Navier-Stokes equations with Navier boundary condition

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work was completed under the guidance of Prof. C. Amrouche at the University of Pau, France and Prof. M. Escobedo at University of Basque country, Bilbao.

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In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. C. Amrouche. **Prof. M. Escobedo.**

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Dedicated to all my teachers.

Contents

CHAPTER 1

Introduction

In this thesis, we consider the motion of an incompressible, viscous, Newtonian fluid in a bounded domain in $\mathbb{R}^3$. The fluid flow is described by the well-known Navier-Stokes equations:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad &\text{in } \Omega\times(0,T) \\ \operatorname{div}\boldsymbol{u} = 0 \quad &\text{in } \Omega\times(0,T). \end{aligned} \tag{1.1}$$

Here $\boldsymbol{u}(t,\boldsymbol{x})$ is the velocity of the fluid particle, passing through $\boldsymbol{x}\in\Omega$ at a time t, π is the pressure of the fluid, ν is the viscosity and $\boldsymbol{f}$ is the external force, acting on the fluid. As we have assumed here an incompressible flow, the density of the fluid is constant.

The French mathematician-engineer C. Navier provided these equations in 1823 in a paper [96] where he added in the Euler equation (where $\nu = 0$) the affects of attraction and repulsion between neighbouring molecules. For Navier, ν was simply a function of the molecular spacing to which he attached no particular physical significance, His seminal paper [96] was presented at the French Académie des Sciences and was well-received.

The equations for the motion of a viscous fluid were rederived by Cauchy in 1828 and by Poisson in 1829. However, the person whose name is now attached with Navier's for the viscous equation is the British mathematician-physicist George G. Stokes. In 1845, he published a derivation of the viscous equations in a manner that is followed in most of the current texts. Unlike Navier, he made it clear that the parameter ν has an important physical meaning: namely, ν measures the magnitude of the viscosity.

For the unsteady model, it is required to impose initial condition in order to define the evolution of the system. It is enough to have the initial condition on the velocity:

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \tag{1.2}$$

Since the pressure can be considered as a Lagrange multiplier associated to the incompressibility condition, there is no mathematical meaning to impose an initial value for it.

As the governing equations in (1.1) are differential equations in a bounded set, to make it well-posed we must precise some boundary condition as well. There are various types of boundaries, but we restrict ourselves to the most common one: the impermeable wall. Therefore the commonly accepted hypothesis reads:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma$$

which says that no fluid pass through the wall. $\boldsymbol{n}$ denotes the exterior unit normal to the boundary Γ. The behavior of the tangential component of the velocity is a more delicate issue. For many years, the *no-slip boundary condition*

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma \tag{1.3}$$

has been the most widely used, given its success in reproducing the standard velocity profiles for incompressible viscous fluid. But even before that, in 1823, Navier proposed a more general boundary condition that allows slip at the interface of the fluid and the solid wall. The boundary condition, which is in the following form:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T) \tag{1.4}$$

states that the tangential component of the velocity is proportional to the tangential stress at the boundary. Here $\mathbb{D} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the symmetric gradient. The proportionality constant α is called the friction coefficient; In literature, it is also customary to call $\frac{1}{\alpha}$ as slip length. Note that α may depend on various parameters, for example, on the boundary (in case of rough or porous wall), on the viscosity (as derived from the kinetic equation [91] and studied in [97]).

In the case where non-linear effects can be neglected in the Navier-Stokes equation (1.1) (*i.e.* when the Reynolds number is sufficiently small), the system reduces to the following Stokes equations:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T) \; ; \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \times (0,T); \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

This problem is linear with constant coefficients. Therefore, a large part of the mathematical study is contained in the steady Stokes problem:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{1.6}$$

in particular, concerning the existence and properties of the velocity and the pressure. In the following, we set $\nu = 1$.

We would like to first discuss some of the results, approaches and works done for the Navier-Stokes equations (1.1) with the Dirichlet boundary condition (1.3).

The mathematical foundations were laid in 1930's in the papers of J. Leray ([87], [88]) and continued by E. Hopf ([69]). The class of solutions

$$L^\infty(0,T;\boldsymbol{L}^2_\sigma(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1_{0,\sigma}(\Omega))$$

is called *Leray-Hopf class*, in their honor. Here the subscript σ denotes the divergence free condition. The Leray-Hopf class plays an important role in the theory of Navier-Stokes equations, as the norm of this space represents the energy of $\boldsymbol{u}$. However, the Leray-Hopf class seems to be quite large to have uniqueness of the weak solution. We mention the recent survey article by H. Jia and V. Šverák ([76]) concerning non-uniqueness results.

Different conditions can be imposed in order to obtain the uniqueness of weak solution. For example, a weak solution $\boldsymbol{u}$ which additionally satisfies the following regularity (Serrin's class)

$$\boldsymbol{u} \in L^s(0,T;\boldsymbol{L}^q(\Omega)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{q} = 1 \quad \text{and} \quad q > 3, s > 2$$

is unique. Thus constructing weak solutions within Serrin's class seems desirable and that leads to study the general L^p-theory of Navier-Stokes equations.

Another example of advantage of additional L^p-estimates for weak solution is shown by L. Escauriaza et al ([43]). In the case $\Omega = \mathbb{R}^3$, the authors proved that a solution in Leray-Hopf class which also belongs to the space

$$L^\infty(0,T;\boldsymbol{L}^3(\mathbb{R}^3))$$

is smooth in $[0,\infty) \times \mathbb{R}^3$. This space is often called as 'critical space' and 'Serrin class' for the Navier-Stokes equations, because the norm in this space is invariant under the natural scaling behavior of solutions of the Navier-Stokes equations. This is interesting in view of the famous open problem by Clay Mathematics Institute [46]: whether solutions of the Navier-Stokes equations, corresponding to an initial velocity field in the Schwartz space are smooth in $[0,\infty) \times \mathbb{R}^3$ and lie in $L^\infty(0,\infty;\boldsymbol{L}^3(\mathbb{R}^3))$ or not.

One basic method to study the L^p-theory is the development of the semigroup theory. If we denote the Stokes operator associated to the system (1.5) (with the boundary condition replaced by (1.3)) by A, the Stokes semigroup is then the solution operator $\left(e^{-tA}\right)_{t\geq 0}$ to (1.5) (boundary condition replaced by (1.3)). With this semigroup, the Navier-Stokes equations can be transformed into an integral equation via the variation of constants formula. A breakthrough to handle Navier-Stokes equations using this integral equation was by T. Kato and H. Fujita [79]. Although this work was done in the L^2-framework, it gave rise to the famous paper of Kato ([78]) which provides some L^p-theory. In this later work, Kato has shown the existence of a global strong solution of the Navier-Stokes equation in the critical space $L^\infty(0,\infty;\boldsymbol{L}^3(\mathbb{R}^3))$ if the initial data is small. To prove such an existence result, Kato defined an approximate scheme and obtained convergence using only the following two properties of the Stokes semigroup:

(i) $L^p - L^q$ estimate

$$\|e^{-tA}\boldsymbol{f}\|_{L^q(\mathbb{R}^3)} \leq Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|\boldsymbol{f}\|_{L^p(\mathbb{R}^3)}, \qquad \frac{3}{2} \leq p \leq q < \infty;$$

(ii) gradient estimate

$$\|\nabla e^{-tA}\boldsymbol{f}\|_{L^3(\mathbb{R}^9)} \leq Ct^{-\frac{1}{2}}\|\boldsymbol{f}\|_{L^3(\mathbb{R}^3)}.$$

Around the same time, Y. Giga proposed another similar iteration scheme [58] which produces strong solutions in the critical space. This scheme is based on the L^p-L^q-estimate of the Stokes semigroup and a different gradient estimate:

(iii) estimates of Stokes semigroup on a divergence form

$$\|e^{-tA}\mathbb{P}\ \mathrm{div}\ \boldsymbol{F}\|_{L^p(\mathbb{R}^3)} \leq Ct^{-\frac{1}{2}}\|\boldsymbol{F}\|_{L^p(\mathbb{R}^9)}, \qquad 1 < p < \infty.$$

Here $\mathbb{P}$ is the Helmholtz projection from $L^p(\mathbb{R}^3)$ onto the solenoidal vector field $L^p_\sigma(\mathbb{R}^3)$. To put (iii) into the picture, Giga used the well-known identity for the solenoidal vector field

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \mathrm{div}\ (\boldsymbol{u}\otimes\boldsymbol{u}).$$

We emphasize that this is a simple structural information that Giga exploited in his proof but was neglected by Kato.

Besides the techniques of Kato and Giga, there is another way to prove the existence of solution of Navier-Stokes problem in L^p-settings. This requires maximal regularity of the Stokes operator *i.e.* the inhomogeneous Stokes system (1.5) (boundary condition replaced by (1.3)) is considered for some $\boldsymbol{f}\in L^q(0,\infty;\boldsymbol{L}^p_\sigma(\Omega))$ and whether each of the term in the left hand side $\frac{\partial \boldsymbol{u}}{\partial t}$, $\Delta\boldsymbol{u}$ and $\nabla\pi$ belong to $L^q(0,\infty;\boldsymbol{L}^p_\sigma(\Omega))$ is investigated. If that is the case, then abstract arguments imply the following type of estimate

$$\|\frac{\partial \boldsymbol{u}}{\partial t}\|_{L^q(0,\infty;\boldsymbol{L}^p_\sigma(\mathbb{R}^3))} + \|\Delta \boldsymbol{u}\|_{L^q(0,\infty;\boldsymbol{L}^p_\sigma(\mathbb{R}^3))} + \|\nabla \pi\|_{L^q(0,\infty;\boldsymbol{L}^p_\sigma(\mathbb{R}^3))} \leq C\|\boldsymbol{f}\|_{L^q(0,\infty;\boldsymbol{L}^p_\sigma(\mathbb{R}^3))}.$$

Knowing that the Stokes operator is maximal regular, a fixed point method can be used to solve the non-linear equation. If A has bounded inverse, $\boldsymbol{u} \in L^q(0,\infty;\boldsymbol{L}^p_\sigma(\mathbb{R}^3))$. Thus if p, q are chosen suitably, one may construct solutions satisfying Serrin's condition. The first work concerning the maximal regularity of Stokes operator on bounded, smooth domain and with $p = q$ was done by Solonnikov ([111]). A new approach is developed in ([50]) by Geissert et al.

A century of agreement between experimental results and theories derived assuming no-slip boundary condition had the consequence that today many textbooks of fluid dynamics fail to mention that the no-slip boundary condition remains an assumption. The validity of the no-slip boundary condition at a fluid-solid interface was debated for some years during the last century. One of the reason is the famous *no-collision paradox*. Consider the free fall of a rigid sphere over a wall, assuming the no-slip condition at the fluid-solid interface. Then the model predicts no collision is possible between the solid and the wall in finite time, no matter what their relative density and the viscosity of the fluid. This no-collision paradox has been known from 1960's, after articles by Cox and Brenner ([24]) and Cooley and O'Neil ([36]) in the context of Stokes equations. Since then, the no-collision paradox has been confirmed at the level of Navier-Stokes equations (see [67], [68]).

Of course such a result is unrealistic as it goes against Archimedes' principle. Many physicists have tried to find an explanation for the paradox. Several refinements were proposed in the past (eg. [40], [86]). One possible explanation is, when the distance between the solids gets very small (below the micrometer), the no-slip condition is no longer accurate and must be replaced by the Navier-slip condition.

Another issue in the debate around boundary conditions, is the irregularity of the solid surface. Its effect is a topic of intense discussion. On one hand, some people argue that it increases the surface of friction and may therefore cause a decrease of the slip. On the other hand, it may generate small scale phenomena favorable to slip. For instance, some rough hydrophobic surfaces seem more slippery due to the trapping of air bubbles in the humps of roughness. Moreover, irregularity creates a boundary layer in its vicinity, meaning high velocity gradients. Thus, even though (1.3) is satisfied at the rough boundary, there may be significant velocities right above. In other words, the no-slip condition may hold at the small scale of the boundary layer but not at the large scale of the mean flow. This phenomena, due to scale separation, is called *apparent slip* in the physics literature.

In parallel to experimental works, several theoretical studies have been carried, to clarify the role of roughness. Many of them relate to the homogenization theory (eg. [27]). First, the irregularity is modelled by small scale variations of the boundary. Then, an asymptotic analysis is performed, as the small scale go to zero. The idea is to replace the constitutive boundary condition at the rough surface by a homogenized or effective boundary condition at the smoothened surface. In this way, one can describe the averaged effect of the roughness. Such homogenized conditions (often called *wall laws*) are also of practical interest in numerical codes. They allow to filter out the small scales of the boundary, which have a high computational cost.

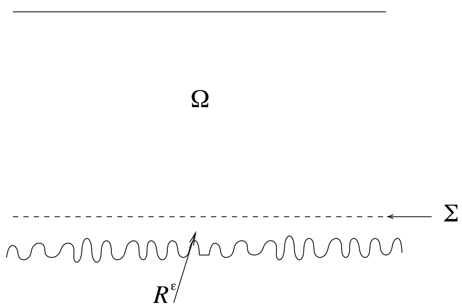

Figure 1.1: The rough domain Ω^ε

Let us briefly describe some interesting mathematical results (cf. [38]) on wall laws. To describe a unified model, consider a two-dimensional rough channel (Figure 1.1)

$$\Omega^\varepsilon := \Omega \cup \Sigma \cup R^\varepsilon$$

where $\Omega = \mathbb{R} \times (0,1)$ is the smooth part, R^ε is the rough part and $\Sigma = \mathbb{R} \times \{0\}$ their interface. Assume that the rough part has size ε that is

$$R^\varepsilon := \varepsilon R \quad \text{with} \quad R := \{x,\ \omega(x_1) < x_2 < 0\}$$

for a Lipschitz function $\omega : \mathbb{R} \to (-1, 0)$. Also let

$$\Gamma^\varepsilon := \varepsilon\Gamma \quad \text{with} \quad \Gamma := \{x,\ x_2 = \omega(x_1)\}.$$

Let $\boldsymbol{u}^\varepsilon$ be a steady flow in this channel, modeled by the stationary Navier-Stokes system, with a prescribed flux ϕ across a vertical cross-section σ^ε of Ω^ε. Moreover, either pure slip, partial slip or no slip condition is prescribed at the rough boundary Γ^ε. This means the constant α in (1.4) can be either zero, positive or ∞. For simplicity, no-slip is assumed at the upper boundary. We get eventually

$$\begin{cases} -\Delta \boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \nabla \boldsymbol{u}^\varepsilon + \nabla \pi^\varepsilon = \boldsymbol{0}, \quad \text{div } \boldsymbol{u}^\varepsilon = 0, \qquad x \in \Omega^\varepsilon, \\ \boldsymbol{u}^\varepsilon|_{x_2=1} = \boldsymbol{0}, \quad \int\limits_{\sigma^\varepsilon} \boldsymbol{u}_1^\varepsilon = \phi, \\ \boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \quad \alpha^\varepsilon \boldsymbol{u}_{\boldsymbol{\tau}}^\varepsilon + [(\mathbb{D}\boldsymbol{u}^\varepsilon)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma^\varepsilon. \end{cases} \tag{1.7}$$

This problem has a singularity in ε, due to the high frequency oscillation of the boundary. Thus, the problem is to replace the singular problem in Ω^ε by a regular problem in Ω. Their idea was to keep the same Navier-Stokes equations

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = 0, \quad \text{div } \boldsymbol{u} = 0, \qquad x \in \Omega, \\ \boldsymbol{u}|_{x_2=1} = 0, \quad \int\limits_{\sigma} \boldsymbol{u}_1 = \phi \end{cases}$$

but with a boundary condition at the artificial boundary Σ which is regular in ε. The goal is to find the most accurate condition.

A series of work has been done in order to answer this question (cf. [1], [2], [72], [73], [9], [17], [64], [52]), starting from the standard Dirichlet condition at Γ^ε ($\alpha^\varepsilon = \infty$ in (1.7)). Vaguely, the following two facts have been noted (for rigorous statements, please consult the afore-mentioned references):

(i) for any roughness profile ω, the Dirichlet condition (1.3) provides an $O(\varepsilon)$ approximation of $\boldsymbol{u}^\varepsilon$ in $L^2_{loc}(\Omega)$.

(ii) for generic roughness profile ω, the Navier condition does better, choosing $\frac{1}{\alpha} = \lambda\varepsilon$ for some good constant λ in (1.4).

These results lead to the fact that the slip length $\lambda\varepsilon$ is related to a boundary layer of amplitude ε near the rough boundary, which is the mathematical expression of the apparent slip mentioned earlier.

Apart from the special case $\alpha^\varepsilon = \infty$, some studies have also dealt with the more general situation $\alpha^\varepsilon \in [0, \infty)$ (we do not go into further detail, for reference, see [30], [26]). The limit $\boldsymbol{u}^0$ of $\boldsymbol{u}^\varepsilon$ and the condition satisfied at Σ have been investigated. Interestingly, the striking result obtained in this later case is that as soon as the boundary is genuinely rough, $\boldsymbol{u}^0$ satisfies a no-slip condition at Σ. This result can be seen as a mathematical justification of the no-slip condition. Indeed, any realistic boundary is rough. If someone is only interested in scales greater than the scale ε of the roughness, then (1.3) is an appropriate boundary condition, whatever the microscopic phenomena behind. Even then, as in the case $\alpha^\varepsilon = \infty$, one may be interested in more quantitative estimates: can the boundary condition be improved, is there possibility of $O(\varepsilon)$ slip? Such questions are important in micro-fluids, a domain in which minimizing wall friction is crucial.

There are different notions of slip and it has been encountered in many different contexts. The possibility of gas slip was first introduced by Maxwell ([92]). The flows of non-Newtonian fluids such as polymer solutions show significant apparent slip in a variety of situations, some of which can lead to a slip-induced instabilities. In the context of Newtonian fluids, molecular slip has been used as a way to remove singularities arising in the motion of contact lines, as reviewed in [16].

Having said that, keeping in mind the growing relevance of the Navier-slip boundary condition, the aim of this thesis is to study thoroughly the non-linear equations (1.1)-(1.2) with the boundary condition (1.4). The friction coefficient α is assumed to be a non-negative and non-smooth scalar function on the boundary (cf. (2.8)). The main two aspects discussed in this thesis are, **(i)** the existence of weak solutions for the system (1.1)-(1.2), (1.4) in L^p-spaces for all $p \in (1, \infty)$ on a bounded domain with minimal regularity of the friction coefficient and the domain and **(ii)** the limit system as α tends to ∞. Throughout the work, we assume Ω is $\mathcal{C}^{1,1}$, unless otherwise specified, since this is the minimum required regularity on a bounded domain for existence of weak solution **for all** $p \in (1, \infty)$.

1.1 Stokes and Navier-Stokes equations with Navier boundary condition

We start with the steady problem with Navier boundary condition in Chapter 1. We first analyze (1.6) in the Hilbert setting. The existence of a weak solution for any fixed friction coefficient α and some given external force $\boldsymbol{f}$ is studied whose proof is straight forward using Lax-Milgram lemma and several Korn and Korn-type inequalities. If $\boldsymbol{f}$ merely belongs to the dual space $[\boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)]'$ (the subscript $\boldsymbol{\tau}$ signifies that the normal component of the vector field at the boundary is zero), the boundary terms can not be well-defined, so we need to consider the data in better space. Here the given external force is assumed in the form $\boldsymbol{f} = \boldsymbol{f}_1 + \text{div}\, \mathbb{F}$ where $\mathbb{F}$ is a 3×3 matrix and $\boldsymbol{f}_1 \in \boldsymbol{L}^{6/5}(\Omega)$, $\mathbb{F} \in \mathbb{L}^2(\Omega)$. Note that this space is larger than $\boldsymbol{L}^2(\Omega)$, but of course a subset of $[\boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)]'$. Also α is considered in $L^2(\Gamma)$.

One interesting case is when the domain Ω is axisymmetric. In that case, $\|\mathbb{D}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ is not any more an equivalent norm in $\boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$ due to the presence of the following non-trivial kernel (corresponding to the case $\alpha = 0$)

$$\boldsymbol{\mathcal{T}}^p(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathbb{D}\boldsymbol{v} = 0 \ \text{ in } \ \Omega;\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma\}. \tag{1.8}$$

Hence the study becomes more involved. The above kernel reduces to zero if Ω is not axisymmetric, but in the case of a domain, obtained by rotating around a constant axis vector say $\boldsymbol{b}$, we have (cf. Remark 2.4.2),

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \text{span}\{\boldsymbol{b} \times \boldsymbol{x}\}, \quad \boldsymbol{x} \in \Omega.$$

Observe that the kernel is independent of p and hence, can be denoted as $\boldsymbol{\mathcal{T}}(\Omega)$. We prove a Korn-type inequality (2.25) involving boundary term in order to show the coercivity of the associated bilinear form for (1.6):

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} \qquad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega).$$

From the variational formulation, we also obtain several estimates which provide uniform bound on the solution (cf. Theorem 2.4.3). Later we will prove the same estimates but in L^p norm, $p \neq 2$. Next the existence of strong solution is deduced (cf. Theorem 2.4.5), provided the data are more regular, namely if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\alpha \in H^{\frac{1}{2}}(\Gamma)$.

Further we move on to study L^p-theory. The existence of weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ can be obtained easily by using the regularity result for the system with $\alpha = 0$ (cf. [11]). But that can not be followed for $p < 2$ since we do not have the existence of a solution in that case. So we choose the more classical tool to show the existence result: inf-sup condition. To our knowledge, there is no established inf-sup condition for the symmetric gradient $\mathbb{D}$ operator. So we use the following relations

$$\Delta \boldsymbol{v} = -\textbf{curl}\ \textbf{curl}\boldsymbol{v} + \nabla \text{div}\ \boldsymbol{v}$$

and

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \textbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v}$$

where $\boldsymbol{\Lambda}\boldsymbol{v} := \sum_{k=1}^2 \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\right) \boldsymbol{\tau}_k$ is a first order term. With these, the following form

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}$$

can be seen as a bilinear form in **curl** operator which enables us to use the well-known inf-sup condition (cf. [13, Lemma 4.4]) for all $p \in (1, \infty)$:

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq \boldsymbol{0}}} \ \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \textbf{curl}\ \boldsymbol{\xi} \cdot \textbf{curl}\ \boldsymbol{\varphi}}{\|\boldsymbol{\xi}\|_{\boldsymbol{V}^p_{\sigma,\tau}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}(\Omega)}} \geq C.$$

Thus we can get the existence of a unique weak solution of (1.6) in $\boldsymbol{W}^{1,p}(\Omega)$ for all $p \in (1, \infty)$ along with the following interesting inf-sup condition:

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \ \sup_{\substack{\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega) \\ \boldsymbol{u} \neq 0}} \frac{2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}}{\|\boldsymbol{u}\|_{\boldsymbol{V}^p_{\sigma,\tau}(\Omega)} \ \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq C. \tag{1.9}$$

Note that the above positive constant C depends on Ω, p and α. Now the question we ask is: whether it is possible to obtain C independent of α, at least under some assumption, or not? Or in other words, the inf-sup condition (1.9) yields the standard estimate on the solution (by Theorem 2.5.1):

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega, p, \alpha)\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)}$$

but can we improve the above bound with respect to the friction coefficient and obtain such estimate, uniform in α? The motivation of arising this question is:

(i) if we let α go to ∞ in the Navier boundary condition (1.4), formally we obtain the Dirichlet condition (1.3) and the estimate on the solution of Stokes equation with Dirichlet boundary condition is independent of α. Therefore, we may expect the same when α tends to ∞.

(ii) in the Hilbert case $p = 2$, we already have obtained some bound on solutions of (1.6), independent of α provided α is large (cf. Theorem 2.4.3).

We try to answer the above question. One of the main approach used in this thesis widely, is to exploit α independent L^2-estimate in each situation to deduce α independent L^p-estimate. We are able to achieve this goal with the help of the L^p-extrapolation theorem of Shen (cf. Lemma 3.3.11) which contains some real variable argument. This extrapolation theorem can be seen as a refined version of the well-known Calderón-Zygmund theory. The celebrated Calderón-Zygmund theory says: if T is a convolution operator with an associated kernel and that kernel satisfies further estimates and the operator is bounded on L^2, then T is bounded on L^p for all $1 < p < \infty$. There are some examples of elliptic partial differential equations where certain associated operators are L^p-bounded for some p's in an interval around two, but not for all p. In order to prove L^p-boundedness for such operators, the classical theory of singular integrals fail as they are not sensitive for the L^p-boundedness on a proper subinterval of $(1, \infty)$. They provide L^p-boundedness either for all p or for no p. The main novelty of Shen's extrapolation theorem is that it is p-sensitive. That means it says that if an operator T is L^2-bounded and satisfies certain estimates depending on a number $p > 2$, then the operator is L^q-bounded for all $2 < q < p$. Although this is not the situation in our case as we work in $\mathcal{C}^{1,1}$ domain, but this is more suitable and easy to apply.

Let us mention that we obtain the uniform bound on the solution for all $\alpha \not\equiv 0$ on Γ when Ω is not axisymmetric; and only for α large when Ω is axisymmetric. This is not surprising and supports the characterization of the kernel (1.8) *i.e.* the fact that in the case $\alpha \equiv 0$, the Stokes problem with slip boundary condition has a non trivial kernel only when Ω is axisymmetric.

Strikingly, the condition (3.39) (without the last term in the right hand side) can be seen as a reverse Hölder inequality. Since the integration is made on different sets, it is named in the literature as *weak reverse Hölder inequality* (wRHI). This weak reverse Hölder estimate, which is originally due to Giaquinta and Modica [54, Proposition 5.1] while the idea was the same as the one by Gehring [49], has a self-improving property. Also the explicit dependence of the constants has been precised in [54, Proposition 5.1] which will be useful for us.

The weak reverse Hölder inequality in the interior of the domain for a weak solution u of an elliptic system of the form

$$-D_k\left(a_{ij}^{kl}(x)D_l u^j\right) = 0$$

is much easier to prove and follows from Caccioppoli inequality along with the Sobolev-Poincaré inequality. But in order to obtain the result up to the boundary is more involved and obviously depends on the boundary condition. In our situation, it is exactly same to get the interior estimate as done in [53, the beginning of Chapter V]; But there remains an extra term for the pressure, when we consider the boundary estimate. So we follow the argument as done for the Stokes system with Dirichlet boundary condition in [55] with necessary modifications. First a suitable pressure estimate is deduced, then some Caccioppoli inequality is derived which allows us to have the required weak reverse Hölder inequality. Note that, the friction coefficient α being non-negative, we can drop the boundary term containing α and the constant in the L^2-estimate is as well α-independent. Thus, the desired α-independent L^p-estimate on the solution of (1.6) is obtained for some p bigger than two. The self-improving property of the weak reverse Hölder inequality gives the result for all $p > 2$. Then the proof is completed in several steps considering the dual problem. Let us mention that it is here that we need to consider the data in the right hand side of the divergence form (as mentioned before). Also one must be careful with the constants involved in weak reverse Hölder inequality as it should be independent of the radius of the underlying sets involved. This shows that the constant in the inf-sup condition (1.9) actually does not depend on α whenever α is large.

Finally, these uniform bounds enable us to study the limit problem (Section 2.7): for each α, if we denote the solution of (1.6) as $\boldsymbol{u}_\alpha$, passing to the limit $\alpha \to \infty$, $\boldsymbol{u}_\alpha$ converges to the solution of the Stokes problem with no-slip boundary condition in the energy space and the convergence rate is same as $1/\alpha$. We also show that when $\alpha \to 0$, $\boldsymbol{u}_\alpha$ converges to the solution of Stokes equation with Navier boundary condition corresponding to $\alpha = 0$ (Theorem 2.7.3).

In Section 2.8, the steady Navier-Stokes equation is discussed. Existence of weak solution in the Hilbert setting (cf. Theorem 2.8.4) can be shown using the classical Galerkin method as for the no-slip boundary condition. And then further regularity (both in integrability and differentiability) of solution is obtained assuming regular data (cf. Corollary 2.8.8) for all $p > \frac{3}{2}$. To prove the existence of weak solution for $p \in (\frac{3}{2}, 2)$, we use the method developed by [104] which works fine in our case too. We also study the limit problem for the non-linear system (cf. Theorem 2.8.9, Theorem 2.8.11).

1.2 Semigroup theory for the Stokes operator with Navier boundary condition on L^p spaces

In this chapter, the evolution problem (1.1) - (1.2), (1.4) is discussed. The purpose of this chapter is two-fold. On one hand, it serves as a collection of properties of the Stokes operator with Navier boundary condition that are well-known in the case of Dirichlet boundary condition so that it can be used as a basis and directly referred whenever necessary to answer other relevant questions. On the other hand, we study the dependence of the solution of the linear (1.5) or the nonlinear system (1.1), (1.4) on the friction coefficient α and how it behaves as α tends to ∞. The reason of this second part is already described above.

The Stokes operator and the Stokes semigroup are one of the central objects in the study of incompressible fluid flow, especially they are fundamental for the approaches by Fujita-Kato and Kato and Giga (as explained in the beginning). So we start with introducing the strong and weak Stokes operators $A_{p,\alpha}$ and $B_{p,\alpha}$, for each fixed α, on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $[\boldsymbol{H}^{p'}_0(\text{div}, \Omega)]'$ and show that they generate analytic semigroups on the respective spaces for all $p \in (1, \infty)$. The proof of this is not very complicated and mostly uses the existence and estimate results studied in the previous chapter for the steady problem. Further we study the imaginary and fractional powers of the Stokes operators. To show that the operators $A_{p,\alpha}$ and $B_{p,\alpha}$ are of *bounded imaginary power* is not straight forward; Here we did not use pseudo-differential operator theory or Fourier multiplier theory as done by Giga [61], but chose a rather different approach: we show that the Stokes operator with (full) slip boundary condition (in its weak form) can be written as a lower order perturbation of the Navier-type boundary condition (cf. (3.52)) for which the result is known (cf. [6, Theorem 6.1]) and then with the help of Amann's interpolation-extrapolation theory [8], we recover the boundedness of imaginary power of $A_{p,\alpha}$. This method has been used in [101], though the ultimate goal was different for them. Next we prove that the Stokes operator has maximal L^q-regularity and establish various types of $L^p - L^q$ estimates which helps to develop an L^p-theory for the Navier-Stokes equations. We have used the abstract theory by Giga [58] for semilinear parabolic equations in L^p to achieve similar existence and regularity of a local strong solution and global weak solution.

But the interesting part is to show the resolvent estimate

$$|\lambda| \|(\lambda I + A_{p,\alpha})^{-1} \boldsymbol{f}\|_{\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)} \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)} \qquad \forall\, \lambda \in \mathbb{C}^* \quad \text{with} \quad \operatorname{Re} \lambda \geq 0$$

where the positive constant C does not depend on α for α sufficiently large. Note that with the usual method, multiplying the equation by $|\boldsymbol{u}|^{p-2}\bar{\boldsymbol{u}}$ (for example, as followed in [5]), one can easily obtain the above estimate but with the constant depending on α. Therefore we had to try some different idea. In the Hilbert case, this follows (cf. Theorem 3.3.4) from the variational formulation as expected. But proving it for $p \neq 2$ is much more delicate. We have used L^p-extrapolation theory of Shen again, but this time with a different operator T. The main difficulty was to satisfy the necessary condition (3.39). In the stationary case, the weak reverse Hölder inequality was one type of gradient estimate which follows using the standard tools at least in the interior of the domain. In the present case, wRHI (3.34) is much more tricky to obtain. Unfortunately, the estimates derived on the steady problem do not help in this situation. The technique we employed here has been used by Shen in [107] to deduce L^p-resolvent estimate for Stokes operator with no-slip boundary condition in Lipschitz domain though the purpose and the situation there was different. It is indeed an interesting idea to deduce similar L^p-estimate from known L^2-estimate.

We want to emphasize here that Mitrea and Wright proved in [94, Lemma 6.7] the Caccioppoli inequality yields some kind of weak reverse Hölder inequality.

Finally in Section 3.9, we discuss the limit problem for both the linear and nonlinear system as $\alpha \to \infty$ and prove various convergence results. If $\boldsymbol{u}_\alpha$ is denoted as the solution of (1.1), (1.4) for a fixed α and $\boldsymbol{u}_\infty$ is a solution of Navier-Stokes equations with no-slip condition, we show that $\boldsymbol{u}_\alpha$ converges to $\boldsymbol{u}_\infty$ in appropriate spaces under assumption of different initial data. We also obtain the rate of convergence.

1.3 Uniform $W^{1,p}$ estimate for elliptic operator with Robin boundary condition in $\mathcal{C}^1$ domain

This last chapter is concerned with estimates for a Laplace-Robin problem. Let us consider the problem in more general form

$$\begin{cases} \operatorname{div}(A\nabla)u = \operatorname{div}\boldsymbol{f} + F & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = \boldsymbol{f} \cdot \boldsymbol{n} + g & \text{on } \Gamma \end{cases} \tag{1.10}$$

which is a second order elliptic operator in divergence form in a bounded domain $\Omega \subset \mathbb{R}^n$ of class $\mathcal{C}^1$, with the Robin boundary condition. The coefficient matrix A is symmetric and in $VMO(\mathbb{R}^3)$ (note that $A \in VMO(\mathbb{R}^3)$ or with small BMO coefficients is a necessary condition for the existence of a solution of the above elliptic problem with non-constant coefficient).

In the above boundary condition, $\alpha = 0$ corresponds to the Neumann condition and $\alpha = \infty$ gives the Dirichlet condition. We wish to find the uniform bound of the solution with respect to the Robin coefficient α, precisely,

$$\|u\|_{W^{1,p}(\Omega)} \leq C(\Omega, p) \left(\|F\|_{L^p(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right)$$

where the positive constant $C(\Omega, p)$ does not depend on α.

The original motivation comes from the problem in the first chapter. Observe that the Stokes equation with slip boundary condition (1.6) reduces to the problem (1.10) in the simplest case, replacing the Stokes operator by the Laplacian and the Navier boundary condition by the Neumann one. Hence, we first started to analyze the above problem for α-independent bound on solutions and see what do we find. The goal was the same as, if we can find such estimates, then passing to the limit on α, we may expect to recover the solution of the Laplace equation with Dirichlet boundary condition. To work with the full Navier-Stokes system with the complicated boundary condition was at the beginning quite cumbersome, thus we concentrated on the simpler scalar version. Surprisingly this is itself an interesting question and still difficult to answer. To our knowledge, there is no work in the literature done concerning the precise dependence of the solution on the Robin coefficient.

Here we consider α a function belonging to some L^q-space (precised below in Chapter 3). Apart from proving existence, uniqueness of weak and strong solutions, we obtain the bound on u, uniform in α for α sufficiently large, in the L^2-setting. Then with the help of Shen's L^p-extrapolation theorem, the same bound is deduced for L^p-norm. Though the proof lies in the same line, it uses some more abstract tool Lemma 2.6.8 (originally developed by Shen [106] and then generalized by Geng [51]) with which the final estimate is obtained directly. Note that the weak reverse Hölder inequality up to the boundary can be obtained with the help of classical boundary Hölder estimate for elliptic operator with constant coefficient as mentioned in [81, after Theorem 4.1], though it is not very clear to us. We have separately studied the two cases: the interior estimate and the boundary estimate to make the main idea clear in the simple set up. The complete proof is done in several Lemmas and Theorems: first the case $p > 2$ and $F = 0, g = 0$ is considered, then it is extended for all $p \in (1, \infty)$; and lastly, for $\boldsymbol{f} = \boldsymbol{0}$ but assuming $F \neq 0, g \neq 0$.

One interesting consequence of this analysis is that we can derive straight forwardly some H^s-estimate, $s \in (0, \frac{1}{2})$, on u for only Lipschitz domains with a very simple proof and also get the α-independent bound. Continuing the same method for the difference quotient, we further deduce the uniform bound on strong solutions as well. Though we have announced all the results in this chapter in $\mathbb{R}^3$, they are true also in $\mathbb{R}^2$.

CHAPTER 2

Stokes and Navier-Stokes equations with Navier boundary condition

This work is done jointly with Paul Acevedo, Chérif Amrouche and Carlos Conca.

Abstract : We study the stationary Stokes and Navier-Stokes equations with non-homogeneous Navier boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. We prove the existence and uniqueness of weak and strong solutions in $\boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{W}^{2,p}(\Omega)$ for all $1 < p < \infty$ considering minimal regularity on the friction coefficient α. Moreover, we deduce estimates to analyze the behavior of the solution with respect to α, in particular when $\alpha \to \infty$.

2.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ, possibly not connected, of class $\mathcal{C}^{1,1}$ (additional smoothness of the boundary will be precised whenever needed). Consider the stationary Stokes equations

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \text{div}\, \boldsymbol{u} = \chi \quad \text{in } \Omega \tag{2.1}$$

and the stationary Navier-Stokes equations

$$-\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \text{div}\, \boldsymbol{u} = \chi \quad \text{in } \Omega \tag{2.2}$$

where $\boldsymbol{u}$ and π are the velocity field and the pressure of the fluid respectively, $\boldsymbol{f}$ is the external force acting on the fluid and χ stands for the compressibility condition.

Concerning these equations, the first thought goes to the classical no-slip Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{2.3}$$

This condition was formulated by G. Stokes in 1845. An alternative was suggested by C.L. Navier [96] even before, in 1823. Along with the usual impermeability condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \tag{2.4}$$

Navier proposed a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangent to the surface, instead of being zero, should be proportional to the rate of strain at the surface i.e.

$$2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on } \Gamma \tag{2.5}$$

where $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit outward normal and tangent vectors on Γ respectively and $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the rate of strain tensor. Here, α is the coefficient which measures the tendency of the fluid to slip on the boundary, called *friction coefficient*.

Although the no-slip hypothesis seems to be in good agreement with experiments, it leads to certain rather surprising conclusions, the most striking one being the absence of collisions of rigid bodies immersed in a linearly viscous fluid (see [67]). In contrast with the no-slip condition, Navier's boundary conditions offer more freedom and are likely to provide a physically acceptable solution at least to some of the paradoxical phenomena, resulting from the no-slip condition (see for instance, [89]). There have been several attempts in the literature to provide a rigorous justification of the no-slip boundary condition, based on the idea that the physical boundary is never smooth but contains small asperities that drive the fluid to rest under the mere impermeability hypothesis! These kind of results have been shown by Casado-Diaz et al. [30] in the case of periodically distributed asperities and the references therein. Conversely, [72] identified the Navier' slip boundary condition as a suitable approximation of the behavior of a viscous fluid out of a boundary layer created by the no-slip condition, imposed on a rough boundary. It is worth noting that these results are by far not contradictory but reflect two conceptually different approaches in mathematical modelling of viscous fluids. Some interesting remarks on the use of this boundary condition can be found in Serrin [105]. Recently this boundary condition has also been identified as appropriate for some large eddy simulation models for turbulent flows, see for instance Galdi and Layton [48]. Note that, in [37], F. Coron has derived rigorously the slip boundary condition (2.5) from the boundary condition at the kinetic level (Boltzmann equation) for compressible fluids. The Navier slip with friction conditions are also used for simulations of flows near rough boundaries, such as in aerodynamics, in weather forecast, in haemodynamics etc.

In this work, we study the existence, uniqueness and regularity of solutions of the system (2.1) and (2.2) with the boundary condition (2.4)-(2.5). Skipping the extremely extensive literature of the well-known no-slip boundary condition, we give a brief overview of some of the available results on the Navier/Navier-type boundary conditions. Concerning the non-stationary Navier-Stokes equation with Navier boundary condition, there are considerably many works, among other reasons, for studying the limiting viscosity case. For example, in 2D, the existence of solutions are studied assuming α in $C^2(\Gamma)$ (see Mikelić et al [33]) and for α in $L^\infty(\Gamma)$ (see Kelliher [80]); whereas in 3D, Beirão da Veiga [20] considered the system with α a positive constant and Iftimie and Sueur [70] have analysed the case with α in $C^2(\Gamma)$. Also we refer to the work of D. Bucur et al. [26] in the periodic boundary case (see the references therein also). We mention here the paper of Monniaux and Ouhabaz [95] where they studied a similar boundary condition in Lipschitz domain but with α depending on both time and space variables. On the contrary, for the stationary problem, comparatively less works are known. The first paper concerning basic existence and regularity result is by Solonnikov and Scadilov [112] where they treated the problem for $\alpha = 0$. They considered the stationary Stokes system with Dirichlet boundary condition on some part and Navier boundary conditions (2.5) (with $\alpha = 0$) on the other part of the boundary and showed existence of weak solution in $\boldsymbol{H}^1(\Omega)$ which is regular (belongs to $\boldsymbol{H}^2_{loc}(\Omega)$) upto some part of the boundary (except in the neighbourhood of the intersection of the two part). Also, it is worth mentioning the work of Beirão da Veiga [19] where he proved existence results of weak and strong solution of the Stokes problem in the L^2-settings, but for more generalized system and again with positive constant α. Also he did not precise in the estimate the dependence of the constant on α. Recently, Berselli [22] gave some result concerning very weak solution in the special case of a flat domain in $\mathbb{R}^3$, in general $\boldsymbol{L}^p(\Omega)$ settings and considering $\alpha = 0$ which is based on the regularity theory of Poisson equation. In the paper of Amrouche and Rejaiba [11], they proved the existence and regularity of weak, strong and very weak solutions in a bounded domain in $\mathbb{R}^3$ for all $1 < p < \infty$ for non-smooth data, but for $\alpha = 0$. In the work of Medková [93], we can find various other forms of Navier problems and the references therein. Furthermore, the numerical study has been done in, e.g. Verfürth [115] (though again for $\alpha = 0$) and John [77]. Also Novotný et al. studied in [75] the steady compressible fluid flow subject to the slip boundary condition with $\alpha \geq 0$ but without precising any dependence of the slip coefficient. They obtained the existence of various solutions without any restriction of the size of the data.

Note that, as shown in (2.11), the two boundary conditions

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \tag{2.6}$$

and

$$2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}$$

are very much similar and in the case of flat boundary and $\alpha = 0$, they are actually equal. Hence, there are several studies concerning this Navier type boundary condition (2.6) as well. We refer to [12] where Amrouche et al. studied the weak, strong and very weak solutions and the references therein.

In both cases of stationary and evolution problems, all the available works have considered α as either a constant or a smooth function, as mentioned before. In this work, we analyse the possible minimal regularity of α for the existence of weak and strong solutions in $L^p(\Omega)$ for all $1 < p < \infty$ [see (2.8)]. Also, it is worth mentioning that the restriction that α is non-negative is usual, in order to ensure the conservation of energy. But mathematically, we can take into account negative values of α as well. Some authors have studied the evolution system with α negative where there was no mathematical difficulty due to the access of Gronwall's inequality. But in the stationary problem, it is not the case.

Another interesting question is the precise dependence of the solution of (S) or (NS) on α and if we let the function α tend to ∞ or 0 in (2.5), how does the solution behave? As per the authors' knowledge, there is no previous work on that even if α is smooth function or a constant. We prove the estimates in Theorem 2.4.3 and Theorem 2.6.11 which show that the solution is uniformly bounded with respect to α. Moreover, in Section 2.7, we show that as α converges to 0, the solution of the Stokes equation with Navier boundary conditions converges strongly to the solution of the Stokes equation corresponding to $\alpha = 0$ and if α goes to ∞, the solution converges strongly to the Stokes equation with Dirichlet boundary condition. This type of convergence is, in a sense, an inverse of the derivation of the Navier boundary conditions from no-slip boundary condition for rough boundaries as we have explained previously also. In [35], Conca studied a similar system in a bounded domain in $\mathbb{R}^2$ with smooth viscosity. There he assumed the well-posedness of the problem (2.1) (or, (2.2)) with (2.4)-(2.5) and proved some convergence results as ε goes to 0, based on homogenization theory, where he considered the domain depends on ε as well. In some sense, our work generalizes the work in [35]. We want to mention that the main purpose of this work is to develop a complete L^p-theory for all $1 < p < \infty$ to deal with the well-posedness for the stationary Stokes and Navier-Stokes equations with full Navier boundary condition, considering general non-regular α and study the limiting cases. Hence, our work can be useful to study the evolution problem and other related issues; for example, the coupled fluid-structure interaction problems are another interesting open problem. Also the estimates obtained in Section 2.6.2, precisely in Theorem 2.6.14 can be of independent interest. The corresponding well-posedness and limiting behavior for non-linear system is studied in Section 2.8. The main results of our work are mentioned below.

2.2 Main results

Let us briefly discuss here the main results of our paper, referring to the next sections for precise definitions and complete proofs. Since the case $\alpha \equiv 0$ in (2.5) has already been studied in [11], here onwards we consider that $\alpha \not\equiv 0$ on Γ. If we do not precise otherwise, we will always assume

$$\alpha \geq 0 \quad \text{on } \Gamma \quad \text{and } \alpha > 0 \quad \text{on some } \Gamma_0 \subset \Gamma \quad \text{with } |\Gamma_0| > 0.$$

Also let

$$\boldsymbol{\beta}(x) = \boldsymbol{b} \times \boldsymbol{x} \tag{2.7}$$

when Ω is axisymmetric with respect to a constant vector $\boldsymbol{b} \in \mathbb{R}^3$.

Note that we can always reduce the non vanishing divergence problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma \end{cases}$$

with the necessary condition $\int_\Omega \chi = \int_\Gamma g$ to the case where $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, by solving the following Neumann problem

$$\Delta \theta = \chi \quad \text{in } \Omega, \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma$$

and hence using the change of unknowns $\boldsymbol{w} = \boldsymbol{u} - \nabla \theta$ and $\Pi = \pi - \chi$. Therefore, it is sufficient to study the following Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{S}$$

The first main result is the existence and uniqueness of weak and strong solution of the Stokes problem (S) which is given in Theorem 2.4.1 in the Hilbert case and in Corollary 2.5.6 and Theorem 2.5.9 for $p \neq 2$. Note that we proved more general existence result in Theorem 2.5.3 than the solution of the problem (S).

For that, we need the following assumption on α:

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{with} \quad \begin{cases} t(p) = 2 & \text{if} \quad p = 2 \\ t(p) > 2 & \text{if} \quad \frac{3}{2} \leq p \leq 3, p \neq 2 \\ t(p) > \frac{2}{3}\max\{p, p'\} & \text{otherwise} \end{cases} \tag{2.8}$$

and where $t(p) = t(p')$. Also we will always assume, unless stated otherwise, $\mathbb{F}$ is a 3×3 matrix and $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and we will not repeat these hypothesis every time.

Theorem 2.2.1 (Existence of weak and strong solutions of Stokes problem). *Let $p \in (1, \infty)$.*
(i) *If*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma)$$

where $t(p)$ as above and $r(p)$ is defined in (2.10)*, then the Stokes problem* (S) *has a unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.*
(ii) *Moreover, if $\mathbb{F} = 0$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \leq \frac{3}{2}$ and $q = p$ otherwise, then the weak solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Also we obtain some uniform bounds for the weak solution of the problem (S) in $\boldsymbol{W}^{1,p}(\Omega)$ for all $p \in (1, \infty)$, which we believe are quite interesting.

Theorem 2.2.2 (Stokes estimates). *Let $p \in (1, \infty)$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ be the weak solution of the Stokes problem (S). Then it satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

(ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \frac{C_p(\Omega)}{\min\{2, \alpha_*\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

In Theorem 2.4.3 and Theorem 2.4.5, we discuss in detail the estimates of weak and strong solutions in the Hilbert case and the proof of Theorem 2.2.2 for general p is done in Theorem 2.6.11.

The next theorem gives existence of weak and strong solutions for the following Navier-Stokes problem and some estimates.

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} + \operatorname{div}\mathbb{F}, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{NS}$$

Theorem 2.2.3 (Existence of weak and strong solutions of Navier-Stokes problem and estimates). *Let $p \in (\frac{3}{2}, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

1. *Then the problem* (NS) *has a weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.*
2. *Also for any $p \in (1, \infty)$, if $\mathbb{F} = 0$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \leq \frac{3}{2}$ and $q = p$ otherwise, then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.
3. *For $p = 2$, the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfies the following estimates:*

a) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

b) *if Ω is axisymmetric and*
(i) $\alpha \geq \alpha_* > 0$ on Γ, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0,$$

then the solution $\boldsymbol{u}$ satisfies $\int_\Gamma \alpha\boldsymbol{u}\cdot\boldsymbol{\beta} = 0$ and

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \|\pi\|^2_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)^2.$$

In particular, if α is a constant, then $\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{\beta} = 0$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

We refer to Theorem 2.8.4 and Corollary 2.8.8 for the proof of the above result, which is similar to that of the Stokes problem. Note that in the above theorem, the existence of weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $\frac{3}{2} < p < 2$ is not trivial.

The last interesting result to mention here, is the strong convergence of (NS) to the Navier-Stokes equations with no-slip boundary condition when α tends to infinity (see Theorem 2.8.11). The proof is essentially based on the estimates obtained above.

Theorem 2.2.4 (Limiting case for Navier-Stokes problem). *Let $p \geq 2$, α be a constant and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be a weak solution of* (NS) *where*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega) \text{ and } \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Then

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega) \quad \text{as} \quad \alpha \to \infty$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ is a solution of the Navier-Stokes problem with Dirichlet boundary condition,

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

2.3 Notations and preliminary results

Before studying the problem (2.1) and (2.2) with (2.4)-(2.5), we review some basic notations and functional framework. We will use the term *axisymmetric* to mean a non-empty set which is generated by rotation around an axis. The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^3$ are denoted by bold font and blackboard bold font respectively. Unless otherwise stated, we follow the convention that C is an unspecified positive constant that may vary from expression to expression, even across an inequality (but not across an equality). Also C depends on Ω generally and the dependence of C on other parameters will be specified within parenthesis when necessary.

