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Seyed-Meghdad Mirebrahimi

INTERACTING STOCHASTIC SYSTEMS WITH INDIVIDUAL AND COLLECTIVE REINFORCEMENT

Directeur de Thèse : **Pierre-Yves Louis**

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Rapporteurs :	**Giacomo Aletti**	Professeur, University of Milan
	Peggy Cénac-Guesdon	Maître de conférences (HDR), Université de Bourgogne
	Cécile Mailler	Lecturer, University of Bath
Examinateurs :	**Marc Arnaudon**	Professeur, Université de Bordeaux
	Hermine Biermé	Professeur, Université de Poitiers
	Pierre-Yves Louis	Maître de conférences (HDR), Université de Poitiers
	Julien Michel	Professeur, Université de Poitiers
	Yousri Slaoui	Maître de conférences (HDR), Université de Poitiers

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To Nasim

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Abstract

The Pólya urn is the paradigmatic example of a reinforced stochastic process. It leads to a random (non degenerated) almost sure (a.s.) time-limit. The Friedman urn is a natural generalization whose a.s. time-limit is not random anymore. Many stochastic models for applications are based on reinforced processes, like urns with their use in adaptive design for clinical trials or economy, stochastic algorithms with their use in non parametric estimation or optimization. In this work, in the stream of previous recent works, we introduce a new family of (finite) systems of reinforced stochastic processes, interacting through an additional collective reinforcement of mean field type. The two reinforcement rules strengths (one componentwise, one collective) are tuned through (possibly) different rates. In the case the reinforcement rates are like n^{-1}, these reinforcements are of Pólya or Friedman type as in urn contexts and lead to limits which may be random or not. We state two kind of mathematical results. Different parameter regimes needs to be considered: type of reinforcement rule (Pólya/Friedman), strength of the reinforcement (γ in the asymptotics behavior of $(r_n)_n$). We study the time-asymptotics and prove that a.s. convergence always holds. Moreover all the system's components share the same time-limit (synchronization). The nature of the limit (random/deterministic) according to the parameters' regime is considered. We then study fluctuations by proving central limit theorems. Rates of convergence (given by scaling coefficients) vary according to the regime considered.

Keywords. Reinforced stochastic processes; Interacting random systems; Almost sure convergence; Central limit theorems; Synchronization; Fluctuations

Abstract

Résumé en français de la thèse

1 Introduction

Un phénomène de *renforcement* dans un processus stochastique se dit lorsque la probabilité d'obtenir un événement augmente avec le nombre de réalisations antérieures. Autrement dit, plus un événement se produit, plus il a des chances de se produire. Ce phénomène est présent en physique, biologie, sciences sociales, réseaux aléatoires [57]. De nombreux modèles aléatoires sont fondés sur des processus avec renforcement comme pour la conception d'essais cliniques au design adaptatif [66], en économie [73], ou pour des algorithmes stochastiques à des fins d'optimisation ou d'estimation non paramétrique [11, 23, 24, 48, 77, 78]. Dans [4, 32] une interprétation comme modèles pour des dynamiques d'opinion est présenté. L'urne de Pólya (à 2 couleurs) est l'exemple typique de processus stochastique avec renforcement [81], [25, Chap. 7]. La limite presque sûre (p.s.) en temps de la proportion d'une couleur existe, est aléatoire et non dégénérée [63]. L'urne de Friedman [47] est une généralisation naturelle dont la proportion limite n'est plus aléatoire [46].

Les modèles d'urnes généralisées peuvent exhiber des comportements différents. Par exemple, on peut considérer un modèle où la proportion Z_n est définie par récurrence par

$$\forall n \in \mathbb{N}, \qquad Z_{n+1} = (1 - r_n)\, Z_n + r_n\, \xi_{n+1}, \quad Z_0 = \frac{1}{2} \tag{1}$$

où la loi (conditionnellement au passé) de la variable aléatoire ξ_{n+1} est une loi de Bernoulli dont le paramètre provient d'une transformation de Z_n telle que

$$\mathbb{P}(\xi_{n+1} = 1 \mid \mathcal{F}_n) = \varphi(Z_n)$$

où φ est une application de $[0,1]$ dans $[0,1]$, et où $r_n := (n+3)^{-1}$ et $\mathcal{F}_n := \sigma\{Z_1, \ldots, Z_n\}$. Il est prouvé dans [56] que si φ est continue alors $(Z_n)_n$ converge a.s. et sa limite Z_∞ est telle que $\varphi(Z_\infty) = Z_\infty$ p.s. On remarque que le cas $\varphi(x) = x$ est celui de l'urne de Pólya et que le cas $\varphi(x) = 1 - x$ correspond à l'évolution de la proportion dans le cas d'une règle de renforcement de type Friedman. Dans [56] est donné un exemple de non convergence (cas où φ est discontinue, présence d'oscillations)

Un autre phénomène présent dans la nature et dans plusieurs champs scientifiques est celui de la *synchronisation*. Dans le cas d'un système de dynamiques, l'interaction entre les composantes peut faire émerger une comportement collectif qui se traduit par des comportements similaires sur les composantes. Dans [36], un système (fini) d'urnes de Pólya

en interaction de type champ moyen (proportion moyenne sur toutes les composantes du système) a été introduit et étudié. Un certain type de synchronisation a été prouvé, dans le sens où il y a convergence presque sûre vers une limite aléatoire commune. Des variations ont été considérées dans [71]. Les vitesses de convergence ont été prouvéees grâce à des théorèmes centraux limite [33] (étude des fluctuations). Le cas des urnes de Friedman en interaction a été considérées dans [89]. Lorsque le taux de renforcement r_n est généralisé, d'un comportement asymptotique $r_n \sim cn^{-1}$ (usuellement dans les contextes d'urnes) à $r_n \sim cn^{-\gamma}$ ($\gamma \in]\frac{1}{2}; 1]$, usuel dans le contexte d'algorithmes stochastiques). Dans [32], différents cas de renforcement (type Pólya/Friedman) sont considérés. La synchronisation p.s. a lieu. Et les vitesses de convergences sont étudiées grâce à des théorèmes centraux limite de type fonctionnel (FCLT). Dans les cas où la limite est aléatoire, la synchronisation a lieu plus rapidement que la convergence à l'asymptotique. L'interaction a été généralisée du champ moyen au cas d'une interaction donnée par un graphe pondéré dans [4]. La synchronisation des moyennes empiriques des variables aléatoires de renforcement $(\xi_n(i))_n$ (éventuellement pondérées) a également été prouvée dans [3, 5].

Dans ce mémoire, inspirés par de nombreux articles récents [3–6, 32, 33, 36, 89], et plus particulièrement sur les modèles d'urnes [33, 36, 61, 89], et en écho avec le théorème de [56] rappelé précédemment pour le modèle avec l'application φ défini par (1), nous introduisons une nouvelle famille de systèmes (finis) $\{(Z_n(i))_n,\ 1 \leq i \leq N\}$ de processus de renforcement (*i.e.* avec auto-renforcement ou renforcement individuel) où l'interaction entre les composantes du système se traduit par un phénomène de renforcement collectif additif, de type champ moyen (proportion moyenne dans le système). Les deux taux de renforcement peuvent être différents : l'un $r_n^l \sim \frac{c_1}{n^{\gamma_1}}$ spécifique à chaque composante (auto-renforcement), et l'autre $r_n^g \sim \frac{c_2}{n^{\gamma_2}}$ collectif (global) est commun à toutes les composantes [1]. Si $\gamma_1 < \gamma_2$ (par exemple), on pourrait s'attendre naïvement, puisque l'interaction/le renforcement collectif, serait négligeable, relativement à l'auto-renforcement, à ce qu'il n'y ait pas de synchronisation. Nous prouvons que cela n'a pas lieu. En effet, la synchronisation a lieu au sens L^2 et presque sûrement.

Plus généralement, nous prouvons deux types de résultats mathématiques. Nous prouvons l'existence d'une limite p.s. commune à toutes les composantes du système (synchronisation). La nature de la limite (aléatoire/déterministe) est étudiée en fonction du régime de paramètres. Différents régimes de paramètres doivent être considérés : type de la règle (brièvement, Pòlya/Friedman), taux r_n^l, r_n^g des renforcement. Nous étudions également les fluctuations en prouvant des théorèmes centraux de la limite. Les changements d'échelle varient en fonction du régime de paramètres considéré. Différentes vitesses de convergence sont ainsi établies et même dans certains cas où la limite commune est déterministe, la synchronisation peut avoir lieu plus rapidement que la convergence à l'équilibre ce qui n'avait pas été observé au préalable.

[1] Pour des raisons techniques, l'hypothèse sur le comportement asymptotique est renforcée par (3.12)

2 Plan de la thèse

La thèse est organisée de la manière suivante. Dans le chapitre 2, nous présentons des modèles d'urne comme point de départ pour les processus stochastiques renforcés et présentons des généralisations en lien avec des applications, en particulier en lien avec les algorithmes stochastiques.

Dans le chapitre 3, nous considérons des systèmes de dynamiques stochastiques de renforcement où l'interaction provient d'un mécanisme supplémentaire de renforcement collectif. Après avoir présenté différentes familles de modèles considérées récemment, nous introduisons une nouvelle famille de systèmes définis par (3.9) et reproduite dans la section suivante.

Dans le chapitre 4, nous rappelons la définition d'un processus stochastique qui est une quasi-martingale. Nous prouvons que les $(Z_n(i)_n$ sont des quasi-martingales grâce à la vitesse de convergence L^2. La convergence presque sûre vers une limite Z_∞, identique pour toutes les composantes $(Z_n(i))_n$ du système est ensuite prouvée. Deux situations doivent être distinguées : Théorèmes 4.3 et 4.4 correspondent au cas où cette limite est déterministe ($Z_\infty = \frac{1}{2}$). Parler de "synchronisation" est dans ce cas abusif. Théorème 4.6 correspond au cas où cette limite Z_∞ est aléatoire (cas $\lambda_1 = \lambda_2 = 0$).

Dans le chapitre 5, dans chacun des différents cas selon que λ_1, λ_2 sont nuls ou non, nous prouvons des théorèmes centraux de la limite qui permettent de caractériser l'échelle des fluctuations aléatoires de $(Z_n - Z_\infty)_n$ et $(Z_n - Z_n(i))_n$. Différents facteurs d'échelles apparaissent. Les théorèmes 5.4 et 5.6 regroupent les résultats lorsque la limite Z_∞ est déterministe. Le théorème 5.7 est dans le cas où la limite Z_∞ est aléatoire.

Dans le chapitre 6, des remarques de conclusions sont présentées ainsi que des développements possibles. En annexe, nous présentons un lemme A.2 important sur le comportement asymptotique d'une suite déterministe définie par récurrence à l'aide d'une fonction affine, non homogène dans le temps.

Les principaux résultats des chapitres 4 et 5 sont l'objet d'un article soumis pour publication [72], actuellement en révision.

3 Famille introduite et étudiée de systèmes de processus stochastiques renforcés

Nous présentons ci après brièvement, en français, les résultats de la thèse. Nous rappelons tout d'abord les résultats suivants à propos des urnes de Pólya et Friedman qui sont redéfinies dans le chapitre 2.

Proposition 1 (Comportement asymptotique en temps de l'urne de Pólya, [63])**.**
La suite de variables aléatoires $(Z_n)_{n\geq 0}$ *qui décrit l'évolution de la proportion d'une couleur dans une urne de Polya bicolore est une martingale bornée qui de ce fait converge presque sûrement (p.s.)*

$$\lim_{n\to\infty} Z_n = Z_\infty \quad p.s.$$

où de plus, Z_∞ suit une loi Beta sur $[0,1]$ de paramètres $(\frac{R_0}{c}, \frac{B_0}{c})$. L'entier c représente le nombre de boules de la couleur tirée, ajoutées dans l'urne à chaque tirage. Les entiers R_0 et B_0 représentent le nombre de boules de chaque couleur, initialement.

Proposition 2 (Comportement asymptotique en temps de l'urne de Friedman, [46])**.** *La suite de variables aléatoires $(Z_n)_{n\geq 0}$ qui décrit l'évolution de la proportion d'une couleur dans une urne de Friedman bicolore satisfait*

$$\lim_{n\to\infty} Z_n := Z_\infty = \frac{1}{2} \quad p.s.$$

Les systèmes stochastiques que nous introduisons et étudions sont définis dans le chapitre 3. Ils sont définis grâce aux équations de récurrence (3.9) qui sont reproduites ici. Chaque modèle est une famille de variables aléatoires $\{(Z_n(i))_n,\ 1 \leq i \leq N\}$ réelles, à valeurs dans $[0,1]$ où, sur chaque composante i, la dynamique $(Z_n(i))_n$ est définie par récurrence :

$$Z_{n+1}(i) = (1 - r_n^l - r_n^g)Z_n(i) + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g, \tag{2}$$

où $Z_0(i) = \frac{1}{2}$ et où $\xi_{n+1}^l(i)$ (resp. ξ_{n+1}^g) sont des variables aléatoires qui traduisent l'effet local (resp. collectif) de renforcement. Les variables aléatoires $(\xi_{n+1}^l(i))_n$ représentent l'auto-renforcement de la dynamique sur la composante i. Sachant les événements $\mathcal{F}_n$ du passé, jusqu'à l'instant n, ces variables aléatoires suivent des lois de Bernoulli indépendantes de paramètres, respectivement donnés par

$$\begin{aligned} \mathbb{P}(\xi_{n+1}^l(i) = 1 \mid \mathcal{F}_n) = \psi_1(Z_n(i)) := (1 - 2\lambda_1)Z_n(i) + \lambda_1, \\ \mathbb{P}(\xi_{n+1}^g = 1 \mid \mathcal{F}_n) = \psi_2(Z_n) := (1 - 2\lambda_2)Z_n + \lambda_2, \end{aligned} \tag{3}$$

où $(\lambda_1, \lambda_2) \in [0,1]^2$. On suppose par la suite, sauf mention contraire, que les suites $(r_n^l)_n$ et $(r_n^g)_n$ satisfont (3.12) :

$$r_n^l = \frac{c_1}{n^{\gamma_1}} + O(\frac{1}{n^{2\gamma_1}}) \quad and \quad r_n^g = \frac{c_2}{n^{\gamma_2}} + O(\frac{1}{n^{2\gamma_2}}).$$

avec $(\gamma_1, \gamma_2) \in]1/2, 1]^2$. En particulier, ces suites vérifient chacune (3.13)

$$\sum r_n = +\infty, \qquad \sum (r_n)^2 < +\infty.$$

4 Résultats de comportement en temps long

Dans le chapitre 4, nous prouvons la convergence au sens L^2 et presque sûre vers une limite $Z_\infty \in [0,1]$, identique pour toutes les composantes i du système (*synchronisation*). Plusieurs situations doivent être distinguées : Théorèmes 4.3 et 4.4 correspondent au cas où cette limite est déterministe ($Z_\infty = \frac{1}{2}$). Comme signalé précédemment, parler de "synchronisation" est dans ce cas un abus de langage.

Théorème 3 (Th. 4.3 et Th. 4.4 dans le mémoire)**.** *Dans tous les cas où* $\lambda_1 = \lambda_2 = 0$ *n'est* pas *vrai, la convergence presque sûre a lieu vers* $Z_\infty = \frac{1}{2}$ *i.e.*

$$\forall i = 1, ..., N, \qquad \lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2} \ p.s.$$

Le théorème 4.6 considère le cas où l'on démontre que Z_∞ est une variable aléatoire non dégénérée.

Théorème 4 (Th. 4.6 dans le mémoire)**.**
On suppose $\lambda_1 = \lambda_2 = 0$*. Le processus de champ moyen* $(Z_n)_n$ *converge presque sûrement une variable aléatoire* Z_∞ *telle que* $\mathbb{V}ar(Z_\infty) > 0$*. De plus, la synchronisation a lieu presque sûrement au sens, où pour tout* $i \in \{1, \dots, N\}$*,*

$$\lim_{n\to\infty} Z_n(i) = Z_\infty \quad p.s.$$

Ces théorèmes établissent également les vitesses de convergence dans L^2 de $(Z_n - Z_\infty)_n$ et $(Z_n(i) - Z_n)_n$. Ces résultats sont utiles pour la suite.

5 Résultats sur les vitesses de convergence (TCL/-fluctuations)

Dans le chapitre 5, dans chacun des différents cas selon que λ_1, λ_2 sont nuls ou non, nous prouvons des théorèmes centraux de la limite (TCL), au sens de la convergence stable (définie dans la secion 5.1), qui permettent de caractériser l'échelle des fluctuations aléatoires de $(Z_n - Z_\infty)_n$ et $(Z_n - Z_n(i))_n$. Différents facteurs d'échelles doivent être considérés. Le théorème 5.4 regroupe les résultats lorsque la limite Z_∞ est déterministe.

Théorème 5 (Th. 5.4 dans le mémoire)**.**
Soient $\lambda_1 > 0, \lambda_2 > 0$*. Soit* $\gamma := \min(\gamma_1, \gamma_2)$*.*

i) on a :

a) si $\gamma_1 \le \gamma_2$, $n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stable} \mathcal{N}\left(0, \tilde{\sigma}_1^2\right)$,

b) si $\gamma_2 < \gamma_1$, $n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stable} \mathcal{N}\left(0, \tilde{\sigma}_2^2\right)$.

ii) Lorsque $\gamma < 1$*, on a :*

$$n^{\frac{\gamma}{2}}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stable} \mathcal{N}\left(0, \hat{\sigma}^2\right).$$

iii) Lorsque $\gamma = 1$ *(ce qui signifie* $\gamma_1 = \gamma_2 = 1$*),*

a) si $(\lambda_1 + \lambda_2) > \frac{1}{4}$, $\sqrt{n}\left(Z_n - \frac{1}{2}\right) \xrightarrow[n\to\infty]{stable} \mathcal{N}\left(0, \sigma_1^{*2}\right)$.

b) si $(\lambda_1+\lambda_2)=\frac{1}{4}$, $\sqrt{\frac{n}{\ln n}}\left(Z_n-\frac{1}{2}\right)\xrightarrow[n\to\infty]{stable}\mathcal{N}\left(0,\sigma_2^{*^2}\right)$.

Le théorème 5.5 considère le cas particulier lorsque $\gamma_1=\gamma_2=1$. Il est à mettre en perspective avec l'urne de Friedman où il existe un régime de fluctuations non gaussiennes. L'hypothèse $\lambda_1+\lambda_2<\frac{1}{4}$ permet d'assurer de ne pas être dans cette situation.

Théorème 6 (Th. 5.5 dans le mémoire)**.**
Soient $\lambda_1>0,\lambda_2>0$ *tels que* $(\lambda_1+\lambda_2)<\frac{1}{4}$. *On suppose* $\gamma_1=\gamma_2=1$. *On a*

$$n^{4(\lambda_1+\lambda_2)}\left(Z_n-\frac{1}{2}\right)\xrightarrow[n\to\infty]{p.s./L^1}\tilde{X},$$

pour une variable aléatoire réelle $\tilde{X}$ *telle que* $\mathbb{P}(\tilde{X}\neq 0)>0$.

Le théorème 5.6 considère le cas où les renforcements individuels et collectifs sont de type différents (Pólya/Friedman dans un sens généralisé, c'est à dire lorsque $r_n\sim n^{-\gamma}$ avec $\gamma\in]\frac{1}{2},1[$).

Théorème 7 (Th. 5.6 dans le mémoire)**.**
Dans les cas suivants : $(\lambda_1>0,\ \lambda_2=0)$ *ou* $(\lambda_1=0,\ \lambda_2>0)$, *on a la convergence* stable *vers une distribution gaussienne pour* $(Z_n-Z_n(i))_n$ *et* $(Z_n-\frac{1}{2})_n$ *avec le changement d'échelle adéquat. La première table considère le cas* $\gamma:=min(\gamma_1,\gamma_2)<1$.

	$\lambda_1\neq 0,\ \lambda_2=0$	$\lambda_1=0,\ \lambda_2\neq 0$
$\gamma_1\leq\gamma_2$	$n^{\frac{\gamma_1}{2}}(Z_n-Z_n(i))\xrightarrow{stable}\mathcal{N}\left(0,\tilde{\sigma}_3^2\right)$ $n^{\frac{\gamma_1}{2}}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\hat{\sigma}_1^2\right)$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stable}\mathcal{N}\left(0,\tilde{\sigma}_4^2\right)$ $n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\hat{\sigma}_2^2\right)$
$\gamma_2<\gamma_1$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stable}\mathcal{N}\left(0,\tilde{\sigma}_5^2\right)$ $n^{\frac{2\gamma_2-\gamma_1}{2}}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\hat{\sigma}_3^2\right)$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stable}\mathcal{N}\left(0,\tilde{\sigma}_6^2\right)$ $n^{\frac{\gamma_2}{2}}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\hat{\sigma}_4^2\right)$

La deuxième table ci-après considère le cas $\gamma=1$.

$\lambda_i=0,\ \lambda_j>\frac{1}{4}$	$\lambda_i=0,\ \lambda_j=\frac{1}{4}$	$\lambda_i=0,\ \lambda_j<\frac{1}{4}$
$\sqrt{n}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\sigma_3^{*^2}\right)$	$\sqrt{\frac{n}{\ln n}}(Z_n-\frac{1}{2})\xrightarrow{stable}\mathcal{N}\left(0,\sigma_4^{*^2}\right)$	$n^{4(\lambda_1+\lambda_2)}(Z_n-\frac{1}{2})\xrightarrow{p.s./L^1}\tilde{\chi}$

Les variances asymptotiques sont précisées dans le chapitre 5.

Le théorème 5.7 considère le cas où les renforcements locaux et le renforcement collectif conduisent à une limite aléatoire (renforcement de type Pólya généralisé en local et en collectif).

Théorème 8 (Th. 5.7 dans le mémoire)**.**
On suppose $\lambda_1=\lambda_2=0$.

(i) On a

$$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stable} \mathcal{N}\Big(0, \vartheta\ Z_\infty(1-Z_\infty)\Big).$$

(ii) Avec $\gamma := \min(\gamma_1, \gamma_2)$*, on a*

$$n^{\frac{2\gamma-1}{2}}(Z_n - Z_\infty) \xrightarrow[n\to\infty]{stable} \mathcal{N}\Big(0, \vartheta\ Z_\infty(1-Z_\infty)\Big).$$

Les sections 5.3, 5.4, 5.5 et 5.6 contiennent les preuves des différents TCL.

Résumé en français de la thèse

Chapter 1

Introduction

A basic model of *self-reinforcement* is the well known Pólya urn. In Physics, Biology or social science and economy, reinforcement is defined as an action which increases the frequency of a certain behavior. We may define a reinforced process as a stochastic process where an event which has occurred many times in the past has a higher probability to occur in the future. A survey on this kind of processes is [81]. There is a big variety of reinforced processes, urn models belong to them. The Pólya urn is the simplest. We briefly describe it below. See Chap. 7 in [25] too for an introduction. At time 0, an urn contains R_0 red balls and B_0 black balls. At each discrete time step $n > 0$, a ball is drawn out and it is replaced in the urn together with c balls of the same color (so called Pólya reinforcement rule). We denote by Z_n the proportion of red balls in the urn at time n. According to the reinforcement rule, it plays a role as parameter of the probability to choose a red ball at time $n + 1$, given the past, which means, given the proportion Z_n of the red balls at time n. One is interested in the distribution of Z_n when n is large. An easy calculation shows that $(Z_n)_{n\geq 0}$ is a bounded *martingale*, thus, according the general martingale's theorems, it converges almost surely (a.s.) to a random variable Z_∞, which is proven to be non degenerate ($\mathbb{V}\mathrm{ar}(Z_\infty) > 0$). Moreover it can be proved that Z_∞ follows a Beta distribution with parameters R_0/c and B_0/c [63]. *Generalized Pólya urns* may exhibit very different behaviors, even when some seemingly slight changes in the reinforcement scheme is made. An example is given by the Friedman urn [47]: at each step the ball selected is replaced by a balls of the same color and b balls of the color not drawn, where $a > b > 0$. A theorem was stated in [46] proving the proportion Z_n of red balls converges almost surely (a.s.) to $1/2$. Thus, modifying the reinforcement scheme may leads to a deterministic limit distribution. In urn models, it is well known that the bicolor Pólya reinforcement rule [81] (reinforcement of the chosen color) leads a.s. to a random asymptotics proportion whereas the Friedman rule (reinforcement of the chosen color as well as the non chosen color) leads to a deterministic limit proportion. This somewhat surprising fact is explained hereafter through a theorem stated in [56].

This work is motivated by the study of systems of interacting stochastic processes (in discrete time) related to the one considered in several recent works [3–6, 32, 33, 36, 89]. These are systems of interacting stochastic processes where the interaction holds through a reinforcement rule. In some special cases (mainly when the rates are such that $r_n \sim n^{-1}$)

these may be interpreted as models of interacting urns, *e.g.* [33, 36, 61, 89].

Synchronization (in a broad sense) occurs in many natural contexts and is a common topic of different scientific fields. This is a general concept for a phenomenon observed in multicomponent dynamical evolutions. The following are constituting aspect: notion of unit (cell, component, individual) with a proper dynamics; finite (possibly large) number of units (here denoted with N); interaction among units which influences their dynamics (here, mean field interaction); ant, after some time, the units adopt the same kind of behavior, each individual behavior being coordinated to a collective common characteristic.

One of the first model of this class was introduced in [36] as a finite system of interacting Pólya urns. Each urn is updated through the Pólya reinforcement rule using its own proportion or using the proportion averaged over the system. Synchronization was proved in the sense of the a.s. convergence (in time) of each system's component to a common random limit denoted by Z_∞. Some variations were considered in [71]. Fluctuations (and thus rate of convergence) were then proved through central limit theorems (CLT) in [33]. The case of interacting Friedman urn was considered in [89]. Generalizing the assumption on the reinforcement rate r_n from $r_n \sim cn^{-1}$ to $r_n \sim cn^{-\gamma}$ leads to systems of interacting stochastic algorithms. In [32] several cases of reinforcement (like Pólya/Friedman) were considered. A.s. synchronization was stated and speed of convergence studied through functional central limit theorems (FCLT). It was proven, that in parameters' regime where the time limit is random ($\mathbb{V}\mathrm{ar}(Z_\infty) > 0$), *synchronization happens quicker than convergence to the time limit*. The kind of interaction was then generalized from mean field to *network-based* interaction in [4]. The empirical means of the reinforcement random variables ($\xi_n(i)$) were studied in [3, 5]. Synchronization was proven too. And statistical applications were stated.

As emphasized in the previous works, there are many applicative contexts these models may be useful for. Urn models are well known [73] to have applications in economy, in clinical trials adaptive design [34,66], random networks [57]. In the general case (rate $(r_n)_n$ satisfying the assumption $\lim_n n^{-\gamma} r_n = c > 0$), each component dynamics is nothing but a *stochastic algorithm* [14, 38] with many applications in the framework of *stochastic optimization* (see for instance [11, 23, 24, 48, 77, 78]). In [4, 32] an interpretation of these processes as opinion dynamics was presented. For related statistical inference, see [4].

In the family of models we are introducing (defined through recurrence equations (3.9)) and studying in this work, we are considering a (finite) system of reinforced stochastic processes. There are two kind of reinforcement, one $\xi^l_{n+1}(i)$ depending only on the single component (*self/auto/local reinforcement*), one ξ^g_{n+1} creating the interaction and depending on the average over all components. This is modeling a *collective reinforcement* effect to be confronted with an individual reinforcement. For the sake of simplicity, we choose to consider a mean field interaction and to symmetrize the model with respect to $1/2$. Each reinforcement is associated to its own rate r^l_n (resp. r^g_n). Each rate may have its own asymptotic behavior: $r^l_n \sim \frac{c_1}{n^{\gamma_1}}$ (resp. $r^g_n \sim \frac{c_2}{n^{\gamma_2}}$) such that (3.13) holds. Thus, unless otherwise specified, we are assuming $(\gamma_1, \gamma_2) \in]1/2, 1]^2$. Pay attention, for technical reasons,

we will assume a slightly more precise behavior with assumption 3.12, which is

$$r_n^l = \frac{c_1}{n^{\gamma_1}} + O\left(\frac{1}{n^{2\gamma_1}}\right), \quad r_n^g = \frac{c_2}{n^{\gamma_2}} + O\left(\frac{1}{n^{2\gamma_2}}\right), \tag{1.1}$$

where $c_1 > 0$, $c_2 > 0$ and $(\gamma_1, \gamma_2) \in]1/2, 1]^2$.

Remark that, if, for instance, $\gamma_1 < \gamma_2$, one could naively expect the collective reinforcement to be negligible. The system could behave like a system with independent components, leading to a possible *absence of synchronization*. We prove later this is not happening. L^2/a.s. synchronization holds. Additional issues we are addressing are: *nature* of the almost sure time limit distribution according to the type of reinforcement (deterministic/random, diffuse or atomic), *fluctuations* with respect to this limit, which are studied through Central Limit Theorem (CLT). This gives insight to the rates of convergence. We prove, according to the parameters' cases, that the rate of synchronization is quicker or the same as the speed of convergence to the time-asymptotics. In the models considered in [32], synchronization quicker than convergence towards the asymptotics value Z_∞ holds only in cases where $\mathbb{V}\mathrm{ar}(Z_\infty) > 0$. In the following models it may happen even when Z_∞ is deterministic ($\mathbb{V}\mathrm{ar}(Z_\infty) = 0$).

The main reinforced stochastic processes, we are interested in, as individual stochastic evolution, are defined through the following recursive equation:

$$\forall n \in \mathbb{N}, \qquad Z_{n+1} = (1 - r_n)\, Z_n + r_n\, \xi_{n+1}, \quad Z_0 = \frac{1}{2}, \tag{1.2}$$

where the law of the random variable ξ_{n+1}, conditionally to the past, is a Bernoulli distribution whose parameter is a transformation of Z_n

$$\mathbb{P}(\xi_{n+1} = 1 \mid \mathcal{F}_n) = \varphi(Z_n). \tag{1.3}$$

The transformation φ is a map from $[0, 1]$ to $[0, 1]$. The reinforcement rate $r_n = (n + 3)^{-1}$ (in case of urn models). Let $\mathcal{F}_n := \sigma\{Z_1, \ldots, Z_n\}$. In this framework, a well known result [56] states the asymptotic behavior. If φ is assumed to be continuous then, the random sequence $(Z_n)_n$ converges a.s. and its limit Z_∞ is such that $\varphi(Z_\infty) = Z_\infty$ a.s. It means the support of the distribution of Z_∞ in included in the set of fix points of φ. Remark the following particular cases.

- Case $\varphi(x) = x$ is the Pólya reinforcement rule in a two-color Pólya urn context. The random time-asymptotics proportion Z_∞ is beta-distributed.
- Case $\varphi(x) = 1 - x$ corresponds to the proportion when a Friedman replacement scheme is used: at each time step, a balls of the chosen color are added to the urn and $b > 0$ balls from the not chosen color. It holds $Z_\infty = \frac{1}{2}$ a.s.

We want to perform such analysis for finite size systems $\{(Z_n(i))_n,\ 1 \le i \le N\}$ of such processes ($N > 1$) when an interaction takes place through an additive reinforcement

mechanism. As it was stated in previous recent works [33, 36, 89], such systems proved to have very interesting properties. More precisely, motivated by [4, 32], we want to study different strength of reinforcement considering $r_n \sim cn^{-\gamma}$ with $\gamma \in]1/2, 1]$ (as it will be explained later). In the case $\gamma = 1$, these dynamics may be interpreted as stochastic model for urn evolution. Like Pólya and Friedman urns. The results are twofold. The first kind of results is dedicated to the asymptotic behavior of reinforced interacting systems: convergence and synchronization, in analogy with those studied in [32] and [36]. The second kind is study the stochastic fluctuations with respect to the time-asymptotics behavior. Tools and methods are based on [33] and [89].

This PhD-thesis is organized as follows. In Chapter 2 we present as a starting point, single urn models of reinforced stochastic processes and some generalizations, properties, applications and in particular connection to stochastic algorithm's framework. In Chapter 3 we consider the context of reinforced stochastic process in which several components (urns) could have an interaction. We give a brief review of a variety of families of interacting stochastic systems, where the interaction is through the reinforcement. Motivated by these examples, we introduce a new family of models (defined through the recurrence equations (3.9)). We study its first properties. In Chapter 4 we first present the definition of quasi-martingale. Using the recurrence Lemma A.2 which is stated in Appendix A, we prove that L^2 and a.s. convergence holds towards a limit Z_∞ shared by all the components (synchronization). Two main cases are distinguished: Theorems 4.3 and 4.4 deals with cases where $Z_\infty = \frac{1}{2}$ (as noticed, the word synchronization is abusive in this situation) whereas Theorem 4.6 deals with Z_∞ random. In Chapter 5, in the different cases, we study the fluctuations $(Z_n - Z_\infty)_n$ and $(Z_n - Z_n(i))_n$ by proving Central Limit Theorems (CLT). Scaling sequences are worth of interest. Theorem 5.4 consider the case where each individual and collective reinforcement rules lead to a deterministic limiting value. Theorem 5.5 consider the special case when $\gamma_1 = \gamma_2 = 1$ reminiscent of the Friedman urn context, where, in some regime, fluctuations are proven not to be gaussian ($\lambda_1 + \lambda_2 < \frac{1}{4}$). Theorem 5.6 deals with the mixed cases where individual and reinforcement type are of different nature. Theorem 5.7 consider the case where the individual and the collective reinforcement leads to a random limit. Finally, the sections 5.3, 5.4, 5.5 and 5.6 deal respectively with the proofs of the CLTs.

We finally remark that main original results of Chapters 4 and 5 are contained in [72] and submitted for publication in an international refereed journal. The paper is on revision.

Chapter 2

Reinforced stochastic processes

The phenomenon of reinforcement is very present in nature, especially in genetics, statistical physics, sociology, psychology, and neuroscience. Generally speaking, by reinforcement in a stochastic dynamics we mean any mechanism for which the probability that a given event occurs has an increasing dependence on the number of times that the same event occurred in the past. This "reinforcement mechanism", related to the "preferential attachment rule", is a key feature governing some dynamics of many biological, economic and social systems [81]. The best known example of reinforced stochastic process is the standard (Eggenberger-)Pólya urn [40, 60, 73], which has been widely studied and generalized (some recent variants can be found in [7–9, 19, 27, 30, 31, 49, 66]).

This Chapter is organized as follows: in section 2.1 we present single urn models as a starting point of reinforced stochastic process and generalizations inside associated with important models: urn models, the random graphs, growing with preferential attachment rule of Barabási-Albert, and reinforced random walks. In section 2.2 we present some application of urn models. In section 2.3 we review the concept of general urn function as introduced in [56] and state its main results. At the end, in section 2.5 we introduce briefly the stochastic algorithm's framework and its connection to reinforced stochastic processes.

2.1 Urns, generalizations of urn models

The Pólya urn is a simple model of reinforcement. It models the fact that success or wealth is self-amplifying over time. A Pólya urn containing balls up to k ($k \in \mathbb{N}$) different colors. The urn evolves in discrete time steps. At each step, we shake the urn well and a ball is sampled uniformly at random (all balls being equally likely). The color of the ball withdrawn is observed, and the ball is returned to the urn. If at any step the color of the ball withdrawn is i ($i = 1, \ldots, k$), then A_{ij} balls of color j are placed in the urn, where A_{ij} follows a discrete probability distribution on a set of integers. Generally speaking, the entries A_{ij} can be deterministic or random, positive or negative. It is customary to represent the urn scheme or reinforcement rule by a square "ball addition" matrix

(reinforcement matrix):

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,k} \\ A_{2,1} & A_{2,2} & \dots & A_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k,1} & A_{k,2} & \dots & A_{k,k} \end{pmatrix}.$$

The rows are indexed by the color of the ball drawn. The columns are indexed by the color of the balls added.

Tenability. In order to have an asymptotic theory, the urn needs to be *tenable i.e.* one from which we can continue the drawing according to a given rule on all possible stochastic trajectory. In a tenable urn rule, it is always possible to draw balls and to follow the replacement rules; we never get "stuck" (empty urn). As we must see, the tenability of an urn rule is a combination of what rule is given as stochastic replacement rules, and the initial conditions. For example, the following reinforcement matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

of B. Friedman's urn is tenable, under whichever nonempty initial state it starts in. In fact an urn is tenable, if all the entries $A_{i,j}$ are nonnegative, under any nonempty starting conditions. Contrarily, an urn rule of white and blue balls with the following reinforcement matrix

$$\begin{pmatrix} -1 & -B \\ 3 & 4 \end{pmatrix},$$

with B being a Bernoulli distributed random variable with parameter $\frac{4}{7}$, may or may not be tenable, depending on the initial conditions. For more details and conditions see [73].

2.1.1 Classical Pólya urn

At time $n = 0$, an urn containing $R_0 > 0$ red balls and $B_0 > 0$ black balls is prepared. To make the configuration of the urn at time $n = 1$, one randomly draws a ball in the urn, then puts the ball drawn back in the urn, and c new balls of the same color. The *replacement matrix* is

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

This mechanism is repeated independently at all times $n \in \mathbb{N}$. Note $Z_n \in [0, 1]$ the proportion of red balls at the moment $n \in \mathbb{N}$. Then, at the moment $n \in \mathbb{N}$, the urn contains $R_0 + B_0 + cn$ balls of which $(R_0 + B_0 + cn)\, Z_n$ are red. Conditionally to Z_n, c red balls are added at time $n + 1$ with probability Z_n. Thus, $Z_0 = R_0/(R_0 + B_0)$, and

$$Z_{n+1} = \frac{(R_0 + B_0 + cn)Z_n + c\ \mathbb{1}_{\{U_{n+1} \le Z_n\}}}{R_0 + B_0 + c(n+1)}$$

where $(U_n)_{n\ge1}$ is a sequence of i.i.d random variable with uniform distribution on $[0, 1]$, and the event $\{U_{n+1} \le Z_n\}$ corresponds to adding a one red ball. This mechanism can be

interpreted as a Bernoulli trial such that the random variable $\xi_n = 1$ ($U_{n+1} \leq Z_n$ occurs) if a red ball is drawn at time n, conditionally to the past, with probability Z_n, and $\xi_n = 0$ otherwise. Indeed,

$$\mathbb{P}(\xi_{n+1} = 1 \mid \xi_n = x_n, ..., \xi_n = x_1) = \frac{R_0 + c \sum_{i=1}^{n} x_i}{R_0 + B_0 + cn} = Z_n \tag{2.1}$$

where $\{x_1, ..., x_n\} \in \{0, 1\}^n$. We note $\mathcal{F}_n = \sigma(\xi_i, 0 \leq i \leq n)$ the σ-field generated by ξ_i up to time n.
Let R_n be the number of red balls at time n. So, the number of red balls at next time $n + 1$ follows

$$R_{n+1} = R_n + c\ \xi_{n+1}.$$

Moreover, the proportion of red balls at time n

$$Z_n = \frac{R_n}{R_0 + B_0 + cn},$$

and therefore,

$$Z_{n+1} = \frac{R_0 + B_0 + cn}{R_0 + B_0 + c(n+1)}\ Z_n + \frac{c}{R_0 + B_0 + c(n+1)}\ \xi_{n+1}. \tag{2.2}$$

Taking $r_n = c\ (R_0 + B_0 + c(n+1))^{-1}$ in (2.2), yields the following recursive equation

$$Z_{n+1} = (1 - r_n)Z_n + r_n\ \xi_{n+1}. \tag{2.3}$$

The random sequence $(Z_n)_{n\geq 0}$ defined through this recurrence equation is both a non-homogeneous Markov chain on the state space $[0, 1]$ and a martingale. Let us recall some definitions and basic results.

Theorem 2.1 (Martingale)**.** *The sequence $(Z_n)_{n\geq 0}$ is a martingale with values in $[0, 1]$ for the filtration $(\mathcal{F}_n)_{n\geq 0}$ defined by $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$ and in particular the average of the proportion of red balls is kept during the time : for all $n \in \mathbb{N}$,*

$$\mathbb{E}(Z_n) = \mathbb{E}(Z_0) = \frac{R_0}{R_0 + B_0}.$$

Moreover, there is a random variable Z_∞ on $[0, 1]$ such that

$$\lim_{n\to\infty} Z_n = Z_\infty \quad a.s.$$

and in L^p for all $p \geq 1$. In particular $\mathbb{E}(Z_\infty) = \mathbb{E}(Z_0) = \frac{R_0}{R_0 + B_0}$.

