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« Modulo ℓ-representations of p-adic groups $\mathrm{SL}_n(F)$ »

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Chapter 1

Résumé

Soit F un corps fini $\mathbb{F}_q$ de caractéristique p ou bien un corps local non archimédien dont le corps résiduel est fini et de caractéristique p. Notons G le groupe des points F-rationnels d'un groupe algébrique réductif connexe défini sur F. Soit k un corps algébriquement clos de caractéristique ℓ (avec $\ell \neq p$). La thèse porte sur les représentation *modulaires* (i.e., $\ell \neq 0$) de G. Tous les représentations considérées dans la thèse pour F non archimédien sont lisses.

La théorie des représentations modulaires a des grandes similitudes avec la théorie complexe mais aussi des différences importantes. Par exemple, la condition $\ell \neq p$ assure l'existence d'une mesure de Haar sur G à valeurs dans k, mais le fait que ℓ soit non nul implique que la mesure d'un sous-groupe ouvert compact de G peut être nulle. D'autre part, les représentations modulaires d'un sous-groupe ouvert compact ne sont pas semi-simples en général et les notions de représentation cuspidale et supercuspidale ne sont pas équivalentes, car il existe des représentation cuspidales qui ne sont pas supercuspidales. Pour ces raisons les méthodes utilisées dans le cas complexe ne sont pas entièrement utilisables dans le cas modulaire.

Dans l'étude de la catégorie des représentations du groupe G, une étape importante est la décomposition de Bernstein, qui affirme que la catégorie $\mathrm{Rep}_k(\mathrm{G})$ des k-représentations lisses de G se décompose en un produit infini de sous-catégories pleines et indécomposables. Toute k-représentation de G se decompose de façon unique en somme directe de sous-représentations, chacune appartenant à un bloc, et tout morphisme entre représentations est alors un produit de morphismes, chacun appartenant à bloc. En conséquence, pour la compréhension de la catégorie, il suffit d'étudier chaque bloc séparément.

La décomposition de Bernstein admet un analogue du "côté galoisien" via la correspondance de Langlands. Fixons un nombre premier r différent de p . Lorsque $\ell > 0$, nous prendrons $r = \ell$. Notons W_F le groupe de Weil de F. Pour $\mathbf{G} = \mathrm{GL}_n$, la correspondance de Langlands a été prouvée pour F de caractéristique p par Laumon, Rapoport et Stuhler [LRS], et pour F de caractéristique 0, indépendamment par Harris et Taylor [HT], par Henniart [Hen], et par Scholze [Sch]. Elle fournit une bijection

canonique de l'ensemble des classes d'isomorphisme de représentations irréductibles r-adiques de $\mathrm{GL}_n(F)$ vers celui des classes d'isomorphisme de représentations de Deligne W_F-semisimple r-adiques de dimension n, qui généralise l'application de réciprocité d'Artin de la théorie du corps de classes. Via la correspondance de Langlands, deux k-représentations irréductibles π et π' appartiennent au même bloc si et seulement si leurs paramètres de Langlands ont des restrictions au sous-groupe d'inertie I_F de W_F isomorphes. Pour $\mathbf{G}$ un groupe réductif connexe quelconque défini sur F, un analogue de la décomposition de Bernstein pour les paramètres de Langlands enrichis a été construit dans [AMS].

On souhaiterait décomposer aussi la catégorie $\mathrm{Rep}_k(\mathrm{G})$ en somme directe de sous-catégories, appelées blocs de Bernstein, lorsque ℓ est non nul. Dans le cas $\ell = 0$, la décomposition de Bernstein repose notamment sur la propriété d'unicité du support supercuspidal, qui affirme que le support supercuspidal d'une k-représentation lisse irréductible π de G est une classe de G-conjugaison d'une paire (M, σ), où M est un sous-groupe de Levi de G et σ est une k-représentation lisse irréductible supercuspidale de M. En particulier, la définition des blocs repose sur la notion de support supercuspidal. Autrement dit, l'unicité du support supercuspidal des k-représentations irréductibles de G impliquerait une décomposition de l'ensemble des classes d'équivalence $\mathrm{Irr}_k(\mathrm{G})$ de représentations lisses irréductibles de G, et cette décomposition impliquerait la décomposition de Bernstein restreinte à l'ensemble des classes d'équivalence $\mathrm{Irr}_k(\mathrm{G})$. Quand $\ell = 0$, une k-représentation cuspidale est supercuspidale donc l'unicité de support supercuspidal est impliquée par l'unicité de support cuspidal, qui est simple à vérifier. Mais dans le cas $\ell \neq 0$, comme on l'a expliqué dans le second paragraphe, la notion de support cuspidal n'est pas suffisante pour résoudre ce problème. Malheuresement, l'unicité de support supercuspidal n'est pas toujours vraie. Un contre-exemple a été exhibé par Jean-François Dat pour $\mathrm{G} = \mathrm{Sp}_8(F)$ où F est non archimédien. Ce contre-exemple est obtenu par relèvement d'un contre-exemple similaire construit par Olivier Dudas pour $\mathrm{G} = \mathrm{Sp}_8(\mathbb{F}_q)$, où $q = p^m, m \in \mathbb{Z}$. Mais pour $\mathrm{G} = \mathrm{GL}_n(F)$, l'unicité de support supercuspidal est un théorème, démontré par Vignéras dans [V2].

Nous supposons désormais $\ell \neq 0$. Soient $W(k)$ l'anneau des vecteurs de Witt de k et $\mathcal{K} = \mathrm{Frac}(W(k))$ le corps des fractions de $W(k)$. Une preuve de la décomposition de Bernstein pour la catégorie des $W(k)[\mathrm{GL}_n(F)]$-modules lisses a été donnée par David Helm dans [Helm]. Vincent Sécherre et Shawn Stevens ont donné une preuve de la décomposition de Bernstein des catégories des k-représentations lisses de $\mathrm{GL}_n(F)$ et de ses formes intérieures dans [SeSt]. Ces preuves reposent de manière cruciale sur l'unicité du support supercuspidal des k-représentations lisses irréductibles de $\mathrm{GL}_n(F)$. La décomposition de Bernstein de $\mathrm{Rep}_k(\mathrm{G})$ n'est pas connue pour un groupe réductif G défini sur F arbitraire.

De plus, dans le cas $\mathrm{G} = \mathrm{GL}_n(F)$, Vignéras a construit dans [V4] une bijection entre l'ensemble des classes d'isomorphisme de représentations irréductibles ℓ-modulaires de $\mathrm{GL}_n(F)$ et l'ensemble des classes d'isomorphisme de représentations

de Deligne W_F-semi-simples ℓ-modulaires de dimension n et d'opérateur de Deligne nilpotent. Il s'ensuit que deux $\overline{\mathbb{F}}_\ell$-représentations irréductibles π et π' de G sont dans le même bloc si et seulement si leurs paramètres de Langlands ont des restrictions à I_F^ℓ (le noyau de l'application canonique $I_F \to \mathbb{Z}_\ell$) qui sont isomorphes, ainsi que l'a observé Dat dans [DaII] §1.2.1.

Dans cette thèse, nous étudions la catégorie des représentations lisses du groupe spécial linéaire $\mathrm{SL}_n(F)$ à coefficients dans un corps k algébriquement clos de caractéristique ℓ avec ℓ différent de p. Le résultat principal de la thèse est la preuve de l'unicité du support supercuspidal pour toutes les k-représentations irréductibles de $\mathrm{SL}_n(F)$, dans le cas où F est soit fini (Théorème 4.1.11), soit local non archimédien (Théorème 6.1.10).

Theorem 1.0.1. *Soient* M' *un sous-groupe de Levi de* $\mathrm{SL}_n(F)$*, et* ρ *une* k*-représentation irréductible de* M'*. Le support supercuspidal de* ρ *est la classe de* M'*-conjugaison d'une paire* (L', τ')*, où* L' *est un sous-groupe de Levi de* M' *et* τ' *est une* k*-représentation irréductible supercuspidale de* L'*.*

Désormais, nous utilisons G pour désigner $\mathrm{GL}_n(F)$ et G' pour $\mathrm{SL}_n(F)$, sauf mention contraire. Cette thèse est constituée de deux parties : la section 4 est consacrée à l'étude des k-représentations des groupes finis, et la section 5 est consacrée au cas où F est non-archimédien. L'unicité du support cuspidal est connue pour toutes les k-représentations irréductibles cuspidales de M', où M' désigne un sous-groupe Levi de G'. Cela nous permet de réduire le problème à celui de l'unicité du support supercuspidal pour un sous-groupe de Levi M' de G'. Dans tous les cas, pour toute k-representation cuspidale irréductible π' de M', il existe une k-représentation irréductible cuspidale π de M (un sous groupe de Levi de G, et $\mathrm{M} \cap \mathrm{G}' = \mathrm{M}'$) telle que π' intervienne dans la k-representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$, qui est semi-simple de longueur finie (voir [Ta] quand $\ell = 0$, et Proposition 5.1.32 quand ℓ est positif). Notre stratégie consiste à étudier π' via l'étude de π.

Dans la première partie nous posons $F = \mathbb{F}_q$. En s'inspirant de travaux de Gerhard Hiss, nous décrivons le support supercuspidal d'une k-représentation irréductible cuspidale π' d'un sous-groupe de Levi M' de G en fonction de son enveloppe projective. En utilisant la théorie de Deligne-Lusztig, on construit l'enveloppe projective $\mathrm{P}_{\pi'}$ de π'. Ensuite, pour π une k-représentation irréductible de M comme ci-dessus, nous remarquons que $\mathrm{P}_{\pi'}$ est une composante indécomposable de la restriction de l'enveloppe projective P_π de π. En considérant les restrictions paraboliques $\mathrm{P}_{\pi'}$ de π' aux sous-groupes de Levi de M', lesquelles ont des propriétés similaires aux restrictions paraboliques P_π aux sous-groupes de Levi de M, nous déduisons l'unicité du support supercuspidal de π' de celle du support supercuspidal de π.

Le reste de cette thèse étudie les k-représentations du groupe $\mathrm{SL}_n(F)$, où F est local non-archimédien, au moyen de la théorie des types. La construction de Bushnell et Kutzko a été généralisée du cas complexe au cas modulaire avec $\ell \neq p$ par Marie-France Vignéras pour $\mathrm{GL}_n(F)$. Pour le groupe $\mathrm{SL}_n(F)$, cette théorie n'avait été

établie que dans le cas complexe (pour les représentations supercuspidales par Colin Bushnell et Philip Kutzko, pour les non-supercuspidales par David Goldberg et Alan Roche).

Nous construisons un *k-type simple maximal cuspidal* pour toute k-représentation cuspidale π' d'un sous-groupe de Levi M' du groupe $\mathrm{G}' = \mathrm{SL}_n(F)$, *i.e.*, un couple $(N_{\mathrm{M}'}(\mu'_{\mathrm{M}}), \tau_{\mathrm{M}'})$, formé d'un sous-groupe ouvert $N_{\mathrm{M}'}(\mu'_{\mathrm{M}})$ de M', compact modulo le centre, et d'une k-représentation irréductible $\tau_{\mathrm{M}'}$ de $N_{\mathrm{M}'}(\mu'_{\mathrm{M}})$, telle que l'induite compacte de $\tau_{\mathrm{M}'}$ à M' soit isomorphe à π'. Comme dans le cas complexe, $(N_{\mathrm{M}'}(\mu'_{\mathrm{M}}, \tau_{\mathrm{M}'}))$ s'obtient à partir d'un k-type simple maximal cuspidal $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ (construit précédemment par Vignéras) de M, où M est un sous-groupe de Levi de $\mathrm{G} = \mathrm{GL}_n(F)$ tel que $\mathrm{M} \cap \mathrm{G}' = \mathrm{M}'$. Nous considérons d'abord le normalisateur projectif $\tilde{J}_{\mathrm{M}}$ de $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ et l'induite compacte irréductible $\tilde{\lambda}_{\mathrm{M}}$ de λ_{M} à $\tilde{J}_{\mathrm{M}}$, qui est une notion introduite dans le cas complexe par Bushnell et Kutzko dans leur construction des types simples maximaux dans $\mathrm{SL}_n(F)$. Soient μ'_{M} une composante irréductible de la restriction de $\tilde{\lambda}_{\mathrm{M}}$ à $\tilde{J}_{\mathrm{M}} \cap \mathrm{G}'$, et $\tau_{\mathrm{M}'}$ une k-représentation irréductible du normalisateur $N_{\mathrm{M}'}(\mu'_{\mathrm{M}})$ de μ'_{M} dans M' telle que $\tau_{\mathrm{M}'}|_{\tilde{J}_{\mathrm{M}} \cap \mathrm{G}'}$ contient μ'_{M}. Nous démontrons que les couples de la forme $(N_{\mathrm{M}'}(\mu'_{\mathrm{M}}), \tau_{\mathrm{M}'})$ sont les k-types simples maximaux cuspidaux de M'. L'un des points délicats est la preuve de l'irréductibilité de l'induite compacte de $\tau_{\mathrm{M}'}$ à M'. Alors que dans le cas complexe, il suffit de montrer que le groupe d'entrelacement de la représentations $\tau_{\mathrm{M}'}$ coïncide avec $N_{\mathrm{M}'}(\mu'_{\mathrm{M}})$, lorsque $\ell \neq 0$, il est nécessaire de prouver qu'une condition technique supplémentaire, qualifiée de *seconde condition pour l'irréductibilité* dans cette thèse, est satisfaite.

Une fois ce résultat obtenu, nous montrons que si $[\mathrm{L}, \sigma]$ est le support supercuspidal de π (π, π' comme ci-dessus), avec L un sous-groupe Levi de M et σ une k-représentation irréductible supercuspidale de L, et σ' est un facteur direct de la restriction de σ sur $\mathrm{L}' = \mathrm{L} \cap \mathrm{G}'$ de σ, alors le support supercuspidal de π' est contenu dans la classe de M-conjugaison de (L', σ'). Il existe une autre méthode pour montrer cette propriété, qui n'utilise pas la théorie des types, consistant à appliquer la méthode de Tadić. Mais notre construction des k-types simples maximaux cuspidaux est intéressante en elle-même: lors de sa démonstration dans [Helm] de la décomposition de Bernstein de $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))$, Helm construit au moyen des k-types simples maximaux cuspidaux une famille d'objets projectifs, qui sont au coeur de la démonstration.

Les autres ingrédients de la preuve de l'unicité de support supercuspidal de π' sont l'étude du modèle de Whittaker de σ' (section 5.2), et la généralisation aux k-représentations de M' d'une formule sur les dérivations obtenue par Bernstein et Zelevinsky dans le cas complexe pour $\mathrm{GL}_n(F)$ (section 6 et appendice). Plus précisement, notons T le sous-groupe formé des matrices diagonales et U celui des matrices strictements triangulaires supérieures. Nous montrons l'existence d'un caractère θ non-dégénéré de $\mathrm{U} \cap \mathrm{M}'$ tel que la plus haute dérivée associée à $\theta|_{\mathrm{U} \cap \mathrm{M}'}$ de π' est non-triviale. Soit (L', σ') contenu dans le support supercuspidal de π'. Nous déduisons de la généralisation de la formule de Bernstein et Zelevinsky que la plus

haute dérivée associée à $\theta|_{U\cap L'}$ est non-triviale. Nous établissons aussi l'existence d'un unique facteur direct de la restriction de σ sur L' tel que la plus haute dérivée associée à $\theta|_{U\cap L'}$ est non-triviale.

Chapter 2

Introduction

2.1 Introduction

Let F be a finite field of characteristic p, or a non-archimedean locally compact field whose residue field is of characteristic p, and $\mathbf{G}$ a reductive connected algebraic group defined over F. We denote by G the group $\mathbf{G}(F)$ of the F-points of $\mathbf{G}$, and endow it with the locally pro-finite topology through F. Let k be an algebraically closed field of characteristic ℓ ($\neq p$), and $W(k)$ its ring of Witt vectors. We use $\mathcal{K} = \mathrm{Frac}(W(k))$ to denote the fraction field of $W(k)$. We denote by $\mathrm{Rep}_k(\mathrm{G})$ the category of smooth k-representations of G, where a k-representation (π, V) (here V is the k-space of representation π) of G is smooth if any element $v \in V$ is stabilised by an open subgroup of G. In the thesis, when we say a k-representation, we always assume it is smooth.

When $\ell = 0$ and F is a non-archimedean locally compact field, the existence of Bernstein decomposition of the category $\mathrm{Rep}_k(\mathrm{G})$ has been proved by Bernstein: The category $\mathrm{Rep}_k(\mathrm{G})$ is equivalent to the direct product of some full-subcategories, which are indecomposable and called blocks. This means that each k-representation is isomorphic to a direct sum of sub representations belonging to different blocks, and each morphism of k-representations is isomorphic to a product of morphisms belonging to different blocks. We say that a full-subcategory of $\mathrm{Rep}_k(\mathrm{G})$ is indecomposable (or a block) if it is not equivalent to a product of any two non-trivial full-subcategories.

This decomposition has a counterpart in the "Galois side" through local Langlands correspondence. Let r be a prime number such that $r \neq p$. When $\ell > 0$, we will take $r = \ell$. Let ${}^L\mathbf{G}$ denote the L-group of $\mathbf{G}$ and W_F the Weil group of F. In the case when $\mathbf{G}$ equals GL_n, the local Langlands correspondence (LLC) was proved when F has characteristic p by Laumon, Rapoport and Sthuler [LRS], and, when F has characteristic 0, independently by Harris and Taylor [HT], by Henniart [Hen], and by Scholze [Sch]. It provides a canonical bijection between the set of isomorphism classes of r-adic irreducible representations of $\mathrm{GL}_n(F)$ and the set

of isomorphism classes of r-adic n-dimensional W_F -semisimple Deligne representations, generalizing the Artin reciprocity map of local class field theory. A nice property of LLC is that the Rankin-Selberg local factors of a pair of irreducible $\overline{\mathbb{Q}}_r$-representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, and the Artin-Deligne local factors of the corresponding tensor product of representations of W_F are equal, and moreover this condition characterizes LLC completely. Under the local Langlands correspondence, two irreducible k-representations π and π' belong to the same block if and only if their Langlands parameters are isomorphic when restricting to the inertial subgroup I_F of W_F. For $\mathbf{G}$ an arbitrary connected reductive group defined over F, an analog of the Bernstein decomposition for (enhanced) Langlands parameters is constructed in [AMS].

When ℓ is positive and F is a non-archimedean locally compact field. Helm gives a proof of the Bernstein decomposition of $\mathrm{Rep}_{\overline{W(\mathbb{F}_\ell)}}(\mathrm{G})$ in [Helm], where $W(\mathbb{F}_\ell)$ denotes the ring of Witt vectors of $\mathbb{F}_\ell$, and this deduces the Bernstein decomposition of $\mathrm{Rep}_{\mathbb{F}_\ell}(\mathrm{G})$. Sécherre and Stevens gave a proof of the Bernstein decomposition of the category of smooth k-representations of $\mathrm{GL}_n(F)$ and its inner forms in [SeSt]. The Bernstein decomposition of $\mathrm{Rep}_k(\mathrm{G})$ is unknown for general reductive groups $\mathbf{G}$ defined over F. In the case where $\mathbf{G}$ equal GL_n, Vignéras constructed in [V4] a bijection between the set of isomorphism classes of ℓ-modular irreducible representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of ℓ-modular n-dimensional W_F- semisimple Deligne representations with nilpotent Deligne operator. Combining with the Bernstein decomposition, it implies that two irreducible $\overline{\mathbb{F}}_\ell$-representations π, π' of G belong to the same block if and only if their Langlands parameters are isomorphic when restricting to I_F^ℓ, which is the kernel of the canonical map $I_F \to \mathbb{Z}_\ell$, as observed by Dat in [DaII] §1.2.1.

The theory of Rankin-Selberg local factors of Jacquet, Shalika and Piatetski-Shapiro has a natural extension at least to generic k-representations of $\mathrm{GL}_n(F)$. However, via the ℓ-modular local Langlands correspondence these factors do not agree with the factors of Artin-Deligne. In [KuMa], Kurinczuk and Matringe classified the indecomposable ℓ-modular W_F-semisimple Deligne representations, extended the definitions of Artin-Deligne factors to this setting, and define an ℓ-modular local Langlands correspondence where in the generic case, the Rankin-Selberg factors of representations on one side equal the Artin-Deligne factors of the corresponding representations on the other.

In this thesis, we study the category $\mathrm{Rep}_k(\mathrm{SL}_n(F))$. The proofs of Helm, Sé cherre and Stevens in [Helm] and [SeSt] of the Bernstein decompositions are based on the fact that the supercuspidal support (Definition 3.1.13) of any irreducible k-representation of $\mathrm{GL}_n(F)$ is unique, which has been proved by Vignéras in [V2]. As the main results of this thesis, we prove the uniqueness of supercuspidal support for $\mathrm{SL}_n(F)$ in both cases that F is finite (Theorem 4.1.11) and F is non-archimedean (Theorem 6.1.10):

Theorem 2.1.1. *Let* M' *be a Levi subgroup of* $\mathrm{SL}_n(F)$*, and* ρ *an irreducible* k*-representation of* M'*. The supercuspidal support of* ρ *is a* M'*-conjugacy class of a pair* (L', τ')*, where* L' *is a Levi subgroup of* M'*, and* τ' *is an irreducible supercuspidal* k*-representation of* L'*.*

However, the uniqueness of supercuspidal support of irreducible k-representations is not always true for general reductive groups when ℓ is positive. A counter-example has been found in [Da] by Dat and Dudas for $\mathrm{Sp}_8(F)$.

From now on, we use G to denote $\mathrm{GL}_n(F)$, and G' to denote $\mathrm{SL}_n(F)$ unless otherwise specified. This manuscript has two parts: in section 2, we study the k-representations of finite groups; next we consider the case that F is non-archimedean locally compact from section 3. There is a fact that for any irreducible k-representation π' of M', a Levi subgroup of G', its cuspidal support (see 3.1.13 for the definition) is unique. Hence we could reduce our problem to the uniqueness of supercuspidal support for irreducible cuspidal k-representations of M', where M' denote any Levi subgroup of G'. In both parts, for any irreducible cuspidal k-representation π' of M', there exists an irreducible cuspidal k-representation π of M such that π' is a component of the semisimple k-representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ which has finite length (see [Ta] when $\ell = 0$, and Proposition 5.1.32 when ℓ is positive). Our strategy is to study π' by considering π, in other words, to reduce the problem of π' to the one of π.

In the first part, we describe the supercuspidal support of an irreducible cuspidal k-representation π' of M' in terms of its projective cover (see the paragraph after Theorem 4.1.11), which has been considered by Hiss [Hiss]. Using Deligne-Lusztig theory, we construct the projective cover $\mathrm{P}_{\pi'}$ of π', which is one of the indecomposable components of the restriction of the projective cover P_π of π to M'. The construction is based on the Gelfand-Graev lattice. We deduce the uniqueness of supercuspidal support of π' by considering the parabolic restrictions of $\mathrm{P}_{\pi'}$ to any Levi subgroup of M'.

The projective covers $\mathrm{P}_{\pi'}$ constructed in this part are interesting in their own right. Let $\bar{\mathcal{K}}$ denote an algebraic closure of $\mathcal{K}$. In the article [Helm], Helm gave the relation between Bernstein decompositions of $\mathrm{Rep}_{\bar{\mathcal{K}}}(\mathrm{GL}_n(F))$ and of $\mathrm{Rep}_k(\mathrm{GL}_n(F))$. One of the key objects of his article is a family of projective objects associated to irreducible cuspidal k-representations. These projective objects are constructed by projective covers of irreducible cuspidal k-representations of finite groups of GL_m type, where m divides n.

In the second part, G and G' are defined over a non-archimedean local field. We prove the uniqueness of the supercuspidal support (Theorem 6.1.10) in two steps. From section 5 to section 7, we construct maximal simple cuspidal k-types of M' (Theorem 5.3.9), where M' denote any Levi subgroup of G'. This gives a first description of the supercuspidal support for any irreducible cuspidal k-representation π' of M'. In section 6, we describe precisely the supercuspidal support of π' by

considering the derivatives of the elements in the supercuspidal support, and deduce that it is unique.

Definition 2.1.2. *An maximal simple cuspidal k-type of M' for an irreducible cuspidal k-representation π' of M' is a pair $(K', \tau_{\mathrm{M}'})$ consisting of an open and compact modulo center subgroup K' of M', and an irreducible k-representation $\tau_{\mathrm{M}'}$ of K', such that:*

$$\mathrm{ind}_{\mathrm{K}'}^{\mathrm{M}'}\tau_{\mathrm{M}'} \cong \pi'. \tag{2.1}$$

Inspired by [BuKuI], [BuKuII], [GoRo] and [Ta], we construct the maximal simple cuspidal k-types of M' from those of M, where M is a Levi subgroup of G such that $\mathrm{M} \cap \mathrm{G}' = \mathrm{M}'$. More precisely, let π be an irreducible cuspidal k-representation of M, such that π' is an component of $\mathrm{res}_{\mathrm{M}'}^{\mathrm{M}}\pi$. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple cuspidal k-type of M of π (we inherit the notations from those of [BuKu]), which means the compact induction $\mathrm{ind}_{\mathrm{K}}^{\mathrm{M}}\Lambda_{\mathrm{M}}$ is isomorphic to π, where K is an open subgroup of M compact modulo center, which contains J_{M} as the unique maximal compact subgroup, and Λ_{M} is an extension of λ_{M} to K. In the equation (2.1), the group K' is also compact modulo centre. Furthermore, the group K' contains $\tilde{J}_{\mathrm{M}} \cap \mathrm{M}'$ as the unique maximal open compact subgroup, and K' is a normal subgroup of $(\mathrm{K} \cap \mathrm{M}')(\tilde{J}_{\mathrm{M}} \cap \mathrm{M}')$ with finite index, where $\tilde{J}_{\mathrm{M}}$ is an open compact subgroup of M containing J_{M}. The irreducible k-representation $\tau_{\mathrm{M}'}$ of K' contains some irreducible component of the semisimple representation $\mathrm{res}_{\tilde{J}_{\mathrm{M}} \cap \mathrm{M}'}^{\tilde{J}_{\mathrm{M}}}\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\lambda_{\mathrm{M}}$. When $\mathrm{M}' = \mathrm{G}'$, the group K' equals $\tilde{J}_{\mathrm{M}} \cap \mathrm{M}'$, and this simple case is considered in section 3, based on which the case for proper Levi is dealt.

In the construction, the technical difficulty is to prove that the compact induction $\mathrm{ind}_{\mathrm{K}'}^{\mathrm{M}'}\tau_{\mathrm{M}'} \cong \pi'$ is irreducible. When $char(k)$ is 0, it is sufficient to prove that the intertwining group of $\tau_{\mathrm{M}'}$ equals to K'. In our case, besides of this condition about intertwining group, we need to verify the second condition explained in section 3.5, which is given by Vignéras in [V3]. After this construction, we give the first description of thesupercuspidal support of any irreducible cuspidal k-representation π' in Proposition 5.3.18:

Proposition 2.1.3. *Let π' be an irreducible cuspidal k-representation of M', and π an irreducible cuspidal k-representation of M such that π contains π'. Let $[\mathrm{L}, \tau]$ be the supercuspidal support of π, where L is a Levi subgroup of M and τ an irreducible supercuspidal k-representation of L. Let τ' be a direct component of $\mathrm{res}_{\mathrm{L}'}^{\mathrm{L}}\tau$. The supercuspidal support of π' is contained in the M-conjugacy class of (L', τ').*

We finish the proof of the uniquess of the supercuspidal support of π' by proving that there is only one irreducible component τ_0' of $\mathrm{res}_{\mathrm{L}'}^{\mathrm{L}}\tau$, such that (L', τ_0') belongs to the supercuspidal support of π'. The idea is to study the Whittaker model of τ' and apply the derivative formula given by Bernstein and Zelevinsky in [BeZe]. For this, we need to generalise their formula to the case of k-representations of M'. In fact, let T_{M} be a fixed maximal split torus of M defined over F and $\mathrm{T}_{\mathrm{M}'} = \mathrm{T}_{\mathrm{M}} \cap \mathrm{M}'$.

Fix $\mathrm{B_M} = \mathrm{T_M U_M}$ a Borel subgroup of M, and $\mathrm{B_{M'}} = \mathrm{T_{M'} U_M}$ a Borel subgroup of M$'$. There is a non-degenerate character $\dot{\theta}$ of $\mathrm{U_M}$ such that the highest derivative of π' according to $\dot{\theta}$ is non-zero. On the other hand, assume that L is a standard Levi subgroup of M, and θ denotes $\dot{\theta}|_{\mathrm{U_L}}$, which is also a non-degenerate character of $\mathrm{U_L}$. There is only one irreducible component τ_0' of $\mathrm{res}^{\mathrm{L}}_{\mathrm{L'}}\tau$, such that the highest derivative of τ_0' according to θ is non-zero. If π' is a subquotient of $i^{\mathrm{M'}}_{\mathrm{L'}}\tau'$ for some irreducible component τ' of $\mathrm{res}^{\mathrm{L}}_{\mathrm{L'}}\tau$, then the highest derivative of $i^{\mathrm{M'}}_{\mathrm{L'}}\tau'$ according to $\dot{\theta}$ is also non-zero (Proposition 6.1.9). Applying the generalised formula of derivative in Corollary 6.1.7 (4), we obtain that the highest derivative of $i^{\mathrm{M'}}_{\mathrm{L'}}\tau'$ according to $\dot{\theta}$ is isomorphic to the highest derivative of τ' according to θ. Hence $\tau' \cong \tau_0'$. This ends this thesis.

Chapter 3

Background

3.1 Modulo ℓ representations of p-adic groups with $p \neq \ell$

Let H be a locally profinite group, and k be an algebraically closed field. Then $\mathrm{Rep}_k(H)$ denotes the category of smooth k-representations of H. In this thesis, a k-representation of H always means a smooth k-representation of H.

3.1.1 Restrictions and compact inductions

Let G denote the group of F-points of a reductive connected algebraic group defined over F, where F is a non-archimedean locally compact field whose residue field is of characteristic p. Let res, ind, Ind denote the functors of restriction, compact induction and induction, respectively.

Proposition 3.1.1 (Mackey's decomposition formulae)**.** *Let H, K be two closed subgroups of* G*, such that the double cosets $HgK, g \in \mathrm{G}$ are open and closed. For any k-representation $\sigma \in \mathrm{Rep}_k(H)$, the restriction on double cosets induces the isomorphisms:*

$$\mathrm{res}^{\mathrm{G}}_K \mathrm{Ind}^{\mathrm{G}}_H(\sigma) \cong \prod_{HgK} I_{K,g(H)}g(\sigma),$$

$$\mathrm{res}^{\mathrm{G}}_K \mathrm{ind}^{\mathrm{G}}_H(\sigma) \cong \bigoplus_{HgK} i_{K,g(H)}g(\sigma),$$

where

$$I_{K,g(H)}g(\sigma) = \mathrm{Ind}^K_{K\cap g(H)}\mathrm{res}^{g(H)}_{K\cap g(H)}g(\sigma), \quad i_{K,g(H)}g(\sigma) = \mathrm{ind}^K_{K\cap g(H)}\mathrm{res}^{g(H)}_{K\cap g(H)}g(\sigma).$$

Remark 3.1.2.

- *The functions in* $\mathrm{res}^{\mathrm{G}}_K\mathrm{ind}^{\mathrm{G}}_H(\sigma)$ *are supported on finitely many double cosets HgK for $g \in \mathrm{G}$, while the functions in* $\mathrm{res}^{\mathrm{G}}_K\mathrm{Ind}^{\mathrm{G}}_H(\sigma)$ *can be supported on infinitely many double cosets HgK for $g \in \mathrm{G}$.*

- *The double cosets $HgK, g \in \mathrm{G}$ are open and closed when H is open or K is open. In fact, the group* G *can be written as a disjoint union of HgK for a*

family of $g \in \mathrm{G}$, and HgK are open, since HgK is a union of left cosets of H as well as a union of right cosets of K, which are open by our hypothesis.

Proposition 3.1.3 (Frobenius reciprocity)**.** *Let H and K be closed subgroups of* G*.*

1. *The restriction functor* $\mathrm{res}_H^{\mathrm{G}} : \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(\mathrm{G})$ *has a right adjoint* $\mathrm{Ind}_H^{\mathrm{G}}$*.*
2. *If K is open, the restriction functor* $\mathrm{res}_K^{\mathrm{G}}$ *has also a left adjoint* $\mathrm{ind}_K^{\mathrm{G}}$*.*

Proof. A proof is given in §5.7 [V1]. □

We take the notations as in Proposition 3.1.3. We have four natural transformations deduced from the Frobenius reciprocity above:

- $v \to f(v) : V \to \mathrm{Ind}_H^{\mathrm{G}}\mathrm{res}_H^{\mathrm{G}}V$ is injective;
- $f \to f(1) : \mathrm{res}_H^{\mathrm{G}}\mathrm{Ind}_H^{\mathrm{G}}W \to W$ is surjective;
- $\omega \to i_\omega : W \to \mathrm{res}_K^{\mathrm{G}}\mathrm{ind}_K^{\mathrm{G}}W$ is injective, and the image of W is a direct factor;
- $f \to \sum_{Kg} g^{-1}f(g) : \mathrm{ind}_K^{\mathrm{G}}\mathrm{res}_K^{\mathrm{G}}V \to V$ is surjective.

To prove the injectivity or surjectivity for these four morphisms, we check directly the definition of these morphisms as in the proof of Proposition 3.1.3. For the third morphism, the fact that the image of W is a direct factor is deduced by applying Proposition 3.1.1, the Mackey's decomposition formulae.

3.1.2 Schur's lemma and Mackey's criterions of irreducibility

Let G denote the group of F-points of some reductive connected algebraic group defined over F, and F be as in Section 3.1.1. Each irreducible k-representation (π, V) of G is admissible (§II, 2.8 in [V1]), hence there exists an open compact subgroup K, such that $\dim(e_K V) = \dim(V^K) < \infty$, where e_K is an idempotent contained in the Heck algebra $H_k(\mathrm{G})$ of G.

Proposition 3.1.4 (Schur's lemma)**.** *Let A be an algebra defined over an algebraically closed field R, and M a simple A-module. Then* $\mathrm{End}_A M = R$ *if condition* (1) *or* (2) *hold:*

1. $\dim_R M < |R|$*,*
2. *there exists an idempotent e of A, such that $eM \neq 0$, and* $\dim_R(eM) < |R|$*, where $|R|$ indicates the cardinality of field R.*

Definition 3.1.5. *The global Hecke algebra $H_k(G)$ of G over k is the algebra formed by k-algebra $k_c^\infty G$, the set of isomorphism classes of locally constant k-functions on G with compact support, endowed with convolution product after fixing a normalised Haar measure.*

Since the category $\mathrm{Rep}_k(\mathrm{G})$ is equivalent to the category of modules over the Hecke algebra $H_k(G)$, irreducible k-representations of G always verify the condition (2) of Proposition 3.1.26.

3.1.3 Intertwining and weakly intertwining

In this section, we suppose that G is a locally pro-finite group, and Z is the centre of G. We identify a compact open subgroup K of G/Z to a open subgroup of G containing Z. For any $x \in \mathrm{G}$, we always denote $x(K)$ to be xKx^{-1}. For any field R and R-representation ρ of K, we always denote $x(\rho)$ as an R-representation of $x(K)$, such that $x(\rho)(g) = \rho(x^{-1}gx)$, for $g \in x(K)$.

Definition 3.1.6. *Let* G *be a locally profinite group, and* R *be an algebraically closed field. Let* K_i *be an open compact subgroup of* G *for* $i = 1, 2$*, and* ρ_i *a* R*-representation of* K_i*. Define* $i_{K_1,K_2}x(\rho_2)$ *to be the induced* R*-representation*

$$\mathrm{ind}_{K_1\cap x(K_2)}^{K_1}\mathrm{res}_{K_1\cap x(K_2)}^{x(K_2)}x(\rho_2),$$

where $x(\rho_2)$ *is the conjugation of* ρ_2 *by* x*.*

- *We say an element* $x \in \mathrm{G}$ *weakly intertwines* ρ_1 *with* ρ_2*, if* ρ_1 *is an irreducible subquotient of* $i_{K_1,K_2}x(\rho_2)$*. And* ρ_1 *is weakly intertwined with* ρ_2 *in* G*, if* ρ_1 *is isomorphic to a subquotient of* $\mathrm{ind}_{K_2}^{\mathrm{G}}\rho_2$*. We denote* $\mathrm{I}_{\mathrm{G}}^{w}(\rho_1, \rho_2)$ *the set of elements in* G*, which weakly intertwines* ρ_1 *with* ρ_2*. When* $\rho_1 = \rho_2$*, we abbreviate* $\mathrm{I}_{\mathrm{G}}^{w}(\rho_1, \rho_2)$ *as* $\mathrm{I}_{\mathrm{G}}^{w}(\rho_1)$*.*

- *We say the element* $x \in \mathrm{G}$ *intertwines* ρ_1 *with* ρ_2*, if the* Hom *set*

  $$\mathrm{Hom}_{kK_1}(\rho_1, i_{K_1,K_2}x(\rho_2)) \neq 0.$$

  Representation ρ_1 *is intertwined with* ρ_2 *in* G*, if the* Hom *set*

  $$\mathrm{Hom}_{k\mathrm{G}}(\mathrm{ind}_{K_1}^{\mathrm{G}}\rho_1, \mathrm{ind}_{K_2}^{\mathrm{G}}\rho_2) \neq 0.$$

  We denote $\mathrm{I}_{\mathrm{G}}(\rho_1, \rho_2)$ *the set of elements in* G*, which intertwine* ρ_1 *with* ρ_2*. When* $\rho_1 = \rho_2$*, we abbreviate* $\mathrm{I}_{\mathrm{G}}(\rho_1, \rho_2)$ *as* $\mathrm{I}_{\mathrm{G}}(\rho_1)$*.*

When ρ_1 *is irreducible, we deduce directly from Mackey's decomposition formulae that* ρ_1 *is (weakly) intertwined with* ρ_2 *in* G *if and only if there exists an element* $x \in \mathrm{G}$*, such that* x *(weakly) intertwines* ρ_1 *with* ρ_2*.*

Proposition 3.1.7 (Mackey's criterions of irreducibility (§I, 8.3 in [V1]))**.** *Let* K, K' *be two compact open subgroups of* G/Z*, where* Z *denotes the centre of* G*, and* σ *and* σ' *are irreducible* k*-representations of* K *and* K' *respectively. We have*

1. $\mathrm{End}_{k[\mathrm{G}]}(\mathrm{ind}_K^{\mathrm{G}}\sigma) = k$ *is equivalent to* $\mathrm{I}_{\mathrm{G}}(\sigma) = K$*, where* $\mathrm{I}_{\mathrm{G}}(\sigma)$ *denotes the intertwining set of* σ *in* G*.*

2. *If* $\mathrm{ind}_K^{\mathrm{G}}\sigma$ *is reducible, then* σ *is a subquotient of* $i_{K,g(K)}g(\sigma)$ *for at least one* $g \notin K$*.*

3. $\mathrm{Hom}_{\mathrm{G}}(\mathrm{ind}_K^{\mathrm{G}}\sigma, \mathrm{ind}_{K'}^{\mathrm{G}}\sigma') \neq 0$ *if and only if there exists* $g \in \mathrm{G}$*, such that* σ *is a sub-representation of* $i_{K,g(K')}g(\sigma')$*.*

4. *If the two* k*-representations* $\mathrm{ind}_K^{\mathrm{G}}\sigma$ *and* $\mathrm{ind}_{K'}^{\mathrm{G}'}\sigma'$ *are isomorphic, there exists* $g \in \mathrm{G}$*, such that* σ *is a direct factor of* $i_{K,g(K')}g(\sigma')$*. The converse is true, if the two* k*-representations* $\mathrm{ind}_K^{\mathrm{G}}\sigma$ *and* $\mathrm{ind}_K^{\mathrm{G}}\sigma'$ *are irreducible.*

Remark 3.1.8. *When* k *is of characteristic* 0*. Mackey's criterions of irreducibility above have a brief version: the induced* k*-representation* $\mathrm{ind}_K^{\mathrm{G}}\sigma$ *is irreducible, if and only if the intertwining set* $\mathrm{I}_{\mathrm{G}}(\sigma) = K$*. This is the criterion of justifying that the induced* k*-representation* $\mathrm{ind}_{E^\times J}^{\mathrm{GL}_n(F)}\Lambda_{\overline{\mathbb{Q}}_\ell}$ *is irreducible, in the theory of* $\overline{\mathbb{Q}}_\ell$*-types of* $\mathrm{GL}_n(F)$ *in [BuKu].*

3.1.4 Parabolic inductions and restrictions

Let G be a locally pro-finite group and H be a closed subgroup of G. We denote $\mathrm{Ind}_H^{\mathrm{G}}$ and $\mathrm{ind}_H^{\mathrm{G}}$ the functors of induction and compact induction respectively, from $\mathrm{Rep}_k(H)$ to $\mathrm{Rep}_k(\mathrm{G})$. Denote $\mathrm{res}_H^{\mathrm{G}}$ the functor of restriction from $\mathrm{Rep}_k(\mathrm{G})$ to $\mathrm{Rep}_k(H)$. Let δ_{G} denote the character of module of G.

Let $\mathrm{P} = \mathrm{MU}$ be a parabolic subgroup of G, where M denotes the Levi subgroup of P and U denotes the unipotent radical of P. We define $i_{\mathrm{M}}^{\mathrm{G}}, r_{\mathrm{M}}^{\mathrm{G}}$ the normalised parabolic induction and restriction:

- Let $\pi \in \mathrm{Rep}_k(\mathrm{M})$. Define $i_{\mathrm{M}}^{\mathrm{G}}\pi$ as $\mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}(\pi \otimes \delta_{\mathrm{P}}^{\frac{1}{2}})$, where we view π as a representation of P by acting U trivially on π.