Note that the vector-valued Laplace operator of a vector-field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined as

$$\Delta \boldsymbol{v} = 2 \operatorname{div} \mathbb{D}\boldsymbol{v} - \textbf{grad} \operatorname{div} \boldsymbol{v}.$$

We denote by $\boldsymbol{\mathcal{D}}(\Omega)$ the set of smooth functions (infinitely differentiable) with compact support in Ω. Define

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$$

and

$$L^p_0(\Omega) := \left\{ v \in L^p(\Omega);\, \int\limits_\Omega v = 0 \right\}.$$

If $p \in [1, \infty)$, p' denotes the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. For $p, r \in [1, \infty)$, we introduce the following space

$$\boldsymbol{H}^{r,p}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{r,p}(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|_{L^p(\Omega)}.$$

It can be shown that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{r,p}(\mathrm{div}, \Omega)$ (cf. [103, Proposition 1.0.2]). The closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^{r,p}(\mathrm{div}, \Omega)$ is denoted by $\boldsymbol{H}^{r,p}_0(\mathrm{div}, \Omega)$ and can be characterized as

$$\boldsymbol{H}^{r,p}_0(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{r,p}(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma\}.$$

Also for $p \in (1, \infty)$, the dual space of $\boldsymbol{H}^{r,p}_0(\mathrm{div}, \Omega)$, denoted by $[\boldsymbol{H}^{r,p}_0(\mathrm{div}, \Omega)]'$, can be characterized as follows (cf. [103, Proposition 1.0.4]):

Proposition 2.3.1. *A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^{r,p}_0(\mathrm{div}, \Omega)]'$ iff there exists $\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$. Moreover, we have the estimate :*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r,p}_0(\mathrm{div},\Omega)]'} \leq \inf_{\boldsymbol{f}=\boldsymbol{\psi}+\nabla\chi} \ \max\{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\chi\|_{L^{p'}(\Omega)}\}.$$

We also recall the following useful result (cf. [13, Theorem 3.5]):

Proposition 2.3.2. *Let $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ with $\mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)$, $\mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v} \cdot \boldsymbol{n} \in W^{1-\frac{1}{p},p}(\Gamma)$. Then $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\left(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}\right).$$

We need to introduce the following spaces also :

$$\boldsymbol{V}^p_{\sigma,\tau}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathrm{div}\ \boldsymbol{v} = 0 \ \text{ in } \Omega \ \text{ and } \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma\right\} =: \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$$

equipped with the norm of $\boldsymbol{W}^{1,p}(\Omega)$ and

$$\boldsymbol{E}^p(\Omega) := \left\{(\boldsymbol{v}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega);\ -\Delta\boldsymbol{v} + \nabla\pi \in \boldsymbol{L}^{r(p)}(\Omega)\right\} \tag{2.9}$$

where

$$\begin{cases} r(p) = \max\left\{1, \frac{3p}{p+3}\right\} & \text{if} \quad p \neq \frac{3}{2} \\ r(p) > 1 & \text{if} \quad p = \frac{3}{2} \end{cases} \tag{2.10}$$

which is a Banach space with the norm

$$\| (\boldsymbol{v}, \pi) \|_{\boldsymbol{E}^p(\Omega)} := \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} + \| - \Delta \boldsymbol{v} + \nabla \pi\|_{\boldsymbol{L}^{r(p)}(\Omega)}.$$

Let us now introduce some notations to describe the boundary. Consider any point P on Γ and choose an open neighbourhood W of P in Γ, small enough to allow the existence of 2 families of $\mathcal{C}^2$ curves on W with the following properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthogonal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves.

With this notations, we have $\boldsymbol{v} = \sum_{k=1}^{2} v_k \boldsymbol{\tau}_k + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k = (\tau_{k1}, \tau_{k2}, \tau_{k3})$ and $v_k = \boldsymbol{v} \cdot \boldsymbol{\tau}_k$. In the sequel, for simplicity of notation, we will use

$$\boldsymbol{\Lambda v} = \sum_{k=1}^{2} \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k} \right) \boldsymbol{\tau}_k \ .$$

We recall the following relations which give the equivalence of the two boundary conditions (2.5) and (2.6) and which will be used extensively to prove some of our main results (for proof, see [11, Appendix A]). Note that Ω being $\mathcal{C}^{1,1}$ is sufficient and there is a sign change in the second relation, compared to [11] and it is the corrected formulation.

Lemma 2.3.3. *For any $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, we have the following equalities:*

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v} \ ,$$

$$\mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} = -\nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v} \ .$$

Remark 2.3.4. In the particular case $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, we obtain, for all $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$,

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v} \quad \text{and} \quad \mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v}$$

which implies that

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda v}. \tag{2.11}$$

Next, we prove the following Green formula to define the trace of the strain tensor of a vector field.

Lemma 2.3.5. *Let Ω be a Lipschitz domain. Then,*
(i) $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{E}^p(\Omega)$ *and*
(ii) *The linear mapping* $(\boldsymbol{v}, \pi) \mapsto [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$, *defined on* $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *can be extended to a linear, continuous map from* $\boldsymbol{E}^p(\Omega)$ *to* $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. *Moreover we have the following relation: for all* $(\boldsymbol{v}, \pi) \in \boldsymbol{E}^p(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$\int_\Omega (-\Delta \boldsymbol{v} + \nabla \pi) \cdot \boldsymbol{\varphi} = 2 \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} - 2 \left\langle [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \right\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \tag{2.12}$$

Proof. **(i)** The proof of the density result is very much similar to [64, Lemma 1.5.3.9]. Let $P : \boldsymbol{W}^{1,p}(\Omega) \to \boldsymbol{W}^{1,p}(\mathbb{R}^3)$ be any continuation operator such that $P\boldsymbol{u}\big|_\Omega = \boldsymbol{u}$. Then for all $\boldsymbol{\ell} \in [\boldsymbol{E}^p(\Omega)]'$, there exists $(\boldsymbol{\varphi}, \lambda, \boldsymbol{\psi}) \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3) \times L^{p'}(\Omega) \times \boldsymbol{L}^{(r(p))'}(\Omega)$ with supp $\boldsymbol{\varphi} \subset \overline{\Omega}$ such that for any $(\boldsymbol{v}, \pi) \in \boldsymbol{E}^p(\Omega)$,

$$\langle \boldsymbol{\ell}, (\boldsymbol{v}, \pi) \rangle = \langle \boldsymbol{\varphi}, P\boldsymbol{v} \rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3) \times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_\Omega \lambda \pi + \int_\Omega \boldsymbol{\psi} \cdot (-\Delta \boldsymbol{v} + \nabla \pi).$$

Thanks to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{\ell}$ which vanishes on $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is actually zero on $\boldsymbol{E}^p(\Omega)$.

Let us suppose that $\boldsymbol{\ell} = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ and let $\widetilde{\lambda} \in L^{p'}(\mathbb{R}^3)$ and $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^{(r(p))'}(\mathbb{R}^3)$ be the extension by zero to $\mathbb{R}^3$. Then for all $(\boldsymbol{V} \times \Pi) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)$, let $\boldsymbol{v} = \boldsymbol{V}\big|_\Omega$ and $\pi = \Pi\big|_\Omega$ so that $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ and

$$\langle \boldsymbol{\varphi}, \boldsymbol{V} \rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3) \times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \widetilde{\lambda} \Pi + \int_{\mathbb{R}^3} \widetilde{\boldsymbol{\psi}} \cdot (-\Delta \boldsymbol{V} + \nabla \Pi) \, \mathrm{d}x = 0$$

since supp $\boldsymbol{\varphi} \subset \overline{\Omega}$ implies $\langle \boldsymbol{\varphi}, \boldsymbol{V} \rangle = \langle \boldsymbol{\varphi}, P\boldsymbol{v} \rangle$. It then follows that

$$\left\langle \boldsymbol{\varphi} - \Delta \widetilde{\boldsymbol{\psi}}, \boldsymbol{V} \right\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^3) \times \boldsymbol{\mathcal{D}}(\mathbb{R}^3)} = 0 \quad \text{and} \quad \left\langle \widetilde{\lambda} - \operatorname{div} \widetilde{\boldsymbol{\psi}}, \Pi \right\rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} = 0.$$

i.e. $\boldsymbol{\varphi} - \Delta \widetilde{\boldsymbol{\psi}} = \boldsymbol{0}$ and $\widetilde{\lambda} - \operatorname{div} \widetilde{\boldsymbol{\psi}} = 0$ in the sense of distribution in $\mathbb{R}^3$. Hence, $\Delta \widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3)$. As a consequence, $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{W}^{1,p'}(\mathbb{R}^3)$ and therefore $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}_0(\Omega)$. Then by density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{W}^{1,p'}_0(\Omega)$, there exists a sequence $(\boldsymbol{\psi}_k)_k \subset \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{\psi}_k \to \boldsymbol{\psi}$ as $k \to \infty$ in $\boldsymbol{W}^{1,p'}(\Omega)$. Also $\Delta \widetilde{\boldsymbol{\psi}}_k \to \Delta \widetilde{\boldsymbol{\psi}}$ in $\boldsymbol{W}^{-1.p'}(\mathbb{R}^3)$. Now, for any $(\boldsymbol{v}, \pi) \in \boldsymbol{E}^p(\Omega)$, we have,

$$\langle \boldsymbol{\ell}, (\boldsymbol{v}, \pi) \rangle$$

$$
\begin{aligned}
&= \langle \boldsymbol{\varphi}, P\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_\Omega \lambda\pi + \int_\Omega \boldsymbol{\psi}\cdot(-\Delta \boldsymbol{v} + \nabla\pi)\\
&= \langle \Delta\widetilde{\boldsymbol{\psi}}, P\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_\Omega \pi \ \mathrm{div}\ \boldsymbol{\psi} + \langle \boldsymbol{\psi}, (-\Delta\boldsymbol{v} + \nabla\pi)\rangle_{\boldsymbol{W}_0^{1,p'}(\Omega)\times \boldsymbol{W}^{-1,p}(\Omega)}\\
&= \lim_{k\to\infty} [\langle \Delta\widetilde{\boldsymbol{\psi}}_k, P\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_\Omega \pi \ \mathrm{div}\ \boldsymbol{\psi}_k +\\
&\qquad + \langle \boldsymbol{\psi}_k, (-\Delta\boldsymbol{v} + \nabla\pi)\rangle_{\boldsymbol{W}_0^{1,p'}(\Omega)\times \boldsymbol{W}^{-1,p}(\Omega)}]\\
&= \lim_{k\to\infty} [\langle \Delta\boldsymbol{\psi}_k, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)\times\boldsymbol{\mathcal{D}}'(\Omega)} - \langle \boldsymbol{\psi}_k, \nabla\pi\rangle_{\boldsymbol{\mathcal{D}}(\Omega)\times\boldsymbol{\mathcal{D}}'(\Omega)} + \langle \boldsymbol{\psi}_k, -\Delta\boldsymbol{v} + \nabla\pi\rangle_{\boldsymbol{\mathcal{D}}(\Omega)\times\boldsymbol{\mathcal{D}}'(\Omega)}]\\
&= 0\ .
\end{aligned}
$$

Thus $\boldsymbol{\ell}$ is identically zero.

(ii) The Green formula for $(\boldsymbol{v},\pi) \in \boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega)$ follows immediately by integration by parts and then we use the density result (cf. [11, Lemma 2.4]). ■

Remark 2.3.6. 1. The following Green formula also can be obtained in the same way as (2.12), which will be used later: for $(\boldsymbol{v},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ such that $-\mathrm{div}(2\mathbb{D}\boldsymbol{v} + \mathbb{F}) + \nabla\pi \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega)$,

$$
\int_\Omega (-\mathrm{div}(2\mathbb{D}\boldsymbol{v} + \mathbb{F}) + \nabla\pi)\cdot\boldsymbol{\varphi} = 2\int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} + \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} - \langle [(2\mathbb{D}\boldsymbol{v} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma . \tag{2.13}
$$

2. In fact, we can obtain Lemma 2.3.5 for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ where

$$
\boldsymbol{E}^p(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{r(p)',p'}(\mathrm{div},\Omega)]' \right\} .
$$

Thus we may extend (2.11) in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ as follows: let Ω be $\mathcal{C}^{1,1}$ and for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\Delta\boldsymbol{v} \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ,

$$
2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{2.14}
$$

We will also need the following density result:

Lemma 2.3.7. *Let Ω be $\mathcal{C}^{2,1}$. The space $\{\boldsymbol{v} \in \boldsymbol{H}^2(\Omega),\ \mathrm{div}\ \boldsymbol{v} = 0$ in $\Omega, \boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\Gamma\}$ is dense in $\boldsymbol{V}_{\sigma,\tau}^2(\Omega)$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}_{\sigma,\tau}^2(\Omega)$. So, there exists a sequence $\boldsymbol{u}_m \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that $\boldsymbol{u}_m \to \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$. Now consider the problem,

$$
\begin{cases}
-\Delta w_m &= \mathrm{div}\ \boldsymbol{u}_m \quad \text{in } \Omega\\
\frac{\partial w_m}{\partial \boldsymbol{n}} &= \boldsymbol{u}_m\cdot\boldsymbol{n} \quad \text{on } \Gamma.
\end{cases}
$$

Since Ω is of class $\mathcal{C}^{2,1}$, the solution of the above problem $w_m \in H^3(\Omega)$. Also $w_m \to w$ in $H^2(\Omega)$ where w is the solution of the following problem,

$$\begin{cases} -\Delta w &= 0 \quad \text{in } \Omega \\ \frac{\partial w}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \Gamma. \end{cases}$$

Now consider, $\boldsymbol{v}_m = \boldsymbol{u}_m - \nabla w_m - \nabla w$. Clearly, $\boldsymbol{v}_m \in \boldsymbol{H}^2(\Omega)$ with div $\boldsymbol{v}_m = \boldsymbol{0}$ in Ω and $\boldsymbol{v}_m \cdot \boldsymbol{n} = \boldsymbol{0}$ on Γ. Also, $\boldsymbol{v}_m \to \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$. ■

Lemma 2.3.8. *Let $p \in (1, \infty)$. For $\alpha \in L^{t(p)}(\Gamma)$ with $t(p)$ defined in (2.8), $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, the integral over the boundary $\int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}$ is well-defined.*

Proof. For convenience, from now on, we consider $\alpha \in L^{t(p)}(\Gamma)$ where we recall that

$$t(p) = \begin{cases} 2 & \text{if} \quad p = 2 \\ 2 + \varepsilon & \text{if} \quad \frac{3}{2} \le p \le 3, p \ne 2 \\ \frac{2}{3}\max\{p, p'\} + \varepsilon & \text{otherwise} \end{cases}$$

for some arbitrary $\varepsilon > 0$ sufficiently small. Since $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, we have $\boldsymbol{\varphi}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma) \hookrightarrow \boldsymbol{L}^m(\Gamma)$ where

$$\frac{1}{m} = \begin{cases} 1 - \frac{3}{2p} & \text{if } p > \frac{3}{2}, \\ \text{any positive real number} < 1 & \text{if } p = \frac{3}{2}, \\ 0 & \text{if } p < \frac{3}{2}. \end{cases} \tag{2.15}$$

Similarly, for $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{L}^s(\Gamma)$ with

$$\frac{1}{s} = \begin{cases} \frac{3}{2p} - \frac{1}{2} & \text{if } p < 3, \\ \text{any positive real number} < 1 & \text{if } p = 3, \\ 0 & \text{if } p > 3. \end{cases} \tag{2.16}$$

We want to show that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{m'}(\Gamma)$.

(i) $p = 2$: Since $\alpha \in L^2(\Gamma)$, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ by Hölder inequality. But $\frac{1}{m'} = 1 - \frac{1}{m} = \frac{3}{4}$ i.e. $q = m'$. So the integral is well-defined.

(ii) $\frac{3}{2} \le p \le 3$, $p \ne 2$: As $\alpha \in L^{2+\varepsilon}(\Gamma)$, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{1}{2+\varepsilon} + \frac{1}{s}$.
So, for $\frac{3}{2} < p < 3$, $\frac{1}{q} = \frac{3}{2p} - \frac{1}{2} + \frac{1}{2+\varepsilon}$. Then clearly, $q > m'$.

For $p = 3$, $\frac{1}{q} = \frac{1}{2+\varepsilon} + \frac{1}{s}$ where s is any number > 1. Then, we can choose s suitably such that $q > m'$.
For $p = \frac{3}{2}$, $\frac{1}{q} = \frac{1}{2+\varepsilon} + \frac{1}{2}$ and we can choose $m > 1$ suitably so that $q > m'$. Hence, in all cases, the integral is well-defined.

(iii) $p > 3$: Since $\alpha \in L^{\frac{2}{3}p+\varepsilon}(\Gamma)$, $\alpha \boldsymbol{u_\tau} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{1}{\frac{2}{3}p+\varepsilon}$. So clearly $q > m'$ and thus the integral is well-defined.

(iv) $p < \frac{3}{2}$: As $\alpha \in L^{\frac{2}{3}p'+\varepsilon}(\Gamma)$, $\alpha \boldsymbol{u_\tau} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{3}{2p} - \frac{1}{2} + \frac{1}{\frac{2}{3}p'+\varepsilon}$. Again $q > m'$ and the integral is well-defined. ■

Definition 2.3.9. Given $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$, a couple $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is called a weak solution of the Stokes system (S) if it satisfies: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u_\tau} \cdot \boldsymbol{\varphi_\tau} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{2.17}$$

Proposition 2.3.10. *Let $p \in (1, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma)$$

with $r(p)$ and $t(p)$ defined by (2.10) *and* (2.8) *respectively. Then the following two statements are equivalent:*
(i) $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ *is a weak solution of (S), in the sense of Definition 2.3.9 and*
(ii)$(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ *satisfies:*

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \ \operatorname{div} \boldsymbol{u} = 0 \quad & \textit{in the sense of distribution} \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad & \textit{in the sense of trace} \\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u_\tau} = \boldsymbol{0} \quad & \textit{in } \boldsymbol{W}^{-1/p,p}(\Gamma). \end{aligned} \tag{2.18}$$

Proof. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega) \times L^p(\Omega)$ is a weak solution of (S). We want to show it implies **(ii)**. Choosing $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as a test function in (2.17), we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi}.$$

So by De Rham's theorem, there exists $\pi \in L^p(\Omega)$, defined uniquely up to an additive constant such that

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega. \tag{2.19}$$

Also $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ implies div $\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Thus it remains to prove the Navier boundary condition. Multiplying equation (2.19) by $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ and using the Green's formula (2.13), we obtain from (2.17),

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega), \qquad \langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\varphi}\rangle_\Gamma + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,. \tag{2.20}$$

Now, let $\mu \in \boldsymbol{W}^{\frac{1}{p},p'}(\Omega)$. There exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that div $\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Then $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ and using (2.20),

$$\begin{aligned}\langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}\rangle_\Gamma &= \langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\rangle_\Gamma \\ &= \langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma = 0.\end{aligned}$$

Hence,

$$[(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} \qquad \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Conversely, using the Green formula (2.13), we can easily deduce that any $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (2.18) is a weak solution of (S), in the sense of Definition 2.3.9. ■

The following two propositions give some Korn-type inequalities which will be useful in the context. '$\simeq$' denotes the equivalence of two norms.

Proposition 2.3.11. *Let Ω be Lipschitz. Then, for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, we have the following equivalence of norms:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} \quad \textit{if } \Omega \textit{ is not axisymmetric}, \tag{2.21}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma_0)} \quad \textit{with } \Gamma_0 \subset \Gamma, |\Gamma_0| > 0 \textit{ if } \Omega \textit{ is axisymmetric}. \tag{2.22}$$

Proof. Since (2.21) follows from [11, Lemma 3.3], we only prove (2.22). Also, it is enough to show that there exists $C > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C \left(\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma_0)} \right)$$

as the other inequality is obvious.

We prove by contradiction. Suppose for any $m \in I\!N$, there exists $\boldsymbol{u}_m \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_m \cdot \boldsymbol{n} = 0$ on Γ and

$$\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)} > m \left(\|\mathbb{D}\boldsymbol{u}_m\|_{\mathbb{L}^2(\Omega)} + \|(\boldsymbol{u}_m)_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma_0)} \right). \tag{2.23}$$

Since $\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)} > 0$, we can define $\boldsymbol{v}_m := \frac{\boldsymbol{u}_m}{\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)}}$ so that $\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} = 1$ for all $m \in I\!N$. Then from (2.23), we have

$$\|\mathbb{D}\boldsymbol{v}_m\|_{\mathbb{L}^2(\Omega)} + \|(\boldsymbol{v}_m)_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma_0)} < \frac{1}{m}.$$

Hence, as $m \to \infty$,

$$\mathbb{D}\boldsymbol{v}_m \to 0 \ \text{ in } \boldsymbol{L}^2(\Omega) \quad \text{ and } \quad (\boldsymbol{v}_m)_{\boldsymbol{\tau}} \to \boldsymbol{0} \ \text{ in } \boldsymbol{L}^2(\Gamma_0).$$

But as $\boldsymbol{v}_m \cdot \boldsymbol{n} = 0$ on Γ, we get $\boldsymbol{v}_m \to \boldsymbol{0}$ in $\boldsymbol{L}^2(\Gamma_0)$. On the other hand, since $\{\boldsymbol{v}_m\}_m$ is bounded in $\boldsymbol{H}^1(\Omega)$, there exists a subsequence, which we still call $\{\boldsymbol{v}_m\}_m$ and $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v}_m \rightharpoonup \boldsymbol{v}$ weakly in $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{v}_m \to \boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$. Thus

$$\mathbb{D}\boldsymbol{v} = 0 \ \text{ in } \Omega \quad \text{ and } \quad \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \Gamma_0$$

which yields $\boldsymbol{v} = \boldsymbol{0}$ in Ω. But this is a contradiction since from Korn's inequality, we have

$$1 = \|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \leq C \left(\|\boldsymbol{v}_m\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{v}_m\|_{\mathbb{L}^2(\Omega)}\right) \to \boldsymbol{0}.$$

Thus (2.22) follows. ∎

Proposition 2.3.12. *Let Ω be Lipschitz. For Ω axisymmetric, we have the following inequalities: for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ,*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta}\right)^2\right] \tag{2.24}$$

and

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta}\right)^2\right]. \tag{2.25}$$

Proof. **(i)** First recall from (2.7) that $\boldsymbol{\beta} \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $\mathbb{D}\boldsymbol{\beta} = 0$ in $\mathbb{R}^3$. Then (2.24) follows from the following result [11, Lemma 3.3]:

$$\inf_{\boldsymbol{w} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\Omega) \left(\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u} \cdot \boldsymbol{n}|^2\right). \tag{2.26}$$

Since, $\boldsymbol{w} = c\boldsymbol{\beta}$ for some $c \in \mathbb{R}$, $\inf_{\boldsymbol{w} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} = \inf_{c \, \in \, \mathbb{R}} \|\boldsymbol{u} + c\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}$ and this infimum is attained at

$$c = -\frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}} \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta}\right).$$

Now,

$$\begin{aligned}
&\left\|\boldsymbol{u} - \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}}\left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right\|^2_{\boldsymbol{L}^2(\Omega)} \\
&= \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{2}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}}\left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\right)^2 + \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}}\left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\right)^2 \\
&= \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}}\left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\right)^2.
\end{aligned}$$

Hence, we obtain (2.24).

(ii) Now we prove the inequality (2.25), by contradiction. Let us denote

$$|||\boldsymbol{u}||| := \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}.$$

Now assume that for all $m \in I\!N$, there exists $\boldsymbol{u}_m \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u}_m \cdot \boldsymbol{n} = 0$ on Γ and $|||\boldsymbol{u}_m||| = 1$ such that

$$1 > m\left[\|\mathbb{D}\boldsymbol{u}_m\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Gamma \boldsymbol{u}_m\cdot\boldsymbol{\beta}\right)^2\right]. \tag{2.27}$$

So $\{\boldsymbol{u}_m\}_m$ is a bounded sequence in $\boldsymbol{H}^1(\Omega)$; Hence there exists a subsequence, we still call $\{\boldsymbol{u}_m\}_m$ and $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ so that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. This gives $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ and $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$. But from (2.27), we have

$$\mathbb{D}\boldsymbol{u}_m \to 0 \;\text{ in }\; \boldsymbol{L}^2(\Omega) \qquad \text{and} \qquad \int_\Gamma \boldsymbol{u}_m\cdot\boldsymbol{\beta} \to 0.$$

Then $\mathbb{D}\boldsymbol{u} = 0$ in Ω which implies $\boldsymbol{u} = c\boldsymbol{\beta}$ for some $c \in \mathbb{R}$. But also $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is compactly embedded in $\boldsymbol{L}^2(\Gamma)$ implies $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Gamma)$. Therefore, we have $\boldsymbol{u}_m\cdot\boldsymbol{\beta} \to \boldsymbol{u}\cdot\boldsymbol{\beta}$ in $\boldsymbol{L}^2(\Gamma)$ which yields $\boldsymbol{u}\cdot\boldsymbol{\beta} = 0$ on Γ. Hence, $\boldsymbol{u} = \boldsymbol{0}$ in Ω. But then

$$1 = |||\boldsymbol{u}_m||| = \|\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}_m\|_{\mathbb{L}^2(\Omega)} \to 0$$

which is a contradiction. ∎

2.4 Stokes equations: L^2-theory

In this section, we study the well-posedness of solutions of the Stokes problem (S) in the Hilbert space. First we prove the existence and uniqueness of the weak solution.

Theorem 2.4.1 (Existence in $\boldsymbol{H}^1(\Omega)$). *Let Ω be Lipschitz and*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ and } \alpha \in L^2(\Gamma)$$

with $\alpha > 0$ on $\Gamma_0 \subseteq \Gamma, |\Gamma_0| > 0$. Then the Stokes problem (S) *has a unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ which satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\alpha) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.28}$$

Proof. Existence of a unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of (S) follows from Lax-Milgram theorem. The bilinear form

$$\forall\, \boldsymbol{u}, \boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega), \qquad \boldsymbol{a}(\boldsymbol{u}, \boldsymbol{\varphi}) = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}$$

is clearly continuous. And from Proposition 2.3.11,

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{u}) = 2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \geq C(\alpha) \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \tag{2.29}$$

which shows that it is also coercive on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Moreover, the linear form $\boldsymbol{\ell} : \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \to \mathbb{R}$, defined as

$$\boldsymbol{\ell}(\boldsymbol{\varphi}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}$$

is continuous on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Hence, Lax-Milgram theorem gives the existence of a unique $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ satisfying

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{\varphi}) = \boldsymbol{\ell}(\boldsymbol{\varphi}) \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega). \tag{2.30}$$

This completes the proof since (2.30) is equivalent to the problem (S) (see Proposition 2.3.10). Existence of the pressure term follows from De Rham theorem.

The estimate (2.28) is obvious from (2.29) and (2.30). ■

Remark 2.4.2. Note that with $\alpha > 0$ on some $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| > 0$, we get the uniqueness of the solution of the Stokes problem (S). But for the case $\alpha \equiv 0$ on Γ, there is a non-trivial kernel when Ω is axisymmetric (see [11, Theorem 3.4]).

Indeed, consider the kernel $\boldsymbol{\mathcal{T}}_\alpha(\Omega)$ of the Stokes operator: $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfying (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$. Then we have the energy estimate

$$2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = 0$$

with $\alpha \geq 0$ on Γ. Hence $\mathbb{D}\boldsymbol{u} = 0$ in Ω implies $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x} + \boldsymbol{c}$ for almost all $\boldsymbol{x} \in \Omega$ (in fact, for all $x \in \overline{\Omega}$ since $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{C}^0(\overline{\Omega})$) where $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ are arbitrary constant vectors. But also $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ gives $\boldsymbol{c} = \boldsymbol{0}$.

a) If $\alpha > 0$ on Γ_0, then $\boldsymbol{b} \times \boldsymbol{x} = \boldsymbol{0}, \boldsymbol{x} \in \Gamma_0$ and thus $\boldsymbol{b} = \boldsymbol{0}$ *i.e.* $\boldsymbol{\mathcal{T}}_\alpha(\Omega) = \{\boldsymbol{0}\}$.

b) If $\alpha \equiv 0$ on Γ, we can verify easily that

i) $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x}$ if Ω is axisymmetric *i.e.* $\boldsymbol{b}$ is co-linear to the axis of Ω and $\dim \boldsymbol{\mathcal{T}}_\alpha(\Omega) = 1$.

ii) $\boldsymbol{u} = \boldsymbol{0}$ if Ω is not axisymmetric *i.e.* $\boldsymbol{\mathcal{T}}_\alpha(\Omega) = \{\boldsymbol{0}\}$.

In the next theorem we improve the estimate (2.28) with respect to α in some particular cases.

Theorem 2.4.3 (Estimates in $\boldsymbol{H}^1(\Omega)$). *With the same assumption on $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ and α as in Theorem 2.4.1, the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of the Stokes problem* (S) *satisfies the following estimates:*

a) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.31}$$

b) *if Ω is axisymmetric and*

(i) $\alpha \geq \alpha_* > 0$ on Γ, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.32}$$

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0 \tag{2.33}$$

then, the solution $\boldsymbol{u}$ satisfies $\int_\Gamma \alpha \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ and

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \|\pi\|^2_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2. \tag{2.34}$$

In particular, if α is a non-zero constant, then $\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.35}$$

Remark 2.4.4. Note that in the case of Ω axisymmetric, if α is a non-zero constant, we can use the estimate (2.32) with $\alpha = \alpha_*$. In particular if $\alpha = \frac{1}{n}, n \in \mathbb{N}^*$, the corresponding solution $(\boldsymbol{u}_n, \pi_n)$ satisfy

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\pi_n\|_{L^2(\Omega)} \leq nC(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right).$$

But this estimate is not optimal, when we suppose (2.33). In fact because of $\int_\Gamma \boldsymbol{u}_n \cdot \boldsymbol{\beta} = 0$, we have by (2.35) the better estimate

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\pi_n\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

where $C(\Omega)$ does not depend on n. That means if $\alpha \to 0$, (2.35) is better estimate than (2.32).

Proof. The solution $\boldsymbol{u}$ satisfies:

$$2\int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}. \tag{2.36}$$

a) If Ω is not axisymmetric, estimate (2.21) shows that the norm $\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and hence from (2.36), it follows

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.37}$$

On the other hand,

$$\begin{aligned} \|\pi\|_{L^2(\Omega)} \leq \|\nabla\pi\|_{\boldsymbol{H}^{-1}(\Omega)} &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \end{aligned} \tag{2.38}$$

The estimate (2.31) then follows from (2.37) and (2.38).

b) If Ω is axisymmetric and
(i) $\alpha \geq \alpha_* > 0$, estimate (2.22) gives,

$$\begin{aligned} \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C(\Omega) \left(2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \alpha_* \|\boldsymbol{u}_{\boldsymbol{\tau}}\|^2_{\boldsymbol{L}^2(\Gamma)} \right) \\ &\leq \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(2\int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \right); \end{aligned} \tag{2.39}$$

Hence estimate (2.32) follows from (2.36).

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition (2.33), then from (2.30), we get

$$2\int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{u} + \langle \boldsymbol{h}, \boldsymbol{u} \rangle_\Gamma$$

$$= \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} + k\boldsymbol{\beta}) - \int_\Omega \mathbb{F} : \nabla(\boldsymbol{u} + k\boldsymbol{\beta}) + \langle \boldsymbol{h}, \boldsymbol{u} + k\boldsymbol{\beta} \rangle_\Gamma \qquad \forall k \in \mathbb{R}$$

$$\leq C(\Omega) \left(\|\boldsymbol{f}_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \inf_{k \in \mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|_{\boldsymbol{H}^1(\Omega)}.$$

Also from Korn inequality and the inequality (2.26), we know,

$$\inf_{k \in \mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|^2_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega) \left(\inf_{k \in \mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \right) \leq C(\Omega)\ \|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)}.$$

which yields

$$2 \int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u_\tau}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}.$$

This in turn implies

$$\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

and then

$$\int_\Gamma \alpha |\boldsymbol{u_\tau}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2$$

and hence the inequality (2.34).

Moreover, if α is a non-zero constant, the variational formulation (2.30) gives,

$$\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0.$$

So now (2.25) shows that the norm $\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ is equivalent to the full norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and (2.35) is a consequence of(2.36). ■

Next we discuss the strong solution of the system (S) and the corresponding bounds, not depending on α.

Theorem 2.4.5 (**Existence and estimate in $\boldsymbol{H}^2(\Omega)$**). *Assume that α is a constant. Then if*

$$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \text{ and } \boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma),$$

the weak solution $(\boldsymbol{u}, \pi)$ of the Stokes problem (S) *with $\mathbb{F} = 0$ belongs to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. Also it satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \right). \tag{2.40}$$

(ii) *if Ω is axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{\min\{2,\alpha\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \right). \tag{2.41}$$

If moreover, $\boldsymbol{f}, \boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \right). \tag{2.42}$$

Remark 2.4.6. 1. We show in Theorem 2.5.9 the existence of $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ for more general α, not necessarily constant.

2. It is not sensible to consider non-zero $\mathbb{F} \in \mathbb{H}^1(\Omega)$ for the strong solution as we are considering any $\boldsymbol{L}^2(\Omega)$ data $\boldsymbol{f}$ in the RHS.

Proof. **Method I:** If α is a constant and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and therefore $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. So using the regularity result for strong solution [11, Theorem 4.1], we get that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.

But concerning the estimate, with this method, using the results in [11], we can not obtain the bound on $\boldsymbol{u}$, independent of α. Thus we need to consider the fundamental but long method, explained below.

Method II: Here we follow the method of difference quotient as in the book of L.C.Evans [44]. Without loss of generality, we consider $\boldsymbol{h} = \boldsymbol{0}$, for ease of notation. Also, let denote the difference quotient by,

$$D_k^h \boldsymbol{u}(x) = \frac{\boldsymbol{u}(x + h\boldsymbol{e}_k) - \boldsymbol{u}(x)}{h}, \quad k = 1, 2, 3, \quad h \in \mathbb{R}.$$

Interior regularity: The unique solution $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of (S) belongs to $\boldsymbol{H}^2_{loc}(\Omega) \times H^1_{loc}(\Omega)$ with the corresponding local estimate (2.40)-(2.42), can be shown in the same way using difference quotient as for the Dirichlet boundary condition, with the help of Theorem 2.4.3, since it does not depend on which boundary condition is considered. Thus we do not repeat it.

Boundary regularity: The solution $(\boldsymbol{u}, \pi)$ satisfies the variational formulation, for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} - \int_\Omega \pi \operatorname{div}\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}. \tag{2.43}$$

Case 1. $\Omega = B(0,1) \cap \mathbb{R}^3_+$ **:** First we consider the case when Ω is a half ball. Set $V := B(0, \frac{1}{2}) \cap \mathbb{R}^3_+$ and choose a cut-off function $\zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{cases} \zeta \equiv 1 \text{ on } B(0,\frac{1}{2}),\ \zeta \equiv 0 \text{ on } \mathbb{R}^3 \setminus B(0,1), \\ 0 \leq \zeta \leq 1. \end{cases}$$

So $\zeta \equiv 1$ on V and vanishes on the curved part of Γ.

i) Tangential regularity of velocity: Let $h > 0$ be small and $\boldsymbol{\varphi} = -D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}), k = 1,2$. Clearly, $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega)$. So substituting $\boldsymbol{\varphi}$ into the identity (2.43) we obtain,

$$\begin{aligned} 2\int_\Omega \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + 2\int_\Omega D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta\nabla\zeta D_k^h \boldsymbol{u} + \int_\Gamma \alpha\zeta^2 |D_k^h \boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\ - \int_\Omega \pi \ \mathrm{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) = \int_\Omega \boldsymbol{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})). \end{aligned} \tag{2.44}$$

Now we estimate the different terms. In this proof, from here onwards, the constant C might depend on ζ which we do not mention. For the second term in the left hand side, using Cauchy's inequality with ϵ, we get

$$\begin{aligned} |\int_\Omega D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta\nabla\zeta D_k^h \boldsymbol{u}| &\leq C \int_\Omega 2\zeta |D_k^h \mathbb{D}\boldsymbol{u}||D_k^h \boldsymbol{u}| \\ &\leq C \left[\epsilon \int_\Omega \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + \frac{1}{\epsilon} \int_\Omega |D_k^h \boldsymbol{u}|^2 \right]. \end{aligned} \tag{2.45}$$

Similarly, for the fourth term in the left hand side, we write,

$$|\int_\Omega \pi \ \mathrm{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))| \leq \epsilon \int_\Omega |\mathrm{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))|^2 + \frac{C}{\epsilon} \int_\Omega |\pi|^2$$

But note that,

$$\begin{aligned} \mathrm{div}(D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) &= D_k^{-h} \mathrm{div}(\zeta^2 D_k^h \boldsymbol{u}) \\ &= D_k^{-h}(2\zeta\nabla\zeta \cdot D_k^h \boldsymbol{u}) + D_k^{-h}(\zeta^2 \underbrace{\mathrm{div}(D_k^h \boldsymbol{u})}_{=0}) \\ &= D_k^{-h}(2\zeta\nabla\zeta) \cdot D_k^h \boldsymbol{u}(x - he_k) + 2\zeta\nabla\zeta \cdot D_k^{-h} D_k^h \boldsymbol{u} \end{aligned}$$

which means

$$\int_\Omega |\mathrm{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))|^2 \leq C \left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |D_k^{-h} D_k^h \boldsymbol{u}|^2 \right)$$

$$\leq C \left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 \right).$$

Therefore,

$$|\int_\Omega \pi \ \mathrm{div}(D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))| \leq \epsilon \left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 \right) + \frac{C}{\epsilon} \int_\Omega |\pi|^2. \tag{2.46}$$

And for the right hand side, proceeding in the same way, we derive,

$$|\int_\Omega \boldsymbol{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))| \leq \epsilon \int_\Omega |D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})|^2 + \frac{C}{\epsilon} \int_\Omega |\boldsymbol{f}|^2.$$

But, since

$$\int_\Omega |D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})|^2 \leq C \int_\Omega |\nabla(\zeta^2 D_k^h \boldsymbol{u})|^2 \leq C \left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 \right)$$

we get,

$$|\int_\Omega \boldsymbol{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))| \leq \epsilon \left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 \right) + \frac{C}{\epsilon} \int_\Omega |\boldsymbol{f}|^2. \tag{2.47}$$

Hence, incorporating (2.45), (2.46) and (2.47) in (2.44) yields,

$$\begin{aligned}
&2 \int_\Omega \zeta^2 |D_k^h \mathbb{D} \boldsymbol{u}|^2 + \int_\Gamma \alpha \ \zeta^2 |D_k^h \boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\
&\leq \epsilon \left(\int_\Omega \zeta^2 |\mathbb{D} D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 \right) + \frac{C_1}{\epsilon} \left(\int_\Omega |\boldsymbol{f}|^2 + \int_\Omega |\pi|^2 \right) \\
&\qquad + C_2 \int_\Omega |D_k^h \boldsymbol{u}|^2 \\
&\leq \epsilon \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 + \frac{C_1}{\epsilon} \left(\int_\Omega |\boldsymbol{f}|^2 + \int_\Omega |\pi|^2 \right) + C_2 \int_\Omega |D_k^h \boldsymbol{u}|^2.
\end{aligned} \tag{2.48}$$

Furthermore, we see that

$$\begin{aligned}
\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C \left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}(\zeta D_k^h \boldsymbol{u})\|^2_{\mathbb{L}^2(\Omega)} \right) \\
&\leq C \left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\zeta \mathbb{D} D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \right) \\
&\leq C \left(\|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\zeta \mathbb{D} D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \right)
\end{aligned}$$

and

$$\|\zeta \nabla D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} = \|\nabla(\zeta D_k^h \boldsymbol{u}) - \nabla\zeta D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \le \|\nabla(\zeta D_k^h \boldsymbol{u})\|^2_{\mathbb{L}^2(\Omega)} + C\ \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\le C\left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right).$$

Combining these inequalities with (2.48), we have,

$$\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \le \epsilon \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{C_1}{\epsilon}\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\pi\|^2_{\boldsymbol{L}^2(\Omega)}\right) + C_2 \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.49}$$

Choosing ϵ small, we obtain,

$$\|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(V)} \le \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \le C\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\pi\|^2_{\boldsymbol{L}^2(\Omega)} + \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right)$$

for $k = 1, 2$ and sufficiently small $|h| \neq 0$; which yields that $\partial^2 \boldsymbol{u}/\partial x_i \partial x_j$ belongs to $\boldsymbol{L}^2(V)$ for all $i, j = 1, 2, 3$ except for $i = j = 3$ with the corresponding estimates, using the estimates in Theorem 2.4.3 for $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$.

ii) Tangential regularity of pressure: Now we deduce the tangential regularity of the pressure in terms of the above derivatives of $\boldsymbol{u}$. Indeed, for $i = 1, 2$, from the Stokes equation, we get,

$$\frac{\partial}{\partial x_i}(\nabla \pi) = \frac{\partial}{\partial x_i}(\boldsymbol{f} + \Delta \boldsymbol{u}) = \frac{\partial \boldsymbol{f}}{\partial x_i} + \operatorname{div}(\nabla \frac{\partial \boldsymbol{u}}{\partial x_i}).$$

Since there is no term of the form $\partial^2 \boldsymbol{u}/\partial x_3^2$, by preceding arguments, we obtain

$$\nabla \frac{\partial \pi}{\partial x_i} = \frac{\partial}{\partial x_i}(\nabla \pi) \in \boldsymbol{H}^{-1}(V).$$

Furthermore, as we already know $\frac{\partial \pi}{\partial x_i} \in H^{-1}(V)$, Nečas inequality implies $\frac{\partial \pi}{\partial x_i} \in L^2(V)$ which also satisfies the usual estimate.

iii) Normal regularity: For the complete regularity of the solution, it remains to study the derivatives of $\boldsymbol{u}$ and π in the direction $\boldsymbol{e}_3$. Differentiating the divergence equation with respect to x_3 gives,

$$\frac{\partial^2 u_3}{\partial x_3^2} = -\sum_{i=1}^{2} \frac{\partial^2 u_i}{\partial x_i \partial x_3} \in L^2(V).$$

Also from the 3rd component of the Stokes equation, we can write,

$$\frac{\partial \pi}{\partial x_3} = f_3 + \Delta u_3 \in L^2(V)$$

which proves that $\pi \in H^1(V)$. Finally, for $i = 1, 2$, writing the ith equation of the system in the form

$$\frac{\partial^2 u_i}{\partial x_3^2} = -\sum_{j=1}^{2} \frac{\partial^2 u_i}{\partial x_j^2} - f_i + \frac{\partial \pi}{\partial x_i} \in L^2(V)$$

gives $u_i \in H^2(V)$. Hence apart from the regularity of $\boldsymbol{u}$ and π, we obtain the existence of a constant $C = C(\Omega) > 0$ independent of α such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} + \|\pi\|_{H^1(V)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

Case 2. General domain: Now we drop the assumption that Ω is a half ball and consider the general case. In this part, we follow the strategy in [112] (same as in [19]). Since Γ is $\mathcal{C}^{1,1}$, for any $x_0 \in \Gamma$, we can assume, upon relabelling the coordinate axes,

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_3 > H(x_1, x_2)\}$$

for some $r > 0$ and $H : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{1,1}$. Let us now introduce the change of variable,

$$y = (x_1, x_2, x_3 - H(x_1, x_2)) := \phi(x)$$

i.e.

$$x = (y_1, y_2, y_3 + H(y_1, y_2)) := \phi^{-1}(y)$$

which flattens the boundary locally. Choose $s > 0$ small so that the half ball $\Omega' := B(0, s) \cap \mathbb{R}^3_+$ lies in $\phi(\Omega \cap B(x_0, r))$. Also define $V' := B(0, s/2) \cap \mathbb{R}^3_+$. We also introduce the new unknown variable

$$\boldsymbol{u}'(y) = \left(u_1(x), u_2(x), u_3(x) - \frac{\partial H}{\partial x_1} u_1(x) - \frac{\partial H}{\partial x_2} u_2(x)\right).$$

It is easy to see

$$\boldsymbol{u}' \in \boldsymbol{H}^1(\Omega')$$

and

$$\boldsymbol{u}' \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega' \cap \partial\mathbb{R}^3_+.$$

The last relation is true since $\boldsymbol{n} = (\partial_1 H, \partial_2 H, -1)$ and $0 = \boldsymbol{u} \cdot \boldsymbol{n} = u_1 \partial_1 H + u_2 \partial_2 H - u_3$ and the fact that $\frac{\partial H}{\partial y_i}(0, 0) = 0, i = 1, 2$. With this transformation, it follows, for $i = 1, 2$ and $j = 1, 2$,

$$\begin{aligned}
\frac{\partial u_i}{\partial x_j} &= \frac{\partial u_i'}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u_i'}{\partial y_3} \\
\frac{\partial u_i}{\partial x_3} &= \frac{\partial u_i'}{\partial y_3} \\
\frac{\partial u_3}{\partial x_j} &= \frac{\partial u_3'}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u_3'}{\partial y_3} + \sum_{k=1}^{2}\left[\frac{\partial u_k'}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u_k'}{\partial y_3}\right] \\
\frac{\partial u_3}{\partial x_3} &= \frac{\partial u_3'}{\partial y_3} + \sum_{k=1}^{2}\frac{\partial H}{\partial y_k}\frac{\partial u_k'}{\partial y_3}.
\end{aligned}$$

Thus div $\boldsymbol{u}' = \text{div } \boldsymbol{u} = 0$. Next, we consider the variational formulation (2.43) under this change of variable. Here onwards, the calculation follows exactly same steps as in [112] (or in [19]), hence we do not repeat it here. Note that the boundary term remains unchanged *i.e.*

$$\int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_{\Gamma'} \alpha \boldsymbol{u}'_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}'_{\boldsymbol{\tau}}.$$

Therefore following the method in [19, page 1099], we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} \leq C(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

where $V = \phi^{-1}(V')$.

Now as Γ is compact, we can cover Γ with finitely many sets $\{V_i\}$ as above. Thus summing the resulting estimates, along with the interior estimate, we get $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ with the bounds, as mentioned in the Theorem. ■

2.5 Stokes equations: L^p-theory

2.5.1 General solution in $W^{1,p}(\Omega)$

In this subsection, we study the regularity of weak solution of the Stokes problem (S). We begin with recalling some useful results. For the following theorem which was introduced independently by Babǔska [14] and Brezzi [25], see [13, Theorem 4.2].

Theorem 2.5.1. *Let X and M be two reflexive Banach spaces and X' and M' be their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the operators defined by*

$$\forall v \in X, \quad \forall w \in M, \quad a(v, w) = \langle Av, w\rangle = \langle v, A'w\rangle$$

and $V = Ker\ A$. Then the following statements are equivalent :

(i) There exists $C = C(\Omega) > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v, w)}{\|v\|_X \ \|w\|_M} \geq C. \tag{2.50}$$

(ii) The operator $A : X/V \mapsto M'$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of A^{-1}.

(iii) The operator $A' : M \mapsto X' \perp V$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of $(A')^{-1}$.

Next we introduce the kernel:

$$\boldsymbol{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \text{div}\ \boldsymbol{v} = 0,\ \textbf{curl}\ \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Thanks to [13, Corollary 4.1], we know that this kernel is trivial iff Ω is simply connected. Otherwise, it is of finite dimension and spanned by the functions $\widetilde{\textbf{grad}}\ q_j^T, 1 \leq j \leq J$, where q_j^T is the unique solution up to an additive constant of the problem:

$$\begin{cases} -\Delta q_j^T & = 0 \text{ in } \Omega^o, \\ \partial_n q_j^T & = 0 \text{ on } \Gamma, \\ [q_j^T]_k & = \text{ constant} \quad \text{and} \quad [\partial_n q_j^T]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \partial_n q_j^T, 1 \right\rangle_{\Sigma_k} & = \delta_{jk}, \quad 1 \leq k \leq J. \end{cases}$$

Recall that Σ_j are the cuts in Ω such that the open set $\Omega^0 = \Omega \backslash \bigcup_{j=1}^{J} \Sigma_j$ is simply connected and $[\cdot]_k$ = jump through Σ_k. For more details, see [13].

Also, recall the following inf-sup condition (see [13, Lemma 4.4]):

Lemma 2.5.2. *There exists a constant $C > 0$, depending only on Ω and p such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq \boldsymbol{0}}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}_{\sigma,\tau}^p(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \textbf{curl}\ \boldsymbol{\xi} \cdot \textbf{curl}\ \boldsymbol{\varphi}}{\|\boldsymbol{\xi}\|_{\boldsymbol{V}_{\sigma,\tau}^p(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}(\Omega)}} \geq C \tag{2.51}$$

where

$$\boldsymbol{V}^{p'}(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega);\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0 \quad \forall\ 1 \leq j \leq J \right\}.$$

Let us now define the bilinear form: for $\boldsymbol{u} \in \boldsymbol{V}_{\sigma,\tau}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega)$,

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}. \tag{2.52}$$

Theorem 2.5.3. *Let Ω be $\mathcal{C}^{2,1}$ and $p \in (1, \infty)$, $\boldsymbol{\ell} \in [\boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega)]'$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the problem:*

$$\textit{find } \boldsymbol{u} \in \boldsymbol{V}_{\sigma,\tau}^p(\Omega) \textit{ such that for any } \boldsymbol{\varphi} \in \boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega), \quad a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle \tag{2.53}$$

has a unique solution.

Proof. Let us consider first $p \geq 2$. Since $[\boldsymbol{V}_{\sigma,\tau}^{p'}(\Omega)]' \hookrightarrow [\boldsymbol{V}_{\sigma,\tau}^2(\Omega)]'$, by Lax-Milgram theorem there exists a unique $\boldsymbol{u} \in \boldsymbol{V}_{\sigma,\tau}^2(\Omega)$ satisfying

$$\forall\ \boldsymbol{\varphi} \in \boldsymbol{V}_{\sigma,\tau}^2(\Omega), \quad a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}_{\sigma,\tau}^2(\Omega)]' \times \boldsymbol{V}_{\sigma,\tau}^2(\Omega)}. \tag{2.54}$$

Now we want to show that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Since the inf-sup condition (2.50) is known for the bilinear form

$$b(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}$$

with suitable spaces X and M (see (2.51)), we will use another formulation of problem (2.54). For that, first note for any $\boldsymbol{\varphi} \in \boldsymbol{H}^2(\Omega)$ with div $\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, we have the relation

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle .$$

Therefore, using the relation (2.11) and integration by parts, we get, for any $\boldsymbol{\varphi} \in \boldsymbol{H}^2(\Omega)$ with div $\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ,

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}x = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^2_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}\, \mathrm{d}s + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi}\, \mathrm{d}s. \quad (2.55)$$

But due to the density result Lemma 2.3.7, the relation (2.55) holds for all $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$.

Now we are in position to prove that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and for that we consider different cases.

