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Modèles intégrables avec fonction twist et modèles de Gaudin affines

Integrable models with twist function and affine Gaudin models

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Foreword

This thesis is the final dissertation of my doctorate, that I prepared under the cosupervision of Marc Magro (at the ENS de Lyon) and Benoit Vicedo (at the University of Hertfordshire). It is mostly based on the articles or preprints I published, together with collaborators, during my PhD. These publications and preprints are listed below:

[P1] **On the Hamiltonian integrability of the bi-Yang-Baxter σ-model**
François Delduc, Sylvain Lacroix, Marc Magro and Benoît Vicedo,
JHEP **1603** (2016) 104, [1512.02462].

[P2] **On q-deformed symmetries as Poisson-Lie symmetries and application to Yang-Baxter type models**
François Delduc, Sylvain Lacroix, Marc Magro and Benoît Vicedo
J. Phys. **A49** (2016) 415402, [1606.01712].

[P3] **Local charges in involution and hierarchies in integrable sigma-models**
Sylvain Lacroix, Marc Magro and Benoît Vicedo
JHEP **1709** (2017) 117 [1703.01951].

[P4] **Cyclotomic Gaudin models, Miura opers and flag varieties**
Sylvain Lacroix and Benoît Vicedo
Ann. Henri Poincaré **19** (2018) 71-139 [1607.07397].

[P5] **Affine Gaudin models and hypergeometric functions on affine opers**
Sylvain Lacroix, Benoît Vicedo and Charles A. S. Young
Preprint arXiv (april 2018): [1804.01480]

[P6] **Cubic hypergeometric integrals of motion in affine Gaudin models**
Sylvain Lacroix, Benoît Vicedo and Charles A. S. Young
Preprint arXiv (april 2018): [1804.06751]

The main ideas of the articles and preprints [P1,P4–P6] are summarised in this thesis, with less technical details. These summaries are meant to be read independently of the articles. For completeness, we add the complete copies of these publications and preprints in the last part V of this thesis. The entirety of the articles [P2,P3] is embodied in the thesis, so we choose not to join them in this last part.

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Chapter 1

Introduction

1.1 Integrability in field theory

Field theories. Field theory has many applications in various domains of physics. For example, electromagnetism, general relativity and hydrodynamics are described by classical field theories. The Standard Model of particle physics, which describes the fundamental components of matter and their interactions, is based on particular models of quantum field theory, the gauge theories. Similarly, field theory models are also used in condensed matter theory and statistical physics. Most methods of field theory are based on perturbative expansions. They apply to models describing fields whose interactions are controlled by a small parameter. The physical observables of these models are then expressed as power series expansions in this parameter.

In quantum field theory, such methods led to the development of Feynman diagrams, which proved to be extremely successful for many applications. For instance, the use of perturbative methods in the Standard Model of particle physics has allowed the computation of many physical observables, such as the probabilities of creation and annihilation of particles. These theoretical predictions agree with the experimental observations with a tremendous accuracy. A particular recent example of the theoretical and experimental success of the Standard Model is the observation of the Higgs boson at the Large Hadron Collider, in 2012.

Despite this success, perturbative methods turn out to be insufficient in other situations. For example, the theory of Quantum Chromodynamics (which is the gauge theory describing the strong nuclear interaction in the Standard Model) possesses a strongly-coupled regime which cannot be described by perturbative expansions (as there is no small perturbation parameter in this case). This motivates the development of non-perturbative methods.

Integrable field theories. Some of these methods belong to the theory of integrability. This theory, which will be described in more details in the next sections of this introduction, allows to compute exactly some physical observables, for certain models called integrable systems. Integrable field theories are scarce, as the requirements for being integrable are quite constraining. In particular, almost all techniques for finding and studying integrable field theories are restricted to two-dimensional field theories. However, integrable field theories are still a domain of growing interest in theoretical research. For example, the existence of exactly solvable models allows one to test and develop further the more usual techniques of field theory, using integrable theories as toy models for a deeper understanding of general field theory. Moreover, the theory of integrability has proven to be mathematically quite rich and has allowed many deep developments at the boundary between physics and mathematics.

A very important development in the domain of integrable field theories came recently with the AdS/CFT correspondence [1–3]. This correspondence conjectures a dictionary between some physical observables of quantum gravity models defined on Anti de-Sitter (AdS) manifolds and observables of dual Conformal Field Theories (CFT). The most studied example of the AdS/CFT correspondence concerns, on the CFT side, the $\mathcal{N} = 4$ super-symmetric Yang-Mills theory in four dimensions. This is a gauge theory, such as Quantum Chromodynamics, which possesses an additional property of invariance

under super-symmetry. Although this (extended) super-symmetry prevents the theory to be a good phenomenological description of particle physics, it shares several similarities with non super-symmetric Yang-Mills theories and is thus a very good toy model for theoretical high-energy physics. The AdS dual of this theory is the Green-Schwarz superstring on the background $AdS_5 \times S^5$ [4], *i.e.* the cartesian product of the five dimensional Anti de-Sitter manifold with the five dimensional sphere.

This superstring theory has been shown to be integrable at the classical level [5–7]. This is the reason why this particular example of AdS/CFT correspondence is the most studied. Indeed, one can then apply integrability techniques to obtain results on both the string model and the four dimensional Super-Yang-Mills theory (see for instance [8]).

1.2 Classical integrable models with twist function

Part of this PhD thesis is devoted to the study of a certain family of classical integrable field theories, called models with twist function. In this section of the introduction, we introduce and motivate this class, particularly through its relation with integrable σ-models, and present the main questions concerning classical integrable models with twist function addressed in this thesis.

1.2.1 Classical integrable field theories

Liouville integrability. The theory of integrability started within the Hamiltonian formalism of classical mechanics, which deals with models with a finite number of degrees of freedom. Such a model is said to be integrable (in the sense of Liouville) if it possesses sufficiently large number of independent conserved quantities in involution, *i.e.* whose Poisson brackets vanish (more precisely, this number must be half the dimension of the phase space of the model). By a theorem of Liouville, such a model can be solved exactly by quadratures. Concretely, this means that the solutions of this model can be formally expressed and can be explicitly computed after performing some algebraic manipulations and integrations.

Integrable field theories. The notion of integrability for a classical field theory is more subtle. As it possesses an infinite number of degrees of freedom, a natural necessary condition for a field theory to be integrable is to admit an infinite number of conserved charges in involution. Yet, one cannot give a precise definition of such a condition as there is not a unique notion of infinity: how can one ensure that the infinity of charges that one has is "enough" to solve the model ? Thus, there is no universally accepted definition of integrability for field theories.

Lax pairs. In this thesis, we will be interested in two-dimensional field theories. We make the choice of defining integrability in this context through the notion of Lax pairs. A model is said to admit a Lax pair [9] if its equations of motion can be recast as a zero curvature equation (or Lax equation) on a two-dimensional connection. The spatial and temporal components of this connection are described by two matrices $\mathcal{L}$ and $\mathcal{M}$, valued in some Lie algebra $\mathfrak{g}$, depending on the fields of the model and on an auxiliary complex parameter λ, called the spectral parameter. The Lax equation then reads

$$\partial_t \mathcal{L}(\lambda) - \partial_x \mathcal{M}(\lambda) + \big[\mathcal{M}(\lambda), \mathcal{L}(\lambda)\big] = 0, \qquad \forall\, \lambda \in \mathbb{C},$$

where $[\cdot,\cdot]$ denotes the Lie bracket of $\mathfrak{g}$. This ensures the existence of an infinite number of conserved charges of the model extracted from the monodromy of the Lax matrix $\mathcal{L}$.

This zero curvature equation can be seen as the compatibility condition of an auxiliary system of linear partial differential equations. This auxiliary system is at the basis of the so-called Inverse Scattering Method [10, 11]. This led to rich developments in the theory of solitons, for example for the Korteweg-de Vries [12], non-linear Schroedinger [13] and sine-Gordon [14] equations (see also the books [15, 16] and the lectures [17]).

Maillet bracket. The existence of a Lax pair allows the construction of an infinite number of conserved charges. However, it does not ensure that these charges have vanishing Poisson brackets, as required for the theory to be integrable. As these charges are constructed from the monodromy of the Lax matrix $\mathcal{L}$, their Poisson brackets are related to the Poisson bracket between the different components of $\mathcal{L}$.

A first sufficient condition for the charges extracted from the monodromy to be in involution is given by the Sklyanin bracket [18]. This condition requires the Poisson bracket of $\mathcal{L}$ to be expressed in terms of commutators of $\mathcal{L}$ with another matrix called the $\mathcal{R}$-matrix, which is skew-symmetric. Moreover, it requires this Poisson bracket to be proportional to the Dirac δ-distribution in the space coordinate (and thus not to contain derivatives of this distribution). Such a bracket is called ultralocal. It describes for example the sine-Gordon model and the non-linear Schroedinger equation.

A generalisation of this condition, which allows the Poisson bracket of $\mathcal{L}$ to contain a term proportional to the first derivative of the δ-distribution, has been found by Maillet [19, 20]. Using the standard tensorial notations, it is given by

$$\begin{aligned}\big\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x),\mathcal{L}_{\underline{\mathbf{2}}}(\mu,y)\big\} = \big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu),\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x)\big]\,\delta(x-y) - \big[\mathcal{R}_{\underline{\mathbf{21}}}(\mu,\lambda),\mathcal{L}_{\underline{\mathbf{2}}}(\mu,x)\big]\,\delta(x-y)\\ - \big(\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)+\mathcal{R}_{\underline{\mathbf{21}}}(\mu,\lambda)\big)\,\partial_x\delta(x-y).\end{aligned}$$

It also involves a $\mathcal{R}$-matrix $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)$, which contrarily to the case of the Sklyanin bracket is not skew-symmetric. As it contains a derivative of δ, the Maillet bracket is said to be non-ultralocal.

1.2.2 Integrable σ-models

A new class of integrable theories. Important examples of integrable two-dimensional field theories are given by the integrable (non-linear) σ-models, such as the Principal Chiral Model (PCM) on a Lie group. They have applications in various domains of physics and mathematics. For instance, one particular integrable σ-model, called the non-linear $O(3)$ model, plays a role in condensed matter as the continuum limit of certain spin chains, describing the magnetic properties of materials [21].

Integrable σ-models also play an important role in high-energy physics and more precisely in string theory. The recent and archetypal example is the $AdS_5\times S^5$ Green-Schwarz superstring mentioned above in the context of the AdS/CFT correspondence. The classical limit of this theory is described as a two-dimensional σ-model with the Metsaev-Tseytlin action [4]. It has been shown by Bena, Polchinski and Roiban in [5] that this model admits a Lax pair representation and by Magro that the Hamiltonian Lax matrix satisfies a Maillet bracket [6].

Integrable σ-models regained interest in the mathematical physics community even more recently with the discovery of integrable deformations of these models. These are models depending on continuous deformation parameters, such that the integrability property is preserved for any value of these parameters. These deformed models thus form a whole new class of integrable field theories, which is quite interesting regarding the rarity of integrable models.

The first examples of such integrable deformations of σ-models were found for particular target spaces of low dimension. For instance, integrable deformations of the PCM on the group $SU(2)$ were discovered by Cherednik in [22], by Balog, Forgács, Horváth and Palla in [23] (see also [24]) and by Fateev in [25]. Similarly, an integrable deformation of the $O(3)$-model was proposed by Fateev, Onofri and Zamolodchikov in [26]. We will now discuss more general integrable deformations.

Yang-Baxter type deformations. A general deformation of the PCM on an arbitrary Lie group, called the Yang-Baxter model, was discovered by Klimčik in [27,28] (for the group $SU(2)$, this deformation coincides with a particular limit of [22]). The extension of this deformation for symmetric-space σ-models was discovered by Delduc, Magro and Vicedo in [29] (recovering [30] in the case of the $O(3)$-model, the deformation proposed in [26]). This deformation was also extended by the same authors to the superstring model on $AdS_5\times S^5$ mentioned above [31,32] (we will come back on this deformation later). In [28], Klimčik proposed a two-parameter deformation of the PCM, the Bi-Yang-Baxter model, which can be seen as a further deformation of the Yang-Baxter model and which was proven to admit

a Lax pair in [33] (the Bi-Yang-Baxter model for $SU(2)$ has been identified in [30] with the deformed model proposed in [25]).

One of the features of these deformations is to break a global symmetry of the undeformed model. It was shown in [29,32] (and before that in [34,35] for the PCM on $SU(2)$) that the conserved charges associated with this symmetry in the undeformed model are deformed in a set of non-local conserved charges satisfying a q-deformed algebra. More precisely, the Poisson brackets of these charges form a Poisson-Hopf algebra, which can be thought of as a classical limit of a quantum group. In the case of the Yang-Baxter model, it was also shown recently [36] that this deformed algebra is part of a bigger, infinite, one: a q-deformed affine Poisson-Hopf algebra (here also, the case for the group $SU(2)$ was already treated in [35,37]).