- Let $\pi \in \mathrm{Rep}_k(\mathrm{G})$. Define $r_{\mathrm{M}}^{\mathrm{G}}\pi$ as $\mathrm{res}_{\mathrm{P}}^{\mathrm{G}}(\pi(\mathrm{U}) \otimes \delta_{\mathrm{P}}^{-\frac{1}{2}})$, where $\pi(\mathrm{U})$ denotes the U-coinvariants of π.

Proposition 3.1.9. *The quotient group* G/P *is compact. The unipotent radical* U *of* P *has a increasing filtration of countably many pro-p open compact subgroups.*

The first property above indicates that the two inductions $\mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}$ and $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ from $\mathrm{Rep}_k(\mathrm{P})$ to $\mathrm{Rep}_k(\mathrm{G})$ coincide. The second property is applied in Proposition 6.1.9.

Proposition 3.1.10.

- *The two functors* $i_{\mathrm{M}}^{\mathrm{G}}, r_{\mathrm{M}}^{\mathrm{G}}$ *are transitive. Let* M_1 *be a Levi subgroup of* G *and* M_2 *be a Levi subgroup of* M_1*, we have:*

$$i_{\mathrm{M}_1}^{\mathrm{G}} \circ i_{\mathrm{M}_2}^{\mathrm{M}_1} = i_{\mathrm{M}_2}^{\mathrm{G}}, r_{\mathrm{M}_1}^{\mathrm{G}} \circ i_{\mathrm{M}_2}^{\mathrm{M}_1} = r_{\mathrm{M}_2}^{\mathrm{G}}.$$

- *The functor* $r_{\mathrm{M}}^{\mathrm{G}}$ *is a left adjoint of* $i_{\mathrm{M}}^{\mathrm{G}}$*.*

- *The two functors* $i_{\mathrm{M}}^{\mathrm{G}}, r_{\mathrm{M}}^{\mathrm{G}}$ *are exact, and respect finite length.*

Lemma 3.1.11. *The two functors* $i_{\mathrm{M}}^{\mathrm{G}}, r_{\mathrm{M}}^{\mathrm{G}}$ *respect direct limits.*

Proof. The functor $r_{\mathrm{M}}^{\mathrm{G}}$ respects direct limits since it has a right adjoint. For the functor $i_{\mathrm{M}}^{\mathrm{G}}$, this property is proved in Proposition A.0.3. □

3.1.5 (Super)cuspidal support

Let F be a finite field of characteristic p or a non-archimedean locally compact field whose residue field is of characteristic p, and k an algebraically closed field of characteristic ℓ where $\ell \neq p$. Let $\mathbb{G}$ be a reductive connected algebraic group defined over F, G be the group of F-points $\mathbb{G}(F)$ of $\mathbb{G}$, and $\mathrm{Rep}_k(\mathrm{G})$ the category of smooth k-representations of G.

Let M be a Levi subgroup of G.

- $i_{\mathrm{M}}^{\mathrm{G}}$ denotes the normalised parabolic induction from the category $\mathrm{Rep}_k(\mathrm{M})$ to the category $\mathrm{Rep}_k(\mathrm{G})$;

- $r_{\mathrm{M}}^{\mathrm{G}}$ denotes the normalised parabolic restriction from the category $\mathrm{Rep}_k(\mathrm{G})$ to the category $\mathrm{Rep}_k(\mathrm{M})$.

Definition 3.1.12. *Let π be an irreducible k-representation of* G.

- *We say that π is cuspidal, if for any proper Levi subgroup* M *of* G *and any irreducible k-representation ρ of* M*, π does not appear as a subrepresentation or a quotient representation of $i_{\mathrm{M}}^{\mathrm{G}}\rho$;*

- *We say that π is supercuspidal, if for any proper Levi subgroup* M *of* G *and any irreducible k-representation ρ of* M*, π does not appear as a subquotient representation of $i_{\mathrm{M}}^{\mathrm{G}}\rho$.*

Definition 3.1.13. *Let π be an irreducible k-representation of* G.

- *Let (M, ρ) be a cuspidal pair of* G*, which means* M *is a Levi subgroup of* G*, and ρ is an irreducible cuspidal k-representation of* M*. We say that (M, ρ) belongs to the cuspidal support of π, if π is a subrepresentation or a quotient representation of $i_{\mathrm{M}}^{\mathrm{G}}\rho$;*

- *Let (M, ρ) be a supercuspidal pair of* G*, which means* M *is a Levi subgroup of* G*, and ρ is an irreducible supercuspidal k-representation of* M*. We say that (M, ρ) belongs to the supercuspidal support of π, if π is a subquotient of $i_{\mathrm{M}}^{\mathrm{G}}\rho$.*

Proposition 3.1.14 (Uniqueness of cuspidal support)**.** *Let π be an irreducible k-representation of* G*, its cuspidal support is unique up to* G*-conjugation.*

Proof. See [V1, 2.20]. □

When $\ell = 0$, the cuspidal support of an irreducible k-representation π of G coincide with the supercuspidal support of π. Hence, in this case, Proposition 3.1.14 is equivalent to say that the supercuspidal support of π is unique up to G-conjugation.

3.1.6 Reduction modulo ℓ

In this section, we suppose that $k = \overline{\mathrm{F}}_\ell$ an algebraically closed field of characteristic $0 < l \neq p$, and $\overline{\mathbb{Z}}_\ell$ the ring of integers of $\overline{\mathbb{Q}}_\ell$. We introduce the $\overline{\mathbb{Z}}_\ell$-integral structure for an $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{G} = \mathrm{GL}_n(F)$.

First we consider the representations with abstract coefficients, that is, let R be a field with valuation and A be its ring of integers. Let (π, V) be an R-representation of G. We define $A[\mathrm{G}]$-lattice of V as following:

Definition 3.1.15. *We say that an R-representation (π, V) is admissible, if for any open compact subgroup K of* G*, the K-invariant subspace V^K of V has finite dimension.*

Definition 3.1.16. *Let (π, V) be an admissible R-representation of* G*, an $A[\mathrm{G}]$-lattice of V is a sub-A-module L of V, who verifies the two equivalent conditions below:*

- *For any open compact subgroup K of* G*, the K-invariant sub-module L^K is an A-lattice of V^K.*
- *L contains an R-basis of V, such that for any open compact subgroup K, L^K is contained in a finite type A-module.*

We say that V is A-integral if it contains an $A[G]$-lattice.

Remark 3.1.17. *When A is a principal ring. Let (π, V) be an R-representation of* G*. A free sub-A-module of V, which generates V and stable under the action of* G*, has the property: for any open compact subgroup K, the A-module L^K is a free sub-A-module of V^K which generates V^K. Hence when (π, V) is admissible, any free sub-A-module of V, which generates V and stable under the action of* G*, verifies the conditions in Definition 3.1.16, and is an $A[\mathrm{G}]$-lattice of V in the sense of Definition 3.1.16.*

Now we come back to $\overline{\mathbb{Q}}_\ell$-representations of G:

Definition 3.1.18. *A $\overline{\mathbb{Q}}$-representation (π, V) of* G *is said to be ℓ-integral if it contains a free sub-$\overline{\mathbb{Z}}_\ell$-module L of V and L generates V, with L stable under the action of* G*.*

We will justify that when (π, V) is irreducible (hence admissible, see §II, 2.8 in [V1]), being ℓ-integral as in Definition 3.1.18 is also being $\overline{\mathbb{Z}}_\ell$-integral as in Definition 3.1.16. According to §I.9.3(vii), III.4.13, 4.14 in [V1], we conclude:

Proposition 3.1.19. *Let (π, V) be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of* G*, and ℓ-integral as in Definition 3.1.18. Then there exits a $\overline{\mathbb{Z}}_\ell$-lattice L, which is free over $\overline{\mathbb{Z}}_\ell$, stable under* G *and finite type as $\overline{\mathbb{Z}}_\ell[\mathrm{G}]$-module. In addition, the lattice L is defined over $\mathrm{O}_\mathcal{E}$, where $\mathcal{E}/\mathbb{Q}_\ell$ is a finite field extension, and $\mathrm{O}_\mathcal{E}$ is the ring of integers of $\mathcal{E}$. This is to say, there exists an $\mathrm{O}_\mathcal{E}[\mathrm{G}]$-lattice $L_\mathcal{E}$ of V, which is free over $\mathrm{O}_\mathcal{E}$ and finite type as $\mathrm{O}_\mathcal{E}[\mathrm{G}]$-module, such that $L \cong L_\mathcal{E} \otimes_{\mathrm{O}_\mathcal{E}} \overline{\mathbb{Z}}_\ell$.*

Let (π, V), L a $\overline{\mathbb{Z}}_\ell$-lattice as in Proposition 3.1.19, and $(\pi_\mathcal{E}, V_\mathcal{E})$ be a $\mathcal{E}$-representation of G, such that $V_\mathcal{E} \cong L_\mathcal{E} \otimes \overline{\mathbb{Q}}_\ell$. The ring of integers $\mathrm{O}_\mathcal{E}$ of $\mathcal{E}$ is principal. Hence by Remark 3.1.17, the lattice $L_\mathcal{E}$ is actually an $\mathrm{O}_\mathcal{E}[\mathrm{G}]$-lattice of $V_\mathcal{E}$ as in Definition 3.1.16. Furthermore, the subspace $L^K = e_K L$ for any open compact subgroup K in G, where e_K denotes the idempotent in the Hecke algebra $H_{\overline{\mathbb{Q}}_\ell}(\mathrm{G})$ according to K (§I, 3.2 in [V1]), which also belongs to the Hecke algebra $H_\mathcal{E}(\mathrm{G})$. Hence

$$L^K = e_K L_\mathcal{E} \otimes \overline{\mathbb{Z}}_\ell = L_\mathcal{E}^K \otimes \overline{\mathbb{Z}}_\ell,$$

and by the same reason, we have

$$V^K = V_\mathcal{E}^K \otimes \overline{\mathbb{Q}}_\ell,$$

which means that we can deduce from the fact that $L_\mathcal{E}^K$ is an $\mathrm{O}_\mathcal{E}$-lattice of $V_\mathcal{E}^K$ that L^K is a $\overline{\mathbb{Z}}_\ell$-lattice of V^K. Hence L is a $\overline{\mathbb{Z}}_\ell[\mathrm{G}]$-lattice as defined in Definition 3.1.18.

Proposition 3.1.20 (Principe of Brauer-Nesbitt)**.** *Let (π, V) be an ℓ-integral irreducible $\overline{\mathbb{Q}}_\ell$-representation of* G*, and L be a $\overline{\mathbb{Z}}_\ell[\mathrm{G}]$-lattice of V which is free over $\overline{\mathbb{Z}}_\ell$ and finite type as $\overline{\mathbb{Z}}_\ell[\mathrm{G}]$-module, defined over a finite field extension of $\mathbb{Q}_\ell$. Then the quotient $L/\ell L$ has finite length and its semi-simplification is independent of the choice of L, where $\ell\overline{\mathbb{Z}}_\ell$ is the maximal ideal of the ring of integers $\overline{\mathbb{Z}}_\ell$.*

Definition 3.1.21 (Reduction modulo ℓ and lift to $\overline{\mathbb{Q}}_\ell$)**.**

- *Let (π, V), L be as in Proposition 3.1.20. The reduction modulo ℓ of (π, V) is the semi-simplification of the k-representation $L \otimes \overline{F}_\ell$.*
- *Let (τ, W) be a k-representation of* G*, we say it can be lift to a $\overline{\mathbb{Q}}_\ell$-representation of* G*, if there is an ℓ-integral $\overline{\mathbb{Q}}_\ell$-representation of* G*, whose reduction modulo ℓ is equivalent with (τ, W).*

3.1.7 Derivatives of k-representations of $\mathrm{GL}_n(F)$

In this section, let k be an algebraically closed field of characteristic $l \neq p$. Let G_n be the group of F-points of reductive groups GL_n defined over F, where F is a non-archimedean locally compact field whose residue field is of characteristic p. Let P_n denote the mirabolic subgroup of G_n, which is the group consisting of matrices whose last row is $(0, \ldots, 0, 1)$ when $n > 1$, and is the trivial group when $n = 1$. When $n > 1$, the unipotent radical V_{n-1} of P_n is the group consisting of matrices upper-triangular by blocks, where the first block on the diagonal is the identity matrix of size $n - 1 \times n - 1$, and the second bloc on the diagonal is the identity matrix of size 1×1. The group V_{n-1} is isomorphic to the additive group F^{n-1}.

We fix ψ a k-quasicharacter of F. When F is finite of characteristic p, we assume that ψ is non-trivial. When F is non-archimedean locally compact, we assume that ψ is trivial on the normalizer π_F of F but is non-trivial on the integer ring O_F of F. Let U_n be the strictly upper-triangular subgroup of G_n. Any k-character θ of U_n

can be written as $\theta(u_{ij}) = \psi(\sum_{i=1}^{n-1} a_i u_{i,i+1})$, where $(u_{ij})_{i,j} \in \mathrm{U}_n$ and $(a_i)_i \in F^{n-1}$. Wa say ψ is non-degenerate, if $a_i \neq 0$ for any $i \in \{1, \ldots, n-1\}$. The non-degenerate characters of U_n are P_n-conjugate.

Let θ be a non-degenerate k-character of U_n. For any k-character ψ of U and any k-representation (π, V) of G_n, where V is the k-space of representation, we use $V(\mathrm{U}, \psi)$ to denote the subrepresentation generated by $g(v) - \psi(g)v$ for any $v \in V$, and $\pi_{\mathrm{U},\theta}$ to denote the quotient-representation on $V_{\mathrm{U},\psi} = V/V(\mathrm{U}, \psi)$, named as (U, ψ)-coinvariants of π.

Let G_n denote $\mathrm{GL}_n(F)$. Define functors:

- $\Psi^- : \mathrm{Rep}_k(P_n) \to \mathrm{Rep}_k(\mathrm{G}_{n-1})$. Let $(\pi, V) \in \mathrm{Rep}_k(P_n)$, then Ψ^- maps (π, V) to $(\pi_{\mathrm{V}_{n-1},1}, V_{\mathrm{V}_{n-1},1})$.

- $\Psi^+ : \mathrm{Rep}_k(\mathrm{G}_{n-1}) \to \mathrm{Rep}_k(P_n)$. Let $(\pi, V) \in \mathrm{Rep}_k(\mathrm{G}_{n-1})$, then Ψ^+ extends (π, V) trivially to V_{n-1};

- $\Phi^- : \mathrm{Rep}_k(P_n) \to \mathrm{Rep}_k(P_{n-1})$. Let $(\pi, V) \in \mathrm{P_n}$, then Φ^- maps (π, V) to $(\pi_{\mathrm{V}_{n-1},\theta}, V_{\mathrm{V}_{n-1},\theta})$.

- $\Phi^+ : \mathrm{Rep}_k(P_{n-1}) \to \mathrm{Rep}_k(P_n)$. We extend (π, V) to the unipotent radical V_{n-1} of P_n as $g(v) = \theta(g)v$ for any $g \in \mathrm{V}_{n-1}, v \in V$, and denote this extended k-representation of $P_{n-1}\mathrm{V}_{n-1}$ as π_0, then $\Phi^+\pi = \mathrm{ind}_{P_{n-1}\mathrm{V}_{n-1}}^{P_n} \pi_0$.

The definition of functors Φ^+, Φ^- is independent of the choice of θ, since any two non-degenerate k-characters of U are G_n-conjugate. This is not true for general reductive groups. For $\mathrm{G}' = \mathrm{SL}_n(F)$, non-degenerate k-characters of U are not always G'-conjugate, for which we indicate the non-degenerate k-character θ we choose when defining Φ^-, Φ^+ as in Section 6.1.1.

Proposition 3.1.22.
- *$\Psi^+, \Psi^-, \Phi^+, \Phi^-$ are exact.*

- *Ψ^- is the left adjoint of Ψ^+, and Φ^- is the left adjoint of Φ^+.*

Definition 3.1.23. *Let π be a k-representation of G_n. For any $k \in \{1, ..., n\}$, the k-th derivative $\pi^{(k)}$ is defined as the k-representation $\Psi^-(\Phi^-)^{(k-1)}\pi$ of G_{n-k}.*

When $k = n$, the n-th derivative is a k-representation of the trivial group, hence can be seen as a k-vector space. Let $(\pi, V) \in \mathrm{Rep}_k(\mathrm{G}_n)$, we denote the k-vector space of representation $\pi^{(n)}$ as $V^{(n)}$, then $V^{(n)} \cong V_{\mathrm{U},\theta}$.

3.1.8 Whittaker models of irreducible k-representations of $\mathrm{GL}_n(F)$

Let G_n, U_n and k be as in Section 3.1.7, and θ be a non-degenerate k-character of U_n.

Definition 3.1.24. *Let τ denote $\mathrm{ind}_{\mathrm{U}_n}^{\mathrm{G}_n}\theta$, and π be an irreducible k-representation of G_n. We say that π has a Whittaker model if $\mathrm{Hom}_{k\mathrm{G}_n}(\tau,\pi)\neq 0$. Hence π has a Whittaker model if its n-th derivative $\pi^{(n)}\neq 0$. Furthermore, we say that π has a unique Whittaker model if the k-dimension of $\pi^{(n)}$ equals 1.*

Theorem 3.1.25 (§III, 5.10 [V1])**.** *Every irreducible cuspidal k-representation π of G_n has a unique Whittaker model.*

Proposition 3.1.26. *Let π be an ℓ-integral irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of G_n. Then it is ℓ-irreducible, which means its reduction modulo ℓ is irreducible*

Proof. A proof is given in §III,1.9 of [V1]. □

3.2 Maximal cuspidal simple $\overline{\mathbb{Q}}_\ell$-types of $\mathrm{GL}_n(F)$

In this section, F denotes a non-archimedean local field, and G denotes the group $\mathrm{GL}_n(F)$. We introduce basic notations and properties of maximal cuspidal distinguished $\overline{\mathbb{Q}}_\ell$-types ([BuKu]) and maximal cuspidal distinguished k-types of $\mathrm{GL}_n(F)$ ([V1]).

3.2.1 Hereditary orders

Let $\mathfrak{o}_F$ denote the ring of integers of F, and $\mathfrak{p}_F$ denote the maximal ideal of $\mathfrak{o}_F$. Let X be a finite-dimensional F-vector space. An $\mathfrak{o}_F$-lattice in X is a finitely generated $\mathfrak{o}_F$-module which contains a basis of X. If X is also an F-algebra which is associative with 1, then an $\mathfrak{o}_F$-order in X is an $\mathfrak{o}_F$-lattice which is also a subring with the same 1 of X. Let V denote a n-dimensional F-vector space, and A denote $\mathrm{End}_F(V)$, or equivalently $M_n(F)$ the F-algebra of F-matrices of size $n\times n$, and $\mathfrak{A}$ an $\mathfrak{o}_F$-order of A. We say $\mathfrak{A}$ is hereditary, if $\mathfrak{A}$ is an $\mathfrak{o}_F$-order of A and any $\mathfrak{A}$-lattice in any finitely generated A-module is $\mathfrak{A}$-projective.

As in §1.1 [BuKu], we can express hereditary $\mathfrak{o}_F$-orders in A in terms of $\mathfrak{o}_F$-lattice chains in F^n. An $\mathfrak{o}_F$-lattice chain in an F-vector space V is a non-empty set $\mathcal{L}=\{L_i : i\in\mathbb{Z}\}$ of $\mathfrak{o}_F$-lattices in V such that:

1. $L_i \supsetneqq L_{i+1}$ for all $i\in\mathbb{Z}$;
2. there exists an integer $e\in\mathbb{Z}$, such that $\mathfrak{p}_F L_i = L_{i+e}$ for any $i\in\mathbb{Z}$.

We could define

$$\mathrm{End}^m_{\mathfrak{o}_F}(\mathcal{L})=\{x\in A : xL_i\subset L_{i+m}, i\in\mathbb{Z}\}$$

for each $m\in\mathbb{Z}$. In particular $\mathrm{End}^0_{\mathfrak{o}_F}$ is an hereditary $\mathfrak{o}_F$-order in A, denoted as $\mathfrak{A}(\mathcal{L})$. Conversely, any hereditary $\mathfrak{o}_F$-order is of this form, for some lattice chain $\mathcal{L}$. In additional, if $(L_i : L_{i+1})=(L_j : L_{j+1})$, for all $i,j\in\mathbb{Z}$, the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}(\mathcal{L})$ is called principal.

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A, and $\mathcal{L}$ a lattice chain associated, which means $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L})$. We denote the Jacobson radical of $\mathfrak{A}$ by

$$\mathfrak{P} = \mathrm{rad}(\mathfrak{A}) = \mathrm{End}^1_{\mathfrak{o}_F}(\mathcal{L}).$$

The ideal $\mathfrak{P}$ is a fractional ideal and invertible. And we have

$$\mathfrak{P}^n = \mathrm{End}^n_{\mathfrak{o}_F}(\mathcal{L}), n \in \mathbb{Z}.$$

We have

$$\mathfrak{P}^n L_i = L_{i+n},$$
$$\mathfrak{p}_F \mathfrak{A} = \mathfrak{P}^e,$$

and we set

$$\mathbf{U}^0(\mathfrak{A}) = \mathbf{U}(\mathfrak{A}) = \mathfrak{A}^\times,$$
$$\mathbf{U}^m(\mathfrak{A}) = 1 + \mathfrak{P}^m, m \geq 1.$$

Since $\mathrm{G} \subset A$, we use $\mathfrak{K}(\mathfrak{A})$ to denote the G-normaliser of $\mathcal{L}$ as

$$\mathfrak{K}(\mathfrak{A}) = \{x \in \mathrm{G} : xL_i \in \mathcal{L}, i \in \mathbb{Z}\}.$$

We define a "valuation" according to a fixed hereditary order $\mathfrak{A}$: For any $x \in A, x \neq 0$

$$\nu_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} : x \in \mathfrak{P}^n\},$$

and $\nu_{\mathfrak{A}}(0) = \infty$.

From now on we fix a continuous $\overline{\mathbb{Q}}_\ell$-character ψ_F of the additive group F which is null on the maximal ideal $\mathfrak{p}_F$ but non-null on the integer ring $\mathfrak{o}_F$ of F. Let m, r be integers such that $0 \leq [\frac{m}{2}] \leq r < m$. There is an isomorphism

$$(\mathbf{U}^{r+1}(\mathfrak{A})/\mathbf{U}^{m+1}(\mathfrak{A}))^\wedge \cong \mathfrak{P}^{-m}/\mathfrak{P}^{-r},$$

where "$\wedge$" denotes the Pontrjagin dual, and the isomorphism is defined by

$$b + \mathfrak{P}^{-r} \mapsto \psi_{A,b} = \psi_b, \quad b \in \mathfrak{P}^{-m}, \quad \text{where}$$
$$\psi_b(1 + x) = \psi_F \circ \mathrm{tr}_{A/F}(bx), \quad x \in \mathfrak{P}^{r+1}.$$

We suppose that E/F is a field extension contained in $A = \mathrm{End}_F(V)$, where V is a n-dimensional F-vector space, then V can be viewed as an E-vector space, via the inclusion $E \to A$. Now we consider hereditary orders relative to such subfield E in A.

Proposition 3.2.1. *Let $\mathfrak{A}$ be a hereditary order in A, with $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L})$, for some $\mathfrak{o}_F$-lattice chain $\mathcal{L} = \{L_i\}$ in V. Let E/F be some subfield of A. The following conditions are equivalent:*

- *$E^\times \subset \mathfrak{K}(\mathfrak{A})$ (i.e. $E^\times$ normalises $\mathfrak{A}$);*

- *each L_i is an $\mathfrak{o}_E$-lattice and there is an integer e' such that $\mathfrak{p}_E L_i = L_{i+e'}$ for all i (i.e. $\mathcal{L}$ is an $\mathfrak{o}_E$-lattice chain in the E-vector space V).*

We suppose that the equivalent conditions of Proposition 3.2.1 hold. Define

$$B = \mathrm{End}_E(V) = \text{the } A\text{-centraliser of } E,$$

$$\mathfrak{B} = \mathfrak{A} \cap B,\ \mathfrak{Q} = \mathfrak{P} \cap B.$$

We could write E as $F[\beta]$ for some element $\beta \in A$, and some times we also denote B as B_β.

Definition 3.2.2. *Let E/F be a field extension of F, and $\mathfrak{A}$ a hereditary order contained in A. Assume that $E^\times \subset \mathfrak{K}(\mathfrak{A})$. Let $\beta \in A$ such that $E = F[\beta]$, and $k \in \mathbb{Z}$. Define*

$$\mathfrak{N}_k = \mathfrak{N}(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} : (\beta x - x\beta) \in \mathfrak{P}^k\}$$

The subset $\mathfrak{N}_k(\beta, \mathfrak{A})$ is a lattice in A, and

$$\mathfrak{N}_k(\beta, \mathfrak{A}) \subset \mathfrak{B} + \mathfrak{P},$$

for all sufficiently large k as proved in §(1.4.4) [BuKu]. While $k \leq \nu_{\mathfrak{A}}(\beta)$, we always have $\mathfrak{A} = \mathfrak{N}_k(\beta, \mathfrak{A})$.

Definition 3.2.3. *Suppose that $E \neq F$. Define*

$$k_0 = k_0(\beta, \mathfrak{A}) = \max\{k \in \mathbb{Z} : \mathfrak{N}_k \not\subset \mathfrak{B} + \mathfrak{P}\}.$$

While $E = F$, we define $k_0(\beta, \mathfrak{A}) = -\infty$. Furthermore, we always have the inequality

$$\nu_{\mathfrak{A}}(\beta) \leq k_0(\beta, \mathfrak{A}),$$

and we say β is minimal over F if $\nu_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A})$.

3.2.2 Simple stratum and simple $\overline{\mathbb{Q}}_\ell$-characters

Definition 3.2.4. *Let $\mathfrak{A}$ be a hereditary order, and $n, r \in \mathbb{Z}$, $b, \beta \in A$.*

- *The 4-tuple $[\mathfrak{A}, n, r, b]$ is called a stratum in A, if $r < n$ and $-n \leq \nu_{\mathfrak{A}}(b)$.*
- *Let $[\mathfrak{A}, n, r, b]$ be a stratum in A. It is simple if*
  1. *the algebra $E = F[\beta]$ is a field,*
  2. *$E^\times \subset \mathfrak{K}(\mathfrak{A})$,*
  3. *$\nu_{\mathfrak{A}}(\beta) = -n$,*
  4. *$r < -k_0(\beta, \mathfrak{A})$.*

Definition 3.2.5. *Let $[\mathfrak{A}_i, n_i, r_i, b_i]$ be strata in A, $i = 1, 2$, and $\mathfrak{P}_i = \mathrm{rad}(\mathfrak{A}_i)$, we say they are equivalent if*

$$b_1 + \mathfrak{P}_1^{-r_1} = b_2 + \mathfrak{P}_2^{-r_2},$$

denoted as

$$[\mathfrak{A}_1, n_1, r_1, b_1] \sim [\mathfrak{A}_2, n_2, r_2, b_2].$$

We now fix a simple stratum $[\mathfrak{A}, n, 0, \beta]$, and set $r = -k_0(\beta, \mathfrak{A})$. We will introduce simple characters, and first we need to define a pair of $\mathfrak{o}_F$-orders (not hereditary) such that

$$\mathfrak{H}(\beta, \mathfrak{A}) \subset \mathfrak{J}(\beta, \mathfrak{A}) \subset \mathfrak{A}.$$

Definition 3.2.6.

1. *Suppose that β is minimal over F, which means $r = n$ or ∞. Put*

$$\mathfrak{H}(\beta) = \mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{P}^{[\frac{n}{2}]+1}.$$

2. *Suppose that $r < n$, and let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Put*

$$\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{H}(\beta) = \mathfrak{B}_\beta + \mathfrak{H}(\gamma) \cap \mathfrak{P}^{[\frac{r}{2}]+1}.$$

Definition 3.2.7.

1. *Suppose that β is minimal over F. Put*

$$\mathfrak{J}(\beta) = \mathfrak{B}_\beta + \mathfrak{P}^{[\frac{n+1}{2}]}.$$

2. *Suppose that β is not minimal over F, and let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Put*

$$\mathfrak{J}(\beta) = \mathfrak{B}_\beta + \mathfrak{J}(\gamma) \cap \mathfrak{P}^{[\frac{r+1}{2}]}.$$

The case 2 of the two definitions above are well defined. Under the assumption of case 2, the stratum $[\mathfrak{A}, n, r, \beta]$ is pure but not simple, and we will define $\mathfrak{H}(\beta)$ and $\mathfrak{J}(\beta)$ by iteration relative to the defining sequence for $[\mathfrak{A}, n, r, \beta]$ as below:

Lemma 3.2.8. *For a given pure stratum $[\mathfrak{A}, n, r, \beta]$. There is a family $[\mathfrak{A}, n, r_i, \gamma_i], 0 \leq i \leq s$, of simple strata, such that*

- $[\mathfrak{A}, n, r_0, \gamma_0] \sim [\mathfrak{A}, n, r, \beta]$;
- $r = r_0 < r_1 < ... < r_s < n$;
- $r_{i+1} = -k_0(\gamma_i, \mathfrak{A})$, *and* $[\mathfrak{A}, n, r_{i+1}, \gamma_{i+1}]$ *is equivalent to* $[\mathfrak{A}, n, r_{i+1}, \gamma_{i+1}], 0 \leq i \leq s-1$;
- $k_0(\gamma_s, \mathfrak{A}) = -n$ *or* $-\infty$;
- *Let $\mathfrak{B}_i$ be the $\mathfrak{A}$-centraliser of γ_i and s_i be a tame corestriction on A relative to $F[\gamma_i]/F$. The derived stratum $[\mathfrak{B}_i, r_i, r_i - 1, s_i(\gamma_{i-1} - \gamma_i)]$ is equivalent ot a simple stratum for $1 \leq i \leq s$.*

We call such family the defining sequence for $[\mathfrak{A}, n, r, \beta]$.

In the last condition of Lemma 3.2.8, a tame corestriction on A relative to the field extension $(F[\beta] =)E/F$, is a (B_β, B_β)-bimodule homomorphism $s : A \to B_\beta$ such that $s(\mathfrak{A}) = \mathfrak{A} \cap B_\beta$ for any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A which is normalised by $E^\times$.

Definition 3.2.9. *We now define two families of open compact subgroups in* G *according to* $\mathfrak{H}(\beta, \mathfrak{A})$ *and* $\mathfrak{J}(\beta, \mathfrak{A})$ *by*

$$\begin{cases} H^m(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap \boldsymbol{U}^m(\mathfrak{A}) \\ J^m(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap \boldsymbol{U}^m(\mathfrak{A}) \end{cases},$$

for $m \geq 0$. *In particular, we always use* $H(\beta, \mathfrak{A})$ *and* $J(\beta, \mathfrak{A})$ *instead of* $H^0(\beta, \mathfrak{A})$ *and* $J^0(\beta, \mathfrak{A})$.

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and $r = -k_0(\beta, \mathfrak{A})$. We now define the set of simple characters $C(\mathfrak{A}, m, \beta)$ for $m \geq 0$.

Definition 3.2.10. *Suppose that* β *is minimal over* F. *For* $0 \leq m \leq n-1$, *let* $C_{\overline{\mathbb{Q}}_\ell}(\mathfrak{A}, m, \beta)$ *denote the set of* $\overline{\mathbb{Q}}_\ell$*-characters* θ *of* $H^{m+1}(\beta)$ *such that*

1. $\theta|_{H^{m+1}(\beta) \cap \boldsymbol{U}^{[\frac{n}{2}]+1}(\mathfrak{A})} = \psi_\beta$;
2. $\theta|_{H^{m+1}(\beta) \cap B_\beta^\times}$ *factors through* $\det_{B_\beta} : B_\beta^\times \to F[\beta]^\times$.

Definition 3.2.11. *Suppose that* $r = -k_0(\beta, \mathfrak{A}), r < n$. *We take a simple stratum* $[\mathfrak{A}, n, r, \gamma]$ *equivalent to* $[\mathfrak{A}, n, r, \beta]$ *(as in 3.2.8).*

- *For* $0 \leq m \leq r-1$, *let* $C_{\overline{\mathbb{Q}}_\ell}(\mathfrak{A}, m, \beta)$ *be the set of* $\overline{\mathbb{Q}}_\ell$*-characters* θ *of* $H^{m+1}(\beta)$ *such that*
  1. $\theta|_{H^{m+1}(\beta) \cap B_\beta^\times}$ *factors through* $\det_{B_\beta} : B_\beta^\times \to F[\beta]^\times$;
  2. θ *is normalised by* $\mathfrak{K}(\mathfrak{B}_\beta)$;
  3. *if* $m' = \max\{m, [\frac{r}{2}]\}$, *the restriction* $\theta|_{H^{m'+1}(\beta)}$ *is of the form* $\theta_0 \psi_c$, *for some* $\theta_0 \in C_{\overline{\mathbb{Q}}_\ell}(\mathfrak{A}, m', \gamma)$, *where* $c = \beta - \gamma$.
- *For* $m \geq r$, *we define* $C_{\overline{\mathbb{Q}}_\ell}(\mathfrak{A}, m, \beta) = C_{\overline{\mathbb{Q}}_\ell}(\mathfrak{A}, m, \gamma)$.

In (3) of Definition 3.2.11, we have the identity of two groups $H^{m'+1}(\beta) = H^{m'+1}(\gamma)$ as in the last paragraph of §3.1.9 [BuKu]. As defined in Definition 3.2.5, the element $c = \beta - \gamma$ belongs to $\mathfrak{P}^r$ and ψ_c is a character of $\mathbf{U}^{[\frac{r}{2}]+1}/\mathbf{U}^{r+1}$, hence is trivial on $H^{r+1}(\beta)$. It therefore defines a character of $H^{m'+1}(\beta)/H^{r+1}(\beta)$.

3.2.3 Simple $\overline{\mathbb{Q}}_\ell$-types of $\mathrm{GL}_n(F)$

Now we introduce simple $\overline{\mathbb{Q}}_\ell$-types. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum of A, E the field $F[\beta]$, B the A-centraliser of E, $H^1(\beta, \mathfrak{A})$ and $J^1(\beta, \mathfrak{A})$ as defined in Definition 3.2.9.

Proposition 3.2.12. *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, and $\theta \in C(\mathfrak{A}, 0, \beta)$. There exists a unique irreducible $\overline{\mathbb{Q}}_\ell$-representation $\eta(\theta)$ of the group $J^1(\beta, \mathfrak{A})$ such that $\eta(\theta)|H^1(\beta, \mathfrak{A})$ contains θ up to isomorphism. Moreover, $\eta(\theta)|H^1(\beta, \mathfrak{A})$ is a multiple of θ, and*

$$\dim(\eta(\theta)) = (J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A}))^{\frac{1}{2}}.$$

The G*-intertwining of $\eta(\theta)$ is $J^1(\beta, \mathfrak{A})B^\times J^1(\beta, \mathfrak{A})$.*

Definition 3.2.13. *A β-extension of η (as defined in Proposition 3.2.12) is a $\overline{\mathbb{Q}}_\ell$-representation κ of $J(\beta, \mathfrak{A})$ such that*

1. *$\kappa|_{J^1(\beta,\mathfrak{A})} = \eta$, and*
2. *κ is intertwined by the whole of $B^\times$,*

where B denotes the A-centraliser of β.

Definition 3.2.14. *A simple $\overline{\mathbb{Q}}_\ell$-type in* G *is one of the following* (1) *or* (2)*:*

1. *An irreducible $\overline{\mathbb{Q}}_\ell$-representation $\lambda = \kappa \otimes \sigma$ of $J = J(\beta, \mathfrak{A})$ where:*
   (a) *$\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in A and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum;*
   (b) *for some $\theta \in C_{\mathbb{C}}(\mathfrak{A}, 0, \beta)$, κ is a β-extension of the unique irreducible $\overline{\mathbb{Q}}_\ell$-representation η of $J^1(\beta, \mathfrak{A})$ which contains θ as in Proposition 3.2.12;*
   (c) *if we write $E = F[\beta], \mathfrak{B} = \mathfrak{A} \cap B$, where $B = \mathrm{End}_E(V)$. So that*

   $$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong \boldsymbol{U}(\mathfrak{B})/\boldsymbol{U}^1(\mathfrak{B}) \cong \mathrm{GL}_f(\mathrm{k}_E)^e,$$

   for certain integers e, f, and σ is the inflation of a $\overline{\mathbb{Q}}_\ell$-representation $\sigma_0 \otimes ... \otimes \sigma_0$ of the group $J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A})$, where σ_0 is an irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_f(\mathrm{k}_E)$.
2. *an irreducible $\overline{\mathbb{Q}}_\ell$-representation σ of $\boldsymbol{U}(\mathfrak{A})$, where*
   (a) *$\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in A;*
   (b) *if we write $\boldsymbol{U}(\mathfrak{A})/\boldsymbol{U}^1(\mathfrak{A}) \cong \mathrm{GL}_f(\mathrm{k}_E)^e$, for certain integers e, f, then σ is the inflation of a $\overline{\mathbb{Q}}_\ell$-representation of $\sigma_0 \otimes ... \otimes \sigma_0$, where σ_0 is an irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_f(\mathrm{k}_F)$.*

3.2.4 Intertwining and extension by a central character

Definition 3.2.15. *Let K_i be open compact subgroups of G, and π_i be irreducible $\overline{\mathbb{Q}}_\ell$-representations of K_i, for $i = 1, 2$, we say that $g \in \mathrm{G}$ intertwines π_1 with π_2, if $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[K_1 \cap g(K_2)]}(\pi_1, g(\pi_2))$ is non-trivial.*

Theorem 3.2.16. *Let $(J_1, \lambda_1), (J_2, \lambda_2)$ be simple $\overline{\mathbb{Q}}_\ell$-types in G, attached to principal $\mathfrak{o}_F$-orders $\mathfrak{A}_1, \mathfrak{A}_2$, respectively. Suppose that $\mathfrak{A}_1 \cong \mathfrak{A}_2$ as $\mathfrak{o}_F$-orders, and that the $\overline{\mathbb{Q}}_\ell$-representations λ_1, λ_2 intertwine in G. Then there exists $x \in \mathrm{G}$ such that $J_2 = x^{-1} J_1 x$ and λ_2 is equivalent to the x-conjugation $x(\lambda_1)$ of λ_1.*

Fixing a simple type (J, λ), we have $J \cap F^\times = \mathfrak{o}_F^\times$ and the restriction $\lambda|_{\mathfrak{o}_F^\times}$ is a multiple of a $\overline{\mathbb{Q}}_\ell$-quasicharacter ω_λ of $\mathfrak{o}_F^\times$. Since $F \cong \mathfrak{o}_F^\times \times \mathbb{Z}$, we can always extend ω_λ as a $\overline{\mathbb{Q}}_\ell$-quasicharacter of $F^\times$. Let ω be some $\overline{\mathbb{Q}}_\ell$-quasicharacter of $F^\times$ such that $\omega|_{\mathfrak{o}_F^\times} = \omega_\lambda$. We can extend λ to a $\overline{\mathbb{Q}}_\ell$-representation λ_ω of $F^\times J$ by

$$\lambda_\omega(zj) = w(z)\lambda(j), j \in J, z \in F^\times.$$

This is well defined since $F^\times$ is the center of G. We now show properties of extension of λ to $E^\times J$.

Proposition 3.2.17. *Let Λ be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $E^\times J$ such that $\Lambda|_J$ contains λ. Then*

1. *there is a unique $\overline{\mathbb{Q}}_\ell$-quasicharacter ω of $F^\times$ such that $\omega|_{\mathfrak{o}_F^\times} = \omega_\lambda$ and $\Lambda|_{F^\times J} = \lambda_\omega$;*
2. *given λ_ω as above, there exists $e(E|F)$ distinct extensions Λ of λ_ω to $E^\times J$;*
3. *given $w' \in \widetilde{\boldsymbol{W}}(\mathfrak{B})$ (see Definition 5.5.9 in [BuKu]) and an extension Λ of λ_ω to $E^\times J$, there is a unique extension Λ' of λ_ω such that w' intertwines Λ with Λ'.*

In particular, if $\mathfrak{B}$ is a maximal order in B, distinct extensions of λ_ω to $E^\times J$ do not intertwine in G.

Proposition 3.2.18. *Let (J, λ) be a simple $\overline{\mathbb{Q}}_\ell$-type in G, and Λ be an extension of λ to $E^\times J$ as in Proposition 3.2.17. Assume that the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached to (J, λ) is maximal. Then the intertwining set $I_{\mathrm{G}}(\Lambda) = E^\times J$.*

3.2.5 Maximal simple cuspidal $\overline{\mathbb{Q}}_\ell$-types of $\mathrm{GL}_n(F)$

Theorem 3.2.19. *Let (J, λ) be a simple type in G, and suppose that there exists an irreducible supercuspidal $\overline{\mathbb{Q}}_\ell$-representation π of G such that $\pi|_J$ contains λ. Then the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached to (J, λ) is maximal.*

Theorem 3.2.20. *Suppose that the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached to (J, λ) is maximal. Then any irreducible $\overline{\mathbb{Q}}_\ell$-representation π of* G *containing λ is supercuspidal. Moreover, for any such $\overline{\mathbb{Q}}_\ell$-representation π, there is a uniquely determined $\overline{\mathbb{Q}}_\ell$-representation Λ of $E^\times J$ such that $\Lambda|_J = \lambda$ and*

$$\pi \cong \mathrm{ind}_{E^\times J}^{\mathrm{G}} \Lambda.$$

The two theorems above and Theorem 3.2.16 imply that there is a bijection between the set of irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representations of G and the set of G-conjugacy class of simple types (J, λ), such that the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached is maximal. Hence we define maximal cuspidal simple $\overline{\mathbb{Q}}_\ell$-types of G as below.

Definition 3.2.21. *Let (J, λ) be a simple $\overline{\mathbb{Q}}_\ell$-type of* G*, we call it maximal cuspidal simple, if the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached to (J, λ) is maximal.*

3.3 Maximal simple cuspidal k-types of $\mathrm{GL}_n(F)$

In this section, we assume that G denotes $\mathrm{GL}_n(F)$. Let k be an algebraically closed field of characteristic $l \neq p$. In §III, 4 of [V1], Vignéras gave a construction of maximal simple cuspidal k-types of G_n, and a relation between maximal simple cuspidal k-types and maximal simple cuspidal k-types of G through the method of reduction modulo ℓ (Definition 3.1.21).

