligned}
&|\mathbb{P}_{\mu,s}(X_t \in S|\tau_X > T) - \mathbb{P}^Y_{\mu,s}(Y_t \in S|\tau_Y > T)| \\
&= \left| \mathbb{P}_{\mu,s}(X_t \in S|\tau_X > T) - \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in S,\tau_X>T}\right)}{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{\tau_X>T}\right)} \right| \\
&\le \left| \frac{\mathbb{P}_{\mu,s}(\tau_X > T)}{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{\tau_X>T}\right)} \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in S,\tau_X>T}\right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in S,\tau_X>T}\right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \right| \\
&\quad + \left| \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in S,\tau_X>T}\right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\mathbb{P}_{\mu,s}(X_t \in S, \tau_X > T)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \right| \\
&\le \underbrace{\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > T)}{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{\tau_X>T}\right)} - 1 \right|}_{=:A_s(\mu,T)} \times \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{\tau_X>T}\right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \\
&\quad + \underbrace{\frac{\left|\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in S,\tau_X>T}\right) - \mathbb{P}_{\mu,s}(X_t \in S, \tau_X > T)\right|}{\mathbb{P}_{\mu,s}(\tau_X > T)}}_{=:C_s(\mu,t,T,S)}
\end{aligned}
$$

In order to show (3.11), we will bound the functions A_s and C_s.

Step 1 : Upper bound for C_s .

For any $0 \le s \le t \le T$, probability measure μ and B measurable set,

$$
C_s(\mu,t,T,S) = \left| \underbrace{\mathbb{E}_{\mu,s}\left[\left(\exp\left(\frac{1}{2}N'_{s,T}\right) - 1\right)\mathbb{1}_{X_t\in S}\middle|\tau_X > T\right]}_{=:f(s,t,T,\mu,S)} \right|
$$

On the event $\{\tau_X > T\}$, $X_u^2 < 1$ for any $0 \le u \le T$. Hence the function f defined as above is bounded as follows

$$
\begin{aligned}
&\exp\left(-\frac{\kappa}{2}(s+1)^{2\kappa-1} - \frac{1}{2}\int_s^T [\kappa(u+1)^{2\kappa-1}]' du\right) - 1 \\
&\le f(s,t,T,\mu,S) \le \exp\left(\frac{1}{2}\kappa(T+1)^{2\kappa-1} + \frac{1}{2}\int_s^T (\kappa(u+1)^{\kappa-1})^2 du\right) - 1
\end{aligned}
$$

In particular, for any $0 \le s \le t \le T$, for any probability measure μ and S

measurable set,

$$|f(s,t,T,\mu,S)| \leq \left(1-\exp\left(-\frac{1}{2}\kappa(s+1)^{2\kappa-1}\right)\right) \vee \left(\exp\left(\frac{1}{2}\left(\kappa+\frac{\kappa^2}{1-2\kappa}\right)\kappa(s+1)^{2\kappa-1}\right)-1\right) =: \phi(s)$$

Hence,

$$C_s(\mu,t,T,S) \leq \phi(s)$$

Step 2 : Upper bound for A_s .

Taking $S=(-1,1)$,

$$C_s(\mu,t,T,(-1,1)) = \frac{|\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{X_t\in(-1,1),\tau_X>T}\right)-\mathbb{P}_{\mu,s}(X_t\in(-1,1),\tau_X>T)|}{\mathbb{P}_{\mu,s}(\tau_X>T)} = \left|\frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,T}\right)\mathbb{1}_{\tau_X>T}\right)}{\mathbb{P}_{\mu,s}(\tau_X>T)}-1\right|$$

According to the previous bound we have shown, for any $s\leq T$, for any probability measure μ on $(-1,1)$,

$$1-\phi(s) \leq \frac{\mathbb{E}_{\mu,s}\left(\exp\left(\frac{1}{2}N'_{s,t}\right)\mathbb{1}_{\tau_X>t}\right)}{\mathbb{P}_{\mu,s}(\tau_X>t)} \leq 1+\phi(s) \tag{3.12}$$

We deduce from this last inequality that

$$A_s(\mu,T) \leq \left(1-\frac{1}{1+\phi(s)}\right)\vee\left(\frac{1}{1-\phi(s)}-1\right) =: \psi(s)$$

We set then, for any $s\geq 0$,

$$F(s)=\phi(s)+\psi(s)(1+\phi(s))$$

which concludes the proof. □

3.4.2 Quasi-stationarity of Y

Now we are interested in the quasi-stationarity of the process Y. Note that, by Dubin-Schwartz's theorem, there exists a Brownian motion $\tilde{B}$ such that for any $t\geq 0$

$$Y_t = \tilde{B}_{\frac{(t+1)^{1-2\kappa}-1}{1-2\kappa}} \tag{3.13}$$

Denote $\tau_{\tilde{B}} := \inf\{t \geq 0 : |\tilde{B}_t| = 1\}$. Then, by (3.13), for any initial law μ and $s \geq 0$,

$$\mathbb{P}^Y_{\mu,s}(Y_t \in \cdot | \tau_Y > t) = \mathbb{P}^{\tilde{B}}_{\mu}\left(\tilde{B}_{\frac{(t+1)^{1-2\kappa}-(s+1)^{1-2\kappa}}{1-2\kappa}} \in \cdot \middle| \tau_{\tilde{B}} > \frac{(t+1)^{1-2\kappa}-(s+1)^{1-2\kappa}}{1-2\kappa}\right)$$

It is well known that a Brownian motion absorbed at $(-1,1)^c$ admits a unique quasi-stationary distribution α_{Bm} whose explicit form is

$$\alpha_{Bm}(dx) := \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)dx$$

and that there exists $\lambda_{Bm} > 0$ such that

$$\mathbb{P}^{\tilde{B}}_{\alpha_{Bm}}(\tau_Y > t) = e^{-\lambda_{Bm}t}, \quad \forall t \geq 0$$

Remark that λ_{Bm} satisfies also

$$\alpha_{Bm}\left(\frac{1}{2}\Delta f\right) = -\lambda_{Bm}\alpha_{Bm}(f), \quad \forall f \in \{g \in \mathcal{C}^2([-1,1]) : g(1) = g(-1) = 0\}$$

The Brownian motion absorbed at $(-1,1)^c$ satisfies the Champagnat-Villemonais condition $(A1)-(A2)$ in [14], which implies the existence of $C_{Bm}, \gamma_{Bm} > 0$ such that for any probability measure μ and any $t \geq 0$,

$$||\mathbb{P}^{\tilde{B}}_{\mu}(\tilde{B}_t \in \cdot | \tau_{\tilde{B}} > t) - \alpha_{Bm}||_{TV} \leq C_{Bm}e^{-\gamma_{Bm}t}$$

Thus, using the Dubins-Schwartz transformation, for any $s \leq t$ and probability measure μ

$$||\mathbb{P}^Y_{\mu,s}(Y_t \in \cdot | \tau_Y > t) - \alpha_{Bm}||_{TV} \leq C_{Bm}\exp\left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa}-(s+1)^{1-2\kappa}}{1-2\kappa}\right) \tag{3.14}$$

In the same way as in the critical case, an analogous version of Propositions 17 and 18 can be stated as follows

Proposition 20. *(i) There exists a non-negative function $\eta_{Bm} : [-1,1] \to \mathbb{R}_+$, positive on $(-1,1)$ and vanishing on $\{-1,1\}$ such that for any $x \in (-1,1)$ and any $s \geq 0$,*

$$\eta_{Bm}(x) = \lim_{t\to\infty} e^{\lambda_{Bm}\frac{(t+1)^{1-2\kappa}-(s+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{P}^Y_{x,s}(\tau_Y > t)$$

where the convergence holds uniformly on $[-1,1]$ and $\alpha_{Bm}(\eta_{Bm}) = 1$.

(ii) There exists a Q-process for Y in the sense of Definition 11 and the family of probability measure $(\mathbb{Q}^Y_{x,s})_{x\in(-1,1),s\geq 0}$ defined by $\mathbb{Q}^Y_{s,x}(Y_{[s,t]} \in \cdot) := \lim_{T\to\infty}\mathbb{P}^Y_{x,s}(X_{[s,t]} \in \cdot | T < \tau_Y)$ satisfies also

$$\mathbb{Q}^Y_{x,s}(X_{[s,t]} \in \cdot) = \mathbb{E}_{x,s}\left(\mathbb{1}_{Y_{[s,t]}\in\cdot, t<\tau_Y}e^{\lambda_{Bm}\frac{(t+1)^{1-2\kappa}-(s+1)^{1-2\kappa}}{1-2\kappa}}\frac{\eta_{Bm}(Y_t)}{\eta_{Bm}(x)}\right)$$

for any $x \in (-1,1)$ and $s \leq t$

(iii) The probability measure β_{Bm} defined by

$$\beta_{Bm} = \eta_{Bm}(x)\alpha_{Bm}(dx)$$

is the unique stationary distribution of Y under $(\mathbb{Q}^Y_{x,s})_{x\in(-1,1),s\geq 0}$ and, for any $x \in (-1,1)$ and $s \geq 0$,

$$||\mathbb{Q}^Y_{x,s}(Y_t \in \cdot) - \beta_{Bm}||_{TV} \leq C_{Bm} \exp\left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}\right)$$

where C_{Bm} and γ_{Bm} are the same as in 3.14.

Proof. The proof is essentially the same as for the proof of Proposition 18. □

3.4.3 Quasi-limiting distribution of X

Now we will use Proposition 19 in order to show the existence of quasi-limiting and quasi-ergodic distribution for the process X. Let us start with stating and proving the existence of the quasi-limiting distribution.

Theorem 17. *For any probability measure μ on $(-1,1)$ and any $0 \leq s \leq t$,*

$$||\mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_X > t) - \alpha_{Bm}||_{TV} \leq F\left(\frac{t}{2}\right) + C_{Bm} \exp\left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa} - (\frac{t}{2}+1)^{1-2\kappa}}{1-2\kappa}\right) \tag{3.15}$$

where the function F is defined in Proposition 19. In particular, for any $\mu \in \mathcal{M}_1((-1,1))$ and any $s \geq 0$,

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_X > t) = \alpha_{Bm}$$

Proof. Let $\mu \in \mathcal{M}_1((-1,1))$. For any $s \leq t$ define

$$\mu_{(s,t)} := \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_X > t)$$

Then, according to Markov property, for any $s \leq t \leq u$,

$$\mu_{(s,u)} = \mathbb{P}_{\mu_{(s,t)},t}(X_u \in \cdot|\tau_X > u)$$

Thus, for any $s \leq t$,

$$\begin{aligned}
||\mu_{(s,2t)} - \alpha_{Bm}||_{TV} &\leq ||\mu_{(s,2t)} - \mathbb{P}^Y_{\mu_{(s,t)},t}(Y_{2t} \in \cdot|\tau_Y > 2t)||_{TV} \\
&\quad + ||\mathbb{P}^Y_{\mu_{(s,t)},t}(Y_{2t} \in \cdot|\tau_Y > 2t) - \alpha_{Bm}||_{TV} \\
&= ||\mathbb{P}_{\mu_{(s,t)},t}(X_{2t} \in \cdot|\tau_X > 2t) - \mathbb{P}^Y_{\mu_{(s,t)},t}(Y_{2t} \in \cdot|\tau_Y > 2t)||_{TV} \\
&\quad + ||\mathbb{P}^Y_{\mu_{(s,t)},t}(Y_{2t} \in \cdot|\tau_Y > 2t) - \alpha_{Bm}||_{TV} \\
&\leq F(t) + C_{Bm} \exp\left(-\gamma_{Bm} \times \frac{(2t+1)^{1-2\kappa} - (t+1)^{1-2\kappa}}{1-2\beta}\right)
\end{aligned}$$

where we used the inequalities (3.11) and (3.14). This shows the inequality (3.15). Now, since $\lim_{t\to\infty} F(t) = 0$ by Proposition 19 and noting that

$$\lim_{t\to\infty} \exp\left(-\gamma_{Bm} \times \frac{(2t+1)^{1-2\kappa} - (t+1)^{1-2\kappa}}{1-2\kappa}\right) = 0$$

because $\kappa < \frac{1}{2}$, this shows that for any $\mu \in \mathcal{M}_1((-1,1))$ and $s \geq 0$,

$$\lim_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \alpha_{Bm}$$

□

3.4.4 Quasi-ergodic distribution

It is shown in [28] and [17] that, if a Markov process satisfies the Champagnat-Villemonais condition stated in [14], then the Markov process admits a unique quasi-ergodic distribution (more generally, it is shown in [9] that if the Q-process of the Markov process is Harris recurrent, then there exists a quasi-ergodic distriution). As a result, there exists $\beta_{Bm} \in \mathcal{M}_1((-1,1))$ such that for any $\mu \in \mathcal{M}_1((-1,1))$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}_\mu(\tilde{B}_s \in \cdot | \tau_{\tilde{B}} > t) ds = \beta_{Bm}$$

and one has (see again [28, 17] for more details)

$$\beta_{Bm}(dx) = \eta_{Bm}(x)\alpha_{Bm}(dx)$$

where we recall that η_{Bm} is defined in Proposition 20 and α_{Bm} is the quasi-stationary distribution.

The following theorem states that the quasi-ergodic distribution of X (in the sense of Definition 12) is the quasi-ergodic distribution of the Brownian motion absorbed at $(-1,1)^c$.

Theorem 18. *for any probability measure μ on $(-1,1)$ and any $s \geq 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}_{\mu,s}(X_u \in \cdot | \tau_X > t) du = \beta_{Bm}$$

Proof. Let $\mu \in \mathcal{M}_1((-1,1))$. We recall the notation

$$\mu_{(s,t)} = \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t), \quad \forall s \leq t$$

For any probability measure μ and $s \leq t$,

$$\begin{aligned}
&\left|\left|\int_0^1 \mathbb{P}_{\mu,s}(X_{s+q(t-s)} \in \cdot | \tau_X > t) dq - \beta_{Bm}\right|\right|_{TV} \\
&\leq \left|\left|\int_0^1 \mathbb{P}_{\mu,s}(X_{s+q(t-s)} \in \cdot) | \tau_X > t) dq - \int_0^1 \mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq\right|\right|_{TV} \\
&\qquad + \left|\left|\int_0^1 \mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm}\right|\right|_{TV} \\
&\leq \left|\left|\int_0^1 \mathbb{P}_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(X_{s+q(t-s)} \in \cdot | \tau_X > t) dq - \int_0^1 \mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq\right|\right|_{TV} \\
&\qquad + \left|\left|\int_0^1 \mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm}\right|\right|_{TV} \\
&\leq \int_0^1 F\left(s + \frac{q}{2}(t-s)\right) dq + \int_0^1 \left|\left|\mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm}\right|\right|_{TV} dq
\end{aligned}$$

By Lebesgue's theorem,

$$\lim_{t\to\infty} \int_0^1 F\left(s + \frac{q}{2}(t-s)\right) dq = 0$$

In order to prove the convergence towards the quasi-ergodic distribution, it remains therefore to show that

$$\lim_{t\to\infty} \int_0^1 \left|\left|\mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s)),\frac{q}{2}(t-s)}}(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm}\right|\right|_{TV} dq = 0 \tag{3.16}$$

The idea of the following reasoning is the same as in the critical case. Similarly one has for any $x \in (-1,1)$, $s \leq t$, $q \in (0,1)$ and f bounded measurable,

$$\begin{aligned}
&e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}(f(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > t}) \\
&= e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)} \left(g_t\left(Y_{s+q(t-s)}\right) \mathbb{1}_{\tau_Y > s+q(t-s)}\right)
\end{aligned}$$

with for any $y \in (-1,1)$

$$g_t(y) := e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+q(t-s)+1)^{1-2\kappa}}{1-2\kappa}} f(y) \mathbb{P}^Y_{y,s+q(t-s)+1}\left[\tau_Y > t+1\right]$$

Also define for any $y \in (-1,1)$,

$$g_\infty(y) := f(y) \eta_{Bm}(y)$$

By Proposition 20 (i), $(g_t)_{t\geq 0}$ converges uniformly on $(-1,1)$ towards g_∞, which implies that the following quantity

$$\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)} \left(|g_t(Y_{s+q(t-s)}) - g_\infty(Y_{s+q(t-s)})| \middle| \tau_Y > s + q(t-s)\right)$$

goes to 0 when t goes to infinity. As a result, if one of the limit in the following equality exists, then the other limit exists also and one has

$$\lim_{t\to\infty} e^{\lambda_{Bm}\frac{(t+1)^{1-2\kappa}-(s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}(f(Y_{s+q(t-s)})\mathbb{1}_{\tau_Y>t})$$
$$= \lim_{t\to\infty} e^{\lambda_{Bm}\frac{(s+q(t-s)+1)^{1-2\kappa}-(s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}\left(g_\infty\left(Y_{s+q(t-s)}\right)\mathbb{1}_{\tau_Y>s+q(t-s)}\right)$$

By the definition of conditional expectation, one has

$$\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}\left(g_\infty\left(Y_{s+q(t-s)}\right)\mathbb{1}_{\tau_Y>s+q(t-s)}\right)$$
$$= \mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}\left(g_\infty\left(Y_{s+q(t-s)}\right)\Big|\tau_Y>s+q(t-s)\right)\mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}(\tau_Y>s+q(t-s))$$

On the one hand, by (3.14),

$$\lim_{t\to\infty}\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}\left(g_\infty\left(Y_{s+q(t-s)}\right)\Big|\tau_Y>s+q(t-s)\right)=\alpha_{Bm}(g_\infty) \tag{3.17}$$

On the other hand, by Proposition 20 (i), the following quantity

$$\left|e^{\lambda_{Bm}\frac{(s+q(t-s)+1)^{1-2\kappa}-(s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}(\tau_Y>s+q(t-s))-\mu_{(s,s+\frac{q}{2}(t-s))}(\eta_{Bm})\right|$$

goes to 0 when t goes to infinity, and again by (3.14),

$$\lim_{t\to\infty}\mu_{(s,s+\frac{q}{2}(t-s))}(\eta_{Bm})=\alpha_{Bm}(\eta_{Bm})=1$$

As a result,

$$\lim_{t\to\infty} e^{\lambda_{Bm}\frac{(s+q(t-s)+1)^{1-2\kappa}-(s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{P}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}(\tau_Y>s+q(t-s))=1 \tag{3.18}$$

Hence we deduce from (3.17) and (3.18) that

$$\lim_{t\to\infty} e^{\lambda_{Bm}\frac{(s+q(t-s)+1)^{1-2\kappa}-(s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}}\mathbb{E}^Y_{\mu_{(s,s+\frac{q}{2}(t-s))},s+\frac{q}{2}(t-s)}\left(g_\infty\left(Y_{s+q(t-s)}\right)\mathbb{1}_{\tau_Y>s+q(t-s)}\right)$$
$$=\alpha_{Bm}(g_\infty)=\int_{(-1,1)}\alpha_{Bm}(dx)f(x)\eta_{Bm}(x)=\beta_{Bm}(f)$$

This last equality implies (3.16), which allows us to conclude the convergence towards the quasi-ergodic distribution. □

3.4.5 Q-process

Existence of the Q-process

Now it remains to prove the existence of the Q-process. More precisely, this subsection is devoted to the proof of the following theorem

Theorem 19. *For any $s \leq t$ and $\mu \in \mathcal{M}_1((-1,1))$, the family of probability measure $(\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | T < \tau_X))_{T>t}$ converges weakly when T goes to infinity towards*

$$\mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot) = \mathbb{E}_{\mu,s}\left(\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t} \frac{\eta_t(X_t)}{\mu(\eta_s)}\right) \tag{3.19}$$

where $(\eta_t)_{t\geq 0}$ is defined in Proposition 21. Moreover, for any $s \leq t$ and $\mu \in \mathcal{M}_1((-1,1))$, one has

$$||\mathbb{Q}_{\mu,s}(X_t \in \cdot) - \mathbb{Q}^Y_{\mu,s}(Y_t \in \cdot)||_{TV} \leq F(s) \tag{3.20}$$

where F is the same function as in Proposition 19 and $\mathbb{Q}^Y$ is as defined in Proposition 20.

Before proving this theorem, let us first state the following key proposition.

Proposition 21. *There exist a family of positive bounded functions $(\eta_s)_{s\geq 0}$ satisfying*

$$\mathbb{E}_{x,s}(\mathbb{1}_{\tau_X > t}\eta_t(X_t)) = \eta_s(x), \qquad \forall x \in (-1,1), \forall s \leq t \tag{3.21}$$

and $H : (0,1) \times \{s,t \in \mathbb{R}_+ : s \leq t\} \to (0,\infty)$ such that, for any $a \in (0,1)$ and $s \geq 0$,

$$\lim_{t\to\infty} H(a,s,t) = 0$$

and that, for any $a \in (0,1)$, for any $s \leq t$, for any $\mu \in \mathcal{M}_1((-1,1))$ and any $\nu \in \mathcal{M}_1((-1,1))$ such that $\nu([-a,a]) > \frac{1}{2}$,

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mu(\eta_s)}{\nu(\eta_s)}\right| \leq H(a,s,t) \tag{3.22}$$

The proof of this proposition is postponed after the proof of Theorem 19.

Proof of Theorem 19. Let $\mu \in \mathcal{M}_1((-1,1))$ and $s \leq t$. We define $\mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)$ as the formula (3.19). Then, for any $T > t$,

$$\begin{aligned}
&||\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | \tau_X > T) - \mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)||_{TV} \\
&\qquad = \left|\left|\mathbb{E}_{\mu,s}\left[\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t}\left(\frac{\mathbb{P}_{X_t,t}(\tau_X > T)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\eta_t(X_t)}{\eta_s(x)}\right)\right]\right|\right|_{TV} \\
&\qquad = \left|\left|\mathbb{E}_{\mu,s}\left[\frac{\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t}}{\mathbb{P}_{\mu,s}(\tau_X > t)}\left(\frac{\mathbb{P}_{X_t,t}(\tau_X > T)}{\mathbb{P}_{\mu_{(s,t)},t}(\tau_X > T)} - \frac{\eta_t(X_t)}{\mu_{(s,t)}(\eta_t)}\right)\right]\right|\right|_{TV}
\end{aligned}$$

where (3.21) was used. Now, by Theorem 17, the family $(\mu_{(s,t)})_{t\geq s}$ converges weakly towards α_{Bm}. Thus, by Prokhorov's theorem, the family $(\mu_{(s,t)})_{t\geq s}$ is tight and there exists $a \in (0,1)$ independent on t (but depending on s) such that, for any $t \geq s$, $\mu_{(s,t)}([-a,a]) \geq \frac{1}{2}$. Hence, by (3.22) in Proposition 21,

$$\begin{aligned}
||\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | \tau_X > T) - \mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)||_{TV} &\leq H(a,t,T)\left|\left|\mathbb{E}_{\mu,s}\left[\frac{\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t}}{\mathbb{P}_{\mu,s}(\tau_X > t)}\right]\right|\right|_{TV} \\
&\leq H(a,t,T)
\end{aligned}$$

Hence, since, for t fixed, $\lim_{T\to\infty} H(a,t,T) = 0$, this implies the weak convergence of $(\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | T < \tau_X))_{T\geq t}$ towards $\mathbb{Q}_{\mu,s}$ defined in (3.19). The inequality (3.20) is a straightforward consequence of (3.11) in Proposition 19, letting $T \to \infty$. □

Proof of Proposition 21

The remainig of the chapter is dedicated to prove Proposition 21. In the proof, two important lemmata are used. So we will start by proving these lemmata before tackling the proof of Propostion 21.

Lemma 4.

- *For any $s \geq 0$ and $a \in (0,1)$, there exists $C_{s,a} > 0$ such that*

$$\inf_{x\in[-a,a]} \mathbb{P}_{x,s}(\tau_X > t) \geq C_{s,a} \sup_{x\in(-1,1)} \mathbb{P}_{x,s}(\tau_X > t), \qquad \forall t \geq 0$$

- *For any $a \in (0,1)$, there exists $C_a > 0$ such that*

$$\inf_{x\in[-a,a]} \mathbb{P}^Y_{x,s}(\tau_Y > t) \geq C_a \sup_{x\in(-1,1)} \mathbb{P}^Y_{x,s}(\tau_Y > t), \qquad \forall t \geq 0$$

Proof.

- Let $a > 0$. To prove this, note that for any $x \in (-1,1)$ and $t \geq s \geq 0$,

$$\mathbb{P}_{x,s}(\tau_X > t) = \mathbb{P}^B_{(s+1)^\kappa x}\left[\tau_B^{(\cdot+s+1)^\kappa} > t - s\right]$$

where, for any $s \geq 0$,

$$\tau_B^{(\cdot+s+1)^\kappa} := \inf\{t \geq 0 : |B_t| = (t+s+1)^\kappa\}$$

So the Harnack inequality to show becomes : for any $t \geq 0$,

$$\inf_{x\in[-a(s+1)^\kappa, a(s+1)^\kappa]} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) \geq C_{s,a} \sup_{x\in(-(s+1)^\kappa,(s+1)^\kappa)} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

Actually, for any $t \geq 0$,

$$\inf_{x\in[-a(s+1)^\kappa, a(s+1)^\kappa]} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) = \mathbb{P}_{a(s+1)^\kappa}(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

and

$$\sup_{x\in(-(s+1)^\kappa,(s+1)^\kappa)} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) = \mathbb{P}_0(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

Then, for any $t \geq 0$,

$$\begin{aligned}
\mathbb{P}_{a(s+1)^\kappa}(\tau_B^{(\cdot+s+1)^\kappa} > t) &\geq \mathbb{P}_{a(s+1)^\kappa}\left(\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}, \tau_B^{(\cdot+s+1)^\kappa} > t + \tau_B^0\right) \\
&= \mathbb{E}_{a(s+1)^\kappa}\left(\mathbb{1}_{\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}} \mathbb{P}_0(\tau_B^{(\cdot+s+v+1)^\kappa} > t)|_{v=\tau_B^0}\right) \\
&\geq \mathbb{P}_{a(s+1)^\kappa}\left(\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}\right)\mathbb{P}_0\left(\tau_B^{(\cdot+s+1)^\kappa} > t\right)
\end{aligned}$$

where

$$\tau_B^0 := \inf\{t \geq 0 : B_t = 0\}$$

Then setting $C_{s,a} := \mathbb{P}_{a(s+1)^\kappa}\left(\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}\right)$, one has $C_{s,a} > 0$ for any $s \geq 0$ and

$$\inf_{x\in[-a,a]} \mathbb{P}_{x,s}(\tau_X > t) \geq C_{s,a} \sup_{x\in(-1,1)} \mathbb{P}_{x,s}(\tau_X > t), \qquad \forall t \geq 0$$

- This is straightforward using the Harnack inequality for a Brownian motion and using the change of time provided by the Dubin-Schwartz's transformation (3.13). □

Now let us state and prove Lemma 5.

