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Stabilité de la Couche Limite en Magnétohydrodynamique

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Préambule

Résumé

Dans cette thèse nous prouvons la stabilité de certains systèmes d'EDP issus de la magnétohydrodynamique et obtenus grâce à des développements de type couche limite. Nous commençons par discuter les conditions aux bords les plus appropriées pour le système MHD dans un contexte géophysique. Nous obtenons un résultat d'existence de solutions à la Leray pour de telles conditions. Ensuite, nous effectuons une analyse des couches limites sur l'équation adimensionnée, en identifiant les différents modèles asymptotiques selon les régimes d'intérêt. Nous étudions le caractère bien ou mal posé de chacun de ces modèles, dans des cadres linéarisé ou complètement non-linéaire. Nous montrons en particulier le caractère stabilisant d'un champ magnétique tangentiel à la paroi. Finalement, nous illustrons avec des simulations numériques la stabilité ou l'instabilité pour certains des modèles introduits.

Mots-clés

EDP, équation de Navier-Stokes, équation d'Euler, dynamique des fluides, couche limite, système de Prandtl, MHD, stabilité.

Abstract

In this thesis we establish the stability of some PDE systems derived from magnetohydrodynamics and obtained through some boundary layer developments. We begin discussing the most appropriate boundary conditions for the MHD system in a geophysics context. We obtain an existence result of Leray solutions for this complete system. Next, we perform a boundary layer analysis on the nondimensionalized equation, identifying the different asymptotic models according to the most interesting regimes. We study the well- or ill-posedness of each model, either in a linearised case or either in a completely non-linear case. In particular, we show the stabilizing effect provided by a magnetic field tangent to the wall. Finally, we illustrate with numerical simulations the comparison between stability and instability for some of those models.

Keywords

PDE, Navier-Stokes equation, Euler equation, fluid dynamics, boundary layers, Prandtl system, MHD, stability.

Contents

Chapter 1

Introduction

In this thesis we establish either the stability or the instability of meaningful boundary layer models in magnetohydrodynamics (MHD). We use different tools from the analysis of PDE's: compactness arguments, multiscale expansion, Sobolev estimates, as well as numerical simulations. In particular, we try to adapt to the MHD context some techniques and ideas that arose from studies concerning the purely hydrodynamic case, that is the Prandtl system. Most strikingly, we are able to obtain - for some asymptotic regimes in which the Prandtl system is unstable - wellposedness and stability under fair regularity hypothesis. Lastly, to emphasize the difference between stable and unstable behaviours, we perform some numerical simulations.

Starting from D'Alembert paradox in 1752, the study of fluid dynamics provided many reasons to take a closer look near the border that confines the flow. Physical experiments as well as mathematical results suggest that in the thin layer near the boundary lies an important part of crucial dynamics. To explain the mechanism causing instabilities in the flow's evolution, Ludwig Prandtl in 1904 proposed a mathematical model for it.

The idea was to inject in the Navier-Stokes equations a particular form of solution, a *boundary layer development*, where the normal variable is rescaled in order to take into account the concentration of streamlines near the border. More precisely, the solution to plug into the Navier-Stokes equation is split into a first term which is solution to the associated Euler equation (that is, the same equation without the diffusion term) plus a boundary layer corrector that only models the behaviour close to the boundary. Once done, one can derive a new equation for the corrector that arises looking at the leading order terms in the Navier-Stokes equation. This equation, completed with the incompressibility condition and the proper boundary and initial conditions, constitutes the Prandtl system, whose study brought a deeper understanding of viscous flows.

One of the main issues that were explored, was the relation between the instability of the flow and the boundary of fluid domain. This stability/instability problem has benefited from strong mathematical developments in the last decade. The purpose of this work is to rely on this recent progress to gain insight and perspective in the study of the magnetohydrodynamic system (MHD), which couples the Navier-Stokes equation (to model viscous flows) and an approximation of the Maxwell equations (for the electromagnetic field). This system describes the evolution of a conducting fluid under the effect of a magnetic field (such as, for instance, the liquid iron in the Earth's core under the influence of the Earth's magnetic field). More specifically, we would like to investigate its possible boundary layer behaviours, which depend on the magnitude of different quantities involved (like the typical length of a flow, the width of the boundary layer itself or the typical intensity of the magnetic field). In this work we will derive some MHD boundary layer systems through a multiscale expansion, we will set them in the right context (recognizing sometimes classical models like the Hartmann layer or the Shercliff layer) and we will determine conditions to establish their stability or instability for Sobolev initial data.

Before presenting our main results, we first recall some results about the purely hydrodynamic case, namely about the Prandtl system. This will provide us with all the right perspectives and techniques to further venture into the study of the MHD models.

1.1 The Euler and Navier Stokes models

Let $\Omega \subset \mathbb{R}^d$ an open set whose boundary $\partial\Omega$ is regular. The two classical systems that model the flow of an incompressible fluid in Ω are the Euler and the Navier-Stokes equations.

The Euler system reads:

$$\begin{cases} \rho\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p = 0 \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \\ u\cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{Euler}$$

where ρ is the density of the fluid, $\mathbf{u} = \mathbf{u}(t,x)$ is its velocity considered at time $t \in \mathbb{R}^+$ at the point $x \in \Omega$ and p is the pressure (which is a scalar quantity depending on $\mathbf{u}$). We will restrict ourselves to the case of a constant ρ, that models an homogeneous fluid. The vector $\mathbf{n}$ is a unit normal vector to the boundary. The boundary condition expresses the impermeability of the boundary.

On the other hand, the Navier-Stokes equation reads

$$\begin{cases} \rho\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p - \mu\Delta\mathbf{u} = 0 \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases} \tag{Navier-Stokes}$$

It differs from the Euler equations by the diffusion term $\mu\Delta\mathbf{u}$, which models viscous flows, μ

being the viscosity. Besides, at the boundary we have now a no-slip condition, appropriate to the parabolic type of the equation and satisfied experimentally in most situations. We will refer to these systems as (E) and (NS), respectively.

Remark 1.1.1. The same equations are also used in the case $\Omega = \mathbb{R}^d$, once completed with a condition at infinity.

For both systems, one can establish local existence results *in the strong sense* for initial data of Sobolev regularity. In particular, for the Navier-Stokes equation one has :

Theorem 1.1.1. *Let* $s > \frac{d}{2} + 1$, $\rho > 0$, $\mu > 0$, $s \in \mathbb{N}$ *and* $\boldsymbol{u}_0 \in H^s_\sigma(\Omega)$ *being* 0 *near* $\partial\Omega$. *Then there exists a time* $T > 0$ *and a unique solution* $\boldsymbol{u} \in L^\infty(0, T; H^s_\sigma(\Omega)) \cap W^{1,\infty}(0, T; H^{s-2}_\sigma(\Omega))$ *for the system*

$$\begin{cases} \rho\partial_t \boldsymbol{u} + \mathbb{P}(\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \mu\Delta\boldsymbol{u}) = 0 \qquad \text{in } (0,T)\times\Omega \\ \boldsymbol{u}|_{\partial\Omega} = 0 \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases}$$

where $\mathbb{P}$ *denotes the Leray projector and* H^s_σ *refers to the solenoidal vector fields in* H^s.

Remark 1.1.2. When $\mathbf{u}_0$ does not vanish near the boundary, additional compatibility conditions are needed: see [4].

Remark 1.1.3. In dimension 2, the strong solution is known to be global: one can take $T = +\infty$.

1.2 The high Reynolds number limit

Let us now consider the Navier-Stokes equation: we want to rewrite it in a dimensionless form. To do so, let us take the typical time scale T (the characteristic time of observation of the phenomenon) and the spatial scale L (the characteristic size of the flow considered). We deduce the characteristic velocity of the problem $U := \frac{L}{T}$, and we denote by Π the characteristic pressure (to be determined). We introduce then, the new variables which are said to be dimensionless or nondimensional, defined by

$$\mathbf{u} = U\mathbf{u}', \quad p = \Pi p', \quad x = Lx', \quad t = \frac{L}{U}t'.$$

We inject them into the Navier-Stokes equation to find:

$$\frac{U^2\rho}{L}\partial_t\mathbf{u}' + \frac{U^2\rho}{L}(\mathbf{u}'\cdot\nabla)\mathbf{u}' + \frac{\pi}{L}\nabla p' - \frac{\mu U}{L^2}\Delta\mathbf{u}' = 0.$$

Dropping the primes and multiplying everything by $\frac{L}{U^2\rho}$ we get

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\pi}{U^2\rho}\nabla p - \frac{\mu}{LU\rho}\Delta\mathbf{u} = 0.$$

We define the constants $\nu = \frac{\mu}{LU\rho}$, $\quad \pi := U^2\rho$, to bring it in the final form:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = 0 \quad \text{ in } \Omega \\ \text{div } \mathbf{u} = 0 \text{ in } \Omega \\ \mathbf{u}|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

The constant ν is called the *inverse Reynolds number*, Re $= \frac{1}{\nu}$ being the *Reynolds number*. Let us focus, then, on some phenomena having place with a very high typical velocity (like a flying plane). In this case, the parameter ν goes to 0, and since the diffusion term depends on it, it becomes more and more negligible. At this moment, knowing that for a regular enough initial datum one has existence and uniqueness of a regular solution, one could be tempted to expect that the solution to (NS) tends toward the solution to (E) with the same initial datum, at least for a small time (independent from ν). Unfortunately, the situation is far less intuitive, since the difference in the boundary conditions of (E) and (NS) affects the solutions' behaviour.
Indeed, in (NS) the boundary condition is

$$\mathbf{u}|_{\partial\Omega} = 0,$$

whereas for (E) one has just

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ where } \mathbf{n} \text{ is the normal vector to } \partial\Omega,$$

and this variation is crucial, because since the boundary conditions are independent from the parameter ν, they are not influenced by the the fact that the two systems, in a way, "tend" toward each other.

Still, apart from this border conditions, the Navier-Stokes equation formally tends to the Euler equation. In the limit $\nu \to 0$, one could guess that the solutions to (NS) and (E) should be similar - at least far from the boundary - except for a certain region close to the boundary of Ω, which we will call *boundary layer*. In fact, one can prove that when considered in the whole space $\mathbb{R}^d$ ($d = 2, 3$), the Navier-Stokes solution tends to the Euler solution as Re $\to \infty$ (see [8], for instance). On the other hand, in the case of an open set Ω, the following result by Kato in [31] establishes a necessary and sufficient condition for the convergence to occur:

Theorem 1.2.1 (Kato, 1984). *Let $\Omega := \{|\boldsymbol{x}| < 1\} \subset \mathbb{R}^2$, $\omega_0 \in \mathcal{C}_c^\infty(\Omega)$ and $\boldsymbol{u}_0 = K[\omega_0]$, where K is the Biot-Savart operator on Ω. Let $\boldsymbol{u}^\nu$ be the unique classical solution of the Navier Stokes equation in Ω with the no-slip condition and initial velocity $\boldsymbol{u}_0$. Let $\boldsymbol{u}$ be* *the unique smooth solution of the Euler equations with* $\boldsymbol{u} \cdot \boldsymbol{x}|_{\{|\boldsymbol{x}|=1\}} = 0$ *and same initial velocity* $\boldsymbol{u}_0$.

Fix $T > 0$. *Then, one has that* $\boldsymbol{u}^\nu \to \boldsymbol{u}$ *as* $\nu \to 0$ *strongly in* $L^\infty(]0,T[;L^2(\Omega))$ *if and only if* $\nu \int_0^T \|\nabla \boldsymbol{u}^\nu(\cdot,t)\|^2_{L^2(\Gamma_{c\nu})} dt \to 0$ *as* $\nu \to 0$, *for some* $c > 0$, *where* $\Gamma_{c\nu} := \{1 - c\nu < |x| < 1\}$.

Remark 1.2.1. This theorem suggests that the typical size of the boundary layer is a strip of width ν near the boundary of Ω. We will see that its size won't always be the same. The next section, in particular, will be devoted to the presentation of an important boundary layer model whose size isn't ν: the Prandtl system.

1.3 The derivation of the Prandtl system

Aiming at a deeper understanding of the boundary layer phenomenon, Ludwig Prandtl proposed in [48] a new way to model it, by introducing an approximation of the fluid's behaviour near the boundary of the domain. The underlying idea was to use a particular asymptotic development in this region. We will provide here a short exposition of this technique, that will be generalized in the following.

Let us consider the upper half-plane $\mathbb{R} \times \mathbb{R}^+$, having boundary $\{y = 0\}$. Using the notation $\mathbf{u}^\nu = (u^\nu, v^\nu)$, one can write down the Navier-Stokes equation component by component:

$$\begin{cases} \partial_t u^\nu + u^\nu \partial_x u^\nu + v^\nu \partial_y u^\nu + \partial_x p^\nu - \nu(\partial_x^2 u^\nu + \partial_y^2 u^\nu) = 0 \\ \partial_t v^\nu + u^\nu \partial_x v^\nu + v^\nu \partial_y v^\nu + \partial_y p^\nu - \nu(\partial_x^2 v^\nu + \partial_y^2 v^\nu) = 0 \\ \partial_x u^\nu + \partial_y v^\nu = 0 \\ u^\nu(t,x,0) = 0, \ v^\nu(t,x,0) = 0 \qquad \text{(boundary conditions for NS)} \end{cases} \tag{1.2}$$

Prandtl's idea is to inject an *ansatz* for the two components of the velocity to express them in a more useful form. Exploiting the fact that the most meaningful part of the flow takes place near the boundary, the *ansatz* provides a re-parametrisation of the vertical variable and of the velocity components.

This asymptotic consists of two different expansions of $\mathbf{u} = (u, v)$ (solution of (NS) for Re tending to $+\infty$), respectively outside and inside the boundary layer:

- outside the boundary layer, no concentration should occur: one should have

$$u(t,\mathbf{x}) \sim u^0(t,\mathbf{x}), \qquad v(t,\mathbf{x}) \sim v^0(t,\mathbf{x});$$

  where (u^0, v^0) is the solution of the Euler equation.

- inside the boundary layer, $\mathbf{u}$ should exhibit strong gradients, transversally to the boundary: more precisely, the asymptotics suggested by Prandtl is

$$u(t,x,y) \sim u^{BL}\left(t,x,\frac{y}{\sqrt{\nu}}\right), \quad v(t,x,y) \sim \sqrt{\nu}v^{BL}\left(t,x,\frac{y}{\sqrt{\nu}}\right)$$

  where $u^{BL} = u^{BL}(t,x,Y)$ and $v^{BL} = v^{BL}(t,x,Y)$ are *boundary layer profiles*, depending on a rescaled variable $Y = \frac{y}{\sqrt{\nu}}$, $Y > 0$.

Therefore, in the Prandtl model, the typical scale of the boundary layer is $\sqrt{\nu}$, as suggested by the heat part of the Navier-Stokes equation. Accordingly, for the divergence-free condition not to degenerate, the vertical amplitude of the velocity is $O(\sqrt{\nu})$. Let us note that this Prandtl size $\sqrt{\nu}$ is very different from the size ν involved in Kato's theorem.

Plugging the expression above in (1.2) and keeping the leading order terms we derive the Prandtl system (denoting y instead of Y):

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = -\frac{\partial p^0}{\partial x}(t,x,0) = (\partial_t u^0 + u^0\partial_x u^0)|_{y=0} \\ \partial_x u + \partial_y v = 0 \\ u|_{y=0} = v|_{y=0} = 0, \\ \lim_{y\to+\infty} u(t,x,y) = u^0(t,x,0) \ \lim_{y\to+\infty} p(t,x,y) = p^0(t,x,0) \end{cases} \tag{Prandtl equations}$$

where $u^0(t,x,0)$ and $p^0(t,x,0)$ are the Euler tangential velocity and pressure at the boundary. The system is completed by both the no-slip condition

$$u|_{y=0} = v|_{y=0} = 0,$$

and by the connection to the Euler flow for $y \to +\infty$ (thus away from the border) which gives

$$\lim_{y\to+\infty} u(t,x,y) = u^0(t,x,0), \qquad \lim_{y\to+\infty} p(t,x,Y) = p^0(t,x,0).$$

Remark 1.3.1. One of the consequences of this formal work is that we eliminated from the system the evolution equation on the vertical component; it is now recovered through the divergence-free condition.

Remark 1.3.2. Before proceeding with our exposition, we would like to underline the fact that the previous construction can be generalized in a very natural way to the case of an open set with a regular boundary $\partial\Omega$ using the Frenet frame, i.e. by decomposing $\mathbf{x} = \tilde{x}(x) + y\mathbf{n}(x)$ (where $\tilde{x} \in \partial\Omega$, x is the new parametrisation of $\partial\Omega$ and $\mathbf{n}$ is the normal vector to $\partial\Omega$ in the point $\tilde{x}$).

1.4 The regularity of the Prandtl solution

One of the most important issues concerning the Prandtl system is its *stability*. Determining which order of regularity on the initial datum is enough to obtain that the system is well-posed or ill-posed is the objective of many recent works. Besides, the justification of the initial ansatz remains a central issue in contemporary research.

Let us focus on the local well-posedness in time.

Without providing all the details (we refer to [16] for a larger introduction about this system), we can briefly synthesize the main results on this subject in this list:

- **[47], [58]**: for $(x, y) \in]0, L[\times\mathbb{R}^+$, for an initial datum that is monotone in the vertical variable y (that is, $\partial_y u > 0$), the Prandtl system is locally well-posed (see [47]). More precisely, thanks to the monotonicity hypothesis, using Crocco's transform Oleinik and Samokhin were able to prove the local well-posedness of both the boundary value and initial value problems. Moreover, Xin and Zhang in [58] proved that the latter problem has a global weak solution when the source is null and $\partial_x P \leqslant 0$ (that is, a favourable pressure gradient, which is physically known to be stabilizing).

- **[7], [5], [6]**: for an initial datum which is analytic in $x \in \mathbb{R}$ and an analytic Euler flow as well, the initial value problem for the Prandtl equation is well-posed (see [7], [5], [6]). This strong regularity hypothesis is natural to balance the loss of regularity due to the term $v\partial_y u$, that leads to an instability (the next paragraph will concern this mechanism).

- **[40], [19]**: for an initial datum of class Gevrey 2 in the horizontal variable x, the Prandtl system is locally well-posed (see [40]). This recent result improves the previous one, demanding less regularity to obtain the well-posedness. The authors use energy estimates that are clever generalizations of ideas exposed in [19] (see section 1.5 for more details on this method).

We want, now, to recall some results about the case of Sobolev initial data. What can be said in this setting? One can show that without the monotonicity hypothesis on the initial data, the system is *ill-posed*, because of a strong instability mechanism at high frequencies. This result and the dynamic it reveals will be important later, during the analysis of the MHD system. We will determine whether this type of instability still occurs in the boundary layer systems we will derive from the MHD equations.

The reader can find all the details in [18], we will only provide here the fundamental notions.

For simplicity, the hypothesis is to have $(x, Y) \in \mathbb{T} \times \mathbb{R}^+$ and $u^0(t, x, 0) = 0$. we will write

y instead of Y. The Prandtl system becomes, then:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u + \partial_y v = 0, & \text{dans } \mathbb{T} \times \mathbb{R}^+ \\ (u,v)|_{y=0} = (0,0),\ \lim_{y\to+\infty} u = 0. \end{cases} \tag{1.3}$$

In order to achieve a deeper comprehension of the latter system, one can *linearise* it around a certain solution (which of course will be chosen to simplify the analysis). A natural choice is a *shear layer flow*: it is a flow whose main characteristic is to be parallel to the boundary (here $\{y = 0\}$) and independent from the horizontal variable x.
To build it, one considers $u_s(t, y)$: a regular solution to the heat equation with initial data U_s, that is

$$\begin{cases} \partial_t u_s - \partial_y^2 u_s = 0 \\ u_s|_{y=0} = 0,\ u_s|_{t=0} = U_s. \end{cases} \tag{1.4}$$

In this way, the shear layer $(u_s, v_s) = (u_s(t, y), 0)$ trivially satisfies (1.3), and as a solution of the latter one can linearise the equation around it, which gives the simplified system

$$\begin{cases} \partial_t u + u_s\partial_x u + v\partial_y u_s - \partial_y^2 u = 0, & \text{dans } \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u + \partial_y v = 0, & \text{in } \mathbb{T} \times \mathbb{R}^+ \\ (u,v)|_{y=0} = (0,0),\ \lim_{y\to+\infty} u = 0. \end{cases} \tag{1.5}$$

Now that the problem has been rewritten in a simpler form, one can turn to the stability properties of (1.5).

To provide a framework where this system is well-posed, we introduce the following notations and functional spaces:

Definition 1.4.1. One defines

- $W^{s,\infty}_\alpha(\mathbb{R}^+) := \{f(y) \text{ such that } e^{\alpha y} f \in W^{s,\infty}(\mathbb{R}^+)\}\ \forall \alpha, s > 0$, with $\|f\|_{W^{s,\infty}_\alpha} := \|e^{\alpha y} f\|_{W^{s,\infty}}$;
- $E_{\alpha,\beta} := \left\{u(x, y) = \sum_{k\in\mathbb{Z}} \hat{u}^k(y)e^{ikx},\ \|\hat{u}^k\|_{W^{0,\infty}_\alpha} \leqslant C_{\alpha,\beta}e^{-\beta|k|},\ \forall\ k\right\}$, $\forall\ \alpha, \beta > 0$ with $\|u\|_{E_{\alpha,\beta}} := \sup_{k\in\mathbb{Z}} e^{\beta|k|}\|\hat{u}^k\|_{W^{0,\infty}_\alpha}$.

Note that the functions of $E_{\alpha,\beta}$ have analytic regularity in x. They have only L_∞ regularity in y, with an exponential weight. More regularity in y could be considered as well. Let $\alpha,\ \beta > 0$: then one has (cfr [18]):

Proposition 1.4.1 (Well-posedness in the analytic setting). *Let $u_s \in C^0(\mathbb{R}_+; W^{1,\infty}_\alpha(R_+))$. There exists $\rho > 0$ such that: for all T with $\beta-\rho T > 0$, and all $\boldsymbol{u}_0 \in E_{\alpha,\beta}$ the linear equation 1.5 has a unique solution $\boldsymbol{u} \in C([0, T); E_{\alpha,\beta-\rho T})$, $\quad \boldsymbol{u}(t, \cdot) \in E_{\alpha,\beta-\rho t}$, $\quad \boldsymbol{u}|_t = 0 = \boldsymbol{u}_0$.*

In short, the Cauchy problem for (1.5) is locally well-posed in the analytic setting.

Definition 1.4.2. We define, then $T(t,s)u_0 := u(t,\cdot)$, where u is solution of (1.5) with $u|_{t=s} = u_0$. As the spaces $E_{\alpha,\beta}$ are dense in the spaces

$$H^m := H^m(\mathbb{T}_x, W^{0,\infty}_\alpha(\mathbb{R}^+_y)),\ m \geqslant 0 \text{ with } \|f\|^2_{H^m} := \sum_{k\in\mathbb{Z}} |k|^{2m} \|\hat{f}_k(y)\|^2_{W^{0,\infty}_\alpha},$$

we introduce for all $T \in \mathcal{L}(E_{\alpha,\beta}, E_{\alpha,\beta'})$

$$\|T\|_{\mathcal{L}(H^{m_1},H^{m_2})} := \sup_{u_0\in E_{\alpha,\beta}} \frac{\|Tu_0\|_{H^{m_2}}}{\|u_0\|_{H^{m_1}}} \in [0,+\infty].$$

The main result about Sobolev ill-posedness of (1.5) follows:

Theorem 1.4.1 (Gérard-Varet, Dormy; 2010)**.** *Let u_s belonging to $\mathcal{C}^0(\mathbb{R}^+; W^{4,\infty}_\alpha(\mathbb{R}^+)) \cap \mathcal{C}^1(\mathbb{R}^+; W^{2,\infty}_\alpha(\mathbb{R}^+))$. Assume that the initial velocity has a non-degenerate critical point over $\mathbb{R}^+$. Then there exists $\sigma > 0$, such that for all $\delta > 0$,*

$$\sup_{0\leqslant s\leqslant t\leqslant\delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t,s)\|_{\mathcal{L}(H^m,H^{m-\mu})} = +\infty,\ \forall\ m \geqslant 0,\ \mu \in [0,1/2).$$

Moreover, one can find solutions u_s of the heat equation and a $\sigma > 0$ such that for all $\delta > 0$ one has:

$$\sup_{0\leqslant s\leqslant t\leqslant\delta} \|e^{-\sigma(t-s)\sqrt{|\partial_x|}} T(t,s)\|_{\mathcal{L}(H^{m_1},H^{m_2})} = +\infty,\ \forall\ m_1, m_2 \geqslant 0.$$

This theorem expresses ill-posedness in the Sobolev setting. The main idea behind the proof is that if the base profile u_s has a non-degenerate critical point, one can find a solution of (1.5) whose k-th Fourier mode explodes like $e^{\sigma_0\sqrt{|k|}t}$.
We give here the main hints of its complete proof.

1. System (1.5) has constant coefficients in t and y. Since we are expecting high frequency instabilities, it is fair (and can be established a posteriori) to replace $u_s(t,y)$ with the initial data $u_s(0,y) = U_s(y)$, in order to have constant coefficients only in y. Then, one can restrict to solutions with frequency k, $k \gg 1$ in x, of the type:

$$\left(u(t,y) = e^{ik(\omega(k)t+x)}\hat{u}^k(y), v(t,y) = ke^{ik(\omega(k)t+x)}\hat{v}^k(y)\right).$$

   After some manipulations (among which a development of the linear operator around the critical point of U_s), we end up with an eigenvalue problem for an ordinary differential equation.

2. One looks for approximate eigensolutions of this system. Thanks to a proper rewriting and to the accurate choice of the functional space, one ends up with the following

result: there exists an eigenvalue of the equivalent system such that its imaginary part is negative. This implies that in the previous development we have an almost instantaneous growth, like $e^{\sigma_0 \sqrt{|k|}t}$.

3. Thanks to the previous steps, it is easy to construct a solution of (1.5) which causes the explosion. The idea is to focus again on the k-th Fourier mode and to propose a solution (made up by a regular part and a shear layer part) such that it allows estimates involving the exponential growth, leading to the ill-posedness.

Remark 1.4.1. This result clearly emphasizes that even in a case of strong regularity of the initial data, we can't expect this property to be preserved.

1.5 Well-posedness in a weighted Sobolev space for a monotone initial profile

The previous section was totally devoted to illustrate a strong instability arising in the Prandtl system; this one, instead, will concern a result which is similar to that of Oleinik: a wellposedness result for a monotone initial data. Although the hypothesis and the results are similar, the interest of this proof lies in its techniques, which differ from those of Oleinik (who used Crocco's transformation). These methods, developed by Masmoudi and Wong in [44], will be our starting line to prove some well-posedness results in the case of magnetohydrodynamics.
Their approach is based on the elimination of the problematic term in the Prandtl equation. Since, as we have recalled in the previous paragraph, this term causes an explosion in high-frequencies thus leading to instability, they neutralize it by performing a clever change of variable. This, at the price of exploiting the monotonicity of the initial data.
Their trick arises from a simple observation. Let us consider the linearized Prandtl equation 1.5 and its derivative along the vertical direction

$$\partial_t u + u_s \partial_x u + v \partial_y u_s - \partial_y^2 u = 0, \tag{1.6}$$

$$\partial_t [\partial_y u] + u_s \partial_x [\partial_y u] + [\partial_y u_s] \partial_x u + v \partial_y [\partial_y u_s] + \partial_y v [\partial_y u_s] - \partial_y^3 u = 0; \tag{1.7}$$

we remark, then, that exploiting the incompressibility condition, one can eliminate two terms in the second equation and that calling $\partial_y u =: \omega$ one can rewrite the system as

$$\partial_t u + u_s \partial_x u + v \partial_y u_s - \partial_y^2 u = 0, \tag{1.8}$$

$$\partial_t \omega + u_s \partial_x \omega + v \partial_y^2 u_s - \partial_y^2 \omega = 0. \tag{1.9}$$

At this point, performing a standard energy estimate on the Prandtl equation would lead to a huge issue: that of bounding the term $v \partial_y u_s$. Indeed, if we rewrite it using the incompressibility condition we have $-\partial_y^{-1}\partial_x u\ \partial_y u_s$, and this term has one extra x derivative, which is not skew-symmetric and does not disappear in the estimate. Now, Masmoudi and Wong's idea consists of combining two ingredients to bypass this problem. The first remark is that the equations in system (1.5) are term by term very similar. The second one is that we can exploit the monotonicity hypothesis on u_s that assures that $\partial_y u_s > 0$ so that multiplying the first equation by $\frac{\partial_y^2 u_s}{\partial_y u_s}$ and afterwards subtracting it from the second we obtain:

$$\partial_t\left(\omega - u\frac{\partial_y^2 u_s}{\partial_y u_s}\right) + u_s\partial_x\left(\omega - u\frac{\partial_y^2 u_s}{\partial_y u_s}\right) - \partial_y^2\omega + \partial_y^2 u\frac{\partial_y^2 u_s}{\partial_y u_s} + u\partial_t\left(\frac{\partial_y^2 u_s}{\partial_y u_s}\right) = 0 \qquad (1.10)$$

which allows to eliminate the term responsible for instability. We will therefore call $g := \omega - u\frac{\partial_y^2 u_s}{\partial_y u_s}$ and then work with the simpler equation

$$\partial_t g + u_s\partial_x g - \partial_y^2 g = \left[\partial_y^2, \frac{\partial_y^2 u_s}{\partial_y u_s}\right](u) - u\partial_t\left(\frac{\partial_y^2 u_s}{\partial_y u_s}\right), \qquad (1.11)$$

on which it becomes easy to get the wished energy estimate.