Theorem 2.2 (Asymptotics of Polya Urn, [63])**.** *The random variable Z_∞ (that appears in the previous theorem) follows the Beta distribution on $[0, 1]$ with parameters $(\frac{R_0}{c}, \frac{B_0}{c})$, i.e. with density with respect to the Lebesgue measure on $[0, 1]$:*

$$u \in [0, 1] \mapsto \frac{u^{\frac{R_0}{c}-1}(1-u)^{\frac{B_0}{c}-1}}{Beta\ (\frac{R_0}{c}, \frac{B_0}{c})}.$$

In particular, if $R_0 = B_0 = 1$ then Z_∞ has a uniform distribution on $[0, 1]$.

Theorem 2.3 (Fluctuations of Pólya urns, Theorem in [55]). *It holds the following central limit theorem for fluctuations of the sequence $(Z_n)_n$ around the random variable Z_∞ such that,*

$$\sqrt{n}\big(Z_n - Z_\infty\big) \xrightarrow{D} \sqrt{Z_\infty - (1 - Z_\infty)}\ \nu$$

where ν is an independent $\mathcal{N}(0,1)$ distributed random variable.

2.1.2 Friedman urn

The Friedman urn model is a generalization of a Pólya urn. Friedman [47] extends the basic model of Pólya to one where the chosen color is reinforced with a balls ($a \geq 0$) of the color sampled and b balls ($b > 0$) of the other color. The replacement matrix is symmetric:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Assume as before the two colors to be red and black, and the respective number of balls of these colors after n draws are denoted with R_n and B_n. For mathematical convenience Friedman stayed with the constant row. We shall generally refer to these urn schemes as *b*alanced. Of course, the case $a = b$ is degenerate, where $R_n = R_0 + a\ n$. This degenerate case is not interesting. The recursive equations, for the number of balls, defining the Friedman urn are

$$\begin{cases} R_{n+1} = R_n + a\ \xi^R_{n+1} + b\ \xi^B_{n+1} \\ B_{n+1} = B_n + b\ \xi^R_{n+1} + a\ \xi^B_{n+1} \\ R_0 = 1, B_0 = 1 \end{cases}$$

where ξ^R_{n+1} and $\xi^B_{n+1} := 1 - \xi^R_{n+1}$ are respectively Bernoulli random variable corresponding to the events of drawing a red or a black ball at the nth step. Freedman [46] developed a theory about the asymptotic behavior of B. Friedman's urn.
As previously, let us define $Z_n := \dfrac{R_n}{R_n + B_n}$ denotes the proportion of red balls after n draws.

Theorem 2.4 (Time-asymptotic behavior of Friedman's urn, [46]).
It holds

$$\lim_{n\to\infty} Z_n := Z_\infty = \frac{1}{2} \quad a.s.$$

The following result states that fluctuations of the Friedman's urn are non-trivial.

Theorem 2.5 (Friedman urn's fluctuations, Theorems in [46]).
Let $\rho := \dfrac{a-b}{a+b}$. It holds

- *if $\rho < \frac{1}{2}$, then $\sqrt{n}\big(Z_n - \frac{1}{2}\big) \xrightarrow{D} \mathcal{N}\big(0, \frac{(a-b)^2}{4(1-2\rho)}\big)$, where $\xrightarrow{D}$ denotes the convergence in distribution;*
- *if $\rho = \frac{1}{2}$, then $\sqrt{\frac{n}{\log n}}\big(Z_n - \frac{1}{2}\big) \xrightarrow{D} \mathcal{N}\big(0, (a-b)^2\big)$;*

- *if* $\rho > \frac{1}{2}$, *then* $n^{1-\rho}\big(Z_n - \frac{1}{2}\big) \xrightarrow{D} \mathcal{L}$, *where the distribution of* $\mathcal{L}$ *is not gaussian.*

For $\rho < \frac{1}{2}$ the limiting distribution is normal under the scaling sequence $\sqrt{n}$, and it is interesting to note that in the case $\rho = \frac{1}{2}$, one needs a different scaling sequence $\frac{\sqrt{n}}{\sqrt{\log n}}$ to get a Gaussian limit distribution. For $\rho > 1/2$ the behavior is different, it holds:

$$\frac{R_n - B_n}{n^\rho} \xrightarrow{\text{D}} Beta\Big(\frac{R_0}{a}, \frac{B_0}{a}\Big).$$

It is interesting to note that in the case $\rho \leq \frac{1}{2}$, considering any initial condition will have no influence asymptotically. Contrarily, in the case $\rho > \frac{1}{2}$, the asymptotic proportion of colors depends critically on the initial conditions.

2.1.3 Random graphs growing with preferential attachment

Random graphs can be used to model plenty of natural phenomena, such as friendship structures in social networks, link structures between pages in the World Wide Web, collaborative structures in artistic and scientific productions, regulation between proteins, the links between machines in the Internet, etc. The Galton-Watson trees constitute a model of random graph adapted to the genealogical structures. The most famous and simple model of random graph is Erdös-Rényi. It is built recursively by adding a new site and then pulling independently its connection with each of the existing sites [41, 50]. This model does not match the reality of social random graphs, for which the new sites preferentially connect to the most important existing sites in the sense of connectivity (degree). This is another example of the reinforcement phenomenon that must be specifically considered.
Indeed, Barabási-Albert's preferential attachment random graph is defined as follows (see [13, 57]): at time $n \geq 1$, the graph contains n sites (vertices) and a number of undirected links (edges) between these sites. The degree of a vertex is the number of edges pointing to that vertex. At time $n = 1$, site 1 is connected to itself. The degree of site 1 at time 1 is therefore 2. To recursively construct the graph from time n to time $n+1$, we consider the degrees $d_{n,1}, \ldots, d_{n,n}$ of the n sites of the graph at time n, and the associated probability distribution

$$\pi_{n,j} = \frac{d_{n,j}}{d_{n,1} + \cdots + d_{n,n}}, \quad (1 \leq j \leq n)$$

then the new $n+1$ site is connected to a site chosen randomly and independently from the n existing sites, with respect to probability law $\pi_{n,\cdot}$. With this mechanism, we get $d_{1,1} = 2$, $d_{2,1} = 3$, $d_{2,2} = 1$, and for all $n \geq 1$, $d_{n,1} + \cdots + d_{n,n} = 2n$ (*i.e.* n edges).

Remark 2.6. *(Urn). We can realize this construction of the sequence* $(d_{n,\cdot})_{n\geq 1}$ *(losing the geometry of the graph) as a generalized Pólya urn model: at time* $n \geq 1$ *the urn contains* $2n$ *balls whose colors can go from* 1 *to* n*, we draw a ball at random, and if* j *is its color, we put in the urn* 2 *balls of color* j *and a new ball of color* $n+1$ *which corresponds to*

reinforcing the color site k and introducing a new site of color $n+1$, in other words the graph gains a vertex $(n+1)$ and an edge $(j \leftrightarrow n+1)$.

Theorem 2.7 (Power law)**.** *Fix $j > 1$. Let $d_{n,j}$ be the degree of the site j in the Barabási-Albert preferential attachment random graph at n sites with $n > j$. Then the proportion $d_{n,j}/(d_{n,1} + \cdots + d_{n,j})$ converges almost surely when $n \to \infty$ to a Beta distributed random variable with parameter $(1, 2j-1)$ on $[0,1]$ of density $u \in [0,1] \mapsto (2j-1)(1-x)^{2(j-1)}$.*

2.1.4 Reinforced random walks on a lattice

In the classical Pólya urn, an urn contains both red and black balls, one is drawn at random and replaced together with another ball of the same color, and this procedure is repeated indefinitely. It is easy to show that, with probability one, infinitely many balls of each color are drawn, regardless of the initial distribution.
Here we state the result that appears in the appendix of [37] and is proved by Rubin's exponential embedding. This model which is known also as *reinforced random walk*, has many generalizations (see for instance [81]) as well as in some recent works [68, 69] in relationship with the concept of reinforced interacting processes that we will discuss more precisely in Chapter 3.

In order to state the Rubin's theorem, consider an urn containing at the beginning with $1-1$ balls. Add one ball at a time, same color of the randomly uniformly chosen one. Let R_n be the number of red balls at time n and B_n, the number of black balls, therefore $Z_n := \frac{R_n}{n+2}$ is the proportion of red ball at time n with the following recursive equation's point of view

$$\begin{cases} R_{n+1} = R_n + \xi_{n+1} \\ R_0 = 1 \end{cases}$$

where $\mathbb{P}(\xi_{n+1} = 1 \mid \mathcal{F}_n) = \dfrac{\varpi(R_n)}{\varpi(R_n) + \varpi(B_n)}$ with the sequence $(\varpi(k))_k$ used for a reinforcement rule. For instance, $\varpi(k) = \varrho^k$, for a given ϱ. In the context of urn models, this would mean an exponential number of red (resp. black) balls to be added at each time step.

Theorem 2.8 (Rubin's construction, Theorem 3.2 in [37])**.**
There is saturation/polarization *in the sense that $Z_\infty \in \{0,1\}$ a.s. if and only if*

$$\sum_k \frac{1}{\varpi(k)} < \infty.$$

2.2 Applications of urn models

Generalized Pólya Urn (GPU) models have been successfully applied in many fields for instance, biology (*e.g.* [60, 64]) and clinical trials (*e.g.* [34]). One interesting applications of urn models is the following probabilistic model of neuron growth. The motivating biological question concerns the mechanisms by which apparently identical cells develop

into different types. Khanin and Khanin [62] examine the development of neurons in two types : *axon and dendrite*. Non clear at first, groups of such cells exhibit periods of growth and retraction until one rapidly elongates to eventually become an axon. They note experimental data suggesting that model is a Markov process with competition between the growing neurites, wherein longer objects have more chances to grow. They propose a multi-colors urn model where at each discrete time one of the existing neurons grows by a constant length ℓ, and the others do not grow. The probability of being selected to grow is proportional to υ-power of it length like the model that was discussed in 2.1.4, for some parameter $\upsilon > 0$. The model is as follows [62]:

Consider k growing objects, neurites. Denote their length at time n by $l_1(n), l_2(n), ..., l_k(n)$, and their initial lengths at time $n = 0$ by $l_s(0) > 0$, $1 \leq s \leq k$. Let us consider a discrete model, where time takes integer values. Suppose also, that for each interval of time $[n, n+1]$ a unit of length ℓ, is added to one of the neurites, while others remain unchanged. Thus, at time $n + 1$ there exist a neurite with number j, $1 \leq j \leq k$ such that

$$l_j(n + 1) = l_j(n) + \ell.$$

For all other neurites, wherein $s \neq j$, as $l_s(n + 1) = l_s(n)$. Clearly, the total length of all neurites at time n is given by

$$L(n) = \sum_{j=1}^{k} l_j(n) = L(0) + n\ \ell,$$

where $L(0) = \sum_{j=1}^{k} l_j(0)$. The probability a neurite l_j grows at time n depends on the current length of this neurite and the lengths of all other neurites. Suppose that the probability that neurite j will grow at time $n + 1$ is proportional to its length l_j at time n to the power υ :

$$\mathbb{P}[l_j(n + 1) = l_j(n) + \ell] \propto l_j^{\upsilon}(n). \tag{2.4}$$

Since at any time n one neurite grows with probability 1, the sum of all probabilities equals to one and we obtain the following formula for the transition probabilities

$$\mathbb{P}[l_j(n + 1) = l_j(n) + \ell] = \frac{l_j^{\upsilon}(n)}{\sum_{s=1}^{k} l_s^{\upsilon}(n)}.$$

Authors give rigorous proofs of the long-term behavior in three cases. When $\upsilon > 1$, they quote Rubin's Theorem from [37] to show that after a certain random time, only one neuron grows (only one color is always chosen after a certain amount of time). When $\upsilon = 1$, they cite results on the classical Pólya urn from [43] to show that the pairwise length ratios have random finite limits. When $\upsilon < 1$, they use embedding methods (through Poisson point processes on $\mathbb{R}$) to show that every pair of lengths has ratio equal to 1 in the limit and to show fluctuations that are Gaussian when $\upsilon < \frac{1}{2}$, Gaussian with a logarithm in the scaling when $\upsilon = \frac{1}{2}$, and differing by a n^{ν} times a random limiting constant when $\upsilon \in (\frac{1}{2}, 1)$.

2.3 General urn function

Let $Z_0 = x$ be the proportion of red balls in an urn containing initially m balls and let φ be a map from the unit interval into itself. Suppose that a red ball is added to the urn with probability $\varphi(x)$ and a black ball is added with probability $1-\varphi(x)$. Let $(Z_n)_{n\geq 0}$ be an *urn process* as defined hereafter. This section explains that $(Z_n)_n$ converges almost surely to a random variable Z_∞ whose support is contained in the set $C = \{p : \varphi(p) = p\}$.

In order to compile the urn models in a general framework (general urn function), we define the evolution of Z_n through the following recursive formula:

$$Z_{n+1} = (1 - r_n)\, Z_n + r_n\, \xi_{n+1}, \tag{2.5}$$

for $n \geq 0$ where $Z_0 = \dfrac{1}{2}$ and $r_n = \dfrac{1}{n+3}$, and where ξ_{n+1} is, given the past, Bernoulli distributed random variable with the following parameter

$$\mathbb{P}(\xi_{n+1} = 1 \mid \mathcal{F}_n) = \varphi(Z_n), \tag{2.6}$$

the law of Bernoulli is a transformation of Z_n, meaning the probability of random variable ξ_{n+1} at $n+1$ given to all the past ($\mathcal{F}_n$ is σ-field generated by $\{\xi_i, 0 \leq i \leq n\}$) is $\varphi(Z_n)$.

Hill, Lane and Sudderth in 1980 [56] have stated theorems explaining the asymptotic behavior of such generalized urn models. The urn process converges almost surely to a limit variable Z_∞. This is a quite general phenomenon. In particular, it happens whenever the set of discontinuities of the urn function φ is nowhere dense in $[0,1]$.

Theorem 2.9 (Corollary 2.1 in [56]).
If the set of discontinuities of φ is nowhere dense in $[0,1]$, then Z_n converges almost surely.

The limit variable Z_∞ has support equal to the crossing set $C = \{p : \varphi(p) = p\}$. For continuous urn functions, the support of Z_∞ is always contained in C.

Theorem 2.10 (Corollary 3.1 in [56]).
Suppose Z_∞ is the almost sure limit of a process corresponding to a continuous function φ. Then $\varphi(Z_\infty) = Z_\infty$ a.s.

Theorem 2.11 concerns a class of urn functions which are not necessarily continuous, but for which the associated urn processes converge to a single point almost surely.

Theorem 2.11 (Theorem 4.1 in [56]).
Suppose φ is a function, and a continuous function g such that $\exists\ p_0$ in $(0,1)$

(i) $\{p_0\} = \{p : g(p) = p\}$,

(ii) *for $p < p_0$, $\varphi(p) \geq g(p)$ and for $p > p_0$, $\varphi(p) \leq g(p)$.*

Then $(Z_n)_n$ converges to p_0 almost surely.

Definition 2.12. *A point p_0 in $[0,1]$ is called an up-crossing (down-crossing) if, for all p in some neighborhood of p_0, $p < p_0$ implies $\varphi(p) < p$ $(\varphi(p) > p)$ and $p > p_0$ implies $\varphi(p) > p$ $(\varphi(p) < p)$. In particular, if f is differentiable at a point p_0 in $\mathbf{C}$, then p_0 is an up-crossing (down-crossing) point if and only if $\varphi'(p_0) > 1$ $(\varphi'(p_0) < 1)$.*

In general, down-crossing points are limit points for urn processes (Theorem 2.13), but urn processes never (probability equal to 0) converge to an up-crossing point (Theorem 2.14).

Theorem 2.13 (Theorem 4.2 in [56])**.** *Suppose φ is a function and it is continuous in a neighborhood of p_0, a down-crossing point of φ. Then $(Z_n)_n$ converges to p_0 with positive probability.*

Theorem 2.14 (Theorem 5.1 in [56])**.** *If p_0 is an up-crossing point, then* $\mathbb{P}(\lim_{n\to\infty} Z_n = p_0) = 0$.

2.4 Some numerical simulations

In this section, based on the theorems as discussed from [56] in previous section 2.3, we present some illustrations by numerical simulations in order to give some visual intuition. We present following examples for some different urn functions φ corresponding to the properties that have been stated in the theorems above like up-crossing and down-crossing points. We generalized the rate of the "urn function" (reinforcement rules) into $\frac{1}{n^\gamma}$ like in stochastic algorithms.

Example 2.15. *Consider the function $\varphi(x) = x$ which has obviously nowhere down-crossing point and nowhere up-crossing point, thus based on the theorems [56] and well known results, that is the evolution of the proportion of one color in a two-color Pólya urn context where Z_∞ is beta-distributed. Figure 2.1 illustrates the long time behavior of component $Z_n(i)$ that evolves through the urn function (reinforcement rules) φ.*

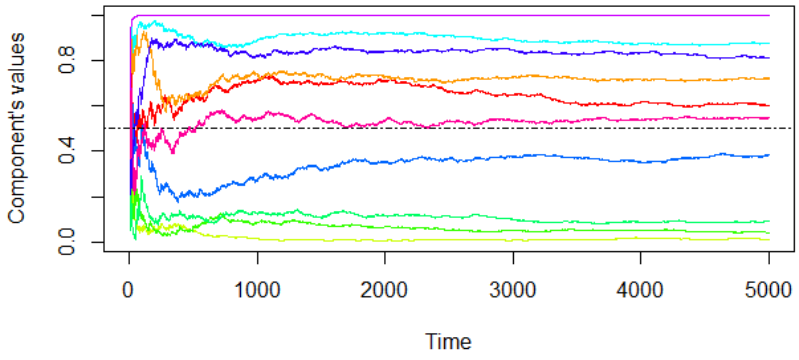


Figure 2.1 – Illustration of $N = 10$ independent components that are considered for the function φ. Each component starts with initial value of $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of the value of the components.

Example 2.16. *Consider the function $\tilde{\varphi}(x) = 1 - x$ where the only down-crossing point is $\{\frac{1}{2}\}$ thus, the case corresponds to the Friedman replacement rules is used that is known $Z_\infty = \frac{1}{2}$ a.s. Figure 2.2 illustrates the long time behavior of component $Z_n(i)$ that evolves through the urn function $\tilde{\varphi}$.*

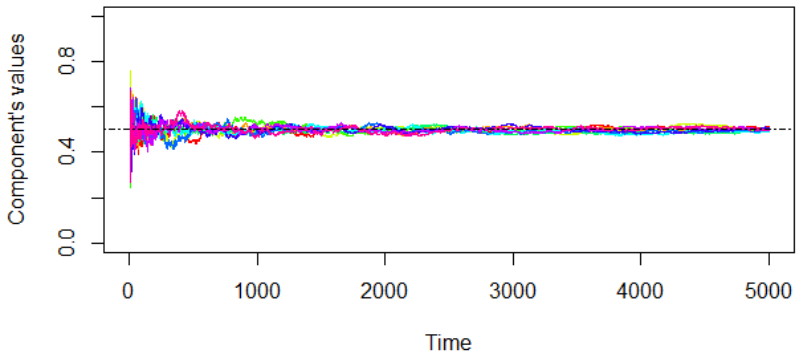
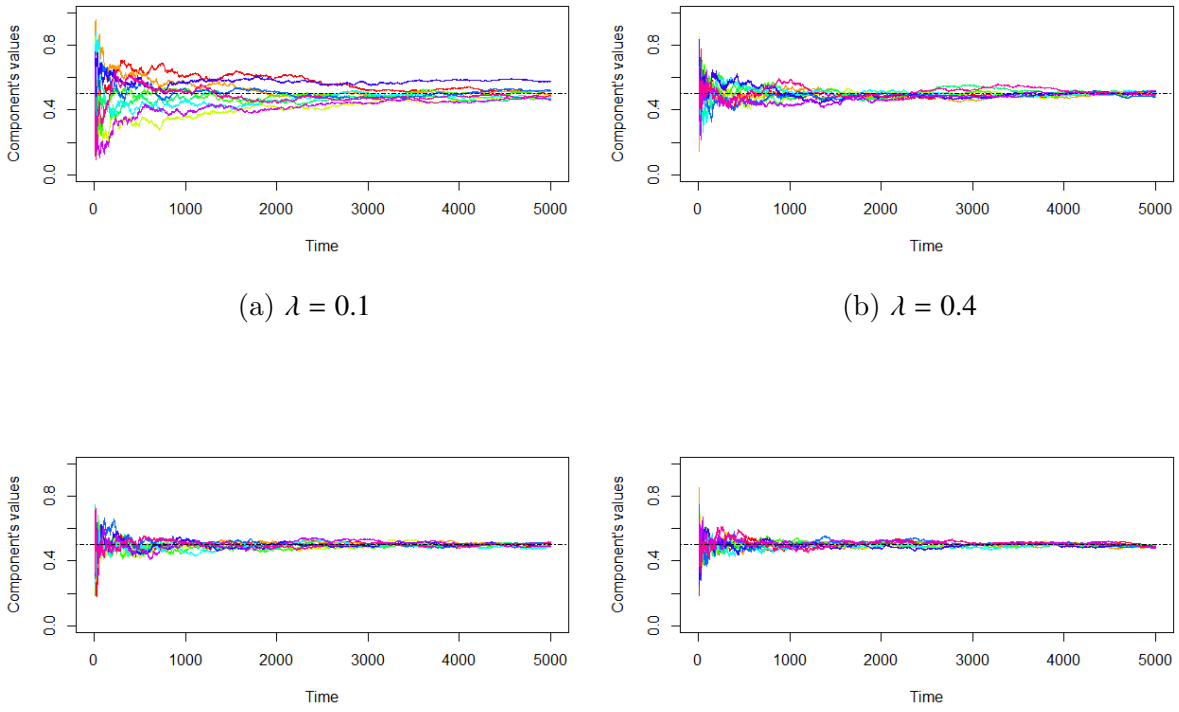


Figure 2.2 – Illustration of $N = 10$ independent components that are considered for the function $\tilde{\varphi}$. Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of the value of the components.

Example 2.17. *Let us consider the function* $\varphi^*(x) = (1-2\lambda)x + \lambda$ *and consider the cases* $\lambda = 0$ *and* $\lambda = 1$. *We come back respectively to Pólya case in Example 2.15 and Friedman case in Example 2.16. Moreover, in the general case* $1 \geq \lambda > 0$, *we still have a family of functions with down-crossing point in* $\{\frac{1}{2}\}$. *Figure 2.3 illustrates the long time behavior of components* $(Z_n(i),\ 1 \leq i \leq N)$ *that evolves through the urn function (reinforcement rules)* φ^* *with some different values* $\lambda > 0$.



(a) $\lambda = 0.1$ (b) $\lambda = 0.4$

(c) $\lambda = 0.6$ (d) $\lambda = 0.8$

Figure 2.3 – Illustration of $N = 10$ independent components that are sampled for the function φ^* with four different values of λ: $\lambda = 0.1$, $\lambda = 0.4$, $\lambda = 0.6$ and $\lambda = 0.8$. Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

Example 2.18. *Consider the following functions :*

a) The function $\varphi_1(x) = \mathbb{1}_{(\frac{1}{2},1]}(x) + \frac{1}{2}\ \mathbb{1}_{\{\frac{1}{2}\}}(x)$ *where the set* $\{0,1\}$ *are the down-crossing points, therefore* $Z_\infty \in \{0,1\}$ *a.s.*

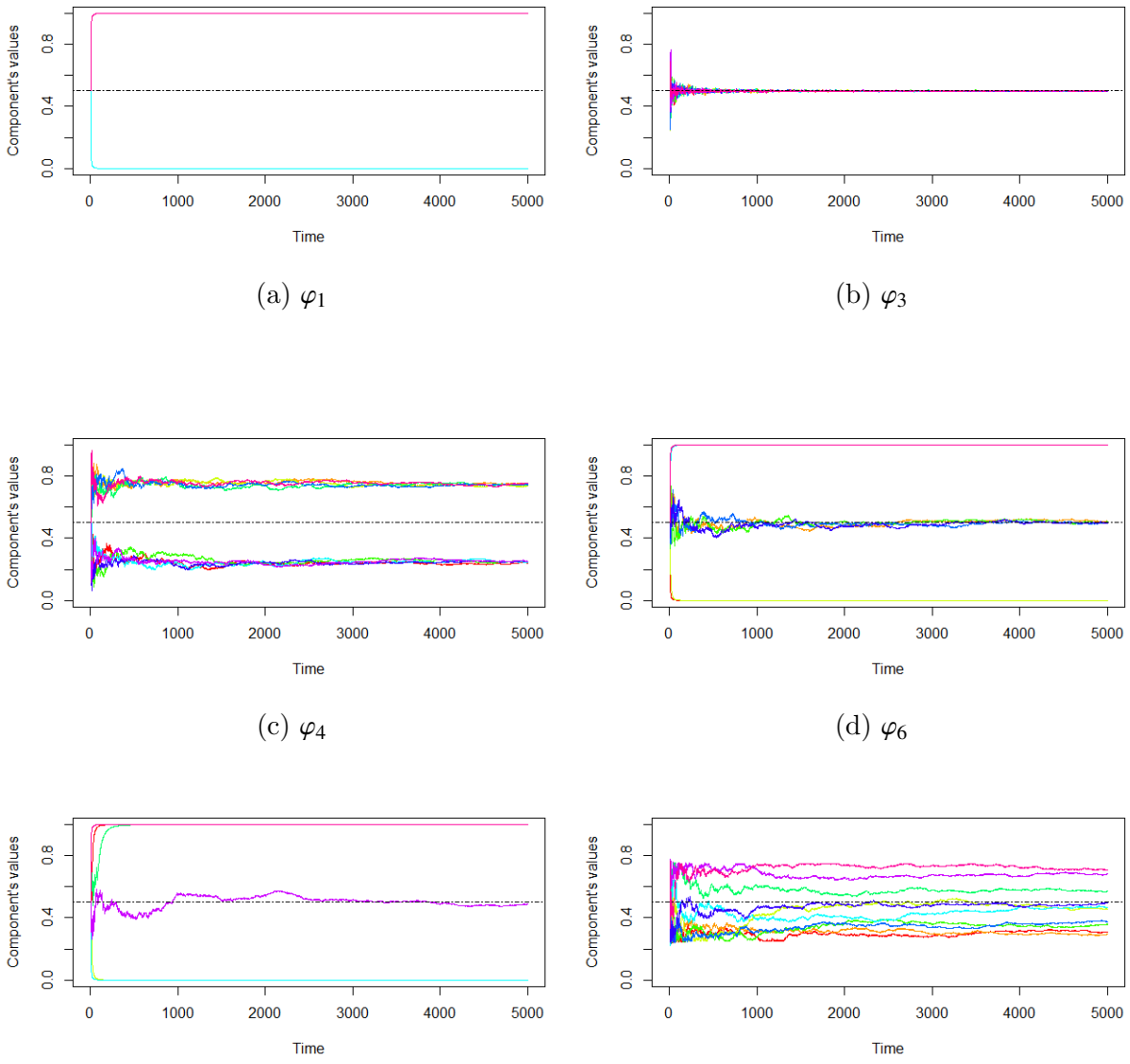
b) The function $\varphi_3(x) = \frac{3}{4}\mathbb{1}_{[0,\frac{1}{2})}(x) + \frac{1}{4}\mathbb{1}_{(\frac{1}{2},1]}(x) + \frac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)$ *where the set* $\{\frac{1}{2}\}$ *is the down-crossing point, therefore* $Z_\infty = \frac{1}{2}$ *a.s.*

c) The function $\varphi_4(x) = \frac{1}{4}\mathbb{1}_{[0,\frac{1}{2})}(x) + \frac{3}{4}\mathbb{1}_{(\frac{1}{2},1]}(x) + \frac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)$ *where the set* $\{\frac{1}{4},\frac{3}{4}\}$ *is the down-crossing points set.*

d) The function $\varphi_6(x) = \frac{1}{2}\ \mathbb{1}_{(\frac{1}{4},\frac{3}{4})}(x) + \mathbb{1}_{(\frac{3}{4},1]}(x)$ *where the set* $\{0,\frac{1}{2},1\}$ *is the down-crossing points set.*

e) The function $\varphi_8(x) = x\mathbb{1}_{(\frac{1}{4},\frac{3}{4})}(x) + \mathbb{1}_{(\frac{3}{4},1]}(x)$ *where the set* $\{0,1\}\cup]\frac{1}{4},\frac{3}{4}[$ *is the down-crossing points set.*

f) The function $\varphi_9(x) = x\mathbb{1}_{(\frac{1}{4},\frac{3}{4})}(x) + \mathbb{1}_{(0,\frac{1}{4}]}(x)$ *where the set* $[\frac{1}{4},\frac{3}{4}]$ *is the down-crossing points set.*
Figure 2.4 illustrates the long time behavior of component $Z_n(i)$ *that evolves through the urn functions (reinforcement rules)* φ_1, φ_3, φ_4, φ_6, φ_8 *and* φ_9.



(a) φ_1 (b) φ_3

(c) φ_4 (d) φ_6

(e) φ_8 (f) φ_9

Figure 2.4 – Illustration of $N = 10$ independent components that are considered for four functions φ_1, φ_3, φ_4, φ_6, φ_8 and φ_9. Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of values of the components.

2.5 Stochastic optimization

Many applied contexts require the solution of optimization problems where the objective to be minimized is defined through some expectations of random quantities. When the expectation can be computed in closed form, conventional numerical optimization algorithms are generally applicable and yield good solutions. On the other hand, when the expectations require evaluation via simulation, the need for *simulation-based optimization* becomes apparent. The difficulties that arise in conventional numerical optimization also manifest themselves in the setting of simulation-based optimization. First of all this is the fact that in the absence of convexity, it is difficult to guarantee that an iterative algorithm will converge to a global optimizer. Instead, the mathematical theory focuses on the less ambitious goal of establishing convergence to local optimizer. One pragmatic approach to compute a global optimizer in the non-convex setting is to run an *iterative search algorithm* from different initial feasible points, with the goal of using the optimal among all the limit points generated by different iterations. The same pragmatic approach is generally followed in the practical applications of simulation-based optimization.

2.5.1 Stochastic approximation algorithms

We consider a class of iterative algorithms that are used in practice for purposes of minimizing (via numerical simulation) an objective function $h(Z) : \mathbb{R}^d \to \mathbb{R}^d$. We assume that $h(.)$ is smooth an that constraints are not binding at the minimizer $z^\star$, so that $\nabla h(z^\star) = 0$ where ∇ as usual denotes the gradient. Stochastic approximation algorithms are iterative simulation-based algorithms that are intended to converge to zero of $\nabla h(Z)$. Given that the algorithm at iteration n has produced an approximation Z_n to $z^\star$, it next generates a random variable Y_{n+1} having an expectation close to $\nabla h(Z_n)$. In particular, if one is able to obtain an unbiased gradient estimator (as sometimes it occurs for instance when likelihood ratio methods are applicable), then the conditional expectation of Y_{n+1} given Z_n is precisely $\nabla h(Z_n)$. On the other hand, for some applications we must satisfy ourselves with finite difference methods, in which case Y_{n+1} has a conditional expectation close to $\nabla h(Z_n)$. Stochastic optimization algorithms that employ finite difference methods are called of *Kiefer-Wolfwitz* type.

Otherwise, the well-known stochastic approximation algorithm is of *Robbins-Monro* type for stochastic search of (isolated) zeros of a given vector field $g : \mathbb{R}^d \to \mathbb{R}^d$. It can be written in the following general recursive form, for any positive (deterministic) sequence $(r_n)_n$ such that $\lim_{n\to\infty} r_n = 0$ and for any starting square-integrable distribution on $\mathbb{R}^d$:

$$Z_{n+1} = Z_n + r_n \left(g(Z_n) + \zeta_{n+1}\right) \tag{2.7}$$

where $(\zeta_n)_n$ is a $\mathbb{R}^d$-valued square-integrable random variables with zero-mean noise *i.e.* $\mathbb{E}(\zeta_{n+1}|\mathcal{F}_n) = 0$ with respect to the filtration $\mathcal{F}_n := \sigma\{Z_0, \ldots, Z_n\}$ (difference of martingale).

The stochastic process defined through (2.7) is in some sense a discrete version (approximation) of the differential flow, solution of the following ODE

$$\frac{dZ_t}{dt} = g(Z_t), \quad t \in \mathbb{R}^+, \quad Z_0 = z_0.$$

Indeed, the following result holds. Let $z^\star$ such that $g(z^\star) = 0$. Let $\sigma : \mathbb{R}^d \to \mathbb{R}$ be defined as

$$\sigma^2(z) := \mathbb{E}(\|\zeta_{n+1}\|^2 \mid Z_n = z),$$

where $\|z\|$ denotes the canonical euclidean norm on $\mathbb{R}^d$ and $\langle , \rangle$ is the associated scalar product. Assuming

$$\begin{array}{ll} (i) & \text{for } z \neq z^\star,\ \langle g(z) - g(z^\star), z - z^\star \rangle < 0, \\ (ii) & \exists K > 0,\ \sigma^2(z) + \|g(z)\|^2 \leq K(1 + \|z\|^2), \\ (iii) & \sum_{n=0}^{\infty} r_n = +\infty \text{ and } \sum_{n=0}^{\infty} r_n^2 < +\infty, \end{array}$$

the almost sure convergence of $(Z_n)_n$ towards $z^\star$ holds. See for instance [38, Theorem 1.4.26]. Remark that condition (iii) is assumed in the following as assumption (3.13) ($\gamma \in]1/2, 1]$).

Definition 2.19. *Let $(u_n)_n$, $(v_n)_n$ be two sequences of reals.*
Let $u_n \sim v_n$ denotes $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$. Let $u_n \simeq v_n$ denotes $\lim_{n\to\infty} \frac{u_n}{v_n} = cst$.

Remark 2.20. *Condition (iii) is standard condition on r_n that appear throughout the stochastic approximation literature. A natural choice for $(r_n)_n$ is to put $r_n \sim cn^{-\gamma}$ for some $\frac{1}{2} < \gamma \leq 1$. This means that $(Z_n)_n$ does not converge in trivial manner by having a path of bounded variation, but that the total variance of increments is finite so that at points where the drift g disappears, $(Z_n)_n$ may converge in the manner of a diffusion whose converges.*

Remark 2.21. *When $d = 1$, the condition (i) means that*

$$z < z^\star \Leftrightarrow g(z) < g(z^\star)$$
$$z > z^\star \Leftrightarrow g(z) > g(z^\star).$$

This method was developed in the framework of optimization, stochastic gradient method and used in many different applicative contexts [16, 38, 48, 65, 90]. In statistical learning contexts the sequence $(r_n)_n$ is called *learning rate sequence*. For instance, analogously recursively defined sequences of random variables contains algorithms used to implement recursive estimation of statistical quantities like max-likelihood, densities, regression functions [88]. Moreover, The Robbins-Monro was the starting point of *stochastic approximation* techniques (see *e.g.* [14, 22]) which are particularly often used in the context of urn models [66].

In order to compare to urn models, let $\gamma = 1$ and recall the Pólya urn containing red and black balls. Let the number of red and black balls at time n be R_n and B_n, respectively, and let $Z_n = R_n/(R_n + B_n)$. Instead of drawing a red ball with probability Z_n draw a red ball with probability $f(Z_n)$, where f is any map from $[0, 1]$ into itself. As discussed previously, it has been shown in [56], under a condition on the discontinuities of f, $(Z_n)_n$ converges almost surely to a random variable Z_∞ for which $f(Z_\infty) = Z_\infty$. The following result gives more precision.

Theorem 2.22 (Theorem 5.1 in [56]).
If p_0 is an up-crossing point, then $\mathbb{P}(\lim_{n\to\infty} Z_n = p_0) = 0$.

To see how this fits into the framework of previous, let $g(t) = f(t) - t$ and note that

$$\begin{aligned} Z_{n+1} &= \left(1 - \frac{1}{R_n + B_n + 1}\right) Z_n + \frac{1}{R_n + B_n + 1}\, \xi_{n+1} \\ &= Z_n + \frac{1}{R_n + B_n + 1}(\xi_{n+1} - Z_n). \end{aligned}$$

Let $Y_{n+1} = \xi_{n+1} - Z_n$ and $\mathbb{E}(\xi_{n+1}|\mathcal{F}_n) = f(Z_n)$, therefore $g(Z_n) = \mathbb{E}(Y_{n+1}|\mathcal{F}_n) = f(Z_n) - Z_n$. So, if $r_n = \dfrac{1}{R_n + B_n + 1}$ then,

$$\begin{aligned} Z_{n+1} &= Z_n + r_n Y_{n+1} \\ &= Z_n + r_n(Y_{n+1} - \mathbb{E}(Y_{n+1}|\mathcal{F}_n)) + r_n\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \\ &= Z_n + r_n g(Z_n) + r_n\zeta_{n+1}, \end{aligned}$$

where $\zeta_{n+1} = Y_{n+1} - \mathbb{E}(Y_{n+1}|\mathcal{F}_n)$ with $\mathbb{E}(\zeta_n|\mathcal{F}_n) = 0$, then the process $(Z_n)_n$ satisfies (2.7).

2.5.2 Motivation: one component, two rates and competing reinforcement rules

In this section we generalize the urn function models following examples where we consider the function which is combined with two different reinforcement rules. We define two reinforcement rates and let them to have "competition" with each others. We sample them with some simulations in order to show that we can not guess the asymptotic behavior of the system $(Z_n(i), 1 \le i \le N)$ anymore in these situations.

Example 2.23. *Consider the following stochastic recursive equation where the reinforcement rule evolves through two random variables ξ^1_{n+1} and ξ^2_{n+1} which are defined by the probability laws φ_1 and φ_2.*

$$
\begin{aligned}
&Z_{n+1} = (1 - r^1_n - r^2_n)\, Z_n + r^1_n\, \xi^1_{n+1} + r^2_n\, \xi^2_{n+1}, \quad (n \geq 0),\\
&\mathbb{P}(\xi^1_{n+1} = 1 \mid \mathcal{F}_n) = \varphi_1(Z_n),\\
&\mathbb{P}(\xi^2_{n+1} = 1 \mid \mathcal{F}_n) = \varphi_2(Z_n),\\
&r^1_n = \frac{1}{n^{\gamma_1}}, \quad r^2_n = \frac{1}{n^{\gamma_2}}, \quad \gamma_1 \in]\tfrac{1}{2}, 1], \quad \gamma_2 \in]\tfrac{1}{2}, 1], \qquad Z_0 = \frac{1}{2},\\
&\varphi_1(x) = \mathbb{1}_{(\frac{1}{2},1]}(x) + \tfrac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)\\
&\varphi_2(x) = \mathbb{1}_{[0,\frac{1}{2})}(x) + \tfrac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)
\end{aligned}
$$

Figure 2.5 illustrates the long time behavior of component $Z_n(i)$ that evolves through the competition (reinforcement rules) between φ_1, φ_2 and their reinforcement rates r^1_n and r^2_n.