3.3.1 Simple k-types of $\mathrm{GL}_n(F)$

As in §III, 4 of [V1], the construction of hereditary orders (§3.2.1), simple strata, simple $\overline{\mathbb{Q}}_\ell$-characters (§3.2.2) and simple $\overline{\mathbb{Q}}_\ell$-types (§3.2.3) can be kept and generalised to the case where the coefficient field is k. In particular, when we fix a simple stratum $[\mathfrak{A}, n, 0, \beta]$, the $\mathfrak{o}_F$-orders $\mathfrak{H}(\beta, \mathfrak{A}), \mathfrak{J}(\beta, \mathfrak{A})$ are independent with the coefficient fields k or $\overline{\mathbb{Q}}_\ell$. And let $C_\ell(\mathfrak{A}, m, \beta)$ be the set of simple k-characters on $H^{m+1}(\beta, \mathfrak{A})$ for each $m \geq 0$.

Definition 3.3.1. *A simple k-type in* G *is one of the following two cases* 1 *or* 2*:*

1. *An irreducible k-representation $\lambda = \kappa \otimes \sigma$ of $J = J(\beta, \mathfrak{A})$ where:*
   - *(a) $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in A and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum;*
   - *(b) for some $\theta \in C_\ell(\mathfrak{A}, 0, \beta)$, κ is a β-extension of the unique irreducible k-representation η of $J^1(\beta, \mathfrak{A})$ which contains θ as in Proposition 3.2.12;*
   - *(c) if we write $E = F[\beta], \mathfrak{B} = \mathfrak{A} \cap B$, where $B = \mathrm{End}_E(V)$. So that*

$$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong \boldsymbol{U}(\mathfrak{B})/\boldsymbol{U}^1(\mathfrak{B}) \cong \mathrm{GL}_f(\mathrm{k}_E)^e,$$

   for certain integers e, f, and σ is the inflation of a k-representation $\sigma_0 \otimes \ldots \otimes \sigma_0$, where σ_0 is an irreducible cuspidal k-representation of $\mathrm{GL}_f(\mathrm{k}_E)$.

2. *an irreducible k-representation σ of $\boldsymbol{U}(\mathfrak{A})$, where*

   (a) *$\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in A;*

   (b) *if we write $\boldsymbol{U}(\mathfrak{A})/\boldsymbol{U}^1(\mathfrak{A}) \cong \mathrm{GL}_f(\mathrm{k}_E)^e$, for certain integers e, f, then σ is the inflation of a k-representation of $\sigma_0 \otimes ... \otimes \sigma_0$, where σ_0 is an irreducible cuspidal k-representation of $\mathrm{GL}_f(\mathrm{k}_F)$.*

When $k = \overline{\mathrm{F}}_\ell$, we consider the operator of reduction modulo ℓ from simple $\overline{\mathbb{Q}}_\ell$-types to simple k-types of G. Otherwise, let $W(k)$ denote the ring of Witt vectors, $\mathcal{K}$ denote the fractional field $\mathrm{Frac}(W(k))$, and $\overline{\mathcal{K}}$ an algebraic closure of $\mathcal{K}$. We consider the operator of reduction modulo ℓ from simple $\overline{\mathcal{K}}$ types to k-simple types of G.

Proposition 3.3.2 (§III, 4.25 in [V1])**.** *The reduction modulo ℓ of an ℓ-integral simple $\overline{\mathbb{Q}}_\ell$-type $(J, \lambda_{\overline{\mathbb{Q}}_\ell})$ of* G *is a simple k-type (J, λ_ℓ) of* G*; conversely, each simple k-type (J, λ_ℓ) of* G *can be lifted to a simple $\overline{\mathbb{Q}}_\ell$-type $(J, \lambda_{\overline{\mathbb{Q}}_\ell})$ of* G*, which means that $(J, \lambda_{\overline{\mathbb{Q}}_\ell})$ is ℓ-integral, and its reduction modulo ℓ is equivalent with (J, λ_ℓ).*

Let (J, λ_ℓ) be a simple k-type of G, and $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, $E = F[\beta]$ be the field extension contained in A attached to (J, λ_ℓ). The representation λ_ℓ can always be extended to the group $E^\times J$ as explained in §III, 4.27.1 of [V1]. Let Λ_ℓ be one of such extension.

Proposition 3.3.3 (§III, 4.29 in [V1])**.** *Let $(J, \lambda_{\overline{\mathbb{Q}}_\ell})$ be a simple $\overline{\mathbb{Q}}_\ell$-type of* G*, and $\Lambda_{\overline{\mathbb{Q}}_\ell}$ be an extension of $\lambda_{\overline{\mathbb{Q}}_\ell}$ to $E^\times J$, then its reduction modulo ℓ is an extension of a simple k-type (J, λ_ℓ) of* G*. Conversely, let (J, λ_ℓ) be a simple k-type of* G*, and Λ_ℓ be an extension of λ_ℓ to $E^\times J$, then Λ_ℓ can be lift to an extension of a simple $\overline{\mathbb{Q}}_\ell$-type of* G*.*

3.3.2 Maximal simple cuspidal k-types of $\mathrm{GL}_n(F)$

In this section, let G denote $\mathrm{GL}_n(F)$, and $k = \overline{\mathrm{F}}_\ell$. We will give a construction of irreducible cuspidal k-representations of G through maximal simple cuspidal k-types, and give a relation between irreducible cuspidal k-representations and irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representations.

Definition 3.3.4. *Let (J, λ_ℓ) be a simple k-type of* G*, we say it is maximal simple cuspidal if the $\mathfrak{o}_E$-order $\mathfrak{B}$ attached to (J, λ_ℓ) is maximal.*

Theorem 3.3.5 (§III, 1.1d), 5.2, 5.3, 5.8 in [V1])**.** *Let (J, λ_ℓ) be a maximal simple cuspidal k-type of* G *and Λ_ℓ be an extension of λ to $E^\times J$. The induction*

$$\mathrm{ind}_{E^\times J}^{\mathrm{G}} \Lambda_\ell$$

is irreducible and cuspidal.

Conversely, let π be an irreducible cuspidal k-representation of G*. There exists a maximal simple cuspidal k-type (J, λ_ℓ) and an extension Λ_ℓ of λ_ℓ to $E^\times J$, such that $\pi \cong \mathrm{ind}_{E^\times J}^{\mathrm{G}} \Lambda_\ell$.*

Furthermore, for each irreducible cuspidal k-representation π, in which the maximal simple cuspidal k-types contained are unique up to G*-conjugation.*

Applying Proposition 3.1.26, 3.3.2, and 3.3.3, we conclude that:

Theorem 3.3.6. *Each irreducible cuspidal k-representation can be lifted to an irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of* G.

3.4 Supercuspidal support of irreducible k-representations of $\mathrm{GL}_n(F)$

Let G_n denote $\mathrm{GL}_n(F)$, where F is a finite field of characteristic $p \neq l$ or a non-archimedean locally compact field whose residue field is of characteristic $p \neq l$. Let k be an algebraically closed field of characteristic l. Vignéras gave a proof of uniqueness of supercuspidal support of irreducible k-representations of G in §V4 in [V2]:

Theorem 3.4.1 (Uniqueness of supercuspidal support)**.** *Let $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_t$ be irreducible supercuspidal k-representations of $\mathrm{G}_{n_1}, ..., \mathrm{G}_{n_r}, \mathrm{G}_{m_1}, ...\mathrm{G}_{m_t}$, where $n = n_1 + ... + n_r = m_1 + ... + m_t$. Then the Jordan-Holder seuquences of*

$$i_{\mathrm{G}_{n_1}\times\cdots\times\mathrm{G}_{n_r}}^{\mathrm{G}_{n_1+\cdots+n_r}} \rho_1 \otimes \cdots \otimes \rho_r$$

and

$$i_{\mathrm{G}_{m_1}\times\cdots\times\mathrm{G}_{m_t}}^{\mathrm{G}_{m_1+\cdots+m_t}} \rho'_1 \otimes \cdots \otimes \rho'_t$$

are equal, if and only if $r = t, m_i = n_i$, and the multisets $\{\rho_1, \dots \rho_r\}$ and $\{\rho'_1, \dots, \rho'_r\}$ are equal. Otherwise, they are disjoint.

In [V2], Vignéras also defined the supercuspidal support of k-simple types of G_n, and give a proof of uniqueness of supercuspidal support of simple k-types of G in §IV, 2.3 of [V2].

3.5 Bernstein center

Let k denote an algebraically closed field of strictly positive characteristic $l \neq p$, and $W(k)$ be its ring of Witt vectors. Let $\mathcal{K}$ be the fractional field of $W(k)$, and $\overline{\mathcal{K}}$ denote an algebraic closure of $\mathcal{K}$.

3.5.1 Bernstein decomposition of $\mathrm{Rep}_k(\mathrm{GL}_n(F))$ and $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_n(F))$

Let G denote $\mathrm{GL}_n(F)$, and M denote a Levi subgroup of G. Let τ be an irreducible cuspidal k-representation of G, and $(J, \lambda \cong \kappa \otimes \sigma)$ be the maximal simple cuspidal k-type associated. Then κ can be lift to a $W(k)$[G]-module, noted as $\tilde{\kappa}$. Let $\mathcal{P}_\sigma$ denote the projective $W(k)$-cover of σ in the category of $\mathrm{Rep}_{W(k)}(J/J^1)$. Define

$\mathcal{P}_{J,\lambda}$ as $\mathrm{ind}_J^{\mathrm{G}}(\tilde{\kappa}\otimes\mathcal{P}_\sigma)$. Let $\delta\cong\delta_1\otimes...\delta_m$ be an irreducible cuspidal k-representation of M, and (J_i,λ_i) the maximal simple cuspidal k-types of δ_i. Then define $\mathcal{P}_{(\mathrm{M},\delta)}$ as $i_{\mathrm{M}}^{\mathrm{G}}\mathcal{P}_{J_1,\lambda_1}\otimes...\otimes\mathcal{P}_{J_m,\lambda_m}$. Let $\delta^\vee$ be the dual of δ, define $I_{(\mathrm{M},\delta)}$ as $\mathcal{P}^\vee_{(\mathrm{M},\delta^\vee)}$. Now we fix a supercuspidal k-pair (M,π) in G. For any simple $W(k)[\mathrm{G}]$-module, whose mod ℓ supercuspidal support is defined by (M,π), its mod ℓ cuspidal support falls into finitely many possible inertially equivalence classes (Definition 3.5.1) defined by cuspidal pairs (M_j,π_j) of G. We define $I_{[\mathrm{M},\pi]}$ as $\oplus_j I_{(\mathrm{M}_j,\pi_j)}$.

Definition 3.5.1. *Let* M_1,M_2 *be two Levi subgroups of* G*, and* π_1,π_2 *be two* k*-representations of* M_1,M_2 *respectively. We say that* (M_1,π_1) *and* (M_2,π_2) *belong to one inertially equivalence class if and only if there exists an unramified* k*-character* χ *of* M_1 *such that* M_1 *is* G *conjugate to* M_2*, and* $\pi_1\otimes\chi$ *is* G *conjugate to* π_2*.*

Theorem 3.5.2 (Theorem 11.8 in [Helm])**.** *Let* $[\mathrm{M},\pi]$ *be an inertially equivalence class of a supercuspidal pair* (M,π) *(see Definition 3.1.13). The full subcategory* $\mathrm{Rep}_{W(k)}(\mathrm{G})_{[\mathrm{M},\pi]}$ *of* $\mathrm{Rep}_{W(k)}(\mathrm{G})$ *consisting of smooth* $W(k)[\mathrm{G}]$*-modules* Π *such that every simple subquotient of* Π *has mod* ℓ *inertial supercuspidal support given by* (M,π) *is a block of* $\mathrm{Rep}_{W(k)}(\mathrm{G})$*. Moreover, every element of* $\mathrm{Rep}_{W(k)}(\mathrm{G})_{[\mathrm{M},\pi]}$ *has a resolution by direct sums of copies of* $I_{[\mathrm{M},\pi]}$*.*

3.5.2 Bernstein center of $\mathrm{Rep}_k(\mathrm{GL}_n(F))$ and $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_n(F))$

In [Helm], Helm gave a relation between the Bernstein centre of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))$ and the Bernstein centre of $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_n(F))$.

Definition 3.5.3. *Let* $\mathcal{A}$ *be an abelian category. The centre of* $\mathcal{A}$ *is the endomorphisms of the identity functor* $\mathrm{Id}:\mathcal{A}\to\mathcal{A}$*.*

Now we come back to the category $\mathrm{Rep}_{W(k)}(\mathrm{G})$, which is equivalent to the direct product of full subcategories $\mathrm{Rep}_{W(k)}(\mathrm{G})_{[\mathrm{M},\pi]}$, where $[\mathrm{M},\pi]$ runs through the inertially equivalence classes of supercuspidal k-pairs of G, as in Theorem 3.5.2.

Proposition 3.5.4 (Proposition 12.1 in [Helm])**.** *Let* $A_{[\mathrm{L},\pi]}$ *be the centre of the category* $\mathrm{Rep}_{W(k)}(\mathrm{G})_{[\mathrm{L},\pi]}$*. There is a natural isomorphism:*

$$A_{[\mathrm{L},\pi]}\otimes\overline{\mathcal{K}}\cong\prod_{(\tilde{\mathrm{M}},\tilde{\pi})}A_{\tilde{M},\tilde{\pi}},$$

where $(\tilde{\mathrm{M}},\tilde{\pi})$ *runs over inertial equivalence classes of pairs in which* $\tilde{\mathrm{M}}$ *is a Levi subgroup of* G *and* $\tilde{\pi}$ *is a cuspidal representation of* M *over* $\overline{\mathcal{K}}$ *whose mod* ℓ *inertial supercuspidal support equals* (L,π)*. This isomorphism is uniquely characterised by the property that for any* Π *in* $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{G})$*, and any* x *in* $A_{[\mathrm{L},\pi]}$*, the action of* x *on* Π *coincides with that of its image in* $\prod_{(\tilde{\mathrm{M}},\tilde{\pi})}A_{\tilde{\mathrm{M}},\tilde{\pi}}$*.*

In the proposition above, $A_{\tilde{\mathrm{M}},\tilde{\pi}}$ denotes the centre of the subcategory

$$\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_n(F))_{\tilde{\mathrm{M}},\tilde{\pi}}$$

of $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_n(F))$, defined by the inertially equivalence class of the cuspidal $\overline{\mathcal{K}}$-pair $(\tilde{\mathrm{M}}, \tilde{\pi})$.

Chapter 4

k-representations of finite groups $\mathrm{SL}_n(F)$

4.1 Representation theory of finite groups

Let $\mathbf{G}'$ and $\mathbf{G}$ be the connected reductive group defined over F_q with type SL_n and type GL_n respectively, where q is a power of a prime number p. Note $\mathrm{G}' = \mathbf{G}'(\mathrm{F}_q)$ and $\mathrm{G} = \mathbf{G}(\mathrm{F}_q)$. We have two main purposes in this section:
- Prove Theorem 4.1.11.
- For any irreducible cuspidal k-representation of G', construct its $W(k)$-projective cover.

Notice that the center of $\mathbf{G}'$ is disconnected but the center of $\mathbf{G}$ is connected, so we want to follow the method of [DeLu] (page 132), which is also applied in [Bon]: consider the regular inclusion $i : \mathbf{G}' \to \mathbf{G}$, then we want to use functor $\mathrm{Res}^{\mathrm{G}}_{\mathrm{G}'}$ to induce properties from G-representations to G′-representations.

4.1.1 Preliminary

Regular inclusion i

We summarize the context we will need in section 2 of [Bon]:

The canonical inclusion i commutes with F and maps F-stable maximal torus to F-stable maximal torus. If we fix one F-stable maximal torus $\mathbf{T}$ of G and note $\mathbf{T}' = i^{-1}(\mathbf{T})$, then i induces a bijection between the root systems of $\mathbf{G}$ and $\mathbf{G}'$ relative to $\mathbf{T}$ and $\mathbf{T}'$. Furthermore, i gives a bijection between standard F-stable parabolic subgroups of $\mathbf{G}$ and $\mathbf{G}'$ with inverse $_ \cap \mathbf{G}'$, which respects subsets of simple roots contained by parabolic subgroups. Besides, restrict i to any F-stable Levi subgroup $\mathbf{L}$ of any F-stable parabolic subgroup of $\mathbf{G}$, it is the canonical inclusion from $\mathbf{L}'$ to $\mathbf{L}$.

From now on, we fix a F-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$, whence fix one of $\mathbf{G}'$ as well, noted as $\mathbf{T}'_0$. For any F-stable standard Levi subgroup $\mathbf{L}$, we always use $\mathbf{L}'$ to denote the F-stable Levi subgroup of $\mathbf{G}'$ under i, and use L and L$'$ to denote the corresponding split Levi subgroups $\mathbf{L}^F$ and $\mathbf{L}'^F$ respectively.

Now we consider the dual groups. Let $(\mathbf{G}^*, \mathbf{T}_0^*, F^*)$ and $(\mathbf{G}'^*, \mathbf{T}'^*, F^*)$ be triples dual to $(\mathbf{G}, \mathbf{T}_0, F)$ and $(\mathbf{G}', \mathbf{T}', F)$ respectively. We can induce from i a surjective morphism $i^* : \mathbf{G}^* \to \mathbf{G}'^*$, which commutes with F^* and maps $\mathbf{T}_0^*$ to $\mathbf{T}_0'^*$. For any F-stable standard parabolic subgroup $\mathbf{P}$ and its F-stable Levi soubgroup $\mathbf{L}$, let us use $\mathbf{P}'$ and $\mathbf{L}'$ to denote the F-stable standard parabolic subgroups $\mathbf{P} \cap \mathbf{G}'$ and Levi subgroups $\mathbf{L} \cap \mathbf{G}'$, then we have:

$$i^*(\mathbf{L}^*) = \mathbf{L}'^*,$$

whence, if we note $\mathbf{L}'^{*^{F^*}} = \mathrm{L}'^*$ and $\mathbf{L}^{*^{F^*}} = \mathrm{L}^*$, then:

$$i^*(\mathrm{L}^*) = \mathrm{L}'^*.$$

Lusztig series and ℓ-blocks

From now on, if we consider a semisimple element $\tilde{s} \in \mathrm{L}^*$ for any split Levi subgroup L* of G*, we always use s to denote $i^*(\tilde{s})$ and $[\tilde{s}]$ (resp. $[s]$) to denote its L*(resp. L$'^*$)-conjugacy class. Notice that the order of $\tilde{s}$ is divisible by the order of s, whence s is ℓ-regular when $\tilde{s}$ is ℓ-regular, where ℓ denotes a prime number different from p.

Let $\mathbf{G}(\mathrm{F}_q)$ be any finite group of Lie type, where $\mathbf{G}$ is a connected reductive group defined over F_q. For any irreducible representation χ of $\mathbf{G}(\mathrm{F}_q)$, let e_χ denote the central idempotent of $\bar{\mathcal{K}}(\mathbf{G}(\mathrm{F}_q))$ associated to χ (see definition in the beginning of section 2 of [BrMi]). Fixing some semisimple element $s \in \mathbf{G}^*(\mathrm{F}_q)$, where $\mathbf{G}^*$ denotes the dual group of $\mathbf{G}$, then $\mathcal{E}(\mathbf{G}(\mathrm{F}_q), (s))$ denotes the Lusztig serie of $\mathbf{G}(\mathrm{F}_q)$ corresponding to the $\mathbf{G}^*(\mathrm{F}_q)$-conjugacy class $[s]$ of s. If s is ℓ-regular (i.e. its order is prime to p), define

$$\mathcal{E}_\ell(\mathbf{G}(\mathrm{F}_q), s) := \bigcup_{t \in (C_{\mathbf{G}^*}(s)^{F^*})_\ell} \mathcal{E}(\mathbf{G}(\mathrm{F}_q), (ts)).$$

Here $(C_{\mathbf{G}^*}(s)^{F^*})_\ell$ denotes the group consisting with all ℓ-elements of $C_{\mathbf{G}^*}(s)^{F^*}$, so ts is still semisimple. Now define:

$$b_s = \sum_{\chi \in \mathcal{E}_\ell(\mathbf{G}(\mathrm{F}_q), s)} e_\chi,$$

which obviously belongs to $\bar{\mathcal{K}}(\mathbf{G}(\mathrm{F}_q))$.

Theorem 4.1.1 (Broué, Michel)**.** *Let $s \in \boldsymbol{G}^*(\mathrm{F}_q)$ be any semisimple ℓ-regular element, and $\mathcal{L}'$ be the set of prime numbers without ℓ. Define $\bar{\mathbb{Z}}_\ell = \bar{\mathbb{Z}}[1/r]_{r \in \mathcal{L}'}$, where $\bar{\mathbb{Z}}$ denotes the ring of algebraic integers, then $b_s \in \bar{\mathbb{Z}}_\ell(\boldsymbol{G}(\mathrm{F}_q))$.*

Remark 4.1.2. *The theorem above tells us that $\mathcal{E}_\ell(\boldsymbol{G}(\mathrm{F}_q), s)$ is an union of ℓ-blocks.*

Let K be a finite field extension of $\mathcal{K}$ which is sufficiently large for G, with valuation ring O, which contains $W(k)$ as a subring. We have $\mathcal{K}$ is complete, implying K is complete. Notice that K is also sufficiently large for any split Levi subgroup L of G, which means all the irreducible $\bar{\mathcal{K}}$-representations of L is defined over K, whence there is a natural bijection:

$$\mathrm{Irr}_{\bar{\mathcal{K}}}(\mathrm{L}) \longleftrightarrow \mathrm{Irr}_K(\mathrm{L}),$$

so we define Lusztig series for $\mathrm{Irr}_K(\mathrm{L})$ through this bijection. Since K is also sufficiently large for any split Levi L′ of G′, we have the same bijection for $\mathrm{Irr}_K(\mathrm{L}')$ and $b_s \in \mathrm{O}[\mathrm{L}']$.

Proposition 4.1.3. *For any split Levi subgroup* L *(resp.* L′*) and any semisimple ℓ-regular element $\tilde{s} \in \mathrm{L}^*$ (resp. $s \in \mathrm{L}'^*$), we have: $b_{\tilde{s}} \in \mathrm{O}[\mathrm{L}]$ (resp. $b_s \in \mathrm{O}[\mathrm{L}']$).*

Proof. We deduce from the analysis above and the definition that $e_\chi \in K(\mathrm{L})$. Combining this with theorem 4.1.1, we conclude that $b_{\tilde{s}} \in \mathrm{O}[\mathrm{L}]$. The same for b_s's. □

Gelfand-Graev lattices and its projective direct summands

For any split Levi subgroup L′ of G′, fix one rational maximal torus T′ and let $\mathrm{B}'_{\mathrm{L}'}$ be a standard split Borel subgroup with unipotent radical $\mathrm{U}'_{\mathrm{L}'}$, then $\mathrm{O}_{\mathrm{U}}(\mathrm{L}')$ denotes the set of non-degenerated characters of $\mathrm{U}'_{\mathrm{L}'}$. Consider any $\mu \in \mathrm{O}_{\mathrm{U}}(\mathrm{L}')$, of which the representation space is 1-dimensional, so it obviously has an $\mathrm{O}[\mathrm{U}'_{\mathrm{L}'}]$-lattice , noted as O_μ. Define $\mathrm{Y}_{\mathrm{L}',\mu} = \mathrm{Ind}_{\mathrm{U}'_{\mathrm{L}'}}^{\mathrm{L}'} \mathrm{O}_\mu$, the **Gelfand-Graev lattice** associated to μ. In fact, we have that $\mathrm{Y}_{\mathrm{L}',\mu}$ is defined up to the T′-conjugacy class of μ. Take any ℓ-regular semisimple element $s \in \mathrm{L}'^*$, define:

$$\mathrm{Y}_{\mathrm{L}',\mu,s} = b_s \cdot \mathrm{Y}_{\mathrm{L}',\mu}.$$

Meanwhile, from definition we have directly that $\sum_{[s]} b_s = 1$, where the sum runs over all the ℓ-regular semisimple L'^*-conjugacy class $[s]$. So:

$$\mathrm{Y}_{\mathrm{L}',\mu} = \bigoplus_{[s]} \mathrm{Y}_{\mathrm{L}',\mu,s}.$$

Since O_μ is projective (free and rank 1) and induction respect projectivity, we see that $\mathrm{Y}_{\mathrm{L}',\mu}$ is a projective $\mathrm{O}[\mathrm{L}']$-module. Proposition 4.1.3 implies that $\mathrm{Y}_{\mathrm{L}',\mu,s}$ are $\mathrm{O}[\mathrm{L}']$-modules and direct components of projective $\mathrm{O}[\mathrm{L}']$-module $\mathrm{Y}_{\mathrm{L}',\mu}$, so we conclude that $\mathrm{Y}_{\mathrm{L}',\mu,s}$ are projective $\mathrm{O}[\mathrm{L}']$-modules.

Let G be the group of F_q points of an algebraic group defined over F_q, and $(\mathcal{K}, \mathcal{O}, k)$ be a splitting ℓ-modular system. We define

$$\mathcal{E}_{\ell'}(\mathrm{G}) := \bigcup_{z \text{ semi-simple, } \ell-\text{regular}} \mathcal{E}(\mathrm{G}, z)$$

Definition 4.1.4 (Gruber, Hiss)**.** *Let* G *be the group of* F_q *points of an algebraic group defined over* F_q*, and* $(\mathcal{K}, \mathcal{O}, k)$ *be a splitting* ℓ*-modular system. Let* Y *be an* $\mathcal{O}[\mathrm{G}]$*-lattice with ordinary character* ψ*. Write* $\psi = \psi_{\ell'} + \psi_{\ell}$*, such that all constituents of* $\psi_{\ell'}$ *and non of* ψ_{ℓ} *belong to* $\mathcal{E}_{\ell'}(\mathrm{G})$*. Then there exists a unique pure sublattice* $V \leq Y$*, such that* Y/V *is an* $\mathcal{O}[\mathrm{G}]$*-lattice whose character is equal to* $\psi_{\ell'}$*. The quotient* Y/V *is called the* ℓ*-regular quotient of* Y *and noted by* $\pi_{\ell'}(Y)$*.*

Corollary 4.1.5. *Let* L' *be a split Levi subgroup of* G'*, and* s *be an* ℓ*-regular semisimple element in* L'^**. For any* $\mu \in \mathrm{O}_{\mathrm{U}}(\mathrm{L}')$*, the module* $\mathrm{Y}_{\mathrm{L}',\mu,s}$ *is indecomposable.*

Proof. Since $\mathrm{Y}_{\mathrm{L}',\mu,s}$ is a projective $\mathrm{O}[\mathrm{L}']$-module, the section §4.1 of [GrHi] or Lemma 5.11(Hiss) in [Geck] tells us that it is indecomposable if and only if its ℓ-regular quotient $\pi_{l'}(\mathrm{Y}_{\mathrm{L}',\mu,s})$ (see §3.3 in [GrHi]) is indecomposable. Inspired by section 5.13. of [Geck], we consider $K \otimes \pi_{l'}(\mathrm{Y}_{\mathrm{L}',\mu,s})$, which is the irreducible sub-representation of $K \otimes \mathrm{Y}_{\mathrm{L}',\mu}$ lying in Lusztig serie $\mathcal{E}(\mathrm{L}', (s))$. The module $\pi_{\ell'}(\mathrm{Y}_{\mathrm{L}',\mu,s})$ is torsion-free, so we deduce that $\pi_{l'}(\mathrm{Y}_{\mathrm{L}',\mu,s})$ is indecomposable. □

Proposition 4.1.6. *Let* L' *be split Levi subgroup of* G'*, and* $\mu \in \mathrm{O}_{\mathrm{U}}(\mathrm{L}')$*. All the projective indecomposable direct summands* $\mathrm{Y}_{\mathrm{L}',\mu,s}$ *of Gelfand-Graev lattice* $\mathrm{Y}_{\mathrm{L}',\mu}$ *are defined over* $W(k)$*, which means there exist projective* $W(k)[\mathrm{L}']$*-modules* $\mathcal{Y}_{\mathrm{L}',\mu,s}$ *such that* $\mathcal{Y}_{\mathrm{L}',\mu,s} \otimes \mathrm{O} = \mathrm{Y}_{\mathrm{L}',\mu,s}$*. In particular,* $\mathcal{Y}_{\mathrm{L}',\mu,s}$ *are indecomposable.*

Proof. Notice that $\mathrm{U}'_{\mathrm{L}'}$ are p-groups, whence μ is defined over $\mathcal{K}$, and there is a $W(k)[\mathrm{U}'_{\mathrm{L}'}]$-module $\mathcal{O}_\mu$ such that $\mathrm{O}_\mu = \mathcal{O}_\mu \otimes_{W(k)[\mathrm{U}'_{\mathrm{L}'}]} \mathrm{O}$. Define a projective $W(k)[\mathrm{L}']$-module $\mathcal{Y}_{\mathrm{L}',\mu} = \mathrm{Ind}_{\mathrm{U}'_{\mathrm{L}'}}^{\mathrm{L}'}(\mathcal{O}_\mu)$. Since k is algebraically closed, then $\bar{\mathrm{Y}}_{\mathrm{L}',\mu}$, the reduction modulo ℓ of $\mathrm{Y}_{\mathrm{L}',\mu}$, coincides with $\bar{\mathcal{Y}}_{\mathrm{L}',\mu}$, the reduction modulo ℓ of $\mathcal{Y}_{\mathrm{L}',\mu}$. Proposition 42 (b) and Lemma 21 (b) in [Ser] imply that the decomposition of $\mathrm{Y}_{\mathrm{L}',\mu}$ gives an decomposition:

$$\bar{\mathrm{Y}}_{\mathrm{L}',\mu} = \sum_{[s]} \bar{\mathrm{Y}}_{\mathrm{L}',\mu,s}.$$

By the same reason, this gives an decomposition by indecomposable projective modules:

$$\mathcal{Y}_{\mathrm{L}',\mu} = \mathcal{Y}_{\mathrm{L}',\mu,s}$$

such that the reduction modulo ℓ of $\mathcal{Y}_{\mathrm{L}',\mu,s}$ equals to $\bar{\mathrm{Y}}_{\mathrm{L}',\mu,s}$. In particular, we can check directly through Proposition 42 (b) of [Ser] that $\mathcal{Y}_{\mathrm{L}',\mu,s} \otimes \mathrm{O} = \mathrm{Y}_{\mathrm{L}',\mu,s}$. □

Remark 4.1.7. *Since* $\mathrm{U}'_{\mathrm{L}'}$ *is also the unipotent radical of* B_{L}*, where* $\mathrm{B}'_{\mathrm{L}'}$ *is the inverse image of the regular inclusion* i *of* B_{L}*. We can repeat the proof for* $\mathrm{Y}_{\mathrm{L},\tilde{s}}$ *and see that they are also defined over* $W(k)$ *in the same manner.*

For split Levi subgroup L of G, we know from [DiMi] that if we fix one rational maximal torus and define $\mathrm{O}_{\mathrm{U}}(\mathrm{L})$, this set of non-degenerate characters consists with only one orbit under conjugation of the fixed torus. So the Gelfand-Graev lattice

is unique, and we note it as $\mathrm{Y_L}$. All the analysis above still work for $\mathrm{Y_L}$, and take some ℓ-regular semisimple element $\tilde{s} \in \mathrm{L}^*$. In particular, we use $\mathrm{Y_{L,\tilde{s}}}$ to denote the indecomposable projective direct summand $b_{\tilde{s}} \cdot \mathrm{Y_L}$.

Corollary 4.1.8. *Let $\tilde{s} \in \mathrm{L}^*$ be a semisimple ℓ-regular element, then:*

$$\mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}) \hookrightarrow b_s \cdot \mathrm{Res^L_{L'}} \mathrm{Y_L}.$$

Proof. We know directly from definition that, for any semisimple ℓ-regular $s' \in \mathrm{G'}^*$:

$$b_{s'} \cdot \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}) \hookrightarrow b_{s'} \cdot \mathrm{Res^L_{L'}} \mathrm{Y_L},$$

Meanwhile $b_{s'} \cdot \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L})$ is a projective $\mathrm{O[G']}$-module, so it is free when considered as O-module. And Proposition 11.7 in [Bon] told us that $b_{s'} \cdot \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}) \otimes \bar{\mathrm{K}} = 0$ if $[s'] \neq [s]$ with $s = i^*(\tilde{s})$, which means $b_{s'} \cdot \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}) = 0$. Combine this with

$$\bigoplus_{[s']} b_{s'} \cdot \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}) = \mathrm{Res^L_{L'}}(b_{\tilde{s}} \cdot \mathrm{Y_L}),$$

we obtain the result. □

Proposition 4.1.9. *For any split Levi subgroup* L *of* G*, let* L′ *denote the split Levi subgroup* $\mathrm{L} \cap \mathrm{G'}$ *of* G′*, and* Z(L)*,* Z(L′) *denote the center of* L *and* L′ *respectively, then we have the equation:*

$$\mathrm{Res^L_{L'}} \mathrm{Y_L} = |\mathrm{Z(L)} : \mathrm{Z(L')}| \bigoplus_{[\mu] \in \mathrm{O_U(L')}} \mathrm{Y_{L',\mu}},$$

where $[\mu]$ *denote the* T′*-orbit of* μ*.*

Proof. Let B be a split Borel subgroup of L and $\mathrm{B'} = \mathrm{B} \cap \mathrm{L'}$ the corresponding split Borel of L′, and U′ denotes the unipotent radical of B′, observing that U′ is also the unipotent radical of B. Fixing one non-degenerate character μ of U′, let O_μ be its $\mathrm{O[U']}$-lattice. By the transitivity of induction, we have:

$$\mathrm{Y_L} = \mathrm{Ind^L_{L'}} \circ \mathrm{Ind^{L'}_{U'}} \mathrm{O}_\mu = \mathrm{Ind^L_{L'}} \mathrm{Y_{L',\mu}}.$$

Since $[\mathrm{T} : \mathrm{T'}] = [\mathrm{L} : \mathrm{L'}]$, by using Mackey formula we have:

$$\mathrm{Res^L_{L'}} \mathrm{Y_L} = \bigoplus_{\alpha_i \in [\mathrm{T:T'}]} ad(\alpha_i)(\mathrm{Y_{L',\mu}}).$$

Furthermore, $ad(\alpha_i)(\mathrm{Ind^{L'}_{U'}} \mathrm{O}_\mu) = \mathrm{Ind^{L'}_{U'}}(ad(\alpha_i)(\mathrm{O}_\mu))$. Notice that after fixing one character of U′, all its $\mathrm{O[U']}$-lattices are equivalent, so $ad(\alpha_i)(\mathrm{Y_{L',\mu}}) = \mathrm{Y_{L',ad(\alpha_i)(\mu)}}$. Whence, let $[\mu]$ denote the T′-orbit of μ in $\mathrm{O_U(L')}$, we have

$$\mathrm{Stab_T}([\mu]) \subset \mathrm{Stab_T}(\mathrm{Y_{L',\mu}}) \subset \mathrm{Stab_T}(\mathrm{Y_{L',\mu}} \otimes \bar{\mathrm{K}}).$$

On the other hand, the proof of lemma 2.3 a) in [DiFl] tells that

$$\mathrm{Stab}_{\mathrm{T}}(\mathrm{Y}_{\mathrm{L}',\mu} \otimes \bar{\mathrm{K}}) \subset \mathrm{Stab}_{\mathrm{T}}([\mu]).$$

So the inclusion above is in fact a bijection. Combine this with the statement of lemma 2.3 a) in [DiFl], we finish our proof. □

Lemma 4.1.10. *Fix a semisimple ℓ-regular $s \in \mathrm{G}'^*$, define $\mathcal{S}_{[s]}$ to be the set of semisimple ℓ-regular $\widetilde{\mathrm{G}}^*$-conjugacy classes $[\tilde{s}] \subset \widetilde{\mathrm{G}}^*$ such that $i^*[\tilde{s}] = [s]$. Then*

$$\bigoplus_{[\tilde{s}] \in \mathcal{S}_{[s]}} \mathrm{Res}^{\mathrm{L}}_{\mathrm{L}'} \mathrm{Y}_{\mathrm{L},\tilde{s}} = |\mathrm{Z}(\mathrm{L}) : \mathrm{Z}(\mathrm{L}')| \bigoplus_{\mu \in \mathrm{O}_{\mathrm{U}'}(\mathrm{L}')} \mathrm{Y}_{\mathrm{L}',\mu,s}$$

Proof. By definition, $\mathrm{Y}_{\mathrm{L},\tilde{s}} = b_{\tilde{s}} \cdot \mathrm{Y}_{\mathrm{L}}$. Multiplying b_s on both sides of the equation in Proposition 4.1.9 and considering Corollary 4.1.8, we conclude that for any ℓ-regular semisimple G'^*-conjugacy class $[s]$, $\bigoplus_{[\tilde{s}] \in \mathcal{S}_{[s]}} \mathrm{Res}^{\mathrm{L}}_{\mathrm{L}'} \mathrm{Y}_{\mathrm{L},\tilde{s}}$ is a projective direct summand of $|\mathrm{Z}(\mathrm{L}) : \mathrm{Z}(\mathrm{L}')| \bigoplus_{\mu \in \mathrm{O}_{\mathrm{U}}(\mathrm{L}')} \mathrm{Y}_{\mathrm{L}',\mu,s}$. Meanwhile, let $\mathcal{S} = \{\mathcal{S}_{[s]} |\ s \in \mathrm{G}'^*, s \text{ semisimlpe } \ell\text{-regular}\}$, then Proposition 4.1.9 can be written as:

$$\bigoplus_{\mathcal{S}} \bigoplus_{[\tilde{s}] \in \mathcal{S}_{[s]}} \mathrm{Res}^{\mathrm{L}}_{\mathrm{L}'} \mathrm{Y}_{\mathrm{L},\tilde{s}} = |\mathrm{Z}(\mathrm{L}) : \mathrm{Z}(\mathrm{L}')| \bigoplus_{[s]} \bigoplus_{\mu \in \mathrm{O}_{\mathrm{U}}(\mathrm{L}')} \mathrm{Y}_{\mathrm{L}',\mu,s}$$

So they equal to each other. □

4.1.2 Uniqueness of supercuspidal support

In this part, we will proof the main theorem 4.1.11 for this section. First we recall the notions of cuspidal, supercuspidal and supercuspidal support.

We always use i and r to denote the functors of parabolic induction and parabolic restriction. Let π be an irreducible k-representation of a finite group of Lie type or a p-adic group G. We say π is cuspidal, if for any proper Levi subgroup L of G, the representation $r^{\mathrm{G}}_{\mathrm{L}}\pi$ is 0. Let τ be any irreducible k-representation of L, if π is not isomorphic to any irreducible subquotient of $i^{\mathrm{G}}_{\mathrm{L}}\tau$ for any pair (L, τ), we say π is supercuspidal. It is clear that a supercuspidal representation is cuspidal. We say (L, τ) is a (super)cuspidal pair, if τ is a (super)cuspidal k-representation of L

The cuspidal (resp. supercuspidal) support of π consists of the cuspidal (resp. supercuspidal) pairs (L, τ), such that π is an irreducible subrepresentation (resp. subquotient) of $i^{\mathrm{G}}_{\mathrm{L}}\tau$.

Theorem 4.1.11. *Let L' be any standard split Levi subgroup of G' and ν be any cuspidal k-representation of L'. Then the supercuspidal support of ν is unique up to L'-conjugation.*

Let P_ν denote the $\mathrm{O}[\mathrm{L}']$-projective cover of ν. To prove the theorem above, we will follow the strategy below:

1. For any standard Levi subgroup M' of L', prove that $r_{\mathrm{M}'}^{\mathrm{L}'}\mathrm{P}_\nu$ is either equal to 0 or indecomposable.

2. Prove that there is only one unique standard split Levi subgroup M' of L', such that $r_{\mathrm{M}'}^{\mathrm{L}'}\mathrm{P}_\nu$ is cuspidal.

Let (M', θ) be a supercuspidal k-pair of L'. From the proof of Proposition 3.2 of [Hiss], we know that (M', θ) belongs to the supercuspidal support of (L', ν), if and only if $\mathrm{Hom}(r_{\mathrm{M}'}^{\mathrm{L}'}\mathrm{P}_\nu, \theta) \neq 0$. Combining this fact with (1), we find that $r_{\mathrm{M}'}^{\mathrm{L}'}\mathrm{P}_\nu$ is the projective cover of θ. Proposition 2.3 of [Hiss] states that an irreducible k-representation of M' is supercuspidal if and only if its projective cover is cuspidal, whence Theorem 4.1.11 is equivalent to (2).

Remark 4.1.12. *- If we consider standard Levi subgroups* L *of* G*, the analysis above is true as well.*

- Proposition 3.2 of [Hiss] concerns $k[\mathrm{L}']$*-projective cover, but from Proposition 42 of [Ser] we know that there is a surjective morphism of* $k[\mathrm{L}']$*-modules from the* $W(k)[\mathrm{L}']$*-projective cover to the* $k[\mathrm{L}']$*-projective cover, and hence obtain the same result for* $W(k)[\mathrm{L}']$*-projective cover.*

Proposition 4.1.13. *Let* ν *be an irreducible cuspidal* k*-representation of* L'*. There exists a simple* $k\mathrm{L}$*-module* $\widetilde{\nu}$*, and a surjective morphism* $\mathrm{Res}_{\mathrm{L}'}^{\mathrm{L}}\widetilde{\nu} \twoheadrightarrow \nu$*. Furthermore, let* $\mathrm{Y}_{\mathrm{L},\tilde{s}}$ *be the projective cover of* $\widetilde{\nu}$*, where* $\tilde{s} \in \mathrm{G}^*$ *is an* ℓ*-regular semisimple element, then there exists* $\mu \in \mathrm{O}_{\mathrm{U}'}(\mathrm{L}')$ *such that the composed morphism:*

$$\mathrm{Y}_{\mathrm{L}',\mu,s} \hookrightarrow \mathrm{Res}_{\mathrm{L}'}^{\mathrm{L}}\mathrm{Y}_{\mathrm{L},\tilde{s}} \twoheadrightarrow \mathrm{Res}_{\mathrm{L}'}^{\mathrm{L}}\widetilde{\nu} \twoheadrightarrow \nu \tag{4.1}$$

is surjective, which means $\mathrm{Y}_{\mathrm{L}',\mu,s}$ *is the* $\mathrm{O}[\mathrm{L}']$*-projective cover of* ν*.*

Proof. By the property of Mackey formula, we can find such $\widetilde{\nu}$.