For the sake of brevity and clarity we have been working with the linearised equation, but one can obtain an energy estimate in weighted in y Sobolev spaces for the Prandtl equation itself. More precisely, the result establishes the following:

Theorem 1.5.1 (Existence and local uniqueness for the Prandtl equation in a weighted Sobolev space)**.** *Let $s \geqslant 5$ an even integer and $\gamma \geqslant 1$ the order of the weight in y. We suppose that the flow in the upper part U (far from the boundary) of the boundary layer satisfies*

$$\sup_t \sum_{l=0}^{\left[\frac{s+9}{2}\right]} \|\partial_t^l U\|_{W^{s-2l+9,\infty}(\mathbb{T})} < +\infty. \qquad (1.12)$$

We also suppose that $u_0 - U$ and ω_0 (the initial data and its initial derivative, respectively) belong in two weighted Sobolev spaces with s adapted derivatives. There exists, then, a time $T > 0$ and a unique classical solution (u, v) to the Prandtl equation such that $u - U \in L^\infty([0,T]; H^{s,\gamma}) \cap C([0,T]; H^s - w)$ and that the curl $\omega := \partial_y u \in L^\infty([0,T]; H^{s,\gamma}_{\sigma,\delta}) \cap C([0,T]; H^s - w)$, where the space $H^{s,\gamma}_{\sigma,\delta}$ is an adapted weighted Sobolev space and by $H^s - w$ we denoted the space H^s considered with its weak topology.

Once again, for a more precise statement and the proof, the reader can find all in [44].

1.6 Conclusion

The Prandtl system has shown through time that it models in a precise way the flow near the boundary, but when studied mathematically, it proves to be ill-posed in most cases. In fact, as soon as its initial datum of Sobolev regularity is *not* monotone (that is, it admits at least one point with zero derivative) the solution is unstable. On the other hand, provided with the monotonicity of the initial data, a Sobolev-type regularity is enough to assure that the system is well-posed. The monotonicity is a key hypothesis: without it, one can't control a loss of horizontal derivative in the energy estimates, which can't be otherwise balanced.

According to this perspective, one could wonder whether the intrinsic instability of the Prandtl equation is to be found in *magnetohydrodynamics* (MHD) as well. This discipline studies the flow of fluids that are subjects to the effects of the electromagnetic field, such as the Earth's iron core or plasma in astrophysics. More specifically, we are interested in the dynamics of an electrically conducting liquid near a wall. This problem can be studied applying some of the boundary layer techniques we previously presented to a different system of partial differential equations. This topic, whose applications range among many domains of active research, such as dynamo theory [13] or nuclear fusion [57], has been of constant interest. The MHD system couples a Maxwell like equation on the evolution of the magnetic field with the Navier-Stokes equation, which gains a source term taking into account the Lorentz force. This system can be studied performing the same boundary layer development we presented in this introduction, and it is therefore natural to investigate the same issues, particularly its stability. Two results should now be mentioned, one suggesting a stabilization of the flow in the MHD case, and one against it.

In [14], the authors prove the Sobolev validity of boundary layer expansions for an MHD system that models a rotating fluid (so that it takes into account the Coriolis force) with an almost constant magnetic field, is well-posed. They consider a fluid contained in the space between two planes of $\mathbb{R}^3$ with Dirichlet boundary conditions on the velocity and the continuity of $(E \times b) \cdot n$ through the boundary (E being the electric field and b the magnetic field). Their proof relies on a boundary layer development, where the role played by the hypothesis of the constant magnetic field (at the first order) is crucial.

On the other hand, in [45], the author considers a two-dimensional domain and studies a shear flow solution of the MHD system with Dirichlet boundary conditions on the velocity and Neumann conditions on the magnetic field. The article concludes that the shear layer profile of the velocity can develop a non degenerate critical point in a finite time. This hypothesis, as we have seen, constitutes the key factor in the proof of the ill-posedness of the Prandtl equation (in [18]). This result would suggest that even for an MHD flow with a monotone initial profile for the velocity, nothing could prevent the formation of a critical point, thus preparing the ground for the very same explosion in high-frequencies described in [18].

1.7 Presentation of the results

Keeping these two works in mind, we present in this paragraph the results we established in order to gain insight on the issue of the MHD boundary layer stability.

In the second chapter we discuss all the preliminary hypothesis and some results about the MHD system. We begin by deriving the system itself from the Maxwell equations and the Navier-Stokes equation. We consider the case of a fluid contained in a domain Ω surrounded by an insulator (a situation physically relevant), in contrast with all previous studies that considered a perfect conductor and thus neglected the magnetic field outside Ω. As a consequence, we derive jump conditions for the magnetic field at the boundary - assuring it to be continuous - and for a no-slip condition on the velocity.
As in most of the literature, we neglect the displacement current, and perform a non-dimensionalisation.
Once derived the MHD system, we introduce a variational formulation for the equation on the evolution of the magnetic field in order to prove a weak existence (in the Leray sense) result for the whole system. The existence result is proved adapting the classical proof of Leray to the MHD system, thus with a Galerkin approximation and then compactness arguments.
Let us stress that the jump conditions and equations on the magnetic field introduce non-standard difficulties in both finding the appropriate variational formulation and solving it.

The third chapter consists of the article [21]. It is divided into two parts.
First part:
It is devoted to a formal *boundary layer analysis for the MHD system.* We work in $\Omega = \mathbb{R}^3_+$ (whose boundary is thus the plan $\mathbb{R}^2$) and we consider the following system

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} = S\mathbf{b}\cdot\nabla\mathbf{b}, \\ \partial_t \mathbf{b} - \mathrm{curl}\,(\mathbf{u}\times b) + \dfrac{1}{\mathrm{Rm}}\mathrm{curl}\,\mathrm{curl}\,\mathbf{b} = 0, \\ \mathrm{div}\,\mathbf{u} = 0, \mathrm{div}\,\mathbf{b} = 0, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{b}|_{\partial\Omega} = \mathbf{e}. \end{cases} \tag{1.13}$$

The parameters Re and Rm are the hydrodynamic and magnetic Reynolds numbers respectively; S is the so-called coupling parameter, given by $S = \frac{\mathrm{Ha}^2}{\mathrm{Re}\,\mathrm{Rm}}$, where Ha is the Hartmann number. Obviously, for $S = 0$ we find again the Navier-Stokes equation. The magnetic field at the boundary equals $\mathbf{e}$, which is a uniform background magnetic field.

From this set of equations, investigate the different possible boundary layer expansions. We plug in the system the following approximations:

$$\begin{aligned} \mathbf{u} &\approx \Big(u'_x\left(t,x,y,\lambda^{-1}z\right),\, u'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, u'_z\left(t,x,y,\lambda^{-1}z\right)\Big), \\ \mathbf{b} &\approx \mathbf{e} + \delta\Big(b'_x\left(t,x,y,\lambda^{-1}z\right),\, b'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, b'_z\left(t,x,y,\lambda^{-1}z\right)\Big), \end{aligned} \tag{1.14}$$

choosing for $\mathbf{b}$ either a tangent magnetic field (that is $\mathbf{e} = e_x$) or transverse magnetic field (that is $\mathbf{e} = e_z$). The value of $\lambda \ll 1$ is the boundary layer size whereas $\delta = O(1)$ is the typical norm of the magnetic perturbation. Retaining only the non-trivial cases, we find the following regimes, that are afterwards analysed:

1. **Hartmann regime**: choosing $\lambda = \mathrm{Ha}^{-1}$, $\mathrm{Ha}^2 \gg \mathrm{Re}$ we find the classical Hartmann layer.

2. **Mixed Prandtl/Hartmann regime**: choosing $\lambda = \mathrm{Ha}^{-1}$, $\mathrm{Ha}^2 \sim \mathrm{Re}$, the evolution equations decouple, and we recognize a damped Prandtl equation.

3. **Shercliff regime**: choosing $\lambda = \mathrm{Ha}^{-1/2}$, $\mathrm{Ha} \gg \mathrm{Re}$ we find the equations describing the Shercliff layer.

4. **Mixed Prandtl/Shercliff regime**: choosing $\lambda = \mathrm{Ha}^{-1/2}$, $\mathrm{Ha} \sim \mathrm{Re} \gg \mathrm{Rm}$ we find a new system, that has features from both the Prandtl and the Shercliff system.

5. **Fully non-linear MHD regime**: In this case, we take $\delta = 1$ and $\mathbf{e} = 0$ (since the perturbation to the constant magnetic field e_x is of size one). Choosing $\lambda = \mathrm{Ha}^{-1/2}$, $\mathrm{Ha} \sim \mathrm{Re} \sim \mathrm{Rm}$ we obtain the fully non-linear layer. It is the only one to feature two evolution equations (one on the velocity and one on the magnetic field).

Second part:
The previous derivation being formal, we study if the reduced boundary layer models are well-posed, at least locally in time, so that boundary layer expansions can be built. From the point of view of well-posedness, the interesting systems are the non-linear ones, that mix Prandtl and magnetic features. They correspond to the mixed Prandtl/Hartmann regime (with background transverse magnetic field $\mathbf{e} = \mathbf{e}_z$), the mixed Prandlt/Shercliff and the fully non-linear (with background tangential magnetic field $\mathbf{e} = \mathbf{e}_x$). On those systems, we perform a linear stability analysis, restricting ourselves to the 2d case in the variables (x, z), with $\mathbf{u} = (u, v)$, $\mathbf{b} = (b, c)$.
For the mixed Prandtl/Hartmann regime, that consists of a damped Prandtl equation, we prove that the dynamic is the same as for the original Prandtl system, developing a strong instability as soon as the initial profile is not monotone.
For the two other cases, instead, we can use the fact that the two equations are coupled to obtain a cancellation of the terms leading to the instability. We consider linearizations

around shear flows first, postponing the discussion of the fully nonlinear models to Chapter 4.

The linearised mixed Prandtl-Shercliff system is the following

$$\begin{cases} \partial_t u + U\partial_x u + vU' - \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 u = \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_x b, \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0,\ (u,b) \to 0 \qquad \text{as } z \to +\infty, \end{cases} \tag{1.15}$$

where $U = U(z)$ connects 0 at $z = 0$ to some constant u^∞ at infinity.

In this system, we have an evolution equation on the velocity and a second equation that provides a useful relation between the velocity and the magnetic field. To obtain the linear stability, we perform two energy estimates respectively on the first equation and its vorticity formulation (obtained simply deriving along the vertical variable). Then, we exploit (1.15b) to conveniently rewrite some terms and summing them up we obtain a final estimate that allows to conclude.

Finally, the linearized MHD layer reads

$$\begin{cases} \partial_t u + U\partial_x u + U'\,v - \partial_z^2 u = S\partial_x b, \\ \partial_t \mathbf{b} - \nabla^\perp(v - U\,c) - \dfrac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2 \mathbf{b} = 0, \\ \partial_x u + \partial_z v = \operatorname{div} \mathbf{b} = 0, \\ u|_{z=0} = v|_{z=0}, \quad \mathbf{b}|_{z=0} = 0, \\ u \to 0, \quad b \to 0, \quad \text{as } z \to +\infty, \end{cases} \tag{1.16}$$

where $\mathbf{u} = (u,v)$ and $\mathbf{b} = (b,c)$ are the perturbations of the reference solution $u = U(z)$, $v = 0$, $\mathbf{b} = \mathbf{e}_x$. After having noticed that there is no loss of generality in assuming that b has zero average in x, we can write $\mathbf{b} = \nabla^\perp\phi$, for some function ϕ which is periodic with zero average in x. This allows to express the second equation of the previous system as

$$\partial_t\phi + U\partial_x\phi - v - \frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2\phi = 0. \tag{1.17}$$

This last equation is a key ingredient in the stability analysis of our system: in fact, combining the evolution equation on u and the evolution equation on ϕ, one can get rid of the term in v, responsible for the loss of one derivative in x (which caused the ill-posedness of the Prandtl system). This idea is reminiscent of article [44] about the classical Prandtl equation, that used the monotonicity of the velocity profile to perform the change of variable. However, the novelty in the present MHD context is that no monotonicity of the velocity profile is needed to obtain well-posedness. We rather consider the following modified velocity $\tilde{u} = u + U'\phi$ that allows a better energy estimate, and we easily conclude that the system is well-posed.

In Chapter 4, we deal with the three non-linear models, particularly with the mixed Prandtl/Shercliff layer. All of them behave like the linearised versions studied in the previous chapter. In this chapter we detail these proofs.

- The fully non-linear MHD model has been proved to be well-posed by Liu, Xie and Yang in [41] while the work for this thesis was being completed. We provide here some more commentaries about one particular case where one can explicitly compute its solutions. Indeed, considering the zero-viscosity case (which means that the diffusion term is absent in the Navier-Stokes equation) and linearising the system, it becomes easy to calculate these solutions.

- The mixed Prandtl/Hartmann case is analysed, putting into light how the damped Prandtl equation is similar to the Prandtl equation. The adjustments needed to apply the ill-posedness theorem of [18] and the well-posedness theorem for initial data of class Gevrey $\frac{7}{4}$ of [19] are provided.

- The new case to treat is the mixed Prandtl/Shercliff model, that is

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 u = \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_x b - \partial_x p^\infty, \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0, \\ u \to U, \quad b \to B, \quad \text{as } z \to +\infty. \end{cases} \tag{1.18}$$

  Mathematically, it presents the same difficulty as the fully non-linear MHD layer, namely the unboundedness of the non-linear term $v\partial_z u$ in the evolution equation on the velocity. We were able to prove that this system is well-posed for Sobolev initial data. To do so, we work with an approximation of (1.18) that adds a non-homogeneous diffusion. Then, instead of defining a new variable that involves both the velocity and the magnetic field as for the fully non-linear MHD layer, we exploit here the second equation (that gives $\partial_x u = -\partial_z^2 b$) to obtain a cancellation in the a priori energy estimate. In order to do so, we define a proper weighted Sobolev space and we perform our energy estimate. Analysing term by term we use the previous equation to rewrite the problematic terms and thus obtain their mutual cancellation.

  It is straightforward to show the existence of the solution of the approximate system, and the energy estimate allows to prove its convergence toward the solution we were looking for in the first place.

Directly following the study of the mixed Prandtl/Shercliff case, we will present in the last

Chapter some numerical simulations that underline how Prandtl instability mechanism, differs from the more stable dynamics of the mixed Prandtl/Shercliff layer. As in [18], where the simulations had put into light the instability of the computed solutions of the ordinary differential equation obtained passing in Fourier on the linearised Prandtl system, we perform similar calculations on the mixed Prandtl/Shercliff case, obtaining stable profiles.

Chapter 2

Preliminaries on the MHD system

To prepare the boundary layer analysis of the next chapter, we provide here a careful derivation of the MHD system, in dimensionless form. Special attention is paid to the conditions satisfied by the magnetic field at the boundary of the fluid domain Ω. We focus on the case where the fluid domain is surrounded by an insulator: for instance, this is relevant to the flow of liquid iron in the Earth's core. This is in contrast with most previous studies on the MHD equations ([15], [53], [22], [23]), which treat the case of a perfect conductor, and neglect the magnetic field outside Ω. Here, on the contrary, the magnetic field is defined in the whole space $\mathbb{R}^3$, with jump conditions at $\partial\Omega$. It induces some changes in the construction of solutions. We provide in this chapter a theorem of existence of weak solutions *a la Leray* for this full MHD system (Theorem 2.2.1). Some further remarks on the notion of strong solutions are provided at the end of the chapter.

2.1 Derivation of the dimensionless MHD equations

We first remind here the classical derivation of the dimensional MHD system, following for instance [24]. We will notably discuss the appropriate boundary conditions. We will then put the equations in dimensionless form, to lay the ground for the following boundary layer developments.

The aim of the MHD system is to describe the motion of a viscous incompressible conducting fluid. We denote by Ω the three dimensional bounded domain occupied by the fluid. The governing equations for the fluid velocity u and pressure p are the classical Navier-Stokes equations, with constant density ρ and kinematic viscosity coefficient ν:

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p - \nu\Delta u = j \times b \ \text{ in } \ \Omega, \\ \operatorname{div} u = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0. \end{aligned} \tag{2.1}$$

With regards to the purely hydrodynamic case, the difference lies in the addition of the Lorentz force, at the right-hand side of the momentum equation. It is given by the cross product of the current density j and the magnetic field b. Hence, to complete the description, one needs to couple this Navier-Stokes dynamic to the one for the magnetic and electric fields b and e. Therefore, we write the Maxwell equations, in which we neglect the displacement currents:

$$\begin{aligned} \partial_t b + \text{curl } e = 0 \quad &\text{in } \mathbb{R}^3, \\ \text{curl } b = \mu_0 \, j \quad &\text{in } \mathbb{R}^3, \\ \text{div } b = 0 \quad &\text{in } \mathbb{R}^3. \end{aligned}$$

Note that these equations hold in the whole space $\mathbb{R}^3$. To close the system, one still has to specify the properties of the current density j. In the fluid domain Ω, we express Ohm's law in a moving medium of electrical conductivity σ, which yields:

$$j = \sigma(e + u \times b) \text{ in } \Omega.$$

Together with the the Maxwell equations, it gives

$$\begin{aligned} \partial_t b &= \text{curl } (u \times b) - \eta \, \text{curl curl } b \text{ in } \Omega, \\ \text{div } b &= 0 \text{ in } \Omega, \end{aligned} \tag{2.2}$$

with magnetic diffusivity $\eta = \frac{1}{\mu_0 \sigma}$. Eventually, we have to specify the properties of the medium surrounding the fluid domain, namely $\Omega' = \mathbb{R}^3 \backslash \overline{\Omega}$. We will restrict here to the case of a perfect insulator, resulting in: $j = 0$ in Ω'. With the second line of the Maxwell equation, it amounts to

$$\begin{aligned} \text{curl } b = 0 \text{ in } \Omega', \\ \text{div } b = 0 \text{ in } \Omega'. \end{aligned} \tag{2.3}$$

The last step of the derivation consists in specifying some interface conditions at $\partial\Omega$. As b is divergence-free over the whole space, one must have $[b \cdot n]|_{\partial\Omega} = 0$, where n is a unit normal vector at $\partial\Omega$, and $[\,\cdot\,]|_{\partial\Omega}$ refers to the jump across $\partial\Omega$. Also, as we do not consider the case of a perfect conductor outside Ω, we do not expect surface currents at $\partial\Omega$, so that the relation curl $b = \mu_0 \, j$ implies that $[b \times n]|_{\partial\Omega} = 0$. Hence, the whole field b is continuous across the interface:

$$[b]|_{\partial\Omega} = 0. \tag{2.4}$$

Equations (2.1)-(2.2)-(2.3)-(2.4) form the full MHD system. Note that it mixes the Navier-Stokes equation, set on Ω only, with equations on the magnetic field that are both set in Ω and Ω'. To the best of our knowledge, this difficulty was not considered in previous mathematical studies devoted to the MHD equations, see [15], [53], [22], [23], [28], [41], [29],

[54]. In those works the study is conducted considering the case of a perfect conductor: only the equation (2.2) is retained for the magnetic field, while the jump condition is replaced by the boundary condition $\operatorname{curl} b \times n|_{\partial\Omega} = 0$. The dynamics of the magnetic field outside Ω is implicitly neglected, which yields more standard initial boundary value problems in Ω. We will come back to this issue in the next section of this chapter. With the upcoming asymptotic analysis of magnetohydrodynamic flows in mind, we wish to provide a dimensionless version of the MHD equation. Therefore, we introduce some typical length L, typical speed U of the fluid flow, as well as some typical amplitude B of the magnetic field. We further introduce the typical advection time $T = \frac{L}{U}$ and the so-called hydrodynamic pressure $\Pi = \rho U^2$. We rescale all unknowns and variables accordingly, more precisely we set:

$$u = Uu', \quad b = Bb', \quad p = \Pi p', \quad x = Lx', \quad t = Tt', \quad \Omega = L\Omega'.$$

Dropping the primes, we obtain:

$$\frac{U}{T}\partial_t u + \frac{U^2}{L}(u \cdot \nabla)u + \frac{\pi}{L\rho}\nabla p - \frac{\nu U}{L^2}\Delta u = \frac{B^2}{\mu_0 \rho L}\operatorname{curl} b \times b,$$

where we have expressed j in terms of b in the right-hand side. Multiplying everything by $\frac{L}{U^2}$ we get

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \frac{1}{\mathrm{Re}}\Delta u = S \operatorname{curl} b \times b,$$

where the *Reynolds number* Re and the *coupling parameter* S are defined by:

$$\mathrm{Re} := \frac{\nu}{LU}, \quad S := \frac{B^2}{\mu_0 U^2 \rho}.$$

We can do the same for the first equation in (2.2), which becomes

$$\frac{UB}{L}\partial_t b - \frac{UB}{L}\operatorname{curl}(u \times b) + \frac{B\eta}{L^2}\operatorname{curl}\operatorname{curl} b = 0$$

so that multiplying by $\frac{L}{UB}$ we get

$$\partial_t b - \operatorname{curl}(u \times b) - \frac{1}{\mathrm{Rm}}\operatorname{curl}\operatorname{curl} b = 0,$$

with the magnetic Reynolds number $\mathrm{Rm} := \frac{UL}{\eta}$. Eventually, the dimensionless version of the full MHD system takes the form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \frac{1}{\mathrm{Re}} \Delta u = \mathrm{curl}\, b \times b & \text{in } \Omega \\ \partial_t b - \mathrm{curl}\, (u \times b) + \frac{1}{\mathrm{Rm}} \mathrm{curl}\, \mathrm{curl}\, b = 0 & \text{in } \Omega \\ \mathrm{div}\, b = 0 & \text{in } \mathbb{R}^3 \\ \mathrm{curl}\, b = 0 & \text{in } \Omega' \\ \mathrm{div}\, u = 0 & \text{in } \Omega \\ \text{with the condition } [b] = 0 & \text{at } \partial\Omega \\ \text{and the no-slip condition } u|_{\partial\Omega} = 0 & \text{at } \partial\Omega. \end{cases} \tag{MHD}$$

Let us note that, taking into account the divergence-free constraint on b, we can rewrite the second order operator in (MHDb) as curl curl $b = -\Delta b$. Nevertheless, the expression in terms of the curl operator emphasizes that the equation div $b = 0$ in Ω is preserved by the evolution: if it is satisfied initially, it is satisfied for all times. It can be seen by taking the divergence of (MHDb), which gives that $\partial_t \mathrm{div}\, b = 0$. Moreover, this expression is more suitable for a variational formulation, and the construction of weak solutions. The existence of solutions for (MHD) will be discussed in the next section.

Remark 2.1.1. Instead of starting from the strong formulation (MHD), one could have considered the slightly different one:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \frac{1}{\mathrm{Re}} \Delta u = \mathrm{curl}\, b \times b & \text{in } \Omega \\ \partial_t b + \mathrm{curl}\, E = 0 & \text{in } \mathbb{R}^3 \\ E = -u \times b + \frac{1}{\mathrm{Rm}} \mathrm{curl}\, b & \text{in } \Omega, \\ \mathrm{div}\, b = 0 & \text{in } \mathbb{R}^3 \\ \mathrm{curl}\, b = 0 & \text{in } \Omega' \\ \mathrm{div}\, u = 0 & \text{in } \Omega \\ \text{with the condition } [b] = 0 & \text{at } \partial\Omega \\ \text{and the no-slip condition } u|_{\partial\Omega} = 0 & \text{at } \partial\Omega. \end{cases} \tag{2.5}$$

This formulation, closer to the Maxwell equations, is *a priori* stronger than (MHD), as its second and third equations seem to contain more information than the second equation in (MHD) (and in particular one more unknown E !). Nevertheless, if b and u are smooth on each side of the interface $\partial\Omega$ (which is always implicitly assumed here), one can show that the formulations are equivalent. Indeed, if b satisfies (MHD), one has div $\partial_t b = \partial_t \mathrm{div}\, b = 0$ over $\mathbb{R}^3$, so that there exists $\bar{E}$ such that

$$\partial_t b = \mathrm{curl}\, \bar{E} \text{ in } \mathbb{R}^3.$$

Note that as b is smooth on each side of the interface and does not jump across it, $\partial_t b$ has at least H^1 regularity in the neighbourhood of the interface. Hence, the field $\bar{E}$ can be chosen with H^2 regularity across the interface. Now, from the second relation in (MHD), we infer that

$$\text{curl } \bar{E} = \text{curl } (-u \times b + \text{curl } b) \text{ in } \Omega$$

so that $\bar{E} = -u \times b + \text{curl } b + \nabla p$ in Ω, for some p which belongs at least to $H^3(\Omega)$. If Ω is smooth enough, one can extend p as a function $\tilde{p}$ in $H^3(\mathbb{R}^3)$. Finally, if we set $E = \bar{E} - \nabla \tilde{p}$, we recover the second and third relations in (2.5).

2.2 Weak solutions of the MHD system

The system (MHD) derived in the previous section is used to describe various magnetohydrodynamic flows, notably in the context of dynamo theory ([24]). Without further assumptions on the magnetic field in the insulator Ω', it is mathematically and numerically challenging, as it couples the dynamics of the fluid in the bounded domain Ω to the dynamics of the magnetic field in the whole space $\mathbb{R}^3$. This creates some difficulties in the well-posedness analysis, for which we could find no references. We will therefore address this issue before turning to the boundary layer analysis. We focus in this section on weak solutions of Leray type. Further remarks on smooth solutions will be made in the next section. For simplicity, we take all parameters Re, Rm and S equal to unity.

2.2.1 Variational formulation and existence result

At first, we must provide a weak formulation for (MHD). The formulation of the Navier-Stokes equation will be standard, and will involve the classical spaces

$$\mathcal{D}_\sigma(\Omega) := \left\{\phi \in \mathcal{C}_c^\infty(\Omega)^3 \text{ such that div } \phi = 0\right\}, \tag{2.6}$$

$$H(\Omega) := \left\{ \text{ the adherence of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega)^3\right\}, \tag{2.7}$$

$$V(\Omega) := \left\{ \text{ the adherence of } \mathcal{D}_\sigma(\Omega) \text{ in } H_0^1(\Omega)^3\right\}. \tag{2.8}$$

In particular, if we multiply (MHDa) by a test field ϕ in $C_c^1((0,T), V(\Omega))$ and integrate by parts over $(0,T) \times \Omega$, we find, assuming enough smoothness on Ω, u and b:

$$\int_0^T \int_\Omega (\partial_t \phi + u \cdot \nabla \phi) \cdot u - \int_0^T \int_\Omega \nabla u \cdot \nabla \phi + \int_0^T \int_\Omega (\text{curl } b \times b) \cdot \phi = 0. \tag{2.9}$$

The key point is to find the appropriate variational formulation of the equations for b. As we will see, an appropriate functional space is

$$B = \left\{\phi \in L^2(\mathbb{R}^3) \mid \text{curl } \phi \in L^2(\mathbb{R}^3), \ \text{curl } \phi = 0 \text{ in } \Omega'\right\}. \tag{2.10}$$

Moreover, we claim that a suitable variational formulation for the magnetic part of the MHD system is:

$$\int_0^T \int_{\mathbb{R}^3} \partial_t \psi \cdot b + \int_0^T \int_\Omega (u \times b) \cdot \mathrm{curl}\, \psi + \int_0^T \int_\Omega \mathrm{curl}\, b \cdot \mathrm{curl}\, \psi = 0, \tag{2.11}$$

for $\psi \in C_c^1((0,T); B)$.

A first justification for this choice is the equivalence between the strong and weak formulation for smooth vector fields. More precisely, we have the following:

Proposition 2.2.1. *Let Ω be a smooth bounded domain, and $\Omega' = \mathbb{R}^3 \backslash \overline{\Omega}$. Let u be a smooth field over $[0,T] \times \overline{\Omega}$ with $u|_{\partial\Omega} = 0$, and b be a smooth field over $[0,T] \times (\overline{\Omega} \cup \overline{\Omega'})$ decaying fast enough at infinity.*

If div $b|_{t=0} = 0$ in $\mathbb{R}^3$, $b(t) \in B$ for all $t \in [0,T]$ and if b satisfies equation (2.11) for all $\psi \in C_c^1((0,T); B)$, then b satisfies (MHDb)-(MHDc)-(MHDd)-(MHDf).

Conversely, if Ω' is simply connected, and if b satisfies the equations (MHDb)-(MHDc)-(MHDd)-(MHDf), then $b(t) \in B$ for all $t \in [0,T]$ and b satisfies equation (2.11) for all $\psi \in C_c^1((0,T); B)$.

Proof. The easiest part of the proposition is the first one. We assume that div $b|_{t=0} = 0$ in $\mathbb{R}^3$, $b(t) \in B$ for all $t \in [0,T]$ and that b satisfies equation (2.11) for all $\psi \in C_c^1((0,T); B)$. Since for all t one has that $b(t) \in B$, equation (MHDd) is trivially satisfied. Then, by choosing ψ in $C_c^1((0,T); \mathcal{D}(\Omega)^3)$ (where $\mathcal{D}(\Omega)^3$ can be seen as a subset of B after extension by zero), we deduce from (2.11) that equation (MHDb) is satisfied as well. Besides, if we take $\psi = \nabla\psi'$, with $\psi' \in C_c^1((0,T); \mathcal{D}(\mathbb{R}^3))$, we find that $\int_0^T \int_\Omega \partial_t b \cdot \nabla\psi' = 0$ i.e. $\partial_t \mathrm{div}\, b = 0$ in a distributional sense. With the assumption div $b|_{t=0} = 0$ in $\mathbb{R}^3$, we recover div $b(t) = 0$ for all t, that is (MHDc). Eventually, the fact that curl $b(t) \in L^2(\mathbb{R}^3)^3$ for all t implies that $[b \times n]|_{\partial\Omega} = 0$, while the fact that div $b(t) \in L^2(\mathbb{R}^3)$ (and is even zero) for all t implies that $[b \cdot n]|_{\partial\Omega} = 0$. Put together, these two conditions yield (MHDe).