We shall refer to all of these deformations as Yang-Baxter type deformations (they are also called η-deformations or $\varkappa$-deformations in the literature), as they are constructed from a solution of the modified Classical Yang-Baxter Equation (mCYBE). One can also construct [38] deformations associated with solutions of the (homogeneous) Classical Yang Baxter Equation. Contrarily to inhomogenous deformations, these models do not possess q-deformed conserved charges. As these charges will be one of the subjects of interest of this thesis, we will use the denomination Yang-Baxter type deformation to refer to deformations associated with solutions of the (inhomogeneous) mCYBE.

Other deformations of σ-models. Another type of integrable deformation of σ-models, called λ-deformation, was also constructed by Sfetsos in [39] for the PCM and Hollowood, Miramontes and Schmidtt in [40] for symmetric space σ-models. This deformation was also extended to the Green-Schwarz superstring model on $AdS_5 \times S^5$ in [41].

The PCM also possesses another type of integrable deformation, which is obtained by adding a Wess-Zumino term (see [42,43] for the construction of this term) to the action of the PCM, multiplied by an arbitrary deformation parameter. This model was first considered in [44] for a particular value of this parameter which makes the model conformal. It was later shown in [45] that this model is integrable for any value of the deformation parameter.

The PCM can be further deformed by combining several types of deformations. For example, one can combine the addition of a Wess-Zumino term with a Yang-Baxter deformation (as shown by Delduc, Magro and Vicedo in [46]) and with a Bi-Yang-Baxter deformation and so called TsT transformations (as shown quite recently by Delduc, Hoare, Magro and Kameyama in [47], recovering a 4-parameter deformation introduced by Lukyanov in [48] for the group $SU(2)$).

Deformations of the $AdS_5 \times S^5$ superstring. As explained above, both the Yang-Baxter type deformation and the λ-deformation have been extended to the Green-Schwarz superstring on background $AdS_5 \times S^5$. These deformations have been the subjects of many recent works in the mathematical physics community.

For the Yang-Baxter deformation of the $AdS_5 \times S^5$ superstring, the existence of q-deformed symmetries mentioned above can also be seen on the tree-level light-cone S-matrix. Indeed, it was shown in [49–51] that this matrix coincides with the large string tension limit of the S-matrix invariant under the quantum group of the centrally-extended $\mathfrak{psu}(2|2)^2$ algebra [52–54]. This matrix is thus a q-deformation of the light-cone S-matrix of the undeformed superstring on $AdS_5 \times S^5$ [55–62] (see also the reviews [63,64]).

An interesting question about these deformations concerns their geometry, which is in particular important to understand whether or not they define a string theory. As a type IIB string theory, the undeformed model defines a background on $AdS_5 \times S^5$ which is a solution of the equations of type IIB supergravity. It is now known [65] that the background of the λ-deformation is also a solution of the equations of type IIB supergravity (see also [66–68] and [69, 70]). Thus, the λ-deformation defines a string theory. However, the Arutyunov-Borsato-Frolov [49–51] background obtained from the (inhomogeneous) Yang-Baxter deformation is not a solution of type IIB supergravity equations [50] but is a solution of generalised type IIB supergravity equations [71, 72] (see also [73]).

Another aspect of interest of these models concerns the AdS/CFT correspondence. As the undeformed superstring model is dual, through this correspondence, to the $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions, it is natural to wonder if the deformed models are dual to a deformation of this gauge theory. This is still an open question (see however [74] and [75–77] for homogeneous Yang-Baxter deformations).

1.2.3 The algebraic structure behind integrable σ-models: the twist function

Integrable σ-models as models with twist function. The σ-models presented above and their deformations possess a Lax matrix (see Subsection 1.2.1 of this introduction). The Hamiltonian analysis of most of these models has been performed, showing their integrability at the Hamiltonian level. More precisely, as explained in details in Chapter 3 of this thesis, it was shown that the Poisson bracket of the Lax matrix of most of these models takes the form of a non-ultralocal Maillet bracket (see Subsection 1.2.1), ensuring the existence of an infinite number of conserved charges in involution for these models.

Moreover, it was observed that the $\mathcal{R}$-matrices governing the Maillet brackets of all these models share a common algebraic structure. They are encoded in a model-dependent rational function φ of the spectral parameter, that we call the twist function. More precisely, they are given by

$$\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu) = \mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu)\varphi(\mu)^{-1},$$

where $\mathcal{R}^0$ is a standard $\mathcal{R}$-matrix on a loop algebra. This observation thus enables one to interpret integrable σ-models as part of a larger class of integrable field theories, the models which possess such a twist function.

Twist function and deformations of σ-models. An interesting aspect of this common algebraic structure concerns the deformations of σ-models. Indeed, one sees that the known deformations of σ-models do not affect the dependence of the Lax matrix of the model on the spectral parameter. The effect of the deformation is in fact to modify the Poisson bracket of this Lax matrix by modifying the twist function of the model. More precisely, one observes that the deformations of σ-models modify the poles of the twist function.

Let us note here that homogeneous Yang-Baxter type deformations exhibit a slightly different behaviour than other deformations in this context. Studying their Hamiltonian structure, one shows that these deformations do not affect the Poisson bracket of the Lax matrix of the model and thus its twist function. In this case, the effect of the deformation is purely to modify the way the Lax matrix depends on the fundamental fields of the model. However, these homogeneous deformations are part of the class of models with twist function and thus, all results which apply to this whole class can be applied to them.

Models with twist function as Affine Gaudin Models. The observation that σ-models are models with twist function led recently to their reinterpretation in terms of so-called classical Affine Gaudin Models (AGM). Gaudin models form a general class of integrable systems which are constructed from certain Lie algebras. In particular, the AGM are the Gaudin models associated with affine Kac-Moody algebras. They were studied recently by Vicedo in [78]. More precisely, it was shown in [78] that AGM can be seen as integrable field theories with twist function. Moreover, in the framework of AGM, the twist function appears quite naturally as part of the data defining the model (in particular, the poles of the twist function are the so-called sites of the Gaudin model).

Conversely, Vicedo also proved in [78] that all known models with twist function (including integrable σ-models and their deformations but also affine Toda field theories) are realisations of some particular AGM. This sheds the light on the algebraic origin of integrable σ-models and exhibits the common mathematical structure behind their integrability.

1.2.4 Goal of this thesis

One of the main goals of my PhD projects was to develop general methods to study integrable σ-models in a model-independent way, using their common algebraic structure as models with twist function.

Such methods would show the key role played by the twist function in the study of integrable σ-models and the interest of treating these models as a whole class. Below, we list the main questions addressed in this thesis following this direction. They form mainly the first part of this thesis.

Local charges in involution. It is known [79,80] that some undeformed integrable σ-models (the PCM and the symmetric space σ-model) possess an infinity of local charges in involution. It is natural to ask whether this is a unique feature of these particular models or if it is shared by other integrable σ-models, as for example deformed ones. We address this question in Chapter 4 of this thesis and show that it is in fact a result shared by all models with twist function, under certain assumptions on this twist function. As such, it applies to all integrable σ-models and their deformations but also more generally to AGM.

Deformed symmetries of Yang-Baxter type deformations. As mentioned in Subsection 1.2.2, the Yang-Baxter type deformations are characterised by the breaking of some global symmetry and the appearance of some q-deformed Poisson-Hopf algebra of conserved charges. However, it is not known whether these deformed conserved charges are associated with a symmetry of the model. We address this question in Chapter 5 and exhibit a Poisson-Lie deformed symmetry associated with these charges. The approach used to derive this result is model-independent and uses the common features of Yang-Baxter type deformations. In particular, a key point of this method is the fact that these deformations have for effect to split a double pole of the undeformed twist function into two simple poles, showing the key role played by the twist function in these deformations.

Twist function of the Bi-Yang-Baxter model. One of my other PhD project presented in this thesis does not concern the development of general results for σ-models based on the twist function. It consists rather of proving that a particular deformed σ-model, the Bi-Yang-Baxter model, is also described by a twist function, as other deformed σ-models. We prove this fact in Chapter 3, Section 3.4 of this thesis (modulo some technical subtleties), thus showing that the Bi-Yang-Baxter model also belongs to the more general class of models with twist function.

1.3 Quantum Gaudin models

A second part of this thesis is devoted to quantum aspects of integrable models, and more precisely of Gaudin models, either finite or affine.

1.3.1 Quantum integrability

Quantum integrable mechanical systems and spin chains. For physical systems with a finite number of degrees of freedom, one can define quantum integrability similarly to classical integrability, replacing conserved charges in involution by conserved commuting operators. Such systems can be quantised mechanical systems, as the Toda chain, or spin chains, as the XXX or XXZ Heisenberg chains.

Given a quantum model, one of the main steps towards its resolution is to find its spectrum, *i.e.* to find the eigenvalues of its Hamiltonian. For an integrable model, this Hamiltonian commutes with a large number of commuting operators. Provided that these operators are diagonalisable, there exists a common basis of eigenvectors for all these operators and one can aim to study the whole spectrum of all these operators at once. For certain quantum integrable models, this can be achieved by various methods. Example of such methods are the coordinate Bethe ansatz or the algebraic Bethe ansatz, which proved quite efficient for the study of quantum integrable spin chains.

Integrable quantum field theory. Defining the notion of integrability for quantum field theories is a subtle question, which does not admit a universally accepted answer. In this thesis, we will choose to define integrability for a quantum field theory as the existence of an infinity of commuting conserved

operators. An important feature of models which admit such a property is that their scattering matrix factorises. More precisely, one shows that there is no possibility of creation and annihilation of particles and that all scattering processes can be reduced to combination of scattering of two particles. Moreover, one can often constrain the form of these two by two scattering matrices using the symmetries of the model. Once these two by two scattering matrices are known, one can extract more results on the theory, such as its spectrum, *via* the Thermodynamic Bethe Ansatz. This is the so-called Factorised Scattering Theory [81].

If one is given a classical integrable field theory, a natural goal is to find a quantisation of this theory which leads to an integrable quantum field theory. The usual approach of quantum field theory based on second quantisation of the canonical bracket and perturbative expansions is in general not well adapted for such a goal. Alternative quantisation methods which preserve integrability by construction have been developed. The main idea of these methods is to discretise the space coordinate (while preserving the integrability), to write the Poisson brackets of the evaluations of the Lax matrix on this discretisation and to quantise these Poisson brackets. These methods of quantisation lead to the so-called Quantum Inverse Scattering Method, allowing the use of the algebraic Bethe ansatz on this theory. One then has to take a continuum limit to obtain a quantum field theory.

These methods were mostly developed [82] (see also the review [83] and the book [84]) for ultralocal theories, satisfying the Sklyanin bracket (see Subsection 1.2.1 of this introduction). Developing a similar method for non-ultralocal theories first necessitates to find a discrete integrable Poisson algebra whose continuum limit coincides with the Maillet bracket. Such an algebra was proposed in [85,86]. However, the corresponding $\mathcal{R}$-matrix arising from its continuum limit satisfies very particular conditions. A similar construction for an arbitrary $\mathcal{R}$-matrix is still lacking at the time.

Quantum integrable σ-models ? In the previous section of this introduction, we presented particular examples of classical integrable field theories, the integrable σ-models. The methods of quantisation by discretisation mentioned above cannot be applied to these models, as the Poisson bracket of their Lax matrix is non-ultralocal and does not satisfy the condition to apply the results of [85,86]. Hence, there is still no proof of the quantum integrability of these models from first principles and one cannot apply the Quantum Inverse Scattering Method to solve them.

However, quantum σ-models and the quantum superstring on $AdS_5 \times S^5$ have been extensively studied in the literature. These works rest on several assumptions, in particular the hypothesis that one can apply the Factorised Scattering Theory to these models, allowing the description of their spectrum through the Thermodynamic Bethe Ansatz. A way to prove this hypothesis would be to show the quantum integrability of these models, which is not done yet (note however that the factorisation of the scattering matrix was proven for particular σ-models [87]). Such a program turned out to be extremely successful in the context of the AdS/CFT correspondence (see for example the review [8]). It led to the study of the so-called Quantum Spectral Curve [88].

Quantum affine Gaudin models. As mentioned in Subsection 1.2.3 of this introduction, classical integrable σ-models have been re-interepreted as realisations of affine Gaudin models in [78]. It was proposed in the same article that this re-interpretation can be used to prove, in the long-term, the integrability of quantum σ-models and to determine their spectrum. This proposition is based on the many results known about another type of quantum Gaudin models, the finite ones (associated with finite Kac-Moody algebras instead of affine ones). Let us now discuss these results.

1.3.2 Quantum finite Gaudin models

The models. Finite Kac-Moody algebras are the semi-simple Lie algebras of finite dimension. The associated finite Gaudin models possess a finite number of degrees of freedom. More precisely, they describe integrable spin chains. They were introduced by Gaudin, first in [89] for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, as a limit of the XXX spin chain, and later in [90] for an arbitrary semi-simple Lie algebra.

In addition to this underlying Lie algebra $\mathfrak{g}$, a Gaudin model is constructed from the data of N representations of $\mathfrak{g}$, which describe the sites of the spin chain, attached to N points of the complex

plane, which represent the positions of these sites.

Using the invariant non-degenerate bilinear form on the semi-simple Lie algebra, one defines a set of N commuting Hamiltonians of the Gaudin models. These Hamiltonians encode quadratic spin-spin interactions between the sites. A specificity of these Hamiltonians, compared to the ones of most other integrable spin chains, is that these interactions are long-range: every site of the chain interacts with all other sites.