(i) $2 < p \leq 3$:

1st Step: Since $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_1}(\Gamma)$ with $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_1}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_1}, p_1}(\Gamma)$ with $p_1 = \frac{3}{2} q_1 \ > 2$ i.e. $\frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right)$.
Therefore, as $\boldsymbol{W}^{\frac{1}{p_1}, p_1'}(\Gamma) \hookrightarrow \boldsymbol{L}^{q_1'}(\Gamma)$ with $\frac{4}{3} < q_1' < 4$ and $\boldsymbol{\Lambda}\boldsymbol{u} \in \boldsymbol{L}^4(\Gamma)$, the mapping

$$\langle \boldsymbol{L}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi} \quad \text{for} \ \ \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega) \quad (2.56)$$

defines an element in the dual space of $\boldsymbol{V}^{s_1'}(\Omega)$ with $s_1 = \min\{p_1, p\}$. Now from the inf-sup condition (2.51) and using Theorem 2.5.1, there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^{s_1}_{\sigma,\tau}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega)\ , \quad \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} = \langle \boldsymbol{L}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{s_1'}(\Omega)]' \times \boldsymbol{V}^{s_1'}(\Omega)} . \quad (2.57)$$

We will show that $\mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{u}$. For that first we extend (2.57) to any test function $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$. Since $\widetilde{\nabla} q_j^T \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \hookrightarrow \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$, using (2.55) we get

$$\begin{aligned}\left\langle \boldsymbol{L}, \widetilde{\nabla} q_j^T \right\rangle &= \left\langle \boldsymbol{\ell}, \widetilde{\nabla} q_j^T \right\rangle - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot (\widetilde{\nabla} q_j^T)_{\boldsymbol{\tau}} + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \widetilde{\nabla} q_j^T \\ &= \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \widetilde{\nabla} q_j^T \ = 0.\end{aligned}$$

Hence, for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$, we set $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \sum_j \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla} q_j^T$ which implies

$$\langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{L}, \tilde{\boldsymbol{\varphi}}\rangle + \sum_j \langle \boldsymbol{\varphi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} \left\langle \boldsymbol{L}, \widetilde{\nabla} q_j^T\right\rangle = \langle \boldsymbol{L}, \tilde{\boldsymbol{\varphi}}\rangle$$

and also $\tilde{\boldsymbol{\varphi}} \in \boldsymbol{V}^{s_1'}(\Omega)$. Therefore (2.57) yields,

$$\int_\Omega \textbf{curl}\ \boldsymbol{v}\cdot\textbf{curl}\ \boldsymbol{\varphi} = \int_\Omega \textbf{curl}\ \boldsymbol{v}\cdot\textbf{curl}\ \tilde{\boldsymbol{\varphi}} = \langle \boldsymbol{L}, \tilde{\boldsymbol{\varphi}}\rangle = \langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle .$$

So finally, we get that $\boldsymbol{v} \in \boldsymbol{V}^{s_1}_{\sigma,\tau}(\Omega)$ satisfies

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega), \qquad \int_\Omega \textbf{curl}\ \boldsymbol{v}\cdot\textbf{curl}\ \boldsymbol{\varphi} = \langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle . \tag{2.58}$$

Now as $\boldsymbol{V}^2_{\sigma,\tau}(\Omega) \hookrightarrow \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$, we deduce from (2.55) that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega), \quad \int_\Omega \textbf{curl}\ \boldsymbol{v}\cdot\textbf{curl}\ \boldsymbol{\varphi} = \int_\Omega \textbf{curl}\ \boldsymbol{u}\cdot\textbf{curl}\ \boldsymbol{\varphi} \tag{2.59}$$

which gives,

$$\textbf{curl}\ \boldsymbol{u} = \textbf{curl}\ \boldsymbol{v} \quad \text{in } \Omega. \tag{2.60}$$

Therefore, as $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega) \hookrightarrow \boldsymbol{L}^{s_1}(\Omega), \textbf{curl}\ \boldsymbol{u} \in \boldsymbol{L}^{s_1}(\Omega), \text{div}\ \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ; from Proposition 2.3.2, we deduce $\boldsymbol{u} \in \boldsymbol{W}^{1,s_1}(\Omega)$. If $s_1 = p$, the proof is complete. Otherwise, $s_1 = p_1$ and we proceed to the next step.

2nd Step: Now $\boldsymbol{u} \in \boldsymbol{W}^{1,p_1}(\Omega)$ implies $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^m(\Gamma)$ where $\frac{1}{m} = \frac{3}{2p_1} - \frac{1}{2}$ (since $p_1 < 3$). Then $\alpha\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_2}(\Gamma)$ where $\frac{1}{q_2} = \frac{1}{m} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_2}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_2},p_2}(\Gamma)$ with $p_2 = \frac{3}{2}q_2 \ > p_1$ i.e. $\frac{1}{p_2} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon} - \frac{1}{2} + \frac{1}{2+\varepsilon}\right) = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right)$. So $\boldsymbol{W}^{\frac{1}{p_2},p_2'}(\Gamma) \hookrightarrow \boldsymbol{L}^{q_2'}(\Gamma)$ with $m' < q_2'$ and $\boldsymbol{\Lambda u} \in \boldsymbol{L}^m(\Gamma)$ and hence the mapping $\boldsymbol{L}$ in (2.56) where now $\langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle$ is the duality between $[\boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)]'$ and $\boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)$, defines an element in the dual of $\boldsymbol{V}^{s_2'}(\Omega)$ with $s_2 = \min\{p_2, p\}$. Therefore, as in the previous step, there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^{s_2}_{\sigma,\tau}(\Omega)$ such that (2.58) holds for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)$ and then (2.60). Thus we get, $\boldsymbol{u} \in \boldsymbol{L}^{p_1^*}(\Omega) \hookrightarrow \boldsymbol{L}^{s_2}(\Omega), \textbf{curl}\ \boldsymbol{u} \in \boldsymbol{L}^{s_2}(\Omega), \text{div}\ \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ which gives $\boldsymbol{u} \in \boldsymbol{W}^{1,s_2}(\Omega)$. If $s_2 = p$, we are done. Otherwise, $s_2 = p_2$ and we proceed next.

(k+1)th Step: Proceeding similarly, we get $\boldsymbol{u} \in \boldsymbol{V}^{p_{k+1}}_{\sigma,\tau}(\Omega)$ with $\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right)$ (where in each step, we assumed that $p_k < 3$) which also satisfies

$$\forall\ \boldsymbol{\varphi} \in \boldsymbol{V}^{p_{k+1}'}_{\sigma,\tau}(\Omega), \quad \int_\Omega \textbf{curl}\ \boldsymbol{u}\cdot\textbf{curl}\ \boldsymbol{\varphi} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle - \int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{\varphi}_{\boldsymbol{\tau}} + 2\int_\Gamma \boldsymbol{\Lambda u}\cdot\boldsymbol{\varphi}.$$

Now choose $k = [\frac{1}{\varepsilon} - \frac{1}{2}] + 1$ such that $p_{k+1} \geq 3 \geq p$ (where $[a]$ stands for the greatest integer less than or equal to a). Hence $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. i.e. for $2 < p \leq 3$, there exists a unique $\boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$, we have (2.55) where the duality bracket $\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)}$ is replaced by $\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}$.

(ii) $p > 3$: From the previous case, we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$ which implies $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{s}(\Gamma)$ for all $s \in (1, \infty)$. Now $\alpha \in L^{\frac{2}{3}p+\varepsilon}(\Gamma)$ gives $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q}(\Gamma)$ where $\frac{1}{q} = \frac{1}{s} + \frac{1}{\frac{2}{3}p+\epsilon}$. Choosing $s > 1$ suitably, we can get $q = \frac{2}{3}p$ and hence $\boldsymbol{L}^{q}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Since $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma) \hookrightarrow \boldsymbol{L}^{q'}(\Gamma)$ with $s' < q'$ and $\boldsymbol{\Lambda u} \in \boldsymbol{L}^{s}(\Gamma)$, the mapping $\boldsymbol{L}$ in (2.56) defines an element in the dual of $\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$. So there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ such that (2.58) holds for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ and thus (2.60). Therefore we obtain similarly $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Hence, $\boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ solves the problem (2.53) for all $2 \leq p < \infty$.

Finally consider the operator $A \in \mathcal{L}(\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega), (\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega))')$, associated to the bilinear form a in (2.52), defined as $\langle A\xi, \varphi \rangle = a(\xi, \varphi)$. As proved above, for $p \geq 2$, the operator A is an isomorphism from $\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ to $(\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega))'$. Then the adjoint operator, which is equal to A is an isomorphism from $\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ to $(\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega))'$ for $p' \leq 2$. This means that the operator A is an isomorphism for $p \leq 2$ also, which ends the proof. ■

As a consequence, the above theorem yields the next important inf-sup condition.

Proposition 2.5.4. *For all* $p \in (1, \infty)$ *and* $\alpha \in L^{t(p)}(\Gamma)$*, there exists a constant* $\gamma = \gamma(\Omega, p, \alpha) > 0$ *such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega) \\ \boldsymbol{u} \neq 0}} \frac{2\int_{\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_{\Gamma} \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}}{\|\boldsymbol{u}\|_{\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)} \, \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq \gamma. \tag{2.61}$$

Also for any $\boldsymbol{\ell} \in [\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'$*, the unique solution* $\boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ *of the variational problem:*

$$\forall\ \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega), \quad 2\int_{\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_{\Gamma} \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle$$

given by Theorem 2.5.3, satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \frac{1}{\gamma} \|\boldsymbol{\ell}\|_{[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'}. \tag{2.62}$$

Remark 2.5.5. The inf-sup condition (2.61) will be improved in Theorem 2.6.14 where we obtain that the above continuity constant γ does not depend on α.

Proof. Using the equivalence (i) and (ii) in Theorem 2.5.1, we obtain the inf-sup condition (2.61) from Theorem 2.5.3. The estimate (2.62) follows immediately from (2.61). ■

Finally Theorem 2.5.3 enables us to obtain the existence of weak solution for the Stokes problem for all $1 < p < \infty$.

Corollary 2.5.6 (Existence in $\boldsymbol{W}^{1,p}(\Omega)$). *Let Ω be $\mathcal{C}^{2,1}$, $p \in (1, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

Then the Stokes problem (S) *has a unique weak solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ *which satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega, \alpha, p) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right). \tag{2.63}$$

Remark 2.5.7. The existence of $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and corresponding estimate as above can be deduced directly using the regularity result in [11, Theorem 3.7] taking $\alpha \boldsymbol{u}_{\boldsymbol{\tau}}$ as the source term in the right hand side, but only for $p > 2$. We need Theorem 2.5.3 to obtain the existence of solution for $p < 2$.

Proof. Let consider

$$\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega).$$

Clearly $\boldsymbol{\ell} \in [\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'$. Then Theorem 2.5.3 yields the existence of a unique $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ which satisfies the variational formulation (2.17).

The estimate (2.63) follows from (2.62) and

$$\|\boldsymbol{\ell}\|_{[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

∎

Remark 2.5.8. i) All the previous and following results where we have assumed $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$ hold true also for $\boldsymbol{f} \in [\boldsymbol{H}^{(r(p))',p'}_0(\mathrm{div}, \Omega)]'$ which is clear from the characterization of the space in Proposition 2.3.1.

ii) We also want to emphasize that in this work, our assumption on α is quite steep. We need this regularity in order to ensure that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-\frac{1}{q},q}(\Gamma)$ for some q so that eventually we can use our tools. But we will see later (Subsection 2.7.4) that we may suppose α less regular in some cases.

iii) Note that in the case $\alpha \equiv 0$, we are considering here more general Stokes problem than in [11]. But all the existence results (and the corresponding estimates) hold for that as well.

2.5.2 Strong solution in $W^{2,p}(\Omega)$

Concerning the existence of a strong solution, we prove the following regularity result.

Theorem 2.5.9 (Existence in $\boldsymbol{W}^{2,p}(\Omega)$). *Let $p \in (1,\infty)$. Then, for*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \le \frac{3}{2}$ and $q = p$ otherwise, the weak solution $(\boldsymbol{u}, \pi)$ of the Stokes problem (S) *with $\mathbb{F} = 0$, given by Corollary 2.5.6 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ which also satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C(\Omega, \alpha, p) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \right).$$

Proof. The proof is done essentially using the existence of weak solution and a bootstrap argument. Clearly, the data $\boldsymbol{f}, \boldsymbol{h}$ and α satisfy the hypothesis of Corollary 2.5.6. Hence there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S).

(i) $1 < p \le \frac{3}{2}$**:** We also have the following embeddings: $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{r(q)}(\Omega), \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{q},q}(\Gamma)$ and $W^{1-\frac{1}{\frac{3}{2}+\epsilon},\frac{3}{2}+\epsilon}(\Gamma) \hookrightarrow L^{2+\varepsilon}(\Gamma)$ where $q = p^*$ i.e. $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ with $q \in (\frac{3}{2}, 3]$ which show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,q}(\Omega) \times L^q(\Omega)$ again using Corollary 2.5.6. Now $\boldsymbol{u} \in \boldsymbol{W}^{1,q}(\Omega) \hookrightarrow \boldsymbol{L}^{q^*}(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$. Also as $\alpha \in W^{1-\frac{1}{\frac{3}{2}+\epsilon},\frac{3}{2}+\epsilon}(\Gamma)$, we can consider $\alpha \in W^{1,\frac{3}{2}+\varepsilon}(\Omega)$, using the lift operator. Hence from Sobolev inequality $\alpha \in L^{(\frac{3}{2}+\varepsilon)^*}(\Omega)$ and $\nabla\alpha \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$. All these implies, for all $i, j = 1, 2, 3$,

$$\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_1}(\Omega) \text{ where } \frac{1}{q_1} = \frac{1}{\frac{3}{2}+\varepsilon} - \frac{1}{3} + \frac{1}{q}$$

and

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_2}(\Omega) \text{ where } \frac{1}{q_2} = \frac{1}{\frac{3}{2}+\varepsilon} + \frac{1}{q^*}.$$

But $q_1 = q_2 > p$ and thus $\frac{\partial}{\partial x_j}(\alpha u_i) = \frac{\partial \alpha}{\partial x_j} u_i + \alpha \frac{\partial u_i}{\partial x_j} \in L^p(\Omega)$. This implies $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ or in other words $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Therefore the (general) regularity result as [11, Theorem 4.1] gives $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Note that it is possible to prove the strong existence result [11, Theorem 4.1] only for $\mathcal{C}^{1,1}$ domain since the problem (S) takes the form of an uniformly elliptic operator with complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [4].

(ii) $p > \frac{3}{2}$**:** First we assume $p < 3$. We have $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega)$ for all $s \in (1, \infty)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Also, since $\alpha \in W^{1-\frac{1}{p},p}(\Gamma)$, we can consider $\alpha \in W^{1,p}(\Omega)$.

Then $\alpha \in L^{p^*}(\Omega)$ and $\nabla\alpha \in \boldsymbol{L}^p(\Omega)$. Therefore for all $i, j = 1, 2, 3$,

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_2}(\Omega) \quad \text{where} \quad \frac{1}{q_2} = \frac{1}{p} + \frac{1}{s}$$

and

$$\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_3}(\Omega) \quad \text{where} \quad \frac{1}{q_3} = \frac{1}{p^*} + \frac{1}{3} = \frac{1}{p} \ .$$

Clearly, $q_2 < q_3$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^{q_2}(\Omega)$ where $q_2 \in (\frac{3}{2}, p)$. That implies $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega)$ and hence $\alpha \boldsymbol{u_\tau} \in \boldsymbol{W}^{1-\frac{1}{q_2},q_2}(\Gamma)$. So again by the regularity result, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega)$ where $q_2 \in (\frac{3}{2}, p)$.

Now $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^{q_2^*}(\Omega)$. So for all $i, j = 1, 2, 3$,

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^p(\Omega) \quad \text{and} \quad \alpha \frac{\partial u_i}{\partial x_j} \in L^{q_4}(\Omega) \quad \text{where} \quad \frac{1}{q_4} = \frac{1}{p^*} + \frac{1}{q_2^*} = \frac{1}{p} + \frac{1}{q_2} - \frac{2}{3}.$$

As $q_4 > p$, $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^p(\Omega)$ which implies $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and thus $\alpha \boldsymbol{u_\tau} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Therefore the regularity result as [11, Theorem 4.1] gives $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

The case for $p \geq 3$ follows exactly in the same way. ■

More generally, we have the following regularity result.

Theorem 2.5.10 (Existence in $\boldsymbol{W}^{m,p}(\Omega)$). *Let $m \geq 3$. Assume Ω is of class $\mathcal{C}^{m-1,1}$ and $p \in (1, \infty)$. Moreover, let*

$$\boldsymbol{f} \in \boldsymbol{W}^{m-2,p}(\Omega), \ \boldsymbol{h} \in \boldsymbol{W}^{m-1-\frac{1}{p},p}(\Gamma) \ \textit{and} \ \alpha \in W^{m-1-\frac{1}{p},p}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \pi)$ of the problem (S) with $\mathbb{F} = 0$, given by Corollary 2.5.6 belongs to $\boldsymbol{W}^{m,p}(\Omega) \times W^{m-1,p}(\Omega)$.

Proof. For ease of understanding, here we prove only the case $m = 3$. For $m \geq 3$, the proof is exactly similar.

First assume $p < 3$. Also we assume that $\alpha \in W^{2,p}(\Omega)$. As $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we obtain $\boldsymbol{u} \in \boldsymbol{W}^{2,p^*}(\Omega)$ from strong regularity result in Theorem 2.5.9. Note that $p^* > \frac{3}{2}$, so $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$. Now $\nabla\alpha \in W^{1,p}(\Omega)$ gives $\frac{\partial \alpha}{\partial x_j} u_i \in W^{1,p}(\Omega)$. Also as $\alpha \in W^{1,p^*}(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,p^*}(\Omega)$, we get $\alpha \frac{\partial u_i}{\partial x_j} \in W^{1,p}(\Omega)$. Thus $\frac{\partial}{\partial x_j}(\alpha u_i) = \frac{\partial \alpha}{\partial x_j} u_i + \alpha \frac{\partial u_i}{\partial x_j} \in W^{1,p}(\Omega)$ shows $\alpha \boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ which implies $\alpha \boldsymbol{u_\tau} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)$. So from the regularity result as [11, Theorem 4.1], we deduce $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{3,p}(\Omega) \times W^{2,p}(\Omega)$. As we have pointed out in the last theorem, it is possible to prove the existence result [11, Theorem 4.1] only for $\mathcal{C}^{1,1}$ regularity of the domain Ω.

For $p \geq 3$, the result follows from bootstrap argument. ■

2.6 Estimates

2.6.1 First estimates

We can deduce estimates giving precise dependence of the weak solution of (S) on the friction coefficient α in some particular cases, which is better than (2.63). Note that the following result is not optimal with respect to α and will be improved in Theorem 2.6.11.

Proposition 2.6.1. *Let $p > 2$. With the same assumptions on $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ and α as in Corollary 2.5.6, the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of problem* (S) *satisfies the following bounds:*

a) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega, p, \|\alpha\|^2_{L^{t(p)}(\Gamma)}) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right). \tag{2.64}$$

b) *if Ω is axisymmetric and*

(*i*) $\alpha \geq \alpha_* > 0$ *on* Γ, *then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \frac{C(\Omega, p, \|\alpha\|^2_{L^{t(p)}(\Gamma)})}{\min\{2, \alpha_*\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

(*ii*) $\boldsymbol{f}, \mathbb{F}$ *and* $\boldsymbol{h}$ *satisfy the condition:*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0,$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega, p, \|\alpha\|^2_{L^{t(p)}(\Gamma)}) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

Proof. We only prove (2.64) since the other inequalities follow in the same way. Assume that Ω is not axisymmetric.

(i) $2 < p < 3$: As in Lemma 2.3.8, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{3}{2p} - \frac{1}{2} + \frac{1}{2+\epsilon} < \frac{3}{2p}$ and $\boldsymbol{L}^q(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Therefore, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. But from the relation,

$$\boldsymbol{L}^q(\Gamma) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \underset{\text{continuous}}{\hookrightarrow} \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$$

we have for any $\delta > 0$, there exists a constant $C(\delta)$ with $C(\delta) \to \infty$ as $\delta \to 0$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \delta \, \|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Gamma)} + C(\delta) \, \|\boldsymbol{v}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \qquad \forall \, \boldsymbol{v} \in \boldsymbol{L}^q(\Gamma). \tag{2.65}$$

Choosing $\boldsymbol{v} = \alpha \boldsymbol{u_\tau}$ in (2.65) and using Hölder inequality and trace theorem, we get

$$\begin{aligned}\|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} &\leq \delta \ \|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{L}^q(\Omega)} + C(\delta) \ \|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{L}^{4/3}(\Gamma)} \\ &\leq \delta \ \|\alpha\|_{L^{2+\epsilon}(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C(\delta) \ \|\alpha\|_{L^2(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Now the generalized regularity result [11, Corollary 3.8] yields

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p_0(\Omega)} \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|\alpha \boldsymbol{u_\tau}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \delta \|\alpha\|_{L^{2+\epsilon}(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\right) \\ &\qquad\qquad + C(\delta) \|\alpha\|_{L^2(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Choosing $\delta > 0$ such that $1 - \delta C \|\alpha\|_{L^{2+\epsilon}(\Gamma)} = \frac{1}{2}$, we obtain

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p_0(\Omega)} \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) + C(\|\alpha\|_{L^{2+\epsilon}(\Gamma)}) \ \|\alpha\|_{L^2(\Gamma)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq C(\|\alpha\|^2_{L^{(2+\epsilon)}(\Gamma)}) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right).\end{aligned}$$

(ii) $p \geq 3$**:** The analysis is exactly similar to the previous case. ■

Remark 2.6.2. We can also extend the above estimates of Proposition 2.6.1 for $p < 2$ by duality argument in the same way as in Proposition 2.6.10 and Proposition 2.6.12.

2.6.2 Second estimates

In this subsection we prove one of the main result of this chapter. We improve the estimates in Proposition 2.6.1 with respect to α and for all $p \in (1, \infty)$.

First we discuss the estimate for $p > 2$ with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$, similar to (2.31) or (2.32).

Theorem 2.6.3 (Estimates in $\boldsymbol{W}^{1,p}(\Omega), p > 2$ for RHS $\mathbb{F}$). *Let $p > 2$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the weak solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ of (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega) \ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \tag{2.66}$$

(ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{2.67}$$

The proof of the above theorem uses the weak Reverse Hölder inequality and is similar to the result done for the Laplace-Robin problem in [10]. Since Ω is $\mathcal{C}^{1,1}$, there exists some $r_0 > 0$ such that for any $x_0 \in \Gamma$, there exists a coordinate system (x', x_3) which is isometric to the usual coordinate system and a $\mathcal{C}^{1,1}$ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ so that

$$B(x_0, r_0) \cap \Omega = \{(x', x_3) \in B(x_0, r_0) : x_3 > \psi(x')\}$$

and

$$B(x_0, r_0) \cap \Gamma = \{(x', x_3) \in B(x_0, r_0) : x_3 = \psi(x')\}\,.$$

In some places, we may write B instead of $B(x_0, r)$ where there is no ambiguity and $aB := B(x_0, ar)$ for $a > 0$. Also for any integrable function f on a domain ω, we use the usual notation to denote the average of it by

$$\fint_\omega f = \frac{1}{|\omega|} \int_\omega f.$$

We require some further results to complete the proof of Theorem 2.6.3. The following lemma is proved in [55, Lemma 0.5].

Lemma 2.6.4. *Let f, g, h be non-negative functions in $L^1(Q_0)$ where Q_0 is a cube in $\mathbb{R}^n$, $Q_R(x_0)$ is a cube centered at x_0 with sides $2R$ and let $\beta \in \mathbb{R}^+$. There exists δ_0 such that if for some $\delta \le \delta_0$, the following inequality*

$$\int_{Q_R(x_0)} f \le C(\delta) \left[R^{-\beta} \int_{Q_{2R}(x_0)} g + \int_{Q_{2R}(x_0)} h \right] + \delta \int_{Q_{2R}(x_0)} f$$

holds for all $x_0 \in Q_0$ and $R < \frac{1}{2} d(x_0, \partial Q_0)$, then there exists a constant $C > 0$ such that

$$\int_{Q_R(x_0)} f \le C \left[R^{-\beta} \int_{Q_{2R}(x_0)} g + \int_{Q_{2R}(x_0)} h \right]$$

for all $x_0 \in Q_0$ and all $R < \frac{1}{2} d(x_0, \partial Q_0)$.

Next we deduce the Caccioppoli inequality for Stokes problem, up to the boundary.

Lemma 2.6.5 (Caccioppoli inequality). *Let $(\boldsymbol{u}, \pi) \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \times L^2(\Omega)$ be a weak solution of the problem*

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} = -\int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} \qquad \forall\, \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega). \tag{2.68}$$

Then there exists a constant $C > 0$, independent of α such that for all $x_0 \in \overline{\Omega}$ and $0 < r < \frac{r_0}{2}$, we have

$$\int_{B\cap\Omega} |\nabla \boldsymbol{u}|^2 \leq C \left(\frac{1}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \int_{2B\cap\Omega} |\mathbb{F}|^2 \right). \tag{2.69}$$

Proof. We will use the identity several times that for any 'smooth enough' $\boldsymbol{v}$ and any symmetric matrix $\mathbb{M}$, $\int_\omega \mathbb{D}\boldsymbol{v} : \mathbb{M} = \int_\omega \nabla \boldsymbol{v} : \mathbb{M}$. In particular, for any 'smooth enough' $\boldsymbol{v}, \boldsymbol{\varphi}$,

$$\int_\omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} = \int_\omega \nabla \boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi}.$$

i) Pressure estimate: Let $\pi_0 = ⨍_{2B\cap\Omega} \pi$. Then we can have

$$\|\pi - \pi_0\|_{L^2(2B\cap\Omega)} \leq C \|\nabla(\pi - \pi_0)\|_{\boldsymbol{H}^{-1}(2B\cap\Omega)} = C\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(2B\cap\Omega)}$$

where $\boldsymbol{v} \in \boldsymbol{H}_0^1(2B\cap\Omega)$ is the weak solution of

$$\int_{2B\cap\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = \int_{2B\cap\Omega} (\pi - \pi_0) \text{div } \boldsymbol{\varphi} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{H}_0^1(2B\cap\Omega).$$

Note that the above constant C depends only on Ω, not on r (cf. [55, 4. Proposition 1.8, Part II]). But from (2.68), we obtain (extending $\boldsymbol{\varphi}$ by 0 outside $2B\cap\Omega$, we may consider $\boldsymbol{\varphi} \in \boldsymbol{H}_0^1(\Omega)$ and replacing π by $\pi - \pi_0$ since $\pi - \pi_0$ also satisfies (S))

$$\int_{2B\cap\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = 2 \int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_{2B\cap\Gamma} \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + \int_{2B\cap\Omega} \mathbb{F} : \nabla \boldsymbol{\varphi} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{H}_0^1(2B\cap\Omega).$$

Now putting $\boldsymbol{\varphi} = \boldsymbol{v}$ yields

$$\|\pi - \pi_0\|_{L^2(2B\cap\Omega)} \leq C(\Omega) \left(\|\nabla \boldsymbol{u}\|_{\mathbb{L}^2(2B\cap\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(2B\cap\Omega)} \right). \tag{2.70}$$

ii) Caccioppoli inequality: Consider a cut-off unction $\eta \in C_c^\infty(2B)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B \quad \text{and} \quad |\nabla \eta| \leq \frac{C}{r} \text{ in } 2B. \tag{2.71}$$

Now choosing $\boldsymbol{\varphi} = \eta^2 \boldsymbol{u}$ in (2.68), we have,

$$2 \int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}(\eta^2 \boldsymbol{u}) + \int_{2B\cap\Gamma} \alpha \eta^2 |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 - \int_{2B\cap\Omega} (\pi - \pi_0) \text{ div}(\eta^2 \boldsymbol{u}) = - \int_{2B\cap\Omega} \mathbb{F} : \nabla(\eta^2 \boldsymbol{u})$$

which gives, using the fact that div $\boldsymbol{u} = 0$ in Ω,

$$\begin{aligned} &2 \int_{2B\cap\Omega} \eta^2 |\mathbb{D}\boldsymbol{u}|^2 + \int_{2B\cap\Gamma} \alpha \eta^2 |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\ =& -4 \int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \eta \nabla \eta \boldsymbol{u} + 2 \int_{2B\cap\Omega} (\pi - \pi_0) \eta \nabla \eta \boldsymbol{u} - \int_{2B\cap\Omega} \mathbb{F} : \eta^2 \nabla \boldsymbol{u} - 2 \int_{2B\cap\Omega} \mathbb{F} : \eta \nabla \eta \boldsymbol{u} \end{aligned}$$

where $\nabla\eta\boldsymbol{u}$ is the matrix $\nabla\eta\otimes\boldsymbol{u}$. Next using Young's inequality on the RHS yields,

$$
\begin{aligned}
& 2\int_{2B\cap\Omega}\eta^2|\mathbb{D}\boldsymbol{u}|^2+\int_{2B\cap\Gamma}\alpha\eta^2|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\
\leq\ & \varepsilon\int_{2B\cap\Omega}\eta^2|\mathbb{D}\boldsymbol{u}|^2+C_\varepsilon\int_{2B\cap\Omega}|\boldsymbol{u}|^2|\nabla\eta|^2+\varepsilon\int_{2B\cap\Omega}\eta^2|\pi-\pi_0|^2+C_\varepsilon\int_{2B\cap\Omega}|\nabla\eta|^2|\boldsymbol{u}|^2 \\
& +\varepsilon\int_{2B\cap\Omega}\eta^2|\nabla\boldsymbol{u}|^2+C_\varepsilon\int_{2B\cap\Omega}\eta^2|\mathbb{F}|^2+\varepsilon\int_{2B\cap\Omega}\eta^2|\mathbb{F}|^2+C_\varepsilon\int_{2B\cap\Omega}|\nabla\eta|^2|\boldsymbol{u}|^2.
\end{aligned}
$$

Upon choosing $\varepsilon>0$ suitably and using the properties (2.71) and that $\alpha\geq 0$, we get then

$$
\int_{B\cap\Omega}|\mathbb{D}\boldsymbol{u}|^2\leq\frac{C}{r^2}\int_{2B\cap\Omega}|\boldsymbol{u}|^2+\varepsilon\int_{2B\cap\Omega}|\pi-\pi_0|^2+\varepsilon\int_{2B\cap\Omega}|\nabla\boldsymbol{u}|^2+C\int_{2B\cap\Omega}|\mathbb{F}|^2
$$

where the constant $C>0$ is independent of α. Now plugging the pressure estimate (2.70) gives,

$$
\int_{B\cap\Omega}|\mathbb{D}\boldsymbol{u}|^2\leq\frac{C}{r^2}\int_{2B\cap\Omega}|\boldsymbol{u}|^2+\varepsilon\int_{2B\cap\Omega}|\nabla\boldsymbol{u}|^2+C\int_{2B\cap\Omega}|\mathbb{F}|^2.
$$

Next adding the term $\int_{B\cap\Omega}|\boldsymbol{u}|^2$ in both sides, choosing $r\leq 1$ (as Ω is bounded, we can do so) hence $\frac{1}{r}\geq 1$ and using Korn inequality, we obtain

$$
\int_{B\cap\Omega}|\nabla\boldsymbol{u}|^2\leq\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(B\cap\Omega)}\leq C(\Omega)\left(\frac{1}{r^2}\int_{2B\cap\Omega}|\boldsymbol{u}|^2+\int_{2B\cap\Omega}|\mathbb{F}|^2\right)+\varepsilon\int_{2B\cap\Omega}|\nabla\boldsymbol{u}|^2.
$$

Therefore, using Lemma 2.6.4 with $\beta=2$, we achieve the desired inequality (2.69). ■

We state the following boundary Hölder estimate:

Proposition 2.6.6. *Let $p>1$ and $\gamma\in(0,1)$. Suppose that*

$$
\begin{cases}
-\Delta\boldsymbol{v}+\nabla z=\boldsymbol{0}, \quad \operatorname{div}\boldsymbol{v}=0 & \text{in } B(x_0,r)\cap\Omega \\
\boldsymbol{v}\cdot\boldsymbol{n}=0, \quad \alpha\boldsymbol{v}_{\boldsymbol{\tau}}+2[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}=\boldsymbol{0} & \text{on } B(x_0,r)\cap\Gamma
\end{cases}
$$

for some $x_0\in\Gamma$ and $0<r<r_0$, Then for any $x,y\in B(x_0,r/2)\cap\Omega$,

$$
|\boldsymbol{v}(x)-\boldsymbol{v}(y)|\leq C\left(\frac{|x-y|}{r}\right)^\gamma\left(\fint_{B(x_0,r)\cap\Omega}|\boldsymbol{v}|^2\right)^{1/2} \tag{2.72}
$$

where $C>0$ depends only on Ω, but independent of α.

Proof. It can be shown in the exact same way as done in [55, Theorem 2.8 (a), Part II], since we have the corresponding Caccioppoli inequality (2.69) as in [55, Theorem 2.2, Part II]. ■

Lemma 2.6.7 (weak reverse Hölder inequality). *Let $p \geq 2$. Then for any $B(x_0, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x_0, 2r) \subset \Omega$ or $x_0 \in \Gamma$, the following weak Reverse Hölder inequality holds:*
(i) *if $B(x_0, 2r) \subset \Omega$,*

$$\left(\fint_{B(x_0,r)} |\nabla \boldsymbol{u}|^p \right)^{1/p} \leq C \left(\fint_{B(x_0,2r)} |\nabla \boldsymbol{u}|^2 \right)^{1/2} \tag{2.73}$$

whenever $v \in H^1(B(x, 2r))$ satisfies $\mathcal{L}v = 0$ in $B(x, 2r)$.
(ii) *if $x \in \Gamma$,*

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla \boldsymbol{u}|^p \right)^{1/p} \leq C \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 \right)^{1/2} \tag{2.74}$$

whenever $\boldsymbol{u} \in \boldsymbol{H}^1(B(x_0, 2r) \cap \Omega)$ satisfies

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \text{div } \boldsymbol{u} = 0 & \text{in } \; B(x_0, 2r) \cap \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \alpha \boldsymbol{u}_{\boldsymbol{\tau}} + 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \; B(x_0, 2r) \cap \Gamma \quad (\text{if } x_0 \in \Gamma). \end{cases}$$

The constant $C > 0$ at most depends on Ω and p.

Proof. **case(i) :** $B(x_0, 2r) \subset \Omega$.
The weak reverse Hölder inequality (2.73) holds for any $p \geq 2$, by the following interior estimates for Stokes operator [65, Theorem 2.7 (a)]:

$$\sup_{B(x_0,r)} |\nabla \boldsymbol{u}| \leq C \left(\fint_{B(x_0,2r)} |\nabla \boldsymbol{u}|^2 \right)^{1/2}.$$

case(ii) : $x_0 \in \Gamma$.
From the interior gradient estimate for Stokes problem, we can write (eg. see [65, Theorem 2.7, (3)])

$$|\nabla \boldsymbol{u}(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{u}|^2 \right)^{1/2}.$$

Now for fixed $y \in B(x, 2c\delta(x))$, let $\boldsymbol{v}(x) = \boldsymbol{u}(x) - \boldsymbol{u}(y)$. Then $-\Delta \boldsymbol{v} + \nabla z = \mathbf{0}$, div $\boldsymbol{v} = 0$ in $B(x, 2c\delta(x))$ and thus we may write from the above argument,

$$|\nabla \boldsymbol{v}(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{v}|^2 \right)^{1/2}$$

which gives, along with the boundary Hölder estimate (2.72),

$$\begin{aligned}
|\nabla \boldsymbol{u}(x)| &\leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{u}(z) - \boldsymbol{u}(y)|^2 \mathrm{d}z \right)^{1/2} \\
&= \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\int_{B(x,c\delta(x))} |\boldsymbol{u}(z) - \boldsymbol{u}(y)|^2 \mathrm{d}z \right)^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left[\int_{B(x,2c\delta(x))} \left(\frac{|z-y|}{r} \right)^{2\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right) \mathrm{d}z \right]^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right)^{1/2} \frac{1}{r^\gamma} \left(\int_{B(x,2c\delta(x))} |z-y|^{2\gamma} \mathrm{d}z \right)^{1/2} \\
&\leq \frac{C_\gamma}{(\delta(x))^{1+\frac{3}{2}}} \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right)^{1/2} \frac{1}{r^\gamma} (\delta(x))^{\gamma+\frac{3}{2}} \\
&= C_\gamma \, \frac{(\delta(x))^{\gamma-1}}{r^\gamma} r^{-3/2} \left(\int_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right)^{1/2} \\
&\leq C_\gamma \, \frac{(\delta(x))^{\gamma-1}}{r^\gamma} r^{1-3/2} \left(\int_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^6 \right)^{1/6} \\
&\leq C_\gamma \left(\frac{r}{\delta(x)} \right)^{1-\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 \right)^{1/2}.
\end{aligned}$$

Since $\gamma \in (0, 1)$ is arbitrary, we thus have,

$$|\nabla \boldsymbol{u}(x)| \leq C_\gamma \left(\frac{r}{\delta(x)} \right)^{\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 \right)^{1/2}.$$

Finally it yields choosing γ so that $p\gamma < 1$,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla \boldsymbol{u}|^p \right)^{1/p} \leq C_p \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 \right)^{1/2}.$$

This completes the proof. ■

With the following abstract lemma which is proved in [51, Theorem 2.2], we are now in a position to prove Theorem 2.6.3.

Lemma 2.6.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $p > 2$. Let $G \in L^2(\Omega)$ and $f \in L^q(\Omega)$ for some $2 < q < p$. Suppose that for each ball B with the property that $|B| \leq \beta|\Omega|$ and either $2B \subset \Omega$ or B centers on Γ, there exist two integrable functions G_B and R_B on $2B \cap \Omega$ such that $|G| \leq |G_B| + |R_B|$ on $2B \cap \Omega$ and*

$$\left(\fint_{2B\cap\Omega} |R_B|^p \right)^{1/p} \leq C_1 \left[\left(\fint_{\gamma B\cap\Omega} |G|^2 \right)^{1/2} + \sup_{B\subset B'} \left(\fint_{B'\cap\Omega} |f|^2 \right)^{1/2} \right] \tag{2.75}$$

and

$$\left(\fint_{2B\cap\Omega} |G_B|^2 \right)^{1/2} \leq C_2 \sup_{B\subset B'} \left(\fint_{B'\cap\Omega} |f|^2 \right)^{1/2} \tag{2.76}$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \gamma$. Then we have,

$$\left(\fint_{\Omega} |G|^q \right)^{1/q} \leq C \left[\left(\fint_{\Omega} |G|^2 \right)^{1/2} + \left(\fint_{\Omega} |f|^q \right)^{1/q} \right] \tag{2.77}$$

where $C > 0$ depends only on $C_1, C_2, n, p, q, \beta, \gamma$ and Ω.

Proof of Theorem 2.6.3. Given any ball B with either $2B \subset \Omega$ or B centers on Γ, let $\varphi \in C_c^\infty(8B)$ be a cut-off function such that $0 \leq \varphi \leq 1$ and

$$\varphi = \begin{cases} 1 & \text{on } 4B \\ 0 & \text{outside } 8B \end{cases}$$

and we decompose $(\boldsymbol{u}, \pi) = (\boldsymbol{v}, \pi_1) + (\boldsymbol{w}, \pi_2)$ where $(\boldsymbol{v}, \pi_1), (\boldsymbol{w}, \pi_2)$ satisfy

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi_1 = \text{div } (\varphi\mathbb{F}), \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = -[(\varphi\mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} \text{ on } \Gamma \end{cases} \tag{2.78}$$

and

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \pi_2 = \text{div}\ ((1-\varphi)\mathbb{F}), \quad \text{div}\ \boldsymbol{w} = 0 \ \ \text{in } \Omega \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{w}_{\boldsymbol{\tau}} = -[((1-\varphi)\mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} \ \ \text{on } \Gamma. \end{cases} \tag{2.79}$$

Multiplying (2.78) by $\boldsymbol{v}$ and integrating by parts, we get,

$$\int_\Omega |\nabla \boldsymbol{v}|^2 + \int_\Gamma \alpha |\boldsymbol{v}_{\boldsymbol{\tau}}|^2 = -\int_\Omega \varphi\mathbb{F} : \nabla \boldsymbol{v}$$

which gives

$$\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} \leq \|\varphi\mathbb{F}\|_{\mathbb{L}^2(\Omega)}. \tag{2.80}$$

(i) First we consider the case $4B \subset \Omega$. We want to apply Lemma 2.6.8 with $G = |\nabla \boldsymbol{u}|, G_B = |\nabla \boldsymbol{v}|$ and $R_B = |\nabla \boldsymbol{w}|$. It is easy to see that

$$|G| \leq |G_B| + |R_B|.$$

Now we verify (2.75) and (2.76). For that, using (2.80) we get,

$$\begin{aligned} \frac{1}{|2B|}\int_{2B} |G_B|^2 = \frac{1}{|2B|}\int_{2B} |\nabla \boldsymbol{v}|^2 \leq \frac{1}{|2B\cap\Omega|}\int_\Omega |\nabla \boldsymbol{v}|^2 &\leq \frac{1}{|2B\cap\Omega|}\int_\Omega |\varphi\mathbb{F}|^2 \\ &\leq \frac{C(\Omega)}{|8B\cap\Omega|}\int_{8B\cap\Omega} |\mathbb{F}|^2 \end{aligned}$$

where in the last inequality, we used that $|8B\cap\Omega| \leq |\Omega|$. This gives the estimate (2.76).

Next, from (2.79), we observe that $-\Delta \boldsymbol{w} + \nabla \pi_2 = \boldsymbol{0}$, div $\boldsymbol{w} = 0$ in $4B$. Hence, by the weak reverse Hölder inequality in Lemma 2.6.7 (using $2B$ instead of B), we have

$$\left(\fint_{2B} |\nabla \boldsymbol{w}|^p\right)^{1/p} \leq C_p(\Omega)\left(\fint_{4B} |\nabla \boldsymbol{w}|^2\right)^{1/2}$$

which implies together with (2.80),

$$\begin{aligned} \left(\fint_{2B} |R_B|^p\right)^{1/p} \leq C_p(\Omega)\left(\fint_{4B} |\nabla \boldsymbol{w}|^2\right)^{1/2} &\leq C_p(\Omega)\left[\left(\fint_{4B} |\nabla \boldsymbol{u}|^2\right)^{1/2} + \left(\fint_{4B} |\nabla \boldsymbol{v}|^2\right)^{1/2}\right] \\ &\leq C_p(\Omega)\left(\fint_{4B} |G|^2\right)^{1/2} + \left(\fint_{8B\cap\Omega} |\mathbb{F}|^2\right)^{1/2}. \end{aligned}$$

This gives (2.75). So from (2.77), it follows that

$$\left(\fint_\Omega |\nabla \boldsymbol{u}|^q\right)^{1/q} \leq C_p(\Omega)\left[\left(\fint_\Omega |\nabla \boldsymbol{u}|^2\right)^{1/2} + \left(\fint_\Omega |\mathbb{F}|^q\right)^{1/q}\right]$$

for any $2 < q < p$ where $C_p(\Omega) > 0$ does not depend on α.

Because of the self-improving property of the weak Reverse Hölder condition (2.73), the above estimate holds for any $q \in (2, \tilde{p})$ for some $\tilde{p} > p$ also and in particular, for $q = p$, which along with the $\boldsymbol{L}^2$-estimate (2.31) implies (2.66).

(ii) Next consider B centers on Γ. We apply again Lemma 2.6.8 with $G = |\nabla \boldsymbol{u}|, G_B = |\nabla \boldsymbol{v}|$ and $R_B = |\nabla \boldsymbol{w}|$, which yields $|G| \leq |G_B| + |R_B|$ and (2.76) as before. Also now $\boldsymbol{w}$ satisfies the problem

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \pi_2 = \boldsymbol{0}, \quad \text{div}\ \boldsymbol{w} = 0 & \text{in } 4B \cap \Omega \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad \alpha \boldsymbol{w}_{\boldsymbol{\tau}} + 2[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } 4B \cap \Gamma. \end{cases}$$

So by the weak reverse Hölder inequality in Lemma 2.6.7 and the estimate (2.80), we obtain (2.75) as in the previous case. Thus (2.77) yields,

$$\left(\fint_\Omega |\nabla \boldsymbol{u}|^q\right)^{1/q} \leq C_p(\Omega)\left[\left(\fint_\Omega |\nabla \boldsymbol{u}|^2\right)^{1/2} + \left(\fint_\Omega |\mathbb{F}|^q\right)^{1/q}\right]$$

for any $2 < q < p$. This completes the proof together with the $\boldsymbol{L}^2$-estimate (2.32). ■

The next proposition will be used to study the complete Stokes problem (S). We will improve the following result in Proposition 2.6.12 where we consider data less regular.

Proposition 2.6.9 (Estimates in $\boldsymbol{W}^{1,p}(\Omega), p > 2$ for RHS $\boldsymbol{f}$). *Let $p > 2$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S) with $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$, satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{2.81}$$

(ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.82}$$

Proof. The result follows using the same argument as in Theorem 2.6.3 and hence we do not repeat it. Note that as $\pi \in L^p_0(\Omega)$,

$$\|\pi\|_{L^p(\Omega)} \leq C\|\nabla\pi\|_{\boldsymbol{W}^{-1,p}(\Omega)} \leq C\left(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}(\Omega)}\right) \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.83}$$

■

Proposition 2.6.10 (Estimates in $\boldsymbol{W}^{1,p}(\Omega)$ for RHS $\mathbb{F}$). *Let $p \in (1,\infty)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega)\ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \tag{2.84}$$

(ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega,\alpha_*)\ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{2.85}$$

Proof. For $p > 2$, the estimates (2.84) and (2.85) are proved in Theorem 2.6.3. Now suppose that $1 < p < 2$. We prove it in two steps. Also without loss of generality, we consider Ω is not axisymmetric.

(i) We first show that

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega)\|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{2.86}$$

For that, we write

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0\neq\mathbb{G}\in\mathbb{L}^{p'}(\Omega)} \frac{|\int_\Omega \nabla\boldsymbol{u} : \mathbb{G}|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}}. \tag{2.87}$$

Now, for any matrix $\mathbb{G} \in (\mathcal{D}(\Omega))^{3\times3}$, let $(\boldsymbol{v},\tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ be the solution of

$$\begin{cases} -\Delta\boldsymbol{v} + \nabla\tilde{\pi} = \operatorname{div}\mathbb{G}, \quad \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{v} + \mathbb{G})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Since $p' > 2$, from Theorem 2.6.3, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}.$$

Also if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the solution of (S) with $\boldsymbol{f} = \boldsymbol{0}, \boldsymbol{h} = \boldsymbol{0}$, using the weak formulation of the problems satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$, we obtain

$$-\int_\Omega \mathbb{F} : \nabla\boldsymbol{v} = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + \int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{v}_{\boldsymbol{\tau}} = -\int_\Omega \mathbb{G} : \nabla\boldsymbol{u}$$

which gives,

$$|\int_\Omega \boldsymbol{G} : \nabla \boldsymbol{u}| \leq \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C_p(\Omega) \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}$$

and hence (2.86) follows from (2.87).

(ii) Next we prove that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{2.88}$$

For that, we write similarly

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}}. \tag{2.89}$$

From Proposition 2.6.9, we get for any $\boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)$, the unique solution $(\boldsymbol{w}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L_0^{p'}(\Omega)$ of the problem

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tilde{\pi} = \boldsymbol{\varphi}, \quad \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{w}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{2.90}$$

satisfies

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\ \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{2.91}$$

Therefore using the weak formulation of the problems satisfied by $\boldsymbol{u}$ and $\boldsymbol{w}$, we get,

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \tilde{\pi}) &= 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} - 2 \int_\Gamma \boldsymbol{u} \cdot (\mathbb{D}\boldsymbol{w})\boldsymbol{n} \\ &= 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = - \int_\Omega \mathbb{F} : \nabla \boldsymbol{w} \end{aligned}$$

which implies (2.88) from the relation (2.89), along with (2.91).

The bound on pressure follows as in the previous proposition. This completes proof. ■

Now we study the complete problem (S).

Theorem 2.6.11 (Complete estimates in $\boldsymbol{W}^{1,p}(\Omega)$). *Let $p \in (1, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \alpha \in L^{t(p)}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L_0^p(\Omega)$ of (S) satisfies the following estimates:
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \tag{2.92}$$

(ii) *if* Ω *is axisymmetric and* $\alpha \geq \alpha_* > 0$*, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right). \tag{2.93}$$

To prove the above theorem, we also need the following proposition:

Proposition 2.6.12 (Estimates in $\boldsymbol{W}^{1,p}(\Omega)$ with RHS $\boldsymbol{f}$ and $\boldsymbol{h}$). *Let* $p \in (1, \infty)$ *and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \alpha \in L^{t(p)}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ *of (S) with* $\mathbb{F} = 0$ *satisfies the following estimates:*
(i) *if* Ω *is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \tag{2.94}$$

(ii) *if* Ω *is axisymmetric and* $\alpha \geq \alpha_* > 0$*, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right). \tag{2.95}$$

Proof. Without loss of generality, we only consider the case Ω is not axisymmetric. The proof is similar to that of Proposition 2.6.9 with obvious modification.
(i) To show

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \tag{2.96}$$

we write

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \mathbb{G} \in \mathbb{L}^{p'}(\Omega)} \frac{|\int_\Omega \nabla \boldsymbol{u} : \mathbb{G}|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}}. \tag{2.97}$$

For any matrix $\mathbb{G} \in (\mathcal{D}(\Omega))^{3\times 3}$, let $(\boldsymbol{v}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ be the solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \tilde{\pi} = \operatorname{div} \mathbb{G}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{v} + \mathbb{G})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

which satisfies estimate (by Proposition 2.6.10)

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega) \|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}.$$

Also, if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ is a solution of (S) with $\mathbb{F} = 0$, from the weak formulation of the problems satisfied by $\boldsymbol{u}$ and $\boldsymbol{v}$, we get

$$-\int_\Omega \mathbb{G} : \nabla \boldsymbol{u} = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma.$$

This implies, together with the embedding $\boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{(r(p))'}(\Omega)$ for all $p \in (1,\infty)$ (follows from the definition of $r(p)$),

$$|\int_\Omega \mathbb{G} : \nabla \boldsymbol{u}| \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^{(r(p))'}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \|\boldsymbol{v}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$$
$$\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)}.$$

Therefore, (2.96) follows from (2.97).