The second part of the proposition remains to be proved, thus assuming (MHDb)-(MHDc)-(MHDd)-(MHDf). First, by condition (MHDd) and by the continuity of the tangential components of b at the interface $\partial\Omega$ (see (MHDf)) we get that $b(t) \in B$ for all t. Now, to recover (2.11), we write:

$$\begin{cases} \partial_t b = \mathrm{curl}\,(E_+) \text{ in } \Omega & \text{with } E_+ = u \times b + \mathrm{curl}\, b \\ \partial_t b = \mathrm{curl}\,(E_-) \text{ in } \Omega' & \text{for some smooth } E_- \text{ over } [0,T] \times \overline{\Omega'}. \end{cases}$$

The first line follows from (MHDb). The second line is a consequence of (MHDc): it implies that $\partial_t b$ is divergence-free over Ω' and as Ω' is supposed to be simply connected, it implies that $\partial_t b$ can be written as a curl. From there, one has that for all test function

$\psi \in C_c^1(0,T;B)$:

$$\int_0^T \int_{\mathbb{R}^3} \partial_t b \cdot \psi = -\int_0^T \int_{\partial\Omega} ((E_+ - E_-) \times n) \cdot \psi + \int_0^T \int_\Omega (u \times b) \cdot (\text{curl } \psi) - \int_0^T \int_\Omega \text{curl } b \cdot \text{curl } \psi.$$

On the other hand, by integration by parts in time, the first term can be rewritten as

$$\int_0^T \int_{\mathbb{R}^3} \partial_t b \cdot \psi = -\int_0^T \int_{\mathbb{R}^3} b \cdot \partial_t \psi.$$

Hence, to obtain (2.11), it is enough to show that $\int_{\partial\Omega}((E_+ - E_-) \times n) \cdot \Psi = 0$ for all Ψ in B. Note that this boundary integral has to be understood in a generalized sense: we have formally $\int_{\partial\Omega}((E_+ - E_-) \times n) \cdot \Psi = \int_{\partial\Omega}((E_+ - E_-) \times n) \cdot (\Psi \times n)$ and the latter has to be interpretated as a duality bracket between an element of $H^{\frac{1}{2}}(\partial\Omega)$ and an element of $H^{-\frac{1}{2}}(\partial\Omega)$. Indeed, as Ψ is in L^2 and curl Ψ is in L^2, the tangential trace $\Psi \times n$ is in $H^{-\frac{1}{2}}$. To establish this identity, we remark that, since curl $\Psi = 0$ in the simply connected domain Ω', we can write $\Psi = \nabla p$ in Ω' for some p in $H^1(\Omega' \cap B(0,R))$ for any R (because Ψ is in $L^2(\Omega')^3$). Hence, it is enough to show that

$$\int_{\partial\Omega} ((E_+ - E_-) \times n) \cdot \nabla p = 0.$$

By a density argument, it is enough to show it for $p \in \mathcal{D}(\mathbb{R}^3)$. We write:

$$\begin{aligned}
&\int_{\partial\Omega} ((E_+ - E_-) \times n) \cdot \nabla p = \int_{\partial\Omega} (E_+ \times n) \cdot \nabla p - \int_{\partial\Omega} (E_- \times n) \cdot \nabla p \\
&= -\int_\Omega (\text{curl } E_+ \cdot \nabla p - E_+ \cdot \text{curl } \nabla p) - \int_{\Omega'} (\text{curl } E_- \cdot \nabla p - E_- \cdot \text{curl } \nabla p) \\
&= -\int_\Omega \text{curl } E_+ \cdot \nabla p \; - \int_{\Omega'} \text{curl } E_- \cdot \nabla p \\
&= +\int_\Omega \text{div curl } E_+ \, p - \int_{\partial\Omega} (\text{curl } E_+ \cdot n) \, p \; + \int_{\Omega'} \text{div curl } E_- \, p + \int_{\partial\Omega} (\text{curl } E_- \cdot n) \, p \\
&= -\int_{\partial\Omega} (\text{curl } E_+ \cdot n) \, p + \int_{\partial\Omega} (\text{curl } E_- \cdot n) \, p = -\int_{\partial\Omega} [\partial_t b \cdot n]|_{\partial\Omega} \, p = 0.
\end{aligned}$$

Note that the second and fourth lines come from integration by parts involving respectively the curl and the gradient operator. Eventually, the last quantity vanishes because of the continuity of $b \cdot n$ across $\partial\Omega$. This concludes the proof. $\square$

We can now state the existence of a weak solution to (MHD), given appropriate initial data. We will consider an initial magnetic field which is divergence-free, namely $b_0 \in \mathbb{P}B \subset B$, with $\mathbb{P}$ being the Leray projector on $\mathbb{R}^3$.

Theorem 2.2.1. *Let Ω be an open bounded domain of $\mathbb{R}^3$ with smooth boundary. Let*

$u_0 \in H(\Omega)$, $b_0 \in \mathbb{P}B$. There exists

$$u \in C_w([0,T]; H(\Omega)) \cap L^2(0,T; V(\Omega)), \quad b \in C_w([0,T]; \mathbb{P}L^2(\mathbb{R}^3)) \cap L^2(0,T,\mathbb{P}B)$$

with initial conditions u_0, b_0, such that

i) u and b satisfy (2.9) *for all ϕ in $C_c^1((0,T), V(\Omega))$,*

ii) u and b satisfy (2.11) *for all ψ in $C_c^1(0,T; B)$,*

iii) one has the following energy inequality: for almost all $t \in \mathbb{R}^+$,

$$\frac{1}{2}\left(|u(t)\|^2_{L^2(\Omega)} + \|b(t)\|^2_{L^2(\mathbb{R}^3)}\right) + \int_0^t \|\nabla u(t)\|^2_{L^2(\Omega)} + \int_0^t \|curl\, b(t)\|^2_{L^2(\Omega)} \leqslant \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} + \|b_0\|^2_{L^2(\mathbb{R}^3)}\right).$$

Let us emphasize that the non-linear terms involving the magnetic field are well-defined in both relations (2.9) and (2.11) (see right below).

2.2.2 Sketch of proof

We give here the main steps leading to Theorem 2.2.1. They follow as a whole the classical proof for Navier-Stokes (as seen for instance in [4]), but we shall underline the novelties brought by the MHD system.

In this respect, we first remark that the non-linear terms involving the magnetic field are well-defined in both relations (2.9) and (2.11). Indeed, it is well-known that $\mathbb{P}B$ is embedded in $H^1(\mathbb{R}^3)$ (the whole gradient is controlled by the curl for divergence-free vector fields). In particular, usual Sobolev imbeddings yield:

$$\left|\int_0^T \int_\Omega (\text{curl } b \times \mathbb{P}b) \cdot \phi\right| \leqslant \|\text{curl } b\|_{L^2(L^2)} \|\mathbb{P}b\|_{L^2(L^6)} \|\phi\|_{L^\infty(L^3)} < +\infty.$$

Similarly,

$$\left|\int_0^T \int_\Omega (u \times \mathbb{P}b) \cdot \text{curl } \psi\right| \leqslant \|u\|_{L^2(L^3)} \|\mathbb{P}b\|_{L^2(L^6)} \|\text{curl } \psi\|_{L^\infty(L^2)} < +\infty.$$

One can remark that in order to show Theorem 2.2.1, it is enough to establish that there exists

$$u \in C_w([0,T]; H(\Omega)) \cap L^2(0,T; V(\Omega)), \quad b \in C_w([0,T]; L^2(\mathbb{R}^3)) \cap L^2(0,T,B)$$

with initial conditions u_0, b_0, such that i') for all $\phi \in C^1_c(0,T;V(\Omega))$,

$$\int_0^T \int_\Omega (\partial_t \phi + u \cdot \nabla \phi) \cdot u - \int_0^T \int_\Omega \nabla u \cdot \nabla \phi - \int_0^T \int_\Omega (\mathrm{curl}\, b \times \mathbb{P} b) \cdot \phi = 0; \tag{2.12}$$

ii') for all ψ in $C^1_c(0,T;B)$,

$$\int_0^T \int_{\mathbb{R}^3} \partial_t \psi \cdot b + \int_0^T \int_\Omega (u \times \mathbb{P} b) \cdot \mathrm{curl}\, \psi + \int_0^T \int_\Omega \mathrm{curl}\, b \cdot \mathrm{curl}\, \psi = 0; \tag{2.13}$$

and the energy inequality iii). In other words, we claim that it is enough to build a solution (u,b) with b which is not *a priori* divergence-free, and to replace b with $\mathbb{P}b$ in the variational formulation.

One can argue as for Proposition 2.2.1: we consider in (2.13) a test field of the form $\psi = \nabla p$, with p with $p \in C^1_c(0,T;\mathcal{D}(\mathbb{R}^3)$, which yields that

$$\int_0^T \int_\Omega b \cdot (\partial_t \nabla p) = 0.$$

This implies in turn that $\partial_t \mathrm{div}\ b = 0$ in the sense of distributions, so that div $b(t)$ is constant in time over $(0,T)$. Using the weak continuity of b in time with values in L^2, we get: div $b(t) =$ div $b_0 = 0$. Thus, we recover the divergence-free condition on b *a posteriori*. Now, as $\mathbb{P}b = b$, equations (2.12) and (2.13) are the same as (2.9) and (2.11).

We now explain how to construct a solution satisfying i')-ii')-iii). We proceed in a standard manner, through a Galerkin approximation and compactness methods.

Galerkin approximation and weak convergence

For the Galerkin approximation, we consider:

- the usual hilbertian basis of $H(\Omega)$, made of the family of eigenvectors $(e_k)_{k\in\mathbb{N}}$ of the Stokes operator. We remind that this basis is a complete orthogonal system of $V(\Omega)$ (endowed with the usual homogeneous H^1 norm).

- a hilbertian basis of B, that we denote by $(e'_k)_{k\in\mathbb{N}}$. We remind that B is a separable Hilbert space, endowed with the natural scalar product:

$$(b|c) = \int_{\mathbb{R}^3} b \cdot c \, + \int_{\mathbb{R}^3} \mathrm{curl}\, b \cdot \mathrm{curl}\, c = \int_{\mathbb{R}^3} b \cdot c \, + \int_\Omega \mathrm{curl}\, b \cdot \mathrm{curl}\, c.$$

We then consider approximations of the form

$$u^n(t,x,y,z) = \sum_{k=0}^n \alpha_k(t) e_k(x,y,z), \qquad b^n(t,x,y,z) = \sum_{k=0}^n \beta_k(t) e'_k(x,y,z),$$

solving

i'-n) for all ϕ^n in $\mathrm{span}(e_0, \dots, e_n)$

$$\int_\Omega (\partial_t u^n + u^n \cdot \nabla u^n) \cdot \phi^n + \int_\Omega \nabla u^n \cdot \nabla \phi^n = \int_\Omega (\mathrm{curl}\, b^n \times \mathbb{P} b^n) \cdot \phi^n = 0, \tag{2.14}$$

ii'-n) for all ψ^n in $\mathrm{span}(e'_0, \dots, e'_n)$,

$$\int_{\mathbb{R}^3} \partial_t b^n \cdot \psi^n - \int_\Omega (u^n \times \mathbb{P} b^n) \cdot \mathrm{curl}\, \psi^n + \int_\Omega \mathrm{curl}\, b^n \cdot \mathrm{curl}\, \psi^n = 0, \tag{2.15}$$

together with the initial conditions:

$$u^n|_{t=0} = P_n u_0, \quad b^n|_{t=0} = P'_n b_0$$

where P_n and P'_n are the orthogonal projections on $\mathrm{span}(e_0, \dots, e_n)$ and $\mathrm{span}(e'_0, \dots, e'_n)$ respectively.

The previous equations yield a finite differential system on $\alpha = (\alpha^1, \dots, \alpha^n)$ and $\beta = (\beta^1, \dots, \beta^n)$ of the form

$$\begin{cases} \frac{d\alpha}{dt} = \mathcal{F}_1(t, \alpha, \beta) \\ \frac{d\beta}{dt} = \mathcal{F}_2(t, \alpha, \beta), \end{cases}$$

where $\mathcal{F}_1, \mathcal{F}_2$ are polynomial. By the Cauchy-Lipschitz theorem, there exists a unique maximal solution $u^n \in \mathcal{C}^\infty([0, T_n[,\ \mathrm{Vect}\,\{e_0, ..., e_n\})$ and $b^n \in \mathcal{C}^\infty([0, T_n[,\ \mathrm{Vect}\,\{e'_0, ..., e'_n\})$ where T_n is the maximal time of existence.

To prove that $T_n = T$ independently from n, we derive an energy estimate: we take $\phi^n = u^n$, $\psi^n = b^n$, and get after integration by parts:

$$\frac{1}{2}\partial_t \int_\Omega |u^n|^2 + \int_\Omega |\nabla u^n|^2 = \int_\Omega (\mathrm{curl}\, b^n) \times \mathbb{P} b^n \cdot u^n, \quad \text{and} \tag{2.16}$$

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |b^n|^2 + \int_{\mathbb{R}^3} |\mathrm{curl}\, b^n|^2 = \int_\Omega (u^n \times \mathbb{P} b^n) \cdot \mathrm{curl}\, b^n. \tag{2.17}$$

If we now sum up the two equations and integrate in time between 0 and $t \in [0, T_n[$, we find (by skew-symmetry of the cross product):

$$\begin{aligned} &\frac{1}{2}\|u^n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|b^n(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \int_\Omega |\nabla u^n|^2 + \int_0^t \int_{\mathbb{R}^3} |\mathrm{curl}\, b^n|^2 \\ &= \frac{1}{2}\int_\Omega |u^n|^2(0, \cdot) + \frac{1}{2}\int_{\mathbb{R}^3} |b^n|^2(0, \cdot) \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|b_0\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{2.18}$$

Note that we have implicitly assumed here that $\|b^n(0, \cdot)\|_{L^2} \leqslant \|b_0\|_{L^2}$. This is obvious if b_0

is zero, and if not this can be easily realized by choosing e_0' colinear to b_0. In particular

$$\limsup_{t\to T_n} \|u^n(t,\cdot)\|_{H(\Omega)} < +\infty, \qquad \limsup_{t\to T_n} \|b^n(t,\cdot)\|_{L^2(\mathbb{R}^3)} < +\infty;$$

so that by standard blow up results for ODEs one has $T_n = +\infty$. Besides, as the bound in (2.18) is uniform in n, it implies the weak convergence of subsequences of (u^n) and (b^n): if we relabel these subsequences (u^n) and (b^n), we have more precisely:

$$\begin{aligned} &u^n \to u \text{ weakly* in } L^\infty(0,T;H(\Omega)), \quad \text{weakly in } L^2(0,T;V(\Omega)), \\ &b^n \to b \text{ weakly* in } L^\infty(0,T;L^2(\mathbb{R}^3)), \quad \text{weakly in } L^2(0,T;B). \end{aligned} \tag{2.19}$$

Bounds on time derivatives

The next step in the classical existence scheme of Leray type solutions is to obtain some equicontinuity in time (possibly in low norm) for (u^n) and (b^n). We start with the standard bound on $\partial_t u^n$, where we only need to pay attention to the additional magnetic term:

Proposition 2.2.2. *For all $T > 0$, there exists a constant $M > 0$ such that for all n,*

$$\int_0^T \|\partial_t u^n\|_{V'(\Omega)}^{4/3} \leqslant M. \tag{2.20}$$

Proof. Let $\phi \in V(\Omega)$, with $\|\nabla\phi\|_{L^2(\Omega)} = 1$. We take $\phi^n = P_n\phi$ in the variational formulation for u^n. We get

$$\begin{aligned} \int_\Omega \partial_t u^n \cdot \phi = \int_\Omega \partial_t u^n \cdot P_n\phi = -\int_\Omega (u^n\cdot\nabla u^n)\cdot P_n\phi - \int_\Omega \nabla u^n \cdot \nabla P_n\phi \\ + \int_\Omega (\text{curl } b^n)\times \mathbb{P}b^n \cdot P_n\phi. \end{aligned}$$

The first two terms appear in the classical Navier-Stokes equation, so that we focus on the third one. Let us remark that $\mathbb{P}b^n$ is divergence-free, so that by standard vectorial identities:

$$\left|\int_\Omega (\text{curl } b^n)\times \mathbb{P}b^n \cdot P_n\phi\right| = \left|\int_\Omega (\text{curl } \mathbb{P}b^n)\times \mathbb{P}b^n \cdot P_n\phi\right| \tag{2.21}$$

$$= \left|-\int_\Omega \nabla\left(\frac{(\mathbb{P}b^n)^2}{2}\right)\cdot P_n\phi + \int_\Omega (\mathbb{P}b^n\cdot\nabla)\mathbb{P}b^n\cdot P_n\phi\right| = \left|-\int_\Omega (\mathbb{P}b^n\cdot\nabla)P_n\phi\cdot\mathbb{P}b^n\right| \tag{2.22}$$

$$\leqslant \int_\Omega |\mathbb{P}b^n|^2|\nabla P_n\phi| \leqslant \|\mathbb{P}b^n\|^2_{L^4(\Omega)}\|\nabla P_n\varphi\|_{L^2(\Omega)} \leqslant \|\mathbb{P}b^n\|^2_{L^4(\Omega)}, \tag{2.23}$$

where we used the orthogonality properties of (e_k) to write

$$||\nabla P_n\phi||_{L^2(\Omega)} \leqslant ||\nabla\phi||_{L^2(\Omega)} = 1.$$

To continue, we need the following

Lemma 2.2.2 (Gagliardo-Nirenberg). *There exists $C > 0$ such that: for all $b \in H^1(\mathbb{R}^d)$ and $\theta_d = \frac{d}{4}$, one has $\|b\|_{L^4(\mathbb{R}^d)} \leqslant C\|b\|_{L^2(\mathbb{R}^d)}^{1-\theta_d}\|\nabla b\|_{L^2(\mathbb{R}^d)}^{\theta_d}$.*

Now we can improve the estimate for

$$\left|\int_\Omega (\text{curl } b^n) \times \mathbb{P}b^n \cdot P_n\phi\right| \leqslant \|\mathbb{P}b^n\|_{L^4(\Omega)}^2 \leqslant \|\mathbb{P}b^n\|_{L^4(\mathbb{R}^3)}^2 \leqslant C\|\mathbb{P}b^n\|_{L^2(\mathbb{R}^3)}^{1/2}\|\nabla\mathbb{P}b^n\|_{L^2(\mathbb{R}^3)}^{3/2}$$
$$\leqslant C\|b^n\|_{L^2(\mathbb{R}^3)}^{1/2}\|\text{curl } \mathbb{P}b^n\|_{L^2(\mathbb{R}^3)}^{3/2} \leqslant C\|b^n\|_{L^2(\mathbb{R}^3)}^{1/2}\|\text{curl } b^n\|_{L^2(\mathbb{R}^3)}^{3/2}.$$

It follows that

$$\|\partial_t u^n\|_{V'(\Omega)} = \sup_{\phi\in V,\ \|\nabla\phi\|_{L^2}=1}\left|\int_\Omega \partial_t u^n \phi\right|$$
$$\leqslant C\left(\|\nabla u^n\|_{L^2(\Omega)} + \|u^n\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla u^n\|_{L^2}^{\frac{3}{2}} + \|b^n\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\text{curl } b^n\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}}\right)$$

from which, integrating in time between $]0, T[$ we get to

$$\int_0^T \|\partial_t u^n\|_{V'(\Omega)}^{\frac{4}{3}} \leqslant C'\left(\int_0^T \|\nabla u^n\|_{L^2(\Omega)}^{\frac{4}{3}} + \int_0^T \|u^n\|_{L^2(\Omega)}^{\frac{2}{3}}\|\nabla u^n\|_{L^2(\Omega)}^2 + \int_0^T \|b^n\|_{L^2(\mathbb{R}^3)}^{\frac{2}{3}}\|\text{curl } b^n\|_{L^2(\mathbb{R}^3)}^2\right)$$
$$\leqslant C'_T\Bigg(\|\nabla u^n\|_{L^2(0,T;L^2(\Omega))}^{4/3} + \|u^n\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{2}{3}}\|\nabla u^n\|_{L^2(0,T;L^2(\Omega))}^2$$
$$+ \|b^n\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^{\frac{2}{3}}\|\text{curl } b^n\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2\Bigg) \leqslant M_T,$$

using (2.18), and this allows us to conclude for (2.20). □

We still have to deal with the uniform equicontinuity in time of (b^n). It turns out that deriving a uniform bound on $\partial_t b^n$ in low norm is not so easy as in the case of $\partial_t u^n$. Namely, the previous method does not work because the basis (e'_k) of B does not enjoy as good orthogonality properties as (e_k) - it is not orthogonal in both L^2 and B. Moreover, we would like a bound on $\partial_t\mathbb{P}b^n$ rather than on $\partial_t b^n$. This is due to the fact that the uniform bound on (b^n) in the space B does not provide a uniform H^1 bound. Hence, we have to restrict to $(\mathbb{P}b^n)$ to obtain equicontinuity in space. To solve these issues, we will rather control a fractional time derivative of $(\mathbb{P}b^n)$, through Fourier transform in time:

Proposition 2.2.3. *Let $T > 0$, and $\widetilde{b^n}$ (resp. $\widetilde{\mathbb{P}b^n}$) the extension by zero of b^n (resp. $\mathbb{P}b^n$) outside $(0, T)$. We denote by $\widehat{b^n}$ (resp. $\widehat{\mathbb{P}b^n}$) the Fourier transform of $\widetilde{b^n}$ (resp. $\widetilde{\mathbb{P}b^n}$) with respect to time. Then, for all $\gamma < \frac{1}{4}$, there exists a constant $M > 0$ such that for all n,*

$$\int_\mathbb{R} |\tau|^{2\gamma}\|\widehat{b^n}(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau \leqslant M, \quad \int_\mathbb{R} |\tau|^{2\gamma}\|\widehat{\mathbb{P}b^n}(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau \leqslant M. \tag{2.24}$$

Proof. The second bound follows from the first by the simple continuity of $\mathbb{P}$ on $L^2(\mathbb{R}^3)$. To achieve the first one, we follow closely the reasoning of [56]. The extension $\widetilde{b^n}$ verifies: for all $j = 0, \dots, n$,

$$\frac{d}{dt}\int_{\mathbb{R}^3} \widetilde{b^n}\, e'_j = \int_\Omega (\widetilde{u^n} \times \widetilde{\mathbb{P}b^n})\,\mathrm{curl}\, e'_j - \int_\Omega \mathrm{curl}\, \widetilde{b^n}\,\mathrm{curl}\, e'_j$$
$$+\delta_0 \int_\Omega b_0^n\, e'_j - \delta_T \int_\Omega b^n|_{t=T}\, e'_j,$$

where δ_0 and δ_T refer to the Dirac masses located at $t = 0$ and $t = T$. Taking the Fourier transform with respect to the time variable, we obtain

$$i\tau \int_{\mathbb{R}^3} \widehat{b^n}(\tau) e'_j = \int_\Omega (\widehat{u^n \times \mathbb{P}b^n}(\tau)) \cdot \mathrm{curl}\, e'_j - \int_\Omega \mathrm{curl}\, \widehat{b^n}(\tau)\,\mathrm{curl}\, e'_j$$
$$+ \int_\Omega b_0^n\, e'_j - e^{-2i\pi Tt} \int_\Omega b^n|_{t=T}\, e'_j$$

where $\widehat{u^n \times \mathbb{P}b^n}$ refers to the Fourier transform in time of $\widetilde{u^n} \times \widetilde{\mathbb{P}b^n}$. We then multiply by the conjugate of $\widehat{\beta}_j(\tau)$ (with obvious notation) and sum over j to obtain

$$i\tau \|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)} = \int_\Omega (\widehat{u^n \times \mathbb{P}b^n}(\tau)) \cdot \mathrm{curl}\, \widehat{b^n}(\tau) - \int_\Omega \mathrm{curl}\, \widehat{b^n}(\tau) \cdot \mathrm{curl}\, \widehat{b^n}(\tau)$$
$$+ \int_\Omega b_0^n \cdot \widehat{b^n}(\tau) - e^{-2i\pi Tt} \int_\Omega b^n|_{t=T} \cdot \widehat{b^n}(\tau).$$

First of all, we remark that $\|b_0^n\|_{L^2(\mathbb{R}^3)}$ and $\|b^n|_{t=T}\|_{L^2(\mathbb{R}^3)}$ are uniformly bounded in n, so that

$$|\tau| \|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)} \leqslant \left| \int_\Omega (\widehat{u^n \times \mathbb{P}b^n}(\tau)) \cdot \mathrm{curl}\, \widehat{b^n}(\tau) \right| - \int_\Omega |\mathrm{curl}\, \widehat{b^n}(\tau)|^2 + C_1 \|\widehat{b^n}(\tau)\|_{L^2(\mathbb{R}^3)}$$
$$\leqslant \sup_{\tau\in\mathbb{R}} \left(\| |(\widehat{u^n \times \mathbb{P}b^n})(\tau)\|_{L^2(\Omega)} + \|\mathrm{curl}\, \widehat{b^n}(\tau)\|_{L^2(\Omega)} \right) \|\mathrm{curl}\, \widehat{b^n}(\tau)\|_{L^2(\Omega)} + C_1 \|\widehat{b^n}(\tau)\|_{L^2(\mathbb{R}^3)}.$$

Now, we write that

$$\sup_{\tau\in\mathbb{R}} \left(\|(\widehat{u^n \times \mathbb{P}b^n})(\tau)\|_{L^2(\Omega)} + \|\mathrm{curl}\, \widehat{b^n}(\tau)\|_{L^2(\Omega)} \right) \tag{2.25}$$
$$\leqslant \int_{\mathbb{R}} \left(\|\widetilde{u^n \times \mathbb{P}b^n}(t)\|_{L^2(\Omega)} + \|\mathrm{curl}\, \widetilde{b^n}(t)\|_{L^2(\Omega)} \right) dt \tag{2.26}$$
$$= \int_0^T \left(\|(u^n \times \mathbb{P}b^n)(t)\|_{L^2(\Omega)} + \|\mathrm{curl}\, b^n(t)\|_{L^2(\Omega)} \right) dt \tag{2.27}$$
$$\leqslant \int_0^T \left(\|u^n(t)\|_{L^4(\Omega)} \|\mathbb{P}b^n(t)\|_{L^4(\Omega)} + \|\mathrm{curl}\, b^n(t)\|_{L^2(\Omega)} \right) dt \tag{2.28}$$
$$\leqslant C \|u^n\|_{L^2(0,T;H^1(\Omega))} \|\mathbb{P}b^n\|_{L^2(0,T;H^1(\Omega))} + \sqrt{T} \|\mathrm{curl}\, b^n\|_{L^2(0,T;L^2(\Omega)} \tag{2.29}$$
$$\leqslant C' \|u^n\|_{L^2(0,T;H^1(\Omega))} \|b^n\|_{L^2(0,T;B)} + \sqrt{T} \|\mathrm{curl}\, b^n\|_{L^2(0,T;L^2(\Omega)} \leqslant M \tag{2.30}$$

using the energy bound. Back to the previous inequality, we find that:

$$|\tau|\|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)} \leqslant C\left(\|\mathrm{curl}\ \widehat{b^n}(\tau)\|_{L^2(\Omega)} + \|\widehat{b^n}(\tau)\|_{L^2(\mathbb{R}^3)}\right) \leqslant C\|\widehat{b^n}(\tau)\|_B. \tag{2.31}$$

From there, we can conclude exactly as in [56]: we notice that for $\gamma < \frac{1}{2}$,

$$|\tau|^{2\gamma} \leqslant C_\gamma \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}$$

so that combining this inequality with estimate (2.31):

$$\begin{aligned}\int_{\mathbb{R}} |\tau|^{2\gamma}\|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau &\leqslant C\int_{\mathbb{R}} \frac{\|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)}}{1+|\tau|^{1-2\gamma}}\,d\tau + C\int_{\mathbb{R}}\frac{\|\widehat{b^n}(\tau)\|_B}{1+|\tau|^{1-2\gamma}}d\tau \\ \leqslant C\int_{\mathbb{R}}\|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau &+ C\left(\int_{\mathbb{R}}\|\widehat{b^n}(\tau)\|^2_B d\tau\right)^{1/2}\left(\int_{\mathbb{R}}\frac{1}{(1+|\tau|^{2-4\gamma})}d\tau\right)^{1/2}.\end{aligned}$$

Note that the last integral at the right-hand side is finite due to the constraint $\gamma < \frac{1}{4}$. Moreover, by Plancherel equality and (2.18), we find that

$$\begin{aligned}&\int_{\mathbb{R}}\|\widehat{b^n}(\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \leqslant C\int_0^T \|b^n(t)\|^2_{L^2(\mathbb{R}^3)}\ dt \leqslant M,\\ &\int_{\mathbb{R}}\|\widehat{b^n}(\tau)\|^2_B d\tau \leqslant C\int_0^T \|b^n(t)\|^2_B\ dt \leqslant M,\end{aligned}$$

for some M independent of n. This concludes the proof of the proposition. □

Compactness

Thanks to the bounds obtained in the previous paragraphs, one can conclude the proof of Theorem 2.2.1 by usual compactness arguments. First, we apply Aubin-Lions lemma ([3],[55]): as in the original Leray setting, we use the following triplet of spaces

$$V(\Omega) \underset{\text{compact}}{\hookrightarrow} H(\Omega) \hookrightarrow (V(\Omega))',$$

so that for $p = 2$ and $q = \frac{4}{3}$ one has $E_{2,\frac{4}{3}} = \left\{v \in L^2(0,T;V(\Omega)),\ \frac{dv}{dt} \in L^{\frac{4}{3}}(0,T;V'(\Omega))\right\}$ and thus

$$E_{2,\frac{4}{3}} \underset{\text{compact}}{\hookrightarrow} L^2(0,T;H(\Omega)),$$

which provides the strong convergence of a subsequence of (u^n) in $L^2((0,T)\times\Omega)$.

As regards the magnetic field, we rely on the following theorem (see [56]):

Theorem 2.2.3. *Let $\gamma > 0$. Let X_0, X, X_1 three Hilbert spaces such that*

$$X_0 \underset{compact}{\hookrightarrow} X \hookrightarrow X_1.$$

Let $K \subset \mathbb{R}$ a compact, and

$$\mathcal{H}^\gamma_K(\mathbb{R}, X_0, X_1) := \left\{v \in L^2(\mathbb{R}, X_0),\ \partial_t^\gamma v \in L^2(\mathbb{R}, X_1), \quad v = 0 \text{ outside } K\right\},$$

which is a Hilbert space endowed with the norm $\|v\|^2_{\mathcal{H}^\gamma_K} = \|v\|^2_{L^2(\mathbb{R},X_0)} + \|\partial_t^\gamma v\|^2_{L^2(\mathbb{R},X_1)}$. Then, for all $\gamma > 0$, the imbedding $\mathcal{H}^\gamma_K(\mathbb{R}, X_0, X_1) \hookrightarrow L^2(\mathbb{R}, X)$ is compact.