Bethe ansatz. An important step towards the resolution of Gaudin models is to diagonalise these Hamiltonians. As they commute, they are simultaneously diagonalisable and one can look for a basis of common eigenvectors of all Hamiltonians. When the representations at the sites of the model are highest-weight representations of $\mathfrak{g}$, one can look for this basis using the algebraic Bethe ansatz. This is a general method used to diagonalise integrable spin chains which admit a monodromy matrix, satisfying certain commutation relations. The Hamiltonian of such a spin chain is extracted from the monodromy matrix. The Bethe ansatz method then constructs eigenvectors of this Hamiltonian, starting from the vacuum state of the model and acting by suitable lowering operators, extracted from the monodromy matrix.

The Bethe ansatz was applied to the Gaudin model on $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ by Gaudin himself, in its original article [89]. For spin chains associated with Lie algebras of higher rank, one usually works recursively on the rank by successive applications of the Bethe ansatz for $\mathfrak{sl}(2, \mathbb{C})$: this is the so-called nested Bethe ansatz. This method has been applied to Gaudin models [91], resulting in a rather intricate algorithm constructing the eigenvectors. Another approach to the Bethe ansatz, which does not use a recursive algorithm, was developed by Babujian and Flume in [92], based on the results [93] of Schechtman and Varchenko about the Knizhnik-Zamolodchikov equation (see also [94]).

Higher Gaudin Hamiltonians. In their pioneering work [95] on Gaudin models, Feigin, Frenkel and Reshetikhin have proved that the quadratic Gaudin Hamiltonians are part of a larger commutative algebra. This larger commutative algebra, called the Gaudin algebra or Bethe algebra [96], contains Hamiltonians of degrees greater than two (for $\mathfrak{g}$ of rank at least two), which then commute between themselves and with the quadratic ones. In [95], this subalgebra is constructed abstractly, using modules of an affine algebra at the critical level. The higher degree Gaudin Hamiltonians were later constructed explicitly for simple algebras of type A [97–99] and of type B, C and D [100] (see also [101] for the uniqueness of these quantum Hamiltonians, proving that these abstract and explicit constructions coincide).

FFR approach. As these higher degree Hamiltonians commute with the quadratic ones and between themselves, they can be simultaneously diagonalised. In the article [95], Feigin, Frenkel and Reshetikhin also proposed a method to describe the eigenvalues of these Hamiltonians. This approach, developed further in [102–108] and to which we shall refer as the FFR approach, does not require the use of the Bethe ansatz and in fact recovers its results. It describes the spectrum of the Gaudin Hamiltonians (including higher degree ones) in terms of differential operators, called opers. These opers are constructed from the so-called Langlands dual of the inital Lie algebra $\mathfrak{g}$, which is also a semi-simple and finite dimensional Lie algebra.

The FFR approach is related [105] to a deep mathematical result called the Geometric Langlands Correspondence. It is part of the so-called Langlands program, which aims at finding relations between various domains of mathematics: geometry, group theory, Galois theory, ... This places the FFR approach at the frontier between physics and pure mathematics.

Generalisations. The results mentioned above concern the "usual" finite Gaudin model, as introduced by Gaudin in [90]. This model can be generalised in various ways.

For example, the Lax matrix of the usual Gaudin model depends rationally on the spectral parameter and has simple poles at the sites of the model. One can consider a generalisation of this model where the Lax matrix possesses poles of arbitrary multiplicities, as introduced by Feigin, Frenkel and

Toledano Laredo in [109]. In the same article, they prove that this model with multiplicities admits quadratic and higher degree Hamiltonians and describe their spectrum in terms of opers. As for the case with no multiplicities, these opers are differential operators, associated with the Langlands dual of $\mathfrak{g}$, the main difference being that they now possess so-called irregular singularities.

One can also construct generalised Gaudin models on a finite algebra $\mathfrak{g}$, associated with arbitrary skew-symmetric solutions of the classical Yang-Baxter equation on the loop algebra of $\mathfrak{g}$. The usual Gaudin model then corresponds to the standard rational solution of this equation. The nested Bethe ansatz for these models has been developed for rational and trigonometric solutions of the classical Yang-Baxter equation in [91] (for classical Lie algebras). The Bethe ansatz for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and elliptic solutions was described in [110].

A similar construction for non-skew symmetric solutions of the classical Yang-Baxter equation has been proposed by Skrypnyk in [111]. In particular, one can construct such solutions from automorphisms of the Lie algebra $\mathfrak{g}$ of finite order. The corresponding generalised Gaudin models are called cyclotomic Gaudin models (particular examples of these models also appear in [112]). In [113], Vicedo and Young proved the existence for cyclotomic models of higher degree Hamiltonians, using a method similar to the one of [95] for the usual Gaudin model. Moreover, they developed the (non-nested) algebraic Bethe ansatz for these models, describing the eigenvectors and eigenvalues of these Hamiltonians (some first results using the nested Bethe ansatz for particular cyclotomic Gaudin models were also found in [114]). The same authors constructed the Hamiltonians of cyclotomic Gaudin models with multiplicities in [115] (including higher degree Hamiltonians) and described their spectrum through the Bethe ansatz (under some restrictions on the multiplicities).

The cyclotomic and non-cyclotomic Gaudin models are complex spin chains. In [78], Vicedo considered similar models at the classical level, called dihedral Gaudin models, which admit additional reality conditions, ensuring in particular that the Hamiltonian of the model is real. So far, dihedral Gaudin models were not considered at the quantum level. It is natural to expect that these models can be quantised into quantum integrable spin chains with self-adjoint Hamiltonian.

1.3.3 Quantum affine Gaudin models

As explained above, the article [78] offers hope that quantum integrable σ-models can be studied as quantum affine Gaudin models. At the moment, such a goal is however quite distant, as not much is known about quantum affine Gaudin models. In particular, in [78], classical integrable σ-models are identified with realisations of dihedral affine Gaudin models, which thus include many generalisations compared to the simplest affine Gaudin models (multiple poles, cyclotomy, reality conditions, ...). These generalisations were never studied at the quantum level for affine Gaudin models (see previous subsection for finite ones). However, there exist promising results about quantum affine Gaudin models with no multiplicities, no cyclotomy and no reality conditions.

Hamiltonians of quantum affine Gaudin models. The work [93] on the Knizhnik-Zamolodchikov equation concerned arbitrary (symmetrisable) Kac-Moody algebras. In particular, it applies also to affine algebras, showing that quantum affine Gaudin models possess quadratic commuting Hamiltonians. Moreover, the Bethe ansatz for these Hamiltonians can be derived from the results of [93], allowing the diagonalisation of these Hamiltonians on tensor product of highest-weight representations of the affine algebra.

These results rely on the algebraic structure shared by all Kac-Moody algebras. In particular, it treats in a very similar way the finite Gaudin models and the affine ones. Considering the extensive number of results one can find on finite Gaudin models, one can hope that some of these results can also be extended to the affine case. For example, it was conjectured by Feigin and Frenkel [116], in the context of the study of the quantum KdV equation, that quantum affine Gaudin models also possess higher degree commuting Hamiltonians (more precisely an infinity of such Hamiltonians).

Affine FFR approach and ODE/IM correspondence. In the article [116], it was also conjectured that the FFR approach describing the spectrum of finite Gaudin models (see previous subsection)

generalises to affine Gaudin models. This is supported by the existence of a notion of affine opers, which is an affine equivalent of the finite opers used in the FFR approach for finite Gaudin models. On the long term, such an affine FFR approach could allow the description of the spectrum of integrable field theories.

One of the motivation for such a program is the so-called ODE/IM correspondence. This correspondence establishes a link between the eigenvalues of Integrals of Motion (IM) of some integrable quantum field theories and the solutions of some Ordinary Differential Equations (ODE). The first example of such a correspondence concerns the quantum KdV equation and was developed by Bazhanov, Lukyanov and Zamolodchikov in [117, 118], following the first insight of Dorey and Tateo [119]. It was shown in [116] that the ODE appearing in this example can be encoded in affine opers associated with the affine algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$. Following this observation, Feigin and Frenkel proposed in that article that the ODE/IM finds its origin in the (conjectural) affine FFR approach, as it would also relate the spectrum of integrable models with affine opers.

Other examples of ODE/IM correspondences have been proposed in the last two decades, including for some integrable field theories with twist function. As these models were reinterpreted as affine Gaudin models in [78], it was proposed, in the same spirit as the conjectures of Feigin and Frenkel, that the ODE/IM correspondence for these models originates from the affine FFR approach. We will discuss this idea in more details in the conclusion of this thesis, as one of its main perspective. Thus, we also refer to the discussion in Section 9.3 for references about the ODE/IM.

1.3.4 Goal of this thesis

During my PhD, I have worked on both finite and affine quantum Gaudin models and the FFR approach. The results I obtained in these projects will be the main subjects of the second part of this thesis.

Quantum affine Gaudin models and affine opers. One of my long-term goals is to prove the existence of higher degree Hamiltonians for quantum affine Gaudin models. In this thesis, I will present several results in this direction. The first one, discussed in Chapter 6, is actually a result at the classical level: applying the results of Chapter 4, we prove the existence of such Hamiltonians for classical affine Gaudin models (under slight restrictions but including real cyclotomic models with arbitrary multiplicities).

In Chapter 8, we develop further the theory of affine opers. Using this result and having in mind the idea of generalising the FFR approach to affine Gaudin models, we give and support conjectures on the form of the higher degree quantum Hamiltonians and their eigenvalues. The main result supporting these conjectures is the explicit construction of the first higher degree Hamiltonian, the cubic one (for untwisted affine algebras of type A).

Towards a cyclotomic FFR approach. In Chapter 7, we discuss a conjectured generalisation of the FFR approach to cyclotomic finite Gaudin models (see Subsection 1.3.2). For that, we define a notion of cyclotomic finite opers and extend known results about non-cyclotomic opers to this new setting.

1.4 Plan of this thesis

This thesis is separated into two parts.

The first one is devoted to the class of integrable classical field theories with twist function. We start by reviewing the Lax formalism of classical integrable field theories in Chapter 2. In particular, we discuss the Maillet bracket and non-ultralocal field theories. We end this chapter with the definition of the models with twist function, which serves as a general formalism for the first part of the thesis.

In Chapter 3, we present the main examples of integrable field theories with twist function, the integrable σ-models and their deformations. Most of this chapter is a review of known results about these models, in particular their Lax matrix and its Hamiltonian structure. The last section of the chapter presents some new results about the Bi-Yang-Baxter model. More precisely, we prove that this model is integrable at the Hamiltonian level and belongs to the class of models with twist function. This section is based on the article [P1], that I wrote during my PhD with F. Delduc, M. Magro and B. Vicedo, and a few subsequent results. In this section, we will give a less technical presentation of the content of [P1], which is meant to be readable independently. For completeness, we join the whole publication [P1] in the last part of this thesis.

In Chapter 4, we show the existence of an infinity of local charges in involution for all integrable models with twist function, under certain assumptions on the twist function. In particular, we apply this result to all integrable σ-models and their deformations. We go on to show that these charges generate compatible integrable equations and thus form an infinite integrable hierarchy. This chapter is mainly similar to the article [P3] of M. Magro, B. Vicedo and myself (we thus choose not to add this publication to this thesis).

Chapter 5 concerns the deformed symmetries of Yang-Baxter type deformations. Considering the poles of the twist function, we show that all these models admit a q-deformed Poisson-Hopf algebra of conserved charges. We then show that these charges are associated with Poisson-Lie symmetries of the model and construct explicitly these symmetries. In particular, we show that these symmetries are non-local. The content of this chapter is almost equivalent to the one of my PhD publication [P2] with F. Delduc, M. Magro and B. Vicedo (which is then not joined to this thesis).

The second part of this thesis concerns Gaudin models, either finite or affine, and both at the classical and the quantum levels.

Chapter 6 is a review about classical Gaudin models. We start by discussing these models for an arbitrary Lie algebra (with a non-degenerate invariant bilinear form). We then specialise to classical affine Gaudin models and explain how they can be interpreted as integrable field theories with twist function. Conversely, we discuss on the example of the Yang-Baxter model how integrable σ-models can be seen as realisations of affine Gaudin models. This chapter is almost entirely a review of known results (mostly the article [78] on affine Gaudin models). However, we present a new result in Subsection 6.2.6 by applying the results of Chapter 4 to affine Gaudin models, proving that they possess an integrable hierarchy.

Chapter 7 concerns quantum finite Gaudin models. We first review the construction of these models and their Hamiltonians. We then discuss their spectrum, presenting two approaches for its description: the Bethe ansatz and the FFR approach. Subsection 7.3.6 concerns the generalisation of the FFR approach to cyclotomic Gaudin models and is a short summary of the article [P4], that I wrote during my PhD with B. Vicedo. We add a copy of the article [P4] at the end of this thesis.