(ii) Next we prove the following bound as done in (2.88):

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \tag{2.98}$$

except that we do not need to assume $p < 2$ here as in (2.88). Having

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}}, \tag{2.99}$$

there exists a unique $(\boldsymbol{w}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L_0^{p'}(\Omega)$ of the problem (2.90) for any $\boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)$ satisfying the estimate (2.91) (For $p < 2$, the estimate (2.91) can be proved by the exact same argument as in Proposition 2.6.10). Thus we can write,

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \tilde{\pi}) = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \langle \boldsymbol{h}, \boldsymbol{w} \rangle_\Gamma$$

which yields (2.98) as before. For the pressure estimate, we proceed as in (2.83). ■

Proof of Theorem 2.6.11. Let $\boldsymbol{u}_1 \in \boldsymbol{W}^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta \boldsymbol{u}_1 + \nabla \pi_1 = \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_1 = 0 & \text{in } \Omega \\ \boldsymbol{u}_1 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_1 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{1\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

given by Proposition 2.6.10 and $\boldsymbol{u}_2 \in \boldsymbol{W}^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta \boldsymbol{u}_2 + \nabla \pi_2 = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_2 = 0 & \text{in } \Omega \\ \boldsymbol{u}_2 \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u}_2)\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{2\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases}$$

given by Proposition 2.6.12. Then $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1, \pi_1) + (\boldsymbol{u}_2, \pi_2)$ and is the solution of the problem (S) which also satisfies the estimate (2.92) and (2.93). ■

Remark 2.6.13. Note that it is also possible to deduce uniform estimate (2.92) in the case when Ω is axisymmetric, α is a constant with no strict positive lower bound α_* and the condition (2.33) is satisfied. Indeed, we may use the L^2 estimate (2.35) in (**??**) and carry forward all consequent results.

In the next result, we improve the dependence of the continuity constant γ of the inf-sup condition (2.61) on the parameters and show that it is actually independent of α.

Theorem 2.6.14. *Let $p \in (1,\infty)$ and $\alpha \in L^{t(p)}(\Gamma)$. We have the following inf-sup condition:*

$$\inf_{\substack{\boldsymbol{u}\in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)\\ \boldsymbol{u}\neq 0}} \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\left|2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u_\tau}\cdot \boldsymbol{\varphi_\tau}\right|}{\|\boldsymbol{u}\|_{\boldsymbol{V}^p_{\sigma,\tau}(\Omega)}\,\|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq \gamma(\Omega,p) \tag{2.100}$$

when either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$.

Proof. It follows the same proof as in Proposition 2.6.12. Indeed, let $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{u} \neq \boldsymbol{0}$. Then by Korn inequality, $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$.

1. We first write

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0\neq \mathbb{G}\in \mathbb{L}^{p'}(\Omega)} \frac{\left|\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}} = \sup_{0\neq \mathbb{G}\in \mathbb{L}^{p'}_s(\Omega)} \frac{\left|\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}} \tag{2.101}$$

where $\mathbb{L}^{p'}_s(\Omega)$ is the space of all symmetric matrices in $\mathbb{L}^{p'}(\Omega)$. For the last equality, note that, for any matrix $\mathbb{G}$, it can be decomposed as $\mathbb{G} = \frac{1}{2}(\mathbb{G} + \mathbb{G}^T) + \frac{1}{2}(\mathbb{G} - \mathbb{G}^T)$ and $\mathbb{D}\boldsymbol{u} : (\mathbb{G} - \mathbb{G}^T) = 0$. Therefore, denoting $\mathbb{K} = \frac{1}{2}(\mathbb{G} + \mathbb{G}^T)$, we have $\mathbb{K} \in \mathbb{L}^{p'}_s(\Omega)$ and $\|\mathbb{K}\|_{\mathbb{L}^{p'}(\Omega)} \leq 2\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}$ which shows that

$$\sup_{0\neq \mathbb{G}\in \mathbb{L}^{p'}(\Omega)} \frac{\left|\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}} \leq \sup_{0\neq \mathbb{K}\in \mathbb{L}^{p'}_s(\Omega)} \frac{\left|\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{K}\right|}{\|\mathbb{K}\|_{\mathbb{L}^{p'}(\Omega)}}.$$

And the reverse inequality in the above relation is obvious.

Now for any $\mathbb{G} \in \mathbb{L}^{p'}_s(\Omega)$, let $(\boldsymbol{\varphi}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \nabla\tilde{\pi} = \operatorname{div}\,\mathbb{G}, \quad \operatorname{div}\,\boldsymbol{\varphi} = 0 & \text{in } \Omega\\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{\varphi} + \mathbb{G})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{\varphi_\tau} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{2.102}$$

Since we have either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, the solution also satisfies estimate (by Proposition 2.6.10),

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}. \tag{2.103}$$

Also taking $\boldsymbol{u}$ as a test function in the weak formulation of (2.102), we obtain

$$2\int_\Omega \mathbb{D}\boldsymbol{\varphi} : \mathbb{D}\boldsymbol{u} + \int_\Gamma \alpha\boldsymbol{\varphi_\tau}\cdot\boldsymbol{u_\tau} = -\int_\Omega \mathbb{G} : \nabla\boldsymbol{u} = -\int_\Omega \mathbb{G} : \mathbb{D}\boldsymbol{u} \tag{2.104}$$

where in the last equality, we used $\mathbb{G}$ is symmetric. Thus, from (2.101), combining (2.103) and (2.104), we get

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \sup_{\substack{\varphi \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) \\ \varphi \neq 0}} \frac{|2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}}.$$

2. Similarly as in (2.88), to show

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \sup_{\substack{\varphi \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) \\ \varphi \neq 0}} \frac{|2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} \tag{2.105}$$

we write

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \boldsymbol{u} \cdot \boldsymbol{w}|}{\|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)}}. \tag{2.106}$$

So for any $\boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)$, the unique solution $(\boldsymbol{\varphi}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ of the Stokes problem

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \nabla\tilde{\pi} = \boldsymbol{w}, \quad \operatorname{div} \boldsymbol{\varphi} = 0 & \text{in } \Omega \\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{\varphi})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{2.107}$$

satisfies (by Proposition 2.6.9)

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\ \|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{2.108}$$

Therefore taking $\boldsymbol{u}$ as a test function in the weak formulation of (2.107), we get,

$$2\int_\Omega \mathbb{D}\boldsymbol{\varphi} : \mathbb{D}\boldsymbol{u} + \int_\Gamma \alpha \boldsymbol{\varphi}_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{w}$$

which yields (2.105) from (2.106) together with (2.108).

Hence, the constant $\gamma(\Omega, p)$ is simply the inverse of $C_p(\Omega)$. ■

Theorem 2.6.15. *Let $p > 2$, α be a constant and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ of (S) with $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimates:*
(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C_p(\Omega)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{2.109}$$

(ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.110}$$

Remark 2.6.16. If we consider the operator of the form $\mathrm{div}(A(x)\nabla)\boldsymbol{u}$ instead of $\Delta\boldsymbol{u}$ in the first equation of the system (S) and continue all the corresponding estimates, we may obtain the above $\boldsymbol{W}^{2,p}$-estimate for all $p \in (1,\infty)$ as done in Theorem 4.3.1.

Proof. The proof follows combining the $\boldsymbol{H}^2$-estimate and the $\boldsymbol{W}^{1,p}$-estimate proved in Theorem 2.4.5 and Theorem 2.6.11 respectively. ∎

2.7 Limiting cases

Our objective in this section is to study the limiting behaviour of the solution of (S) when the friction coefficient α goes to 0 and ∞.

2.7.1 α tends to 0

Theorem 2.7.1. *Let $p \in (1,\infty)$, Ω be not axisymmetric and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the weak solution of* (S) *where*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \alpha \in L^{t(p)}(\Gamma).$$

Then as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$, we have the convergence,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_0, \pi_0) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$$

where $(\boldsymbol{u}_0, \pi_0)$ is a solution of the following Stokes problem with Navier boundary condition corresponding to $\alpha = 0$,

$$\begin{cases} -\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f} + \mathrm{div}\,\mathbb{F}, \quad \mathrm{div}\,\boldsymbol{u}_0 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{2.111}$$

Proof. Let $\alpha \to 0$ in $L^{t(p)}(\Gamma)$. That means there does not exist any $\alpha_* > 0$ such that $\alpha \geq \alpha_*$ on Γ. Now from the estimates (2.31) and estimates (2.64), it is clear that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ for all $p \in (1,\infty)$. Then there exists $(\boldsymbol{u}_0, \pi_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0) \quad \text{weakly in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

It can be easily shown that $(\boldsymbol{u}_0, \pi_0)$ is the unique solution of the Stokes problem with Navier boundary condition, corresponding to $\alpha = 0$,

$$\begin{cases} -\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f} + \mathrm{div}\,\mathbb{F}, \quad \mathrm{div}\,\boldsymbol{u}_0 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{2.112}$$

Indeed, since $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is the solution of (S), it satisfies the weak formulation (2.17). Now as in Lemma 2.3.8, $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ implies

$$\boldsymbol{u}_{\alpha\boldsymbol{\tau}} \rightharpoonup \boldsymbol{u}_{0\boldsymbol{\tau}} \quad \text{in } \ \boldsymbol{L}^s(\Gamma)$$

where s satisfies (2.16) and because $\alpha \to 0$ in $L^{t(p)}(\Gamma)$,

$$\alpha \boldsymbol{u}_{\alpha\boldsymbol{\tau}} \rightharpoonup \boldsymbol{0} \quad \text{in } \ \boldsymbol{L}^{m'}(\Gamma)$$

with m satisfies (2.15). Hence in the weak formulation (2.17), the boundary term in the left hand side goes to 0. So finally, passing the limit, we deduce,

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega), \quad 2\int_\Omega \mathbb{D}\boldsymbol{u}_0 : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,.$$

Now by taking difference between the system (S) and the limiting system (2.112), we get,

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) + \nabla(\pi_\alpha - \pi_0) = \boldsymbol{0}, \quad \operatorname{div}\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) = 0 & \text{in } \ \Omega\,, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\cdot\boldsymbol{n} = 0, \quad 2\left[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)_{\boldsymbol{\tau}} = -\alpha\boldsymbol{u}_{0\boldsymbol{\tau}} & \text{on } \ \Gamma\,. \end{cases}$$

Once again using the estimates of Theorem 2.4.3 and Proposition 2.6.1 for the above system and also using Hölder inequality and trace theorem, we obtain

$$\begin{aligned} & \|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} \\ & \leq C(\Omega)\ \|\alpha\boldsymbol{u}_{0\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \\ & \leq C(\Omega)\ \|\alpha\|_{L^{t(p)}(\Gamma)}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}\ . \end{aligned}$$

Therefore, $\boldsymbol{u}_\alpha - \boldsymbol{u}_0$ and $\pi_\alpha - \pi_0$ both tend to zero in the same rate as α. ■

Remark 2.7.2. We can prove also the above theorem for Ω axisymmetric and α constant, provided the compatibility condition (2.33), with the help of the estimates (2.35) and Remark 2.6.13. Indeed, to expect the limiting system to be (2.111), we must assume the compatibility condition since this is the necessary condition for the existence of a solution of the system (2.111).

2.7.2 α tends to ∞

Next we study the behaviour of $\boldsymbol{u}_\alpha$ where α is a constant and grows to ∞.

Theorem 2.7.3. *Let $p \in (1,\infty)$ and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the weak solution of* (S) *where*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \ \textit{and } \alpha \textit{ constant.}$$

i) *Then as $\alpha \to \infty$, we have the convergence,*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ is the unique solution of the Stokes problem with Dirichlet boundary condition,

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \textit{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \textit{in } \Omega, \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \textit{on } \Gamma. \end{cases} \tag{2.113}$$

ii) *Moreover, for any $q \in (2, p)$, we obtain the strong convergence*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad \boldsymbol{W}^{1,q}(\Omega) \times L^q_0(\Omega).$$

Proof. As $\alpha \to \infty$, we can consider $\alpha \geq 1$.

i) From estimates (2.92) or (2.93), we see that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$, hence there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

But we can also write the system (S) as follows :

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\alpha = 0 & \text{in } \Omega \\ \boldsymbol{u}_\alpha = \frac{1}{\alpha} \left(\boldsymbol{h} - [(2\mathbb{D}\boldsymbol{u}_\alpha + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} \right) & \text{on } \Gamma. \end{cases} \tag{2.114}$$

Passing to the limit in the above system as $\alpha \to \infty$, we obtain that $(\boldsymbol{u}_\infty, \pi_\infty)$ is the solution of the Stokes problem with Dirichlet boundary condition (2.113).

Indeed, passing to the limit in the first two equations of (2.114) is easy. And for the boundary condition, since $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{E}^p(\Omega)$, by the Green formula (2.12), $2[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$ is bounded in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Hence taking limit as $\alpha \to \infty$ in the boundary condition of (2.114), we obtain the boundary condition of (2.113).

ii) Now taking the difference between the systems (S) and (2.113), we get

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + \nabla(\pi_\alpha - \pi_\infty) = \boldsymbol{0}, \quad \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) = 0 & \text{in } \Omega \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \cdot \boldsymbol{n} = 0, \quad 2\left[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}} = \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \end{cases}$$

Multiplying the above system by $\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty$ and integrating by parts yields,

$$2\int_\Omega |\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)|^2 + \alpha \int_\Gamma |(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}}|^2 = \langle \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

As $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_\infty$ in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ weakly and $\boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, this shows the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$. And the strong convergence for the pressure term follows from the following estimate

$$\|\pi_\alpha - \pi_\infty\|_{L^2(\Omega)} \leq \|\nabla(\pi_\alpha - \pi_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C\|\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

Next, for $p > 2$, we write for any $q \in (2, p)$,

$$\|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\mathbb{L}^q(\Omega)} \leq \|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|^{\gamma}_{\mathbb{L}^2(\Omega)}\|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|^{1-\gamma}_{\mathbb{L}^p(\Omega)}$$

where $\frac{1}{q} = \frac{\gamma}{2} + \frac{1-\gamma}{p}, \gamma \in (0,1)$. Now using the fact that $\nabla\boldsymbol{u}_\alpha \to \nabla\boldsymbol{u}_\infty$ in $\mathbb{L}^2(\Omega)$ and $\|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\mathbb{L}^p(\Omega)}$ is bounded, we obtain that

$$\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \to 0 \quad \text{in} \quad \mathbb{L}^q(\Omega).$$

This completes the proof. ■

2.7.3 α less regular

Theorem 2.7.4. *Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ and } \alpha \in L^{\frac{4}{3}}(\Gamma).$$

Then the Stokes problem (S) *has a weak solution* $(\boldsymbol{u}, \pi)$ *in* $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$.

Proof. Without loss of generality, we assume $\boldsymbol{h} = \boldsymbol{0}$.

i) First let us consider Ω is not axisymmetric. Let $\alpha_k \in \mathcal{D}(\Gamma)$ such that $\alpha_k \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$. Now if $(\boldsymbol{u}_k, \pi_k) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ is the solution of the problem (S) corresponding to α_k, from the estimates (2.31) satisfied by $(\boldsymbol{u}_k, \pi_k)$, we can see, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ such that

$$(\boldsymbol{u}_k, \pi_k) \rightharpoonup (\boldsymbol{u}, \pi) \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \times L^2(\Omega).$$

Also it is easy to show that

$$-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma.$$

Now from the Green formula (2.12), we obtain

$$2\left[(\mathbb{D}\boldsymbol{u}_k)\boldsymbol{n}\right]_{\boldsymbol{\tau}} \rightharpoonup 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} \quad \text{in} \quad \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Moreover, as $\alpha_k \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$ and $\boldsymbol{u}_{k\boldsymbol{\tau}} \rightharpoonup \boldsymbol{u}_{\boldsymbol{\tau}}$ in $L^4(\Gamma)$, it gives

$$\alpha_k \boldsymbol{u}_{k\boldsymbol{\tau}} \rightharpoonup \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \quad \text{in} \quad \boldsymbol{L}^1(\Gamma).$$

Therefore since $\boldsymbol{u}_k$ satisfies the boundary condition

$$[(2\mathbb{D}\boldsymbol{u}_k + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha_k\, \boldsymbol{u}_{k\boldsymbol{\tau}} = \boldsymbol{0} \ \text{ on } \Gamma,$$

passing the limit, it yields

$$[(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \ \text{ on } \Gamma.$$

Hence, $(\boldsymbol{u}, \pi)$ becomes the solution of the Stokes problem (S).

ii) Note that, when Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, we can find a sequence $\alpha_k \in \mathcal{D}(\Gamma)$ such that $\alpha_k \geq \alpha_*$ and $\alpha_k \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$. So we can make use of the estimate (2.32) and obtain the same result. ■

Remark 2.7.5. For $\alpha \in L^{\frac{4}{3}}(\Gamma)$ and $\boldsymbol{h} = \boldsymbol{0}$, the solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfies the extra property: $[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{L}^1(\Gamma)$.

2.8 Navier-Stokes equations

Finally we consider the non-linear problem and study the existence of generalized and strong solutions for the Navier-Stokes system (NS).

Definition 2.8.1. Given $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$, a couple $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is called a weak solution of the Navier-Stokes system (NS) if it satisfies: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma \tag{2.115}$$

where $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \boldsymbol{w}$.

2.8.1 Existence and regularity

Theorem 2.8.2. *Let* $p \in (1, \infty)$ *and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \ \textit{and} \ \alpha \in L^{t(p)}(\Gamma)$$

where $r(p)$ *and* $t(p)$ *are defined by* (2.10) *and* (2.8) *respectively. Then the following two statements are equivalent:*

(i) $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ *is a weak solution of (NS), in the sense of Definition 2.8.1 and,*
(ii) $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ *satisfies:*

$$\begin{aligned} -\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} + \operatorname{div}\,\mathbb{F},\ \operatorname{div}\,\boldsymbol{u} = 0 &\quad \textit{in the sense of distribution} \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 &\quad \textit{in the sense of trace} \\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} &\quad \textit{in } \boldsymbol{W}^{-1/p,p}(\Gamma). \end{aligned} \tag{2.116}$$

The proof is standard and very similar to that of Proposition 2.3.10, hence we omit it. To facilitate the work, we give some properties of the operator b but skip the proof (cf. [11, Lemma 7.2]).

Lemma 2.8.3. *The trilinear form b is defined and continuous on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega) \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Also, we have*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{0} \tag{2.117}$$

and

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega).$$

Moreover

$$b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\beta}) = 0 \quad \textit{and} \quad b(\boldsymbol{\beta}, \boldsymbol{\beta}, \boldsymbol{u}) = 0.$$

Now we can prove the existence of the generalized solution of the Navier-Stokes problem (NS). First we study the Hilbert case.

Theorem 2.8.4. *Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \textit{ and } \alpha \in L^2(\Gamma).$$

Then the problem (NS) *has a weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfying the following estimates:*
a) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{2.118}$$

b) *if Ω is axisymmetric and*

(i) $\alpha \ge \alpha_* > 0$ *on* Γ, *then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le \frac{C(\Omega)}{\min\{2, \alpha_*\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{2.119}$$

(ii) $\boldsymbol{f}, \boldsymbol{h}$ *satisfy the condition:*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

then the solution $\boldsymbol{u}$ *satisfies* $\int_\Gamma \alpha \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u_\tau}|^2 + \|\pi\|^2_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2 . \quad (2.120)$$

In particular, if α *is a constant, then* $\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) . \quad (2.121)$$

Remark 2.8.5. Note that in the case of $\boldsymbol{u} \cdot \boldsymbol{n} \neq 0$ on Γ when Ω has multiply connected boundary, the existence of solution of Navier-Stokes equation with Dirichlet boundary condition is not yet clear in full generality, e.g. see [84]. So we do not consider that case here for the Navier boundary condition also.

Proof. **1. Existence:** The existence of solution of (2.115) can be shown using standard arguments i.e. by Galerkin method, we construct an approximate solution and then pass to the limit. Nonetheless, we state it briefly for completeness.

For each fixed integer $m \geq 1$, define an approximate solution $\boldsymbol{u}_m$ of (2.115) by

$$\boldsymbol{u}_m = \sum_{i=1}^m \xi_{i,m} \boldsymbol{v}_i, \quad \xi_{i,m} \in \mathbb{R}$$

$$2 \int_\Omega \mathbb{D}\boldsymbol{u}_m : \mathbb{D}\boldsymbol{v}_k + b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v}_k) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau} m} \cdot \boldsymbol{v}_{\boldsymbol{\tau} k} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_k - \int_\Omega \mathbb{F} : \nabla \boldsymbol{v}_k + \langle \boldsymbol{h}, \boldsymbol{v}_k \rangle_\Gamma , \quad \text{for } k = 1, ..., m \quad (2.122)$$

and $V_m := \langle \boldsymbol{v}_1, ..., \boldsymbol{v}_m \rangle$ is the space spanned by the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_m$ and $\{\boldsymbol{v}_i\}_i$ is a Hilbert basis of $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Note that V_m is equipped with the scalar product $(,)$ induced by $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Let us also define the mapping $P_m : V_m \to V_m$ as for all $\boldsymbol{w}, \boldsymbol{v} \in V_m$,

$$(P_m(\boldsymbol{w}), \boldsymbol{v}) = 2 \int_\Omega \mathbb{D}\boldsymbol{w} : \mathbb{D}\boldsymbol{v} + b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) + \int_\Gamma \alpha \boldsymbol{w_\tau} \cdot \boldsymbol{v_\tau} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_\Omega \mathbb{F} : \nabla \boldsymbol{v} - \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma .$$

The continuity of the mapping is obvious. Also, using (2.117), we get

$$\begin{aligned}(P_m(\boldsymbol{w}), \boldsymbol{w}) &= 2\|\mathbb{D}\boldsymbol{w}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{w_\tau}|^2 - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \int_\Omega \mathbb{F} : \nabla \boldsymbol{w} - \langle \boldsymbol{h}, \boldsymbol{w} \rangle_\Gamma \\ &\geq C(\alpha) \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \left\{ \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} - C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \right\} .\end{aligned}$$

Hence, $(P_m(\boldsymbol{w}), \boldsymbol{w}) > 0$ for all $\|\boldsymbol{w}\|_{V_m} = k$ where $k > C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$. Therefore, the hypothesis of Brower's theorem is satisfied and there exists a solution $\boldsymbol{u}_m$ of (2.122).

Next as $\boldsymbol{u}_m$ is a solution of (2.122), we have

$$2\|\mathbb{D}\boldsymbol{u}_m\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau} m}|^2 = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_m - \int_\Omega \mathbb{F} : \nabla \boldsymbol{u}_m + \langle \boldsymbol{h}, \boldsymbol{u}_m \rangle_\Gamma$$

which yields the a priori estimate

$$\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)} \leq C(\alpha) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right).$$

Since the sequence $\boldsymbol{u}_m$ remains bounded in $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$, there exists some $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ and a subsequence which we still call $\boldsymbol{u}_m$ such that

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{V}^2_{\sigma,\tau}(\Omega).$$

Now due to the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, we can pass to the limit in (2.122) and obtain, for any $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{v} + \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma$$

and thus $\boldsymbol{u}$ is a solution of (2.115).

2. Estimates: The estimates can be shown as for the Stokes problem in Theorem 2.4.3. ■

Proposition 2.8.6. *The weak solution of the problem (NS), given by Theorem 2.8.4 is unique provided*

$$\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} < \frac{1}{C(\alpha, \Omega)} \tag{2.123}$$

where the constant $C(\alpha, \Omega)$ depends on the continuity constant of the linear form b and the equivalence constant of $\boldsymbol{H}^1(\Omega)$ norm, precised in the proof.

Remark 2.8.7. Interestingly, in the case of $\alpha \equiv 0$, there is no uniqueness of the solution of the system (NS) even for small data. But in our case, when $\alpha \neq 0$ on some $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| > 0$, there is indeed uniqueness of the solution under the assumption of small data as in the case of Dirichlet boundary condition. The reason of this behaviour is the presence of a non-trivial kernel of the Stokes operator for $\alpha \equiv 0$.

Proof. Taking $\boldsymbol{\varphi} = \boldsymbol{u}$ in (2.115) and using (2.117), we obtain that any solution of (2.115) satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\alpha, \Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \tag{2.124}$$

where we used the following norm equivalence:

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq C(\alpha, \Omega) \left(2\|\mathbb{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Omega \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \right).$$

Note that if Ω is non-axisymmetric, $C(\alpha, \Omega)$ only depends on Ω and if Ω is axisymmetric, $C(\alpha, \Omega) = C(\Omega) \min\{2, \alpha\}$. Now if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two different solutions of (2.115), let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and subtracting the equations (2.115) corresponding to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we get

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega), \quad 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}_1, \boldsymbol{u}, \boldsymbol{\varphi}) + b(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{\varphi}) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = 0. \tag{2.125}$$

Taking $\boldsymbol{\varphi} = \boldsymbol{u}$ in (2.125) and once again using (2.117) implies

$$2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = -b(\boldsymbol{u}, \boldsymbol{u}_2, \boldsymbol{u})$$

which yields, using the continuity of b and the estimate (2.124) for $\boldsymbol{u}_2$,

$$\begin{aligned} \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C(\Omega) C(\alpha, \Omega) \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq C(\Omega) (C(\alpha, \Omega))^2 \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \end{aligned}$$

Thus considering the condition (2.123), the above inequality implies $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0$ that is $\boldsymbol{u}_1 = \boldsymbol{u}_2$. ■

Next we prove the existence of solution of the system (NS) in $\boldsymbol{W}^{1,p}(\Omega)$ using the Hilbert case and the Stokes regularity result.

Corollary 2.8.8. *Let Ω be $\mathcal{C}^{2,1}$, $p > \frac{3}{2}$ and $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ and α satisfy the assumptions as in Theorem 2.8.2.*

i) *There exists a weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (NS).*

ii) *Moreover, for any $p \in (1, \infty)$, if $\mathbb{F} = 0$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \leq \frac{3}{2}$ and $q = p$ otherwise, then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Proof. **i)** First let us consider $p > 2$. Then we have existence of a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ from Theorem 2.8.4 using the Hilbert case. Since $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, the non-linear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{r(p)}(\Omega)$ if $p \le 3$. Hence, using the regularity result for Stokes problem in Corollary 2.5.6, we obtain that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. For $p > 3$, repeating the same argument with $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$, we deduce the required regularity.

And to obtain existence result for $p \in (\frac{3}{2}, 2)$, we follow the exact same construction as in [104, Theorem 1.1]. Note that we replaced the space $\boldsymbol{W}^{-1,p}(\Omega)$ for the given data in [104] by $\boldsymbol{L}^{r(p)}(\Omega)$. For example, we used the following lemma instead of [104, Lemma 1.2]:

If there exists $(\boldsymbol{v}, \tilde{\pi}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \tilde{\pi} - \boldsymbol{f} & \in \boldsymbol{L}^{r(p)}(\Omega) \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \\ 2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

for $p \le q \le 2$, then there exists $(\boldsymbol{w}, \bar{\pi}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{w} + \nabla \bar{\pi} - \boldsymbol{f} & \in \boldsymbol{L}^{r(s)}(\Omega) \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \\ 2\left[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{w}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

where $\frac{1}{s} = \frac{1}{q} + \frac{1}{p} - \frac{2}{3}$ (thus $s > q$).
The rest of the proof follows the same argument as in [104] without any further changes.

ii) Next to prove the strong regularity result, consider the more regular data. For $p \in (1, \frac{3}{2}]$, since the Sobolev exponent $p^* \in (\frac{3}{2}, 3]$ and thus $r(p^*) = p$, we have $\boldsymbol{f} \in \boldsymbol{L}^{r(p^*)}(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p^*},p^*}(\Gamma)$ and hence using the above regularity result for weak solution of (NS), we obtain $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p^*}(\Omega) \times L^{p^*}(\Omega)$. Now for $p \in (1, \frac{3}{2})$, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^s(\Omega)$ with

$$\frac{1}{s} = \frac{2}{p} - 1$$

which implies $s > p$ and thus using Theorem 2.5.9 again, we obtain $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. For $p = \frac{3}{2}$, since $\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^m(\Omega)$ for any $m \in (1, \infty)$, we have $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^s(\Omega)$ with $\frac{1}{s} = \frac{1}{3} + \frac{1}{m}$, so choosing $m > 3$ gives $s > \frac{3}{2}$ and thus $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$.

Now for $p > \frac{3}{2}$, having $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ gives $\sum_i u_i \partial_i u \in \boldsymbol{L}^{3-\epsilon}(\Omega)$ which yields $\boldsymbol{u} \in \boldsymbol{W}^{2,3-\epsilon}(\Omega)$. Further repeating the argument, we get $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. ■

Finally we discuss the limiting behaviour of the Navier-Stokes system (NS) as α goes to 0 or ∞.

2.8.2 Limiting cases

Theorem 2.8.9. *Let $p \geq 2$, Ω be not axisymmetric and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be a weak solution of* (NS) *where*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

Then as $\|\alpha\|_{L^{t(p)}(\Gamma)} \to 0$, we have the convergence,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_0, \pi_0) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$$

where $(\boldsymbol{u}_0, \pi_0)$ is a solution of the following Navier-Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_0 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{2.126}$$

Proof. **i)** We assume for ease of calculation, $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. As $\alpha \to 0$ in $L^{t(p)}(\Gamma)$, there does not exist any $\alpha_* > 0$ such that $\alpha \geq \alpha_*$ on $\Gamma_0 \subseteq \Gamma$. Therefore, $(\boldsymbol{u}_\alpha, \pi_\alpha)$ satisfies the estimates (2.118). For $2 < p \leq 3$, using the Stokes estimate (2.64), we obtain

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq C(\Omega) \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \end{aligned}$$

and for $p > 3$,

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{W}^{1,3}(\Omega)} \right) \\ &\leq C(\Omega) \left[1 + \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right)^2 \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right] \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)}. \end{aligned}$$

Then $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ with respect to α. So there exists $(\boldsymbol{u}_0, \pi_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0) \text{ weakly in } \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

Now, as in Theorem 2.7.1, passing the limit as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$ in the variational formulation satisfied by $(\boldsymbol{u}_\alpha, \pi_\alpha)$, we get that $\boldsymbol{u}_0$ satisfies the equation

$$2 \int_\Omega \mathbb{D}\boldsymbol{u}_0 : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{\varphi}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \qquad \forall\, \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega).$$

Indeed, $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0$ weakly in $\boldsymbol{W}^{1,p}(\Omega)$ implies $\boldsymbol{u}_\alpha \to \boldsymbol{u}_0$ in $\boldsymbol{L}^s(\Omega)$ where

$$s \in \begin{cases} (1, p^*) & \text{if} \quad p < 3 \\ (1, \infty) & \text{if} \quad p = 3 \\ (1, \infty] & \text{if} \quad p > 3. \end{cases} \tag{2.127}$$

Also, $\nabla \boldsymbol{u}_\alpha \rightharpoonup \nabla \boldsymbol{u}_0$ weakly in $\boldsymbol{L}^p(\Omega)$. Therefore, $\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0$ weakly in $\boldsymbol{L}^q(\Omega)$ where

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{s}$$

and note that, $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{q'}(\Omega)$. Hence, $b(\boldsymbol{u}_\alpha, \boldsymbol{u}_\alpha, \boldsymbol{\varphi}) \to b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{\varphi})$ as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$. Therefore, $(\boldsymbol{u}_0, \pi_0)$ is a solution of the problem (2.126).

ii) Next we show that the convergence that $(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0)$ weakly in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ occurs in fact in strong sense. Taking the difference between the systems (NS) and (2.126), we get,

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) + \nabla(\pi_\alpha - \pi_0) = \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 - \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha & \text{in } \Omega \\ \operatorname{div}\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) = 0 & \text{in } \Omega \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \cdot \boldsymbol{n} = 0, \quad 2\,[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\alpha\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

Note that $\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 - \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha = \operatorname{div}(\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha - \boldsymbol{u}_0 \otimes \boldsymbol{u}_0)$. Thus using the Stokes estimate (2.64) for the above system gives

$$\begin{aligned} &\|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} \\ &\leq C\left(\|\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha - \boldsymbol{u}_0 \otimes \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} + \|\alpha \boldsymbol{u}_{0\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &= C\left(\|(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \otimes \boldsymbol{u}_\alpha + \boldsymbol{u}_0 \otimes (\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\|_{\boldsymbol{L}^p(\Omega)} + \|\alpha \boldsymbol{u}_{0\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\leq C\left[\|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{L}^s(\Omega)}\left(\|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}\right) + \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}\right] \end{aligned}$$

where s is defined as in (2.127). Now since $\boldsymbol{u}_\alpha$ is bounded in $\boldsymbol{W}^{1,p}(\Omega)$ and by compactness, $\boldsymbol{u}_\alpha \to \boldsymbol{u}_0$ in $\boldsymbol{L}^s(\Omega)$, we obtain the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ as $\alpha \to 0$. ■

Remark 2.8.10. As in the Stokes case, we can prove also the above theorem for Ω axisymmetric and α constant, provided the compatibility condition (2.33), with the help of the estimates (2.120) and Remark 2.6.13. Indeed, to expect the limiting system to be (2.126), we must assume the above compatibility condition since this is the necessary condition for the existence of a solution of the system (2.126).

Theorem 2.8.11. *Let $p \geq 2$ and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the weak solution of* (NS) *with $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and α a constant. Then as $\alpha \to \infty$, we have the convergence,*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad \boldsymbol{W}^{1,q}(\Omega) \times L^q_0(\Omega) \quad \textit{for any} \quad q \in (2, p)$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ is a solution of the Navier-Stokes problem with Dirichlet boundary condition,

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \textit{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \textit{in } \Omega \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \textit{on } \Gamma. \end{cases} \tag{2.128}$$

Proof. **i)** Without loss of generality, assume $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. As $\alpha \to \infty$, we can consider $\alpha \geq 1$ and then we have estimates (2.118) and (2.119). Also as done in Theorem 2.8.9, using the Stokes estimate (2.92) and (2.93), we can write, for $2 < p \leq 3$,

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq C_p(\Omega) \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \end{aligned}$$

and then similar for $p > 3$ as well. This shows that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for all $p \geq 2$. Hence there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

Now rewriting the system (NS) as,

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\alpha = 0 & \text{in } \Omega \\ \boldsymbol{u}_\alpha = -\frac{2}{\alpha} [(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \end{cases} \tag{2.129}$$

and as in Theorem 2.7.3 [note that here $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in \boldsymbol{L}^{r(p)}(\Omega)$], letting $\alpha \to \infty$ in the above system, we obtain that $(\boldsymbol{u}_\infty, \pi_\infty)$ satisfies the Navier-Stokes problem (2.128).

ii) Therefore, $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ satisfies the system

$$\begin{cases} -\Delta (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + \nabla (\pi_\alpha - \pi_\infty) = \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty - \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha & \text{in } \Omega, \\ \operatorname{div} (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \cdot \boldsymbol{n} = 0, \quad 2 [\mathbb{D} (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}} = -2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma. \end{cases}$$

Multiplying by $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ and integrating by parts, we get,

$$\begin{aligned} &2\int_\Omega |\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)|^2 + \alpha \int_\Gamma |(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}}|^2 \\ =& b\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \boldsymbol{u}_\infty, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\right) - \langle 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

But as $\alpha \to \infty$, by compactness $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{L}^4(\Omega)$ and thus

$$b\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \boldsymbol{u}_\infty, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\right) \leq \|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|^2_{\boldsymbol{L}^4(\Omega)} \|\nabla \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0.$$

Also since $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_\infty$ weakly in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, it implies,

$$\langle 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \to 0.$$

Therefore along with the fact that $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{L}^2(\Omega)$, we obtain the strong convergence $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$. For the pressure term, we can write

$$\begin{aligned} \|\pi_\alpha - \pi_\infty\|_{L^2(\Omega)} &\leq C\|\nabla(\pi_\alpha - \pi_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)} \\ &\leq C\| - \Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + (\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)} \\ &\leq C\|(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty \otimes \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

and thus $\pi_\alpha \to \pi_\infty$ in $L^2(\Omega)$.

Now similar to the Stokes case, for $p > 2$, we write for any $q \in (2, p)$,

$$\|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\mathbb{L}^q(\Omega)} \leq \|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|^\gamma_{\mathbb{L}^2(\Omega)} \|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|^{1-\gamma}_{\mathbb{L}^p(\Omega)}$$

where $\frac{1}{q} = \frac{\gamma}{2} + \frac{1-\gamma}{p}, \gamma \in (0,1)$. Now using the fact that $\nabla \boldsymbol{u}_\alpha \to \nabla \boldsymbol{u}_\infty$ in $\mathbb{L}^2(\Omega)$ and $\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ is bounded in $\mathbb{L}^p(\Omega)$, we obtain

$$\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \to 0 \quad \text{in} \quad \mathbb{L}^q(\Omega).$$

This completes the proof. ■

CHAPTER 3

Semigroup theory for the Stokes operator with Navier boundary condition on L^p spaces

This is a joint work with Chérif Amrouche and Miguel Escobedo.

Abstract : We consider the Navier-Stokes equation in a bounded domain with $\mathcal{C}^{1,1}$ boundary, completed with slip boundary condition. Apart from studying the general semi-group theory related to the Stokes operator with Navier boundary condition where the slip coefficient α is a non-smooth scalar function, our main goal is to obtain estimate on the solutions, independent of α. We show that for α large, the weak and strong solutions of both the linear and non-linear system are bounded uniformly with respect to α. This justifies mathematically that the solution of the Navier-Stokes problem with slip condition converges in the energy space to the solution of the Navier-Stokes with no-slip boundary condition as $\alpha \to \infty$.

3.1 Introduction

In this article we prove the existence of solutions to the following problem for the Navier Stokes equation:

$$
\begin{cases}
\dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{0} & \text{in } \Omega\times(0,T) \quad (3.1)\\
\operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega\times(0,T) \quad (3.2)\\
\boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma\times(0,T) \quad (3.3)\\
2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma\times(0,T) \quad (3.4)\\
\boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega \quad (3.5)
\end{cases}
$$

where Ω is a bounded domain of $\mathbb{R}^3$, possibly multiply connected, whose boundary Γ is of class $C^{1,1}$. The initial velocity $\boldsymbol{u}_0$ and the (scalar) friction coefficient α are given functions. The external unit normal vector on Γ is denoted by $\boldsymbol{n}$, $\mathbb{D}\boldsymbol{u} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$ denotes the strain tensor and the subscript $\boldsymbol{\tau}$ denotes the tangential component i.e. $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ for any vector field $\boldsymbol{v}$. The functions $\boldsymbol{u}$ and π describe respectively the velocity and the pressure of a viscous incompressible fluid.

The boundary condition (3.4) was introduced by H. Navier in [96], taking into account the molecular interactions with the boundary, and is called Navier boundary condition. It may be deduced from kinetic theory considerations, as first described in [92] and rigorously proved under suitable conditions in [91]. It has been widely studied in recent years, because of its significance in modeling and simulations of flows and fluid-solid interaction problems (cf. [70], [52], [98] and references therein). In that context the function α is, up to some constant, the inverse of the slip length. We then impose the condition $\alpha \geq 0$ in all the remaining of this work.

The problem with Dirichlet boundary condition has deserved a lot of attention. In particular, a good semigroup theory has been developed in a series of work by Giga (cf. [56], [58], [59], [61]). Here we wish to establish a similar framework for $\alpha \not\equiv 0$. We will study two different types of solutions for (3.6), (3.2)-(3.5): strong solutions which belong to $L^p(0,T,\boldsymbol{L}^q(\Omega))$ type spaces and weak solutions (in a suitable sense) that may be written for a.e. $t>0$, as $\boldsymbol{u}(t) = \boldsymbol{v}(t) + \nabla w(t)$ where $\boldsymbol{v}\in L^p(0,T;\boldsymbol{L}^q(\Omega))$ and $w\in L^p(0,T;L^q(\Omega))$. We will also consider different hypothesis on regularity of the function α. In particular, we collect some of the relevant results available for the Navier-Stokes problem with no-slip condition based on semigroup properties and prove them for the system (3.1)-(3.5) for the sake of completeness, so that this paper can be used as a basis for further work.

Let us briefly mention here some related works. The system (3.1)-(3.5) has been studied in [34] in 2D where $\alpha\geq 0$ is a function in $C^2(\Gamma)$, with the main objective to analyse vanishing viscosity limit where the existence of weak and strong solutions have been established. Also in [32], the authors have studied stochastic Navier-Stokes equation with Navier boundary condition, similar to (3.1)-(3.5) where they considered same assumption that $\alpha \geq 0$ is in $C^2(\Gamma)$ and proved existence of weak solution. Beirão da Veiga [18] has considered the same problem in 3D in $\mathcal{C}^{2,1}$ domain with $\alpha \geq 0$ constant and first show that the Stokes operator with Navier boundary condition A is a maximal monotone, self-adjoint operator on $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$ which generates an analytic semigroup of contraction and thus obtain strong solution of Stokes problem; Also by identifying the domain of $A^{1/2}$, he obtain the global strong solution of Navier-Stokes equation under the assumption of small data as in the no-slip boundary condition. The system (3.1)-(3.5) has also been studied by Tanaka et al. [71] in Sobolev-Slobodetskii spaces in point of view to analyze asymptotic behavior of the unsteady solution to the steady solution where they have considered Γ of class $W_2^{\frac{5}{2}+l}$ and $\alpha \in (0,1)$ belongs to $W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}((0,\infty)\times\Gamma)$ with $l \in (\frac{1}{2},1)$ and proved existence of local in time, strong solution and global in time, strong solution for small data. Note that in this work, α depends on both time and space variable. We want to mention further the works of [70, 90] where though the main goal is again to study viscosity limit, in the first paper, Iftimie and Sueur show existence of global in time weak solution in $C([0,\infty);\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap L^2_{loc}([0,\infty);\boldsymbol{H}^1(\Omega))$ for $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$ initial value; by classical approach: first deriving some energy estimate and then using Galerkin method. There they considered α a scalar function of class $\mathcal{C}^2$, positive or negative. And in the second paper, Masmoudi and Rousset worked on anistropic conormal Sobolev spaces considering smooth domain and $|\alpha| \leq 1$. In the paper [95], the authors aimed to prove the existence of global in time, strong solution of a similar problem assuming small data, in $\mathcal{C}^{1,1}$ domain and α non-negative, Hölder continuous in time. Also we mention the work [116] where Lagrangian Navier-Stokes problem (as a regularization system of classical Navier-Stokes equations) with vorticity slip boundary condition (which is close to the boundary condition (3.4)) has been studied for non-negative smooth function α and existence of weak solution, global in time is obtained.

Further, in [21], Beneš has established a unique weak solution, local in time for the Navier-Stokes system with mixed boundary condition: on some part of the boundary Navier condition with $\alpha = 0$ is considered and on other part, Neumann type boundary condition. Some similar result for Navier-Stokes problem with Navier-type boundary condition (which corresponds to $\alpha = 0$) has been studied in, for example [85]. Also for $\alpha = 0$ the semigroup associated to equation (3.6) with (3.2)-(3.5) has been studied in [7]. We also mention the interesting work [41] where the low Mach number limit is studied for the Navier-Stokes system describing the motion of a compressible viscous fluid subjected to the slip boundary condition, with the friction coefficient inversely proportional to a certain power of the Mach number, in an unbounded (exterior) domain with compact boundary. We also mention the works by Shimada [110], [108], [109] which discuss the Stokes operator with Robin boundary condition, in particular resolvent estimates, $L^p - L^q$ regularity and asymptotic analysis.

In this work, we wish to study the general semigroup theory for any $p \in (1, \infty)$ for the Stokes operator with Navier boundary condition (NBC) with (possibly) minimal regularity on α which gives us existence, uniqueness and regularity of both strong and weak solutions of (3.6), (3.2)-(3.5). Though the method to deduce the resolvent estimate for general $p \neq 2$ is close to that in [7] but the main difficulty in our case is to obtain some estimate on the pressure term in suitable space, due to the presence of the friction α which is not smooth. As explained in [18], for the case $p = 2$, the operator behaves well and the semigroup framework is easy to establish which is not the situation for non-Hilbert cases. To achieve the uniform estimate on the solutions with respect to α which is one of the main goal of this work, we use the L^p-extrapolation theorem by Shen [107]. We want to mention the work in [45] where the Stokes resolvent system has been considered in a general unbounded domain with $\mathcal{C}^3$ boundary and Navier slip boundary condition (where $\alpha \geq 0$ a constant). The authors studied the resolvent estimate in function spaces of the type $L^q \cap L^2$ when $q \leq 2$ and $L^q + L^2$ when $1 < q < 2$ (though without any further information concerning the behavior with respect to the friction coefficient).

Next using the properties of the Stokes operator with NBC and analyzing the fractional and imaginary powers, we obtain the $L^p - L^q$ estimates and the maximal regularity for non-homogeneous linear problem as for the classical Dirichlet boundary condition. But again here the difficulty is to obtain bounds for the imaginary powers of the operator since our operator does not reduce to an elliptic operator as in [6]. And for the no-slip boundary condition, Giga [56] used the theory of pseudo-differential operators for bounded domain, while in the case of full space, Giga and Sohr [60] used Fourier multiplier method which is not adaptable in our case. So here we use a perturbation argument adapting the theory of interpolation-extrapolation, established in [8]. In [101], the authors use the same method to establish that the Stokes operator with NBC for $\alpha > 0$ constant, possesses a bounded $\mathcal{H}^\infty$-calculus on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ which implies that the operator has bounded imaginary powers. Also in our work, we consider optimal regularity of the domain unlike the other works.

It seems very natural to let $\alpha \to \infty$ in some sense in order to obtain the Dirichlet boundary condition on Γ from the condition (3.4). One of the goals of the present article is to study the behavior of the solutions of the unsteady Stokes and Navier-Stokes equation with NBC with respect to α, in particular what happens when α goes to ∞. This problem is considered by Kelliher [80] in 2D where the author shows in Theorem 9.2 that, for $\boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega)$, when $\|1/\alpha\|_{L^\infty(\Gamma)} \to 0$, the solution of problem (3.1)-(3.5) converges to the solution of the Navier-Stokes problem with Dirichlet boundary condition in suitable spaces (cf. Section 3.9 for details). In our work, we first deduce uniform resolvent estimates with respect to some suitable norm of the function α which in turn provides α independent estimates for the solutions of non-stationary Stokes problem with NBC. This enables us to consider the case where α is a constant function and $\alpha \to \infty$ in both the linear problem (3.6), (3.2)-(3.5) and the nonlinear problem (3.1)-(3.5). We show that the solutions of the problems with NBC converge strongly in the energy space to the solutions of corresponding problem with Dirichlet boundary condition as α goes to ∞.

We state now our main results, for which the following notations are needed:

$$\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\ \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

equipped with the norm of $\boldsymbol{L}^p(\Omega)$ and

$$\mathbf{D}(A_{p,\alpha}) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega);\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\right\}.$$

The space $\mathbf{D}(A_{p,\alpha})$ is nothing but the domain of $A_{p,\alpha}$, the Stokes operator on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ with the boundary conditions (3.3)-(3.4).

Theorem 3.1.1. *Suppose that $\alpha \in W^{1-\frac{1}{r_1},r_1}(\Gamma)$ for some $r_1 \geq 3$ and $\alpha \geq 0$. Then, for every $\boldsymbol{u}_0 \in \boldsymbol{L}^{r_2}_{\sigma,\boldsymbol{\tau}}(\Omega)$ with $r_2 \geq 3$, there exists a unique solution $\boldsymbol{u}$ of (3.1)–(3.5) defined on a maximal time interval $[0, T_\star)$ such that*

$$\boldsymbol{u} \in C([0,T_\star); \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap L^q(0,T_\star; \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega))$$

$$t^{1/q}\boldsymbol{u} \in C([0,T_\star); \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \quad \textit{and} \quad t^{1/q}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \to 0 \textit{ as } t \to 0$$

with $r = \min(r_1, r_2)$, $p > r$, $q > r$ and $\frac{2}{q} = \frac{3}{r} - \frac{3}{p}$. Moreover,

$$\boldsymbol{u} \in C((0,T_\star), \mathbf{D}(A_{r,\alpha})) \cap C^1((0,T_\star);\ \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)).$$

If $r > 3$ and $T_\star < \infty$,

$$\|u(t)\|_{\boldsymbol{L}^r(\Omega)} \geq C(T_\star - t)^{(3-r)/2r}$$

where C is independent of $T_\star$ and t.

Also, there exists a constant $\varepsilon > 0$ such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^3(\Omega)} < \varepsilon$, then $T_\star = \infty$.

Under weaker conditions on Ω and α, a similar Theorem holds for initial data in the space of distributions $\boldsymbol{u}_0 = \boldsymbol{\psi} + \nabla\chi$ where $\boldsymbol{\psi} \in L^r(\Omega)$ and $\chi \in L^r(\Omega)$ (denoted by $[\mathbf{H}^{r'}_0(\mathrm{div},\Omega)]'$, cf. Proposition 3.2.1), with $r \geq 3$.

The proof of these results is based on a careful study of the semigroup associated to the linear equation

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \tag{3.6}$$

with conditions (3.2)-(3.5). For that we first study the strong and weak Stokes operators $A_{p,\alpha}$ and $B_{p,\alpha}$ and deduce that both of them have bounded inverse on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $[\mathbf{H}^{p'}_0(\mathrm{div},\Omega)]'$ respectively for all $p \in (1,\infty)$. Also $-A_{p,\alpha}$ and $-B_{p,\alpha}$ generate bounded analytic semigroups on their respective spaces (cf. Theorem 3.3.13 and Theorem 3.4.3) and their pure imaginary powers are uniformly bounded as well (cf. Theorem 3.5.1). We obtain the following theorems, if $\boldsymbol{f} = 0$:

Theorem 3.1.2. *Let $1 < p < \infty$ and $\alpha \geq 0$ be as in (3.18). Then for $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$, the problem (3.6), (3.2)-(3.5) with $\boldsymbol{f} = \mathbf{0}$ has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0,\infty), \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap C((0,\infty), \mathbf{D}(A_{p,\alpha})) \cap C^1((0,\infty), \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega))$$

and

$$\boldsymbol{u} \in C^k((0,\infty), \mathbf{D}(A^l_{p,\alpha})) \quad \forall\, k \in \mathbb{N},\ \forall\, l \in \mathbb{N}\backslash\{0\}.$$

Also, for all $t > 0$ and $q \geq p$, $\boldsymbol{u}(t) \in L^q(\Omega)$ and there exists $\delta > 0$ independent of t and q such that:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\ e^{-\delta t} t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, the following estimates also hold

$$\|\mathbb{D}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\ e^{-\delta t} t^{-3/2(1/p-1/q)-1/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},$$

$$\forall\, m,n \in \mathbb{N}, \quad \|\frac{\partial^m}{\partial t^m} A^n_{p,\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\ e^{-\delta t} t^{-(m+n)-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

For $\boldsymbol{f} \not\equiv 0$, and if we denote E_q the following real interpolation space:

$$E_q \equiv (D(A_{p,\alpha}), \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega))_{\frac{1}{q},q}$$

we have the result:

Theorem 3.1.3. *Let $1 < p,q < \infty$. Also assume that $0 < T \leq \infty$ and $\alpha \geq 0$ be as in (3.18). Then for $\boldsymbol{f} \in L^q(0,T;\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega))$ and $\boldsymbol{u}_0 \in E_q$, there exists a unique solution $(\boldsymbol{u},\pi)$ of (3.6) with (3.2)-(3.5) satisfying:*

$$\boldsymbol{u} \in L^q(0,T_0;\boldsymbol{W}^{2,p}(\Omega)) \ \ \text{for all}\ \ T_0 \leq T \ \ \text{if}\ \ T < \infty \ \ \text{and}\ \ T_0 < \infty \ \ \text{if}\ \ T = \infty,$$

$$\pi \in L^q(0,T;W^{1,p}(\Omega)/\mathbb{R}), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T,\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)),$$

$$\int_0^T \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}t + \int_0^T \|\boldsymbol{u}\|^q_{\boldsymbol{W}^{2,p}(\Omega)} \mathrm{d}t + \int_0^T \|\pi\|^q_{W^{1,p}(\Omega)} \mathrm{d}t \leq C \left(\int_0^T \|\boldsymbol{f}\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}t + \|\boldsymbol{u}_0\|^q_{E_q} \right) \quad (3.7)$$

where $C > 0$ is independent of $\boldsymbol{f}, \boldsymbol{u}_0$ and T.