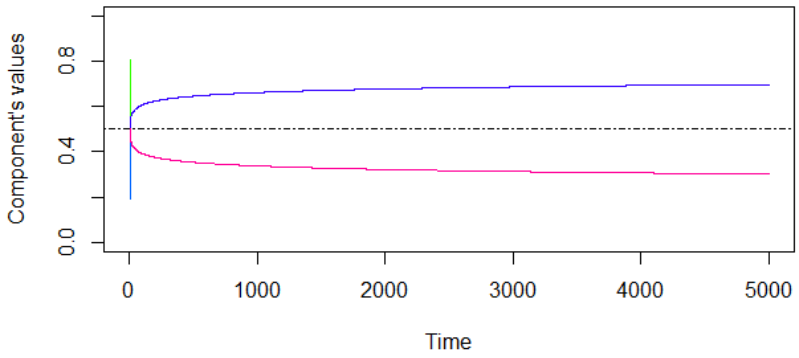


Figure 2.5 – Illustration of $N = 10$ independent components that are considered for the evolution of $(Z_n)_n$ in Example 2.23 with the reinforcement rates $\gamma_1 = 0.8$ and $\gamma_2 = 0.9$. Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

Example 2.24. *Consider the following stochastic recursive equation where the reinforcement rule evolves through two random variables* ξ^1_{n+1} *and* ξ^2_{n+1} *which are defined by the probability laws* φ_3 *and* φ_4.

$$
\begin{aligned}
&Z_{n+1} = (1 - r^1_n - r^2_n)\, Z_n + r^1_n\, \xi^1_{n+1} + r^2_n\, \xi^2_{n+1}, \quad (n \geq 0),\\
&\mathbb{P}(\xi^1_{n+1} = 1 \mid \mathcal{F}_n) = \varphi_3(Z_n),\\
&\mathbb{P}(\xi^2_{n+1} = 1 \mid \mathcal{F}_n) = \varphi_4(Z_n),\\
&r^1_n = \frac{1}{n^{\gamma_1}}, \quad r^2_n = \frac{1}{n^{\gamma_2}}, \quad \gamma_1 \in]\tfrac{1}{2}, 1], \quad \gamma_2 \in]\tfrac{1}{2}, 1], \qquad Z_0 = \frac{1}{2},\\
&\varphi_3(x) = \tfrac{3}{4}\mathbb{1}_{[0,\frac{1}{2})}(x) + \tfrac{1}{4}\mathbb{1}_{(\frac{1}{2},1]}(x) + \tfrac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)\\
&\varphi_4(x) = \tfrac{1}{4}\mathbb{1}_{[0,\frac{1}{2})}(x) + \tfrac{3}{4}\mathbb{1}_{(\frac{1}{2},1]}(x) + \tfrac{1}{2}\mathbb{1}_{\{\frac{1}{2}\}}(x)
\end{aligned}
$$

Figure 2.6 illustrates the long time behavior of component $Z_n(i)$ that evolves through the "competition" between the reinforcement rules defined with φ_3, φ_4 and their reinforcement rates r^1_n and r^2_n.

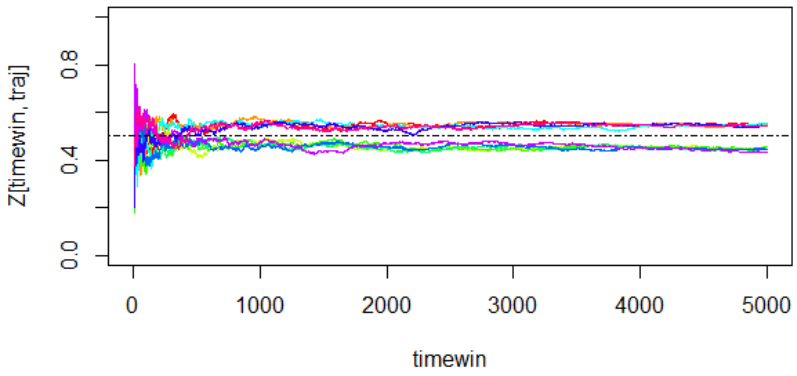


Figure 2.6 – Illustration of $N = 10$ independent components that are considered for the evolution Z_n in Example 2.24 with the reinforcement rates $\gamma_1 = 0.9$ and $\gamma_2 = 0.8$. Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

Chapter 3

Stochastic systems with interaction through reinforcement

In the chapter 2 we considered the reinforcement phenomenon for one dynamics/one component like a single urn. Now we want to think about a more complex issue, when many components are interacting.

Systems of stochastic processes are of wide interest in many fields, from theoretical perspective as well as for application purposes. For instance, social networks and economic deal with agents that make decisions under the influence of other agents. In social life, preferences and beliefs are partly transmitted by means of various forms of social interaction and opinions are driven by the tendency of individuals to become more similar when they interact. Hence, a collective phenomenon, that we call "synchronization", reflects the result of the interactions among different individuals. The main idea is that individuals have opinions that change through the influence of other individuals realize to in an inclination of collective behavior. In particular, there exists a growing interest in systems of interacting urn models (*e.g.* [6,15,26,28,33,36,45,54,70,80]) and their variants and generalizations (*e.g.* [3,4,32,61]). In this chapter, we refer to, specifically, the class of the so-called interacting reinforced stochastic processes considered in [3,4] with a general network-based interaction and in [32] with a mean-field interaction. We present their some main results. Based on these works, hereafter, we introduce and study a new family of reinforced interacting systems.

Consider a probability space with $\Omega = [0,1]^{\mathbb{N}^2}$, $\mathcal{F} = \mathcal{B}([0,1])^{\otimes\mathbb{N}^2}$ and $\mathbb{P} = \mu^{\otimes\mathbb{N}^2}$ in which a family $\{U_n(i); n, i \in \mathbb{N}\}$ of i.i.d random variables with uniform distribution on $[0,1]$ is defined. Take $\mathcal{F}_n = \sigma(U_j(i); 0 \leq j \leq n, i \in \mathbb{N})$. We are interested in studying systems $\{(Z_n(i))_n,\ 1 \leq i \leq N\}$ of such processes for $N > 1$ where an interaction takes place through the *reinforcement* mechanism. A Reinforced Stochastic Process (RSP) can be defined as a stochastic process in which, along the time-steps, an agent performs an action chosen in the set $\{0,1\}$ in such a way that the probability of adopting "action 1" at a certain time-step has an increasing dependence on the number of times that the agent adopted "action 1" in the previous actions. Formally, they are the stochastic processes $\xi(i) = \{\xi_n(i) : n \geq 1\}$ taking values in $\{0,1\}$ such that the random variable $\xi_n(i) = \mathbb{1}_{\{U_n(i) \leq \varphi(Z_n(i))\}}$. The random

variables $\xi_n(i)$ are conditionally independent given $\mathcal{F}_n$ and have conditional distribution which is Bernoulli with parameter given by

$$\mathbb{P}(\xi_{n+1}(i) = 1 \mid \mathcal{F}_n) = \varphi(Z_n(i)) \tag{3.1}$$

where φ is a map from unit interval into itself as it was discussed in previous section about general urn functions, with

$$Z_{n+1}(i) = (1 - r_n)Z_n(i) + r_n\ \xi_{n+1}(i) \tag{3.2}$$

where $Z_n(i)$ is a random variable with values in $[0, 1]$ and $(r_n)_{n\geq 0}$ is a sequence of real numbers in $(0, 1)$ such that

$$r_n = \frac{c}{n^\gamma} + O(\frac{1}{n^{2\gamma}}). \tag{3.3}$$

In particular, $\lim_n n^\gamma r_n = c > 0$.

Definition 3.1. *Let $(u_n)_n$, $(v_n)_n$ be two sequences of complex numbers. The notation*

$$u_n = O(v_n)$$

denotes $|u_n| \leq C|v_n|$ for a suitable constant $C > 0$ and n large enough.

This Chapter is organized as follows: In section 3.1 we recall the concept of synchronization. In section 3.2 we review variety of families of interacting reinforced stochastic systems, such as interacting Pólya urns, interacting Friedman urns and interacting reinforced random walks. In section 3.3 we review the results of the general model, network-based interaction. Finally, in section 3.4 we introduce a new family of models (3.9) and study its properties.

3.1 Synchronization phenomenon

Synchronization is a common phenomenon in different scientific fields. This is a general concept that occurs in many natural contexts in multicomponent dynamical evolutions with a large variety perspectives (*e.g.* [1,10,82,91]). It was also considered in some recent mathematical works by [18,20,29,58]. The following are constituting aspect given in [71]:
• notion of unit (cell, component, individual) with a proper dynamics.
• finite (possibly large) number of units,
• interaction among units which influences their dynamics,
• the units after some time adopt the same kind of behavior, each individual behavior being coordinated to global common characteristic.

A Pólya urn is a basic model of self-reinforcement process. However, a reinforcement process can be influenced by environment, which lead us to study systems of interacting urns. Recently, systems of interacting urns have been considered by *e.g.* [68, 69, 80]. In particular, in [68, 69] the models have a strong reinforcement mechanism and the conditional probability of drawing a color depends on the frequency with which that color was

drawn in the past both in the given urn and in the whole system. Under the conditions on these probabilities, it has been shown that there is a phenomenon of fixation, i.e., depending on the strength of interaction, all or part of the urns draw eventually the same color. So, if the interaction among urns is sufficiently strong, the urns synchronize, which means that the proportion of a given color in the urns converges a.s. to the same random variable which take values in the set $\{0, 1\}$. In following section, we survey some families of interacting systems with a weaker reinforcement which are considered in various recent works (*e.g.* [3, 4, 32, 33, 36]). We see that synchronization holds whatever the strength of interaction is. Indeed, synchronization phenomenon for such systems is defined, in the sense

$$\forall i \in \{1, \dots, N\}, \quad \lim_{n\to\infty} Z_n(i) = Z_\infty \text{ a.s.} \tag{3.4}$$

3.2 Mean field interaction

3.2.1 Interacting Pólya urns

A system of N Pólya-type urns containing balls of two colors in which the reinforcement of each urn is occured by both the content of the urn and a group interaction, the average content of all urns, so-called a mean field interacting [36].
Let $R_0 + B_0$ be the total number of balls in a single urn at time 0. We denote $R_n(i)$ and

$$Z_n(i) = \frac{R_n(i)}{n + R_0(i) + B_0(i)}$$

respectively the number and proportion of red balls in the urn i at time $n \geq 0$, where $i \in \{1, \dots, N\}$ and $Z_0(i) = \frac{R_0(i)}{R_0(i)+B_0(i)}$. Let Z_n be the *mean field* defined with

$$Z_n := \frac{1}{N}\sum_{i=1}^{N} Z_n(i) = \frac{\sum_{i=1}^{N} R_n(i)}{N(R_0(i) + B_0(i) + n)}.$$

The evolution of i-th single urn is as follow

$$Z_{n+1}(i) = \left(1 - \frac{1}{R_0(i) + B_0(i) + n + 1}\right) Z_n(i) + \frac{1}{R_0(i) + B_0(i) + n + 1}\, \xi_{n+1}$$

where $\xi_n(i) = 1_{\{U_n(i) \leq \alpha Z_n + (1-\alpha) Z_n(i)\}}$ with the interaction parameter $\alpha \in [0, 1]$.
Indeed, at each time $n > 0$, given $\mathcal{F}_n$, independently between the urns, in each urn a red ball is added with a probability $\alpha Z_n + (1 - \alpha)Z_n(i)$ (Bernoulli distribution). It can be checked that $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n$ therefore, Z_n is a bounded martingale. Consequently, $(Z_n)_n$ converges a.s. and L^p to random variable such as Z_∞. Moreover, if $\alpha = 0$ or not, we have two cases as follows.

- Case $\alpha = 0$: N independent Pólya urns, each converging a.s. to its own random limit $Z_\infty(i)$ obviously.
- Case $\alpha > 0$: $(Z_n(i))_n$ is not a martingale, but $(Z_n)_n$ is still a bounded martingale.

In [36], it has been shown that, as soon as $\alpha > 0$, the urns synchronize in L^2 i.e.

$$\lim_{n\to\infty} \mathbb{E}[(Z_n(i) - Z_n)^2] = 0.$$

Then, using bounds on L^2 rate of convergence, it has been derived the a.s. synchronization. The following Theorem describe them.

Theorem 3.2 (Theorems 1 and 2 in [36]). *The following asymptotic estimates hold:*

$$\mathbb{E}[(Z_n(i) - Z_n)^2] = \begin{cases} O(n^{-2\alpha}) & for \quad 0 < \alpha < \frac{1}{2} \\ O(n^{-1}\log n) & for \quad \alpha = \frac{1}{2} \\ O(n^{-1}) & for \quad \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Moreover, for each $i = 1, \dots, N$,

$$\lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = Z_\infty \qquad a.s.$$

Furthermore, in [33] the fluctuations around the synchronized regime, in detail the fluctuations of $(Z_n(i) - Z_n)_n$ around zero has been studied. In the sense, some central limit theorems has been stated for them. The scaling of these fluctuations depends on the parameter α. In particular the standard scaling $n^{-1/2}$ appears only for $\alpha > \frac{1}{2}$. For $\alpha \geq \frac{1}{2}$, it is also determined the limit distribution of the rescaled, they converge in distribution to a mixture of centered Gaussian distribution, whose random variance is an explicit function of the limit random variable Z_∞. The main results are as follows.

Theorem 3.3 (Theorem 3.1 in [33]).
The fluctuations of Z_n around its limit random variable Z_∞. For all α, it holds

$$\sqrt{n}(Z_n - Z_\infty) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_N\ Z_\infty(1 - Z_\infty)\Big).$$

Theorem 3.4 (Theorem 3.3 and 3.4 in [33]). *(i) Consider the stochastic process $(Z_n - Z_n(i))_n$. It holds*

(a) For $\frac{1}{2} < \alpha \leq 1$, we have

$$\sqrt{n}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{N,\alpha}\ Z_\infty(1 - Z_\infty)\Big).$$

(b) For $\alpha = \frac{1}{2}$, we have

$$\frac{\sqrt{n}}{\sqrt{\log n}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_N\ Z_\infty(1 - Z_\infty)\Big).$$

(c) For $0 < \alpha < 1/2$, we have

$$n^\alpha(Z_n(i) - Z_n) \xrightarrow[n\to\infty]{a.s./L^1} \Lambda,$$

for some real random variable Λ such that $\mathbb{P}(\Lambda \neq 0) > 0$.

(ii) Consider the stochastic process $(Z_n(i) - Z_\infty)_n$. *It holds*

(a) For $\frac{1}{2} < \alpha \leq 1$, *we have*

$$\sqrt{n}(Z_n(i) - Z_\infty) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{N,\alpha}\ Z_\infty(1 - Z_\infty)\Big).$$

(b) For $\alpha = \frac{1}{2}$, *we have*

$$\frac{\sqrt{n}}{\sqrt{\log n}}(Z_n(i) - Z_\infty) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_N\ Z_\infty(1 - Z_\infty)\Big).$$

(c) For $0 < \alpha < 1/2$,

$$n^\alpha(Z_n(i) - Z_\infty) \xrightarrow[n\to\infty]{P} \tilde{\Lambda},$$

3.2.2 Interacting Friedman urns

In [89] model of N interacting bicolor Friedman urns was considered. It has been shown that the urns synchronize a.s. and that the fraction of balls of each color converges to the deterministic limit $\frac{1}{2}$, which matches with the limit known for a single Friedman urn. Furthermore, they are obtained some limit theorems for fluctuations around the synchronization limit.
Consider at time $n = 0$ each urn contains $R_0(i) > 0$ red and $B_0(i) > 0$ black balls. Let $M_0(i) = R_0(i) + B_0(i)$ denote the total number of balls at the beginning and let $R_n(i)$ and $B_n(i)$ respectively denote the number of red and black balls in urn i at time n. Starting with the same number of balls (denoted by M_0) in each urn and add $a + b$ balls with probability 1 at each time step. Thus, $M_n = n(a + b) + M_0$ for $n \geq 1$ and

$$Z_n := \frac{1}{N}\sum_{i=1}^{N} Z_n(i) = \frac{\sum_{i=1}^{N} R_n(i)}{NM_n}.$$

For fix $a, b \in \mathbb{N}$, consider the following reinforcement rule

$$\mathbb{P}(\xi_{n+1}(i) = \omega \mid \mathcal{F}_n) = \begin{cases} \alpha Z_n + (1-\alpha)Z_n(i) & for \quad \omega = a \\ 1 - \alpha Z_n - (1-\alpha)Z_n(i) & for \quad \omega = b \end{cases}$$

for $\alpha \in [0, 1]$, is called the interaction parameter. Then the evolution of $(Z_n(i))_n$ is as follows

$$Z_{n+1}(i) = \frac{1}{M_{n+1}}[M_n\ Z_n(i) + \xi_{n+1}(i)].$$

The main results are as follows.

Theorem 3.5 (Theorem 1 in [89]).
Set $\rho = (a-b)/(a+b) > 0$. *For every* $i \in \{1, ..., N\}$, *the following asymptotic behaviors hold:*

$$\mathbb{E}[(Z_n(i) - Z_n)^2] = \begin{cases} O(n^{2\rho-2\rho\alpha-2}) & for \quad \rho > \dfrac{1}{2(1-\alpha)} \\ O(n^{-1}\log n) & for \quad \rho = \dfrac{1}{2(1-\alpha)} \\ O(n^{-1}) & for \quad \rho < \dfrac{1}{2(1-\alpha)}. \end{cases}$$

Moreover, for every $i \in \{1, ..., N\}$*, it holds*

$$\lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2} \qquad a.s.$$

The fluctuation results are as follows.

Theorem 3.6 (Theorems 3, 4 and 5 in [89])**.**
Let $\rho = (a-b)/(a+b)$*. Then the following statements holds*

(i) Consider the stochastic process $(Z_n(i) - \frac{1}{2})_n$*.*

(a) For $0 < \rho < \frac{1}{2}$*,*

$$\sqrt{n}(Z_n(i) - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{\rho,N}\Big).$$

(b) For $\rho = \frac{1}{2}$*,*

$$\frac{\sqrt{n}}{\sqrt{\log n}}(Z_n(i) - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{\rho,N}\Big).$$

(c) For $\rho > \frac{1}{2}$ *and* $u = 1 - \rho$*,*

$$n^u(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{a.s./L^1} \Upsilon,$$

for some real random variable Υ *such that* $\mathbb{P}(\Upsilon \neq 0) > 0$*.*

(ii) Consider the stochastic process $(Z_n - Z_n(i))_n$*.*

(a) For $0 < \rho < \frac{1}{2(1-\alpha)}$*,*

$$\sqrt{n}(Z_n(i) - Z_n) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{\rho,N,\alpha}\Big).$$

(b) For $\rho = \frac{1}{2(1-\alpha)}$*,*

$$\frac{\sqrt{n}}{\sqrt{\log n}}(Z_n(i) - Z_n) \xrightarrow[n\to\infty]{stably} \mathcal{N}\Big(0, C_{\rho,N}\Big).$$

(c) For $\rho > \frac{1}{2(1-\alpha)}$ *and* $u = 1 - (1-\alpha)\rho$*,*

$$n^u(Z_n(i) - Z_n) \xrightarrow[n\to\infty]{a.s./L^1} \tilde{\Upsilon},$$

for some real random variable $\tilde{\Upsilon}$ *such that* $\mathbb{P}(\tilde{\Upsilon} \neq 0) > 0$*.*

3.2.3 Interacting reinforced random walks

Going from two colors to a general finite number of colors: as in [32] the framework we are using can be generalized as follows. Let $\mathcal{P}(\Omega)$ the simplex of probability on a finite set Ω such that

$$\mathcal{P}(\Omega) := \left\{\mu : \Omega \longrightarrow [0,1] : \sum_{\omega\in\Omega} \mu(\omega) = 1\right\}$$

and consider a model as time-inhomogeneous random walk on $\mathcal{P}(\Omega)$ and arise in urn contexts where Ω is a set of multi-color balls in urn as was considered in several recent works (*e.g.* [67, 74, 75]). Consider the stochastic evolutions on $\mathcal{P}(\Omega)$ such that

$$Z_{n+1} = (1 - r_n)Z_n + r_n K_n(\xi_{n+1}), \tag{3.5}$$

where $K_n : \Omega \longrightarrow \mathcal{P}(\Omega)$ and $(\xi_{n+1})_{n\geq 1}$ is a sequence of Ω-valued random variable

$$\mathbb{P}(\xi_{n+1} = \omega \mid \mathcal{F}_n) = Z_n(\omega).$$

In particular case $K_n(\omega) = \delta_\omega$ with δ_ω denoting the Dirac measure at $\omega \in \Omega$, the larger $Z_n(\omega)$, the higher the probability of increasing it at the next step.

Now, consider the systems of N interacting random walks in which, to N evolutions as in (3.5), add an interaction term of mean-field type. Assume $\Omega = \{0, 1\}$ and $K_n(\omega) = K(\omega)$ to be independent of time. The most general function $K : \Omega \longrightarrow \mathcal{P}(\Omega)$ can be written in the form

$$K(\omega) = \beta\delta_\omega + (1 - \beta)q,$$

for some $\rho \in [0, 1]$ and a given $q \in \mathcal{P}(\Omega)$. The evolution of the i-th walk is therefore given by

$$Z_{n+1}(i) = (1 - r_n)Z_n(i) + r_n(\beta\xi_{n+1}(i) + (1 - \beta)q),$$

with

$$\mathbb{P}(\xi_{n+1}(i) = 1 \mid \mathcal{F}_n) = (1 - \alpha)Z_n(i) + \alpha Z_n$$

where $\alpha \in [0, 1]$ the interaction parameter, Z_n is mean-field.

In this model $r_n \sim \frac{c}{n^\gamma}$ with $\frac{1}{2} < \gamma < 1$, therefore

$$\sum_n r_n = +\infty \text{ and } \sum_n r_n^2 < +\infty \tag{3.6}$$

The following Theorem sates that synchronization takes place either interaction holds ($\alpha > 0$) or the limit of Z_n is deterministic ($\beta < 1$). Moreover, describes the fluctuations of $(Z_n)_n$ around its limit Z_∞ and the rate of synchronization $(Z_n(i) - Z_n)_n$, in term of a *functional* Central Limit Theorem.

Theorem 3.7 (Consequences of Theorems 2.3, 2.5, 2.6, 2.8 and 2.9 in [32])**.**

(i) *Suppose that (3.6) holds and* $\beta(1-\alpha) < 1$. *Let* Z_∞ *be the almost sure limit of* $(Z_n)_n$ *(note that for* $\beta < 1$, *by Theorem 2.1 in [32],* $Z_\infty = q$*). Then for all* $i \in \{1, 2, ..., N\}$

$$Z_n(i) - Z_n \longrightarrow 0 \qquad a.s.$$

In particular $Z_n(i) \to Z_\infty$ *a.s.*

(ii) *For* $\beta = 1$ *(and so* $\alpha > 0$*), it holds* $Z_n(i) - Z_n = O(n^{-\gamma/2}) \ll O(n^{-(\gamma-1/2)}) = Z_n - Z_\infty$ *for* $\frac{1}{2} < \gamma < 1$.

(iii) *For* $\beta < 1$ *and* $q \in \{0, 1\}$,

(a)

$$Z_n(i) - Z_n = O(n^{-\gamma/2})$$

(b)

$$Z_n - q = O(n^{-\gamma/2}).$$

Remark 3.8. *Functional Central Limit Theorem like those proven in [32] were stated for many urn models, mainly when the time-limit is deterministic. In particular [12, 51] and [92] contain results for Friedman urn models and Pólya urn models respectively.*

3.3 Network-based interaction

In this section we refer to general network-based interaction models which have been introduce in [4] and developed in several works after (*e.g.* [3, 5, 61]) in which is interested in the analysis of a system of $N \geq 2$ interacting reinforced stochastic processes $\{\xi(i) = (\xi_n(i))_{n\geq 1} : 1 \leq i \leq N\}$ positioned at the vertices of a weighted directed graph $G = (V, E, W)$, where $V := \{1, ..., N\}$ denotes the set of vertices, $E \subset V \times V$ the set of edges and $W = [w_{h,i}]_{h,i\in V\times V}$ the weighted adjacency matrix with $w_{h,i} \geq 0$ for each pair of vertices. The presence of the edge $(h, i) \in E$ indicates a "direct influence" that the vertex h has on the vertex i and it corresponds to a strictly positive element $w_{h,i}$ of W, that represents a weight quantifying this influence. We assume the weights to be normalized so that $\sum_{h=1}^N w_{h,i} = 1$ for each $j \in V$. The interaction between the processes $\{\xi(i) : i \in V\}$ is explicitly inserted in (3.1) and it is modeled as follows: for any $n \geq 0$, the random variables $\{\xi_{n+1}(i) : i \in V\}$ are conditionally independent given $\mathcal{F}_n$ with

$$\mathbb{P}(\xi_{n+1}(i) = 1 \mid \mathcal{F}_n) = \sum_{h=1}^N w_{h,i} Z_n(h) = w_{ii} Z_n(i) + \sum_{h\neq i} w_{h,i} Z_n(h), \tag{3.7}$$

where $\mathcal{F}_n := \sigma(Z_n(h))$ and for each $h \in V$, the evolution of the single process $(Z_n(h))_{n\geq 0}$ is the same as in (3.2), that is

$$Z_{n+1}(h) = (1 - r_n) Z_n(h) + r_n \xi_{n+1}(h) \tag{3.8}$$

where $Z_n(h)$ is a random variable taking value in $[0,1]$ and $r_n = \frac{c}{n^{-\gamma}} + O(n^{-2\gamma})$.
To express the above dynamics in a compact form, we define the vectors $\mathbf{X}_n = (\xi_n(1), ..., \xi_n(N))^T$ and $\mathbf{Z}_n = (Z_n(1), ..., Z_n(N))^T$. Hence, the dynamics can be expressed as follows:

$$\mathbb{E}[\mathbf{X}_{n+1} \mid \mathcal{F}_n] = W^T \mathbf{Z}_n,$$

where

$$\mathbf{Z}_{n+1} = (1 - r_n)\mathbf{Z}_n + r_n \mathbf{X}_{n+1}$$

and the assumption about the normalization of the matrix W can be written as $W^T \mathbf{1} = \mathbf{1}$.

Moreover, we must consider the following assumptions on matrix W.

- The weighted adjacency matrix W is irreducible.

This condition reflects a situation in which all the vertices are connected among each others and hence there are no sub-systems with independent dynamics.

- The weighted adjacency matrix W is diagonalizable.

This assumption implies that there exists a non-singular matrix $\tilde{U}$ such that $\tilde{U}^T W (\tilde{U}^T)^{-1}$ is diagonal with elements $\eta_j \in \mathrm{Sp}(W)$. Notice that each column $\mathbf{u}_j$ of $\tilde{U}$ is a left eigenvector of W associated to η_j. Without loss of generality, we set $\|\mathbf{u}_j\| = 1$. Moreover, when the multiplicity of some η_j is bigger than one, we set the corresponding eigenvectors to be orthogonal. Then, if we define $\tilde{V} = (\tilde{U}^T)^{-1}$, we have that each column $\mathbf{v}_j$ of $\tilde{V}$ is a right eigenvector of W associated to η_j.
Denoting η^* an eigenvalue belonging to $\mathrm{Sp}(W) \setminus \{1\}$ such that

$$Re(\eta^*) = \max\{Re(\eta_j) : \eta_j \in \mathrm{Sp}(W) \setminus \{1\}\}.$$

Theorem 3.9 (Synchronization. Theorem 3.1 in [3]). *It holds*

$$\mathbf{Z}_n \xrightarrow[n\to\infty]{a.s.} Z_\infty \mathbf{1}.$$

Theorem 3.10 (Central Limit Theorem, Theorem 3.2 in [3]). *It holds*

(a) For $\frac{1}{2} < \gamma < 1$

$$n^{\gamma - \frac{1}{2}}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\tilde{\Gamma}_\gamma)$$

where $\tilde{\Gamma}_\gamma := \dfrac{c^2 \|\mathbf{v}_1\|^2}{N(2\gamma - 1)} \mathbf{1}\mathbf{1}^T$.

(b) For $\gamma = 1$ *and* $Re(\eta^*) < 1 - (2c)^{-1}$,

$$\sqrt{n}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\tilde{\Gamma}_1 + \hat{\Gamma}_1)$$

where $\hat{\Gamma}_1 := U \hat{S} U^T$ *with* $[\hat{S}]_{h,j} := \dfrac{c^2}{2c - c(\eta_h + \eta_j) - 1} (\mathbf{v}_h^T \mathbf{v}_j)$.

(c) For $\gamma = 1$ *and* $Re(\eta^*) = 1 - (2c)^{-1}$,

$$\frac{\sqrt{n}}{\sqrt{\ln n}}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\hat{\Gamma}_1^*)$$

where, $\hat{\Gamma}_1^* := U\hat{S}_1^* U^T$ *with* $[\hat{S}_1^*]_{h,j} := c^2(\mathbf{v}_h^T \mathbf{v}_j)$ *if* $(\eta_h + \eta_j) = 2 - c^{-1}$.

Theorem 3.11 (Central Limit Theorem, Theorem 3.3 in [3])**.**
It holds

(a) For $\frac{1}{2} < \gamma < 1$

$$n^{\frac{\gamma}{2}}(Z_n(i) - Z_n(k)) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\Gamma_{\gamma,i,k})$$

where $\Gamma_{\gamma,i,k} := [\tilde{\Gamma}_\gamma]_{j,j} + [\tilde{\Gamma}_\gamma]_{k,k} - 2[\tilde{\Gamma}_\gamma]_{j,k}$, $\hat{\Gamma} := U\hat{S}_\gamma U^T$ *with* $[\hat{S}]_{h,j} := \dfrac{c}{2 - (\eta_h + \eta_j) - 1}(\mathbf{v}_h^T \mathbf{v}_j)$.

(b) For $\gamma = 1$ *and* $Re(\eta^*) < 1 - (2c)^{-1}$,

$$\sqrt{n}(Z_n(i) - Z_n(k)) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\Gamma_{1,i,k})$$

where, $\Gamma_{1,i,k} := [\tilde{\Gamma}_1]_{j,j} + [\tilde{\Gamma}_1]_{k,k} - 2[\tilde{\Gamma}_1]_{j,k}$.

(c) For $\gamma = 1$ *and* $Re(\eta^*) = 1 - (2c)^{-1}$,

$$\sqrt{\frac{n}{\ln n}}(\mathbf{Z}_n - Z_\infty \mathbf{1}) \xrightarrow[n\to\infty]{stably} \mathcal{N}(0,\ Z_\infty(1 - Z_\infty)\hat{\Gamma}_1^*)$$

where, $\Gamma_{1,i,k}^* := [\tilde{\Gamma}_1^*]_{j,j} + [\tilde{\Gamma}_1^*]_{k,k} - 2[\tilde{\Gamma}_1^*]_{j,k}$.

Remark 3.12. Mean-field interaction as particular case of network-based interaction.

This kind of interaction can be expressed in terms of a particular weighted adjacency matrix W *as follows: for any* $1 \le i, k \le N$,

$$w_{i,k} = \frac{\alpha}{N} + \delta_{i,k}(1 - \alpha)$$

with $\alpha \in [0, 1]$ *where* $\delta_{i,k}$ *is equal to* 1 *when* $i = k$ *and to* 0 *otherwise. Note that* W *is irreducible for* $\alpha > 0$.

3.4 New family of models with individual and collective reinforcement rules

In the family of models, defined through (3.9), we are introducing and studying in this work, we are considering a (finite) system of reinforced stochastic processes. There are two kind of reinforcement, one depending only on the component $\xi_{n+1}^l(i)$, one creating the interaction ξ_{n+1}^g and depending on the average over all components. This is modeling a collective reinforcement effect to be confronted with an individual reinforcement. For the sake of simplicity, we choose to consider a mean field interaction. Each reinforcement has its own rate r_n^l (resp. r_n^g). Each rate may have its own asymptotic behavior: $r_n^l \sim c_1 n^{-\gamma_1}$ (resp. $r_n^g \sim c_2 n^{-\gamma_2}$).

3.4.1 Definition of the model

Let us define the following new model. For $i \in \{1, ..., N\}$ and $n \in \mathbb{N}$, we consider the stochastic dynamics defined through the recursive relation

$$Z_{n+1}(i) = (1 - r_n^l - r_n^g)Z_n(i) + r_n^l \xi_{n+1}^l(i) + r_n^g \xi_{n+1}^g, \tag{3.9}$$

where $Z_0(i) = \frac{1}{2}$ and where $\xi_{n+1}^l(i)$ and ξ_{n+1}^g denote local and collective reinforcements random variables. Given $\mathcal{F}_n$, they have independent Bernoulli distributions with

$$\begin{aligned} \mathbb{P}(\xi_{n+1}^l(i) = 1 \mid \mathcal{F}_n) = \psi_1(Z_n(i)) := (1 - 2\lambda_1)Z_n(i) + \lambda_1, \\ \mathbb{P}(\xi_{n+1}^g = 1 \mid \mathcal{F}_n) = \psi_2(Z_n) := (1 - 2\lambda_2)Z_n + \lambda_2, \end{aligned} \tag{3.10}$$

where $\psi_k : [0,1] \longrightarrow [0,1]$ $(k \in \{1,2\})$ with

$$Z_n := \frac{1}{N} \sum_{i=1}^{N} Z_n(i) \tag{3.11}$$

(so called *mean field*) and where $\lambda_1, \lambda_2 \in [0,1]$ are parameters. The local (resp. collective) reinforcement rate are such that for real numbers
$0 \leq r_n^l < 1$ and $0 \leq r_n^g < 1$, $r_n^l \sim \frac{c_1}{n^{\gamma_1}}$ (resp. $r_n^g \sim \frac{c_2}{n^{\gamma_2}}$).

Assumption 3.13. *There exist real constants γ_1, γ_2 and $c > 0$ such that*

$$r_n^l = \frac{c_1}{n^{\gamma_1}} + O(\frac{1}{n^{2\gamma_1}}) \quad \textit{and} \quad r_n^g = \frac{c_2}{n^{\gamma_2}} + O(\frac{1}{n^{2\gamma_2}}). \tag{3.12}$$

In particular, it follows

$$\lim_n n^{\gamma_1} r_n^l = c_1 > 0 \quad \textit{and} \quad \lim_n n^{\gamma_2} r_n^g = c_2 > 0.$$

In all results in Chapter 4 and 5„ we assume that $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$ in order to satisfy the usual assumptions about $(r_n^l)_n$ (same respectively for $(r_n^g)_n$):

$$\sum r_n^l = +\infty, \qquad \sum (r_n^l)^2 < +\infty. \tag{3.13}$$

Indeed, in order to distinguish the values of γ, we consider three regions as follows:
• Taking $\gamma > 1$ yields r_n goes to 0 quickly ($\sum_n r_n < \infty$) and therefore not an interesting case.
• The region $\frac{1}{2} < \gamma_i \leq 1$, that is the case (moderated reinforcement) which we are interested in, in connection with the stochastic algorithm's framework. Moreover, the special case $\gamma = 1$ is related to urn models context, and may be related to some additional assumptions to have a gaussian fluctuation regime.
• The case $\gamma \leq \frac{1}{2}$ causes to have a very strong reinforcement, in the sense $\sum_n r_n^l = +\infty$ and $\sum_n (r_n^l)^2 = +\infty$. That is the case gives as usual the saturation in time limit behavior *i.e.* $Z_\infty \in \{0,1\}$ (*e.g.* [32]). We have obtained some initial results in this case that present them later as a perspective.

We are choosing the transformations ψ_1 and ψ_2 as linear. For the sake of simplicity, we are choosing to symmetrize the model with respect to $\frac{1}{2}$. For the same reason, according to the previously cited works, it is enough to consider the starting conditions all equal to $\frac{1}{2}$.

Remark 3.14. *In the particular case when $\gamma_1 = \gamma_2 = \gamma$, we can rewrite the model such as*

$$Z_{n+1}(i) = (1 - 2r_n)Z_n(i) + r_n\tilde{\xi}_{n+1}(i),$$

where $\tilde{\xi}_{n+1}(i) = \xi^l_{n+1}(i) + \xi^g_{n+1}$, therefore $\tilde{\xi}_{n+1}(i) \in \{0, 1, 2\}$ such that

$$\begin{aligned}\mathbb{P}(\tilde{\xi}_{n+1}(i) = 2|\mathcal{F}_n) &= \mathbb{P}(\xi^l_{n+1}(i) = 1|\mathcal{F}_n) \times \mathbb{P}(\xi^g_{n+1} = 1|\mathcal{F}_n) \\ &= [(1 - 2\lambda_1)Z_n(i) + \lambda_1] \times [(1 - 2\lambda_2)Z_n + \lambda_2].\end{aligned}$$

The other probabilities may be computed in an analogous way. The reinforcement rate remains $r_n \sim cn^{-\gamma}$.

Remark 3.15. *As mentioned earlier, we can have two different behaviors of time limit distribution with defining a proper reinforcement rule which comes from the role of parameters λ_js which in play. Indeed, nullity of parameters λ_js (being equal 0 or not) let the transformation ψ_js to make a different character of reinforcement (Pólya or Friedman respectively).*

Remark 3.16. *In this work the parameters λ_1, λ_2 are kept fixed. Cases where λ_j may converge to 0 depending on n, N are work in progress and will be considered in a forthcoming work. In special case when $\lambda_1 = \lambda_2 = \frac{1}{2}$, there is no reinforcement in the model anymore because the law of evolution is not depending on Z_n and $Z_n(i)$ which is always fixed probability $\frac{1}{2}$ as well as a random walk. Moreover, in the case when $\lambda_1 \neq \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$, although we still have reinforcement at each component individually, we lose the interaction in the sense the collective rule is not effective.*

This is straightforward, to compute the following relationships that will be used frequently in this paper:

$$\mathbb{E}[Z_{n+1}(i)|\mathcal{F}_n] = (1 - 2\lambda_1 r^l_n - r^g_n)Z_n(i) + r^g_n(1 - 2\lambda_2)Z_n + \lambda_1 r^l_n + \lambda_2 r^g_n, \tag{3.14}$$

with its increment point of view

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = \lambda_1 r^l_n\Big(1 - 2Z_n(i)\Big) + \lambda_2 r^g_n\Big(1 - 2Z_n\Big) + r^g_n\Big(Z_n - Z_n(i)\Big), \tag{3.15}$$

Hence,

$$\begin{aligned}\mathbb{E}(Z_{n+1}(i)) &= (1 - 2\lambda_1 r^l_n - r^g_n)\mathbb{E}(Z_n(i)) + (1 - 2\lambda_2)r^g_n\mathbb{E}(Z_n) + \lambda_1 r^l_n + \lambda_2 r^g_n \\ &= \mathbb{E}(Z_n(i)) - 2\lambda_1 r^l_n\mathbb{E}(Z_n(i)) - r^g_n\Big[\mathbb{E}(Z_n(i)) - (1 - 2\lambda_2)\mathbb{E}(Z_n)\Big] + \lambda_1 r^l_n + \lambda_2 r^g_n,\end{aligned} \tag{3.16}$$

and by averaging over i in $\{1, \ldots, N\}$, we have

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = (1 - 2\lambda_1 r^l_n - 2\lambda_2 r^g_n)Z_n + \lambda_1 r^l_n + \lambda_2 r^g_n. \tag{3.17}$$

Then,

$$\mathbb{E}(Z_{n+1}) = (1 - 2\lambda_1 r^l_n - 2\lambda_2 r^g_n)\mathbb{E}(Z_n) + \lambda_1 r^l_n + \lambda_2 r^g_n. \tag{3.18}$$

Remark that when $\lambda_1 = \lambda_2 = 0$, $(Z_n)_n$ is a martingale; however, $(Z_n(i))_n$ is not a martingale anymore.