We discuss quantum affine Gaudin models and affine opers in Chapter 8, based on the preprints [P5] and [P6] of B. Vicedo, C.A.S. Young and myself. We first show that functions on affine opers take the form of hypergeometric integrals. Based on this result, we give a series of conjectures about quantum higher order Hamiltonians of affine Gaudin models and their spectrum. We support these conjectures by various observations, in particular the construction of the cubic Hamiltonian. The content of this chapter is mostly a less technical summary of [P5], which is meant to be readable independently (although some of the results of [P5] are not discussed in this thesis). For brevity, the main result of the preprint [P6] (the construction of the cubic Hamiltonian) is simply mentioned in Chapter 8, without any details or proofs. We add the whole preprints [P5, P6] in the last part of this thesis.

We conclude this thesis by a brief overview and some perspectives in Chapter 9. Technical appendices about Lie algebras, Poisson and symplectic geometries and $\mathcal{R}$-matrices form the last chapters A, B and C.

Part I

Integrable field theories with twist function

Chapter 2

Lax formalism and twist function

This chapter is a self-contained review about classical integrable hamiltonian field theories. The goal is to introduce the class of non-ultralocal integrable field theories with twist function, through the key notion of Lax matrix. We consider a two-dimensional Minkowski space-time, with time coordinate t and space coordinate x, which can be taken on the real line $\mathbb{R}$ or on the circle $\mathbb{S}^1$. We are interested in classical field theories on this space-time, given as a set of dynamical fields ϕ_i, depending on the coordinates (x,t), together with partial differential equations describing the time evolution of these fields. If x is taken in $\mathbb{R}$, we suppose that the fields $\phi_i(x,t)$ decrease sufficiently fast when x goes to $\pm\infty$. If x is taken on the circle $\mathbb{S}^1 \simeq [0,2\pi]$, we suppose that the fields $\phi_i(x,t)$ satisfy periodic boundary conditions $\phi_i(x=0,t) = \phi_i(x=2\pi,t)$.

Such a theory is said to be integrable if it admits an infinite number of conserved charges, in involution in the hamiltonian sense. We will discuss the hamiltonian framework and the involution properties in section 2.2. We first focus on a condition of existence of an infinite number of conserved charges through the Lax formalism.

2.1 Lax matrices and monodromies

Let us consider a pair of matrices $\mathcal{L}(\lambda,x,t)$ and $\mathcal{M}(\lambda,x,t)$, depending on the fields $\phi_i(x,t)$ of the model and on a complex auxiliary parameter $\lambda \in \mathbb{C}$ called the spectral parameter. More precisely, we will consider that $\mathcal{L}$ and $\mathcal{M}$ are valued in some finite dimensional Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot,\cdot]$ (see the Appendix A for conventions and definitions on Lie algebras). We say that $(\mathcal{L},\mathcal{M})$ forms a **Lax pair** of the model if the equations of motion of the fields ϕ_i can be rewritten as the **Lax equation** [9]

$$\partial_t \mathcal{L}(\lambda,x,t) - \partial_x \mathcal{M}(\lambda,x,t) + [\mathcal{M}(\lambda,x,t), \mathcal{L}(\lambda,x,t)] = 0, \quad \forall \lambda \in \mathbb{C}. \tag{2.1.1}$$

Note that this equation can be reformulated more geometrically as the zero curvature equation

$$[\nabla_t, \nabla_x] = 0$$

of the two-dimensional $\mathfrak{g}$-connection

$$\nabla = (\nabla_x, \nabla_t) = (\partial_x + \mathcal{L}(\lambda,x,t), \partial_t + \mathcal{M}(\lambda,x,t))\,.$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (for example the adjoint group of $\mathfrak{g}$). We define the transfer matrices of the connection ∇ between the points x and y as the path-ordered exponential

$$T(\lambda\,;x,y\,;t) = \overleftarrow{\mathrm{Pexp}}\left(-\int_y^x \mathrm{d}z\ \mathcal{L}(\lambda,z,t)\right),$$

valued in the group G. We refer to the Appendix A.5 for the definition and properties of path-ordered exponentials. In particular, it is proven in this appendix that the Lax equation (2.1.1) implies that

$$\partial_t T(\lambda\,;x,y;t) = T(\lambda\,;x,y;t)\mathcal{M}(\lambda,y,t) - \mathcal{M}(\lambda,x,t)T(\lambda\,;x,y;t), \tag{2.1.2}$$

for all $\lambda \in \mathbb{C}$. We will now distinguish the cases where x is taken on the real line or on the circle.

Real line coordinate. Let us suppose that x is taken on the real line $\mathbb{R}$. We define the **monodromy matrix** $T(\lambda,t)$ as the transfer matrix from $-\infty$ to $+\infty$, *i.e.* as

$$T(\lambda,t)=T(\lambda\,;+\infty,-\infty\,;t).$$

If we suppose that the field $\mathcal{M}(\lambda,x,t)$ decreases sufficiently fast at $x=\pm\infty$, we get from equation (2.1.2) that

$$\partial_t T(\lambda,t)=0,\quad \forall\lambda\in\mathbb{C}.$$

Thus, a consequence of the Lax equation (2.1.1) is that the whole monodromy matrix is conserved. Note that this equation is true for any value of the spectral parameter $\lambda\in\mathbb{C}$. Thus, varying λ, we get an infinite number of conserved charges. This was the reason for the introduction of the spectral parameter dependence in the Lax pair $(\mathcal{L},\mathcal{M})$: as G is a finite dimensional manifold, a monodromy matrix without a dependence on an arbitrary auxiliary parameter λ would produce only a finite number of conserved quantities.

Circle coordinate. Let us now suppose that the coordinate x is on the circle $\mathbb{S}^1\simeq[0,2\pi]$. We then define the monodromy matrix as

$$T(\lambda,t)=T(\lambda\,;0,2\pi\,;t).$$

Assuming periodic boundary conditions $\mathcal{M}(\lambda,0,t)=\mathcal{M}(\lambda,2\pi,t)$, the equation (2.1.2) becomes

$$\partial_t T(\lambda,t)=\left[T(\lambda,t),\mathcal{M}(\lambda,0,t)\right],\quad \forall\lambda\in\mathbb{C}.$$

In contrast with the case of a coordinate on the real line, the monodromy matrix $T(\lambda,t)$ is then not conserved. However, as the time evolution of $T(\lambda,t)$ is given by a commutator, one have

$$\partial_t\Phi\big(T(\lambda,t)\big)=0,\quad \forall\lambda\in\mathbb{C},$$

for any function $\Phi:G\to\mathbb{R}$ which is invariant under conjugacy. Thus, the quantity $\Phi\big(T(\lambda,t)\big)$ is conserved along the time evolution of the model. Here also, this equation of conservation is true for any value of $\lambda\in\mathbb{C}$, hence the existence of an infinite number of conserved quantities.

We now come back to the general case, with x either on the real line or on the circle. In both cases, the quantities $\Phi\big(T(\lambda,t)\big)$, for Φ a conjugacy invariant function on G, are conserved (for $x\in\mathbb{R}$, the whole monodromy $T(\lambda,t)$ is conserved, hence so is any function of $T(\lambda,t)$). A generic way of constructing such invariant functions Φ on G is to consider traces of powers in a matricial representation $\rho:G\to GL(d)$ of G. Indeed, the function

$$\begin{array}{rccc}\Phi_n^\rho: & G & \longrightarrow & \mathbb{R}\\ & g & \longmapsto & \mathrm{Tr}\big(\rho(g)^n\big)\end{array}$$

for $n\in\mathbb{N}$, is invariant under conjugacy transformation $g\mapsto hgh^{-1}$. Moreover, if $\mathfrak{g}$ is semi-simple (which will almost always be the case in the following chapters) and if we fix a representation ρ of G, it is known that the functions Φ_n^ρ generate all invariant functions on G. We can then consider the conserved charges

$$\mathcal{Q}_n(\lambda,t)=\Phi_n^\rho\big(T(\lambda,t)\big)=\mathrm{Tr}\big(\rho(T(\lambda,t))^n\big).$$

Let us finish this section by discussing light-cone coordinates. For relativistic field theories, it can be useful to work with the light-cone coordinates $x^\pm=\frac{1}{2}\left(t\pm x\right)$ instead of the usual space-time coordinates (x,t). The derivatives with respect to $x^\pm$ are then given by $\partial_\pm=\partial_t\pm\partial_x$. Introducing the light-cone components

$$\mathcal{L}_\pm(\lambda,x^\pm)=\mathcal{M}(\lambda,x,t)\pm\mathcal{L}(\lambda,x,t)\tag{2.1.3}$$

of the Lax pair $(\mathcal{L},\mathcal{M})$, the Lax equation (2.1.1) can be rewritten as

$$\partial_+\mathcal{L}_-(\lambda,x^\pm)-\partial_-\mathcal{L}_+(\lambda,x^\pm)+\left[\mathcal{L}_+(\lambda,x^\pm),\mathcal{L}_-(\lambda,x^\pm)\right]=0.\tag{2.1.4}$$

For relativistic field theories, it will be often simpler to find a Lax pair in light-cone coordinates. However, to construct conserved quantities, one then has to come back to the space-time Lax pair $(\mathcal{L},\mathcal{M})$ (more precisely, one needs the Lax matrix $\mathcal{L}$, *i.e.* the spatial component of the Lax pair).

2.2 Hamiltonian integrability and non-ultralocal Poisson brackets

As explained above, we say that a model is integrable if it possesses an infinite number of conserved quantities in involution (in the hamiltonian sense). In the previous section, we have seen how one can construct an infinite number of conserved charges using Lax pairs and monodromy matrices. It is then natural to ask under which conditions these quantities are in involution (as in this case we have a proof of the integrability of the model). To discuss this, we first need to discuss the hamiltonian formulation of classical field theories.

Hamiltonian field theories. A hamiltonian field theory is given by a phase space M, a Poisson bracket $\{\cdot,\cdot\}$ and a Hamiltonian $\mathcal{H}$. The phase space M describes the possible space configurations of the dynamical fields of the model. As we are in the hamiltonian formalism, we consider the fields as depending on the space coordinate x (the time t is no longer a coordinate of the field but is considered as a hamiltonian flow, as explained below). The phase space M is a Poisson manifold, meaning that the space $\mathcal{F}[M]$ of functionals on M is equipped with a Poisson bracket

$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{F}[M]\times\mathcal{F}[M] & \longrightarrow & \mathcal{F}[M] \\ & (f,g) & \to & \{f,g\} \end{array}.$$

The Poisson bracket is a skew-symmetric bilinear derivation satisfying the Jacobi identity:

$$\forall f,g,h\in\mathcal{F}[M],\quad \{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0.$$

The dynamic of the model on M is encoded in a functional $\mathcal{H}\in\mathcal{F}[M]$, called the Hamiltonian. More precisely, for any functional $f\in\mathcal{F}[M]$ (we restrict here to functionals which do not depend explicitly on the time t), its time evolution is given by the hamiltonian flow of $\mathcal{H}$:

$$\partial_t f=\{\mathcal{H},f\}\,. \tag{2.2.1}$$

We say that two functionals f and g in $\mathcal{F}[M]$ are in involution if their Poisson bracket vanishes, *i.e.* if

$$\{f,g\}=0.$$

In particular, by equation (2.2.1), f is in involution with the Hamiltonian $\mathcal{H}$ if and only if f is conserved, *i.e.* if $\partial_t f=0$.

Let us also introduce here the momentum $\mathcal{P}$ of the theory. It is also a functional in $\mathcal{F}[M]$, whose Hamiltonian flow generates the derivative with respect to the space coordinate x, *i.e.* such that

$$\forall f\in\mathcal{F}[M],\quad \{\mathcal{P},f\}=\partial_x f.$$

Poisson bracket of the Lax matrix. We will now use the definitions and results of Section 2.1. In particular, we suppose that we have a Lax pair $\big(\mathcal{L}(\lambda,x),\mathcal{M}(\lambda,x)\big)$ satisfying the Lax equation (2.1.1), which can be rewritten in the hamiltonian formalism as

$$\{\mathcal{H},\mathcal{L}(\lambda,x)\}-\{\mathcal{P},\mathcal{M}(\lambda,x)\}+[\mathcal{M}(\lambda,x),\mathcal{L}(\lambda,x)]=0. \tag{2.2.2}$$

One can then construct the monodromy matrix $T(\lambda)$ associated with this Lax pair and obtain an infinity of conserved charges $\Phi\big(T(\lambda)\big)$, with Φ invariant functions on G.

As we are interested in integrable models, it is natural to ask under which condition these conserved charges are in involution. They are defined from the monodromy matrix $T(\lambda)$, which itself is defined from the Lax matrix $\mathcal{L}(\lambda,x)$. Thus, the Poisson brackets of these charges depend on the Poisson brackets between the components of the Lax matrix.

To discuss Poisson brackets of Lie algebra-valued functionals in a compact way, we will use tensorial notations. For X a functional valued in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, we define

$$X_{\underline{\mathbf{1}}}=X\otimes\mathrm{Id}\qquad\text{and}\qquad X_{\underline{\mathbf{2}}}=\mathrm{Id}\otimes X,$$

which belong to $U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes\mathcal{F}[M]$, *i.e.* are $U(\mathfrak{g})\otimes U(\mathfrak{g})$-valued functionals.