Lemma 5. *Let $a > 0$. Then there exists a function $\chi_a : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $s \leq t$, for any $\mu \in \mathcal{M}_1((-1,1))$ and any $\nu \in \mathcal{M}_1((-1,1))$ such that $\nu([-a,a]) > \frac{1}{2}$,*

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}^Y_{\mu,s}(\tau_Y > t)}{\mathbb{P}^Y_{\nu,s}(\tau_Y > t)}\right| \leq \chi_a(s)$$

with $\chi_a(s) \to 0$ when s goes to infinity

Proof. Let $s \leq t$. Remind the equality for any $\mu \in \mathcal{M}_1((-1,1))$,

$$\mathbb{P}_{\mu,s}(\tau_Y > t) = \left(\frac{s+1}{t+1}\right)^{\frac{\kappa}{2}} \mathbb{E}_{\mu,s}\left[\exp\left(\frac{1}{2}N'_{s,t}\right)\mathbb{1}_{\tau_X>t}\right] \tag{3.23}$$

where we recall that $N'_{s,t}$ is defined in the proof of Proposition 19, and by (3.12), for any $\mu \in \mathcal{M}_1((-1,1))$,

$$1 - \phi(s) \leq \frac{\mathbb{E}_{\mu,s}(\exp\left(\frac{1}{2}N'_{s,t}\right)1_{\tau_X>t})}{\mathbb{P}_{\mu,s}(\tau_X > t)} \leq 1 + \phi(s)$$

where the function ϕ is also defined in the proof of Proposition 19. Thus, by (3.23), for any $\mu \in \mathcal{M}_1((-1,1))$,

$$\left(\frac{s+1}{t+1}\right)^{\frac{\kappa}{2}}(1-\phi(s)) \leq \frac{\mathbb{P}_{\mu,s}(\tau_Y > t)}{\mathbb{P}_{\mu,s}(\tau_X > t)} \leq \left(\frac{s+1}{t+1}\right)^{\frac{\kappa}{2}}(1+\phi(s))$$

and, since $\phi(s) < 1$ for any $s \geq 0$, one has also,

$$\left(\frac{t+1}{s+1}\right)^{\frac{\kappa}{2}} \frac{1}{1+\phi(s)} \leq \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\mu,s}(\tau_Y > t)} \leq \left(\frac{t+1}{s+1}\right)^{\frac{\kappa}{2}} \frac{1}{1-\phi(s)}$$

Thus, for any $\mu, \nu \in \mathcal{M}_1((-1,1))$,

$$\frac{1-\phi(s)}{1+\phi(s)} \leq \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\mu,s}(\tau_Y > t)}\frac{\mathbb{P}_{\nu,s}(\tau_Y > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \leq \frac{1+\phi(s)}{1-\phi(s)} \tag{3.24}$$

Thus, it is deduced from (3.24) that, for any $\mu, \nu \in \mathcal{M}_1((-1,1))$,

$$\begin{aligned}
&\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}^Y_{\mu,s}(\tau_Y > t)}{\mathbb{P}^Y_{\nu,s}(\tau_Y > t)}\right| \\
&\qquad \le \frac{\mathbb{P}^Y_{\mu,s}(\tau_Y > t)}{\mathbb{P}^Y_{\nu,s}(\tau_Y > t)} \left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \frac{\mathbb{P}_{\nu,s}(\tau_Y > t)}{\mathbb{P}_{\mu,s}(\tau_Y > t)} - 1\right| \\
&\qquad \le \frac{\mathbb{P}^Y_{\mu,s}(\tau_Y > t)}{\mathbb{P}^Y_{\nu,s}(\tau_Y > t)} \left[\left(\frac{1+\phi(s)}{1-\phi(s)} - 1\right) \vee \left(1 - \frac{1-\phi(s)}{1+\phi(s)}\right)\right]
\end{aligned}$$

Now, if ν is such that $\nu([-a,a]) > \frac{1}{2}$, by Lemma 4,

$$\begin{aligned}
\mathbb{P}^Y_{\mu,s}(\tau_Y > t) &\le \sup_{x \in (-1,1)} \mathbb{P}^Y_{x,s}(\tau_Y > t) \\
&\le \frac{1}{C_a} \inf_{x \in [-a,a]} \mathbb{P}^Y_{x,s}(\tau_Y > t) \\
&\le \frac{1}{C_a \nu([-a,a])} \mathbb{P}^Y_{\nu,s}(\tau_Y > t) \\
&\le \frac{2}{C_a} \mathbb{P}^Y_{\nu,s}(\tau_Y > t)
\end{aligned}$$

As a result,

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}^Y_{\mu,s}(\tau_Y > t)}{\mathbb{P}^Y_{\nu,s}(\tau_Y > t)}\right| \le \frac{2}{C_a} \left[\left(\frac{1+\phi(s)}{1-\phi(s)} - 1\right) \vee \left(1 - \frac{1-\phi(s)}{1+\phi(s)}\right)\right]$$

It remains to set $\chi_a(t) := \frac{2}{C_a}\left(\frac{1+\phi(s)}{1-\phi(s)} - 1\right) \vee \left(1 - \frac{1-\phi(t)}{1+\phi(t)}\right)$. Then, since $\phi(s) \to 0$ when $s \to \infty$, χ_a goes also to 0 when s goes to infinity. □

Now we can prove Proposition 21.

Proof of Proposition 21. Let $a \in (0,1)$. Let $\mu \in \mathcal{M}_1((-1,1))$ and $\nu \in \mathcal{M}_1((-1,1))$ such that $\nu([-a,a]) > \frac{1}{2}$. Finally, let $s \le t$ and $T \ge 0$. Then, using Markov's property,

$$\frac{\mathbb{P}_{\mu,s}(\tau_X > t+T)}{\mathbb{P}_{\nu,s}(\tau_X > t+T)} - \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} = \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \left(\frac{\mathbb{P}_{\mu_{(s,t)},t}(\tau_X > t+T)}{\mathbb{P}_{\nu_{(s,t)},t}(\tau_X > t+T)} - 1\right)$$

Using the same argument as in the proof of Lemma 5, by Lemma 4 one has

$$\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \le \frac{2}{C_{s,a}}$$

Thus,

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t+T)}{\mathbb{P}_{\nu,s}(\tau_X > t+T)} - \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)}\right| \le \frac{2}{C_{s,a}} \left|\frac{\mathbb{P}_{\mu_{(s,t)},t}(\tau_X > t+T)}{\mathbb{P}_{\nu_{(s,t)},t}(\tau_X > t+T)} - 1\right|$$

Using Lemma 5, one has

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t+T)}{\mathbb{P}_{\nu,s}(\tau_X > t+T)} - \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)}\right| \leq \frac{2}{C_{s,a}}\left(\chi_a(t) + \left|\frac{\mathbb{P}^Y_{\mu_{(s,t)},t}(\tau_Y > t+T)}{\mathbb{P}^Y_{\nu_{(s,t)},t}(\tau_Y > t+T)} - 1\right|\right)$$

Now,

$$\begin{aligned}\left|\frac{\mathbb{P}^Y_{\mu_{(s,t)},t}(\tau_Y > t+T)}{\mathbb{P}^Y_{\nu_{(s,t)},t}(\tau_Y > t+T)} - 1\right| &= \frac{|\mathbb{P}^Y_{\mu_{(s,t)},t}(\tau_Y > t+T) - \mathbb{P}^Y_{\nu_{(s,t)},t}(\tau_Y > t+T)|}{\mathbb{P}^Y_{\nu_{(s,t)},t}(\tau_Y > t+T)} \\ &\leq \frac{\sup_{x\in(-1,1)}\mathbb{P}^Y_{x,t}(\tau_Y > t+T)}{\mathbb{P}^Y_{\nu_{(s,t)},t}(\tau_Y > t+T)}||\mu_{(s,t)} - \nu_{(s,t)}||_{TV} \\ &\leq \frac{4}{C_a} \times \left(F\left(\frac{t}{2}\right) + C_{Bm}\exp\left(-\gamma_{Bm}\frac{(t+1)^{1-2\kappa} - (\frac{t}{2}+1)^{1-2\kappa}}{1-2\kappa}\right)\right)\end{aligned}$$

where we used Lemma 4 and (3.15).
We conclude from all these computations that $t \to \frac{\mathbb{P}_{\mu,s}(\tau_X>t)}{\mathbb{P}_{\nu,s}(\tau_X>t)}$ is a Cauchy sequence, hence converges as $t \to \infty$. Denote by $h(s,\mu,\nu)$ the limit and set

$$H(a,s,t) := \frac{2}{C_{s,a}}\left[\chi_a(t) + \frac{4}{C_a} \times \left(F\left(\frac{t}{2}\right) + C_{Bm}\exp\left(-\gamma_{Bm}\frac{(t+1)^{1-2\kappa} - (\frac{t}{2}+1)^{1-2\kappa}}{1-2\kappa}\right)\right)\right]$$

One has therefore, for any $\mu,\nu \in \mathcal{M}_1((-1,1))$ such that $\nu([-a,a]) > \frac{1}{2}$,

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - h(s,\mu,\nu)\right| \leq H(a,s,t)$$

and $\lim_{t\to\infty} H(a,s,t) = 0$.
In order to complete the proof, we will inspire by the proof of Proposition 3.1. in [18]. We define for any $s \geq 0$

$$\eta_s : x \to h(s,\delta_x,\delta_0)$$

Remark that one has necessarily $\delta_0([-a,a]) = 1 > \frac{1}{2}$. Since, on the one hand,

$$\lim_{t\to\infty}\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{0,s}(\tau_X > t)} = h(s,\mu,\delta_0)$$

and, on the other hand, by Lebesgue's theorem,

$$\lim_{t\to\infty}\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{0,s}(\tau_X > t)} = \mu(\eta_s),$$

then, for any $\mu,\nu \in \mathcal{M}_1((-1,1))$ with $\nu([-a,a]) > \frac{1}{2}$,

$$h(s,\mu,\nu) = \lim_{t\to\infty}\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} = \lim_{t\to\infty}\frac{\mathbb{P}_{\mu,s}(\tau_X > t)/\mathbb{P}_{0,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)/\mathbb{P}_{0,s}(\tau_X > t)} = \frac{\mu(\eta_s)}{\nu(\eta_s)}$$

Moreover, for any $s \leq t \leq u$,

$$\mathbb{E}_{x,s}\left(\mathbb{1}_{\tau_X>t}\frac{\mathbb{P}_{X_t,t}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)}\right) = \frac{\mathbb{P}_{x,s}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)}$$
$$= \frac{\mathbb{P}_{x,s}(\tau_X > u)}{\mathbb{P}_{0,s}(\tau_X > u)}\mathbb{E}_{0,s}\left(\mathbb{1}_{\tau_X>t}\frac{\mathbb{P}_{X_t,t}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)}\right)$$

For any $\mu \in \mathcal{M}_1((-1,1))$, integrating both sides of the equation with respect to μ, letting $u \to \infty$ and using Lebesgue's theorem, we deduce that, for any $s \leq t$, there exists a positive constant $c_{s,t}$ which does not depend on μ such that

$$c_{s,t} = \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X>t}\eta_t(X_t))}{\mu(\eta_s)}$$

In addition, for any $s \leq t \leq u$ and for any $\nu \in \mathcal{M}_1((-1,1))$,

$$c_{s,t}c_{t,u} = \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X>t}\eta_t(X_t))}{\mu(\eta_s)}\frac{\mathbb{E}_{\nu,s}(\mathbb{1}_{\tau_X>t}\eta_t(X_t))}{\nu(\eta_s)}$$

Choosing $\nu = \mu_{(s,t)}$ and using Markov's property, we obtain

$$c_{s,t}c_{t,u} = \frac{\mathbb{P}_{\mu,s}(\tau_X > t)\mathbb{P}_{\nu,t}(\mathbb{1}_{\tau_X>u}\eta_u(X_u))}{\mu(\eta_s)} = \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X>u}\eta_u(X_u))}{\eta_s(x)} = c_{s,u}$$

Because of the last equality, replacing for all $s \geq 0$ the function $\eta_s(x)$ by $\eta_s(x)/c_{0,s}$ entails (3.21). □

Chapter 4

Quasi-stationarity for time-inhomogeneous sub-Markovian semi-groups through Lyapunov criteria

This chapter consists in a joint work with Nicolas Champagnat and Denis Villemonais under finalization.

Abstract

This chapter will tackle the notion of *quasi-stationarity* applied to time-(in)homogeneous Markov process absorbed by moving boundaries. Considering an absorbing element which is immobile, the *quasi-stationarity* for a Markov process concerns the asymptotic behavior of its marginal law conditioned not to be absorbed by this absorbing element. Our aim in this chapter is also to do the same considering an absorbing subset which can move. We provide therefore assumptions implying a weak ergodicity for the conditioned process. These assumptions provide furthermore the existence of the so-called Q-process and some results about the existence of a quasi-ergodic distribution.

Notation

- $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$
- $\mathcal{M}_1(F)$: Set of the probability measure whose the support is included in F
- $\mathcal{B}(F)$: Set of the bounded measurable function defined on F

- For $\mu \in \mathcal{M}_1(F)$ and $f \in \mathcal{B}(F)$,

$$\mu(f) := \int_F f(x)\mu(dx)$$

4.1 Statement of the general results

4.1.1 Statement of the general assumption

Let I be $\frac{1}{N}\mathbb{Z}_+$ ($N \in \mathbb{N}$) or $\mathbb{R}_+$. Let $(\Omega, (\mathcal{F}_{s,t})_{s\leq t}, (X_t)_{t\in I}, (\mathbb{P}_{x,s})_{x\in E, s\in I})$ be a (possibly) time-inhomogeneous Markov process defined on a state space $(E, \mathcal{E})$, where $\mathcal{E}$ is a σ-field of E. Define $(A_t)_{t\in I}$ a family of measurable subsets of E and denote by

$$\tau_A := \inf\{t \in I : X_t \in A_t\}.$$

For any $t \in I$, denote by E_t the complement of A_t. Denote $(P_{s,t})_{s\leq t}$ the semi-group defined by, for any μ probability measure on E and f bounded measurable function on E,

$$\mu P_{s,t} f := \int_E \mu(dx)\mathbb{E}_{x,s}(f(X_t)\mathbb{1}_{\tau_A > t}),$$

where $\mathbb{E}_{x,s}$ is the expectation associated to $\mathbb{P}_{x,s}$. In this chapter, the main concern will be the asymptotic behavior of $t \to \mu\phi_{s,t}$ when t goes to infinity, where $(\phi_{s,t})_{s\leq t}$ is defined as

$$\mu\phi_{s,t} f := \mathbb{E}_{\mu,s}(f(X_t)|\tau_A > t) = \frac{\mu P_{s,t} f}{\mu P_{s,t}\mathbb{1}}$$

for any $s \leq t$, $\mu \in \mathcal{M}_1(E_s)$ and $f \in \mathcal{B}(E_t)$, where $\mathbb{E}_{\mu,s}$ is the expectation associated to $\mathbb{P}_{\mu,s} := \int_{E_s} \mu(dx)\mathbb{P}_{x,s}$. In particular, for any $s \leq t$ and $\mu \in \mathcal{M}_1(E_s)$, one denote

$$\mu\phi_{s,t} := \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_A > t).$$

Let us now introduce assumption (F') as follows :

Assumption (F'). There exist positive real constants $\gamma_1, \gamma_2, c_1, c_2$ and c_3, $t_1, t_2 \in I$, a family of measurable functions $\psi_{1,s} : E_s \to [1, +\infty)$, and a family of probability measure $(\nu_s)_{s\in I}$ on a measurable subset $L_s \subset E_s$ such that

(F'0) *(A strong Markov property).* Defining

$$\tau_L := \inf\{t \in I : X_t \in L_t\}, \tag{4.1}$$

assume that for all $s \in I$ and $s \in I$, $X_{\tau_L} \in L_{\tau_L}$, $\mathbb{P}_{x,s}$-almost surely on the event $\{\tau_L < \infty\}$ and for all $t > 0$ and all measurable $f : E \to \mathbb{R}_+$,

$$\mathbb{E}_{x,s}\left[f(X_t)\mathbb{1}_{\tau_L\leq t<\tau_A}\right] = \mathbb{E}_{x,s}\left[\mathbb{1}_{\tau_L\leq t\wedge\tau_A}\mathbb{E}_{X_{\tau_L},\tau_L}\left[f(X_t)\mathbb{1}_{t<\tau_A}\right]\right].$$

(F'1) *(Local Dobrushin coefficient).* $\forall x \in L_{s+t_1}$,

$$\mathbb{P}_{x,s}(X_{s+t_1} \in \cdot, \tau_A > s + t_1) \geq c_1\nu_{s+t_1}(\cdot \cap L_{s+t_1}).$$

(F'2) *(Global Lyapunov criterion).* We have $\gamma_1 < \gamma_2$ and

$$\begin{aligned}
&\mathbb{E}_{x,s}(\psi_{1,s+t_2}(X_{s+t_2})\mathbb{1}_{s+t_2<\tau_L\wedge\tau_A}) \leq \gamma_1^{t_2}\psi_{1,s}(x),\ \forall x \in E_s\\
&\mathbb{E}_{x,s}(\psi_{1,t}(X_t)\mathbb{1}_{t<\tau_A}) \leq c_2,\ \forall x \in L_s,\ \forall t \in [s, s+t_2] \cap I,\\
&\inf_{s\in I, x\in L_s} \gamma_2^{-(t-s)}\mathbb{P}_{x,s}(X_t \in L_t, \tau_A > t) \xrightarrow[t\to+\infty]{} +\infty.
\end{aligned}$$

(F'3) *(Local Harnack inequality).* There exists $c_3 > 0$ such that, for any $r \leq s \leq t$, $x \in L_r$ and $y \in L_s$,

$$\mathbb{P}_{x,r}(\tau_A > t) \leq c_3\mathbb{P}_{y,s}(\tau_A > t)$$

These assumptions are directly inspired from the assumptions (F) introduced in [15, p.15-16]. A first difference to note with the time-homogeneous setting is the dependance in time of the Lyapunov function $\psi_{1,s}$, the measurable set L_s and the probability measure ν_s. However, it is important, for the following, to keep the times (t_1 and t_2) and the constants c_1, c_2, c_3, γ_1 and γ_2 in order to get our results. The time-dependance of these quantities are harder to manage and set an open question to solve.

Moreover, the Harnack inequality (F'3) is sightly different from the time-homogeneous Harnack inequality (F3) introduces in [15] because of the necessity to deal with three times $r \leq s \leq t$. In the time-homogeneous setting, we do not need it, because this statement is naturally implied by the increase of $r \mapsto \mathbb{P}_x(\tau_\partial > t - r)$ for any x and $t \geq 0$. In the time-inhomogeneous setting, the increase of $r \mapsto \mathbb{P}_{x,r}(\tau_A > t)$ is not necessarily true. However, it is not excluded that the increase of $r \mapsto \mathbb{P}_{x,r}(\tau_A > t)$ is a consequence of some weaker assumptions.

In this chapter, three main results are proved. They are respectively stated in the subsections 4.1.2, 4.1.3 and 4.1.4.

4.1.2 Mixing property

A *quasi-limiting distribution* is defined as the limiting probability measure (for the weak convergence) of $t \to \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_A > t)$ for some $\mu \in \mathcal{M}_1(E_s)$ and $s \geq 0$. In the time-inhomogeneous setting, the existence of a quasi-limiting distribution is usually not expected. However, one can expect a *mixing property*, that is that, for any $s \geq 0$ and any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$, one has

$$||\mathbb{P}_{\mu_1,s}(X_t \in \cdot|\tau_A > t) - \mathbb{P}_{\mu_2,s}(X_t \in \cdot|\tau_A > t)||_{TV} \xrightarrow[t\to\infty]{} 0.$$

This property is actually the first results which is proved in this chapter :

Theorem 20. *Under Assumption (F'), there exists $C, \gamma > 0$ such that, for any $s \leq t \in I$, for any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,*

$$||\mathbb{P}_{\mu_1,s}(X_t \in \cdot|\tau_A > t) - \mathbb{P}_{\mu_2,s}(X_t \in \cdot|\tau_A > t)||_{TV} \leq Ce^{-\gamma t}\left(\frac{\mu_1(\psi_{1,s})}{\mu_1(\psi_{2,s})} + \frac{\mu_2(\psi_{1,s})}{\mu_2(\psi_{2,s})}\right),$$

where, for any $s \geq 0$ and $x \in E_s$,

$$\psi_{2,s}(x) := \frac{\gamma_2^{-t_2} - 1}{\gamma_2^{-n_0 t_2} - 1} \sum_{i=0}^{n_0 - 1} \gamma_2^{-it_2} \mathbb{P}_{x,s}(X_{s+it_2} \in L, \tau_A > s + it_2)$$

and n_0 is an integer large enough.

4.1.3 Existence of a Q-process

It is possible to go further showing the existence of a *Q-process* for the process $(X_t)_{t \in I}$ absorbed at $(A_t)_{t \in I}$. More precisely, the *Q-process* is the law of the process $(X_t)_{t \in I}$ under $(\mathbb{Q}_{x,s})_{x \in E_s, s \in I}$, where, for any $s \leq t$ and any $x \in E_s$,

$$\mathbb{Q}_{x,s}(\Gamma) := \lim_{T \to \infty} \mathbb{P}_{x,s}(\Gamma | \tau_A > T), \quad \forall \Gamma \in \mathcal{F}_{s,t} \tag{4.2}$$

In other words, the Q-process can be interpreted as the law of X conditioned never to be absorbed at $(A_t)_{t \in I}$. In order to prove the convergence (4.2), it is necessary to prove the following proposition :

Proposition 22. *There exists a family of function $(\eta_s)_{s \geq 0}$ such that, for any $x \in E_s$,*

$$\left| \frac{\mathbb{P}_{x,s}(\tau_A > t)}{\mathbb{P}_{\nu_s,s}(\tau_A > t)} - \eta_s(x) \right| \leq C e^{-\gamma(t-s)} \psi_{1,s}(x),$$

where $C, \gamma > 0$ and ν is the probability measure defined in $(F'1)$.

For any $s \in I$, define

$$E'_s := \{x \in E_s : \eta_s(x) \neq 0\}$$

Then it is shown in this chapter that, under Assumption (F'), the convergence (4.2) goes exponentially fast with respect to the total variation norm.

Theorem 21. *Let $(\mathbb{Q}_{x,s})_{x \in E'_s, s \in I}$ be the family of probability measures defined by*

$$\mathbb{Q}_{x,s}(\Gamma) := \mathbb{E}_{x,s}\left(\mathbb{1}_{\Gamma, \tau_A > t} \frac{\eta_t(X_t)}{\mathbb{E}_{x,s}(\eta_t(X_t) \mathbb{1}_{\tau_A > t})} \right), \quad \forall \Gamma \in \mathcal{F}_{s,t}$$

Then, for any $s \leq t \leq T$, for any $x \in E'_s$,

$$||\mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) - \mathbb{P}_{x,s}(X_{[s,t]} \in \cdot | T < \tau_A)||_{TV} \leq C e^{-\gamma(T-t)} \left(1 + \frac{\psi_{1,s}(x)}{\eta_s(x)}\right) \frac{\psi_{1,s}(x)}{\eta_s(x)}.$$

Moreover, defining, for any $s \geq 0$ and $\mu \in \mathcal{M}_1(E'_s)$,

$$\mathbb{Q}_{\mu,s} := \int_{E'_s} \mathbb{Q}_{x,s} \mu(dx),$$

one has, for any $s \leq t \leq T$ and μ such that $\mu(\eta_s) > 0$,

$$||\mathbb{Q}_{\eta_s * \mu, s}(X_{[s,t]} \in \cdot) - \mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | T < \tau_A)||_{TV} \leq C e^{-\gamma(T-t)} \left(1 + \frac{\mu(\psi_{1,s})}{\mu(\eta_s)}\right) \frac{\mu(\psi_{1,s})}{\mu(\eta_s)},$$

with

$$\eta_s * \mu(dx) = \frac{\eta_s(x) \mu(dx)}{\mu(\eta_s)}.$$

4.1.4 Existence of a quasi-ergodic distribution

It could be also interesting to deal with the *quasi-ergodic distribution*, which is the limit law of

$$\frac{1}{t}\int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | \tau_A > t) du$$

when t goes to infinity. As the quasi-limiting distribution, it is not necessarily that the quasi-ergodic distribution exists in the time-inhomogeneous setting. However, Theorem 21 implies the following corollary :

Corollary 3. *For any $s \leq t$, for any μ such that $\mu(\eta_s) > 0$,*

$$\left\| \frac{1}{t}\int_s^t \mathbb{Q}_{\eta_s * \mu, s}(X_u \in \cdot) du - \frac{1}{t}\int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | t < \tau_A) du \right\|_{TV} \leq \frac{C}{t}\left(1 + \frac{\mu(\psi_{1,s})}{\mu(\eta_s)}\right)\frac{\mu(\psi_{1,s})}{\mu(\eta_s)}$$

In particular, if

$$\frac{1}{t}\int_s^t \mathbb{Q}_{\eta_s * \mu, s}(X_u \in \cdot) du \xrightarrow[t\to\infty]{\mathcal{L}} \beta$$

for some measure μ such that $\mu(\psi_{1,s}/\mu(\eta_s) < \infty$, then one has

$$\frac{1}{t}\int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | t < \tau_A) du \xrightarrow[t\to\infty]{\mathcal{L}} \beta.$$

In order to prove these results, we will first state some alternative condition (E') relevant for the discrete-time setting. The proofs in the following section will be the same as written in [15] for the time-inhomogeneous setting. Once finishing to deal with the discrete-time setting, we show that the Assumptions (F') imply the Assumptions (E') for a subsequence of the Markov process $(X_t)_{t\in I}$. All the proofs showing the mixing property is quite close to the proofs written in the time-homogeneous setting in [15]; however, for the sake of completeness, we give all the details for proving the mixing property. The main novelty concerns the proposition 22 since, contrary to our case, the proof provided in [15] in the time-homogeneous setting strongly relies on the existence of a quasi-stationary distribution, which is not well-defined in the most of the interesting cases (for time-homogeneous Markov process absorbed by moving boundaries, see Proposition 2.1. in [45]). For our purpose, we use instead the family of probability measure $(\nu_s)_{s\in I}$.