For the second part of this proposition, since Res respect projectivity, we know the fact that $\mathrm{Res}_{\mathrm{L}'}^{\mathrm{L}}\mathrm{Y}_{\mathrm{L},\tilde{s}}$ is a projective direct summand of $\mathrm{Res}_{\mathrm{L}'}^{\mathrm{L}}\mathrm{Y}_{\mathrm{L}}$, and contained in $|\mathrm{Z}(\mathrm{L}) : \mathrm{Z}(\mathrm{L}')| \bigoplus_{\mu \in \mathrm{O}_{\mathrm{U}'}(\mathrm{L}')} \mathrm{Y}_{\mathrm{L}',\mu,s}$ as lemma 4.1.10 proved, whence all the projective indecomposable direct summands belong to $\{\mathrm{Y}_{\mathrm{L}',\mu,s}\}_{\mathrm{O}_{\mathrm{U}'}(\mathrm{L}')}$. Then there must exists $\mu \in \mathrm{O}_{\mathrm{U}'}(\mathrm{L}')$ such that the composed morphism: $\mathrm{Y}_{\mathrm{L}',\mu,s} \to \nu$ is non trivial hence a surjection. □

Remark 4.1.14. *We induce from Proposition 4.1.6 that* $\mathcal{Y}_{\mathrm{L}',\mu,s}$ *is the* $W(k)[\mathrm{L}']$*-projective cover of* ν*, noted as* $\mathcal{P}_\nu$*.*

Let M' be any standard split Levi subgroup of L'. It is clear that $\mu \downarrow_{\mathrm{M}'}$ belongs to $\mathrm{O}_{\mathrm{U}'}(\mathrm{M}')$. Now consider the intersection $[s] \cap \mathrm{M}'^*$. As in the paragraph above Proposition 5.10 of [Helm], $[\tilde{s}] \cap \mathrm{M}^*$ consists of one M^*-conjugacy class or is empty, so does $[s] \cap \mathrm{M}'^*$. For the first case, notation $\mathrm{Y}_{\mathrm{M}',\mu\downarrow_{\mathrm{M}'},[s]\cap\mathrm{M}'^*}$ is well defined, and for the second case, we define it to be 0. From now on, we will always use $\mathrm{Y}_{\mathrm{M}',\mu,s}$ to simplify $\mathrm{Y}_{\mathrm{M}',\mu\downarrow_{\mathrm{M}'},[s]\cap\mathrm{M}'^*}$. We use the same manner to define $\mathrm{Y}_{\mathrm{M},\tilde{s}}$.

Proposition 4.1.15. *Let ν be an irreducible cuspidal $k\mathrm{L}'$-representation, and $\widetilde{\nu}$, $\mathrm{Y}_{\mathrm{L}',\mu,s}$, $\mathrm{Y}_{\mathrm{L},\tilde{s}}$ be as in Proposition* 4.1.13. *Then $r^{\mathrm{L}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}$ is equal to* 0 *or indecomposable and isomorphic to $\mathrm{Y}_{\mathrm{M}',\mu,s}$ as $\mathrm{O}[\mathrm{M}']$-module.*

Proof. In the proof of lemma 4.1.10 we know that $\mathrm{Y}_{\mathrm{L}',\mu,s}$ is a direct summand of $\mathrm{Res}^{\mathrm{L}}_{\mathrm{L}'}(\mathrm{Y}_{\mathrm{L},\tilde{s}})$. Observing that the unipotent radical of M' is also the unipotent radical of M, we deduce directly from the definition that $r^{\mathrm{L}'}_{\mathrm{M}'}(\mathrm{Res}^{\mathrm{L}}_{\mathrm{L}'}(\mathrm{Y}_{\mathrm{L},\tilde{s}})) = \mathrm{Res}^{\mathrm{M}}_{\mathrm{M}'}(r^{\mathrm{L}}_{\mathrm{M}}(\mathrm{Y}_{\mathrm{L},\tilde{s}}))$, and Proposition 5.10 in [Helm] states that $r^{\mathrm{L}}_{\mathrm{M}}(\mathrm{Y}_{\mathrm{L},\tilde{s}}) = \mathrm{Y}_{\mathrm{M},\tilde{s}}$. The statements above, combining with the fact that parabolic restriction is exact and respects projectivity, derive that $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}$ is a projective direct summand of $\mathrm{Res}^{\mathrm{M}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{M},\tilde{s}}$. As what we have mentioned, $[\tilde{s}]\cap\mathrm{M}^*$ is empty or consists of one M^*-conjugacy class, so does $[s]\cap\mathrm{M}'^*$. In the first case $\mathrm{Y}_{\mathrm{M},\tilde{s}} = 0$, whence $r^{\mathrm{L}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s} = 0$, so the result.

Now considering the second case: let $\tilde{s}' \in \mathrm{M}^*$ and $[\tilde{s}']$ denote the M^*-conjugacy class equals to $[\tilde{s}]\cap\mathrm{M}^*$. Let μ' denote the character $\mathrm{Res}^{\mathrm{U}'_{\mathrm{M}'}}_{\mathrm{U}'_{\mathrm{L}'}}\mu$, where $\mathrm{U}'_{\mathrm{L}'}$ and of $\mathrm{U}'_{\mathrm{M}'}$ denote the unipotent radical of L' and M' respectively, which is non-degenerate by definition. Corollary 15.15 in [Bon] gives an equation:

$$r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}\otimes\bar{\mathrm{K}} = \mathrm{Y}_{\mathrm{M}',\mu',s'}\otimes\bar{\mathrm{K}}.$$

which means the ℓ-regular quotient of $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}$ is indecomposable, and by using the criterion of [Geck, lemma 5.11] we conclude that $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}$ is indecomposable. Note that Corollary 15.11 in [Bon] tells that the sub-representation of $\mathrm{Res}^{\mathrm{M}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{M},\tilde{s}}\otimes\bar{\mathrm{K}}$ corresponding to $[s']$ is without multiplicity, and the equation above says that the irreducible sub-representations corresponding to $[s']$ of $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}\otimes\bar{\mathrm{K}}$ and $\mathrm{Y}_{\mathrm{M}',\mu',s'}\otimes\bar{\mathrm{K}}$ coincide, whence these two projective direct summands of $\mathrm{Res}^{\mathrm{M}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{M},\tilde{s}}$ coincide each other.

□

We have finished the first step to prove Theorem 4.1.11. Remark 4.1.12 tells that the statement of step 2 is true for L, whence there only left the proposition below to finish our proof:

Proposition 4.1.16. *Let $\mathrm{Y}_{\mathrm{L}',\mu,s}$, $\mathrm{Y}_{\mathrm{L},\tilde{s}}$, $\widetilde{\nu}$ be as in Proposition 4.1.13, then for any standard split Levi M' of L', we have $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s} = \mathrm{Y}_{\mathrm{M}',\mu,s}$ is cuspidal if and only if $r^{\mathrm{L}}_{\widetilde{M}}\mathrm{Y}_{\mathrm{L},\tilde{s}} = \mathrm{Y}_{\mathrm{M},\tilde{s}}$ is cuspidal.*

Proof. Since $\mathrm{L}' \hookrightarrow \mathrm{L}$ is a bijection preserving partial order between standard Levi subgroups of G and G′, the statement in the proposition is equivalent to say that for any split Levi M' of L',

$$r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s} = 0 \iff r^{\mathrm{L}}_{\mathrm{M}}\mathrm{Y}_{\mathrm{L},\tilde{s}} = 0.$$

The proof of Proposition 4.1.15 tells us

$$r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s} \hookrightarrow \mathrm{Res}^{\mathrm{M}}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{M},\tilde{s}},$$

whence "⇒" is clear.

Now consider the other direction. Notice that $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}$ is an $\mathrm{O}[\mathrm{M}']$-lattice, and definition 5.9 in [Geck] tells us that $r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s} = 0$ if and only if its ℓ-regular quotient $\pi_{l'}(r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}) = 0$. By definition $(\pi_{l'}(r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}))\otimes\bar{\mathrm{K}}$ is the sum of all simple $\bar{\mathrm{K}}[\mathrm{M}']$-submodules of $r^{\mathrm{L}'}_{\mathrm{M}'}(\mathrm{Y}_{\mathrm{L}',\mu,s}\otimes\bar{\mathrm{K}})$ which lie in the Lusztig series corresponding to ℓ-regular semisimple M'^*-conjugacy classes. [Bon, Corollary 15.15] states that, in fact $\bar{\mathrm{K}}[\mathrm{M}']$-module $(\pi_{l'}(r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}))\otimes\bar{\mathrm{K}}$ is the sum of all irreducible $\bar{\mathrm{K}}[\mathrm{M}']$-submodules of Gelfand-Graev representation $\mathrm{Ind}^{\mathrm{M}'}_{\mathrm{U}'_{\mathrm{M}'}}\mu$ lying in the Lusztig series corresponding to $[s]\cap\mathrm{M}'^*$, where $[s]$ denotes the L'^*-conjugacy class. We have now $(\pi_{l'}(r^{\mathrm{L}'}_{\mathrm{M}'}\mathrm{Y}_{\mathrm{L}',\mu,s}))\otimes\bar{\mathrm{K}} = 0$ implies $[s]\cap\mathrm{M}'^* = 0$, which means $[\tilde{s}]\cap\mathrm{M}^* = 0$, whence $\mathrm{Y}_{\mathrm{M},\tilde{s}} = 0$.

□

Chapter 5

Maximal simple cuspidal k-types

5.1 Construction of cuspidal k-representations of G'

From this section until the end of this manuscript, we assume that the field F is non-archimedean locally compact, whose residue field is of characteristic $p(\neq l)$. Let G' denote $\mathrm{SL}_n(\mathrm{F})$ and G denote $\mathrm{GL}_n(\mathrm{F})$. Let $\mathrm{Rep}_k(\mathrm{G}')$ denote the category of smooth k-representations of G'.

In this section, we want to prove that for any k-irreducible cuspidal representation π' of G', there exists an open compact subgroup $\tilde{J}'$ of G' and an irreducible representation $\tilde{\lambda}'$ of $\tilde{J}'$ such that π' is isomorphic to $\mathrm{ind}_{\tilde{J}'}^{\mathrm{G}'}\tilde{\lambda}'$ (5.1.30, 5.1.34 and 5.2.7).

5.1.1 Types $(J, \lambda \otimes \chi \circ \det)$

Let (J, λ) be a maximal simple cuspidal k-type of G, and we need to check that the type $(J, \lambda \otimes \chi \circ \det)$ is also a maximal simple cuspidal k-type of G, which will be used in the proof of Proposition 5.1.12. This has been proved in appendix of [BuKuII] in the case of characteristic 0, and by using the following two lemmas, we observe the same results for the case of characteristic ℓ by reduction modulo ℓ.

Definition 5.1.1. *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and θ be a simple k-character or a simple $\overline{\mathbb{Q}}_\ell$ character of H^1, and η the unique irreducible k-representation of J^1 which contains θ, and κ an β-extension of η to J. Let (J, λ) be a simple k-type or a simple $\overline{\mathbb{Q}}_\ell$-type of* G*. And also all the notations used: $\mathfrak{H}(\beta, \mathfrak{A})$, $\mathfrak{J}(\beta, \mathfrak{A})$ are defined in Section 3.2 and Section 3.3, as in [BuKu].*

Proposition 5.1.2 (Vignéras, IV.1.5 in [V2])**.** *The reduction modulo ℓ of any maximal simple cuspidal $\bar{\mathbb{Q}}_\ell$-type of* G *is a maximal simple cuspidal k-type. And conversely, any maximal simple cuspidal k-type is the reduction modulo ℓ of a maximal simple cuspidal $\bar{\mathbb{Q}}_\ell$-type of* G*.*

Lemma 5.1.3. *Let K be a compact subgroup of a p-adic reductive group. Any $\overline{\mathbb{Q}}_\ell$-character of K is ℓ-integral, and reduction modulo ℓ gives a surjection from the set of $\overline{\mathbb{Q}}_\ell$-characters of K to the set of $\bar{\mathbb{F}}_\ell$-characters of K*

Proof. Fix an isomorphism from $\mathbb{C}$ to $\bar{\mathbb{Q}}_\ell$ of fields and until the end of this proof, we identify the two fields through this isomorphism. Let $\chi_{\mathbb{C}}$ be any $\mathbb{C}$-character of K. The smoothness implies that there exists a finite field extension $\mathrm{E}/\mathbb{Q}_\ell$ such that $\chi_{\mathbb{C}}$ is defined over E and we can find an $\mathrm{O}_{\mathrm{E}}[K]$-lattice of χ. Hence we can define reduction modulo ℓ for $\chi_{\mathbb{C}}$ and denote is as $\bar{\chi}_{\mathbb{C}}$. On the other hand, let χ_ℓ be any $\bar{\mathbb{F}}_\ell$-character of K. It is clear that χ_ℓ is defined over a finite field extension $\bar{\mathrm{E}}/\mathbb{F}_\ell$. Notice that the quotient group $K/\mathrm{Ker}(\chi_\ell)$ is a finite abelian group with order prime to ℓ. Lemma 10 of section 14.4 in [Ser] implies that χ_ℓ is projective as $\bar{\mathrm{E}}[K/\mathrm{Ker}(\chi_\ell)]$-module. Then Proposition 42 of [Ser] states that χ_ℓ can be lift to $\mathrm{O}_{\bar{\mathrm{E}}}$, where the fractional field $\mathrm{frac}(\mathrm{O}_{\bar{\mathrm{E}}})$ is a finite field extension of $\mathbb{Q}_\ell$ and its residual field is isomorphic to $\bar{\mathrm{E}}$. Hence we finish the proof. □

Corollary 5.1.4. *Let $\bar{\chi}$ be any $k^\times$-character of $\mathrm{F}^\times$, then it can be always lifted to a $\bar{\mathbb{Q}}_\ell$-character χ of $\mathrm{F}^\times$.*

Proof. We could write $F^\times \cong \mathbb{Z} \times \mathrm{O}_F^\times$ and $\bar{\chi}$ is uniquely defined by $\bar{\chi}|_{\mathbb{Z}}$ and $\bar{\chi}|_{\mathrm{O}_F^\times}$. The part $\bar{\chi}|_{\mathrm{O}_F^\times}$ can be lift to a $\bar{\mathbb{Q}}_\ell$-character by lemma 5.1.3. It is left to consider the restriction $\bar{\chi}|_{\mathbb{Z}}$, of which the image is a finite group of order prime to ℓ. Thus we could find a finite field extension K of $\mathbb{Q}_\ell$ such that there is an embedding from $\bar{\chi}(\mathbb{Z})$ to the quotient ring O_K/p_K, where p_K is the uniformizer of O_K. □

Recall the equivalence

$$(\mathrm{U}^{[\frac{1}{2}n]+1}(\mathfrak{A})/\mathrm{U}^{n+1}(\mathfrak{A}))^\wedge \cong \mathfrak{P}^{-n}/\mathfrak{P}^{-([\frac{1}{2}n]+1)},$$

where $(\mathrm{U}^{[\frac{1}{2}n]+1}(\mathfrak{A})/\mathrm{U}^{n+1}(\mathfrak{A}))^\wedge$ denote the Pontrjagin dual. Let $\beta \in \mathfrak{P}^{-n}/\mathfrak{P}^{-([\frac{1}{2}n]+1)}$, we use ψ_β to denote the character on $\mathrm{U}^{[\frac{1}{2}n]+1}(\mathfrak{A})/\mathrm{U}^{n+1}(\mathfrak{A})$ induced through the equivalence above (or consult Section 3.2.1). Recall that in Section 3.2.1 we fixed an additive character ψ_F from F to $\mathbb{C}^\times$, let $\bar{\psi_\beta}$ to denote the reduction modulo ℓ of ψ_β according to the choice of ψ_F.

Lemma 5.1.5. *Let (J, λ) be a maximal simple cuspidal k-type of G, if (J, λ) is of level zero or $\beta \in F$, then $(J, \lambda \otimes \chi \circ \det)$ is also a maximal simple cuspidal k-type of G, where χ is any k-quasicharacter of $F^\times$. In particular, while χ is not trivial on $\mathrm{U}^1(\mathfrak{A})$. Let $n_0 \geq 1$ is the least integer such that $\chi \circ \det$ is trivial on $\mathrm{U}^{n_0+1}(\mathfrak{A})$, and $c \in \mathfrak{P}^{-n_0}$ such that $\chi \circ \det$ coincides with $\bar{\psi_c}$ on $\mathrm{U}^{[\frac{1}{2}n_0]+1}(\mathfrak{A})$. Then*

$$\mathfrak{H}(\beta + c, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{J}(\beta + c, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{A}$$

Proof. While (J, λ) is of level zero, we only need to prove $\chi \circ \det$ is a simple character on $\mathrm{U}^1(\mathfrak{A})$. While $\beta \in F$, we only need to prove the character $\theta = \bar{\psi_\beta} \otimes \chi \circ \det$ is a simple character on $\mathrm{U}^1(\mathfrak{A})$. This is directly induced by the results in the appendix in [BuKuII] for the complex case, because the definition of simple stratum in the case of characteristic ℓ is the same as the case of characteristic 0. And the definition 2.2.2 of [MS] implies that simple k-characters are reduction modulo ℓ of simple $\mathbb{C}$-characters. □

Lemma 5.1.6. *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a simple stratum in* A *with* $\beta \notin \mathrm{F}$, $n \geq 1$. *Let* $c \in \mathrm{F}$, *and* $n_0 = -v_{\mathfrak{A}}(c)$, $n_1 = -v_{\mathfrak{A}}(\beta + c)$,

1. *The stratum* $[\mathfrak{A}, n, 0, \beta+c]$ *is a simple stratum in* A, *and we have* $\mathfrak{H}(\beta+c, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A})$ *and* $\mathfrak{J}(\beta + c, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A})$.

2. *Let* $\chi_{\mathbb{C}}$ *be a* $\mathbb{C}$*-quasicharacter of* $\mathrm{F}^\times$ *such that* $\chi_{\mathbb{C}} \circ \det$ *agrees with* ψ_c *on* $\mathrm{U}^{[\frac{1}{n_0}]+1}(\mathfrak{A})$. *Then we have an equivalence of simple* $\mathbb{C}$*-characters:*

$$C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c) = C_{\mathbb{C}}(\mathfrak{A}, 0, \beta) \otimes \chi_{\mathbb{C}} \circ \det.$$

3. *Let* χ_ℓ *be any* k*-quasicharacter of* $F^\times$ *such that* $\chi_\ell \circ \det$ *agrees with* $\bar{\psi}_c$ *on* $\mathrm{U}^{[\frac{1}{n_0}]+1}(\mathfrak{A})$. *Then we have an equivalence of simple* k*-characters:*

$$C_\ell(\mathfrak{A}, 0, \beta + c) = C_\ell(\mathfrak{A}, 0, \beta) \otimes \chi_\ell \circ \det.$$

Proof. The first two assertions are the lemma in appendix of [BuKuII], so we only need to proof the last assertion. Recall that we fixed an additive character ψ_F from F to $\mathbb{C}^\times$ in Section 3.2.1. Lemma 5.1.3 implies that every simple $\mathbb{C}$-character in $C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c)$ is ℓ-integral and has a reduction modulo ℓ. According to the definition 2.2.2 of [MS], the reduction modulo ℓ gives a bijection between simple $\mathbb{C}$-characters $C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c)$ to $C_\ell(\mathfrak{A}, 0, \beta + c)$. Notice that this bijection is dependent with the choice of ψ_F. Apparently,

$$C_\ell(\mathfrak{A}, 0, \beta + c) = \overline{C_{\mathbb{C}}(\mathfrak{A}, 0, \beta)} \otimes \bar{\chi}_{\mathbb{C}} \circ \det,$$

where $\bar{\chi}_{\mathbb{C}}$ denote the reduction modulo ℓ of $\chi_{\mathbb{C}}$, and $\overline{C_{\mathbb{C}}(\mathfrak{A}, 0, \beta)}$ denote the set of k-characters, which are reduction modulo ℓ of characters in $C_{\mathbb{C}}(\mathfrak{A}, 0, \beta)$. By definition $\overline{C_{\mathbb{C}}(\mathfrak{A}, 0, \beta)} = C_l(\mathfrak{A}, 0, \beta)$, and hence

$$C_\ell(\mathfrak{A}, 0, \beta + c) = C_\ell(\mathfrak{A}, 0, \beta) \otimes \bar{\chi}_{\mathbb{C}} \circ \det.$$

Applying Corollary 5.1.4 to χ_ℓ, there exists a $\bar{\mathbb{Q}}_\ell$-quasicharacter $\tau_{\mathbb{C}}$ of $F^\times$, such that $\chi_\ell \circ \det$ is isomorphic to the reduction modulo ℓ of $\tau_{\mathbb{C}} \circ \det$. Notice that simple characters in $C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c)$ are defined on $\mathrm{H}^1 = \mathfrak{H}(\beta, \mathfrak{A}) \cap \mathrm{U}^1(\mathfrak{A})$, which is a pro-$p$-subgroup of G. The reduction modulo ℓ of $\tau_{\mathbb{C}} \circ \det$ is isomorphic to $\bar{\psi}_c$ on $\mathrm{H}^1(\beta) \cap \mathrm{U}^{[\frac{1}{2}n_0]+1}(\mathfrak{A})$, which implies that $\tau_{\mathbb{C}} \circ \det$ is isomorphic to ψ_c on $\mathrm{H}^1(\beta) \cap \mathrm{U}^{[\frac{1}{2}n_0]+1}(\mathfrak{A})$. The assertion (1) and (2) tell that

$$C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c) = C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c) \otimes \chi_{\mathbb{C}}^{-1} \otimes \tau_{\mathbb{C}} \circ \det.$$

We deduce directly that

$$C_l(\mathfrak{A}, 0, \beta + c) = \overline{C_{\mathbb{C}}(\mathfrak{A}, 0, \beta + c)} = C_l(\mathfrak{A}, 0, \beta + c) \otimes \chi_\ell \circ \det,$$

as required. □

Corollary 5.1.7. *Let (J, λ) be a maximal simple cuspidal k-type of* G*, and χ a k-quasicharacter of $F^\times$. Then the k-type $(J, \lambda \otimes \chi \circ \det)$ is also a maximal simple cuspidal k-type.*

Proof. Let $(J, \lambda_{\mathbb{C}})$ be an ℓ-integral maximal simple cuspidal $\mathbb{C}$-type of G, whose reduction modulo ℓ is isomorphic to (J, λ). Let $\chi_{\mathbb{C}}$ be a $\mathbb{C}$-quasicharacter of $F^\times$ whose reduction modulo ℓ is isomorphic to χ (by 5.1.4). Then by the appendix of [BuKuII], the ℓ-integral type $(J, \lambda_{\mathbb{C}} \otimes \chi_{\mathbb{C}} \circ \det)$ is also maximal cuspidal simple. Thus its reduction modulo ℓ is maximal simple cuspidal k-type by Proposition 5.1.2. Let $c \in \mathrm{F}$ be the element corresponding to a $\mathbb{C}$-lifting of χ, and β corresponding to a simple character θ (this is well-defined, because $\mathrm{H}^1(\beta)$ is pro-p) contained in $(J, \lambda_{\mathbb{C}})$ (as in lemma 5.1.5 or 5.1.6), then the two lemmas above imply that the reduction modulo ℓ of $\theta_{\mathbb{C}} \otimes \chi \circ \det$ is a simple character contained in $(J, \lambda \otimes \chi \circ \det)$. And $\mathrm{H}^1(\beta + c) = \mathrm{H}^1(\beta)$, where $\mathrm{H}^1 = \mathfrak{H}(\beta + c, \mathfrak{A}) \cap \mathrm{U}^1(\mathfrak{A})$. □

Remark 5.1.8. *Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple cuspidal k-type of* M*, where* M *is a Levi subgroup of* G*. Then $\lambda_{\mathrm{M}} \cong \lambda_1 \otimes \cdots \otimes \lambda_r$ for some $r \in \mathbb{N}^*$, where (J_i, λ_i) are maximal simple cuspidal k-types of $\mathrm{GL}_{n_i}(F)$. Hence for any k-quasicharacter χ of $F^\times$, then new k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}} \otimes \chi \circ \det)$ is also maximal simple cuspidal of* M*.*

5.1.2 Intertwining and weakly intertwining

In this section, for any closed subgroup H of G, we always use H' to denote its intersection with G'. Let M denote any Levi subgroup of G.

Proposition 5.1.9. *Let K be a compact subgroup of* M*, and ρ be an irreducible k-representation of K. The restriction $\mathrm{res}^K_{K'}\rho$ is semisimple.*

Proof. Let O denote the kernel of ρ, which is a normal subgroup of K. The subgroup $O \cdot K'$ is also a compact open normal subgroup of K, hence with finite index in K. We deduce that the restriction $\mathrm{res}^K_{O \cdot K'}\rho$ is semisimple by Clifford theory, furthermore the restriction $\mathrm{res}^K_{K'}\rho$ is semisimple. □

Proposition 5.1.10. *Let K be a compact open subgroup of* M*, ρ an irreducible smooth representation of K, and ρ' an irreducible component of the restriction $\mathrm{res}^K_{K'}\rho$. Let $\bar{\rho}$ be an irreducible representation of K such that $\mathrm{res}^K_{K'}\bar{\rho}$ also contains ρ'. Then there exists a k-quasicharacter χ of $F^\times$ such that $\rho \cong \bar{\rho} \otimes \chi \circ \det$.*

Proof. Let U be any pro-p normal subgroup of K contained in the kernel of ρ, hence with finite index. Let's consider $\mathrm{Ind}^U_{U'}(1)$, which is semisimple, thus by lemma of Schur it is a direct sum of characters in the form of $\chi \circ \det|_U$. Since χ can be extended to a quasicharacter of $F^\times$, and we note the extended quasicharacter as χ as well, then we write $\chi \circ \det|_U$ as $(\chi \circ \det)|_U$. The fact that $\mathrm{res}^K_{U'}\rho$ contains the trivial character induces the same property for $\mathrm{res}^K_{U'}\bar{\rho}$. By Frobenius reciprocity, we know that $\mathrm{res}^K_U \bar{\rho}$ contains a character in the form of $(\chi \circ \det)|_U$, and the irreducibility

implies that it is in fact a multiple of this character. We can hence assume that $\bar{\rho}$ is trivial on U.

By the Clifford theory, the restriction of ρ (resp. $\bar{\rho}$) to $K'U$ is semisimple. Hence $\mathrm{Hom}_{K'U}(\rho, \bar{\rho}) \neq 0$. Applying Frobenius reciprocity, we see that ρ is a subrepresentation of $\mathrm{ind}_{K'U}^{K}\mathrm{res}_{K'U}^{K}\bar{\rho}$, which is equivalent to $\bar{\rho} \otimes \mathrm{ind}_{K'U}^{K}(1)$ by 5.2 d, chapitre I, [V1]. In fact, the Jordan-Hölder factors are $\bar{\rho} \otimes \chi \circ \det$: Any irreducible factor τ of $\mathrm{ind}_{K'U}^{K}(1)$ can be view an irreducible k-representation of the quotient group $K/K'U$, which is isomorphic to a subgroup of the finite abelian group $\mathcal{O}_F^{\times}/\det U$, where $\mathcal{O}_F$ indicates the ring of integers of F. Hence τ must be a k-character of $K/K'U$ and can be extended to $F^{\times}$, since we can first extend τ to $\mathcal{O}_F^{\times}/\det U$ (hence to $\mathcal{O}_F^{\times}$) and $F^{\times} \cong \mathcal{O}_F^{\times} \times \mathbb{Z}$. We denote this extension by $\tilde{\tau}$. It is clear that $\tau \cong \tilde{\tau} \circ \det|_K$.

□

Group M is the group of F-points of a reductive group defined over F. Let K_i be open compact subgroups of M for $i = 1, 2$, and ρ_i a representation of K_i. Recall that $i_{K_1,K_2}x(\rho_2)$ to be the induced representation $\mathrm{ind}_{K_1\cap x(K_2)}^{K_1}\mathrm{res}_{K_1\cap x(K_2)}^{x(K_2)}x(\rho_2)$ (see Definition 3.1.6), where $x(\rho_2)$ is the conjugation of ρ_2 by x. For any element $x \in \mathrm{M}$, we say x intertwines (weakly intertwines) ρ_1 with ρ_2 as defined in Definition 3.1.6.

Proposition 5.1.11. *For $i = 1, 2$, let K_i be a compact open subgroup of* M*, and ρ_i an irreducible representation of K_i and ρ_i' be an irreducible component of* $\mathrm{res}_{K_i'}^{K_i}\rho_i$*. Let $x \in \mathrm{M}$ that weakly intertwines ρ_1' with ρ_2'. Then there exists a k-quasicharacter χ of $F^{\times}$ such that x weakly intertwines ρ_1 with $\rho_2 \otimes \chi \circ \det$.*

Proof. By Mackey's decomposition formula, $i_{K_1',K_2'}x(\rho_2')$ is a subrepresentation of $i_{K_1,K_2}x(\rho_2)$. Since $i_{K_1,K_2}x(\rho_2)$ has finite length, the uniqueness of Jordan-Hölder factors implies that there exists an irreducible subquotient of $i_{K_1,K_2}x(\rho_2)$, whose restriction to K_1' contains ρ_1' as a direct components. By 5.1.10, this irreducible subquotient is isomorphic to $\rho_1 \otimes \chi \circ \det$, where χ is a quasicharacter. By definition, this means ρ_1 is weakly intertwined with $\rho_2 \otimes \chi^{-1} \circ \det$ by x. □

Now we begin to consider the maximal simple cuspidal k-types of $\mathrm{G} = \mathrm{GL}_n(F)$.

Proposition 5.1.12. *Let (J, λ) be a maximal simple cuspidal k-type of* G*, and χ a k-quasicharacter of $F^{\times}$. If $(J, \lambda \otimes \chi \circ \det)$ is weakly intertwined with (J, λ), then they are intertwined. And there exists an element $x \in \mathrm{U}(\mathfrak{A})$ such that $x(J) = J$ and $x(\lambda) \cong \lambda \otimes \chi \circ \det$.*

Proof. There is a surjection from $\mathrm{res}_{H^1}^{J}\lambda$ to θ_1. By Frobenius reciprocity, there is an injection from λ to $\mathrm{ind}_{H^1}^{J}\theta_1$, and exactness of the functors ensure that there exists an injection: $\mathrm{res}_{H^1}^{G}\mathrm{ind}_{J}^{G}\lambda \hookrightarrow \mathrm{res}_{H^1}^{G}\mathrm{ind}_{H^1}^{G}\theta_1$. Whence, by hypothesis, $\mathrm{res}_{H^1}^{J}\lambda \otimes \chi \circ \det$ is a subquotient of $\mathrm{res}_{H^1}^{G}\mathrm{ind}_{H^1}^{G}\theta_1$. After Corollary 5.1.7, the groups $\mathrm{H}^1(\beta + c) = \mathrm{H}^1(\beta)$. Hence $\mathrm{res}_{\mathrm{H}^1}^{\mathrm{G}}(\lambda \otimes \chi \circ \det)$ is a multiple of θ_2, from which we deduce that θ_2 is a subquotient of $\mathrm{res}_{\mathrm{H}^1}^{\mathrm{G}}\mathrm{ind}_{\mathrm{H}^1}^{\mathrm{G}}\theta_1$.

Notice that H^1 is a prop-p group, and any smooth representation of H^1 is semisimple. It follows that θ_2 is a sub-representation of $\mathrm{res}^{\mathrm{G}}_{\mathrm{H}^1}\mathrm{ind}^{\mathrm{G}}_{\mathrm{H}^1}\theta_1$, which is equivalent to say that θ_2 is intertwined with θ_1 in G. Let $i = 1, 2$ and $\theta_{i_{\mathbb{C}}}$ be $\mathbb{C}$-simple characters whose reduction modulo ℓ is isomorphic to θ_i, then $\theta_{1_{\mathbb{C}}}$ is intertwined with $\theta_{2_{\mathbb{C}}}$ in G cause H^1 is pro-p. It follows that the nonsplit fundamental strata $[\mathfrak{A}, n, n-1, \beta]$ and the nonsplit fundamental strata $[\mathfrak{A}, m, m-1, \beta + c]$ are intertwined. We deduce that $n = m$ by 2.3.4 and 2.6.3 of [BuKu]. Then we apply Theorem 3.5.11 of [BuKu]: there exists $x \in \mathrm{U}(\mathfrak{A})$ such that $x(\mathrm{H}^1) = \mathrm{H}^1$, $C(\mathfrak{A}, 0, \beta) = C(\mathfrak{A}, 0, x(\beta + c))$ and $x(\theta_{2_{\mathbb{C}}}) = \theta_{1_{\mathbb{C}}}$, hence $x(\theta_2) = \theta_1$. In particular, $x(J)$ is a subset of $\mathcal{I}_{\mathrm{U}(\mathfrak{A})}(\theta_1)$. Meanwhile, the 2.3.3 of [MS] and 3.1.15 of [BuKu] implies that $\mathcal{I}_{\mathrm{G}}(\theta_1) \cap \mathrm{U}(\mathfrak{A}) = J$, then $x(J) = J$. Proposition 2.2 of [MS] shows the uniqueness of η_1, hence $x(\eta_2) \cong \eta_1$. From [V3, Corollary 8.4] we know that the η_1-isotypic part of $\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J(\lambda)$ can be viewed as a representation of J, which is a direct factor of $\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J(\lambda)$ and is multiple of λ when (J, λ) is maximal simple. Since $x(\lambda \otimes \chi \circ \det)$ could only be a subquotient of the η_1-isotypic part of $\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J(\lambda)$ and $\mathrm{ind}^{\mathrm{G}}_J\lambda \cong \mathrm{ind}^{\mathrm{G}}_J x(\lambda)$, we deduce from above that $\mathrm{Hom}_{kJ}(\lambda \otimes \chi \circ \det, \mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda) \neq 0$. □

Corollary 5.1.13. *For any $g \in \mathrm{G}$, if g weakly intertwines $(J, \lambda \otimes \chi \circ \det)$ and (J, λ), then g intertwines $(J, \lambda \otimes \chi \circ \det)$ and (J, λ).*

Proof. By Mackey's decomposition formula $i_{J,g(J)}g(\lambda)$ is a direct factor of $\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J(\lambda)$. On the other hand, we notice that the $\mathrm{ind}^{\mathrm{G}}_J\lambda$ is isomorphic to $x(\mathrm{ind}^{\mathrm{G}}_J\lambda)$ as G-representation, so they are equivalent after restricting to J. Hence the $x(\eta_1)$-isotypic part $(\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda)^{x(\eta_1)}$ is isomorphic to $x(\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda)^{x(\eta_1)}$ as J representation. The later one is isomorphic to $x((\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda)^{\eta_1})$, which is a multiple of $x(\lambda)$. In the proof of 5.1.12, there exists $x \in \mathrm{U}(\mathfrak{A})$ such that $x(\eta_1) \cong \eta_2$, $x(\lambda) \cong \lambda \otimes \chi \circ \det$. By hypothesis $\lambda \otimes \chi \circ \det^{x(\eta_1)}$ is a subquotient of $i_{J,g(J)}g(\lambda)$, hence a subquotient of $(i_{J,g(J)}g(\lambda))^{x(\eta)}$. And $(i_{J,g(J)}g(\lambda))^{x(\eta)}$ is a sub-representation of $(\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda)^{x(\eta)}$, whence a multiple of $x(\lambda)$ as well. So $\lambda \otimes \chi \circ \det^{x(\eta_1)}$ is a sub-representation of $(i_{J,g(J)}g(\lambda))^{x(\eta)}$. We finish the proof. □

5.1.3 Decomposition of $\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda$

In this subsection, we need to do some computation to obtain the decomposition in 5.1.14, which plays a key role in the proof of Proposition 5.1.25. And this consists half of the proof of Theorem 5.1.30.

Theorem 5.1.14. *Let (J, λ) be a maximal simple cuspidal k-type of G. There exists an integer m and a decomposition:*

$$\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda \cong (\oplus_{i=1}^{m}\ x_i(\Lambda(\lambda))) \oplus W$$

where $x_i \in \mathrm{U}(\mathfrak{A})$, and $x_1 = 1$. The representation $\Lambda(\lambda)$ is semisimple, and a multiple of λ. For each x_i, the representation $x_i(\Lambda(\lambda))$ is the x_i-conjugation of $\Lambda(\lambda)$. The

elements x_i's satisfy that $x_i(\eta) \ncong x_j(\eta)$ if $i \neq j$ (see Definition 5.1.1 for η), and let λ' be any irreducible sub-representation of $\mathrm{res}^J_{J'}\lambda$, then λ' is not equivalent to any irreducible subquotient of $\mathrm{res}^J_{J'}W$.

Remark 5.1.15. *From now on, let (J, λ) be any maximal simple cuspidal k-type of G. We always use Λ_λ to denote $\oplus_{i=1}^m x_i(\Lambda(\lambda))$, where $\Lambda(\lambda)$ has been defined in Theorem 5.1.14, and we could write the decomposition in Theorem 5.1.14 as:*

$$\mathrm{res}^{\mathrm{G}}_J \mathrm{ind}^{\mathrm{G}}_J \lambda \cong \Lambda_\lambda \oplus W.$$

To prove Theorem 5.1.14, we need the following lemmas:

Lemma 5.1.16. *Let K_1, K_2 be two compact open subgroups of G such that $K_1 \subset K_2$. Then the compact induction $\mathrm{ind}_{K_1}^{K_2}$ respect infinite direct sum.*

Proof. Let I be an index set, and (V_i, π_i) be k-representations of K_1. Define $\pi = \oplus_{i\in I}\pi_i$. By definition of compact induction, the representation space of $\mathrm{ind}_{K_1}^{K_2}\pi$ are the smooth vectors of the k-vector space consisting with function $f : K_2 \to V$ such that $f(hg) = \pi(h)f(g)$, where $h \in K_1$, $g \in K_2$, and K_2 acts as right transition. Notice first that every function satisfied the condition above is smooth. In fact, the quotient group K_1/K_2 is finite, of which let $a_1, \ldots, a_m$ be a set of representatives in K_2. Then there is an bijection from the vector space, consisting of the functions on K_2 verified the condition above, to V^m, which is sending f to $f(a_1), \ldots, f(a_m)$. Now let f be any such function on K_2. For any $j \in \{1, \ldots, m\}$, there exists an open subgroup $H_j \subset K_1$ which stabilizes v_j. Let $g \in K_2$, the value $(a_j^{-1}(g)(f))(a_j) = f(a_j)$. Hence the open compact subgroup $H = \cap_{j=1}^m a_j^{-1}(H_j)$ stablizes f, so f is smooth. Notice that $\oplus_{j=1}^m(\oplus_{i\in I}V_i) \cong \oplus_{i\in I}(\oplus_{j=1}^m V_i)$ as vector spaces, which the result follows. □

Lemma 5.1.17. *Let K be a compact open subgroup of M, where M is a Levi subgroup of G, and $K' = K \cap \mathrm{G}'$. Let π be a k-representation of K. If τ' is an irreducible subquotient of the restricted representation $\mathrm{res}^K_{K'}\pi$, then there exists an irreducible subquotient τ of π, such that τ' is an irreducible direct component of $\mathrm{res}^K_{K'}\tau$.*

Proof. Let H be a pro-p open compact subgroup of K. The representation $\mathrm{res}^K_H\pi$ is semisimple, which can be written as $\oplus_{i\in I}\pi_i$, where I is an index set. There is an injection from π to $\mathrm{ind}^K_H\mathrm{res}^K_H\pi$, and the lemma 5.1.16 implies that $\mathrm{ind}^K_H\mathrm{res}^K_H\pi \cong \oplus_{i\in I}\mathrm{ind}^K_H\pi_i$. Notice that for each $i \in I$, the representation $\mathrm{ind}^K_H\pi_i$ has finite length. Let W', V' be two sub-representations of $\pi' = \mathrm{res}^K_{K'}\pi$, such that $\tau' \cong W'/V'$. When τ' is non-trivial, there exists $x \in W'$ such that $x \notin V'$. Since $\mathrm{ind}^K_H\mathrm{res}^K_H\pi$ is isomorphic to a direct sum of $\mathrm{ind}^K_H\pi_i$, there exists a finite index set $\{i_1, \ldots, i_m\} \subset I$, where $m \in \mathbb{N}^*$, such that $x \in \oplus_{i_1,\ldots,i_m}\mathrm{ind}^K_H\pi_i$. We have:

$$0 \neq (W' \cap \oplus_{i_1,\ldots,i_m}\mathrm{ind}^K_H\pi_i)/(V' \cap \oplus_{i_1,\ldots,i_m}\mathrm{ind}^K_H\pi_i) \hookrightarrow W'/V',$$

Since W'/V' is irreducible, the injection above is an isomorphism, and we conclude that

$$(W' \cap \oplus_{i_1,\ldots,i_m} \mathrm{ind}_H^K \pi_i)/(V' \cap \oplus_{i_1,\ldots,i_m} \mathrm{ind}_H^K \pi_i) \cong W'/V' \cong \tau'.$$

Since the restricted representation $\mathrm{res}^K_{K'} \oplus_{i_1,\ldots,i_m} \mathrm{ind}_H^K \pi_i$ has finite dimension hence finite length, by the uniqueness of Jordan-Hölder factors, there exists an irreducible subquotient of $\oplus_{i_1,\ldots,i_m} \mathrm{ind}_H^K \pi_i$, whose restriction to K' is semisimple (by Proposition 5.1.9) and containing τ' as a subrepresentation. □

Now we look back Theorem 5.1.14.