Let $\{I^a\}$ be a basis of the Lie algebra $\mathfrak{g}$ and $X, Y \in \mathfrak{g}\otimes\mathcal{F}[M]$ be $\mathfrak{g}$-valued functionals. We decompose X and Y in this basis as

$$X = X_a I^a \qquad \text{and} \qquad Y = Y_a I^a,$$

where we used an implicit summation convention on repeated indices. The coefficients X_a and Y_a are then scalar-valued functionals in $\mathcal{F}[M]$. Let us define the Poisson bracket

$$\{X_{\underline{\mathbf{1}}}, Y_{\underline{\mathbf{2}}}\} = \{X_a, Y_b\}\, I^a \otimes I^b,$$

valued in $\mathfrak{g}\otimes\mathfrak{g}$. One checks that it is independent of the choice of basis $\{I^a\}$ of $\mathfrak{g}$. In particular, one can consider

$$\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda, x), \mathcal{L}_{\underline{\mathbf{2}}}(\mu, y)\}, \tag{2.2.3}$$

which encodes all Poisson brackets between the components of the Lax matrix.

Ultralocal Sklyanin bracket. As explained above, we are looking for conditions on the Poisson bracket (2.2.3) for the conserved charges $\Phi\big(T(\lambda)\big)$ and $\Psi\big(T(\mu)\big)$ to be in involution for all spectral parameters $\lambda, \mu \in \mathbb{C}$ and all invariant functions Φ and Ψ. A first sufficient condition for this has been found by Sklyanin in [18] (see also [120] and [15]). It requires the bracket (2.2.3) to be of the form

$$\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda, x), \mathcal{L}_{\underline{\mathbf{2}}}(\mu, y)\} = \big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu), \mathcal{L}_{\underline{\mathbf{1}}}(\lambda, x) + \mathcal{L}_{\underline{\mathbf{2}}}(\mu, x)\big]\, \delta_{xy}, \tag{2.2.4}$$

where δ_{xy} is the Dirac δ-distribution and $\mathcal{R}_{\underline{\mathbf{12}}}$ is a matrix in $\mathfrak{g}\otimes\mathfrak{g}$ depending on the spectral parameters $\lambda, \mu \in \mathbb{C}$. The skew-symmetry of the Poisson bracket (2.2.4) requires this matrix to be skew-symmetric, in the sense that

$$\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu) = -\mathcal{R}_{\underline{\mathbf{21}}}(\mu, \lambda). \tag{2.2.5}$$

The Poisson bracket (2.2.4) is said to be ultralocal, as it contains only a Dirac δ-distribution and not its derivatives. As we shall see in chapter 3, this is not enough to describe the hamiltonian integrability of various models, for which there exists a Lax matrix whose Poisson bracket also contains derivatives of δ-distribution.

Non-ultralocal Maillet bracket. A generalisation of the ultralocal Sklyanin bracket (2.2.4) has been presented by Maillet in [19,20]. It is still a sufficient condition on the Poisson bracket of the Lax matrix for the conserved charges extracted from the monodromy to be in involution. However, this condition allows for this Poisson bracket to contain a derivative δ'_{xy} of the Dirac distribution. It takes the form

$$\begin{aligned}\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda, x), \mathcal{L}_{\underline{\mathbf{2}}}(\mu, y)\} = {} & \big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu), \mathcal{L}_{\underline{\mathbf{1}}}(\lambda, x)\big]\, \delta_{xy} - \big[\mathcal{R}_{\underline{\mathbf{21}}}(\mu, \lambda), \mathcal{L}_{\underline{\mathbf{2}}}(\mu, x)\big]\, \delta_{xy} \\ & - \big(\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu) + \mathcal{R}_{\underline{\mathbf{21}}}(\mu, \lambda)\big)\, \delta'_{xy},\end{aligned} \tag{2.2.6}$$

where here also $\mathcal{R}$ is a $\mathfrak{g}\otimes\mathfrak{g}$-valued function of two spectral parameters λ and μ. This bracket is called **Maillet bracket**, or **non-ultralocal bracket** (due to the presence of the δ' term), and will play a major role in this thesis.

Let us note that here, we supposed that $\mathcal{R}$ was not dynamical, *i.e.* that it did not depend on x trough a dependence on the fields of the model. The original articles [19,20] of Maillet also included the case of dynamical $\mathcal{R}$ matrices. However, as we will not need this generalisation in this thesis, we will restrain to the case of non-dynamical $\mathcal{R}$ for simplicity.

It is worth noticing that the skew-symmetry of the Poisson bracket (2.2.6) is automatically satisfied, without requiring any further condition on the matrix $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu)$. If one considers the case of a skew-symmetric $\mathcal{R}$, *i.e.* which satisfies equation (2.2.5), the Maillet bracket (2.2.6) reduces to the Sklyanin bracket (2.2.4). In general, the matrix $\mathcal{R}$ of a Maillet bracket can possess a skew-symmetric part r and a symmetric part s:

$$r_{\underline{\mathbf{12}}}(\lambda, \mu) = \frac{1}{2}\big(\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu) - \mathcal{R}_{\underline{\mathbf{21}}}(\mu, \lambda)\big) \qquad \text{and} \qquad s_{\underline{\mathbf{12}}}(\lambda, \mu) = \frac{1}{2}\big(\mathcal{R}_{\underline{\mathbf{12}}}(\lambda, \mu) + \mathcal{R}_{\underline{\mathbf{21}}}(\mu, \lambda)\big).$$

The bracket (2.2.6) then takes the following form (this is in fact the original form proposed in the article [19]):

$$\begin{aligned}\big\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x),\mathcal{L}_{\underline{\mathbf{2}}}(\mu,y)\big\} = &\big[r_{\underline{\mathbf{12}}}(\lambda,\mu),\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x)+\mathcal{L}_{\underline{\mathbf{2}}}(\mu,x)\big]\,\delta_{xy}\\ &+\big[s_{\underline{\mathbf{12}}}(\lambda,\mu),\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x)-\mathcal{L}_{\underline{\mathbf{2}}}(\mu,x)\big]\,\delta_{xy}-2s_{\underline{\mathbf{12}}}(\lambda,\mu)\,\delta'_{xy},\end{aligned} \tag{2.2.7}$$

somtimes called a r/s-system. The non-ultralocality of the bracket, *i.e.* the term proportional to δ'_{xy}, is then controlled by the symmetric part s of $\mathcal{R}$.

2.3 Yang-Baxter equation, $\mathcal{R}$-matrices and twist function

Classical Yang-Baxter equation. As we have seen in the previous sections, a sufficient condition to have an integrable field theory is the existence of a Lax matrix $\mathcal{L}(\lambda,x)$ satisfying a non-ultralocal Poisson bracket (2.2.6), controlled by a $\mathfrak{g}\otimes\mathfrak{g}$-valued matrix $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)$. A natural question to ask at this point is whether this matrix can be anything or if there are some constraints on it, coming from the properties that a Poisson bracket must satisfy. As we have seen, the skew-symmetry of the bracket (2.2.6) is automatically verified, without requiring any additional constraint on $\mathcal{R}$. However, this bracket must also satisfy the Jacobi identity, *i.e.* we must have

$$\begin{aligned}\big\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda_1,x_1),\big\{\mathcal{L}_{\underline{\mathbf{2}}}(\lambda_2,x_2),\mathcal{L}_{\underline{\mathbf{3}}}(\lambda_3,x_3)\big\}\big\}+\big\{\mathcal{L}_{\underline{\mathbf{2}}}(\lambda_2,x_2),\big\{\mathcal{L}_{\underline{\mathbf{3}}}(\lambda_3,x_3),\mathcal{L}_{\underline{\mathbf{1}}}(\lambda_1,x_1)\big\}\big\}\\ +\big\{\mathcal{L}_{\underline{\mathbf{3}}}(\lambda_3,x_3),\big\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda_1,x_1),\mathcal{L}_{\underline{\mathbf{2}}}(\lambda_2,x_2)\big\}\big\}=0,\end{aligned} \tag{2.3.1}$$

which is to be understood as an identity in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$. Using the expression (2.2.6), one can check that this is equivalent to

$$\big[\mathcal{Y}_{\underline{\mathbf{123}}}(\lambda_1,\lambda_2,\lambda_3),\mathcal{L}_{\underline{\mathbf{1}}}(\lambda_1,x)\big]+\big[\mathcal{Y}_{\underline{\mathbf{231}}}(\lambda_2,\lambda_3,\lambda_1),\mathcal{L}_{\underline{\mathbf{2}}}(\lambda_2,x)\big]+\big[\mathcal{Y}_{\underline{\mathbf{312}}}(\lambda_3,\lambda_1,\lambda_2),\mathcal{L}_{\underline{\mathbf{3}}}(\lambda_3,x)\big]=0, \tag{2.3.2}$$

where

$$\mathcal{Y}_{\underline{\mathbf{123}}}(\lambda_1,\lambda_2,\lambda_3)=\big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]+\big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}_{\underline{\mathbf{23}}}(\lambda_2,\lambda_3)\big]+\big[\mathcal{R}_{\underline{\mathbf{32}}}(\lambda_3,\lambda_2),\mathcal{R}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]\,.$$

Thus, a sufficient condition for the Jacobi identity to be verified is the so-called **Classical Yang-Baxter Equation (CYBE)**:

$$\big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]+\big[\mathcal{R}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}_{\underline{\mathbf{23}}}(\lambda_2,\lambda_3)\big]+\big[\mathcal{R}_{\underline{\mathbf{32}}}(\lambda_3,\lambda_2),\mathcal{R}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]=0. \tag{2.3.3}$$

The matrices that satisfy this equation are called **$\mathcal{R}$-matrices**. Although it is not a necessary condition to have a Jacoby identity, we shall restrict ourselves from now on to matrices $\mathcal{R}$ which satisfy this stronger condition. Indeed, the CYBE (2.3.3) is much more easy to manipulate than the general condition (2.3.2), as it depends only on $\mathcal{R}$ and not on the Lax matrix $\mathcal{L}$ anymore. Moreover, the CYBE is an algebraic, non-dynamical, equation in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, which only depends on the Lie bracket over $\mathfrak{g}$. As we shall see in the next paragraph, this allows general schemes of constructions of solutions of the CYBE. For more details on the algebraic interpretations of the CYBE, we refer to the appendix C.

Standard $\mathcal{R}$-matrices. In this paragraph, we present some particular solutions of the CYBE called the standard $\mathcal{R}$-matrices. Let us suppose that $\mathfrak{g}$ is a semi-simple Lie algebra (see appendix A.2 for definitions and details). In particular, it possesses a non-degenerate invariant bilinear form κ. Let us consider the so-called split Casimir of $\mathfrak{g}$:

$$C_{\underline{\mathbf{12}}}=\kappa_{ab}I^a\otimes I^b\in\mathfrak{g}\otimes\mathfrak{g},$$

which is independent of the choice of basis $\{I^a\}$ of $\mathfrak{g}$. It is symmetric ($C_{\underline{\mathbf{12}}}=C_{\underline{\mathbf{21}}}$) and satisfies the following identity (see Appendix A.2.4):

$$\big[C_{\underline{\mathbf{12}}},X_{\underline{\mathbf{1}}}+X_{\underline{\mathbf{2}}}\big]=0,\quad\forall X\in\mathfrak{g}.$$

In particular, this implies that

$$\left[C_{\underline{\mathbf{12}}}, C_{\underline{\mathbf{23}}}\right] = \left[C_{\underline{\mathbf{32}}}, C_{\underline{\mathbf{13}}}\right] = -\left[C_{\underline{\mathbf{12}}}, C_{\underline{\mathbf{13}}}\right]. \tag{2.3.4}$$

Using this identity in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, together with the circle lemma

$$\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} - \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} - \frac{1}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)} = 0, \tag{2.3.5}$$

we get that

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda} \tag{2.3.6}$$

is a solution of the CYBE (2.3.3). We refer the reader to the appendix C for a more algebraic interpretation of this solution of the CYBE, related to the non-twisted loop algebra $\mathcal{L}(\mathfrak{g})$. We call it **the standard (non-twisted) $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$**. Let us note that, as the quadratic Casimir is symmetric, the standard non-twisted $\mathcal{R}$-matrix $\mathcal{R}^0$ is skew-symmetric.

Suppose now that we are given an automorphism σ of $\mathfrak{g}$, of finite order T (see the appendix A.4 for more details on these automorphisms). It is a standard result that it preserves the Killing form κ. As a consequence, the quadratic Casimir satisfies

$$\sigma_{\underline{\mathbf{1}}}\sigma_{\underline{\mathbf{2}}}C_{\underline{\mathbf{12}}} = C_{\underline{\mathbf{12}}}, \tag{2.3.7}$$

where $\sigma_{\underline{\mathbf{1}}}$ and $\sigma_{\underline{\mathbf{2}}}$ act respectively on the first and second tensor factor of $\mathfrak{g}\otimes\mathfrak{g}$. Let us chose a primitive T^{th} root of unity ω. We then define the so-called **standard twisted $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$** as

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\sum_{k=0}^{T-1}\frac{\sigma^k_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}}}{\mu-\omega^{-k}\lambda}. \tag{2.3.8}$$

Using the invariance (2.3.7) of the Casimir under σ and the fact that σ is an automorphism of $\mathfrak{g}$, one checks that $\mathcal{R}^0$ is a solution of the CYBE (2.3.3). We refer to appendix C for a more algebraic interpretation of this solution in terms of the twisted loop algebra $\mathcal{L}(\mathfrak{g},\sigma)$. Let us note that, contrarily to the non-twisted $\mathcal{R}$-matrix, the twisted standard $\mathcal{R}$-matrix is not skew-symmetric.