3.4.2 First results about the variances

Using (3.14) and (3.17), we compute recursive equations, (3.32) and (3.33), satisfied by the variances of $Z_n(i)$ and Z_n.

$$\mathbb{V}\mathrm{ar}(Z_{n+1}(i)|\mathcal{F}_n) = \mathbb{V}\mathrm{ar}\Big[(1-r_n^l-r_n^g)Z_n + r_n^l\xi_{n+1}^l(i) + r_n^g\xi_{n+1}^g\Big|\mathcal{F}_n\Big] \tag{3.19}$$

$$= (r_n^l)^2\mathbb{V}\mathrm{ar}(\xi_{n+1}^l(i)|\mathcal{F}_n) + (r_n^g)^2\mathbb{V}\mathrm{ar}(\xi_{n+1}^g|\mathcal{F}_n) \tag{3.20}$$

$$= (r_n^l)^2\left[(1-2\lambda_1)^2(Z_n(i)-Z_n(i)^2)+\lambda_1-\lambda_1^2\right] + (r_n^g)^2\left[(1-2\lambda_2)^2(Z_n-Z_n^2)+\lambda_2-\lambda_2^2\right], \tag{3.21}$$

then using the *law of total variance* $(*)$, we have

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &\overset{*}{=} \mathbb{E}[\mathbb{V}\mathrm{ar}(Z_{n+1}(i)|\mathcal{F}_n)] + \mathbb{V}\mathrm{ar}[\mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n)]\\ &= (r_n^l)^2\left[(1-2\lambda_1)^2\Big(\mathbb{E}(Z_n(i)) - \mathbb{E}(Z_n(i)^2)\Big)+\lambda_1-\lambda_1^2\right] + (r_n^g)^2\left[(1-2\lambda_2)^2\Big(\mathbb{E}(Z_n)-\mathbb{E}(Z_n^2)\Big)+\lambda_2-\lambda_2^2\right]\\ &\quad + (1-2\lambda_1 r_n^l - r_n^g)^2\mathbb{V}\mathrm{ar}(Z_n(i)) + (r_n^g)^2(1-2\lambda_2)^2\mathbb{V}\mathrm{ar}(Z_n) + 2(1-2\lambda_1 r_n^l - r_n^g)r_n^g(1-2\lambda_2)\mathbb{V}\mathrm{ar}(Z_n).\end{aligned} \tag{3.22}$$

Where in the last equation we used the fact that $\mathrm{Cov}(Z_n(i), Z_n) = \mathbb{V}\mathrm{ar}(Z_n)$. Indeed by symmetry

$$\mathbb{E}(Z_n(i)Z_n) = \frac{1}{N}\sum_{j=1}^N \mathbb{E}(Z_n(j)Z_n) = \mathbb{E}\Big[\Big(\frac{1}{N}\sum_{j=1}^N (Z_n(j)\Big)Z_n\Big] = \mathbb{E}(Z_n^2).$$

Thus,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}|\mathcal{F}_n)) &= \frac{(r_n^l)^2}{N^2}\sum_{i=1}^N \mathbb{V}\mathrm{ar}(\xi_{n+1}^l(i)|\mathcal{F}_n) + (r_n^g)^2\mathbb{V}\mathrm{ar}(\xi_{n+1}^g|\mathcal{F}_n)\\ &= \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(Z_n - \frac{1}{N}\sum_{i=1}^N Z_n(i)^2\Big)+\lambda_1-\lambda_1^2\right] + (r_n^g)^2\left[(1-2\lambda_2)^2(Z_n-Z_n^2)+\lambda_2-\lambda_2^2\right]\end{aligned}$$

therefore,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1-2\lambda_1 r_n^l - 2\lambda_1 r_n^g)^2\mathbb{V}\mathrm{ar}(Z_n)\\ + \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(\mathbb{E}(Z_n) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)\Big)+\lambda_1-\lambda_1^2\right]\\ + (r_n^g)^2\left[(1-2\lambda_2)^2(\mathbb{E}(Z_n)-\mathbb{E}(Z_n^2))+\lambda_2-\lambda_2^2\right].\end{aligned} \tag{3.23}$$

Furthermore we can rewrite the (3.9) as

$$Z_{n+1}(i) - Z_n(i) = r_n^l[\xi_{n+1}(i) - Z_n(i)] + r_n^g[\xi_{n+1} - Z_n] \tag{3.24}$$

and therefore

$$Z_{n+1} - Z_n = r_n^l\left(\frac{1}{N}\sum_{i=1}^N \xi_{n+1}^l(i) - Z_n\right) + r_n^g\left(\xi_{n+1}^g - Z_n\right), \tag{3.25}$$

also we can obtain easily

$$\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^{l}(i)\Big|\mathcal{F}_n\right) = (1-2\lambda_1)Z_n + \lambda_1,$$

$$\mathbb{E}(\xi_{n+1}^{g}|\mathcal{F}_n) = (1-2\lambda_2)Z_n + \lambda_2.$$

Finally, note that defining the following quantities

$$\Delta M_{n+1}^{l} := \frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^{l}(i) - \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^{l}(i)|\mathcal{F}_n\right)$$

$$\Delta M_{n+1}^{g} := \xi_{n+1}^{g} - \mathbb{E}(\xi_{n+1}^{g}|\mathcal{F}_n),$$

which means

$$\Delta M_{n+1}^{l} = \frac{1}{N}\sum_{i=1}^{N}\xi_{n+1}^{l}(i) - [(1-2\lambda_1)Z_n + \lambda_1], \tag{3.26}$$

and

$$\Delta M_{n+1}^{g} = \xi_{n+1}^{g} - [(1-2\lambda_2)Z_n + \lambda_2]. \tag{3.27}$$

and similarly,

$$\Delta \hat{M}_{n+1}^{l} = \xi_{n+1}^{l}(i) - [(1-2\lambda_1)Z_n(i) + \lambda_1], \tag{3.28}$$

and

$$\Delta \hat{M}_{n+1}^{g} = \xi_{n+1}^{g} - [(1-2\lambda_2)Z_n + \lambda_2]. \tag{3.29}$$

Using (3.25), (3.26) and (3.27) it leads to

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2}\right)\left[1 - 2r_n^l\lambda_1 - 2r_n^g\lambda_2\right] + r_n^l\Delta M_{n+1}^{l} + r_n^g\Delta M_{n+1}^{g}. \tag{3.30}$$

Moreover, using (3.24), (3.28) and (3.29) it leads to

$$Z_{n+1}(i) - \frac{1}{2} = \left(Z_n(i) - \frac{1}{2}\right)\left[1 - 2r_n^l\lambda_1 - 2r_n^g\lambda_2\right] + r_n^l\Delta \hat{M}_{n+1}^{l} + r_n^g\Delta \hat{M}_{n+1}^{g}. \tag{3.31}$$

Remark 3.17. *The assumption is* $\forall i \in \{1,..N\}\ Z_0(i) = \frac{1}{2}$. *It is a fix point in the equation between expectations. Thus* $\forall n \in \mathbb{N}$, $\mathbb{E}(Z_n) = \mathbb{E}(Z_n(i)) = \frac{1}{2}$. *Indeed,*

$$\mathbb{E}(Z_{n+1}(i)) = (1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g)\mathbb{E}(Z_n(i)) + (\lambda_1 r_n^l + \lambda_2 r_n^g).$$

Then, when $\lambda_1 \neq 0, \lambda_2 \neq 0$, *the only unique point is* $\frac{1}{2}$. *Same situation holds when* $\lambda_1 = \lambda_2 = 0$.

Thus, from (3.23), we get

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) &= \left[1-4\Big(\lambda_1 r_n^l+\lambda_2 r_n^g-2\lambda_1\lambda_2 r_n^l r_n^g-\lambda_1^2(r_n^l)^2-\lambda_2^2(r_n^g)^2+\frac{(r_n^g)^2}{4}(1-2\lambda_2)\Big)\right]\mathbb{V}\mathrm{ar}(Z_n)\\ &+ \frac{(r_n^l)^2}{N}\left[(1-2\lambda_1)^2\Big(\frac{1}{2}-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\Big)+\lambda_1-\lambda_1^2\right]+\frac{(r_n^g)^2}{4}. \qquad (3.32)\end{aligned}$$

And from (3.22) we get

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \Big[(1-2\lambda_1 r_n^l-r_n^g)^2-(r_n^l)^2(1-2\lambda_1)^2)\Big]\mathbb{V}\mathrm{ar}(Z_n(i))\\ &+ \frac{(r_n^l)^2}{4}+\frac{(r_n^g)^2}{4}\\ &+ 2(1-2\lambda_1 r_n^l-r_n^g)r_n^g(1-2\lambda_2)\mathbb{V}\mathrm{ar}(Z_n). \qquad (3.33)\end{aligned}$$

Remark 3.18. *When $\gamma_1=\gamma_2=\gamma$ we have,*

$$\begin{aligned}\mathbb{E}(Z_{n+1}(i)|\mathcal{F}_n) &= (1-2r_n)Z_n(i)+r_n\mathbb{E}[\tilde{\xi}_{n+1}(i)|\mathcal{F}_n]\\ &= (1-(1+2\lambda_1)r_n)Z_n(i)+(1-2\lambda_2)r_nZ_n+(\lambda_1+\lambda_2)r_n\end{aligned}$$

since,

$$\begin{aligned}\mathbb{E}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n) &= \mathbb{P}(\xi_{n+1}^l(i)=1|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=0|\mathcal{F}_n)+\mathbb{P}(\xi_{n+1}^l(i)=0|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=1|\mathcal{F}_n)\\ &+ 2\mathbb{P}(\xi_{n+1}^l(i)=1|\mathcal{F}_n)\times\mathbb{P}(\xi_{n+1}^g=1|\mathcal{F}_n)\\ &= (1-2\lambda_1)Z_n(i)+(1-2\lambda_2)Z_n+\lambda_1+\lambda_2.\end{aligned}$$

Moreover, using same property, we get

$$\begin{aligned}\mathbb{E}(Z_{n+1}|\mathcal{F}_n) &= (1-2r_n)Z_n+\frac{r_n}{N}\sum_{i=1}^{n}\mathbb{E}(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n)\\ &= \Big(1-2(\lambda_1+\lambda_2)r_n\Big)Z_n+(\lambda_1+\lambda_2)r_n.\end{aligned}$$

In order to characterize $(\mathbb{V}ar(Z_n(i)))_n$, consider

$$\begin{aligned}\mathbb{V}ar(Z_{n+1}(i)|\mathcal{F}_n) &= r_n^2\Big(\Big[(1-2\lambda_1)Z_n(i)+\lambda_1\Big]-\Big[(1-2\lambda_1)^2Z_n(i)^2+\lambda_1^2+2\lambda_1(1-2\lambda_1)Z_n(i)\Big]\\ &+ \Big[(1-2\lambda_2)Z_n+\lambda_2\Big]-\Big[(1-2\lambda_2)^2Z_n^2+\lambda_2^2+2\lambda_2(1-2\lambda_2)Z_n\Big]\Big)\end{aligned}$$

therefore,

$$\begin{aligned}\mathbb{V}ar(Z_{n+1}(i)) &\overset{*}{=} r_n^2[(1-2\lambda_1)^2(\mathbb{E}(Z_n(i))-\mathbb{E}(Z_n(i)^2))+\lambda_1-\lambda_1^2]+[(1-2\lambda_2)^2(\mathbb{E}(Z_n)-\mathbb{E}(Z_n^2))+\lambda_2-\lambda_2^2]\\ &+ (1-r_n(2\lambda_1+1))^2\mathbb{V}ar(Z_n(i))+r_n^2(1-2\lambda_2)^2\mathbb{V}ar(Z_n)+(1-r_n(2\lambda_1+1))(1-2\lambda_2)\mathbb{V}ar(Z_n),\end{aligned}$$

thus,

$$\begin{aligned}\mathbb{V}ar(Z_{n+1}(i)) &= [(1-r_n(2\lambda_1+1))^2 - r_n(1-2\lambda_1)^2]\mathbb{V}ar(Z_n(i)) + \frac{r_n^2}{2} \\ &+ [(1-2\lambda_2)^2 + r_n(1-r_n(2\lambda_1+1))(1-2\lambda_1)]\mathbb{V}ar(Z_n). \end{aligned} \tag{3.34}$$

Furthermore, for $\mathbb{V}ar(Z_n)$ *we have,*

$$\mathbb{V}ar(Z_{n+1}|\mathcal{F}_n) = \frac{r_n^2}{N^2}\sum_{i=1}^{N}\mathbb{V}ar(\tilde{\xi}_{n+1}(i)|\mathcal{F}_n) = r_n^2\left[\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{V}ar(\xi^l_{n+1}(i)|\mathcal{F}_n) + \mathbb{V}ar(\xi^g_{n+1}|\mathcal{F}_n)\right]$$

therefore,

$$\begin{aligned}\mathbb{V}ar(Z_{n+1}) = \Big(1 - 4[(\lambda_1+\lambda_2)r_n - (\lambda_1+\lambda_2)^2 r_n^2] - (1-2\lambda_2)^2 r_n^2\Big)\mathbb{V}ar(Z_n) \\ + \frac{r_n^2}{N}\left[(1-2\lambda_1)^2(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)) + \lambda_1 - \lambda_1^2 + \frac{N}{4}\right]\end{aligned} \tag{3.35}$$

Moreover,

$$Z_{n+1} - \frac{1}{2} = \left(Z_n - \frac{1}{2}\right)[1 - 2r_n(\lambda_1+\lambda_2)] + r_n\Delta\tilde{M}_{n+1}(i), \tag{3.36}$$

where $\Delta\tilde{M}_{n+1}(i) = \Delta M^l_{n+1}(i) + \Delta M^g_{n+1}$. *This concludes this remark.*

We are now proving some intermediate results useful for proofs of next chapter.

Lemma 3.19. *When* $\lambda_1 = \lambda_2 = 0$*, it holds*

$$\lim_{n\to\infty}\mathbb{V}ar(Z_n) < \frac{1}{4}.$$

Moreover, it follows

$$\sup_n \mathbb{E}(Z_n^2) < \frac{1}{2}.$$

Remark, this implies

$$\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n^2(i))\right) > 0. \tag{3.37}$$

Proof.
Since for all i, $\mathbb{E}(Z_n^2(i)) \le \mathbb{E}(Z_n(i)) = \frac{1}{2}$, it holds obviously $\mathbb{V}\mathrm{ar}(Z_n) \le \frac{1}{4}$. Using (3.32) with $\lambda_1 = \lambda_2 = 0$ gives:

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = \Big(1 - (r_n^g)^2\Big)\mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^g)^2}{4} + \frac{(r_n^l)^2}{N}\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\right).$$

From above equalities, we have

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) \leq \Big(1-(r_n^g)^2\Big)\mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^g)^2}{4} + \frac{(r_n^l)^2}{2N},$$

since

$$\left|\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n^2(i))\right| \leq \frac{1}{2}. \tag{3.38}$$

Let $x_n := \frac{1}{4} - \mathbb{V}\mathrm{ar}(Z_n) \geq 0$, it is equivalent to

$$x_{n+1} \geq \Big(1-(r_n^g)^2\Big)x_n$$

from which it follows

$$x_n \geq x_0 \prod_{k=0}^{n-1}\Big(1-(r_k^g)^2\Big).$$

Since $\sum_n (r_n^g)^2 < +\infty$, we obtain $\lim_{n\to\infty} x_n > 0$.

Moreover, it holds $\mathbb{E}(Z_{n+1}^2|\mathcal{F}_n) = Z_n^2 + \mathbb{V}\mathrm{ar}(Z_{n+1}|\mathcal{F}_n)$. Then $\mathbb{E}(Z_{n+1}^2|\mathcal{F}_n) \geq Z_n^2$ so, $(Z_n^2)_n$ is a bounded submartingale. Consequently, $\sup_n \mathbb{E}(Z_n^2) = \lim_n \mathbb{E}(Z_n^2) < \frac{1}{2}$. □

For the three other cases about (λ_1, λ_2), let us prove the following lemma.

Lemma 3.20. *If at least one of the following conditions holds true: $\lambda_1 > 0$, $\lambda_2 > 0$, then it holds $\lim_{n\to\infty} \mathbb{V}ar(Z_n) = 0$. Thus in particular, $\lim_{n\to\infty} \mathbb{V}ar(Z_n) < 1/4$ and*

$$\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n^2(i))\right) > 0.$$

Proof.

To prove this, use (3.32), synthetically written as:

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1-4\varepsilon_n)\mathbb{V}\mathrm{ar}(Z_n) + K_n^1(r_n^l)^2 + \frac{1}{4}(r_n^g)^2$$

where

$$\varepsilon_n := \lambda_1 r_n^l + \lambda_2 r_n^g - 2\lambda_1\lambda_2 r_n^l r_n^g - \lambda_1^2(r_n^l)^2 - \lambda_2^2(r_n^g)^2 + \frac{(r_n^g)^2}{4}(1-2\lambda_2)$$

and

$$K_n^1 := \frac{1}{N}\left[(1-2\lambda_1)^2\Big(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n^2(i))\Big) + \lambda_1 - \lambda_1^2\right].$$

It holds $\sum_n \varepsilon_n = +\infty$ in all the considered cases, due to the fact it is assumed $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. Using (3.38), it holds

$$0 \leq K_n^1 \leq \frac{1}{2}(1-2\lambda_1)^2 + \lambda_1 - \lambda_1^2 \leq 1.$$

It follows $\mathbb{V}\mathrm{ar}(Z_n) \leq y_n$ where $(y_n)_n$ is the sequence defined in appendix'lemma A.1 through (A.3) with the same ε_n and $K\delta_n$ is $\frac{1}{N}(r_n^l)^2 + \frac{(r_n^s)^2}{4}$. Thus, using Lemma A.1, we get $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) = 0$.

Remark, that using the same argument as previously, $((Z_n^2))_{n\geq 0}$ is a sub-martingale, thus we get in these cases too

$$\sup_n \mathbb{E}(Z_n^2) < \frac{1}{2}.$$

□

Chapter 4

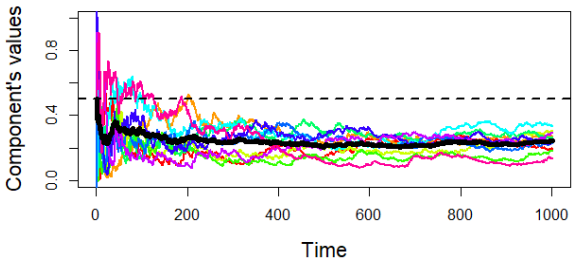
Almost sure synchronization

In this chapter we study the convergence of $(Z_n)_n$ and the synchronization phenomenon. Indeed, we obtain different kind of time-limit (deterministic or random) for $(Z_n)_n$ according to the nullity of λ_1, λ_2. Moreover L^2 and a.s. synchronization are proven to always hold. For those, we need the tools such as Lemma A.2 and concept of quasi-martingale that we explain them below.
This Chapter is organized as follows: First of all, in section 4.1 we represent some numerical simulations in order to have more intuition about the results. In section 4.2 we present the concept and definition of quasi-martingale. In section 4.3 we obtain the L^2-synchronization and in section 4.4 prove that a.s. convergence holds towards a limit Z_∞ shared by all the components (synchronization). Two main cases are to be distinguished : Theorems 4.3 and 4.4 deals with cases where $Z_\infty = \frac{1}{2}$ (the word synchronization is abusive in this situation) where as Theorem 4.6 deals with Z_∞ random.

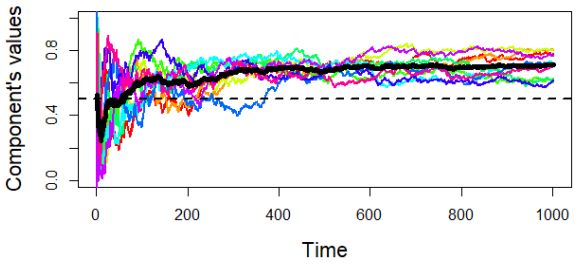
4.1 Numerical simulations

In this section we present some illustrations by numerical simulation of evolution of system that was defined in the model through (3.9). We consider the all cases ($\lambda_1 = 0$, $\lambda_2 = 0$), ($\lambda_1 > 0$, $\lambda_2 = 0$), ($\lambda_1 = 0$, $\lambda_2 > 0$) and ($\lambda_1 > 0$, $\lambda_2 > 0$) for some chosen (γ_1, γ_2). We can see that the realizations are coherent with the theorem that we have stated above. Eventually, we selected six samples for each cases.

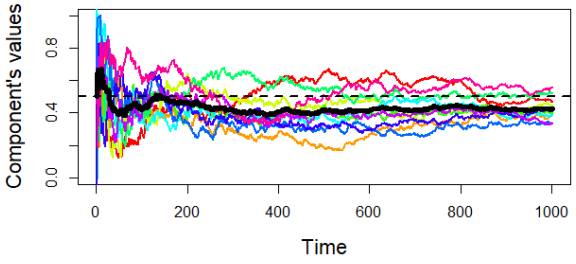
In particular, first we present in Figure 4.1 the case $\lambda_1 = 0$ and $\lambda_2 = 0$ in where by Theorems 4.6 and 4.10 the time-limit converges to a random limit Z_∞ however the synchronization still holds a.s.



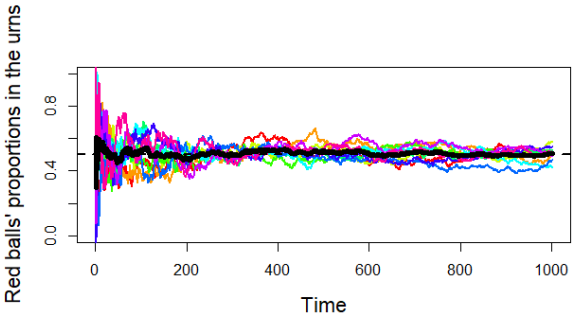
(a) Sample path 1



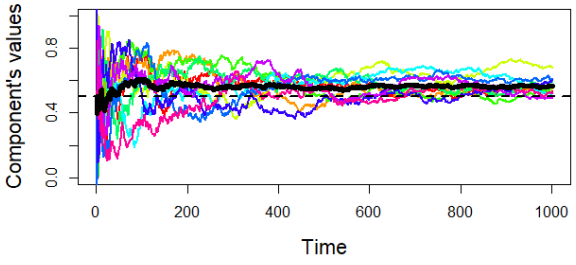
(b) Sample path 2



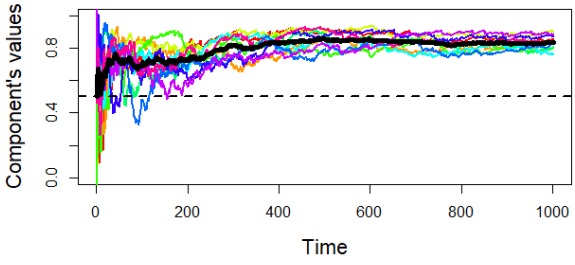
(c) Sample path 3



(d) Sample path 4



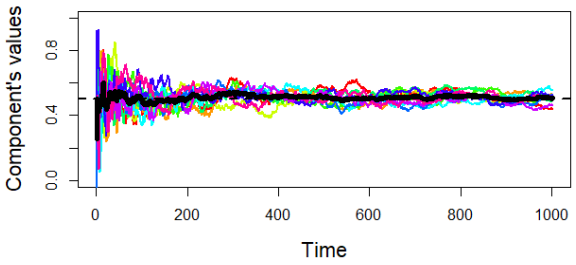
(e) Sample path 5



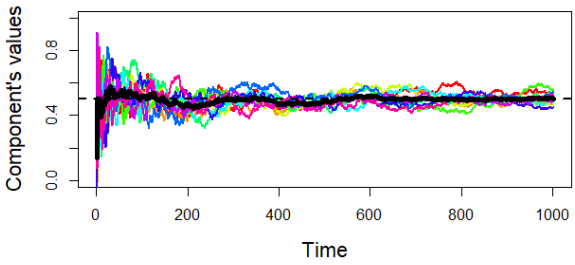
(f) Sample path 6

Figure 4.1 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0$ and $\lambda_2 = 0$. With the reinforcement rates $\gamma_1 = 0.7$ and $\gamma_2 = 0.9$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

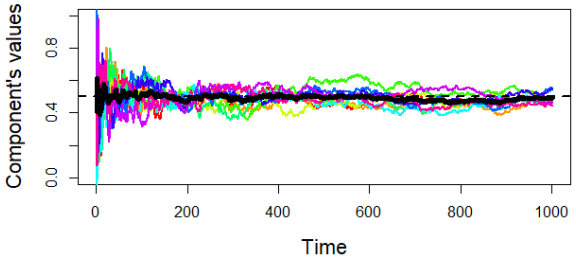
Below, we present in Figure 4.2 the case $\lambda_1 > 0$ and $\lambda_2 > 0$ in where by Theorems 4.3 and 4.8 the time-limit converges to a deterministic limit 1/2 and the synchronization phenomenon still holds a.s.



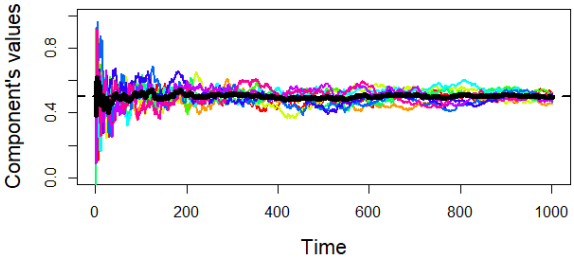
(a) Sample path 1



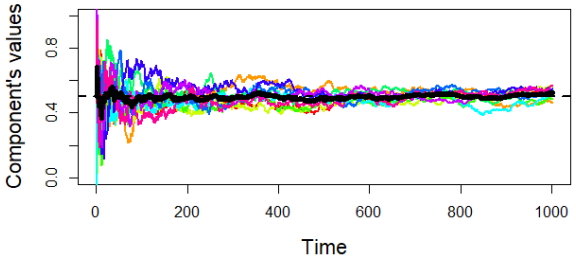
(b) Sample path 2



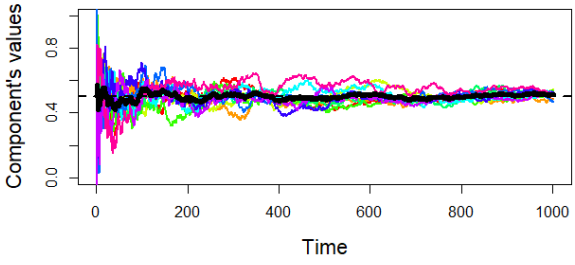
(c) Sample path 3



(d) Sample path 4



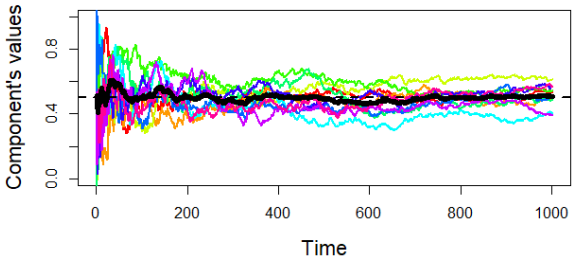
(e) Sample path 5



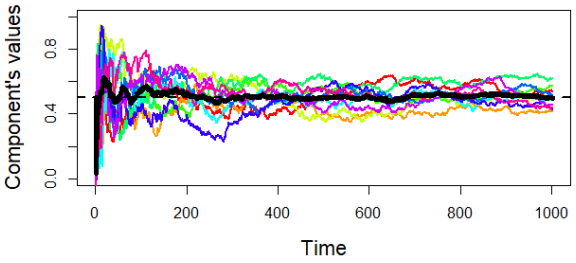
(f) Sample path 4

Figure 4.2 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$ ($\lambda_j > 0$). With the reinforcement rates $\gamma_1 = 0.7$ and $\gamma_2 = 0.9$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

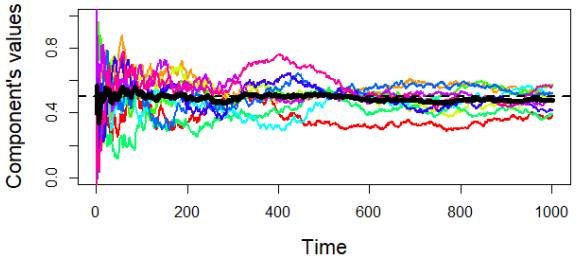
Below, we present in Figure 4.3 the case $\lambda_1 = 0$ and $\lambda_2 > 0$ in where by Theorems 4.4 and 4.9 the time-limit converges to the limit $1/2$ and the synchronization phenomenon still holds a.s.



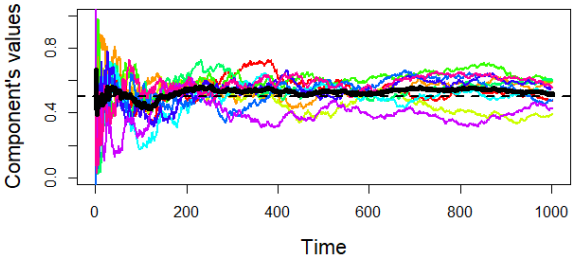
(a) Sample path 1



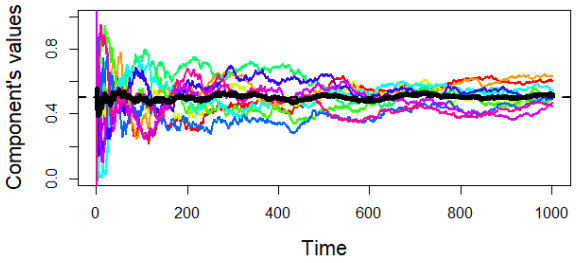
(b) Sample path 2



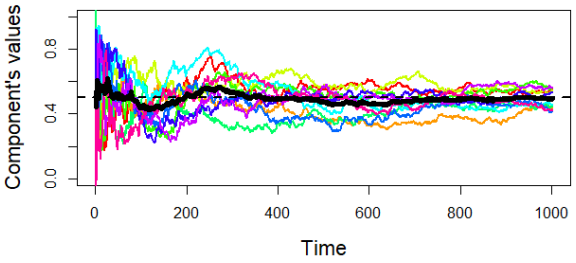
(c) Sample path 3



(d) Sample path 4



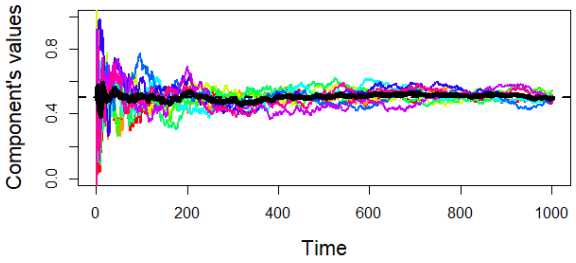
(e) Sample path 5



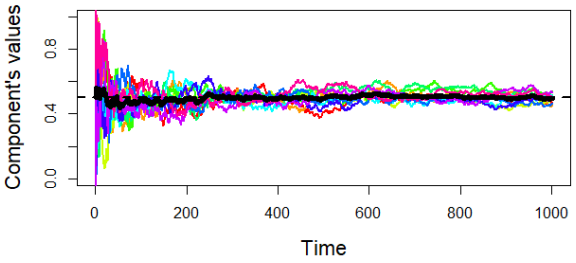
(f) Sample path 6

Figure 4.3 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0$ and $\lambda_2 = 0.6$ ($\lambda_1 = 0$ and $\lambda_2 > 0$). With the reinforcement rates $\gamma_1 = 0.7$ and $\gamma_2 = 0.9$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

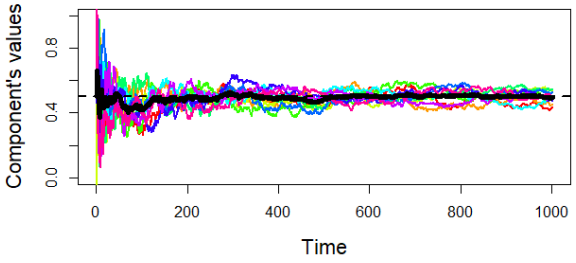
Below, we present in Figure 4.4 the case $\lambda_1 = 0$ and $\lambda_2 > 0$ in where by Theorems 4.4 and 4.9 the time-limit converges to the limit $1/2$ and the synchronization phenomenon still holds a.s.



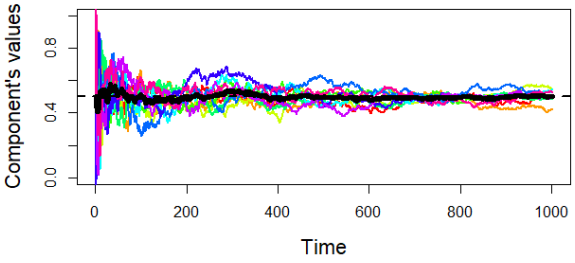
(a) Sample path 1



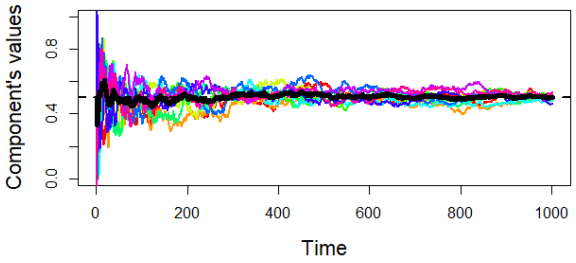
(b) Sample path 2



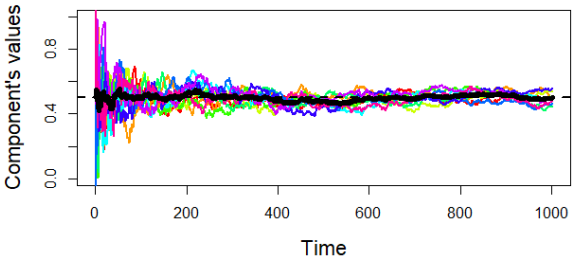
(c) Sample path 3



(d) Sample path 4



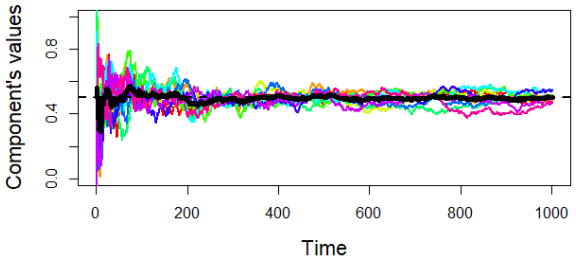
(e) Sample path 5



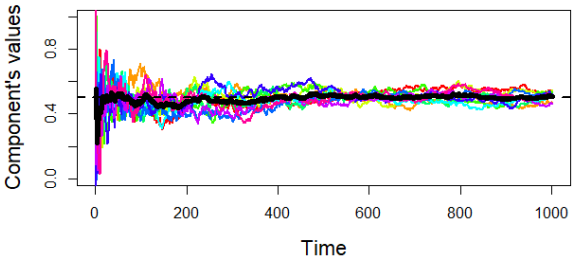
(f) Sample path 6

Figure 4.4 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0$ and $\lambda_2 = 0.6$ ($\lambda_1 = 0$ and $\lambda_2 > 0$). With the reinforcement rates $\gamma_1 = 0.9$ and $\gamma_2 = 0.7$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

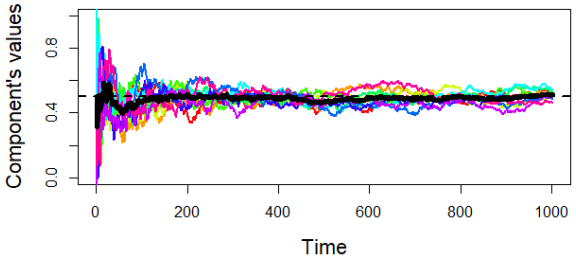
Below, we present in Figure 4.5 the case $\lambda_1 > 0$ and $\lambda_2 = 0$ in where by Theorems 4.4 and 4.9 the time-limit converges to the limit $1/2$ and the synchronization phenomenon still holds a.s.



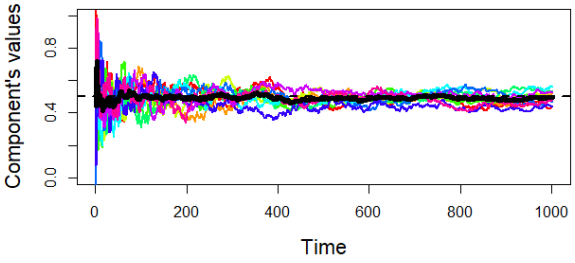
(a) Sample path 1



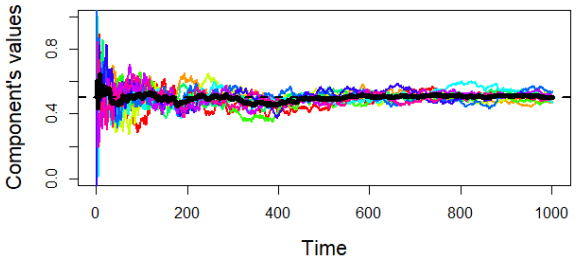
(b) Sample path 2



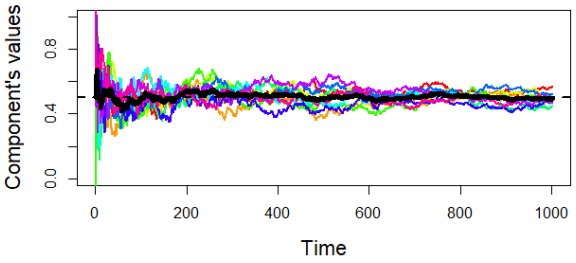
(c) Sample path 3



(d) Sample path 4



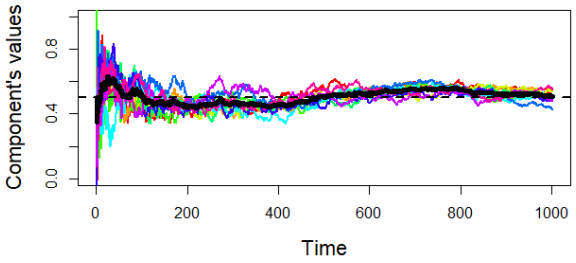
(e) Sample path 5



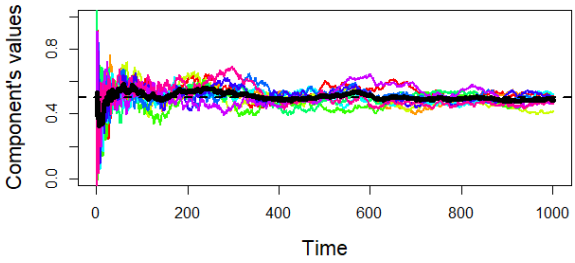
(f) Sample path 6

Figure 4.5 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0.4$ and $\lambda_2 = 0$ ($\lambda_1 > 0$ and $\lambda_2 = 0$). With the reinforcement rates $\gamma_1 = 0.7$ and $\gamma_2 = 0.9$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

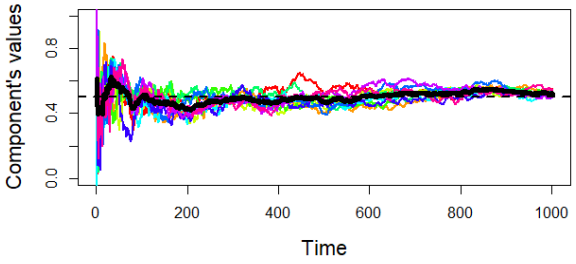
Below, we present in Figure 4.6 the case $\lambda_1 > 0$ and $\lambda_2 = 0$ in where by Theorems 4.4 and 4.9 the time-limit converges to the limit $1/2$ and the synchronization phenomenon still holds a.s.



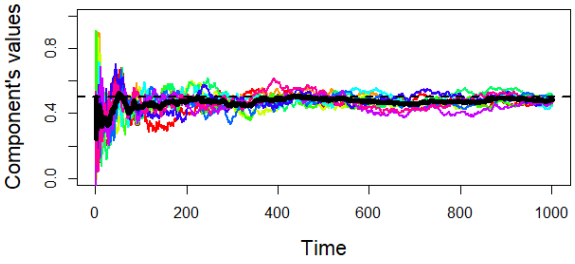
(a) Sample path 1



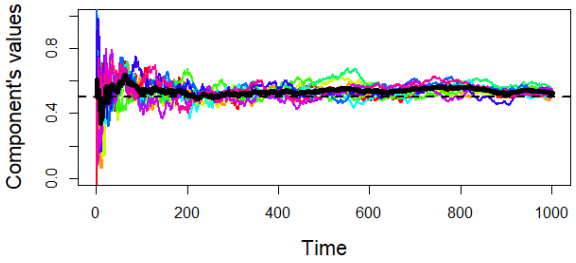
(b) Sample path 2



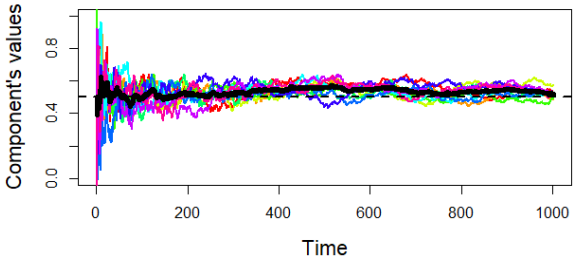
(c) Sample path 3



(d) Sample path 4



(e) Sample path 5



(f) Sample path 6

Figure 4.6 – Illustration of 6 sample paths; $N = 10$ components through mean-field (black curve) interacting are considered for the case $\lambda_1 = 0.4$ and $\lambda_2 = 0$ ($\lambda_1 > 0$ and $\lambda_2 = 0$). With the reinforcement rates $\gamma_1 = 0.9$ and $\gamma_2 = 0.7$ ($\frac{1}{2} < \gamma_j \leq 1$). Each component starts with initial value $Z_0(i) = \frac{1}{2}$ (the black horizontal dotted line). The time is the x-axis. Represented along the y-axis are the trajectories of value of components.