Proof. of 5.1.14:

By [V3, Corollary 8.4], we can decompose $\mathrm{res}^{\mathrm{G}}_J \mathrm{ind}^{\mathrm{G}}_J \lambda \cong \Lambda(\lambda) \oplus W_1$, where any irreducible subquotient of W_1 is not isomorphic to λ. Let λ' be an irreducible subrepresentation of the semisimple k-representation $\mathrm{res}^J_{J'} \lambda$. If λ' is an irreducible subquotient of $\mathrm{res}^J_{J'} W_1$, by Lemma 5.1.17 and Propositon 5.1.10, there exists a k-quasicharacter χ of $F^\times$ such that $\lambda \otimes \chi \circ \det$ is an irreducible subquotient of W_1. This follows that $\lambda \otimes \chi \circ \det$ is weakly intertwined with λ. By Proposition 5.1.12, they are intertwined and there exists $x \in \mathrm{U}(\mathfrak{A})$ such that $\lambda \otimes \chi \circ \det \cong x(\lambda)$. The fact that $\lambda \otimes \chi \circ \det$ is a subquotient of W_1 implies that $x(\eta) \ncong \eta$. As in the proof of Corollary 5.1.13, we have:

$$(\Lambda(\lambda))^{x(\eta)} \oplus W_1^{x(\eta)} \cong (\mathrm{res}^{\mathrm{G}}_J \mathrm{ind}^{\mathrm{G}}_J \lambda)^{x(\eta)} \cong x((\mathrm{res}^{\mathrm{G}}_J \mathrm{ind}^{\mathrm{G}}_J \lambda)^{\eta}),$$

and the later one is isomorphic to $x(\Lambda(\lambda))$, which is a direct sum of $x(\lambda)$. Since $x(\eta) \ncong \eta$, we have $(\Lambda(\lambda))^{x(\eta)} = 0$. As for W_1, thus we can decompose W_1 as $W_1^{x(\eta)} \oplus W_2$. Hence $W_1^{x(\eta)} \cong x(\Lambda(\lambda))$. Now we obtain an isomorphism:

$$\mathrm{res}^{\mathrm{G}}_J \mathrm{ind}^{\mathrm{G}}_J \lambda \cong \Lambda(\lambda) \oplus x(\Lambda(\lambda)) \oplus W_2,$$

where $W_2^{x(\eta)} = 0$ and $W_2^{\eta} = 0$. This implies any irreducible subquotient of W_2 is not isomorphic to λ neither $x(\lambda)$. If λ' is an irreducible subquotient of W_2, we can repeat the steps above, then find a k-quasicharacter χ_2, an element $x_2 \in \mathrm{U}(\mathfrak{A})$, and decompose W_2 as $x_2(\Lambda(\lambda)) \oplus W_3$, where any $W_3^{x_2(\eta)} = 0$. Furthermore, any irreducible representation of J, whose restriction to J' contains λ' as a subrepresentation, is $\mathrm{U}(\mathfrak{A})$-conjugate to λ. The quotient group $\mathrm{U}(\mathfrak{A})/J$ is finite, hence the set of irreducible representations $\{x(\lambda)\}_{x \in \mathrm{U}(\mathfrak{A})}$ is finite, which means after repeat the steps above for finite times, we could obtain the decomposition as required. □

5.1.4 Projective normalizer $\tilde{J}$ and its subgroups

Now we will recall one definition and some propositions given by Bushnell and Kutzko in [BuKuII] when they consider the $\overline{\mathbb{Q}}_\ell$-representations of G'.

Definition 5.1.18 (Bushnell,Kutzko)**.** *We define the projective normalizer* $\tilde{J} = \tilde{J}(\lambda)$ *of* (J, λ)*. Let* $\mathfrak{A}$ *be the principal order attached to* (J, λ)*. Then define* $\tilde{J}$ *to be the group of all* $x \in \mathrm{U}(\mathfrak{A})$ *such that:*

- $xJx^{-1} = J$*, and*
- *there exists a k-quasicharacter* χ *of* F^* *such that* $x(\lambda) \cong \lambda \otimes \chi \circ \det$*.*

Proposition 5.1.19. *Let* (J, λ) *be a simple type in* G *as in definition 5.1.18, and* χ *be a k-quasicharacter of* F^**. The following are equivalent:*

1. $\lambda \cong \lambda \otimes \chi \circ \det$,
2. $\chi \circ \det|_{J^1}$ *is trivial and* $\sigma \otimes \chi \circ \det|_{\mathrm{U}(\mathfrak{B})} \cong \sigma$,
3. $\chi \circ \det|_{J^1}$ *is trivial, and* λ, $\lambda \otimes \chi \circ \det$ *are intertwined in* G.

Proof. The proof in Proposition 2.3 [BuKuII] still works in our case, and we write it here to ensure it in the modulo ℓ case. We prove this proposition in the order of $(2) \to (1) \to (3) \to (2)$.

Since $J/J' \cong \mathrm{U}(\mathfrak{B}/\mathrm{U}^1(\mathfrak{B}))$, the implication $(2) \to (1)$ is trivial. Now let us assume λ is equivalent to $\lambda \otimes \chi \circ \det$. Restricting to H^1, we see that the simple character $\theta \cong \theta \otimes \chi \circ \det|_{H^1}$, which implies $\chi \circ \det|_{H^1}$ is trivial. Now assume (3) holds. Proposition 5.1.12 gives an element $x \in \mathrm{U}(\mathfrak{A})$ such that $x(J) = J$ and $x(\lambda) = \lambda \otimes \chi \circ \det$. We have $\chi \circ \det$ is trivial on J^1. Combining this fact with the uniqueness of η corresponding to any fixed simple character of H^1 (see Proposition 2.2 of [?]), we have $x(\eta) \cong \eta$. In particular, $x \in \mathcal{I}_{\mathrm{G}}(\theta) = J^1 B^{\times} J^1$, by IV.1.1 in [V2], hence $x \in J^1 B^* J^1 \cap \mathrm{U}(\mathfrak{A}) = J$. Whence $\lambda \cong x(\lambda) \cong \lambda \otimes \chi \circ \det$. We therefore have $\kappa \otimes \sigma \cong \kappa \otimes \sigma \otimes \chi \circ \det$, where κ is a β-extension of η to J.

As indicate in the proof of Bushnell and Kutzko, from now on, we use the technique in Proposition 5.3.2 of [BuKu]: Let X denote the representation space of κ and Y the representation space of σ, which can be identified with the representation space of $\sigma \otimes \chi \circ \det$. Let ϕ be the isomorphism between $\kappa \otimes \sigma$ and $\kappa \otimes \sigma \otimes \chi \circ \det$. We may write ϕ as $\sum_j S_j \otimes T_j$ where $S_j \in \mathrm{End}_k(X)$ and $T_j \in \mathrm{End}_k(Y)$, and where $\{T_j\}$ are linearly independent. Let $g \in J^1$, we have $\kappa \otimes \sigma(g) \circ \phi = \phi \circ (\kappa \otimes \sigma \otimes \chi \circ \det)(g)$. Since $J^1 \subset \ker(\sigma) = \ker(\sigma \otimes \chi \circ \det)$, this relation reads:

$$(\eta(g) \otimes 1) \circ \sum_j S_j \otimes T_j = (\sum_j S_j \otimes T_j) \circ (\eta(g) \otimes 1),$$

which is equivalent to say that:

$$\sum_j (\eta(g) \circ S_j - S_j \circ \eta(g)) \otimes T_j = 0.$$

The linearly independence of T_j implies that $S_j \in \mathrm{End}_{kJ^1}(\eta) = k^*$, by the lemma of Schur. Hence $\phi = 1 \otimes \sum_j S_j \cdot T_j$. Now note $T = \sum_j S_j \cdot T_j$ and take $g \in J$, the

morphism relation reads:

$$(\kappa(g)\otimes\sigma(g))\circ(1\otimes T)=\kappa(g)\otimes(\sigma(g)\circ T)=\kappa(g)\otimes(T\circ\sigma\otimes\chi\circ\det(g))$$

$$=(1\otimes T)\circ(\kappa(g)\otimes\sigma\otimes\chi\circ\det(g)),$$

which says $T\in\mathrm{Hom}_{kJ}(\sigma,\sigma\otimes\chi\circ\det)\neq 0$. We finish the proof. □

Corollary 5.1.20 (Bushnell,Kutzko)**.** *Let $x\in\tilde{J}(\lambda)$, and let χ be a quasicharacter of F^* such that $x(\lambda)\cong\lambda\otimes\chi\circ\det$. Then:*

1. *the map $x\mapsto\chi\circ\det|_{J^1}$ is an injective homomorphism $\tilde{J}/J\to(\det(J^1))^\wedge$. The later one denotes the dual group of the subgroup $\det(J^1)$ of $F^\times$;*
2. *$\tilde{J}/J$ is a finite abelian p-group, where p is the residual characteristic of F.*

Proof. For (1). Let $x\in\tilde{J}$. Suppose there exist two k-quasicharacter χ_1,χ_2 of F^*, such that $x(\lambda)\cong\lambda\otimes\chi_1\circ\det$ and $\lambda\otimes\chi_1\circ\det\cong\lambda\otimes\chi_2\circ\det$. This is equivalent to say that

$$\lambda\cong\lambda\otimes(\chi_1\circ\det)\otimes(\chi_2^{-1}\circ\det)\cong\lambda\otimes(\chi_1\otimes\chi_2^{-1})\circ\det.$$

The equivalence between (1) and (2) of Proposition 5.1.19 implies that $(\chi_1\otimes\chi_2^{-1})\circ\det|_{J^1}$ is trivial. Hence $\chi_1\circ\det|_{J^1}\cong\chi_2\circ\det|_{J^1}$. So the map is well defined, and is clearly a morphism between groups. Now suppose that $x\in\tilde{J}$ and χ is a k-quasicharacter of $F^\times$ which is trivial on $\det(J^1)$, such that $x(\lambda)\cong\lambda\otimes\chi\circ\det$. As in the Proposition 5.1.19, the equivalence of conditions means that $\lambda\cong\lambda\otimes\chi\circ\det$. Thus x intertwined λ to itself. Whence the element x belongs to $JB^\times J\cap\mathrm{U}(\mathfrak{A})=J$.

For (2). Since J^1 is a pro-p group, this is induced directly from (1). □

5.1.5 Two conditions for irreducibility

In this section, let (J,λ) be any maximal simple cuspidal k-type of G. We will construct a compact subgroup M_λ of G' and a family of irreducible representations λ'_{M_λ} of M_λ, such that the induced representation $\mathrm{ind}_{M_\lambda}^{\mathrm{G}'}\lambda'_{M_\lambda}$ is irreducible and cuspidal (Theorem 5.1.30). And in the next section, we will see that any irreducible cuspidal k-representation π' of G' can be constructed in this manner.

To check the irreducibility of this induced representation, we only need to calculate its intertwining set in G', when considering representations in characteristic 0, but this is not sufficient in the case of modulo ℓ. As noted in lemma 4.2 in article [V3], Vignéras presents a criterium of irreducibility in modulo ℓ cases:

Lemma 5.1.21 (criterium of irreducibility by Vignéras)**.** *Let K be an open compact subgroup of G', and π' be a k-irreducible representation of K. The induced representation $\mathrm{ind}_K^{\mathrm{G}'}\pi'$ is irreducible, when*

1. $\mathrm{End}_{k\mathrm{G}'}(\mathrm{ind}_K^{\mathrm{G}'}\pi')=k$,

2. *for any k-irreducible representation ν of G', if π' is contained in $\mathrm{res}^{\mathrm{G}'}_K\nu$ then there is a surjection which maps $\mathrm{res}^{\mathrm{G}'}_K\nu$ to π'.*

As in 8.3 chapter I of [V1], the first criterion of irreducibility is equivalent to say that the intertwining set $\mathrm{I}_{\mathrm{G}'}(\pi') = K$.

Corollary 5.1.22. *Let (J,λ) be a maximal simple cuspidal k-type in G. The induced k-representation $\mathrm{ind}^{\tilde{J}}_J\lambda$ is irreducible.*

Proof. Lemma 5.1.21 can be applied in this case after changing G' to any locally pro-finite group. First, we calculate $\mathrm{End}_{k\tilde{J}}(\mathrm{ind}^{\tilde{J}}_J\lambda)$, which equals to k since the intertwining group $\mathrm{I}_{\tilde{J}}(\lambda) = J$. Now we consider the second condition. Let ν be an irreducible k-representation of $\tilde{J}$, such that

$$\lambda \hookrightarrow \mathrm{res}^{\tilde{J}}_J\lambda.$$

By Frobenius reciprocity and the exactness of functors ind and res, we have a surjection:

$$\mathrm{res}^{\mathrm{G}}_J\mathrm{ind}^{\mathrm{G}}_J\lambda \to \mathrm{res}^{\tilde{J}}_J\nu.$$

The (J^1,κ)-isotypic part ν^κ of $\mathrm{res}^{\tilde{J}}_J\nu$ is a direct component as J representation, and ν^κ is a quotient of the (J^1,κ)-isotypic part λ^κ of λ as J representation. The later one is a multiple of λ by Corollary 8.4 of [V3]. Hence the surjection required in the second condition of Lemma 5.1.21 exists. □

Theorem 5.1.23. *Let λ' be a subrepresentation of $\mathrm{res}^J_{J'}\lambda$. Then λ' verifies the second condition of irreducibility. This is to say that for any irreducible representation π' of G', if there is an injection: $\lambda' \hookrightarrow \mathrm{res}^{\mathrm{G}'}_{J'}\pi'$, then there is a surjection: $\mathrm{res}^{\mathrm{G}'}_{J'}\pi' \twoheadrightarrow \lambda'$.*

Proof. Since J is open, every double coset $\mathrm{G}'gJ$ is open and closed, hence we could apply Mackey's decomposition formula:

$$\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\mathrm{ind}^{\mathrm{G}}_J\lambda \cong \oplus_{a\in J\backslash \mathrm{G}/\mathrm{G}'}\mathrm{ind}^{\mathrm{G}'}_{\mathrm{G}'\cap a(J)}\mathrm{res}^{a(J)}_{\mathrm{G}'\cap a(J)}a(\lambda).$$

We take $a = 1$, then $\mathrm{ind}^{\mathrm{G}'}_{J'}\mathrm{res}^J_{J'}\lambda$ is a direct factor of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\mathrm{ind}^{\mathrm{G}}_J\lambda$. The hypothesis $\lambda' \hookrightarrow \mathrm{res}^{\mathrm{G}'}_{J'}\pi'$ implies a surjection from $\mathrm{ind}^{\mathrm{G}'}_{J'}\lambda'$ to π' by Frobenius reciprocity. Since $\mathrm{res}^J_{J'}\lambda$ is semisimple with finite length by Proposition 5.1.9 and the functor $\mathrm{ind}^{\mathrm{G}'}_{J'}$ respects finite direct sum, we have an surjection:

$$\mathrm{ind}^{\mathrm{G}'}_{J'}\mathrm{res}^J_{J'}\lambda \twoheadrightarrow \pi',$$

hence we obtain a surjection:

$$\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\mathrm{ind}^{\mathrm{G}}_J\lambda \twoheadrightarrow \pi'.$$

Now consider the surjection:

$$\iota : \mathrm{res}^{\mathrm{G}}_{J'}\mathrm{ind}^{\mathrm{G}}_{J}\lambda \twoheadrightarrow \mathrm{res}^{\mathrm{G}'}_{J'}\pi'.$$

Meanwhile, by Theorem 5.1.14, we could decompose $\mathrm{res}^{\mathrm{G}}_{J}\mathrm{ind}^{\mathrm{G}}_{J}\lambda \cong \Lambda_\lambda \oplus W$. We have $\Lambda_\lambda \oplus W/\ker\iota \cong \mathrm{res}^{\mathrm{G}'}_{J'}\pi'$. If the image of the injection $\lambda' \hookrightarrow \Lambda_\lambda \oplus W/\ker(\iota)$ is contained in $W + \ker(\iota)/\ker(\iota)$, then λ' is an irreducible subquotient of W, which is contradicted with Theorem 5.1.14. Hence the image of the composed morphism:

$$\lambda' \hookrightarrow \Lambda_\lambda \oplus W/\ker(\iota) \twoheadrightarrow \Lambda_\lambda \oplus W/(W + \ker(\iota)) \cong \Lambda_\lambda/(\Lambda_\lambda \cap (W + \ker(\iota)))$$

is non-trivial. Since $\Lambda_\lambda/(\Lambda_\lambda \cap (W + \ker(\iota)))$ is a quotient of Λ_λ, and the functor $\mathrm{res}^{J}_{J'}$ maps any irreducible representation of J to a semisimple representation with finite length of J', the representation $\mathrm{res}^{J}_{J'}\Lambda_\lambda$ is semisimple with finite length of J'. So does the quotient $\Lambda_\lambda/(\Lambda_\lambda + \ker(\iota))$, of which λ' is an irreducible direct component. This implies a surjection: $\mathrm{res}^{\mathrm{G}'}_{J'}\pi' \twoheadrightarrow \lambda'$. □

In the theorem above, we proved that λ' verifies the second condition of irreducible criterium of irreducibility in lemma 5.1.21. Unfortunately, (J', λ') does not satisfies the first condition. This is also false for representations of characteristic 0. A natural idea is to construct a open compact subgroup of G', which is bigger than J'. In the case of characteristic 0, Bushnell and Kutzko calculated in [BuKuII]. This group is $\tilde{J}'_{\mathbb{C}} = \tilde{J}_{\mathbb{C}} \cap \mathrm{G}'$, the intersection of projective normalizer of a $\overline{\mathbb{Q}}_\ell$-maximal cuspidal simple type and G'. We will see in Proposition 5.2.6 and definition 5.2.7 that this group is $\tilde{J}' = \tilde{J} \cap \mathrm{G}'$ in the case of modulo ℓ.

Proposition 5.1.24. *Let L be any subgroup of $\tilde{J}' = \tilde{J} \cap \mathrm{G}'$ such that $J' \subset L \subset \tilde{J}'$, and λ' an irreducible subrepresentation of $\lambda|_{J'}$. Then the induced representation $\mathrm{ind}^{L}_{J'}\lambda'$ is semisimple.*

Proof. By Mackey's decomposition formula, the induced representation $\mathrm{ind}^{\tilde{J}'}_{J'}\lambda'$ is a subrepresentation of $\mathrm{res}^{\tilde{J}}_{\tilde{J}'}\mathrm{ind}^{\tilde{J}}_{J}\lambda$. Applying Mackey's decomposition formula, we have

$$\mathrm{res}^{\tilde{J}}_{\tilde{J}'}\mathrm{ind}^{\tilde{J}}_{J}\lambda \cong \oplus_{g\in J\backslash\tilde{J}}\mathrm{res}^{J}_{J'}g(\lambda),$$

since $\tilde{J}$ normalises J and J'. Hence $\mathrm{res}^{\tilde{J}}_{\tilde{J}'}\mathrm{ind}^{\tilde{J}}_{J}\lambda$ is semisimple by Proposition 5.1.9 and the fact that $\mathrm{ind}^{\tilde{J}}_{J}\lambda$ is irreducible. Since L is a normal open subgroup of $\tilde{J}'$, the index of L in $\tilde{J}'$ is finite. Hence the restricted representation $\mathrm{res}^{\tilde{J}'}_{L}\mathrm{ind}^{\tilde{J}'}_{J'}\lambda'$ is semisimple by Clifford theory, of which $\mathrm{ind}^{L}_{J'}\lambda'$ is a subrepresentation. Now we obtain the result. □

Proposition 5.1.25. *Let λ' be an irreducible subrepresentation of $\mathrm{res}^{J}_{J'}\lambda$, and λ'_L an irreducible subrepresentation of $\mathrm{ind}^{L}_{J'}\lambda'$. Then λ'_L verifies the second condition of irreducibility. This is to say that for any irreducible representation π' of G', if there is an injection $\lambda'_L \hookrightarrow \mathrm{res}^{\mathrm{G}'}_{L'}\pi'$, then there exists a surjection $\mathrm{res}^{\mathrm{G}'}_{L'}\pi' \twoheadrightarrow \lambda'_L$, where $L' = L \cap \mathrm{G}'$.*

Proof. We have proved in Proposition 5.1.24 that $\mathrm{ind}_{J'}^{L}\lambda'$ is semisimple. Hence the injection from λ'_L to $\mathrm{res}_{L}^{\mathrm{G}'}\pi'$ induces a non-trivial homomorphism $\mathrm{ind}_{J'}^{L}\lambda' \twoheadrightarrow \mathrm{res}_{L}^{\mathrm{G}'}\pi'$. By Frobenius reciprocity, we obtain an injection from λ' to $\mathrm{res}_{J'}^{\mathrm{G}'}\pi'$. Thus there exists a non-trivial homomorphism $\mathrm{res}_{J'}^{J}\lambda \to \mathrm{res}_{J'}^{\mathrm{G}'}\pi'$. After applying Frobenius reciprocity and the exactness of the functor $\mathrm{res}_{J'}^{\mathrm{G}'}$, we obtain a surjection:

$$\mathrm{res}_{J'}^{\mathrm{G}'}\mathrm{ind}_{J'}^{\mathrm{G}'}\mathrm{res}_{J'}^{J}\lambda \twoheadrightarrow \mathrm{res}_{J'}^{\mathrm{G}'}\pi'.$$

By Mackey's decomposition formula, the k-representation $\mathrm{ind}_{J'}^{\mathrm{G}'}\mathrm{res}_{J'}^{J}\lambda$ is a direct component of $\mathrm{res}_{\mathrm{G}'}^{\mathrm{G}}\mathrm{ind}_{J}^{\mathrm{G}}\lambda$, and combining this fact with the exactness of functor $\mathrm{res}_{J'}^{\mathrm{G}'}$, the k-representation $\mathrm{res}_{J'}^{\mathrm{G}'}\mathrm{ind}_{J'}^{\mathrm{G}'}\mathrm{res}_{J'}^{J}\lambda$ is a direct component of $\mathrm{res}_{J'}^{\mathrm{G}}\mathrm{ind}_{J}^{\mathrm{G}}\lambda$. Hence the surjection above implies a non-trivial homomorphism:

$$\mathrm{res}_{J'}^{\mathrm{G}}\mathrm{ind}_{J}^{\mathrm{G}}\lambda \twoheadrightarrow \mathrm{res}_{J'}^{\mathrm{G}'}\pi'.$$

By Proposition 5.1.14, the left hand side isomorphic to $\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{res}_{J'}^{J}W$, where any irreducible subquotient of $\mathrm{res}_{J'}^{J}W$ is not isomorphic to any irreducible subrepresentation of $\mathrm{res}_{J'}^{J}\lambda$. Now we obtain an equivalence of $\mathrm{res}_{J'}^{\mathrm{G}'}\pi'$ with $(\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{res}_{J'}^{J}W)/K$, where K (a k-representation of J') is the kernel of the surjection above.

We have:

$$\lambda'_L \hookrightarrow \mathrm{res}_{L}^{\mathrm{G}'}\pi' \hookrightarrow \mathrm{ind}_{J'}^{L}\mathrm{res}_{J'}^{\mathrm{G}'}\pi',$$

and the last factor is isomorphic to $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{res}_{J'}^{J}W)/K)$. We note this composed homomorphism from λ'_L to $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{res}_{J'}^{J}W)/K)$ as τ.

Since the functor $\mathrm{ind}_{J'}^{L}$ is exact, the right side is isomorphic to $(\mathrm{ind}_{J'}^{L}\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{ind}_{J'}^{L}\mathrm{res}_{J'}^{J}W)/\mathrm{ind}_{J'}^{L}K$. And we consider the representation $\mathrm{ind}_{J'}^{L}(\mathrm{res}_{J'}^{J}W+K)/\mathrm{ind}_{J'}^{L}K$, which is isomorphic to $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K)$. We assume the image $\tau(\lambda'_L)$ in $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}\Lambda_\lambda \oplus \mathrm{res}_{J'}^{J}W)/K)$ is contained in $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K)$. Then τ is a non-trivial morphism from λ'_L to $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K)$. By Frobenius reciprocity, we deduce a non-trivial morphism from $\mathrm{res}_{J'}^{L}\lambda'_L$ to $(\mathrm{res}_{J'}^{J}W + K)/K$. Notice that

$$\mathrm{res}_{J'}^{L}\lambda'_L \hookrightarrow \mathrm{res}_{J'}^{\tilde{J}}\mathrm{ind}_{J}^{\tilde{J}}\lambda \cong \oplus_{a\in J\backslash\tilde{J}/J'}\mathrm{res}_{J'}^{J}a(\lambda),$$

and by definition of $\tilde{J}$, the representation $a(\lambda) \cong \lambda\otimes\chi\circ\det$, for some k-quasicharacter χ of $F^\times$. Hence

$$\oplus_{a\in J\backslash\tilde{J}/J'}\mathrm{res}_{J'}^{J}a(\lambda) \cong \oplus_{J\backslash\tilde{J}/J'}\mathrm{res}_{J'}^{J}\lambda.$$

Thus there exists an irreducible direct component λ'' of $\mathrm{res}_{J'}^{J}\lambda$, from which there is an injective morphism to $(\mathrm{res}_{J'}^{J}W + K)/K \cong \mathrm{res}_{J'}^{J}W/(\mathrm{res}_{J'}^{J}W \cap K)$. Hence λ'' is isomorphic to a subquotient of $\mathrm{res}_{J'}^{J}W$. This is contradicted to Theorem 5.1.14. So the image $\tau(\lambda'_L)$ is not contained in $\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K)$. We deduce that the composed map:

$$\lambda'_L \hookrightarrow \mathrm{res}_{L}^{\mathrm{G}'}\pi' \to \mathrm{ind}_{J'}^{L}(\mathrm{res}_{J'}^{J}(\Lambda_\lambda \oplus W)/K)/\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K),$$

is non-trivial. The right hand side factor:

$$\mathrm{ind}_{J'}^{L}(\mathrm{res}_{J'}^{J}(\Lambda_\lambda \oplus W)/K)/\mathrm{ind}_{J'}^{L}((\mathrm{res}_{J'}^{J}W + K)/K)$$

$$\cong \mathrm{ind}_{J'}^{L}(\mathrm{res}_{J'}^{J}(\Lambda_\lambda \oplus W)/(\mathrm{res}_{J'}^{J}W + K)).$$

Notice that $\mathrm{res}_{J'}^{J}(\Lambda_\lambda \oplus W)/(\mathrm{res}_{J'}^{J}W + K)$ is a quotient, hence isomorphic to a subrepresentation of $\mathrm{res}_{J'}^{J}\Lambda_\lambda$. The representation $\mathrm{res}_{J'}^{J}\Lambda_\lambda$ is semisimple, with irreducible direct components in the form of $x(\lambda')$, where $x \in \mathrm{U}(\mathfrak{A})$. Furthermore, since $\mathrm{ind}_{J'}^{L}x(\lambda')$ is a direct component of $\mathrm{res}_{L}^{\tilde{J}'}\mathrm{ind}_{J'}^{\tilde{J}'}x(\lambda)$, as in the proof of Proposition 5.1.24, it is semisimple. After lemma 5.1.16, we deduce that $\mathrm{ind}_{J'}^{L}\mathrm{res}_{J'}^{J}\Lambda_\lambda$ is semisimple, and so is the subrepresentation $\mathrm{ind}_{J'}^{L}(\mathrm{res}_{J'}^{J}(\Lambda_\lambda \oplus W)/((\mathrm{res}_{J'}^{J}W) + K))$, of which λ'_L is a direct factor. Hence we finish the proof. □

Definition 5.1.26. *Let (J, λ) be a maximal simple cuspidal k-type of* G*, and λ' be any irreducible subrepresentation of* $\mathrm{res}_{J'}^{J}\lambda$*. Define M_λ to be the subgroup of $\tilde{J}'$ consisting with all the elements $x \in \tilde{J}'$, such that $x(\lambda') \cong \lambda'$.*

Remark 5.1.27. *Since M_λ normalizes J and the intersection $M_\lambda \cdot J \cap \mathrm{G}'$ equals to M_λ, we deduce that J normalizes M_λ. Notice that irreducible subrepresentations of* $\mathrm{res}_{J'}^{J}\lambda$ *are J-conjugate. Hence the group M_λ depends only on λ.*

We will prove at the end of this section, that the couple $(M_\lambda, \lambda'_{M_\lambda})$ verifies the two criterium of irreducibility. The first criterion has been checked in Proposition 5.1.25. And we will calculate its intertwining group in G' in two steps. First is to prove $\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda') \subset \mathrm{U}(\mathfrak{A})'$ (Proposition 5.1.28), and then prove that $\mathrm{I}_{\mathrm{U}(\mathfrak{A})'}\lambda'_{M_\lambda} = M_\lambda$ (Theorem 5.1.30).

Proposition 5.1.28. *Let λ' be an irreducible subrepresentation of* $\mathrm{res}_{J'}^{J}\lambda$*, then the intertwining set* $\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda')$ *is contained in* $\mathrm{U}(\mathfrak{A})'$*.*

Proof. Let τ denote the irreducible representation $\mathrm{ind}_{J}^{\mathrm{U}(\mathfrak{A})}\lambda$. The induced representation $\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda'$ is a subrepresentation of $\mathrm{res}_{\mathrm{U}(\mathfrak{A})'}^{\mathrm{U}(\mathfrak{A})}\tau$, thus is semisimple with finite direct components. We write $\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda'$ as $\oplus_{i\in I}\tau'_i$, where τ'_i are irreducible direct components and I is a finite index set. Let $g \in \mathrm{G}$, we have an equality:

$$\mathrm{I}_g(\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda') = \bigcup_{i\in I, j\in I} \mathrm{Hom}(\tau'_i, i_{\mathrm{U}(\mathfrak{A})', g(\mathrm{U}(\mathfrak{A})')}\tau'_j).$$

Hence we have:

$$\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda') = \bigcup_{i\in I, j\in I} \mathrm{I}_{\mathrm{G}'}(\tau'_i, \tau'_j).$$

Now we assume that $g \in \mathrm{G}'$ intertwines τ'_i with τ'_j. Since τ'_i and τ'_j are direct components of $\mathrm{res}_{\mathrm{U}(\mathfrak{A})'}^{\mathrm{U}(\mathfrak{A})}\tau$, by Proposition 5.1.11, there exists a k-quasicharacter χ of $F^\times$ such that g weakly intertwines τ with $\tau \otimes \chi \circ \det$. This implies that τ is a

subquotient of $i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}\tau\otimes\chi\circ\det$, hence the restriction $\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}\tau$ is a subquotient of $\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}\tau\otimes\chi\circ\det$. Since λ is a subrepresentation of $\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}\tau$, it is hence a subquotient of $\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}\tau\otimes\chi\circ\det$. We have

$$\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}\tau\otimes\chi\circ\det$$

$$=\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}\mathrm{ind}_{\mathrm{U}(\mathfrak{A})\cap g(\mathrm{U}(\mathfrak{A}))}^{\mathrm{U}(\mathfrak{A})}\mathrm{res}_{\mathrm{U}(\mathfrak{A})\cap g(\mathrm{U}(\mathfrak{A}))}^{g(\mathrm{U}(\mathfrak{A}))}\mathrm{ind}_{g(J)}^{g(\mathrm{U}(\mathfrak{A}))}g(\lambda)\otimes\chi\circ\det$$

Applying Mackey's decomposition formula two times to the later factor, we obtain that $\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}(\tau\otimes\chi\circ\det)$ is isomorphic to a finite direct sum, whose direct components are in the form of $\mathrm{ind}_{J\cap y(J)}^{J}\mathrm{res}_{J\cap y(J)}^{y(J)}y(\lambda\otimes\chi\circ\det)$, where $y\in\mathrm{U}(\mathfrak{A})g\mathrm{U}(\mathfrak{A})$. More precisely,

$$\mathrm{res}_J^{\mathrm{U}(\mathfrak{A})}i_{\mathrm{U}(\mathfrak{A}),g(\mathrm{U}(\mathfrak{A}))}(\tau\otimes\chi\circ\det)$$

$$=\bigoplus_{\beta\in\mathrm{U}(\mathfrak{A})\cap\alpha g(J)\backslash\mathrm{U}(\mathfrak{A})/J}\ \bigoplus_{\alpha\in g(J)\backslash g(\mathrm{U}(\mathfrak{A}))/\mathrm{U}(\mathfrak{A})\cap g(\mathrm{U}(\mathfrak{A}))}\mathrm{ind}_{J\cap y(J)}^{J}\mathrm{res}_{J\cap y(J)}^{\beta(\mathrm{U}(\mathfrak{A}))\cap y(J)}y(\lambda)\otimes\chi\det$$

where $y=\beta\alpha g$.

After the uniqueness of Jordan-Hölder factors, the representation λ is weakly intertwined with $\lambda\otimes\chi\circ\det$ by some $y\in\mathrm{U}(\mathfrak{A})g\mathrm{U}(\mathfrak{A})$. Hence y intertwines λ with $\lambda\otimes\chi\circ\det$ by Corollary 5.1.13, and there exists $x\in\mathrm{U}(\mathfrak{A})$ such that $x(\lambda\otimes\chi\circ\det)\cong\lambda$ by Proposition 5.1.12. The element yx^{-1} intertwines λ to itself, and hence lies in $E^\times J$. Therefore $g\in\mathrm{U}(\mathfrak{A})\mathrm{E}^\times J\mathrm{U}(\mathfrak{A})\cap\mathrm{G}'$. Furthermore, we have $\mathrm{U}(\mathfrak{A})\mathrm{E}^\times J\mathrm{U}(\mathfrak{A})\cap\mathrm{G}=\mathrm{U}(\mathfrak{A})'$, because $E^\times$ normalises $\mathrm{U}(\mathfrak{A})$ and for any $e\in E^\times$, $\det(e)\in\mathfrak{o}_F^\times$ if and only $e\in\mathfrak{o}_E^\times$, where $\mathfrak{o}_F$,$\mathfrak{o}_E$ denote the ring of integers of F,E respectively. Hence for any $a\in\mathrm{U}(\mathfrak{A})$, $\det(ea)=1$ if and only if $ea\in\mathrm{U}(\mathfrak{A})\cap\mathrm{G}'$. From which, we deduce that $\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{J'}^{\mathrm{U}(\mathfrak{A})'}\lambda')=\mathrm{U}(\mathfrak{A})'$. □

Lemma 5.1.29. *Let λ' be an irreducible component of $\mathrm{res}_{J'}^{J}\lambda$, and let $x\in\mathrm{U}(\mathfrak{A})'$ intertwines λ'. Then $x\in\tilde{J}'$.*

Proof. If $x\in\mathrm{U}(\mathfrak{A})'$ intertwines λ', then by Proposition 5.1.11 the element x weakly intertwines λ with $\lambda\otimes\chi\circ\det$ for some quasicharacter χ of $F^\times$. Then Corollary 5.1.13 implies that x intertwines λ with $\lambda\otimes\chi\circ\det$, and Proposition 5.1.12 implies that there exists an element $y\in\mathrm{U}(\mathfrak{A})$ such that $y(J)=J$ and $y(\lambda)\cong\lambda\otimes\chi\circ\det$. By definition of $\tilde{J}$, this element y is clearly contained in $\tilde{J}$. The element xy^{-1} therefore intertwines λ, §IV.1.1 in [V2] says that $x\in E^\times Jy\cap\mathrm{U}(\mathfrak{A})'$. However, $E^\times J\cap\mathrm{U}(\mathfrak{A})=J$ and $y\in\mathrm{U}(\mathfrak{A})$. We deduce that $x\in Jy\cap\mathrm{U}(\mathfrak{A})'\subset\tilde{J}'$. □

Theorem 5.1.30. *Let λ'_{M_λ} be an irreducible subrepresentation of $\mathrm{ind}_{J'}^{M_\lambda}\lambda'$. Then the induced representation $\mathrm{ind}_{M_\lambda}^{\mathrm{G}'}\lambda'_{M_\lambda}$ is irreducible and cuspidal.*

Proof. We only need to verify that $(M_\lambda,\lambda'_{M_\lambda})$ satisfies the two conditions of irreducibility. We have proved in Proposition 5.1.25 that $(M_\lambda,\lambda'_{M_\lambda})$ verifies the second

condition. It is left only to prove the intertwining set of λ'_{M_λ} in G' equals to M_λ, i.e. $\mathrm{I}_{\mathrm{G}'}(\lambda'_{M_\lambda}) = M_\lambda$. In Lemma 5.1.29, we have proved that $\mathrm{I}_{\mathrm{U}(\mathfrak{A})'}\lambda' \subset \tilde{J}'$. Since $\tilde{J}'$ normalizes J', then $x \in \mathrm{I}_{\mathrm{U}(\mathfrak{A})}\lambda'$ verifying $\mathrm{Hom}_{J'}(\lambda', x(\lambda)') \neq 0$, which is equivalent to say that $x(\lambda') \cong \lambda$. Hence $\mathrm{I}_{\mathrm{U}(\mathfrak{A})}\lambda' \subset M_\lambda$. By the Proposition 3 in 8.10, chapter I of [V1], let $g \in \mathrm{G}'$ and X a finite set of G' such that $M_\lambda g M_\lambda = \cup_{x\in X} J'xJ'$, then there is an k-isomorphism:

$$\mathrm{I}_{g^{-1}}(\mathrm{ind}_{J'}^{M_\lambda}\lambda') \cong \oplus_{j\in X}\mathrm{I}_{(gj^{-1})}(\lambda'). \tag{5.1}$$

Furthermore, we have:

$$\mathrm{I}_{\mathrm{U}(\mathfrak{A})'}(\mathrm{ind}_{J'}^{M_\lambda}\lambda') = M_\lambda, \tag{5.2}$$

Hence $\mathrm{I}_{\mathrm{U}(\mathfrak{A})'}(\lambda'_{M_\lambda}) = M_\lambda$ follows by the inclusion:

$$\mathrm{I}_{\mathrm{U}(\mathfrak{A})'}(\lambda'_{M_\lambda}) \subset \mathrm{I}_{\mathrm{U}(\mathfrak{A})'}(\mathrm{ind}_{J'}^{M_\lambda}\lambda').$$

Whence, there left to prove that

$$\mathrm{I}_{\mathrm{G}'}(\lambda'_{M_\lambda}) \subset \mathrm{U}(\mathfrak{A})'.$$

Notice that $\mathrm{ind}_{M_\lambda}^{\mathrm{U}(\mathfrak{A})'}\lambda'_{M_\lambda}$ is a subrepresentation of $\mathrm{res}^{\mathrm{U}(\mathfrak{A})}_{\mathrm{U}(\mathfrak{A})'}\tau$, where $\tau = \mathrm{ind}_J^{\mathrm{U}(\mathfrak{A})}\lambda$ (as in the proof of Proposition 5.1.28). We have:

$$\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{M_\lambda}^{\mathrm{U}(\mathfrak{A})}\lambda'_{M_\lambda}) \subset \mathrm{I}_{\mathrm{G}'}(\mathrm{res}^{\mathrm{U}(\mathfrak{A})}_{\mathrm{U}(\mathfrak{A})'}\tau),$$

since $\mathrm{res}^{\mathrm{U}(\mathfrak{A})}_{\mathrm{U}(\mathfrak{A})'}\tau$ is semisimple. We obtain then

$$\mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{M_\lambda}^{\mathrm{U}(\mathfrak{A})}\lambda'_{M_\lambda}) \subset \mathrm{U}(\mathfrak{A})' \tag{5.3}$$

by Proposition 5.1.28. Now use one more time Proposition 3 in 8.10, chapter I of [V1] as equation (5.1) and equation (5.2): Let $h \in \mathrm{G}'$ and Y a finite set of G' such that $\mathrm{U}(\mathfrak{A})h\mathrm{U}(\mathfrak{A}) = \cup_{y\in Y} M_\lambda y M_\lambda$, then there is an k-isomorphism:

$$\mathrm{I}_{h^{-1}}(\mathrm{ind}_{M_\lambda}^{\mathrm{U}(\mathfrak{A})'}\lambda'_{M_\lambda}) \cong \oplus_{s\in Y}\mathrm{I}_{(hs^{-1})}(\lambda'_{M_\lambda}).$$

Hence we have:

$$\mathrm{I}_{\mathrm{G}'}(\lambda'_{M_\lambda}) \subset \mathrm{I}_{\mathrm{G}'}(\mathrm{ind}_{M_\lambda}^{\mathrm{U}(\mathfrak{A})'}\lambda'_{M_\lambda}).$$

Combining with the equation 5.3, we deduce the result. □

5.1.6 Cuspidal k-representations of G'

Let M denote a Levi subgroup of G, and $\mathrm{M}' = \mathrm{M} \cap \mathrm{G}'$. In this section, we consider the restriction functor $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}$, which has been studied by Tadić in [Ta] for representations with characteristic 0. In his article, he proved that any irreducible complex representation of M' is contained in an irreducible complex representation of M, and they are cuspidal simultaneously. His method can be adapted for the case of modulo ℓ.

Proposition 5.1.31. *Let K be a locally pro-finite group, and $K' \subset K$ is a closed normal subgroup of K with finite index. Let (π, V) be an irreducible k-representation of K, then the restricted representation $\mathrm{res}^K_{K'}\pi$ is semisimple with finite length.*

Proof. The proof is the same as §6.12,II in [V1]. We repeat it again is to check that we can drop the condition that $[K : K']$ is inversible in k.

The restricted representation $\mathrm{res}^K_{K'}\pi$ is finitely generated, hence has an irreducible quotient. Let V_0 be the sub-representation such that V/V_0 is irreducible. Let $\{k_1, ..., k_m\}, m \in \mathbb{N}$ be a family of representatives of the quotient K/K'. Now we consider the kernel of the non-trivial projection from $\mathrm{res}^K_{K'}\pi$ to $\oplus_{i=1}^m V/k_i(V_0)$, which is K-stable, hence equals to 0 since π is irreducible. We deduce that $\mathrm{res}^K_{K'}\pi$ is a sub-representation of $\oplus_{i=1}^m V/k_i(V_0)$ hence is semisimple. □

Proposition 5.1.32. *Let π be any irreducible k-representation of* M*, then the restriction $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ is semisimple with finite length, and the direct components are* M*-conjugate. Conversely, let π' be any irreducible k-representation of* M′*, then there exists an irreducible representation π of* M*, such that π' is a direct component of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$.*

Proof. For the first part of this proposition. The method of Silberger in [Si] when $\ell = 0$ can be generalised to our case that ℓ is positive. We first assume that π is cuspidal. Let Z denote the center of M, and the quotient $\mathrm{M}/Z\mathrm{M}'$ is compact. Since for any vector v in the representation space of π the stabiliser $\mathrm{Stab}_{\mathrm{M}}(v)$ is open, the image of $\mathrm{Stab}_{\mathrm{M}}(v)$ has finite index in the quotient group $\mathrm{M}/Z\mathrm{M}'$. Combining with Schur's lemma, the restricted k-representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ is finitely generated. By §2.7,II in [V1] the restricted representation is $Z' = Z \cap \mathrm{M}'$-compact.