Now, given an arbitrary smooth bounded domain $\mathcal{O}$ of $\mathbb{R}^3$, we take $K = [0, T]$, $X_0 = H^1(\mathcal{O})$, $X = X_1 = L^2(\mathcal{O})$. Note that the compact imbedding of X_0 into X is ensured by the Rellich theorem (which is valid as soon as $\mathcal{O}$ is Lipschitz). From the analysis of the previous paragraphs, we have, for $\gamma < \frac{1}{4}$ and some $M > 0$, that

$$\|\widetilde{\mathbb{P}b^n}\|_{L^2(\mathbb{R};X_0)} \leqslant \|\mathbb{P}b^n\|_{L^2(0,T;H^1(\mathbb{R}^3))} \leqslant \|b^n\|_{L^2(0,T;B)} \leqslant M \tag{2.32}$$

$$\|\partial_t^\gamma \widetilde{\mathbb{P}b^n}\|_{L^2(\mathbb{R};X_1)} \leqslant M. \tag{2.33}$$

Applying the previous theorem (and a standard diagonal argument), we obtain a subsequence of b^n that converges strongly in $L^2((0,T) \times \mathcal{O})$ for any bounded domain $\mathcal{O}$, notably in $\mathcal{O} = \Omega$.

This strong convergence of (u^n) and (b^n) in $L^2((0,T) \times \Omega)$ (up to a subsequence) allows to take the limit of the non-linear terms:

$$\int_0^T \left(\int_\Omega (\text{curl } b^n) \times \mathbb{P}b^n \cdot e_j\right) \chi(t)dt = \int_0^T \left(\int_\Omega \mathbb{P}b^n \cdot (\mathbb{P}b^n \cdot \nabla e_j)\right) \chi(t)dt,$$

and

$$\int_0^T \left(\int_\Omega (u^n \times \mathbb{P}b^n) \cdot \text{curl } e'_j\right) \chi(t)dt$$

for any $\chi \in C^1_c(0, T)$, for any fixed j. From there, the recovering of the variational formulations i')-ii'), of the energy inequality iii), and of the initial conditions is very standard and left to the reader.

2.3 Additional remarks on the well-posedness of (MHD)

We devoted most of this chapter to the construction of Leray type solutions to the system (MHD). We tried to emphasize the issues and tools associated with the treatment of the different domains occupied by u and b, that are Ω and $\mathbb{R}^3$. Such issues and tools were not met in the treatment of the other MHD type systems found in the literature. Let us stress that our analysis extends also without difficulty to the two-dimensional case, with the usual changes. Note that in the 2d setting, the non-linearity curl $b \times b$ has to be replaced by $(\text{curl } b)\, b^\perp = (\text{curl } b) \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix}$, where the 2d curl is the scalar operator defined by curl $b = \partial_1 b_2 - \partial_2 b_1$. Similarly, the non-linearity curl $(u \times b)$ has to be replaced by $-\nabla^\perp (u \cdot b^\perp)$.

It would be interesting to check if the usual results for Navier-Stokes subsist in this MHD setting: uniqueness of the Leray type solutions and global regularity in 2d, local existence of strong solutions in 3d. Nevertheless, we do not push further the analysis here: in the next sections, we will concentrate on a refined asymptotic analysis of the system, focusing on MHD boundary layer stability. Notably, in the next chapter, we will consider a simplified setting in which:

- the external magnetic field is assumed to be constant: $b = e$ in Ω^c, where e is a unit vector.
- the fluid domain is a half-space: $\Omega = \mathbb{R}^3_+$.

In particular, with the first assumption, the dynamics of the magnetic field will be restricted to the fluid domain Ω:

$$\begin{aligned} \partial_t b - \text{curl } (u \times b) + \frac{1}{\text{Rm}} \text{curl curl } b &= 0 \text{ in } \Omega \\ \text{div } b &= 0 \text{ in } \Omega \\ b|_{\partial\Omega} &= e. \end{aligned} \tag{2.34}$$

We will build approximate solutions of boundary layer type for this set of equations (coupled with the Navier-Stokes one). Let us note that, u being given, solutions to (2.34) are limited, as these linear equations form an overdetermined system. The reason is that the second operator curl curl is not elliptic, because it vanishes on gradient vector fields. In particular, the boundary value problem formed by the first equation and the full Dirichlet condition is in general ill-posed. One could argue that as we restrict to divergence-free vector fields, the term curl curl b in (2.34a) can be replaced by the full Laplacian Δb, and that the boundary value problem

$$\begin{aligned} \partial_t b - \text{curl } (u \times b) - \frac{1}{\text{Rm}} \Delta b &= 0 \text{ in } \Omega \\ b|_{\partial\Omega} &= e. \end{aligned}$$

is well-posed. However, if one works with this formulation, then the preservation of the divergence-free condition on b is not guaranteed anymore. Indeed, one finds in this case that

$$\partial_t \text{div } b - \Delta \text{div } b = 0 \text{ in } \Omega, \quad (\text{ and div } b|_{t=0}) = 0$$

but the boundary condition for div b does not *a priori* remain homogeneous, so that one can not conclude that div $b = 0$. This is in sharp contrast with what happens for the boundary conditions curl $b \cdot n|_{\partial\Omega} = 0$, $b \cdot n|_{\partial\Omega} = 0$, see [15].

Hence, the analysis to come has to be thought as a first step in the discussion of boundary layers for the full system (MHD). Moreover we hope that it can be useful to study the case of a perfect conductor as well.

Chapter 3

Formal Derivation and Stability Analysis of Boundary Layer Models in MHD

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We provide a systematic derivation of boundary layer models in magnetohydrodynamics (MHD), through an asymptotic analysis of the incompressible MHD system. We recover classical linear models, related to the famous Hartmann and Shercliff layers, as well as nonlinear ones, that we call magnetic Prandtl models. We perform their linear stability analysis, emphasizing the stabilizing effect of the magnetic field.

3.1 Introduction

The dynamics of an electrically conducting liquid near a wall has been a topic of constant interest, at least since the pioneering work of Hartmann [26]. It is relevant to many domains of active research, such as dynamo theory [13] or nuclear fusion [57].

An appropriate starting point to describe such dynamics is the classical incompressible MHD system. It is set in an open subset Ω of $\mathbb{R}^3$, modeling the fluid domain. It reads in dimensionless form [12, 23]:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} = S \mathbf{b} \cdot \nabla \mathbf{b}, \\ \partial_t \mathbf{b} - \mathrm{curl}\, (\mathbf{u} \times \mathbf{b}) + \dfrac{1}{\mathrm{Rm}} \mathrm{curl}\, \mathrm{curl}\, \mathbf{b} = 0, \\ \mathrm{div}\, \mathbf{u} = 0, \mathrm{div}\, \mathbf{b} = 0, \quad t > 0, \quad \mathbf{x} \in \Omega. \end{cases} \tag{3.1}$$

The parameters Re and Rm are the hydrodynamic and magnetic Reynolds numbers respectively. The parameter S is the so-called coupling parameter. It is given by

$$S = \frac{B_0^2}{\mu\rho U^2} = \frac{\mathrm{Ha}^2}{\mathrm{Re}\,\mathrm{Rm}}, \quad \text{where } \mathrm{Ha} = B_0 L \left(\frac{\sigma}{\eta}\right)^{1/2}$$

is the Hartmann number. Here, B_0 and U are typical amplitudes for the magnetic and velocity fields, L is a typical length scale of the flow, ρ is the density of the fluid, μ is its magnetic permeability and η is the viscosity coefficient.

Equations in Ω^c and boundary conditions at the interface $\partial\Omega$ depend on the electrical properties of the surrounding medium Ω^c. We focus here on the case of an insulator, so that

$$\mathrm{curl}\,\mathbf{b} = 0, \quad \mathrm{div}\,\mathbf{b} = 0 \quad \text{in } \Omega^c. \tag{3.2}$$

The boundary conditions at $\partial\Omega$ are

$$\mathbf{u} = 0, \quad [\mathbf{b}] = 0 \quad \text{at } \partial\Omega, \tag{3.3}$$

where the bracket refers to the jump of $\mathbf{b}$ across the boundary $\partial\Omega$ (see [24] for more).

For simplicity, we assume a uniform background magnetic field, meaning that $\mathbf{b} = \mathbf{e}$ in Ω^c for some constant vector $\mathbf{e}$. This relation is satisfied for all times if it is satisfied initially. Under this assumption, the MHD system can be recast in Ω only:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} = S\mathbf{b}\cdot\nabla\mathbf{b}, \\ \partial_t \mathbf{b} - \mathrm{curl}\,(\mathbf{u}\times b) + \dfrac{1}{\mathrm{Rm}}\mathrm{curl}\,\mathrm{curl}\,\mathbf{b} = 0, \\ \mathrm{div}\,\mathbf{u} = 0, \mathrm{div}\,\mathbf{b} = 0, \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{b}|_{\partial\Omega} = \mathbf{e}. \end{cases} \tag{3.4}$$

Many MHD flows are characterized by a large hydrodynamic Reynolds number, $\mathrm{Re} \gg 1$. It generates a *boundary layer* near $\partial\Omega$, that is a thin zone of high velocity gradients. The understanding of the boundary layer is a major problem in hydrodynamics, notably in relation to drag computation, or vortex generation. For purely hydrodynamic flows ($S = 0$ in 3.4), a classical model for the boundary layer is the celebrated Prandtl system [48]. However, this model is known to be highly unstable. It is especially true in the presence of an adverse pressure gradient, where reverse flow and boundary layer separation can occur.

It is then very natural to investigate the effect of a magnetic field on such instabilities. The existing results on this issue go both ways:

- On one hand, stabilizing effects were stressed out. For instance, in the context of ideal MHD and plane parallel flows, the action of a parallel magnetic field tightens the region of possible unstable wave speeds [28]. Another more mathematical example is the well-posedness of inviscid hydrostatic equations between two planes, that is restored under the action of a parallel magnetic field [49]. As regards dissipative MHD, similar stability results are known. For instance, in the regime $\mathrm{Ha} \gg 1$, transverse magnetic fields generate boundary layers of Hartmann type, which behave much better than the Prandtl ones [26, 1, 50].

- On the other hand, it was shown that magnetic fields can favour the appearance of inflexion points in the velocity profile [29, 45]. By this loss of concavity, they may generate instabilities, and one could expect earlier separation in the boundary layers.

The purpose of this note is to gain some insight into the analysis of MHD boundary layer models. It is primarily intended to mathematicians, either applied or interested in the theory of fluid PDE's. The goal is twofold. First, we wish to provide a clear picture of the various models available, depending on the asymptotics under consideration, and the orientation of the background field with respect to the wall. Then, we wish to emphasize the stabilizing effect of the magnetic field, through partial linear stability analysis. We hope that this work will serve as a starting point for more complete mathematical and numerical analysis.

The outline of the paper is as follows. We consider the case of a half-space $\Omega = \mathbb{R}^3_+$, and consider both the case of a transverse and tangent background magnetic fields: $\mathbf{e} = \mathbf{e}_z$ and $\mathbf{e} = \mathbf{e}_x$, with $\mathbf{x} = (x, y, z)$. The first part of the paper is a systematic derivation of MHD boundary layer models, depending on the relative scalings of Re, Rm and S. We obtain in this way different sets of equations. They include linear systems, related to the classical Hartmann and Shercliff layers, but also nonlinear ones, that we call *magnetic Prandtl models.*

Such magnetic Prandtl models marry features of the Prandtl equations and the Hartmann/Shercliff ones. They are interesting mathematically, because their well-posedness is unclear. Indeed, contrary to Navier-Stokes, such asymptotic models do not retain tangential diffusion. Therefore, the control of high tangential frequencies is an issue. Note that this difficulty already occurs in the classical Prandtl equation, whose well-posedness properties have been satisfactorily understood only recently [18, 20, 44, 2, 19, 40, 10]. In particular, for general smooth initial data, without monotonicity assumption, local well-posedness fails: it only holds under Gevrey regularity in x of the data, that is under strong localization in frequency.

In light of these results, we discuss in the second part of the paper the well-posedness of the magnetic Prandtl models. Namely, we study linearizations around shear flows, and their stability with respect to high frequencies. *We notably show that for tangential magnetic fields, linearizations around non-monotonic shear flows are well-posed in Sobolev spaces.* This is in sharp contrast with the Prandtl equation, which is known to be ill-posed in Sobolev spaces. Hence, while tangential magnetic fields create inflexion points in the velocity profiles, as advocated in [29, 45], they may at the same time suppress hydrodynamic instabilities.

3.2 Derivation of MHD layers

We wish to study solutions of 3.4 that are of boundary layer type, and to find which reduced models they satisfy, depending on the relative values of parameters Re, Rm and S. Obviously, we always assume that $\mathrm{Re} \gg 1$, which is necessary for the generation of a boundary layer. In the case $S = 0$, that is in the purely hydrodynamic regime, it is well-known that a formal asymptotics leads to the so-called Prandtl equation. But of course, our goal here is rather to emphasize the role of magnetic effects in the boundary layer: we are interested in models that couple equations on $\mathbf{u}$ and $\mathbf{b}$. Let us also stress that in most applications, the magnetic Reynolds number is usually smaller than the hydrodynamic one, so that we always assume:

$$\mathrm{Re} \gg 1, \quad \mathrm{Rm} \lesssim \mathrm{Re}\,. \tag{3.5}$$

For simplicity, we further restrict to a simple geometry, namely the half-space $\Omega = \{z > 0\}$. Nevertheless, we believe that our analysis could extend to curved bondaries (through the introduction of curvilinear and transverse coordinates near the boundary). We distinguish between the case of a transverse background magnetic field $\mathbf{e} = \mathbf{e}_z$ and a tangent background magnetic field, say $\mathbf{e} = \mathbf{e}_x$.

3.2.1 Layers under a transverse magnetic field

We consider here solutions of 3.4 behaving like:

$$\begin{aligned} \mathbf{u} &\approx \Big(u'_x\left(t,x,y,\lambda^{-1}z\right),\, u'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, u'_z\left(t,x,y,\lambda^{-1}z\right)\Big), \\ \mathbf{b} &\approx \mathbf{e} + \delta\Big(b'_x\left(t,x,y,\lambda^{-1}z\right),\, b'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, b'_z\left(t,x,y,\lambda^{-1}z\right)\Big) \end{aligned} \tag{3.6}$$

and similarly for the pressure. The parameter $\lambda \ll 1$ denotes the size of the boundary layer: the profiles $\mathbf{u}' = (\mathbf{u}'_h, u'_z) = (u'_x, u'_y, u'_z)$, p' and $\mathbf{b}' = (\mathbf{b}'_h, b'_z) = (b'_x, b'_y, b'_z)$ depend on a rescaled variable $z' = \lambda^{-1}z$. The parameter $\delta = O(1)$ denotes the typical norm of the magnetic perturbation. Note the rescaling of the vertical components by a factor λ: it is consistent with the divergence-free conditions on u and b.

We insert the expressions (3.6) into 3.4. After dropping the primes, we get

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u} \cdot \nabla \mathbf{u}_h + \nabla_h p - \dfrac{1}{\mathrm{Re}} \left(\Delta_h + \lambda^{-2} \partial_z^2\right) \mathbf{u}_h = \dfrac{S\delta}{\lambda} \partial_z \mathbf{b}_h + S\delta^2 \mathbf{b} \cdot \nabla \mathbf{b}_h, \\ \partial_t u_z + \mathbf{u} \cdot \nabla u_z + \lambda^{-2} \partial_z p - \dfrac{1}{\mathrm{Re}} \left(\Delta_h + \lambda^{-2} \partial_z^2\right) u_z = \dfrac{S\delta}{\lambda} \partial_z b_z + S\delta^2 \mathbf{b} \cdot \nabla b_z, \\ \partial_t \mathbf{b}_h - (\delta\lambda)^{-1} \partial_z \mathbf{u}_h - \left(\mathrm{curl}\, (\mathbf{u} \times \mathbf{b})\right)_h + \dfrac{1}{\mathrm{Rm}} \nabla_h \mathrm{div}\, \mathbf{b} - \dfrac{1}{\mathrm{Rm}} \left(\Delta_h + \lambda^{-2} \partial_z^2\right) \mathbf{b}_h = 0, \\ \partial_t b_z - (\delta\lambda)^{-1} \partial_z u_z - \left(\mathrm{curl}\, (\mathbf{u} \times \mathbf{b})\right)_z + \dfrac{1}{\mathrm{Rm}\, \lambda^2} \partial_z \mathrm{div}\, \mathbf{b} - \dfrac{1}{\mathrm{Rm}} \left(\Delta_h + \lambda^{-2} \partial_z^2\right) b_z = 0, \\ \mathrm{div}\, \mathbf{u} = \mathrm{div}\, \mathbf{b} = 0, \end{cases} \tag{3.7}$$

where the substrict h above refers to horizontal components or variables:

$$\mathbf{f}_h = (f_x, f_y), \quad \nabla_h = (\partial_x, \partial_y), \quad \Delta_h = \partial_x^2 + \partial_y^2.$$

The equations are completed by the Dirichlet conditions

$$\mathbf{u} = \mathbf{b} = 0 \text{ at } z = 0. \tag{3.8}$$

Moreover, we expect vertical variations of the boundary layer solutions to be localized near $z = 0$. Therefore, we impose that $\mathbf{u}_h$ and $\mathbf{b}_h$ have a limit as $z \to +\infty$. We denote by $\mathbf{u}_h^\infty$ and $\mathbf{b}_h^\infty$ such limits. We also impose that the z derivatives of $\mathbf{u}_h$ and $\mathbf{b}_h$ decay to zero:

$$(\mathbf{u}_h, \mathbf{b}_h) \to (\mathbf{u}_h^\infty, \mathbf{b}_h^\infty), \quad \partial_z^k (\mathbf{u}_h, \mathbf{b}_h) \to (0, 0), \quad \forall k \geqslant 1, \quad \text{as } z \to +\infty. \tag{3.9}$$

Note that once $\mathbf{u}_h$ and $\mathbf{b}_h$ are determined, the divergence-free conditions and Dirichlet conditions (3.8) fully determine u_z and b_z. From the condition (3.9), they should be at most $O(z)$ at infinity. Note that equivalently, b_z can be determined by equation (3.7d). This follows easily from the well-known fact that the divergence-free condition is preserved by the evolution equations (3.7c,d). Indeed, taking the divergence of (3.7c,d), we get $\partial_t \mathrm{div}\, \mathbf{b} = 0$ in Ω.

Hartmann regime. The first case is when $\frac{S\delta}{\lambda} \gg 1$. Then, the term $\frac{S\delta}{\lambda} \partial_z \mathbf{b}_h$ in (3.7a) is diverging. It must be balanced by the term coming from diffusion in z. We must also keep *a priori* the horizontal pressure gradient, whose amplitude in the layer is unknown. Retaining these leading order terms, we get

$$\nabla_h p - \frac{1}{\mathrm{Re}} \lambda^{-2} \partial_z^2 \mathbf{u}_h = \frac{S\delta}{\lambda} \partial_z \mathbf{b}_h, \tag{3.10}$$

which yields in particular that

$$\frac{1}{\mathrm{Re}} \lambda^{-2} \sim \frac{S\delta}{\lambda}. \tag{3.11}$$

With this balance and the assumption Re $\gg 1$, the second equation (3.7b) yields at leading order: $\partial_Z p = 0$. We recover the classical fact that the pressure is constant in boundary layers. Back to (3.10), we can send z to infinity and use (3.9) to deduce that $\nabla_h p = 0$ and

$$-\frac{1}{\mathrm{Re}}\lambda^{-2}\partial_z^2\mathbf{u}_h = \frac{S\delta}{\lambda}\partial_z\mathbf{b}_h. \tag{3.12}$$

Similarly, in (3.7c), the only term that can balance $\lambda^{-1}\partial_z\mathbf{u}_h$ is the term coming from diffusion in z. Retaining these two terms we get

$$-(\delta\lambda)^{-1}\partial_z\mathbf{u}_h - \frac{1}{\mathrm{Rm}}\lambda^{-2}\partial_z^2\mathbf{b}_h = 0, \tag{3.13}$$

so that

$$(\delta\lambda)^{-1} \sim \frac{1}{\mathrm{Rm}}\lambda^{-2}. \tag{3.14}$$

Combining (3.11) and (3.14), we get

$$\lambda^2 \sim \frac{1}{\mathrm{Re}\ \mathrm{Rm}\, S} \sim \mathrm{Ha}^{-2}.$$

Hence, *the typical size of the layer is Ha^{-1}*. We set

$$\lambda = \mathrm{Ha}^{-1}, \quad \delta = \mathrm{Rm}\,\mathrm{Ha}^{-1}. \tag{3.15}$$

The previous equations (3.12)-(3.13) on $\mathbf{u}_h, \mathbf{b}_h$ simplify into

$$\partial_z^2\mathbf{u}_h + \partial_z\mathbf{b}_h = 0, \quad \partial_z\mathbf{u}_h + \partial_z^2\mathbf{b}_h = 0 \tag{3.16}$$

which yields

$$-\partial_z^3\mathbf{u}_h = -\partial_z\mathbf{u}_h.$$

From the boundary conditions, we deduce

$$\mathbf{u}_h \;=\; (1-e^{-z})\mathbf{u}_h^\infty, \quad \mathbf{b}_h = (1-e^{-z})\mathbf{u}_h^\infty \tag{3.17}$$

or

$$\mathbf{u}_h \;=\; (1-e^{-\mathrm{Ha}\,z})\mathbf{u}_h^\infty, \quad \mathbf{b}_h = (1-e^{-\mathrm{Ha}\,z})\mathbf{u}_h^\infty.$$

in the original z variable. These are the classical Hartmann profiles.

Remark 3.2.1. As the focus of our note is on boundary layers, we do not adress the dynamics of the limits at infinity $\mathbf{u}_h^\infty(t,x,y)$ and $\mathbf{b}_h^\infty(t,x,y)$. In a full analysis of 3.4, these limits appear as the boundary values of velocity and magnetic fields $\mathbf{u}_h^{int}$ and $\mathbf{b}_h^{int}$, describing the (horizontal) dynamics away from the boundary layer. Hence, they are not arbitrary, but constrained by equations 3.4 and the solvability of the boundary layer. For instance, in the Hartmann case, we see from (3.17) that one condition is $\mathbf{b}_h^\infty = \mathbf{u}_h^\infty$.

Remark 3.2.2. To be consistent, the derivation of the Hartmann boundary layer requires *a priori* some assumptions on the parameters. The first requirement is of course that the size λ of the layer be small, or equivalently $\mathrm{Ha} \gg 1$. Also, we assumed that $\frac{S\delta}{\lambda} \gg 1$, that is $\frac{\mathrm{Ha}^2}{\mathrm{Re}} \gg 1$. Eventually, the condition $\delta = O(1)$ means $\mathrm{Rm}\,\mathrm{Ha}^{-1} = O(1)$. Note however that this last condition on δ is not needed in the derivation of the Hartmann equations: a sufficient condition is that $\frac{S\delta}{\lambda} \gg S\delta^2$ and $(\delta\lambda)^{-1} \gg 1$. Both conditions come down to $\mathrm{Ha}^2 \gg \mathrm{Rm}$, which is automatically satisfied if $\mathrm{Ha}^2 \gg \mathrm{Re}$ and $\mathrm{Rm} \lesssim \mathrm{Re}$ (see (3.5)). Note also that these assumptions can be sometimes relaxed. For instance, in the case where $\mathbf{u}_h^\infty$ is constant, one can check that the Hartmann profiles (3.17) are exact solutions of the full system (3.7) (with $u_z = b_z = 0$).

Mixed Prandtl/Hartmann regime. The second case is when $\frac{S\delta}{\lambda} \sim 1$. In this case, the convective term in the equation for $\mathbf{u}_h$ can no longer be neglected. Hence, the leading order dynamics reads:

$$\partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \frac{1}{\mathrm{Re}}\lambda^{-2}\partial_z^2\mathbf{u}_h = \frac{S\delta}{\lambda}\partial_z \mathbf{b}_h, \tag{3.18}$$

Meanwhile, the induction equation still yields the same balance:

$$-(\delta\lambda)^{-1}\partial_z\mathbf{u}_h - \frac{1}{\mathrm{Rm}}\lambda^{-2}\partial_z^2\mathbf{b}_h = 0,$$

or after integration in z:

$$-(\delta\lambda)^{-1}(\mathbf{u}_h - \mathbf{u}_h^\infty) - \frac{1}{\mathrm{Rm}}\lambda^{-2}\partial_z\mathbf{b}_h = 0. \tag{3.19}$$

As before, we can take $\lambda = \mathrm{Ha}^{-1}$. Note that $\frac{1}{\mathrm{Re}}\lambda^{-2} \sim \frac{S\delta}{\lambda} \sim 1$, giving the extra condition

$$\lambda^2 \sim \mathrm{Re}^{-1}, \text{ or } \mathrm{Ha} \sim \sqrt{\mathrm{Re}}.$$

Moreover, the equation for the vertical velocity component gives at leading order: $\partial_z p = 0$. Eventually, substituting (3.19) in (3.18), we obtain the system

$$\begin{cases} \partial_t\mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \frac{\mathrm{Ha}^2}{\mathrm{Re}}\partial_z^2\mathbf{u}_h + \frac{\mathrm{Ha}^2}{\mathrm{Re}}(\mathbf{u}_h - \mathbf{u}_h^\infty) = 0, \\ \partial_z p = 0, \\ \mathrm{div}_h\,\mathbf{u}_h + \partial_z u_z = 0. \end{cases} \tag{3.20}$$

We recognize a nonlinear Prandtl type equation, *with an extra magnetic damping term.* This model belongs to what we called in the introduction *magnetic Prandtl models*, mixing features of Prandtl and Hartmann dynamics.

3.2.2 Layers in a tangent magnetic field.

In this section, we consider the case of a tangent background magnetic field $\mathbf{b} = \mathbf{e}_x$. As the MHD system is invariant through horizontal rotation, the choice of $\mathbf{e}_x$ is no loss of generality. Proceeding as before, we look for approximate solutions of the type

$$\begin{aligned} \mathbf{u} &\approx \Big(u'_x\left(t,x,y,\lambda^{-1}z\right),\, u'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, u'_z\left(t,x,y,\lambda^{-1}z\right)\Big), \\ \mathbf{b} &\approx \mathbf{e}_x + \delta\Big(b'_x\left(t,x,y,\lambda^{-1}z\right),\, b'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, b'_z\left(t,x,y,\lambda^{-1}z\right)\Big). \end{aligned} \tag{3.21}$$

By plugging these approximations in the MHD equations, we have this time:

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \dfrac{1}{\mathrm{Re}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)\mathbf{u}_h = S\delta\partial_x \mathbf{b}_h + S\delta^2 \mathbf{b}\cdot\nabla\mathbf{b}_h, \\ \partial_t u_z + \mathbf{u}\cdot\nabla u_z + \lambda^{-2}\partial_z p - \dfrac{1}{\mathrm{Re}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)u_z = S\delta\partial_x b_z + S\delta^2 \mathbf{b}\cdot\nabla b_z, \\ \partial_t \mathbf{b}_h - \delta^{-1}\partial_x \mathbf{u}_h - \left(\mathrm{curl}\,(\mathbf{u}\times\mathbf{b})\right)_h + \dfrac{1}{\mathrm{Rm}}\nabla_h \mathrm{div}\,\mathbf{b} - \dfrac{1}{\mathrm{Rm}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)\mathbf{b}_h = 0, \\ \partial_t b_z - \delta^{-1}\partial_x u_z - \left(\mathrm{curl}\,(\mathbf{u}\times\mathbf{b})\right)_z + \dfrac{1}{\mathrm{Rm}\,\lambda^2}\partial_z \mathrm{div}\,\mathbf{b} - \dfrac{1}{\mathrm{Rm}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)b_z = 0, \\ \mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{b} = 0. \end{cases} \tag{3.22}$$

This system is still completed by (3.8)-(3.9). Note that when $\delta \sim 1$, the last two terms at the right-hand side of (3.22a,b) have the same amplitude. The same remark applies to the terms $\delta^{-1}\partial_x \mathbf{u}$ and curl $(\mathbf{u}\times\mathbf{b})$, see the third and fourth equations. In other words, when $\delta \sim 1$, the perturbative writing (3.21b) is somehow artificial, and should be replaced by

$$\mathbf{b} \approx \Big(b'_x\left(t,x,y,\lambda^{-1}z\right),\, b'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, b'_z\left(t,x,y,\lambda^{-1}z\right)\Big).$$

We shall consider this non perturbative regime at the end of the section.

Shercliff regime. We consider here that

$$\delta \ll 1, \quad S\delta \gg 1.$$

In the equation for $\mathbf{u}_h$, the diffusion in z and the horizontal pressure gradient can balance the linearized Lorentz force $S\delta\partial_x \mathbf{b}_h$. The reduced dynamics reads

$$\nabla_h p - \frac{1}{\mathrm{Re}}\lambda^{-2}\partial_z^2 \mathbf{u}_h = S\delta\partial_x \mathbf{b}_h \tag{3.23}$$

and in particular

$$\frac{1}{\mathrm{Re}}\lambda^{-2} \sim S\delta. \tag{3.24}$$

Like in the Hartmann regime, the second equation yields at leading order $\partial_z p = 0$. Taking into account (3.9), we then rewrite equation (3.23) as

$$S\delta\partial_x \mathbf{b}_h^\infty - \frac{1}{\mathrm{Re}}\lambda^{-2}\partial_z^2 \mathbf{u}_h = S\delta\partial_x \mathbf{b}_h. \tag{3.25}$$

Similarly, in the equation for $\mathbf{b}_h$, only the magnetic diffusion in z can balance $-\delta^{-1}\partial_x \mathbf{u}_h$. We find

$$\delta^{-1}\partial_x \mathbf{u}_h + \frac{1}{\mathrm{Rm}}\lambda^{-2}\partial_z^2 \mathbf{b}_h = 0$$

and in particular

$$\delta^{-1} \sim \frac{1}{\mathrm{Rm}}\lambda^{-2}. \tag{3.26}$$

Combining (3.24) and (3.26) yields $\lambda^4 \sim \mathrm{Ha}^{-2}$. We set

$$\lambda = \mathrm{Ha}^{-1/2}.$$

The previous equations resume to

$$\partial_x(\mathbf{b}_h - \mathbf{b}_h^\infty) + \partial_z^2 \mathbf{u}_h = 0, \quad \partial_x \mathbf{u}_h + \partial_z^2 \mathbf{b}_h = 0. \tag{3.27}$$

These equations describe the so-called Shercliff layer, of typical size $\mathrm{Ha}^{-1/2}$ [54]. In the half-space case, they can be solved by taking the Fourier transform in x. Accounting for (3.8)-(3.9), we find

$$\begin{aligned}
\widehat{\mathbf{u}_h}(\xi, z) &= -i\frac{\xi}{|\xi|}\widehat{\mathbf{b}_h^\infty}(\xi)e^{-\sqrt{\frac{|\xi|}{2}}z}\sin\left(\sqrt{\frac{|\xi|}{2}}z\right), \\
\widehat{\mathbf{b}_h}(\xi, z) &= \widehat{\mathbf{b}_h^\infty}(\xi)\left(1 - e^{-\sqrt{\frac{|\xi|}{2}}z}\cos\left(\sqrt{\frac{|\xi|}{2}}z\right)\right).
\end{aligned}$$

Remark 3.2.3. In this derivation, we assumed implicitly that $\lambda \ll 1$, that is $\mathrm{Ha} \gg 1$. Also, we assumed that $\delta \ll 1$, which amounts to $\mathrm{Rm}\,\mathrm{Ha}^{-1} \ll 1$, as well as $S\delta \gg 1$, which amounts to $\mathrm{Ha} \gg \mathrm{Re}$. Taking (3.5) into account, the constraint $\mathrm{Ha} \gg \mathrm{Re}$ is the more stringent.