Both the twisted and non-twisted standard $\mathcal{R}$-matrices are singular at $\lambda=\mu$. Moreover, the asymptotic behaviour of the twisted matrix around this singularity is the one of the non-twisted one (divided by T), as we have

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda} + O\big((\lambda-\mu)^0\big).$$

As $\sigma^T=\mathrm{Id}$, the eigenvalues of σ are of the form ω^p where, by convention, we take $p\in\{0,\dots,T-1\}$. We denote by $\mathfrak{g}^{(p)}$ the corresponding eigenspace and by $\pi^{(p)}$ the projection on $\mathfrak{g}^{(p)}$ in the direct sum $\mathfrak{g}=\bigoplus_{p=0}^{T-1}\mathfrak{g}^{(p)}$ (see the appendix A.4 for more details about these eigenspaces). Defining $C^{(p)}_{\underline{\mathbf{12}}} = \pi^{(p)}_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}} = C^{(-p)}_{\underline{\mathbf{21}}}$, we can rewrite $\mathcal{R}^0$ as

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \sum_{p=0}^{T-1}\frac{\lambda^p\mu^{T-1-p}}{\mu^T-\lambda^T}C^{(p)}_{\underline{\mathbf{12}}}. \tag{2.3.9}$$

Twist function. We end this section by a quick but crucial remark [121–123]. Let us fix a solution $\mathcal{R}^0$ of the CYBE (2.3.3). Typically, in this thesis, we will choose $\mathcal{R}^0$ to be a non-twisted or twisted standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$, as described in the previous paragraph. It is easy to check that for any function φ from $\mathbb{C}$ to itself, the matrix $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu) = \mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\varphi(\mu)^{-1}$ is also a solution of the CYBE (2.3.3). Indeed, the left-hand side of the CYBE for $\mathcal{R}$ is simply the one for $\mathcal{R}^0$ multiplied by $\varphi(\lambda_2)^{-1}\varphi(\lambda_3)^{-1}$, so the CYBE for $\mathcal{R}^0$ implies the one for $\mathcal{R}$. This function, that is often supposed to be a rational function of the spectral parameter, is called **the twist function** [123]. We refer the reader to the appendix C for the loop algebra interpretation of the corresponding $\mathcal{R}$-matrix.

2.4 Integrable models with twist function

We define an integrable model with twist function to be a Hamiltonian field theory such that:

- its equations of motion admit a **Lax pair** $\big(\mathcal{L}(\lambda,x),\mathcal{M}(\lambda,x)\big)$, valued in $\mathfrak{g}$ and depending rationally on the spectral parameter ;
- the Lax matrix $\mathcal{L}$ satisfies a **non-ultralocal Maillet bracket** of the form (2.2.6) ;
- the $\mathcal{R}$-matrix underlying the Maillet bracket is of the form

$$\mathcal{R}_{\underline{12}}(\lambda,\mu)=\mathcal{R}^0_{\underline{12}}(\lambda,\mu)\varphi(\mu)^{-1} \tag{2.4.1}$$

  with $\mathcal{R}^0$ a **standard $\mathcal{R}$-matrix** on $\mathcal{L}(\mathfrak{g})$ and φ a rational function, the **twist function** ;
- if the standard matrix $\mathcal{R}^0$ is twisted by an automorphism σ, the Lax matrix and the twist function satisfy some **equivariance properties**, described below ;
- the Lax matrix and the automorphism σ satisfy some **reality conditions**, described below.

Equivariance properties. As explained above, if the standard matrix $\mathcal{R}^0$ is twisted by an automorphism σ of order T, we require some additional equivariance condition on $\mathcal{L}$ and φ. In this case, we speak of a **cyclotomic model**.

As the order of σ is T, it defines an action of the cyclic group $\mathbb{Z}_T=\mathbb{Z}/T\mathbb{Z}$ on $\mathfrak{g}$. On the other hand, $\mathbb{Z}_T$ can be seen as acting on the complex numbers $\mathbb{C}$ *via* multiplication by ω. We then remark that the first tensor factor of the matrix $\mathcal{R}^0$ is equivariant under these two actions, in the sense that

$$\sigma_{\underline{1}}\mathcal{R}^0_{\underline{12}}(\lambda,\mu)=\mathcal{R}^0_{\underline{12}}(\omega\lambda,\mu). \tag{2.4.2}$$

We will suppose that the Lax matrix $\mathcal{L}$ possesses a similar equivariance property, namely

$$\sigma\big(\mathcal{L}(\lambda,x)\big)=\mathcal{L}(\omega\lambda,x). \tag{2.4.3}$$

Moreover, using the invariance (2.3.7) of the Casimir under σ, one gets, on the second tensor factor,

$$\sigma_{\underline{2}}\mathcal{R}^0_{\underline{12}}(\lambda,\mu)=\omega\mathcal{R}^0_{\underline{12}}(\lambda,\omega\mu). \tag{2.4.4}$$

The compatibility of these properties with the Poisson bracket (2.2.6) imposes that

$$\varphi(\omega\lambda)=\omega^{-1}\varphi(\lambda), \tag{2.4.5}$$

from which we deduce that $\lambda\varphi(\lambda)$ is invariant under the action of $\mathbb{Z}_T$. Thus, there exists a rational function ζ such that

$$\lambda\varphi(\lambda)=\zeta(\lambda^T). \tag{2.4.6}$$

Let us note that the equivariance properties (2.4.2) and (2.4.4) of $\mathcal{R}^0$, together with the one of the twist function (2.4.5), implies

$$\sigma_{\underline{1}}\mathcal{R}_{\underline{12}}(\lambda,\mu)=\mathcal{R}_{\underline{12}}(\omega\lambda,\mu)\qquad\text{and}\qquad\sigma_{\underline{2}}\mathcal{R}_{\underline{12}}(\lambda,\mu)=\mathcal{R}_{\underline{12}}(\lambda,\omega\mu).$$

Thus, the matrix $\mathcal{R}$ is equivariant both in the first and the second tensor factors.

Reality condition. In non-ultralocal models with twist function, we suppose that the Lax matrix depends rationally on the spectral parameter λ, which is a complex number. Thus, the Lie algebra $\mathfrak{g}$ in which the Lax matrix is valued is a complex Lie algebra. However, many field theories are described by real valued fields: we keep track of this in the Lax matrix through reality conditions.

Let $\mathfrak{g}_0$ be a real form of the complex Lie algebra $\mathfrak{g}$ (see the Appendix A.3 for definitions and conventions on real forms). It can be seen as the algebra of fixed points under an anti-automorphism involution τ of $\mathfrak{g}$. We will suppose that the Lax matrix satisfies the following **reality condition**:

$$\tau\big(\mathcal{L}(\lambda,x)\big) = \mathcal{L}\left(\bar{\lambda},x\right), \tag{2.4.7}$$

where $\bar{\lambda}$ is the complex conjugate of the spectral parameter λ.

If the model is cyclotomic, *i.e.* the standard matrix $\mathcal{R}^0$ is twisted by an automorphism σ, we will assume that σ satisfies the following property:

$$\sigma\circ\tau = \tau\circ\sigma^{-1}, \tag{2.4.8}$$

that we call the **dihedrality condition**. The quadratic Casimir satisfies

$$\tau_{\underline{1}}C_{\underline{12}} = \tau_{\underline{2}}C_{\underline{12}} = C_{\underline{12}}.$$

Combining this equation with the dihedrality condition (2.4.8), we get the following reality conditions on the standard $\mathcal{R}$-matrix:

$$\tau_{\underline{1}}\mathcal{R}^0_{\underline{12}}(\lambda,\mu) = \tau_{\underline{2}}\mathcal{R}^0_{\underline{12}}(\lambda,\mu) = \mathcal{R}^0_{\underline{12}}\left(\bar{\lambda},\bar{\mu}\right). \tag{2.4.9}$$

The compatibility of the Poisson bracket (2.2.6) with the reality conditions (2.4.7) and (2.4.9) requires that the twist function is a rational function with real coefficients, *i.e.* that

$$\overline{\varphi(\lambda)} = \varphi(\bar{\lambda}). \tag{2.4.10}$$

Combining the two conditions (2.4.9) and (2.4.10), we find

$$\tau_{\underline{1}}\mathcal{R}_{\underline{12}}(\lambda,\mu) = \tau_{\underline{2}}\mathcal{R}_{\underline{12}}(\lambda,\mu) = \mathcal{R}_{\underline{12}}\left(\bar{\lambda},\bar{\mu}\right). \tag{2.4.11}$$

Dihedrality. Let us end this section by explaining the name of the dihedrality condition (2.4.8). We consider the abstract group Γ_T defined by the following presentation:

$$\Gamma_T = \left\langle \mathrm{r},\mathrm{s} \;\middle|\; \mathrm{r}^T = \mathrm{Id},\ \mathrm{s}^2 = \mathrm{Id},\ \mathrm{rs} = \mathrm{sr}^{-1}\right\rangle, \tag{2.4.12}$$

i.e. the group generated by two abstract generators r and s satisfying the relations above. It is a classical result from group theory that Γ_T is then of order $2T$ and is isomorphic to the **dihedral group** D_T, defined as the group of symmetries of the regular polygon with T edges (r then represents the rotation of angle $\frac{2\pi}{T}$ and s the symmetry with respect to an axis of the polygon).

Let $\omega = \exp\left(\frac{2i\pi}{T}\right)$ be a primitive T^{th}-root of unity. There is a natural action of the dihedral group Γ_T on the complex plan as the multiplication by ω and the complex conjugation:

$$\forall\,\lambda\in\mathbb{C},\quad \mathrm{r}.\lambda = \omega\lambda \quad\text{ and }\quad \mathrm{s}.\lambda = \overline{\lambda}. \tag{2.4.13}$$

In the same way, there is an action of Γ_T on the Lie algebra $\mathfrak{g}$, defined from the automorphism σ of order T and the antilinear involutive automorphism τ introduced above:

$$\forall\,X\in\mathfrak{g},\quad \mathrm{r}.\mathrm{X} = \sigma(\mathrm{X}) \quad\text{ and }\quad \mathrm{s}.\mathrm{X} = \tau(\mathrm{X}),$$

as σ and τ satisfy the dihedrality condition (2.4.8). The cyclotomic property (2.4.3) and the reality condition (2.4.7) imposed on the Lax matrix $\mathcal{L}$ can be understood as the equivariance of $\mathcal{L}$ under the action of Γ_T:

$$\forall\,\mathrm{u}\in\Gamma_T,\quad \mathrm{u}.\mathcal{L}(\lambda,x) = \mathcal{L}(\mathrm{u}.\lambda,x).$$

Chapter 3

Examples of models with twist function: integrable σ-models

In the previous chapter, we introduced the class of integrable models with twist function. In the present one, we discuss various examples of such models. All the examples considered here are integrable σ-models but there exist other theories with twist function, such as the Korteweg-de Vries (KdV) equation or affine Toda field theories, that we shall not discuss here (see [78]).

The first three sections of this chapter are reviews about σ-models which are known to possess a twist function. The first section discusses the general framework of integrable σ-models. The second section concerns the Principal Chiral Model (PCM), which is the simplest example of integrable σ-models, and the $\mathbb{Z}_T$-coset models. We review here their Lax formulation and the Poisson bracket of their Lax matrix, exhibiting the structure of a non-ultralocal model with twist function.

The third section concerns integrable deformations of the PCM and the $\mathbb{Z}_T$-coset models. Indeed, it has been discovered in the past decades that these models admit continuous deformations (controlled by real parameters) which still admit a Lax pair. Moreover, the hamiltonian analysis of these models has been conducted and it has been found that the Poisson bracket of their Lax matrix is also governed by a twist function. More precisely, it has been observed that the effect of the deformation is to deform the poles of the twist function of the model. In the third section, we review different results on this subject that exist in the literature and present an overall view of the panorama of integrable deformed σ-models and their twist function.

The fourth and final section is about a particular two-parameter deformation of the PCM, the so-called Bi-Yang-Baxter model. This model was introduced by Klimčik in [28], who also proved in [33] that its equations of motion can be recast as a Lax equation, thus proving the existence of an infinite number of conserved charges for this model. However, the hamiltonian analysis of the Bi-Yang-Baxter model was never carried out. My first PhD project was the study of the Poisson bracket of the Lax matrix of the Bi-Yang-Baxter model. In Section 3.4, after recalling the construction of the model and of its Lax pair, I present the results I found in [P1] concerning the hamiltonian integrability of the Bi-Yang-Baxter model. In particular, we shall see that the Bi-Yang-Baxter model also enters the class of models with twist function.