Let $(v_1, ..., v_m), m \in \mathbb{N}$ be a family of generators of the representation space of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$. For any compact open subgroup K of M′, we want to prove the space V^K is finitely dimensional. We could always assume that K stabilises $v_i, i = 1, ..., m$, and consider the map

$$\alpha_i : g \hookrightarrow e_K g v_i, i = 1, ...m,$$

where e_K is the idempotent associated K in the Heck algebra of M′. Apparently, the space V^K is generated by $\mathcal{L} = \{e_K g v_i, g \in \mathrm{M}', i = 1, ..., m\}$. If the dimension of V^K is infinite, we can choose a infinite subset $\mathcal{L}'$ of $\mathcal{L}$ which forms a basis of V^K, especially there exists $i_0 \in \{1, ..., m\}$ such that $\mathrm{M}'_{i_0} = \{g \in \mathrm{M}', e_K g v_{i_0} \in \mathcal{L}'\}$ is an infinite set. In particular, cosets $gK, g \in \mathrm{M}'_{i_0}$ are disjoint since K stabilizes v_{i_0}. Furthermore, since the center Z' of M′ acts as a character on $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$, which means Z' stabilises each v_i, the images of cosets $gK, g \in \mathrm{M}'_{i_0}$ are disjoint in the quotient M'/Z'. Let $v^*_{i_0}$ be an k-linear form of V^K which equals to 1 on the set $\mathcal{L}'$. The above analysis implies that the image of the support of coefficient $\langle v^*_{i_0} e_K, g v_{i_0}\rangle = \langle v^*_{i_0}, e_K g v_{i_0}\rangle$ in M'/Z' contains infinite disjoint cosets $gK, g \in \mathrm{M}'_{i_0}$, which contradicts with the assumption that $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ is Z'-compact. We conclude that $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ is finitely generated and admissible, hence has finite length.

Now we come back to the general case:π is irreducible representation of M. We first prove that $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ has finite length, then we prove it is semisimple. For the first part, it is sufficient to prove the restricted representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ is finitely generated and admissible. Let (L,σ) be a cuspidal pair in M such that π is a sub-representation of $i^{\mathrm{M}}_{\mathrm{L}}\sigma$. Applying Theorem A.0.4, we have $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}i^{\mathrm{M}}_{\mathrm{L}}\sigma \cong i^{\mathrm{M}'}_{\mathrm{L}'=\mathrm{L}\cap\mathrm{M}'}\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\sigma$. We have proved that $\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\sigma$ is admissible and finitely generated. Since normalised parabolic induction $i^{\mathrm{M}'}_{\mathrm{L}'}$ respect admissibility and finite generality, the k-representation $\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\sigma$ is also admissible and finitely generated, and hence has finite length. So does its sub-representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$. For the semi-simplicity, let W be an irreducible sub-representation of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$, of which gW is also an irreducible sub-representation for $g\in\mathrm{M}$. Let $W'=\sum_{g\in\mathrm{M}'}g(W)$, which is a semisimple (by the equivalence condition in §A.VII. of [Re]) sub-representation of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$. Obviously, M stabilises W', hence $W'=\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ by the irreducibility of π.

Now we consider the second part of this proposition, and apply the proof of Proposition §2.2 in [Ta] in our case. Let π' be any irreducible k-representation of M', and S the subgroup of Z generated by the scalar matrix ϖ_F, where ϖ_F is the uniformizer of the ring of integers of $\mathfrak{o}_F$. It is clear that the intersection $\mathrm{S}\cap\mathrm{M}'=\{\mathbb{1}\}$. Hence we could let $\tilde{\pi}$ denote the extension of π to SM', where S acts as identity. The quotient group M/SM' is compact, hence the induced representation $\mathrm{ind}^{\mathrm{M}}_{\mathrm{SM}'}\tilde{\pi}$ is admissible (see the formula in §I,5.6 of [V1]). For any M-subrepresentation τ of $\mathrm{ind}^{\mathrm{M}}_{\mathrm{SM}'}\tilde{\pi}$, there is a surjective morphism from $\mathrm{res}^{\mathrm{M}}_{\mathrm{SM}'}\tau$ to $\tilde{\pi}$, defined as $f\mapsto f(\mathbb{1})$. This induced a surjective morphism from $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\tau$ to π'.

Now let π_1 be a finitely generated subrepresentation of $\mathrm{ind}^{\mathrm{M}}_{\mathrm{SM}'}\tilde{\pi}$. Since π_1 is finite type and admissible, it has finite length containing an irreducible subrepresentation noted as π. And there is a surjective morphism from $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ to π'. Combining this with the first part above, the representation π is the one we want. □

Corollary 5.1.33. *Let π be an irreducible k-representation of* M. *If the restricted representation* $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ *contains an irreducible cuspidal k-representation of* M', *then π is cuspidal. This is to say that any cuspidal k-representation of* M' *is a subrepresentation of* $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ *for some cuspidal k-representation π of* M.

Proof. For the first part above, we know that the direct components of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$ are M-conjugate by Proposition 5.1.32. Let $\mathrm{P}'=\mathrm{L}'\cdot\mathrm{U}$ be any proper parabolic subgroup of M' and $\mathrm{P}=\mathrm{L}\cdot\mathrm{U}$ the proper parabolic subgroup of M such that $\mathrm{P}\cap\mathrm{M}'=\mathrm{P}'$ and $\mathrm{L}\cap\mathrm{M}'=\mathrm{L}'$. Let π'_0 be any direct component of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi\cong\oplus_{i\in I}\pi'_i$, where I is a finite index set and π'_i are irreducible representations of M', and for each $i\in I$ let $a_i\in\mathrm{M}$ such that $\pi'_i\cong a_i(\pi')$. In particular, we could assume that $\{a_i\}_{i\in I}$ is a subset of L. We have:

$$\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}r^{\mathrm{M}}_{\mathrm{L}}\pi\cong\oplus_{i\in I}r^{\mathrm{M}'}_{\mathrm{L}'}\pi'_i.$$

Meanwhile, since the unipotent radical U is normal in L, we deduce that:

$$r^{\mathrm{M}'}_{\mathrm{L}'}\pi'_i\cong a_i(r^{\mathrm{G}'}_{\mathrm{L}'}\pi')\cong 0.$$

Hence π is cuspidal as required. □

Corollary 5.1.34. *For any irreducible cuspidal k-representation π' of G', there exists a maximal simple cuspidal k-type (J,λ) of G, and M_λ as in definition 5.1.26. There exists a direct component λ'_{M_λ} of $\mathrm{ind}_{J'}^{M_\lambda}\mathrm{res}_{J'}^{J}\lambda$ such that π' is isomorphic to the induced representation $\mathrm{ind}_{M_\lambda}^{\mathrm{G}'}\lambda'_{M_\lambda}$.*

Proof. Applying Corollary 5.1.33, let π be an irreducible cuspidal k-representation of G' which contains π' as a sub-G'-representation. Let (J_0,λ_0) be the maximal simple cuspidal k-type of G corresponding to π, and $(M_{\lambda_0},\lambda'_{M_{\lambda_0}})$ as in Theorem 5.1.30. We know that π is isomorphic to $\mathrm{ind}_{E^\times J_0}^{\mathrm{G}}\Lambda_0$, where Λ_0 is an extension of λ to $E^\times J_0$, and the intersection $E^\times J_0\cap\mathrm{G}'=J'_0$. Then after applying Mackey's decomposition formula to $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\pi$, we obatin that of which the representation $\mathrm{ind}_{J'_0}^{\mathrm{G}'}\mathrm{res}_{J'_0}^{J}\lambda_0$ is a subrepresentation. Hence, $\mathrm{ind}_{M_{\lambda_0}}^{\mathrm{G}'}\lambda'_{M_{\lambda_0}}$ is isomorphic to some direct component of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\pi$, which is isomorphic to $g(\pi')$ for some $g\in\mathrm{G}$ by Proposition 5.1.32. This implies that π' contains $g^{-1}(\lambda'_{M_{\lambda_0}})$. Notice that $g^{-1}(M_{\lambda_0})=M_{g^{-1}(\lambda_0)}$ and $g^{-1}(\lambda'_{M_{\lambda_0}})$ is a direct component of $\mathrm{ind}_{g^{-1}(J')}^{M_{g^{-1}(\lambda_0)}}g^{-1}(\lambda')$, so we could write is as $\lambda'_{M_{g^{-1}(\lambda_0)}}$. Hence by Frobenius reciprocity and Theorem 5.1.30, this implies that $\pi'\cong\mathrm{ind}_{M_{g^{-1}(\lambda_0)}}^{\mathrm{G}'}g^{-1}(\lambda'_{M_{g^{-1}(\lambda_0)}})$. And $(g^{-1}(J_0),g^{-1}(\lambda))$ is the required maximal simple cuspidal k-type. □

5.2 Whittaker models and maximal simple cuspidal k-types of G'

5.2.1 Uniqueness of Whittaker models

In this section, we will see that the subgroup M_λ of $\tilde{J}'=\tilde{J}(\lambda)\cap\mathrm{G}'$ in the definition 5.2.7 actually coincides with $\tilde{J}'$. In other words, we will prove that for any element $x\in\mathrm{U}(\mathfrak{A})$, if x normalises J and $x(\lambda)\cong\lambda\otimes\chi\circ\det$ for some k-quasicharacter χ of $F^\times$, then $x(\lambda')\cong\lambda'$, for any irreducible direct component λ' of $\lambda|_{J'}$.

Let $\mathrm{U}=\mathrm{U}_n(F)$ be the group consisting with those strictly upper triangular matrices in G. A non-degenerate character ψ of U is a k-quasicharacter defined on U. Let $P_n=\mathrm{P}_n(F)$ be the mirabolic subgroup of $\mathrm{GL}_n(F)$, and $P'_n=P_n\cap\mathrm{SL}_n(F)$. We denote the unipotent radical of P_n as V_{n-1}, which is an abelian group isomorphic to the additive group F^{n-1}. The unipotent radical of P'_n is also V_{n-1}.

Definition 5.2.1. *1. $r_{\mathrm{id}}:=r_{\mathrm{G}_{n-1},P_n}$ the functor of V_{n-1}-coinvariants of representations of P_n, $r_{\mathrm{id}'}:=r_{\mathrm{G}'_{n_1},P'_n}$ the functor of V_{n-1}-coinvariants of representations of P'_n.*

2. $r_\psi:=r_{\psi,P_{n-1},P_n}$ the functor of (V_{n-1},ψ)-coinvariants of representations of P_n, $r'_\psi:=r_{\psi,P'_{n-1},P'_n}$ the functor of (V_{n-1},ψ)-coinvariants of representations of P'_n.

Definition 5.2.2. *Let $1 \leq k \leq n$ and $\pi \in \mathrm{Mod}_k P_n$, $\pi' \in \mathrm{Mod}_k P'_n$. We define the k-th derivative of π to be the representation $\pi^{(k)} := r_{\mathrm{id}} r_\psi^{k-1} \pi$, and the k-th derivative of π' relative to ψ to be the representation $\pi'^{(\psi,k)} := r'_{\mathrm{id}} r'^{k-1}_\psi \pi'$.*

Remark 5.2.3. *The unipotent radical of P_n and P'_n coincide and $\mathrm{U} \subset \mathrm{G}'$, so $\mathrm{res}^{\mathrm{G}_{n-k}}_{\mathrm{G}'_{n-k}} \pi^{(k)}$ is equivalent to $(\mathrm{res}^{\mathrm{G}_n}_{\mathrm{G}'_n} \pi)^{(k)}$, where $\pi \in \mathrm{Mod}_n \mathrm{G}_n$.*

Proposition 5.2.4. *Let π be a cuspidal k-representation of G, then the restriction $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'} \pi$ is multiplicity free.*

Proof. We have proved in Proposition 5.1.32, that the restriction $\mathrm{res}^{\mathrm{G}_n}_{\mathrm{G}'_n} \pi$ is semisimple with finite direct components. Hence we could write it as $\oplus_{i=1}^m \pi_i$, where $m \in \mathbb{N}$ and π_i's are irreducible k-cuspidal representations of G'_n. Let ψ be any non-degenerate character of U. As in 1.7 chapter III of [V1], we obtain that $\dim \pi^{(n)} = 1$. We apply Remark 5.2.3 above, then

$$\dim(\mathrm{res}^{\mathrm{G}_n}_{\mathrm{G}'_n} \pi)^{(n)} = \oplus_{i=1}^m \dim(\pi_i)^{(\psi,n)} = 1.$$

So there exists one unique components π_{i_0}, where $1 \leq i_0 \leq m$, such that $\pi_{i_0}^{(\psi,n)}$ is non-trivial. And we deduce the result. □

Corollary 5.2.5. *Let π' be an irreducible cuspidal k-representation of G'. Then there exists a non-degenerate character ψ of U, such that $\dim \pi'^{(g(\psi),n)} = 1$.*

Proof. This is deduced from Corollary 5.1.33. In fact the direct components of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'} \pi$ is Corollary 5.1.33 are conjugated to each other by diagonal matrices, and the conjugation of non-degenerate characters of U by any diagonal matrix is also a non-degenerate character of U. □

5.2.2 Distinguished cuspidal k-types of G'

Proposition 5.2.6. *Let (J, λ) be a maximal simple cuspidal k-type of G, and $\tilde{J}$ the projective normalizer of λ. Then the subgroup M_λ in definition 5.1.26 of $\tilde{J}'$ coincides with $\tilde{J}'$.*

Proof. Let Λ be an extension of λ to $E^\times J$. Then $\mathrm{ind}^{\mathrm{G}}_{E^\times J} \Lambda$ is an irreducible cuspidal representation of G, we denote it as π. The restricted representation $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'} \pi$ is semisimple and its direct components are cuspidal. By Theorem 5.1.30, there exists a direct component π' of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'} \pi$, such that π' is isomorphic to $\mathrm{ind}^{\mathrm{G}'}_{M_\lambda} \lambda'_{M_\lambda}$, for some λ'_{M_λ}. In the proof of Theorem 5.1.30, we have showed that the intertwining subgroup $\mathrm{I}_{\mathrm{G}'}(\lambda'_{M_\lambda})$ equals to M_λ. If $\tilde{J}' \neq M_\lambda$, and let x be an element belonging to $\tilde{J}'$ but not to M_λ. Then $x(\lambda'_{M_\lambda})$ is not isomorphic to λ'_{M_λ}. However $x(\pi') \cong \pi'$, so $\mathrm{res}^{\mathrm{G}'}_{M_\lambda} \pi'$ contains $x(\lambda_{M_\lambda})$, from which we deduce that

$$\mathrm{ind}^{\mathrm{G}'}_{M_\lambda} x(\lambda'_{M_\lambda}) \cong \mathrm{ind}^{\mathrm{G}'}_{M_\lambda} \lambda'_{M_\lambda} \cong \pi'. \tag{5.4}$$

Meanwhile, by Mackey's decomposition formula

$$\mathrm{res}^{\tilde{J}'}_{M_\lambda}\mathrm{ind}^{\tilde{J}'}_{J'}\mathrm{res}^{J}_{J'}\lambda \cong \oplus_{J'\backslash\tilde{J}'/M_\lambda}\mathrm{ind}^{M_\lambda}_{J'}\mathrm{res}^{J}_{J'}\lambda,$$

hence $x(\lambda'_{M_\lambda})$ is another direct component of $\mathrm{ind}^{M_\lambda}_{J'}\mathrm{res}^{J}_{J'}\lambda$. Since we could change order of the functor $\mathrm{ind}^{\mathrm{G}'}_{M_\lambda}$ and finite direct sum, and $\mathrm{ind}^{\mathrm{G}'}_{J'}\mathrm{res}^{J}_{J'}\lambda$ is a subrepresentation of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\pi$, the two representations $\mathrm{ind}^{\mathrm{G}'}_{M_\lambda}x(\lambda'_{M_\lambda})$ and $\mathrm{ind}^{\mathrm{G}'}_{M_\lambda}\lambda'_{M_\lambda} \cong \pi'$ are two different direct components of $\mathrm{res}^{\mathrm{G}}_{\mathrm{G}'}\pi$. By Proposition 5.2.4, they are not isomorphic, which is contradicted to the equivalence 5.4. Hence $\tilde{J}' = M_\lambda$. □

Definition 5.2.7. *Let (J,λ) be a maximal simple cuspidal k-type of* G *and $\tilde{J}' = \tilde{J}\cap \mathrm{G}'$ as in definition 5.1.18, and $\tilde{\lambda}'$ any direct component of* $\mathrm{ind}^{\tilde{J}'}_{J'}\mathrm{res}^{J}_{J'}\lambda$. *We define the couples $(\tilde{J}',\tilde{\lambda}')$ to be maximal simple cuspidal k-types of* G'. *By Corollary 5.1.34 and Proposition 5.2.6, for any irreducible cuspidal k-representation of* G', *there exists a maximal simple cuspidal k-type $(\tilde{J}',\tilde{\lambda}')$ of* G' *such that π' is isomorphic to* $\mathrm{ind}^{\mathrm{G}'}_{\tilde{J}'}\tilde{\lambda}'$.

5.3 Maximal simple cuspidal k-types for Levi subgroups of G'

5.3.1 Intertwining and weakly intertwining

In this section, let M denote any Levi subgroup of G and for any closed subgroup H of G, we always use H' to denote its intersection with G'. We will consider the maximal simple cuspidal k-types of M. Recall that Proposition 5.1.9, Proposition 5.1.10, Definition 3.1.6 and Proposition 5.1.11 will be used in this section.

Proposition 5.3.1. *Let $(J_\mathrm{M},\lambda_\mathrm{M})$ be a maximal simple cuspidal k-type of* M, *and χ a k-quasicharacter of $F^\times$. If $(J_\mathrm{M},\lambda_\mathrm{M}\otimes\chi\circ\det)$ is weakly intertwined with $(J_\mathrm{M},\lambda_\mathrm{M})$, then they are intertwined. There exists an element $x\in \mathrm{U}(\mathfrak{A}_\mathrm{M}) = \mathrm{U}(\mathfrak{A}_1)\times\cdots\times \mathrm{U}(\mathfrak{A}_r)$ such that $x(J_\mathrm{M}) = J_\mathrm{M}$ and $x(\lambda_\mathrm{M})\cong\lambda_\mathrm{M}\otimes\chi\circ\det$, where $\mathfrak{A}_i$ is a hereditary order associated to (J_i,λ_i) $(i=1,...,r)$. Furthermore, for any $g\in\mathrm{G}$, if g weakly intertwines $(J_\mathrm{M},\lambda_\mathrm{M}\otimes\chi\circ\det)$ and $(J_\mathrm{M},\lambda_\mathrm{M})$, then g intertwines $(J_\mathrm{M},\lambda_\mathrm{M}\otimes\chi\circ\det)$ and $(J_\mathrm{M},\lambda_\mathrm{M})$.*

Proof. By definition, write M as a product $\mathrm{GL}_{n_1}\times\cdots\times\mathrm{GL}_{n_r}$, then $J_\mathrm{M} = J_1\times\cdots\times J_r$ and $\lambda_\mathrm{M}\cong\lambda_1\times\cdots\times\lambda_r$, where (J_i,λ_i) are k-maximal cuspidal simple type of GL_{n_i} for $i\in\{1,\ldots,r\}$. The group $\mathrm{U}(\mathfrak{A}_\mathrm{M}) = \mathrm{U}(\mathfrak{A}_1)\times\cdots\times\mathrm{U}(\mathfrak{A}_r)$. Hence the two results are directly deduced by 5.1.12 and 5.1.13.

□

Definition 5.3.2. *Let $(J_\mathrm{M},\lambda_\mathrm{M})$ be a k-maximal cuspidal simple type of* M. *We define the group of projective normalizer $\tilde{J_\mathrm{M}}$ a subgroup of J_M. An element $x\in\mathrm{U}(\mathfrak{A}_\mathrm{M})$, where $\mathfrak{A}_\mathrm{M} = \mathfrak{A}_1\times\cdots\times\mathfrak{A}_r$, belongs to $\tilde{J_\mathrm{M}}$, if $x(J_\mathrm{M}) = J_\mathrm{M}$, and there exists a k-quasicharacter χ of $F^\times$ such that $x(\lambda_\mathrm{M})\cong\lambda_\mathrm{M}\otimes\chi\circ\det$.*

The induced k-representation $\tilde{\lambda}_{\mathrm{M}} = \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\lambda_{\mathrm{M}}$ is irreducible by Corollary 5.1.22, and according to 5.1.9, the restriction $\mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\tilde{\lambda}_{\mathrm{M}}$ is semisimple. Let μ_{M} denote one of its irreducible component.

Lemma 5.3.3. *Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a k-maximal cuspidal simple type of M, and ν_{M} and μ_{M} be two irreducible components of the restricted representation $\mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\tilde{\lambda}_{\mathrm{M}}$. Then :*

$$\mathrm{I}^w_{\mathrm{M}'}(\nu_{\mathrm{M}}, \mu_{\mathrm{M}}) = \{m \in \mathrm{M}' : m(\nu_{\mathrm{M}}) \cong \mu_{\mathrm{M}}\},$$

whence $\mathrm{I}^w_{\mathrm{M}'}(\nu_M, \mu_M) = \mathrm{I}_{\mathrm{M}'}(\nu_M, \mu_M)$. In particular, $\mathrm{I}_{\mathrm{M}'}(\mu_{\mathrm{M}})$ is the normalizer group of μ_{M} in M'. Moreover, this group is independent of the choice of μ_{M}.

Proof. Let $m \in \mathrm{M}'$ weakly intertwines μ_{M} with ν_{M}. Then by 5.1.11, the element m weakly intertwines $\tilde{\lambda}_{\mathrm{M}}$ with $\tilde{\lambda}_{\mathrm{M}} \otimes \chi \circ \det$ for some k-quasicharacter χ of $F^\times$. By definition

$$\tilde{\lambda}_{\mathrm{M}}|_{J_{\mathrm{M}}} \cong \oplus_{x \in \tilde{J}_{\mathrm{M}}/J_{\mathrm{M}}} x(\lambda_{\mathrm{M}}) \cong \oplus_{x \in \tilde{J}_{\mathrm{M}}/J_{\mathrm{M}}} \lambda_{\mathrm{M}} \otimes \xi_x \circ \det.$$

Since the induced representation $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\lambda_{\mathrm{M}} \otimes \xi_x \circ \det \cong \tilde{\lambda}_{\mathrm{M}} \otimes \xi_x \circ \det$, by Frobenius reciprocity, we have $\tilde{\lambda}_{\mathrm{M}} \otimes \xi_x \circ \det \cong \tilde{\lambda}_{\mathrm{M}}$ for every $x \in \tilde{J}_{\mathrm{M}}/J_{\mathrm{M}}$. It follows that for some $g \in \tilde{J}_{\mathrm{M}}$, the element gm weakly intertwines λ_{M} with $\lambda_{\mathrm{M}} \otimes \xi_x \cdot \chi \circ \det$ for some $x \in \tilde{J}_{\mathrm{M}}/J_{\mathrm{M}}$. Applying 5.3.1, the element gm intertwines λ_{M} with $\lambda_{\mathrm{M}} \otimes \xi_x \cdot \chi \circ \det$, and there exists an element $y \in \tilde{J}_{\mathrm{M}}$ such that $y(\lambda_{\mathrm{M}}) \cong \lambda_{\mathrm{M}} \otimes \xi_x \cdot \chi \circ \det$. Inducing this isomorphism to $\tilde{J}_{\mathrm{M}}$, we see tha $\tilde{\lambda}_{\mathrm{M}} \cong \tilde{\lambda}_{\mathrm{M}} \otimes \chi \circ \det$, whence m intertwines $\tilde{\lambda}_{\mathrm{M}}$.

Furthermore, the intertwining set $\mathrm{I}_{\mathrm{M}}(\lambda_{\mathrm{M}}) = N_{\mathrm{M}}(\lambda_{\mathrm{M}})$, the latter group is the normalizer of λ_{M}, which also normalizes $\mathrm{U}(\mathfrak{A}_{\mathrm{M}})$, hence normalizes $\tilde{J}_{\mathrm{M}}$. We deduce that $\mathrm{I}_{\mathrm{M}}(\tilde{\lambda}_{\mathrm{M}}) = \tilde{J}_{\mathrm{M}}N_{\mathrm{M}}(\lambda_{\mathrm{M}})$. Then each element of $\mathrm{I}^w_{\mathrm{M}}(\mu_{\mathrm{M}}, \nu_{\mathrm{M}})$ normalizes $\tilde{\lambda}_{\mathrm{M}}$ and the group $\tilde{J}'_{\mathrm{M}}$. This gives the first two assertions.

To prove the third assertion, observe that the irreducible components of $\tilde{\lambda}_{\mathrm{M}}|_{\tilde{J}'_{\mathrm{M}}}$ form a single $\tilde{J}_{\mathrm{M}}$-conjugacy class. We have to show therefore that $\tilde{J}_{\mathrm{M}}$ normalizes $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$.

The quotient group $N_{\mathrm{M}}(\tilde{\lambda}_{\mathrm{M}})/\tilde{J}_{\mathrm{M}}$ is abelian. In fact, as we have proved above, it is a subgroup of $N_{\mathrm{M}}(\lambda_{\mathrm{M}})/J_{\mathrm{M}}$. The latter group is abelian, since $N_{\mathrm{M}}(\lambda_{\mathrm{M}})$ can be written as $E_1^\times J_1 \times \cdots \times E_r^\times J_r$, where $E_1, \ldots, E_r$ are field extensions of F. Now let $x \in \tilde{J}_{\mathrm{M}}$ and $y \in N_{\mathrm{M}'}(\mu_{\mathrm{M}})$, we have $x^{-1}yx = y \cdot m$ for some $m \in \tilde{J}'_{\mathrm{M}}$. Therefore:

$$x^{-1}yx(\mu_{\mathrm{M}}) \cong y(\mu_{\mathrm{M}}) \cong \mu_{\mathrm{M}},$$

as required. □

Remark 5.3.4. *To be more detailed, we proved that the intertwining group $\mathrm{I}_{\mathrm{M}'}(\mu_{\mathrm{M}})$ is the stabilizer group $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$, which is a subgroup of $E_1^\times \tilde{J}_1 \times \cdots \times E_r^\times \tilde{J}_r \cap \mathrm{M}'$, hence a compact group modulo center.*

5.3.2 Maximal simple cuspidal k-types of M'

In this section, we construct maximal simple cuspidal k-types of M' (5.3.10). This means that for any irreducible cuspidal k-representation π', there exists an irreducible component μ_{M} of $\mathrm{res}^{\tilde{J}_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}}\tilde{\lambda}_{\mathrm{M}}$, and an irreducible k-representation $\tau_{\mathrm{M}'}$ of $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$ containing μ_{M}, such that $\pi' \cong \mathrm{ind}^{\mathrm{M}'}_{N_{\mathrm{M}'}(\mu_{\mathrm{M}})}\tau_{\mathrm{M}'}$. We follow the same method as in the case of G', which is to calculate the intertwining group and verify the second condition of irreducibility (5.1.21).

Lemma 5.3.5. *As in the case when* $\mathrm{M} = \mathrm{G}$*, we have a decomposition:*

$$\mathrm{res}^{\mathrm{M}}_{J_{\mathrm{M}}}\mathrm{ind}^{\mathrm{M}}_{J_{\mathrm{M}}}\lambda_{\mathrm{M}} \cong \Lambda_{\lambda_{\mathrm{M}}} \oplus W_{\mathrm{M}},$$

where $\Lambda_{\lambda_{\mathrm{M}}}$ *is semisimple, of which each irreducible component is isomorphic to* $\lambda_{\mathrm{M}} \otimes \chi \circ \det$ *for some* k*-quasicharacter* χ *of* $F^{\times}$*. Non of irreducible subquotient of* W_{M} *is contained in* $\Lambda_{\lambda_{\mathrm{M}}}$*.*

Proof. This is directly deduced from the decomposition in 5.1.14. □

Proposition 5.3.6. *Let* μ_{M} *be an irreducible* k*-subrepresentation of* $\mathrm{res}^{\tilde{J}_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}}\mathrm{ind}^{\tilde{J}_{\mathrm{M}}}_{J_{\mathrm{M}}}\lambda_{\mathrm{M}}$*. Then* μ_{M} *verifies the second condition of irreducibility (5.1.21). This means that for any irreducible representation* π' *of* M'*, if there is an injection* $\mu_{\mathrm{M}} \to \mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}}\pi'$*, then there exists a surjection from* $\mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}}\pi'$ *to* μ_{M}*.*

Proof. A same proof can be given as in the case while $\mathrm{M}' = \mathrm{G}'$ (5.1.25).

□

Proposition 5.3.7. *Let* $\tau_{\mathrm{M}'}$ *be an irreducible representation of* $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$ *containing* μ_{M}*. Then* $\tau_{\mathrm{M}'}$ *verifies the second condition of irreducibility.*

Proof. Let $N_{\mathrm{M}'}$ denote $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$, then we have:

$$\mathrm{res}^{\mathrm{M}'}_{N_{\mathrm{M}'}}\mathrm{ind}^{\mathrm{M}'}_{N_{\mathrm{M}'}}\tau_{\mathrm{M}'} \cong \oplus_{N_{\mathrm{M}'}\backslash \mathrm{M}'/N_{\mathrm{M}'}}\mathrm{ind}^{N_{\mathrm{M}'}}_{N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})}\mathrm{res}^{a(N_{\mathrm{M}'})}_{N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})}a(\tau_{\mathrm{M}'}).$$

Notice that $N_{\mathrm{M}'}$ has a unique maximal open compact subgroup $\tilde{J}'_{\mathrm{M}}$, hence $\tilde{J}'_{\mathrm{M}} \cap ba(N_{\mathrm{M}'}) = \tilde{J}'_{\mathrm{M}} \cap ba(\tilde{J}'_{\mathrm{M}})$, for any $b, a \in \mathrm{M}'$. Hence we have the following equivalence:

$$\mathrm{res}^{N_{\mathrm{M}'}}_{\tilde{J}'_{\mathrm{M}}}\mathrm{ind}^{N_{\mathrm{M}'}}_{N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})}\mathrm{res}^{a(N_{\mathrm{M}'})}_{N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})}a(\tau_{\mathrm{M}'})$$

$$\cong \oplus_{b\in N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})\backslash N_{\mathrm{M}'}/\tilde{J}'_{\mathrm{M}}}\mathrm{ind}^{\tilde{J}'_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}\cap ba(N_{\mathrm{M}'})}\mathrm{res}^{ba(N_{\mathrm{M}'})}_{\tilde{J}'_{\mathrm{M}}\cap ba(N_{\mathrm{M}'})}ba(\tau_{\mathrm{M}'})$$

$$\cong \oplus_{b\in N_{\mathrm{M}'}\cap a(N_{\mathrm{M}'})\backslash N_{\mathrm{M}'}/\tilde{J}'_{\mathrm{M}}}\mathrm{ind}^{\tilde{J}'_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}\cap ba(\tilde{J}'_{\mathrm{M}})}\mathrm{res}^{ba(\tilde{J}'_{\mathrm{M}})}_{\tilde{J}'_{\mathrm{M}}\cap ba(\tilde{J}'_{\mathrm{M}})}(\oplus\mu_{\mathrm{M}}),$$

where $\oplus\mu_{\mathrm{M}}$ denotes a finite multiple of μ_{M}.

Let $a \notin N_{\mathrm{M}'}$, then ba is an element of $N_{\mathrm{M}'} \cdot a$, and $N_{\mathrm{M}'} \cdot a \cap N_{\mathrm{M}'} = \emptyset$. By 5.3.3, this means $ba \notin \mathrm{I}^{w}_{\mathrm{M}'}(\mu_{\mathrm{M}})$. This implies that non of irreducible subquotient

of $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}\cap ba(N_{\mathrm{M}'})}^{\tilde{J}'_{\mathrm{M}}}\mathrm{res}_{\tilde{J}'_{\mathrm{M}}\cap ba(N_{\mathrm{M}'})}^{ba(N_{\mathrm{M}'})} ba(\tau_{\mathrm{M}'})$ is isomorphic to μ_{M}. Now combining with the first equivalence in this proof above, we obtain a decomposition:

$$\mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\mathrm{ind}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\tau_{\mathrm{M}'} \cong \tau_{\mathrm{M}'} \oplus W_{N_{\mathrm{M}'}},$$

non of irreducible subquotient of $W_{N_{\mathrm{M}'}}$ is isomorphic to $\tau_{\mathrm{M}'}$.

Now we verify the second condition of $\tau_{\mathrm{M}'}$. Let π' be any irreducible k-representation of M'. If there is an injection $\tau_{\mathrm{M}'} \hookrightarrow \mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\pi'$, then $\mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\pi'$ is isomorphic to a quotient representation $(\mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\mathrm{ind}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\tau_{\mathrm{M}'})/W_0$. And the image of the composed morphism below:

$$\tau_{\mathrm{M}'} \hookrightarrow \mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\pi' \cong (\mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\mathrm{ind}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\tau_{\mathrm{M}'})/W_0$$

is not contained in $(W_{N_{\mathrm{M}'}} + W_0)/W_0$ by the analysis above. Then we have a non trivial morphism:

$$\mathrm{res}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\pi' \to (\tau_{\mathrm{M}'} \oplus W_{N_{\mathrm{M}'}})/(W_{N_{\mathrm{M}'}} + W_0) \cong \tau_{\mathrm{M}'}.$$

Hence we finish the proof.

□

Lemma 5.3.8. *Let* G *be a locally pro-finite group, and* K_1, K_2 *two open subgroups of* G*, where* K_1 *is the unique maximal open compact subgroup in* K_2*. Let* π *be an irreducible* k*-representation of* K_2*, and* τ *an irreducible* k*-representation of* K_1*. Assume that* $\pi|_{K_1}$ *is a multiple of* τ*. If* $x \in \mathrm{G}$ *(weakly) intertwines* π*, then there exists an element* $y \in K_2$ *such that* yx *(weakly) intertwines* τ*.*

Proof. Since π is isomorphic to a subquotient of $\mathrm{ind}_{K_2\cap x(K_2)}^{K_2}\mathrm{res}_{K_2\cap x(K_2)}^{x(K_2)}x(\pi)$, the restriction $\mathrm{res}_{K_1}^{K_2}\pi$ is isomorphic to a subquotient of $\mathrm{res}_{K_1}^{K_2}\mathrm{ind}_{K_2\cap x(K_2)}^{K_2}\mathrm{res}_{K_2\cap x(K_2)}^{x(K_2)}x(\pi)$. Applying Mackey's decomposition formula, we have

$$\mathrm{res}_{K_1}^{K_2}\mathrm{ind}_{K_2\cap x(K_2)}^{K_2}\mathrm{res}_{K_2\cap x(K_2)}^{x(K_2)}x(\pi) \cong \bigoplus_{a\in K_2\cap x(K_2)/K_2\backslash K_1} \mathrm{ind}_{K_1\cap ax(K_2)}^{K_1}\mathrm{res}_{K_1\cap ax(K_2)}^{ax(K_2)}ax(\pi).$$

Since $K_1 \cap ax(K_2)$ is open compact in $ax(K_2)$, by the uniqueness of open compact subgroup in $ax(K_2)$, the intersection $K_1 \cap ax(K_2) \subset ax(K_1)$, hence $K_1 \cap ax(K_2) = K_1 \cap ax(K_1)$. Write $\mathrm{res}_{K_1}^{K_2}\pi \cong \bigoplus_I \tau$, where I is an index set. We have an equivalence

$$\mathrm{res}_{K_1\cap ax(K_1)}^{ax(K_1)}ax(\pi) \cong \bigoplus_I \mathrm{res}_{K_1\cap ax(K_1)}^{ax(K_1)}ax(\tau)$$

Since functors ind, res can change order with infinite direct sum, we reform the first equivalence in this proof

$$\mathrm{res}_{K_1}^{K_2}\mathrm{ind}_{K_2\cap x(K_2)}^{K_2}\mathrm{res}_{K_2\cap x(K_2)}^{x(K_2)}x(\pi)$$

$$\cong \bigoplus_{I} \bigoplus_{a\in K_2\cap x(K_2)/K_2\backslash K_1} \mathrm{ind}_{K_1\cap ax(K_1)}^{K_1}\mathrm{res}_{K_1\cap ax(K_1)}^{ax(K_1)}ax(\tau).$$

As in the proof of Lemma 5.1.17, this implies that there exists at least one $y \in K_2$ such that τ is an subquotient of $\mathrm{ind}_{K_1\cap yx(K_1)}^{K_1}\mathrm{res}_{K_1\cap yx(K_1)}^{yx(K_1)}yx(\tau)$. □

Theorem 5.3.9. *The induced k-representation $\mathrm{ind}_{N_{\mathrm{M}'}(\mu_{\mathrm{M}})}^{\mathrm{M}'}\tau_{\mathrm{M}'}$ is cuspidal and irreducible. Conversely, any irreducible cuspidal representation π' of M' contains an irreducible k-representation $\tau_{\mathrm{M}'}$ of $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$, and $\pi' \cong \mathrm{ind}_{N_{\mathrm{M}'}(\mu'_{\mathrm{M}})}^{\mathrm{M}'}\tau_{\mathrm{M}'}$, where $\tau_{\mathrm{M}'}$ and $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$ are defined as in Proposition 5.3.7 of some maximal simple cuspidal k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ of* M.

Proof. For the first assertion, we only need to verify the two condition of irreducibility. The second condition has been checked in 5.3.7. By 5.3.8 and 5.3.3, we obtain that the induced k-representation $\mathrm{ind}_{N_{\mathrm{M}'}(\mu_{\mathrm{M}})}^{\mathrm{M}'}\tau_{\mathrm{M}'}$ is irreducible. Let π' be the induced k-representation, and π the k-irreducible representation as in 5.1.32. We deduce from 5.1.17 and the fact that π' contains $(J'_{\mathrm{M}}, \lambda'_{\mathrm{M}})$, that π contains $(J_{\mathrm{M}}, \lambda_{\mathrm{M}} \otimes \chi \circ \det)$. Hence π is cuspidal (5.1.7) and this implies that π' is cuspidal. Conversely, let π' be an irreducible cuspidal k-representation of M', and π be the irreducible cuspidal k-representation of M which contains π'. Then there exists a maximal simple cuspidal k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, and an extension Λ_{M} of λ_{M} to $N_{\mathrm{M}}(\lambda_{\mathrm{M}})$ such that $\pi \cong \mathrm{ind}_{N_{\mathrm{M}}(\lambda_{\mathrm{M}})}^{\mathrm{M}}\Lambda_{\mathrm{M}}$. Let $\mu = \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\lambda_{\mathrm{M}}$, and $N_{\mathrm{M}}(\mu)$ be the normalizer of μ in M. By the transitivity of induction:

$$\pi \cong \mathrm{ind}_{N_{\mathrm{M}}(\mu)}^{\mathrm{M}} \circ \mathrm{ind}_{N_{\mathrm{M}}(\lambda_{\mathrm{M}})}^{N_{\mathrm{M}}(\mu)}\Lambda_{\mathrm{M}}.$$

Denote $\mathrm{ind}_{N_{\mathrm{M}}(\lambda_{\mathrm{M}})}^{N_{\mathrm{M}}(\mu)}\Lambda_{\mathrm{M}}$ as τ_{M}, which is an irreducible representation containing μ.

Till the end of this proof, we denote μ_{M} as a direct component of $\mu|_{\tilde{J}'_{\mathrm{M}}}$, N as $N_{\mathrm{M}}(\mu)$, N' as $N \cap \mathrm{M}'$, and $N_{\mathrm{M}'}$ as $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$. Let K be an open compact subgroup of $\tilde{J}_{\mathrm{M}}$ contained in the kernel of τ_{M}, and Z be the center of M. Since the quotient $(Z \cdot N')/N$ is compact and the image of K in this quotient is open, we deduce that $Z \cdot N' \cdot K$ is a normal subgroup with finite index of N. Hence the restriction $\mathrm{res}_{Z\cdot N'\cdot K}^{N}\tau_{\mathrm{M}}$ is semisimple with finite length as in the first part of proof of 5.1.32, from which we deduce that the restriction $\mathrm{res}_{N'}^{N}\tau_{\mathrm{M}}$ is semisimple with finite length as well. After conjugate by an element m in M, the cuspidal representation π' contains a direct component of this restricted representation. We can assume that m is identity, and denote this direct component as τ'. Applying Frobenius reciprocity, the representation $\mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{N}\mathrm{ind}_{\tilde{J}_{\mathrm{M}}}^{N}\mu$ is semisimple, consisting of $\mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\mu$, hence τ' contains a μ_{M}. Notice that $N_{\mathrm{M}'}$ is a normal subgroup with finite index in N'. In fact, the group $N_{\mathrm{M}'}$ contains $Z\cdot\tilde{J}'_{\mathrm{M}}$. And as we have discussed after the proof of 5.3.3, we could write N' as a subgroup of $E_1^{\times}\tilde{J}_1 \times \cdots \times E_r^{\times}\tilde{J}_r \cap \mathrm{M}' = (E_1^{\times} \times \cdots \times E_r^{\times} \cap \mathrm{M}')(\tilde{J}'_{\mathrm{M}})$. Hence $\mathrm{res}_{N_{\mathrm{M}'}}^{N'}\tau'$ is semisimple with finite length, and there must be one direct component $\tau_{\mathrm{M}'}$ containing μ_{M}. Since π' contains $\tau_{\mathrm{M}'}$, we have:

$$\pi' \cong \mathrm{ind}_{N_{\mathrm{M}'}}^{\mathrm{M}'}\tau_{\mathrm{M}'}.$$

This ends the proof. □

Definition 5.3.10. *Let* $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ *be a maximal simple cuspidal* k*-type of* M*, and* μ_{M} *be an irreducible component of* $\mathrm{res}^{\tilde{J}_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}}\tilde{\lambda}_{\mathrm{M}}$*, where* $\tilde{J}_{\mathrm{M}}$ *and* $\tilde{\lambda}_{\mathrm{M}}$ *are defined as in 5.3.2. Let* $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$ *be the normalizer group of* μ_{M} *in* M′*, and* $\tau_{\mathrm{M}'}$ *an irreducible* k*-representation of* $N_{\mathrm{M}'}(\mu_{\mathrm{M}})$ *containing* μ_{M}*. We define the couples in forms of* $(N_{\mathrm{M}'}(\mu_{\mathrm{M}}), \tau_{\mathrm{M}'})$ *are the maximal simple cuspidal* k*-types of* M′*.*

5.3.3 The k-representations π

In this section π' is an irreducible cuspidal k-representation of M′. We study the irreducible cuspidal k-representations π of M, which contains π' as a common component, and we prove that any two of them are different by a k-character of M factor through determinant (Lproposition 9). This is the key to give the first description of supercuspidal support of π' in the next section.