4.2 Proof of the mixing property : the discrete-time case

In this section, assume that $I = \mathbb{Z}_+$. Let us introduce the following assumption.

Assumption (E'). For any $k \in \mathbb{Z}_+$, there exist positive integer n_1 and n_2, positive real constants $\theta_1, \theta_2, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5$ two functions $\varphi_{1,k}, \varphi_{2,k} : E \to \mathbb{R}_+$ and a probability measure ν_k on a measurable subset $K_k \subset E_k$ such that

(E'1) $\forall x \in K_k$,

$$\delta_x P_{k,k+n_1} \geq \tilde{c}_1 \nu_{k+n_1}$$

(E'2) We have $\theta_1 < \theta_2$ and

$$\inf_{x\in E_k} \varphi_{1,k}(x) \geq 1, \sup_{x\in K_k} \sup_{k\in\mathbb{Z}_+} \varphi_{1,k}(x) < \infty$$

$$\inf_{x\in K_k} \inf_{k\in\mathbb{Z}_+} \varphi_{2,k}(x) > 0, \sup_{x\in E_k} \varphi_{2,k}(x) \leq 1$$

$$P_{k,k+1}\varphi_{1,k+1}(x) \leq \theta_1\varphi_{1,k}(x) + \tilde{c}_2\mathbb{1}_K(x), \forall x \in E_k$$

$$P_{k,k+1}\varphi_{2,k+1}(x) \geq \theta_2\varphi_{2,k}(x), \forall x \in E_k$$

(E'3) There exists $\tilde{c}_3$ such that, for any $k \leq l \leq n$, $x \in K_k$ and $y \in K_l$,

$$\mathbb{P}_{x,k}(\tau_A > n) \leq \tilde{c}_3\mathbb{P}_{y,l}(\tau_A > n)$$

(E'4) There exists $n_\nu \in \mathbb{N}$ such that, for all $n \geq n_\nu$,

$$\inf_{k\in\mathbb{Z}_+} \mathbb{P}_{\nu_k,k}(X_{n+k} \in K_{n+k}, \tau_A > n+k) > 0$$

The first aim in this section is to prove the following theorem.

Theorem 22. *There exists $C > 0$ and $\alpha \in (0,1)$ such that, for any $m \leq n \in \mathbb{Z}_+$ and $\mu_1, \mu_2 \in \mathcal{M}_1(E_m)$,*

$$||\mathbb{P}_{\mu_1,m}(X_n \in \cdot|\tau_A > n) - \mathbb{P}_{\mu_2,m}(X_n \in \cdot|\tau_A > n)||_{TV} \leq C\alpha^{n-m}\left(\frac{\mu_1(\varphi_{1,m})}{\mu_1(\varphi_{2,m})} + \frac{\mu_2(\varphi_{1,m})}{\mu_2(\varphi_{2,m})}\right)$$

Then, the version of Proposition 22 for discrete-time Markov process is as follows.

Proposition 23. *Assume Assumption (E'). Then there exists $C > 0$ such that, for any $m \leq n \leq p$ and $x \in E_m$,*

$$\left|\frac{\mathbb{P}_{x,m}(\tau_A > n)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} - \frac{\mathbb{P}_{x,m}(\tau_A > p)}{\mathbb{P}_{\nu_m,m}(\tau_A > p)}\right| \leq C\alpha^{n-m}\varphi_{1,m}(x)$$

where α is the same constant as in Theorem 1 and ν_m is as (E'1). In particular, there exists a funcion η_m such that, for any $x \in E_m$,

$$\left|\frac{\mathbb{P}_{x,m}(\tau_A > n)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} - \eta_m(x)\right| \leq C\alpha^{n-m}\varphi_{1,m}(x)$$

4.2.1 Preliminary lemmata

As in time-homogeneous setting, we will first show five lemmata corresponding to the lemmata 9.5-9.9 in [15]. This subsection is entirely devoted to this purpose.

Define, for any $m \in \mathbb{Z}_+$,

$$T_K^{(m)} := \inf\{n \geq m : X_n \in K_n\}$$

We start by proving the version of Lemma 9.5.

Lemma 6. *For any $x \in E_k \setminus K_k$ and $k \leq n$,*

$$\mathbb{P}_{x,k}(n < T_K^{(k)} \wedge \tau_A) \leq \mathbb{E}_{x,k}[\varphi_{1,n}(X_n)\mathbb{1}_{n<T_K^{(k)}\wedge\tau_A}] \leq \theta_1^{n-k}\varphi_{1,k}(x).$$

For any $x \in E_k$ and $k \leq n$,

$$\mathbb{P}_{x,k}(n < \tau_A) \geq \mathbb{E}_{x,k}[\varphi_{2,n}(X_n)\mathbb{1}_{n<\tau_A}] \geq \theta_2^{n-k}\varphi_{2,k}(x).$$

Proof of Lemma 6. By Markov's property, for any $k \leq n$ and $x \in E_k$, one has

$$\mathbb{E}_{x,k}[\varphi_{1,n}(X_n)\mathbb{1}_{n<T_K^{(k)}\wedge\tau_A}] = \mathbb{1}_{x\in E_k\setminus K_k} P_{k,k+1}\left[\mathbb{E}_{\cdot,k+1}\left(\varphi_{1,n}(X_n)\mathbb{1}_{n<T_K^{(k+1)}\wedge\tau_A}\right)\right](x).$$

As a result, the first inequality of the lemma can be shown by induction using this previous equality and the first line of (E'2). In a same way, the second statement of the lemma can be proved with the Lyapunov condition (E'2) related with $\varphi_{2,n}$. □

The next lemma correponds to the Lemma 9.6. in [15] and states that the expectation of $\varphi_{1,n}(X_n)$ is controlled by the expectation of $\varphi_{2,n}(X_n)$ uniformly in time.

Lemma 7. *For any $\theta \in (\theta_1/\theta_2, 1]$, there exists a finite constant $D_\theta > 0$ such that, for any $k \leq T$ and any probability measure μ on E_k satisfying $\mu(\varphi_{1,k})/\mu(\varphi_{2,k}) < \infty$,*

$$\frac{\mu P_{k,T}\varphi_{1,T}}{\mu P_{k,T}\varphi_{2,T}} \leq \left(\theta^{T-k}\frac{\mu(\varphi_{1,k})}{\mu(\varphi_{2,k})}\right) \vee D_\theta. \tag{4.3}$$

Proof of Lemma 7. It follows from (E'2) that

$$\mu P_{k,T+1}\varphi_{1,T+1} \leq \theta_1 \mu P_{k,T}\varphi_{1,T} + \tilde{c}_2 \mu P_{k,T}\mathbb{1}_K$$

and

$$\mu P_{k,T+1}\varphi_{2,T+1} \geq \theta_2 \mu P_{k,T}\varphi_{2,T}.$$

Hence

$$\begin{aligned}\frac{\mu P_{k,T+1}\varphi_{1,T+1}}{\mu P_{k,T+1}\varphi_{2,T+1}} &\leq \frac{\theta_1 \mu P_{k,T}\varphi_{1,T} + \tilde{c}_2 \mu P_{k,T}\mathbb{1}_K(x)}{\theta_2 \mu P_{k,T}\varphi_{2,T}} \\ &\leq \frac{\theta_1}{\theta_2}\frac{\mu P_{k,T}\varphi_{1,T}}{\mu P_{k,T}\varphi_{2,T}} + \frac{\tilde{c}_2}{\theta_2 \inf_{y\in K_T}\varphi_{2,T}(y)}.\end{aligned}$$

Since $\theta_1/\theta_2 < \theta$, these arithmetico-geometric inequalities entail (4.3). □

We now give an irreducibility inequality.

Lemma 8. *For any $C \geq 1$, there exists a time $n_5(C) \in \mathbb{N}$ such that,*

$$a_5(C) := \inf_{k\in\mathbb{Z}_+} \inf_{\mu\in\mathcal{M}_1(E_k) \text{ s.t. } \mu(\varphi_{1,k})\leq C\mu(\varphi_{2,k})} \mathbb{P}_{\mu,k}(X_{k+n_5(C)} \in K_{k+n_5(C)}, \tau_A > k + n_5(C)) > 0. \tag{4.4}$$

Proof of Lemma 8. It follows from (E'4) that there exists a time $n_\nu \in \mathbb{N}$ such that for all $n \geq n_\nu$, $\inf_{k\in\mathbb{Z}_+} \mathbb{P}_{\nu_k,k}(X_{n+k} \in K_{n+k}, \tau_A > n + k) > 0$, and, using (E'1), such that for all $n \geq n_\nu + n_1$,

$$\inf_{x\in K} \inf_{k\in\mathbb{Z}_+} \mathbb{P}_{x,k}(X_{n+k} \in K_{n+k}, \tau_A > n+k) \geq \tilde{c}_1 \inf_{k\in\mathbb{Z}_+} \mathbb{P}_{\nu_{n_1+k},n_1+k}(X_{n+k} \in K_{n+k}, \tau_A > n+k) > 0.$$

Fix $k \in \mathbb{Z}_+$. Let $C \geq 1$ and μ be such that $\mu(\varphi_{1,k}) \leq C\mu(\varphi_{2,k})$. It follows from Lemma 6 that, for any $k \leq n$,

$$\mathbb{P}_{\mu,k}(T_K^{(k)} \wedge \tau_A > n) \leq \mathbb{E}_{\mu,k}\left[\varphi_{1,n}(X_n)\mathbb{1}_{T_K^{(k)}\wedge\tau_A>n}\right] \leq \theta_1^{n-k}\mu(\varphi_{1,k}) \leq C\theta_1^{n-k}\mu(\varphi_{2,k}).$$

and

$$\mathbb{P}_{\mu,k}(n < \tau_A) \geq \theta_2^{n-k}\mu(\varphi_{2,k}).$$

Therefore,

$$\mathbb{P}_{\mu,k}(T_K^{(k)} \leq n < \tau_A) \geq \left(\theta_2^{n-k} - C\theta_1^{n-k}\right)\mu(\varphi_{2,k}).$$

Choosing $n(C) = \lceil 2C/\log(\theta_2/\theta_1)\rceil$, we deduce that

$$\mathbb{P}_{\mu,k}(T_K^{(k)} \leq k + n(C) < \tau_A) \geq \frac{\theta_2^{n(C)}}{2}\mu(\varphi_{2,k}) \geq \frac{\theta_2^{n(C)}}{2C}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}_{\mu,k}(X_{k+n(C)+n_\nu+n_1} \in K_{k+n(C)+n_\nu+n_1}, \tau_A > k + n(C) + n_\nu + n_1)\\
&\quad\geq \mathbb{E}_{\mu,k}\left[\mathbb{1}_{T_K^{(k)}\leq k+n(C)<\tau_A}\mathbb{P}_{X_{T_K^{(k)}},T_K^{(k)}}(X_{k+n(C)+n_\nu+n_1} \in K_{k+n(C)+n_\nu+n_1}, \tau_A > k + n(C) + n_\nu + n_1)\right]\\
&\quad\geq \min_{0\leq l\leq n(C)} \inf_{x\in K} \mathbb{P}_{x,k+l}(X_{k+n_\nu+n_1+n(C)} \in K_{k+n(C)+n_\nu+n_1}, \tau_A > k + n(C) + n_\nu + n_1)\,\frac{\theta_2^{n(C)}}{2C}\\
&\quad\geq \min_{0\leq l\leq n(C)} \inf_{k\in\mathbb{Z}_+} \inf_{x\in K} \mathbb{P}_{x,k+l}(X_{k+n_\nu+n_1+n(C)} \in K_{k+n(C)+n_\nu+n_1}, \tau_A > k + n(C) + n_\nu + n_1)\,\frac{\theta_2^{n(C)}}{2C}\\
&\quad=: a_5(C) > 0.
\end{aligned}$$

Hence we have proved Lemma 8 with $n_5(C) = n_\nu + n_1 + n(C)$. Note that $n_5(C)$ does not depend on k. □

The next lemma corresponds to the Lemma 9.8. in [15].

Lemma 9. *There exists a time $n_6 \in \mathbb{N}$ such that for any $k \in \mathbb{Z}_+$,*

$$\inf_{T \geq n_6} \inf_{x \in K_k} \mathbb{P}_{x,k}(X_{k+T} \in K_{k+T} \mid k+T < \tau_A) > 0.$$

Proof of Lemma 9. Fix $k \in \mathbb{Z}_+$. Since $\varphi_{1,k}/\varphi_{2,k}$ is bounded over K_k, we deduce from Lemma 7 that, setting $C := D_1 + \sup_{k \in \mathbb{Z}_+} \sup_{x \in K_k} \frac{\varphi_{1,k}(x)}{\varphi_{2,k}(x)}$, we have for all $x \in K_k$ and all $T \geq n_5(C)$,

$$\frac{P_{k,k+T-n_5(C)}\varphi_{1,k+T-n_5(C)}(x)}{P_{k,k+T-n_5(C)}\varphi_{2,k+T-n_5(C)}(x)} \leq C. \tag{4.5}$$

Using Lemma 8 applied to $\mu = \frac{\delta_x P_{k,k+T-n_5(C)}}{\delta_x P_{k,k+T-n_5(C)} \mathbb{1}_E}$, we deduce that, for all $x \in K_k$ and $T \geq n_5(C)$,

$$\mathbb{P}_{x,k}(X_{k+T} \in K_{k+T} \mid k+T < \tau_A) = \frac{\mu P_{k+T-n_5(C),k+T} \mathbb{1}_{K_{k+T}}}{\mu P_{k+T-n_5(C),k+T} \mathbb{1}_E} \geq \mu P_{k+T-n_5(C),k+T} \mathbb{1}_{K_{k+T}} \geq a_5(C).$$

□

Finally, the next lemma states that survival probabilities are controlled by $\varphi_{1,k}$.

Lemma 10. *For any $\theta \in (\theta_1, 1)$, $n \geq k+1$ and $x \in E_k$,*

$$\mathbb{P}_{x,k}(n < T_K^{(k)} \wedge \tau_A) \leq \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta} \theta^n. \tag{4.6}$$

There exists a constant $C > 0$ such that, for any $x \in E_k$ and $n \geq k+1$,

$$\mathbb{P}_{x,k}(n < \tau_A) \leq C \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta_2} \inf_{y \in K_k} \mathbb{P}_{y,k}(n < \tau_A). \tag{4.7}$$

Proof of Lemma 10. We first prove (4.6). It follows from Lemma 6 that, for all $\theta > \theta_1$ and $x \in E_k \setminus K_k$,

$$\mathbb{E}_{x,k}(\theta^{-T_K^{(k)} \wedge \tau_A}) \leq \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta}. \tag{4.8}$$

By Markov inequality, for all $x \in E_k \setminus K_k$,

$$\mathbb{P}_{x,k}(n < T_K^{(k)} \wedge \tau_A) \leq \mathbb{E}_{x,k}(\theta^{-T_K^{(k)} \wedge \tau_A}) \theta^n \leq \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta} \theta^n.$$

The inequality is trivial if $x \in K_k$.

Using (4.6), the second inequality of Lemma 6 and (E'3), we have for all $x \in E_k$

$$\begin{aligned}
\mathbb{P}_{x,k}(n < \tau_A) &= \mathbb{P}_{x,k}(n < T_K^{(k)} \wedge \tau_A) + \mathbb{P}_{x,k}(T_K^{(k)} \wedge \tau_A \leq n < \tau_A) \\
&\leq \theta_2^n \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta_2} + \sum_{l=k}^{n} \mathbb{P}_{x,k}(T_K^{(k)} \wedge \tau_A = l) \sup_{y \in K_l} \mathbb{P}_{y,l}(n < \tau_A) \\
&\leq \frac{\inf_{z\in K_k} \mathbb{P}_{z,k}(n+k < \tau_A)}{\inf_{z\in K_k} \varphi_{2,k}(z)} \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta_2} \\
&\qquad + \tilde{c}_3 \sum_{l=k}^{n} \mathbb{P}_{x,k}(T_K^{(k)} \wedge \tau_A = l) \inf_{y \in K_l} \mathbb{P}_{y,l}(n < \tau_A) \\
&\leq C \inf_{z\in K_k} \mathbb{P}_{z,k}(n+k < \tau_A) \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta_2} \\
&\qquad + C \inf_{z\in K_k} \mathbb{P}_{z,k}(n < \tau_A) \sum_{l=k}^{n} \mathbb{P}_{x,k}(T_K^{(k)} \wedge \tau_A = l) \theta_2^{-(l-k)}, \qquad (4.9)
\end{aligned}$$

where we used the fact that, for some constant $C > 0$, for all $n \geq k \geq 0$ and all $z \in K_k$,

$$\mathbb{P}_{z,k}(n < \tau_A) \geq C\theta_2^{l-k} \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A). \qquad (4.10)$$

This is proved using the three following equations. For all $n \geq l \geq k+n_6$ and all $z \in K_k$, by Lemmata 9 and 6,

$$\begin{aligned}
\mathbb{P}_{z,k}(n < \tau_A) &\geq \mathbb{P}_{z,k}(X_l \in K_l \mid l < \tau_A) \mathbb{P}_{z,k}(l < \tau_A) \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A) \\
&\geq C\theta_2^{l-k} \varphi_{2,k}(z) \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A) \\
&\geq C\theta_2^{l-k} \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A).
\end{aligned}$$

Also, using (E'3), for all $n \geq k + n_6 \geq l$,

$$\begin{aligned}
\mathbb{P}_{z,k}(n < \tau_A) &\geq C\theta_2^{n_6+k} \inf_{y\in K_{n_6+k}} \mathbb{P}_{y,n_6+k}(n < \tau_A) \\
&\geq C\theta_2^{n_6+k} \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A) \\
&\geq (C\theta_2^{n_6+k})\, \theta_2^{l-k} \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A).
\end{aligned}$$

Finally, for all $l \leq n < n_6 + k$,

$$\mathbb{P}_{z,k}(n < \tau_A) \geq \mathbb{P}_{z,k}(n_6 + k < \tau_A) \geq C\theta_2^{n_6+k} \geq (C\theta_2^{n_6+k})\, \theta_2^{l-k} \inf_{y\in K_l} \mathbb{P}_{y,l}(n < \tau_A),$$

and (4.10) is proved.

Now it follows from (4.8) and the inequality $\theta_2 > \theta_1$ that, for all $x \in E_k \setminus K_k$,

$$\mathbb{E}_{x,k}(\theta_2^{-(T_K^{(k)} \wedge \tau_A - k)}) \leq \frac{\varphi_{1,k}(x)}{1-\theta_1/\theta_2}.$$

Since the inequality is trivial for $x \in K_k$, plugging this inequality in (4.9) ends the proof of Lemma 10. □

4.2.2 Proof of Theorem 22

For any $T \in \mathbb{Z}_+$, we consider the law of the process X conditioned to not be absorbed before time T. We introduce the linear operators $(S^T_{m,n})_{0\le m\le n\le T}$ defined by

$$S^T_{m,n} f(x) = \mathbb{E}_{x,m}(f(X_n) \mid T < \tau_A) = \frac{P_{m,n}\left(f P_{n,T} \mathbb{1}_E\right)(x)}{P_{m,T}\mathbb{1}_E(x)}.$$

It is well-known that $(S^T_{m,n})_{0\le m\le n\le T}$ forms a time-inhomogeneous semigroup (i.e. $S^T_{m,n}S^T_{n,p} = S^T_{m,p}$ for all $m \le n \le l \le T$) and that the process $(X_n, 0 \le n \le T)$ under $\mathbb{P}_x^{S^T_{0,\cdot}}$ is a (time-inhomogeneous) Markov process, where we denote by $\mathbb{P}_x^{S^T_{0,\cdot}}$ the law of the process $(X_n, 0 \le n \le T)$ conditionally on $T < \tau_A$ and $X_0 = x$.

The following proof is divided in four steps :

- In a first time, a collection of Lyapunov functions $(\psi_{k,T})_{0\le k\le T}$ for the semi-group $(S^T_{m,n})_{0\le m\le n\le T}$ is constructed.
- Then, a Dobrushin-type property for the time-inhomogeneous semi-group is obtained.
- These two last steps are used in order to obtain a mixing property along a subsequence and starting from two Dirac measures.
- Finally, we prove the final mixing property from the key proposition obtained in the third step.

Construction of a Lyapunov function for $(S^T_{m,n})_{0\le m\le n\le T}$

Fix $\theta \in (\theta_1/\theta_2, 1)$. For any $k \le T$, we set, for $x \in E$,

$$\psi_{k,T}(x) = \mathbb{E}_{x,k}(\theta^{-T_K^{(k)}\wedge T+k} \mid T < \tau_A)$$

The following proposition provides a Lyapunov-type property for the inhomogeneous semigroup S.

Proposition 24. *There exists a constant $\bar{C} > 0$ such that, for all $0 \le m < T$ and $1 \le k \le T - m$,*

$$S^T_{m,m+k}\psi_{m+k,T}(x) \le \theta^k \psi_{m,T}(x) + \bar{C}, \quad \forall x \in E. \tag{4.11}$$

Proof. Markov's property implies that, for all $x \in E_k \setminus K_k$ and $T, m \ge 1$,

$$S^T_{m,m+1}\psi_{m+1,T}(x) = \theta\psi_{m,T}(x). \tag{4.12}$$

Indeed,

$$\begin{aligned}\theta\psi_{m,T}(x) &= \frac{\mathbb{E}_{x,m}(\theta^{1-T_K^{(m)}\wedge T+m}\mathbb{1}_{T<\tau_A})}{\mathbb{P}_{x,m}(T<\tau_A)}\\ &= \frac{\mathbb{E}_{x,m}\left[\mathbb{1}_{m<\tau_A}\mathbb{E}_{X_{m+1},m+1}(\theta^{-T_K^{(m+1)}\wedge T+m+1}\mid T<\tau_A)\mathbb{P}_{X_{m+1},m+1}(T<\tau_A)\right]}{\mathbb{P}_{x,m}(T<\tau_A)}\\ &= S^T_{m,m+1}\psi_{m+1,T}(x).\end{aligned}$$

Similarly, for all $x\in K_k$,

$$S^T_{m,m+1}\psi_{m+1,T}(x) = \theta\mathbb{E}_{x,m}(\theta^{-\sigma_K^{(m)}\wedge T+m}\mid T<\tau_A), \tag{4.13}$$

where

$$\sigma_K^{(m)} := \min\{n\geq m+1,\ X_n\in K_n\}$$

is the first return time in $(K_k)_{k\in\mathbb{Z}_+}$. Setting

$$C := \sup_{0\leq m\leq T}\sup_{x\in K_m}\mathbb{E}_{x,m}(\theta^{-\sigma_K\wedge(T-m)}\mid T<\tau_A),$$

which is finite (see Lemma 11), we can apply recursively (4.12) and (4.13) to obtain

$$\begin{aligned}S^T_{m,m+k}\psi_{m+k,T} &= S^T_{m,m+k-1}\left(\mathbb{1}_{E_k\setminus K_k}S^T_{m+k-1,m+k}(\psi_{m+k,T})\right)\\ &\quad + S^T_{m,m+k-1}\left(\mathbb{1}_{K_k}S^T_{m+k-1,m+k}(\psi_{m+k,T})\right)\\ &\leq \theta S^T_{m,m+k-1}\psi_{m+k-1,T} + C\theta\\ &\leq \ldots \leq \theta^k\psi_{m,T}(x) + C\sum_{\ell=1}^{k}\theta^\ell.\end{aligned}$$

Hence Proposition 24 follows from the next lemma. □

Lemma 11. *For all $\theta\in(\theta_1/\theta_2,1)$,*

$$\sup_{0\leq m\leq T}\sup_{x\in K_k}\mathbb{E}_{x,m}(\theta^{-\sigma_K^{(m)}\wedge T+m}\mid T<\tau_A)<\infty.$$

Proof of Lemma 11. Fix $x\in K_k$. On the one hand, by Lemma 10, we have for any $1\leq n<T$,

$$\begin{aligned}\mathbb{P}_{x,m}(n<\sigma_K^{(m)}\text{ and }T<\tau_A) &= \mathbb{E}_{x,m}(\mathbb{1}_{n<\sigma_K^{(m)}\wedge\tau_A}\mathbb{P}_{X_n,n}(T<\tau_A))\\ &\leq C\inf_{y\in K_n}\mathbb{P}_{y,n}(\tau_A>T)\mathbb{E}_{x,m}(\mathbb{1}_{n<\sigma_K^{(m)}\wedge\tau_A}\varphi_{1,n}(X_n)).\end{aligned}$$

Using (E'2) and Markov's property as in the proof of Lemma 6, we deduce

$$\mathbb{P}_{x,m}(n < \sigma_K^{(m)} \text{ and } T < \tau_A) \leq C \inf_{y \in K_n} \mathbb{P}_{y,n}(\tau_A > T)\theta_1^{n-m-1} P_{m,m+1}\varphi_{1,m+1}(x) \quad (4.14)$$

$$\leq C \inf_{y \in K_n} \mathbb{P}_{y,n}(\tau_A > T)\theta_1^{n-m}. \quad (4.15)$$

On the other hand, Lemma 9 implies the existence of a constant $C > 0$ such that, for all $x \in K$ and all $n \geq m + n_6$,

$$\mathbb{P}_{x,m}(X_n \in K_n, \tau_A > n) \geq C\mathbb{P}_{x,m}(n < \tau_A).$$

We deduce from Markov's property and Lemma 6 that

$$\begin{aligned}\mathbb{P}_{x,m}(T < \tau_A) &\geq \mathbb{P}_{x,m}(X_n \in K_n) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \\ &\geq C\mathbb{P}_{x,m}(n < \tau_A) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \\ &\geq C\theta_2^{n-m} \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A).\end{aligned}$$

Combining this with (4.14), we finally deduce that there exists a constant $C > 0$ such that, for all $x \in K$ and all $T \geq n \geq m + n_6$,

$$\mathbb{P}_{x,m}(n < \sigma_K^{(m)} \mid T < \tau_A) \leq C\left(\frac{\theta_1}{\theta_2}\right)^{n-m}. \quad (4.16)$$

The conclusion follows. □

Dobrushin property for $(S_{m,n}^T)_{0 \leq m \leq n \leq T}$

The next proposition provides a Dobrushin coefficient-type property for the inhomogeneous semigroup S.