4.2 Quasi-martingales

Quasi-martingales are a generalization of martingales, super-martingales and sub-martingales. They were first introduced by Donald L. Fisk in 1965 [44] in order to extend the Doob-Meyer decomposition to a larger class of processes, showing that continuous quasi-martingales can be decomposed into martingale and finite variation terms. In 1967, this was later extended to right-continuous processes by Orey [79]. The way in which quasi-martingales relate to super-martingales and sub-martingales is very similar to how functions of finite variation relate to increasing and decreasing functions. In particular, by the Jordan decomposition, any finite variation function on an interval decomposes as the sum of an increasing and a decreasing function. Similarly, a stochastic process is a quasi-martingale if and only if it can be written as the sum of a sub-martingale and a super-martingale. This result was shown by Rao in 1969 [86], and means that much of the theory of sub-martingales can be extended without much work to also cover quasi-martingales. Often, given a process, it shall to show that it is a semi-martingale so that the techniques of stochastic calculus can be applied. If there is no obvious decomposition into local martingale and finite variation terms, then, one way of doing this is to show that it is a quasi-martingale.

Definition 4.1. *Consider the filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\geq 0}, \mathbb{P})$. *The mean variation of an integrable stochastic process* $(Z_n)_n$ *on an interval* $[0,T]$ *is*

$$MV_T(Z) = \sup_{0\leq n\leq T} \mathbb{E}\left[\sum_{k=1}^{n} \mid \mathbb{E}(Z_k - Z_{k-1} \mid \mathcal{F}_{k-1}) \mid \right].$$

A quasi-martingale, then, is a process with finite mean variation on each bounded interval.

Theorem 4.2 (Almost sure convergence of quasi-martingales, Theorem 9.4 in [76])**.** *Let* $(Z_n)_{n\in\mathbb{N}}$ *be a real adapted process which satisfies the following conditions:*

1) $\sum_n \mathbb{E}\Big[\mid \mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) - Z_n \mid \Big] < \infty$,

2) $\sup_n \mathbb{E}(Z_n^-) < \infty$,

where $Z_n^- = \max\{-Z_n, 0\}$ *and therefore* $\mathbb{E}(Z_n^-) = \int_{-\infty}^{0} P(Z_n < x)dx$.

Then the sequence $(Z_n)_{n\in\mathbb{N}}$ *converges a.s. towards an integrable variable* Z_∞ *and*

$$\mathbb{E}|Z_\infty| \leq \liminf_n \mathbb{E}|Z_n| < \infty.$$

4.3 L^2-synchronization

4.3.1 Case of a deterministic time-asymptotics

We call *deterministic*, the case when the time limit $Z_\infty(i)$ of $(Z_n(i))_n$ $(n \to \infty)$ is not random. Moreover, it will be proven that $Z_\infty(i)$ is not depending on i. This behavior corresponds to cases where at least one of the following assumptions is true $\lambda_1 > 0$ or $\lambda_2 > 0$. The mean field process $(Z_n)_n$ is not a martingale. In order to investigate the behavior of the interacting system, we first consider the time limits of the variances $\mathbb{V}\mathrm{ar}(Z_n(i))$ and $\mathbb{V}\mathrm{ar}(Z_n)$. Second we show that L^2-synchronization holds *i.e.* $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n(i) - Z_n) = 0$ for $N > 1$. We get the rates of convergence as well as synchronization one too and compare them. Finally, in next section 4.4, we prove that synchronization holds almost surely and the deterministic limit is $Z_\infty := \frac{1}{2}$ a.s. consequently.

In the following results, Lemma A.2 will be used many times, to state the rates of convergence. Please find, in A.3, a list summarizing all the specific sequences used in the different cases, according to the recurrence equations considered.

Theorem 4.3. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^s)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. For any $\lambda_1 > 0$ and $\lambda_2 > 0$ following results hold:*

i) asymptotics of variances $(n \to \infty)$:
$\mathbb{V}ar(Z_n) = O(\frac{1}{n^\gamma})$ and $\mathbb{V}ar(Z_n(i)) = O(\frac{1}{n^\gamma})$ where $\gamma := \min(\gamma_1, \gamma_2)$;

ii) behavior of the L^2-distance between Z_n and $Z_n(i)$ when $n \to \infty$:

a) if $\gamma_1 \le \gamma_2$, then $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{\gamma_1}})$,

b) if $\gamma_2 < \gamma_1$, then $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{2\gamma_1 - \gamma_2}})$.

Proof.
((i) of Theorem 4.3)

First consider the equation (3.32) which behaves like

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4\lambda r_n + \mathbf{o}(r_n))\mathbb{V}\mathrm{ar}(Z_n) + K_n (r_n)^2 + \mathbf{o}((r_n)^2),$$

where $\lambda = \begin{cases} \lambda_1 & \text{if } \gamma_1 < \gamma_2 \\ \lambda_2 & \text{if } \gamma_1 > \gamma_2 \end{cases}$.

- If $\gamma_1 < \gamma_2$ then $A = 4\lambda_1$ and

$$K_n = \frac{1}{N}\left((1 - 2\lambda_1)^2\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}(Z_n^2(i))\right) + \lambda_1 - \lambda_1^2\right)$$

is bounded and $\lim_{n\to\infty} K_n > 0$. Indeed, since $\mathbb{E}(Z_n^2) < \frac{1}{2}$, we get

$$\sum_{i=1}^{N} \mathbb{E}(Z_n(i)^2) = \mathbb{E}(Z_n^2(j)) + \sum_{i=1,\ i\neq j}^{N-1} \mathbb{E}(Z_n^2(i)) < \frac{1}{2} + \frac{N-1}{2} = \frac{N}{2}$$

thus, by Lemma A.2, it holds $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{\gamma_1}})$.

- If $\gamma_1 > \gamma_2$ then $A = 4\lambda_2$ and $K_n = \frac{1}{4}$ thus we get $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{\gamma_2}})$.

- Hence in both cases, it holds $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) = 0$. Remark, this was already proved in previous chapter. Here we precise the rate of convergence, using Lemma A.2.

In order to investigate the behavior of $\mathbb{V}\mathrm{ar}(Z_n(i))$, consider (3.33) in the three following cases

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \left[1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda^2 (r_n^l)^2 + 4\lambda_1 r_n^l r_n^g - (r_n^l)^2(1-2\lambda_1)^2\right]\mathbb{V}\mathrm{ar}(Z_n(i)) \\ &+ \frac{(r_n^l)^2}{4} + \frac{(r_n^g)^2}{4} + 2(1 - 2\lambda_1 r_n^l - r_n^g) r_n^g (1-2\lambda_2)\mathbb{V}\mathrm{ar}(Z_n).\end{aligned}$$

- When $\gamma_1 < \gamma_2$, since $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{\gamma_1}})$ thus,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= \left[1 - 4\lambda_1 r_n^l + \mathbf{o}(r_n^l)\right]\mathbb{V}\mathrm{ar}(Z_n(i)) \\ &+ \frac{(r_n^l)^2}{4} + \mathbf{o}((r_n^l)^2)\end{aligned}$$

then $A = 4\lambda_1$ and $K_n = \frac{1}{4}$ which implies $\mathbb{V}\mathrm{ar}(Z_n(i)) = O(\frac{1}{n^{\gamma_1}})$.

- When $\gamma_2 < \gamma_1$ we have $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{\gamma_2}})$. Thus,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) &= [1 - 2r_n^g + \mathbf{o}(r_n^g)]\mathbb{V}\mathrm{ar}(Z_n(i)) \\ &+ \frac{(r_n^g)^2}{4} + 2(1-2\lambda_2)(r_n^g)^2\end{aligned}$$

then $A = 2$ and $K_n = \left[\frac{1}{4} + \frac{2(1-2\lambda_2)}{16\lambda_2}\right] = \frac{1}{8\lambda_2}$. It implies by Lemma A.2, that $\mathbb{V}\mathrm{ar}(Z_n(i)) = O(\frac{1}{n^{\gamma_2}})$.

- When $\gamma_1 = \gamma_2$, using (3.35) we have

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4(\lambda_1+\lambda_2)r_n - N(1-2\lambda_2)^2 r_n^2)\mathbb{V}\mathrm{ar}(Z_n) + r_n^2 K_n$$

where $A = 4(\lambda_1 + \lambda_2)$ and

$$K_n = \frac{1}{N}\left((1-2\lambda_1)^2\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)\right) + \lambda_1 - \lambda_1^2 + \frac{N}{4}\right),$$

which is bounded and $\lim_{n\to\infty} K_n > 0$, which implies by Lemma A.2 $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^\gamma})$. In the case when $\gamma = 1$ and $\lambda_1 + \lambda_2 = \frac{1}{4}$, $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{\log n}{n})$. Moreover, using (3.34) and the fact $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^\gamma})$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}(i)) \;=\; \Big[(1-r_n(2\lambda_1+1))^2 - r_n(1-2\lambda_1)^2\Big]\mathbb{V}\mathrm{ar}(Z_n(i)) + r_n^2 K_n$$

then $A = 3+4\lambda_1^2$ and $K_n = \frac{1}{2} + [(1-2\lambda_2)^2 + r_n(1-r_n(2\lambda_1+1))(1-2\lambda_1)]$ which, using Lemmas at the end of the last chapter, implies by Lemma A.2 $\mathbb{V}\mathrm{ar}(Z_n(i)) = O(\frac{1}{n^\gamma})$. □

Proof.
(proof of (ii) Theorem 4.3)

Consider the following recursive equation satisfied, for any $i \in \{1,\dots,N\}$ by the L^2-distance between one component and the mean field. With

$$x_n := \mathbb{E}[(Z_n(i)-Z_n)^2] = \mathbb{V}\mathrm{ar}(Z_n(i)-Z_n), \tag{4.1}$$

it holds

$$\begin{aligned}
x_{n+1} \;&\overset{*}{=}\; \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left[(1-r_n^l-r_n^g)(Z_n(i)-Z_n) + r_n^l(\xi_n^l(i) - \frac{1}{N}\sum_j \xi_n^l(j))\Big|\mathcal{F}_n\right]\right)\\
&+\; \mathbb{V}\mathrm{ar}\Big[Z_n(i) - 2\lambda_1 r_n^l Z_n(i) + r_n^g(Z_n(i) - (1-2\lambda_2)Z_n) - Z_n(1-2\lambda_1 r_n^l - r_n^g)\Big]\\
&=\; (r_n^l)^2\mathbb{E}\Big[\mathbb{V}\mathrm{ar}(\xi_n^l(i) - \frac{1}{N}\sum_i \xi_n^l(i)|\mathcal{F}_n)\Big] + \mathbb{V}\mathrm{ar}\Big[(1-2\lambda_1 r_n^l - r_n^g)(Z_n(i)-Z_n)\Big]\\
&=\; (1-2\lambda_1 r_n^l - r_n^g)^2\mathbb{V}\mathrm{ar}(Z_n(i)-Z_n)\\
&+\; (r_n^l)^2\Big((1-\frac{1}{N})^2 + (\frac{N-1}{N^2})\Big)\mathbb{E}\Big[\mathbb{V}\mathrm{ar}(\xi_n^l(i)|\mathcal{F}_n)\Big]\\
&=\; (1-2\lambda_1 r_n^l - r_n^g)^2 x_n\\
&+\; \frac{N-1}{N}(r_n^l)^2\left[\Big[(1-2\lambda_1)\mathbb{E}(Z_n(i)) + \lambda_1\Big] - \Big[\frac{(1-2\lambda_1)^2}{N}\mathbb{E}(Z_n(i)^2) + \lambda_1^2 + 2\lambda_1(1-2\lambda_1)\mathbb{E}(Z_n)\Big]\right].
\end{aligned}$$

Therefore we get

$$x_{n+1} \;=\; \Big(1 - 4\lambda_1 r_n^l - 2r_n^g + 4\lambda_1^2(r_n^l)^2 + (r_n^g)^2 + 2\lambda_1 r_n^l r_n^g\Big)x_n + (r_n^l)^2 J_n, \tag{4.2}$$

where $J_n = \frac{N-1}{N}\left(\frac{1}{2} - [\frac{(1-2\lambda_1)^2}{N}\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2]\right)$ is bounded and not equal zero for $N > 1$.

(a) For $\gamma_1 < \gamma_2$ the relation (4.2) gives $x_{n+1} = [1-4\lambda_1 r_n^l - \mathbf{o}(r_n^l)]x_n + (r_n^l)^2 J_n$ which implies by Lemma A.2, $x_n = O(\frac{1}{n^{\gamma_1}})$ where $A = 4\lambda_1$.

Also for $\gamma_1 = \gamma_2$ we have from (4.2)

$$x_{n+1} \;=\; [(1-r_n-2\lambda_1 r_n)^2]x_n + r_n^2 J_n = [1-(2+4\lambda_1)r_n + \mathbf{o}(r_n)]x_n + r_n^2 J_n$$

which implies by Lemma A.2, $x_n = O(\frac{1}{n^\gamma})$ where $A = 2+4\lambda_1$.

(b) When $\gamma_2 < \gamma_1$, $x_{n+1} = (1-2r_n^g + \mathbf{o}(r_n^g))x_n + J_n(r_n^l)^2$ where $A = 2$ implies by Lemma A.2, $x_n = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$.

Thus in all cases, $\lim_{n\to\infty}\mathbb{E}[(Z_n(i)-Z_n)^2] = 0$. □

Two others regimes of parameters $\lambda_1,\ \lambda_2$ lead to the following results.

Theorem 4.4. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. In the following cases: either ($\lambda_1 > 0$ and $\lambda_2 = 0$) or ($\lambda_1 = 0$ and $\lambda_2 > 0$), the following table summarizes the L^2 speed of convergence with $\gamma := \min(\gamma_1, \gamma_2)$ and where C denotes a generic constant.*

	$\lambda_1 \neq 0,\ \lambda_2 = 0$	$\lambda_1 = 0,\ \lambda_2 \neq 0$
$\gamma_1 \leq \gamma_2$	$\mathbb{V}ar(Z_n) = O(\frac{1}{n^{\gamma_1}})$ $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{\gamma_1}})$	$\mathbb{V}ar(Z_n) = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$ $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$
$\gamma_2 < \gamma_1$	$\mathbb{V}ar(Z_n) = O(\frac{1}{n^{2\gamma_2-\gamma_1}})$ $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$	$\mathbb{V}ar(Z_n) = O(\frac{1}{n^{\gamma_2}})$ $\mathbb{E}\left([Z_n - Z_n(i)]^2\right) = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$

Proof.
(Theorem 4.4)

We shall consider two different situations of λ_j and different relationships between γ_1 and γ_2.

- **Case $\lambda_1 \neq 0, \lambda_2 = 0$.**

First consider to the recursive equation (3.32) of $\mathbb{V}\text{ar}(Z_n)$.

- When $\gamma_1 < \gamma_2$,

$$\mathbb{V}\text{ar}(Z_{n+1}) = [1 - 4\lambda_1 r_n^l + \mathbf{o}(r_n^l)]\mathbb{V}\text{ar}(Z_n) + K_n(r_n^l)^2,$$

where

$$K_n = \frac{1}{N}\left(\frac{1}{2} - \frac{(1-2\lambda_1)^2}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2) + (\lambda_1 - \lambda_1^2)\right),$$

and $A = 4\lambda_1$, it implies $\mathbb{V}\text{ar}(Z_n) = O(\frac{1}{n^{\gamma_1}})$. It thus means $(Z_n)_n$ converges to a constant.

To study the synchronization, consider to the L^2-distance (4.2) which behaves as follows

$$x_{n+1} = (1 - 4\lambda_1 r_n^l + \mathbf{o}(r_n^l))x_n + J_n(r_n^l)^2,$$

where $J_n = \frac{N-1}{N}\left(\frac{1}{2} - [\frac{(1-2\lambda_1)^2}{N}\mathbb{E}(Z_n(i)^2) + \lambda_1 - \lambda_1^2]\right)$ and where $A = 4\lambda_1$. One can then derive $x_n = O(\frac{1}{n^{\gamma_1}})$ which implies $\lim_{n\to\infty} x_n = 0$.

- When $\gamma_1 = \gamma_2 (=: \gamma)$, so (3.32)

$$\mathbb{V}\text{ar}(Z_{n+1}) = (1 - 4\lambda_1 r_n + \mathbf{o}(r_n))\mathbb{V}\text{ar}(Z_n) + K_n r_n^2,$$

where $A = 4\lambda_1$ which implies $\mathbb{V}\text{ar}(Z_n) = O(\frac{1}{n^{\gamma}})$ ($\frac{\log n}{n}$ when $\gamma = 1$ and $\lambda_1 = \frac{1}{4}$). To study the L^2-distance's behavior, consider (4.2)

$$x_{n+1} = (1 - (2 + 4\lambda_1)r_n + \mathbf{o}(r_n))\, x_n + J_n r_n^2,$$

where $A = (2 + 4\lambda_1)$ which implies $x_n = O(\frac{1}{n^{\gamma}})$.

- When $\gamma_2 < \gamma_1$. Let us consider the recursive equation (3.32),

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4\lambda_1 r_n^l)\mathbb{V}\mathrm{ar}(Z_n) + K_n(r_n^g)^2,$$

where $A = 4\lambda_1$ which implies $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{2\gamma_2-\gamma_1}})$ which implies $(Z_n)_n$ converges to a constant. Moreover considering the L^2-distance's (4.2) behaves,

$$x_{n+1} = (1 - 2r_n^g + \mathbf{o}(r_n^g))x_n + J_n(r_n^l)^2,$$

where $A = 2$ which implies $x_n = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$.

- **Case $\lambda_1 = 0, \lambda_2 \neq 0$.**
- When $\gamma_1 < \gamma_2$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4\lambda_2 r_n^g + \mathbf{o}(r_n^g))\mathbb{V}\mathrm{ar}(Z_n) + K_n(r_n^l)^2,$$

where $A = 4\lambda_2$ and $K_n = \frac{1}{N}\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)\right)$ which implies $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$, thus it means $(Z_n)_n$ converges to a constant. To study the synchronization, consider the L^2-distance which behaves as follows

$$x_{n+1} = (1 - 2r_n^g + \mathbf{o}(r_n^g))x_n + J_n(r_n^l)^2,$$

where $A = 2$ which implies $x_n = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$.

- When $\gamma_2 < \gamma_1$. Let us consider the recursive equation (3.32),

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4\lambda_2 r_n^g + \mathbf{o}(r_n^g))\mathbb{V}\mathrm{ar}(Z_n) + K_n(r_n^g)^2,$$

where $A = 4\lambda_2$ which implies $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^{\gamma_2}})$ and thus that $(Z_n)_n$ converges to a constant.To study the synchronization, consider the L^2-distance which behaves as follows

$$x_{n+1} = (1 - 2r_n^g)x_n + J_n(r_n^l)^2,$$

where $A = 2$ which implies $x_n = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$.

- When $\gamma_1 = \gamma_2$. It holds

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - 4\lambda_2 r_n + \mathbf{o}(r_n))\mathbb{V}\mathrm{ar}(Z_n) + K_n r_n^2,$$

where $A = 4\lambda_2$ which implies $\mathbb{V}\mathrm{ar}(Z_n) = O(\frac{1}{n^\gamma})$ ($\frac{\log n}{n}$ when $\gamma = 1$ and $\lambda_2 = \frac{1}{4}$).

To study the L^2-distance's behavior,

$$x_{n+1} = (1 - 2r_n + \mathbf{o}(r_n)x_n + J_n r_n^2,$$

which implies $x_n = O\left(\frac{1}{n^\gamma}\right)$. □

Remark 4.5. *(Comparison of convergence rates and synchronization rates)*
In the case $\lambda_1 > 0$, $\lambda_2 > 0$, when $\gamma_1 < \gamma_2$, the L^2 convergence rate of $(Z_n)_n$ to $\frac{1}{2}$ and the L^2 rate of convergence of $(Z_n(i) - Z_n)_n$ to 0 are the same. However, when $\gamma_2 < \gamma_1$, we obtain that synchronization happen faster than convergence.

Moreover in the case $\lambda_1 > 0$, $\lambda_2 = 0$ and when $\gamma_1 < \gamma_2$, the speed of convergence and synchronization are the same $(n^{-\gamma_1})$. While when $\gamma_2 < \gamma_1$, the synchronization is faster than convergence.

Similarly, in the case $\lambda_1 = 0$, $\lambda_2 > 0$ and when $\gamma_2 < \gamma_1$, the speed of convergence and synchronization are the same $(n^{-(2\gamma_1-\gamma_2)})$, while when $\gamma_2 < \gamma_1$, the speed of synchronization is faster than convergence $n^{-(2\gamma_1-\gamma_2)}$ and $n^{-\gamma_2}$ respectively).

4.3.2 Case of a common shared random time-asymptotics

Differently to the previous cases, the case $\lambda_1 = \lambda_2 = 0$ yields $(Z_n)_n$ is a martingale. We will prove it leads to a random time-asymptotics Z_∞ ($\mathbb{V}\mathrm{ar}(Z_\infty) > 0$). We will study the system's time-asymptotics behavior in a similar way as in the previous cases. First we show that $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) \neq 0$. Second we prove that L^2-synchronization holds. Third we state the almost sure synchronization holds.

Theorem 4.6. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. For $\lambda_1 = \lambda_2 = 0$,*

(i) it holds $(n \to \infty)$ $\mathbb{V}ar(Z_n) > 0$. In particular $(Z_n)_n$ converges a.s. to a non-degenerated $(\mathbb{V}ar(Z_\infty) > 0)$ random limit denoted by Z_∞.

(ii) The L^2-distance between the mean field Z_n and each component $Z_n(i)$ behaves as follows,

$$\mathbb{E}\left([Z_n(i) - Z_n]^2\right) = O\left(\frac{1}{n^{2\gamma_1-\gamma_2}}\right).$$

Proof.
((i) of Theorem 4.6)
When $\lambda_1 = \lambda_2 = 0$ then $(Z_n)_n$ is a bounded martingale. Therefore, it converges a.s. to a random variable Z_∞. On the other hand, by Lemma 3.19, $\mathbb{V}\mathrm{ar}(Z_\infty) < \frac{1}{4}$. Let us consider for $\gamma = \min(\gamma_1, \gamma_2)$ and consequently $r_n = r_n^l \vee r_n^g$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - (r_n^g)^2)\mathbb{V}\mathrm{ar}(Z_n) + K_n(r_n)^2,$$

where

a) If $\gamma_1 < \gamma_2$ then $A = 1$ and $K_n = \frac{1}{N}\left[\left(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(Z_n(i)^2)\right)\right]$ is bounded $\lim_n K_n > 0$. Using Lemma A.2, we get $\lim_n \mathbb{V}\mathrm{ar}(Z_n) > 0$.

b) If $\gamma_2 < \gamma_1$ then, $A = 1$ and $K_n = \frac{1}{4}$, again by Lemma A.2 $\lim_n \mathbb{V}\mathrm{ar}(Z_n) > 0$.

Also when $\gamma_1 = \gamma_2$,

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = (1 - r_n^2)\mathbb{V}\mathrm{ar}(Z_n) + K_n r_n^2,$$

where $K_n = \left[(\frac{1}{2} - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(Z_n(i)^2)) + \frac{N}{4}\right]$ which using Lemma A.2 implies $\lim_n \mathbb{V}\mathrm{ar}(Z_n) > 0$, where $A = 1$.

□

Proof.
((ii) Theorem 4.6) To study the synchronization phenomenon, consider the L^2-distance

$$x_{n+1} = (1 - 2r_n^g + (r_n^g)^2)x_n + (r_n^l)^2 J_n$$

where $J_n = \frac{N-1}{N}\left(\frac{1}{2} - [\frac{1}{N}\mathbb{E}(Z_n(i)^2)]\right)$ is bounded and $\lim_n J_n > 0$ for $N > 1$ and then $A = 2$. Thus, Lemma A.2 yields $x_n = O(\frac{1}{n^{2\gamma_1-\gamma_2}})$, hence the L^2-synchronization holds as $n \to \infty$. Moreover when $\gamma_1 = \gamma_2$, $x_{n+1} = (1 - 2r_n)x_n + J_n r_n^2$ which implies $x_n = O\left(\frac{1}{n^\gamma}\right)$. □

4.4 Almost sure synchronization

4.4.1 Case of a deterministic time-asymptotics

In this section we prove that the synchronization holds almost surely and the deterministic limit is $Z_\infty := \frac{1}{2}$.

Proposition 4.7. *Assume* $Z_n \xrightarrow{a.s.} \frac{1}{2}$. *For any* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *it holds:*
$\mathbb{E}[(Z_n(i) - \frac{1}{2})^2] = O(\frac{1}{n^{\gamma_1}})$ *when* $\gamma_1 \le \gamma_2$
and $\mathbb{E}[(Z_n(i) - \frac{1}{2})^2] = O(\frac{1}{n^{2\gamma_2-\gamma_1}})$ *when* $\gamma_2 < \gamma_1$.

Proof.
From (3.31) we have

$$\begin{aligned}
\mathbb{E}[(Z_{n+1}(i) - \frac{1}{2})^2|\mathcal{F}_n] &= (Z_n(i) - \frac{1}{2})^2[1 - 4r_n^l\lambda_1 + 4\lambda_1^2(r_n^l)^2] \\
&+ (r_n^l)^2\{\mathbb{E}[(\Delta\hat{M}_{n+1}^l)^2|\mathcal{F}_n]\} \\
&+ (r_n^g)^2\{4\lambda_1^2((Z_n - \frac{1}{2})^2) + \mathbb{E}[(\Delta\hat{M}_{n+1}^g)^2|\mathcal{F}_n]\} \\
&- (Z_n(i) - \frac{1}{2})(Z_n - \frac{1}{2})[4r_n^g\lambda_2 - 4r_n^l r_n^g\lambda_1\lambda_2].
\end{aligned}$$

Therefore, by assumption $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$,

$$\begin{aligned}
\mathbb{E}\{\mathbb{E}[(Z_{n+1}(i) - \frac{1}{2})^2|\mathcal{F}_n]\} &= \mathbb{E}(Z_n(i) - \frac{1}{2})^2[1 - 4r_n^l\lambda_1 + 4\lambda_1^2(r_n^l)^2] \\
&+ (r_n^l)^2\mathbb{E}[(\Delta\hat{M}_{n+1}^l)^2] \\
&+ (r_n^g)^2\mathbb{E}[(\Delta\hat{M}_{n+1}^g)^2]
\end{aligned}$$

Let $y_n = \mathbb{E}[(Z_n(i) - \frac{1}{2})^2]$, so

$$y_{n+1} = [1 - 4r_n^l\lambda_1 + 4\lambda_1^2(r_n^l)^2]y_n + (r_n^l)^2K_{n+1}^l + (r_n^g)^2K_{n+1}^g,$$

where $0 < K_{n+1}^l = \mathbb{E}[(\Delta\hat{M}_{n+1}^l)^2] \leq K$ and $0 < K_{n+1}^g = \mathbb{E}[(\Delta\hat{M}_{n+1}^g)^2] \leq K$. Then using lemma A.2 implies that $y_n = \mathcal{O}(\frac{1}{n^{\gamma_1}})$ when $\gamma_1 \leq \gamma_2$ and $y_n = \mathcal{O}(\frac{1}{n^{2\gamma_2-\gamma_1}})$ when $\gamma_2 < \gamma_1$. When $\gamma_1 = \gamma_2 = 1$ and $\lambda_1 = \frac{1}{4}$ then $y_n = \mathcal{O}(\frac{\log n}{n})$. □

Theorem 4.8. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. For any $\lambda_1 > 0$ and $\lambda_2 > 0$ almost sure convergence holds i.e.*

$$\forall i = 1, ..., N, \qquad \lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2} \ a.s.$$

Proof.
Now to prove that the convergence's value is equal $\frac{1}{2}$ a.s, using (3.30) and let

$$\begin{aligned}
&\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2}\right)^2 [1 + 4(r_n^l)^2\lambda_1^2 + 4(r_n^g)^2\lambda_2^2 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + 4r_n^lr_n^g\lambda_1\lambda_2] \\
&\quad + (r_n^l)^2\mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n] + (r_n^g)^2\mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + \mathbf{o}(r_n^l) + \mathbf{o}(r_n^g)\right] \\
&\quad + (r_n^l)^2 \left[4\lambda_1^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right] + (r_n^g)^2 \left[4\lambda_2^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right]
\end{aligned}$$

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + (r_n^l)^2W_n^l + (r_n^g)^2W_n^g$ where

$$W_n^l := \left[4\lambda_1^2\left(Z_n - \frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right],$$

$$W_n^g := \left[4\lambda_2^2\left(Z_n - \frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right].$$

By assumption $\frac{1}{2} < \gamma_1 \leq 1$ and $\frac{1}{2} < \gamma_2 \leq 1$, so $(Z_{n+1} - \frac{1}{2})^2$ is a positive almost supermartingale and almost sure convergence holds.

It is enough to consider L^2 convergence in order to identify the (deterministic) limit.

$$\begin{aligned}
\mathbb{E}\left(\mathbb{E}\left[\left(Z_{n+1} - \frac{1}{2}\right)^2|\mathcal{F}_n\right]\right) &= \mathbb{E}\left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + 4r_n^lr_n^g\lambda_1\lambda_2\right] \\
&+ (r_n^l)^2K_n^l + (r_n^g)^2K_n^g.
\end{aligned}$$

Let $y_n := \mathbb{E}(Z_n - \frac{1}{2})^2$ thus,

$$y_{n+1} = \Big(1 - 4r_n^l\lambda_1 - 4r_n^g\lambda_2 + \lambda_1^2(r_n^l)^2 + \lambda_2^2(r_n^g)^2 + 4r_n^l r_n^g\lambda_1\lambda_2\Big)y_n + (r_n^l)^2K_{n+1}^l + (r_n^g)^2K_{n+1}^g \quad (4.3)$$

where $0 < K_{n+1}^l := \{\mathbb{E}[(\Delta M_{n+1}^l)^2]\} \leq 1$, $0 < K_{n+1}^g := \{\mathbb{E}[(\Delta M_{n+1}^g)^2]\} \leq 1$. By lemma A.2 $\lim_{n\to\infty} y_n = 0$. So, $\lim \mathbb{E}(Z_n - \frac{1}{2})^2 = 0$. Using the fact that $(Z_n)_n$ converges almost surely, then $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$.

The proof of the case when $\gamma_1 = \gamma_2$ is essentially the same as above using (3.36). Indeed,

$$\begin{aligned}\mathbb{E}[\Big(Z_{n+1} - \frac{1}{2}\Big)^2 |\mathcal{F}_n] &= \Big(Z_n - \frac{1}{2}\Big)^2 [1 - 2r_n(\lambda_1 + \lambda_2)]^2 + r_n^2\mathbb{E}[\Delta\tilde{M}_{n+1}(i)^2|\mathcal{F}_n] \\ &+ 2\Big(Z_n - \frac{1}{2}\Big)[1 - 2r_n(\lambda_1 + \lambda_2)]r_n\mathbb{E}[\Delta\tilde{M}_{n+1}(i)|\mathcal{F}_n].\end{aligned}$$

So $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + r_n^2\tilde{W}_n$, where

$$\tilde{W}_n = r_n^2\Big(4(\lambda_1 + \lambda_2)^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta\tilde{M}_{n+1}(i))^2|\mathcal{F}_n]\Big).$$

To prove the a.s. synchronization, since L^2 synchronization holds, it is enough to show a.s convergence exists for $Z_n(i)$. We can obtain that $(Z_n(i))_n$ is quasi-martingale. This holds using the fact that

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = \lambda_1 r_n^l\Big(1 - 2Z_n(i)\Big) + \lambda_2 r_n^g\Big(1 - 2Z_n\Big) + r_n^g\Big(Z_n - Z_n(i)\Big).$$

According to $\mathbb{E}|1 - 2Z_n| = 2\mathbb{E}|Z_n - \frac{1}{2}|$ and $\mathbb{E}|1 - 2Z_n(i)| = 2\mathbb{E}|Z_n(i) - \frac{1}{2}|$, we obtain the convergence of the series of $\mathbb{E}|\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)|$ using the following inequality and bounding from above the first moment with the second moment,

$$\begin{aligned}\sum_n \mathbb{E}|\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)| &\leq 2\lambda_1\sum_n r_n^l\mathbb{E}\left|Z_n(i) - \frac{1}{2}\right| + 2\lambda_2\sum_n r_n^g\mathbb{E}\left|Z_n - \frac{1}{2}\right| + \sum_n r_n^g\mathbb{E}\,|Z_n - Z_n(i)| \\ &\leq 2\lambda_1\sum_n r_n^l\Big(\mathbb{E}\left|Z_n(i) - \frac{1}{2}\right|^2\Big)^{1/2} + 2\lambda_2\sum_n r_n^g\Big(\mathbb{E}\left|Z_n - \frac{1}{2}\right|^2\Big)^{1/2} \\ &+ \sum_n r_n^g\Big(\mathbb{E}\,|Z_n - Z_n(i)|^2\Big)^{1/2} < +\infty\end{aligned}$$

since by Lemma A.2 $\mathbb{E}[(Z_n - \frac{1}{2})^2] = O\Big(\frac{1}{n^\gamma}\Big)$, $\mathbb{E}[(Z_n - Z_n(i))^2] = O(\frac{1}{n^{\gamma_1}})$ (resp. $\frac{1}{n^{2\gamma_1-\gamma_2}}$) when $\gamma_1 \leq \gamma_2$ (resp. $\gamma_2 < \gamma_1$) and by Lemma 4.7 $\mathbb{E}[(Z_n(i) - \frac{1}{2})^2] = O(\frac{1}{n^{\gamma_1}})$ (resp. $\frac{1}{n^{2\gamma_2-\gamma_1}}$) when $\gamma_1 \leq \gamma_2$ (resp. $\gamma_2 < \gamma_1$) □

Theorem 4.9. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. In the following cases: either ($\lambda_1 > 0$ and $\lambda_2 = 0$) or ($\lambda_1 = 0$ and $\lambda_2 > 0$) it holds*

$$\lim_{n\to+\infty} Z_n(i) = \lim_{n\to+\infty} Z_n = \frac{1}{2} \quad a.s.$$

Proof.
• **Case $\lambda_1 \neq 0, \lambda_2 = 0$.**

Using similar argument in proof of Theorem 4.8, we can proof that $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. To show it, using (3.30) and let $\lambda_2 = 0$, then the result is obtained. Indeed,

$$\begin{aligned}
&\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \\
&= \left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1 + \mathbf{o}(r_n^l) + \mathbf{o}(r_n^g)\right] \\
&\quad + (r_n^l)^2 \left[4\lambda_1^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right] + (r_n^g)^2 \left[\mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right]
\end{aligned}$$

Thus, $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n - \frac{1}{2})^2 + (r_n^l)^2 W_n^l + (r_n^g)^2 W_n^g$ where

$$W_n^l := \left[4\lambda_1^2\left(Z_n - \frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right],$$

$$W_n^g := \left[\mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right].$$

By assumption $\frac{1}{2} < \gamma_1 \leq 1$ and $\frac{1}{2} < \gamma_2 \leq 1$, so $(Z_{n+1} - \frac{1}{2})^2$ is a positive almost supermartingale and almost sure convergence holds.

It is enough to consider L^2 convergence in order to identify the (deterministic) limit.

$$\begin{aligned}
\mathbb{E}\left(\mathbb{E}\left[\left(Z_{n+1} - \frac{1}{2}\right)^2 |\mathcal{F}_n\right]\right) &= \mathbb{E}\left(Z_n - \frac{1}{2}\right)^2 \left[1 - 4r_n^l\lambda_1\right] \\
&+ (r_n^l)^2 K_n^l + (r_n^g)^2 K_n^g.
\end{aligned}$$

Let $y_n := \mathbb{E}(Z_n - \frac{1}{2})^2$, so

$$y_{n+1} = \left(1 - 4r_n^l\lambda_1 + \lambda_1^2(r_n^l)^2\right)y_n + (r_n^l)^2 K_{n+1}^l + (r_n^g)^2 K_{n+1}^g \tag{4.4}$$

where $0 < K_{n+1}^l := \{\mathbb{E}[(\Delta M_{n+1}^l)^2]\} \leq 1$, $0 < K_{n+1}^g := \{\mathbb{E}[(\Delta M_{n+1}^g)^2]\} \leq 1$, using $\lambda_1 \leq 1$, same for λ_2 and by lemma A.2 $\lim_{n\to\infty} y_n = 0$. So, $\lim \mathbb{E}(Z_n - \frac{1}{2})^2 = 0$. Using the fact that $(Z_n)_n$ converges almost surely, then $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$.

The proof of the case when $\gamma_1 = \gamma_2$ is essentially the same as above using (3.36). Indeed,

$$\begin{aligned}\mathbb{E}[\Big(Z_{n+1} - \frac{1}{2}\Big)^2 |\mathcal{F}_n] &= \Big(Z_n - \frac{1}{2}\Big)^2 [1 - 2r_n\lambda_1]^2 + r_n^2\mathbb{E}[\Delta\tilde{M}_{n+1}(i)^2|\mathcal{F}_n] \\ &+ 2\Big(Z_n - \frac{1}{2}\Big)[1 - 2r_n\lambda_1]r_n\mathbb{E}[\Delta\tilde{M}_{n+1}(i)|\mathcal{F}_n]\end{aligned}$$

So $\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \le (Z_n - \frac{1}{2})^2 + r_n^2\tilde{W}_n$, where

$$\tilde{W}_n = r_n^2\Big(4\lambda_1^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta\tilde{M}_{n+1}(i))^2|\mathcal{F}_n]\Big).$$

To prove the a.s. synchronization, since L^2 synchronization is hold, it is enough to show a.s convergence exists for $Z_n(i)$. We can obtain that $(Z_n(i))_n$ is quasi-martingale. Indeed using the fact that

$$\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = \lambda_1 r_n^l\Big(1 - 2Z_n(i)\Big) + r_n^g\Big(Z_n - Z_n(i)\Big).$$

According to $\mathbb{E}|1 - 2Z_n(i)| = 2\mathbb{E}|Z_n(i) - \frac{1}{2}|$, we obtain the convergence of the series of $\mathbb{E}|\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)|$ using the following inequality and bounding the expectation with the second moment,

$$\begin{aligned}\sum_n \mathbb{E}|\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)| &\le 2\lambda_1\sum_n r_n^l\mathbb{E}|Z_n(i) - \frac{1}{2}| + \sum_n r_n^g\mathbb{E}|Z_n - Z_n(i)| \\ &\le 2\lambda_1\sum_n r_n^l\Big(\mathbb{E}|Z_n(i) - \frac{1}{2}|^2\Big)^{1/2} + \sum_n r_n^g\Big(\mathbb{E}|Z_n - Z_n(i)|^2\Big)^{1/2} < +\infty\end{aligned}$$

since $\mathbb{E}[(Z_n - Z_n(i))^2] = O(\frac{1}{n^{\gamma_1}})$ (resp. $\frac{1}{n^{2\gamma_1-\gamma_2}}$) when $\gamma_1 \le \gamma_2$ (resp. $\gamma_2 < \gamma_1$) and by Lemma 4.7, it holds $\mathbb{E}[(Z_n(i) - \frac{1}{2})^2] = O(\frac{1}{n^{\gamma_1}})$ (resp. $\frac{1}{n^{2\gamma_2-\gamma_1}}$) when $\gamma_1 \le \gamma_2$ (resp. $\gamma_2 < \gamma_1$).