Similar results hold for less regular data (cf. Theorem 3.6.4 and Theorem 3.7.1). And the last interesting result of our work is the following limit problem which improves the result in [80, Theorem 9.2]:

Theorem 3.1.4. *Let* Ω *be* $\mathcal{C}^{2,1}$ *and* α *be a constant. Also let* $(\boldsymbol{u}_\alpha, \pi_\alpha)$ *be a solution of the problem (3.1)-(3.5) where* $\boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega)$ *with* $\operatorname{div} \boldsymbol{u}_0 = 0$ *in* Ω *and* $(\boldsymbol{u}_\infty, \pi_\infty) \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \times L^2(0,T;L^2(\Omega))$ *a solution of the following Navier-Stokes problem with Dirichlet boundary condition*

$$\begin{cases} \dfrac{\partial \boldsymbol{u}_\infty}{\partial t} - \Delta \boldsymbol{u}_\infty + (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \boldsymbol{u}_\infty(0) = \boldsymbol{u}_0 & \text{in } \Omega \end{cases} \tag{3.8}$$

(whose existence has been proved in [113, Theorem 3.1, Chapter III]). Then for any $T < T_*$ *(where* T_* *is defined in Theorem 3.1.1),*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad L^2(0,T;\boldsymbol{H}^1(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \times L^2(0,T;L^2(\Omega)) \quad \text{as} \quad \alpha \to \infty$$

and

$$\int_0^T \int_\Gamma |\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty|^2 \leq \frac{C}{\alpha}. \tag{3.9}$$

Similar result holds for the linear problem as well as for initial data in L^p-spaces (cf. Section 3.9).

3.2 Preliminaries

First, we review some basic notations and functional framework, we will use in the study. Throughout the work, if we do not specify otherwise, Ω is an open bounded set in $\mathbb{R}^3$ with boundary Γ of class $\mathcal{C}^{1,1}$ possibly multiply connected. Also if we do not precise otherwise, we will always assume

$$\alpha \geq 0 \quad \text{on } \Gamma \quad \text{and } \alpha > 0 \quad \text{on some } \Gamma_0 \subset \Gamma \quad \text{with } |\Gamma_0| > 0.$$

We follow the convention that C is an unspecified positive constant that may vary from expression to expression, even across an inequality (but not across an equality); Also C depends on Ω and p generally and the dependence on other parameters will be specified in the parenthesis when necessary.

The vector-valued Laplace operator of a vector field $\boldsymbol{v}$ can be equivalently defined by

$$\Delta \boldsymbol{v} = 2 \text{ div } \mathbb{D}\boldsymbol{v} - \textbf{grad} \text{ (div } \boldsymbol{v}).$$

We will denote by $\mathscr{D}(\Omega)$ the set of smooth functions (infinitely differentiable) with compact support in Ω. Define

$$\mathscr{D}_\sigma(\Omega) := \{\boldsymbol{v} \in \mathscr{D}(\Omega) : \text{div } \boldsymbol{v} = 0 \text{ in } \Omega\}$$

and

$$L^p_0(\Omega) := \left\{ v \in L^p(\Omega) : \int_\Omega v = 0 \right\}.$$

For $p \in [1, \infty)$, p' denotes the conjugate exponent of p i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We also introduce the following space:

$$\boldsymbol{H}^p(\text{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) : \text{div } \boldsymbol{v} \in L^p(\Omega)\}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\text{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\text{div } \boldsymbol{v}\|_{L^p(\Omega)}.$$

It can be shown that $\mathscr{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\text{div}, \Omega)$ for all $p \in [1, \infty)$. The closure of $\mathscr{D}(\Omega)$ in $\boldsymbol{H}^p(\text{div}, \Omega)$ is denoted by $\boldsymbol{H}^p_0(\text{div}, \Omega)$ and it can be characterized by

$$\boldsymbol{H}^p_0(\text{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\text{div}, \Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}. \tag{3.10}$$

For $p \in (1, \infty)$, we denote by $[\boldsymbol{H}^p_0(\text{div}, \Omega)]'$, the dual space of $\boldsymbol{H}^p_0(\text{div}, \Omega)$, which can be characterized as (for details, see [103, Proposition 1.0.4]):

Proposition 3.2.1. *A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^p_0(\text{div}, \Omega)]'$ iff there exists $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$. Moreover, we have the estimate :*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^p_0(\text{div},\Omega)]'} \leq \inf_{\boldsymbol{f}=\boldsymbol{\psi}+\nabla\chi} \max\{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}, \|\chi\|_{L^{p'}(\Omega)}\}.$$

Next we introduce the spaces:

$$\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) : \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

equipped with the norm of $\boldsymbol{L}^p(\Omega)$;

$$\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) : \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

equipped with the norm of $\boldsymbol{W}^{1,p}(\Omega)$ and $\mathbf{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega) := \mathbf{W}^{1,2}_{\sigma,\boldsymbol{\tau}}(\Omega)$. Also let us define

$$\mathbf{E}^p(\Omega) := \left\{(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega); -\Delta\boldsymbol{u} + \nabla\pi \in \boldsymbol{L}^{r(p)}(\Omega)\right\}$$

where

$$\begin{cases} r(p) = \max\left\{1, \frac{3p}{p+3}\right\} & \text{if } \ p \neq \frac{3}{2} \\ r(p) > 1 & \text{if } \ p = \frac{3}{2}. \end{cases}$$

Let us now introduce some notations to describe the boundary. Consider any point P on Γ and choose an open neighborhood W of P in Γ, small enough to allow the existence of 2 families of $\mathcal{C}^2$ curves on W with the following properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthogonal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves.

With these notations, we have $\boldsymbol{v} = \sum_{k=1}^{2} v_k \boldsymbol{\tau}_k + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k = (\tau_{k1}, \tau_{k2}, \tau_{k3})$ and $v_k = \boldsymbol{v} \cdot \boldsymbol{\tau}_k$. For simplicity of notation, we will denote,

$$\Lambda \boldsymbol{v} = \sum_{k=1}^{2} \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k} \right) \boldsymbol{\tau}_k \ . \tag{3.11}$$

Here we state a relation between the Navier boundary condition and another type of boundary condition involving **curl** (often called as 'Navier type boundary condition') which will be used in later work. For proof, see [11, Appendix A].

Lemma 3.2.2. *For any $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, we have the following equalities:*

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v}$$

and

$$\mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} = -\nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v}.$$

Remark 3.2.3. In the particular case $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, we have the following equalities for all $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$,

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v} \quad \text{and} \quad \mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v}$$

which implies that

$$2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda v}. \tag{3.12}$$

Note that on a flat boundary, $\boldsymbol{\Lambda} = \mathbf{0}$ and $2\left[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}\right]_{\boldsymbol{\tau}}$ is actually equal to $\mathbf{curl}\ \boldsymbol{v} \times \boldsymbol{n}$.

Let us recall the Green's formula that plays an important role in this work, which is proved in Lemma 2.3.5.

Theorem 3.2.4. *Let $\Omega \subset \mathbb{R}^3$ be a $C^{0,1}$ bounded domain. Then,*

(i) $\boldsymbol{\mathscr{D}}(\overline{\Omega}) \times \mathscr{D}(\overline{\Omega})$ is dense in $\mathbf{E}^p(\Omega)$.

(ii) The linear mapping $(\boldsymbol{v}, \pi) \mapsto [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$, defined on $\boldsymbol{\mathscr{D}}(\overline{\Omega}) \times \mathscr{D}(\overline{\Omega})$ can be extended to a linear, continuous map from $\boldsymbol{E}^p(\Omega)$ to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Moreover, we have the following relation: for all $(\boldsymbol{v}, \pi) \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\int_\Omega (-\Delta \boldsymbol{v} + \nabla \pi) \cdot \boldsymbol{\varphi} = 2 \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} - 2 \left\langle [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \right\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \tag{3.13}$$

Finally, we recall that the infinitesimal generator of an analytic semigroup can be characterized by the following theorem [15, Theorem 3.2, Chapter I]:

Theorem 3.2.5. *Let A be a densely defined linear operator in a Banach space $\mathscr{E}$. Then A generates an analytic semigroup on $\mathscr{E}$ if and only if there exists $M > 0$ such that*

$$\| (\lambda I - A)^{-1} \|_{\mathcal{L}(\mathscr{E})} \leq \frac{M}{|\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$.

3.3 Stokes operator on $L^p_{\sigma,\tau}(\Omega)$: Strong solutions

It is known that the closure of $\boldsymbol{\mathscr{D}}_\sigma(\Omega)$ in $\boldsymbol{L}^p(\Omega)$ is the Banach space $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ [113, Theorem 1.4]. We introduce now the unbounded operator $(A_{p,\alpha}, \mathbf{D}(A_{p,\alpha}))$ on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ whose definition depends on the regularity of the function α.

1. If $\alpha \in L^{t(p)}(\Gamma)$ with

$$t(p) = \begin{cases} \frac{2}{3}p' + \rho & \text{if} \quad 1 < p < \frac{3}{2} \\ 2 + \rho & \text{if} \quad \frac{3}{2} \leq p \leq 3, p \neq 2 \\ 2 & \text{if} \quad p = 2 \\ \frac{2}{3}p + \rho & \text{if} \quad p > 3 \end{cases} \tag{3.14}$$

with $\rho > 0$ arbitrarily small, we define the Stokes operator $A_{p,\alpha}$ on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ as:

$$\begin{cases} \mathbf{D}(A_{p,\alpha}) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) : \Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega), 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma \right\} & (3.15) \\ A_{p,\alpha}(\boldsymbol{u}) = -P(\Delta \boldsymbol{u}) \quad \text{for } \boldsymbol{u} \in \mathbf{D}(A_{p,\alpha}) & (3.16) \end{cases}$$

where $P : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ is a projection on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. More precisely, for all $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$, $P(\boldsymbol{\psi}) = \boldsymbol{\psi} - \nabla\pi$ where $\pi \in W^{1,p}(\Omega)$ is a weak solution of the following weak Neumann problem

$$\begin{cases} \operatorname{div}(\nabla\pi - \boldsymbol{\psi}) = 0 \text{ in } \Omega \\ (\nabla\pi - \boldsymbol{\psi}) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma. \end{cases} \tag{3.17}$$

Notice that, when $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$ but $\boldsymbol{u} \notin \boldsymbol{W}^{2,p}(\Omega)$, the boundary term $[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ is still well defined as shown in [11, Lemma 2.4].

2. If α is such that

$$\alpha \in \begin{cases} W^{1-\frac{1}{\frac{3}{2}+\rho}, \frac{3}{2}+\rho}(\Gamma) & \text{if} \quad 1 < p \leq \frac{3}{2} \\ W^{1-\frac{1}{p},p}(\Gamma) & \text{if} \quad p > \frac{3}{2} \end{cases} \tag{3.18}$$

with $\rho > 0$ arbitrarily small, then we define the Stokes operator $A_{p,\alpha}$ on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ as

$$\begin{cases} \mathbf{D}(A_{p,\alpha}) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\right\} & (3.19) \\ A_{p,\alpha}(\boldsymbol{u}) = -P(\Delta\boldsymbol{u}) \quad \text{for } \boldsymbol{u} \in \mathbf{D}(A_{p,\alpha}). & (3.20) \end{cases}$$

Remark 3.3.1. If α satisfies (3.18), then $\alpha \in L^{t(p)}(\Gamma)$ as well.

Remark 3.3.2. When α satisfies (3.18) and $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$, $A_{p,\alpha}\boldsymbol{u} = A\boldsymbol{u}$ where A is the following operator:

$$\begin{aligned} & A \in \mathcal{L}\left(\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega), \left(\mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)\right)'\right) \\ \text{defined by} \quad & \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}) \\ \text{where} \quad & a(\boldsymbol{u}, \boldsymbol{v}) = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\bar{\boldsymbol{v}} + \int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \bar{\boldsymbol{v}}_{\boldsymbol{\tau}}. \end{aligned}$$

More precisely, using Green's formula (3.13) and the relation $2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\alpha\boldsymbol{u}_{\boldsymbol{\tau}}$ on Γ, we can deduce $\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \langle A_{p,\alpha}\boldsymbol{u}, \boldsymbol{v}\rangle$ for any $\boldsymbol{v} \in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$.

3.3.1 Analyticity

In order to show that $(\mathbf{D}(A_{p,\alpha}), A_{p,\alpha})$ generates an analytic semi group on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$, some estimate on the resolvent $(\lambda I + A_p)^{-1}, \lambda \in \mathbb{C}$ is needed.

Suppose that $\alpha \in L^{t(p)}(\Gamma)$ and $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. Then by (3.15)-(3.16), $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$ and $(\lambda I + A_{p,\alpha})\boldsymbol{u} = \boldsymbol{f}$ is equivalent to $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega & (3.21)\\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega & (3.22)\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma & (3.23)\\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma & (3.24) \end{cases}$$

for some $\pi \in L^p(\Omega)$.

If on the other hand, α satisfies (3.18), $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$ and $(\lambda I + A_{p,\alpha})\boldsymbol{u} = \boldsymbol{f}$ for $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$, is equivalent to $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ satisfying (3.21)-(3.24).

Proposition 3.3.3. *Suppose that $\alpha \in L^{t(p)}(\Gamma)$, $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\text{div}, \Omega)]'$ with $p \in (1, \infty)$ and $\lambda \in \mathbb{C}$. Then, $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solves (3.21)-(3.24) for some $\pi \in L^p(\Omega)$ is equivalent to $\boldsymbol{u} \in \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$ satisfies:*

$$\begin{cases} a_\lambda(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \bar{\boldsymbol{\varphi}} \rangle \quad \forall\ \boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega) \\ \textit{where:} \\ a_\lambda(\boldsymbol{u}, \boldsymbol{\varphi}) = \lambda \displaystyle\int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{\varphi}} + 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\bar{\boldsymbol{\varphi}} + \langle \alpha \boldsymbol{u}_{\boldsymbol{\tau}}, \bar{\boldsymbol{\varphi}}_{\boldsymbol{\tau}} \rangle_\Gamma \end{cases} \tag{3.25}$$

and $\langle , \rangle_\Gamma$ denotes the duality product between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$.

Proof. It follows from the trace theorem and Hölder inequality that for all $\boldsymbol{u} \in \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \bar{\boldsymbol{\varphi}}_{\boldsymbol{\tau}} \in L^1(\Gamma)$ (cf. Lemma 2.3.8) and

$$\int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \bar{\boldsymbol{\varphi}}_{\boldsymbol{\tau}} \leq C \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}.$$

It easily then follows that $a_\lambda(\cdot,\cdot)$ is a continuous sesqui-linear form on $\mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) \times \mathbf{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$. When $\lambda = 0$, it is the sesqui-linear form $a(\cdot,\cdot)$ introduced in the previous chapter. The proof of this proposition then follows exactly the same steps and uses the same arguments as in Proposition 2.3.10. ■

The following theorem gives the existence of a unique solution of the resolvent problem and also the resolvent estimate.

Theorem 3.3.4. *For any $\varepsilon \in (0, \pi)$, let $\lambda \in \Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \pi - \varepsilon\}$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\alpha \in L^2(\Gamma)$. Then,*

1. *the problem (3.21)-(3.24) has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$.

2. *there exists a constant* $C_\varepsilon > 0$, *independent of* $\boldsymbol{f}$, α *and* λ, *such that the solution* $\boldsymbol{u}$ *satisfies the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{C_\varepsilon |\lambda|} \, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.26}$$

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{C_\varepsilon \sqrt{|\lambda|}} \, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.27}$$

and

$$\|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left[1 + \frac{1}{C_\varepsilon}\left(1 + \frac{1}{\sqrt{|\lambda|}}\right)\right] \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.28}$$

3. *moreover, if either (i)* Ω *is not axisymmetric or (ii)* Ω *is axisymmetric and* $\alpha \geq \alpha_* > 0$, *then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(1 + \frac{1}{C_\varepsilon}\right) \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.29}$$

and if α *is a constant, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega)\left(1 + \frac{1}{C_\varepsilon}\right) \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.30}$$

where $C(\Omega)$ *does not depend on* α.

Remark 3.3.5. Note that the estimates (3.26) and (3.27) give better decay for λ large and enables us having a good semigroup theory, in general. On the other hand, estimate (3.29) gives uniform bound on the solution, especially when λ is small.

Proof. **1.** In view of Proposition 3.3.3, it is enough to prove the existence and uniqueness of a solution to (3.25). Also the case $\lambda = 0$ corresponds to the existence result for the stationary Stokes problem Theorem 2.4.1. So we may consider $\lambda \neq 0$.

By Korn's inequality, $\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ is an equivalent norm on $\mathbf{H}^1(\Omega)$. Then using the following inequality (cf. [6, Lemma 4.1]):

$$\forall\ \lambda \in \Sigma_\varepsilon, \forall\ a, b > 0, \qquad |\lambda a + b| \geq C_\varepsilon \left(|\lambda| a + b\right) \text{ for some constant } C_\varepsilon > 0$$

and $\alpha \geq 0$, we get,

$$|a_\lambda(\boldsymbol{u}, \boldsymbol{u})| \geq C_\varepsilon \left(|\lambda| \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbb{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \right)$$

$$\begin{aligned}&\geq C_\varepsilon \ \min(|\lambda|, 2) \left(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\ &\geq C_\varepsilon \ \min(|\lambda|, 2)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Hence, for all $\lambda \in \Sigma_\varepsilon$, a_λ is coercive on $\mathbf{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$ and therefore, by Lax-Milgram lemma, we get a unique solution in $\mathbf{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$ of the problem (3.25) which proves **1.**

2. From the variational formulation, we have,

$$a_\lambda(\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{u}}$$

which gives

$$|a_\lambda(\boldsymbol{u}, \boldsymbol{u})| \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

But we also have,

$$|a_\lambda(\boldsymbol{u}, \boldsymbol{u})| \geq C_\varepsilon \left(|\lambda|\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbb{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right).$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{C_\varepsilon|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

and then

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \ \leq \ \frac{1}{2\ C_\varepsilon}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \ \leq \ \frac{1}{2\ |\lambda|\ C_\varepsilon^2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}$$

prove the inequalities (3.26) and (3.27).

From the equation (3.21), we may write, using (3.26) and (3.27),

$$\begin{aligned}\|\pi\|_{L^2(\Omega)} \leq \|\nabla\pi\|_{\boldsymbol{H}^{-1}(\Omega)} &\leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} + |\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right)\\ &\leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + |\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right)\\ &\leq C(\Omega)\left[1 + \frac{1}{C_\varepsilon}\left(1 + \frac{1}{\sqrt{2|\lambda|}}\right)\right]\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\end{aligned}$$

which gives (3.28).

3. Moreover, writing (3.21) as $-\Delta\boldsymbol{u}+\nabla\pi = \boldsymbol{f}-\lambda\boldsymbol{u}$, we deduce from the Stokes estimate Theorem 2.4.3 in the case either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, the existence of a constant $C > 0$ which depends only on Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + |\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right) \leq C(\Omega)\left(1 + \frac{1}{C_\varepsilon}\right)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

This provides the better bound (3.29) on $\boldsymbol{u}$ and π when λ is small.

Similarly, for constant α, the H^2 estimate of the Stokes problem Theorem 2.4.5 yields

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega)\left(1 + \frac{1}{C_\varepsilon}\right)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

■

In the next theorem we prove the analyticity of the semigroup generated by the Stokes operator with Navier boundary condition on $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$.

Theorem 3.3.6. *For any $\alpha \in L^2(\Gamma)$, the operator $-A_{2,\alpha}$, defined in (3.15)-(3.16) with $p = 2$, generates a bounded analytic semigroup on $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$.*

Proof. Obviously $\mathbf{D}(A_{2,\alpha})$ is dense in $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$. Therefore, according to Theorem 3.2.5, it is enough to prove the resolvent estimate. Now, by definition and from the previous theorem, we have,

$$\|(\lambda I + A_{2,\alpha})^{-1}\| = \sup_{\substack{\boldsymbol{f}\in \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)\\ \boldsymbol{f}\neq \boldsymbol{0}}} \frac{\|(\lambda I + A_{2,\alpha})^{-1}\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}} = \sup_{\boldsymbol{f}\in \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)} \frac{\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}} \leq \frac{1}{C_\varepsilon |\lambda|}.$$

Hence, the result. ∎

Next we extend the results of Theorem 3.3.4 for all $p \in (1, \infty)$.

Theorem 3.3.7. *Suppose that $p \in (1, \infty)$, α satisfies (3.18) and $\lambda \in \Sigma_\varepsilon$ where Σ_ε is defined as in Theorem 3.3.4. Then for every $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the problem (3.21)-(3.24) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \cap L^p_0(\Omega)$.*

Proof. **case (i):** $p > 2$. Since from the assumption, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\alpha \in L^2(\Gamma)$, there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of (3.21)-(3.24) by Theorem 3.3.4. Now writing the equation (3.21) as $-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} - \lambda \boldsymbol{u}$ and since $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have $\boldsymbol{f} - \lambda \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ for all $p \leq 6$. Thus, using the regularity result in Theorem 2.5.9, we obtain $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ for all $p \leq 6$.

Now for $p > 6$, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Hence $\boldsymbol{f} - \lambda \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and by the same regularity result, we get $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ for all $p > 6$.

case (ii): $1 < p < 2$. We first claim that $(\lambda I + A)$ is an isomorphism from $\boldsymbol{W}^{1,q}_{\sigma,\boldsymbol{\tau}}(\Omega)$ to $(\boldsymbol{W}^{1,q'}_{\sigma,\boldsymbol{\tau}}(\Omega))'$ for all $q \geq 2$. Then the adjoint operator, $\lambda I + A^*$ is also an isomorphism from $\boldsymbol{W}^{1,q'}_{\sigma,\boldsymbol{\tau}}(\Omega)$ to $(\boldsymbol{W}^{1,q}_{\sigma,\boldsymbol{\tau}}(\Omega))'$ with $q' \leq 2$. Then, for any $\boldsymbol{f} \in \mathbf{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) \subset (\boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega))'$, there exists a unique $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$ such that $(\lambda I + A^*)\boldsymbol{u} = \boldsymbol{f}$. Our second claim is that, since $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ it follows that $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$ and $A^*\boldsymbol{u} = A_{p,\alpha}\boldsymbol{u}$. This finally implies that $(\lambda I + A_{p,\alpha})\boldsymbol{u} = \boldsymbol{f}$.

First claim: For $q > 2$ and $\boldsymbol{\ell} \in (\boldsymbol{W}^{1,q'}_{\sigma,\boldsymbol{\tau}}(\Omega))' \subset (\boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega))'$, from Lax-Milgam lemma there is a unique $\boldsymbol{u} \in \boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$ such that $a_\lambda(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle$ for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$. Then, $a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell} - \lambda \boldsymbol{u}, \boldsymbol{\varphi} \rangle$ with $\boldsymbol{\ell} - \lambda \boldsymbol{u} \in (\boldsymbol{W}^{1,q'}_{\sigma,\boldsymbol{\tau}}(\Omega))'$. On the other hand, by Theorem 2.5.3 there exists a unique $\boldsymbol{w} \in \boldsymbol{W}^{1,q}_{\sigma,\boldsymbol{\tau}}(\Omega) \subset \boldsymbol{W}^{1,2}_{\sigma,\boldsymbol{\tau}}(\Omega)$ such that

$$a(\boldsymbol{w}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell} - \lambda \boldsymbol{u}, \boldsymbol{\varphi} \rangle \ \ \forall \boldsymbol{\varphi} \in \boldsymbol{W}^{1,q'}_{\sigma,\boldsymbol{\tau}}(\Omega).$$

It then follows that

$$a(\boldsymbol{w}-\boldsymbol{u},\boldsymbol{\varphi}) = 0 \ \ \forall \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$$

and by the uniqueness result Theorem 2.5.3, $\boldsymbol{u} = \boldsymbol{w}$ in $\boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$ and thus $\boldsymbol{u} \in \boldsymbol{W}^{1,q}_{\sigma,\boldsymbol{\tau}}(\Omega)$.

Second claim: If $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ with $1 < p < 2$ and $(\lambda I + A^*)\boldsymbol{u} = \boldsymbol{f}$ with $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$, then $A^*\boldsymbol{u} = \boldsymbol{f} - \lambda\boldsymbol{u} =: \boldsymbol{g} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ which means $a(\boldsymbol{u},\boldsymbol{\varphi}) = (\boldsymbol{g},\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$. It then follows by the regularity result Theorem 2.5.9 that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ with π defined by (3.17) for $\boldsymbol{\psi} = \Delta\boldsymbol{u}$ and $\boldsymbol{u}$ satisfies the boundary condition. In particular $\boldsymbol{u} \in \mathbf{D}(A_{p,\alpha})$. Then, using Remark 3.3.2, $A_p\boldsymbol{u} = A^*\boldsymbol{u}$. ■

Remark 3.3.8. Notice that though the two boundary conditions

$$\textbf{curl}\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \ \ \text{on } \Gamma \tag{3.31}$$

and

$$2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \ \ \text{on } \Gamma \tag{3.32}$$

are very much similar, as described in (3.12), but in the case of the Stokes problem with Navier type boundary condition (3.31) on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ the pressure is constant and hence does not appear in the operator (see [5, Proposition 3.1]). On the contrary, the pressure term does appear in the Stokes operator with Navier boundary condition (3.32).

Next we deduce the $\boldsymbol{L}^p$-resolvent estimate for all $p \in (1,\infty)$.

Theorem 3.3.9. *Let $\lambda \in \mathbb{C}^*$ with $\operatorname{Re}\lambda \geq 0$ and $p \in (1,\infty)$. Then for $\alpha \in L^{t(p)}(\Gamma)$ and for any $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ of (3.21)-(3.24) satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$$

where C depends at most on p and Ω.

To prove the above theorem, we need to establish the following weak Reverse Hölder estimate (as in [107, Lemma 6.2]):

Lemma 3.3.10. *Let $x_0 \in \overline{\Omega}$ and $0 < r < c$ diam(Ω). Let $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(B(x_0,2r)\cap\Omega) \times L^2(B(x_0,2r)\cap\Omega)$ satisfies the Stokes system*

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \quad \operatorname{div}\boldsymbol{u} = 0 & \textit{in } B(x_0,2r)\cap\Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \textit{on } B(x_0,2r)\cap\Gamma \end{cases} \tag{3.33}$$

where $\lambda \in \mathbb{C}^*$ *with* $\operatorname{Re}\lambda \geq 0$ *and* $\alpha \in L^{t(p)}(\Gamma)$*. Then, for any* $p \geq 2$*,*

$$\left(\frac{1}{r^3}\int\limits_{\Omega\cap B(x_0,r)} |\boldsymbol{u}|^p\right)^{1/p} \leq C\left(\frac{1}{r^3}\int\limits_{\Omega\cap B(x_0,2r)} |\boldsymbol{u}|^2\right)^{1/2} \tag{3.34}$$

where $C > 0$ *depends only on* p *and* Ω*.*

Note that if $B(x_0, 2r) \cap \Gamma = \emptyset$, then we do not consider any boundary condition.

Proof. By a geometric consideration (since Γ is compact), we only need to establish the estimate in two cases : (i) $x_0 \in \Omega$ and $B(x_0, 3r) \subset \Omega$; (ii) $x_0 \in \Gamma$.

The first case (i) follows from interior estimate (cf. [107, estimate 5.22]). The second case (ii) concerns a boundary estimate. Here onwards, we denote the ball $B(x_0, R)$ by B_R and $⨍_\omega f = \frac{1}{|\omega|}\int_\omega f$.

1. Let $r \leq s < t \leq 2r$ and consider a cut-off unction $\eta \in C_c^\infty(B_t)$ such that

$$\eta \equiv 1 \text{ on } B_s, \quad 0 \leq \eta \leq 1 \quad \text{and} \quad |\nabla\eta| \leq \frac{C}{t-s}.$$

Multiplying (3.33) by $\eta^2\boldsymbol{u}$ and integrating by parts, we get,

$$0 = \lambda\int\limits_{B_{2r}\cap\Omega} \eta^2|\boldsymbol{u}|^2 + 2\int\limits_{B_{2r}\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}(\eta^2\boldsymbol{u}) + \int\limits_{B_{2r}\cap\Gamma} \alpha\eta^2|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 - \int\limits_{B_{2r}\cap\Omega} (\pi - \pi_0)\operatorname{div}(\eta^2\boldsymbol{u})$$

where $\pi_0 = ⨍_{B_t\cap\Omega}\pi$. This gives, using the fact that div $\boldsymbol{u} = 0$ in Ω,

$$\lambda\int\limits_{B_{2r}\cap\Omega} \eta^2|\boldsymbol{u}|^2 + 2\int\limits_{B_{2r}\cap\Omega} \eta^2|\mathbb{D}\boldsymbol{u}|^2 + \int\limits_{B_{2r}\cap\Gamma} \alpha\eta^2|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = -4\int\limits_{B_{2r}\cap\Omega} \mathbb{D}\boldsymbol{u} : \eta\nabla\eta\bar{\boldsymbol{u}} + 2\int\limits_{B_{2r}\cap\Omega} (\pi - \pi_0)\eta\nabla\eta\bar{\boldsymbol{u}}$$

where $\nabla\eta\bar{\boldsymbol{u}}$ is the matrix $\nabla\eta \otimes \bar{\boldsymbol{u}}$. Equating real and imaginary parts and using Cauchy's inequality, this implies

$$\begin{aligned}
&\operatorname{Re}\lambda\int\limits_{B_{2r}\cap\Omega} \eta^2|\boldsymbol{u}|^2 + 2\int\limits_{B_{2r}\cap\Omega} \eta^2|\mathbb{D}\boldsymbol{u}|^2 + \int\limits_{B_{2r}\cap\Gamma} \alpha\eta^2|\boldsymbol{u}_{\boldsymbol{\tau}}|^2\\
&= -4\operatorname{Re}\int\limits_{B_{2r}\cap\Omega} \mathbb{D}\boldsymbol{u} : \eta\nabla\eta\bar{\boldsymbol{u}} + 2\operatorname{Re}\int\limits_{B_{2r}\cap\Omega} (\pi - \pi_0)\eta\nabla\eta\bar{\boldsymbol{u}}\\
&\leq \varepsilon\int\limits_{\Omega\cap B_{2r}} \eta^2|\mathbb{D}\boldsymbol{u}|^2 + C_\varepsilon\int\limits_{\Omega\cap B_{2r}} |\nabla\eta|^2|\boldsymbol{u}|^2 + \varepsilon\int\limits_{\Omega\cap B_{2r}} \eta^2|\pi - \pi_0|^2 + C_\varepsilon\int\limits_{\Omega\cap B_{2r}} |\nabla\eta|^2|\boldsymbol{u}|^2 \quad (3.35)
\end{aligned}$$

and

$$\operatorname{Im}\lambda\int\limits_{B_{2r}\cap\Omega} \eta^2|\boldsymbol{u}|^2$$

$$= -4\,\mathrm{Im}\int_{B_{2r}\cap\Omega} \mathbb{D}\boldsymbol{u} : \eta\nabla\eta\bar{\boldsymbol{u}} + 2\,\mathrm{Im}\int_{B_{2r}\cap\Omega} (\pi-\pi_0)\eta\nabla\eta\bar{\boldsymbol{u}}$$

$$\leq \varepsilon\int_{\Omega\cap B_{2r}} \eta^2|\mathbb{D}\boldsymbol{u}|^2 + C_\varepsilon\int_{\Omega\cap B_{2r}} |\nabla\eta|^2|\boldsymbol{u}|^2 + \varepsilon\int_{\Omega\cap B_{2r}} \eta^2|\pi-\pi_0|^2 + C_\varepsilon\int_{\Omega\cap B_{2r}} |\nabla\eta|^2|\boldsymbol{u}|^2. \quad (3.36)$$

Now adding (3.35) and (3.36) gives, since $\alpha > 0$,

$$(\mathrm{Re}\,\lambda + |\,\mathrm{Im}\,\lambda|)\int_{B_{2r}\cap\Omega} \eta^2|\boldsymbol{u}|^2 + 2\int_{B_{2r}\cap\Omega} \eta^2|\mathbb{D}\boldsymbol{u}|^2$$

$$\leq \varepsilon\int_{\Omega\cap B_{2r}} \eta^2|\mathbb{D}\boldsymbol{u}|^2 + C_\varepsilon\int_{\Omega\cap B_{2r}} |\nabla\eta|^2|\boldsymbol{u}|^2 + \varepsilon\int_{\Omega\cap B_{2r}} \eta^2|\pi-\pi_0|^2.$$

Incorporating the properties of η, we obtain

$$|\lambda|\int_{B_s\cap\Omega} |\boldsymbol{u}|^2 + \int_{B_s\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \leq \frac{C(\Omega)}{(t-s)^2}\int_{\Omega\cap B_t} |\boldsymbol{u}|^2 + \varepsilon\int_{\Omega\cap B_t} |\pi-\pi_0|^2. \quad (3.37)$$

2. Next to estimate the pressure term, we write

$$\|\pi-\pi_0\|_{L^2(B_t\cap\Omega)} \leq C\|\nabla(\pi-\pi_0)\|_{\boldsymbol{H}^{-1}(B_t\cap\Omega)} = C\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(B_t\cap\Omega)}$$

where $\boldsymbol{v} \in \boldsymbol{H}_0^1(B_t\cap\Omega)$ is the weak solution of

$$\int_{B_t\cap\Omega} \nabla\boldsymbol{v} : \nabla\boldsymbol{\varphi} = \int_{B_t\cap\Omega} (\pi-\pi_0)\mathrm{div}\,\boldsymbol{\varphi} \qquad \forall\boldsymbol{\varphi}\in\boldsymbol{H}_0^1(B_t\cap\Omega).$$

Note that the above constant C depends only on Ω, not on t (cf. [55, 4. Proposition 1.8, Part II]). But multiplying (3.33) by $\boldsymbol{\varphi}$ yields (replacing π by $(\pi-\pi_0)$ and extending $\boldsymbol{\varphi}$ by 0 outside $\Omega\cap B_t$),

$$\lambda\int_{B_t\cap\Omega} \boldsymbol{u}\cdot\boldsymbol{\varphi} + 2\int_{B_t\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} = \int_{B_t\cap\Omega} (\pi-\pi_0)\ \mathrm{div}\ \boldsymbol{\varphi}.$$

Hence,

$$\int_{B_t\cap\Omega} \nabla\boldsymbol{v} : \nabla\boldsymbol{\varphi} = \lambda\int_{B_t\cap\Omega} \boldsymbol{u}\cdot\boldsymbol{\varphi} + 2\int_{B_t\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} \qquad \forall\boldsymbol{\varphi}\in\boldsymbol{H}_0^1(B_t\cap\Omega).$$

Now putting $\boldsymbol{\varphi} = \boldsymbol{v}$ gives

$$\|\pi-\pi_0\|_{L^2(B_t\cap\Omega)} \leq C(\Omega)\left(|\lambda|\|\boldsymbol{u}\|_{L^2(B_t\cap\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{L^2(B_t\cap\Omega)}\right).$$

Plugging this estimate in (3.37), we obtain

$$|\lambda|\int_{B_s\cap\Omega} |\boldsymbol{u}|^2 + \int_{B_s\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \leq \frac{C(\Omega)}{(t-s)^2}\int_{\Omega\cap B_t} |\boldsymbol{u}|^2 + \varepsilon\int_{\Omega\cap B_t} |\mathbb{D}\boldsymbol{u}|^2 + \varepsilon|\lambda|\int_{\Omega\cap B_t} |\boldsymbol{u}|^2.$$

From here, we deduce the following Caccioppoli inequality for Stokes system

$$\int\limits_{B_s\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \le \frac{C(\Omega)}{(t-s)^2}\int\limits_{B_t\cap\Omega} |\boldsymbol{u}|^2$$

which follows from the general result [55, Lemma 0.5] setting $f = |\lambda||\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2, g = |\boldsymbol{u}|^2, h = 0$ and $\alpha = 2$.

3. Therefore [94, Lemma 6.7] enables us to have the following reverse Hölder inequality,

$$\left(\fint\limits_{\Omega\cap B_r} |\boldsymbol{u}|^2\right)^{1/2} \le C(q)\left(\fint\limits_{\Omega\cap B_{2r}} |\boldsymbol{u}|^q\right)^{1/q} \quad \text{for any } q > 0. \tag{3.38}$$

Finally we claim that the above inequality implies (3.34). To prove that, let us define an operator

$$T : \boldsymbol{L}^{p'}(\Omega\cap B_{2r}) \to \boldsymbol{L}^2(\Omega\cap B_r)$$
$$\boldsymbol{v} \mapsto \boldsymbol{v}.$$

Now for the adjoint map

$$T^* : \boldsymbol{L}^2(\Omega\cap B_r) \to \boldsymbol{L}^p(\Omega\cap B_{2r})$$

by definition, we can write,

$$\begin{aligned}\langle T^*\boldsymbol{u}, \boldsymbol{f}\rangle_{\boldsymbol{L}^p(\Omega\cap B_{2r})\times\boldsymbol{L}^{p'}(\Omega\cap B_{2r})} &= \langle \boldsymbol{u}, T\boldsymbol{f}\rangle_{\boldsymbol{L}^2(\Omega\cap B_r)\times\boldsymbol{L}^2(\Omega\cap B_r)} \\ &\le C(p)r^{3(\frac{1}{2}-\frac{1}{p'})}\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega\cap B_{2r})}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega\cap B_r)}\end{aligned}$$

where the last inequality comes from (3.38). This shows

$$\|T^*\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega\cap B_{2r})} \le C(p)r^{3(\frac{1}{2}-\frac{1}{p'})}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega\cap B_r)}.$$

Since $T^*\boldsymbol{u} = \boldsymbol{u}$ on $\boldsymbol{L}^p(\Omega\cap B_r)$, we then obtain

$$\left(\int\limits_{\Omega\cap B_r} |\boldsymbol{u}|^p\right)^{1/p} \le C(p)r^{3(\frac{1}{2}-\frac{1}{p'})}\left(\int\limits_{\Omega\cap B_r} |\boldsymbol{u}|^2\right)^{1/2}$$

which concludes (3.34) upon using $\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega\cap B_r)} \le \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega\cap B_{2r})}$. ■

The following lemma, due to Shen [107, Lemma 6.3], contains the real variable argument needed to complete the proof of Theorem 3.3.9.

Lemma 3.3.11. *Let $p > 2$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose that:*
(1) T is a bounded sublinear operator in $L^2(\Omega;\mathbb{C}^m)$ and $\|T\|_{L^2\to L^2} \leq C_0$;
(2) There exist constants $0 < \beta < 1$ and $N > 1$ such that for any bounded measurable f with supp$f \subset \Omega \setminus 3B$,

$$\left(\fint_{\Omega\cap B} |Tf|^p\right)^{1/p} \leq N \left\{ \left(\fint_{\Omega\cap 2B} |Tf|^2\right)^{1/2} + \sup_{B'\supset B} \left(\fint_{B'} |f|^2\right)^{1/2} \right\} \tag{3.39}$$

where $B = B(x,r)$ is a ball with $x \in \overline{\Omega}$ and $0 < r < \beta$ diam(Ω). Then T is bounded on $L^q(\Omega;\mathbb{C}^m)$ for any $2 < q < p$. Moreover, $\|T\|_{L^q\to L^q}$ is bounded by a constant depending on at most $d, m, \beta, N, C_0, p, q$ and the Lipschitz character of Ω.

The definition of the Lipschitz character (used above) of a bounded Lipschitz domain Ω can be found in [100, Sec. 5]. For convenience, we recall it.

A bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$ is called a bounded Lipschitz domain if there exists $r_0, M > 0$ such that for each point $x \in \partial\Omega$, there exists a Lipschitz continuous function $\eta_x : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\eta_x(0) = 0$ and $\|\nabla\eta_x\|_{\boldsymbol{L}^\infty(\mathbb{R}^{n-1})} \leq M$ and a rotation $R_x : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $0 < r < r_0$,

$$R_x(\Omega - \{x\}) \cap D(r) = D_{\eta_x}(r)$$
$$R_x(\partial\Omega - \{x\}) \cap D(r) = I_{\eta_x}(r)$$

where

$$D(r) := \{(x', x_n) : |x'| < r, |x_n| < 10n(M+1)r\}$$
$$D_{\eta_x}(r) := \{(x', x_n) : |x'| < r, \eta_x(x') < x_n < 10n(M+1)r\}$$
$$I_{\eta_x}(r) := \{(x', x_n) : |x'| < r, \eta_x(x') = x_n\}.$$

Now let $x_1, ..., x_N \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^N$ covers $\partial\Omega$ where $U_{x,r} := \{x\} + R_x^{-1}D(r)$. Then a constant $C > 0$ is said to depend on *the Lipschitz character of* Ω if it depends on M and N.

Proof of Theorem 3.3.9. By rescaling, we may assume that diam$(\Omega) = 1$. Then for $\lambda \in \mathbb{C}^*$ with $\mathrm{Re}\,\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfying (3.21)-(3.24). Also

$$|\lambda| \int_\Omega |\boldsymbol{u}|^2 + 2\int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \leq C \int_\Omega |\boldsymbol{f}||\boldsymbol{u}|$$

where C depends only on Ω. By Hölder's inequality, this implies

$$|\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq C_0\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

where C_0 depends only on Ω. Let us now define the operator T_λ by $T_\lambda(\boldsymbol{f}) = |\lambda|\boldsymbol{u}$. Then T_λ is a bounded linear operator on $\boldsymbol{L}^2(\Omega)$ and $\|T_\lambda\|_{\boldsymbol{L}^2\to\boldsymbol{L}^2} \leq C_0$ which is the assumption (1) in Lemma 3.3.11. We will show $\|T_\lambda\|_{\boldsymbol{L}^q\to\boldsymbol{L}^q} \leq C$ for $2 < q < p$ using Lemma 3.3.11.

To verify the assumption (2) in Lemma 3.3.11, let $B = B(x_0, r)$ where $x_0 \in \overline{\Omega}$ and $0 < r < c$. Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ with $\mathrm{supp}\boldsymbol{f} \subset \Omega \setminus 3B$. Since

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \;\; \mathrm{div}\ \boldsymbol{u} = 0 & \text{in } \ \Omega \cap 3B \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \cap 3B \end{cases}$$

we may then apply Lemma 3.3.10 to obtain,

$$\left(\frac{1}{r^3}\int_{\Omega\cap B}|\boldsymbol{u}|^p\right)^{1/p} \leq C\left(\frac{1}{r^3}\int_{\Omega\cap 2B}|\boldsymbol{u}|^2\right)^{1/2}.$$

It follows that

$$\left(\fint_{\Omega\cap B}|T_\lambda(\boldsymbol{f})|^p\right)^{1/p} \leq C\left(\fint_{\Omega\cap 2B}|T_\lambda(\boldsymbol{f})|^2\right)^{1/2}$$

where C depends only on p and Ω. Therefore by Lemma 3.3.11, we conclude that the operator T_λ is bounded on $\boldsymbol{L}^q(\Omega)$ for any $2 < q < p$ and that $\|T_\lambda\|_{\boldsymbol{L}^q\to\boldsymbol{L}^q} \leq C_q$ where C_q depends on at most q and Ω. Thus in view of the definition of T_λ, we have shown that for any $q > 2$,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq \frac{C_q}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)}.$$

By duality, the estimate also holds for $q \in (1, 2)$. ■

As a conclusion, we have the following theorem:

Theorem 3.3.12. *Let $p \in (1,\infty)$ and α as in* (3.18). *Then for all $\lambda \in \mathbb{C}^*$ with* $\mathrm{Re}\,\lambda \geq 0$ *and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, there exists a constant $C > 0$ depending on at most p and Ω such that the unique solution $(\boldsymbol{u}, \pi) \in \mathbf{D}(A_{p,\alpha}) \times W^{1,p}(\Omega) \cap L^p_0(\Omega)$ of* (3.21)-(3.24) *satisfies:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.40}$$

If moreover α is a constant and either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.41}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C \ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.42}$$

Proof. Estimate (3.40) follows from Theorem 3.3.9.

Let us now prove the estimate (3.41). From Gagliardo-Nirenberg inequality [3, Chapter IV, Theorem 4.14, Theorem 4.17] and regularity estimate for the stationary Stokes problem Theorem 2.6.15, we have,

$$\begin{aligned}\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C \ \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C \ \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{W}^{2,p}(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\ &\leq C \, \|\boldsymbol{f} - \lambda\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\ &\leq C \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + |\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\right)^{1/2} \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\end{aligned}$$

and estimate (3.41) follows using (3.40).

To prove (3.42) we again use Theorem 2.6.15 and (3.40) to obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C \ \|\boldsymbol{f} - \lambda\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C \ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

■

Finally we obtain our first main result:

Theorem 3.3.13. *Let α be as in* (3.18)*. The operator $-A_{p,\alpha}$ generates a bounded analytic semigroup on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ for all $1 < p < \infty$.*

Proof. In view of Theorem 3.3.12, to apply Theorem 3.2.5 it remains to check that $\mathbf{D}(A_{p,\alpha})$ is dense in $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. But this is immediate since $\boldsymbol{\mathscr{D}}_\sigma(\Omega) \hookrightarrow \mathbf{D}(A_{p,\alpha}) \hookrightarrow \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and by definition $\boldsymbol{\mathscr{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. ■

3.4 Stokes operator on $[H^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$

We first recall that if $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'$ (defined in (3.10)) and is such that $\mathrm{div}\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ for some $p \in (1,\infty)$, then its normal trace $(\boldsymbol{f}\cdot\boldsymbol{n})_{|\Gamma}$ is well defined and belongs to $W^{-1-\frac{1}{p},p}(\Gamma)$ [6, Corollary 3.7].

Let $\mathscr{B}$ be the closure of $\boldsymbol{\mathscr{D}}_\sigma(\Omega)$ in $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'$. Then, it can be shown that

$$\mathscr{B} = [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}} := \left\{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]' : \mathrm{div}\ \boldsymbol{f} = 0 \text{ in } \Omega, \boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}$$

which is a Banach space with the norm of $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ [6, Proposition 3.9]. Let $Q : [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \to \mathscr{B}$ be a projection on $\mathscr{B}$. We define the Stokes operator $B_{p,\alpha}$ on $\mathscr{B}$, as

$$\begin{cases} \mathbf{D}(B_{p,\alpha}) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \cap \mathscr{B} : \Delta\boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]', 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\right\}; \\ B_{p,\alpha}(\boldsymbol{u}) = -Q(\Delta\boldsymbol{u}) \quad \text{for } \boldsymbol{u} \in \mathbf{D}(B_{p,\alpha}) \end{cases}$$

with $\alpha \in L^{t(p)}(\Gamma)$ where $t(p)$ is defined in (3.14).

3.4.1 Analyticity

As in the previous section, we will now discuss the analyticity of the semigroup generated by the Stokes operator $B_{p,\alpha}$ on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}}$.

Theorem 3.4.1. *Let $p \in (1,\infty)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then, for all $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{Re}\,\lambda \geq 0$, and all $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, the problem (3.21)-(3.24) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L_0^p(\Omega)$ satisfying*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq \frac{C}{|\lambda|} \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \tag{3.43}$$

for some constant C independent of λ and α.

Proof. **1. Existence:** The proof of the existence and uniqueness of the solution follows similar arguments as in the proof of Theorem 3.3.4 and Theorem 3.3.7. For $p = 2$ the existence and uniqueness of solution comes from the Lax-Milgram lemma and de Rham theorem for the pressure.

When $p > 2$, since $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \subset [\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'$ and $\alpha \in L^{t(p)}(\Gamma) \subset L^2(\Gamma)$, we have the existence of the unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L_0^2(\Omega)$. We can now apply the regularity result for weak solution of the stationary Stokes system Corollary 2.5.6, since $\boldsymbol{f} - \lambda\boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ to obtain $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.

For $p < 2$, the proof follows in the same way as in Corollary 2.5.6.

2. Estimate: Now to prove the estimate (3.43), consider the problem,

$$\begin{cases} \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla\theta = \boldsymbol{F}, \quad \mathrm{div}\,\boldsymbol{v} = 0 \quad \text{in } \Omega \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} + \tilde{\alpha}\boldsymbol{v}_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on } \Gamma \end{cases}$$

where $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)$ and $\tilde{\alpha}$ as in (3.18). Thanks to Theorem 3.3.7 and the estimate (3.40), there exists unique $(\boldsymbol{v},\theta) \in \boldsymbol{W}^{2,p'}(\Omega) \times (W^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$ with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} .$$

As a result, we get,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \leq \frac{C}{|\lambda|} \, \|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}.$$

Now, for the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ of the problem (3.21)-(3.24), we get,

$$\begin{aligned}
\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} &= \sup_{\substack{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\\ \boldsymbol{F}\neq \boldsymbol{0}}} \frac{|\langle \boldsymbol{u}, \boldsymbol{F}\rangle|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} = \sup_{\substack{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\\ \boldsymbol{F}\neq \boldsymbol{0}}} \frac{|\langle \boldsymbol{u}, \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla\theta\rangle|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&= \sup_{\substack{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\\ \boldsymbol{F}\neq \boldsymbol{0}}} \frac{|\langle \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} = \sup_{\substack{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\\ \boldsymbol{F}\neq \boldsymbol{0}}} \frac{|\langle \boldsymbol{f}, \boldsymbol{v}\rangle|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&\leq \frac{C}{|\lambda|} \, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}
\end{aligned}$$

which is the required estimate. ■

Theorem 3.4.2. *Let $p \in (1, \infty)$ and $\alpha \in L^{t(p)}(\Gamma)$ with $t(p)$ defined in* (3.14). *Then, for $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'$, the unique solution $(\boldsymbol{u}, \pi)$ of (3.21)-(3.24) in $\mathbf{D}(B_{p,\alpha}) \times L_0^p(\Omega)$ is such that the pressure π satisfies the following estimate in the case when (i) either Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$:*

$$\|\pi\|_{L^p(\Omega)} \leq C \, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

for some constant C independent of λ and α.