Proposition 25. *There exists a constant $\alpha_0 \in (0,1)$ such that, for all $R > 0$, there exists $k_R \geq 1$ such that, for any $k \in \mathbb{Z}_+$, for all $T \geq k + k_R$ and all $x, y \in E$ such that $\psi_{k,T}(x) + \psi_{k,T}(y) \leq R$, we have*

$$\left\|\delta_x S_{k,k+k_R}^T - \delta_y S_{k,k+k_R}^T\right\|_{TV} \leq 2(1 - \alpha_0).$$

Proof. We start by stating a lemma proved at the end of this subsection.

Lemma 12. *For all $x \in K_k$, $k \in \mathbb{Z}_+$ and $k + n_1 + n_6 \leq n \leq T$,*

$$\mathbb{P}_{x,k}(X_n \in \cdot \mid T < \tau_A) \geq c_1' \nu_n, \quad (4.17)$$

where the measure ν_n and the integer n_1 are the one of Condition (E'1), the integer n_6 is from Lemma 8 and $c_1' > 0$ is independent of x, n and T.

Fix $\theta \in (\theta_1/\theta_2, 1)$ and set $k_R = \lceil \log(2R)/\log(1/\theta) \rceil + n_1 + n_6$ and fix $T \geq k + k_R$. For all $x \in E$ such that $\psi_{k,T}(x) \leq R$, Markov's inequality implies that

$$\mathbb{P}_{x,k}(T_K^{(k)} > k + k_R - n_1 - n_6 \mid T < \tau_A) \leq \frac{R}{\theta^{-k_R+n_1+n_6}} \leq \frac{1}{2}.$$

It follows from Lemma 12 that, for all measurable $A \subset E_{k+k_R}$,

$$\begin{aligned}
\mathbb{P}_{x,k}\left(X_{k+k_R} \in A | T < \tau_A\right) &\geq \frac{\mathbb{E}_{x,k}\left[\sum_{l=k}^{k+k_R-n_1-n_6} \mathbb{1}_{T_K^{(k)}=l} \mathbb{P}_{X_l,l}(X_{k+k_R} \in A,\, T < \tau_A)\right]}{\mathbb{P}_{x,k}(T < \tau_A)} \\
&\geq c_1' \nu_{k+k_R}(A) \frac{\mathbb{E}_{x,k}\left[\sum_{l=k}^{k+k_R-n_1-n_6} \mathbb{1}_{T_K^{(k)}=l} \mathbb{P}_{X_l,l}(T < \tau_A)\right]}{\mathbb{P}_{x,k}(T < \tau_A)} \\
&= c_1' \nu_{k+k_R}(A) \mathbb{P}_{x,k}(T_K^{(k)} \leq k + k_R - n_1 - n_6 \mid T < \tau_A) \\
&\geq \frac{1}{2} c_1' \nu_{k+k_R}(A).
\end{aligned}$$

This concludes the proof of Proposition 25 with $\alpha_0 = c_1'/2$. □

Proof of Lemma 12. By Markov's property, for any $T \geq n \geq k + n_1 + n_6$, $x \in K_k$ and $A \subset K_n$,

$$\begin{aligned}
\mathbb{P}_{x,k}(X_n \in A,\, T < \tau_A) &\geq \mathbb{E}_{x,k}\left[\mathbb{1}_{X_{n-n_1} \in K_{n-n_1}, \tau_A > n-n_1} \mathbb{E}_{X_{n-n_1}, n-n_1}\left(\mathbb{1}_{X_n \in A, \tau_A > n} \mathbb{P}_{X_n,n}(T < \tau_A)\right)\right] \\
&\geq \mathbb{E}_{x,k}\left[\mathbb{1}_{X_{n-n_1} \in K_{n-n_1}, \tau_A > n-n_1} \mathbb{P}_{X_{n-n_1}, n-n_1}(X_n \in A, \tau_A > n)\right] \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \\
&\geq \tilde{c}_1 \nu_n(A) \mathbb{P}_{x,k}\left(X_{n-n_1} \in K_{n-n_1}, \tau_A > n - n_1\right) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A),
\end{aligned} \tag{4.18}$$

where we used (E'1). Now, let us note that, for any $n \in \mathbb{Z}_+$, $y \in K_{n-n_1}$ and $T \geq n - n_1$,

$$\begin{aligned}
\mathbb{P}_{y,n-n_1}(\tau_A > T) &\geq \mathbb{E}_{y,n-n_1}(\mathbb{1}_{X_{T-n_1} \in K, \tau_A > T-n_1} \mathbb{P}_{X_{T-n_1}, T-n_1}(\tau_A > T)) \\
&\geq \tilde{c}_1 \mathbb{P}_{y,n-n_1}(X_{T-n_1} \in K, \tau_A > T - n_1) \\
&\geq \tilde{c}_1 \tilde{c}_6 \mathbb{P}_{y,n-n_1}(\tau_A > T - n_1)
\end{aligned}$$

where we used (E'1) and Lemma 9. Thus, using (E'3), one has that there exists a constant $C > 0$ such that, for any $n \geq n_1$, $x \in K_n$, $y \in K_{n-n_1}$ and $T \geq n$,

$$\mathbb{P}_{x,n}(\tau_A > T) \geq C \mathbb{P}_{y,n-n_1}(\tau_A > T - n_1).$$

Thus, using this last inequality and Lemma 10, we deduce that there exists a constant $c > 0$ such that

$$\begin{aligned}
\mathbb{P}_{x,k}(T < \tau_A) \leq \mathbb{P}_{x,k}(T - n_1 < \tau_A) &= \mathbb{E}_{x,k}\left(\mathbb{1}_{n-n_1 < \tau_A} \mathbb{P}_{X_{n-n_1}, n-n_1}(T - n_1 < \tau_A)\right) \\
&\leq c \mathbb{E}_{x,k}\left(\mathbb{1}_{n-n_1 < \tau_A} \varphi_{1,n-n_1}(X_{n-n_1})\right) \inf_{y \in K_n} \mathbb{P}_{y,n-n_1}(T - n_1 < \tau_A) \\
&\leq c \mathbb{E}_{x,k}\left(\mathbb{1}_{n-n_1 < \tau_A} \varphi_{1,n-n_1}(X_{n-n_1})\right) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A)
\end{aligned}$$

Since $\varphi_{1,n-n_1}(x)/\varphi_{2,n-n_1}(x)$ is uniformly bounded over $x \in K_{n-n_1}$, Lemma 7 implies that there exists a constant $c' > 0$ such that, for all $x \in K_k$,

$$\mathbb{E}_{x,k}\left[\mathbb{1}_{n-n_1<\tau_A}\varphi_{1,n-n_1}(X_{n-n_1})\right] \le c'\mathbb{E}_{x,k}\left[\mathbb{1}_{n-n_1<\tau_A}\varphi_{2,n-n_1}(X_{n-n_1})\right] \le c'\mathbb{P}_{x,k}\left(n-n_1<\tau_A\right).$$

But $n - n_1 \ge k + n_6$, hence Lemma 9 entails that there exists a constant $c'' > 0$ such that, for all $x \in K_k$,

$$\mathbb{P}_{x,k}\left(n-n_1<\tau_A\right) \le c''\mathbb{P}_{x,k}(X_{n-n_1} \in K_{n-n_1}, \tau_A > n-n_1).$$

Hence we obtain

$$\mathbb{P}_{x,k}(T<\tau_A) \le cc'c''\,\mathbb{P}_{x,k}\left(X_{n-n_1} \in K_{n-n_1}\right) \inf_{y\in K_n} \mathbb{P}_{y,n}(T<\tau_A).$$

Combining this with (4.18), we obtain

$$\mathbb{P}_{x,k}(X_n \in A \mid T<\tau_A) \ge \frac{\tilde{c}_1}{cc'c''}\,\nu_n(A).$$

This ends the proof of Lemma 12. □

Key proposition

The following property is a consequence of the two previous ones.

Proposition 26. *There exist constants $n_0 \ge 1$, $C > 0$ and $\alpha \in (0,1)$ such that, $\forall n \ge 1, k \in \mathbb{Z}_+$ and all $x, y \in E$,*

$$\left\|\delta_x S^{n_0 n}_{0,n_0 n} - \delta_y S^{n_0 n}_{0,n_0 n}\right\|_{TV} \le C\alpha^n(2+\psi_{0,n_0 n}(x)+\psi_{0,n_0 n}(y)).$$

Proof. We transpose the ideas of [25] to the time-inhomogeneous setting. We fix the constants $R = 4\bar{C}/(1-\theta)$ and $\beta = \alpha_0/2\bar{C}$, where $\bar{C}$ is the constant of Proposition 24. For all $T \ge 0$ and all $\varphi : E \to \mathbb{R}$, we set

$$|||\varphi|||_{k,T} = \sup_{x,y\in E} \frac{|\varphi(x)-\varphi(y)|}{2+\beta\psi_{k,T}(x)+\beta\psi_{k,T}(y)}.$$

Fix n and $T \ge 0$ such that $(n+1)k_R \le T$ and let φ be such that $|||\varphi|||_{(n+1)k_R,T} \le 1$. Then, replacing φ by $\varphi + c$ for some appropriate constant c, one has $|\varphi| \le 1 + \beta\psi_{(n+1)k_R,T}$ (see Lemma 3.8 p.14 in [25]).

If $\psi_{nk_R,T}(x) + \psi_{nk_R,T}(y) > R$, then, using Proposition 24,

$$\begin{aligned}
&\left|S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y)\right| \\
&\qquad\le 2+\theta\beta\psi_{nk_R,T}(x)+\theta\beta\psi_{nk_R,T}(y)+2\beta\bar{C} \\
&\qquad\le 2+(\theta+(1-\theta)/2)\left(\beta\psi_{nk_R,T}(x)+\beta\psi_{nk_R,T}(y)\right) \\
&\qquad\qquad -(R\beta)(1-\theta)/2+2\beta\bar{C} \\
&\qquad\le (1-\alpha_1)(2+\beta\psi_{nk_R,T}(x)+\beta\psi_{nk_R,T}(x)),
\end{aligned}$$

where $\alpha_1 \in (0,1)$ is such that $2 + (\theta + (1-\theta)/2)\, y \leq (1-\alpha_1)(2+y)$ for all $y \geq \beta R$. If $\psi_{nk_R,T}(x) + \psi_{nk_R,T}(y) \leq R$, then, considering

$$\varphi = \varphi' + \varphi'',$$

with $|\varphi'| \leq 1$ and $|\varphi''| \leq \beta \psi_{(n+1)k_R,T}$, Propositions 24 and 25 entail

$$\left| S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y) \right| \\ \leq 2(1-\alpha_0) + \beta\theta\psi_{nk_R,T}(x) + \beta\theta\psi_{nk_R,T}(y) + 2\beta\bar{C}.$$

Our choice $\beta = \alpha_0/2\bar{C}$ implies that

$$\left| S^T_{nk_R,(n+1)k_R}\varphi(x) - S^T_{nk_R,(n+1)k_R}\varphi(y) \right| \leq (1-\alpha_2)(2 + \beta\psi_{nk_R,T}(x) + \beta\psi_{nk_R,T}(y)).$$

for the constant $\alpha_2 = \frac{\alpha_0}{2} \wedge (1-\theta) > 0$.

Hence, we obtained

$$|||S^T_{nk_R,(n+1)k_R}\varphi|||_{nk_R,T} \leq (1 - \alpha_1 \wedge \alpha_2)|||\varphi|||_{(n+1)k_R,T},$$

which implies by iteration that

$$|||S^{nk_R}_{0,nk_R}\varphi|||_{0,nk_R} \leq (1-\alpha_1\wedge\alpha_2)^n |||\varphi|||_{nk_R,nk_R} \leq (1-\alpha_1\wedge\alpha_2)^n \|\varphi\|_\infty 2/(2+2\beta).$$

This concludes the proof of Proposition 26. □

Proof of Theorem 22

Let us now deduce Theorem 22 from this last proposition. We have, for all $x, y \in E$,

$$\left\| \delta_x P_{0,nn_0} - \delta_x P_{0,nn_0}\mathbb{1}_E\, \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \\ \leq C\alpha^n \left(2\delta_x P_{0,nn_0}\mathbb{1}_E + \mathbb{E}_{x,0}\left(\theta^{-T_K^{(0)}\wedge nn_0}\mathbb{1}_{nn_0<\tau_A} \right) + \psi_{0,n_0 n}(y)\delta_x P_{0,nn_0}\mathbb{1}_E \right).$$

Hence, for any probability measure μ on E, integrating the above inequality over $\mu(dx)$ leads to

$$\left\| \mu P_{0,nn_0} - \mu P_{0,nn_0}\mathbb{1}_E\, \delta_y S^{n_0 n}_{0,n_0 n} \right\|_{TV} \\ \leq C\alpha^n \left(2\mu P_{0,nn_0}\mathbb{1}_E + \mathbb{E}_{\mu,0}\left(\theta^{-T_K^{(0)}\wedge nn_0}\mathbb{1}_{nn_0<\tau_A} \right) + \psi_{0,n_0 n}(y)\mu P_{0,nn_0}\mathbb{1}_E \right).$$

We make use of the following lemma.

Lemma 13. *For all $\theta \in (\theta_1/\theta_2, 1)$, there exists a constant C such that, for all $0 \leq m \leq T$ and all probability measure μ over E such that $\mu(\varphi_2) > 0$,*

$$\mathbb{E}_{\mu,0}\left(\theta^{-T_K^{(0)}\wedge T}\mathbb{1}_{T<\tau_A} \right) \leq C\, \frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})} \mathbb{P}_{\mu,0}\left(T < \tau_A\right).$$

This implies that, for all μ such that $\mu(\varphi_{2,0}) > 0$,

$$\left\|\mu P_{0,nn_0} - \mu P_{0,nn_0}\mathbb{1}_E\,\delta_y S^{n_0 n}_{0,n_0 n}\right\|_{TV} \leq C\alpha^n\left(2\mu P_{0,nn_0}\mathbb{1}_E + \frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})}\mathbb{P}_{\mu,0}\left(n_0 n < \tau_A\right) + \psi_{0,n_0 n}(y)\mu P_{0,nn_0}\mathbb{1}_E\right).$$

Hence

$$\left\|\frac{\mu P_{0,nn_0}}{\mu P_{0,nn_0}\mathbb{1}_E} - \delta_y S^{n_0 n}_{0,n_0 n}\right\|_{TV} \leq C\alpha^n\left(2 + \frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})} + \psi_{0,n_0 n}(y)\right).$$

Using the same procedure w.r.t. y, we deduce that, for any probability measures μ_1 and μ_2 on E,

$$\left\|\frac{\mu_1 P_{0,nn_0}}{\mu_1 P_{0,nn_0}\mathbb{1}_E} - \frac{\mu_2 P_{0,nn_0}}{\mu_2 P_{0,nn_0}\mathbb{1}_E}\right\|_{TV} \leq C\alpha^n\left(\frac{\mu_1(\varphi_{1,0})}{\mu_1(\varphi_{2,0})} + \frac{\mu_2(\varphi_{1,0})}{\mu_2(\varphi_{2,0})}\right),$$

where we used the fact that $\mu(\varphi_{1,0})/\mu(\varphi_{2,0}) \geq 1$ for all probability measure μ on E.

Because of Lemma 7 below, we deduce that, for some constant $D_1 > 0$ and for all $0 \leq k < n_0$,

$$\begin{aligned}\left\|\frac{\mu_1 P_{0,nn_0+k}}{\mu_1 P_{0,nn_0+k}\mathbb{1}_E} - \frac{\mu_2 P_{0,nn_0+k}}{\mu_2 P_{0,nn_0+k}\mathbb{1}_E}\right\|_{TV} &\leq C\alpha^n\left(\frac{\mu_1 P_{0,k}\varphi_{1,0}}{\mu_1 P_{0,k}\varphi_{2,0}} + \frac{\mu_2 P_{0,k}\varphi_{1,0}}{\mu_2 P_{0,k}\varphi_{2,0}}\right)\\ &\leq C\alpha^n\left(\frac{\mu_1(\varphi_{1,0})}{\mu_1(\varphi_{2,0})} \vee D_1 + \frac{\mu_2(\varphi_{1,0})}{\mu_2(\varphi_{2,0})} \vee D_1\right).\end{aligned}$$

Therefore, up to a change in the constant C and replacing α by α^{1/n_0}, we deduce that, for all probability measures μ_1 and μ_2 on E such that $\mu_1(\varphi_{2,0}) > 0$ and $\mu_2(\varphi_{2,0}) > 0$ and for all $n \geq 0$,

$$\left\|\frac{\mu_1 P_{0,n}}{\mu_1 P_{0,n}\mathbb{1}_E} - \frac{\mu_2 P_{0,n}}{\mu_2 P_{0,n}\mathbb{1}_E}\right\|_{TV} \leq C\alpha^n\left(\frac{\mu_1(\varphi_{1,0})}{\mu_1(\varphi_{2,0})} + \frac{\mu_2(\varphi_{1,0})}{\mu_2(\varphi_{2,0})}\right). \tag{4.19}$$

To conclude, it is enough to apply the result (4.19) to the time-inhomogeneous semigroup $(P^{(m)}_{n,p})_{n\leq p}$ defined by

$$P^{(m)}_{n,p} := P_{m+n,m+p}$$

If $(P_{n,p})_{n\leq p}$ satisfies the assumptions (E'), then $(P^{(m)}_{n,p})_{n\leq p}$ satisfies also the assumptions (E') replacing the Lyapunov functions $(\varphi_{1,n})_{n\in\mathbb{Z}_+}$ and $(\varphi_{2,n})_{n\in\mathbb{Z}_+}$ by

$$\varphi^{(m)}_{1,n} := \varphi_{1,n+m}, \quad \varphi^{(m)}_{2,n} := \varphi_{2,n+m}$$

Thus, by (4.19) applied to $(P_{n,p}^{(m)})_{n\le p}$, for any $m \le n \in \mathbb{Z}_+$ and $\mu_1, \mu_2 \in \mathcal{M}_1(E_m)$,

$$\begin{aligned}
||\mathbb{P}_{\mu_1,m}(X_n \in \cdot | \tau_A > n) - \mathbb{P}_{\mu_2,m}(X_n \in \cdot | \tau_A > n)||_{TV} &= \left\| \frac{\mu_1 P_{0,n-m}^{(m)}}{\mu_1 P_{0,n-m}^{(m)} \mathbb{1}_E} - \frac{\mu_2 P_{0,n-m}^{(m)}}{\mu_2 P_{0,n-m}^{(m)} \mathbb{1}_E} \right\|_{TV} \\
&\le C\alpha^{n-m} \left(\frac{\mu_1(\varphi_{1,0}^{(m)})}{\mu_1(\varphi_{2,0}^{(m)})} + \frac{\mu_2(\varphi_{1,0}^{(m)})}{\mu_2(\varphi_{2,0}^{(m)})} \right) \\
&= C\alpha^{n-m} \left(\frac{\mu_1(\varphi_{1,m})}{\mu_1(\varphi_{2,m})} + \frac{\mu_2(\varphi_{1,m})}{\mu_2(\varphi_{2,m})} \right).
\end{aligned}$$

Proof of Lemma 13. This lemma in a generalization of Lemma 11. Its proof is based on similar computations. We give the details for sake of completeness.

For all probability measure μ on E, for any $0 \le n < T$, using Lemma 10 for the second inequality and Lemma 6 for the third inequality, we have

$$\begin{aligned}
\mathbb{P}_{\mu,0}(n < T_K^{(0)} \text{ and } T < \tau_A) &\le \mathbb{E}_{\mu,0}(\mathbb{1}_{n<T_K^{(0)}} \mathbb{P}_{X_n,n}(T < \tau_A)) \\
&\le C \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \mathbb{E}_{\mu,0}(\mathbb{1}_{n<T_K^{(0)}} \varphi_{1,n}(X_n)) \\
&\le C \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \theta_1^n \mu(\varphi_{1,0}). \qquad (4.20)
\end{aligned}$$

For all integer $n \ge n_\mu$, where

$$n_\mu := \left\lceil n_5(D_\theta) + \frac{\log \frac{\mu(\varphi_{1,0})}{D_\theta \mu(\varphi_{2,0})}}{\log(1/\theta)} \right\rceil,$$

it follows from Lemma 7 that

$$\frac{\mu P_{n_5(D_\theta),n} \varphi_{1,n}}{\mu P_{n_5(D_\theta),n} \varphi_{2,n}} \le D_\theta$$

and from Lemma 8 that

$$\frac{\mu P_{0,n} \mathbb{1}_{K_n}}{\mu P_{0,n} \mathbb{1}_{E_n}} \ge a_5(D_\theta) > 0.$$

Therefore, we obtain from the Markov property and Lemma 6 that

$$\begin{aligned}
\mathbb{P}_{\mu,0}(T < \tau_A) &\ge \mathbb{P}_{\mu,0}(X_n \in K_n, \tau_A > n) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \\
&\ge a_5(D_\theta) \mathbb{P}_{\mu,0}(n < \tau_A) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A) \\
&\ge a_5(D_\theta) \theta_2^n \mu(\varphi_{2,0}) \inf_{y \in K_n} \mathbb{P}_{y,n}(T < \tau_A).
\end{aligned}$$

Combining this with (4.20), we obtain

$$\mathbb{P}_{\mu,0}(n < T_K^{(0)} \text{ and } T < \tau_A) \le \frac{C}{a_5(D_\theta)} \left(\frac{\theta_1}{\theta_2} \right)^n \frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})} \mathbb{P}_{\mu,0}(T < \tau_A).$$

Hence

$$\mathbb{E}_{\mu,0}\left(\theta^{-T_K^{(0)}\wedge T}\mathbb{1}_{T_K^{(0)}\geq n_\mu,\ T<\tau_A}\right)\leq C\frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})}\mathbb{P}_{\mu,0}\left(T<\tau_A\right).$$

We deduce that

$$\mathbb{E}_{\mu,0}\left(\theta^{-T_K^{(0)}\wedge T}\mathbb{1}_{T<\tau_A}\right)\leq\left(C\frac{\mu(\varphi_{1,0})}{\mu(\varphi_{2,0})}+\theta^{-n_\mu}\right)\mathbb{P}_{\mu,0}\left(T<\tau_A\right).$$

Since $\theta^{-n_\mu}\leq\frac{\theta^{-(n_5(D_\theta)+1)}\mu(\varphi_{1,0})}{D_\theta\mu(\varphi_{2,0})}$, we have proved Lemma 13. □

4.2.3 Proof of Proposition 23

Let $\nu_m\in\mathcal{M}_1(K_m)$ be as (E'1). For any $x\in E_m$, denote

$$\eta_{m,n}(x):=\frac{\mathbb{P}_{x,m}(\tau_A>n)}{\mathbb{P}_{\nu_m,m}(\tau_A>n)}$$

For any $x\in E_m$ and $m\leq n\leq p$, by Markov's property,

$$\eta_{m,p}(x)=\eta_{m,n}(x)\frac{\mathbb{P}_{\delta_x\phi_{m,n},n}(\tau_A>p)}{\mathbb{P}_{\nu_m\phi_{m,n},n}(\tau_A>p)}=\eta_{m,n}(x)\frac{\delta_x\phi_{m,n}(\eta_{n,p})}{\nu_m\phi_{m,n}(\eta_{n,p})}=\frac{\delta_xP_{m,n}(\eta_{n,p})}{\nu_mP_{m,n}(\eta_{n,p})}$$

By the mixing property, for any $x\in K_m$,

$$\begin{aligned}|\eta_{m,p}(x)-\eta_{m,n}(x)|&=\eta_{m,n}(x)\left|\frac{\mathbb{P}_{\delta_x\phi_{m,n},n}(\tau_A>p)}{\mathbb{P}_{\nu_m\phi_{m,n},n}(\tau_A>p)}-1\right|\\&=\eta_{m,n}(x)\left|\frac{\delta_x\phi_{m,n}(\eta_{n,p})-\nu_m\phi_{m,n}(\eta_{n,p})}{\nu_m\phi_{m,n}(\eta_{n,p})}\right|\\&\leq\eta_{m,n}(x)\frac{\|\eta_{n,p}\|_\infty}{\nu_m\phi_{m,n}(\eta_{n,p})}C\alpha^{n-m},\end{aligned}$$

where we used that $\sup_{x\in K_m}\frac{\varphi_{1,m}(x)}{\varphi_{2,m}(x)}<\infty$. Moreover,

$$\nu_m\phi_{m,n}(\eta_{n,p})=\frac{\mathbb{P}_{\nu_m,m}(\tau_A>p)}{\mathbb{P}_{\nu_m,m}(\tau_A>n)\mathbb{P}_{\nu_n,n}(\tau_A>p)}\tag{4.21}$$

Using the definition of ν_m and Lemma 4,

$$\begin{aligned}\mathbb{P}_{\nu_m,m}(\tau_A>p)&=\mathbb{E}_{\nu_m,m}(\mathbb{1}_{\tau_A>n-n_0}\mathbb{E}_{X_{n-n_0},n-n_0}(\mathbb{1}_{\tau_A>n}\mathbb{P}_{X_n,n}(\tau_A>p)))\\&\geq\tilde{c}_1\mathbb{P}_{\nu_m,m}(\tau_A>n-n_0,X_{n-n_0}\in K_{n-n_0})\mathbb{P}_{\nu_n,n}(\tau_A>p)\\&\geq C\mathbb{P}_{\nu_m,m}(\tau_A>n-n_0)\mathbb{P}_{\nu_n,n}(\tau_A>p)\\&\geq C\mathbb{P}_{\nu_m,m}(\tau_A>n)\mathbb{P}_{\nu_n,n}(\tau_A>p)\end{aligned}$$

So, finally,

$$|\eta_{m,p}(x) - \eta_{m,n}(x)| \leq C\eta_{m,n}(x)\alpha^{n-m}\|\eta_{n,p}\|_\infty$$

By Lemma 5, for any $x \in K_m$,

$$\eta_{m,n}(x) \leq C\varphi_{1,m}(x)\frac{\inf_{y\in K_m}\mathbb{P}_{y,m}(\tau_A > n)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \leq C$$

In particular, defining $\|f\|_{L^\infty(A)} := \sup_{x\in A}|f(x)|$ for all measurable $A \subset E$ and all bounded measurable function f on A, we deduce that for all $n \geq m$,

$$\|\eta_{m,n} - \eta_{m,n+1}\|_{L^\infty(K_m)} \leq C\alpha^{n-m}\|\eta_{n,n+1}\|_{L^\infty(E)} \leq \frac{C\tilde{c}_3}{\theta_2\nu_n(\varphi_{2,n})}\,\alpha^{n-m}. \tag{4.22}$$

where the inequality $||\eta_{n,n+1}||_\infty = \sup_{x\in E_n}\frac{\mathbb{P}_{x,n}(\tau_A>n+1)}{\mathbb{P}_{\nu_n,n}(\tau_A>n+1)} \leq \frac{1}{\theta_2\nu_n(\varphi_{2,n})}$ is used. Hence the sequence $\eta_{m,n}$ is Cauchy in $L^\infty(K_m)$ and converges to some η_m. We define for all $x \in E_m \setminus K_m$

$$\eta_m(x) := \mathbb{E}_{x,m}\left(\mathbb{1}_{n<\tau_A}\eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right)\frac{\nu_{T_K^{(m)}\wedge\tau_A}P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu,m}(\tau_A > n)}\right).$$

Note that, since η_m is bounded on K_m and since

$$\nu_m P_{m,T_K^{(m)}\wedge\tau_A}\mathbb{1}_E \geq \theta_2^{T_K^{(m)}\wedge\tau_A}\nu_m(\varphi_{2,m})$$

almost surely, (4.8) implies that $\eta_m(x) < \infty$ for all $x \in E$.