3.1 Generalities about integrable σ-models

3.1.1 Lie group valued field and currents

Before going into the details of the different models, let us discuss some general aspects regarding integrable σ-models. A σ-model (not necessarily integrable) is a two-dimensional field theory with dynamical fields $\phi : \Sigma \to M$, where Σ is the two-dimensional Minkowski space-time with coordinates (x, t), called the worldsheet, and M is a Riemannian manifold, which we call the target space. We are interested in some of these models, which have the additional property of being integrable. These share the property that their target space is a real Lie group G_0 or one of its cosets, *i.e.* the quotient G_0/H of G_0 by a particular subgroup H. We shall suppose here that G_0 is a connected semi-simple

Lie group. We will denote $G = G_0^{\mathbb{C}}$ the complexification of G_0, which is then a complex semi-simple Lie group (see Appendix A.3).

As we will see in more details in this chapter, we shall not describe the coset models with fields directly valued in G_0/H. Instead, we shall consider an equivalent formulation with a field valued in G_0, together with a gauge symmetry in H, which eliminates the additional degrees of freedom. As a result, all the models we shall consider are defined by a field $g : \Sigma \to G_0$, valued in the Lie group G_0. We shall use extensively the so-called **left currents**

$$j_0^L = g^{-1}\partial_t g = -\big(\partial_t g^{-1}\big)g \quad \text{and} \quad j_1^L = g^{-1}\partial_x g = -\big(\partial_x g^{-1}\big)g,$$

which are $\mathfrak{g}_0$-valued fields, with $\mathfrak{g}_0$ the Lie algebra of G_0. These are called left currents as they are invariant under the left multiplication

$$L_h : g \longmapsto hg$$

of g by a constant element h of G_0. There are also **right currents**

$$j_0^R = g\partial_t g^{-1} = -\big(\partial_t g\big)g^{-1} \quad \text{and} \quad j_1^R = g\partial_x g^{-1} = -\big(\partial_x g\big)g^{-1},$$

invariant under the right multiplication $R_h : g \mapsto gh$. They are related to the left currents by a conjugacy transformation:

$$j_\mu^R = -g j_\mu^L g^{-1}, \quad \text{for } \mu = 0, 1. \tag{3.1.1}$$

The left and right currents are flat, *i.e.* they satisfy the zero curvature equation:

$$\partial_t j_1^{L,R} - \partial_x j_0^{L,R} + \Big[j_0^{L,R}, j_1^{L,R}\Big] = 0, \tag{3.1.2}$$

called **the Maurer-Cartan equation**. Finally let us also introduce the light-cone currents:

$$j_\pm^L = g^{-1}\partial_\pm g = j_0^L \pm j_1^L \quad \text{and} \quad j_\pm^R = g\partial_\pm g^{-1} = j_0^R \pm j_1^R. \tag{3.1.3}$$

In light-cone coordinates, the Maurer-Cartan equation reads

$$\partial_+ j_-^{L,R} - \partial_- j_+^{L,R} + \Big[j_+^{L,R}, j_-^{L,R}\Big] = 0. \tag{3.1.4}$$

3.1.2 Hamiltonian formulation

Conjugate momenta and phase space. The σ-models are naturally defined as Lagrangian field theories with an action $S[g]$ depending on the field $g(x,t)$. As we are interested in the integrability properties of σ-models, we will need to consider them in the Hamiltonian formalism. We thus pass from the field g valued in G_0 and depending on (x,t) to fields depending only on the space coordinate x and valued in the cotangent bundle T^*G_0. These fields form the phase space of the model and contain the G_0-valued coordinate-field $g(x)$, together with some conjugate momentum fields.

Let us be more explicit about that. We fix some local coordinates $\psi^i : G_0 \to \mathbb{R}$ $(i = 1, \cdots, n)$ of G_0, where n is the dimension of G_0 (more precisely, the coordinates ψ^i could be defined only on an open subset of G_0). Locally, one can describe the field g as n real-valued fields $\phi^i(x) = \psi^i\big(g(x)\big)$. The Lagrangian density of the model can then be written as a function $L\big(\phi^i, \partial_\mu \phi^i\big)$ of the fields ϕ^i's and their space-time derivatives. In the Hamiltonian formalism, these coordinate fields are paired with conjugate momenta $\pi_i(x)$, obtained from the Lagrangian density L as

$$\pi_i = \frac{\partial L}{\partial(\partial_t \phi^i)}.$$

The fields ϕ^i's and π_i's then describe the whole phase space of the model and satisfy the canonical Poisson brackets

$$\big\{\phi^i(x), \phi^j(y)\big\} = 0, \tag{3.1.5a}$$
$$\big\{\pi_i(x), \pi_j(y)\big\} = 0, \tag{3.1.5b}$$
$$\big\{\pi_i(x), \phi^j(y)\big\} = \delta_i^j \delta_{xy}, \tag{3.1.5c}$$

where δ_i^j is the Kronecker symbol and δ_{xy} is the Dirac δ-distribution.

We now want a coordinate-free description of this phase-space. The coordinate fields ϕ^i are equivalent to the G_0-valued field g. Thus, there is left to find a way to encode the conjugate momenta π_i in a coordinate-free way. Let ∂_i denote the derivative with respect to the local coordinate ψ^i: $g^{-1}\partial_i g$ then belongs to the Lie algebra $\mathfrak{g}_0$. Let us fix a basis $\{I_a\}_{a=1,\cdots,n}$ of $\mathfrak{g}_0$ and define $L^a{}_i$ such that

$$g^{-1}\partial_i g = L^a{}_i I_a.$$

The matrix $\left(L^a{}_i\right)_{i,a=1,\cdots,n}$ is then invertible. We shall write $\left(L^i{}_a\right)_{i,a=1,\cdots,n}$ its inverse. It verifies

$$L^a{}_i L^i{}_b = \delta^a_b \quad \text{and} \quad L^i{}_a L^a{}_j = \delta^i_j,$$

where the summation on repeated indices is implied. We then define the $\mathfrak{g}_0$-valued field

$$X = L^i{}_a \pi_i \, \kappa^{ab} I_b, \tag{3.1.6}$$

where κ^{ab} is the Killing form of $\mathfrak{g}_0$ in the basis $\{I^a\}$, dual to $\{I_a\}$. One can then check that this field is invariant under a change of the coordinates $\{\psi^i\}$ on G_0 or of basis $\{I^a\}$ of $\mathfrak{g}_0$.

Thus, the phase-space of the model is parametrised by the G_0-valued field $g(x)$ and the $\mathfrak{g}_0$-valued field $X(x)$. Together, they form a field valued in the cotangent bundle T^*G_0. One can check that the canonical brackets (3.1.5) can be rewritten in terms of the fields g and X as

$$\{g_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} = 0, \tag{3.1.7a}$$
$$\{X_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} = g_{\underline{\mathbf{2}}}(x) C_{\underline{\mathbf{12}}} \delta_{xy}, \tag{3.1.7b}$$
$$\{X_{\underline{\mathbf{1}}}(x), X_{\underline{\mathbf{2}}}(y)\} = -\left[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}(x)\right] \delta_{xy}, \tag{3.1.7c}$$

where $C_{\underline{\mathbf{12}}}$ is the quadratic Casimir of $\mathfrak{g}_0$ (see Appendix A).

Left and right multiplication. Let us investigate what is the Hamiltonian flow generated by the integral of the field X. Let us define

$$m^R = \int \mathrm{d}x \; X(x), \tag{3.1.8}$$

where the integral is either on the real line $\mathbb{R}$ or on the circle $\mathbb{S}^1$, depending on the space-time of the theory. This is a $\mathfrak{g}_0$-valued quantity, which can be seen as $\mathfrak{g}_0^*$-valued *via* the duality between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ induced by the Killing form κ. Moreover, we deduce from the Poisson bracket (3.1.7c) that it satisfies the Kostant-Kirillov bracket (see Appendix B.2).

$$\{m^R_{\underline{\mathbf{1}}}, m^R_{\underline{\mathbf{2}}}\} = -\left[C_{\underline{\mathbf{12}}}, m^R_{\underline{\mathbf{2}}}\right].$$

Thus, it can be chosen as the moment map of an infinitesimal action of $\mathfrak{g}_0$ (see Appendix B.4). For $\epsilon \in \mathfrak{g}_0$ infinitesimal, the corresponding action on an observable $\mathcal{O} \in \mathcal{F}[M]$ is given by

$$\delta^R_\epsilon \mathcal{O} = \kappa\big(\epsilon, \{m^R, \mathcal{O}\}\big).$$

Using the completeness relation (A.2.7), one finds that the action δ^R_ϵ on the phase space parametrised by the fields g and X is given by

$$\delta^R_\epsilon g(x) = g(x)\epsilon \quad \text{and} \quad \delta^R_\epsilon X(x) = -\left[\epsilon, X(x)\right].$$

Thus, the action of δ^R_ϵ on g is the infinitesimal transformation induced by the right multiplication $g \mapsto gh$ by constant elements h of G_0. One can also describe the left multiplication action by considering gXg^{-1} instead of X. Indeed, the canonical bracket can be also written as

$$\{g_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} = 0, \tag{3.1.9a}$$
$$\left\{\left(gXg^{-1}\right)_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\right\} = C_{\underline{\mathbf{12}}} \, g_{\underline{\mathbf{2}}}(x) \delta_{xy}, \tag{3.1.9b}$$
$$\left\{\left(gXg^{-1}\right)_{\underline{\mathbf{1}}}(x), \left(gXg^{-1}\right)_{\underline{\mathbf{2}}}(y)\right\} = \left[C_{\underline{\mathbf{12}}}, \left(gXg^{-1}\right)_{\underline{\mathbf{2}}}(x)\right] \delta_{xy}. \tag{3.1.9c}$$

The moment map

$$m^L = \int \mathrm{d}x\; g(x)X(x)g(x)^{-1} \tag{3.1.10}$$

generates the infinitesimal left multiplication on g:

$$\delta^L_\epsilon g(x) = \epsilon g(x) \quad \text{and} \quad \delta^L_\epsilon\big(gXg^{-1}\big)(x) = \big[\epsilon, \big(gXg^{-1}\big)(x)\big]\,.$$

One can remark that X and gXg^{-1} are respectively invariant under the left and right actions:

$$\delta^L_\epsilon X(x) = 0 \quad \text{and} \quad \delta^R_\epsilon\big(gXg^{-1}\big)(x) = 0.$$

Momentum and Hamiltonian. Let us consider the current $j^L = j^L_1 = g^{-1}\partial_x g$ (when working in Hamiltonian formalism, we shall drop the indices $\mu = 1$ as we dropped out the time dependence and kept only the space dependence). We will use the following Poisson brackets, which are a consequence of (3.1.7):

$$\big\{g_{\underline{\mathbf{1}}}(x), j^L_{\underline{\mathbf{2}}}(y)\big\} = 0, \tag{3.1.11a}$$

$$\big\{j^L_{\underline{\mathbf{1}}}(x), j^L_{\underline{\mathbf{2}}}(y)\big\} = 0, \tag{3.1.11b}$$

$$\big\{X_{\underline{\mathbf{1}}}(x), j^L_{\underline{\mathbf{2}}}(y)\big\} = -\big[C_{\underline{\mathbf{12}}}, j^L_{\underline{\mathbf{2}}}(x)\big]\delta_{xy} - C_{\underline{\mathbf{12}}}\delta'_{xy}. \tag{3.1.11c}$$

Using these Poisson brackets, we find that the total momentum of the theory is given by

$$\mathcal{P} = \int \mathrm{d}x\; \kappa\big(X, j^L\big).$$

Indeed, one checks that

$$\{\mathcal{P}, g(x)\} = g(x)j^L(x) = \partial_x g(x) \quad \text{and} \quad \{\mathcal{P}, X(x)\} = \partial_x X(x).$$

To conclude this subsection, let us say a few words about the Hamiltonian of the system. If we choose a set of coordinate fields ϕ^i, as we have done above, the Hamiltonian $\mathcal{H}$ is related to the Lagragian density L by the Legendre transformation

$$\mathcal{H} = \int \mathrm{d}x\; \big(\pi_i \partial_t \phi^i - L\big),$$

with π_i the conjugate momenta. These momenta are encoded in a coordinate-free way in the current X defined as (3.1.6). One can then rewrite the Hamiltonian also in a coordinate-free way as

$$\mathcal{H} = \int \mathrm{d}x\; \Big(\kappa\big(X, g^{-1}\partial_t g\big) - L\Big). \tag{3.1.12}$$

3.2 Undeformed integrable σ-models

3.2.1 The Principal Chiral Model

Action and equations of motion. The Principal Chiral Model (PCM) is the simplest example of an integrable σ-model. Its target space is a real Lie group G_0, which we assume to be connected and semi-simple, equipped with the Killing metric. It is defined by the action

$$S_{\mathrm{PCM}}[g] = \frac{K}{2}\int_\Sigma \mathrm{d}x\,\mathrm{d}t\; \Big(\kappa(g^{-1}\partial_t g, g^{-1}\partial_t g) - \kappa(g^{-1}\partial_x g, g^{-1}\partial_x g)\Big),$$

where K is a global constant factor and κ is the Killing form on $\mathfrak{g}_0$. This can be reexpressed in terms of the left or right currents $j^{L,R}_\mu$ as

$$S_{\mathrm{PCM}}[g] = \frac{K}{2}\int_\Sigma \mathrm{d}x\,\mathrm{d}t\; \Big(\kappa(j^L_0, j^L_0) - \kappa(j^L_1, j^L_1)\Big) = \frac{K}{2}\int_\Sigma \mathrm{d}x\,\mathrm{d}t\; \Big(\kappa(j^R_0, j^R_0) - \kappa(j^R_1, j^R_1)\Big), \tag{3.2.1}$$

where the expression with the right currents j_μ^R follows from equation (3.1.1) and the invariance of κ under conjugacy transformation.