Mixed Prandtl/Shercliff regime. We still assume here that $\delta \ll 1$, but $S\delta \sim 1$. One must then retain all terms of order one in the equation for $\mathbf{u}_h$, namely

$$\partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla \mathbf{u}_h + \nabla_h p - \frac{1}{\mathrm{Re}}\lambda^{-2}\partial_z^2 \mathbf{u}_h = S\delta\partial_x \mathbf{b}_h.$$

The leading order terms in the equation for $\mathbf{b}_h$ remain the same:

$$\delta^{-1}\partial_x \mathbf{u}_h + \frac{1}{\mathrm{Rm}}\lambda^{-2}\partial_z^2 \mathbf{b}_h = 0.$$

It is therefore legitimate to maintain the same definition for the boundary layer size, that is $\lambda = \mathrm{Ha}^{-1/2}$. As $\frac{1}{\mathrm{Re}}\lambda^{-2} \sim S\delta \sim 1$, the regime that we investigate here corresponds to

$$\mathrm{Re} \sim \mathrm{Ha}\,.$$

We finally obtain the following boundary layer system:

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 \mathbf{u}_h = \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_x \mathbf{b}_h, \\ \partial_z p = 0, \\ \partial_x \mathbf{u}_h + \partial_z^2 \mathbf{b}_h = 0, \\ \mathrm{div}\, \mathbf{u} = 0. \end{cases} \tag{3.28}$$

This is a mixed Prandtl/Shercliff system.

Fully nonlinear MHD layer. We eventually consider the case where the perturbation to the constant magnetic field $\mathbf{e}_x$ is of size one. In such setting, distinguishing between $\mathbf{e}_x$ and its perturbation is artificial. One rather looks directly for

$$\mathbf{b} \approx \Big(b'_x\left(t,x,y,\lambda^{-1}z\right),\, b'_y\left(t,x,y,\lambda^{-1}z\right),\, \lambda\, b'_z\left(t,x,y,\lambda^{-1}z\right)\Big).$$

We plug this new expansion into (3.4), to obtain

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \dfrac{1}{\mathrm{Re}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)\mathbf{u}_h = S\mathbf{b}\cdot\nabla\mathbf{b}_h, \\ \partial_t u_z + \mathbf{u}\cdot\nabla u_z + \lambda^{-2}\partial_z p - \dfrac{1}{\mathrm{Re}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)u_z = S\mathbf{b}\cdot\nabla b_z, \\ \partial_t \mathbf{b}_h - \left(\mathrm{curl}\,(\mathbf{u}\times\mathbf{b})\right)_h + \dfrac{1}{\mathrm{Rm}}\nabla_h \mathrm{div}\,\mathbf{b} - \dfrac{1}{\mathrm{Rm}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)\mathbf{b}_h = 0, \\ \partial_t b_z - \left(\mathrm{curl}\,(\mathbf{u}\times\mathbf{b})\right)_z + \dfrac{1}{\mathrm{Rm}\,\lambda^2}\partial_z \mathrm{div}\,\mathbf{b} - \dfrac{1}{\mathrm{Rm}}\left(\Delta_h + \lambda^{-2}\partial_z^2\right)b_z = 0, \\ \mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{b} = 0. \end{cases} \tag{3.29}$$

We stress that the Dirichlet conditions are now

$$\mathbf{u} = 0, \quad \mathbf{b} = \mathbf{e}_x \quad \text{at } z = 0. \tag{3.30}$$

Let us first consider the case $S \gg 1$. On one hand, the contribution of the Lorentz force diverges in (3.29a), and is expected to be balanced by the diffusion in z, resulting in

$$\frac{1}{\mathrm{Re}}\lambda^{-2} \sim S \gg 1.$$

On the other hand, looking at the equation (3.29c), we see that

$$\frac{1}{\mathrm{Rm}}\lambda^{-2} \lesssim 1$$

otherwise the dynamics of $\mathbf{b}_h$ would be trivial. But the constraints $\frac{1}{\mathrm{Re}}\lambda^{-2} \gg 1$ and $\frac{1}{\mathrm{Rm}}\lambda^{-2} \lesssim 1$ are incompatible with (3.5).

The only relevant case is therefore $S \sim 1$: the case $S \ll 1$, leading to the usual Prandtl equation, does not exhibit any magnetic effect. To be consistent with the Dirichlet conditions, the reduced boundary layer model should contain diffusion terms for both the velocity and the magnetic field. This is possible under the two conditions

$$\frac{1}{\mathrm{Re}}\lambda^{-2} \sim S \sim 1, \quad \frac{1}{\mathrm{Rm}}\lambda^{-2} \sim 1$$

which imply

$$\mathrm{Re} \sim \mathrm{Rm} \sim \mathrm{Ha}\,, \quad \lambda \sim \frac{1}{\sqrt{\mathrm{Re}}}.$$

We set $\lambda = \frac{1}{\sqrt{\mathrm{Re}}}$. We find the MHD boundary layer system

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \partial_z^2 \mathbf{u}_h = S\mathbf{b}\cdot\nabla\mathbf{b}_h, \\ \partial_z p = 0, \\ \partial_t \mathbf{b}_h - (\mathrm{curl}\,(\mathbf{u}\times\mathbf{b}))_h - \frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2 \mathbf{b}_h = 0, \\ \partial_t b_z - (\mathrm{curl}\,(\mathbf{u}\times\mathbf{b}))_z + \frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z \mathrm{div}\,\mathbf{b} - \frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2 b_z = 0, \\ \mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{b} = 0. \end{cases}$$

As discussed before, the divergence-free condition on $\mathbf{b}$ is preserved by the evolution equation on $(\mathbf{b}_h, b_z)$, so that we can get rid of the equation $\mathrm{div}\,\mathbf{b} = 0$ in the previous system. On the contrary, if we keep this equation, we can set the term $\frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z \mathrm{div}\,\mathbf{b}$ to zero in the equation for b_z, and the MHD boundary layer system then reads

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \nabla_h p - \partial_z^2 \mathbf{u}_h = S\mathbf{b}\cdot\nabla\mathbf{b}_h, \\ \partial_z p = 0, \\ \partial_t \mathbf{b} - (\mathrm{curl}\,(\mathbf{u}\times\mathbf{b})) - \frac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2 \mathbf{b} = 0, \\ \mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{b} = 0. \end{cases} \tag{3.31}$$

Remark 3.2.4. The derivation of (3.31) as an asymptotic boundary layer model is only valid under stringent assumptions on the coupling parameter and the Reynolds numbers:

$$\mathrm{Re} \sim \mathrm{Rm} \sim \mathrm{Ha} \gg 1.$$

	Linear models	Nonlinear models	
Transverse field (Layer size Ha^{-1})	$\mathrm{Ha}^2 \gg \mathrm{Re}$ Hartmann, *cf* (3.16).	$\mathrm{Ha}^2 \sim \mathrm{Re}$ damped Prandtl, *cf* (4.1).	
Tangent field (Layer size $\mathrm{Ha}^{-1/2}$)	$\mathrm{Ha} \gg \mathrm{Re}$ Shercliff , *cf* (3.27).	$\mathrm{Ha} \sim \mathrm{Re} \gg \mathrm{Rm}$ mixed Prandtl/Shercliff, *cf* (3.28).	$\mathrm{Ha} \sim \mathrm{Re} \sim \mathrm{Rm}$ fully nonlinear, *cf* (3.31).

Still, compared to the two models derived earlier (the Shercliff and Prandtl/Shercliff systems), it is the one that retains most terms from the original system (3.4). The other two can be seen as degeneracies from it.

3.2.3 Summary of the formal derivation

To gather the results of the previous paragraphs, we draw the following table, that relates the various boundary layer models to the various asymptotic regimes and to the orientation of the magnetic field.

3.3 Linear Stability

The previous derivation is of course formal. It assumes the existence of solutions of (3.4) that take the approximate form (3.6) and (3.21). To ground this idea on rigorous arguments, two further steps are needed:

- To show that the reduced boundary layer models are well-posed, at least locally in time, so that boundary layer expansions can be built.
- To show that once they are built, these expansions are good approximations of exact MHD solutions, over some reasonable time. This is a stability issue within the MHD system 3.4.

We shall provide here elements for the first step only. For simplicity, we will assume invariance with respect to y, and restrict in this way to two-dimensional boundary layer models: $x \in \mathbb{T}$, $z > 0$. Let us note that for the classical 2D Prandtl system, with velocity field $\mathbf{u} = (u, v)$,

$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u + \partial_x p &= 0, \\ \partial_z p &= 0, \\ \partial_x u + \partial_z v &= 0, \\ u|_{z=0} = v|_{z=0} &= 0, \\ u \to u^\infty, \quad p \to p^\infty \quad &\text{as } z \to +\infty, \end{aligned} \tag{3.32}$$

the well-posedness theory is already difficult, and was only recently well-understood. To explain the underlying difficulties, it is worth considering simple linearizations, say around shear flows: $u = U(z), v = 0$. Linearized Prandtl then reads

$$\begin{aligned} \partial_t u + U \partial_x u + vU' - \partial_z^2 u &= 0, \\ \partial_x u + \partial_z v &= 0, \\ u|_{z=0} = v|_{z=0} &= 0, \\ u \to 0 \quad &\text{as } z \to +\infty, \end{aligned} \tag{3.33}$$

where (u, v) now refers to the perturbation. The main problem comes from the term vU': indeed, in the Prandtl model, v is recovered from u through the divergence-free condition: $v = -\int_0^z \partial_x u$. This is a first order term in u (with respect to variable x), and contrary to the transport term $U\partial_x u$ it has no hyperbolic structure. Hence, no basic energy estimate can be achieved. Indeed, it turns out that the L^2 type well-posedness of (3.33) requires a monotonicity assumption on the velocity profile U. Let us stress that a similar monotonicity assumption is needed on the initial data for the nonlinear system (3.32) to be well-posed in Sobolev spaces, see for instance [44]. On the contrary, when U has a non-degenerate critical point a, system (3.33) is ill-posed in L^2 or Sobolev regularity: it has solutions that behave like

$$u \approx e^{ikx} e^{i\omega(k)t} U_k(z), \qquad \text{with } \omega(k) = -kU(a) + \sqrt{|k|}\tau, \quad \Im\tau < 0, \quad |k| \gg 1,$$

see [11, 18]. Hence, it admits unstable modes whose growth rate is proportional to the square root of the wave number k. As a consequence, the only functional settings that can be preserved by the Prandtl evolution in small time are made of functions highly localized in frequency: their Fourier mode k in x should decay at least like $e^{-\delta\sqrt{|k|}}$ for some $\delta > 0$. This corresponds to Gevrey 2 regularity in x. Accordingly, local well-posedness results in such Gevrey classes were obtained recently for the full Prandtl system: see [19, 40, 10].

On the basis of these results in the hydrodynamic case, it is very interesting to investigate the effect of the magnetic field on boundary layer stability, and notably the well-posedness of MHD boundary layer models. Following the previous sections, we can distinguish between linear and nonlinear models. The two linear models that we have derived are the Hartmann system (3.16) and the Shercliff system (3.27). They do not raise any mathematical difficulty. System (3.16) is made of ODEs in variable z, and can be solved explicitly. The same is true for (3.27) after Fourier transform in variable x. The variable t is only a parameter and appears through the functions $\mathbf{u}_h$ and $\mathbf{b}_h$, that is through the dynamics outside the boundary layer.

From the point of view of well-posedness, the interesting systems are the nonlinear ones, that mix Prandtl and magnetic features. We call them magnetic Prandtl models. They correspond to equations (4.1) (with background transverse magnetic field $\mathbf{e} = \mathbf{e}_z$), (3.28) and (3.31) (with background tangential magnetic field $\mathbf{e} = \mathbf{e}_x$). We shall discuss their well-posedness properties in the next section. As explained above, *we shall restrict to the 2D case in variables* (x, z)*, with* $\mathbf{u} = (u, v)$*,* $\mathbf{b} = (b, c)$. The 3D case could carry additional difficulties, see [42] in the classical Prandtl case.

3.3.1 Mixed Prandtl/Hartmann regime

The 2D version of (4.1) reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \dfrac{\mathrm{Ha}^2}{\mathrm{Re}}\partial_z^2 u + \dfrac{\mathrm{Ha}^2}{\mathrm{Re}}(u - u^\infty) = -\partial_x p^\infty, \\ \partial_x u + \partial_z v = 0, \\ u \to u^\infty \text{ as } z \to +\infty, \quad u|_{z=0} = v|_{z=0} = 0. \end{cases} \tag{3.34}$$

We recall that u^∞, p^∞ are known functions of t and x, which are the trace of an Euler flow: they satisfy

$$\partial_t u^\infty + u^\infty \partial_x u^\infty = -\partial_x p^\infty.$$

The only difference with the usual Prandtl system is the damping $\frac{\mathrm{Ha}^2}{\mathrm{Re}}(u - u^\infty)$. This damping does not affect the usual well-posedness theory (or in other words the stability properties of high frequencies). A close look at papers [19, 40, 18] shows that both the Gevrey well-posedness results and the Sobolev ill-posedness results apply to (3.34).

3.3.2 Mixed Prandtl/Shercliff regime

The 2D version of (3.28) reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 u = \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_x b - \partial_x p^\infty, \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0, \\ u \to u^\infty, \quad b \to b^\infty, \quad \text{as } z \to +\infty. \end{cases} \tag{3.35}$$

Contrary to the simple damping term due to a transverse magnetic field, the effect created by a tangential magnetic field is more subtle. Strikingly, in the context of (4.5), it is stabilizing. To provide a clear illustration of this fact, we restrict ourselves to a simple linearization, namely around

$$u = U(z), \quad v = 0, \quad b = b^\infty \text{ constant.}$$

We assume that U connects 0 at $z = 0$ to some constant u^∞ at infinity. The linearized system reads

$$\begin{cases} \partial_t u + U\partial_x u + vU' - \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 u = \dfrac{\mathrm{Ha}}{\mathrm{Re}}\partial_x b, \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0, \ (u,b) \to 0 \qquad \text{as } z \to +\infty. \end{cases} \tag{3.36}$$

Our aim is to prove good *a priori* estimates for this linear system, in the Sobolev framework. Therefore, we introduce the analogue of vorticity, which in the boundary layer context is simply $\omega = \partial_z u$. Differentiating the first equation with respect to z, we find

$$\partial_t \omega + U\partial_x \omega + vU'' - \frac{\mathrm{Ha}}{\mathrm{Re}}\partial_z^2 \omega = \frac{\mathrm{Ha}}{\mathrm{Re}}\partial_x\partial_z b.$$

We remark that $\partial_z \omega|_{z=0} = \partial_z^2 u|_{z=0} = 0$, as can be seen from evaluating (3.36a) at $z = 0$. Multiplication by ω and integration over $\Omega = \mathbb{T} \times \mathbb{R}_+$ give

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \frac{\mathrm{Ha}}{\mathrm{Re}}\|\partial_z \omega\|_{L^2}^2 = -\int_\Omega U''v\omega + \frac{\mathrm{Ha}}{\mathrm{Re}}\int_\Omega \partial_x\partial_z b\,\omega.$$

The first term at the r.h.s. is bounded by

$$\left|\int_\Omega U''v\omega\right| \leqslant \left\|U''\int_0^z \partial_x u\right\|_{L^2} \|\omega\|_{L^2} \ \leqslant\ 2\|zU''\|_{L^\infty}\|\partial_x u\|_{L^2}\|\omega\|_{L^2},$$

where we assumed implicitly that $z \to zU''$ is bounded and applied the Hardy inequality to the first factor. As regards the additional term, we use the second equation to get

$$\int_\Omega \partial_x\partial_z b\,\omega = -\int_\Omega \partial_x\partial_z^2 b\,u = \int_\Omega \partial_x^2 u\,u = -\int_\Omega |\partial_x u|^2.$$

Hence, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \frac{\mathrm{Ha}}{\mathrm{Re}}\left(\|\partial_z \omega\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2\right) \leqslant 2\|zU''\|_{L^\infty}\|\partial_x u\|_{L^2}\|\omega\|_{L^2}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \frac{\mathrm{Ha}}{2\mathrm{Re}}\left(\|\partial_z \omega\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2\right) \leqslant C\|\omega\|_{L^2}^2, \tag{3.37}$$

with $C = 2\|zU''\|_{L^\infty}^2\frac{\mathrm{Re}}{\mathrm{Ha}}$. To have some information on u itself rather than ω, we perform another energy estimate directly on (3.36a), which gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{\mathrm{Ha}}{\mathrm{Re}}\|\partial_z u\|_{L^2}^2 = -\int_\Omega U'v\,u + \frac{\mathrm{Ha}}{\mathrm{Re}}\int_\Omega \partial_x b\,u.$$

As previously, we have

$$\left|\int_\Omega U' v\, u\right| \leqslant 2\|zU'\|_{L^\infty}\|\partial_x u\|_{L^2}\|u\|_{L^2}, \quad \int_\Omega \partial_x b\, u = -\int_\Omega |\partial_z b|^2$$

and we end up with

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \frac{\mathrm{Ha}}{\mathrm{Re}}\left(\|\partial_z u\|_{L^2}^2 + \|\partial_z b\|_{L^2}^2\right) \leqslant \|zU'\|_{L^\infty}\left(\|\partial_x u\|_{L^2}^2 + \|u\|_{L^2}^2\right). \tag{3.38}$$

Combining with inequality (3.37), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + (1+\alpha)\|\omega\|_{L^2}^2\right) + \frac{\mathrm{Ha}}{2\mathrm{Re}}\left(\|\partial_z b\|_{L^2}^2 + \|\partial_z \omega\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2\right) \leqslant C'\left(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\right),$$

where $\alpha = \frac{2\mathrm{Re}}{\mathrm{Ha}}\|zU'\|_{L^\infty}$, and $C' = \max(\|zU'\|_{L^\infty}, C(1+\alpha))$. Eventually, with Gronwall inequality:

$$\|\omega(t)\|^2 + \|u(t)\|_{L^2}^2 + \int_0^t (\|\partial_z b\|_{L^2}^2 + \|\partial_z \omega\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2) \leqslant M(\|\omega_0\|_{L^2}^2 + \|u_0\|_{L^2}^2)e^{Mt}, \quad \forall t \geqslant 0, \tag{3.39}$$

where $M > 0$ is large enough. Eventually, to have some more information on b, one can multiply (3.36a) by $\partial_t u$. Integrating over Ω and over $[0,t]$, we get after straightforward manipulations:

$$\int_0^t \|\partial_t u\|_{L^2}^2 - \frac{\mathrm{Ha}}{\mathrm{Re}}\int_0^t\int_\Omega \partial_x b\,\partial_t u \leqslant C\int_0^t (\|\partial_x u\|_L^2 + \|\partial_z \omega\|_{L^2})\|\partial_t u\|_{L^2}.$$

Using (3.36b), we find

$$\int_0^t\int_\Omega \partial_x b\,\partial_t u = \int_0^t\int_\Omega b\,\partial_t\partial_z^2 b = \frac{1}{2}\|\partial_z b(t)\|_{L^2}^2 - \frac{1}{2}\|\partial_z b_0\|_{L^2}^2,$$

and we can conclude that

$$\int_0^t \|\partial_t u\|_{L^2}^2 + \frac{\mathrm{Ha}}{\mathrm{Re}}\|\partial_z b(t)\|_{L^2}^2 \leqslant \frac{\mathrm{Ha}}{\mathrm{Re}}\|\partial_z b_0\|_{L^2}^2 + C^2 M(\|\omega_0\|_{L^2}^2 + \|u_0\|_{L^2}^2)e^{Mt}, \quad \forall t \geqslant 0. \tag{3.40}$$

Let us stress that, from the bounds (3.39) and (3.40), all terms at the l.h.s. of (3.36a) belong to $L^2_{loc}(\mathbb{R}_+, L^2(\Omega))$, and therefore so does the r.h.s. $\partial_x b$. Moreover, $\partial_z b$ belongs to $L^\infty_{loc}(\mathbb{R}_+, L^2(\Omega))$, as seen from (3.40). We recall that b has zero average in $x \in \mathbb{T}$, as deduced easily from (3.36b) and the Dirichlet condition b. It follows that b belongs to $L^2_{loc}(\mathbb{R}_+, H^1(\Omega))$.

These *a priori* estimates, combined with a classical approximation procedure, allow to state the following well-posedness result:

Proposition 3.3.1. *Assume that $U \in W^{2,\infty}(\mathbb{R}_+)$, $zU', zU'' \in L^\infty(\mathbb{R}_+)$. Let $u_0 \in L^2(\Omega)$ s.t.*

$\omega_0 = \partial_z u_0 \in L^2(\Omega)$, $u_0|_{z=0} = 0$. Let $b_0 \in L^2_{loc}(\Omega)$ s.t. $\partial_z b_0 \in L^2(\Omega)$, $b_0|_{z=0} = 0$ and with zero average in x. Then there there exists a unique solution (u, v, b) of (3.36) satisfying (3.39)-(3.40), $(u, b)|_{t=0} = (u_0, b_0)$.

Remark 3.3.1. The main point of the proposition is that it does not involve any monotonicity assumption on the velocity profile U. This is in sharp contrast with the usual Prandtl system and its linearizations. In particular, when U has a non-degenerate critical point, system (3.33) does not admit this kind of solutions, see [20]. The difference comes from the control of $\partial_x u$ provided by the relation of Shercliff type. Let us stress that there is even a regularization effect in x, as no regularity in x is required at initial time.

3.3.3 Fully nonlinear MHD layer

In the specific regime in which Re $\sim$ Rm $\sim$ Ha, the formal model governing the boundary layer is (3.31). Its 2D version reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u = S\mathbf{b}\cdot\nabla b - \partial_x p^\infty, \\ \partial_t \mathbf{b} - \nabla^\perp(\mathbf{u}\times\mathbf{b}) - \dfrac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2\mathbf{b} = 0, \\ \partial_x u + \partial_z v = \mathrm{div}\,\mathbf{b} = 0, \\ u|_{z=0} = v|_{z=0}, \quad \mathbf{b}|_{z=0} = \mathbf{e}_x, \\ u \to u^\infty, \quad b \to b^\infty, \qquad \text{as } z \to +\infty. \end{cases} \tag{3.41}$$

We recall that $\mathbf{u} = (u, v)$ and $\mathbf{b} = (b, c)$ are the 2D velocity and magnetic fields respectively. We also recall that the cross product of $\mathbf{u}$ and $\mathbf{b}$ is a scalar function: $\mathbf{u}\times\mathbf{b} = uc - bv$. To investigate the stability properties of this system, we consider once more a simple linearization, around

$$u = U(z), \quad v = 0, \quad \mathbf{b} = \mathbf{e}_x. \tag{3.42}$$

The linearized equations are

$$\begin{cases} \partial_t u + U\partial_x u + U'\,v - \partial_z^2 u = S\partial_x b, \\ \partial_t \mathbf{b} - \nabla^\perp(v - U\,c) - \dfrac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2\mathbf{b} = 0, \\ \partial_x u + \partial_z v = \mathrm{div}\,\mathbf{b} = 0, \\ u|_{z=0} = v|_{z=0}, \quad \mathbf{b}|_{z=0} = 0, \\ u \to 0, \quad b \to 0, \qquad \text{as } z \to +\infty. \end{cases} \tag{3.43}$$

Here, $\mathbf{u} = (u, v)$ and $\mathbf{b} = (b, c)$ are the perturbations of the reference solution (3.42).

Note that by the conditions $\partial_x b + \partial_z c = 0$, $c|_{z=0} = 0$, c has zero average in x. Moreover, the evolution of the x-average of b is decoupled and solves

$$\partial_t \int_{\mathbb{T}} b - \frac{\mathrm{Re}}{\mathrm{Rm}} \partial_z^2 \int_{\mathbb{T}} b = 0.$$

Hence, *there is no loss of generality in assuming that b has zero average in x as well.* With regards to the divergence-free condition, this means we can write $\mathbf{b} = \nabla^\perp \phi$, for some function ϕ which is periodic with zero average in x. We can then write the second component of (3.43b) as

$$\partial_t \partial_x \phi - \partial_x (v - U\, \partial_x \phi) - \frac{\mathrm{Re}}{\mathrm{Rm}} \partial_z^2 \partial_x \phi = 0$$

or equivalently

$$\partial_t \phi + U \partial_x \phi - v - \frac{\mathrm{Re}}{\mathrm{Rm}} \partial_z^2 \phi = 0. \tag{3.44}$$

This last equation is a key ingredient in the stability analysis of (3.43). The idea is that, combining the equation (3.43a) on u and (3.44), one can get rid of the bad term in v, responsible for the possible loss of one derivative in x. This idea is reminiscent of article [44] about the classical Prandtl equation. In [44], a similar cancellation of the v term was obtained combining the equations on u and $\omega = \partial_y u$. In the linearized setting, the appropriate combination was $g = \omega - \frac{U''}{U'} u$. However, some monotonicity of the velocity profile was needed, in order to divide by U'. *The main point in the present MHD context is that no monotonicity of the velocity profile is needed to obtain well-posedness.* We rather consider the following modified velocity:

$$\tilde{u} = u + U' \phi.$$

Summing (3.43a) and $U' \times$(3.44), we get

$$\partial_t \tilde{u} + U \partial_x \tilde{u} - \partial_z^2 \tilde{u} = S \partial_x b + \frac{\mathrm{Re}}{\mathrm{Rm}} U' \partial_z^2 \phi - \partial_z^2 \left(U' \phi \right), \tag{3.45}$$

while the equation on $b = \mathbf{b} \cdot \mathbf{e}_x$ can be written as

$$\partial_t b + U \partial_x b - \partial_x \tilde{u} - \frac{\mathrm{Re}}{\mathrm{Rm}} \partial_z^2 b = 0. \tag{3.46}$$

Formulation (3.45)-(3.46) is much better behaved than the original formulation, and will allow to establish stability. Indeed, a standard energy estimate yields

$$\frac{d}{dt} \left(\frac{1}{2} \|\tilde{u}\|_{L^2}^2 + \frac{S}{2} \|b\|_{L^2}^2 \right) \; + \; \|\partial_z \tilde{u}\|_{L^2}^2 + \frac{S\mathrm{Re}}{\mathrm{Rm}} \|\partial_z b\|_{L^2}^2 \leqslant \frac{\mathrm{Re}}{\mathrm{Rm}} \int_\Omega U' (\partial_z^2 \phi) \tilde{u} - \int_\Omega \partial_z^2 \left(U' \phi \right) \, \tilde{u},$$

where we have used the identity

$$-S \int_\Omega \partial_x \tilde{u}\, b = S \int_\Omega \partial_x b\, \tilde{u}.$$

To control the r.h.s., we then use that $\partial_z \phi = -b$. In particular,

$$\|\partial_z \phi\|_{L^2} = \|b\|_{L^2}, \quad \|\partial_z^2 \phi\|_{L^2} = \|\partial_z b\|_{L^2}, \quad \|z^{-1}\phi\|_{L^2} \leqslant 2\|b\|_{L^2}.$$

Hence,

$$\frac{d}{dt}\left(\frac{1}{2}\|\tilde{u}\|_{L^2}^2 + \frac{S}{2}\|b\|_{L^2}^2\right) + \|\partial_z \tilde{u}\|_{L^2}^2 + \frac{S\mathrm{Re}}{\mathrm{Rm}}\|\partial_z b\|_{L^2}^2 \leqslant C(\|b\|_{L^2} + \|\partial_z b\|_{L^2})\|\tilde{u}\|_{L^2},$$

where the constant C depends implicitly on $\|U'\|_{L^\infty}$, $\|U''\|_{L^\infty}$, $\|zU'''\|_{L^\infty}$. After application of Young's inequality:

$$\frac{d}{dt}\left(\frac{1}{2}\|\tilde{u}\|_{L^2}^2 + \frac{S}{2}\|b\|_{L^2}^2\right) + \|\partial_z \tilde{u}\|_{L^2}^2 + \frac{S\mathrm{Re}}{2\mathrm{Rm}}\|\partial_z b\|_{L^2}^2 \leqslant C'\left(\|\tilde{u}\|_{L^2}^2 + \|b\|_{L^2}^2\right),$$

Gronwall inequality yields

$$\|\tilde{u}(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left(\|\partial_z \tilde{u}\|_{L^2}^2 + \|\partial_z b\|_{L^2}^2\right) \leqslant M\left(\|\tilde{u}(0)\|_{L^2}^2 + \|b(0)\|_{L^2}^2\right)e^{Mt}, \quad \forall t \geqslant 0$$

where $M > 0$ is large enough. Using $\|(U', U'')\phi\|_{L^2} \leqslant 2\|z(U', U'')\|_{L^\infty}\|b\|_{L^2}$, it follows that

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left(\|\partial_z u\|_{L^2}^2 + \|\partial_z b\|_{L^2}^2\right) \leqslant M'\left(\|u(0)\|_{L^2}^2 + \|b(0)\|_{L^2}^2\right)e^{M't} \quad \forall t \geqslant 0 \tag{3.47}$$

for some M' large enough.