For all $x \in E_m \setminus K_m$, we have

$$\begin{aligned}
|\eta_{m,n}(x) - \eta_m(x)| \leq{}& \left|\eta_{m,n}(x) - \frac{\mathbb{P}_{x,m}(T_K^{(m)}\wedge\tau_A \leq n < \tau_A)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)}\right| \\
&+ \left|\frac{\mathbb{P}_x(T_K^{(m)}\wedge\tau_A \leq n < \tau_A)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} - \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A\leq n<\tau_A}\eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right)\frac{\nu_{T_K^{(m)}\wedge\tau_A}P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu,m}(\tau_A > n)}\right)\right| \\
&+ \left|\mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A\leq n<\tau_A}\eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right)\frac{\nu_{T_K^{(m)}\wedge\tau_A}P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu_m,m}(\tau_A > n)}\right) - \eta_m(x)\right|.
\end{aligned}$$

We shall control each term of the right hand side. For the first one, we deduce from Lemma 10 that, for some constant $C > 0$,

$$\eta_{m,n}(x) - \frac{\mathbb{P}_{x,m}(T_K^{(m)}\wedge\tau_A \leq n < \tau_A)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} = \frac{\mathbb{P}_{x,m}(n < T_K^{(m)}\wedge\tau_A)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \leq C\left(\frac{\theta_1}{\theta_2}\right)^{n-m}\varphi_{1,m}(x).$$

For the second one, Markov's property implies that

$$\mathbb{P}_{x,m}(T_K^{(m)}\wedge\tau_A \leq n < \tau_A) = \mathbb{E}_{x,m}\left(\sum_{k=1}^{n}\mathbb{1}_{T_K^{(m)}\wedge\tau_A=k}\mathbb{P}_{X_k,k}(n \leq \tau_A)\right).$$

Now, (4.22) entails that, for all $m \le k \le n$ and $x \in K_k$,

$$|\mathbb{P}_{x,k}(n < \tau_A) - \eta_k(x)\mathbb{P}_{\nu_k,k}(n < \tau_A)| \le C\alpha^{n-k}\mathbb{P}_{\nu_k,k}(\tau_A > n)$$

Therefore,

$$\begin{aligned}
&\left| \frac{\mathbb{P}_x(T_K^{(m)} \wedge \tau_A \le n < \tau_A)}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} - \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A \le n < \tau_A} \eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right) \frac{\nu_{T_K^{(m)}\wedge\tau_A} P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \right) \right| \\
&\le \frac{1}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \left| \mathbb{E}_{x,m}\left(\sum_{k=m}^{n} \mathbb{1}_{T_K^{(m)}\wedge\tau_A = k, T_K^{(m)} < \tau_A} \left(\mathbb{P}_{X_k,k}(\tau_A > n) - \eta_k(X_k)\mathbb{P}_{\nu_k,k}(\tau_A > n) \right) \right) \right| \\
&\qquad \le \frac{C\alpha^n}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K\wedge\tau_A \le n} \alpha^{-T_K\wedge\tau_A} \mathbb{P}_{\nu,T_k\wedge\tau_A}(\tau_A > n) \right) \\
&\qquad\qquad \le C\alpha^n \mathbb{E}_{x,m}\left(\frac{\mathbb{1}_{T_K\wedge\tau_A \le n}\alpha^{-T_K\wedge\tau_A}}{\nu_m P_{m,T_K^{(m)}\wedge\tau_A}\mathbb{1}_E} \right) \\
&\qquad\qquad\qquad \le C\alpha^n \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K\wedge\tau_A \le n} (\theta_2\alpha)^{-T_K\wedge\tau_A} \right) \le C\alpha^n \varphi_{1,m}(x),
\end{aligned}$$

where we used (4.8) in the last inequality.

For the third term, using the a.s. inequality $\mathbb{1}_{T_K\wedge\tau_A > n} \le \alpha^{n - T_K\wedge\tau_A}$, we have

$$\begin{aligned}
&\eta_m(x) - \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A \le n < \tau_A} \eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right) \frac{\nu_{T_K^{(m)}\wedge\tau_A} P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \right) \\
&\quad = \mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A > n} \eta_{T_K^{(m)}\wedge\tau_A}\left(X_{T_K^{(m)}\wedge\tau_A}\right) \frac{\nu_{T_K^{(m)}\wedge\tau_A} P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \right) \\
&\quad \le C\mathbb{E}_{x,m}\left(\mathbb{1}_{T_K^{(m)}\wedge\tau_A > n} \frac{\nu_{T_K^{(m)}\wedge\tau_A} P_{T_K^{(m)}\wedge\tau_A,n}\mathbb{1}_E}{\mathbb{P}_{\nu_m,m}(\tau_A > n)} \right) \\
&\quad \le C\mathbb{E}_{x,m}\left((\alpha\theta_2)^{-T_K\wedge\tau_A} \right) \alpha^n \\
&\quad \le C\alpha^n \varphi_{1,m}(x),
\end{aligned}$$

Thus, for all $x \in E_m \setminus K_m$,

$$|\eta_{m,n}(x) - \eta_m(x)| \le C\alpha^n \varphi_{1,m}(x), \tag{4.23}$$

which concludes the proof of the convergence of $\eta_{m,n}$ to η_m in $L^\infty(\varphi_{1,m})$.

4.3 Proof of Theorem 20

We assume that Assumption (F') is satisfied. In Subsection 4.3.1, we prove that Assumption (E') holds true for the sub-Markovian semigroup $(P^{(s)}_{m,n})_{m\le n}$ of the absorbed Markov process $(X_{s+nt_2}, n \in \mathbb{Z}_+)$, for $s \in I$. In Subsection 4.3.2, we prove the existence of the weak quasi-ergodicity for $(X_t)_{t\in I}$ with the claimed properties.

4.3.1 Proof of (E') for the chain $(X_{s+nt_2})_{n\in\mathbb{Z}_+}$, for $s \in I$

We fix $\theta_1 \in (\gamma_1^{t_2}, \gamma_2^{t_2})$ and set $\theta_2 = \gamma_2^{t_2}$. Using the last line of (F'2), there exists a number $n_0 \in \mathbb{N}$ large enough so that, for any $t \geq (n_0-1)t_2$ and for any $s \in I$,

$$\inf_{x\in L_s} \gamma_2^{-t}\mathbb{P}_{x,s}(X_{t+s} \in L_{t+s}, \tau_A > t+s) \geq 1 \vee \frac{c_2}{\theta_1 - \gamma_1^{t_2}}$$

and we set

$$\varphi_{1,k}^{(s)} = \psi_{1,s+kt_2} \quad \text{and} \quad \varphi_{2,k}^{(s)} = \gamma_2^{kt_2}\frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\sum_{i=k}^{k+n_0-1}\gamma_2^{-it_2}P_{k,i}^{(s)}\mathbb{1}_L.$$

Step 1. Proof of (E'2), (E'4) and (E'1).

For all $x \in E_{s+kt_2} \setminus L_{s+kt_2}$, it follows from (F'0) and the second line of (F'2) that

$$\begin{aligned}
P_{k,k+1}^{(s)}\varphi_{1,k+1}^{(s)}(x) &= \mathbb{E}_{x,s+kt_2}\left(\psi_{1,s+(k+1)t_2}(X_{s+(k+1)t_2})\mathbb{1}_{s+(k+1)t_2<\tau_L\wedge\tau_A}\right)\\
&\quad + \mathbb{E}_{x,s+kt_2}\left(\mathbb{1}_{\tau_L\leq s+(k+1)t_2}\mathbb{E}_{X_{\tau_L},\tau_L}(\mathbb{1}_{s+(k+1)t_2<\tau_A}\psi_{1,s+(k+1)t_2}(X_{s+(k+1)t_2}))\right)\\
&\leq \gamma_1^{t_2}\psi_{1,s+kt_2}(x) + \mathbb{P}_{x,s+kt_2}(\tau_L \leq s+(k+1)t_2)c_2.
\end{aligned}$$

We define $K_n^{(s)} = \left\{y \in E_{s+nt_2}, \mathbb{P}_{y,s+nt_2}(\tau_L \leq s+(n+1)t_2)/\psi_{1,s+nt_2}(y) \geq (\theta_1-\gamma_1^{t_2})/c_2\right\}$. The second line of (F'2) at time $t = s$ and the fact that $\theta_1 - \gamma_1^{t_2} < 1$ imply that $L_{s+nt_2} \subset K_n^{(s)}$. Moreover, we have, for all $x \notin K_k^{(s)}$,

$$P_{k,k+1}^{(s)}\varphi_{1,k+1}^{(s)}(x) \leq \theta_1\varphi_{1,k}^{(s)}(x). \tag{4.24}$$

Hence, for all $x \in E_{s+kt_2}$,

$$P_{k,k+1}^{(s)}\varphi_{1,k+1}^{(s)}(x) \leq \theta_1\varphi_{1,k}^{(s)}(x) + c_2\mathbb{1}_{K_k^{(s)}}(x). \tag{4.25}$$

Note that it immediately follows from the definition of $K_k^{(s)}$ that

$$\sup_{k\in\mathbb{Z}_+}\sup_{x\in K_k^{(s)}}\varphi_{1,k}^{(s)}(x) < \infty.$$

In particular, the first and third lines of (E'2) are proved.

Moreover, using the Markov property provided by (F'0) and the definition of n_0, we deduce that, for all $t \geq s+(k+n_0-1)t_2$,

$$\inf_{x\in K}\gamma_2^{-(t-s-kt_2)}\mathbb{P}_{x,s+kt_2}(X_t \in L_{s+kt_2}, \tau_A > t) \tag{4.26}$$

$$\geq \inf_{x\in K_{s+kt_2}}\mathbb{P}_{x,s+kt_2}(\tau_L \leq s+(k+1)t_2)\inf_{u\in[s+kt_2,s+(k+1)t_2]}\inf_{y\in L_u}\gamma_2^{-(t-u)}\mathbb{P}_{y,u}\left(X_t \in L_t, \tau_A > t\right) \tag{4.27}$$

$$\geq 1, \tag{4.28}$$

where we used the fact that, for all $x \in K_{s+kt_2}$, $\mathbb{P}_{x,s+kt_2}(\tau_L \le s+(k+1)t_2) \ge \frac{\theta_1-\gamma_1^{t_2}}{c_2}$. In particular,

$$\begin{aligned} P^{(s)}_{k,k+1}\varphi^{(s)}_{2,k+1} &= \gamma_2^{t_2}\varphi^{(s)}_{2,k} + \gamma_2^{(k+1)t_2}\frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\left(\gamma_2^{-(k+n_0)t_2}P^{(s)}_{k,k+n_0}\mathbb{1}_{L_{s+k+n_0}} - \gamma_2^{-kt_2}\mathbb{1}_{L_k}\right) \\ &\ge \gamma_2^{t_2}\varphi^{(s)}_{2,k} = \theta_2\varphi^{(s)}_{2,k}. \end{aligned}$$

In addition, for all $x \in K^{(s)}_k$, $s \in I$ and $k \in \mathbb{Z}_+$,

$$\varphi^{(s)}_{2,k}(x) \ge \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}\gamma_2^{-(n_0-1)t_2}P^{(s)}_{k,k+n_0-1}\mathbb{1}_{L_{s+k+n_0-1}}(x) \ge \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0t_2}-1}.$$

Hence (E'2) is proved.

Fix $n_1 \ge 1$ such that $n_1t_2 - t_1 \ge n_0t_2$. Condition (F'1) and then (4.26) imply that, for all $x \in K^{(s)}_k$,

$$\begin{aligned} &\mathbb{P}_{x,s+kt_2}(X_{s+(k+n_1)t_2} \in \cdot \cap K^{(s)}_k, \tau_A > s+(k+n_1)t_2) \\ &\ge \mathbb{P}_{x,s+kt_2}(X_{s+(k+n_1)t_2-t_1} \in L_{s+(k+n_1)t_2-t_1}, \tau_A > s+(k+n_1)t_2-t_1)c_1\nu_{s+(k+n_1)t_2-t_1}(\cdot \cap L_{s+(k+n_1)t_2-t_1}) \\ &\ge \gamma_2^{n_1t_2-t_1}c_1\nu_{s+(k+n_1)t_2-t_1}(\cdot \cap L_{s+(k+n_1)t_2-t_1}). \end{aligned}$$

Extending $\nu_{s+(k+n_1)t_2-t_1}$ as a probability measure on $K_{s+(k+n_1)t_2-t_1}$, we obtain (E'1) with $\tilde{c}_1 = \gamma_2^{n_1t_2-t_1}c_1$.

In order to prove (E'4), it is enough to note that, by the last line of (F'2), there exists a constant $C > 0$ (not depending on s) such that, for any $k \in \mathbb{Z}_+$, for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}_{\nu_{s+kt_2},s+kt_2}(X_{s+(n+k)t_2} &\in K^{(s)}_{n+k}, \tau_A > s+(n+k)t_2) \\ &\ge \mathbb{P}_{\nu_{s+kt_2},s+kt_2}(X_{s+(n+k)t_2} \in L_{s+(n+k)t_2}, \tau_A > s+(n+k)t_2) \\ &\ge C\gamma_2^{nt_2} > 0. \end{aligned}$$

Step 2. Estimation of the survival probability.

Our goal here is to prove a version of Lemma 10, where (4.7) is replaced by

$$\mathbb{P}_{x,s+kt_2}(s+nt_2 < \tau_A) \le C\frac{\varphi^{(s)}_{1,k}(x)}{1-\theta_1/\theta_2}\inf_{y \in L_{s+kt_2}}\mathbb{P}_{y,s+kt_2}(s+nt_2 < \tau_A), \quad \forall x \in E_{s+kt_2}, \forall n \in \mathbb{N}. \tag{4.29}$$

Since the proof is similar, we only highlight the main differences. First, Lemma 9 only uses (E'1), (E'2) and (E'4), so that there exist $n_6 \ge 1$ and $\zeta_1 > 0$ such that, for all $k \in \mathbb{Z}_+$, $x \in K_k$ and all $n \ge n_6 + k$,

$$\delta_x P^{(s)}_{k,k+n}\mathbb{1}_{K^{(s)}_{k+n}} \ge \zeta_1\delta_x P^{(s)}_{k,k+n}\mathbb{1}_{E_{k+n}}.$$

Hence, for all $x \in K_k^{(s)}$ and all $N \geq n_0 + n_6$, using (4.26),

$$\delta_x P^{(s)}_{k,k+N} \mathbb{1}_{L_{s+(k+N)t_2}} \geq \gamma_2^{n_0 t_2}\, \delta_x P^{(s)}_{k,k+N-n_0} \mathbb{1}_{K^{(s)}_{k+N-n_0}} \geq \zeta_1 \gamma_2^{n_0 t_2}\, \delta_x P^{(s)}_{k,k+N-n_0} \mathbb{1}_E \geq \zeta_1 \gamma_2^{n_0 t_2}\, \delta_x P^{(s)}_{k,k+N} \mathbb{1}_E.$$

Hence,

$$\inf_{N \geq n_0+n_6} \inf_{x \in K_k^{(s)}} \mathbb{P}_{x,s+kt_2}(X_{s+(k+N)t_2} \in L_{s+(k+N)t_2} \mid s + (k+N)t_2 < \tau_A) > 0. \tag{4.30}$$

Third, it follows from (F'2) that, for all $x \in E_{s+kt_2} \setminus L_{s+kt_2}$,

$$\mathbb{P}_{x,s+kt_2}(s + nt_2 < \tau_L \wedge \tau_A) \leq \gamma_1^{(n-k)t_2} \psi_{1,s+kt_2}(x) \leq \theta_1^{n-k} \varphi^{(s)}_{1,k}(x). \tag{4.31}$$

and from (E'2) that, for all $x \in E_{s+kt_2}$,

$$\mathbb{P}_{x,s+kt_2}(s + nt_2 < \tau_A) \geq \gamma_2^{(n-k)t_2} \varphi^{(s)}_{2,k}(x). \tag{4.32}$$

Therefore, following the same lines as in (4.9) (replacing K with L), we deduce from (4.31) and (4.32) that, for all $x \in E_{s+kt_2}$

$$\begin{aligned}
\mathbb{P}_{x,s+kt_2}(s + nt_2 < \tau_A) &\leq \theta_1^{n-k} \varphi^{(s)}_{1,k}(x) \\
&\qquad + c_3 \int_{kt_2}^{nt_2} \inf_{y \in L_{s+\lceil u/t_2 \rceil t_2}} \mathbb{P}_{y,s+\lceil u/t_2 \rceil t_2}\left(s + nt_2 < \tau_A\right) \mathbb{P}_{x,s+kt_2}(\tau_L \wedge \tau_A \in du) \\
&\leq C \inf_{z \in L} \mathbb{P}_{z,s+kt_2}(s + nt_2 < \tau_A) \varphi^{(s)}_{1,k}(x) \\
&\qquad + \frac{c_3 \gamma_2^{-t_2}}{c} \inf_{z \in L_{s+kt_2}} \mathbb{P}_{z,s+kt_2}(s + nt_2 < \tau_A) \mathbb{E}_{x,s+kt_2}\left(\gamma_2^{-\tau_L \wedge \tau_A}\right),
\end{aligned}$$

which entails (4.29), where we used in the second inequality the fact that, for any $k \leq l \leq n$

$$\mathbb{P}_{x,s+kt_2}(s + nt_2 < \tau_A) \geq c \gamma_2^{(l-k)t_2} \inf_{y \in L_{s+lt_2}} \mathbb{P}_{y,s+lt_2}\left(s + nt_2 < \tau_A\right), \quad \forall x \in L_{s+kt_2},$$

which is deduced from (4.30) exactly as in Lemma 10.

Step 4. Proof of (E'3).

Writing (4.29) with the semi-group $(P^{(s)}_{k,n})_{k \leq n}$, one has

$$\delta_x P^{(s)}_{k,n} \mathbb{1} \leq C \frac{\varphi^{(s)}_{1,k}(x)}{1 - \theta_1/\theta_2} \delta_y P^{(s)}_{k,n} \mathbb{1}$$

Applying $\delta_x P^{(s)}_{k,l}$ to this last inequality, for any $k \leq l \leq n$, $x \in K_k^{(s)}$ and $y \in L_{s+kt_2}$,

$$\delta_x P^{(s)}_{k,n} \mathbb{1} \leq C \frac{\delta_x P^{(s)}_{k,l} \varphi^{(s)}_{1,l}}{1 - \theta_1/\theta_2} \delta_y P^{(s)}_{l,n} \mathbb{1}$$

Then, by Lemma 7 applied to $(P^{(s)}_{m,n})_{m\le n}$, there exists a constant $C>0$ (which is not the same as in the previous inequality) such that, for any $k\le l\le n$, $x\in K^{(s)}_k$ and $y\in L_{s+kt_2}$,

$$\delta_x P^{(s)}_{k,n}\mathbb{1}\le C\delta_y P^{(s)}_{l,n}\mathbb{1}$$

Moreover, using the Markov property at time n_0t_2 and (4.26), we deduce that, for all $t\ge s+kt_2$,

$$\inf_{x\in K^{(s)}_k}\mathbb{P}_{x,s+kt_2}(t<\tau_A)\ge \inf_{x\in K^{(s)}_k}\mathbb{P}_{x,s+kt_2}(t+n_0t_2<\tau_A)\ge \gamma_2^{n_0t_2}\inf_{y\in L_{s+kt_2}}\mathbb{P}_{y,s+kt_2}(t<\tau_A).$$

These inequalities imply (E'3).

4.3.2 Proof of Theorem 20

Fix $\rho_1\in(\theta_1^{1/t_2},\gamma_2)$. For any $s\in I$, define

$$\varphi'_{1,s}(x):=\mathbb{E}_{x,s}\left(\rho_1^{-\tau_L\wedge\tau_A+s}\right). \tag{4.33}$$

It follows from the inequality (4.31) generalized on the space of time I that there exists a constant $C>0$ such that, for any $s\in I$ and $x\in E_s$,

$$\varphi'_{1,s}(x)\le C\,\psi_{1,s}(x).$$

In addition, for all $s\in I$ and $x\in E_s\setminus L_s$,

$$\begin{aligned}\mathbb{E}_{x,s}\left(\mathbb{1}_{s+t_2<\tau_L\wedge\tau_A}\varphi'_{1,s+t_2}(X_{s+t_2})\right)&=\rho_1^{t_2}\mathbb{E}_{x,s}\left(\mathbb{1}_{s+t_2<\tau_L\wedge\tau_A}\rho_1^{-\tau_L\wedge\tau_A+s}\right)\\&\le\rho_1^{t_2}\varphi'_{1,s}(x)\end{aligned} \tag{4.34}$$

and the inequality is trivial for $x\in L_s$. In addition, for any $s\in I$, $t\in[s,s+t_2]$, $k\in\mathbb{Z}_+$ and all $x\in L_s$, $\mathbb{E}_{x,s}\left(\varphi'_{1,t+s}(X_{t+s})\mathbb{1}_{s+t<\tau_A}\right)\le C\mathbb{E}_{x,s}\left(\psi_{1,s+t}(X_{t+s})\mathbb{1}_{t+s<\tau_A}\right)\le Cc_2$. Hence Condition (F') is satisfied replacing γ_1 with ρ_1 and $\psi_{1,s}$ with $\varphi'_{1,s}$. Therefore, we can apply Step 1 to prove that (E') is satisfied with $\varphi'^{(s)}_{1,k}:=\varphi'_{1,s+kt_2}$ and $\varphi'^{(s)}_{2,k}$ where

$$\varphi'^{(s)}_{2,k}=\gamma_2^{kt_2}\frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n_0't_2}-1}\sum_{i=k}^{k+n_0'-1}\gamma_2^{-it_2}P^{(s)}_{k,i}\mathbb{1}_{L_{s+it_2}}.$$

for an integer n_0' that can be chosen larger than n_0.