Proof. By the regularity result of the stationary Stokes problem Theorem 2.6.11, we can write

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C\|\boldsymbol{f} - \lambda\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

and the result follows using estimate (3.43). ■

The analyticity of the semigroup generated by the Stokes operator $B_{p,\alpha}$ on $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\boldsymbol{\tau}}$ is now easily deduced from Theorem 3.4.1.

Theorem 3.4.3. *Let $\alpha \in L^{t(p)}(\Gamma)$ with $t(p)$ defined in* (3.14)*. The operator $-B_{p,\alpha}$ generates a bounded analytic semigroup on $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\boldsymbol{\tau}}$ for all $1 < p < \infty$.*

Proof. In view of Theorem 3.4.1, to apply Theorem 3.2.5 it remains to check that $\mathbf{D}(B_{p,\alpha})$ is dense in $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\boldsymbol{\tau}}$. But this is immediate since $\boldsymbol{\mathscr{D}}_\sigma(\Omega) \hookrightarrow \mathbf{D}(B_{p,\alpha}) \hookrightarrow [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\boldsymbol{\tau}}$ and by definition $\boldsymbol{\mathscr{D}}_\sigma(\Omega)$ is dense in $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\boldsymbol{\tau}}$. ■

3.5 Imaginary and Fractional powers

3.5.1 Imaginary powers

Our main purpose in this section is to prove local bounds on pure imaginary powers $A_{p,\alpha}^{is}$ and $B_{p,\alpha}^{is}$ of the Stokes operators defined in Section 3.3 and Section 3.4 respectively. A complete theory of fractional powers of an operator (bounded or unbounded) can be found in Komatsu [83].

Since these operators are non-negative operators, it then follows from the results in [83] and in [114] that their powers are well, densely defined and closed linear operators on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ with domain $\mathbf{D}(A_{p,\alpha}^{is})$ and $\mathbf{D}(B_{p,\alpha}^{is})$ respectively.

Notice that in [5], it was comparatively straight forward to obtain the bounds on pure imaginary powers, since with Navier type boundary condition, the Stokes operator actually reduces to Laplace operator and thus they could borrow the well-established theory for elliptic operators, which is not our case. Therefore we use the theory of interpolation-extrapolation to make use of the established theory for similar operators and implement a perturbation argument.

Theorem 3.5.1. *Let α be as in (3.18) and if $p \in (1,3]$ suppose also that $\alpha \in L^\infty(\Gamma)$. Then there exists an angle $0 < \theta < \pi/2$ and a constant $C > 0$ such that for any $s \in \mathbb{R}$,*

$$\|A_{p,\alpha}^{is}\| \leq C\ e^{|s|\theta}. \tag{3.44}$$

Similarly, for $\alpha \in L^\infty(\Gamma)$, there exists an angle $0 < \theta' < \pi/2$ and a constant $C' > 0$, such that for any $s \in \mathbb{R}$,

$$\|B_{p,\alpha}^{is}\| \leq C'\ e^{|s|\theta'}. \tag{3.45}$$

Proof. Since the proof of (3.45) is exactly similar to that of (3.44), we only show (3.44). The proof of (3.44) is based on the theory of interpolation-extrapolation scales from [8]. A similar approach has been followed in [101], considering the perturbation of a different operator than ours and for α constant.

1. Let us define $X_0 := \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $A_0 := \lambda I + A_{NT}$, for $\lambda > 0$ and where A_{NT} is the Stokes operator with Navier-type boundary condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma.$$

i.e.

$$\begin{cases} \mathbf{D}(A_{NT}) = \left\{\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)\cap\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), \mathbf{curl}\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\right\} \\ A_{NT}(\boldsymbol{u}) = -P(\Delta\boldsymbol{u}) \quad \text{for } \boldsymbol{u}\in\mathbf{D}(A_{NT}). \end{cases}$$

$$\mathbf{D}(\lambda I + A_p) = \mathbf{D}(A_p) \quad \text{for } \lambda > 0.$$

As indicated in the Introduction to this Section, the powers A_0^a of the operator A_0 are well, densely defined and closed linear operators on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ with domain $\mathbf{D}(A_0^a)$.

Now by [8, Theorems V.1.5.1 and V.1.5.4], (X_0, A_0) generates an interpolation-extrapolation scale $(X_a, A_a), a \in \mathbb{R}$ with respect to the complex interpolation functor since A_0 is a closed operator on X_0 with bounded inverse (cf. [5, Theorem 4.8]). More precisely, for every $a \in \mathbb{R}$, X_a is a Banach space, $X_a \hookrightarrow X_{a-1}$ and A_a is an unbounded linear operator on X_a with domain X_{a+1} and for $a > 0$:

(i) $X_a = (\mathbf{D}(A_0^a), \|A_0^a \cdot\|)$

(ii) A_a is the restriction of A_0 on X_a.

Moreover, for any $b \in (a, a+1)$,

$$X_b = [X_a, X_{a+1}]_\theta \quad \text{where} \quad \frac{1}{b} = \frac{1-\theta}{a} + \frac{\theta}{a+1}.$$

Similarly, let $X_0^\sharp := (X_0)' = \boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$, $A_0^\sharp := (A_0)'$. Then $(X_0^\sharp, A_0^\sharp)$ generates another interpolation-extrapolation scale $(X_a^\sharp, A_a^\sharp)$, the dual scale by [8, Theorem V.1.5.12] and

$$(X_a)' = X^\sharp_{-a} \quad \text{and} \quad (A_a)' = A^\sharp_{-a} \quad \text{for } a \in \mathbb{R}$$

where A' denotes the dual of A. In the particular case $a = -1/2$, we obtain by definition, an operator $A_{-1/2} : X_{-1/2} \to X_{-1/2}$ with

$$\mathbf{D}(A_{-1/2}) = X_{1/2} = [X_0, X_1]_{1/2}. \tag{3.46}$$

We now claim that

$$[X_0, X_1]_{1/2} = \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) \tag{3.47}$$

and then,

$$X_{-1/2} = \left[\boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)\right]' \tag{3.48}$$

will follow.

To prove (3.47), one inclusion is obvious. Indeed,

$$[X_0, X_1]_{1/2} \subset [L^p_{\sigma,\boldsymbol{\tau}}(\Omega), \boldsymbol{W}^{2,p}_{\sigma,\boldsymbol{\tau}}(\Omega)]_{1/2} = \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega).$$

And for the other inclusion, by (3.46) it is enough to prove that $\boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) \subset \mathbf{D}(A_0^{1/2})$. To this end, first consider the operator $(A_0^\sharp)^{1/2}$ on $\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$. Since A_0 has a bounded inverse,

$(A_0^\sharp)^{1/2}$ is an isomorphism from $\mathbf{D}((A_0^\sharp)^{1/2})$ to $\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$ [114, Theorem 1.15.2, part(e)] and thus, for any $\mathbf{F} \in \boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$, there exists a unique $\boldsymbol{v} \in \mathbf{D}((A_0^\sharp)^{1/2})$ such that $(A_0^\sharp)^{1/2}\boldsymbol{v} = \mathbf{F}$. So, for all $\boldsymbol{u} \in \mathbf{D}(A_0)$,

$$
\begin{aligned}
\|A_0^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)} = \sup_{\substack{\mathbf{F}\in \boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)\\ \mathbf{F}\neq \mathbf{0}}} \frac{\left|\left\langle A_0^{1/2}\boldsymbol{u}, \mathbf{F}\right\rangle\right|}{\|\mathbf{F}\|_{\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}} &= \sup_{\substack{\boldsymbol{v}\in \mathbf{D}(A_0^{1/2})\\ \boldsymbol{v}\neq \mathbf{0}}} \frac{\left|\left\langle A_0^{1/2}\boldsymbol{u}, (A_0^\sharp)^{\frac{1}{2}}\boldsymbol{v}\right\rangle\right|}{\|(A_0^\sharp)^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}} \\
&= \sup_{\substack{\boldsymbol{v}\in \mathbf{D}(A_0^{1/2})\\ \boldsymbol{v}\neq \mathbf{0}}} \frac{|\langle A_0\boldsymbol{u}, \boldsymbol{v}\rangle|}{\|(A_0^\sharp)^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}} \\
&= \sup_{\substack{\boldsymbol{v}\in \mathbf{D}(A_0^{1/2})\\ \boldsymbol{v}\neq \mathbf{0}}} \frac{|\int_\Omega \lambda \boldsymbol{u}\cdot\boldsymbol{v} + \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}|}{\|(A_0^\sharp)^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}_{\sigma,\boldsymbol{\tau}}(\Omega)}} \\
&\leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \qquad (3.49)
\end{aligned}
$$

Now as $\mathbf{D}(A_0)$ is dense in $\boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$, we get the inequality (3.49) for all $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$ which gives the required embedding.

Now from [6, Theorem 6.1], we know that there exist constants $M > 0$ and $\theta \in (0, \frac{\pi}{2})$ such that

$$\forall s \in \mathbb{R}, \ \ \|A_0^{is}\|_{\mathcal{L}(X_0)} \leq Me^{|s|\theta}.$$

It then follows from [8, Theorem V.1.5.5 (ii)] that

$$\forall s \in \mathbb{R}, \ \ \| \left(A_{-1/2}\right)^{is} \|_{\mathcal{L}(X_{-1/2})} \leq Me^{|s|\theta}.$$

We call the operator $A_{-1/2}$ the weak Stokes operator subject to Navier-type boundary condition. Since $A_{-1/2}$ is the closure of A_0 in $X_{-1/2}$ and $X_1 \hookrightarrow X_{1/2}$, it follows that $A_{-1/2}\boldsymbol{u} = A_0\boldsymbol{u}$ for $\boldsymbol{u} \in X_1$ and thus, for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\left\langle \boldsymbol{v}, A_{-1/2}\boldsymbol{u}\right\rangle_{(X_{-1/2})'\times X_{-1/2}} = \langle \boldsymbol{v}, A_0\boldsymbol{u}\rangle = \lambda \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}$$

where we only used integration by parts. Now using the density of X_1 in $X_{1/2}$, we obtain the relation, for all $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) \times \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\left\langle A_{-1/2}\boldsymbol{u}, \boldsymbol{v}\right\rangle = \lambda \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}. \qquad (3.50)$$

2. Next let us define an unbounded operator $A_{N,w}$ on $X_{-1/2}$, with domain $X_{1/2}$, as, for all $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega) \times \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\langle A_{N,w}\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + \langle \Lambda\boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma + \int_\Gamma \alpha\boldsymbol{u}\cdot\boldsymbol{v} \qquad (3.51)$$

where Λ is defined in (3.11). We call the operator $A_{N,w}$ the weak Stokes operator subject to Navier boundary conditions. Comparing (3.51) with (3.50) implies

$$\langle (\lambda I + A_{N,w})\, \boldsymbol{u}, \boldsymbol{v}\rangle = \left\langle A_{-1/2}\boldsymbol{u}, \boldsymbol{v}\right\rangle + \langle \Lambda_\alpha \boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma \tag{3.52}$$

where the linear operator $\Lambda_\alpha : X_{-1/2} \to X_{-1/2}$, given by,

$$\langle \Lambda_\alpha \boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma = \langle \Lambda \boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma + \int_\Gamma \alpha \boldsymbol{u}\cdot\boldsymbol{v}$$

is a lower order perturbation of $A_{-1/2}$. Therefore, as $\alpha \in L^\infty(\Gamma)$, it follows from [101, Proposition 3.3.9],

$$\forall s \in \mathbb{R}: \quad \|\, [(\lambda I + A_{N,w})]^{is}\, \|_{\mathcal{L}(X_{-1/2})} \le M e^{|s|\theta_A}$$

for some constant $\theta_A \in (0, \pi/2)$. Since, from Corollary 2.5.6, $A_{N,w}$ has a bounded inverse it follows from [101, Proposition 3.3.9] again, that

$$\|A^{is}_{N,w}\|_{\mathcal{L}(X_{-1/2})} \le M e^{|s|\theta_A}.$$

3. Now we want to transfer this 'bounded imaginary power' property to the strong Stokes operator $A_{p,\alpha}$ with Navier boundary condition, defined in (3.19)-(3.20) on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. For that we will apply again Amann's theory of interpolation-extrapolation scales. Let $X^w_0 := \left[\boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)\right]'$, $A^w_0 := A_{N,w}$ and $X^w_1 := \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$. By [8, Theorems V.1.5.1 and V.1.5.4], the pair (X^w_0, A^w_0) generates an interpolation-extrapolation scale $(X^w_a, A^w_a), a \in \mathbb{R}$ with respect to the complex interpolation functor and by [8, Theorem V.1.5.5 (ii)], for any $a \in \mathbb{R}$,

$$\forall s \in \mathbb{R}, \quad \|(A^w_a)^{is}\|_{\mathcal{L}(X^w_a)} \le M e^{|s|\theta_A}.$$

We will show in the remaining part of this proof that the operator $A^w_{1/2} : X^w_{3/2} \subset X^w_{1/2} \to X^w_{1/2}$ coincides with $A_{p,\alpha}$ where the strong Stokes operator $A_{p,\alpha} : \mathbf{D}(A_{p,\alpha}) \subset \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) \to \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ is defined in (3.19)-(3.20). Observe that, by (3.46), (3.48),

$$X^w_0 = X_{-1/2} \quad \text{and} \quad X^w_1 = X_{1/2}.$$

Therefore,

$$X^w_{1/2} = [X^w_0, X^w_1]_{1/2} = \left[X_{-1/2}, X_{1/2}\right]_{1/2} = X_0 = \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$$

and the operator $A^w_{1/2}$ is the restriction of A^w_0 on $X^w_{1/2}$. Hence, $A^w_{1/2}\boldsymbol{u} = A^w_0\boldsymbol{u} = A_{N,w}\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbf{D}(A^w_{1/2}) = X^w_{3/2}$ and then, for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\left\langle \boldsymbol{\varphi}, A^w_{1/2}\boldsymbol{u}\right\rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}} = \langle \boldsymbol{\varphi}, A_{N,w}\boldsymbol{u}\rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}} \tag{3.53}$$

$$= \int_{\Omega} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} + \langle \Lambda \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\Gamma} + \int_{\Gamma} \alpha \boldsymbol{u} \cdot \boldsymbol{\varphi}. \tag{3.54}$$

On the other hand, for any $(\boldsymbol{v}, \boldsymbol{\varphi}) \in \mathbf{D}(A_{p,\alpha}) \times \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$, it follows from integration by parts that

$$\langle \boldsymbol{\varphi}, A_{p,\alpha} \boldsymbol{v} \rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}} = \int_{\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} + \langle \Lambda \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Gamma} + \int_{\Gamma} \alpha \boldsymbol{v} \cdot \boldsymbol{\varphi}. \tag{3.55}$$

Now for any given $\boldsymbol{u} \in \mathbf{D}(A^w_{1/2})$, $A^w_{1/2} \boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and then there exists a unique $\boldsymbol{v} \in \mathbf{D}(A_{p,\alpha})$ such that

$$A_{p,\alpha} \boldsymbol{v} = A^w_{1/2} \boldsymbol{u}$$

since $A_{p,\alpha}$ is onto. Thus it follows from (3.53) that for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}_{\sigma,\boldsymbol{\tau}}(\Omega)$,

$$\langle \boldsymbol{\varphi}, A_{p,\alpha} \boldsymbol{v} \rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}} = \langle \boldsymbol{\varphi}, A_{N,w} \boldsymbol{u} \rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}}$$

This in turn implies by (3.54) and (3.55) that

$$\langle \boldsymbol{\varphi}, A_{N,w} \boldsymbol{u} \rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}} = \langle \boldsymbol{\varphi}, A_{N,w} \boldsymbol{v} \rangle_{\left(X^w_{1/2}\right)' \times X^w_{1/2}}.$$

Hence, $\boldsymbol{v} = \boldsymbol{u}$ by injectivity of $A_{N,w}$. Similarly, if $\boldsymbol{v} \in \mathbf{D}(A_{p,\alpha})$ is given, then there exists a unique $\boldsymbol{u} \in \mathbf{D}(A^w_{1/2})$ such that $A^w_{1/2} \boldsymbol{u} = A_{p,\alpha} \boldsymbol{v}$ since $A_{p,\alpha} \boldsymbol{v} \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $A^w_{1/2}$ is onto. By the same argument as above, we obtain $\boldsymbol{u} = \boldsymbol{v}$ showing that $\mathbf{D}(A_{p,\alpha}) = \mathbf{D}(A^w_{1/2})$ and $A_{p,\alpha} = A^w_{1/2}$. Thus finally we get that,

$$\forall s \in \mathbb{R}, \quad \| A^{is}_{p,\alpha} \|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega))} \leq M e^{|s| \theta_A}.$$

■

3.5.2 Fractional powers

The above result allows us to study the domains of $A^{\beta}_{p,\alpha}$, $\beta \in \mathbb{R}$. It can be shown that $\boldsymbol{D}(A^{\beta}_{p,\alpha})$ is a Banach space with the graph norm which is equivalent to the norm $\| A^{\beta}_{p,\alpha} \cdot \|_{\boldsymbol{L}^p(\Omega)}$, since $A_{p,\alpha}$ has bounded inverse. Note that for any $\beta \in \mathbb{R}$, the map $\boldsymbol{u} \to \| A^{\beta}_{p,\alpha} \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)}$ defines a norm on $\boldsymbol{D}(A^{\beta}_{p,\alpha})$ due to the injectivity of $A^{\beta}_{p,\alpha}$.

Theorem 3.5.2. *For all $p \in (1, \infty)$, $\mathbf{D}(A^{1/2}_{p,\alpha}) = \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$ with equivalent norms.*

Proof. Since the pure imaginary power of $A_{p,\alpha}$ is bounded and satisfies estimate (3.44), using the result [114, Theorem 1.15.3], we get that

$$\mathbf{D}(A^{1/2}_{p,\alpha}) = [\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), \mathbf{D}(A_{p,\alpha})]_{\frac{1}{2}}.$$

Then it is enough to show that

$$[\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), \mathbf{D}(A_{p,\alpha})]_{\frac{1}{2}} = \mathbf{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)$$

with equivalent norms, which is already proved in (3.47). ■

Remark 3.5.3. If Ω is not obtained by rotation around an axis i.e. if Ω is not axisymmetric, the norms $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ and $\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ are equivalent for $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ, as shown in Proposition 2.3.11. As a result we have the following equivalence for all $\boldsymbol{u} \in \mathbf{D}(A^{1/2}_{p,\alpha})$:

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|A^{1/2}_{p,\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Our next result is an embedding theorem of Sobolev type for domains of fractional powers which will be applied to deduce the so-called $L^p - L^q$ estimates for the solution of the evolutionary Stokes equation.

Theorem 3.5.4. *For all $1 < p < \infty$ and for all $\beta \in \mathbb{R}$ such that $0 < \beta < \frac{3}{2p}$, the following embedding holds :*

$$\mathbf{D}(A^{\beta}_{p,\alpha}) \hookrightarrow \boldsymbol{L}^q(\Omega) \quad \textit{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{2\beta}{3}.$$

Proof. First observe that for $0 \le \theta \le 1$, by the result [114, Theorem 1.15.3] and the estimate (3.44), we can write

$$\mathbf{D}(A^{\theta}_{p,\alpha}) = [\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), \mathbf{D}(\lambda I + A_{p,\alpha})]_\theta \hookrightarrow [\boldsymbol{L}^p(\Omega), \boldsymbol{W}^{2,p}(\Omega)]_\theta \hookrightarrow \boldsymbol{W}^{2\theta,p}(\Omega) \hookrightarrow \boldsymbol{L}^{\tilde{q}}(\Omega) \quad (3.56)$$

where

$$\frac{1}{\tilde{q}} = \frac{1}{p} - \frac{2\theta}{3} \quad \text{when} \quad p < \frac{3}{2\theta}.$$

Now since $\beta < \frac{3}{2}$, we write $\beta = \theta + k$ with $0 \le \theta < 1$ and $k = 0$ or 1. If $k = 0$, the result follows from (3.56). If $k = 1$, consider m large so that $\mathbf{D}(A^m_{p,\alpha}) \subset \mathbf{D}(A^\beta_{q,\alpha})$ where $\frac{1}{q} = \frac{1}{p} - \frac{2\beta}{3}$. If we set $\frac{1}{q_0} = \frac{1}{p} - \frac{2\theta}{3}$, then $\frac{1}{q} = \frac{1}{q_0} - \frac{2}{3}$ and $q_0 < \frac{3}{2}$ by assumptions on p and β. Hence as the consequence of the embedding (3.56), we get

$$\mathbf{D}(A^{\theta}_{p,\alpha}) \hookrightarrow \boldsymbol{L}^{q_0}(\Omega) \;\text{ and }\; \mathbf{D}(A_{q_0,\alpha}) \hookrightarrow \boldsymbol{L}^q(\Omega).$$

Thus it follows that for all $\boldsymbol{u} \in \mathbf{D}(A^m_{p,\alpha})$,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \le C\|A^k_{q_0,\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^{q_0}(\Omega)} \le C\|A^\beta_{p,\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally by density of $\mathbf{D}(A^m_{p,\alpha})$ in $\mathbf{D}(A^\beta_{p,\alpha})$ (since $\mathbf{D}(A^\beta_{q,\alpha}) \subset \mathbf{D}(A^\beta_{p,\alpha})$ by the definition of q), we complete the proof. ■

3.6 The homogeneous Stokes problem

In this section, with the help of the semigroup theory, we solve the homogeneous time dependent Stokes problem:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega \end{cases} \tag{3.57}$$

for which the analyticity of the semigroups, considered before give a unique solution satisfying the usual regularity.

3.6.1 Strong solution

We start with the strong solution of the problem (3.57).

Theorem 3.6.1. *Let $p \in (1,\infty)$ and α be as in (3.18). Then for $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$, the problem (3.57) has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0,\infty), \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap C((0,\infty), \mathbf{D}(A_{p,\alpha})) \cap C^1((0,\infty), \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \tag{3.58}$$

and if Ω is C^∞, then

$$\boldsymbol{u} \in C^k((0,\infty), \mathbf{D}(A^l_{p,\alpha})) \quad \forall\, k \in \mathbb{N}, \ \forall\, l \in \mathbb{N}\backslash\{0\}. \tag{3.59}$$

Also we have the estimates, for some constant $C > 0$ independent of α,

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{3.60}$$

and

$$\left\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.61}$$

Moreover, if α is a constant and either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then

$$\|\mathbb{D}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{\sqrt{t}} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{3.62}$$

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \frac{C}{t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{3.63}$$

and

$$\|\nabla \pi\|_{L^p(\Omega)} \leq \frac{C}{t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.64}$$

Proof. Since $-A_{p,\alpha}$ generates an analytic semigroup for every $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$, the initial value problem (3.57) has a unique solution $\boldsymbol{u}(t) = T(t)\boldsymbol{u}_0$, by [99, Corollary 1.5, Chapter 4, page 104]. Also, from [99, Theorem 7.7, Chapter 1, Page 30], we get that

$$\|T(t)\| \leq C \quad \text{for some constant } C > 0, \text{ independent of } \alpha.$$

As a result, we obtain the estimate (3.60). Also, with the help of [99, point (d), Theorem 5.2, Chapter 2], we get the estimate (3.61). To prove the estimate (3.62), we need to proceed as in the proof of (3.41), hence we skip it. The estimate (3.63) follows from (3.61) and using the fact that $\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|A_{p,\alpha}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ for $\boldsymbol{v} \in \mathbf{D}(A_{p,\alpha})$.

Further, using the usual regularity properties of semi group and by [99, Lemma 4.2, chapter 2], we can deduce the regularity (3.58) and (3.59). The estimate on the pressure term (3.64) can be deduced from the equation using (3.61) and (3.63). ■

The estimates (3.60), (3.61) and (3.62) allow us to deduce the following regularity result.

Corollary 3.6.2. *Let* $p \in (1, \infty)$ *and* α *be as in* (3.18). *Moreover,* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega), 0 < T < \infty$ *and* $(\boldsymbol{u}, \pi)$ *be the unique solution of problem* (3.57) *given by theorem 3.6.1. Then, for all* $1 \leq q < 2$*, we have,*

$$\boldsymbol{u} \in L^q(0,T; \boldsymbol{W}^{1,p}(\Omega)), \pi \in L^q(0,T; L^p_0(\Omega)) \text{ and } \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]').$$

Proof. Since we have the Korn inequality

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbb{D}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}$$

and $\boldsymbol{u}(t)$ satisfies the estimates (3.60) and (3.62), we get,

$$\|\boldsymbol{u}(t)\|^q_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + t^{-q/2})\|\boldsymbol{u}_0\|^q_{\boldsymbol{L}^p(\Omega)}$$

which implies $\boldsymbol{u} \in L^q(0,T; \boldsymbol{W}^{1,p}(\Omega))$ only for $1 \leq q < 2$ and for all $0 < T < \infty$.

Moreover, as the operator $B_{p,\alpha} : \boldsymbol{D}(B_{p,\alpha}) \to [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'$ is an isomorphism, we have the equivalence of norm, for any $\boldsymbol{v} \in \boldsymbol{D}(B_{p,\alpha})$, $\|B_{p,\alpha}\boldsymbol{v}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \simeq \|\boldsymbol{v}\|_{\boldsymbol{D}(B_{p,\alpha})}$ and here $B_{p,\alpha}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t}$. Thus $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]')$.

Finally from the equation $\nabla\pi = \Delta\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial t}$, the regularity of π follows. ■

Theorem 3.6.3. *Let* α *satisfy (3.18). Then for all* $p \leq q < \infty$ *and* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$*, there exists* $\delta > 0$ *such that the unique solution* $\boldsymbol{u}(t)$ *of the problem (3.57) belongs to* $\boldsymbol{L}^q(\Omega)$ *and satisfies, for all* $t > 0$ *:*

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\ e^{-\delta t} t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.65}$$

Moreover, the following estimates also hold

$$\|\mathbb{D}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\ e^{-\delta t}t^{-3/2(1/p-1/q)-1/2}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.66}$$

$$\forall\ m,n\in\mathbb{N},\quad \|\frac{\partial^m}{\partial t^m}A^n_{p,\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\ e^{-\delta t}t^{-(m+n)-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.67}$$

Note that all the above constants $C(\Omega,p)$ are independent of α.

Proof. First observe that in the case of $p=q$, the estimates (3.65), (3.66) and (3.67) follow from the classical semi group theory and the result that $\|T(t)\|\leq Me^{-\delta t}$ ([99, Theorem 6.13, Chapter 2]).

Suppose that $p\neq q$. Let $s\in\mathbb{R}$ such that $\frac{3}{2}(\frac{1}{p}-\frac{1}{q})<s<\frac{3}{2p}$ and set $\frac{1}{p_0}=\frac{1}{p}-\frac{2s}{3}$. It is clear that $p<q<p_0$. Since for all $t>0$ and for all $l\in\mathbb{R}^+$, $\boldsymbol{u}(t)\in\mathbf{D}(A^l_{p,\alpha})$, thanks to Theorem 3.5.4, $\boldsymbol{u}(t)\in\mathbf{D}(A^s_{p,\alpha})\hookrightarrow\boldsymbol{L}^{p_0}(\Omega)$. Now $\frac{1}{q}=\frac{\theta}{p_0}+\frac{1-\theta}{p}$ for $\theta=\frac{1/p-1/q}{1/p-1/p_0}\in(0,1)$. Thus $\boldsymbol{u}(t)\in\boldsymbol{L}^q(\Omega)$ and

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)}\leq C\|\boldsymbol{u}(t)\|^{\theta}_{\boldsymbol{L}^{p_0}(\Omega)}\|\boldsymbol{u}(t)\|^{1-\theta}_{\boldsymbol{L}^p(\Omega)} &\leq C\|A^s_{p,\alpha}T(t)\boldsymbol{u}_0\|^{\theta}_{\boldsymbol{L}^p(\Omega)}\|T(t)\boldsymbol{u}_0\|^{1-\theta}_{\boldsymbol{L}^p(\Omega)}\\ &\leq C\,e^{-\delta t}t^{-\theta s}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}\end{aligned}$$

where the last estimate follows from [99, Chapter 2, Theorem 6.13].

In order to prove (3.67) we first obtain from (3.58): $\frac{\partial^m}{\partial t^m}A^n_{p,\alpha}\boldsymbol{u}(t)\in\boldsymbol{L}^q(\Omega)$ for any $m,n\in\mathbb{N}$ and then

$$\|\frac{\partial^m}{\partial t^m}A^n_{p,\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)}=\|A^{(m+n)}_{p,\alpha}T(t)\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)}\leq Ce^{-\delta t}t^{-(m+n)-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

To prove estimate (3.66), we first deduce from (3.67) for $m=0$ and $n=1$:

$$\|A_{p,\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)}\leq Ce^{-\delta t}t^{-1-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{3.68}$$

Then, from Gagliardo Niremberg's inequality, we obtain:

$$\begin{aligned}\|\mathbb{D}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\ \|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,q}(\Omega)}\leq C\ \|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{2,q}(\Omega)}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{L}^q(\Omega)}\\ &\leq C\ \|\boldsymbol{u}(t)\|^{1/2}_{\mathbf{D}(A_{q,\alpha})}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{L}^q(\Omega)}\leq C\ \|A_{q,\alpha}\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{L}^q(\Omega)}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{L}^q(\Omega)}\end{aligned}$$

Thus (3.66) follows from (3.65) and (3.68). ■

Proof of Theorem 3.1.2. This essentially follows from Theorem 3.6.1 and Theorem 3.6.3. ■

3.6.2 Weak solution

The following result says that if the initial data is in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, we have the weak solution for the homogeneous problem (3.57). Here, as in Theorem 3.6.1, we use the analyticity of the semi group generated by the operator $B_{p,\alpha}$ and the fact that $\|T(t)\| \leq Me^{-\delta t}$.

Theorem 3.6.4. *Let $1 < p < \infty$ and $\alpha \in L^{t(p)}(\Gamma)$ where $t(p)$ defined in* (3.14)*. Then, for all $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, the problem* (3.57) *has a unique solution $\boldsymbol{u}(t)$ with the regularity*

$$\boldsymbol{u} \in C([0,\infty),[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]') \cap C((0,\infty),\mathbf{D}(B_{p,\alpha})) \cap C^1((0,\infty),[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$$

and

$$\boldsymbol{u} \in C^k((0,\infty),\mathbf{D}(B_{p,\alpha}^l)) \quad \forall\, k \in \mathbb{N},\ \forall\, l \in \mathbb{N}\backslash\{0\}.$$

Also there exists constants $C > 0$, independent of α and $\delta > 0$ such that for all $t > 0$,

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq Ce^{-\delta t}\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

and

$$\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C\ \frac{e^{-\delta t}}{t}\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

Moreover, if (i) either Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\ \frac{e^{-\delta t}}{t}\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

and

$$\|\nabla \pi\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C\ \frac{e^{-\delta t}}{t}\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

In the same way as we deduced in Corollary 3.6.2, we can have the following regularity result from Theorem 3.6.4.

Corollary 3.6.5. *Let $p \in (1,\infty)$ and $\alpha \in L^{t(p)}(\Gamma)$ where $t(p)$ defined in* (3.14)*. Moreover, suppose $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]', 0 < T < \infty$ and $(\boldsymbol{u},\pi)$ be the unique solution of problem* (3.57) *given by theorem 3.6.4. Then, for all $1 \leq q < 2$, we have*

$$\boldsymbol{u} \in L^q(0,T;\boldsymbol{L}^p(\Omega)).$$

Proof. We know the interpolation inequality

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

Now using the estimates in Theorem 3.6.4 and the fact that $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$ in the above inequality, we get the result. ∎

3.7 The non-homogeneous Stokes problem

Here we discuss the non-homogeneous Stokes problem:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \text{div } \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{3.69}$$

It is known that if $-\mathcal{A}$ generates a bounded analytic semigroup on a Banach space X, then we can construct a strong solution of

$$u' + \mathcal{A}u = f \ \text{ for a.e. } t \in (0,T), \quad u(0) = a \tag{3.70}$$

if f is Hölder continuous in time with values in X. But the analyticity of $e^{-t\mathcal{A}}$ is not sufficient to deduce the existence of solutions of (3.70) for general $f \in L^p(0,T;X)$ unless X is a Hilbert space. Therefore, we use the result on abstract Cauchy problem by Giga and Sohr [61, Theorem 2.3] which used the notion of ζ-convexity. For completeness, we recall the definition of ζ-convexity (see [23], also refer to [102]):

A Banach space X is said to be ζ-convex if there exists a symmetric biconvex function ζ on $X \times X$ such that $\zeta(0,0) > 0$ and

$$\zeta(x,y) \le \|x+y\| \quad \text{if} \quad \|x\| \le 1 \le \|y\|.$$

The concept of ζ-convexity is stronger than that of reflexivity. For application purpose, it is important to recall [61] that X is ζ-convex iff for some $1 < s < \infty$, the truncated Hilbert transform

$$(H_\varepsilon f)(t) = \frac{1}{\pi} \int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau, \quad f \in L^s(\mathbb{R}, X)$$

converges as $\varepsilon \to 0$ for almost all $t \in \mathbb{R}$ and there is a constant $C = C(s,x)$ independent of f such that

$$\|Hf\|_{L^s(\mathbb{R},X)} \le C\|f\|_{L^s(\mathbb{R},X)}$$

where $(Hf)(t) = \lim\limits_{\varepsilon \to 0}(H_\varepsilon f)(t)$.

The result in [61, Theorem 2.3] is useful in two senses : (i) it can be used even when $\mathcal{A}$ does not have a bounded inverse (though in our case, both A_p and B_p have bounded inverse) and (ii) the constant in the estimate is independent of time T, hence gives global in time results.

Here we introduce the notation for the space, for any $1 < p, q < \infty$,

$$D_{\mathcal{A}}^{\frac{1}{q},p} = \left\{ v \in X : \|v\|_{D_{\mathcal{A}}^{\frac{1}{q},p}} = \|v\|_X + \left(\int_0^\infty \|t^{1-\frac{1}{q}} \mathcal{A} e^{-t\mathcal{A}} v\|_X^p \frac{dt}{t} \right)^{1/p} < \infty \right\}$$

which actually agrees with the real interpolation space $(D(\mathcal{A}), X)_{1-1/q,p}$ when $e^{-t\mathcal{A}}$ is an analytic semigroup. First we deduce the strong solution of the Stokes system (3.69) and obtain $L^p - L^q$ estimates.

Proof of Theorem 3.1.3. Since $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ is ζ-convex [61, page 81] and $A_{p,\alpha}$ satisfies the estimate (3.44), all the assumptions of [61, Theorem 2.3] are fulfilled with $\mathcal{A} = A_{p,\alpha}$ and $X = \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. As a result, the regularity of $\boldsymbol{u}$ and $\frac{\partial \boldsymbol{u}}{\partial t}$ follow. The regularity of π comes from the fact that

$$\nabla\pi = \boldsymbol{f} - \frac{\partial \boldsymbol{u}}{\partial t} + \Delta\boldsymbol{u}. \tag{3.71}$$

Also we get the estimate

$$\int_0^T \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^q_{\boldsymbol{L}^p(\Omega)} dt + \int_0^T \|A_{p,\alpha}\boldsymbol{u}\|^q_{\boldsymbol{L}^p(\Omega)} dt \leq C\left(\int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt + \|\boldsymbol{u}_0\|^q_{D^{1-\frac{1}{q},q}_{A_{p,\alpha}}}\right)$$

which yields (3.7) using the fact that $A_{p,\alpha}\boldsymbol{u} = -\Delta\boldsymbol{u} + \nabla\pi$ and the fact that $\|A_{p,\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}$. ■

In the same way, using $\mathcal{A} = B_{p,\alpha}$ and $X = [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}}$ in [61, Theorem 2.3], since $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}}$ is ζ-convex [6, Proposition 2.16] and $B_{p,\alpha}$ satisfies the estimate on the pure imaginary power (3.45), we get the weak solution of the problem (3.69) with corresponding estimates as follows:

Theorem 3.7.1. *Let $0 < T \leq \infty, 1 < p, q < \infty$ and $\alpha \in L^{t(p)}(\Gamma)$ where $t(p)$ be defined in* (3.14)*. Then for every $\boldsymbol{f} \in L^q(0,T;[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}})$ and $\boldsymbol{u}_0 \in D^{1-\frac{1}{q},q}_{B_{p,\alpha}}$ there exists a unique solution $(\boldsymbol{u},\pi)$ of* (3.69) *satisfying the properties :*

$$\boldsymbol{u} \in L^q(0,T_0;\boldsymbol{W}^{1,p}(\Omega)) \quad \textit{for all} \quad T_0 \leq T \quad \textit{if} \quad T<\infty \quad \textit{and} \quad T_0 < \infty \quad \textit{if} \quad T = \infty,$$

$$\pi \in L^q(0,T;L^p_0(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T,[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\boldsymbol{\tau}}),$$

$$\int_0^T \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^q_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} dt + \int_0^T \|\boldsymbol{u}\|^q_{\boldsymbol{W}^{1,p}(\Omega)} dt + \int_0^T \|\pi\|^q_{L^p(\Omega)/\mathbb{R}} dt$$
$$\leq C\left(\int_0^T \|\boldsymbol{f}\|^q_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} dt + \|\boldsymbol{u}_0\|^q_{D^{1-\frac{1}{q},q}_{B_{p,\alpha}}}\right).$$

3.8 Nonlinear problem

In this section, we consider the initial value problem for the Navier-Stokes system with Navier boundary condition :

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{3.72}$$

The semigroup theory formulated in Section 3.3, 3.4 and 3.5 for the Stokes operator provides the necessary properties with which we can obtain some existence, uniqueness and regularity result for the non-linear problem as well. Here we want to employ the results of [58] for the abstract semilinear parabolic equation of the form

$$u_t + \mathcal{A}u = Fu, \quad u(0) = a \tag{3.73}$$

where Fu represents the nonlinear part and $\mathcal{A}$ is an elliptic operator. This abstract theory gives the existence of a local solution $u(t)$ for certain class of Fu. The solution can be extended globally also, provided the norm of the initial data is sufficiently small. Moreover, this solution belongs to $L^q(0,T;L^p)$ with suitably chosen p,q. Since, $u \in L^q(0,T;L^p)$ is equivalent of saying $\|u(t)\|_{L^p(\Omega)} \in L^q(0,T)$, this gives the asymptotic behaviour of $\|u(t)\|_{L^p(\Omega)}$ as $t \to 0$ and $t \to \infty$. Also we mention the interesting article [29] for a discussion on different types of solutions of incompressible Navier-Stokes equations with Dirichlet boundary condition and different approaches to obtain those.

To apply [58, Theorem 1 and Theorem 2], we need to verify the hypothesis therein, which we state below for convenience:

For a closed subspace E^p of $L^p(\Omega)$, let $P : L^p(\Omega) \to E^p$ be a continuous projection for $p \in (1,\infty)$ such that the restriction of P on $C_c(\Omega)$, the space of continuous functions with compact support, is independent of p and $C_c(\Omega) \cap E^p$ be dense in E^p. Let $e^{-t\mathcal{A}}$ be a strongly continuous operator on E^p for all $p \in (1,\infty)$. Also, there exists constants $n, m \geq 1$ such that for a fixed $T \in (0,\infty)$, the estimate

(A) $$\|e^{-t\mathcal{A}} f\|_{L^p(\Omega)} \leq M \|f\|_{L^s(\Omega)}/t^{\sigma}, \quad f \in E^s, \quad t \in (0,T)$$

holds with $\sigma = (\frac{1}{s} - \frac{1}{p})\frac{n}{m}$ for $p \geq s > 1$ and constant M depending only on p, s, T.

Moreover, let Fu be written as

$$Fu = LGu$$

where L is a closed, linear operator, densely defined from $L^p(\Omega)$ to E^q for some $q > 1$ such that for some $\gamma, 0 \leq \gamma < m$, the estimate

$$\textbf{(N1)} \qquad \|e^{-t\mathcal{A}} Lf\|_{L^p(\Omega)} \leq N_1 \|f\|_{L^p(\Omega)}/t^{\gamma/m}, \quad f \in E^p, \quad t \in (0,T)$$

holds with N_1 depending only on T and p, for all $p \in (1,\infty)$ and G is a nonlinear mapping from E^p to $L^h(\Omega)$ such that for some $\beta > 0$, the estimate

$$\textbf{(N2)} \qquad \|Gv - Gw\|_{L^h(\Omega)} \leq N_2 \|v-w\|_{L^p(\Omega)} \left(\|v\|^\beta_{L^p(\Omega)} + \|w\|^\beta_{L^p(\Omega)}\right), \quad G(0) = 0$$

holds with $1 \leq h = p/(1+\beta)$ and N_2 depending only on p, for all $p \in (1,\infty)$.

With these assumptions, the next Theorem follows directly from [58, Theorem 1 and Theorem 2].

Theorem 3.8.1. *For $\boldsymbol{u}_0 \in \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $\alpha \in W^{1-\frac{1}{r},r}(\Gamma), r \geq 3$, $\alpha \geq 0$, there exists $T_0 > 0$ and a unique solution $\boldsymbol{u}(t)$ of (3.72) on $[0, T_0)$ such that*

$$\boldsymbol{u} \in C([0,T_0); \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap L^q(0,T_0; \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \tag{3.74}$$

$$t^{1/q}\boldsymbol{u} \in C([0,T_0); \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \quad \text{and} \quad t^{1/q}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \to 0 \text{ as } t \to 0$$

with $\frac{2}{q} = \frac{3}{r} - \frac{3}{p}, p, q > r$. Moreover, there exists a constant $\varepsilon > 0$ such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)} < \varepsilon$, then T_0 can be taken as infinity for $r = 3$.

Let $(0, T_\star)$ be the maximal interval such that $\boldsymbol{u}$ solves (3.72) in $C((0,T_\star); \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)), r > 3$. Then

$$\|u(t)\|_{\boldsymbol{L}^r(\Omega)} \geq C(T_\star - t)^{(3-r)/2r}$$

where C is independent of $T_\star$ and t.

Proof. As our Stokes operator has all the same properties and estimates satisfied by the Stokes operator with Dirichlet boundary condition, we are exactly in the same set up as in [58] and hence the proof goes similar to that. However, we briefly review it for completeness.

Let E^p be $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $P : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ be the Helmholtz projection, defined in (3.17). It is trivial to see that P is independent of $p \in (1,\infty)$ on $C_c(\Omega)$ and $C_c(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. The Stokes operator $A_{p,\alpha}$ on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ is defined in (3.19)-(3.20) with dense domain and $-A_{p,\alpha}$ generates bounded analytic semigroup on $\boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$ for all $p \in (1,\infty)$ also (cf. Theorem 3.3.13). Applying P on both sides of the Navier-Stokes system (3.72) gives

$$\boldsymbol{u}_t + A_{p,\alpha}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0$$

which is obviously in the form (3.73) with $Fu = -P(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$.

We now need to verify the assumptions (**A**), (**N1**) and (**N2**). Since $-A_{p,\alpha}$ generates a bounded analytic semigroup with bounded inverse, we have (cf. [99, Chapter 2, Theorem 6.13])

$$\forall \boldsymbol{f} \in \boldsymbol{L}^s_{\sigma,\boldsymbol{\tau}}(\Omega), \qquad \|A^\sigma_{s,\alpha} e^{-tA_{s,\alpha}} \boldsymbol{f}\|_{\boldsymbol{L}^s(\Omega)} \leq M \|\boldsymbol{f}\|_{\boldsymbol{L}^s(\Omega)} / t^\sigma.$$

As $\boldsymbol{D}(A^\sigma_{s,\alpha})$ is continuously embedded in $\boldsymbol{W}^{2\sigma,s}(\Omega)$, this together with the Sobolev embedding theorem yields (**A**) with $m = 2, n = 3$.

Next we want to write the nonlinear term Fu. Since div $\boldsymbol{u} = 0$, we have $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_i = \sum_{j=1}^3 \nabla_j (u_j u_i)$. If we define $g : \mathbb{R}^3 \to \mathbb{R}^9$ by

$$(g(x))_{ij} = -x_i x_j$$

and $G : \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) \to (L^p(\Omega))^9$ by

$$G\boldsymbol{u}(x) = g(\boldsymbol{u}(x))$$

and $L : (L^p(\Omega))^9 \to \boldsymbol{L}^q_{\sigma,\boldsymbol{\tau}}(\Omega)$ by

$$Lg_{ij} = \sum_{j=1}^{3} P\nabla_j g_{ij}$$

which is a linear operator, it implies $Fu = LG\boldsymbol{u}$. Also it is easy to see from Hölder inequality that

$$|g(y) - g(z)| \leq N_2 |y - z|(|y| + |z|), \quad g(0) = 0$$

which gives in turn (**N2**) with $\beta = 1$.

Finally

$$\|e^{-tA_{p,\alpha}} L\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} = \|A^{1/2}_{p,\alpha} e^{-tA_p} A^{-1/2}_{p,\alpha} L\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{t^{1/2}} \|A^{-1/2}_{p,\alpha} L\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$$

and since $A^{-1/2}_{p,\alpha} L$ is bounded in $\boldsymbol{L}^p(\Omega)$ (cf. [59, Lemma 2.1]), the assumption (**N1**) is verified for $\gamma = 1$. This completes the proof. ∎

Next we show that the solution of (3.72) given in the above Theorem in the integral form is actually regular enough and satisfies (3.72).

Theorem 3.8.2. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega), r \geq 3$ and $\boldsymbol{u}(t)$ be the unique solution of (3.72) given by Theorem 3.8.1. Then*

$$\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_{r,\alpha})) \cap C^1((0, T_*]; \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega))$$

Proof. As in the previous Theorem, the proof follows exactly the same way as in the case of the Dirichlet boundary condition in [59]. Since the Stokes operator with Navier boundary condition, defined in (3.19)-(3.20), has all the same properties as the Stokes operator with Dirichlet boundary condition, [59, Theorem 2.5] gives (with $f = 0$) that $\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_{p,\alpha}))$. And $\boldsymbol{u} \in C^1((0, T_*]; \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega))$ follows from [47, Lemma 2.14] (with $f = 0$). ■

Next we show that regular solutions satisfy energy inequality provided the initial condition is in $\boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$.

Proposition 3.8.3. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$ and $\boldsymbol{u}$ be a regular solution of (3.72) on $(0, T_0)(T_0 < \infty)$ satisfying (3.74). Then*

$$\boldsymbol{u} \in L^\infty(0, T_0; \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap L^2(0, T_0; \boldsymbol{H}^1(\Omega))$$

and satisfies the energy equality

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}(t)|^2 + 2\int_0^t\int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_0^t\int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2.$$

Proof. The proof follows the same reasoning as in [58, Proposition 1, Section 5]. ■

3.9 The limit as $\alpha \to \infty$

Let us denote now u_α the solutions of the unsteady Stokes or Navier-Stokes equation with NBC for a given slip coefficient $\alpha \geq 0$ and a fixed initial data u_0. A very formal argument suggests that when $\alpha \to \infty$, we may expect that $u_\alpha \to u_\infty$ in some sense, where u_∞ is the solution of the same equation, with the same initial data, but with Dirichlet boundary condition. The existence of solutions u_∞ for such a problem, for a suitable set of initial data has been proved for example in [113, Theorem 1.1, Chapter III] for the Stokes equation and [113, Theorem 3.1, Chapter III] for the Navier Stokes equation.

This question has already been considered in [80] for Ω a two dimensional domain and $\frac{1}{\alpha} \in L^\infty(\Gamma)$. The author proves in Theorem 9.2 that when $\|\frac{1}{\alpha}\|_{L^\infty(\Gamma)} \to 0$ and $\boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$, the solution of problem (3.1)-(3.5) converges to the solution of the Navier-Stokes problem with Dirichlet boundary condition in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \dot{H}^1(\Omega)) \cap L^2(0, T; L^2(\Gamma))$ ([80, Theorem 9.2]).

Our results on this problem are based on the uniform estimates of the solutions with NBC with respect to the parameter α proved in the previous Sections. Then, we may only consider the case where the function α is a non negative constant. But, on the other hand, our convergence result in the Hilbert case, with the same rate of convergence as in [80] only needs the initial data to satisfy $\boldsymbol{u}_0 \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$ and we also obtain convergence results for the non Hilbert cases.

In the first two results of this Section, we prove that when α is a constant and the initial data is such that $\boldsymbol{u}_0 \in \mathbf{D}(A_{p,\alpha})$ for all α sufficiently large, the solutions of the Stokes equation with Navier boundary conditions (3.57) converge in the energy space to the solutions of the Stokes equation with Dirichlet boundary condition obtained in [57]. Moreover, we also obtain estimates on the rates of convergence.