Lemma 5.3.11. *Let* $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ *be a maximal simple cuspidal* k*-type of* M*, and* $\mu = \mathrm{ind}^{\tilde{J}_{\mathrm{M}}}_{J_{\mathrm{M}}}\lambda_{\mathrm{M}}$*. Let* τ *be any irreducible* k*-representation of* $N = N_{\mathrm{M}}(\mu)$ *containing* μ*, then* $\mathrm{res}^{N}_{N'}\tau$ *is semisimple with finite length.*

Proof. By the definition of N, we know the center Z of M is contained in N. Since $F^{\times}/\det Z$ is compact, the quotient group $\det N/\det Z$ is compact as well. Notice that $\tau|_{\tilde{J}}$ is a multiple of μ, then any open subgroup contained in the kernel of μ is also contained in the kernel $\ker(\tau)$ of τ, which implies $\ker(\tau)$ is open. Hence $Z \cdot N' \cdot \ker(\tau)$ is a normal subgroup with finite index in N. Applying Proposition 5.1.31, the restricted k-representation $\mathrm{res}^{N}_{Z\cdot N'}\tau$ is semisimple with finite length, and by Schur's lemma we deduce that $\mathrm{res}^{N}_{N'}\tau$ is semisimple with finite length. □

Lemma 5.3.12. *If* c_1, c_2 *two characters of* Z *and they coincide on* $Z \cap \mathrm{M}'$*, where* Z *denotes the center of* M*. Then* $c_1 \circ c_2^{-1}$ *can be extended to a character on* M *which factor through* det*.*

Proof. First, we extend $c_1 \circ c_2^{-1}$ to $Z \cdot \mathrm{M}'$: For any $a \in Z, b \in \mathrm{M}'$, define $c_0(ab) = c_1 \circ c_2^{-1}(a)$. This is well defines, since for any a', b' such that $a'b' = ab$, then $a^{-1}a' \in Z \cap \mathrm{M}'$. Hence $c_1 \circ c_2^{-1}(a^{-1}a') = 1$, which implies $c_0(ab) = c_0(a'b')$. Now consider $\mathrm{Ind}^{\mathrm{M}}_{Z\cdot\mathrm{M}'}c_0$, which has finite length. There is a surjection from $\mathrm{res}^{\mathrm{M}}_{Z\cdot\mathrm{M}'}\mathrm{Ind}^{\mathrm{M}}_{Z\cdot\mathrm{M}'}c_0$ to c_0, then of which there exists an irreducible k-subquotient c containing c_0, by the uniqueness of Jordan-Hölder factors. According to the fact that M′ is normal in M and c_0 is trivial on M′, the k-representation $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\mathrm{Ind}^{\mathrm{M}}_{Z\cdot\mathrm{M}'}c_0$ is a trivial. Hence c is trivial on M′ as well, and hence factor throught $F^{\times} \cong \mathrm{M}/\mathrm{M}'$. Then by Schur's lemma, c is a character factor through det. □

Lemma 5.3.13. *Let* τ_1, τ_2 *be two irreducible* k*-representations of* N *(notion as in 5.3.11). Assume that* $\mathrm{res}^{N}_{N'}\tau_1$ *and* $\mathrm{res}^{N}_{N'}\tau_2$ *have one direct component in common, then there exists a* k*-quasicharacter of* $F^{\times}$ *such that* $\tau_1 \cong \tau_2 \otimes \chi \circ \det$*.*

Proof. The group N is compact modulo center, and $\tilde{J}_{\mathrm{M}}$ is the unique maximal open compact subgroup of N. Hence every irreducible k-representation of N is finite dimensional, of which the kernel is always open. Let U be an open compact subgroup contained in $\mathrm{Ker}\tau_1 \cap \mathrm{Ker}\tau_2 \cap \tilde{J}_{\mathrm{M}}$. Let c_1, c_2 be the central characters of τ_1 and τ_2 respectively. According to 5.3.12, there exists a k-quasicharacter χ of $F^\times$ such that $c_1 \cong c_2 \otimes \chi \circ \det$. After tensoring $\chi \circ \det$, we could assume that $c_1 \cong c_2$. Hence:

$$\mathrm{Hom}_{Z\cdot N'\cdot U}(\mathrm{res}^N_{Z\cdot N'\cdot U}\tau_1, \mathrm{res}^N_{Z\cdot N'\cdot U}\tau_2) \neq 0.$$

Then:

$$\mathrm{Hom}_N(\tau_1, \mathrm{ind}^N_{Z\cdot N'\cdot U}\mathrm{res}^N_{Z\cdot N'\cdot U}\tau_2) \neq 0.$$

Since $|N : Z \cdot N' \cdot U|$ is finite, the later factor above has finite length and

$$\mathrm{ind}^N_{Z\cdot N'\cdot U}\mathrm{res}^N_{Z\cdot N'\cdot U}\tau_2 \cong \tau_2 \otimes \mathrm{ind}^N_{Z\cdot N\cdot U}1.$$

Notice that any Jordan-Holder factor of $\mathrm{ind}^N_{Z\cdot N'\cdot U}1$ is a character factor through $\det|_N$, and $|F^\times : \det(N)|$ is finite. By the same reason as in the proof of 5.3.12, we could extend each of them as a character of M factor through det. Hence there exists a k-quasicharacter χ of $F^\times$, such that $\tau_1 \cong \tau_2 \otimes \chi \circ \det$. □

Proposition 5.3.14. *Let π' be an irreducible cuspidal k-representation of M'. If π_1, π_2 two irreducible cuspidal k-representations of M, such that π' appears as a direct component of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_1$ and $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_2$ in common, then there exists a k-quasicharacter of $F^\times$ verifying that $\pi_1 \cong \pi_2 \otimes \chi \circ \det$.*

Remark 5.3.15. *We will apply the proposition 5.3.14 in the proof of the proposition 5.3.18, which is the first part of the uniqueness of supercuspidal support of $\mathrm{SL}_n(F)$. We will state two proofs of the proposition 5.3.14 as below. The first proof is given through type theory while the second proof does not concern about type theory, which induce two parallel proofs of uniqueness of supercuspidal support of $\mathrm{SL}_n(F)$, with and without type theory respectively. The second proof is similar to that of the proposition in* §VI.3.2. *in [Re], and also the proposition* 2.4 *in [Ta].*

proof version 1. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple cuspidal k-type of M contained in π_1, and $\mu = \mathrm{ind}^{\tilde{J}_{\mathrm{M}}}_{J_{\mathrm{M}}}\lambda_{\mathrm{M}}$. Then there is an extension τ of μ to $N = N_{\mathrm{M}}(\mu)$ such that $\pi_1 \cong \mathrm{ind}^{\mathrm{M}}_N\tau$. Let N' denote $N \cap \mathrm{M}'$. As in the proof of 5.3.9, there exists a direct component μ_{M} of $\mathrm{res}^{\tilde{J}_{\mathrm{M}}}_{\tilde{J}'_{\mathrm{M}}}\mu$ such that $\pi' \cong \mathrm{ind}^{\mathrm{M}'}_{N'}\tau'$, where τ' is a direct component of $\mathrm{res}^N_{N'}\tau$ and $\tau' \cong \mathrm{ind}^{N'}_{N_{\mathrm{M}'}(\mu_{\mathrm{M}})}\tau'_{\mathrm{M}}$. Here τ'_{M} is an irreducible k-representation containing μ_{M}. By 5.1.17 π_2 contains $(J_{\mathrm{M}}, \lambda_{\mathrm{M}} \otimes \chi_0 \circ \det)$ for some k-quasicharacter χ_0 of $F^\times$. Hence there is an extension Λ_{M} of λ_{M} on $N_{\mathrm{M}}(\lambda_{\mathrm{M}})$ such that $\pi_2 \cong \mathrm{ind}^{\mathrm{M}}_{N_{\mathrm{M}}(\lambda_{\mathrm{M}})}\Lambda_{\mathrm{M}} \otimes \chi_0 \circ \det$. Let τ_2 denote $\mathrm{ind}^N_{N_{\mathrm{M}}(\lambda_{\mathrm{M}})}\Lambda_{\mathrm{M}} \otimes \chi_0 \circ \det$, which is an extension of $\mu \otimes \chi_0 \circ \det$. After tensor $\chi_0^{-1} \circ \det$, we could assume that τ_2 is an extension of μ. Now we want to study the relation between τ and τ_2.

First consider $\mathrm{res}^{\mathrm{M}}_{N}\pi_2$:

$$\mathrm{res}^{\mathrm{M}}_{N}\mathrm{ind}^{\mathrm{M}}_{N}\tau_2 \cong \oplus_{N\backslash \mathrm{M}/N}\mathrm{ind}^{N}_{N\cap a(N)}\mathrm{res}^{a(N)}_{N\cap a(N)}a(\tau_2).$$

Since $[N : N \cap a(N)]$ is finite, the representation above is a direct sum of k-representations with finite length, and of which τ' is a sub-representation. Hence there exists an irreducible sub-quotient τ_1 of $\mathrm{res}^{\mathrm{M}}_{N}\pi_2$ such that τ' is a direct component of $\mathrm{res}^{N}_{N'}\tau_1$. By lemma 5.3.13, there is a k-quasicharacter χ of $F^\times$ such that $\tau_1 \cong \tau \otimes \chi \circ \det$.

We will prove that $\tau_1 \cong \tau_2$. Assume that τ_1 and τ_2 are not isomorphic, then there exists $a \notin N$ such that τ_1 is an irreducible subquotient of $\mathrm{ind}^{N}_{N\cap a(N)}\mathrm{res}^{a(N)}_{N\cap a(N)}a(\tau_2)$, which means a weakly intertwines τ_1 with τ_2. Hence there exists $b \in N$ such that ba weakly intertwines $\mu\otimes\chi\circ\det$ with μ and $c \in \tilde{J}_{\mathrm{M}}, d \in cba(\tilde{J}_{\mathrm{M}})$ such that $dcba$ weakly intertwines $\lambda_{\mathrm{M}}\otimes\chi\circ\det$ with λ_{M}. Hence there is $g \in \tilde{J}_{\mathrm{M}}$ such that $g(\lambda_{\mathrm{M}}\otimes\chi\circ\det) \cong \lambda_{\mathrm{M}}$. This implies that $\mu \otimes \chi \circ \det \cong \mu$. Then the element ba weakly intertwines μ to itself. Then λ_{M}, as a subrepresentation of $\mathrm{res}^{\tilde{J}_{\mathrm{M}}}_{J_{\mathrm{M}}}\mu$, is a subquotient of:

$$\mathrm{ind}^{\tilde{J}_{\mathrm{M}}}_{\tilde{J}_{\mathrm{M}}\cap ba(\tilde{J}_{\mathrm{M}})}\mathrm{res}^{ba(\tilde{J}_{\mathrm{M}})}_{\tilde{J}_{\mathrm{M}}\cap ba(\tilde{J}_{\mathrm{M}})}\mathrm{ind}^{ba(\tilde{J}_{\mathrm{M}})}_{ba(J_{\mathrm{M}})}ba(\lambda_{\mathrm{M}})$$

$$\cong \oplus_{ba(J_{\mathrm{M}})\backslash ba(\tilde{J}_{/rM})/\tilde{J}_{\mathrm{M}}\cap ba(\tilde{J}_{/rM})}\mathrm{ind}^{cba(J_{\mathrm{M}})}_{\tilde{J}_{\mathrm{M}}\cap cba(J_{\mathrm{M}})}cba(\lambda_{\mathrm{M}}).$$

Hence there is $c_0 \in \tilde{J}_{\mathrm{M}}$, such tha $bac_0 \in \mathrm{I}^{w}_{\mathrm{M}}(\lambda_{\mathrm{M}}) = \mathrm{I}_{\mathrm{M}'}(\lambda_{\mathrm{M}}) \subset N$, which is contradicted to our assumption that $a \notin N$. Hence $\tau_1 \cong \tau_2$. We conclude that:

$$\pi_2 \cong \mathrm{ind}^{\mathrm{M}}_{N}\tau_1 \cong (\mathrm{ind}^{\mathrm{M}}_{N}\tau)\otimes\chi\circ\det \cong \pi_1\otimes\chi\circ\det.$$

□

proof version 2. The assumption implies that the set $\mathrm{Hom}_{\mathrm{M}'}(\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_1, \mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_2)$ is non-trivial. The group M acts on this Hom set by

$$g\cdot f := \pi_1(g)\circ f\circ\pi_2(g)^{-1}, f \in \mathrm{Hom}_{\mathrm{M}'}(\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_1, \mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_2), g \in \mathrm{M}.$$

This action factors through M', hence induces an action of the abelian quotient group $\mathrm{M}\backslash\mathrm{M}'$ on this Hom set, which is a finitely dimensional k-vector space, since $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_1$ and $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_2$ are semisimple with finite length. Then the elements of $\mathrm{M}\backslash\mathrm{M}'$ forms a family of commutative linear operators on a finitely dimensional k-vector space, hence they have one common eigenvector. This is to say that there is an k-quasicharacter χ_0 of $\mathrm{M}\backslash\mathrm{M}'$ such that $g\cdot f = \chi_0(g)f$ for each element $g \in \mathrm{M}$, hence χ_0 can be written as $\chi\circ\det$ for some k-quasicharacter χ of $F^\times$. Notice $f \in \mathrm{Hom}_{\mathrm{M}'}(\pi_1\otimes\chi^{-1}\circ\det, \pi_2)$, by irreducibility, the k-representation $\pi_1\otimes\chi^{-1}\circ\det$ coincides with π_2. □

5.3.4 First description of supercuspidal support

Let π and π' be as in 5.1.32. The supercuspidal support of π is unique up to M-conjugate ([V2]). We prove in this section, that the supercuspidal support of π' is also unique up to M-conjugation (5.3.18), which is the first description of supercuspidal support. Eventually, we will prove that the supercuspidal support of π' is unique up to M'-conjugation in the next chapter.

Lemma 5.3.16. *Let π be an irreducible k-representation. If $\pi \otimes \chi \circ \det$ is supercuspidal for some k-quasicharacter χ of $F^\times$, then π is supercuspidal.*

Proof. If $\pi \otimes \chi \circ \det$ is supercuspidal, then it contains a maximal simple cuspidal k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$. Hence π contains $(J_{\mathrm{M}}, \lambda_{\mathrm{M}} \otimes \chi^{-1} \circ \det)$, which is also a maximal simple cuspidal k-type. Hence π is cuspidal. Now assume that there is a supercuspidal representation τ of some proper Levi L of M such that π is an irreducible subquotient of $i_{\mathrm{L}}^{\mathrm{M}}\tau$. Then $\pi \otimes \chi \circ \det$ is a subquotient of $i_{\mathrm{L}}^{\mathrm{M}}\tau \otimes \chi \circ \det$. In fact, we have

$$i_{\mathrm{L}}^{\mathrm{M}}\tau \otimes \chi \circ \det \cong (i_{\mathrm{L}}^{\mathrm{M}}\tau) \times \chi \circ \det.$$

To obtain the equivalence above, we could apply [§I,5.2,d)][V1], by noticing that for any parabolic subgroup containing L, its unipotent radical is a subset of the kernel of det. □

Lemma 5.3.17. *Let π' be an irreducible cuspidal k-representation of* M', *and π an irreducible k-representation of* M *containing π'. Then π' is supercuspidal if and only if π is supercuspidal.*

A similar result has been proved when π' is cuspidal in Corollary 5.1.33.

Proof. Applying 5.3.9, there exists a maximal simple cuspidal k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ and a direct component λ'_{M} of $\lambda_{\mathrm{M}}|_{\mathrm{M}'}$, such that π' contains λ'_{M}. Hence by 5.1.17, the irreducible representation π contains $\lambda_{\mathrm{M}} \otimes \chi \circ \det$ for some k-quasicharacter χ of $F^\times$. Then by §IV1.2, 1.3 in [V2] and 5.1.7, this implies that π is an irreducible cuspidal k-representation.

We assume that π is non-supercuspidal, which means there exists a supercuspidal representation τ of a proper Levi subgroup L of M, the representation π is a subquotient of the parabolic induction $i_{\mathrm{L}}^{\mathrm{M}}\tau$. Now by §5.2 [BeZe], we obtain:

$$\mathrm{res}_{\mathrm{M}'}^{\mathrm{M}} i_{\mathrm{L}}^{\mathrm{M}}\tau \cong\ i_{\mathrm{L}'}^{\mathrm{M}'} \mathrm{res}_{\mathrm{L}'}^{\mathrm{L}}\tau.$$

There must be a direct component τ' of $\mathrm{res}_{\mathrm{L}'}^{\mathrm{L}}\tau$, and π' be an irreducible subquotient of $i_{\mathrm{L}'}^{\mathrm{M}'}\tau'$. Hence π' is not supercuspidal.

□

Proposition 5.3.18. *Let π' be an irreducible cuspidal k-representation of* M', *and π an irreducible cuspidal k-representation of* M *such that π contains π'. Let* $[\mathrm{L}, \tau]$ *be*

the supercuspidal support of π*, where* L *is a Levi subgroup of* M *and* τ *an irreducible supercuspidal* k*-representation of* L*. Let* τ' *be a direct component of* $\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\tau$*. Any element in the supercuspidal support of* π' *is contained in the* M*-conjugacy class of* (L', τ')*.*

Proof. Let L'_0 be a Levi subgroup of M' and τ'_0 an irreducible supercuspidal k-representation of L'_0. Let τ_0 be an irreducible k-representation of L_0 containing τ'_0, hence τ_0 is supercuspidal as in Lemma 5.3.17.

If π' is an irreducible subquotient of $i^{\mathrm{M}'}_{\mathrm{L}'_0}\tau'_0$. By the same reason as in the proof of Lemma 5.3.17, we know that there must be an irreducible subquotient of $i^{\mathrm{M}}_{\mathrm{L}_0}\tau_0$, noted as π_0, such that π' is a direct component of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi_0$. From 5.3.14, there exists a k-quasicharacter χ of $F^\times$ such that $\pi_0 \cong \pi \otimes \chi \circ \det$. On the other hand, the supercuspidal support of $\pi \otimes \chi \circ \det$ is the M-conjugacy class of $(\mathrm{L}, \tau \otimes \chi \circ \det)$. We assume that $\mathrm{L}_0 = \mathrm{L}$ and $\tau_0 \cong \tau \otimes \chi \circ \det$. Then τ'_0 is a direct component of $\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\tau \otimes \chi \circ \det \cong \mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\tau$. □

Chapter 6

Supercuspidal support

6.1 Uniqueness of supercuspidal support

6.1.1 The n-th derivative and parabolic induction

Let $n_1, \ldots, n_m$ be a family of integers, and $\mathrm{M}_{n_1,\ldots,n_m}$ denote the product $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_m}$, which can be canonically embedded into $\mathrm{GL}_{n_1+\cdots+n_m}$. Let $\mathrm{M}'_{n_1,\ldots,n_m}$ denote $\mathrm{M}_{n_1,\ldots,n_m} \cap \mathrm{SL}_{n_1+\cdots+n_m}$, and P_{n_1} the mirabolic subgroup of GL_{n_1}.

Definition 6.1.1. *Let $n_1, \ldots, n_m$ be a family of positive integers, and $s \in \{1, \ldots, m\}$. We define:*

- *the mirabolic subgroup at place s of $\mathrm{M}_{n_1,\ldots,n_m}$, as $P_{(n_1,\ldots,n_m),s} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_{s-1}} \times P_{n_s} \times \mathrm{GL}_{n_{s+1}} \times \cdots \times \mathrm{GL}_{n_m}$;*
- *the mirabolic subgroup at place s of $\mathrm{M}'_{n_1,\ldots,n_m}$, as $P'_{(n_1,\ldots,n_m),s} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_{s-1}} \times P_{n_s} \times \mathrm{GL}_{n_{s+1}} \times \cdots \times \mathrm{GL}_{n_m} \cap \mathrm{M}'_{n_1,\ldots,n_m}$.*

For any $i \in \{1, \ldots, m\}$, let U_{n_i} be the subset of GL_{n_i}, consisted with upper-triangular matrix with 1 on the diagonal. We fix θ_i a non-degenerate character of U_{n_i}. It is clear that $\mathrm{U}_{n_1,\ldots,n_m} = \mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_m}$ is a subgroup of $P_{(n_1,\ldots,n_m),s}$ and $P'_{(n_1,\ldots,n_m),s}$ for any $s \in \{1, \ldots, m\}$. Let V_{n_s-1} denote the additive group of k-vector space with dimension $n_s - 1$, which can be embedded canonically as a normal subgroup in $\mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_m}$. The subgroup V_{n_s-1} is normal both in $P_{(n_1,\ldots,n_m),s}$ and $P'_{(n_1,\ldots,n_m),s}$, furthermore, we have $P_{(n_1,\ldots,n_m),s} = \mathrm{M}_{n_1,\ldots,n_s-1,\ldots,n_m} \cdot V_{n_s-1}$ and $P'_{(n_1,\ldots,n_m),s} = \mathrm{M}'_{n_1,\ldots,n_s-1,\ldots,n_m} \cdot V_{n_s-1}$.

Note γ be any character of $\mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_m}$. For any k-representation $(E, \rho) \in \mathrm{Rep}_k(P'_{(n_1,\ldots,n_m),s})$, let $E_{s,\gamma}$ denote the subspace of E generated by elements in form of $\rho(g)a - \gamma(h)a$, where $g \in V_{n_s-1}, a \in E$. We define the coinvariants of (E, ρ) according to θ as $E/E_{s,\gamma}$, and note it as $E(\gamma, s)$, and view $E(\gamma, s)$ as a k-representation of $\mathrm{M}'_{n_1,\ldots,n_s-1,\ldots,n_m}$.

Definition 6.1.2. *Fix a non-degenerate character θ of $\mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_m}$.*

- *Let* $(E,\rho)\in \mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s})$,

$$\Psi_s^-:\mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s})\to \mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_s-1,\dots,n_m}),$$

which maps E *to* $E(\mathbb{1},s)$*;*

- *Let* $(E,\rho)\in \mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_s-1,\dots,n_m})$,

$$\Psi_s^+:\mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_s-1,\dots,n_m})\to \mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s}).$$

Write $P'_{(n_1,\dots,n_m),s}=\mathrm{M}'_{n_1,\dots,n_s-1,\dots,n_m}\cdot V_{n_s-1}$*. Define* $\Psi_s^+(E,\rho)=(E,\Psi^{+,s}(\rho))$ *by* $\Psi_s^+(\rho)(mg)(a)=\rho(m)(a)$*, for any* $m\in \mathrm{M}'_{n_1,\dots,n_s-1,\dots,n_m}, g\in V_{n_s-1}$ *and* $a\in E$*;*

- *Let* $(E,\rho)\in \mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s})$,

$$\Phi_{\theta,s}^-:\mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s})\to \mathrm{Rep}_k(P'_{(n_1,\dots,n_s-1,\dots,n_m),s}),$$

which maps E *to* $E(\theta,s)$*;*

- *Let* $(E,\rho)\in \mathrm{Rep}_k(P'_{(n_1,\dots,n_s-1,\dots,n_m),s})$,

$$\Phi_{\theta,s}^+:\mathrm{Rep}_k(P'_{(n_1,\dots,n_s-1,\dots,n_m),s})\to \mathrm{Rep}_k(P'_{(n_1,\dots,n_m),s}),$$

by $\Phi_{\theta,s}^+(\rho)=\mathrm{ind}_{P'_{(n_1,\dots,n_s-1,\dots,n_m),s}\cdot V_{n_s-1}}^{P'_{(n_1,\dots,n_m),s}}\rho_\theta$*, where* $\rho_\theta(pg)(a)=\theta(g)\rho(p)(a)$*, for any* $p\in P'_{(n_1,\dots,n_s-1,\dots,n_m),s}, g\in V_{n_s-1}$ *and* $a\in E$*.*

Remark 6.1.3. *By the reason that for any* $m\in\mathbb{Z}$ *the group* V_m *is a limite of pro-p open compact subgroups, the four functors defined above are exact. In the definition of* $\Phi_{\theta,s}^+$*, we view* $P'_{(n_1,\dots,n_s-1,\dots,n_m),s}$ *as a subgroup of* $P'_{(n_1,\dots,n_m),s}$*.*

The notion of derivatives is well defined for k-representations of G, now we consider the parallel operator of derivatives for Levi subgroups of G′.

Definition 6.1.4. *We fix a non-degenerate character* θ *of* $\mathrm{U}_{n_1}\times\cdots\times\mathrm{U}_{n_m}$*. Let* $(E,\rho)\in\mathrm{Rep}_k(P'_{(n_1,\dots,n_m,s)})$*, for any interger* $s\in\{1,\dots,m\}$ *and* $1\le d\le n_1+\ldots+n_s$*, we define the derivative* $\rho_{\theta,s}^{(d)}$*:*

- *when* $1\le d\le n_s$*,* $\rho_{\theta,s}^{(d)}=\Psi_s^-\circ(\Phi_{\theta,s}^-)^{d-1}\rho$*;*
- *when* $n_s+1\le d=n_s+\ldots+n_{s-l}+n'$*, where* $0\le l\le s-1$ *and* $1\le n'\le n_{s-l-1}$*, then* $\rho_{\theta,s}^{(d)}=\Psi_{s-l-1}^-\circ(\Phi_{\theta,s-l-1})^{n'-1}\circ(\Phi_{\theta,s-l})^{n_{s-l}-1}\circ\ldots\circ(\Phi_{\theta,s}^-)^{n_s-1}\rho$

Definition 6.1.5. *To simplify our notations, we need to introduce* $\mathrm{ind}_m^{m-1}:\mathrm{Rep}_k(\mathrm{G}_1)\to\mathrm{Rep}_k(\mathrm{G}_2)$ *according to different cases:*

- *When* $\mathrm{G}_1=\mathrm{M}'_{n_1,\dots,n_m}$ *and* $\mathrm{G}_2=\mathrm{M}'_{n_1,\dots,n_{m-1}+n_m}$*, we embed* G_1 *into* G_2 *as in the figure case* I*, and* ind_m^{m-1} *is defined as* $i_{\mathrm{U},1}$*, and the later one is defined as in* §1.8 *of [BeZe];*

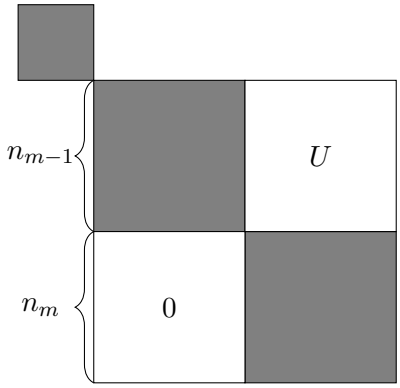


Figure 6.1: Case I

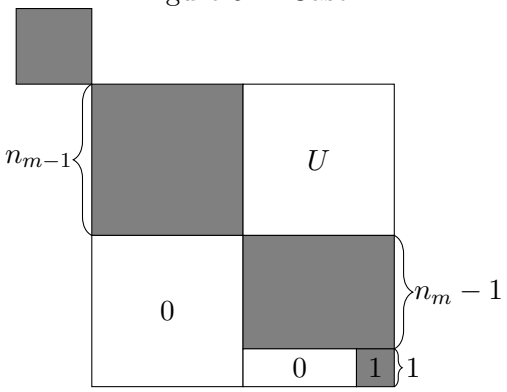


Figure 6.2: Case II

- *When* $\mathrm{G}_1 = P'_{(n_1,\ldots,n_m),m}$ *and* $\mathrm{G}_2 = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}$*, we embed* G_1 *into* G_2 *as in the figure case* II*, and* ind_m^{m-1} *is defined as* $i_{\mathrm{U},1}$*;*
- *When* $\mathrm{G}_1 = P'_{(n_1,\ldots,n_m),m-1}$ *and* $\mathrm{G}_2 = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}$*, we embed* G_1 *into* G_2 *as in the figure case* III*, and* ind_m^{m-1} *is defined as* $i_{\mathrm{U},1} \circ \varepsilon$*. Here* ε *is a character of* $P'_{(n_1,\ldots,n_m),m-1}$*. Write* $g \in P'_{(n_1,\ldots,n_m),m-1} \subset \mathrm{M}'_{n_1,\ldots,n_m}$ *as* $(g_1,\ldots,g_m)$*, define* $\varepsilon(g) = |\det(g_m)|$*, the absolute value of* $\det(g_m)$*. This* k*-character is well defined since* $p \neq l$*.*

Proposition 6.1.6. *Assume that* $\rho_1 \in \mathrm{Rep}_k(\mathrm{M}'_{n_1,\ldots,n_m}), \rho_2 \in \mathrm{Rep}_k(P'_{(n_1,\ldots,n_m),m})$*, and* $\rho_3 \in \mathrm{Rep}_k(P'_{(n_1,\ldots,n_m),m-1})$*. The functor* ind_m^{m-1} *is defined as in 6.1.5 according to different cases.*

1. *In* $\mathrm{Rep}_k(P'_{(n_1,\ldots,n_{m-1}+n_m),m-1})$*, there exists an exact sequence:*

$$0 \to \mathrm{ind}_m^{m-1}(\rho_1|_{P'_{m,m-1}}) \to (\mathrm{ind}_m^{m-1}\rho_1)|_{P'_{m-1,m-1}} \to \mathrm{ind}_m^{m-1}(\rho_1|_{P'_{m,m}}) \to 0,$$

where $P'_{m,m-1} = P'_{(n_1,\ldots,n_m),m-1}, P'_{m-1,m-1} = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}$*, and* $P'_{m,m} = P'_{(n_1,\ldots,n_m),m}$*.*

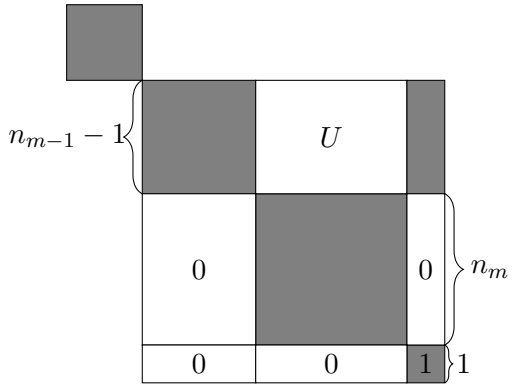


Figure 6.3: Case III

2. *When* $2 \leq m$, *let* $\dot{\theta}$ *be a non-degenerate character of* $\mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_{m-2}} \times \mathrm{U}_{n_{m-1}+n_m}$, *such that* $\dot{\theta}|_{U_{n_1} \times \cdots \times \mathrm{U}_{n_m}} \cong \theta$. *We have equivalences:*

   - $\mathrm{ind}_m^{m-1} \circ \Psi_m^- \rho_2 \cong \Psi_{m-1}^- \circ \mathrm{ind}_m^{m-1} \rho_2$;
   - $\mathrm{ind}_m^{m-1} \circ \Phi_{\theta,m}^- \rho_2 \cong \Phi_{\dot{\theta},m-1}^- \circ \mathrm{ind}_m^{m-1} \rho_2$.

3. *We have an equivalence:*

$$\Psi_{m-1}^- \circ \mathrm{ind}_m^{m-1} \rho_3 \cong \mathrm{ind}_m^{m-1} \circ \Psi_{m-1}^- \rho_3,$$

   and an exact sequence:

$$0 \to \mathrm{ind}_m^{m-1} \circ \Phi_{\theta,m-1}^- \rho_3 \to \Phi_{\dot{\theta},m-1}^- \circ \mathrm{ind}_m^{m-1} \rho_3 \to \mathrm{ind}_m^{m-1}((\Psi_{m-1}^- \rho_3)|_{P'}) \to 0,$$

   where $P' = P'_{(n_1,\ldots,n_{m-1}-1,n_m),m}$.

Proof. As proved in the Appendix, Theorem 5.2 in [BeZe] holds for k-representations of M'. And let $n = n_1 + \ldots + n_m$

For (1): Let $\mathrm{M}' = \mathrm{M}'_{n_1,\ldots,n_m}$ be embedded into $\mathrm{G}' = \mathrm{M}'_{n_1,\ldots,n_{m-1}+n_m}$ as in definition 6.1.5, figure I. Define functor F as $\mathrm{F}(\rho_1) = \rho_1|_{P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}}$, where the functor F is defined as in 5.1 [BeZe] in the following situation:

$$\mathrm{U} = \mathrm{U}_{n_{m-1}}, \vartheta = 1, \mathrm{N} = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}, \mathrm{V} = \{e\}.$$

To compute F, we apply theorem 5.2 [BeZe]. Condition $(1), (2)$ and $(*)$ from 5.1 [BeZe] hold trivially. Let T be the group of diagonal matrix, the Q-orbits on $X = \mathrm{P}\backslash\mathrm{G}$ is actually the $\mathrm{T} \cdot \mathrm{N}$-orbits, and the group $\mathrm{T} \cdot \mathrm{N}$ is a parabolic subgroup. By Bruhat decomposition $\mathrm{T} \cdot \mathrm{N}$ has two orbits: the closed orbit Z of point $\mathrm{P} \cdot e \in X$ and the open orbit Y of the point $\mathrm{P} \cdot \omega^{-1} \in X$, where ω is the matrix of the cyclic permutation $sgn(\sigma)\mathbb{1}_{n_m} \cdot \sigma$, where

$$\sigma = (N_1 + \cdots + n_{m-1} \to n \to n-1 \to \cdots \to n_1 + \cdots + n_{m-1}),$$

and $sgn(\sigma)\mathbb{1}_{n_m}$ denote an element in $\mathrm{M}'_{n_1,\ldots,n_m}$, which equals to identity on the first $m-1$ blocs, and $sgn(\sigma)$ times identity on the last bloc, and $sgn(\sigma)$ denote the signal of σ. Now we consider condition (4) from 5.1 [BeZe]:

- Since $V = \{e\}$, it is clear that $\omega(\mathrm{P}), \omega(\mathrm{M})$ and $\omega(U)$ are decomposable with respect to (N, V);
- Let us consider $\omega^{-1}(\mathrm{Q}) = \omega^{-1}(\mathrm{N})$.

To study the intersection $\omega^{-1}(\mathrm{N}) \cap (\mathrm{M} \cdot \mathrm{U})$, first we consider the Levi subgroup $\mathrm{M}'_{n_1,\ldots,n_{m-1}+n_m-1,1}$ and the corresponding standard parabolic subgroup

$$\mathrm{P}' = \mathrm{M}'_{n_1,\ldots,n_{m-1}+n_m-1,1} \cdot \mathrm{V}_{n_{m-1}+n_m-1},$$

where $\mathrm{V}_{n_{m-1}+n_m-1}$ denotes the unipotent radical of P'. We have $\mathrm{N} \subset \mathrm{P}'$, hence $\omega^{-1}(\mathrm{N}) \subset \omega^{-1}(\mathrm{P}')$. As in 6.1 of [BeZe], after fix a system Ω of roots, and denote Ω^+ the set of positive roots. Then by Proposition in 6.2 [BeZe], we could write $\omega^{-1}(\mathrm{P}') = \mathrm{G}(\mathcal{S})$ and $\mathrm{P} = \mathrm{G}(\mathcal{P}), \mathrm{U} = \mathrm{U}(\mathcal{M})$ in the manner as in 6.1[BeZe], where $\mathcal{S}, \mathcal{P}$ and $\mathcal{M}$ are convex subset of Ω. So by Proposition in 6.1 [BeZe], we have:

$$\omega^{-1}(\mathrm{P}') \cap \mathrm{P} = \mathrm{G}(\mathcal{S} \cap \mathcal{P});$$

$$\omega^{-1}(\mathrm{P}') \cap \mathrm{U} = \mathrm{U}(\mathcal{S} \cap \mathcal{P} \backslash \mathcal{M});$$

$$\omega^{-1}(\mathrm{P}') \cap \mathrm{M} = \mathrm{G}(\mathcal{S} \cap \mathcal{M}).$$

Hence

$$\omega^{-1}(\mathrm{P}') \cap \mathrm{P} = (\omega^{-1}(\mathrm{P}') \cap \mathrm{M}) \cdot (\omega^{-1}(\mathrm{P}') \cap \mathrm{U}).$$

Notice that $\omega^{-1}(\mathrm{P}') \cap \mathrm{U} = \omega^{-1}(\mathrm{N}) \cap \mathrm{U}$, we deduce that:

$$\omega^{-1}(\mathrm{N}) \cap \mathrm{P} = (\omega^{-1}(\mathrm{N}) \cap \mathrm{M}) \cdot (\omega^{-1}(\mathrm{N}) \cap \mathrm{U}).$$

In the formula of Φ_Z in 5.2 [BeZe], since $\mathrm{U} \cap \omega^{-1}(\mathrm{N}) = \mathrm{U}$, the characters $\varepsilon_1 = \varepsilon_2 = 1$. Hence we obtain the exact sequence desired.

For (2). In this part, the functor ind_m^{m-1} is always defined as the case II in 6.1.5. First we consider the case Ψ_m^-. Define functor F as $\Psi_{m-1}^- \circ \mathrm{ind}_m^{m-1}$. We write F as in §5.1 [BeZe] in the situation:

$$\mathrm{G} = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}, \mathrm{M} = P'_{(n_1,\ldots,n_m),m},$$

$$\mathrm{N} = \mathrm{M}'_{n_1,\ldots,n_{m-1}+n_m-1}, \mathrm{V} = \mathrm{V}_{n_{m-1}+n_m-1},$$

and U are defined in the 6.1.5 case II. Condition (1) and (2) of §5.1 [BeZe] is clear. Since $\mathrm{Q} = \mathrm{G}$, and there is only one Q-orbit on $X = \mathrm{P}\backslash\mathrm{G}$, conditions $(3), (4)$ hold trivially. Thus we obtain the equivalence:

$$\mathrm{ind}_m^{m-1} \circ \Psi_m^- \rho_2 \cong \Psi_{m-1}^- \circ \mathrm{ind}_m^{m-1} \rho_2.$$

For the case $\Phi^-_{\dot{\theta},m-1}$: define functor F as $\Phi^-_{\dot{\theta},m-1}\circ \mathrm{ind}_m^{m-1}$. We write F as in §5.1 [BeZe] in the situation:

$$\mathrm{G} = P'_{(n_1,\dots,n_{m-1}+n_m),m-1},$$

$$\mathrm{N} = P'_{(n_1,\dots,n_{m-1}+n_m-1),m-1}, \mathrm{V} = \mathrm{V}_{n_{m-1}+n_m-1},$$

and M, U are defined as the case II of 6.1.5. Conditions $(1), (2)$ of §5.1 [BeZe] hold, and

$$\mathrm{P}\backslash\mathrm{G}/\mathrm{Q} \cong \mathrm{M}'_{n_1,\dots,n_{m-1},n_m-1}\backslash\mathrm{M}'_{n_1,\dots,n_{m-1}+n_m-1}/P'_{(n_1,\dots,n_{m-1}+n_m-1),m-1}.$$

Hence as proved in (1), the group Q has two orbits on $X = \mathrm{P}\backslash\mathrm{G}$: the closed orbit of $\mathrm{P}\cdot e$ and the open orbit $\mathrm{P}\cdot\omega_0^{-1}$, where ω_0 is the matrix $sgn(\sigma_0)\mathbb{1}_{n_m-1}\cdot\sigma_0$. The matrix σ_0 corresponding to the cyclic permutation:

$$(n_1+\cdots+n_{m-1}\to n\to n-1\to\cdots\to n_1+\cdots+n_{m-1}).$$

Now we check the condition (4) of §5.1 [BeZe]. Since

$$\mathrm{P} = \mathrm{M}'_{n_1,\dots,n_{m-1}+n_m}\cdot\mathrm{V},$$

$$\mathrm{M} = \mathrm{M}'_{n_1,\dots,n_m-1}\cdot\mathrm{V}_{n_m-1},$$

and $\omega_0(\mathrm{V}) = \mathrm{V}$, hence $\omega_0(\mathrm{P})$ and $\omega_0(\mathrm{M})$ are decomposable with respect to (N,V). We deduce that $\omega_0(\mathrm{U})$ is decomposable with respect to (N,V) by noticing that $\omega_0(\mathrm{U}) = (\omega_0(\mathrm{U})\cap\mathrm{N})\cdot(\omega_0(\mathrm{U})\cap\mathrm{V})$. Now consider $\omega_0^{-1}(\mathrm{Q})$, $\omega_0^-(\mathrm{N})$ and $\omega_0^-(\mathrm{V})$. Since $\omega_0^{-1}(\mathrm{V}) = \mathrm{V}$, which is decomposable with respect to (M,U) clearly. Notice that $\omega_0^-(\mathrm{V})\cap\mathrm{P} = \omega_0^-(\mathrm{V})$, and $\omega_0^-(\mathrm{N})$ is decomposable with respect to (M,U) by (1). We deduce that $\omega_0^-(\mathrm{Q})$ is decomposable with respect to (M,U). And the condition $(*)$ does not hold for the orbit $\mathrm{P}\cdot\sigma_0$. Then by §5.2 [BeZe], we obtain the equivalence

$$\mathrm{ind}_m^{m-1}\circ\Phi^-_{\theta,m} \cong \Phi^-_{\dot{\theta},m-1}\circ\mathrm{ind}_m^{m-1},$$

for every $\rho_2\in\mathrm{Rep}_k(P'_{(n_1,\dots,n_m),m})$.