As in the case of system (3.36), we can combine the previous estimate with a standard approximation procedure, and obtain the well-posedness of (3.43):

Proposition 3.3.2. *Assume that $U \in W^{3,\infty}(\mathbb{R}_+)$, $zU', zU'', zU''' \in L^\infty(\mathbb{R}_+)$.Let $u_0 \in L^2(\Omega)$. Let $\phi_0 \in L^2_{loc}(\Omega)$, such that $b_0 = \partial_y \phi_0 \in L^2(\Omega)$, $\phi_0|_{z=0} = 0$ and with zero average in x. Then there exists a unique solution of* (3.43) *satisfying* (3.47)*, $u|_{t=0} = 0$, $\mathbf{b}|_{t=0} = -\nabla^\perp \phi_0$.*

Remark 3.3.2. The velocity and magnetic vertical components v and c provided by this well-posedness proposition have weak regularity with respect to x. For instance, $v = -\int_0^y \partial_x u$ has to be understood as the x derivative of a function in $L^2(\mathbb{T}, H^2_{loc}(\mathbb{R}_+))$. For more regularity, one should impose more x regularity on the data.

Remark 3.3.3. While completing the writing of this work, we got aware of the independent recent work [41] by Cheng-Jie Liu, Feng Xie and Tong Yang. These authors consider the same system as (3.41), with the insulating boundary replaced by a conducting one, which amounts to replacing the condition $b|_{z=0} = 0$ by $\partial_z b|_{z=0} = 0$. They establish well-posedness in Sobolev spaces for the nonlinear system, through a change of unknowns which is a nonlinear analogue of our $\tilde{u}$.

3.4 Conclusion

We achieved a formal derivation and stability analysis of boundary layer models in MHD. This work was motivated by some contradictory results on the stabilizing or destabilizing role of the magnetic field, notably when it is tangent to the boundary. The boundary layer models are in most regimes linear, but for some asymptotics of the parameters, the role of the nonlinearities can not be ignored, leading to models of Prandtl type with extra magnetic features. We investigated the stability to high frequencies of these nonlinear models, restricting to simple linearizations. Our analysis shows that in the case of tangent magnetic fields, the growth rate of high tangential frequencies is no longer growing with the wave number, contrary to what happens for the Prandtl system when the velocity has inflexion points. It favours the idea of stabilization by the magnetic field.

Chapter 4

Analysis of the nonlinear models

In this chapter, following the classification of the MHD boundary layer models of the previous chapter, we detail the study of the three nonlinear ones, namely the mixed Prandtl/Hartmann system, the mixed Prandtl/Shercliff system and the fully nonlinear magnetic Prandtl model (for which we only provide some commentaries, the model having been studied in [41], [42]). Most notably, we establish a well-posedness result for the mixed Prandtl/Shercliff model.

4.1 Mixed Prandtl/Hartmann regime

We go back briefly to system (4.1), that we rewrite for the reader's convenience:

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{u} \cdot \nabla \mathbf{u}_h + \nabla_h p - \dfrac{\mathrm{Ha}^2}{\mathrm{Re}} \partial_z^2 \mathbf{u}_h + \dfrac{\mathrm{Ha}^2}{\mathrm{Re}} (\mathbf{u}_h - \mathbf{u}_h^\infty) = 0, \\ \partial_z p = 0, \\ \mathrm{div}_h\, \mathbf{u}_h + \partial_z u_z = 0. \end{cases} \tag{4.1}$$

Despite the damping term $\frac{\mathrm{Ha}^2}{\mathrm{Re}}(\mathbf{u}_h - \mathbf{u}_h^\infty)$ generated by the magnetic interaction, the strong linear instabilities of the Prandtl system persist. More precisely, one can restrict to the two dimensional case and consider a simple linearization around $u = (u_s(t,z), 0)$ in the spirit of [18]. Here u_s is such that $\partial_t u_s - \partial_{zz} u_s = 0$ and the system becomes

$$\begin{cases} \partial_t u + u_s \partial_x u + v \partial_z u_s - \partial_{zz} u + \frac{\mathrm{Ha}}{\mathrm{Re}} u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ \partial_x u + \partial_y z = 0 & \text{in } \mathbb{T} \times \mathbb{R}^+, \\ (u,v)|_{z=0} = (0,0), \qquad \lim\limits_{z \to +\infty} u = 0. & \end{cases} \tag{4.2}$$

In this context, one has

Proposition 4.1.1. *The same ill-posedness statement as in Theorem 1.4.1 (from [18]) holds for system (4.2).*

Proof. The proof is a straightforward adaptation of the one about the classical Prandtl system. The first step is to consider the linearized problem where u_s is replaced by its initial value. We set $U_s(z) = u_s(0, z)$ for which we assume the existence of a non-degenerate critical point. The equations become

$$\begin{cases} \partial_t u + U_s \partial_x u + v U_s' - \partial_{zz} u + u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u + \partial_z v = 0 & \text{in } \mathbb{T} \times \mathbb{R}^+ \\ (u, v)|_{z=0} = (0, 0), \qquad \lim_{z \to +\infty} u = 0. \end{cases} \tag{4.3}$$

Here, we have assumed $\frac{\mathrm{Ha}}{\mathrm{Re}} = 1$ for notational simplicity. Just as in [18], we then look for an approximate solution of this equation under the form

$$u(t, x, y) = e^{ik(\omega(k)t + x)} \hat{u}^k(y) \qquad v(t, x, y) = k e^{ik(\omega(k)t + x)} \hat{v}^k(y), \qquad k > 0.$$

We carry a high frequency analysis, namely with $\varepsilon = \frac{1}{k} \ll 1$. Simple calculations lead to

$$(\omega(\varepsilon) + U_s - i\varepsilon)\, v_\varepsilon'(y) - U_s' v_\varepsilon(y) + i\varepsilon v_\varepsilon^{(3)}(y) = 0 \tag{4.4}$$

which differs from the corresponding equation in [18] only for the term $-i\varepsilon v_\varepsilon'(y)$. This just corresponds to a shift in the eigenvalue. Namely, by setting $\tilde{\omega}(\varepsilon) = \omega(\varepsilon) - i\varepsilon$, we can reproduce exactly the same spectral analysis as in [18]. In particular, we can find an approximate solution with $\tilde{\omega}(\varepsilon) = -U_s(a) + |\varepsilon|^{1/2}\tau$, where a is the critical point of U_s and τ is a fixed complex number with negative imaginary part. It follows that

$$\Im(\omega) = \Im(\tau)|\varepsilon|^{1/2} + \varepsilon \sim \Im(\tau)|\varepsilon|^{1/2}$$

as $\varepsilon \to 0$. Eventually, this provides us with an approximate solution with growth rate proportional to $k^{1/2}$, where $k \gg 1$ is the x-frequency. From here, to take into account the time dependence of u_s and obtain an ill-posedness result in Sobolev spaces, one proceeds exactly as in [18]. □

Remark 4.1.1. Thanks to the similarity between model (4.1) and the usual Prandtl equation, a great deal of results (such as [19] and [40]) on the latter are true on the former, as we just showed for the illposedness in Sobolev spaces in the context of non monotonic profiles. For instance, following [19] we can state the local wellposedness of system (4.1) for data that belong to the Gevrey class $\frac{7}{4}$ in the horizontal variable x. Indeed, our extra damping adds favourable terms to the estimates in [19].

4.2 Mixed Prandtl/Shercliff regime

We focus here on the two-dimensional version of the mixed Prandtl/Shercliff system, in the domain $\Omega = \mathbb{T} \times \mathbb{R}^+$. For notational simplicity, we set Ha $=$ Re $= 1$, but the result would equally apply for general values of these parameters. The system reads

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u = \partial_x(b - b^\infty), \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0, \\ u \to 0, \quad b \to b^\infty \quad \text{as } z \to +\infty. \end{cases} \tag{4.5}$$

Remark 4.2.1. One could consider a general boundary condition of the form $u \to u^\infty$ as $z \to \infty$. But from the second equation in (4.5), one notices the consistency condition $\partial_x u^\infty = 0$. Hence, there is no big loss in assuming $u^\infty = 0$.

Our goal is to establish a local Sobolev well-posedness result without any structural assumption on the data, as opposed to the usual boundary layer results. To this end, we introduce the following functional setting:

Definition 4.2.1.

$$\Omega := \left\{(x,z) : x \in \mathbb{T},\ z \in \mathbb{R}^+\right\}, \tag{4.6}$$

$$L^2_\gamma := \left\{ f(x,z) : \Omega \to \mathbb{R},\ \|f\|_{L^2_\gamma(\Omega)} := \left(\int_\Omega \langle z\rangle^{2\gamma} |f(x,z)|^2 dxdz\right)^{\frac{1}{2}} < +\infty \right\},\ \langle z\rangle := \sqrt{1+z^2}, \tag{4.7}$$

$$H^m_\gamma := \left\{ f(x,z) : \Omega \to \mathbb{R},\ \|f\|_{H^m_\gamma(\Omega)} := \left(\sum_{s_1+s_2 \leqslant m} \|\langle z\rangle^{\gamma+s_2} \partial_x^{s_1} \partial_z^{s_2} f\|^2_{L^2(\Omega)} < +\infty \right)^{\frac{1}{2}} \right\}. \tag{4.8}$$

We will prove

Theorem 4.2.1 (Local existence for mixed Prandtl/Shercliff equations)**.** *Let $m \geqslant 4$ be an even integer, $T > 0$. Let $b^\infty = b^\infty(t,x)$ smooth over $[0,T] \times \mathbb{T}$, with $\int_\mathbb{T} b^\infty dx = 0$. Let $u_0 \in H^m(\Omega)$, $\partial_z u_0 \in H^m_1(\Omega)$.*

Under suitable compatibility conditions on u_0 (see remark below), there exists a time $T > 0$ and a unique solution (u,b) of (4.5) such that

$$u \in L^\infty(0,T; H^m(\Omega)) \cap C_w([0,T], H^m(\Omega)), \quad \partial_x u \in L^2(0,T; H^m_1(\Omega)),$$
$$\partial_z u \in L^\infty(0,T; H^m_1(\Omega)), \quad u|_{t=0} = u_0.$$

Remark 4.2.2. As usual in parabolic type equations in bounded domains, compatibility conditions are needed on the initial data u_0 to obtain local smooth solutions as in the statement of the theorem. Here, it amounts to various identities satisfied by u_0 and its derivatives at $y = 0$. We refer to [59, Proposition 2.2 and Remark 3.4], in which the right compatibility conditions for the Prandtl equations are given and discussed thoroughly.

Remark 4.2.3. The hypothesis that m is even is related to the control of the boundary terms, and it appears during the proof of Lemma 4.2.2.

Remark 4.2.4. Let us stress that the second equation in (4.5) fully determines b in terms of u. More precisely, it is implicit that b is given by

$$b(t, x, z) = \int_0^z \int_{z'}^{+\infty} \partial_x u(t, x, z'')dz''dz' \tag{4.9}$$

which is consistent with the conditions $\partial_z b \to 0$ as $z \to \infty$ and $b|_{z=0} = 0$. The condition $b \to b^\infty$ can then be deduced from the first equation of (4.5). Indeed, sending $z \to +\infty$, taking into account the decay of u and its derivatives at infinity, we find that $\lim_{z\to+\infty} \partial_x(b - b^\infty) = 0$. By assumption b^∞ has zero mean in x, and the same is true for b by formula (4.9). Hence, $\lim_{z\to+\infty} b - b^\infty = 0$.

Remark 4.2.5. The fact that u belongs to $L^\infty(0, T; H^m(\Omega))$ while $\omega = \partial_z u \in L^\infty(0, T; H^m_1(\Omega))$ is coherent with the Hardy inequality:

$$\|f\|_{L^2(\mathbb{R}_+)} \leqslant C\|(1 + z)\partial_z f\|_{L^2(\mathbb{R}_+)} \tag{4.10}$$

valid for functions $f = f(z)$ going to zero at infinity. A more original fact is the bound on $\partial_x u$, namely $\partial_x u \in L^2(0, T; H^m_1(\Omega))$. This corresponds to both a gain of regularity in x (which is crucial to obtain stability estimates for our magnetic Prandtl model) and a gain of decay. This implies notably that $\partial_z b = -\int_z^{+\infty} \partial_x u$ belongs to $L^2(0, T; H^m(\Omega))$, still by Hardy inequality.

The rest of the chapter will be devoted to the proof of Theorem (4.2.1). We will work with a parabolic approximation of (4.5), establish uniform estimates for this approximation and finally send the approximation parameter to zero.

4.2.1 Regularized system

Precisely, the approximation of (4.5) consists of adding a diffusion term in x. For ε arbitrarily small we consider

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_z u^\varepsilon - \partial_z^2 u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon = \partial_x (b^\varepsilon - b^\infty), \\ \partial_x u^\varepsilon + \partial_z^2 b^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \\ u^\varepsilon|_{z=0} = v^\varepsilon|_{z=0} = 0, \qquad b^\varepsilon|_{z=0} = 0, \\ u^\varepsilon \to 0, \quad b^\varepsilon \to b^\infty, \qquad \text{as } z \to +\infty. \end{cases} \tag{4.11}$$

The associated vorticity formulation is

$$\begin{cases} \partial_t \omega^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \partial_z^2 \omega^\varepsilon - \varepsilon^2 \partial_x^2 \omega^\varepsilon = \partial_x \partial_z b^\varepsilon, \\ \partial_x u^\varepsilon + \partial_z^2 b^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \\ u^\varepsilon|_{z=0} = v^\varepsilon|_{z=0} = 0, \qquad b^\varepsilon|_{z=0} = 0, \\ u^\varepsilon \to 0, \quad b \to b^\infty, \qquad \text{as } z \to +\infty. \end{cases} \tag{4.12}$$

We will not detail the existence and uniqueness of smooth solutions to the approximate system (4.11). A very similar system is considered in [44] in the classical Prandtl case. The main point of this approximation is that standard Sobolev estimates can be applied: thanks to the tangential diffusion $-\varepsilon^2 \partial_x^2$, the convection term $v^\varepsilon \partial_z u^\varepsilon$ (or $v^\varepsilon \partial_z \omega^\varepsilon$), responsible for a loss of x-derivative in the Prandtl equation, is under control. More precisely, the construction of solutions can be achieved through an iterative scheme, where the quadratic term is treated as a source (explicitly in the language of numerical analysis). In other words, the construction of solutions comes down to the solvability of the linear inhomogeneous system:

$$\begin{cases} \partial_t u^\varepsilon - \partial_z^2 u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon = \partial_x (b^\varepsilon - b^\infty) + F, \\ \partial_x u^\varepsilon + \partial_z^2 b^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \\ u^\varepsilon|_{z=0} = v^\varepsilon|_{z=0} = 0, \qquad b^\varepsilon|_{z=0} = 0, \\ u^\varepsilon \to 0, \quad b^\varepsilon \to b^\infty, \qquad \text{as } z \to +\infty. \end{cases}$$

Such a linear system is easily solvable as soon as *a priori estimates* are available (for instance through Laplace-Fourier transform in t and x). As we will carry such *a priori* estimates at the level of the full system (4.11), we do not give further details. We have:

Proposition 4.2.1 (Local existence for regularized mixed Prandtl/Shercliff equations)**.** *Let $m \geqslant 4$ be an even integer and $\varepsilon \in [0, 1]$. Under the same assumptions as in Theorem*

4.2.1, there exists a time $T_\varepsilon > 0$ and a solution $u^\varepsilon, b^\varepsilon$ of (4.11) such that

$$u^\varepsilon \in L^\infty(0,T_\varepsilon; H^m(\Omega)) \cap C_w([0,T_\varepsilon], H^m(\Omega)), \quad \partial_x u^\varepsilon \in L^2(0,T_\varepsilon; H^m_1(\Omega)),$$
$$\partial_z u^\varepsilon \in L^\infty(0,T_\varepsilon; H^m_1(\Omega)), \quad u^\varepsilon|_{t=0} = u_0.$$

Remark 4.2.6. Actually, one must apply this last proposition with a compactly supported mollification u_0^ε of u_0 and an index m higher than the one considered in Theorem 4.2.1. In this way, one works with an approximate solution $(u^\varepsilon, b^\varepsilon)$ which is smooth and decaying enough so that all subsequent estimates are legitimate.

4.2.2 Uniform estimates

The main part of the analysis is the derivation of estimates on a time interval independent of ε. We shall need to combine estimates on the vorticity (system (4.12)) and on the velocity (system (4.11)). We will first prove:

Proposition 4.2.2 (L^2 controls on $\langle z\rangle^{1+s_2} D^s \omega$ for $|s| \leqslant m$)**.** *Let $m \geqslant 4$ be an even integer and $\varepsilon \in [0,1]$. There exists $C > 0$ (independent of ε) depending on m and b^∞ such that*

$$\begin{aligned}
&\frac{d}{dt}\|\omega\|^2_{H^m_1(\Omega)} + \varepsilon^2\|\partial_x\omega\|^2_{H^m_1(\Omega)} + \|\partial_z\omega\|^2_{H^m_1(\Omega)} + \|\partial_x u\|^2_{H^m_1(\Omega)} \\
&\leqslant C\left(1 + \|\omega(t,\cdot)\|^m_{H^m_1(\Omega)} + \sum_{0\leqslant|s|\leqslant m}\int_\Omega |\partial_z\partial_x^{|s|} b|^2\right).
\end{aligned} \tag{4.13}$$

Remark 4.2.7. By the equation $\partial_x u + \partial_z^2 b = 0$, we know that b has zero average in x, so that the term $\sum_{0\leqslant|s|\leqslant m}\int_\Omega |\partial_z\partial_x^{|s|} b|^2$ can be replaced by $\int_\Omega |\partial_z\partial_x^m b|^2$.

Remark 4.2.8. This estimate is not closed due to the last term at the right-hand side, and will have to be combined with a velocity estimate.

Throughout this analysis, we will denote the variables (u, v, ω, b) instead of $(u^\varepsilon, v^\varepsilon, \omega^\varepsilon, b^\varepsilon)$ to light up the ensemble. Let $s = (s_1, s_2)$ with $|s| \leqslant m$. We differentiate the vorticity equation (4.12) s_1 times with respect to x and s_2 times with respect to z. We integrate in space over $\Omega = \mathbb{T} \times \mathbb{R}^+$ after multiplying by $\langle z\rangle^{2+2s_2} D^s\omega$. We obtain after straightforward manipulations:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}^2 = -\int_\Omega u\partial_x D^s\omega(D^s\omega)\langle z\rangle^{2+2s_2} - \int_\Omega v\partial_z(D^s\omega)(D^s\omega)\langle z\rangle^{2+2s_2}\\
&-\int_\Omega D^s v\,\partial_z\omega\, D^s\omega\,\langle z\rangle^{2+2s_2} + \int_\Omega \partial_z^2(D^s\omega)(D^s\omega)\langle z\rangle^{2+2s_2} + \varepsilon^2\int_\Omega \partial_x^2(D^s\omega)(D^s\omega)\langle z\rangle^{2+2s_2}\\
&-\sum_{1\leqslant|\alpha|,\,\alpha\leqslant s}\binom{s}{\alpha}\int_\Omega D^\alpha u D^{s-\alpha}\partial_x\omega(D^s\omega)\langle z\rangle^{2+2s_2}\\
&-\sum_{1\leqslant|\alpha|\leqslant|s|-1}\binom{s}{\alpha}\int_\Omega D^\alpha v D^{s-\alpha}\partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2} + \int_\Omega \partial_x\partial_z D^s(b-b^\infty)(D^s\omega)\langle z\rangle^{2+2s_2}.
\end{aligned}
$$

We now proceed through a term by term analysis. We will then sum over $0 \leqslant |s| \leqslant m$.

1.
$$
-\int_\Omega u\partial_x D^s\omega(D^s\omega)\langle z\rangle^{2+2s_2} = \frac{1}{2}\int_\Omega \partial_x u(D^s\omega)^2\langle z\rangle^{2+2s_2} \leqslant C\|\omega\|^3_{H^m_1(\Omega)}.
$$

We have used the inequality $\|\partial_x u\|_{L^\infty} \lesssim \|\omega\|_{H^m_1(\Omega)}$, which can be proved as follows. First, we remind the Sobolev type imbedding (see [44, Lemma B.2])

$$
\|f\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)} \leqslant C\left(\|f\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\partial_x f\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\partial_z^2 f\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\right), \tag{4.14}
$$

which leads to

$$
\|\partial_x u\|_{L^\infty} \leqslant C(\|\partial_x u\|_{L^2} + \|\partial_x^2 u\|_{L^2} + \|\partial_x\partial_z\omega\|_{L^2}).
$$

Then, combining this inequality with (4.10), we end up with

$$
\|\partial_x u\|_{L^\infty} \leqslant C(\|\langle z\rangle\partial_x\omega\|_{L^2} + \|\langle z\rangle\partial_x^2\omega\|_{L^2}\|_{L^2} + \|\partial_x\partial_z\omega\|_{L^2}) \leqslant C'\|\omega\|_{H^m_1(\Omega)}.
$$

2.
$$
\begin{aligned}
&-\int_\Omega v\partial_z(D^s\omega)(D^s\omega)\langle z\rangle^{2+2s_2} = \frac{1}{2}\int_\Omega \partial_z v(D^s\omega)^2\langle z\rangle^{2+2s_2} + \int_\Omega v(D^s\omega)^2(1+s_2)\langle z\rangle^{2s_2}z\\
&\leqslant +\frac{1}{2}\int_\Omega \partial_z v(D^s\omega)^2\langle z\rangle^{2+2s_2} + (1+s_2)\|v\,\langle z\rangle^{-1}\|_{L^\infty}\|D^s\omega\langle z\rangle^{1+s_2}\|^2_{L^2}\\
&\leqslant \frac{1}{2}\int_\Omega \partial_z v(D^s\omega)^2\langle z\rangle^{2+2s_2} + C\|\omega\|^3_{H^m_1(\Omega)}.
\end{aligned}
$$

Here, we have used the bound $\|v\,\langle z\rangle^{-1}\|_{L^\infty} \lesssim \|\omega\|_{H^m_1(\Omega)}$. Indeed, (4.14) implies

$$
\begin{aligned}
\|v\,\langle z\rangle^{-1}\|_{L^\infty} &\leqslant C\Big(\|v\,\langle z\rangle^{-1}\|_{L^2} + \|\partial_x v\,\langle z\rangle^{-1}\|_{L^2} + \|\partial_x\omega\,\langle z\rangle^{-1}\|_{L^2} + \|\partial_x u\,\langle z\rangle^{-2}\|_{L^2}\Big)\\
&\leqslant C\Big(\|\partial_x u\|_{L^2} + \|\partial_x^2 u\,\|_{L^2} + \|\partial_x\omega\,\langle z\rangle^{-1}\|_{L^2}\Big) \leqslant C\|\omega\|_{H^m_1(\Omega)}
\end{aligned} \tag{4.15}
$$

as soon as $m \geqslant 2$, where the last two inequalities come from the usual Hardy inequality and its variation (4.10).

3. We now turn to the bad term containing $D^s v$, and responsible for a potential loss of x-derivative. Remembering that $s = (s_1, s_2)$, we distinguish between two cases.

Case 1: $s_2 > 0$. We write, still using (4.10):

$$
\begin{aligned}
-\int_\Omega D^s v \partial_z \omega (D^s\omega) \langle z\rangle^{2+2s_2} &= \int_\Omega D^{(s_1,s_2-1)}\partial_x u \partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2}\\
= \int_\Omega \langle z\rangle^{s_2-1} D^{(s_1,s_2-1)}\partial_x u \ \langle z\rangle^2 \ \partial_z\omega(D^s\omega)\langle z\rangle^{1+s_2}&\\
\leqslant \|\langle z\rangle^2\partial_z\omega\|_{L^\infty}\ \|\langle z\rangle^{s_2-1}D^{(s_1,s_2-1)}\partial_x u\|_{L^2}\ \|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}&\\
\leqslant \|\langle z\rangle^2\partial_z\omega\|_{L^\infty}\ \|\langle z\rangle^{s_2}D^{(s_1,s_2-1)}\partial_x \omega\|_{L^2}\ \|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}&\\
\leqslant \|\langle z\rangle^2\partial_z\omega\|_{L^\infty}\ \|\langle z\rangle^{s_2}D^{(s_1+1,s_2-1)}\omega\|_{L^2}\ \|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2} &\leqslant C\|\omega\|^3_{H^m_1(\Omega)}
\end{aligned}
$$

where the control of the first factor is deduced from (4.14), as soon as $m \geqslant 3$.

Case 2: $s_2 = 0$. We take care of this case using (4.10)

$$
\begin{aligned}
-\int_\Omega D^s v\, \partial_z\omega(D^s\omega)\,\langle z\rangle^2 &= \int_\Omega \partial_x^{|s|}\partial_z^{-1}\partial_x u\, \partial_z\omega\,(D^s\omega)\langle z\rangle^2\\
\leqslant\ \|\langle z\rangle^{-1}\partial_z^{-1}\partial_x^{|s|+1}u\|_{L^2}\ \|\langle z\rangle^2\partial_z\omega\|_{L^\infty}\ \|D^s\omega\,\langle z\rangle\|_{L^2}&\\
\leqslant\ C\,\|\partial_x^{|s|+1}u\|_{L^2}\ \|\langle z\rangle^2\partial_z\omega\|_{L^\infty}\ \|D^s\omega\,\langle z\rangle\|_{L^2}&\\
\leqslant \begin{cases} C\|\omega\|^3_{H^m_1(\Omega)} & \text{if } |s| = s_1 < m,\\ \eta\|\partial_x^{m+1}u\|^2_{L^2} + C_\eta\|\omega\|^4_{H^m_1(\Omega)} & \text{if } |s| = s_1 = m, \text{ for all } \eta > 0. \end{cases}&
\end{aligned}
$$

Remark 4.2.9. In the case $|s| = s_1 = m$, the extra term $\|\partial_x^{m+1}u\|^2_{L^2}$ can not be controlled by $\|\omega\|_{H^m_1(\Omega)}$. This is the typical problem that arises when studying the Prandtl equation, and can only be solved in some particular cases, thanks to some extra hypothesis on the data. For instance, in [44], the Prandtl system is shown to be well-posed in the Sobolev setting if the initial velocity satisfies a *monotonicity* hypothesis; besides, if this hypothesis is not verified, well-posedness requires at least Gevrey regularity, as emphasized for instance in [19]. When the MHD system is considered, it is a remarkable result that the *contributions of the magnetic field* can grant the well-posedness. This is what we show here in the context of the Prandtl-Shercliff model.

4. We now turn to the diffusive term in z, for which integration by parts creates bound-

ary terms. Taking into account the decay at infinity, we find

$$\int_\Omega \partial_z^2(D^s\omega)(D^s\omega)\langle z\rangle^{2+2s_2} = -\int_{\mathbb{T}} \partial_z(D^s\omega)|_{z=0}(D^s\omega)|_{z=0}$$
$$+\,2(1+s_2)\int_\Omega \partial_z D^s\omega(D^s\omega)\langle z\rangle^{2s_2} z + \|\partial_z D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}^2.$$

We have to study the first two of these three terms. The second one is the easiest:

$$2(1+s_2)\left|\int_\Omega \partial_z D^s\omega(D^s\omega)\langle z\rangle^{2s_2} z\right| \leqslant \eta\|\partial_z D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}^2 + C_\eta\|D^s\omega\langle z\rangle^{s_2}\|_{L^2}^2,$$

using Young's inequality. We now focus on the boundary term. For $|s| \leqslant m-1$, we can rely on the simple trace inequality

$$\|f|_{z=0}\|_{L^2} \leqslant C\|f\|_{L^2}^{1/2}\,\|\partial_z f\|_{L^2}^{1/2}$$

to obtain

$$\begin{aligned}-\int_{\mathbb{T}} \partial_z(D^s\omega)|_{z=0}(D^s\omega)|_{z=0} &\leqslant C\|\partial_z(\partial_z D^s\omega)\|_{L^2}^{1/2}\,\|\partial_z(D^s\omega)\|_{L^2}\|\,\|D^s\omega\|^{1/2}\\ &\leqslant \eta\|\partial_z\omega\|_{H_1^m(\Omega)}^2 + C_\eta\|\omega\|_{H_1^m(\Omega)}^2.\end{aligned} \tag{4.16}$$

In the case $|s| = m$, the trace inequality is not enough. We rely on the analysis of boundary conditions carried in [44], except that we account for the additional magnetic term. Our analogue of Lemma 5.9 in [44] is the following:

Lemma 4.2.2 (Boundary reduction)**.** *We have at the boundary $y=0$:*

$$\begin{aligned}\partial_z\omega|_{z=0} &= \partial_x b^\infty\\ \partial_z^3\omega|_{z=0} &= (\partial_t - \varepsilon^2\partial_x^2)\partial_x b^\infty + \omega\partial_x\omega|_{z=0}.\end{aligned} \tag{4.17}$$

For any $2 \leqslant k \leqslant [\frac{s}{2}]$, there are some constants $C_{k,l,\rho_1,\ldots,\rho_j}$ which do not depend on ε such that

$$\begin{aligned}\partial_z^{2k+1}\omega|_{z=0} &= \partial_x^2\partial_z^{2k-3}\omega|_{z=0}\\ &\quad + (\partial_t - \varepsilon^2\partial_x)^k\partial_x b^\infty + \sum_{l=0}^{k-1}\varepsilon^{2l}\sum_{j=2}^{\max(2,k-l)} C_{k,l,\rho_1,\ldots,\rho_j}\prod_{i=1}^{j} D^{\rho_i}\omega|_{z=0},\end{aligned} \tag{4.18}$$

where $A_{k,l}^j := \{\rho = (\rho_1,\ldots,\rho_j) \in \mathbb{N}^{2j},\ 3\sum_{i=1}^j \rho_i^1 + \sum_{i=1}^j \rho_2^i = 2k+4l+1,$ $\sum_{i=1}^j \rho_i^1 \leqslant k+2l-1,\ \sum_{i=1}^j \rho_i^2 \leqslant 2k-2l-2$ and $|\rho^i| \leqslant 2k-l-1$ for all $i = 1,\ldots,j\}$.