Theorem 3.9.1. *Let* $\boldsymbol{u}_0 \in \boldsymbol{H}^1_0(\Omega)$ *with* $\operatorname{div} \boldsymbol{u}_0 = 0$ *in* Ω*,* $\alpha > 0$ *be a constant and* $T_\alpha(t) : \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega) \to \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$ *the semigroup generated by the Stokes operator* $A_{2,\alpha}$*, defined in (3.15)-(3.16). Then for any* $T < \infty$*,*

$$T_\alpha(t)\boldsymbol{u}_0 \to T_\infty(t)\boldsymbol{u}_0 \quad \textit{in} \quad L^2(0,T;\boldsymbol{H}^1(\Omega)) \quad \textit{as} \quad \alpha \to \infty \tag{3.75}$$

where $T_\infty(t)$ *is the semigroup generated by the Stokes operator with Dirichlet boundary condition [57]. Also we have,*

$$\int_0^t \int_\Omega |\mathbb{D}(T_\alpha(t)\boldsymbol{u}_0 - T_\infty(t)\boldsymbol{u}_0)|^2 + \int_0^T \int_\Gamma |T_\alpha(t)\boldsymbol{u}_0 - T_\infty(t)\boldsymbol{u}_0|^2 \leq \frac{C}{\alpha}. \tag{3.76}$$

Proof. **i)** Let us denote $\boldsymbol{u}_\alpha := T_\alpha(t)\boldsymbol{u}_0$. Then $\boldsymbol{u}_\alpha$ is the solution of the Stokes problem with Navier boundary condition (3.57), given by Theorem 3.6.1 where π_α is the associated pressure. So $\boldsymbol{u}_\alpha$ satisfies the following energy equality

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}_\alpha(T)|^2 + 2\int_0^T\int_\Omega |\mathbb{D}\boldsymbol{u}_\alpha|^2 + \alpha\int_0^T\int_\Gamma |\boldsymbol{u}_{\alpha\boldsymbol{\tau}}|^2 = \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2$$

which shows that as $\alpha \to \infty$,

$$\mathbb{D}\boldsymbol{u}_\alpha \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Omega))$$

and

$$\boldsymbol{u}_{\alpha\boldsymbol{\tau}} \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Gamma)).$$

Therefore, $\boldsymbol{u}_\alpha$ is bounded in $L^2(0,T;\boldsymbol{H}^1(\Omega))$. Hence π_α is also bounded in $L^2(0,T;\boldsymbol{L}^2(\Omega))$ since $\|B_2\boldsymbol{v}\|_{[\boldsymbol{H}^2_0(\operatorname{div},\Omega)]'} \simeq \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}$ for all $\boldsymbol{v} \in \mathbf{D}(B_2)$. This deduces that $\frac{\partial \boldsymbol{u}_\alpha}{\partial t}$ is as well bounded in $L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$. So there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in L^2(0,T;\boldsymbol{H}^1(\Omega)) \times L^2(0,T;L^2(\Omega))$ with $\frac{\partial \boldsymbol{u}_\infty}{\partial t} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$ such that up to a subsequence,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \text{ weakly in } L^2(0,T;\boldsymbol{H}^1(\Omega)) \times L^2(0,T;L^2(\Omega)) \quad \text{as } \alpha \to \infty$$

and

$$\frac{\partial \boldsymbol{u}_\alpha}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{u}_\infty}{\partial t} \text{ weakly in } L^2(0,T;\boldsymbol{H}^{-1}(\Omega)).$$

Also, by Aubin-Lions Lemma, we have

$$\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty \quad \text{in} \quad L^2(0,T;\boldsymbol{L}^{6-\varepsilon}(\Omega)) \text{ for any } \varepsilon > 0.$$

Next we claim that $(\boldsymbol{u}_\infty, \pi_\infty)$ satisfies the following Dirichlet problem

$$(3.77) \qquad \begin{cases} \dfrac{\partial \boldsymbol{u}_\infty}{\partial t} - \Delta \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega \times (0,T) \\ \boldsymbol{u}_\infty = \boldsymbol{0} & \text{on } \Gamma \times (0,T) \\ \boldsymbol{u}_\infty(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases}$$

Indeed, for any $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}^1_{0,\sigma}(\Omega))$ where we denote $\boldsymbol{H}^1_{0,\sigma}(\Omega) = \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$, the weak formulation satisfied by $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is,

$$(3.78) \qquad \int_0^T \left\langle \frac{\partial \boldsymbol{u}_\alpha}{\partial t}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}^1_0(\Omega)} + 2\int_0^T \int_\Omega \mathbb{D}\boldsymbol{u}_\alpha : \mathbb{D}\boldsymbol{v} = 0.$$

Then passing to the limit as $\alpha \to \infty$, we obtain

$$(3.79) \qquad \int_0^T \left\langle \frac{\partial \boldsymbol{u}_\infty}{\partial t}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}^1_0(\Omega)} + 2\int_0^T \int_\Omega \mathbb{D}\boldsymbol{u}_\infty : \mathbb{D}\boldsymbol{v} = 0.$$

Also writing the boundary condition satisfied by $\boldsymbol{u}_\alpha$ in the following way,

$$\boldsymbol{u}_{\alpha\boldsymbol{\tau}} = -\frac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$$

and since $[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$ is bounded in $C((0,T),\boldsymbol{H}^{1/2}(\Gamma))$ [due to (3.58)], passing to the limit as $\alpha \to \infty$ shows that $\boldsymbol{u}_\infty = \boldsymbol{0}$ on $\Gamma \times (0,T)$.

In order to show that $\boldsymbol{u}_\infty(0) = \boldsymbol{u}_0$, we can write from (3.78), for any $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}^1_{0,\sigma}(\Omega))$ with $\boldsymbol{v}(T) = 0$,

$$-\int_0^T \int_\Omega \boldsymbol{u}_\alpha \cdot \frac{\partial \boldsymbol{v}}{\partial t} + 2\int_0^T \int_\Omega \mathbb{D}\boldsymbol{u}_\alpha : \mathbb{D}\boldsymbol{v} = \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{v}(0)$$

and similarly from (3.79),

$$-\int_0^T \int_\Omega \boldsymbol{u}_\infty \cdot \frac{\partial \boldsymbol{v}}{\partial t} + 2\int_0^T \int_\Omega \mathbb{D}\boldsymbol{u}_\infty : \mathbb{D}\boldsymbol{v} = \int_\Omega \boldsymbol{u}_\infty(0) \cdot \boldsymbol{v}(0).$$

As $\boldsymbol{v}(0)$ is arbitrary, we thus conclude that $\boldsymbol{u}_\infty(0) = \boldsymbol{u}_0$. This proves the claim that $(\boldsymbol{u}_\infty, \pi_\infty)$ solves the Dirichlet problem (3.77).

ii) It remains to prove the estimates and the strong convergence of $(\boldsymbol{u}_\alpha, \pi_\alpha)$ to $(\boldsymbol{u}_\infty, \pi_\infty)$. Note that $(\boldsymbol{v}_\alpha, p_\alpha) := (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \pi_\alpha - \pi_\infty)$ satisfies the following problem

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_\alpha}{\partial t} - \Delta \boldsymbol{v}_\alpha + \nabla p_\alpha = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{v}_\alpha = 0 & \text{in } \Omega \times (0,T) \\ \boldsymbol{v}_\alpha \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{v}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\alpha\boldsymbol{\tau}} = -2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \times (0,T) \\ \boldsymbol{v}_\alpha(0) = \boldsymbol{0} & \text{in } \Omega. \end{cases} \tag{3.80}$$

Multiplying the above system by $\boldsymbol{v}_\alpha$, we obtain the following energy estimate, for any $T < \infty$,

$$\frac{1}{2}\int_\Omega |\boldsymbol{v}_\alpha(t)|^2 + 2\int_0^T\int_\Omega |\mathbb{D}\boldsymbol{v}_\alpha|^2 + \int_0^T\int_\Gamma \alpha |\boldsymbol{v}_{\alpha\boldsymbol{\tau}}|^2 = 2\int_0^T \langle [(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v}_{\alpha\boldsymbol{\tau}}\rangle_\Gamma . \tag{3.81}$$

Since $\boldsymbol{u}_0 \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^2_{\sigma,\boldsymbol{\tau}}(\Omega)$, then $\boldsymbol{u}_\infty \in L^2(0,T;\boldsymbol{H}^2(\Omega))$ (cf. [113, Proposition 1.2, Chapter III]) and thus the term in the right hand side can be estimated as

$$\int_0^T\int_\Gamma [(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\alpha\boldsymbol{\tau}} \le \|\boldsymbol{u}_\infty\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))}\|\boldsymbol{v}_\alpha\|_{L^2(0,T;\boldsymbol{L}^2(\Gamma))}.$$

As $\boldsymbol{v}_\alpha \to 0$ in $L^2(0,T;\boldsymbol{L}^2(\Gamma))$ (since $\boldsymbol{v}_\alpha \rightharpoonup 0$ in $L^2(0,T;\boldsymbol{H}^{\frac{1}{2}}(\Gamma))$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is compact in $\boldsymbol{L}^2(\Gamma)$), the above estimate shows that the rate of convergence of $\|\boldsymbol{v}_\alpha\|_{L^2(0,T;\boldsymbol{L}^2(\Gamma))}$ is $\frac{1}{\alpha}\|\boldsymbol{u}_\infty\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))}$. Hence the strong convergence result (3.75) and the estimate (3.76) follow. ■

In order to prove the convergence result for any $p \in (1,\infty)$, we need to use some compactness argument. But due to unavailability of the energy estimate for general $p \neq 2$, some more regularity of the solution of (3.57), hence more regular initial data is required.

Theorem 3.9.2. *Let $p \in (1,\infty)$, $T_\alpha(t), T_\infty(t)$ be defined as in Theorem 3.9.1 and $\boldsymbol{u}_0 \in \boldsymbol{W}^{2,p}_0(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. Then for any $T < \infty$, as $\alpha \to \infty$,*

$$T_\alpha(t)\boldsymbol{u}_0 \to T_\infty(t)\boldsymbol{u}_0 \quad \textit{in} \quad C([0,T];\mathbf{D}(A^{1/2}_{p,\alpha})) \tag{3.82}$$

and

$$T_\alpha(t)\boldsymbol{u}_0 \to T_\infty(t)\boldsymbol{u}_0 \quad \textit{in} \quad C^k(0,T;\mathbf{D}(A^l_{p,\alpha})) \quad \forall k \in \mathbb{N}, \ \forall l \in \mathbb{N}\setminus\{0\}. \tag{3.83}$$

Also if we assume $\boldsymbol{u}_0 \in \mathbf{D}(A^2_{p,\alpha})$, then the following convergence rate can be obtained, for any $1 < q < \infty$,

$$\|\frac{\partial}{\partial t}\left(T_\alpha(t)\boldsymbol{u}_0 - T_\infty(t)\boldsymbol{u}_0\right)\|_{L^q(0,T;\boldsymbol{L}^p(\Omega))} + \|T_\alpha(t)\boldsymbol{u}_0 - T_\infty(t)\boldsymbol{u}_0\|_{L^q(0,T;\boldsymbol{W}^{2,p}(\Omega))} \le \frac{C}{\alpha}. \tag{3.84}$$

Proof. First we claim the following set theoretic equality

$$\boldsymbol{W}_0^{2,p}(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) = \bigcap_{\alpha \geq 1} \mathbf{D}(A_{p,\alpha}).$$

Indeed, it is obvious to see that

$$\bigcap_{\alpha \geq 1} \mathbf{D}(A_{p,\alpha}) = \{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega) : [(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\}.$$

Let us denote the set on the right hand side of the above relation by E. It is then enough to show $E \subseteq \boldsymbol{W}_0^{2,p}(\Omega) \cap \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)$. To simplify, assuming $\Omega = \mathbb{R}^3_+$, we get that if $\boldsymbol{u} = \mathbf{0}$ on Γ, then $[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ iff $(\frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3}, 0) = \mathbf{0}$. Taking into account the fact that div $\boldsymbol{u} = 0$ in Ω, this implies $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \mathbf{0}$ on Γ. Hence the claim.

i) Since $\boldsymbol{u}_0 \in \mathbf{D}(A_{p,\alpha})$ for every α large enough,

$$\boldsymbol{u}_\alpha \in C([0,T]; \boldsymbol{W}^{2,p}_{\sigma,\boldsymbol{\tau}}(\Omega)) \cap C^1([0,T]; \boldsymbol{L}^p_{\sigma,\boldsymbol{\tau}}(\Omega)) \text{ is bounded as } \alpha \to \infty.$$

So by compactness, there exists $\boldsymbol{u}_\infty \in C[0,T; \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega))$ such that up to a subsequence,

$$\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty \text{ in } C([0,T]; \boldsymbol{W}^{1,p}_{\sigma,\boldsymbol{\tau}}(\Omega)).$$

Also by de Rham's theorem, there exists $\pi_\infty \in C([0,T]; L^p(\Omega))$ such that $\pi_\alpha \to \pi_\infty$ in $C([0,T]; L^p(\Omega))$. In fact, as $\boldsymbol{u}_\alpha \in C^k(0,T; \mathbf{D}(A_p^l))$ is bounded uniformly for any $k \in \mathbb{N}$ and $l \in \mathbb{N}\backslash\{0\}$, we get that $\boldsymbol{u}_\infty \in C^k(0,T; \mathbf{D}(A_p^l))$. We now claim that $(\boldsymbol{u}_\infty, \pi_\infty)$ satisfies the Stokes equation with Dirichlet boundary condition. Indeed if we write the system (3.57) as

$$\begin{cases} \dfrac{\partial \boldsymbol{u}_\alpha}{\partial t} - \Delta \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \mathbf{0}, \quad \text{div } \boldsymbol{u}_\alpha = 0 & \text{in } \Omega \times (0,T) \\ \boldsymbol{u}_\alpha \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u}_{\alpha\boldsymbol{\tau}} = -\dfrac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \times (0,T) \\ \boldsymbol{u}_\alpha(0) = \boldsymbol{u}_0 & \text{in } \Omega \end{cases} \tag{3.85}$$

passing the limit $\alpha \to \infty$, we obtain that $\boldsymbol{u}_\infty$ satisfies the system (3.77) which is given by $T_\infty(t)\boldsymbol{u}_0$, with associated pressure π_∞. Note that we use $\boldsymbol{u}_\alpha$ is continuous up to $t = 0$ to show that $\boldsymbol{u}_\infty$ satisfies the initial data. Thus, by the hypothesis on $\boldsymbol{u}_0$, the convergence result (3.82) and (3.83) follow.

ii) Next to deduce the rate of convergence, consider the difference between the systems (3.85) and (3.77). The system satisfied by $(\boldsymbol{v}_\alpha, p_\alpha) := (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \pi_\alpha - \pi_\infty)$ is therefore

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_\alpha}{\partial t} - \Delta \boldsymbol{v}_\alpha + \nabla p_\alpha = \mathbf{0}, \quad \text{div } \boldsymbol{v}_\alpha = 0 & \text{in } \Omega \times (0,T) \\ \boldsymbol{v}_\alpha \cdot \boldsymbol{n} = 0, \quad \boldsymbol{v}_{\alpha\boldsymbol{\tau}} = -\dfrac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \times (0,T) \\ \boldsymbol{v}_\alpha(0) = \mathbf{0} & \text{in } \Omega. \end{cases} \tag{3.86}$$

We then reduce the system (3.86) to a problem with homogeneous boundary data to apply the known estimates for Stokes problem for example, [61, Theorem 2.8]. For any $\alpha > 0$ and $t \geq 0$, let $(\boldsymbol{w}_\alpha(t), z_\alpha(t))$ satisfies the system

$$-\Delta \boldsymbol{w}_\alpha(t) + \nabla z_\alpha(t) = \mathbf{0}, \quad \text{div } \boldsymbol{w}_\alpha(t) = 0, \quad \boldsymbol{w}_\alpha(t)|_\Gamma = \boldsymbol{u}_\alpha|_\Gamma(t).$$

Since we have assumed $\boldsymbol{u}_0 \in \mathbf{D}(A^2_{p,\alpha})$, we get the improved regularity [82, Theorem 4.4.7, Chapter 4] $\boldsymbol{u}_\alpha \in C^1([0,\infty); \boldsymbol{W}^{2,p}_{\sigma,\boldsymbol{\tau}}(\Omega))$ which gives $\frac{\partial \boldsymbol{u}_\alpha}{\partial t}|_\Gamma(t) \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Omega)$ for all $t \geq 0$. Thus the regularity result for stationary Stokes system with Dirichlet boundary condition [31] yields that $\frac{\partial}{\partial t}\boldsymbol{w}_\alpha(t) \in \boldsymbol{W}^{2,p}(\Omega)$ for all $t \geq 0$. We also obtain the estimate

$$(3.87) \quad \|\boldsymbol{w}_\alpha\|_{C^1([0,T];\boldsymbol{W}^{2,p}(\Omega))} \leq C\|\boldsymbol{u}_\alpha\|_{C^1([0,T];\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma))} \leq \frac{C}{\alpha}\|[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}\|_{C^1([0,T];\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma))}.$$

Note that the above continuity constant C is independent of α. Then the substitution $V_\alpha := \boldsymbol{v}_\alpha - \boldsymbol{w}_\alpha$ reduces the system (3.86) to,

$$\begin{cases} \dfrac{\partial V_\alpha}{\partial t} - \Delta V_\alpha + \nabla p_\alpha = -\dfrac{\partial \boldsymbol{w}_\alpha}{\partial t}, \quad \text{div } V_\alpha = 0 & \text{in } \Omega \times (0,T) \\ V_\alpha = \mathbf{0} & \text{on } \Gamma \times (0,T) \\ V_\alpha(0) = -\boldsymbol{w}_\alpha(0) & \text{in } \Omega. \end{cases}$$

Hence the maximal regularity result [61, Theorem 2.8] leads us to,

$$V_\alpha \in L^q(0,T;\boldsymbol{W}^{2,p}_0(\Omega)) \quad \text{with} \quad \frac{\partial V_\alpha}{\partial t} \in L^q(0,T;\boldsymbol{L}^p_\sigma(\Omega)) \quad \text{for any } 1 < q < \infty$$

and the estimate

$$\int_0^T \left\|\frac{\partial V_\alpha}{\partial t}\right\|^q_{\boldsymbol{L}^p(\Omega)} + \int_0^T \|-\Delta V_\alpha + \nabla p_\alpha\|^q_{\boldsymbol{L}^p(\Omega)} \leq C\left(\int_0^T \|\frac{\partial \boldsymbol{w}_\alpha}{\partial t}\|^q_{\boldsymbol{L}^p_\sigma(\Omega)} + \|\boldsymbol{w}_\alpha(0)\|_{\boldsymbol{W}^{2,p}(\Omega)}\right)$$

with $C = C(\Omega,p,q) > 0$ independent of α. Therefore, together with (3.87), we obtain

$$\int_0^T \left\|\frac{\partial \boldsymbol{v}_\alpha}{\partial t}\right\|^q_{\boldsymbol{L}^p(\Omega)} + \int_0^T \|-\Delta \boldsymbol{v}_\alpha + \nabla p_\alpha\|^q_{\boldsymbol{L}^p(\Omega)} \leq \frac{C}{\alpha}$$

since $\|[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}\|_{C^1([0,T];\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma))}$ is bounded for all α large. This concludes the proof. ■

We prove now our two results on the convergence as $\alpha \to \infty$ of the solutions to the Navier Stokes equation and begin with the case where $\boldsymbol{u}_0$ belongs to an L^2-type space.

Proof of Theorem 3.1.4. We proceed as in the linear case.

i) From Proposition 3.8.3 we can see that as $\alpha \to \infty$,

$$\mathbb{D}\boldsymbol{u}_\alpha \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Omega))$$

and

$$\boldsymbol{u}_{\alpha\boldsymbol{\tau}} \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2(\Gamma)).$$

Therefore, $\boldsymbol{u}_\alpha$ is bounded in $L^2(0,T;\boldsymbol{H}^1(\Omega))$. And since $\|B_{2,\alpha}\boldsymbol{v}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} \simeq \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}$ for any $\boldsymbol{v} \in \mathbf{D}(B_{2,\alpha})$, π_α is as well bounded in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. This implies $\frac{\partial \boldsymbol{u}_\alpha}{\partial t}$ is also bounded in $L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$. Hence, there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in L^2(0,T;\boldsymbol{H}^1(\Omega)) \times L^2(0,T;L^2(\Omega))$ with $\frac{\partial \boldsymbol{u}_\infty}{\partial t} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$ such that up to a subsequence,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in} \quad L^2(0,T;\boldsymbol{H}^1(\Omega)) \times L^2(0,T;L^2(\Omega)) \quad \text{as } \alpha \to \infty$$

and

$$\frac{\partial \boldsymbol{u}_\alpha}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{u}_\infty}{\partial t} \quad \text{weakly in} \quad L^2(0,T;\boldsymbol{H}^{-1}(\Omega)).$$

Also, by Aubin-Lions compactness result,

$$\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty \quad \text{in} \quad L^2(0,T;\boldsymbol{L}^{6-\varepsilon}(\Omega)) \quad \text{for any} \quad \varepsilon > 0.$$

Next we show that $(\boldsymbol{u}_\infty, \pi_\infty)$ satisfies the Dirichlet problem (3.8). Indeed, for any $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}^1_{0,\sigma}(\Omega))$, the weak formulation of the problem (3.72) is, for $0 < t \leq T$,

$$\text{(3.88)} \qquad \int_0^t \left\langle \frac{\partial \boldsymbol{u}_\alpha}{\partial t}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} + 2\int_0^t \int_\Omega \mathbb{D}\boldsymbol{u}_\alpha : \mathbb{D}\boldsymbol{v} + \int_0^t \int_\Omega (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha \cdot \boldsymbol{v} = 0.$$

Then passing limit as $\alpha \to \infty$, we obtain

$$\text{(3.89)} \qquad \int_0^t \left\langle \frac{\partial \boldsymbol{u}_\infty}{\partial t}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)\times \boldsymbol{H}_0^1(\Omega)} + 2\int_0^t \int_\Omega \mathbb{D}\boldsymbol{u}_\infty : \mathbb{D}\boldsymbol{v} + \int_0^t \int_\Omega (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty \cdot \boldsymbol{v} = 0.$$

To pass to the limit in the non-linear term, we used the standard relation

$$\int_\Omega (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha \cdot \boldsymbol{v} = -\int_\Omega (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{u}_\alpha.$$

Also writing the boundary condition satisfied by $\boldsymbol{u}_\alpha$ in the following way,

$$\boldsymbol{u}_{\alpha\boldsymbol{\tau}} = -\frac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$$

and since $[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$ is bounded in $L^2(0,T;\boldsymbol{H}^{1/2}(\Gamma))$ [since $\boldsymbol{u}_0 \in \boldsymbol{H}^1_{\sigma,\boldsymbol{\tau}}(\Omega)$ implies $\boldsymbol{u}_\alpha$ is bounded in $L^2(0,T;\boldsymbol{H}^2(\Omega))$ (the deduction follows the proof as in [113, Theorem 3.11, Chapter 3], due to the fact that $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le C(\Omega)\|A_{2,\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)})$], passing to the limit as $\alpha \to \infty$ shows that $\boldsymbol{u}_\infty = \boldsymbol{0}$ on $\Gamma \times (0,T)$.

In order to prove $\boldsymbol{u}_\infty(0) = \boldsymbol{u}_0$, we write from (3.88), for any $\boldsymbol{v} \in C^1([0,T];\boldsymbol{H}^1_{0,\sigma}(\Omega))$ with $\boldsymbol{v}(T) = 0$,

$$-\int_0^T\int_\Omega \boldsymbol{u}_\alpha \cdot \frac{\partial \boldsymbol{v}}{\partial t} + 2\int_0^T\int_\Omega \mathbb{D}\boldsymbol{u}_\alpha : \mathbb{D}\boldsymbol{v} + \int_0^T\int_\Omega (\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{v}(0)$$

and similarly from (3.89),

$$-\int_0^T\int_\Omega \boldsymbol{u}_\infty \cdot \frac{\partial \boldsymbol{v}}{\partial t} + 2\int_0^T\int_\Omega \mathbb{D}\boldsymbol{u}_\infty : \mathbb{D}\boldsymbol{v} + \int_0^T\int_\Omega (\boldsymbol{u}_\infty\cdot\nabla)\boldsymbol{u}_\infty \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{u}_\infty(0) \cdot \boldsymbol{v}(0).$$

As $\boldsymbol{v}(0)$ is arbitrary, we thus conclude $\boldsymbol{u}_\infty(0) = \boldsymbol{u}_0$.

ii) To show the estimates and the strong convergence of $(\boldsymbol{u}_\alpha, \pi_\alpha)$ to $(\boldsymbol{u}_\infty, \pi_\infty)$, setting $\boldsymbol{v}_\alpha = \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty$ and $p_\alpha = \pi_\alpha - \pi_\infty$, it solve the following problem

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_\alpha}{\partial t} - \Delta\boldsymbol{v}_\alpha + (\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha - (\boldsymbol{u}_\infty\cdot\nabla)\boldsymbol{u}_\infty + \nabla p_\alpha = \boldsymbol{0}, \quad \operatorname{div}\,\boldsymbol{v}_\alpha = 0 & \text{in } \Omega\times(0,T) \\ \boldsymbol{v}_\alpha\cdot\boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{v}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{v}_{\alpha\boldsymbol{\tau}} = -2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma\times(0,T) \\ \boldsymbol{v}_\alpha(0) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying the above system by $\boldsymbol{v}_\alpha$ and integrating by parts over $\Omega\times(0,T)$, we deduce

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{v}_\alpha(t)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\int_0^t \|\mathbb{D}\boldsymbol{v}_\alpha\|^2_{\mathbb{L}^2(\Omega)} + \alpha\int_0^t \|\boldsymbol{v}_{\alpha\boldsymbol{\tau}}\|^2_{\boldsymbol{L}^2(\Gamma)} \\ &= -2\int_0^t \langle[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v}_{\alpha\boldsymbol{\tau}}\rangle_\Gamma - \int_0^t \langle(\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha - (\boldsymbol{u}_\infty\cdot\nabla)\boldsymbol{u}_\infty, \boldsymbol{v}_\alpha\rangle_\Omega. \end{aligned} \tag{3.90}$$

We estimate suitably the right hand side of (3.90). As $\boldsymbol{v}_\alpha \to \boldsymbol{0}$ in $L^2(0,T;\boldsymbol{L}^4(\Omega))$ and $\boldsymbol{u}_0 \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ implies $\boldsymbol{u}_\infty \in L^2(0,T;\boldsymbol{H}^2(\Omega)) \cap L^\infty(0,T;\boldsymbol{H}^1_{0,\sigma}(\Omega))$, thus

$$\begin{aligned} \int_0^t \langle(\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha - (\boldsymbol{u}_\infty\cdot\nabla)\boldsymbol{u}_\infty, \boldsymbol{v}_\alpha\rangle_\Omega &= -\int_0^t\int_\Omega (\boldsymbol{v}_\alpha\cdot\nabla)\boldsymbol{u}_\infty\cdot\boldsymbol{v}_\alpha \\ &\le \int_0^t \|\boldsymbol{v}_\alpha\|^2_{\boldsymbol{L}^4(\Omega)}\|\nabla\boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \|\boldsymbol{v}_\alpha\|^2_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{u}_\infty\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} \to 0. \end{aligned}$$

Similarly, since $\boldsymbol{v}_\alpha \to \mathbf{0}$ in $L^2(0,T;\boldsymbol{L}^2(\Gamma))$ (as $\boldsymbol{v}_\alpha \rightharpoonup \mathbf{0}$ in $L^2(0,T;\boldsymbol{H}^1(\Omega))$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is compact in $\boldsymbol{L}^2(\Gamma)$) and $[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \in L^2(0,T;\boldsymbol{L}^2(\Gamma))$,

$$\int_0^t \langle [(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v}_{\alpha\boldsymbol{\tau}}\rangle_\Gamma \leq C\|\boldsymbol{u}_\infty\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))}\|\boldsymbol{v}_\alpha\|_{L^2(0,T;\boldsymbol{L}^2(\Gamma))} \to 0.$$

Therefore, convergence in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and in $L^2(0,T;\boldsymbol{H}^1(\Omega))$ follow immediately, along with the estimate (3.9). The strong convergence for the pressure term follows from the equation. ■

When the initial data belongs to an L^r-type space with $r \geq 3$, we have the following:

Theorem 3.9.3. *Let $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the solution of the problem (3.72) with $\boldsymbol{u}_0 \in \boldsymbol{W}_0^{2,r}(\Omega) \cap \boldsymbol{L}^r_{\sigma,\boldsymbol{\tau}}(\Omega)$ where $r \geq 3$. Suppose also that $(\boldsymbol{u}_\infty, \pi_\infty) \in C([0,T];\boldsymbol{W}^{2,r}(\Omega)) \cap C^1([0,T];\boldsymbol{L}^r(\Omega)) \times C([0,T];\boldsymbol{W}^{1,r}(\Omega))$ is the solution of (3.8) with the same initial data $\boldsymbol{u}_0$ whose existence and uniqueness has been proved in [58, Theorem 4]. Then as $\alpha \to \infty$,*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad C([0,T);\boldsymbol{W}^{1,s}(\Omega)) \times C([0,T);L^s(\Omega)) \tag{3.91}$$

where $s \in [1,\infty)$ if $r = 3$ and $s = \infty$ if $r > 3$.

Moreover, if $\boldsymbol{u}_0 \in \mathbf{D}(A^2_{p,\alpha})$ for α sufficiently large, then we obtain the following rate of convergence, for any $m \in (1,\infty)$,

$$\|\frac{\partial}{\partial t}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{L^m(0,T;\boldsymbol{L}^s(\Omega))} + \|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|_{L^m(0,T;\boldsymbol{W}^{2,s}(\Omega))} \leq \frac{C}{\alpha}. \tag{3.92}$$

Proof. **i)** As explained in the beginning of the proof of Theorem 3.9.2, $\boldsymbol{u}_0 \in \mathbf{D}(A_{r,\alpha})$ for all α large enough and hence

$$\boldsymbol{u}_\alpha \quad \text{is bounded in} \quad C([0,T];\boldsymbol{W}^{2,r}(\Omega)) \cap C^1([0,T];\boldsymbol{L}^r(\Omega)).$$

Thus by compactness, there exists $\boldsymbol{u}_\infty \in C([0,T];\boldsymbol{W}^{1,s}(\Omega))$ with s as defined in the theorem such that, up to a subsequence,

$$\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty \text{ in } C([0,T];\boldsymbol{W}^{1,s}(\Omega)).$$

This implies by de Rham theorem, $\pi_\alpha \to \pi_\infty$ in $C([0,T];L^s(\Omega))$. Now to show that the limit $(\boldsymbol{u}_\infty, \pi_\infty)$ actually satisfies the Navier-Stokes problem with homogeneous Dirichlet boundary data, we write the system (3.72) in the following form

$$\begin{cases} \dfrac{\partial \boldsymbol{u}_\alpha}{\partial t} - \Delta\boldsymbol{u}_\alpha + (\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha + \nabla\pi_\alpha = \mathbf{0}, \quad \operatorname{div}\boldsymbol{u}_\alpha = 0 & \text{in } \Omega\times(0,T) \\ \boldsymbol{u}_\alpha\cdot\boldsymbol{n} = 0, \quad \boldsymbol{u}_{\alpha\boldsymbol{\tau}} = -\dfrac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma\times(0,T) \\ \boldsymbol{u}_\alpha(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{3.93}$$

Since $(\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty = (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \cdot \nabla \boldsymbol{u}_\alpha + \boldsymbol{u}_\infty \cdot \nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$, the regularity implies

$$\begin{aligned} &\|(\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty\|_{\boldsymbol{L}^s(\Omega)} \\ \leq\ & \|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\boldsymbol{L}^s(\Omega)} \\ \leq\ & \|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|_{\boldsymbol{W}^{1,s}(\Omega)} \left(\|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,s}(\Omega)} + \|\boldsymbol{u}_\infty\|_{\boldsymbol{W}^{1,s}(\Omega)} \right) \end{aligned}$$

which shows that $(\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha \to (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty$ in $C([0,T; \boldsymbol{L}^s(\Omega))$ as $\alpha \to \infty$. Hence, passing limit in the other terms of the above system yields that indeed $\boldsymbol{u}_\infty$ is a solution of the problem (3.8). Note that we use the continuity of $\boldsymbol{u}_\alpha$ up to $t = 0$ to obtain $\boldsymbol{u}_\infty$ satisfies the initial data.

Next to deduce the rate of convergence, taking difference between the two systems (3.93) and (3.8) and denoting by $\boldsymbol{v}_\alpha = \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty$ and $p_\alpha = \pi_\alpha - \pi_\infty$, we obtain

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_\alpha}{\partial t} - \Delta \boldsymbol{v}_\alpha + \nabla p_\alpha = (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty - (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha, \quad \operatorname{div} \boldsymbol{v}_\alpha = 0 & \text{in } \Omega \times (0,T) \\ \boldsymbol{v}_\alpha \cdot \boldsymbol{n} = 0, \quad \boldsymbol{v}_{\alpha\boldsymbol{\tau}} = -\dfrac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma \times (0,T) \\ \boldsymbol{v}_\alpha(0) = 0 & \text{in } \Omega. \end{cases}$$

But notice that $(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty - (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha \to \boldsymbol{0}$ in $L^m(0,T; \boldsymbol{L}^s(\Omega))$ for any $m \in [1, \infty)$ and $[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$ is bounded in $L^m(0,T; \boldsymbol{W}^{1-\frac{1}{s},s}(\Gamma))$. Therefore as we have done for the linear problem, using the lift operator and then the maximal regularity for the Stokes system with non-homogeneous initial data [61, Theorem 2.8] leads us to the following

$$\begin{aligned} &\int_0^T \left\| \frac{\partial \boldsymbol{v}_\alpha}{\partial t} \right\|^m_{\boldsymbol{L}^s(\Omega)} + \int_0^T \| - \Delta \boldsymbol{v}_\alpha + \nabla p_\alpha\|^m_{\boldsymbol{L}^s(\Omega)} \\ \leq C & \left(\int_0^T \|(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty - (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha\|^m_{\boldsymbol{L}^s(\Omega)} + \frac{1}{\alpha} \|[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}\|^m_{\boldsymbol{W}^{2-\frac{1}{s},s}(\Gamma)} \right). \end{aligned}$$

This finally shows that both the terms in the right hand side go to 0 as $\alpha \to \infty$. Hence the result. ■

CHAPTER 4

Uniform $W^{1,p}$ estimate for elliptic operator with Robin boundary condition in $\mathcal{C}^1$ domain

This is a joint work with Chérif Amrouche, Carlos Conca and Tuhin Ghosh.

Abstract : We consider the Robin boundary value problem $\operatorname{div}(A\nabla u) = \operatorname{div}\boldsymbol{f} + F$ in Ω, $\mathcal{C}^1$ domain, with $(A\nabla u - \boldsymbol{f})\cdot\boldsymbol{n} + \alpha u = g$ on Γ, where the matrix A belongs to $VMO(\mathbb{R}^3)$ and discover the uniform estimates on $\|u\|_{W^{1,p}(\Omega)}$, with $1 < p < \infty$, independent on α. With the contrast with the case $p = 2$, which is much simpler, we use here the weak reverse Hölder inequality. This estimate shows that the solution of Robin problem converges strongly to the solution of Dirichlet (resp. Neumann) problem in corresponding spaces when the parameter α tends to ∞ (resp. 0).

4.1 Introduction and statement of main result

This paper is concerned with the second order elliptic problem of divergence form with Robin boundary condition. In a bounded domain (open, connected set) Ω in $\mathbb{R}^n$ with $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$, consider the following problem

$$\begin{cases} \mathcal{L}u = \operatorname{div}\boldsymbol{f} + F & \text{in } \Omega, \\ (A\nabla u - \boldsymbol{f})\cdot\boldsymbol{n} + \alpha u = g & \text{on } \Gamma \end{cases} \tag{4.1}$$

where

$$\mathcal{L} = \operatorname{div}(A\nabla) \tag{4.2}$$

with $A(x) = (a_{ij}(x))$ is a 3×3 matrix with real-valued, bounded, measurable entries satisfying the following uniform ellipticity condition

$$\mu|\xi|^2 \leq A(x)\xi \cdot \xi \leq \frac{1}{\mu}|\xi|^2 \quad \text{for all} \quad \xi,\ x \in \mathbb{R}^n \quad \text{and some } \mu > 0.$$

Here $\boldsymbol{n}$ is the outward unit normal vector on the boundary.

We want to study the well-posedness of the problem (4.1), precisely, the existence, uniqueness of weak solution of (4.1) in $W^{1,p}(\Omega)$ for any $p \in (1, \infty)$ and the bound on the solution, uniform in α. Assuming $\alpha \geq 0$ a constant or a smooth function, the proof of existence of a unique solution provided $A \in VMO(\mathbb{R}^3)$ uses Neumann regularity results for elliptic problems; the interested reader is referred to [42] for details. The case $\alpha \leq 0$ corresponds to the Steklov eigen value problem (for a recent survey on this topic, see [63] and the references therein). That being said, our main interest is to obtain some precise estimate on the solution, in particular estimates uniform in α.

Note that, formally, $\alpha = \infty$ corresponds to the Dirichlet boundary condition whereas $\alpha = 0$ gives the Neumann boundary condition. In both Dirichlet and Neumann cases, we have the classical $W^{1,p}$ estimate of the solution. And so for the Robin problem as follows:

$$\|u\|_{W^{1,p}(\Omega)} \leq C(\alpha) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right),$$

where $C(\alpha)$ depends also on p and on Ω. Such well-posedness results on the Robin boundary value problem for arbitrary domains can be found, for example, in [39]. But the continuity constant depends on α whereas the constant in Dirichlet (and Neumann) estimate has no α. So it is natural to expect we may obtain α-independent bound of the solution of problem (4.1). That is, if we let α tend to ∞, we show rigorously that we get back the solution of the Dirichlet problem. The case when α goes to 0 is relatively easier to handle (though not trivial) assuming the compatibility condition of the Neumann problem.

Among the vast literature on Robin boundary value problem and various related questions to study, we did not find any reference concerning the question of *behavior of the solution on the parameter* α in the existing literature so far, even for Laplacian. Hence the purpose of this article is to address that issue, in particular to estimate the continuity constant $C(\alpha)$ uniformly with respect to α.

Here is our main result. Throughout this work, the following assumption on α will be considered which we do not mention each time:

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{and} \quad \alpha \geq \alpha_* > 0 \quad \text{on } \Gamma \tag{4.3}$$

where $t(p)$ defined by

$$\begin{cases} t(p) = 2 & \text{if} \quad p = 2 \\ t(p) = 2 + \varepsilon & \text{if} \quad \frac{3}{2} \le p \le 3, p \neq 2 \\ t(p) = \frac{2}{3}\max\{p, p'\} + \varepsilon & \text{otherwise} \end{cases}$$

where $\varepsilon > 0$ is arbitrary, satisfies $t(p) = t(p')$.

Also let $F \in L^{r(p)}(\Omega)$ where

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if} \quad p > \frac{3}{2} \\ \text{any arbitrary real number} > 1 & \text{if} \quad p = \frac{3}{2} \\ 1 & \text{if} \quad p < \frac{3}{2}. \end{cases}$$

Theorem 4.1.1. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \in (1, \infty)$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are symmetric and in $VMO(\mathbb{R}^3)$. Then the weak solution $u \in W^{1,p}(\Omega)$ of (4.1) satisfies the following estimate:*

$$\|u\|_{W^{1,p}(\Omega)} \le C_p(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right) \tag{4.4}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Notice that, with above estimate result, we immediately get that the solution of the Robin problem (4.1) converges strongly to the solution of Dirichlet boundary problem in the corresponding spaces as α goes to ∞. To prove the above theorem, we first obtain the result for $F = 0$, $g = 0$ and $p > 2$ and then for $p < 2$ using duality argument; And finally for $F \neq 0$, $g \neq 0$. Essentially we want to utilise the α-independent L^2 gradient estimate (which follows from the variational formulation) to yield L^p gradient estimate. The main tool in the proof for $p > 2$ is a weak reverse Hölder inequality (wRHI) for gradient satisfied by the solution of the homogeneous problem, shown in Theorem 4.2.11. Note that for Lipschitz domain, the weak reverse Hölder inequality is only true for certain values of p, even for the Dirichlet boundary condition. It was first proved by Giaquinta [53, Proposition 1.1, Chapter V] in the case of Dirichlet condition, on smooth domain and for Laplace operator which follows from an argument by Gehring [49]. The wRHI in the case of $B(x, r) \subset \Omega$ follows from the classical interior estimate for harmonic functions. But in the case when $x \in \Gamma$, some suitable boundary Hölder estimate is required. In the present paper, to treat the operator in divergence form with VMO-coefficients, we use an approximation argument from the constant coefficient operator case, found in [28]. In the case of Neumann problem and for general second order elliptic operator, the proof of wRHI has been done in [51, section 4] in Lipschitz domain; Whereas the sketch of the proof for Neumann problem in smooth domain has been given in [81, p. 914].

We obtain a similar result for H^s-bound (on Lipschitz domain) for $s \in (0, \frac{1}{2})$ in Theorem 4.2.17 and $W^{2,p}$-estimate (on $\mathcal{C}^{1,1}$ domain) in Theorem 4.3.1.

4.2 Related results and Proof of Theorem 4.1.1

To prove Theorem 4.1.1, we start with the definition of a weak solution and studying the existence result. Note that, we consider here only the case $n = 3$ for the sake of clarity but all the results are true for $n = 2$ as well and exactly the same proof follows with the necessary modifications.

Definition 4.2.1. Given $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$, we say $u \in W^{1,p}(\Omega)$ is a weak solution of (4.1) if it satisfies:

$$\forall \varphi \in W^{1,p'}(\Omega), \quad \int_\Omega A(x)\nabla u \cdot \nabla\varphi + \int_\Gamma \alpha u\, \varphi = \int_\Omega \boldsymbol{f} \cdot \nabla\varphi - \int_\Omega F\varphi + \langle g, \varphi\rangle_\Gamma \tag{4.5}$$

where $\langle \cdot, \cdot\rangle_\Gamma$ denotes the duality between $W^{-\frac{1}{p},p}(\Gamma)$ and $W^{\frac{1}{p},p'}(\Gamma)$.

Note that the boundary integral $\int_\Gamma \alpha u\, \varphi$ is well defined.

Theorem 4.2.2 (Existence result in $\boldsymbol{W^{1,p}(\Omega), p \geq 2}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $p \geq 2$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are symmetric and in $VMO(\mathbb{R}^3)$. Then for any $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $F \in L^{r(p)}(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$, there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of Problem (4.1).*

Remark 4.2.3. Note that for $p = 2$, Ω Lipschitz is sufficient to show the existence of solution $u \in H^1(\Omega)$.

Proof. For $p = 2$, the bilinear form

$$\forall\, u, \varphi \in H^1(\Omega), \quad a(u, \varphi) = \int_\Omega A(x)\nabla u \cdot \nabla\varphi + \int_\Gamma \alpha u\, \varphi$$

is clearly continuous. Also, due to the ellipticity hypothesis on $A(x)$ and by Friedrichs' inequality and the assumption $\alpha \geq \alpha_* > 0$ on Γ, we may have

$$a(u, u) = \int_\Omega A(x)\nabla u \cdot \nabla u + \int_\Gamma \alpha |u|^2 \geq C(\alpha_*)\ \|u\|^2_{H^1(\Omega)}$$

which shows that the bilinear form is coercive on $H^1(\Omega)$. And the right hand side of (4.5) defines an element in the dual of $H^1(\Omega)$. Thus, by Lax-Milgram lemma, there exists a unique $u \in H^1(\Omega)$ satisfying (4.5). So we obtain the existence of a unique weak solution of (4.1) in $H^1(\Omega)$.

Now for $p > 2$, since $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega), L^{r(p)}(\Omega) \hookrightarrow L^{6/5}(\Omega), W^{-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$ and $L^{t(p)}(\Gamma) \hookrightarrow L^2(\Gamma)$, there exists a unique $u \in H^1(\Omega)$ solving (4.1). It remains to show that $u \in W^{1,p}(\Omega)$.

(i) $2 < p \leq 3$**.** Since $u \in H^1(\Omega) \hookrightarrow L^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha u \in L^{q_1}(\Gamma)$ where $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But using the Sobolev embedding $L^{q_1}(\Gamma) \hookrightarrow W^{-\frac{1}{p_1},p_1}(\Gamma)$ with $p_1 = \frac{3}{2}q_1$ (since $q_1 > \frac{4}{3}$)

$$\text{i.e.} \quad \frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right),$$

Neumann regularity result (cf. [42, Theorem 5]) implies $u \in W^{1,p_1}(\Omega)$ since Ω is $\mathcal{C}^1$. If $p_1 \geq p$, we are done. Otherwise, $u \in W^{1,p_1}(\Omega)$. Hence, $u \in L^{s_1}(\Gamma)$ where

$$\frac{1}{s_1} = \frac{1}{p_1} - \frac{1 - \frac{1}{p_1}}{2} = \frac{3}{2p_1} - \frac{1}{2}$$

as $p_1 < p \leq 3$. Then $\alpha u \in L^{q_2}(\Gamma)$ where $\frac{1}{q_2} = \frac{1}{s_1} + \frac{1}{2+\varepsilon}$. But, $L^{q_2}(\Gamma) \hookrightarrow W^{-\frac{1}{p_2},p_2}(\Gamma)$ with $p_2 = \frac{3}{2}q_2$ *i.e.*

$$\frac{1}{p_2} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon} - \frac{1}{2} + \frac{1}{2+\varepsilon}\right) = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right).$$

If $p_2 \geq p$, then as before, we have $u \in W^{1,p}(\Omega)$. Otherwise, $u \in W^{1,p_2}(\Omega)$. Proceeding similarly, we get $u \in W^{1,p_{k+1}}(\Omega)$ with

$$\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right).$$

(where in each step, we assumed that $p_k < 3$). Now choosing $k = \lfloor \frac{1}{\varepsilon} - \frac{1}{2} \rfloor + 1$ such that $p_{k+1} \geq 3 \geq p$ (where $\lfloor a \rfloor$ stands for the greatest integer less than or equal to a), we obtain $u \in W^{1,p}(\Omega)$.

(ii) $p > 3$**.** From the previous case, we obtain $u \in W^{1,3}(\Omega)$ which gives $u \in L^q(\Gamma)$ for all $1 < q < \infty$. But $\alpha \in L^{\frac{2}{3}p+\varepsilon}(\Gamma)$ implies $\alpha u \in L^{\frac{2}{3}p}(\Gamma) \hookrightarrow W^{-\frac{1}{p},p}(\Gamma)$. Therefore, using same reasoning as before, from the Neumann regularity result, we get $u \in W^{1,p}(\Omega)$. ■

Remark 4.2.4. Note that in the above proof, we do not use explicitly the VMO property of A, but this is indeed necessary for the existence of solution of the Neumann problem.

Next we discuss the estimate of the solution of problem (4.1) for $p > 2$ with $F = 0$ and $g = 0$, independent of α.

Theorem 4.2.5 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate, $\boldsymbol{p \geq 2}$ with RHS $\boldsymbol{f}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \geq 2$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are symmetric and in $VMO(\mathbb{R}^3)$. Then the weak solution $u \in W^{1,p}(\Omega)$ of (4.1) with $F = 0$ and $g = 0$, satisfies the following estimate:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.6}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

The proof of the above theorem is very much similar to that of the Neumann problem [51], once we have the wRHI. Since Ω is $\mathcal{C}^1$, there exists some $r_0 > 0$ such that for any $x_0 \in \Gamma$, there exists a coordinate system (x', x_3) which is isometric to the usual coordinate system and a $\mathcal{C}^1$ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ so that,

$$B(x_0, r_0) \cap \Omega = \{(x', x_3) \in B(x_0, r_0) : x_3 > \psi(x')\}$$

and

$$B(x_0, r_0) \cap \Gamma = \{(x', x_3) \in B(x_0, r_0) : x_3 = \psi(x')\}.$$

In some places, we may write B instead of $B(x, r)$ where there is no ambiguity and $aB := B(x, ar)$ for $a > 0$.

We first prove the following weak reverse Hölder inequality for some $p = 2 + \varepsilon$, $\varepsilon > 0$ whose proof is straight forward but this is not sufficient to deduce Theorem 4.2.5.

Lemma 4.2.6. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $\mathcal{L}$ be the operator defined in (4.2) with constant coefficients. For any $B(x, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x, 2r) \subset \Omega$ or $x \in \Gamma$, the following weak Reverse Hölder inequalities hold: for some $\varepsilon > 0$,*
(i) if $B(x, 2r) \subset \Omega$,

$$\left(\frac{1}{r^3} \int_{B(x,r)} |\nabla v|^{2+\varepsilon}\right)^{1/2+\varepsilon} \leq C \left(\frac{1}{r^3} \int_{B(x,2r)} |\nabla v|^2\right)^{1/2}$$

whenever $v \in H^1(B(x, 2r))$ satisfies $\mathcal{L}v = 0$ in $B(x, 2r)$.
(ii) if $x \in \Gamma$,

$$\left(\frac{1}{r^3} \int_{B(x,r)\cap\Omega} (|v|^2 + |\nabla v|^2)^{\frac{2+\varepsilon}{2}}\right)^{1/2+\varepsilon} \leq C \left(\frac{1}{r^3} \int_{B(x,2r)\cap\Omega} |v|^2 + |\nabla v|^2\right)^{1/2}$$

whenever $v \in H^1(B(x,2r)\cap\Omega)$ *satisfying*

$$\begin{cases} \mathcal{L}v & = 0 \quad \text{in} \quad B(x,2r)\cap\Omega \\ A\nabla v\cdot \boldsymbol{n} + \alpha v & = 0 \quad \text{on} \quad B(x,2r)\cap\Gamma. \end{cases} \tag{4.7}$$

The constants $C>0$ *in the above estimates are independent of* α.

Proof. The proof of the weak Reverse Hölder inequality for Robin problem follows the similar argument as for the Dirichlet problem, established in [53].

case(i) : $2B\subset\Omega$.

Since v satisfies the equation $\operatorname{div}(A\nabla)v = 0$ in $2B$, we can have the following Caccioppoli inequality,

$$\int_B |\nabla v|^2 \le \frac{C}{r^2}\int_{2B}|v-\bar{v}|^2, \qquad \bar{v} = \frac{1}{|2B|}\int_{2B} v$$

for some constant $C>0$ independent of α. Now using the following Sobolev-Poincaré inequality, for any $v\in W^{1,p}(\Omega)$, $p>1$,

$$\|v-\bar{v}\|_{L^{p^*}(\Omega)} \le C\|\nabla v\|_{L^p(\Omega)}, \qquad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v$$

where p^* is the Sobolev exponent, we obtain,

$$\int_B |\nabla v|^2 \le \frac{C}{r^2}\left(\int_{2B}|\nabla v|^{\tilde{q}}\right)^{2/\tilde{q}}$$

with $\tilde{q} = 6/5$ (this value comes from the dimension $n=3$). Upon normalizing both sides, we can write,

$$\left(\frac{1}{r^3}\int_B |\nabla v|^2\right)^{1/2} \le C\left(\frac{1}{r^3}\int_{2B}|\nabla v|^{\tilde{q}}\right)^{1/\tilde{q}}.$$

Here note that in $\mathbb{R}^3$, $|B| = cr^3$. Then setting $g = |\nabla v|^{\tilde{q}}$ and $q = 5/3 = 2/\tilde{q}$, we have,

$$\frac{1}{r^3}\int_B g^q \le C\left(\frac{1}{r^3}\int_{2B} g\right)^q.$$

Hence, [53, Proposition 1.1] with $f=0$ and $\theta=0$ implies, for some $\varepsilon>0$,

$$\left(\frac{1}{r^3}\int_B |\nabla v|^{2+\varepsilon}\right)^{1/2+\varepsilon} \le C\left(\frac{1}{r^3}\int_{2B}|\nabla v|^2\right)^{1/2}$$

which completes the proof in the first case.

case(ii) : $x \in \Gamma$.