- **Case $\lambda_1 = 0, \lambda_2 \neq 0$.**

Using similar argument in proof of Theorem 4.8, we can proof that $\lim_{n\to\infty} Z_n = \frac{1}{2}$ a.s. To show it, using (3.30) and let $\lambda_1 = 0$, then the result is obtained. Indeed,

$$\begin{aligned}&\mathbb{E}[(Z_{n+1} - \frac{1}{2})^2|\mathcal{F}_n] \\ &= \Big(Z_n - \frac{1}{2}\Big)^2\Big[1 - 4r_n^g\lambda_2 + \mathbf{o}(r_n^l) + \mathbf{o}(r_n^g)\Big] \\ &\quad + (r_n^l)^2\Big[\mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\Big] + (r_n^g)^2\Big[4\lambda_2^2(Z_n - \frac{1}{2})^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\Big]\end{aligned}$$

Thus, $\mathbb{E}[(Z_{n+1}-\frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n-\frac{1}{2})^2 + (r_n^l)^2 W_n^l + (r_n^g)^2 W_n^g$ where

$$W_n^l := \left[\mathbb{E}[(\Delta M_{n+1}^l)^2|\mathcal{F}_n]\right],$$

$$W_n^g := \left[4\lambda_2^2\left(Z_n-\frac{1}{2}\right)^2 + \mathbb{E}[(\Delta M_{n+1}^g)^2|\mathcal{F}_n]\right].$$

By assumption $\frac{1}{2} < \gamma_1 \leq 1$ and $\frac{1}{2} < \gamma_2 \leq 1$, so $\left(Z_{n+1}-\frac{1}{2}\right)^2$ is a positive almost supermartingale and almost sure convergence holds.

It is enough to consider L^2 convergence in order to identify the (deterministic) limit.

$$\begin{aligned}\mathbb{E}\left(\mathbb{E}\left[\left(Z_{n+1}-\frac{1}{2}\right)^2|\mathcal{F}_n\right]\right) &= \mathbb{E}\left(Z_n-\frac{1}{2}\right)^2\left[1-4r_n^g\lambda_2\right] \\ &+ (r_n^l)^2K_n^l + (r_n^g)^2K_n^g.\end{aligned}$$

Let $y_n := \mathbb{E}(Z_n-\frac{1}{2})^2$, so

$$y_{n+1} = \left(1-4r_n^g\lambda_2+\lambda_2^2(r_n^g)^2\right)y_n + (r_n^l)^2K_{n+1}^l + (r_n^g)^2K_{n+1}^g \tag{4.5}$$

where $0 < K_{n+1}^l := \{\mathbb{E}[(\Delta M_{n+1}^l)^2]\} \leq 1$, $0 < K_{n+1}^g := \{\mathbb{E}[(\Delta M_{n+1}^g)^2]\} \leq 1$, by $\lambda_i \leq K$ and by lemma A.2 $\lim_{n\to\infty} y_n = 0$. So, $\lim \mathbb{E}(Z_n-\frac{1}{2})^2 = 0$. Using the fact that $(Z_n)_n$ converges almost surely, then $Z_n \xrightarrow{\text{a.s.}} \frac{1}{2}$.

The proof of the case when $\gamma_1 = \gamma_2$ is essentially the same as above using (3.36). Indeed,

$$\begin{aligned}\mathbb{E}[\left(Z_{n+1}-\frac{1}{2}\right)^2|\mathcal{F}_n] &= \left(Z_n-\frac{1}{2}\right)^2[1-2r_n\lambda_2]^2 + r_n^2\mathbb{E}[\Delta\tilde{M}_{n+1}(i)^2|\mathcal{F}_n] \\ &+ 2\left(Z_n-\frac{1}{2}\right)[1-2r_n\lambda_2]r_n\mathbb{E}[\Delta\tilde{M}_{n+1}(i)|\mathcal{F}_n]\end{aligned}$$

So $\mathbb{E}[(Z_{n+1}-\frac{1}{2})^2|\mathcal{F}_n] \leq (Z_n-\frac{1}{2})^2 + r_n^2\tilde{W}_n$, where

$$\tilde{W}_n = r_n^2\left(4\lambda_2^2(Z_n-\frac{1}{2})^2 + \mathbb{E}[(\Delta\tilde{M}_{n+1}(i))^2|\mathcal{F}_n]\right).$$

To prove the a.s. synchronization, since L^2 synchronization was proven to hold, it is enough to show a.s convergence exists for $(Z_n(i))_n$, which we can get by proving that $(Z_n(i))_n$ is quasi-martingale.

According to $\mathbb{E}|1-2Z_n| = 2\mathbb{E}|Z_n-\frac{1}{2}|$, we obtain the convergence of the series of $\mathbb{E}|\mathbb{E}(Z_{n+1}(i)-Z_n(i)|\mathcal{F}_n)|$ using the following inequality and bounding the expectation with the second moment,

$$\begin{aligned}\sum_n \mathbb{E}|\mathbb{E}(Z_{n+1}(i)-Z_n(i)|\mathcal{F}_n)| &\leq 2\lambda_2\sum_n r_n^g\mathbb{E}|Z_n-\frac{1}{2}| + \sum_n r_n^g\mathbb{E}|Z_n-Z_n(i)| \\ &\leq 2\lambda_2\sum_n r_n^g\left(\mathbb{E}|Z_n-\frac{1}{2}|^2\right)^{1/2} + \sum_n r_n^g\left(\mathbb{E}|Z_n-Z_n(i)|^2\right)^{1/2} < +\infty\end{aligned}$$

since $\mathbb{E}[(Z_n-\frac{1}{2})^2] = \mathcal{O}(\frac{1}{n^{2\gamma_1-\gamma_2}})$ (resp. $\frac{1}{n^{2\gamma_2}}$) when $\gamma_1 \leq \gamma_2$ (resp. $\gamma_2 < \gamma_1$) and $\mathbb{E}[(Z_n - Z_n(i))^2] = \mathcal{O}(\frac{1}{n^{2\gamma_1-\gamma_2}})$ when $\gamma_1 \leq \gamma_2$ (and $\gamma_2 < \gamma_1$). □

4.4.2 Case of a common shared random time-asymptotics

In this section we state the almost sure synchronization holds.

Theorem 4.10. *Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1,\gamma_2) \in]\frac{1}{2},1]^2$. For $\lambda_1 = \lambda_2 = 0$, $(Z_n)_n$ converges a.s. to a non-degenerated random limit denoted by Z_∞ ($\mathbb{V}ar(Z_\infty) > 0$). and synchronization holds almost surely. It means, for all $i \in \{1,\dots,N\}$,*

$$\lim_{n\to\infty} Z_n(i) = Z_\infty \qquad a.s.$$

Proof.
If $\lambda_1 = \lambda_2 = 0$ then $(Z_n)_n$ is a bounded martingale. Therefore, it converges a.s. to a random variable Z_∞.

To show that synchronization holds a.s., we shall show that the a.s. limit of $(Z_n(i))_n$ exists. We observe that $(Z_n(i))_n$ is a quasi-martingale. Indeed,
since $\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n) = r_n^g\Big(Z_n - Z_n(i)\Big)$ thus,

$$\begin{aligned}\sum_n \mathbb{E}|\mathbb{E}(Z_{n+1}(i) - Z_n(i)|\mathcal{F}_n)| &\leq \sum_n r_n^g \mathbb{E}|Z_n - Z_n(i)| \\ &\leq \sum_n r_n^g\Big(\mathbb{E}|Z_n - Z_n(i)|^2\Big)^{1/2} < +\infty\end{aligned}$$

since, $\mathbb{E}[(Z_n - Z_n(i))^2] = \mathcal{O}(\frac{1}{n^{2\gamma_1-\gamma_2}})$ (resp. $\frac{1}{n^\gamma}$) when $\gamma_1 \neq \gamma_2$ (resp. $\gamma_1 = \gamma_2$). As bounded quasi-martingale, it converges a.s. □

Chapter 5

Fluctuations through central limit theorems (CLT)

In this Chapter, in the different cases, we prove central limit theorems about the fluctuations of $(Z_n - Z_\infty)_n$ and $(Z_n - Z_n(i))_n$. Scaling factors are worth of interest.
This Chapter is organized as follows: In section 5.1 we present the concept of stable convergence for CLT. In section 5.2 we state the fluctuations results. Theorem 5.4 consider the case where each individual and collective reinforcement leads to a deterministic limiting value. Theorem 5.5 consider the special case when $\gamma_1 = \gamma_2 = 1$ reminiscent of the Friedman urn context, in the regime where fluctuations are not gaussian ($\lambda_1 + \lambda_2 < \frac{1}{4}$). Theorem 5.6 deals with the mixed cases where individual and reinforcement type are of different nature. Theorem 5.7 consider the case where the individual and the collective reinforcement leads to a random limit. Finally, the sections 5.3, 5.4, 5.5 and 5.6 deal respectively with the proofs of the CLTs.

5.1 CLT and stable convergence

Stable convergence has been introduced by Rényi in [87] and subsequently investigated by various authors, *e.g.* [2, 35, 42]. It is a strong form of convergence in distribution, in the sense that it is intermediate between the simple convergence in distribution and the convergence in probability.

In this section we recall the concept of stable convergence and its basic definition. First of all we make the motivation for the study of this concept from [53].

Our starting point is the classical central limit theorem. For this, let $(T_k)_{k\geq 1}$ be a sequence of independent and identically distributed real random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume T_1 has a finite variance and set $\theta = \mathbb{E}(T_1)$ and $\tau^2 = Var(T_1)$. To exclude the trivial case of almost surely constant variables, assume also $\tau^2 > 0$. Let N be a random variable which "realizes" the standard normal distribution $\mathcal{N}(0, 1)$.

Writing, as usual, $\bar{T}_n = \frac{1}{n}\sum_{k=1}^n T_k$ for the sample mean of $T_1, ..., T_n$, an equivalent formulation of the classical central limit theorem is

$$n^{1/2}(\bar{T}_n - \theta) \xrightarrow[n\to\infty]{\mathrm{D}} \tau N$$

which means that $\bar{T}_n$ considered as an estimator for θ is asymptotically normal, where the asymptotic distribution $\mathcal{N}(0,1)$ of τN is the centered normal distribution with variance τ^2. If in a statistical setting θ and τ^2 are supposed to be unknown and θ is the parameter of interest and τ^2 is not, *i.e.* τ^2 is a so-called nuisance parameter, then τ has to be removed from the limit theorem by replacing it by a suitable consistent estimator, if the limit theorem is to be used for statistical inference. The proper tool for doing this is

Theorem 5.1. *(Cramér-Slutzky) Let* $(X_n)_n \geq 1$ *and* $(Y_n)_n \geq 1$ *be sequences of real random variables. If*

$$X_n \xrightarrow[n\to\infty]{D} X_\infty$$

for some real random variable X *and*

$$Y_n \xrightarrow[n\to\infty]{P} c$$

for some $c \in \mathbb{R}$*, then*

$$X_n Y_n \xrightarrow[n\to\infty]{D} cX_\infty.$$

For the sample variance $\hat{\tau}_n^2$ of $T_1, ..., T_n$ we have $\hat{\tau}_n^2 \xrightarrow[n\to\infty]{} \tau^2$ almost surely by the strong law of large numbers, and Theorem 5.1 gives

$$n^{1/2}\frac{(\bar{T}_n - \theta)}{\hat{\tau}_n} \xrightarrow[n\to\infty]{\mathrm{D}} N.$$

This convergence result can now be used in asymptotic statistical inference about θ because it is free from the unknown nuisance parameter τ. The situation is different in the following setting in which the limit is a variance mixture of centered normals with non-constant mixing law is as follows. Let $(X_k)_{k\geq 1}$ be a martingale difference sequence w.r.t. an increasing sequence $(\mathcal{F}_k)_k \geq 0$ of sub-σ-fields of $\mathcal{F}$. If $(X_k)_{k\geq 1}$ is also stationary and $X_1 \in L^2$, then the following version of the central limit theorem is true:

$$\frac{1}{n^{1/2}}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{\mathrm{D}} \mathbb{E}(X_1^2|\mathcal{I}_X)^{1/2}N$$

where $\mathcal{I}_X$ is the σ-field of the invariant sets of $X = (X_k)_{k\geq 1}$, N is a random variable with a standard normal distribution and the random variables $\mathbb{E}(X_1^2|\mathcal{I}_X)$ and N are independent. It is important to note that $\mathbb{E}(X_1^2|\mathcal{I}_X)$ is in general indeed a proper random variable so that the limit distribution is a variance mixture of centered normals again. Therefore, though we have

$$\frac{1}{n}\sum_{k=1}^{n} X_k^2 \xrightarrow[n\to\infty]{\mathrm{a.s.}} \mathbb{E}(X_1^2|\mathcal{I}_X)$$

by the ergodic theorem, we cannot derive

$$\Big(\sum_{k=1}^{n} X_k^2\Big)^{-1/2} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{\mathrm{D}} N$$

by an application of Theorem 5.1 thus removing the mixing variable $\mathbb{E}(X_1^2|\mathcal{I}_X)$ from the limit theorem by a random norming, because for a proper application $\frac{1}{n}\sum_{k=1}^{n} X_k^2$ would have to converge (in probability) to a constant, which is not the case in general (unless the stationary sequence $(X_k)_{k\geq 1}$ is ergodic, of course).

The concept of convergence in distribution is not strong enough to allow for a version of the Theorem 5.1 in which the constant factor c in the limit variable is replaced by a proper random variable. There is, however, a stronger notion of convergence for which such a stronger version of the Cramér-Slutzky theorem is true, and this is stable convergence. For a brief exposition of its main features let $(X_n)_{n\geq 1}$ be a sequence of real random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let S be a Polish space, endowed with its Borel σ-field. A kernel on S, or a random probability measure on S, is a random variable κ with values in a set of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto \kappa(\omega)(f) := \int f(x)\ \kappa(\omega)\ (dx)$$

is $\mathcal{F}$-measurable.

On $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{G}$ as a sub-σ-field of $\mathcal{F}$, let $(X_n)_{n\geq 1}$ be a sequence of S-valued random variables and let κ be a $\mathcal{G}$-measurable Markov kernel from Ω to S. Then we say that X_n converges $\mathcal{G}$-stably to κ, and we write

$$X_n \xrightarrow[n\to\infty]{} \kappa \quad \mathcal{G}-\textit{stably},$$

if

$$\mathbb{P}\big(X_n \in .|\mathcal{G}\big) \xrightarrow[n\to\infty]{\text{weakly}} \mathbb{E}\big(\kappa(.)|\mathcal{G}\big) \qquad \textit{for all} \quad \mathcal{G} \in \mathcal{F} \quad \textit{with} \quad \mathbb{P}(\mathcal{G}) > 0.$$

To get a feeling for the difference between convergence in distribution and stable convergence, recall that convergence in distribution of random variables X_n towards a distribution ν is in fact weak convergence of the distributions $\mathbb{P}_{X_n}$ towards the distribution ν, *i.e.* the underlying concept is that of weak convergence of probability measures. Now the distributions $\mathbb{P}_{X_n}$ may obviously be interpreted as the conditional distributions $\mathbb{P}_{X_n|\{\emptyset,\Omega\}}$ of the random variables X_n given the trivial σ-field $\{\emptyset, \Omega\}$. In the concept of stable convergence this trivial σ-field is replaced by some larger sub-σ-field $\mathcal{G}$ of the σ-field $\mathcal{F}$ in $(\Omega, \mathcal{F}, \mathbb{P})$, and the limit distribution ν is replaced by the $\mathcal{G}$-measurable Markov kernel κ. Note that $\mathcal{G}$-stable convergence always implies convergence in distribution (take $f = 1$ in the definition of stable convergence). As for convergence in distribution it can be convenient to "realize" the limit kernel κ through a random variable X_∞ which satisfies $\mathbb{P}_{X|\mathcal{G}}$. Such a random variable does always exist on a suitable extension of $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, if $(X_n)_n \geq 1$ and X_∞ are real random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

and $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-field, we say that $(X_n)_n \geq 1$ converges $\mathcal{G}$-stably to X_∞ as $n \to \infty$, written as

$$X_n \xrightarrow[n\to\infty]{} X_\infty \quad \mathcal{G} - stably,$$

if X_n converges $\mathcal{G}$-stably to the conditional distribution $\mathbb{P}_{X|\mathcal{G}}$.

Therefore, in the example discussed above, we can show that

$$\frac{1}{n^{1/2}} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{} \mathbb{E}(X_1^2|\mathcal{I}_X)^{1/2} N \quad \mathcal{G} - stably,$$

where $\mathcal{G} = \sigma(X_n, n \geq 1)$ and N is independent of $\mathcal{G}$. Consequently, the generalized Cramér-Slutzky theorem implies the desired limit theorems

$$\Big(\sum_{k=1}^{n} X_k^2\Big)^{-1/2} \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{\mathrm{D}} N.$$

Based on the concept of the stable convergence as discussed above and following Lemma 5.2, we recall later Theorem 3.2 in [52] which make bases of our arguments to prove the central limit theorems for fluctuations in next sections.

Lemma 5.2.
Let $\mathcal{G}$ be an (increasing) filtration and (V_k) be an $\mathcal{G}$-adapted sequence of real random variables such that $\mathbb{E}[V_k|\mathcal{G}_{k-1}] \to V$ a.s. and assume $\forall k$, $\mathbb{E}(V_k^2) < \infty$ for some real random variable V. Moreover, let (a_k) and (b_k) be two sequences of strictly positive real numbers such that

$$b_k \uparrow +\infty, \ \sum_{k=1}^{\infty} \frac{\mathbb{E}[V_k^2]}{a_k^2 b_k^2} < +\infty.$$

Then we have:

a) If $\frac{1}{b_n}\sum_{k=1}^{n} \frac{1}{a_k} \to \vartheta$ for some constant ϑ, then $\frac{1}{b_n}\sum_{k=1}^{n} \frac{V_k}{a_k} \to \vartheta V$.

b) If $b_n \sum_{k\geq n} \frac{1}{a_k b_k^2} \to \vartheta$ for some constant ϑ, then $b_n \sum_{k\geq n} \frac{V_k}{a_k b_k^2} \to \vartheta V$.

Theorem 5.3 (Theorem 3.2 in [52])**.**
Let $\{S_{n,k}, \mathcal{F}_{n,k} : 1 \leq k \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences $U_{n,k}$, and let σ^2 be an a.s. finite random variable. Suppose that

1) $\max_{1\leq k\leq k_n} |U_{n,k}| \xrightarrow{P} 0$;

2) $\mathbb{E}[\max_{1\leq k\leq k_n} U_{n,k}^2]$ is bounded in n;

3) $\sum_{k=1}^{k_n} U_{n,k}^2 \xrightarrow{P} \sigma^2$

and the σ-fields are nested, i.e. $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$ for $1 \leq k \leq k_n$, $n \geq 1$. Then $S_{n,k_n} = \sum_{k=1}^{k_n} U_{n,k}$ converges stably to a random variable with characteristic function $\phi(u) = \mathbb{E}[\exp(-\sigma^2 u^2/2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \sigma^2)$.

5.2 Fluctuations' results

In this section we study the fluctuations of $(Z_n(i) - Z_n)_n$ (synchronization) w.r.t 0 and also fluctuations of $(Z_n)_n$ w.r.t its limit Z_∞. These are studied by stating Central Limit Theorems. Pay attention different scaling hold according to (γ_1, γ_2) relationship. We follow the proof's techniques initiated for these models in [33] based on Theorem 5.3 in Appendix leading to stable convergence results. See [17] too.

We first study cases where $Z_\infty = \frac{1}{2}$. The Theorems 5.4, 5.5 deal with the case $\lambda_1 > 0$ and $\lambda_2 > 0$. Moreover, we show that there is a non gaussian limit distribution for some special regime when $0 < (\lambda_1 + \lambda_2) < \frac{1}{4}$. The Theorem 5.6 describe the results of the cases where exactly one of the λ_j is 0.

Finally we state the behavior when $\mathbb{V}\mathrm{ar}(Z_\infty) > 0$. with Theorem 5.7. Let $\mathcal{N}\left(0, \sigma^2\right)$ denote the gaussian distribution with parameters 0 and $\sigma^2 \in \mathbb{R}^+$.

Theorem 5.4.
Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^s)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. Let $\lambda_1 > 0, \lambda_2 > 0$; let $\gamma := \min(\gamma_1, \gamma_2)$. The following statements hold, where σ^2 denotes the variances (depending on N and λ_j) are specified in proofs.

i) It holds

a) when $\gamma_1 \leq \gamma_2$, $n^{\frac{\gamma_1}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\left(0, \tilde{\sigma}_1^2\right)$,

b) when $\gamma_2 < \gamma_1$, $n^{\frac{2\gamma_1 - \gamma_2}{2}}(Z_n - Z_n(i)) \xrightarrow[n\to\infty]{stably} \mathcal{N}\left(0, \tilde{\sigma}_2^2\right)$.

ii) For $\gamma < 1$, it holds

$$n^{\frac{\gamma}{2}}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\left(0, \hat{\sigma}^2\right).$$

iii) For $\gamma = 1$ (meaning $\gamma_1 = \gamma_2 = 1$),

a) for $(\lambda_1 + \lambda_2) > \frac{1}{4}$, $\sqrt{n}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\left(0, \sigma_1^{*2}\right)$.

b) for $(\lambda_1 + \lambda_2) = \frac{1}{4}$, $\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{stably} \mathcal{N}\left(0, \sigma_2^{*2}\right)$.

Theorem 5.5.
Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^s)_n$ satisfy (3.12), (3.13) with $(\gamma_1, \gamma_2) \in]\frac{1}{2}, 1]^2$. Let $\lambda_1 > 0, \lambda_2 > 0$. When $\gamma_1 = \gamma_2 = 1$ and when $(\lambda_1 + \lambda_2) < \frac{1}{4}$, the following statement holds

$$n^{4(\lambda_1+\lambda_2)}(Z_n - \frac{1}{2}) \xrightarrow[n\to\infty]{a.s./L^1} \tilde{X},$$

for some real random variable $\tilde{X}$ such that $\mathbb{P}(\tilde{X} \neq 0) > 0$.

Two other main cases leads to following results.

Theorem 5.6.
Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1,\gamma_2)\in]\frac{1}{2},1]^2$. In the following cases: either ($\lambda_1>0$, $\lambda_2=0$) or ($\lambda_1=0$, $\lambda_2>0$), the stable convergence towards some Gaussian distribution holds for the quantities $(Z_n-Z_n(i))_n$ and $(Z_n-\frac{1}{2})_n$. The following tables summarizes the different scaling of convergence according to the relationship between γ_1, γ_2 and where σ^2 denotes the variances (depending on N and λ_j) are specified in proofs. The first table deals with $\gamma:=min(\gamma_1,\gamma_2)<1$.

	$\lambda_1\neq 0,\ \lambda_2=0$	$\lambda_1=0,\ \lambda_2\neq 0$
$\gamma_1\leq\gamma_2$	$n^{\frac{\gamma_1}{2}}(Z_n-Z_n(i))\xrightarrow{stably}\mathcal{N}\left(0,\tilde{\sigma}_3^2\right)$ $n^{\frac{\gamma_1}{2}}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\hat{\sigma}_1^2\right)$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stably}\mathcal{N}\left(0,\tilde{\sigma}_4^2\right)$ $n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\hat{\sigma}_2^2\right)$
$\gamma_2<\gamma_1$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stably}\mathcal{N}\left(0,\tilde{\sigma}_5^2\right)$ $n^{\frac{2\gamma_2-\gamma_1}{2}}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\hat{\sigma}_3^2\right)$	$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow{stably}\mathcal{N}\left(0,\tilde{\sigma}_6^2\right)$ $n^{\frac{\gamma_2}{2}}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\hat{\sigma}_4^2\right)$

The following second table deals with $\gamma=1$.

$\lambda_i=0,\ \lambda_j>\frac{1}{4}$	$\lambda_i=0,\ \lambda_j=\frac{1}{4}$	$\lambda_i=0,\ \lambda_j<\frac{1}{4}$
$\sqrt{n}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\sigma_3^{*^2}\right)$	$\frac{\sqrt{n}}{\sqrt{\ln n}}(Z_n-\frac{1}{2})\xrightarrow{stably}\mathcal{N}\left(0,\sigma_4^{*^2}\right)$	$n^{4(\lambda_1+\lambda_2)}(Z_n-\frac{1}{2})\xrightarrow{a.s./L^1}\tilde{\chi}$

In the following, let $\mathcal{N}\left(0,\vartheta\ Z_\infty(1-Z_\infty)\right)$ denote the gaussian kernel (mixture of gaussian distributions w.r.t random variance),

$$\mathcal{N}\left(0,\vartheta\ Z_\infty(1-Z_\infty)\right):=\int\mathcal{N}\left(0,\vartheta\ z(1-z)\right)\mathbb{P}_{Z_\infty}(dz).$$

Theorem 5.7.
Consider the model defined through (3.9), (3.10), asuuming the reinforcement rates $(r_n^l)_n$ and $(r_n^g)_n$ satisfy (3.12), (3.13) with $(\gamma_1,\gamma_2)\in]\frac{1}{2},1]^2$. Assume $\lambda_1=\lambda_2=0$.

(i) It holds

$$n^{\frac{2\gamma_1-\gamma_2}{2}}(Z_n-Z_n(i))\xrightarrow[n\to\infty]{stably}\mathcal{N}\left(0,\vartheta\ Z_\infty(1-Z_\infty)\right).$$

(ii) With $\gamma:=\min(\gamma_1,\gamma_2)$, it holds

$$n^{\frac{2\gamma-1}{2}}(Z_n-Z_\infty)\xrightarrow[n\to\infty]{stably}\mathcal{N}\left(0,\vartheta\ Z_\infty(1-Z_\infty)\right).$$

Where ϑ denotes constants are specified in proofs.

Remark 5.8. *(analogous to Theorem 3.2 in [33]).*
We have $\mathbb{P}(Z_\infty = 0) + \mathbb{P}(Z_\infty = 1) < 1$ *and* $\mathbb{P}(Z_\infty = z) = 0$ *for each* $z \in (0,1)$. *Indeed, it guarantees that these limit Gaussian kernels are not degenerate.*

Proof.
The first part immediately follows from the relation $\mathbb{E}[Z_\infty^2] < \mathbb{E}[Z_\infty]$ by Lemma 3.19. The second part is a consequence of the almost sure conditional convergence stated in Theorem 5.7 (ii). Details are essentially same as first step of proof of Theorem 2.5 in [32]. Indeed,
if we denote by Q_n a version of the conditional distribution of $n^{\gamma-\frac{1}{2}}(Z_n - Z_\infty)$ given $\mathcal{F}_n$, then there exists an event A such that $\mathbb{P}(A) = 1$ and for each $\omega \in A$,

$$\lim_n Z_n(\omega) = Z_\infty(\omega) \qquad \textit{and} \qquad Q_n(\omega) \xrightarrow[n\to\infty]{\text{weakly}} \mathcal{N}\Big(0, \vartheta\ (Z_\infty(\omega) - Z_\infty^2(\omega))\Big).$$

Assume now, by contradiction, that there exists $z \in (0,1)$ with $\mathbb{P}(Z_\infty = z) > 0$ and set $A^{'} = A \cap \{Z_\infty = z\}$ and define B_n as the $\mathcal{F}_n$-measurable random set $\{n^{\gamma-\frac{1}{2}}(Z_n - Z_\infty)\}$. Then $\mathbb{P}(A^{'}) > 0$ and since $\mathbb{E}[I_{\{Z_\infty = z\}} \mid \mathcal{F}_n]$ converges almost surely to $I_{\{Z_\infty = z\}}$, there exists an event $A^{''}$ such that $\mathbb{P}(A^{''}) > 0$, $A^{''} \subseteq A^{'}$ and for each $\omega \in A^{''}$,

$$Q_n(\omega)(B_n(\omega)) = \mathbb{E}\Big[I_{\{n^{\gamma-\frac{1}{2}}(Z_n - Z_\infty)\}}\Big(n^{\gamma-\frac{1}{2}}(Z_n - Z_\infty)\Big)|\mathcal{F}_n\Big](\omega) = \mathbb{E}[I_{\{Z_\infty = z\}} \mid \mathcal{F}_n](\omega) \longrightarrow I_{\{Z_\infty = z\}}(\omega) = 1.$$

On the other hand, we observe that $Z_\infty(\omega) - Z_\infty^2(\omega) \neq 0$ when $\omega \in A^{'}$. Hence, if D is the discrepancy metric defined by

$$D[\mu, \nu] = \sup_{B \in \{closed balls of \mathbb{R}\}} |\mu(B) - \nu(B)|,$$

which is a metric for the weak convergence of a sequence of probability distribution on $\mathbb{R}$ in the case when the limit distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then for each $\omega \in A^{'}$, we have

$$\begin{aligned} Q_n(\omega)(B_n(\omega)) &= |Q_n(\omega)(B_n(\omega)) - \mathcal{N}\Big(0, \vartheta\ (Z_\infty(\omega) - Z_\infty^2(\omega))\Big)(B_n(\omega))| \\ &\leq D\Big[Q_n(\omega), \mathcal{N}\Big(0, \vartheta\ (Z_\infty(\omega) - Z_\infty^2(\omega))\Big)\Big] \longrightarrow 0. \end{aligned}$$

This contradicts the previous fact and the proof follows. □

5.3 Proofs of the CLTs (Theorem 5.4)

We now prove the central limit theorems. Recall we are using the notation $a_n \simeq b_n$ when $\lim_n \frac{a_n}{b_n} = cst$.

Define $X_k = Z_k - Z_k(i)$. Set $L_0 = X_0$ and define

$$L_n := X_n - \sum_{k=0}^{n-1} (\mathbb{E}[X_{k+1}|\mathcal{F}_k] - X_k). \tag{5.1}$$

Then it follows:

$$X_{n+1} = [1 - 2\lambda_1 r_n^l - r_n^g]X_n + \Delta L_{n+1} \tag{5.2}$$

where $\Delta L_{n+1} = L_{n+1} - L_n$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write

$$X_n = c_{1,n}X_1 + \sum_{k=1}^{n-1} c_{k+1,n}\Delta L_{k+1} \tag{5.3}$$

where $c_{n,n} = 1$ and $c_{k,n} = \prod_{h=k}^{n-1}[1 - 2\lambda_1 r_h^l - r_h^g]$ for $k < n$.

Proof.
((i)(a) Theorem 5.4)

If $\gamma_1 < \gamma_2$, it is easy to check that $\lim_{n\to\infty} n^{\frac{\gamma_1}{2}} c_{1,n} = 0$ since,

$$\begin{aligned}
c_{1,n} = \prod_{h=1}^{n-1}[1 - 2\lambda_1 r_h^l - r_h^g] &= \prod_{h=1}^{n-1}[1 - \frac{2\lambda_1 c_1}{h^{\gamma_1}} - \frac{c_2}{h^{\gamma_2}} - O(\frac{1}{h^{2\gamma_1}})] \\
&= \exp[-\sum_{h=1}^{n-1}\frac{2\lambda_1 c_1}{h^{\gamma_1}} - \sum_{h=1}^{n-1}\frac{c_2}{h^{\gamma_2}} + O(1)] \\
&= O\Big(\exp[-\frac{-2\lambda_1 c_1}{1-\gamma_1}n^{1-\gamma_1}(1 - \frac{c_2}{1-\gamma_2}\frac{1-\gamma_1}{2\lambda_1 c_1}\frac{1}{n^{\gamma_2-\gamma_1}})]\Big) \\
&= O\Big(\exp\Big(\frac{-2\lambda_1}{1-\gamma_1}n^{1-\gamma_1}\Big)\Big)
\end{aligned}$$

and so with Lemma A.3, we get

$$c_{k,n} = O\Big(\exp[\frac{-2\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})]\Big).$$

Thus it is enough to prove the convergence $n^{\frac{\gamma_1}{2}}\sum_k c_{k+1,n}\Delta L_{n+1} \to \mathcal{N}(0, (1 - 1/N)/16\lambda_1)$. First, let us define $U_{n,k} = n^{\frac{\gamma_1}{2}} c_{k+1,n}\Delta L_{k+1}$ and $\mathcal{G}_{n,k} = \mathcal{F}_{k+1}$. Thus $\{U_{n,k}, \mathcal{G}_{n,k} : 1 \le k \le n\}$ is a square-integrable martingale difference array.
Indeed we have $\mathbb{E}(U_{n,k}^2) < +\infty$ and $\mathbb{E}(U_{n,k+1}|\mathcal{G}_{n,k}) = n^{\frac{\gamma_1}{2}} c_{k+1,n}\mathbb{E}(\Delta L_{k+1}|\mathcal{F}_{k+1}) = 0$ a.s.

Then, using the Theorem recalled as 5.3, through the following statements for

$$U_{n,k} = n^{\frac{\gamma_1}{2}} c_{k+1,n}\Delta L_{k+1}.$$

1) $\max_{1\le k\le n} |U_{n,k}| \to 0$.

2) $\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2]$ is bounded in n.

3) $\sum_{k=1}^n U_{n,k}^2 \to (1 - 1/N)/16\lambda_1$ a.s.

Considering 1), since $\Delta L_{n+1} - (X_{n+1} - X_n) = 2\lambda_1 X_n / n^{\gamma_1}$, $|\Delta L_{n+1}| = O(n^{-\gamma_1})$ a.s.

To prove 2), using 1) to obtain

$$\begin{aligned}
\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2] &\le \mathbb{E}[\sum_{k=1}^{n} U_{n,k}^2] \\
&= n^{\gamma_1} \sum_{k=1}^{n} c_{k+1,n}^2 \mathbb{E}[(\Delta L_{k+1})^2] \\
&\simeq n^{\gamma_1} \sum_{k=1}^{n} e^{\frac{-4\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})} O(k^{-2\gamma_1}) \\
&= n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}} \sum_{k=1}^{n-1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} O(k^{-2\gamma_1}) + \frac{n^2 O(n^{-2\gamma_1})}{n}.
\end{aligned}$$

Thus, $\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2]$ is bounded.

Let us now consider 3). We have

$$\sum_{k=1}^{n} U_{n,k}^2 = n^{\gamma_1} \sum_{k} c_{k+1,n}^2 (\Delta L_{n+1})^2 \simeq n^{\gamma_1} \sum_{k=1}^{n} \frac{k^{-2\gamma_1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}}{e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}} (\Delta L_{k+1})^2 k^{2\gamma_1}.$$

From 1) we obtain

$$\begin{aligned}
\Delta L_{k+1}^2 &= (X_{k+1} - X_k + 2\lambda_1 r_k^l X_k)^2 \\
&= ((Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i)) + 2\lambda_1 r_k^l (Z_k - Z_k(i)))^2 \\
&= [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + 4\lambda_1^2 (r_k^l)^2 (Z_k - Z_k(i))^2 \\
&+ (r_k^l)(Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))].
\end{aligned}$$

Since $Z_n - Z_n(i) \to 0$ a.s. and $r_k^l X_k^2 = O(k^{-2\gamma_1})$ so,

$$\sum_{k=1}^{n} U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^{n} c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

Let $V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$ and setting the $b_n = \frac{1}{n^{\gamma_1}} e^{+4\lambda_1 \frac{n^{1-\gamma_1}}{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-4\lambda_1 \frac{k^{1-\gamma_1}}{1-\gamma_1}}$. Hence, by Lemma 5.2, it holds $\frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{a_k} \xrightarrow[n\to\infty]{} \frac{1}{4\lambda_1}$. Indeed,

$$\frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{a_k} = \frac{n^{\gamma_1}}{e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}} \sum_{k=1}^{n} k^{-2\gamma_1} e^{\frac{4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}} = n^{\gamma_1} O(n^{-\gamma_1})$$

$$\begin{aligned}
\limsup_n \frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{a_k} &= \limsup_n n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \int_1^n u^{-2\gamma_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \\
&= \limsup_n n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \int_1^n \frac{u^{-\gamma_1}}{4\lambda_1} 4\lambda_1 u^{-\gamma_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \\
&= \limsup_n n^{\gamma_1} e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \Big[\Big(\frac{u^{-\gamma_1}}{4\lambda_1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} \Big)_1^n + \frac{\gamma_1}{4\lambda_1} \int_1^n u^{-\gamma_1-1} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du \Big] \\
&= \limsup_n \frac{1}{4\lambda_1} + \frac{\gamma_1 n^{\gamma_1}}{4\lambda_1} e^{-4\lambda_1} \frac{n^{1-\gamma_1}}{1-\gamma_1} \int_1^n \frac{1}{u^{1+\gamma_1}} e^{4\lambda_1 \frac{u^{1-\gamma_1}}{1-\gamma_1}} du = \frac{1}{4\lambda_1}.
\end{aligned}$$

The same holds for $\liminf_n$ then, $\lim_n = \frac{1}{4\lambda_1}$, implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \to V$. Indeed, we compute

$$\begin{aligned}
&\mathbb{E}\left(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2 | \mathcal{F}_k\right) = \\
&k^{2\gamma_1}\Big((r_k^l)^2 \mathbb{E}\left((\xi_{k+1}^l(i) - Z_k(i))^2 | \mathcal{F}_k\right) + (r_k^g)^2 \mathbb{E}\left((\xi_{k+1}^g - Z_k(i))^2 | \mathcal{F}_k\right) \\
&\quad + 2r_k^l r_k^g \mathbb{E}\Big((\xi_{k+1}^l(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i)) | \mathcal{F}_k\Big)\Big) \\
&= k^{2\gamma_1}\Big((r_k^l)^2 \left(\mathbb{V}\mathrm{ar}(\xi_{k+1}^l(i)|\mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^l(i) - Z_k(i))^2 | \mathcal{F}_k)\right) + (r_k^g)^2 \Big(\mathbb{V}\mathrm{ar}(\xi_{k+1}^g|\mathcal{F}_k) + \mathbb{E}((\xi_{k+1}^g - Z_k(i))^2 | \mathcal{F}_k)\Big) \\
&\quad + 2r_k^l r_k^g \mathbb{E}\Big((\xi_{k+1}^l(i) - Z_k(i))(\xi_{k+1}^g - Z_k(i)) | \mathcal{F}_k\Big)\Big)
\end{aligned}$$

behaves like $k^{2\gamma_1}(\frac{(r_k^l)^2}{4} + \frac{(r_k^g)^2}{4})$ when $k \to \infty$. Similarly,

$$\begin{aligned}
\mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)^2 | \mathcal{F}_k] &= k^{2\gamma_1}\Big((r_k^l)^2 \mathbb{E}\Big[(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k)^2 | \mathcal{F}_k\Big] + (r_k^g)^2 \mathbb{E}\Big[(\xi_{k+1}^g - Z_k)^2 | \mathcal{F}_k\Big] \\
&+ 2r_k^l r_k^g \mathbb{E}\Big[(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k)(\xi_{k+1}^g - Z_k) | \mathcal{F}_k\Big]\Big) \\
&= k^{2\gamma_1}\Big((r_k^l)^2 \Big(\mathbb{V}\mathrm{ar}[\frac{1}{N}\sum_i \xi_{k+1}^l(i) | \mathcal{F}_k] + \mathbb{E}^2(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k | \mathcal{F}_k)\Big) \\
&+ (r_k^g)^2 \Big(\mathbb{V}\mathrm{ar}[\xi_{k+1}^g | \mathcal{F}_k] + \mathbb{E}^2(\xi_{k+1}^g - Z_k | \mathcal{F}_k)\Big) \\
&+ 2r_k^l r_k^g \mathbb{E}\Big[(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k)(\xi_{k+1}^g - Z_k) | \mathcal{F}_k\Big]\Big)
\end{aligned}$$

behaves like $k^{2\gamma_1}(\frac{(r_k^l)^2}{4N} + \frac{(r_k^g)^2}{4})$ when $k \to \infty$; and

$$\begin{aligned}
&\mathbb{E}[k^{2\gamma_1}(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i)) | \mathcal{F}_k] = \\
&k^{2\gamma_1}\left((r_k^l)^2 \mathbb{E}\Big[(\xi_{k+1}^l(i) - Z_k(i))(\frac{1}{N}\sum_i \xi_{k+1}^l(i) - Z_k) | \mathcal{F}_k\Big] + (r_k^g)^2 \mathbb{E}\Big[(\xi_{k+1}^g - Z_k(i))(\xi_{k+1}^g - Z_k) | \mathcal{F}_k\Big]\right)
\end{aligned}$$

behaves like $k^{2\gamma_1}\left(\frac{(r_k^l)^2}{4N}+\frac{(r_k^g)^2}{4}\right)$. It follows

$$\mathbb{E}(V_{k+1}|\mathcal{F}_k) = k^{2\gamma_1}\Bigg[(r_k^l)^2\Big(\mathbb{V}\mathrm{ar}[\xi_{k+1}^l(i)|\mathcal{F}_k] + \mathbb{V}\mathrm{ar}[\frac{1}{N}\sum_i \xi_{k+1}^l(i)|\mathcal{F}_k] \\ - 2\mathbb{E}\Big[(\xi_{k+1}^l(i)-Z_k(i))\left(\frac{1}{N}\sum_i \xi_{k+1}^l(i)-Z_k\right)\Big|\mathcal{F}_k\Big]\Big)\Bigg] \xrightarrow{a.s} \frac{1}{4}\left(1-\frac{1}{N}\right).$$

Thus, $V_k \xrightarrow{a.s} \frac{1}{4}(1-\frac{1}{N})$ and therefore, $\tilde{\sigma}_1^2 = \frac{(1-\frac{1}{N})}{16\lambda_1}$.