For part (3). In the case of $\mathrm{F} = \Psi^-_{m-1}\circ\mathrm{ind}_m^{m-1}$, we have (in the manner of §5.1 [BeZe]):

$$\mathrm{G} = P'_{(n_1,\dots,n_m),m},$$

$$\mathrm{N} = \mathrm{M}'_{n_1,\dots,n_m-1}, \mathrm{V} = \mathrm{V}_{n_m-1},$$

and M,U as in 6.1.5 case II. There is only one Q-orbit on $\mathrm{P}\backslash\mathrm{G}$, and condition $(1)-(4)$ and $(*)$ in §5.1 [BeZe] hold. Notice that $\varepsilon\circ\Psi^-_{m-1}\cong\Psi^-_{m-1}$ (ε is defined in 6.1.5). After applying theorem 5.2 of [BeZe], we obtain the equivalence:

$$\Psi^-_{m-1}\circ\mathrm{ind}_m^{m-1}\rho_3 \cong \mathrm{ind}_m^{m-1}\circ\Psi^-_{m-1}\rho_3.$$

For the case $\mathrm{F} = \Phi^-_{\dot{\theta},m-1}\circ\mathrm{ind}_m^{m-1}$. We have (in the manner of §5.1 [BeZe]):

$$\mathrm{G} = P'_{(n_1,\dots,n_m),m},$$

$$\mathrm{N} = P'_{(n_1,\dots,n_{m-1}+n_m-1),m-1}, \mathrm{V} = \mathrm{V}_{n_{m-1}+n_m-1},$$

and M, U are defined as the case II of 6.1.5. As in the proof of part (2), the group Q has two orbits on $\mathrm{P}\backslash\mathrm{G}$: the closed one $\mathrm{P}\cdot e$ and the open one $\mathrm{P}^{-1}\cdot\omega_0$. The condition (4) can be justified as part (2), and condition $(*)$ is clear since $\omega_0(\mathrm{U})\cap\mathrm{V} = \mathbb{1}$. Now we apply theorem 5.2 [BeZe]. The functor corresponds to the orbit $\mathrm{P}\cdot e$ is $\mathrm{ind}_m^{m-1}\circ\Phi^-_{\theta,m-1}$ by noticing $\varepsilon\circ\Phi^-_{\theta,m-1}\cong\Phi_{\theta,m-1}\circ\varepsilon$. Now we consider the functor corresponds to the orbit $\mathrm{P}\cdot\omega_0^{-1}$. Following the notation as §5.1 [BeZe], the character $\psi' = \omega_0^{-1}(\psi)|_{\mathrm{M}\cap\omega_0^{-1}(\mathrm{V})}$ is trivial. The character ε_1 is trivial, and $\varepsilon_2\cong\varepsilon^{-1}$. Hence the functor corresponded to the fixed orbit is

$$\mathrm{ind}_m^{m-1}\circ\mathrm{res}_{P'}^{\mathrm{M}'_{n_1,\dots,n_{m-1},n_m}}\circ\Psi^-_{m-1},$$

from which we deduce the exact sequence desired. □

Corollary 6.1.7. *1. Let $\rho\in\mathrm{Rep}_k(P'_{(n_1,\dots,n_m),m})$. Assume that $1\le i\le n_m$, then $(\mathrm{ind}_m^{m-1}\rho)^{(i)}_{\dot\theta,m-1}\cong\mathrm{ind}_m^{m-1}\rho^{(i)}_{\theta,m}$;*

2. Let $\rho\in\mathrm{Rep}_k(P'_{(n_1,\dots,n_m),m-1})$. Assume that $1\le i\le n_{m-1}+n_m$, then $(\mathrm{ind}_m^{m-1}\rho)^{(i)}_{\dot\theta,m-1}$ is filtrated by $\mathrm{ind}_m^{m-1}((\rho^{(i-j)}_{\theta,m})^{(j)}_{\theta,m-1})$, where $i-n_m\le j\le i$;

3. Let $\rho\in\mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_m})$. Assume that $i\ge 0$, then $(\mathrm{ind}_m^{m-1}\rho)^{(i)}_{\dot\theta,m-1}$ is filtrated by $\mathrm{ind}_m^{m-1}((\rho^{(i-j)}_{\theta,m})^{(j)}_{\theta,m-1})$, where $i-n_m\le j\le i$;

4. Let $\rho\in\mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_m})$, there is an equivalence:

$$(\mathrm{ind}_2^1\circ\cdots\circ\mathrm{ind}_{m-1}^{m-2}\circ\mathrm{ind}_m^{m-1}\rho)^{(n_1+\cdots+n_m)}_{\dot\theta,1}\cong(\cdots((\rho^{(n_m)}_{\theta,m})^{(n_{m-1})}_{\theta,m-1})\cdots)^{(n_1)}_{\theta,1}.$$

Proof. Part (1) follows from the exactness of $\Phi^-_{\theta,m}, \Psi^-_m$ and 6.1.6 (2); (2) from (1) and 6.1.6 (3), (3) from (1), (2) and 6.1.6 (1). Part (4) follows from (3), by noticing that

$$\mathrm{ind}_2^1\circ\cdots\circ\mathrm{ind}_{m-1}^{m-2}\circ\mathrm{ind}_m^{m-1}\rho\cong i_{\mathrm{M}_{n_1,\dots,n_m}}^{\mathrm{GL}_{n_1+\cdots+n_m}}\rho.$$

In fact, this is the transitivity of parabolic induction. □

6.1.2 Uniqueness of supercuspidal support

Proposition 6.1.8. *Let $\tau\in\mathrm{Rep}_k(\mathrm{M}'_{n_1,\dots,n_m})$, and θ a non-degenerate character of $\mathrm{U}_{n_1,\dots,n_m}$. Then $\tau^{(n_1+\dots+n_m)}_{\theta,m}\neq 0$ is equivalent to say that $\mathrm{Hom}_{k[\mathrm{U}_{n_1,\dots,n_m}]}(\tau,\theta)\neq 0$. In particular, this is equivalent to say that $(\mathrm{U}_{n_1,\dots,n_m},\theta)$-coinvariants of τ is non-trivial.*

Proof. In this proof, we use U to denote $\mathrm{U}_{n_1,\dots,n_m}$. For the first equivalence, notice that $\Phi^-_{\theta,m}(\tau)\neq 0$ is equivalent to say that (V_{n_m-1},θ)-coinvariants of τ is non-trivial. For $1\le s\le n_m-1$, let V_s denote the subgroup of U consisting with the matrices

with non-zero coefficients only on the $(s+1)$-th line and the diagonal. Let W denote the representation space of τ. The space of $\tau_{\theta,m}^{(n_1+\cdots+n_m)}$ is isomorphic to the quotient of W by the subspace W_θ generated by $g_s(w)-\theta(g_s)w$, for every s and $g_s \in V_s$, $w \in W$. Meanwhile, since the subgroups V_s's generate U, and θ is determined by $\theta|_{V_s}$ while considering every s, the subspace W_θ of W is isomorphic to the subspace generated by $g(w)-\theta(g)w$, where $g \in \mathrm{U}$. Hence $\tau_{\theta,m}^{(n_1+\cdots+n_m)} \neq 0$ is equivalent to say that $(\mathrm{U}_{n_1,\ldots,n_m},\theta)$-coinvariants of τ is non-trivial. The second equivalence is clear, since the (U,θ)-coinvariants of τ is the largest quotient of τ such that U acts as a multiple of θ. □

Proposition 6.1.9. *Let $\tau \in \mathrm{Rep}_k(\mathrm{M}'_{n_1,\ldots,n_m})$, and ρ be a subquotient of τ. Let θ be a non-degenerate character of $\mathrm{U}_{n_1,\ldots,n_m}$, and $\rho_{\theta,m}^{(n_1+\cdots+n_m)}$ is non-trivial, then $\tau_{\theta,m}^{(n_1+\cdots+n_m)}$ is non-trivial.*

Proof. We consider the $(n_1+\ldots+n_m)$-th derivative functor corresponding to the non-degenerate character θ, from the category $\mathrm{Rep}_k(\mathrm{M}'_{n_1,\ldots,n_m})$ to the category of k-vector spaces, which maps τ to $\tau_{\theta,m}^{(n_1+\cdots+n_m)}$. By Definition 6.1.4 and Remark 6.1.3, this functor is a composition of functors $\Psi_\cdot^-$ and $\Phi_{\theta,\cdot}^-$, hence is exact. Let ρ_0 be a sub-representation of τ such that ρ is a quotient representation of ρ_0. The exactness of derivative functor implies first that $\rho_{0_{\theta,m}}^{n_1+\ldots+n_m} \neq 0$, and apply again the exactness we conclude that $\rho_{\theta,m}^{n_1+\ldots+n_m} \neq 0$.

□

Theorem 6.1.10. *Let M' be a Levi subgroup of G', and ρ an irreducible k-representation of M'. The supercuspidal support of ρ is a M'-conjugacy class of one unique supercuspidal pair.*

Proof. Since the cuspidal support of irreducible k-representation is unique, to prove the uniqueness of supercuspidal support, it is sufficient to assume that ρ is cuspidal. Let π be an irreducible cuspidal k-representation of M, such that ρ is a sub-representation of $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi$. Let (L,τ) be a supercuspidal pair of M, and $[\mathrm{L},\tau]$ consists the supercuspidal support of π. By 5.1.32, we have $\mathrm{res}^{\mathrm{L}}_{\mathrm{L}'}\tau \cong \oplus_{i\in I}\tau_i$, where I is a finite index set. According to 5.3.18, the supercuspidal support of (M',π') is contained in the union of M'-conjugacy class of (L',τ_i), for every $i \in I$. To finish the proof of our theorem, it remains to prove that there exists one unique $i_0 \in I$ such that (L',τ_{i_0}) is contained in the supercuspidal support of (M',ρ).

After conjugation by G', we could assume that $\mathrm{M}' = \mathrm{M}'_{n_1,\ldots,n_m}$ and $\mathrm{L}' = \mathrm{M}'_{k_1,\ldots,k_l}$ for a familly of integers $m, l, n_1, \ldots, n_m, k_1, \ldots, k_l \in \mathbb{N}^*$. There exists a non-degenerate character θ of $\mathrm{U} = \mathrm{U}_{n_1,\ldots,n_m}$, such that $\rho_{\theta,m}^{(n_1+\cdots+n_m)} \neq 0$. In fact, let θ be any non-degenerate character of U and we write $\mathrm{res}^{\mathrm{M}}_{\mathrm{M}'}\pi \cong \oplus_{s\in S}\pi_s$, where S is a finite index set. We have:

$$\pi^{(n_1+\ldots+n_m)} \cong (\pi|_{\mathrm{M}'})_{\theta,m}^{(n_1+\ldots+n_m)} \cong \oplus_{s\in S}(\pi_s)_{\theta,m}^{(n_1+\ldots+n_m)},$$

where $\pi^{(n_1+...+n_m)}$ indicates the $(n_1+...+n_m)$-th derivative of π. As in Section [§III, 5.10, 3)] of [V1], $\dim(\pi^{(n_1+...+n_m)}) = 1$, hence there exists one element $s_0 \in S$ such that $(\pi_{s_0})^{(n_1+...+n_m)}_{\theta,m} \neq 0$. Notice that τ are isomorphic to some π_s, hence there exists a diagonal element $t \in \mathrm{M}$, such that the t-conjugation $t(\pi_{s_0}) \cong \tau$. The character $t(\theta)$ is also non-degenerate of U, and we have $(t(\pi_{s_0}))^{(n_1+...+n_m)}_{t(\theta),m} \cong (\pi_{s_0})^{(n_1+...+n_m)}_{\theta,m}$ as k-vector spaces. We conclude that $\dim\tau^{(n_1+...+n_m)}_{t(\theta),m} = 1$. To simplify the notations, we assume $t = 1$.

If ρ is a subquotient of $i^{\mathrm{M}'}_{\mathrm{L}'}\tau_i$ for some $i \in I$. By (4) of 6.1.7 and 6.1.9, the derivative $\tau^{(k_1+\cdots+k_l)}_{i_{\theta,l}} \neq 0$. Note $\mathrm{U} \cap \mathrm{L}'$ as $\mathrm{U}_{\mathrm{L}'}$. By section [§III, 5.10, 3)] of [V1], the derivative $\tau^{(k_1+\cdots+k_l)}_{\theta,l} = 1$, which means the dimension of $(\mathrm{U}_{\mathrm{L}'}, \theta)$-coinvariants of τ is 1 (by 6.1.8). Notice that the $(\mathrm{U}_{\mathrm{L}'}, \theta)$-coinvariants of τ is the direct sum of $(\mathrm{U}_{\mathrm{L}'}, \theta)$-coinvariants of τ_i for every $i \in I$. This implies that there exists one unique $i_0 \in I$ whose $(\mathrm{U}_{\mathrm{L}'}, \theta)$-coinvariants is non-zero with dimension 1. By 6.1.8 and 6.1.9, this is equivalent to say that there exists one unique $i_0 \in I$, such that the derivative $\tau^{(k_1+\cdots+k_l)}_{i_{\theta,l}} \neq 0$. □

Appendix A

Theorem 5.2 of Bernstein and Zelevinsky

We need the results of Theorem 5.2 of [BeZe] in the case of k-representations. In fact, the proof in [BeZe] is in the language of ℓ-sheaves, which can be translated to a representation theoretical proof and be applied to our case. In the reason for being self-contained, I rewrite the proof following the method in [BeZe].

Let G be a locally compact totally disconnected group, P, M, U, Q, N, V are closed subgroups of G, and θ, ψ be k-characters of U and V respectively. Suppose that they verify conditions $(1)-(4)$ in §5.1 of [BeZe], and denote $\mathrm{X}=\mathrm{P}\backslash\mathrm{G}$. The numbering we choose in condition (3) is $\mathrm{Z}_1,...,\mathrm{Z}_k$ of Q-orbits on X, and for any orbit $\mathrm{Z}\subset X$, we choose $\overline{\omega}\in\mathrm{G}$ and ω as in condition (3) of [BeZe].

We introduce condition $(*)$:

$(*)$ The characters $\omega(\theta)$ and ψ coincide when restricted to the subgroups $\omega(\mathrm{U})\cap\mathrm{V}$.

We define Φ_{Z} equals 0 if $(*)$ does not hold, and define Φ_{Z} as in §5.1 [BeZe] if $(*)$ holds.

Definition A.0.1. *Let* M,U *be closed subgroups of* G*, and* $\mathrm{M}\cap\mathrm{U}=\{e\}$*, and the subgroup* $\mathrm{P}=\mathrm{MU}$ *is closed in* G*. Let* θ *be a* k*-character of* U *normalized by* M*.*

- *Define functor* $i_{\mathrm{U},\theta}\mathrm{Rep}_k(\mathrm{M})\to\mathrm{Rep}_k(\mathrm{G})$*. Let* $\rho\in\mathrm{Rep}_k(\mathrm{M})$*, then* $i_{\mathrm{U},\theta}(\rho)$ *equals* $\mathrm{ind}_{\mathrm{P}}^{\mathrm{G}}\rho_{\mathrm{U},\theta}$*, where* $\rho_{\mathrm{U},\theta}\in\mathrm{Rep}_k(\mathrm{P})$*, such that*

$$\rho_{\mathrm{U},\theta}(mu)=\theta(u)\mathrm{mod}_{\mathrm{U}}^{\frac{1}{2}}(m)\rho(m)$$

  .

- *Define functor* $r_{\mathrm{U},\theta}\mathrm{Rep}_k(\mathrm{G})\to\mathrm{Rep}_k(\mathrm{M})$*. Let* $\pi\in\mathrm{Rep}_k(\mathrm{G})$*, then* $r_{\mathrm{U},\theta}(\pi)$ *equals* $\mathrm{mod}_{\mathrm{U}}^{-\frac{1}{2}}(\mathrm{res}_{\mathrm{P}}^{\mathrm{G}}\pi)/(\mathrm{res}_{\mathrm{P}}^{\mathrm{G}}\pi)(\mathrm{U},\theta)$*, where* $(\mathrm{res}_{\mathrm{P}}^{\mathrm{G}}\pi)(\mathrm{U},\theta)\subset\mathrm{res}_{\mathrm{P}}^{\mathrm{G}}\pi$*, generated by elements* $\pi(u)w-\theta(u)w$*, for any* $w\in W$*, where* W *is the space of* k*-representation* π*.*

Remark A.0.2. *By replacing* ind *to* Ind*, we could define* $I_{\mathrm{U},\theta}$*. Notice that* $r_{\mathrm{U},\theta}$ *is left adjoint to* $I_{\mathrm{U},\theta}$*.*

Proposition A.0.3. *The functors* $i_{\mathrm{U},\theta}$ *and* $r_{\mathrm{U},\theta}$ *commute with inductive limits.*

Proof. The functor $r_{\mathrm{U},\theta}$ commutes with inductive limits since it has a right adjoint as in A.0.2.

For $i_{\mathrm{U},\theta}$. Let $(\pi_\alpha, \alpha \in \mathcal{C})$ be a inductive system, where $\mathcal{C}$ is a directed pre-ordered set. We want to prove that $i_{\mathrm{U},\theta}(\varinjlim \pi_\alpha) \cong \varinjlim(i_{\mathrm{U},\theta}\pi_\alpha)$. The inductive limit $\varinjlim \pi_\alpha$ is defined as $\oplus_{\alpha\in\mathcal{C}}\pi_\alpha/\sim$, where $\sim$ denotes an equivalent relation: When $\alpha \prec \beta, x \in W_\alpha, y \in W_\beta$, $x \sim y$ if $\phi_{\alpha,\beta}(x) = y$, where W_α denotes the space of k-representation π_α, and $\phi_{\alpha,\beta}$ denotes the morphism from π_α to π_β defined in the inductive system.

First, we prove that $i_{\mathrm{U},\theta}$ commutes with direct sum. By definition, $\oplus_{\alpha\in\mathcal{C}} i_{\mathrm{U},\theta}\pi_\alpha$ is a subrepresentation of $i_{\mathrm{U},\theta} \oplus_{\alpha\in\mathcal{C}} \pi_\alpha$, and the natural embedding is a morphism of k-representations of G. We will prove that the natural embedding is actually surjective. For any $f \in \pi := i_{\mathrm{U},\theta} \oplus_{\alpha\in\mathcal{C}} \pi_\alpha$, there exists an open compact subgroup K of G such that f is constant on each right K coset of MU\G. Furthermore, the function f is non-trivial on finitely many right K cosets. Hence there exists a finite index subset $J \subset \mathcal{C}$, such that $f(g) \in \oplus_{j\in J} W_j$, which means $f \in i_{\mathrm{U},\theta} \oplus_{j\in J} \pi_j$. Since $i_{\mathrm{U},\theta}$ commutes with finite direct sum, we finish this case.

The functor $i_{\mathrm{U},\theta}$ is exact, we have:

$$i_{\mathrm{U},\theta}(\varinjlim \pi_\alpha) \cong i_{\mathrm{U},\theta}(\oplus_{\alpha\in\mathcal{C}}\pi_\alpha)/i_{\mathrm{U},\theta}\langle x - y\rangle_{x\sim y}.$$

Notice that $\varinjlim i_{\mathrm{U},\theta}\pi_\alpha \cong \oplus i_{\mathrm{U},\theta}\pi_\alpha/\sim$, where $\sim$ denotes the equivalent relation: When $\alpha \prec \beta, f_\alpha \in V_\alpha, f_\beta \in V_\beta$, where V_α is the space of k-representation $i_{\mathrm{U},\theta}\pi_\alpha$, then $f_\alpha \sim f_\beta$ if $i_{\mathrm{U},\theta}(\phi_{\alpha,\beta})(f_\alpha) = f_\beta$, which is equivalent to say that $\phi_{\alpha,\beta}(f_\alpha(g)) = f_\beta(g)$ for any $g \in \mathrm{G}$. In left to prove that the natural isomorphism from $\oplus_{\alpha\in\mathcal{C}}(i_{\mathrm{U},\theta}\pi_\alpha)$ to $i_{\mathrm{U},\theta}(\oplus_{\alpha\in\mathcal{C}}\pi_\alpha)$, induces an isomorphism from $\langle f_\alpha - f_\beta\rangle_{f_\alpha\sim f_\beta}$ to $i_{\mathrm{U},\theta}\langle x - y\rangle_{x\sim y}$. This can be checked directly through definition as in the case of direct sum above. □

Theorem A.0.4 (Bernstein, Zelevinsky)**.** *Under the conditions above, the functor* $\mathrm{F} = r_{\mathrm{V},\psi} \circ i_{\mathrm{U},\theta} : \mathrm{Rep}_k(\mathrm{M}) \to \mathrm{Rep}_k(\mathrm{N})$ *is glued from the functor* Z *runs through all* Q*-orbits on* X*. More precisely, if orbits are numerated so that all sets* $\mathrm{Y}_i = \mathrm{Z}_1 \cup ... \cup \mathrm{Z}_i$ $(i = 1, ..., k)$ *are open in* X*, then there exists a filtration* $0 = \mathrm{F}_0 \subset \mathrm{F}_1 \subset ... \subset \mathrm{F}_k = \mathrm{F}$ *such that* $\mathrm{F}_i/\mathrm{F}_{i-1} \cong \Phi_{\mathrm{Z}_i}$*.*

The quotient space $\mathrm{X} = \mathrm{P}\backslash\mathrm{G}$ is locally compact totally disconnected. Let Y be a Q-invariant open subset of X. We define the subfunctor $\mathrm{F}_\mathrm{Y} \subset \mathrm{F}$. Let ρ be a k-representation of M, and W be its representation space. We denote $i(W)$ the representation k-space of $i_{\mathrm{U},\theta}(\rho)$. Let $i_\mathrm{Y}(W) \subset i(W)$ the subspace consisting of functions which are equal to 0 outside the set PY, and τ, τ_Y be the k-representations of Q on $i(W)$ and $i_\mathrm{Y}(W)$. Put $\mathrm{F}_\mathrm{Y}(\rho) = r_{\mathrm{V},\psi}(\tau_\mathrm{Y})$, which is a k-representation of N. The functor F_Y is a subfunctor of F due to the exactness of $r_{\mathrm{V},\psi}$.

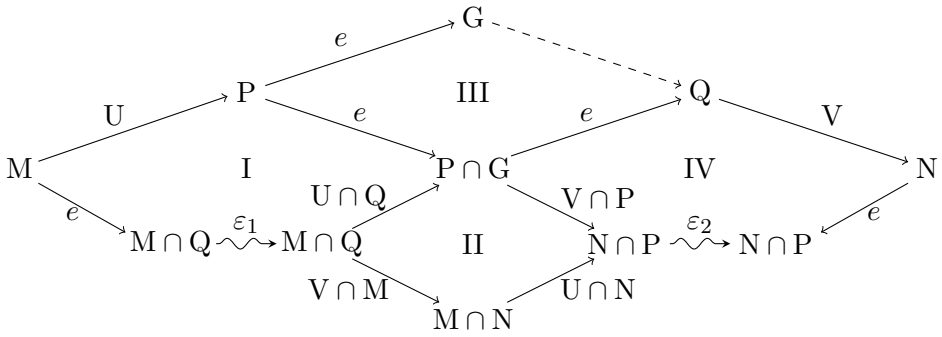


Figure A.1: BZ

Proposition A.0.5. *Let* Y, Y′ *be two* Q*-invariant open subset in* X*, we have:*

$$\mathrm{F}_{\mathrm{Y}\cap\mathrm{Y}'} = \mathrm{F}_{\mathrm{Y}} \cap \mathrm{F}_{\mathrm{Y}'}, \quad \mathrm{F}_{\mathrm{Y}\cup\mathrm{Y}'} = \mathrm{F}_{\mathrm{Y}} + \mathrm{F}_{\mathrm{Y}'}, \quad \mathrm{F}_{\emptyset} = 0, \quad \mathrm{F}_{\mathrm{X}} = \mathrm{F}.$$

Proof. Since $r_{\mathrm{V},\psi}$ is exact, it is sufficient to prove similar formulae for τ_{Y}. The only non-trivial one is the equality $\tau_{\mathrm{Y}\cup\mathrm{Y}'} = \mathrm{F}_{\mathrm{Y}} + \mathrm{F}_{\mathrm{Y}'}$. As in §1.3 [**?**], set K a compact open subgroup of $\mathrm{Y}\cup\mathrm{Y}'$, there exists φ and φ', which are idempotent k-function on Y and Y′, such that $(\varphi+\varphi')|_K = 1$. We deduce the result from this fact. □

Let Z be any Q-invariant locally closed set in X, we define the functor

$$\Phi_{\mathrm{Z}} : \mathrm{Rep}_k(\mathrm{M}) \to \mathrm{Rep}_k(\mathrm{N})$$

to be the functor $\mathrm{F}_{\mathrm{Y}\cup\mathrm{Z}}/\mathrm{F}_{\mathrm{Y}}$, where Y can be any Q-invariant open set in X such that $\mathrm{Y}\cup\mathrm{Z}$ is open and $\mathrm{Y}\cap\mathrm{Z} = \emptyset$. Let $\mathrm{Z}_1, ..., \mathrm{Z}_k$ be the numeration of Q-orbits on X as in A.0.4, and let $\mathrm{F}_i = \mathrm{F}_{\mathrm{Y}_i}$ $(i = 1, ..., k)$, which is a filtration of the functor F be the definition. To prove Theorem A.0.4, it is sufficient to prove that $\mathrm{F}_{\mathrm{Z}_i} \cong \Phi_{\mathrm{Z}_i}$.

By replace P to $\omega(\mathrm{P})$, we could assume that $\omega = \mathbb{1}$. Now we consider the diagram in figure BZ.

This is the same diagram as in §5.7 [BeZe], in which a group a point H means $\mathrm{Rep}_k(\mathrm{H})$, an arrow $\overset{\mathrm{H}}{\nearrow}$ means the functor $i_{\mathrm{H},\theta}$, an arrow $\underset{\mathrm{H}}{\searrow}$ means the functor $i_{\mathrm{H},\psi}$, and an arrow $\overset{\varepsilon}{\rightsquigarrow}$ means the functor ε (consult §5.1 [BeZe] for the definition of ε). Notice that G ⇢ Q does not mean any functor, but the functor P → G ⇢ Q is well-defined as explained above A.0.5. The composition functors along the highest path is of this diagram is F_{Z}, and if the condition $(*)$ holds, the composition functors along the lowest path is Φ_{Z}. We prove Theorem A.0.4 by showing that this diagram is commutative if condition $(*)$ holds, and F_{Z} equals 0 otherwise. Notice that parts I, II, III, IV are four cases of A.0.4, and we prove the statements through verifying them under the four cases respectively.

Let ρ be any k-representation of M, and W is its representation space. We use π to denote $\mathrm{F}_{\mathrm{Z}}(\rho)$, and τ to denote $\Phi_{\mathrm{Z}}(\rho)$.

Case I: $\mathrm{P} = \mathrm{G}, \mathrm{V} = \{e\}$. The k-representations π and τ act on the same space W, and the quotient group $\mathrm{M}\backslash(\mathrm{P} \cap \mathrm{Q})$ is isomorphic to $(\mathrm{M} \cap \mathrm{Q})\backslash(\mathrm{P} \cap \mathrm{Q})$. We verify directly by definition that $\pi \cong \tau$.

Case II: $\mathrm{P} = \mathrm{G} = \mathrm{Q}$. The representation space of π is still W. We have the equation:

$$r_{\mathrm{V},\psi}(W) \cong r_{\mathrm{V}\cap\mathrm{M},\psi}(r_{\mathrm{V}\cap\mathrm{U},\psi}(W)).$$

If $\theta|_{\mathrm{U}\cap\mathrm{V}} \neq \psi|_{\mathrm{U}\cap\mathrm{V}}$, then $\pi = 0$ since $\mathrm{U} \cap \mathrm{V} = \mathrm{U} \cap \mathrm{Q} \cap \mathrm{V} \cap \mathrm{P}$ and $r_{\mathrm{V}\cap\mathrm{U},\psi}(W) = 0$. This means that after proving the diagrams of cases $\mathrm{I}, \mathrm{III}, \mathrm{IV}$ are commutative, the functor $\mathrm{F_Z}$ equals 0 if condition $(*)$ does not hold.

Now we assume that $(*)$ holds. The k-representations π and τ act on the same space $W/W(\mathrm{V} \cap \mathrm{M}, \psi)$, because the fact $r_{\mathrm{V}\cap\mathrm{U},\psi}(i_{\mathrm{V}\cap\mathrm{U},\theta}(W)) = W$ and the equation above. Notice that we have equations for k-character mod:

$$\mathrm{mod_U} = \mathrm{mod}_{\mathrm{U}\cap\mathrm{M}} \cdot \mathrm{mod}_{\mathrm{U}\cap\mathrm{V}}, \quad \mathrm{mod_V} = \mathrm{mod}_{\mathrm{V}\cap\mathrm{N}} \cdot \mathrm{mod}_{\mathrm{V}\cap\mathrm{U}},$$

from which we deduce that $\pi \cong \tau$ when condition $(*)$ holds.

Case III: $\mathrm{U} = \mathrm{V} = \{e\}$. Let $i(W)$ be the representation space of $i_{\mathrm{P}}^{\mathrm{G}}\rho$, then π acts on a quotient space W_1 of $i(W)$. Let:

$$E = \{f \in i(W) | f(\overline{\mathrm{PQ}}\backslash\mathrm{PQ}) = 0\},$$

$$E' = \{f \in i(W) | f(\overline{\mathrm{PQ}}) = 0\},$$

then $W_1 \cong E/E'$. The k-representation τ acts on $i(W)'$, which is the representation space of $i_{\mathrm{P}\cap\mathrm{Q}}^{\mathrm{Q}}\rho$. By definition,

$$i(W)' = \{h : \mathrm{Q} \to W | h(pq) = \rho(p)h(q), p \in \mathrm{P} \cap \mathrm{Q}, q \in \mathrm{Q}\}.$$

We define a morphism γ from W_1 to $i(W)'$, by sending f to $f|_{\mathrm{Q}}$, which respects Q-actions and is actually a bijection. For injectivity, let $f_1, f_2 \in W_1$ and $f_1|_{\mathrm{Q}} = f_2|_{\mathrm{Q}}$, then $f_1 - f_2$ is trivial on PQ, hence $f_1 - f_2$ is trivial on $\overline{\mathrm{PQ}}$ by the definition of E. This means $f_1 - f_2 = 0$ in W_1. Now we prove γ is surjective. Let $h \in i(W)'$, there exists an open compact subgroup K' of $(\mathrm{P} \cap \mathrm{Q})\backslash\mathrm{Q}$ such that h is constant on the right K' cosets of $(\mathrm{P} \cap \mathrm{Q})\backslash\mathrm{Q}$, and denote S the compact support of h. Let K be an open compact subgroup of $\mathrm{P}\backslash\mathrm{G}$ such that $(\mathrm{P} \cap \mathrm{Q})\backslash(\mathrm{Q} \cap K) \subset K'$, and $S \cdot K \cap (\overline{\mathrm{PQ}}/\mathrm{PQ}) = \emptyset$. We define f such that f is constant on the right K cosets of $\mathrm{P}\backslash\mathrm{G}$, and $f|_{(\mathrm{P}\cap\mathrm{Q})\backslash\mathrm{Q}} = h$. The function f is smooth with compact support on the complement of $\overline{\mathrm{PQ}}/\mathrm{PQ}$, hence belongs to E, and $\gamma(f) = h$ as desired.

Case IV: $\mathrm{U} = \{e\}, \mathrm{Q} = \mathrm{G}$. We divide this case into two cases IV_1 and IV_2 as in the diagram of figure CaseIV.

Case IV_1: $\mathrm{U} = \{e\}, \mathrm{Q} = \mathrm{G}, \mathrm{V} \subset \mathrm{M} = \mathrm{P}$. The k-representation π acts on $i(W)^+ = i(W)/i(W)(\mathrm{V}, \psi)$, where

$$i(W)_{\mathrm{V},\psi} = \langle vf - \psi(v)f, \forall f \in i(W), v \in \mathrm{V}\rangle.$$

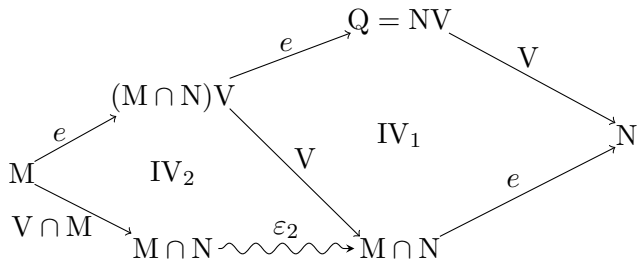


Figure A.2: Case IV

The k-representations τ acts on $i(W^+)$, which is the smooth functions with compact support on $(\mathrm{M}\cap\mathrm{N})\backslash\mathrm{N}$ defined as below:

$$\{h:\mathrm{N}\to W/W(\mathrm{V},\psi)|f(mn)=\rho(m)f(n),\forall m\in\mathrm{M}\cap\mathrm{N},n\in\mathrm{N}\}.$$

There is a surjective projection from $i(W)$ to $i(W^+)$, which projects $f(n)$ in $W^+ = W/W_{\mathrm{V},\psi}$, for any $f\in i(W)$. In fact, let $h\in i(W^+)$, there exists an open compact subgroup K of $\mathrm{P}\backslash\mathrm{G}\cong(\mathrm{M}\cap\mathrm{N})\backslash\mathrm{N}$, such that $f=\sum_{i=1}^m h_i$, $m\in\mathbb{N}$, where $h_i\in i(W^+)$ is nontrivial on one right K coset a_iK of $\mathrm{P}\backslash\mathrm{G}$. We have $h_i\equiv\overline{w_i}$ on a_iK, where $w_i\in W$ and $\overline{w_i}\in W^+$. Define $f=\sum_{i=1}^m f_i$, where $f_i\equiv w_i$ on a_iK, and equals 0 otherwise. The function $f\in i(W)$, and the projection image is h.

It is clear that this projection induces a morphism from $i(W)^+$ to $i(W^+)$, and we prove this morphism is injective. Let $f,f'\in i(W)^+$, and $f=f'$ in $i(W^+)$. As in the proof above, there exists an open compact subgroup K_0 of $\mathrm{P}\backslash\mathrm{G}$, and $f_j\in i(W)^+$ such that f_j is non-trivial on one right K_0 coset of $\mathrm{P}\backslash\mathrm{G}$ and $f-f'=\sum_{j=1}^s f_j$. Furthermore, the supports of f_j's are two-two disjoint. Hence the image of f_j on its support is contained in W^+, since f_j is constant on its support, it equals 0 in $i(W)^+$, whence $f-f'$ equals 0 in $i(W)^+$. We conclude that this morphism is bijection, and the diagram case IV_1 is commutative.

Case IV_2: $\mathrm{U}=\{e\},\mathrm{G}=\mathrm{Q},\mathrm{N}\subset\mathrm{M}$. In this case:

$$\mathrm{X}=\mathrm{NV}'\backslash\mathrm{NV}\cong\mathrm{V}'\mathrm{V},$$

where $\mathrm{V}'=\mathrm{V}\cap\mathrm{M}$. We choose one Haar measures μ of X (the existence see §I, 2.8, [V1]). Let W^+ denote the quotient $W/W(\mathrm{V}',\psi)$ and p the canonical projection $p:W\to W^+$. Let $i(W)$ be the space of $k-$representation $\tau=i_{\{e\},1}(\rho)$.

Define $\overline{A}$ a morphism of k-vector spaces from $i(W)$ to W^+ by:

$$\overline{A}f=\int_{\mathrm{V}'\backslash\mathrm{V}}\psi^{-1}(v)p(f(v))\mathrm{d}\mu(v).$$

This is well defined since the function $\psi^{-1}f$ is locally constant with compact support of V'/V, and the integral is in fact a finite sum. Since μ is stable by right translation, we have for any $v\in\mathrm{V}$:

$$\overline{A}(\tau(v,f))=\psi(v)\overline{(A)}(f).$$

$$\begin{array}{ccc} i(W)'/i(W)'(\mathrm{V},\psi) & \xrightarrow{A} & W^+ \\ \cong\downarrow & & \downarrow\cong \\ i(W)/i(W)(\mathrm{V},\psi) & \xrightarrow{A} & W^+ \end{array}$$

Figure A.3: IV_2

Hence $\overline{A}$ induces a morphism of k-vector spaces:

$$A : i(W)/i(W)(V,\psi) \to W^+.$$

Now we justify that $A \in \mathrm{Hom}_{k[\mathrm{N}]}(\pi,\tau)$, where k-representations $\pi = r_{\mathrm{V},\psi}(\tau)$ equals $\mathrm{F}(\rho)$, and $\tau = \varepsilon_2 \cdot r_{\mathrm{V}',\psi}(\rho)$ equals $\Phi(\rho)$. For any $n \in \mathrm{N}$:

$$A(\pi(n)f) = \mathrm{mod}_{\mathrm{V}}^{-\frac{1}{2}}(n)\int_{\mathrm{V}'/\mathrm{V}} \psi^{-1}(v)p(f(vn))\mathrm{d}\mu(v) \tag{A.1}$$

$$= \mathrm{mod}_{\mathrm{V}}^{-\frac{1}{2}}(n)\sigma(n)\mathrm{mod}_{\mathrm{V}'}^{\frac{1}{2}}(n)\varepsilon_2^{-1}\cdot\int_{\mathrm{V}'/\mathrm{V}} \psi^{-1}(v)p(f(n^{-1}vn))\mathrm{d}\mu(v) \tag{A.2}$$

By replacing $v' = n^{-1}vn$, the equation above equals to:

$$\sigma(n)\int_{\mathrm{V}'/\mathrm{V}} \psi^{-1}(v')p(f(v'))\mathrm{d}\mu(v') = \sigma(n)A(f).$$

Therefore A belongs to $\mathrm{Hom}_{k[N]}(\pi,\tau)$, and hence a morphism from functor F to Φ. Now we prove that A is an isomorphism.

Let ρ' be the trivial representations of $\{e\}$ on W, then $i(W)'$ the space of k-representation $\mathrm{ind}_{\mathrm{V}'}^{\mathrm{V}}\rho'$ is isomorphic to $i(W)$ the space of k-representation $i_{\{e\},1}\rho$. And the diagram A IV_2 is commutative, where A indicates the morphism of k-vector spaces associated to the functor A. Hence it is sufficient to suppose that $\mathrm{N} = \{e\}, \mathrm{M} = \mathrm{V}'$. Replacing ρ by $\psi^{-1}\rho$, we can suppose that $\psi = 1$.

First of all, we consider $\rho = i_{\{e\},1}\mathbb{1} = \mathrm{ind}_e^{\mathrm{V}'}\mathbb{1}$ the regular k-representation on $S(\mathrm{V}')$, which is the space of locally constant functions with compact support on V'. Then $\tau = i_{\{e\},1}\rho$ is the regular k-representation of V on $S(\mathrm{V})$ by the transitivity of induction functor. Any k-linear form on $r_{\mathrm{V}',1}(S(\mathrm{V}'))$ gives a Haar measure on V', and conversely any Haar measure gives a k-linear form on $S(\mathrm{V}')$, whose kernel is $S(\mathrm{V}')(\mathrm{V}',1)$, hence the two spaces is isomorphic, and the uniqueness of Haar measures implies that the dimension of $r_{\mathrm{V}',1}(S(\mathrm{V}'))$ equals one. We obtain the same result to $r_{\mathrm{V},1}(S(\mathrm{V}'))$. Since in this case the morphism A is non-trivial, then it is an isomorphism. The functors $i_{\{e\},1}, r_{\mathrm{V},\psi}, r_{\mathrm{V}',\psi}$ commute with direct sum (as in A.0.3), and the morphism A between k-vector spaces also commutes with direct sum, hence $A : \pi \to \tau$ is an isomorphism when ρ is free, which means ρ is a direct sum of regular k-representations of V'. Notice that any ρ can be viewed as a module over Heck algebra, then ρ is a quotient of some free k-representation. Hence ρ has a free resolution. The exactness of F and Φ implies that $A : \mathrm{F}(\rho) \to \Phi(\rho)$ is an isomorphism for any ρ.

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE RENNES 1

Titre : Représentations modulo ℓ des groupes p-adiques $\mathrm{SL}_n(F)$

Mot clés : représentations modulo ℓ ; groupes spéciaux linéaires p-adiques ; support supercuspidal ; types de Bushnell-Kutzko.

Resumé : Fixons un nombre premier p. Soit k un corps algébriquement clos de caractéristique $\ell \neq p$. Nous construisons les k-types maximaux simples cuspidaux des sous-groupes de Levi M' de $\mathrm{SL}_n(F)$, où F est un corps local non archimédien de caractéristique résiduelle p. Nous montrons que le support supercuspidal des k-représentations lisses irréductibles de M' est unique à M'-conjugaison près, quand F est soit un corps fini de caractéristique p soit un corps local non-archimédien de caractéristique résiduelle p.

Title : Modulo ℓ-representations of p-adic groups $\mathrm{SL}_n(F)$

Keywords : modular ℓ representations ; p-adic special linear groups ; supercuspidal support ; Bushnell-Kutzko types

Abstract : Fix a prime number p. Let k be an algebraically closed field of characteristic $\ell \neq p$. We construct maximal simple cuspidal k-types of Levi subgroups M' of $\mathrm{SL}_n(F)$, where F is a non-archimedean locally compact field of residual characteristic p. And we show that the supercuspidal support of irreducible smooth k-representations of Levi subgroups M' of $\mathrm{SL}_n(F)$ is unique up to M'-conjugation, when F is either a finite field of characteristic p or a non-archimedean locally compact field of residual characteristic p.