Moreover, by Markov property, we have for all $x\in E_s$ and $t\ge s$,

$$\begin{aligned}\varphi'_{1,s}(x)&=\mathbb{E}_{x,s}\left[\mathbb{1}_{t<\tau_L\wedge\tau_A}{\rho_1}^{-\tau_L\wedge\tau_A+s}\right]+\mathbb{E}_{x,s}\left[\mathbb{1}_{t\ge\tau_L\wedge\tau_A}{\rho_1}^{-\tau_L\wedge\tau_A+s}\right]\\&\le{\rho_1}^{-(t-s)}\mathbb{E}_{x,s}\left[\mathbb{1}_{t<\tau_L\wedge\tau_A}\varphi'_{1,t}(X_t)\right]+{\rho_1}^{-(t-s)}\mathbb{P}_{x,s}(t\ge\tau_L\wedge\tau_A)\\&\le{\rho_1}^{-(t-s)}\left(\mathbb{E}_{x,s}[\mathbb{1}_{t<\tau_A}\varphi'_{1,t}(X_t)]+1\right)\end{aligned} \tag{4.35}$$

Let $t \geq t_2 + s$ be fixed and define $k \in \mathbb{N}$ such that $0 \leq t - kt_2 - s < t_2$. It follows from the fact that $P_{s,s+t_2}\varphi'_{1,s+t_2} \leq \bar{C}\varphi'_{1,s}$ and from (4.35) that

$$\begin{aligned}
\mathbb{E}_{x,s}[\mathbb{1}_{t<\tau_A}\varphi'_{1,t}(X_t)] &\leq \bar{C}^k \mathbb{E}_{x,s}\left[\mathbb{1}_{t-kt_2<\tau_A}\varphi'_{1,t-kt_2}(X_{t-kt_2})\right] \\
&\leq \bar{C}^k {\rho_1}^{-(k+1)t_2-s+t}\mathbb{E}_{x,s}\left[\mathbb{1}_{s+t_2<\tau_A}\varphi'_{1,s+t_2}(X_{s+t_2}) + \mathbb{1}_{t-kt_2<\tau_A}\right] \\
&\leq C\bar{C}^k {\rho_1}^{-(k+1)t_2-s+t}\mathbb{E}_{x,s}\left[\mathbb{1}_{s+t_2<\tau_A}\psi_{1,s+t_2}(X_{s+t_2}) + 1\right] \\
&\leq C\bar{C}^k {\rho_1}^{-(k+1)t_2-s+t}(\theta_1 + c_2 + 1)\psi_{1,s}(x). \qquad (4.36)
\end{aligned}$$

Now, fix $s \in I$ and let μ be a probability measure such that, for any $k \in \mathbb{Z}_+$, $\mu(\varphi^{(s)}_{1,k}) < \infty$ and $\mu(\varphi^{(s)}_{2,k}) > 0$. Then, for all $t \geq n_0 t_2$, it follows from (4.26) that, for all $k \geq 0$,

$$\begin{aligned}
\mathbb{P}_{\mu,s}(X_{t+kt_2+s} \in L_{t+kt_2+s}) &\geq \mathbb{P}_{\mu,s}(X_{s+kt_2} \in L_{s+kt_2}) \inf_{y \in L_{s+kt_2}} \mathbb{P}_{y,s+kt_2}(X_{t+s+kt_2} \in L_{t+s+kt_2}) \\
&\geq \gamma_2^t \mathbb{P}_{\mu,s}(X_{s+kt_2} \in L_{s+kt_2}). \qquad (4.37)
\end{aligned}$$

Now define, for any $t \in I$,

$$\psi_{2,t} := \frac{\gamma_2^{-t_2} - 1}{\gamma_2^{-n_0 t_2} - 1} \sum_{i=0}^{n_0-1} \gamma_2^{-it_2} P_{t,t+it_2}\mathbb{1}_{L_{t+it_2}}$$

and note that, for any $k \in \mathbb{Z}_+$,

$$\psi_{2,s+kt_2} = \varphi^{(s)}_{2,k}.$$

Therefore, for all $t \in [n_0 t_2, (n_0+1)t_2]$,

$$\begin{aligned}
\mu P_{s,s+t}\psi_{2,s+t} &= \frac{\gamma_2^{-t_2} - 1}{\gamma_2^{-n_0 t_2} - 1} \sum_{k=0}^{n_0-1} \gamma_2^{-kt_2} \mu P_{s,s+t+kt_2}\mathbb{1}_{L_{s+t+kt_2}} \\
&\geq \frac{\gamma_2^{-t_2} - 1}{\gamma_2^{-n_0 t_2} - 1} \gamma_2^{(n_0+1)t_2} \sum_{k=0}^{n_0-1} \gamma_2^{-kt_2} \mu P_{s,s+kt_2}\mathbb{1}_{L_{s+kt_2}} = \gamma_2^{(n_0+1)t_2}\mu(\psi_{2,s}).
\end{aligned}$$

This and inequality (4.36) imply that (using that $n'_0 \geq n_0$), for all $t \in [n_0 t_2, (n_0+1)t_2]$ and for a constant $C > 0$ that may change from line to line,

$$\frac{\mu P_{s,t}\varphi'_{1,t}}{\mu P_{s,t}\varphi'_{2,t}} \leq C\frac{\mu P_{s,t}\varphi'_{1,t}}{\mu P_{s,t}\varphi_{2,t}} \leq C\frac{\mu(\psi_{1,s})}{\mu(\psi_{2,s})}, \qquad (4.38)$$

Using Markov property, we deduce that for any $s \leq t$,

$$\|\mathbb{P}_{\mu_1,s}(X_t \in \cdot \mid t < \tau_A) - \mathbb{P}_{\mu_2,s}(X_t \in \cdot \mid t < \tau_A)\|_{TV} \leq C\alpha^{\lfloor \frac{t-s}{t_2} \rfloor}\left(\frac{\mu_1(\varphi_{1,s})}{\mu_1(\varphi_{2,s})} + \frac{\mu_2(\varphi_{1,s})}{\mu_2(\varphi_{2,s})}\right).$$

4.4 Proof of the existence of the Q-process

4.4.1 Proof of Proposition 22

Our goal is to prove Proposition 22, where the convergence is exponential in $L^\infty(\psi_{1,s})$ for a given $s \in I$. Since we proved that (E') holds true for the semigroup $(P^{(s)}_{m,n})_{m\leq n}$ and for the functions $\varphi'_{1,s}$ and $\varphi_{2,s}$, it follows from Proposition 23 that there exist $\alpha \in (0,1)$ and $C > 0$ such that, for all $y \in E$,

$$\left| \frac{\mathbb{P}_{y,s}(s + nt_2 < \tau_A)}{\mathbb{P}_{\nu_s,s}(s + nt_2 < \tau_A)} - \eta_s(y) \right| \leq C\alpha^n \varphi'_{1,s}(y). \tag{4.39}$$

For any $t \in [s + t_2, s + 2t_2]$, integrating this inequality with respect to $\mathbb{P}_{x,s}(X_t \in dy; t < \tau_A)$, we deduce from (4.36) that

$$\left| \frac{\mathbb{P}_{x,s}(nt_2 + t < \tau_A)}{\mathbb{P}_{\nu_t,t}(nt_2 + t < \tau_A)} - \mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{t<\tau_A}) \right| \leq C\alpha^n \psi_{1,s}(x) \tag{4.40}$$

for a constant C independent of $t \in [s+t_2, s+2t_2]$. In particular, integrating over $\nu_s(dx)$,

$$\left| \frac{\mathbb{P}_{\nu_s,s}(nt_2 + t < \tau_A)}{\mathbb{P}_{\nu_t,t}(nt_2 + t < \tau_A)} - \mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{t<\tau_A}) \right| \leq C\alpha^n \nu_s(\psi_{1,s}) \leq C'\alpha^n$$

where $C' := C \sup_{s\in I, x\in K_s} \psi_{1,s}(x)$. Then, for any $n \geq n_{s,t} := \frac{1}{\alpha} \log\left(\frac{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}{C'}\right)$,

$$\frac{1}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C'\alpha^n} \leq \frac{\mathbb{P}_{\nu_t,t}(\tau_A > t + nt_2)}{\mathbb{P}_{\nu_s,s}(\tau_A > t + nt_2)} \leq \frac{1}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) - C'\alpha^n}$$

As a result, for any $s \in I$, $t \in [s + t_2, s + 2t_2] \cap I$ and $n \geq n_{s,t}$,

$$\frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) - C\alpha^n\psi_{1,s}(x)}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C'\alpha^n} \leq \frac{\mathbb{P}_{x,s}(nt_2 + t < \tau_A)}{\mathbb{P}_{\nu_s,s}(\tau_A > t + nt_2)} \leq \frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C\alpha^n\psi_{1,s}(x)}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) - C'\alpha^n} \tag{4.41}$$

Now, let us state the following lemma, whose the proof is postponed after this proof :

Lemma 14. *One has*

$$c_6 := \inf_{s\in I, t\in[s+t_2,s+2t_2]\cap I} \mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) > 0.$$

As a result, for any $s \leq t$, $n_{s,t} \geq n_0 := \frac{1}{\alpha}\log\left(\frac{c_6}{C'}\right)$, and setting $\tilde{\eta}_{s,t}(x) := \frac{\mathbb{E}_{x,s}\left[\eta_t(X_t)\mathbb{1}_{t<\tau_A}\right]}{\mathbb{E}_{\nu_s,s}\left[\eta_t(X_t)\mathbb{1}_{t<\tau_A}\right]}$, for all $s \in I$ and $t \in [s + t_2, s + 2t_2] \cap I$, for any $n \geq n_0$,

$$\begin{aligned}
\frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C\alpha^n\psi_{1,s}(x)}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) - C'\alpha^n} &= \frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C\alpha^n\psi_{1,s}(x)}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}\left[1 + \frac{C'}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}\alpha^n + o(\alpha^n)\right] \\
&\leq \frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) + C\alpha^n\psi_{1,s}(x)}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}\left[1 + \frac{C'}{c_6}\alpha^n + o(\alpha^n)\right] \\
&\leq \tilde{\eta}_{s,t}(x) + \alpha^n\left(\frac{C}{c_6}\psi_{1,s}(x) + \tilde{\eta}_{s,t}(x)\frac{C'}{c_6}\right) + o(\alpha^n)
\end{aligned} \tag{4.42}$$

where the term $o(\alpha^n)$ is the sum of terms like $(C_k\psi_{1,s}(x) + C'_k\tilde{\eta}_{s,t}(x))\alpha^{kn}$. Now, using Lemma 14,

$$\begin{aligned}\tilde{\eta}_{s,t}(x) &= \frac{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}{\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})} \\ &\leq \frac{1}{c_6}\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}),\end{aligned}$$

and, by (4.38), one has, for any n,

$$\begin{aligned}\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) &\leq C\alpha^n\psi_{1,s}(x) + \frac{\mathbb{P}_{x,s}(\tau_A > t+nt_2)}{\mathbb{P}_{\nu_t,t}(\tau_A > t+nt_2)} \\ &\leq C\alpha^n\psi_{1,s}(x) + \mathbb{E}_{x,s}\left[\frac{\mathbb{P}_{X_t,t}(\tau_A > t+nt_2)}{\mathbb{P}_{\nu_t,t}(\tau_A > t+nt_2)}\mathbb{1}_{\tau_A>t}\right] \\ &\leq C\alpha^n\psi_{1,s}(x) + C\mathbb{E}_{x,s}[\varphi'_{1,s}(X_t)\mathbb{1}_{\tau_A>t}] \\ &\leq C\alpha^n\psi_{1,s}(x)\end{aligned}$$

where the constant C has changed from line to line. Finally, combining this last inequality with (4.42), keeping in mind the previous remark about $o(\alpha^n)$ and doing the same reasoning for the lower bound in there exists $C > 0$ (not the same as the previous inequalities) such that, for any $x \in E_s$, $s \in I$ and $t \in [s+t_2, s+2t_2] \cap I$,

$$\left|\frac{\mathbb{P}_{x,s}(\tau_A > t+nt_2)}{\mathbb{P}_{\nu_s,s}(\tau_A > t+nt_2)} - \tilde{\eta}_{s,t}(x)\right| \leq C\alpha^n\psi_{1,s}(x).$$

Moreover, using the fact that, for any $s \leq t$ and any $n \in \mathbb{Z}_+$,

$$\frac{\mathbb{P}_{x,s}(\tau_A > nt_2+t)}{\mathbb{P}_{\nu_s,s}(\tau_A > nt_2+t)} = \frac{\mathbb{P}_{x,s}(\tau_A > nt_2+s)}{\mathbb{P}_{\nu_s,s}(\tau_A > nt_2+s)} \times \frac{\delta_x\phi_{s,s+nt_2}(\mathbb{P}_{\cdot,s+nt_2}(\tau_A > nt_2+t))}{\nu_s\phi_{s,s+nt_2}(\mathbb{P}_{\cdot,s+nt_2}(\tau_A > nt_2+t))},$$

and mimicking the beginning of the proof of Proposition 23 for the discrete-time case, one can show that $\tilde{\eta}_{s,t} = \eta_s$. As a result, integrating over $\mu(dx)$ for any $\mu \in \mathcal{M}_1(E_s)$,

$$\left|\frac{\mathbb{P}_{\mu,s}(\tau_A > t)}{\mathbb{P}_{\nu_s,s}(\tau_A > t)} - \mu(\eta_s)\right| \leq C\alpha^{\lfloor\frac{t-s}{t_2}\rfloor}\mu(\psi_{1,s}) \tag{4.43}$$

This conclude the proof of Proposition 22.

Proof of the Lemma 14. Fix $s \in I$ and $t \in [s+t_2, s+2t_2]$. Then, applying Lemma 7 to the semi-group $(P^{(t)}_{m,n})_{m\leq n}$, there exists a constant $c > 0$ such that, for any $x \in E_t$ and $n \in \mathbb{Z}_+$,

$$\frac{\mathbb{P}_{x,t}(\tau_A > t+nt_2)}{\mathbb{P}_{\nu_t,t}(\tau_A > t+nt_2)} > c\varphi'_{2,t}(x),$$

where we recall that $\varphi'_{2,t}(x) = \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-n'_0 t_2}-1}\sum_{k=0}^{n'_0-1}\gamma_2^{-kt_2}P_{t,t+kt_2}\mathbb{1}_{L_{t+kt_2}}(x)$ with n'_0 can be chosen greater than n_0. For example, it is possible to choose $n'_0 = n_1 + n_0$. Integrating this equality over $\mathbb{P}_{\nu_s,s}(X_t \in \cdot, \tau_A > t)$ and using (4.38), there exists $c' > 0$ (not depending on s or t) such that

$$\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t}) > c'\mathbb{E}_{\nu_s,s}(\varphi'_{2,t}(X_t)\mathbb{1}_{\tau_A>t}).$$

Finally, taking $n'_0 = n_1 + n_0$,

$$\begin{aligned}
\mathbb{E}_{\nu_s,s}(\varphi'_{2,t}(X_t)\mathbb{1}_{\tau_A>t}) &= \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-(n_1+n_0)t_2}-1}\sum_{k=0}^{n_1+n_0-1}\gamma_2^{-kt_2}\nu_s P_{s,t+kt_2}\mathbb{1}_{L_{t+kt_2}} \\
&\geq \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-(n_1+n_0)t_2}-1}\gamma_2^{-(n_1+n_0-1)t_2}\nu_s P_{s,t+(n_0-1)t_2+n_1t_2}\mathbb{1}_{L_{t+(n_0-1)t_2+n_1t_2}} \\
&\geq \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-(n_1+n_0)t_2}-1}\gamma_2^{-(n_1+n_0-1)t_2}\gamma_2^{t-s+(n_0-1)t_2}\nu_s P_{s,s+n_1t_2}\mathbb{1}_{L_{s+n_1t_2}} \\
&\geq \frac{\gamma_2^{-t_2}-1}{\gamma_2^{-(n_1+n_0)t_2}-1}\gamma_2^{-(n_1-1)t_2}\tilde{c}_1,
\end{aligned}$$

which concludes the proof. □

4.4.2 Proof of Theorem 21

Let $x \in E'_s$, $s \leq t \leq T$ and $\Gamma \in \mathcal{F}_{s,t}$. Then,

$$\begin{aligned}
|\mathbb{P}_{x,s}(\Gamma|T<\tau_A) - \mathbb{Q}_{x,s}(\Gamma)| &= \left|\frac{\mathbb{E}_{x,s}(\mathbb{1}_{\Gamma,t<\tau_A}\mathbb{P}_{X_t,t}(T<\tau_A))}{\mathbb{P}_{x,s}(T<\tau_A)} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{\Gamma,t<\tau_A}\eta_t(X_t))}{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}\right| \\
&= \left|\frac{\mathbb{P}_{\nu_t,t}(\tau_A>T)}{\mathbb{P}_{\nu_s,s}(\tau_A>T)}\frac{\mathbb{E}_{x,s}(\mathbb{1}_{\Gamma,t<\tau_A}\eta_{t,T}(X_t))}{\eta_{s,T}(x)} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{\Gamma,t<\tau_A}\eta_t(X_t))}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}\right| \\
&= \underbrace{\eta_{s,t}(x)\left|\frac{\mathbb{E}_x(\mathbb{1}_\Gamma\eta_{t,T}(X_k))|t<\tau_A)}{\eta_{s,T}(x)\nu\phi_{s,t}(\eta_{t,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma\eta_t(X_t)|t<\tau_A)}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)|\tau_A>t)}\right|}_{(*)}
\end{aligned}$$

where $(\eta_{s,t})_{s\leq t}$ is defined as, for any $y \in E$,

$$\eta_{s,t}(y) := \frac{\mathbb{P}_{y,s}(\tau_A>t)}{\mathbb{P}_{\nu_s,s}(\tau_A>t)}$$

By the equation (4.43), there exist $C, \gamma > 0$ such that, for any $\mu \in \mathcal{M}_1(E'_s)$,

$$|\mu(\eta_{s,t}) - \mu(\eta_s)| \leq Ce^{-\gamma(t-s)}\mu(\psi_{1,s}) \tag{4.44}$$

As a consequence, for any $\mu \in \mathcal{M}_1(E'_s)$,

$$\left| \frac{1}{\mu(\eta_{s,t})} - \frac{1}{\mu(\eta_s)} \right| \leq Ce^{-\gamma(t-s)} \frac{\mu(\psi_{1,s})}{\mu(\eta_{s,t})\mu(\eta_s)}$$

and

$$\begin{aligned}
(*) &\leq \eta_{s,t}(x) \left| \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_{s,T}(x)\nu_s\phi_{s,t}(\eta_{t,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_s(x)\nu_s\phi_{s,t}(\eta_{t,T})} \right| \\
&\quad + \eta_{s,t}(x) \left| \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_s(x)\nu_s\phi_{s,t}(\eta_{t,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)|\tau_A > t)} \right| \\
&\quad + \eta_{s,t}(x) \left| \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)|\tau_A > t)} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_t(X_t)|t < \tau_A)}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)|\tau_A > t)} \right| \\
&\leq \eta_{s,t}(x) \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\nu_s\phi_{s,t}(\eta_{t,T})} Ce^{-\gamma(T-s)} \frac{\psi_{1,s}(x)}{\eta_{s,T}(x)\eta_s(x)} \\
&\quad + \eta_{s,t}(x) \frac{\mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A)}{\eta_s(x)} Ce^{-\gamma(T-t)} \frac{\nu_s\phi_{s,t}(\psi_{1,t})}{\nu_s\phi_{s,t}(\eta_{t,T})\nu_s\phi_{s,t}(\eta_t)} \\
&\quad + \frac{\eta_{s,t}(x)}{\eta_s(x)\mathbb{E}_{\nu_s,s}(\eta_t(X_t)|\tau_A > t)} \left| \mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_{t,T}(X_t))|t < \tau_A) - \mathbb{E}_{x,s}(\mathbb{1}_\Gamma \eta_t(X_t)|t < \tau_A) \right| \\
&\leq Ce^{-\gamma(T-s)} \frac{\psi_{1,s}(x)}{\eta_s(x)} + e^{-\gamma(T-t)} \left(1 + Ce^{-\gamma(T-s)} \frac{\psi_{1,s}(x)}{\eta_s(x)} \right) \frac{\nu_s\phi_{s,t}\psi_{1,t}}{\nu_s\phi_{s,t}\eta_t} \\
&\quad + Ce^{-\gamma(T-t)} \frac{\mathbb{E}_{x,s}\left(\psi_{1,t}(X_t)\mathbb{1}_{t<\tau_A}\right)}{\mathbb{E}_{x,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}
\end{aligned}$$

As it was written previously, it is possible to replace $\psi_{1,s}$ by $\varphi'_{1,s}$ defined by (4.33). Doing this, the last inequality becomes

$$|\mathbb{Q}_{x,s}(\Gamma) - \mathbb{P}_{x,s}(\Gamma|\tau_A > T)| \leq Ce^{-\gamma(T-t)} \left(\frac{\varphi'_{1,s}(x)}{\eta_s(x)} + \left(1 + Ce^{-\gamma(T-s)} \frac{\varphi'_{1,s}(x)}{\eta_s(x)} \right) \frac{\nu_s P_{s,t}\varphi'_{1,t}}{\nu_s P_{s,t}\eta_t} + \frac{P_{s,t}\varphi'_{1,t}(x)}{P_{s,t}\eta_t(x)} \right)$$

Now, by (E'1) applied to the chain $(X_{s+nt_2})_{n\in\mathbb{Z}_+}$ and using that $\nu_s(\eta_s) = 1$ for any $s \in I$, one has

$$\nu_s P_{s,s+n_1t_2}\eta_{s+n_1t_2} \geq \tilde{c}_1 \nu_{s+n_1t_2}(\eta_{s+n_1t_2}) = \tilde{c}_1.$$

where we recall that n_1 is chosen such that $n_1t_2 - t_1 \geq n_0t_2$. Thus, using the equality $\eta_s = \frac{\mathbb{E}_{\cdot,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}{\mathbb{E}_{\nu,s}(\eta_t(X_t)\mathbb{1}_{\tau_A>t})}$ for any $s \leq t$, $k \in \mathbb{Z}_+$ and $x \in E_{s+kt_2}$,

$$P_{s,s+n_1t_2}\eta_{s+n_1t_2}(x) = \nu_s P_{s,s+n_1t_2}\eta_{s+n_1t_2} \times \eta_s(x) \geq \tilde{c}_1\eta_s(x).$$

Note moreover that there exists a constant $C > 0$ such that, for any $s \in I$, $\epsilon > 0$ and for any $x \in E$

$$P_{s,s+n_1t_2}\varphi'_{1,s+n_1t_2}(x) \leq \rho_1^{n_1t_2}\varphi'_{1,s}(x) + C\mathbb{1}_{K_s^{\epsilon,n_1}}(x),$$

where

$$K_s^{\epsilon,n_1} := \left\{ x \in E_s : \exists t \text{ s.t. } \frac{t-s}{t_2} \in \{0,\dots,n_1-1\}, P_{s,t}\mathbb{1}_{K_t}(x) > 0 \text{ and } \psi_{2,s}(x) \geq \epsilon \right\}.$$

In particular, when $\epsilon < \inf_{x\in K_s, s\in I} \psi_{2,s}(x)$, $K_s \subset K_s^{\epsilon,n_1}$. Since it is possible to choose n_0 large enough such that $\rho_1^{n_0 t_2} < \tilde{c}_1$ $(= \gamma_2^{n_0 t_2 - t_1} c_1)$, the proof of Lemma 2 can be adapted to show that there exists $C > 0$ such that for any $s \in I$, $k \in \mathbb{Z}_+$ and $\mu \in \mathcal{M}_1(E'_s)$,

$$\frac{\mu P_{s,s+kn_1t_2}\varphi'_{1,s+kn_1t_2}}{\mu P_{s,s+kn_1t_2}\eta_{s+kn_1t_2}} \leq C\frac{\mu(\varphi'_{1,s})}{\mu(\eta_s)}$$

As a result, for any $s \leq u$, taking $t = u - \lfloor \frac{u-s}{n_1t_2} \rfloor n_1 t_2$,

$$\frac{\mu P_{s,u}\varphi'_{1,u}}{\mu P_{s,u}\eta_u} = \frac{(\mu\phi_{s,t})P_{t,u}\varphi'_{1,u}}{(\mu\phi_{s,t})P_{t,u}\eta_u} \leq C\frac{\mu P_{s,t}\varphi'_{1,t}}{\mu P_{s,t}\eta_t}$$

Finally, since $\mu P_{s,t}\eta_t = \nu_s P_{s,t}\eta_t \times \mu(\eta_s)$, $\nu_s P_{s,t}\eta_t > c_6$ by Lemma 14, and using (4.36), the last inequality implies that there exists $C > 0$ (different from the last inequality) such that for any $s \leq s + t_2 \leq u$, for any $\mu \in \mathcal{M}_1(E'_s)$,

$$\frac{\mu P_{s,u}\varphi'_{1,u}}{\mu P_{s,u}\eta_u} \leq C\frac{\mu(\psi_{1,s})}{\mu(\eta_s)}$$

To sum up, one has : there exist $C \geq 0$ such that for any $x \in E'_s$, $s \leq t \leq T$,

$$||\mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) - \mathbb{P}_{x,s}(X_{[s,t]} \in \cdot | T < \tau_A)||_{TV} \leq Ce^{-\gamma(T-t)}\left(1 + \frac{\varphi_{1,s}(x)}{\eta_s(x)}\right)\frac{\varphi_{1,s}(x)}{\eta_s(x)}$$

For the generalization to any probability measure satisfying $\mu(\eta_s) > 0$, it is enough to take back the same previous reasoning replacing the Dirac measure δ_x by μ, and one has the following inequality :

$$||\mathbb{Q}_{\eta_s*\mu,s}(X_{[s,t]} \in \cdot) - \mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | T < \tau_A)||_{TV} \leq Ce^{-\gamma(T-t)}\left(1 + \frac{\mu(\psi_{1,s})}{\mu(\eta_s)}\right)\frac{\mu(\psi_{1,s})}{\mu(\eta_s)}.$$

4.4.3 Proof of the existence of a quasi-ergodic distribution

Let us prove the Corollary 3 for $I = \mathbb{Z}_+$. The reader could realize that the following computation can easily be adapted for a general space of time I.

For any $m \leq n \in \mathbb{Z}_+$, for any μ such that $\mu(\eta_m) > 0$,

$$\begin{aligned}
&\left|\left|\frac{1}{n}\sum_{k=m}^{n} \mathbb{P}_{\mu,m}(X_k \in \cdot | \tau_A > n) - \frac{1}{n}\sum_{k=m}^{n} \mathbb{Q}_{\eta_m * \mu, m}(X_k \in \cdot)\right|\right|_{TV} \\
&\qquad \leq \frac{1}{n}\sum_{k=m}^{n} ||\mathbb{Q}_{\eta_m * \mu, m}(X_k \in \cdot) - \mathbb{P}_{\mu,m}(X_k \in \cdot | n < \tau_A)||_{TV} \\
&\qquad \leq \frac{1}{n}\sum_{k=m}^{n} C\alpha^{n-k}\left(1 + \frac{\mu(\varphi_{1,m})}{\mu(\eta_m)}\right)\frac{\mu(\varphi_{1,m})}{\mu(\eta_m)} \\
&\qquad = \frac{C\alpha^n\left(1 + \frac{\mu(\varphi_{1,m})}{\mu(\eta_m)}\right)\frac{\mu(\varphi_1)}{\mu(\eta_m)}}{n}\sum_{k=m}^{n}\alpha^{-k} \\
&\qquad = \frac{C\alpha^{n-m}\left(1 + \frac{\mu(\varphi_1)}{\mu(\eta_m)}\right)\frac{\mu(\varphi_1)}{\mu(\eta_m)}}{n}\frac{\alpha^{-(n-m+1)} - 1}{\alpha^{-1} - 1} \\
&\qquad \leq \frac{C}{n}\left(1 + \frac{\mu(\varphi_{1,m})}{\mu(\eta_m)}\right)\frac{\mu(\varphi_{1,m})}{\mu(\eta_m)}
\end{aligned}$$

where C has changed from the penultimate to the last line.

4.5 Time-homogeneous Markov processes absorbed by moving boundaries

In this subsection, only time-homogeneous Markov processes $(X_t)_{t\in I}$ will be considered. The aim is to understand for which behavior of moving boundaries it is still possible to define quasi-limiting or quasi-ergodic distribution (it is already shown in [45] that the quasi-stationary distribution is not well-defined). Following the paper [44], we will focus on two types of movement :

1. Stabilizing boundaries
2. Periodic boundaries

4.5.1 Existence of a quasi-limiting distribution when the moving boundaries converge

Let $(A_t)_{t\in I}$ be a moving boundary such that there exists E_{min} such that

$$E_{min} \subset E_t, \quad \forall t \in I$$

. This implies that We will say that the process $(X_t)_{t\in I}$ satisfies the Assumption 5 if:

Assumption 5. *There exists a measurable subset $A_\infty \subset E$ and $x \in E_{min}$ such that*

$$\sup_{t\geq s}\left|\frac{\mathbb{P}_{x,s}(\tau_A > t)}{\mathbb{P}_{x,s}(\tau_{A_\infty} > t)} - 1\right| \underset{s\to\infty}{\longrightarrow} 0$$

where

$$\tau_{A_\infty} := \inf\{t \geq 0 : X_t \in A_\infty\}$$

Denote by E_∞ the complement of A_∞. In what follows, we will state the existence of a quasi-limiting distribution under these assumptions.