The proof of next parts and the other theorems follows along the same lines as previously. We sketch the essential arguments below.

- **Case $\gamma_1 = \gamma_2 (=: \gamma)$.** We obtain with the same argument as before that

$$c_{1,n} = O\Big(\exp[\frac{-(1+2\lambda_1)}{1-\gamma}n^{1-\gamma}]\Big).$$

Therefore $\lim_{n\to\infty} n^{\frac{\gamma}{2}} c_{1,n} = 0$. So,

$$c_{k,n} = O\Big(\exp[\frac{-(1+2\lambda_1)}{1-\gamma}(n^{1-\gamma}-k^{1-\gamma})]\Big)$$

and that 1), 2) (as in proof of (i)(a)) hold.
So it is enough to prove that $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)/4(1+2\lambda_1)$. By Lemma 5.2 and letting $b_n = \frac{1}{n^\gamma}e^{\frac{2(1+2\lambda_1)}{1-\gamma}n^{1-\gamma}}$ and $a_k = \frac{k^{2\gamma}}{c_{1,n}^2}e^{-\frac{2(1+2\lambda_1)}{1-\gamma}k^{1-\gamma}}$, thus $\frac{1}{b_n}\sum_{k=1}^n \frac{1}{a_k} \to \frac{1}{2(1+2\lambda_1)}$.
Then consider

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\mathbb{E}[(\tilde{\xi}_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k] \xrightarrow{a.s} \\ \frac{(1-2\lambda_1)}{2}+\lambda_1(1-\lambda_1)-\frac{(1-2\lambda_1)^2}{4}-\lambda_1(1-2\lambda_1) \\ +\frac{(1-2\lambda_2)}{2}+\lambda_2(1-\lambda_2)-\frac{(1-2\lambda_2)^2}{4}-\lambda_2(1-2\lambda_2) = \frac{1}{2}.$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1}-Z_k)^2|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\mathbb{E}[(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k] \xrightarrow{a.s} \frac{1}{2N}$, and $\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k) = k^{2\gamma}r_k^{2\gamma}\mathbb{E}[(\tilde{\xi}_{k+1}(i)-Z_k(i))(\frac{1}{N}\sum_i \tilde{\xi}_{k+1}(i)-Z_k)|\mathcal{F}_k] \xrightarrow{a.s} \frac{1}{2N}$, thus $V_k \xrightarrow{a.s} \frac{1}{2}(1-\frac{1}{N})$ and therefore, $\tilde{\sigma}_1^2 = \frac{(1-\frac{1}{N})}{4(1+2\lambda_1)}$.

- **Case $\gamma_1 = \gamma_2 = 1$.**

We obtain $c_{1,n} := \prod_h^n[1-(1+2\lambda_1)r_h] = O(n^{-(1+2\lambda_1)})$. Then $\sqrt{n}c_{1,n} \to 0$. So it is enough to prove that $\sqrt{n}\sum_k c_{k+1,n}\Delta L_{n+1} \to \mathcal{N}(0,(1-1/N)/2(1+4\lambda_1))$. This can be proved by

usual three conditions for $U_{n,k+1} = \sqrt{n}\sum_k c_{k+1,n}\Delta L_{n+1}$: 1), 2) (as in previous proofs) and 3) $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)/2(1+4\lambda_1)$.

To investigate these conditions, first consider to 1). Since $\Delta L_{n+1} = X_{n-1} - X_n + (1+2\lambda_1)X_n/n$, $|\Delta L_{n+1}| = O(n^{-1})$ a.s.

For 2), using 1) to obtain

$$\begin{aligned}\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2] &\le \mathbb{E}[\sum_{k=1}^n U_{n,k}^2] \\ &\simeq \frac{1}{n^{1+4\lambda_1}}\sum_{k=1}^{n-1}\frac{k^2O(k^{-2})}{k^{-4\lambda_1}} + \frac{n^2O(n^{-2})}{n}.\end{aligned}$$

Thus, $\mathbb{E}[\max_{1\le k\le n} U_{n,k}^2]$ is bounded in n. Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n\sum_k c_{k+1,n}^2(\Delta L_{n+1})^2 \simeq \frac{1}{n^{1+4\lambda_1}}\sum_{k=1}^n \frac{k^2(\Delta L_{n+1})^2}{k^{-4\lambda_1}}.$$

From 1) we get

$$\begin{aligned}\Delta L_{n+1}^2 \simeq [(Z_{k+1}-Z_k)-(Z_{k+1}(i)-Z_k(i))]^2 + r_k^2(Z_k-Z_k(i))^2 \\ + r_k^2(Z_k-Z_k(i))[(Z_{k+1}-Z_k)-(Z_{k+1}(i)-Z_k(i))]\end{aligned}$$

Since $Z_n - Z_n(i) \to 0$ a.s. and $r_k^2X_k^2 = O(k^{-2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 \xrightarrow{a.s} n\sum_{k=1}^n c_{k+1,n}^2[(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))].$$

We use Lemma 5.2 with $b_n = n^{1+4\lambda_1}$ and $a_k = k^{-4\lambda_1}$.
Let $V_k = k^2[(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))]$.
So $\frac{1}{b_n}\sum_{k=1}^n \frac{1}{a_k} \to \frac{1}{1+4\lambda_1}$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $\frac{V}{1+4\lambda_1}$ a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Indeed, $\mathbb{E}(k^2(Z_{k+1}(i)-Z_k(i))^2|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2}$. Similarly, $\mathbb{E}(k^2(Z_{k+1}-Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2N}$, and $\mathbb{E}(k^2(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k) \xrightarrow{a.s} \frac{1}{2N}$.
Thus, $V_k \xrightarrow{a.s} \frac{1}{2}(1-\frac{1}{N})$ and therefore, $\tilde{\sigma}_1^2 = \frac{(1-\frac{1}{N})}{2(1+4\lambda_1)}$.

[(i)(b) Theorem 5.4]

• **Case $\gamma_2 < \gamma_1$.** Since $c_{1,n} = \prod_{h=1}^{n-1}[1-2\lambda_1 r_h^l - r_h^g] = O(\exp[\frac{-1}{1-\gamma_2}n^{1-\gamma_2}])$ therefore, $n^{\gamma_1-\frac{\gamma_2}{2}}c_{1,n} \to 0$. Thus

$$c_{k,n} = O(\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2}-k^{1-\gamma_2})])$$

and that 1) and 2) (as in proof of theorem) hold. So it is enough to prove that $\sum_{k=1}^n U_{n,k}^2 \to (1-1/N)/4$. We have

$$\begin{aligned}(\Delta L_{n+1})^2 &\simeq [(Z_{k+1}-Z_k)-(Z_{k+1}(i)-Z_k(i))]^2 + (r_k^g)^2(Z_k-Z_k(i))^2 \\ &+ (r_k^g)^2(Z_k-Z_k(i))[(Z_{k+1}-Z_k)-(Z_{k+1}(i)-Z_k(i))].\end{aligned}$$

Since $Z_n - Z_n(i) \to 0$ a.s. and $(r_k^g)^2 X_k^2 = O(k^{-2\gamma_2})$ a.s. so,

$$\sum_{k=1}^{n} U_{n,k}^2 = n^{2\gamma_1-\gamma_2} \sum_{k=1}^{n} c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))].$$

We use Lemma 5.2 with $b_n = \frac{1}{n^{2\gamma_1-\gamma_2}} e^{\frac{2}{1-\gamma_2} n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2} k^{1-\gamma_2}}$ thus, $\frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{a_k} \to \frac{1}{2}$. Let $V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. This implies that $\sum_{k=1}^{n} U_{n,k}^2$ converges to V a.s., where V is deterministic such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Since we completed this computation in the proof of the previous part of theorem, we know that in this case $V_k \xrightarrow{a.s} \frac{1}{4}(1 - \frac{1}{N})$ and therefore, $\tilde{\sigma}_2^2 = \frac{1}{8}(1 - \frac{1}{N})$.

[(ii) Theorem 5.4]

- When $\gamma_1 < \gamma_2$, let $X_k = Z_k - \frac{1}{2}$ so,

$$L_n = X_n - \sum_{k=0}^{n-1} \left(\mathbb{E}(Z_{k+1} - \frac{1}{2}|\mathcal{F}_k) - (Z_k - \frac{1}{2})\right) = X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$$

and $X_{n+1} = [1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g] X_n + \Delta L_{n+1}$. So $c_{1,n} = O(\exp[\frac{-2\lambda_1}{1-\gamma_1} n^{1-\gamma_1}])$ and therefore $n^{\frac{\gamma_1}{2}} c_{1,n} \longrightarrow 0$. Then

$$c_{k,n} = O(\exp[\frac{-2\lambda_1}{1-\gamma_1}(n^{1-\gamma_1} - k^{1-\gamma_1})]).$$

It is enough to show that $\sum_{k=1}^{n} U_{n,k}^2 = n^{\gamma_1} \sum_{k=1}^{n} c_{k+1,n}^2 k^{-2\gamma_1} (\Delta L_{k+1})^2 k^{2\gamma_1}$ is a constant. Using Lemma 5.2, $b_n = \frac{1}{n^{\gamma_1}} e^{\frac{4\lambda_1}{1-\gamma_1} n^{1-\gamma_1}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{\frac{-4\lambda_1}{1-\gamma_1} k^{1-\gamma_1}}$. Therefore $\frac{1}{b_n} \sum_k \frac{1}{a_k} \to \frac{1}{4\lambda_1}$. Also $(\Delta L_{n+1})^2 = (Z_{k+1} - Z_k + 2\lambda_1 r_n^l (Z_k - \frac{1}{2}))^2 = (Z_{k+1} - Z_k)^2$. Then $k^{2\gamma_1} \mathbb{E}((Z_{k+1} - Z_k)^2|\mathcal{F}_k) = \frac{1}{4}$. and $\hat{\sigma}^2 = \frac{1}{16\lambda_1}$.

- When $\gamma_2 < \gamma_1$, set $X_k = Z_k - \frac{1}{2}$ then $L_n = X_n + 2(\lambda_1 r_n^l + \lambda_2 r_n^g) \sum_{k=0}^{n-1} X_k$.

So $X_{n+1} = [1 - 2\lambda_1 r_n^l - 2\lambda_2 r_n^g] X_n + \Delta L_{n+1}$. Thus, $c_{1,n} = O(\exp[\frac{-2\lambda_2}{1-\gamma_2} n^{1-\gamma_2}])$ and therefore $n^{\frac{\gamma_2}{2}} c_{1,n} \longrightarrow 0$. Then

$$c_{k,n} = O(\exp[\frac{-2\lambda_2}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})])$$

It is enough to show that $\sum_{k=1}^{n} U_{n,k}^2 = n^{\gamma_2} \sum_{k=1}^{n} c_{k+1,n}^2 k^{-2\gamma_2} (\Delta L_{k+1})^2 k^{2\gamma_2}$ is a constant. Using Lemma 5.2 with $b_n = \frac{1}{n^{\gamma_2}} e^{\frac{4\lambda_2}{1-\gamma_2} n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_2}}{c_{1,n}^2} e^{\frac{-4\lambda_2}{1-\gamma_2} k^{1-\gamma_2}}$. Therefore $\frac{1}{b_n} \sum_k \frac{1}{a_k} \to \frac{1}{4\lambda_2}$. Also

$$\begin{aligned}
(\Delta L_{n+1})^2 &= (X_{n+1} X_n - 2\lambda_2 r_n^g X_n)^2 \\
&= (Z_{k+1} - Z_k + 2\lambda_2 r_n^g (Z_k - \frac{1}{2}))^2 \\
&= (Z_{k+1} - Z_k)^2.
\end{aligned}$$

Thus, $k^{2\gamma_2}\mathbb{E}((Z_{k+1}-Z_k)^2|\mathcal{F}_k)=\frac{1}{4}$ and $\hat{\sigma}^2=\frac{1}{16\lambda_2}$

• When $\gamma_1=\gamma_2(=:\gamma)$, set $X_k=Z_k-\frac{1}{2}$ then $X_{n+1}=[1-2r_n(\lambda_1+\lambda_2)]X_n+\Delta L_{n+1}$ and $c_{1,n}=O(\exp[\frac{-2(\lambda_1+\lambda_2)}{1-\gamma}n^{1-\gamma}])$ and therefore $n^{\frac{\gamma}{2}}c_{1,n}\to 0$. Then

$$c_{k,n}=O(\exp[\frac{-2(\lambda_1+\lambda_2)}{1-\gamma}(n^{1-\gamma}-k^{1-\gamma})])$$

It is enough to show that $\sum_{k=1}^n U_{n,k}^2=n^\gamma\sum_{k=1}^n c_{k+1,n}^2k^{-2\gamma}(\Delta L_{k+1})^2k^{2\gamma}$ is a constant. Using Lemma 5.2 with $b_n=\frac{1}{n^\gamma}e^{\frac{4(\lambda_1+\lambda_2)}{1-\gamma}n^{1-\gamma_1}}$ and $a_k=\frac{k^{2\gamma}}{c_{1,n}^2}e^{\frac{-4(\lambda_1+\lambda_2)}{1-\gamma}k^{1-\gamma}}$.
Therefore $\frac{1}{b_n}\sum_k\frac{1}{a_k}\to\frac{1}{4(\lambda_1+\lambda_2)}$. Also $(\Delta L_{n+1})^2=(Z_{k+1}-Z_k)^2$ and so $k^{2\gamma}\mathbb{E}((Z_{k+1}-Z_k)^2|\mathcal{F}_k)=\frac{1}{4}$ and $\hat{\sigma}^2=\frac{1}{16(\lambda_1+\lambda_2)}$

[(iii) Theorem 5.4]

When $\gamma_1=\gamma_2=1$, it holds $c_{1,n}=\prod_h^n[1-2(\lambda_1+\lambda_2)r_h]=O(n^{-2(\lambda_1+\lambda_2)})$ so, $\sqrt{n}\,c_{1,n}=n^{-2(\lambda_1+\lambda_2)+\frac{1}{2}}\longrightarrow 0$ for $(\lambda_1+\lambda_2)>\frac{1}{4}$. Then we can obtain

$$c_{k,n}=O\left((\frac{k}{n})^{2(\lambda_1+\lambda_2)}\right).$$

Moreover, $\sum_k U_{k,n}^2=n\sum_k(\frac{k}{n})^{4(\lambda_1+\lambda_2)}(\Delta L_{k+1})^2k^2k^{-2}$ and therefore by using 5.2 with taking a_n and b_n, $\frac{1}{b_n}\sum_k\frac{1}{a_k}\to\frac{1}{1-4(\lambda_1+\lambda_2)}$ and thus, $(\Delta L_{n+1})^2=(Z_{k+1}-Z_k)^2$ then $k^2\mathbb{E}((Z_{k+1}-Z_k)^2|\mathcal{F}_k)=\frac{1}{4}$ and therefore, $\sigma_1^{*^2}=\frac{1}{4(1-4(\lambda_1+\lambda_2))}$.

When $(\lambda_1+\lambda_2)=\frac{1}{4}$, $\sqrt{n}(\ln n)^{-\frac{1}{2}}c_{1,n}\longrightarrow 0$. So $c_{k,n}=(\frac{k}{n})^{\frac{1}{2}}$ and $U_{k,n}=\frac{\sqrt{n}}{\sqrt{\log n}}c_{k+1,n}\Delta L_{k+1}$ thus, $\sum_k U_{k,n}^2=\frac{n}{\log n}\sum_k(\frac{k}{n})(\Delta L_{k+1})^2=\frac{1}{\log n}\sum_k\frac{1}{k}k^2(\Delta L_{k+1})^2=k^2(\Delta L_{k+1})^2$ therefore $b_n=\log n$ and $a_k=k$, so $\frac{1}{\log n}\sum_k\frac{1}{k}\to 1$. Thus, $\sigma_2^{*^2}=\frac{1}{4}$. □

5.4 Proofs of the CLTs (Theorem 5.5)

Proof.

Define $\tilde{X}_n=n^{4(\lambda_1+\lambda_2)}(Z_n-\frac{1}{2})$. It holds $\mathbb{E}[\tilde{X}_n^2]<\infty$. In order to show that $\tilde{X}_n$ is a quasi-martingale, we check

$$\begin{aligned}
\sum_k \mathbb{E}\left(|\mathbb{E}[\tilde{X}_{k+1}|\mathcal{F}_k] - \tilde{X}_k]|\right) &= \sum_k \mathbb{E}\Big[\Big|(k+1)^{4(\lambda_1+\lambda_2)}(1-2(\lambda_1+\lambda_2)r_k)X_k - \tilde{X}_k\Big| \\
&= \sum_k \mathbb{E}\Big(\Big|(1+\frac{1}{k})^{4(\lambda_1+\lambda_2)}(1-2(\lambda_1+\lambda_2)r_k)\tilde{X} - \tilde{X}_k\Big|\Big) \\
&= \sum_k \mathbb{E}\Big(\Big|[(1+\frac{1}{k})^{4(\lambda_1+\lambda_2)}(1-2(\lambda_1+\lambda_2)r_k) - 1]\tilde{X}_k\Big|\Big) \\
&= \sum_k \Big|(\frac{1}{k})^{4(\lambda_1+\lambda_2)}(-2(\lambda_1+\lambda_2)r_k)\Big|\mathbb{E}(|\tilde{X}_k|) \\
&= \sum_k O(\frac{1}{k^2})8(\lambda_1+\lambda_2)^2\mathbb{E}(|\tilde{X}_k|) < +\infty.
\end{aligned}$$

Thus $(\tilde{X}_n)_n$ is an $\mathcal{F}$-quasi-martingale. Moreover, from the computations carried out in the proof of Theorem 5.4, $\mathbb{E}(\tilde{X}_n^2) < +\infty$ and so it converges $a.s$ and in mean to some real random variable $\tilde{X}$.
In order to prove that $\mathbb{P}(\tilde{X} \neq 0) > 0$, we will prove that $(\tilde{X}_n^2)_n$ is bounded in L^p for a suitable $p > 1$. Indeed this fact implies that $\tilde{X}_n^2$ converges in mean to $\tilde{X}^2$ and so we obtain $\mathbb{E}(\tilde{X}^2) = \lim_n \mathbb{E}(\tilde{X}_n^2) = \lim_n n^{4(\lambda_1+\lambda_2)}\mathbb{E}(X_n^2) > 0$. To this purpose, we set $p = 1 + \epsilon/2$, with $\epsilon > 0$ and $x_n = \mathbb{E}(|X_n|^{2+\epsilon})$. Using the following recursive equation:

$$X_{n+1} = (1-2r_n)Z_n + \frac{r_n}{N}\sum_{i=1}^{N}\tilde{\xi}_{k+1}(i) - \frac{1}{2}$$

$$\begin{aligned}
x_{n+1} = \mathbb{E}(|X_n|^{2+\epsilon}) &- (2+\epsilon)r_n 2Z_n\mathbb{E}(|X_n|^{1+\epsilon}) \\
&+ (2+\epsilon)r_n\mathbb{E}\left[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (X_n)(\frac{1}{N}\sum_{i=1}^{N}\tilde{\xi}_{k+1}(i))\right] + R_n
\end{aligned}$$

where $R_n = O(n^{-2})$. Now, since

$$\mathbb{E}[\frac{1}{N}\sum_i \tilde{\xi}_{n+1}(i)|\mathcal{F}_n] = 2Z_n - 2(\lambda_1+\lambda_2)(Z_n - \frac{1}{2}),$$

we get

$$\begin{aligned}
x_{n+1} &= \mathbb{E}(|X_n|^{2+\epsilon}) - 2(2+\epsilon)r_n Z_n\mathbb{E}(|X_n|^{1+\epsilon}) \\
&\quad + (2+\epsilon)r_n\mathbb{E}[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (2Z_n - 2(\lambda_1+\lambda_2))X_n] + R_n \\
&= \mathbb{E}(|X_n|^{2+\epsilon}) - (2+\epsilon)r_n 2(\lambda_1+\lambda_2)\mathbb{E}[|X_n|^{1+\epsilon}\ \mathrm{sign}(X_n)\ (X_n)X_n] + R_n \\
&= \mathbb{E}(|X_n|^{2+\epsilon}) - (2+\epsilon)r_n 2(\lambda_1+\lambda_2)\mathbb{E}\Big(|X_n|^{2+\epsilon}\Big) + R_n \\
&= \Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big)x_n + g(n),
\end{aligned}$$

with $g(n) = O(n^{-2})$. Therefore, we have

$$x_{n+1} = \Big(1 - 2(\lambda_1 + \lambda_2)(2+\epsilon)r_n\Big)x_n + g(n).$$

Since, for $\epsilon > 0$ sufficiently small, we have $\alpha(2+\epsilon) < 1$ and as $n \to \infty$

$$\begin{aligned}\prod_{k=0}^{n-1}\Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)r_n\Big) &= \exp[\sum_{k=0}^{n-1}(\ln\Big(1 - 2(\lambda_1+\lambda_2)(2+\epsilon)\frac{c}{n^\gamma} + O(\frac{1}{n^{2\gamma}})\Big)] \\ &= O(\exp[-2(\lambda_1+\lambda_2)(2+\epsilon)\sum_{k=0}^{n-1}\ln n]) \\ &= O(n^{-2(\lambda_1+\lambda_2)(2+\epsilon)}).\end{aligned}$$

So,

$$\mathbb{E}[|X_n|^{(2+\epsilon)}] = O(\frac{1}{n^{2(\lambda_1+\lambda_2)(2+\epsilon)}})$$

and so it implies that $\tilde{X}^2$ is bounded in $L^{1+\frac{\epsilon}{2}}$. □

5.5 Proofs of the CLTs (Theorem 5.6)

Proof.
(Theorem 5.6)

We organize the proof following two main cases, depending on $\lambda_1,\ \lambda_2$ nullity.

- **Case $\lambda_1 \neq 0, \lambda_2 = 0$.**

In order to study the evolution of $(Z_n - Z_n(i))_n$, we consider two cases.

- When $\gamma_1 \leq \gamma_2$,

$X_{n+1} = (1 - 2\lambda_1 r_n^I)X_n + \Delta L_{n+1}$ and the proof follows essentially the same as the part (i)(a) of Theorem 5.4 with $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{16\lambda_1}$ when $\gamma_1 < \gamma_2$, $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{4(1+2\lambda_1)}$ when $\gamma_1 = \gamma_2 = \gamma$ and $\tilde{\sigma}_3^2 = \frac{(1-\frac{1}{N})}{2(1+4\lambda_1)}$ when $\gamma_1 = \gamma_2 = 1$.

- When $\gamma_2 < \gamma_1$,

$X_{n+1} = (1 - r_n^g)X_n + \Delta L_{n+1}$, then the proof follows essentially the same as part (i)(b) of Theorem 5.4 with $\tilde{\sigma}_5^2 = \frac{1}{8}(1 - \frac{1}{N})$.

In order to study $\left(Z_n - \frac{1}{2}\right)_n$:

- When $\gamma_1 \leq \gamma_2$,

$X_{n+1} = (1 - r_n^g)X_n + \Delta L_{n+1}$, then the proof follows essentially the same as the part (ii) of Theorem 5.4 with $\hat{\sigma}_1^2 = \frac{1}{16\lambda_1}$ when $\gamma_1 < \gamma_2$ and $\hat{\sigma}_1^2 = \frac{1}{16(\lambda_1+\lambda_2}$ when $\gamma_1 = \gamma_2 = \gamma$.

- When $\gamma_2 < \gamma_1$,

the proof follows along the same lines as previous. We sketch essential argument below. We have

$$X_{n+1} = (1 - 2\lambda_1 r_n^J)X_n + \Delta L_{n+1}.$$

therefore, $c_{1,n} = O(e^{\frac{-2\lambda_1}{1-\gamma_1}n^{1-\gamma_1}})$ and thus, $n^{\gamma_2-\frac{\gamma_1}{2}}c_{1,n} \to 0$. Following the same steps as in the previous proof, it be can verified that 1) and 2) hold. Only showing that

$$\sum_{k=1}^{n} U_{n,k}^2 = n^{2\gamma_2-\gamma_1}e^{\frac{-4\lambda_1}{1-\gamma_1}n^{1-\gamma_1}} \sum_{k}^{n} k^{-2\gamma_2}e^{\frac{4\lambda_1 k^{1-\gamma_1}}{1-\gamma_1}}k^{2\gamma_2}(\Delta L_{k+1})^2$$

tends to a constant.

It is easy to derive by Lemma 5.2 that $\frac{1}{b_n}\sum_k \frac{1}{a_k} \to \frac{1}{4\lambda_1}$ and

$$k^{2\gamma_2}(\Delta L_{k+1})^2 \to k^{2\gamma_2}(Z_{k+1} - Z_k)^2 = \frac{1}{4}.$$

Therefore, $\hat{\sigma}_3^2 = \frac{1}{16\lambda_1}$.

- **Case $\lambda_1 = 0, \lambda_2 \neq 0$.**

Concerning the evolution of $(Z_n - Z_n(i))_n$, for both cases $\gamma_1 \leq \gamma_2$ and $\gamma_2 < \gamma_1$, it is proved analogously as the part (i)(b) of Theorem 5.4 with $\tilde{\sigma}_4^2 = \tilde{\sigma}_6^2 = \frac{1}{8}(1 - \frac{1}{N})$.

Concerning $\left(Z_n - \frac{1}{2}\right)_n$:

• When $\gamma_1 \leq \gamma_2$, the proof follows along the same lines as previous. We sketch essential argument below. We have

$$X_{n+1} = (1 - 2\lambda_2 r_n^g)X_n + \Delta L_{n+1},$$

therefore it holds $c_{1,n} = O(e^{\frac{-2\lambda_2}{1-\gamma_2}n^{1-\gamma_2}})$ and thus, $n^{\gamma_1-\frac{\gamma_2}{2}}c_{1,n} \to 0$.

Following the same steps as in the previous proof, it be can verified that 1) and 2) hold. It is enough to show that

$$\sum_{k=1}^{n} U_{n,k}^2 = n^{2\gamma_1-\gamma_2}e^{\frac{-4\lambda_2}{1-\gamma_2}n^{1-\gamma_1}} \sum_{k}^{n} k^{-2\gamma_1}e^{\frac{4\lambda_2 k^{1-\gamma_2}}{1-\gamma_2}}k^{2\gamma_1}(\Delta L_{k+1})^2$$

goes to a constant.

It is easy to derive by Lemma 5.2 that $\frac{1}{b_n}\sum_k \frac{1}{a_k} \to \frac{1}{4\lambda_2}$ and

$$k^{2\gamma_1}(\Delta L_{k+1})^2 \to k^{2\gamma_1}(Z_{k+1} - Z_k)^2 = \frac{1}{4}.$$

Therefore, $\hat{\sigma}_2^2 = \frac{1}{16\lambda_2}$.

The proof when $\gamma_2 < \gamma_1$ goes the same as the part (ii) of Theorem 5.4 ($\gamma_2 < \gamma_1$) with $\hat{\sigma}_4^2 = \frac{1}{16\lambda_2}$.

• **The case** $\gamma_1 = \gamma_2 = 1$ is proven similarly as in part (*iii*) Theorem 5.4 with $\sigma_3^{*^2} = \frac{1}{4(1-4(\lambda_1+\lambda_2))}$ when $\lambda_1 + \lambda_2 > \frac{1}{4}$, $\sigma_4^{*^2} = \frac{1}{4}$ when $\lambda_1 + \lambda_2 = \frac{1}{4}$ and Theorem 5.5 when $\lambda_1 + \lambda_2 < \frac{1}{4}$. □

5.6 Proofs of the CLTs (Theorem 5.7)

Proof.
[(i) Theorem 5.7]

- **Case $\gamma_1 \neq \gamma_2$.**

Define $X_k = Z_k - Z_k(i)$. Set $L_0 = X_0$ and let us rewrite

$$\begin{aligned} L_n &= X_n - \sum_{k=0}^{n-1}(\mathbb{E}[X_{k+1}|\mathcal{F}_n] - X_k) \\ &= X_n - \sum_{k=0}^{n-1}([1 - r_k^g](Z_k - Z_k(i)) - (Z_k - Z_k(i))) = X_n + \sum_{k=0}^{n-1} r_k^g X_k. \end{aligned}$$

Then $X_{n+1} = [1 - r_n^g]X_n + \Delta L_{n+1}$. Note that $(L_n)_n$ is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1 - r_h^g]$ for $k \leq n$. It holds $c_{1,n} = \prod_{h=1}^n [1 - r_h^g] = O(\exp[\frac{-1}{1-\gamma_2} n^{1-\gamma_2}])$.

Then $n^{\gamma_1 - \frac{\gamma_2}{2}} c_{1,n} \to 0$ and

$$c_{k,n} = O\left(\exp[\frac{-1}{1-\gamma_2}(n^{1-\gamma_2} - k^{1-\gamma_2})]\right).$$

So it is enough to prove that $n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n}\Delta L_{n+1} \to \mathcal{N}\big(0,\ (1 - 1/N)(Z_\infty - Z_\infty^2)\big)$. Again, this can be proved using following conditions for $U_{n,k+1} = n^{\gamma_1 - \frac{\gamma_2}{2}} \sum_k c_{k+1,n}\Delta L_{n+1}$.
1) $\max_{1\leq k\leq n} |U_{n,k}| \to 0$.
2) $\mathbb{E}[\max_{1\leq k\leq n} U_{n,k}^2]$ is bounded in n.
3) $\sum_{k=1}^n U_{n,k}^2 \to (1 - 1/N)(Z_\infty - Z_\infty^2)$.

It is easy to check that conditions 1) and 2) hold. Let us now consider 3). We have

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_k c_{k+1,n}^2 (\Delta L_{n+1})^2 \simeq n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n \frac{k^{-2\gamma_1} e^{\frac{1}{1-\gamma_2} k^{1-\gamma_2}}}{e^{\frac{1}{1-\gamma_2} n^{1-\gamma_2}}} (\Delta L_{n+1})^2 k^{2\gamma_1}.$$

From 1) we obtain

$$\begin{aligned} (\Delta L_{n+1})^2 &\simeq [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + (r_k^g)^2 (Z_k - Z_k(i))^2 \\ &+ (r_k^g)^2 (Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]. \end{aligned}$$

Since $Z_n - Z_n(i) \xrightarrow{a.s} 0$ and $(r_k^g)^2 X_k^2 = O(k^{-2\gamma_2})$ so,

$$\sum_{k=1}^n U_{n,k}^2 = n^{2\gamma_1 - \gamma_2} \sum_{k=1}^n c_{k+1,n}^2 [(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$$

where we use Lemma 5.2 with $b_n = \frac{1}{n^{2\gamma_1-\gamma_2}} e^{\frac{2}{1-\gamma_2}n^{1-\gamma_2}}$ and $a_k = \frac{k^{2\gamma_1}}{c_{1,n}^2} e^{-\frac{2}{1-\gamma_2}k^{1-\gamma_2}}$.
Let $V_k = k^{2\gamma_1}[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$.
Thus $\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^n \frac{1}{a_k} = \frac{1}{2}$. This implies that $\sum_{k=1}^n U_{n,k}^2$ converges to $V/2$ a.s., where V is such that $\mathbb{E}(V_{k+1}|\mathcal{F}_k) \longrightarrow V$. Indeed,

$$\begin{aligned}
\mathbb{E}\left(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k\right) &= k^{2\gamma_1}(r_k^l)^2\mathbb{E}[(\xi_{k+1}^l(i) - Z_k(i))^2|\mathcal{F}_k] \\
&= k^{2\gamma_1}(r_k^l)^2\mathbb{V}\mathrm{ar}[\xi_{k+1}^l(i)|\mathcal{F}_k] \\
&= k^{2\gamma_1}(r_k^l)^2\left(Z_k - Z_k^2\right) \xrightarrow{a.s} Z_\infty - Z_\infty^2.
\end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma_1}(Z_{k+1} - Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s} Z_\infty - Z_\infty^2$, and

$$\mathbb{E}(k^{2\gamma_1}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k)|\mathcal{F}_k) \xrightarrow{a.s} \frac{Z_\infty - Z_\infty^2}{N}.$$

Thus, $\lim_{k\to\infty} U_k^2 = \vartheta 2(1 - \frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = \frac{1}{2}$.

Consider the case $\gamma_1 = \gamma_2 (=: \gamma)$. Since $L_n = X_n + \sum_{k=0}^{n-1} r_n X_k$, it holds
$L_{n+1} - L_n = X_{n+1} - (1 - r_n)X_n$. So $X_{n+1} = (1 - r_n)X_n + \Delta L_{n+1}$. Note that $(L_n)_n$ is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write
$X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n (1 - r_h)$ for $k \le n$. We get $c_{1,n} = \prod_{h=1}^n [1 - r_h] = \mathcal{O}(\exp[\frac{-1}{1-\gamma} n^{1-\gamma}])$. Then $n^{\frac{\gamma}{2}} c_{1,n} \to 0$.
Moreover $\lim_{n\to\infty} \frac{1}{b_n} \sum_k \frac{1}{a_k} = \frac{1}{2}$,

$$\begin{aligned}
\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k) &= k^{2\gamma} r_k^{2\gamma}\mathbb{E}[(\tilde{\xi}_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k] \\
&= k^{2\gamma} r_k^{2\gamma}\mathbb{V}\mathrm{ar}[\tilde{\xi}_{k+1}(i)|\mathcal{F}_k] \xrightarrow{a.s} 2(Z_\infty - Z_\infty^2).
\end{aligned}$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1} - Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s} 2(Z_\infty - Z_\infty^2)$, and
$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k)|\mathcal{F}_k) \xrightarrow{a.s} \frac{2(Z_\infty - Z_\infty^2)}{N}$
Thus, $\lim_{k\to\infty} U_k^2 = \vartheta 4(1 - \frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = \frac{1}{2}$.

- **Case** $\gamma_1 = \gamma_2 = 1$.

Consider $L_n = X_n + \sum_{k=0}^{n-1} r_n X_k$.
Then $L_{n+1} - L_n = X_{n+1} - [1 - r_n]X_n$ so, $X_{n+1} = [1 - r_n]X_n + \Delta L_{n+1}$. Note that L_n is an $\mathcal{F}_n$-martingale by construction. Iterating the above relation, we can write $X_n = c_{1,n}X_1 + \sum_{k=1}^n c_{k+1,n}\Delta L_{n+1}$ where $c_{n+1,n} = 1$ and $c_{k,n} = \prod_{h=k}^n [1 - r_h]$ for $k \le n$. $c_{1,n} = \prod_{h=1}^n [1 - r_h] = \mathcal{O}(n^{-1})$.
Then $\sqrt{n}\, c_{1,n} \to 0$.
Choosing $b_n = n$ and $a_k = 1$, $\lim_{n\to\infty} \frac{1}{b_n} \sum_k \frac{1}{a_k} = 1$. it holds

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i) - Z_k(i))^2|\mathcal{F}_k) \xrightarrow{a.s} 2(Z_\infty - Z_\infty^2).$$

Similarly, $\mathbb{E}(k^{2\gamma}(Z_{k+1}-Z_k)^2|\mathcal{F}_k) \xrightarrow{a.s} 2(Z_\infty - Z_\infty^2)$, and

$$\mathbb{E}(k^{2\gamma}(Z_{k+1}(i)-Z_k(i))(Z_{k+1}-Z_k)|\mathcal{F}_k) \xrightarrow{a.s} \frac{2(Z_\infty - Z_\infty^2)}{N}.$$

Thus, $\lim_{k\to\infty} U_k^2 = \vartheta 4(1-\frac{1}{N})(Z_\infty - Z_\infty^2)$ a.s. where $\vartheta = 1$.

[(ii) Theorem 5.7]

- **Case $\gamma_1 < \gamma_2$.**

The process $(Z_n)_n$ is a martingale. Indeed, by

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1-2\lambda_1 r_n^l - 2\lambda_2 r_n^g)Z_n + \lambda_1 r_n^l + \lambda_2 r_n^g,$$

we have $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n$ when $\lambda_1 = 0, \lambda_2 = 0$, therefore $(Z_n)_n$ converges a.s.

To this purpose, satisfies the following two conditions:

1. $\mathbb{E}\Big[\sup_k k^{\gamma_1-\frac{1}{2}}|Z_{k+1}-Z_k|\Big] < +\infty$;
2. $n^{2\gamma_1-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \xrightarrow{a.s} \frac{1}{N(2\gamma_1-1)}(Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1}-Z_k| = |r_n^l(\frac{1}{N}\sum_{i=1}^{N}\xi_{k+1}^l(i) - Z_k) + r_n^g(\xi_{k+1}^g - Z_k)| = O(k^{-\gamma_1}).$$

Regarding the second condition, we observe that

$$n^{2\gamma_1-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n^{2\gamma_1-1}\sum_{k\geq n}k^{-2\gamma_1}(r_k^l)^2(\frac{\sum_i \xi_{k+1}(i)}{N} - Z_k)^2 k^{2\gamma_1},$$

and so the desired convergence follows by lemma 5.2 with $a_k = k^{-2\gamma_1+2}$, $b_n = n^{2\gamma_1-1}$ and

$$U_k = k^{2\gamma_1}(r_k^l)^2(\frac{\sum_k \xi_{k+1}(i)}{N} - Z_k)^2,$$

$\lim_{n\to\infty} b_n \sum_{k\geq n}\frac{1}{a_k b_k^2} = -\frac{1}{1-2\gamma_1}$ so,

$$\mathbb{E}\left((\frac{\sum_k \xi_{k+1}(i)}{N} - Z_k)^2\Big|\mathcal{F}\right) = \mathbb{V}\mathrm{ar}\left(\frac{\sum_k \xi_{k+1}(i)}{N}|\mathcal{F}_n\right) = \frac{1}{N}\left(Z_\infty - Z_\infty^2\right).$$

Finally, we take $\vartheta = \frac{1}{(2\gamma_1-1)}$.

- **Case $\gamma_2 < \gamma_1$.**

To this end, we use the following two conditions:

1) $\mathbb{E}\Big(\sup_k k^{\gamma_2-\frac{1}{2}}|Z_{k+1}-Z_k|\Big) < +\infty$;

2) $n^{2\gamma_2-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \overset{a.s}{\longrightarrow} \frac{1}{(2\gamma_2-1)}(Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1}-Z_k| = |r_n^l(\frac{1}{N}\sum_{i=1}^{N}\xi_{k+1}^l(i) - Z_k) + r_n^g(\xi_{k+1}^g - Z_k)| = O(k^{-\gamma_2}).$$

Regarding the second condition, we observe that

$$n^{2\gamma_2-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n^{2\gamma_2-1}\sum_{k\geq n}(r_k^l)^2k^{-2\gamma_2}(\frac{\sum_i \xi_{k+1}(i)}{N} - Z_k)^2k^{2\gamma_2}$$

and the desired convergence follows by lemma 5.2 with $a_k = k^{2\gamma_2+2}$, $b_n = n^{2\gamma_2-1}$ and $U_k = k^{2\gamma_2}(r_k^g)^2(\xi_{k+1}^g - Z_k)^2$, $\lim_{n\to\infty} b_n\sum_{k\geq n}\frac{1}{a_kb_k^2} = -\frac{1}{1-2\gamma_2}$ and

$$\mathbb{E}(\xi_{k+1}^g(i) - Z_k)^2|\mathcal{F}) = \mathbb{V}\mathrm{ar}(\xi_{k+1}^g|\mathcal{F}) = (Z_\infty - Z_\infty^2).$$

Finally, we take $\vartheta = \frac{1}{(2\gamma_2-1)}$.