Proof. The arguments used in [44] to prove Lemma 5.9 remain the same except for two differences. First, the function b^∞ substitutes to the usual Prandtl pressure term

$p(t,x)$. Second, when differentiating $2k-1$ times ($k \geqslant 1$) the vorticity equation and taking its trace to obtain a formula for $\partial_z^{2k+1}\omega|_{z=0}$, one has to account for the extra term $-\partial_z^{2k-1}\partial_x\partial_z b|_{z=0}$. Thanks to the relation $\partial_x u + \partial_z^2 b = 0$, it simplifies into $\partial_x^2\partial_z^{2k-2}u|_{z=0}$, which is zero for $k=1$, and equal to $\partial_x^2\partial_z^{2k-3}\omega|_{z=0}$ for $k \geqslant 2$. This explains the first term at the right-hand side of (4.18). □

Corollary 4.2.3. *For all $k, s_1 \in \mathbb{N}$, there exists $C = C(b^\infty, s_1, k)$ and $C' = C(k, s_1)$ such that*

$$\|\partial_x^{s_1}\partial_z^{2k+1}\omega|_{z=0}\|_{L^2(\mathbb{T})} \leqslant C + C'\|\omega\|_{H^{s_1+2k}(\Omega)}^{\max(2,k)}.$$

Proof. For $k=0,1$, the inequality follows easily from the first two relations in Lemma 4.2.2. For $k \geqslant 2$, the inequality can be shown inductively, using relation (4.18). The point is to bound $\|\prod_{i=1}^j \partial_x^{\alpha_i}D^{\rho_i}\omega\|_{L^2(\mathbb{T})}$, where $(\rho_1,\dots,\rho_j)$ belongs to a set of type $A_{k,l}^j$, and $\alpha_1+\dots+\alpha_j = s_1$. One first notices that $j \leqslant \max(2,k) = k$, so that the product has at most k terms. Then, the constraint $|\rho_i| \leqslant 2k-l-1$ shows that necessarily $\alpha_i + |\rho_i| \leqslant s_1 + 2k - 1$ for all $1 \leqslant i \leqslant j$. Finally, the constraint $\sum(\rho_i^1+\rho_i^2) \leqslant 4k-3$ shows that only one of the integers $\alpha_i + |\rho_i|$ (say for $i=I$) can take the value s_1+2k-1, while the others are less than s_1+2k-2. We can then write

$$\left\|\prod_{i=1}^j \partial_x^{\alpha_i}D^{\rho_i}\omega\right\|_{L^2(\mathbb{T})} \leqslant \|\partial_x^{\alpha_I}D^{\rho_I}\omega\|_{L^2(\mathbb{T})}\prod_{i\neq I}\|\partial_x^{\alpha_i}D^{\rho_i}\omega\|_{L^\infty(\mathbb{T})}.$$

The inequality follows by applying the standard trace inequality to the first factor, and the Sobolev embedding $H^2 \subset L^\infty$ to the others. □

The previous lemma and corollary will allow us to handle the boundary integral in the case $|s| = m$. We remind that we chose m to be an even integer, see Theorem 4.2.1.

Case $s = (m,0)$. Let us first consider the case where all the derivatives are along the first variable. We use the first statement in Lemma 4.2.2, which gives

$$\begin{aligned}
&-\int_{\mathbb{T}} \partial_z(D^s\omega)|_{z=0}(D^s\omega)|_{z=0} = -\int_{\mathbb{T}} \partial_z\partial_x^m\omega|_{z=0}\partial_x^m\omega|_{z=0}\\
&= -\int_{\mathbb{T}} \partial_x^{m+1}b^\infty\partial_x^m\omega|_{z=0}\\
&\leqslant \|\partial_x^{m+1}b^\infty\|_{L^2(\mathbb{T})}\|\partial_z\partial_x^m\omega\|_{L^2}^{1/2}\|\partial_x^m\omega\|_{L^2}^{1/2} \leqslant \eta\|\partial_z\omega\|_{H_m^1(\Omega)}^2 + C_\eta(b^\infty)\|\omega\|_{H_m^1(\Omega)}^2,
\end{aligned}$$

where $\eta > 0$ is arbitrarily small (and $C_\eta(b^\infty)$ depends on b^∞).

Case $|s| = m$, $s_1 \leqslant m-1$. We make use of the previous corollary, distinguishing between two possibilities:

- s_2 is even.

  When $s_2 = 2k$ for some $k \in \mathbb{N}$, we obtain (note that $\max(2,k) \leqslant \frac{m}{2}$).

$$
\begin{aligned}
-\int_{\mathbb{T}} \partial_z (D^s\omega)|_{z=0} (D^s\omega)|_{z=0} &\leqslant \|\partial_x^{s_1}\partial_z^{2k+1}\omega|_{z=0}\|_{L^2(\mathbb{T})} \|D^s\omega|_{z=0}\|_{L^2(\mathbb{T})} \\
&\leqslant \left(C + C'\|\omega\|^{\frac{m}{2}}_{H_1^m(\Omega)}\right) \|\partial_z D^s\omega\|_{L^2}^{1/2} \|D^s\omega\|_{L^2}^{1/2} \\
&\leqslant \eta\|\partial_z\omega\|^2_{H_1^m(\Omega)} + C_\eta(1 + \|\omega\|^m_{H_1^m(\Omega)}),
\end{aligned}
$$

  where C_η depends implicitly on b^∞.

- s_2 is odd

  When $s_2 = 2k+1$ for some $k \in \mathbb{N}$, since $s_1 + s_2 = m$ is even, we know that $s_1 \geqslant 1$. Using integration by parts in x, we have that

$$
-\int_{\mathbb{T}} \partial_z (D^s\omega)|_{z=0} (D^s\omega)|_{z=0} = \int_{\mathbb{T}} \partial_x^{s_1-1}\partial_z^{s_2+1}\omega|_{z=0} \partial_x D^s\omega|_{z=0}. \tag{4.19}
$$

  Now, the term $\partial_x D^s\omega|_{z=0} = \partial_x^{s_1+1}\partial_z^{2k+1}\omega|_{z=0}$ has an odd number of z derivatives, so that we can apply again the previous corollary. We obtain as previously

$$
-\int_{\mathbb{T}} \partial_z (D^s\omega)|_{z=0} (D^s\omega)|_{z=0} \leqslant \eta\|\partial_z\omega\|^2_{H_1^m(\Omega)} + C_\eta(1 + \|\omega\|^m_{H_1^m(\Omega)}),
$$

  where C_η still depends on b^∞.

5. We turn to the treatment of the first trilinear term, starting with

$$
I_u := -\sum_{1\leqslant|\alpha|,\ \alpha\leqslant s} \binom{s}{\alpha} \int_\Omega D^\alpha u D^{s-\alpha}\partial_x\omega (D^s\omega)\langle z\rangle^{2+2s_2}.
$$

Of course we can study this sum term by term, and separate the following cases

Case $1 \leqslant |\alpha| \leqslant \frac{m}{2}$ *:*

$$
\begin{aligned}
&-\int D^\alpha u D^{s-\alpha}\partial_x\omega (D^s\omega)\langle z\rangle^{2+2s_2} \\
&\leqslant \|D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^\infty} \cdot \|D^{s-\alpha}\partial_x\omega\langle z\rangle^{1+s_2-\alpha_2}\|_{L^2} \cdot \|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2} \\
&\leqslant \|D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^\infty} \cdot \|\omega\|^2_{H_1^m(\Omega)}.
\end{aligned}
$$

Note that we were able to bound the second factor by $\|\omega\|_{H_1^m(\Omega)}$ since $|\alpha| \geqslant 1$. Combining (4.14) together with the Hardy-type inequality (4.10), we deduce

$$
\begin{aligned}
&-\int D^\alpha u D^{s-\alpha}\partial_x\omega (D^s\omega)\langle z\rangle^{2+2s_2} \\
&\leqslant C\left(\|D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^2} + \|\partial_x D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^2} + \|\partial_z^2(D^\alpha u\langle z\rangle^{\alpha_2})\|_{L^2}\right) \|\omega\|^2_{H_1^m(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\big(\|D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^2} + \|\partial_x D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^2} \\
&\quad + \|D^\alpha \omega\langle z\rangle^{\alpha_2-1}\|_{L^2} + \|\partial_z D^\alpha \omega\langle z\rangle^{\alpha_2-2}\|_{L^2}\big)\,\|\omega\|^2_{H^m_1(\Omega)} \\
&\leqslant C\left(\|D^\alpha \omega\langle z\rangle^{1+\alpha_2}\|_{L^2} + \|\partial_x D^\alpha \omega\langle z\rangle^{1+\alpha_2}\|_{L^2} + \|\partial_z D^\alpha \omega\langle z\rangle^{\alpha_2+2}\|_{L^2}\right)\,\|\omega\|^2_{H^m_1(\Omega)} \\
&\leqslant C\|\omega\|^3_{H^m_1(\Omega)}.
\end{aligned}
$$

Case $\frac{m}{2} < |\alpha| \leqslant m$, *so that* $|s-\alpha| \leqslant \frac{m}{2} - 1$ *:*

$$
\begin{aligned}
&-\int D^\alpha u D^{s-\alpha}\partial_x\omega (D^s\omega)\langle z\rangle^{2+2s_2} && (4.20)\\
&\leqslant \|D^\alpha u\langle z\rangle^{\alpha_2}\|_{L^2}\cdot\|D^{s-\alpha}\partial_x\omega\langle z\rangle^{s_2-\alpha_2+1}\|_{L^\infty}\,\|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2} && (4.21)\\
&\leqslant \|D^{s-\alpha}\partial_x\omega\langle z\rangle^{s_2-\alpha_2+1}\|_{L^\infty}\,\|\omega\|^2_{H^m_1(\Omega)}. && (4.22)
\end{aligned}
$$

The same steps as for the previous case can be followed, resulting in

$$
\|D^{s-\alpha}\partial_x\omega\langle z\rangle^{s_2-\alpha_2+1}\|_{L^\infty} \leqslant C\|\omega\|_{H^1_m(\Omega)}
$$

and eventually $I_u \leqslant C\|\omega\|^3_{H^1_m(\Omega)}$.

6. Let now consider

$$
I_v := -\sum_{1\leqslant|\alpha|\leqslant|s|-1}\binom{s}{\alpha}\int_\Omega D^\alpha v D^{s-\alpha}\partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2}.
$$

Case $1 \leqslant |\alpha| \leqslant \frac{m}{2}$*:*

$$
\begin{aligned}
&-\int D^\alpha v D^{s-\alpha}\partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2} \\
&\leqslant \|D^\alpha v\,\langle z\rangle^{\alpha_2-1}\|_{L^\infty}\,\|D^{s-\alpha}\partial_z\omega\,\langle z\rangle^{2+s_2-\alpha_2}\|_{L^2}\,\|D^s\omega\,\langle z\rangle^{1+s_2}\|_{L^2}.
\end{aligned}
$$

The first factor can be handled again through (4.14) and Hardy inequalities (see (4.15) in the case $\alpha = 0$). We find $\|D^\alpha v\,\langle z\rangle^{\alpha_2-1}\|_{L^\infty} \lesssim \|\omega\|_{H^m_1(\Omega)}$ and

$$
-\int D^\alpha v D^{s-\alpha}\partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2} \leqslant C\|\omega\|^3_{H^m_1(\Omega)}.
$$

Case $\frac{m}{2} < |\alpha| \leqslant m-1$, *so that* $|s-\alpha| \leqslant \frac{m}{2} - 1$*:*

$$
\begin{aligned}
&-\int D^\alpha v D^{s-\alpha}\partial_z\omega(D^s\omega)\langle z\rangle^{2+2s_2} \\
&\leqslant \|v\langle z\rangle^{\alpha_2-1}\|_{L^2}\|D^{s-\alpha}\partial_z\omega\langle z\rangle^{s_2-\alpha_2+2}\|_{L^\infty}\|D^s\omega\langle z\rangle^{1+s_2}\|_{L^2}.
\end{aligned}
$$

Note that $|\alpha| \leqslant m-1$, so that

$$\|v\langle z\rangle^{\alpha_2-1}\|_{L^2} \lesssim \|\partial_x u\langle z\rangle^{\alpha_2}\|_{L^2} \lesssim \|\partial_x \omega\langle z\rangle^{\alpha_2+1}\|_{L^2} \lesssim \|\omega\|_{H^m_1(\Omega)}.$$

With similar ideas, we derive the bound

$$-\int D^\alpha v D^{s-\alpha}\partial_z\omega (D^s\omega)\langle z\rangle^{2+2s_2} \leqslant C\|\omega\|^3_{H^m_1(\Omega)}.$$

and eventually $I_v \leqslant C\|\omega\|^3_{H^m_1(\Omega)}$.

7. We now turn to the crucial magnetic term

$$\int \partial_x\partial_z D^s b(D^s\omega)\langle z\rangle^{2+2s_2}.$$

We write $D^s\omega = \partial_z D^s u$ and integrate by parts in z, so that

$$\begin{aligned}
&\int \partial_x\partial_z D^s b(D^s\omega)\langle z\rangle^{2+2s_2}\\
&= -\int \partial_x\partial_z^2 D^s b(D^s u)\langle z\rangle^{2+2s_2} - (2+2s_2)\int \partial_x\partial_z D^s b(D^s u)\langle z\rangle^{2s_2} z\\
&= \int \partial_x^2 D^s u(D^s u)\langle z\rangle^{2+2s_2} - (2+2s_2)\int \partial_x\partial_z D^s b(D^s u)\langle z\rangle^{2s_2} z\\
&= -\|\partial_x D^s u\langle z\rangle^{1+s_2}\|^2_{L^2} - (2+2s_2)\int \partial_x\partial_z D^s b(D^s u)\langle z\rangle^{2s_2} z.
\end{aligned}$$

Here, we used the identity $\partial_x u + \partial_z^2 b = 0$ to go from the second to the third inequality, followed by an integration by parts in x. The treatment of the last integral at the right-hand side depends on the value of s_2.

Case 1: $s_2 > 0$. We write

$$\begin{aligned}
\int \partial_x\partial_z D^s b(D^s u)\langle z\rangle^{2s_2} z &= \int \partial_x\partial_z^2 D^{s_1,s_2-1} b\,(D^{s_1,s_2-1}\omega)\langle z\rangle^{2s_2} z\\
&= -\int \partial_x D^{s_1+1,s_2-1} u\,(D^{s_1,s_2-1}\omega)\langle z\rangle^{2s_2} z\\
&\leqslant \eta\|\partial_x D^{s_1+1,s_2-1} u\langle z\rangle^{1+s_2}\|^2_{L^2} + C_\eta\|\omega\|^2_{H^m_1(\Omega)}.
\end{aligned}$$

Case 2: $s_2 = 0$. We just rewrite

$$\begin{aligned}
\int \partial_x\partial_z D^s b(D^s u)\langle z\rangle^{2s_2} z &= -\int \partial_z\partial_x^{|s|} b\,\partial_x^{|s|+1} u z\\
&= \int \partial_z\partial_x^{|s|} b\,\partial_z^2\partial_x^{|s|} b z = \frac{1}{2}\int \partial_z|\partial_z\partial_x^{|s|} b|^2 z\\
&= -\frac{1}{2}\int |\partial_z\partial_x^{|s|} b|^2.
\end{aligned}$$

Note that we used the fact that z vanishes at the boundary, so that no boundary term came out from the integration by parts.

Eventually, we put together all previous inequalities, sum over all $s = (s_1, s_2)$ with $0 \leqslant |s| \leqslant m$. By taking η small enough, we obtain the inequality in the proposition. This concludes its proof. The next step is to control the magnetic term $\int_\Omega |\partial_z \partial_x^m b|^2$ (see the right-hand side of (4.25) and the remark just below. Note that $\partial_z \partial_x^m b = \int_z^{+\infty} \partial_x^{m+1} u$: through Hardy inequality (4.10), one can control it by $C' \|\partial_x^{m+1} u \langle z \rangle\|_{L^2}$, but we can not ensure that the constant C' is small enough so as to absorb it by the right-hand side. This is why we will use a velocity estimate. It will be crucial that $\int_\Omega |\partial_z \partial_x^m b|^2$ contains no weight in z, otherwise commutation with the weight would create a bad term, reminiscent of the magnetic term we have here at the level of the vorticity estimate.

Proposition 4.2.3. *Under the same assumptions as in the previous proposition, we have for any $\eta > 0$, a constant C depending on η, m and b^∞ such that*

$$\begin{aligned} &\frac{d}{dt} \|\partial_x^m u\|^2_{L^2(\Omega)} + \varepsilon^2 \|\partial_x^{m+1} u\|^2_{L^2(\Omega)} + \|\partial_z \partial_x^m b\|^2_{L^2(\Omega)} \\ &\leqslant C_\eta \left(1 + \|\omega(t,\cdot)\|^4_{H_1^m(\Omega)}\right) + \eta \|\partial_x^{m+1} u \langle z \rangle\|^2_{L^2}. \end{aligned} \tag{4.23}$$

We will be quick on the proof of this proposition, as most ingredients are similar to those used in the previous proof. We differentiate m times the velocity equation with respect to x, test against $\partial_x^m u$ and integrate to obtain that

$$\begin{aligned} &\frac{d}{dt} \|\partial_x^m u\|^2_{L^2(\Omega)} + \varepsilon^2 \|\partial_x^{m+1} u\|^2_{L^2(\Omega)} + \|\partial_z u\|^2_{L^2(\Omega)} = -\int_\Omega \partial_x^m v \, \omega \, \partial_x^m u \\ &- \int_\Omega \sum_{1 \leqslant |\alpha| \leqslant m} \binom{m}{\alpha} \partial_x^\alpha u \, \partial_x^{m+1-\alpha} u \, \partial_x^m u - \int_\Omega \sum_{1 \leqslant |\alpha| \leqslant m-1} \binom{m}{\alpha} \partial_x^\alpha v \, \partial_x^{m-\alpha} \, \omega \partial_x^m u + \\ &\int_\Omega \partial_x \partial_x^m (b - b_\infty) \, \partial_x^m u. \end{aligned}$$

The first term at the right-hand side is bounded through

$$\begin{aligned} -\int_\Omega \partial_x^m v \, \omega \, \partial_x^m u &\leqslant \|\partial_x^m v \langle z \rangle^{-1}\|_{L^2} \|\omega \langle z \rangle_{L^\infty} \|\partial_x^m u\|_{L^2} \\ &\leqslant C \|\partial_x^{m+1} u\|_{L^2} \|\omega\|^2_{H_1^m(\Omega)} \leqslant \eta \|\partial_x^{m+1} u\|^2_{L^2} + C_\eta \|\omega\|^4_{H_1^m(\Omega)}. \end{aligned}$$

The second and third terms can be treated with the same ideas as I_u and I_v above. We find

$$-\int_\Omega \sum_{1 \leqslant |\alpha| \leqslant m} \binom{m}{\alpha} \partial_x^\alpha u \, \partial_x^{m+1-\alpha} u \, \partial_x^m u - \int_\Omega \sum_{1 \leqslant |\alpha| \leqslant m-1} \binom{m}{\alpha} \partial_x^\alpha v \, \partial_x^{m-\alpha} \, \omega \partial_x^m u \leqslant C \|\omega\|^3_{H_1^m(\Omega)}.$$

It remains to handle the magnetic term. We use the identity $\partial_x^{m+1} u = -\partial_x^m \partial_z^2 b$ as follows:

$$\begin{aligned}\int_\Omega \partial_x \partial_x^m (b - b^\infty)\, \partial_x^m u &= -\int_\Omega \partial_x^m (b - b_\infty)\, \partial_x^{m+1} u = \int_\Omega \partial_x^m (b - b^\infty) \partial_x^m \partial_z^2 b \\ &= -\int_\Omega |\partial_z \partial_x^m b|^2 + \int_{\mathbb{T}} \partial_x^m b^\infty \, \partial_z \partial_x^m b|_{z=0}.\end{aligned}$$

Finally, to control the second term at the left-hand side, we write

$$\int_{\mathbb{T}} \partial_x^m b^\infty \, \partial_z \partial_x^m b|_{z=0} \leqslant \|\partial_x^m b^\infty\|_{L^2(\mathbb{T})} \|\partial_z \partial_x^m b|_{z=0}\|_{L^2(\mathbb{T})}$$

and write

$$\|\partial_z \partial_x^m b|_{z=0}\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} \left| \int_0^{+\infty} \partial_x^{m+1} u \, dz \right|^2 dx \leqslant \int_0^{+\infty} \langle z \rangle^{-2} dz \, \|\partial_x^{m+1} u \, \langle z \rangle\|_{L^2}^2.$$

By Young's inequality, we have for any $\eta > 0$:

$$\int_\Omega \partial_x \partial_x^m (b - b_\infty)\, \partial_x^m u \leqslant -\int_\Omega |\partial_z \partial_x^m b|^2 + \eta \|\partial_x^{m+1} u \, \langle z \rangle\|_{L^2}^2 + C_\eta.$$

Gathering the previous bounds yields the proposition.

Corollary 4.2.4. *Let $m \geqslant 4$ be an even integer and $\varepsilon \in [0,1]$. There exists $C, C' > 0$ (independent of ε) depending on m and b^∞ such that*

$$\begin{aligned}&\frac{d}{dt}\|\omega\|_{H_1^m(\Omega)}^2 + \varepsilon^2 \|\partial_x \omega\|_{H_1^m(\Omega)}^2 + \|\partial_z \omega\|_{H_1^m(\Omega)}^2 + \frac{1}{2}\|\partial_x u\|_{H_1^m(\Omega)}^2 \\ &+ C\Big(\frac{d}{dt}\|\partial_x^m u\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\partial_z \partial_x^m b\|_{L^2(\Omega)}^2\Big) \leqslant C' \left(1 + \|\omega(t,\cdot)\|_{H_1^m(\Omega)}^m\right).\end{aligned} \tag{4.24}$$

Proof. By the previous propositions (and the remark after Proposition 4.2.2): there exists $\mathcal{C}$ such that

$$\begin{aligned}&\frac{d}{dt}\|\omega\|_{H_1^m(\Omega)}^2 + \varepsilon^2 \|\partial_x \omega\|_{H_1^m(\Omega)}^2 + \|\partial_z \omega\|_{H_1^m(\Omega)}^2 + \|\partial_x u\|_{H_1^m(\Omega)}^2 \\ &\leqslant \mathcal{C} \left(1 + \|\omega(t,\cdot)\|_{H_1^m(\Omega)}^m + \|\partial_z \partial_x^m b\|_{L^2}^2\right)\end{aligned} \tag{4.25}$$

and for all $\eta > 0$, there exists C_η such that

$$\frac{d}{dt}\|\partial_x^m u\|_{L^2(\Omega)}^2 + \|\partial_z \partial_x^m b\|_{L^2(\Omega)}^2 \leqslant C_\eta \left(1 + \|\omega(t,\cdot)\|_{H_1^m(\Omega)}^4\right) + \eta \|\partial_x^{m+1} u \langle z \rangle\|_{L^2}^2.$$

We multiply the second inequality by $2\mathcal{C}$ and add it to the first so that

$$\begin{aligned}&\frac{d}{dt}\|\omega\|_{H_1^m(\Omega)}^2 + \varepsilon^2 \|\partial_x \omega\|_{H_1^m(\Omega)}^2 + \|\partial_z \omega\|_{H_1^m(\Omega)}^2 + \|\partial_x u\|_{H_1^m(\Omega)}^2 + 2\mathcal{C}\frac{d}{dt}\|\partial_x^m u\|_{L^2(\Omega)}^2 + \mathcal{C}\|\partial_z \partial_x^m b\|_{L^2(\Omega)}^2 \\ &\leqslant \mathcal{C}\left(1 + \|\omega(t,\cdot)\|_{H_1^m(\Omega)}^m + \|\partial_z \partial_x^m b\|_{L^2}^2\right) + 2\mathcal{C}\left(C_\eta \left(1 + \|\omega(t,\cdot)\|_{H_1^m(\Omega)}^4\right) + \eta \|\partial_x^{m+1} u \langle z \rangle\|_{L^2}^2\right).\end{aligned}$$

Then, one takes η small enough so that the term $2\mathcal{C}\eta\|\partial_x^{m+1}u\langle z\rangle\|_{L^2}^2$ can be absorbed by $\|\partial_x u\|_{H_1^m(\Omega)}^2$. $\square$

4.2.3 Well-posedness for the mixed Prandtl/Shercliff Equations

We will, in this paragraph, re-use the notations $(u^\varepsilon, v^\varepsilon, b^\varepsilon, \omega^\varepsilon)$ to distinguish the approximate system from the original one. We remember that all the calculations above were performed with these quantities.

Existence: The existence of a solution to the system (4.5) is a standard consequence of the uniform estimates established on the approximate system (4.11). From such estimates and the Gronwall lemma, one can easily show that there exists a time $T > 0$ and some $M > 0$ such that for all $t \in [0, \min(T, T_\varepsilon)]$ (where T_ε is the maximal time of existence of the approximate solution), one has

$$\|\omega^\varepsilon(t)\|_{H_1^m(\Omega)}^2 + \|u^\varepsilon(t)\|_{H^m(\Omega)}^2 + \int_0^t \left(\varepsilon^2\|\partial_x\omega^\varepsilon(t)\|_{H_1^m(\Omega)}^2 + \|u^\varepsilon\|_{H_m^1(\Omega)}^2 + \|\partial_z\partial_x^m b^\varepsilon\|_{L^2(\Omega)}^2\right) \leqslant M. \tag{4.26}$$

This constant M depends on $\|\omega_0\|_{H_1^m(\Omega)}$, $\|u_0\|_{H^m(\Omega)}$ and b^∞. In particular, if one had $T_\varepsilon < T$, the standard blow-up criterion that goes with Proposition 4.2.1, namely

$$\limsup_{t\to T_\varepsilon} \|\omega^\varepsilon(t)\|_{H_1^m(\Omega)} + \|u^\varepsilon(t)\|_{H^m(\Omega)} = +\infty$$

would yield a contradiction. Thus, $T_\varepsilon \geqslant T$, namely, there exists a time T independent on ε on which the approximate systems are all well-posed. From there, existence of a solution to the exact system (4.5) follows by classical compactness arguments relying on Aubin-Lions lemma. We do not provide further details.

Uniqueness: Uniqueness of the Cauchy problem is a straightforward consequence of the following

Proposition 4.2.4 (L^2 Comparison Principle)**.** *For any $m \geqslant 4$, and smooth $b^\infty = b^\infty(t,x)$, let $(u_i, v_i, \omega_i, b_i)$, $i = 1, 2$ two solutions of (4.5). Let us define the following variables:*

$$\tilde{u} := u_1 - u_2, \quad \tilde{v} := v_1 - v_2, \quad \tilde{\omega} := \omega_1 - \omega_2, \quad \tilde{b} := b_1 - b_2.$$

Then we have:

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\omega}\|_{L^2}^2\right) \leqslant C\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\omega}\|_{L^2}^2\right).$$

Proof. The equations on $u_{1,2}$ and on $\omega_{1,2}$ are

$$\partial_t u_i + u_i \partial_x u_i + v \partial_z u_i - \frac{\mathrm{Ha}}{\mathrm{Re}} \partial_z^2 u_i = \partial_x (b_i - b^\infty) \tag{4.27}$$

$$\partial_t \omega_i + u_i \partial_x \omega_i + v \partial_z \omega_i - \partial_z^2 \omega_i = \partial_x \partial_z b_i. \tag{4.28}$$

From there,

$$\partial_t \tilde{u} - \partial_z^2 \tilde{u} = \partial_x \tilde{b} - \tilde{u} \partial_x u_1 - \tilde{v} \partial_z u_1, \tag{4.29}$$

$$\partial_t \tilde{\omega} - \partial_z^2 \tilde{\omega} = \partial_x \partial_z \tilde{b} - u_1 \partial_x \tilde{\omega} - \tilde{u} \partial_x \omega_2 - v_1 \partial_z \tilde{\omega} - \tilde{v} \partial_z \omega_2. \tag{4.30}$$

Multiplication (respectively) by $\tilde{u}$ and $\tilde{\omega}$, followed by integration over $\Omega = \mathbb{T} \times \mathbb{R}_+$ and standard integration by parts leads to

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \|\partial_z \tilde{u}\|_{L^2}^2 = \int_\Omega \partial_x \tilde{b}\, \tilde{u} + - \int_\Omega \tilde{u} \partial_x u_1 \tilde{u} - \int_\Omega \tilde{v} \partial_z u_1 \tilde{u}, \tag{4.31}$$

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\omega}\|_{L^2}^2 + \|\partial_z \tilde{\omega}\|_{L^2}^2 = \int_\Omega \partial_x \partial_z \tilde{b}\, \tilde{\omega} - \int_\Omega \tilde{u} \partial_x \omega_2 \tilde{\omega} - \int_\Omega \tilde{v} \partial_z \omega_2 \tilde{\omega}. \tag{4.32}$$

The two magnetic terms in the two energy estimates can be turned into

$$\int_\Omega \partial_x \tilde{b}\, \tilde{u} = - \int_\Omega \tilde{b}\, \partial_x \tilde{u} = \int_\Omega \tilde{b}\, \partial_z^2 \tilde{b} = - \int_\Omega |\partial_z \tilde{b}|^2, \tag{4.33}$$

$$\int_\Omega \partial_x \partial_z \tilde{b}\, \tilde{\omega} = - \int_\Omega \partial_x \partial_z^2 \tilde{b}\, \tilde{u} = + \int_\Omega \partial_x^2 \tilde{u}\, \tilde{u} = - \int_\Omega |\partial_x \tilde{u}|^2. \tag{4.34}$$

As regards the right-hand side terms, we find:

$$\left| \int_\Omega \tilde{u} \partial_x u_1 \tilde{u} \right| \leqslant \|\partial_x u_1\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 \leqslant C \|\tilde{u}\|_{L^2}^2,$$

$$\left| \int_\Omega \tilde{v} \partial_z u_1 \tilde{u} \right| = \left| \int_\Omega \frac{\tilde{v}}{\langle z \rangle} \partial_z u_1 \langle z \rangle \tilde{u} \right| \leqslant \|\omega_1 \langle z \rangle\|_{L^\infty} \left\| \frac{\tilde{v}}{\langle z \rangle} \right\|_{L^2} \|\tilde{u}\|_{L^2} \leqslant C \left\| \partial_x \tilde{u} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2},$$

while

$$\left| - \int_\Omega \tilde{u} \partial_x \omega_2 \tilde{\omega} \right| \leqslant \|\partial_x \omega_2\|_{L^\infty} \|\tilde{u}\|_{L^2} \|\tilde{\omega}\|_{L^2} \leqslant C \|\partial_x \omega_2\|_{L^\infty} \|\tilde{u}\|_{L^2} \|\tilde{\omega}\|,$$

$$\left| - \int_\Omega \tilde{v} \partial_z \omega_2 \tilde{\omega} \right| = \left| \int_\Omega \frac{\tilde{v}}{\langle z \rangle} \partial_z \omega_2 \langle z \rangle \tilde{\omega} \right| \leqslant \|\partial_z \omega_2 \langle z \rangle\|_{L^\infty} \left\| \frac{\tilde{v}}{\langle z \rangle} \right\|_{L^2} \|\tilde{\omega}\|_{L^2} \leqslant C \left\| \partial_x \tilde{u} \right\|_{L^2} \left\| \tilde{\omega} \right\|_{L^2}.$$

Hence, we get

$$\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \|\partial_z \tilde{u}\|_{L^2}^2 + \|\partial_z \tilde{b}\|_{L^2}^2 \leqslant C \left(\|\tilde{u}\|_{L^2}^2 + \left\| \partial_x \tilde{u} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2} \right) \\ \dfrac{1}{2} \dfrac{d}{dt} \|\tilde{\omega}\|_{L^2}^2 + \|\partial_z \tilde{\omega}\|_{L^2}^2 + \|\partial_x \tilde{u}\|_{L^2}^2 \leqslant C \left(\|\tilde{u}\|_{L^2} \|\tilde{\omega}\|_{L^2} + \left\| \partial_x \tilde{u} \right\|_{L^2} \left\| \tilde{\omega} \right\|_{L^2} \right). \end{cases} \tag{4.35}$$

Using Young's inequality, we get that for all $\eta > 0$,

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\tilde{u}\|_{L^2}^2 + \|\partial_z \tilde{u}\|_{L^2}^2 + \|\partial_z \tilde{b}\|_{L^2}^2 \leqslant \eta \|\partial_x \tilde{u}\|_{L^2}^2 + C_\eta \|\tilde{u}\|_{L^2}^2 \\ \frac{1}{2}\frac{d}{dt}\|\tilde{\omega}\|_{L^2}^2 + \|\partial_z \tilde{\omega}\|_{L^2}^2 + \|\partial_x \tilde{u}\|_{L^2}^2 \leqslant C\|\tilde{u}\|_{L^2}^2 + \eta \|\partial_x \tilde{u}\|_{L^2}^2 + C_\eta \|\tilde{\omega}\|_{L^2}^2, \end{cases} \tag{4.36}$$

where C_η depends on η, but C does not. If we sum up the two equations and take η small enough, the inequality of the proposition follows. □

This concludes the well-posedness analysis of the Prandtl-Shercliff system.