The proof is very much similar to the above interior estimate. First we want to prove a Caccioppoli type inequality for the problem (4.7) up to the boundary. For that, let $\eta \in C_c^\infty(2B)$ be a cut-off function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B \quad \text{and} \quad |\nabla \eta| \le \frac{C}{r}.$$

Now multiplying (4.7) by $\eta^2 v$ and integrating by parts, we get,

$$\int_{2B\cap\Omega} A\nabla v \cdot \nabla(\eta^2 v) + \int_{\partial(2B\cap\Omega)} \alpha \eta^2 v^2 = 0$$

which yields,

$$\mu \int_{2B\cap\Omega} \eta^2 |\nabla v|^2 + \int_{2B\cap\Gamma} \alpha\eta^2 v^2 \le \int_{2B\cap\Omega} \eta^2 A\nabla v \cdot \nabla v + \int_{2B\cap\Gamma} \alpha\eta^2 v^2 = -2 \int_{2B\cap\Omega} \eta v\, A\nabla v \cdot \nabla \eta.$$

Using Cauchy's inequality on the right hand side, we obtain,

$$\int_{2B\cap\Omega} |\nabla v|^2 \eta^2 + \int_{2B\cap\Gamma} \alpha\eta^2 v^2 \le C \left[\frac{1}{4} \int_{2B\cap\Omega} \eta^2 |\nabla v|^2 + 4 \int_{2B\cap\Omega} v^2 |\nabla \eta|^2 \right].$$

The above constant C depends on A only. Simplifying the above estimate gives

$$\int_{2B\cap\Omega} |\nabla v|^2 \eta^2 + \int_{2B\cap\Gamma} \alpha\eta^2 v^2 \le C \int_{2B\cap\Omega} v^2 |\nabla \eta|^2,$$

which yields the Caccioppoli-type inequality, up to the boundary,

$$\int_{B\cap\Omega} |\nabla v|^2 + \int_{B\cap\Gamma} \alpha v^2 \le \int_{2B\cap\Omega} |\nabla v|^2 \eta^2 + \int_{2B\cap\Gamma} \alpha\eta^2 v^2 \le \frac{C}{r^2} \int_{2B\cap\Omega} v^2. \tag{4.8}$$

But we also have,

$$\|v\|^2_{H^1(B\cap\Omega)} \le C \left(\int_{B\cap\Omega} |\nabla v|^2 + \int_{B\cap\Gamma} v^2 \right) \le C(\alpha_*) \left(\int_{B\cap\Omega} |\nabla v|^2 + \int_{B\cap\Gamma} \alpha v^2 \right).$$

Hence, using (4.8), we obtain,

$$\int_{B\cap\Omega} (|v|^2 + |\nabla v|^2) \le \frac{C(\alpha_*)}{r^2} \int_{2B\cap\Omega} |v|^2 \le \frac{C(\alpha_*)}{r^2} \left(\int_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2} \right)^{2/\tilde{q}}$$

with $\tilde{q} = 6/5$ so that $(\tilde{q})^* = 2$. Thus,

$$\frac{1}{r^3}\int_{B\cap\Omega} (|v|^2 + |\nabla v|^2) \le \frac{C(\alpha_*)}{r^5}\left(\int_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2}\right)^{2/\tilde{q}}$$
$$= C(\alpha_*)\left(\frac{1}{r^3}\int_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2}\right)^{2/\tilde{q}}.$$

Now if we set,

$$g(y) = \begin{cases} (|v|^2 + |\nabla v|^2)^{\tilde{q}/2} & \text{if } \ y \in 2B \cap \Omega \\ 0 & \text{if } \ y \in 2B \setminus \Omega \end{cases}$$

and $q = 2/\tilde{q}$, we obtain,

$$\frac{1}{r^3}\int_B g^q \le C(\alpha_*)\left(\frac{1}{r^3}\int_{2B} g\right)^q.$$

Once again [53, Proposition 1.1] with $f = 0$ and $\theta = 0$ implies, for some $\varepsilon > 0$,

$$\left(\frac{1}{r^3}\int_B g^{q+\varepsilon}\right)^{1/q+\varepsilon} \le C\left(\frac{1}{r^3}\int_{2B} g^q\right)^{1/q}$$

i.e.

$$\left(\frac{1}{r^3}\int_{B\cap\Omega} (|v|^2 + |\nabla v|^2)^{(q+\varepsilon)\tilde{q}/2}\right)^{1/q+\varepsilon} \le C\left(\frac{1}{r^3}\int_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)\right)^{\tilde{q}/2}$$

or equivalently, for some $s > 2$,

$$\left(\frac{1}{r^3}\int_{B\cap\Omega} (|v|^2 + |\nabla v|^2)^{s/2}\right)^{1/s} \le C\left(\frac{1}{r^3}\int_{2B\cap\Omega} (|v|^2 + |\nabla v|^2)\right)^{1/2}$$

which finishes the proof. ■

Next to prove wRHI for all $p > 2$, we require the following boundary Hölder estimate for $\mathcal{L}$ under Robin boundary condition.

Theorem 4.2.7. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p > 1$ and $\gamma \in (0,1)$. Suppose that the operator $\mathcal{L}$ defined in (4.2) has constant and symmetric coefficients and*

$$\begin{cases} \mathcal{L}v & = 0 \quad \text{in } \ B(Q,r)\cap\Omega \\ A\nabla v\cdot \boldsymbol{n} + \alpha v & = 0 \quad \text{on } \ B(Q,r)\cap\Gamma \end{cases}$$

for some $Q \in \Gamma$ and $0 < r < r_0$, Then for any $x, y \in B(Q, r/2) \cap \Omega$,

$$|v(x) - v(y)| \leq C \left(\frac{|x-y|}{r} \right)^{\gamma} \left(\fint_{B(Q,r)\cap\Omega} |v|^p \right)^{1/p} \tag{4.9}$$

where $C > 0$ depends only on Ω, p and the ellipticity constant μ, but independent of α.

Proof. Follows from classical regularity theory (for example, see [62, Theorem 8.27]). ■

Now the weak reverse Hölder inequality for any $p > 2$ is proved in the case of constant coefficients.

Lemma 4.2.8. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $p \geq 2$. Suppose that $\mathcal{L}$, defined in (4.2), has constant and symmetric coefficients. Then for any $B(x, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x, 2r) \subset \Omega$ or $x \in \Gamma$, the following weak Reverse Hölder inequalities hold:*
(i) if $B(x, 2r) \subset \Omega$,

$$\left(\fint_{B(x,r)} |\nabla v|^p \right)^{1/p} \leq C \left(\fint_{B(x,2r)} |\nabla v|^2 \right)^{1/2} \tag{4.10}$$

whenever $v \in H^1(B(x, 2r))$ satisfies $\mathcal{L}v = 0$ in $B(x, 2r)$.
(ii) if $x \in \Gamma$,

$$\left(\fint_{B(x,r)\cap\Omega} |\nabla v|^p + |v|^p \right)^{1/p} \leq C \left(\fint_{B(x,2r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2} \tag{4.11}$$

whenever $v \in H^1(B(x, 2r) \cap \Omega)$ satisfies

$$\begin{cases} \mathcal{L}v & = 0 \quad \text{in } \; B(x, 2r) \cap \Omega \\ A\nabla v \cdot \boldsymbol{n} + \alpha v & = 0 \quad \text{on } \; B(x, 2r) \cap \Gamma(\textit{ if } x \in \Gamma). \end{cases}$$

The constant $C > 0$ at most depends on Ω, p and the ellipticity constant μ.

Proof. Since A is symmetric and positive definite, by a change of coordinate system, we may assume that $\mathcal{L} = \Delta$ (although we may consider the full operator and all the results hold true as well).

The proof we will follow has been used for elliptic equations with Neumann boundary condition in [81], just after the statement of Theorem 4.1.

case(i) : $B(x_0, 2r) \subset \Omega$.

The weak reverse Hölder inequality (4.10) holds for any $p \geq 2$, by the following well-known interior estimates for harmonic functions, even when Ω is Lipschitz:

$$\sup_{B(x_0,r)} |\nabla v| \leq C \left(\fint_{B(x_0,2r)} |\nabla v|^2 \right)^{1/2}.$$

case(ii) : $x_0 \in \Gamma$.

From the interior gradient estimate for harmonic function, we can write (e.g. see [66, Lemma 1.10])

$$|\nabla v(x)| \leq \frac{3}{\delta(x)} \sup_{B(x,c\delta(x))} |v|$$

for any $x \in B(x_0, r) \cap \Omega$ where $\delta(x) = d(x, \Gamma)$ and $c > 0$ is chosen such that $B(x, 2c\delta(x)) \subsetneq B(x_0, 2r) \cap \Omega$. From [66, Remark 1.19], we may then write

$$|\nabla v(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |v|^2 \right)^{1/2}.$$

Now for fixed $y \in B(x, 2c\delta(x))$, let $u(x) = v(x) - v(y)$. Then $\mathcal{L}u = 0$ in $B(x, 2c\delta(x))$ and thus we may write from the above argument,

$$|\nabla u(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |u|^2 \right)^{1/2}$$

which gives, along with the boundary Hölder estimate (4.9),

$$\begin{aligned}
|\nabla v(x)| &\leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |v(z) - v(y)|^2 \mathrm{d}z \right)^{1/2} \\
&= \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\int_{B(x,c\delta(x))} |v(z) - v(y)|^2 \mathrm{d}z \right)^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\int_{B(x,2c\delta(x))} \left(\frac{|z-y|}{r} \right)^{2\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |v|^2 \right) \mathrm{d}z \right)^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\fint_{B(x_0,2r)\cap\Omega} |v|^2 \right)^{1/2} \frac{1}{r^\gamma} \left(\int_{B(x,2c\delta(x))} |z-y|^{2\gamma} \mathrm{d}z \right)^{1/2}
\end{aligned}$$

$$\le \frac{C_\gamma}{(\delta(x))^{1+\frac{3}{2}}} \left(\fint_{B(x_0,2r)\cap\Omega} |v|^2 \right)^{1/2} \frac{1}{r^\gamma} (\delta(x))^{\gamma+\frac{3}{2}}$$

$$= C_\gamma \frac{(\delta(x))^{\gamma-1}}{r^\gamma} r^{-3/2} \left(\int_{B(x_0,2r)\cap\Omega} |v|^2 \right)^{1/2}$$

$$\le C_\gamma \frac{(\delta(x))^{\gamma-1}}{r^\gamma} r^{1-3/2} \left(\int_{B(x_0,2r)\cap\Omega} |v|^6 \right)^{1/6} \le C_\gamma \left(\frac{r}{\delta(x)} \right)^{1-\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2}.$$

We have used Sobolev estimate in the last inequality. Since $\gamma \in (0,1)$ is arbitrary, we thus have,

$$|\nabla v(x)| \le C_\gamma \left(\frac{r}{\delta(x)} \right)^\gamma \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2}.$$

Finally it yields choosing γ so that $p\gamma < 1$,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla v|^p \right)^{1/p} \le C_p \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2}.$$

This completes the proof. ∎

A function f is said to be in the space $BMO(\mathbb{R}^n)$ (bounded mean oscillation) if $f \in L^1_{loc}(\mathbb{R}^n)$ satisfies

$$\sup_{x\in\mathbb{R}^n} \frac{1}{r^n} \int_{B(x,r)} |f - \overline{f}| < \infty$$

where $\overline{f} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f$.

Moreover a function f in $BMO(\mathbb{R}^n)$ is said to be in $VMO(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{r^n} \int_{B(x,r)} |f - \overline{f}\,| \, \mathrm{d}x = 0.$$

To treat the elliptic operator with VMO coefficients, we prove the following approximation argument, found in [28].

Lemma 4.2.9. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$. Suppose that the coefficients of operator $\mathcal{L}$, defined in (4.2), are symmetric and in $VMO(\mathbb{R}^3)$. Then there exists a function $h(r)$ and some constants $C > 0, c > 0$ with the following properties:*

i) $\lim\limits_{r\to 0} h(r) = 0$;
ii) for any $v \in H^1$ *solution of*

$$\begin{cases} \mathcal{L}v & = 0 \quad \text{in} \;\; B(x, 8r) \cap \Omega \\ A\nabla v \cdot \boldsymbol{n} + \alpha v & = 0 \quad \text{on} \;\; B(x, 8r) \cap \Gamma \end{cases} \tag{4.12}$$

with $x \in \overline{\Omega}$ *and* $0 < r < cr_0$, *there exists a function* $w \in W^{1,p}(B(x, r) \cap \Omega)$ *such that for any* $p > 2$,

$$\left(\fint_{B(x,r)\cap\Omega} |\nabla v - \nabla w|^2 + |v - w|^2 \right)^{1/2} \leq h(r) \left(\fint_{B(x,8r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2} \tag{4.13}$$

$$\left(\fint_{B(x,r)\cap\Omega} |\nabla w|^p + |w|^p \right)^{1/p} \leq C \left(\fint_{B(x,8r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2}, \tag{4.14}$$

where the constant $C > 0$ *depends at most on* Ω, p, α_*, μ *and* A.

Proof. Let us fix $x_0 \in \overline{\Omega}$ and $0 < r < cr_0$ where $0 < c << 1$ is such that Lemma 4.2.8 can be applied suitably. Let $v \in H^1(B(x_0, 8r) \cap \Omega)$ be a weak solution of (4.12). Consider

$$\begin{cases} \operatorname{div}(B\nabla w) = 0 & \text{in} \;\; B(x_0, 4r) \cap \Omega \\ (B\nabla w) \cdot \boldsymbol{n} + \alpha w = (A\nabla v) \cdot \boldsymbol{n} + \alpha v & \text{on} \;\; \partial(B(x_0, 4r) \cap \Omega) \end{cases} \tag{4.15}$$

where $B = (b_{ij})_{1\leq i,j\leq 3}$ are the constants given by

$$b_{ij} = \frac{1}{|B(x_0, 8r)|} \int_{B(x_0,8r)} a_{ij}(x) \, \mathrm{d}x.$$

So, $w \in H^1(B(x_0, 4r) \cap \Omega)$ is a weak solution of (4.15) if for all $\varphi \in H^1(B(x_0, 4r) \cap \Omega)$,

$$\int_{B(x_0,4r)\cap\Omega} B\nabla w \cdot \nabla\varphi + \int_{\partial(B(x_0,4r)\cap\Omega)} \alpha w \, \varphi = \int_{B(x_0,4r)\cap\Omega} A\nabla v \cdot \nabla\varphi + \int_{\partial(B(x_0,4r)\cap\Omega)} \alpha v \, \varphi.$$

The existence of $w \in H^1(B(x_0, 4r) \cap \Omega)$ follows immediately from the regularity of v. It then follows

$$\int_{B(x_0,4r)\cap\Omega} B\nabla(v - w) \cdot \nabla\varphi + \int_{\partial(B(x_0,4r)\cap\Omega)} \alpha(v - w)\varphi = \int_{B(x_0,4r)\cap\Omega} (B - A)\nabla v \cdot \nabla\varphi.$$

Next we show that w satisfies estimates (4.13) and (4.14).

To see (4.13), choosing $\varphi = v - w$, by ellipticity and Cauchy inequality, we obtain

$$\begin{aligned} &\mu \int_{B(x_0,4r)\cap\Omega} |\nabla(v-w)|^2 + \int_{\partial(B(x_0,4r)\cap\Omega)} \alpha|v-w|^2 \\ &\le \int_{B(x_0,4r)\cap\Omega} |(B-A)\nabla v||\nabla(v-w)| \\ &\le C_\varepsilon \int_{B(x_0,4r)\cap\Omega} |(B-A)\nabla v|^2 + \varepsilon \int_{B(x_0,4r)\cap\Omega} |\nabla(v-w)|^2. \end{aligned}$$

But we also have the equivalence of norm,

$$\begin{aligned} \|v-w\|^2_{H^1(B(x_0,4r)\cap\Omega)} &\le C\left(\int_{B(x_0,4r)\cap\Omega} |\nabla(v-w)|^2 + \int_{B(x_0,4r)\cap\Gamma} |v-w|^2\right) \\ &\le C(\alpha_*)\left(\int_{B(x_0,4r)\cap\Omega} |\nabla(v-w)|^2 + \int_{B(x_0,4r)\cap\Gamma} \alpha|v-w|^2\right) \end{aligned}$$

where the above constant $C>0$ depends on Ω and α_* but is independent of r and α. This gives

$$\begin{aligned} &\left(\fint_{B(x_0,4r)\cap\Omega} |\nabla v - \nabla w|^2 + |v-w|^2\right)^{1/2} \\ &\le C\left(\fint_{B(x_0,4r)\cap\Omega} |(B-A)\nabla v|^2\right)^{1/2} \\ &\le C\left(\fint_{B(x_0,4r)\cap\Omega} |\nabla v|^{2q}\right)^{1/2q}\left(\fint_{B(x_0,4r)\cap\Omega} |B-A|^{2q'}\right)^{1/2q'}. \end{aligned}$$

Defining

$$h(r) = C\sup_{x_0\in\overline{\Omega}}\left(\fint_{B(x_0,4r)\cap\Omega} |B-A|^{2q'}\right)^{1/2q'}$$

the last inequality yields

$$\begin{aligned} \left(\fint_{B(x_0,4r)\cap\Omega} |\nabla v - \nabla w|^2 + |v-w|^2\right)^{1/2} &\le h(r)\left(\fint_{B(x_0,4r)\cap\Omega} |\nabla v|^{2q}\right)^{1/2q} \\ &\le h(r)\left(\fint_{B(x_0,8r)\cap\Omega} |\nabla v|^2 + |v|^2\right)^{1/2}. \end{aligned}$$

Note that in the last line, we used $L^{2+\varepsilon}$ weak reverse Hölder inequality (*i.e.* for some $q > 1$) for v which follows from Lemma 4.2.6. It is known from John-Nirenberg inequality that $h(r) \to 0$ as $r \to 0$. Indeed, John-Nirenberg inequality says, for any BMO-function f,

$$\int_B e^{M|f-\overline{f}|} \leq Cr^n$$

for some constant $C > 0$ depending only on n. Since $A \in VMO(\mathbb{R}^3)$, by definition we get that $h(r) \to 0$.

Finally, to see (4.14), note that $(B\nabla w) \cdot \boldsymbol{n} + \alpha w = 0$ on $B(x_0, 4r) \cap \Gamma$. Thus, by Lemma 4.2.8, we obtain, for any $p \geq 2$,

$$\begin{aligned}
&\left(\fint_{B(x_0,r)\cap\Omega} |\nabla w|^p \right)^{1/p} \\
&\leq C \left(\fint_{B(x_0,4r)\cap\Omega} |\nabla w|^2 + |w|^2 \right)^{1/2} \\
&\leq C \left(\fint_{B(x_0,4r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2} + C \left(\fint_{B(x_0,4r)\cap\Omega} |\nabla (v-w)|^2 + |v-w|^2 \right)^{1/2} \\
&\leq C \left(\fint_{B(x_0,8r)\cap\Omega} |\nabla v|^2 + |v|^2 \right)^{1/2}.
\end{aligned}$$

This shows that in fact $w \in W^{1,p}(B(x_0, r) \cap \Omega)$ which completes the proof. ■

With Lemma 4.2.9 at our hand, we may use the following approximation theorem, motivated from the paper of Caffarelli and Peral [28] and proved in [51], to finish the proof of the weak reverse Hölder inequality for VMO coefficient.

Theorem 4.2.10. *Let $E \subset \mathbb{R}^n$ be any open set and $F : E \to \mathbb{R}^n$ locally square integrable. Let $p > 2$. Suppose there exists some constants $\beta > 1$, $C > 1$ and $\varepsilon > 0$ such that for every cube Q with $2Q = Q(x_0, 2r) \subset E$, there exists a measurable function R_Q on $2Q$ satisfying*

$$\left(\fint_Q |R_Q|^p \right)^{1/p} \leq C \left(\fint_{\beta Q} |F|^2 \right)^{1/2} \tag{4.16}$$

and

$$\left(\fint_Q |F - R_Q|^2 \right)^{1/2} \leq \varepsilon \left(\fint_{\beta Q} |F|^2 \right)^{1/2}. \tag{4.17}$$

Let $2 < q < p$. Then, there exists $\varepsilon_0 = \varepsilon_0(C, n, p, q, \beta)$ such that if $\varepsilon < \varepsilon_0$, we have

$$\left(\fint_Q |F|^q \right)^{1/q} \leq C_1 \left(\fint_{2Q} |F|^2 \right)^{1/2} \tag{4.18}$$

where $C_1 > 0$ depends only on C, n, p, q, β.

Theorem 4.2.11. *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$ and $p \geq 2$. Suppose that the coefficients of operator $\mathcal{L}$, defined in (4.2), are symmetric and in $VMO(\mathbb{R}^3)$. Then for any $B(x, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x, 2r) \subset \Omega$ or $x \in \Gamma$, the weak Reverse Hölder inequalities (4.10) and (4.11) hold with constant $C > 0$ independent of α.*

Proof. Let $h(r)$ be same as in Lemma 4.2.9 and choose q such that $2 < q < p$. Let ε_0 be the same as in Theorem 4.2.10 and then we choose r_0 small enough such that $\sup\limits_{0<r<r_0} h(r) < \varepsilon_0$.

Let $v \in H^1(B(x_0, 8r) \cap \Omega)$ be a weak solution of

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B(x_0, 8r) \cap \Omega \\ A\nabla v \cdot \boldsymbol{n} + \alpha v = 0 & \text{on } B(x_0, 8r) \cap \Gamma \end{cases}$$

where $0 < r < \frac{r_0}{8}$ and either $B(x_0, 2r) \subset \Omega$ or $x_0 \in \Gamma$. To apply Theorem 4.2.10, take $E = B(x_0, 8r)$ and $B(x, r)$ is any ball with $B(x, 2r) \subset E$. Then the proof divides in the following cases:

i) if $B(x, r) \cap \Omega = \emptyset$, we take $F = 0 = R_B$,

ii) if $B(x, r) \subset \Omega$, set $F = \nabla v$ and $R_B = \nabla w$,

iii) if $B(x, r) \cap \Omega \neq \emptyset$ and $B(x, r) \cap (\overline{\Omega})^c \neq \emptyset$, we further consider the two situations:

– if $x \in \overline{\Omega}$, set

$$F = (\nabla v, v)\chi_\Omega \quad \text{and} \quad R_B = \begin{cases} (\nabla w, w) & \text{on } B(x, r) \cap \Omega, \\ 0 & \text{on } B(x, r) \cap \Omega^c; \end{cases}$$

– if $x \notin \overline{\Omega}$, by a geometric observation, it is easy to find a ball $\tilde{B} = B(y, 2r)$ such that $y \in \Gamma$ and $B \subset \tilde{B} \subset E$, we then set

$$F = (\nabla v, v)\chi_\Omega \quad \text{and} \quad R_{\tilde{B}} = \begin{cases} (\nabla w, w) & \text{on } B(y, 2r) \cap \Omega, \\ 0 & \text{on } B(y, 2r) \cap \Omega^c. \end{cases}$$

The estimates (4.16) and (4.17) now follow from (4.14) and (4.13). Hence the proof finishes from (4.18). ∎

Now to complete the proof of Theorem 4.2.5, we will use Lemma 2.6.8.

Proof of Theorem 4.2.5. Given any ball B with either $2B \subset \Omega$ or B centers on Γ, let $\varphi \in C_c^\infty(8B)$ is a cut-off function such that $0 \le \varphi \le 1$ and

$$\varphi = \begin{cases} 1 & \text{on } 4B \\ 0 & \text{outside } 8B \end{cases}$$

and we decompose $u = v + w$ where v, w satisfy

$$\begin{cases} \operatorname{div}(A(x)\nabla v) = \operatorname{div}(\varphi \boldsymbol{f}) & \text{in } \Omega \\ (A\nabla v - \varphi \boldsymbol{f}) \cdot \boldsymbol{n} + \alpha v = 0 & \text{on } \Gamma \end{cases} \tag{4.19}$$

and

$$\begin{cases} \operatorname{div}(A(x)\nabla w) = \operatorname{div}\left((1-\varphi)\boldsymbol{f}\right) & \text{in } \Omega \\ (A\nabla w - (1-\varphi)\boldsymbol{f}) \cdot \boldsymbol{n} + \alpha w = 0 & \text{on } \Gamma. \end{cases} \tag{4.20}$$

Multiplying (4.19) by v and integrating by parts, we get,

$$\int_\Omega A(x)\nabla v \cdot \nabla v + \int_\Gamma \alpha |v|^2 = \int_\Omega \varphi \boldsymbol{f} \cdot \nabla v$$

which gives

$$\|\nabla v\|_{\boldsymbol{L}^2(\Omega)} \le \frac{1}{\mu} \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.21}$$

and since $\alpha \ge \alpha_* > 0$ on Γ,

$$\|v\|^2_{H^1(\Omega)} \le C(\Omega, \alpha_*) \left(\|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma \alpha |v|^2 \right) \le C(\Omega, \alpha_*)\ \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla v\|_{\boldsymbol{L}^2(\Omega)}.$$

This yields the complete L^2-estimate

$$\|v\|_{H^1(\Omega)} \le C(\Omega, \alpha_*)\ \|\varphi \boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.22}$$

(i) First we consider the case $4B \subset \Omega$. We want to apply Lemma 2.6.8 with $G = |\nabla u|, G_B = |\nabla v|$ and $R_B = |\nabla w|$. It is easy to see that

$$|G| \le |G_B| + |R_B|.$$

Now we verify (2.75) and (2.76). For that, using (4.21) we get,

$$\frac{1}{|2B|}\int_{2B}|G_B|^2 = \frac{1}{|2B|}\int_{2B}|\nabla v|^2 \leq \frac{1}{|2B\cap\Omega|}\int_{\Omega}|\nabla v|^2 \leq \frac{C(\Omega,\alpha_*)}{|2B\cap\Omega|}\int_{\Omega}|\varphi \boldsymbol{f}|^2$$
$$\leq \frac{C(\Omega,\alpha_*)}{|8B\cap\Omega|}\int_{8B\cap\Omega}|\boldsymbol{f}|^2$$

where in the last inequality, we used that $|8B\cap\Omega| \leq |\Omega|$. This gives the estimate (2.76).

Next, from (4.20), we observe that $\operatorname{div}(A(x)\nabla)w = 0$ in $4B$. Hence, by the wRHI in Theorem 4.2.11 (using $2B$ instead of B), we have

$$\left(\fint_{2B}|\nabla w|^p\right)^{1/p} \leq C\left(\fint_{4B}|\nabla w|^2\right)^{1/2}$$

which implies together with (4.21),

$$\left(\fint_{2B}|R_B|^p\right)^{1/p} \leq C\left(\fint_{4B}|\nabla w|^2\right)^{1/2} \leq C\left[\left(\fint_{4B}|\nabla u|^2\right)^{1/2} + \left(\fint_{4B}|\nabla v|^2\right)^{1/2}\right]$$
$$\leq C\left(\fint_{4B}|G|^2\right)^{1/2} + C(\Omega,\alpha_*)\left(\fint_{8B\cap\Omega}|\boldsymbol{f}|^2\right)^{1/2}.$$

This gives (2.75). So from Lemma 2.6.8, it follows that

$$\left(\fint_{\Omega}|\nabla u|^q\right)^{1/q} \leq C_p(\Omega)\left[\left(\fint_{\Omega}|\nabla u|^2\right)^{1/2} + \left(\fint_{\Omega}|\boldsymbol{f}|^q\right)^{1/q}\right]$$

for any $2 < q < p$ where $C_p(\Omega) > 0$ does not depend on α.

Because of the self-improving property of the weak Reverse Hölder condition (4.10), the above estimate holds for any $q \in (2,\tilde{p})$ for some $\tilde{p} > p$ also and in particular, for $q = p$, which clearly implies (4.6).

(ii) Next consider B centers on Γ. We apply Lemma 2.6.8 now with $G = |u| + |\nabla u|, G_B = |v| + |\nabla v|$ and $R_B = |w| + |\nabla w|$. Obviously, $|G| \leq |G_B| + |R_B|$ and again by (4.22),

$$\frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega}|G_B|^2 \leq \frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega}(|v|^2 + |\nabla v|^2) \leq \frac{1}{|2B\cap\Omega|}\|v\|^2_{H^1(\Omega)}$$
$$\leq \frac{C(\Omega,\alpha_*)}{|2B\cap\Omega|}\int_{\Omega}|\varphi\boldsymbol{f}|^2$$
$$\leq \frac{C(\Omega,\alpha_*)}{|8B\cap\Omega|}\int_{8B\cap\Omega}|\boldsymbol{f}|^2$$

which yields (2.76). Also w satisfies the problem

$$\begin{cases} \operatorname{div}(A(x)\nabla w) = 0 & \text{in } 4B \cap \Omega \\ (A\nabla w) \cdot \boldsymbol{n} + \alpha w = 0 & \text{on } 4B \cap \Gamma. \end{cases}$$

So by the wRHI in Theorem 4.2.11 and the estimate (4.22), we can write,

$$\begin{aligned}
\left(\fint_{2B\cap\Omega} |R_B|^p \right)^{1/p} &\leq \left(\fint_{2B\cap\Omega} ((|w| + |\nabla w|)^2)^{p/2} \right)^{1/p} \\
&\leq C \left(\fint_{4B\cap\Omega} (|w|^2 + |\nabla w|^2) \right)^{1/2} \\
&\leq C \left[\left(\fint_{4B\cap\Omega} (|u|^2 + |\nabla u|^2) \right)^{1/2} + \left(\fint_{4B\cap\Omega} (|v|^2 + |\nabla v|^2) \right)^{1/2} \right] \\
&\leq C \left(\fint_{4B\cap\Omega} |G|^2 \right)^{1/2} + C(\Omega, \alpha_*) \left(\fint_{8B\cap\Omega} |\boldsymbol{f}|^2 \right)^{1/2}
\end{aligned}$$

which yields (2.75). Thus we have,

$$\left(\fint_{\Omega} (|u| + |\nabla u|)^q \right)^{1/q} \leq C_p(\Omega, \alpha_*) \left[\left(\fint_{\Omega} |u|^2 + |\nabla u|^2 \right)^{1/2} + \left(\fint_{\Omega} |\boldsymbol{f}|^q \right)^{1/q} \right]$$

for any $2 < q < p$ where $C_p(\Omega, \alpha_*) > 0$ does not depend on α. This completes the proof together with the previous case. ■

The next proposition will be used to study the complete estimate of the Robin problem (4.1). The result is not optimal and will be improved in Proposition 4.2.14.

Proposition 4.2.12 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate, $\boldsymbol{p > 2}$ with RHS $\boldsymbol{F}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p > 2$, and $F \in L^p(\Omega)$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are also symmetric and in $VMO(\mathbb{R}^3)$. Then the unique weak solution $u \in W^{1,p}(\Omega)$ of (4.1) with $\boldsymbol{f} = \boldsymbol{0}$ and $g = 0$, satisfies the following estimate:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \; \|F\|_{L^p(\Omega)}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Proof. The result follows using the same argument as in Theorem 4.2.5 and hence we do not repeat it. ■

Proposition 4.2.13 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate with RHS $\boldsymbol{f}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p \in (1, \infty)$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are also symmetric and in $VMO(\mathbb{R}^3)$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of (4.1) with $F = 0$ and $g = 0$, satisfying the following estimate:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \; \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.23}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Proof. The existence of a unique solution and the corresponding estimate for $p > 2$ is done in Theorem 4.2.2 and Theorem 4.2.5 respectively. Now suppose that $1 < p < 2$. We first discuss the estimate and then the existence of a solution.

(i) Estimate I: Let $\boldsymbol{g} \in C_0^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega)$ be the weak solution of $\text{div}(A(x)\nabla)v = \text{div}\, \boldsymbol{g}$ in Ω and $(A\nabla)v \cdot \boldsymbol{n} + \alpha v = 0$ on Γ. Since $p' > 2$, from Theorem 4.2.5, we have

$$\|v\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Also if $u \in W^{1,p}(\Omega)$ is a solution of (4.1) with $F = 0, g = 0$, using the weak formulation of the problems satisfied by u and v, we have

$$\int_\Omega \boldsymbol{f} \cdot \nabla v = \int_\Omega \boldsymbol{g} \cdot \nabla u$$

which gives,

$$|\int_\Omega \boldsymbol{g} \cdot \nabla u| \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\nabla v\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|v\|_{W^{1,p'}(\Omega)}$$

and hence,

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{g} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \nabla u \cdot \boldsymbol{g}|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

(ii) Estimate II: Next we prove that

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.24}$$

For that, from Proposition 4.2.12, we get for any $\varphi \in L^{p'}(\Omega)$, the unique weak solution $w \in W^{1,p'}(\Omega)$ of the problem

$$\begin{cases} \text{div}(A(x)\nabla w) = \varphi & \text{in } \Omega \\ (A\nabla)w \cdot \boldsymbol{n} + \alpha w = 0 & \text{on } \Gamma \end{cases}$$

satisfies

$$\|w\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*) \; \|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Therefore using the weak formulation of the problems satisfied by u and w, we obtain,

$$\int_\Omega u\,\varphi = \int_\Omega u\,\mathrm{div}(A(x)\nabla)w = -\int_\Omega A(x)\nabla u\cdot\nabla w + \int_\Gamma u\frac{\partial w}{\partial \boldsymbol{n}} = -\int_\Omega \boldsymbol{f}\cdot\nabla w$$

which implies

$$\|u\|_{L^p(\Omega)} = \sup_{0\neq\varphi\in L^{p'}(\Omega)} \frac{|\int_\Omega u\,\varphi|}{\|\varphi\|_{L^{p'}(\Omega)}} \leq C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

This completes proof of the estimate (4.23).

(iii) Existence and uniqueness: The uniqueness of solution of (4.1) follows from (4.23). For the existence, we will use a limit argument. Let $\{\boldsymbol{f}_k\}\in C_0^\infty(\Omega)$ such that

$$\boldsymbol{f}_k \to \boldsymbol{f} \quad \text{ in } \boldsymbol{L}^p(\Omega)$$

and $u_k \in W^{1,p'}(\Omega)$ be the unique weak solution of

$$\begin{cases} \mathrm{div}(A(x)\nabla u_k) = \mathrm{div}\ \boldsymbol{f_k} & \text{in } \Omega \\ (A\nabla)u_k\cdot\boldsymbol{n} + \alpha u_k = 0 & \text{on } \Gamma \end{cases}$$

Note that $u_k \in W^{1,p}(\Omega)$ since $p' > 2$. Also from **(i)** we have,

$$\|u_k\|_{W^{1,p}(\Omega)} \leq C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}_k\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\|u_k - u_l\|_{W^{1,p}(\Omega)} \leq C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}_k - \boldsymbol{f}_l\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus it follows $u_k - u_l \to 0$ in $W^{1,p}(\Omega)$ as $k,l\to\infty$ i.e. $\{u_k\}$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Then as $W^{1,p}(\Omega)$ is a Banach space, there exists $u\in W^{1,p}(\Omega)$ such that

$$u_k \to u \text{ in } W^{1,p}(\Omega)$$

satisfying

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Clearly u also solves the system (4.1). ■

Proposition 4.2.14 ($\boldsymbol{W^{1,p}(\Omega)}$ estimate with RHS $\boldsymbol{F}$). *Let Ω be a $\mathcal{C}^1$ bounded domain in $\mathbb{R}^3$, $p\in(1,\infty)$, $F\in L^{r(p)}(\Omega)$ and $g\in W^{-\frac{1}{p},p}(\Gamma)$. Suppose that the coefficients of the operator $\mathcal{L}$, defined in (4.2), are also symmetric and in $VMO(\mathbb{R}^3)$. Then the weak solution $u\in W^{1,p}(\Omega)$ of the problem*

$$\begin{cases} \mathcal{L}u = F & \textit{in } \Omega \\ (A\nabla u)\cdot\boldsymbol{n} + \alpha u = g & \textit{on } \Gamma \end{cases} \tag{4.25}$$

satisfies the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right)$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Proof. It suffices to prove the estimate since the existence and uniqueness of u follows from the same argument as in Proposition 4.2.13.

(i) Estimate I: Let $\boldsymbol{f} \in C_0^\infty(\Omega)$ and $v \in W^{1,p'}(\Omega)$ be the weak solution of $\operatorname{div}(A(x)\nabla v) = \operatorname{div} \boldsymbol{f}$ in Ω and $(A\nabla)v \cdot \boldsymbol{n} + \alpha v = 0$ on Γ. By Proposition 4.2.13, we then have

$$\|v\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*) \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Also, if $u \in W^{1,p}(\Omega)$ is a solution of (4.25), from the weak formulation of the problems satisfied by u and v, we get

$$\int_\Omega \boldsymbol{f} \cdot \nabla u = \int_\Omega A(x)\nabla u \cdot \nabla v + \int_\Gamma \alpha u v = -\int_\Omega F v + \langle g, v \rangle_\Gamma .$$

This implies

$$\begin{aligned} |\int_\Omega \boldsymbol{f} \cdot \nabla u| &\leq \|F\|_{L^{r(p)}(\Omega)} \|v\|_{L^{(r(p))'}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \|v\|_{W^{\frac{1}{p},p'}(\Gamma)} \\ &\leq C_p(\Omega) \left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right) \|v\|_{W^{1,p'}(\Omega)} \end{aligned}$$

since $\frac{1}{(p')_*} = \frac{1}{p'} - \frac{1}{3} = \frac{1}{(r(p))'}$ for $p > \frac{3}{2}$ and $W^{1,p'}(\Omega) \hookrightarrow L^\infty(\Omega)$ when $p < \frac{3}{2}$. Thus,

$$\|\nabla u\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \nabla u \cdot \boldsymbol{f}|}{\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right).$$

(ii) Estimate II: Next we prove the following bound as done in (4.24):

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right) \tag{4.26}$$

except that we do not need to assume $p < 2$ here as in (4.24). For any $\varphi \in L^{p'}(\Omega)$, there exists a unique weak $w \in W^{1,p'}(\Omega)$ solving the problem

$$\begin{cases} \operatorname{div}(A(x)\nabla w) = \varphi & \text{in } \Omega \\ (A\nabla)w \cdot \boldsymbol{n} + \alpha w = 0 & \text{on } \Gamma \end{cases}$$

and satisfying

$$\|w\|_{W^{1,p'}(\Omega)} \leq C_p(\Omega, \alpha_*) \|\varphi\|_{L^{p'}(\Omega)}.$$

(For $p < 2$ the above estimate can be proved by the exact same argument as in Proposition 4.2.13). Finally we can write,

$$\int_\Omega u\ \varphi = \int_\Omega u\Delta w = \int_\Omega \Delta u\ w - \int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}} w + \int_\Gamma u\frac{\partial w}{\partial \boldsymbol{n}} = \int_\Omega Fw - \langle g, w\rangle_\Gamma$$

which yields as before

$$\|u\|_{L^p(\Omega)} \le C_p(\Omega, \alpha_*)\left(\|F\|_{L^{r(p)}(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)}\right)$$

and thus we obtain (4.26). ∎

Proof of Theorem 4.1.1. Let $u_1 \in W^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} \operatorname{div}(A(x)\nabla u_1) = \operatorname{div}\boldsymbol{f} & \text{in } \Omega \\ (A\nabla u_1 - \boldsymbol{f})\cdot\boldsymbol{n} + \alpha u_1 = 0 & \text{on } \Gamma \end{cases}$$

given by Proposition 4.2.13 and $u_2 \in W^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} \operatorname{div}(A(x)\nabla u_2) = F & \text{in } \Omega \\ (A\nabla)u_2\cdot\boldsymbol{n} + \alpha u_2 = g & \text{on } \Gamma \end{cases}$$

given by Proposition 4.2.14. Then $u = u_1 + u_2$ is the solution of the problem 4.1 which also satisfies the estimate (4.4). ∎

Next we prove uniform H^s bound for $s \in (0, \frac{1}{2})$.

Proposition 4.2.15. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^3$, $g \in L^2(\Gamma)$ and α is a constant. Then the problem*

$$\begin{cases} \Delta u = 0 & \textit{in } \Omega \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = g & \textit{on } \Gamma \end{cases} \tag{4.27}$$

has a weak solution $u \in H^{\frac{3}{2}}(\Omega)$ which also satisfies the estimate

$$\|u\|_{H^{\frac{3}{2}}(\Omega)} \le C(\Omega)\|g\|_{L^2(\Gamma)}. \tag{4.28}$$

Proof. A solution $u \in H^1(\Omega)$ of the problem (4.27) satisfies the variational formulation:

$$\forall \varphi \in H^1(\Omega), \qquad \int_\Omega \nabla u\cdot\nabla\varphi + \int_\Gamma \alpha u\varphi = \int_\Gamma g\varphi.$$

Multiplying the above relation by α and substituting $\varphi = u$, we get

$$\alpha \int_\Omega |\nabla u|^2 + \|\alpha u\|^2_{L^2(\Gamma)} = \alpha \int_\Gamma gu \leq \|g\|_{L^2(\Gamma)} \|\alpha u\|_{L^2(\Gamma)}$$

and thus

$$\|\alpha u\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)}.$$

Now from the regularity result for Neumann problem [74, Theorem 2], we obtain

$$\|u\|_{H^{\frac{3}{2}}(\Omega)} \leq C(\Omega)\|g - \alpha u\|_{L^2(\Gamma)} \leq C(\Omega)\|g\|_{L^2(\Gamma)}$$

which gives the required estimate. ∎

Remark 4.2.16. We do not know any reference of $H^s(\Omega)$ regularity results (in particular, $s = 3/2$) for the general second order elliptic operator of divergence form with Neumann boundary condition, instead of Laplacian. That is the reason, we have stated the above Proposition for Laplacian.

Theorem 4.2.17 ($H^s(\Omega)$ estimate). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^3$, $s \in (0, \frac{1}{2})$ and α is a constant. Then for $g \in H^{s-\frac{1}{2}}(\Gamma)$, the problem (4.27) has a solution $u \in H^{1+s}(\Omega)$ which also satisfies the estimate*

$$\|u\|_{H^{1+s}(\Omega)} \leq C(\Omega)\|g\|_{H^{s-\frac{1}{2}}(\Gamma)}.$$

Proof. We obtain the result by interpolation between $H^1(\Omega)$ and $H^{\frac{3}{2}}(\Omega)$ regularity results in Theorem 4.2.2 and Proposition 4.2.15 respectively. ∎

4.3 Estimate for strong solution

Theorem 4.3.1 ($W^{2,p}(\Omega)$ estimate). *Let Ω be a $\mathcal{C}^{1,1}$ bounded domain in $\mathbb{R}^3$, $p \in (1, \infty)$ and α be a constant. Then for $F \in L^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, the weak solution u of the problem*

$$\begin{cases} \Delta u = F & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = g & \text{on } \Gamma \end{cases} \tag{4.29}$$

belongs to $W^{2,p}(\Omega)$ and satisfies the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \right) \tag{4.30}$$

where the constant $C_p(\Omega, \alpha_*) > 0$ is independent of α.

Remark 4.3.2. We can in fact show the existence of $u \in W^{2,p}(\Omega)$ for more general α, not necessarily constant; in particular for $\alpha \in W^{1-\frac{1}{q},q}(\Gamma)$ with $q > \frac{3}{2}$ if $p \leq \frac{3}{2}$ and $q = p$ otherwise.

Proof. For the given data, there exists a unique solution u of (4.29) in $W^{1,p}(\Omega)$, by Theorem 4.1.1. Then it can be shown that in fact u belongs to $W^{2,p}(\Omega)$ by Neumann regularity result using bootstrap argument. But concerning the estimate, we do not obtain a α independent bound on u, using the estimate for Neumann problem. So we consider the following argument.

As Γ is compact and of class $\mathcal{C}^{1,1}$, there exists an open cover U_i *i.e.* $\Gamma \subset \cup_{i=1}^k U_i$ and bijective maps $H_i : Q \to U_i$ such that

$$H_i \in \mathcal{C}^{1,1}(\overline{Q}),\ J^i := H_i^{-1} \in \mathcal{C}^{1,1}(\overline{U_i}),\ H_i(Q_+) = \Omega \cap U_i \ \text{ and } \ H_i(Q_0) = \Gamma \cap U_i$$

where we denote

$$\begin{aligned} Q &= \{x = (x', x_3); |x'| < 1 \text{ and } |x_3| < 1\} \\ Q_+ &= Q \cap \mathbb{R}^3_+ \\ Q_0 &= \{x = (x', 0); |x'| < 1\}. \end{aligned}$$

Then we consider the partition of unity θ_i corresponding to U_i with supp $\theta_i \subset U_i$. So we can write $u = \sum_{i=0}^k \theta_i u$ where $\theta_0 \in C_c^\infty(\Omega)$. It is easy to see that $v_i = \theta_i u \in W^{2,p}(\Omega \cap U_i)$ and satisfies:

$$\begin{cases} \Delta v_i = \theta_i F + 2\nabla\theta_i \nabla u + u\Delta\theta_i =: f_i & \text{in} \quad \Omega \cap U_i \\ \dfrac{\partial v_i}{\partial \boldsymbol{n}} + \alpha v_i = g + \dfrac{\partial \theta_i}{\partial \boldsymbol{n}} u =: h_i & \text{on} \quad \partial(\Omega \cap U_i). \end{cases}$$

Precisely, we have, for all $\varphi \in W^{1,p'}(\Omega \cap U_i)$,

$$\int_{\Omega\cap U_i} \nabla v_i \cdot \nabla\varphi + \alpha \int_{\Gamma\cap U_i} v_i \varphi = - \int_{\Omega\cap U_i} f_i \varphi + \int_{\Gamma\cap U_i} h_i \varphi \tag{4.31}$$

where $f_i \in L^p(\Omega)$ and $h_i \in W^{1-\frac{1}{p},p}(\Gamma)$. Now to transfer $v_i|_{\Omega\cap U_i}$ to Q_+, set $w_i(y) = v_i(H_i(y))$ for $y \in Q_+$. Then,

$$\frac{\partial v_i}{\partial x_j} = \sum_k \frac{\partial w_i}{\partial y_k} \frac{\partial J_k^i}{\partial x_j}.$$

Also let $\psi \in H^1(Q_+)$ and set $\varphi(x) = \psi(J^i(x))$ for $x \in \Omega \cap U_i$. Then $\varphi \in H^1(\Omega \cap U_i)$ and

$$\frac{\partial \varphi}{\partial x_j} = \sum_l \frac{\partial \psi}{\partial y_l} \frac{\partial J_l^i}{\partial x_j}.$$

Thus, putting these in (4.31), we obtain under this change of variable, for all $\psi \in H^1(Q_+)$,

$$\int_{Q_+} a_{kl}(x) \frac{\partial w_i}{\partial y_k} \frac{\partial \psi}{\partial y_l} + \alpha \int_{Q_0} w_i \psi = - \int_{Q_+} \tilde{f}_i \psi + \int_{Q_0} \tilde{h}_i \psi \tag{4.32}$$

with $a_{kl}(x) = \sum_j \frac{\partial J_k^i}{\partial x_j} \frac{\partial J_l^i}{\partial x_j} |\det Jac\ H_i|$, $\tilde{f}_i = f_i \circ J^i$ and $\tilde{h}_i = h_i \circ J^i$. Here $\det Jac\ H_i$ denotes the determinant of the Jacobian matrix of H_i. Note that $a_{kl} \in \mathcal{C}^{0,1}(\overline{Q_+})$, $\tilde{f}_i \in L^p(Q_+)$ and $\tilde{h}_i \in W^{1/p',p}(Q_0)$. Also (4.32) is a Robin problem of the form (4.1) for w_i on Q_+, since w_i vanishes in a neighbourhood of $\partial Q_+ \setminus Q_0$, the coefficient matrix satisfying the assumptions.

For notational convenience, in this last part, we omit the index i, *i.e.* we simply write w instead of w_i. Now denoting $\partial_j = \frac{\partial}{\partial x_j}$, we see that $z_i := \partial_i w, i = 1, 2$ solves the following problem

$$\begin{cases} \operatorname{div}(A(x)\nabla z_i) = \operatorname{div}(\tilde{f} \boldsymbol{e}_i) - \operatorname{div}(\partial_i A(x) \nabla) w & \text{in } Q_+ \\ \dfrac{\partial z_i}{\partial \boldsymbol{n}} + \alpha z_i = \tilde{f} \boldsymbol{e}_i \cdot \boldsymbol{n} - (\partial_i A(x) \nabla) w \cdot \boldsymbol{n} + \partial_i \tilde{h} & \text{on } Q_0 \end{cases} \tag{4.33}$$

where $\boldsymbol{e}_i$ is the unit vector with 1 in i^{th} position. Thus, we can apply Theorem 4.1.1 for the above system and may conclude

$$\|z_i\|_{W^{1,p}(Q_+)} \leq C_p(Q_+) \left(\|\tilde{f}\|_{L^p(Q_+)} + \|\partial_i A(x) \nabla w\|_{\boldsymbol{L}^p(Q_+)} + \|\partial_i \tilde{h}\|_{W^{-\frac{1}{p},p}(Q_0)} \right)$$

which yields, for all $i, j = 1, 2, 3$ except $i = j = 3$,

$$\|\partial_{ij}^2 w\|_{L^p(Q_+)} \leq C_p(Q_+) \left(\|\tilde{f}\|_{L^p(Q_+)} + \|w\|_{W^{1,p}(Q_+)} + \|\tilde{h}\|_{W^{\frac{1}{p'},p}(Q_0)} \right). \tag{4.34}$$

Now to show the estimate for $\partial_{33}^2 w$, we can write from the equation (4.32) (omitting the index i),

$$\partial_{33}^2 w = \frac{1}{a_{33}} \left(\tilde{f} - a_{ij}\ \partial_{ij}^2 w - \partial_i a_{ij}\ \partial_j w \right) \quad \text{in} \quad Q_+.$$

But since J is an one-one map, $a_{33} \neq 0$ and thus together with (4.34), we obtain the same estimate (4.34) for $\partial_{33}^2 w$. Therefore, we can conclude, for all $i = 1, ..., k$,

$$\|v\|_{W^{2,p}(\Omega \cap U_i)} \leq C_p(\Omega) \left(\|F\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|u\|_{W^{1,p}(\Omega)} \right)$$

and consequently (4.30), using $W^{1,p}$-estimate result. ∎

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