- **Case $\gamma_1 = \gamma_2 (=: \gamma)$.**

The process $(Z_n)_n$ is a martingale and converges a.s. Indeed,

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1-2r_n)Z_n + r_n\mathbb{E}\left(\frac{\sum_i\tilde{\xi}_{n+1}(i)}{N}|\mathcal{F}_n\right) = Z_n.$$

To this purpose, we need to check the following two conditions:

1) $\mathbb{E}\Big[\sup_k k^{\gamma-\frac{1}{2}}|Z_{k+1}-Z_k|\Big] < +\infty$;

2) $n^{2\gamma-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 \overset{a.s}{\longrightarrow} \frac{2}{N(2\gamma-1)}(Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1}-Z_k| = |r_n\left(\frac{1}{N}\sum_i\tilde{\xi}_{k+1}(i) - 2Z_k\right)| = O(k^{-\gamma}).$$

Regarding the second condition, we observe that

$$n^{2\gamma-1}\sum_{k\geq n}(Z_{k+1}-Z_k)^2 = n^{2\gamma-1}\sum_{k\geq n}r_k^2k^{-2\gamma}\left(\frac{\sum_i\tilde{\xi}_{k+1}(i)}{N} - Z_k\right)^2k^{2\gamma}$$

and so the desired convergence follows by lemma with $a_k = k^{-2\gamma+2}$, $b_n = n^{2\gamma-1}$ and

$$U_k = k^{2\gamma}r_k^2\left(\frac{\sum_i\tilde{\xi}_{k+1}(i)}{N} - 2Z_k\right)^2,$$

$\lim_{n\to\infty} b_n \sum_{k\geq n} \frac{1}{a_k b_k^2} = -\frac{1}{1-2\gamma}$, $\mathbb{E}\left((\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2 | \mathcal{F}_k\right) = \frac{2}{N}(Z_\infty - Z_\infty^2)$.

Finally, we take $\vartheta = \frac{1}{(2\gamma-1)}$.

- **Case $\gamma_1 = \gamma_2 = 1$.**

To this purpose, we check the following two conditions hold:

1) $\mathbb{E}\Big[\sup_k k^{\frac{1}{2}} |Z_{k+1} - Z_k|\Big] < +\infty$;

2) $n \sum_{k\geq n} (Z_{k+1} - Z_k)^2 \xrightarrow{a.s} \frac{2}{N}(Z_\infty - Z_\infty^2)$.

Indeed, the first condition immediately follows from

$$|Z_{k+1} - Z_k| = |r_n(\frac{1}{N}\sum_{i=1}^{N} \tilde{\xi}_{k+1}(i) - 2Z_k))| = O(k^{-1}).$$

To deal with the second condition, we observe that

$$n \sum_{k\geq n} (Z_{k+1} - Z_k)^2 = n \sum_{k\geq n} r_k^2 k^{-2} k^2 \left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - Z_k\right)^2$$

and thus the desired convergence follows by lemma 5.2 with $a_k = 1$, $b_n = n$ and $U_k = k^2 r_k^2 (\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2$, $\lim_{n\to\infty} b_n \sum_{k\geq n} \dfrac{1}{a_k b_k^2} = 1$.

$$\mathbb{E}\left(\frac{\sum_i \tilde{\xi}_{k+1}(i)}{N} - 2Z_k)^2 | \mathcal{F}\right) = \frac{2}{N} Z_\infty (1 - Z_\infty).$$

Finally, we take $\vartheta = 1$. □

Chapter 6

Conclusion and perspectives

This work was concerned with new family of finite size system of N interacting reinforced stochastic processes that we introduced as in (3.9), in which to N evolutions, we add an interaction term of mean-field type. We are particularly interested in the phenomenon of synchronization as discussed in Chapter 3 could be defined as the tendency of different components to adopt a common large-time behavior. Indeed we studied the almost sure time-asymptotic behavior of $(Z_n)_n$ which is random or deterministic corresponding to λ_j values for $j \in \{1, 2\}$. We showed that under suitable conditions on r_n, $(\gamma_1, \gamma_2) \in]1/2, 1]^2$, either $(Z_n)_n$ converges to random limit or deterministic one, the synchronization always holds. Moreover, we studied the fluctuations theorems in which obtained Central Limit Theorems in stable convergence form.

Based on what we have done, some issues that were mentioned here such as "finite-size", "mean-field" and "CLT" motivate us to provide some interesting issues as the generalizations and perspectives which we express them in individual following sections.

6.1 Generalizations

6.1.1 Strong reinforcement case $(0 \leq \gamma \leq \frac{1}{2})$

As discussed in Chapter 3 when the new model was introduced , we said that region $\gamma \leq \frac{1}{2}$ can be considered separately according to has some special properties such that the saturation. Here, we present some initial results to show that the property of saturation holds in our model under some conditions.

Theorem 6.1. *If $\lambda_1 = \lambda_2 = 0$, then $(Z_n)_n$ converges to a random variable Z_∞ a.s. Moreover, when $\gamma \leq \frac{1}{2}$ then*

$$\mathbb{P}(Z_\infty \in \{0, 1\}) = 1$$

Proof.
By assumption $\lambda_1 = \lambda_2 = 0$, we immediately get that (Z_n) is a bounded martingale. Therefore, it converges a.s. (and in L^p) to a random variable Z_∞, with values in $[0, 1]$ and, since by assumption $\mathbb{E}(Z_0(i)) = \frac{1}{2}$ for every i, $\mathbb{E}(Z_\infty) = \frac{1}{2}$.

Moreover $\mathbb{P}(Z_\infty \in \{0,1\}) = 1$ if

$$\mathbb{V}\mathrm{ar}(Z_\infty) = \lim_{n\to\infty} \mathbb{V}\mathrm{ar}(Z_n) = \frac{1}{4}.$$

Using (3.32) with $\lambda_1 = \lambda_2 = 0$ gives:

$$\mathbb{V}\mathrm{ar}(Z_{n+1}) = \mathbb{V}\mathrm{ar}(Z_n) - (r_n^g)^2\mathbb{V}\mathrm{ar}(Z_n) + \frac{(r_n^g)^2}{4} + (r_n^l)^2 K_n,$$

where $K_n = \dfrac{1}{2N} - \left(\dfrac{1}{N}\sum_{i=1}^{N} \mathbb{E}(Z_n(i)^2)\right)$. Therefore

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) &= \mathbb{V}\mathrm{ar}(Z_n) - (r_n^g)^2\Big(\mathbb{V}\mathrm{ar}(Z_n) - \frac{1}{4}\Big) + (r_n^l)^2 K_n \\ &\geq \mathbb{V}\mathrm{ar}(Z_n) - (r_n^g)^2\Big(\mathbb{V}\mathrm{ar}(Z_n) - \frac{1}{4}\Big) - \frac{1}{4} + \frac{1}{4}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Z_{n+1}) - \frac{1}{4} &\geq \Big(\mathbb{V}\mathrm{ar}(Z_n) - \frac{1}{4}\Big) - (r_n^g)^2\Big(\mathbb{V}\mathrm{ar}(Z_n) - \frac{1}{4}\Big) \\ &= [1 - (r_n^g)^2]\Big(\mathbb{V}\mathrm{ar}(Z_n) - \frac{1}{4}\Big).\end{aligned}$$

Defining $x_n := \frac{1}{4} - \mathbb{V}\mathrm{ar}(Z_n) \geq 0$, it is equivalent to

$$x_{n+1} \leq \Big(1 - (r_n^g)^2\Big)x_n$$

from which it follows

$$x_n \leq x_0 \prod_{k=0}^{n-1} \Big(1 - (r_k^g)^2\Big).$$

Since by assumption, $\sum_n (r_n^g)^2 = +\infty$, we obtain $\lim_{n\to\infty} x_n = 0$. □

6.1.2 Scaling with the system's size λ_N

As we introduced the new family in (3.9), we studied a reinforced stochastic system with a finite size in which the local and collective reinforcement random variables ξ_n^l and ξ_n^g follow law of Bernoulli independently with parameters as defined in (3.10). We also assumed that λ_1 and λ_2 are fixed and as was mentioned in Remark 3.15, we have two different behaviors of time limit distribution. Indeed, nullity of parameters λ_js (being equal 0 or not) let the transformation ψ_js to make a different character of reinforcement (Pólya or Friedman respectively). However, we can further study the asymptotic distribution of Z_n in the limit as number of individual evolutions, N, goes to ∞. Moreover, in order to have more practical model, we let the parameters λ_j's depend on N as the system's size

dependent parameters $\lambda_{j,N}$. Based on the work was done in [36], we are interested to characterize the stochastic process

$$H_n^N := \sqrt{N}\left(Z_n^N - \frac{1}{2}\right)$$

which may converges weakly as $N \to \infty$ to Gauss-Markov process solution of the recursion

$$H_{n+1} = H_n + \sigma_n \; B_{n+1}$$

where $\sigma_n \propto \; x_n^\infty$ with

$$x_n^\infty := \lim_{N\to\infty} \mathbb{E}[(Z_n^N(i) - Z_n^N)^2],$$

the limit of L^2-distance as $N \to \infty$.

6.2 Perspectives

6.2.1 Functional Central Limit Theorem (FCLT)

In chapter 5, we studied the fluctuations of both case, Z_n around Z_∞ and the synchronization one. We proved some Central Limit Theorems in the sense of stable convergence *i.e.*, we found some suitable scaling for fluctuations quantities which state that they converge to some Gaussian random variable stably. Functional Central Limit Theorem is an another approach in order to study the fluctuations which has been used for various urn models (*e.g.* [12,51,59,92]). In particular [12,51] and [92] contain results for Friedman urn models and Pólya urn models respectively, that in our model correspond to the case $N = 1$ and $\gamma = 1$. The Functional central Limit Theorem also known as Donsker's invariance principle [39], states that, as random walk $Z : (Z_n)_{n\in\mathbb{N}}$ taking values in Skorokhod space $\mathcal{D}[0,1]$, the random function

$$W^{(n)}(t) := \frac{Z_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1]$$

converges in distribution to a Brownian motion $W := (W(t))_{t\in[0,1]}$ as $n \to \infty$ (see [21] for details). It can be adapted in our model based on recent work [32] in which has been used to study the fluctuations for the interacting reinforced random walks that we have presented its results in Chapter 3.

6.2.2 Network based-interaction and empirical means

We can see our model that may be generalized with network based-interaction which has been introduced in several recent works [3–5]. In Chapter 3, we represented some main results of these kind of models in particular in [4]. Moreover, in these series works, in particular [3], the asymptotic behavior of stochastic process of the "actions" $(\xi_n(i))_n$ has

been considered. Indeed, it has been studied by characterizing the asymptotic behavior of empirical means

$$N_n(i) = \frac{1}{n} \sum_{k=1}^{n} \xi_k(i) \tag{6.1}$$

in the sense studying of the almost sure synchronization and the central limit theorems. Furthermore, in particular in [5], the asymptotic behavior of weighted empirical means

$$N_n(i) = \sum_{k=1}^{n} q_n(k) \xi_k(i) \tag{6.2}$$

has been characterized. Based on what have been done in these works, we can work further on the stochastic processes, the local $\xi_n^l(i)$ and the ξ_n^g global one, in our model.

Appendix A

Lemma for asymptotic behavior of recurrence

A.1 Lemma for limit of a sequence defined through recursive equation

Following Lemma is adapted from [32] to the more general cases considered in this work. It is used with $\varepsilon_n = ar_n^l$ or $\varepsilon_n = ar_n^g$ and $\delta_n = (r_n^l)^2$ or $\delta_n = (r_n^g)^2$. It is used in subsection 3.4.2.

Lemma A.1. *Let (x_n) be a sequence of positive numbers that satisfies the following equation:*

$$x_{n+1} = (1 - \varepsilon_n)x_n + K_n\delta_n \tag{A.1}$$

where $a > 0$, $\varepsilon_n \geq 0$ and $0 \leq K_n \leq K$. Assume that $(\varepsilon_n)_n$ and $(\delta_n)_n$ are positive sequences of reals

$$\sum_n \varepsilon_n = +\infty, \quad \sum_n \varepsilon_n^2 < +\infty, \quad \text{and} \quad \sum_n \delta_n < +\infty. \tag{A.2}$$

Then

$$\lim_{n\to+\infty} x_n = 0.$$

Proof. The case $K = 0$ is well-known and so we will prove the statement with $K > 0$. Let m_0 be such that $\varepsilon_n < 1$ for all $n \geq m_0$. Then for $n \geq m_0$ we have $x_n \leq y_n$, where

$$\begin{cases} y_{n+1} &= (1 - \varepsilon_n)y_n + K\delta_n \\ y_l &= x_l \end{cases} \tag{A.3}$$

It holds

$$y_n = y_l \prod_{i=l}^{n-1}(1 - \varepsilon_i) + K\sum_{i=l}^{n-1}\delta_i \prod_{j=i+1}^{n-1}(1 - \varepsilon_j). \tag{A.4}$$

Using the fact that $\sum_n \varepsilon_n = +\infty$, and $\sum_n \varepsilon_n^2 < +\infty$ it follows that

$$\prod_{i=l}^{n-1}(1-\varepsilon_i) \longrightarrow 0.$$

Moreover, for every $m \geq m_0$,

$$\begin{aligned}\sum_{i=l}^{n-1}\delta_i \prod_{j=i+1}^{n-1}(1-\varepsilon_j) &= \sum_{i=l}^{m-1}\delta_i \prod_{j=i+1}^{n-1}(1-\varepsilon_j) + \sum_{i=m}^{n-1}\delta_i \prod_{j=i+1}^{n-1}(1-\varepsilon_j) \\ &\leq \prod_{j=m}^{n-1}(1-\varepsilon_j)\sum_{i=l}^{m-1}\delta_i + \sum_{i=m}^{+\infty}\delta_i.\end{aligned} \tag{A.5}$$

Using the fact that $\prod_{j=m}^{n-1}(1-\varepsilon_j) \longrightarrow 0$ and that $\sum_n \delta_n < +\infty$, letting first $n \to +\infty$ and then $m \to +\infty$ in (A.5) the conclusion follows.

A.2 Lemma for asymptotic behavior of recurrence

We now present an extended version of the previous result. Following Lemma is adapted from [32]. In order to go from L^2 results to a.s., we need to know the order of the speed of convergence. This is done using following Lemma. This is in agreement with [5, Lemma A.1], [3, Lemma A.1] as Lemma A.3 is stating.

Lemma A.2. *Let $(z_n)_n$ be a sequence of positive reals satisfying the following equation:*

$$z_{n+1} = (1 - A\varepsilon_n)\, z_n + K_n\delta_n, \tag{A.1}$$

where $A > 0$ and $\forall n \in \mathbb{N}$, $0 < K_n \leq K$. Assume that $(\varepsilon_n)_n$ and $(\delta_n)_n$ are positive sequences of reals

$$\sum_n \varepsilon_n^2 < +\infty \quad \textit{and} \quad \sum_n \delta_n < +\infty. \tag{A.2}$$

Then it holds,

$$\lim_{n\to+\infty} z_n = 0 \Leftrightarrow \sum_n \varepsilon_n = +\infty.$$

In particular, assume $\liminf_n K_n > 0$ and

$$\varepsilon_n = \frac{c_1}{n^{\kappa_1}} + O(\frac{1}{n^{2\kappa_1}}), \tag{A.3}$$

$$\lim_n n^{\kappa_2}\delta_n = c_2 > 0 \tag{A.4}$$

where $\frac{1}{2} < \kappa_1 \leq 1 < \kappa_2$ then,

$$x_n = \begin{cases} O(\frac{1}{n^{\kappa_2-\kappa_1}}) & if \quad \kappa_1 < 1, \\ O(\frac{\log n}{n^A}) & if \quad \kappa_1 = 1 \quad and \quad \kappa_2 - A = 1, \\ O(\frac{1}{n^{\kappa_2-1}}) & if \quad \kappa_1 = 1 \quad and \quad \kappa_2 - A < 1 \\ O(\frac{1}{n^A}) & if \quad \kappa_1 = 1 \quad and \quad \kappa_2 - A > 1. \end{cases} \tag{A.5}$$

Proof.
The case $K = 0$ is well-known and we will prove the statement $K > 0$. Let l be such that $A\varepsilon_n < 1$ for all $n \geq l$. Then for $n \geq l$ we have $z_n \leq y_n$, where

$$\begin{cases} y_{n+1} = (1 - A\varepsilon_n)\, y_n + K\delta_n, \\ y_l = z_l. \end{cases}$$

Set $\varepsilon_n^{'} = A\varepsilon_n$ and $\delta_n^{'} = K\delta_n$. It holds

$$y_n = y_l \prod_{h=l}^{n-1}(1 - \varepsilon_h^{'}) + \sum_{h=l}^{n-1} \delta_h^{'} \prod_{k=h+1}^{n-1}(1 - \varepsilon_k^{'}).$$

Since $\sum_n \varepsilon_n = +\infty$, then $\lim_{n\to\infty} \prod_{h=l}^{n-1}(1 - \varepsilon_h^{'}) = 0$. Moreover, for every $m \geq l$,

$$\begin{aligned} \sum_{h=l}^{n-1} \delta_h^{'} \prod_{k=h+1}^{n-1}(1 - \varepsilon_k^{'}) &= \sum_{h=l}^{m-1} \delta_h^{'} \prod_{k=h+1}^{n-1}(1 - \varepsilon_k^{'}) + \sum_{h=m}^{n-1} \delta_h^{'} \prod_{k=h+1}^{n-1}(1 - \varepsilon_k) \\ &\leq \prod_{k=m}^{n-1}(1 - \varepsilon_k^{'}) \sum_{h=l}^{m-1} \delta_h^{'} + \sum_{h=m}^{+\infty} \delta_h^{'}. \end{aligned}$$

Using the fact that $\prod_{k=m}^{n-1}(1 - \varepsilon_k^{'}) \longrightarrow 0$ and that $\sum_n \delta_n < +\infty$, letting first $n \longrightarrow +\infty$ and then $m \longrightarrow +\infty$, the conclusion follows. We are left to prove if $\sum_n \varepsilon_n < +\infty$ then $\lim_n z_n \neq 0$. From (A.1) we have

$$z_{n+1} \geq \left(1 - \varepsilon_n^{'}\right) z_n$$

from which it follows

$$z_n \geq z_0 \prod_{k=0}^{n-1} \left(1 - \varepsilon_n^{'}\right).$$

Since by assumption, $\sum_n \varepsilon_n < +\infty$, we obtain $\lim_{n\to\infty} z_n > 0$.

Thus, $\lim_{n\to+\infty} z_n = 0 \Leftrightarrow \sum_n \varepsilon_n = +\infty$ $(\kappa_1 \leq 1)$. Otherwise, if $\sum_n \varepsilon_n < +\infty$ $(\kappa_1 > 1)$, then $\lim_{n\to+\infty} z_n \neq 0$.

In particular,

• When $\kappa_1 < 1$. Let $x_{l,n} := \sum_{h=l}^{n-1} \delta_h^{'} \prod_{k=h+1}^{n-1}(1 - \varepsilon_k^{'})$, thus, assuming l is large enough to replace ε_n and δ_n with their asymptotics, and using the monotonicity of their asymptotics

,

$$\begin{aligned} x_{l,n} &= O\Big(\int_l^n \frac{1}{s^{\kappa_2}} \exp\Big(-\int_s^n \frac{1}{u^{\kappa_1}} du\Big) ds\Big) \\ &= O\Big(\int_l^n \frac{1}{s^{\kappa_2}} \exp -\Big[\frac{1}{(1-\kappa_1)u^{\kappa_1-1}}\Big]_s^n ds\Big) \\ &= O\Big(\int_l^n \frac{1}{s^{\kappa_2}} \exp[\frac{1}{1-\kappa_1}(\frac{1}{s^{\kappa_1-1}} - \frac{1}{n^{\kappa_1-1}})] ds\Big) \\ &= O\Big(e^{-\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}} \int_l^n \frac{1}{s^{\kappa_2}} e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}} ds\Big) \\ &= O\Big(\frac{1}{n^{\kappa_2}} \frac{\int_l^n s^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}} ds}{n^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}}}\Big). \end{aligned}$$

Letting $n \to \infty$, using de L'Hôpital's rule, it holds

$$\begin{aligned} x_{l,n} &= O\Big(\frac{1}{n^{\kappa_2}} \frac{n^{-\kappa_2} e^{\frac{1}{(1-\kappa_1)n^{\kappa_1-1}}}}{[(-\kappa_2)n^{-\kappa_2-1} + n^{-\kappa_2} n^{-\kappa_1}] e^{\frac{1}{(1-\kappa_1)s^{\kappa_1-1}}}}\Big) \\ &= O\Big(\frac{1}{n^{\kappa_2}} \frac{1}{\frac{-\kappa_2}{n} + \frac{1}{n^{\kappa_1}}}\Big) \\ &= O\Big(\frac{1}{n^{\kappa_2-\kappa_1}} \Big(\frac{1}{1 - \frac{\kappa_2 n^{\kappa_1}}{n}}\Big)\Big). \end{aligned}$$

- When $\kappa_1 = 1$, set

$$f_n := \frac{z_n}{\prod_{k=0}^{n-1}(1-\varepsilon_k')}.$$

By (A.1) we obtain,

$$f_{n+1} = f_n + F(n)$$

where $F(n) = \frac{\delta_n'}{\prod_{k=0}^{n}(1-\varepsilon_k')}$. So, observing that $f_0 = z_0 = 0$, we obtain

$$f_n = \sum_{h=0}^{n-1} F(h),$$

or equivalently,

$$z_n = \Big[\prod_{k=0}^{n-1}(1-\varepsilon_k')\Big] \sum_{h=0}^{n-1} F(h).$$

Since $\prod_{k=0}^{n-1}(1-\varepsilon_k^{'}) = O(\frac{1}{n^A})$ and therefore $F(n) = O(n^{A-\kappa_2})$, then

$$\begin{aligned} z_n = \Big[\prod_{k=0}^{n-1}(1-\varepsilon_k^{'})\Big]\sum_{h=0}^{n-1} F(h) &= O\Big(\frac{\sum_{h=0}^{n-1}\frac{1}{h^{\kappa_2-A}}}{n^A}\Big) \\ &= O\Big(\frac{\int_1^n h^{A-\kappa_2}}{n^A}\Big) \\ &\overset{n\to\infty}{=} \begin{cases} O(\frac{\log n}{n^A}) & if \quad \kappa_2 - A = 1, \\ O(\frac{1}{n^{b-1}}) & if \quad \kappa_2 - A < 1, \\ O(\frac{1}{n^A}) & if \quad \kappa_2 - A > 1 \end{cases} \end{aligned}$$

The conclusion follows. □

As mentioned, it agrees with the following Lemma. Please note the following Lemma is proved in more generality as Lemma A.4 in [4].

Lemma A.3. *Let γ be a real in $]\frac{1}{2}, 1]$, and $c > 0$. Let $(r_n)_n$ be a sequence of real numbers such that $0 < r_n < 1$ and $a < \frac{1}{r_{m_0}}$. Let*

$$r_n = \frac{c}{n^\gamma} + O\left(\frac{1}{n^{2\gamma}}\right).$$

$a < \frac{1}{r_{m_0}}$. Let $a > 0$. Denote with $l \geq 2$ an integer such that $\forall m \geq m_0$, $a < \frac{1}{r_{m_0}}$. Let

$$p_{m_0,n} := \prod_{m=m_0}^{n}(1 - ar_m) \text{ and } l_{m_0,n} = p_{m_0,n}^{-1}.$$

It holds

$$p_{m_0,n} = \begin{cases} O\left(\exp\left[-\frac{ca}{1-\gamma}n^{1-\gamma}\right]\right) & if\ \frac{1}{2} < \gamma < 1 \\ O\left(n^{-ca}\right) & if\ \gamma = 1 \end{cases}$$

and

$$l_{m_0,n} = \begin{cases} O\left(\exp\left[\frac{ca}{1-\gamma}n^{1-\gamma}\right]\right) & if\ \frac{1}{2} < \gamma < 1 \\ O\left(n^{ca}\right) & if\ \gamma = 1 \end{cases}$$

Thus, setting

$$F_{k+1,n} := \frac{p_{m_0,n}}{p_{m_0,k}} \qquad for\ m_0 \leq k \leq n\,,$$

one gets

$$F_{k+1,n} = \begin{cases} O\left(\exp\left(\frac{a}{1-\kappa_1}(k^{1-\kappa_1} - n^{1-\kappa_1})\right)\right) & for\ 1/2 < \kappa_1 < 1 \\ O\left(\left(\frac{k}{n}\right)^a\right) & for\ \kappa_1 = 1. \end{cases}$$

A.3 Overview of specific sequences when using the previous Lemma

The previous Lemma is used for Theorem 4.3 to prove asymptotics of variance of Z_n and $Z_n(i)$ and for synchronization in L^2 (and speed of convergence). The specific values are collected here for overview and reader's convenience.

- For $\mathbb{V}\mathrm{ar}(Z_n)$
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 4\lambda_2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_2$;
  - case $\gamma_2 = \gamma_1$: $a = 4(\lambda_2 + \lambda_1)$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- For $\mathbb{V}\mathrm{ar}(Z_n(i))$:
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_2$;
  - case $\gamma_2 = \gamma_1$: $a = (3 + 4\lambda_2^2)$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- for speed of synchronisation
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 = \gamma_1$: $a = 2(2\lambda_1 + 1)$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- for $Z_n(i)$ goes to $\frac{1}{2}$
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_2$;
- for Z_n goes to $\frac{1}{2}$
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 4\lambda_2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_2$;
  - case $\gamma_2 = \gamma_1$: $a = 4(\lambda_1 + \lambda_2)$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;

The previous Lemma is used for Theorem 4.4 for asymptotics of variance of Z_n and $Z_n(i)$ and for synchronization in L^2 (and speed of convergence). The specific values are collected here for overview and reader's convenience.

case $\lambda_1 \neq 0,\ \lambda_2 = 0$

- For $\mathbb{V}\mathrm{ar}(Z_n)$
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_2$;
  - case $\gamma_2 = \gamma_1$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- For synchronization
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_1$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$;

- case $\gamma_2 < \gamma_1$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
- case $\gamma_2 = \gamma_1$: $a = 2(1 + 2\lambda_1)$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.

<u>case $\lambda_1 = 0,\ \lambda_2 \neq 0$</u>

- For $\mathbb{V}\mathrm{ar}(Z_n)$
  - case $\gamma_1 < \gamma_2$: $a = 4\lambda_2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 4\lambda_2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_2$;
  - case $\gamma_2 = \gamma_1$: $a = 4\lambda_2$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- For synchronization
  - case $\gamma_1 < \gamma_2$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 = \gamma_1$: $a = 4\lambda_2$, $\kappa_1 = \gamma_1$, $\kappa_2 = 2\gamma_1$.
- For $\mathbb{V}\mathrm{ar}(Z_n)$
  - case $\gamma_1 < \gamma_2$: $a = 1$, $\kappa_1 = 2\gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 1$, $\kappa_1 = 2\gamma_2$, $\kappa_2 = 2\gamma_2$
  - case $\gamma_2 = \gamma_1$: $a = 1$, $\kappa_1 = 2\gamma$, $\kappa_2 = 2\gamma$
- For synchronization
  - case $\gamma_1 < \gamma_2$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$;
  - case $\gamma_2 < \gamma_1$: $a = 2$, $\kappa_1 = \gamma_2$, $\kappa_2 = 2\gamma_1$
  - case $\gamma_2 = \gamma_1$: $a = 2$, $\kappa_1 = \gamma$, $\kappa_2 = 2\gamma$

Appendix B

Simulations' code

These programs, used for simulations in Chapter 2 and Chapter 4, were implemented using the R statistical programming language.

B.1 Programs of numerical simulations in Chapter 2

B.1.1 One component, non-interacting case/i.i.d. sample

Example 2.15. Generalized Pólya urn case: $\lambda = 0$, $r_n \sim n^{-\gamma}$ with $\gamma = 1$.

```
 ## One component, Hill Lane Sudderth, with generalized rate 1/n^gamma
 ## 10 "urns" independent
N=10; # independent sample
Tf=5000; # time end

gamma=1

#lambda=0.4 # local
lambda=0 # Polya
#lambda= 1 # Friedman 1-x

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi=function(u){(1-2*lambda) * u + lambda}
varphi=Vectorize(varphi)

for (t in 2:(Tf+1))    ## t-1 th time
{
  p1=varphi(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
```

```
timewin=1:(Tf+1)
traj=1:N # index   trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.16. Generalized Friedman urn case: $\lambda = 1$, $r_n \sim n^{-\gamma}$ with $\gamma = 1$.

```
## One component, Hill Lane Sudderth, with generalised rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

#lambda=0.4 # local
#lambda=0 # Polya
lambda= 1 # Friedman 1-x

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi=function(u){(1-2*lambda) * u + lambda}
varphi=Vectorize(varphi)

for (t in 2:(Tf+1))    ## t-1 th time
{
  p1=varphi(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.17. Generalized Friedman urn case: $\lambda = 0.4$, $r_n \sim n^{-\gamma}$ with $\gamma = 1$.

```
## One component, Hill Lane Sudderth, with generalised rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
```

```
Tf=5000; # time end

gamma=1

lambda=0.4 # local
#lambda=0 # Polya
#lambda= 1 # Friedman 1-x

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)


renf=1/(((1:Tf)+3)^gamma)

varphi=function(u){(1-2*lambda) * u + lambda}
varphi=Vectorize(varphi)

for (t in 2:(Tf+1))    ## t-1    time
{
  p1=varphi(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # index  from components
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

B.1.2 One component, others urn functions in example 2.24

Example 2.18. a)

```
## One component, Hill Lane Sudderth, with generalised rate 1/n^gamma
## 10 "urns" independent

N=10;  independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi1=function(u){1*(u>0.5)+0.5*(u==0.5)}
varphi1=Vectorize(varphi1)

for (t in 2:(Tf+1))   ## t-1 th time
{
  p1=varphi1(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.18. b)

```
## One component, Hill Lane Sudderth, with generalized rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi3=function(u){3/4*(u<0.5)+1/4*(u>0.5)+0.5*(u==0.5)}
```

```
varphi3=Vectorize(varphi3)

for (t in 2:(Tf+1))    ## t-1 th time
{
  p1=varphi3(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.18. c)

```
## One component, Hill Lane Sudderth, with generalized rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi4=function(u){3/4*(u>0.5)+1/4*(u<0.5)+0.5*(u==0.5)}
varphi4=Vectorize(varphi4)


for (t in 2:(Tf+1))    ## t-1    time
{
  p1=varphi4(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.18. d)

```
## One component, Hill Lane Sudderth, with generalized rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi6=Vectorize(function(u){0.5*((u>0.25)&(u<0.75))+1*(u>0.75)})
varphi6=Vectorize(varphi6)

for (t in 2:(Tf+1))    ## t-1 th time
{
  p1=varphi6(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.18. e)

```
## One component, Hill Lane Sudderth, with generalized rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi8=Vectorize(function(u){u*((u>0.25)&(u<0.75))+1*(u>0.75)})
varphi8=Vectorize(varphi8)

for (t in 2:(Tf+1))    ## t-1 th time
```

```
{
  p1=varphi(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.18. f)

```
## One component, Hill Lane Sudderth, with generalised rate 1/n^gamma
## 10 "urns" independent

N=10; # independent sample
Tf=5000; # time end

gamma=1

Z=matrix(0,ncol=N,nrow=Tf+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf=1/(((1:Tf)+3)^gamma)

varphi9=Vectorize(function(u){u*((u>0.25)&(u<0.75))+1*(u<0.25)})
varphi9=Vectorize(varphi9)

for (t in 2:(Tf+1))   ## t-1 th time
{
  p1=varphi(Z[t-1,]);
  Z[t,]=(1-renf[t-1])*Z[t-1,]+renf[t-1]*(runif(N,min=0,max=1)<p1)
}

couleur=rainbow(N)
timewin=1:(Tf+1)
traj=1:N # indic trajectory/urn
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.23.

```
## One component
## Two competing reinforcement
```

```
N=10; # independent sample
T=5000; # time end

gamma1=0.8 #
gamma2=0.9 #

Z=matrix(0,ncol=N,nrow=T+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf1=1/(((1:T)+3)^gamma1)
renf2=1/(((1:T)+3)^gamma2)

varphi1=Vectorize(function(u){1*(u>0.5)+0.5*(u==0.5)}) # goes to 0 and 1

varphi2=Vectorize(function(u){1*(u<0.5)+0.5*(u==0.5)}) # goes to 0.5

for (t in 2:(T+1)) # attention: at t are computed Y_t, and then Z_t
  ## t-1   time
{
  p1=varphi1(Z[t-1,]);
  p2=varphi2(Z[t-1,]);

  Z[t,]=(1-renf1[t-1]-renf2[t-1])*Z[t-1,]+renf1[t-1]*(runif(N,min=0,max=1)<
    p1)+renf2[t-1]*(runif(N,min=0,max=1)<p2)
}

couleur=rainbow(N)
timewin=1:(T+1)
traj=1:N # index  from components
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

Example 2.24.

```
## One component
## Two competing reinforcement

N=10; # independent sample
T=5000; # time end

gamma1=0.8 #
gamma2=0.9 #

Z=matrix(0,ncol=N,nrow=T+1); # proportion red/total

Z[1,] <- rep(0.5,N)

renf1=1/(((1:T)+3)^gamma1)
renf2=1/(((1:T)+3)^gamma2)
```

```
varphi3=Vectorize(function(u){3/4*(u<0.5)+1/4*(u>0.5)+0.5*(u==0.5)})

varphi4=Vectorize(function(u){3/4*(u>0.5)+1/4*(u<0.5)+0.5*(u==0.5)})

for (t in 2:(T+1)) # attention: at t are computed Y_t, and then Z_t
  ## t-1 th time
{
  p1=varphi3(Z[t-1,]);
  p2=varphi4(Z[t-1,]);

  Z[t,]=(1-renf1[t-1]-renf2[t-1])*Z[t-1,]+renf1[t-1]*(runif(N,min=0,max=1)<
    p1)+renf2[t-1]*(runif(N,min=0,max=1)<p2)
}

couleur=rainbow(N)
timewin=1:(T+1)
traj=1:N # index from components
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values")
abline(h=0.5,col="black",lty=4)
```

B.2 Programs of numerical simulations in Chapter 4

We do consider in the following the interacting stochastic systems introduced and studied in this work.

B.2.1 Case $\lambda_1 = 0$, $\lambda_2 = 0$.

```
## 10 interacting components
## Two reinforcement, lambda1=lambda2=0

N=10;
T=1000;
gamma1=0.7;
gamma2=0.9;
lambda1=0;
lambda2=0;
Z=matrix(0,ncol=N,nrow=T+1);
Zs=matrix(0,ncol=1,nrow=T+1);

Z[1,]=rep(0.5,N)
Zs[1]=mean(Z[1,]);

for  (t in 2:(T+1))
{
  U=runif(N,min=0,max=1);
  V=runif(N,min=0,max=1);

  Rl=(1/t)^gamma1;
  Rg=(1/t)^gamma2;
  Sl=((1-(2*lambda1))*Z[t-1,])+lambda1;
  Sg=((1-(2*lambda2))*Zs[t-1])+lambda2;
  Z[t,]=((1-Rl-Rg)*Z[t-1,])+(Rl*(U<=Sl))+(Rg*(V<=Sg));
  Zs[t]=mean(Z[t,]);
}
couleur=rainbow(N)
timewin=1:(T+1)
traj=1:N # index  from components
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values",main="",cex.lab=1.4,lwd=2)
lines(timewin,Zs[timewin],col="black",lwd=5)
abline(h=Zs[1],col="black",lty=2,lwd=2)
```

B.2.2 Cases $(\lambda_1 > 0,\ \lambda_2 = 0)$, $(\lambda_1 = 0,\ \lambda_2 > 0)$ and $(\lambda_1 > 0,\ \lambda_2 > 0)$

```
## 10 interacting components
## Two reinforcement, lambda1 > 0, lambda2 > 0

N=10;
T=1000;
gamma1=0.7;
gamma2=0.9;
lambda1=0.4;
# lambda1=0;
lambda2=0.6;
# lambda2=0;
Z=matrix(0,ncol=N,nrow=T+1);
Zs=matrix(0,ncol=1,nrow=T+1);

Z[1,]=rep(0.5,N)
Zs[1]=mean(Z[1,]);

for  (t in 2:(T+1))
{
  U=runif(N,min=0,max=1);
  V=runif(N,min=0,max=1);

  Rl=(1/t)^gamma1;
  Rg=(1/t)^gamma2;
  Sl=((1-(2*lambda1))*Z[t-1,])+lambda1;
  Sg=((1-(2*lambda2))*Zs[t-1])+lambda2;
  Z[t,]=((1-Rl-Rg)*Z[t-1,])+(Rl*(U<=Sl))+(Rg*(V<=Sg));
  Zs[t]=mean(Z[t,]);
}
couleur=rainbow(N)
timewin=1:(T+1)
traj=1:N # index  from components
matplot(timewin,Z[timewin,traj],type="l",ylim=c(0,1),col=couleur,lty=1,xlab
    ="Time",ylab="Component's values",main="",cex.lab=1.4,lwd=2)
lines(timewin,Zs[timewin],col="black",lwd=5)
abline(h=Zs[1],col="black",lty=2,lwd=2)
```

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Abstract

The Pólya urn is the paradigmatic example of a reinforced stochastic process. It leads to a random (non degenerated) almost sure (a.s.) time-limit of the proportion. The Friedman urn is a natural generalization whose a.s. time-limit is not random anymore. In this work, in the stream of previous recent works, we introduce a new family of (finite) systems of reinforced stochastic processes, interacting through an additional collective reinforcement of mean field type. The two reinforcement rules strengths (one component-wise, one collective) are tuned through (possibly) different rates. In the case the reinforcement rates are like n^{-1}, these reinforcements are of Pólya or Friedman type as in urn contexts and lead to limits which may be random or not. We state two kind of mathematical results. Different parameter regimes needs to be considered: type of reinforcement rule (Pólya/Friedman), strength of the reinforcement. We study the time-asymptotics and prove that a.s. convergence always holds. Moreover all the components share the same time-limit (synchronization). The nature of the limit (random/deterministic) according to the parameters' regime is considered. We then study fluctuations by proving central limit theorems. Scaling coefficients vary according to the regime considered. This gives insights into different rates of convergence.

Keywords. Reinforced stochastic processes; Interacting random systems; Almost sure convergence; Central limit theorems; Synchronization; Fluctuations

Résumé

L'urne de Polya est l'exemple typique d'un processus stochastique avec renforcement. La limite presque sûre (p.s.) en temps de la proportion existe, est aléatoire et non dégénérée. L'urne de Friedman est une généralisation naturelle dont la limite (proportion asymptotique en temps) n'est plus aléatoire. De nombreux modèles aléatoires sont fondés sur des processus de renforcement comme pour la conception d'essais cliniques au design adaptatif, en économie, ou pour des algorithmes stochastiques à des fins d'optimisation ou d'estimation non paramétrique. Dans ce mémoire, inspirés par de nombreux articles récents, nous introduisons une nouvelle famille de systèmes (finis) de processus de renforcement où l'interaction se traduit par un phénomène de renforcement collectif additif, de type champ moyen. Les deux taux de renforcement (l'un spécifique à chaque composante, l'autre collectif et commun à toutes les composantes) sont possiblement différents. Nous prouvons deux types de résultats mathématiques. Différents régimes de paramètres doivent être considérés : type de la règle (brièvement, Pòlya/Friedman), taux du renforcement. Nous prouvons l'existence d'une limite p.s. commune à toutes les composantes du système (synchronisation). La nature de la limite (aléatoire/déterministe) est étudiée en fonction du régime de paramètres. Nous étudions également les fluctuations en prouvant des théorèmes centraux de la limite. Les changements d'échelle varient en fonction du régime considéré. Différentes vitesses de convergence sont ainsi établies.

Mots-clefs : Processus stochastiques de renforcement ; systèmes aléatoires en interaction ; convergence presque sûre ; théorème central de la limite ; synchronisation ; fluctuations