4.3 Fully nonlinear MHD layer

To conclude this chapter, we discuss briefly the fully nonlinear regime of the MHD layers, described by

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u = S\mathbf{b}\cdot\nabla b - \partial_x p^\infty, \\ \partial_t \mathbf{b} - \nabla^\perp(\mathbf{u}\times\mathbf{b}) - \dfrac{\mathrm{Re}}{\mathrm{Rm}}\partial_z^2 \mathbf{b} = 0, \\ \partial_x u + \partial_z v = \operatorname{div} \mathbf{b} = 0, \\ u|_{z=0} = v|_{z=0}, \quad \mathbf{b}|_{z=0} = \mathbf{e}_x, \\ u \to u^\infty, \quad b \to b^\infty, \quad \text{as } z \to +\infty, \end{cases} \tag{4.37}$$

over $\mathbb{R}\times\mathbb{R}^+$ or $\mathbb{T}\times\mathbb{R}^+$. We have already studied a simple linearisation of (4.37) in the previous chapters and seen how the magnetic term restores Sobolev well-posedness for such linearization. This stabilizing effect of the magnetic field has been nicely emphasized in recent and extensive studies by Cheng-Jie Liu, Feng Xie and Tong Yang. They have notably established the local in time Sobolev well-posedness of such boundary layer systems, under positivity of the tangential component of the initial magnetic field. Moreover, beyond this well-posedness of the boundary layer system, they have established the stability of the associated boundary layer expansions in the MHD equations [41, 42], in sharp contrast with the situation observed for the traditional problem of stability of the Prandtl expansions within the Navier-Stokes equations.

As a little complement, we provide a calculation for the inviscid version of the linearised system.

4.3.1 The linearised zero viscosity case

The purpose of this last section is to detail an explicit calculation of the inviscid case. Interestingly, when we consider the inviscid case (that is, without the diffusion terms in both equations) for the linearised MHD system, we find that the latter can be explicitly solved.

We will consider the system to have been linearised around

$$U(z) \text{ and } B(z),$$

under the hypothesis $U, B \in W^{1,\infty}(\mathbb{R}^+)$.

In this section, we will denote the two components of the velocity as (u, v) and those for the magnetic field as (b, c).

As a preliminary remark, we underline that the initial conditions will always be noted as

$$u(0,x,z) = u_0(x,z), \quad v(0,x,z) = v_0(x,z), \qquad b(0,x,z) = b_0(x,z), \quad c(0,x) = c_0(x,z), \tag{4.38}$$

and that the boundary conditions are the following (remembering that the boundary is $\{z = 0\}$)

$$u(t,x,0) = 0, \quad v(t,x,0) = 0, \qquad [b]|_{z=0} = 0, \quad [c]|_{z=0} = 0. \tag{4.39}$$

As anticipated, using these notations, we rewrite the system without the two diffusion terms, that is

$$\begin{cases} \partial_t u + U\partial_x u + vU_z - B\partial_x b - cB_z = 0 & (1) \\ \partial_t b + U\partial_x b + vB_z - B\partial_x u - cU_z = 0 & (2) \\ \partial_x u + \partial_z v = 0, \qquad \partial_x b + \partial_z c = 0, \end{cases} \tag{4.40}$$

and we Fourier transform in the x variable

$$\begin{cases} i\xi\hat{u} + \partial_z\hat{v} = 0, \qquad i\xi\hat{b} + \partial_z\hat{c} = 0 \quad \text{so that we have} \quad \hat{u} = -\frac{\partial_z\hat{v}}{i\xi}, \qquad \hat{b} = -\frac{\partial_z\hat{c}}{i\xi} \\ \partial_t\hat{u} + Ui\xi\hat{u} + \hat{v}U_z - Bi\xi\hat{b} - \hat{c}B_z = 0 & (1) \\ \partial_t\hat{b} + Ui\xi\hat{b} + \hat{v}B_z - Bi\xi\hat{u} - \hat{c}U_z = 0. & (2) \end{cases} \tag{4.41}$$

Now we rewrite everything in function of $\hat{v}$ and $\hat{c}$

$$\begin{cases} \hat{u} = -\frac{\partial_z\hat{v}}{i\xi}, \qquad \hat{b} = -\frac{\partial_z\hat{c}}{i\xi} \\ -\frac{\partial_t\partial_z\hat{v}}{i\xi} - U\partial_z\hat{v} + \hat{v}U_z + B\partial_z\hat{c} - \hat{c}B_z = 0 & (1) \\ -\frac{\partial_t\partial_z\hat{c}}{i\xi} - U\partial_z\hat{c} + \hat{v}B_z + B\partial_z\hat{v} - \hat{c}U_z = 0, & (2) \end{cases} \tag{4.42}$$

that is the same as

$$\begin{cases} \hat{u} = -\frac{\partial_z\hat{v}}{i\xi}, \qquad \hat{b} = -\frac{\partial_z\hat{c}}{i\xi} \\ -\frac{\partial_t\partial_z\hat{v}}{i\xi} - U^2\partial_z\left(\frac{\hat{v}}{U}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 & (1) \\ -\frac{\partial_t\partial_z\hat{c}}{i\xi} - \partial_z(U\hat{c}) + \partial_z(\hat{v}B) = 0. & (2) \end{cases} \tag{4.43}$$

Let us now consider equation (2)

$$-\frac{\partial_t\partial_z\hat{c}}{i\xi} - \partial_z(U\hat{c}) + \partial_z(\hat{v}B) = 0 \tag{4.44}$$

which we can integrate between 0 and z in the vertical variable to obtain

$$-\frac{\partial_t\hat{c}}{i\xi} - U\hat{c} + \hat{v}B = 0, \tag{4.45}$$

using the fact that $\hat{c}$ is the boundary layer vertical component, which is 0 at $z = 0$. As a consequence, we can express $\hat{v}$ in function of $\hat{c}$ and therefore work on the first equation with just one unknown function:

$$\begin{cases} \hat{v} = \frac{\partial_t\hat{c}}{i\xi B} + U\frac{\hat{c}}{B} \\ -\frac{\partial_t\partial_z\hat{v}}{i\xi} - U^2\partial_z\left(\frac{\hat{v}}{U}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 \quad (1) \\ \hat{u} = -\frac{\partial_z\hat{v}}{i\xi}, \qquad \hat{b} = -\frac{\partial_z\hat{c}}{i\xi}. \end{cases} \tag{4.46}$$

We only deal with equation (1), that is

$$-\frac{\partial_t\partial_z}{i\xi}\left(\frac{\partial_t\hat{c}}{i\xi B} + U\frac{\hat{c}}{B}\right) - U^2\partial_z\left(\frac{1}{U}\frac{\partial_t\hat{c}}{i\xi B} + \frac{\hat{c}}{B}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 \tag{4.47}$$

$$-\frac{\partial_t^2}{(i\xi)^2}\partial_z\left(\frac{\hat{c}}{B}\right) - \frac{\partial_t}{i\xi}\partial_z\left(U\frac{\hat{c}}{B}\right) - U^2\frac{\partial_t}{i\xi}\partial_z\left(\frac{1}{U}\frac{\hat{c}}{B}\right) - U^2\partial_z\left(\frac{\hat{c}}{B}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 \tag{4.48}$$

and writing explicitly all the derivatives one finds

$$-\frac{\partial_t^2}{(i\xi)^2}\partial_z\left(\frac{\hat{c}}{B}\right) - \frac{\partial_t}{i\xi}U\partial_z\left(\frac{\hat{c}}{B}\right) - \frac{\partial_t}{i\xi}U_z\left(\frac{\hat{c}}{B}\right) - U\frac{\partial_t}{i\xi}\partial_z\left(\frac{\hat{c}}{B}\right) + U^2\left(\frac{U_z}{U^2}\right)\frac{\partial_t}{i\xi}\left(\frac{\hat{c}}{B}\right) - \tag{4.49}$$

$$-U^2\partial_z\left(\frac{\hat{c}}{B}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 \tag{4.50}$$

which can be simplified into

$$-\frac{\partial_t^2}{(i\xi)^2}\partial_z\left(\frac{\hat{c}}{B}\right) - 2U\frac{\partial_t}{i\xi}\partial_z\left(\frac{\hat{c}}{B}\right) - U^2\partial_z\left(\frac{\hat{c}}{B}\right) + B^2\partial_z\left(\frac{\hat{c}}{B}\right) = 0 \tag{4.51}$$

$$\frac{\partial_t^2}{(i\xi)^2}\partial_z\left(\frac{\hat{c}}{B}\right) + 2U\frac{\partial_t}{i\xi}\partial_z\left(\frac{\hat{c}}{B}\right) + (U^2 - B^2)\partial_z\left(\frac{\hat{c}}{B}\right) = 0. \tag{4.52}$$

Now, we can easily solve this system in a direct way, and write the exact solutions. From the latter expression one gets $\partial_z\left(\frac{\hat{c}}{B}\right) = c_1(z)e^{(-U+B)i\xi t} + c_2(z)e^{(-U-B)i\xi t}$, and solving it with the initial data gives:

$$\hat{c} = B\int_0^z \left[\frac{\partial_z(\hat{c}_0 - \hat{v}_0)}{2B} - \frac{\hat{c}_0}{B}\left(\frac{B_z - U_z}{2B}\right)\right] e^{-(U+B)ti\xi} dz' + \tag{4.53}$$

$$+ B\int_0^z \left[\frac{\partial_z(\hat{c}_0 + \hat{v}_0)}{2B} - \frac{\hat{c}_0}{B}\left(\frac{B_z + U_z}{2B}\right)\right] e^{-(U-B)ti\xi} dz', \tag{4.54}$$

$$\hat{v} = B\int_0^z \left[\frac{\partial_z(\hat{c}_0 - \hat{v}_0)}{2B} - \frac{\hat{c}_0}{B}\left(\frac{B_z - U_z}{2B}\right)\right] (U(z) - U(z') - B(z'))e^{-(U+B)ti\xi} dz' + \tag{4.55}$$

$$+ B\int_0^z \left[\frac{\partial_z(\hat{c}_0 + \hat{v}_0)}{2B} - \frac{\hat{c}_0}{B}\left(\frac{B_z + U_z}{2B}\right)\right] (U(z) - U(z') + B(z'))e^{-(U-B)ti\xi} dz', \tag{4.56}$$

which are the explicit solutions.

Chapter 5

Numerical Study

In this chapter we carry on two simulations on the Prandtl system and the mixed Prandtl / Shercliff system, in order to check numerically the stabilizing effect of the magnetic field that was established mathematically in the previous chapters. For both systems, we will use the perfect conductor boundary conditions, that allow more efficient computations. We begin with some remarks on the good properties of the linearisation, and we present the simulations afterwards.

5.1 Mixed Prandtl/Shercliff system

To begin, we want to rewrite the linearised mixed Prandtl/Shercliff system in an appropriate form to perform some simulations. We consider the 2D nonlinear version of the system in the upper half space $\mathbb{R} \times \mathbb{R}^+$. We take all the constants involved of size 1, and we model the boundary as perfect conductor. The system reads:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u = \partial_x b - \partial_x p^\infty, \\ \partial_x u + \partial_z^2 b = 0, \\ \partial_x u + \partial_z v = 0, \\ u|_{z=0} = v|_{z=0} = b|_{z=0} = 0, \\ u \to u^\infty, \quad \partial_z b \to 0, \qquad \text{as } z \to +\infty. \end{cases} \tag{5.1}$$

The first step is to consider a linearisation around

$$u = U(z), \quad v = 0, \quad b = b^\infty \text{ constant}.$$

We assume that U connects 0 at $z = 0$ to some constant u^∞ at infinity. The linearized system reads (we will only write the two first equations in the following)

$$\begin{cases} \partial_t u + U\partial_x u + vU' - \partial_z^2 u = \partial_x b \\ \partial_x u + \partial_z^2 b = 0. \end{cases} \tag{5.2}$$

Consistently, one must have $\partial_x p^\infty = 0$. We will now begin to rewrite the first equation of the former system as an evolution equation in terms of v, using the incompressibility condition $\partial_x u + \partial_z v = 0$.

System (5.2) has constant coefficients in t and x, so that we can perform a Fourier analysis: we look for solutions in the form

$$u(t,x,z) = e^{ikx}\hat{u}(kt,z), \qquad v(t,x,z) = ke^{ikx}\hat{v}(kt,z); \tag{5.3}$$
$$b(t,x,z) = e^{ikx}\hat{b}(kt,z), \qquad c(t,x,z) = ke^{ikx}\hat{c}(kt,z). \tag{5.4}$$

Note that we are rescaling the vertical component to be coherent with the incompressibility condition. We then express the whole first equation in terms of $\hat{v}$, using that same condition

$$i\hat{u}(kt,z) = -\partial_z\hat{v}(kt,z),$$

which we substitute in the evolution equation

$$\begin{cases} ik\partial_t\partial_z\hat{v} - kU\partial_z\hat{v} + k\hat{v}U' - i\partial_z^2\partial_z\hat{v} = ik\hat{b} \\ -k\partial_z\hat{v} + \partial_z^2\hat{b} = 0, \end{cases} \tag{5.5}$$

we then divide by ik the first equation and rewrite the second

$$\begin{cases} \partial_t\partial_z\hat{v} + iU\partial_z\hat{v} - i\hat{v}U' - \dfrac{1}{k}\partial_z^3\hat{v} = \hat{b} \\ k\partial_z^{-2}\partial_z\hat{v} = \hat{b}. \end{cases} \tag{5.6}$$

Deriving the first equation to obtain an even number of derivatives and injecting the second one we get

$$\partial_t\partial_z^2\hat{v} + iU'\partial_z\hat{v} + iU\partial_z^2\hat{v} - i\hat{v}U'' - i\partial_z\hat{v}U' - \frac{1}{k}\partial_z^4\hat{v} = k\partial_z\partial_z^{-2}\partial_z\hat{v}. \tag{5.7}$$

Thanks to the boundary conditions on the magnetic field, one has $k\partial_z(\partial_z^{-2})\partial_z\hat{v} = k\hat{v}$, which yields the expression we will use to perform our numerical simulations:

$$(\partial_t + iU)\partial_z^2\hat{v} - \left(iU'' + k\right)\hat{v} - \frac{1}{k}\partial_z^4\hat{v} = 0. \tag{5.8}$$

Before presenting the numerics, let us stress that we can recover a stabilizing estimate from (5.8). We multiply it by $\bar{\hat{v}}$ to get

$$
\begin{aligned}
&\int \partial_t \partial_z^2 \hat{v}\bar{\hat{v}} + i\int U\partial_z^2\hat{v}\bar{\hat{v}} - i\int \hat{v}U''\bar{\hat{v}} - \frac{1}{k}\int \partial_z^4\hat{v}\bar{\hat{v}} - k\int \hat{v}\bar{\hat{v}} = 0 && (5.9)\\
&-\frac{1}{2}\int \partial_t|\partial_z\hat{v}|^2 - i\int U|\partial_z\hat{v}|^2 - i\int U'\partial_z\hat{v}\bar{\hat{v}} - i\int U''|\hat{v}|^2 + \frac{1}{k}\int \partial_z^3\hat{v}\partial_z\hat{v} - k\int |\hat{v}|^2 = 0 && (5.10)\\
&\frac{1}{2}\int \partial_t|\partial_z\hat{v}|^2 + i\int U|\partial_z\hat{v}|^2 + i\int U'\partial_z\hat{v}\bar{\hat{v}} + i\int U''|\hat{v}|^2 + \frac{1}{k}\int |\partial_z^2\hat{v}|^2 + k\int |\hat{v}|^2 = 0 && (5.11)\\
&\frac{1}{2}\frac{d}{dt}\|\partial_z\hat{v}\|_{L^2}^2 + \frac{1}{k}\|\partial_z^2\hat{v}\|_{L^2}^2 + k\|\hat{v}\|_{L^2}^2 = i\int U'\partial_z\hat{v}\hat{v} - i\int U|\partial_z\hat{v}|^2 - i\int U''|\hat{v}|^2 && (5.12)\\
&\frac{1}{2}\frac{d}{dt}\|\partial_z\hat{v}\|_{L^2}^2 + \frac{1}{k}\|\partial_z^2\hat{v}\|_{L^2}^2 + k\|\hat{v}\|_{L^2}^2 \leqslant && (5.13)\\
&\leqslant \|U'\|_{L^\infty}\frac{1}{2\eta}\|\partial_z\hat{v}\|_{L^2}^2 + \|U'\|_{L^\infty}\frac{\eta}{2}\|\hat{v}\|_{L^2}^2 + \|U\|_{L^\infty}\|\partial_z\hat{v}\|_{L^2}^2 + \|U''\|_{L^\infty}\|\hat{v}\|_{L^2}^2, && (5.14)
\end{aligned}
$$

so that taking $\eta = \frac{2k\mathrm{Ha} - 2\mathrm{Re}\,\|U''\|_{L^\infty}}{\mathrm{Re}\,\|U'\|_{L^\infty}}$ we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_z\hat{v}\|_{L^2}^2 + \frac{1}{k}\|\partial_z^2\hat{v}\|_{L^2}^2 \leqslant \left(\frac{\|U'\|_{L^\infty}}{2\eta} + \|U\|_{L^\infty}\right)\|\partial_z\hat{v}\|_{L^2}^2 \tag{5.15}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|\partial_z\hat{v}\|_{L^2}^2 \leqslant C\|\partial_z\hat{v}\|_{L^2}^2, \tag{5.16}$$

which implies thanks to Gronwall inequality that

$$\|\partial_z\hat{v}\|_{L^2}^2 \leqslant 2C\|\partial_z\hat{v}_0\|_{L^2}^2 e^{Ct}, \tag{5.17}$$

which closes the estimate and the stability result.

5.2 Simulation of the stability mechanism and comparison

Let us now go back to (5.8)

$$\begin{cases} (\partial_\theta + iU)\partial_z^2\hat{v} - \left(iU'' + k\right)\hat{v} - \frac{1}{k}\partial_z^4\hat{v} = 0, \\ i\hat{u}(kt,z) = -\partial_z\hat{v}(kt,z), \end{cases} \tag{5.18}$$

where we remember that $\hat{v} = \hat{v}(\theta, z)$, with $\theta := kt$. We have performed some numerical simulations on this equation, which we can easily compare to the Prandtl system (cfr [18]). Indeed, taking the linearisation of the Prandtl equation around shear layer flows:

$u = U(z), v = 0$ and denoting the velocity components as (r, w)

$$\begin{cases} \partial_t r + U\partial_x r + wU' - \partial_z^2 r = 0, \\ \partial_x r + \partial_z w = 0, \\ r|_{z=0} = w|_{z=0} = 0, \\ r \to 0 \quad \text{as } z \to +\infty, \end{cases} \tag{5.19}$$

and once performed the very same operations exposed in the previous paragraph to obtain (5.8), the two first equations are rewritten as

$$\begin{cases} (\partial_\theta + iU)\partial_z^2 \hat{w} - iU''\hat{w} - \dfrac{1}{k}\partial_z^4 \hat{w} = 0 \\ i\hat{r}(kt, z) = -\partial_z \hat{w}(kt, z), \end{cases} \tag{5.20}$$

that is, comparing term by term, an expression identical to (5.18) except for the term $k\hat{v}$. We will, now, calculate numerical solutions for both systems and compare them[1]. We already expect to find quite different behaviours, knowing that the Prandtl system is ill-posed and that the mixed Prandtl/Shercliff is well-posed.

We take for the base velocity profile: $U(z) = 2ze^{-z^2}$. For each, we have discretized these equations in space using finite differences on a stretched grid and in time through a Crank-Nicholson scheme. We will provide a brief sketch of the algorithm.

Let us define the linear operators L and L', from expression (5.18) and expression (5.20) respectively

$$L := -\left(iU'' + k\right)\mathrm{Id} + iU\partial_z^2 - \frac{1}{k}\partial_z^4,$$

$$L' := -iU''\mathrm{Id} + iU\partial_z^2 - \frac{1}{k}\partial_z^4.$$

From this point on, the steps are exactly the same. We provide here some details for (5.18), that thus becomes

$$\partial_\theta \partial_z^2 \hat{v} = L(\hat{v}).$$

We rewrite it using the Crank-Nicholson method, which gives

$$\frac{\partial_z^2 \hat{v}_{n+1} - \partial_z^2 \hat{v}_n}{d\theta} = \frac{L(\hat{v}_{n+1})}{2} + \frac{L(\hat{v}_n)}{2},$$

so that, calling $A_2 := \frac{\partial_z^2}{d\theta} - \frac{L}{2}$ and $A_1 := \frac{\partial_z^2}{d\theta} + \frac{L}{2}$, one has

$$A_2 \hat{v}_{n+1} = A_1 \hat{v}_n.$$

[1]The author would like to thank Emmanuel Dormy for his precious programming advices on the code.

To perform the calculations we need to inverse the matrix A_2, which can be done exploiting the fact that, thanks to the even number of derivatives involved, the matrix is pentadiagonal. This allows to easily solve it as a linear system (the technique is similar to the tridiagonal case).

Starting from initial random data we compute the time evolution of (5.18) and (5.20) for different values of $k = \varepsilon^{-1}$ that range from 100 to 35000. For sufficiently large times, one observes that both numerical solutions $\hat{v}^{num}, \hat{w}^{num}$ behave like:

$$\hat{v}^{num}(\theta, z) \approx e^{i\omega_1^{num}(k)\theta}\hat{V}^{num}(z), \qquad \hat{w}^{num}(\theta, z) \approx e^{i\omega_2^{num}(k)\theta}\hat{W}^{num}(z), \tag{5.21}$$

in the sense that both

$$i\omega_1^{num}(k) = \frac{\hat{v}^{num}(\theta + \Delta\theta, z) - \hat{v}^{num}(\theta, z)}{\Delta\theta\hat{v}^{num}(\theta, z)} \text{ and } i\omega_2^{num}(k) = \frac{\hat{w}^{num}(\theta + \Delta\theta, z) - \hat{w}^{num}(\theta, z)}{\Delta\theta\hat{w}^{num}(\theta, z)} \tag{5.22}$$

get independent of θ and z. Of course, since

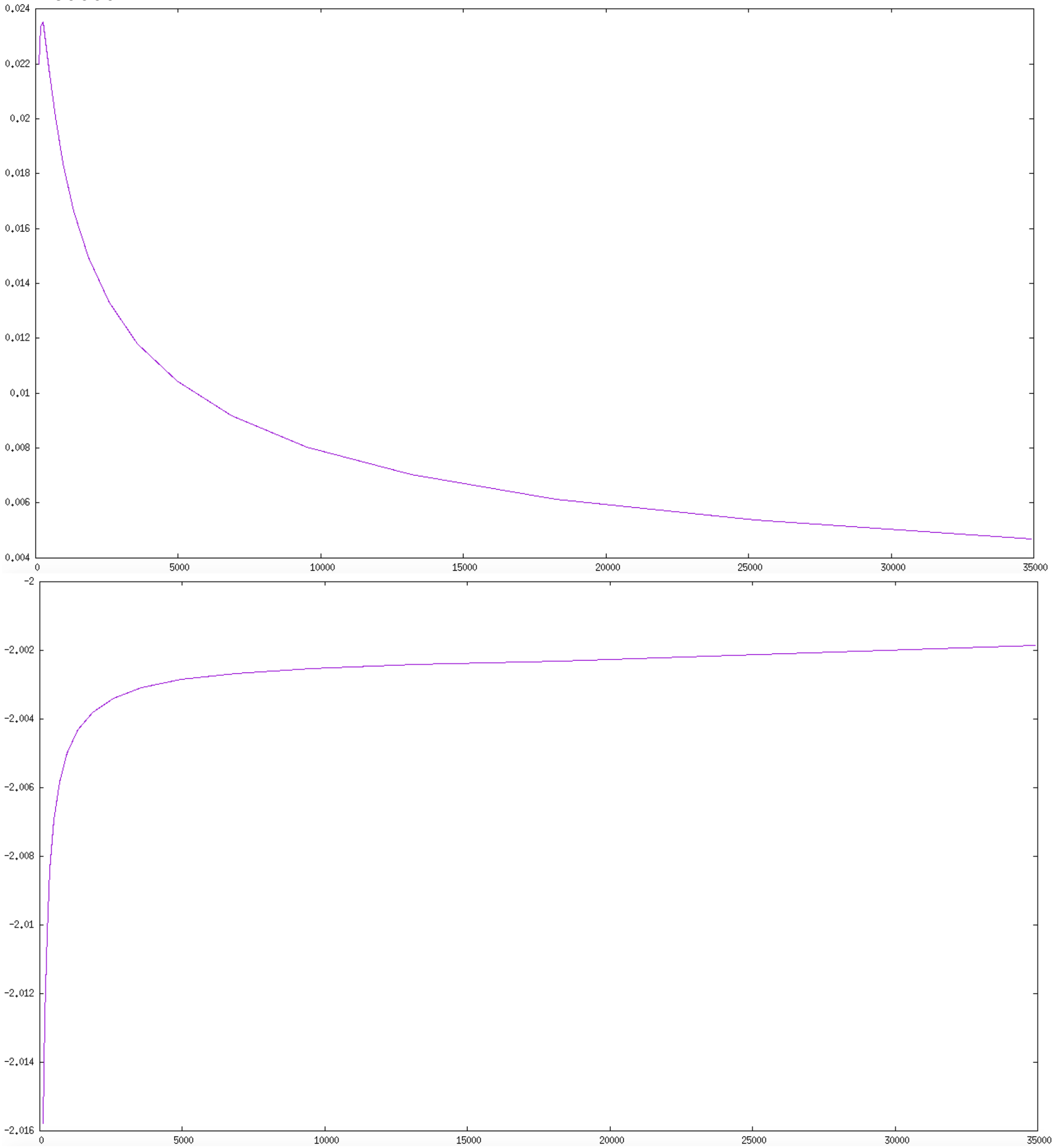
$$e^{i\omega_1^{num}(k)\theta}\hat{V}^{num}(z) = e^{i\Re(\omega_1^{num}(k))\theta}e^{-\Im(\omega_1^{num}(k))\theta}\hat{V}^{num}(z), \tag{5.23}$$

$$e^{i\omega_2^{num}(k)\theta}\hat{W}^{num}(z) = e^{i\Re(\omega_2^{num}(k))\theta}e^{-\Im(\omega_2^{num}(k))\theta}\hat{W}^{num}(z), \tag{5.24}$$

when we look at the sign of $-\Im(\omega_{1,2}^{num})$, we can deduce whether this Fourier mode will grow or decay exponentially with time. If we plot those two values (see figure 5.1) we find that $-\Im(\omega_2^{num})$, that is the one corresponding to the Prandtl equation is positive and decreasing to 0, whereas $-\Im(\omega_1^{num})$, corresponding to the Shercliff system, is negative and tends toward -2. A positive value will lead to the explosion of the Fourier mode, whereas a negative one will be exponentially small as time grows.

These calculations correspond to our expectations: the Shercliff flow presents a decreasing in time Fourier mode and the Prandtl flow an explosing Fourier mode in time.

Figure 5.1: Top: $-\Im(\omega_2^{num})$; bottom: $-\Im(\omega_1^{num})$; for both, we let k range between 100 and 35000.



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