First we state the following proposition.

Proposition 27. *Assume Assumptions (F'). Then, for any B measurable subset, for any $s \in [0, t_2] \cap I$, the quantities*

$$\limsup_{t\to\infty} \mathbb{P}_{\mu,s+mt_2}(X_t \in B|\tau_A > t) \text{ and } \liminf_{t\to\infty} \mathbb{P}_{\mu,s+mt_2}(X_t \in B|\tau_A > t)$$

do not depend on any couple (m, μ) such that $\mu \in \mathcal{M}_1(E_{s+mt_2})$ and such that $\frac{\mu(\psi_{1,s+mt_2})}{\mu(\psi_{2,s+mt_2})} < \infty$.

Proof. Let B be a measurable subset of E and $s \in [0, t_2] \cap I$. Denoting $\mathcal{A}_{s+mt_2} := \{\mu \in \mathcal{M}_1(E_{s+mt_2}) : \mu(\psi_{1,s+mt_2})/\mu(\psi_{2,s+mt_2}) < \infty\}$, for any $\mu_1, \mu_2 \in \mathcal{A}_{s+mt_2}$,

$$\lim_{t\to\infty} ||\mathbb{P}_{\mu_1,s+mt_2}(X_t \in \cdot|\tau_A > t) - \mathbb{P}_{\mu_2,s+mt_2}(X_t \in \cdot|\tau_A > t)||_{TV} = 0.$$

This implies that, for any $m \in \mathbb{Z}_+$ and $\mu_1, \mu_2 \in \mathcal{A}_{s+mt_2}$,

$$\limsup_{t\to\infty} \mathbb{P}_{\mu_1,s+mt_2}(X_t \in B|\tau_A > t) = \limsup_{t\to\infty} \mathbb{P}_{\mu_2,s+mt_2}(X_t \in B|\tau_A > t) \tag{4.45}$$

Now, by Markov property, for any $m \leq n \in \mathbb{Z}_+$ and $\mu \in \mathcal{A}_{s+mt_2}$,

$$\limsup_{t\to\infty} \mathbb{P}_{\mu,s+mt_2}(X_t \in B|\tau_A > t) = \limsup_{t\to\infty} \mathbb{P}_{\mu\phi_{s+mt_2,s+nt_2},s+nt_2}(X_t \in B|\tau_A > t) \tag{4.46}$$

However, by Lemma 7, there exists a constant $C > 0$ such that

$$\frac{\mu\phi_{s+mt_2,s+nt_2}\psi_{1,s+nt_2}}{\mu\phi_{s+mt_2,s+nt_2}\psi_{2,s+nt_2}} \leq C\frac{\mu(\psi_{1,s+mt_2})}{\mu(\psi_{2,s+mt_2})}$$

In other words, $\mu \in \mathcal{A}_{s+mt_2}$ implies that $\mu\phi_{s+mt_2,s+nt_2} \in \mathcal{A}_{s+nt_2}$. As a result, using the equality (4.46), then using the equality (4.45) taking $\mu_1 = \mu\phi_{s+mt_2,s+nt_2}$ and $\mu_2 = \xi$ where $\xi \in \mathcal{A}_{s+nt_2}$, we deduce that for any $m \leq n \in \mathbb{Z}_+$, $\mu \in \mathcal{A}_{s+mt_2}$ and $\xi \in \mathcal{A}_{s+nt_2}$,

$$\limsup_{t\to\infty} \mathbb{P}_{\mu,s+mt_2}(X_t \in B|\tau_A > t) = \limsup_{t\to\infty} \mathbb{P}_{\xi,s+nt_2}(X_t \in B|\tau_A > t),$$

which conclude the proof for the lim sup. We deduce the result applied to the lim inf in the same way.

□

Before tackling the quasi-limiting distribution, let us state the following proposition providing a uniform-in-time convergence of the time-inhomogeneous conditioned semi-group towards the time-homogeneous limit semi-group.

Proposition 28. *Assume that the process $(X_t)_{t\in I}$ satisfies Assumption 5 for a given $x \in E_{min}$. Then*

$$\lim_{s\to\infty} \sup_{s\leq t\leq T} ||\mathbb{P}_{x,s}(X_t \in \cdot|\tau_A > T) - \mathbb{P}_{x,s}(X_t \in \cdot|\tau_{A_\infty} > T)||_{TV} = 0 \tag{4.47}$$

Proof. Let $s \in I$. Moreover, let us introduce the following notation :

- $Z_{s,t} = \mathbb{1}_{\tau_A>t}$
- $\tilde{Z}_{s,t} = \mathbb{1}_{\tau_{A_\infty}>t}$
- $\mu K^T_{s,t} := \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_A > T)$
- $\mu \tilde{K}^T_{s,t} := \mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_{A_\infty} > T)$

Then for any $s \leq t \leq T$,

$$\begin{aligned}
&\left|\left|\delta_x K^T_{s,t} - \delta_x \tilde{K}^T_{s,t}\right|\right|_{TV} \\
&= \left|\left|\frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} Z_{s,T})}{\mathbb{E}_{x,s}(Z_{s,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} \tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}\right|\right|_{TV} = \left|\left|\frac{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(Z_{s,T})}\frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} Z_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} \tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}\right|\right|_{TV} \\
&\leq \left|\left|\frac{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(Z_{s,T})}\frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} Z_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} Z_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}\right|\right|_{TV} + \left|\left|\frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} Z_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})} - \frac{\mathbb{E}_{x,s}(\mathbb{1}_{X_t\in\cdot} \tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}\right|\right|_{TV} \\
&\leq \left|\frac{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(Z_{s,T})} - 1\right| \frac{\mathbb{E}_{x,s}(Z_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})} + \left|\frac{\mathbb{E}_{x,s}(Z_{s,T}) - \mathbb{E}_{x,s}(\tilde{Z}_{s,T})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,T})}\right| \\
&\leq (2 + f(x,s)) \left|\frac{\mathbb{E}_{x,s}(Z_{s,t})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,t})} - 1\right| \leq [2 + f(x,s)] f(x,s)
\end{aligned}$$

where

$$f(x,s) := \sup_{t\geq s} \left|\frac{\mathbb{E}_{x,s}(Z_{s,t})}{\mathbb{E}_{x,s}(\tilde{Z}_{s,t})} - 1\right| = \sup_{t\geq s} \left|\frac{\mathbb{P}_{x,s}(\tau_A > t)}{\mathbb{P}_{x,s}(\tau_{A_\infty} > t)} - 1\right|.$$

By this last inequality, one note that Assumption 5 implies (4.47). □

Now, the existence of a quasi-limiting distribution for the process $(X_t)_{t\in I}$ absorbed by the moving boundary $(A_t)_{t\in I}$ is stated and proved.

Theorem 23. *Assume that the process $(X_t)_{t\in I}$ satisfies Assumptions (F') and Assumption 5. Assume moreover that the process $(X_t)_{t\geq 0}$ absorbed at A_∞ satisfies the homogeneous assumptions (F) introduced in [15] and let $\psi_{1,\infty}, \psi_{2,\infty}$ be the two Lyapunov functions of these assumptions.*

Then, for any $s \in I$ and $\mu \in \mathcal{M}_1(E_s)$ satisfying $\mu(\psi_{1,s})/\mu(\psi_{2,s}) < \infty$,

$$\mathbb{P}_{\mu,s}(X_t \in \cdot|\tau_A > t) \xrightarrow[t\to\infty]{\mathcal{L}} \alpha_\infty,$$

where α_∞ is the unique quasi-stationary distribution of $(X_t)_{t\in I}$ absorbed at A_∞ such that $\alpha_\infty(\psi_{1,\infty})/\alpha_\infty(\psi_{2,\infty}) < \infty$.

Proof. Fix a measurable subset $B \subset E$ and let α_∞ as defined in the statement of the theorem. Then, by Theorem 3.5. in [15], for any $\mu \in \mathcal{M}_1(E_\infty)$ satisfying $\mu(\psi_{1,\infty})/\mu(\psi_{2,\infty} < \infty$,

$$\limsup_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_{A_\infty} > t) = \liminf_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_{A_\infty} > t) = \alpha_\infty(B).$$

By Proposition 27, for a given $s \in I$, $\limsup_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t)$ and $\liminf_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t)$ do not depend on $\mu \in \mathcal{M}_1(E_s)$ satisfying $\mu(\psi_{1,s})/\mu(\psi_{2,s}) < \infty$. Denote therefore by F_{sup} and F_{inf} the functions defined by, for any $s \geq 0$ and any $\mu \in \mathcal{M}_1(E_s)$ satisfying $\mu(\psi_{1,s})/\mu(\psi_{2,s}) < \infty$,

$$F_{sup}(s) := \limsup_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t) = \limsup_{t\to\infty} \mathbb{P}_{x,s}(X_t \in B|\tau_A > t)$$

$$F_{inf}(s) := \liminf_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t) = \liminf_{t\to\infty} \mathbb{P}_{x,s}(X_t \in B|\tau_A > t)$$

where x is an element of E_{min} satisfying the statement of Assumption 5. Then, by Proposition 27, the functions F_{sup} and F_{inf} are t_2-periodic. As a result, for any $s \geq 0$,

$$F_{sup}(s) = \lim_{n\to\infty} F_{sup}(s + nt_2)$$

$$F_{inf}(s) = \lim_{n\to\infty} F_{inf}(s + nt_2)$$

Moreover, by the uniform convergence (4.47) of Proposition 28,

$$\begin{aligned}
\lim_{n\to\infty} F_{sup}(s + nt_2) &= \lim_{n\to\infty} \limsup_{t\to\infty} \mathbb{P}_{x,s+nt_2}(X_{s+nt_2+t} \in B|\tau_A > s + nt_2 + t) \\
&= \limsup_{t\to\infty} \lim_{n\to\infty} \mathbb{P}_{x,s+nt_2}(X_{s+nt_2+t} \in B|\tau_A > s + nt_2 + t) \\
&= \limsup_{t\to\infty} \mathbb{P}_{x,0}(X_t \in B|\tau_{A_\infty} > t) \\
&= \alpha_\infty(B).
\end{aligned}$$

Similarly,

$$\lim_{n\to\infty} F_{inf}(s + nt_2) = \alpha_\infty(B)$$

Hence, for any $s \geq 0$ and $\mu \in \mathcal{M}_1(E_s)$ satisfying satisfying $\mu(\psi_{1,s})/\mu(\psi_{2,s}) < \infty$,

$$\limsup_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t) = \liminf_{t\to\infty} \mathbb{P}_{\mu,s}(X_t \in B|\tau_A > t) = \alpha_\infty(B).$$

□

4.5.2 Existence of a quasi-ergodic distribution when the boundaries move periodically

In this subsection, we will work on periodic moving boundaries and we will assume that the Markov process $(X_t)_{t\geq 0}$ satisfies the Assumption (F').

Theorem 24. *Assume A is γ-periodic with $\gamma > 0$, and assume that $\gamma = t_2$. Then for any $x \in E_0$,*

$$\frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t)ds \xrightarrow[t\to\infty]{(d)} \frac{1}{\gamma}\int_0^\gamma \mathbb{Q}_{0,\beta_\gamma}(X_s \in \cdot)ds$$

where β_γ is the invariant measure of $(X_{n\gamma})_{n\in\mathbb{N}}$ under $\mathbb{Q}_{0,\cdot}$, i.e.

$$\forall n \in \mathbb{N}, \quad \beta_\gamma = \mathbb{Q}_{0,\beta_\gamma}(X_{n\gamma} \in \cdot)$$

Proof. Now we want to show an ergodic theorem for the time-inhomogeneous Markov process $(X_t)_{t\geq 0}$ under $(\mathbb{Q}_{x,s})_{s\geq 0, x\in E_s}$. Since $(A_t)_{t\geq 0}$ is γ-periodic, for any $0 \leq s \leq t$, for any $x \in E_s$,

$$\mathbb{Q}_{x,s+k\gamma}(X_{t+k\gamma} \in \cdot) = \mathbb{Q}_{x,s}(X_t \in \cdot), \quad \forall k \in \mathbb{Z}_+ \tag{4.48}$$

Moreover, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathbb{Q}_{x,0}(X_{n\gamma} \in \cdot) &= \lim_{t\to\infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > t) \\
&= \lim_{m\in\mathbb{Z}_+, m\to\infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > m\gamma) \\
&= \lim_{m\in\mathbb{Z}_+, m\to\infty} \mathbb{P}_x(Y_n \in \cdot | \tau_\partial > m)
\end{aligned}$$

where τ_∂ is defined by

$$\tau_\partial = \begin{cases} \inf\{n \geq 1 : \exists t \in ((n-1)\gamma, n\gamma], X_t \in A_t\} & \text{if } Y_0 \in E_0 \\ 0 & \text{if } Y_0 \in A_0 \end{cases}$$

and $(Y_n)_{n\in\mathbb{Z}_+}$ is the time-homogeneous Markov chain defined by

$$Y_n = \begin{cases} X_{n\gamma} \text{ for } n < \tau_\partial \\ \partial \quad \text{otherwise} \end{cases}$$

where ∂ plays the role of an absorbing state for $(Y_n)_{n\in\mathbb{Z}_+}$. In other words, τ_∂ is an absorbing time for $(Y_n)_{n\in\mathbb{Z}_+}$ and, under $(\mathbb{Q}_{x,0})_{x\in E_0}$, the chain $(X_{n\gamma})_{n\in\mathbb{Z}_+}$ is the Q-process of $(Y_n)_{n\in\mathbb{Z}_+}$. Since Assumption (F') holds and $\gamma = t_2$, this implies that the chain $(X_{s+nt_2})_{n\in\mathbb{Z}_+}$ satisfies the Assumption (E). Hence, by Theorem 2.1 in [15], there exists $\beta_\gamma \in \mathcal{M}_1(E_0)$, $C > 0$ and $\rho \in (0,1)$ such that for any $n \in \mathbb{Z}_+$,

$$||\mathbb{Q}_{\mu,0}(X_{n\gamma} \in \cdot) - \beta_\gamma||_{TV} \leq C\rho^n \frac{\mu(\psi_{1,\gamma})}{\mu(\psi_{2,\gamma})}, \quad \forall \mu \in \mathcal{M}_1(E_0)$$

This implies that, under $\mathbb{Q}_{\cdot,0}$, $(X_{n\gamma})_{n\in\mathbb{N}}$ is Harris recurrent. We can therefore apply Theorem 2.1 in [29] and deduce that, for any nonnegative function f,

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}_{\beta_\gamma,0}\left(\frac{1}{\gamma}\int_0^\gamma f(X_s)ds\right), \quad \mathbb{Q}_{x,0}\text{-almost surely}, \ \forall x \in E_0$$

where $\mathbb{E}^{\mathbb{Q}}_{0,\mu}(G) = \int G d\mathbb{Q}_{0,\mu}$ for any measurable nonnegative function G and $\mu \in \mathcal{M}_1(E_0)$. It extends to $f \in \mathcal{B}(E)$ using $f = f_+ - f_-$ with f_+, f_- non negative functions. Thus, by bounded Lebesgue's convergence theorem, for any $x \in E_0$ and for any $f \in \mathcal{B}(E)$,

$$\frac{1}{t}\int_0^t \mathbb{E}^{\mathbb{Q}}_{x,0}(f(X_s))ds \underset{t\to\infty}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}_{\beta_\gamma,0}\left(\frac{1}{\gamma}\int_0^\gamma f(X_s)ds\right)$$

□

4.6 Example : Ornstein-Uhlenbeck process absorbed by a stabilizing boundary

In this section, $(X_t)_{t\geq 0}$ is a one-dimensional Ornstein-Uhlenbeck process whose the SDE is

$$dX_t = dB_t - \lambda X_t dt \tag{4.49}$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\lambda > 0$. Denote by h a positive decreasing function converging towards 0.

Since the process $(X_t)_{t\geq 0}$ is time-homogeneous, let $(\mathbb{P}_\mu)_{\mu\in\mathcal{M}_1(\mathbb{R})}$ a family of probability measure satisfying $\mathbb{P}_\mu(X_0 \in \cdot) = \mu$. Moreover, for any $s \geq 0$, define θ_s the shift operator, that is the operator such that

$$\theta_s : f \to f(s + \cdot), \quad \forall f \text{ function.}$$

and define also

$$\tau_{h\circ\theta_s} := \inf\{t \geq 0 : X_t \leq h(t + s)\}.$$

Moreover, define, for any $a \in \mathbb{R}$,

$$\tau_a := \inf\{t \geq 0 : X_t = a\}.$$

Our aim is therefore to study the quasi-stationarity of this process conditioned not to hit h. As suggested by Theorem 23, a quasi-limiting distribution is expected due to the convergence of h. However, in the non-moving case, it is well known (see [39]) that the Ornstein-Uhlenbeck process admits an infinity of quasi-limiting distribution, thus an infinity of possible behavior according to the initial measure, which makes the generalization to the moving boundaries difficult.

First, let us focus on the satisfaction of the Assumptions (F') by the process $(X_t)_{t\geq 0}$.

4.6.1 Assumptions (F') are satisfied

The strong Markov property (F'0) is naturally satisfied for the Ornstein-Uhlenbeck process. We will therefore focus on the other conditions.

Before proving that the Ornstein-Uhlenbeck process satisfies the conditions (F'), let us introduce some notation : first of all, let us define $(Q_{s,t})_{s\leq t}$ the penalized semi-group of the time-inhomogeneous Markov process $(X_t - h(t))_{t\geq 0}$ absorbed by 0. More exactly,

$(Q_{s,t})_{s\leq t}$ is defined as follows : for any $s \leq t$, $x \in (0,\infty)$ and f bounded measurable defined on $(0,\infty)$,

$$\delta_x Q_{s,t} f := \mathbb{E}_{x+h(s)} \left[f\left(X_{t-s} - h(t)\right) \mathbb{1}_{\tau_{h\circ\theta_s} > t-s} \right]. \tag{4.50}$$

Moreover, for any $k \in \mathbb{N}$, define $K_k = [1/k, k]$.

Step 1 : Proof of (F'1)

Let us focus on the condition (F'1). To prove this condition, the Theorem 1.1 of Krylov and Safonov [37] will be useful. Fix $t > 10$ and f a bounded positive measurable function, and define, for any $s \in [0,t]$ and $x \in (0,\infty)$,

$$u(s,x) := \delta_x Q_{t-s,t} f$$

Then, for any $s \leq t$ and $x \in (0,\infty)$, one has,

$$\partial_s u = \mathcal{L}_s u$$

where, for any $s \geq 0$ and $g \in C^2$,

$$\mathcal{L}_s g : x \mapsto \frac{1}{2} g''(x) - [\lambda(x + h(s)) + h'(s)] g'(x) \tag{4.51}$$

is the generator of the time-inhomogeneous semi-group $Q_{s,t}$. As a result, the Theorem 1.1 in [37] is applicable to the function u and, reproducing the formula (4.10) in [16], for any $\delta \in (0,2)$ and $k \in \mathbb{N}$, there exists a constant N_k not depending on f and t such that

$$u(\delta + \delta^2, x) \leq N_k u(\delta + 2\delta^2, y), \qquad \forall x, y \in K_k, \quad |x - y| \leq \delta \tag{4.52}$$

Now let $s \geq 0$ and $k > 1$. Fix $x_k \in (k, k+1)$. Let $\nu_t^{(k)}$ denote the conditional law $\delta_{x_k} Q_{t-\delta-\delta^2,t} / \delta_{x_k} Q_{t-\delta-\delta^2,t} \mathbb{1}_E$, where $E := (0,\infty)$. Then, for all measurable $A \subset [0,\infty)$, Harnack's inequality (4.52) with $f = \mathbb{1}_A$ entails that, for all $x \in (0,\infty)$ such that $|x - x_k| < \delta \wedge d(x_k, E \setminus K_{k+1})$,

$$\delta_x Q_{t-\delta-2\delta^2,t} \mathbb{1}_A \geq \frac{\mathbb{P}_{x_k}(\delta + \delta^2 < \tau_{h\circ\theta_{t-\delta-\delta^2}})}{N_{k+1}} \nu_t^{(k)}(A).$$

Denoting $B := B(x_k, \delta \wedge d(x_k, E \setminus K_{k+1}))$, since the diffusion is locally elliptic and E is connected, then, for any $t_1 \geq \delta + 2\delta^2$,

$$\inf_{x \in K_k} \delta_x Q_{s,s+t_1-\delta-2\delta^2} \mathbb{1}_B =: d_{s,k} > 0.$$

This and Markov's property entail that, for all $x \in K_k$,

$$\delta_x Q_{s,s+t_1} \mathbb{1}_A \geq d_{s,k} \frac{\mathbb{P}_{x_k}(\delta + \delta^2 < \tau_{h\circ\theta_{s+t_1-\delta-\delta^2}})}{N_{k+1}} \nu_{s+t_1}^{(k)}(A).$$

Since, for any $(x,y) \in K_k \times B$, $x < y$, then one has,

$$d_{s,k} = \delta_{1/k} Q_{s,s+t_1-\delta-2\delta^2} \mathbb{1}_B,$$

and, since the function h is decreasing, one has, for any $s \geq 0$ and $k > 1$,

$$d_{s,k} \geq \delta_{1/k} Q_{0,t_1-\delta-2\delta^2} \mathbb{1}_B =: d_k.$$

Moreover, since $\tau_{h\circ\theta_{s+t_1-\delta-\delta^2}} \geq \tau_{h\circ\theta_{t_1-\delta-\delta^2}}$ $\mathbb{P}_{x_k}$-almost surely,

$$\mathbb{P}_{x_k}(\delta+\delta^2 < \tau_{h\circ\theta_{s+t_1-\delta-\delta^2}}) \geq \mathbb{P}_{x_k}(\delta+\delta^2 < \tau_{h\circ\theta_{t_1-\delta-\delta^2}})$$

To sum up, for any $k \in \mathbb{N}$, there exists a constant $c_{1,k} > 0$ such that for any $x \in K_k$ and for any $s \geq 0$,

$$\delta_x Q_{s,s+t_1} \mathbb{1}_A \geq c_{1,k} \nu^{(k)}_{s+t_1}(A).$$

Step 2 : Proof of (F'3)

For any $x \in E$ and $u \geq 0$, define $Y^{x,u}$ the Markov process such that $Y^{x,u}_0 = x$ $\mathbb{P}$-almost surely and, for any $t \geq 0$,

$$Y^{x,u}_t := X_t - h(u+t).$$

Let $r \leq s \leq t$. Since h is decreasing, for any $x \in (0,\infty)$, the process $Y^{x,s}$ stochastically dominates the process $Y^{x,r}$.

This stochastic dominance implies therefore that the hitting time of 0 by the process $Y^{x,s}$, that we denote $\tau^{x,s}$, stochastically dominates the one of the process $Y^{x,r}$, that we denote $\tau^{x,r}$. However, the law of $\tau^{x,r}$ (respectively $\tau^{x,s}$) is the law of $\tau_{h\circ\theta_r}$ (respectively $\tau_{h\circ\theta_s}$) under $\mathbb{P}_{x+h(r)}$ (respectively $\mathbb{P}_{x+h(s)}$). In other words, one has

$$\mathbb{P}_{x+h(r)}(\tau_{h\circ\theta_r} > u) \leq \mathbb{P}_{x+h(s)}(\tau_{h\circ\theta_s} > u), \qquad \forall x \in (0,\infty), u \geq 0$$

Thus, using this last inequality and that $u \mapsto \mathbb{P}_{x+h(s)}(\tau_{h\circ\theta_s} > u)$ is non-increasing, for any $x \in E$,

$$\begin{aligned}
\delta_x Q_{r,t} \mathbb{1}_E &= \mathbb{P}_{x+h(r)}(\tau_{h\circ\theta_r} > t-r) \\
&\leq \mathbb{P}_{x+h(s)}(\tau_{h\circ\theta_s} > t-r) \\
&\leq \mathbb{P}_{x+h(s)}(\tau_{h\circ\theta_s} > t-s) \\
&= \delta_x Q_{s,t} \mathbb{1}_E. \qquad (4.53)
\end{aligned}$$

Now, let us remark that, for any $s \leq t$,

$$\inf_{x\in K_k} \delta_x Q_{s,t} \mathbb{1}_E = \delta_{1/k} Q_{s,t} \mathbb{1}_E$$

and

$$\sup_{x\in K_k} \delta_x Q_{s,t} \mathbb{1}_E = \delta_k Q_{s,t} \mathbb{1}_E.$$

Denoting $a := 1/k$, $b := k$ and $\tau_b^{h\circ\theta_s} := \inf\{t \geq 0 : X_t - h(s+t) = b\}$, for any $s \leq t$,

$$\begin{aligned}\mathbb{P}_{a+h(s)}(\tau_{h\circ\theta_s} > t) &\geq \mathbb{P}_{a+h(s)}(\tau_b^{h\circ\theta_s} \leq t < \tau_{h\circ\theta_s}) \\ &= \mathbb{E}_{a+h(s)}(\mathbb{1}_{\tau_b^{h\circ\theta_s}\leq t} f(\tau_b^{h\circ\theta_s}))\end{aligned}$$

where

$$f : u \mapsto \mathbb{P}_{b+h(u)}(\tau_{h\circ\theta_u} > t-u) = \delta_b Q_{u,t}\mathbb{1}_E$$

Using again (4.53), for any $u \in [s,t]$,

$$f(u) \geq f(s).$$

Thus, for any $s \leq t$,

$$\mathbb{P}_{a+h(s)}(\tau_{h\circ\theta_s} > t) \geq \mathbb{P}_{a+h(s)}(\tau_b^{h\circ\theta_s} \leq t)\mathbb{P}_{b+h(s)}(\tau_{h\circ\theta_s} > t).$$

As a result, for any $s \leq t$,

$$\mathbb{P}_{a+h(s)}(\tau_{h\circ\theta_s} > t) \geq C_{3,s,k}\mathbb{P}_{b+h(s)}(\tau_{h\circ\theta_s} > t)$$

where

$$C_{3,s,k} := \inf_{u\in[0,1]} \frac{\mathbb{P}_{a+h(s)}(\tau_{h\circ\theta_s} > u)}{\mathbb{P}_{b+h(s)}(\tau_{h\circ\theta_s} > u)} \wedge \mathbb{P}_{a+h(s)}(\tau_b^{h\circ\theta_s} \leq 1)$$

By comparison, for any $s \geq 0$,

$$C_{3,s,k} \geq C_{3,k} := \mathbb{P}_a(\tau_0 > 1) \wedge \mathbb{P}_{a+h(0)}(\tau_b \leq 1) > 0.$$

In conclusion, denoting $c_{3,k} := 1/C_{3,k}$ for a given $k \in \mathbb{N}$, for any $r \leq s \leq t$, for any $x, y \in K_k$,

$$\delta_x Q_{r,t}\mathbb{1}_E \leq \delta_x Q_{s,t}\mathbb{1}_E \leq c_{3,k}\delta_y Q_{s,t}\mathbb{1}_E.$$

Step 3 : Proof of (F'2)

For any $k \in \mathbb{N}$, denote the moving subset $\tilde{K}_k : s \mapsto h(s) + K_k$ and, for any $s \geq 0$,

$$\tau_{\tilde{K}_k\circ\theta_s} := \inf\{t \geq 0 : X_t \in \tilde{K}_k(s+t)\}.$$

Then, by continuity of $x, k \mapsto \mathbb{P}_x(s_1 < \tau_{\tilde{K}_k\circ\theta_s} \wedge \tau_{h\circ\theta_s})$, one has, for any fixed $s_1 \geq 0$,

$$\sup_{x\in D, s\geq 0} \mathbb{P}_{x+h(s)}(s_1 < \tau_{\tilde{K}_k\circ\theta_s} \wedge \tau_{h\circ\theta_s}) \xrightarrow[k\to\infty]{} 0, \tag{4.54}$$

Hence, set $L = K_{k_0}$ with k_0 large enough so that $\nu_s^{(k_0)}(K_{k_0}) > 0$ for any $s \geq 0$ and such that, for any $s \geq 0$ and $x \in D$,

$$\mathbb{P}_{x+h(s)}(s_1 < \tau_{\tilde{K}_k\circ\theta_s} \wedge \tau_{h\circ\theta_s}) \leq e^{-(\rho_1' + C/p)t_2},$$

and define also $L_s := h(s) + L$. Define

$$\lambda_0 := \inf\left\{\lambda > 0 : \liminf_{t\to+\infty} e^{\lambda t} \inf_{s\geq 0} \mathbb{P}_{x_0+h(s)}(X_t \in L_{t+s}, \tau_{h\circ\theta_s} > t) > 0\right\}. \tag{4.55}$$

for a given $x_0 \in L$. For the following, we need to ensure that $\lambda_0 < \infty$. This property is stated in the following lemma :

Lemma 15.

$$\lambda_0 < \infty.$$

This lemma will be proved at the end of the subsection.

Let $\mu > 0$ and define

$$\varphi : x \in (0, \infty) \mapsto \exp(\mu x).$$

Then, recalling that $\mathcal{L}_s$ is the generator of the semi-group $(Q_{s,t})_{s,t}$ (defined in (4.51)), for any $s \geq 0$ and $x \in (0, \infty)$,

$$\begin{aligned}\mathcal{L}_s\varphi(x) &= \frac{\mu^2}{2}\varphi(x) - (\lambda(x + h(s)) + h'(s))\mu\varphi(x) \\ &\leq \left(\frac{\mu}{2} - (\lambda x + \min_{s\geq 0} h'(s))\right)\mu\varphi(x)\end{aligned}$$

Thus, for any $\lambda_1 > \lambda_0$, one has, for any $s \geq 0$ and $x \in (0, \infty)$,

$$\mathcal{L}_s\varphi(x) \leq -\lambda_1\varphi(x) + C\mathbb{1}_D(x)$$

with

$$D := \left(0, \frac{1}{\lambda}\left(\frac{\mu}{2} + \frac{\lambda_1}{\mu} - \min_{s\geq 0} h'(s)\right)\right]$$

and $C := \left(\frac{\mu}{2} - \min_{s\geq 0} h'(s)\right)\mu < \infty$.

Fix now $\rho_1 \in (\lambda_0, \lambda_1)$, $\rho_2 \in (\lambda_0, \rho_1)$ and $p \in (1, \lambda_1/\rho_1)$ and define

$$\psi_{1,s}(x) = \varphi_s(x)^{1/p}, \ \forall s \geq 0, x \in (h(s), \infty).$$

where $\varphi_s(x) := \varphi(x - h(s))$. Fix $\rho_1' \in (\rho_1, \lambda_1/p)$ and

$$t_2 \geq \frac{2s_1(C + \lambda_1)}{\lambda_1 - p\rho_1'} \vee \frac{\log 2}{\rho_1' - \rho_1},$$

Recalling the Harnack inequality (4.52), for any $t \geq 10$, for any bounded measurable function f, and for any $x, y \in L$ such that, $|x - y| < \delta$,

$$\delta_x Q_{t-\delta-\delta^2,t} f \leq N\delta_y Q_{t-\delta-2\delta^2,t} f.$$

Using again this Harnack inequality applied to $f = Q_{t-\delta^2,t} f$ and $t = t-\delta^2$, for any $t \geq 14$, for any bounded measurable function f, and for any $x, y \in L$ such that, $|x - y| < \delta$,

$$\delta_x Q_{t-\delta-2\delta^2,t} f = \delta_x Q_{t-\delta-2\delta^2,t-\delta^2} Q_{t-\delta^2,t} f \leq N\delta_y Q_{t-\delta-3\delta^2,t} f.$$

This implies therefore, using this last inequality and (4.52), that for any $t \geq 14$, f bounded measurable and $x, y \in L$ such that $|x - y| < 2\delta$,

$$\delta_x Q_{t-\delta-\delta^2,t} f \leq N^2\delta_y Q_{t-\delta-3\delta^2,t} f.$$

Hence, by induction, and since the diameter of L is finite, we deduce that there exists $n_L \in \mathbb{N}$ such that, for any f bounded measurable, for any $t \geq 2 + 4n_L$ for any $x, y \in L$,

$$\delta_x Q_{t-\delta-\delta^2,t} f \leq N^{n_L-1} \delta_y Q_{t-\delta-n_L\delta^2,t} f. \tag{4.56}$$

Now, our aim is to show that, for any $s \geq 0$,

$$\liminf_{t\to\infty} e^{\rho_2 t} \inf_{x\in L, s\geq 0} \delta_x Q_{s,s+t} \mathbb{1}_L = \infty$$

To do this, we use the inequality (4.56) applied to $f = \mathbb{1}_L$ and $s = t - \delta - n_L\delta^2$: for any $s \geq 0$ and $t \geq 2 + 4n_L$,

$$\inf_{x\in L} \delta_x Q_{s,t} \mathbb{1}_L \geq \frac{1}{N^{n_L-1}} \delta_{x_0} Q_{s+(n_L-1)\delta^2,t} \mathbb{1}_L$$

where x_0 is the same as in (4.55). From the definition of λ_0, we deduce that

$$\liminf_{t\to+\infty} e^{\rho_2 t} \inf_{s\geq 0,\ x\in L_s} \mathbb{P}_x(X_t \in L_{s+t}, \tau_{h\circ\theta_s} > t) = +\infty,$$

and hence the last line of (F'2) is proved with $\gamma_2 = e^{-\rho_2}$.

By Kolmogorov's equation and using that $\mathcal{L}_t\varphi(x) \leq C\varphi(x)$ for any t and x,

$$\partial_t Q_{s,t}\varphi(x) = Q_{s,t}\mathcal{L}_t\varphi(x) \leq C Q_{s,t}\varphi(x).$$

So, by Gronwall's lemma, for any $s \leq t$,

$$Q_{s,t}\varphi(x) \leq e^{C(t-s)}\varphi(x).$$

This last inequality can be written with φ_s as follows : for any $x > h(s)$, $s \geq 0$ and $t \geq 0$,

$$\mathbb{E}_x(\varphi_{t+s}(X_t)\mathbb{1}_{\tau_{h\circ\theta_s}>t}) \leq e^{Ct}\varphi_s(x). \tag{4.57}$$

For all $x \in D_s$, we have $\psi_{1,s}(x) \leq \sup_{x\in D_s} \varphi_s^{1/p}(x) < +\infty$. Hence, defining $\tau_{L\circ\theta_s} := \inf\{t \geq 0 : X_t \in L_{s+t}\}$, for all $t \in [s_1, t_2]$, using Hölder's inequality and (4.57),

$$\begin{aligned}
\mathbb{E}_x\left(\psi_{1,t+s}(X_t)\mathbb{1}_{t<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}}\right) &\leq \mathbb{E}_x\left(\mathbb{1}_{t<\tau_{h\circ\theta_s}}\varphi_{t+s}(X_t)\right)^{1/p} \mathbb{P}_x(t < \tau_{L\circ\theta_s} \wedge \tau_{h\circ\theta_s})^{\frac{p-1}{p}} \\
&\leq \varphi_s(x)^{1/p} e^{Ct_2/p} \mathbb{P}_x(s_1 < \tau_{L\circ\theta_s} \wedge \tau_{h\circ\theta_s})^{\frac{p-1}{p}} \\
&\leq e^{-\rho_1' t_2} \leq e^{-\rho_1 t_2}\psi_{1,s}(x).
\end{aligned} \tag{4.58}$$

This proves the first line of (F'2) for all $x \in D$ and $\gamma_1 = e^{-\rho_1}$.

Now, for all $x \in (h(s), \infty) \setminus [h(s) + D]$, since D is closed in $(0, \infty)$, it follows from the strong Markov property at time τ_D that

$$\begin{aligned}
&\mathbb{E}_x\left(\psi_{1,s+t_2}(X_{t_2})\mathbb{1}_{t_2<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}}\right) \\
&\quad= \mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}\wedge\tau_D}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_{1,s+t_2}(X_{s_1})\mathbb{1}_{s_1<\tau_{L\circ\theta_{s+t_2-s_1}}\wedge\tau_{h\circ\theta_{s+t_2-s_1}}}\right)\right) \\
&\qquad+ \mathbb{E}_x\left(\mathbb{1}_{\tau_D\leq t_2-s_1}\mathbb{E}_{X_{\tau_D}}\left(\psi_{1,s+t_2}(X_{t_2-u})\mathbb{1}_{t_2-u<\tau_{h\circ\theta_{s+u}}\wedge\tau_{L\circ\theta_{s+u}}}\right)|_{u=\tau_D}\right).
\end{aligned} \tag{4.59}$$

Using Hölder's inequality and (4.57), we deduce that, for all $y \in D$,

$$\begin{aligned}\mathbb{E}_{y+h(s)}\left(\psi_{1,s+t_2}(X_{s_1})\mathbb{1}_{s_1<\tau_{L\circ\theta_{s+t_2-s_1}}\wedge\tau_{h\circ\theta_{s+t_2-s_1}}}\right) &\leq \mathbb{E}_{y+h(s)}\left(\varphi_{s+t_2}(X_{s_1})\mathbb{1}_{s_1<\tau_{h\circ\theta_{s+t_2-s_1}}}\right)^{1/p}\\ &\leq e^{\frac{s_1C}{p}}\,\varphi_{s+t_2-s_1}(y)^{1/p}\\ &= e^{\frac{s_1C}{p}}\,\psi_{1,s+t_2-s_1}(y).\end{aligned}$$

Hence, the first term in the right-hand side of (4.59) satisfies

$$\begin{aligned}\mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}\wedge\tau_{D_0}}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_{1,s+t_2}(X_{s_1})\mathbb{1}_{s_1<\tau_{L\circ\theta_{s+t_2-s_1}}\wedge\tau_{h\circ\theta_{s+t_2-s_1}}}\right)\right)&\\ \leq e^{\frac{s_1C}{p}}\mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}\wedge\tau_D}\psi_{1,s+t_2-s_1}(X_{t_2-s_1})\right).&\end{aligned}$$

As a consequence, using again Hölder's inequality and applying as above the Kolmogorov's equation and the Gronwall's lemma using that $\mathcal{L}_s\varphi(x) \leq -\lambda_1\varphi(x)$ for all $x \notin D$, one has

$$\begin{aligned}\mathbb{E}_x\left(\mathbb{1}_{t_2-s_1<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}\wedge\tau_D}\mathbb{E}_{X_{t_2-s_1}}\left(\psi_{1,s+t_2}(X_{s_1})\mathbb{1}_{s_1<\tau_{L\circ\theta_{s+t_2-s_1}}\wedge\tau_{h\circ\theta_{s+t_2-s_1}}}\right)\right) &\leq e^{-\lambda_1\frac{t_2-s_1}{p}}e^{\frac{s_1C}{p}}\,\varphi_s(x)^{1/p}\\ &\leq e^{-t_2\frac{\rho_1'+\lambda_1/p}{2}}\psi_{1,s}(x),\end{aligned}$$

where we used in the last inequality that $t_2 \geq \frac{2s_1(C+\lambda_1)}{\lambda_1-p\rho_1'}$. Moreover, using (4.58), we obtain that the second term in the right-hand side of (4.59) satisfies

$$\begin{aligned}\mathbb{E}_x\left(\mathbb{1}_{\tau_D\leq t_2-s_1}\mathbb{E}_{X_{\tau_D}}\left(\psi_{1,s+t_2}(X_{t_2-u})\mathbb{1}_{t_2-u<\tau_{h\circ\theta_{s+u}}\wedge\tau_{L\circ\theta_{s+u}}}\right)|_{u=\tau_D}\right).&\\ \leq e^{-\rho_1't_2}\mathbb{P}_x(\tau_D \leq t_2-s_1) \leq e^{-\rho_1't_2}\psi_{1,s}(x).&\end{aligned}$$

We finally deduce from (4.59) and from the definition of $L = K_{k_0}$ that

$$\mathbb{E}_x\left(\psi_{1,s+t_2}(X_{t_2})\mathbb{1}_{t_2<\tau_{L\circ\theta_s}\wedge\tau_{h\circ\theta_s}}\right) \leq 2e^{-\rho_1't_2}\psi_{1,s}(x) \leq e^{-\rho_1t_2}\psi_{1,s}(x),$$

where we used that $t_2 \geq \log 2/(\rho_1' - \rho_1)$. This concludes the proof that the first line of (F'2) holds true.

Since φ is locally bounded, $\sup_L \varphi < \infty$, and hence, using again (4.57), we deduce that, for all $t \geq 0$,

$$\sup_{x\in L_s}\ \mathbb{E}_x(\psi_{1,s+t}(X_t)\mathbb{1}_{t<\tau_{h\circ\theta_s}}) \leq \sup_{x\in L_s}\ \mathbb{E}_x(\varphi_{s+t}(X_t)\mathbb{1}_{t<\tau_{h\circ\theta_s}}) \leq e^{Ct}\sup_{x\in L_s}\varphi_s(x) < \infty,$$

which implies the second line of Assumption (F'2).

Proof of Lemma 15. Let us remind that, for any $s, t \geq 0$,

$$\mathbb{P}_{x_0+h(s)}(X_t \in L_{s+t}, \tau_{h\circ\theta_s} > t) = \delta_{x_0}Q_{s,s+t}\mathbb{1}_L.$$

Define

$$f : t \to \inf_{s\geq 0}\delta_{x_0}Q_{s,s+t}\mathbb{1}_L.$$

Let $t \geq 0$ and $u \geq \delta + n_L \delta^2$, where n_L is defined in Step 3. Then, for any $s \geq 0$,

$$\begin{aligned}
\delta_{x_0} Q_{s,s+t+u} \mathbb{1}_L &= \delta_{x_0} Q_{s,s+t} Q_{s+t,s+t+u} \mathbb{1}_L \\
&\geq \mathbb{E}_{x_0+h(s)}(\mathbb{1}_{X_t \in L_{t+s}, \tau_{h\circ\theta_s} > t} Q_{s+t,s+t+u} \mathbb{1}_L(X_t)) \\
&\geq C \delta_{x_0} Q_{s,s+t} \mathbb{1}_L \times \delta_{x_0} Q_{s+t+\delta+n_L\delta^2,s+t+u} \mathbb{1}_L \\
&\geq C f(t) \times f(u - \delta - n_L \delta^2)
\end{aligned}$$

where we used the inequality (4.56), with $C = 1/N^{n_L - 1}$. Then, denoting $u_n := f(n(\delta + n_L\delta^2))$, we deduce that for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}
u_{n+1} &\geq C u_0 u_n \\
&\geq C^{n+1} u_0
\end{aligned}$$

since $u_0 = 1$. This last inequality implies that $\lambda_0 < \infty$. □

4.6.2 Existence of a quasi-limiting distribution

Now we will show that the Assumption 5 holds for the Ornstein-Uhlenbeck process with $A_\infty = (-\infty, 0]$. To do so, we will follow the same reasoning as the proof of Proposition 3 in [44] and conclude that Assumption 5 thanks to the Dini's theorem for decreasing functions.

Let $x > h(0)$. It is clear that, for any $t \geq 0$,

$$1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)}{\mathbb{P}_x(\tau_0 > t)} \underset{s\to\infty}{\longrightarrow} 0.$$

Moreover, since h is non-increasing, for any $s < s'$ and any $t \geq 0$,

$$1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)}{\mathbb{P}_x(\tau_0 > t)} \geq 1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_{s'}} > t)}{\mathbb{P}_x(\tau_0 > t)}.$$

Finally, in order to use Dini's theorem, it is necessary to extend the definition of each function $t \to 1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)}{\mathbb{P}_x(\tau_0 > t)}$ to the Alexandroff extension $\mathbb{R}_+ \cup \{\infty\}$. To do so, we have to check that, for any $s \geq 0$, the quantity $1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)}{\mathbb{P}_x(\tau_0 > t)}$ converges when t goes to infinity.

By the strong Markov property , for any $t \geq 0$ and $s \geq 0$, one has

$$\mathbb{P}_x(\tau_{h\circ\theta_s} \leq t < \tau_0) = \mathbb{E}_x(\mathbb{1}_{\tau_{h\circ\theta_s} \leq t} \phi(X_{\tau_{h\circ\theta_s}}, \tau_{h\circ\theta_s}, t))$$

where $\phi(\cdot,\cdot,\cdot)$ is defined as follows

$$\forall z \in (0,\infty), \forall 0 \leq u \leq t, \quad \phi(z,u,t) = \mathbb{P}_z(\tau_0 > t - u).$$

As a result, for any $s, t \geq 0$,

$$\begin{aligned}
1 - \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} > t)}{\mathbb{P}_x(\tau_0 > t)} &= \frac{\mathbb{P}_x(\tau_{h\circ\theta_s} \leq t < \tau_0)}{\mathbb{P}_x(\tau_0 > t)} \\
&= \mathbb{E}_x\left(\mathbb{1}_{\tau_{h\circ\theta_s} \leq t} \frac{\phi(X_{\tau_{h\circ\theta_s}}, \tau_{h\circ\theta_s}, t)}{\mathbb{P}_x(\tau_0 > t)}\right)
\end{aligned}$$

Since the Ornstein-Uhlenbeck process absorbed at 0 satisfies the Assumptions (F) of [15] for some Lyapunov functions ψ_1 and ψ_2, there exists a quasi-stationary distribution α_0 such that, for any probability measure μ,

$$||\mathbb{P}_\mu(X_t \in \cdot | \tau_0 > t) - \alpha_0||_{TV} \leq Ce^{-\gamma t}\frac{\mu(\psi_1)}{\mu(\psi_2)}$$

for some constant C and $\gamma > 0$. Then there exists (see [42, 19]) a constant $\lambda_0 > 0$ such that

$$\mathbb{P}_{\alpha_0}(\tau_0 > t) = e^{-\lambda_0 t}, \quad \forall t \geq 0$$

In [15] it is shown (Theorem 3.5.) that there exists a function η_0 defined on $(0, \infty)$ such that

$$\eta_0(x) = \lim_{t\to\infty} e^{\lambda_0 t}\mathbb{P}_x(\tau_0 > t), \quad \forall x \in (0, \infty) \tag{4.60}$$

where the convergence is exponential in $L^\infty(\psi_1)$, that is there exists $C', \gamma' > 0$ such that, for any $x \in (0, \infty)$,

$$|\eta_0(x) - e^{\lambda_0 t}\mathbb{P}_x(\tau_0 > t)| \leq C'e^{-\gamma' t}\psi_1(x).$$

In fact, we know that η_0 is a linear function since η_0 is the eigenfunction of the Ornstein-Uhlenbeck generator associated to the Perron-Frobenius eigenvalue $-\lambda_0$. In particular, for any $x \in (0, \infty)$, $\eta_0(x) \neq 0$.

Thus $\mathbb{1}_{\tau_h \circ \theta_s \leq t}\frac{\phi(X_{\tau_h\circ\theta_s}, \tau_h\circ\theta_s, t)}{\mathbb{P}_x(\tau_0 > t)}$ is uniformly bounded and converges almost surely towards $\mathbb{1}_{\tau_h\circ\theta_s<\infty}e^{\lambda_0\tau_h\circ\theta_s}\frac{\eta_0(X_{\tau_h\circ\theta_s})}{\eta_0(x)}$. Then, by the bounded Lebesgue's convergence theorem, for any $s \geq 0$,

$$\begin{aligned}\lim_{t\to\infty} 1 - \frac{\mathbb{P}_x(\tau_h\circ\theta_s > t)}{\mathbb{P}_x(\tau_0 > t)} &= \lim_{t\to\infty}\mathbb{E}_x\left(\mathbb{1}_{\tau_h\circ\theta_s\leq t}\frac{\phi(X_{\tau_h\circ\theta_s}, \tau_h\circ\theta_s, t)}{\mathbb{P}_x(\tau_0 > t)}\right)\\ &= \mathbb{E}_x\left(\mathbb{1}_{\tau_A\circ\theta_s<\infty}e^{\lambda_\infty\tau_A\circ\theta_s}\frac{\eta_\infty(X_{\tau_A\circ\theta_s})}{\eta_\infty(x)}\right)\end{aligned}$$

This shows that we can extend the functions $f_s : x \to 1 - \frac{\mathbb{P}_x(\tau_h\circ\theta_s>t)}{\mathbb{P}_x(\tau_0>t)}$ on $\mathbb{R}_+ \cup \{\infty\}$ setting for any $s \geq 0$,

$$f_s(\infty) = \mathbb{E}_x\left(\mathbb{1}_{\tau_A\circ\theta_s<\infty}e^{\lambda_\infty\tau_A\circ\theta_s}\frac{\eta_\infty(X_{\tau_A\circ\theta_s})}{\eta_\infty(x)}\right).$$

Then, like any $t \in \mathbb{R}$, $f_s(\infty)$ is non-increasing and since η_0 vanishes on 0 and η_0 is continuous,

$$\lim_{s\to\infty} f_s(\infty) = 0$$

As a result, the Dini's theorem (for the non-increasing sequences of functions) is applicable and we deduce Assumption 5.

Then, since the assumptions (F') and Assumption 5 hold for the Orsntein-Uhlenbeck process, we can apply the Theorem 23 :

Theorem 25. *The Ornstein-Uhlenbeck process defined in* (4.49) *absorbed by the decreasing function h admits a quasi-limiting distribution.*

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Quasi-stationarité avec frontières mobiles

RÉSUMÉ

Cette thèse étudie les comportements asymptotiques de processus de Markov conditionnés à ne pas atteindre de frontières mobiles. Le premier chapitre s'intéresse à cette question pour des chaînes de Markov à temps discret définies sur un espace d'état fini en considérant des frontières périodiques. Si les notions de distributions quasi-stationnaires et de distributions quasi-limites sont mal définies dans ce cas, l'existence de distributions quasi-ergodiques et d'un Q-processus est démontré. Dans le deuxième chapitre, les résultats précédents sont étendus à des processus de Markov satisfaisant des conditions inhomogènes globales introduites par N.Champagnat et D.Villemonais. Dans le cas de frontières périodiques, nous obtenons l'existence et l'unicité d'une distribution quasi-ergodique. Dans le cas où la frontière absorbante se stabilise à l'infini, nous obtenons en plus l'existence et l'unicité d'une distribution quasi-limite. Le troisième chapitre s'intéresse à la quasi-stationnarité du mouvement brownien dit "renormalisé" absorbé en $\{-1, 1\}$. Ce processus dépend d'un paramètre κ et sa quasi-stationnarité présente une transition de phase de paramètre critique égal à $1/2$. Enfin, le dernier chapitre étend les résultats du deuxième à des processus satisfaisant des critères plus faibles que les conditions globales de Champagnat-Villemonais. On y démontre notamment une propriété de mélange, l'existence du Q-processus et d'une distribution quasi-ergodique pour certains comportements de frontières mobiles.

ABSTRACT

This thesis studies the asymptotic behaviors for Markov processes conditioned not to hit moving boundaries. The first chapter deals with this problem for discrete-time Markov chains defined on finite state space considering periodic boundaries. Even if the notions of quasi-stationary distributions and quasi-limiting distributions are not well-defined considering moving boundaries, the existence of a quasi-ergodic distribution and the Q-process are shown. In the second chapter, the previous results are extended to Markov processes satisfying some global inhomogeneous conditions introduced by N. Champagnat and D. Villemonais. In the periodic case, the existence and uniqueness of a quasi-ergodic distribution are proved. When the boundary stabilizes at infinity, we obtain moreover the existence and uniqueness of a quasi-limiting distribution. The third chapter deals with the quasi-stationarity for the "renormalized" Brownian motion absorbed at $\{-1, 1\}$. The law of this process depends on a parameter κ and a phase transition is observed for its quasi-stationarity, whose the critical parameter is equal to $1/2$. Finally, the last chapter extend the results obtained in the second chapter to Markov processes satisfying some criteria weaker than the global Champagnat-Villemonais conditions. In particular, we obtain under these conditions a mixing property, the existence of the Q-process and the existence of a quasi-ergodic distribution for some behaviors of the boundary.