It will be useful to express the action of the PCM using light-cone coordinates $x^\pm$ and the light-cone currents $j_\pm^{L,R}$. One then finds

$$S_{\mathrm{PCM}}[g] = K\int_\Sigma \mathrm{d}x^+\,\mathrm{d}x^-\,\kappa\big(j_+^L, j_-^L\big) = K\int_\Sigma \mathrm{d}x^+\,\mathrm{d}x^-\,\kappa\big(j_+^R, j_-^R\big) \tag{3.2.2}$$

Varying this action with respect to g, one finds that the equations of motion of the PCM can be written as the conservation equations

$$\partial_t j_0^{L,R} - \partial_x j_1^{L,R} = \frac{1}{2}\Big(\partial_+ j_-^{L,R} + \partial_- j_+^{L,R}\Big) = 0. \tag{3.2.3}$$

The equation of conservation of j_μ^L is equivalent to the one of j_μ^R, using the equation (3.1.1).

Global symmetries. As we have just observed, the equations of motion of the PCM take the form of the conservation equations of the $\mathfrak{g}_0$-valued currents $j_\mu^{L,R}$. Thus, the quantities

$$K\int \mathrm{d}x\; j_0^L(x,t) \qquad \text{and} \qquad K\int \mathrm{d}x\; j_0^R(x,t) \tag{3.2.4}$$

are conserved (the global factor K is here for future convenience). By the Noether theorem, these conserved charges are associated with global symmetries of the model.

Let us recall that the right and left currents are respectively invariant under the right multiplication $g \mapsto gh$ and the left multiplication $g \mapsto hg$ of g by a constant element h of G_0. As the Lagrangian density of the PCM can be entirely written in terms of either one of these currents, it is clear that these transformations are global symmetries of the PCM. Applying the Noether theorem, one finds that the conserved quantities associated with these symmetries are the ones of equation (3.2.4). It is worth noticing that the charge associated with the right multiplication is the integral of the left current and vice-versa.

Lax equation. Let us now investigate the integrability properties of the PCM, following the ideas of Chapter 2. In particular, let us start by exhibiting the Lax representation of the equations of motion of the PCM, using a general scheme developped by Zakharov and Mikhailov [124] (some first results were also found by Pohlmeyer in [125]). The latter relies on the existence of a conserved and flat current (*i.e.* satisfying a conservation equation and a zero curvature equation). In the case of the PCM, we have seen that the equations of motion (3.2.3) take the form of the conservation equations of the currents j_μ^L and j_μ^R. Moreover, according to the Maurer-Cartan equation (3.1.2), these currents are flat. Let us then define the following light-cone Lax pair

$$\mathcal{L}_\pm^{\mathrm{PCM}}(\lambda, x, t) = \frac{j_\pm^L}{1 \mp \lambda}, \tag{3.2.5}$$

depending on the spectral parameter $\lambda \in \mathbb{C}$ (one could also have considered a Lax matrix associated with right currents, but we shall focus here on this particular choice). One then has

$$\partial_+\mathcal{L}_-^{\mathrm{PCM}}(\lambda) - \partial_-\mathcal{L}_+^{\mathrm{PCM}}(\lambda) + \big[\mathcal{L}_+^{\mathrm{PCM}}(\lambda), \mathcal{L}_-^{\mathrm{PCM}}(\lambda)\big] = \frac{1}{1-\lambda^2}\Big(\partial_+ j_-^L - \partial_- j_+^L + \big[j_+^L, j_-^L\big] - \lambda\left(\partial_+ j_-^L + \partial_- j_+^L\right)\Big).$$

The constant term in the bracket exactly vanishes according to the (light-cone) Maurer-Cartan equation (3.1.2). It is worth noticing that this equation is true off-shell, *i.e.* without using the equation of motion of the model. The above equation then reduces to

$$\partial_+\mathcal{L}_-^{\mathrm{PCM}}(\lambda) - \partial_-\mathcal{L}_+^{\mathrm{PCM}}(\lambda) + \big[\mathcal{L}_+^{\mathrm{PCM}}(\lambda), \mathcal{L}_-^{\mathrm{PCM}}(\lambda)\big] = \frac{\lambda}{\lambda^2-1}\left(\partial_+ j_-^L + \partial_- j_+^L\right).$$

Thus, we see that the equations of motion (3.2.3) of the PCM are equivalent to the (light-cone) Lax equation (2.1.4). This proves the existence of a Lax pair representation of the PCM. Moreover, we see that this procedure for constructing a Lax pair generalises to every model which possesses a flat and conserved current.

Let us end this paragraph by exhibiting the Lax pair $(\mathcal{L}_{\text{PCM}}, \mathcal{M}_{\text{PCM}})$ corresponding to the space-time coordinates (x,t). Inverting the relation (2.1.3) and using the expression (3.2.5) of $\mathcal{L}_{\pm}^{\text{PCM}}$, one finds

$$\mathcal{L}_{\text{PCM}}(\lambda) = \frac{j_1^L + \lambda j_0^L}{1-\lambda^2} \quad \text{and} \quad \mathcal{M}_{\text{PCM}}(\lambda) = \frac{j_0^L + \lambda j_1^L}{1-\lambda^2}. \tag{3.2.6}$$

Hamiltonian analysis. We aim to show that the PCM belongs to the class of non-ultralocal models with twist function: thus, we want to compute the Poisson bracket of the Lax matrix with itself. For that, we first need to pass from the Lagrangian to the Hamiltonian formalism. The description of the phase space of the PCM, which is common to all integrable σ-models, was presented in the subsection 3.1.2. It is encoded in the G_0-valued field $g(x)$ and the $\mathfrak{g}_0$-valued current $X(x)$, satisfying the canonical brackets (3.1.7). Choosing a set of coordinates ϕ^i and computing the corresponding conjugate momenta π_i, we compute the expression of the field X, as defined in (3.1.6). For the PCM, one simply finds

$$X = Kg^{-1}\partial_t g = Kj_0^L. \tag{3.2.7}$$

As expected, one finds that X is independent of the choice of coordinates ϕ^i. We note that the right moment map (3.1.8) can then be reexpressed as

$$m^R = K\int \mathrm{d}x\; j_0^L(x),$$

which is equal to the Noether charge (3.2.4) associated with the right multiplication symmetry of the PCM. This is natural, as the moment map m^R generates the infinitesimal right multiplication, as seen in the subsection 3.1.2. In the same way, the left multiplication moment map m^L coincides with the second Noether charge in equation (3.2.4).

Using the equation (3.1.12) together with the expression (3.2.7) of X, we express the Hamiltonian of the PCM in terms of X and $j^L = j_1^L$:

$$\mathcal{H}_{\text{PCM}} = \frac{1}{2}\int \mathrm{d}x\; \left(\frac{1}{K}\kappa(X,X) + K\kappa(j^L, j^L)\right).$$

As a consistency check, one can compute that

$$\partial_t j_0^L = \frac{1}{K}\{\mathcal{H}_{\text{PCM}}, X\} = \partial_x j^L,$$

so that we recover the Lagrangian equation of motion (3.2.3).

Maillet bracket and twist function. Finally, let us compute the Poisson bracket of the Lax matrix of the PCM and show that it is governed by a twist function, as done by Maillet in [121]. We start by re-expressing the Lax matrix (3.2.6) in terms of the currents X and j^L, whose Poisson bracket we know. One simply finds

$$\mathcal{L}_{\text{PCM}}(\lambda, x) = \frac{j^L(x) + \lambda K^{-1}X(x)}{1-\lambda^2}. \tag{3.2.8}$$

We now have all the ingredients to compute the Poisson bracket of the Lax matrix with itself, starting from the above expression and the brackets (3.1.7) and (3.1.11). After a few computations, involving also the circle lemma (2.3.5), one finds that the Lax matrix satisfies a non-ultralocal Maillet bracket (2.2.6). The corresponding $\mathcal{R}$-matrix is given by

$$\mathcal{R}_{\underline{\mathbf{12}}}^{\text{PCM}}(\lambda,\mu) = \mathcal{R}_{\underline{\mathbf{12}}}^{0}(\lambda,\mu)\varphi_{\text{PCM}}(\mu)^{-1},$$

where $\mathcal{R}^0$ is the non-twisted standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ and the twist function is given by

$$\varphi_{\mathrm{PCM}}(\lambda) = K\left(\frac{1}{\lambda^2} - 1\right) = K\frac{1-\lambda^2}{\lambda^2}. \tag{3.2.9}$$

As announced, this proves that the PCM is a non-ultralocal model with twist function. According to the definition of such a model given in Section 2.4, we also have to verify the reality condition (2.4.7): it is obvious from Equation (3.2.8), as j^L and X belongs to $\mathfrak{g}_0$ and thus are invariant under the anti-involution τ (note that we do not have to verify the equivariance conditions of Section 2.4, as the standard $\mathcal{R}$-matrix is non-twisted).

Although it is not clear yet why we are interested in this, let us study the analytical properties of the twist function φ_{PCM}. More precisely, let us look at the poles and the zeros of the 1-form $\varphi_{\mathrm{PCM}}(\lambda)\mathrm{d}\lambda$. It is clear that it possesses two simple zeros at $+1$ and -1. Moreover, it possesses a double pole at 0. Under the inversion of parameter $\alpha = \lambda^{-1}$, this 1-form transforms as

$$\varphi_{\mathrm{PCM}}(\lambda)\,\mathrm{d}\lambda = -\varphi_{\mathrm{PCM}}\left(\frac{1}{\alpha}\right)\frac{\mathrm{d}\alpha}{\alpha^2} = K\left(\frac{1}{\alpha^2} - 1\right)\mathrm{d}\alpha.$$

This has a double pole in $\alpha = 0$, so the 1-form $\varphi_{\mathrm{PCM}}(\lambda)\mathrm{d}\lambda$ has a double pole at infinity. As a conclusion, the twist function of the PCM has double poles at zero and infinity and simple zeros at $+1$ and -1, as represented in Figure 3.1. Let us note also that the two zeros ± 1 of the twist function are also poles of the Lax matrix $\mathcal{L}(\lambda, x)$ (we shall use this property later).

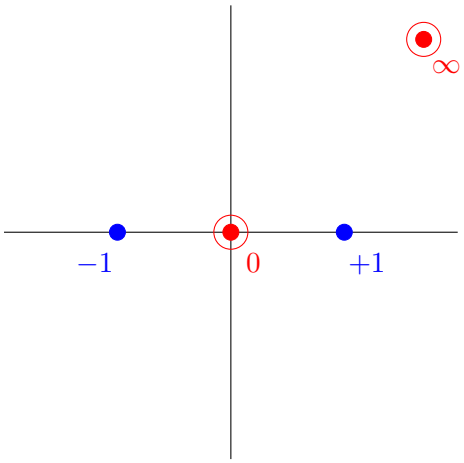


Figure 3.1: Poles and zeros of the twist function of the PCM.

3.2.2 $\mathbb{Z}_T$-coset σ-models

$\mathbb{Z}_T$-grading. We shall now review the $\mathbb{Z}_T$-coset σ-models and their integrable structure. They are σ-models with target space G_0/H, where G_0 is a connected semi-simple Lie group and H is a particular subgroup of G_0. More precisely, we suppose that H is such that the corresponding subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is equal to the 0th-grade $\mathfrak{g}_0^{(0)}$ of a $\mathbb{Z}_T$-grading of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \bigoplus_{p=0}^{T-1} \mathfrak{g}_0^{(p)}, \qquad \text{with} \qquad \left[\mathfrak{g}_0^{(p)}, \mathfrak{g}_0^{(q)}\right] \subset \mathfrak{g}^{(p+q \bmod T)}.$$

According to Corollary A.4.4, such gradings are in one-to-one correspondence with automorphisms $\sigma \in \mathrm{Aut}(\mathfrak{g})$ of order T satisfying the dihedrality condition (2.4.8). We will use the notations introduced in Section 2.3 for the construction of the standard $\mathcal{R}$-matrix twisted by σ (we refer the reader to the Appendix A.4 for more details about finite order automorphisms of Lie algebras). In particular, ω is

a primitive T^{th}-root of unity and $\mathfrak{g}^{(p)}$ $(p = 0, \cdots, T-1)$ is the eigenspace of σ of eigenvalue ω^p. For $X \in \mathfrak{g}$, we define $X^{(p)} = \pi^{(p)}X$ the projection of X on $\mathfrak{g}^{(p)}$ in the decomposition $\bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)}$.

For simplicity, we shall only detail here the case $T = 2$ and just summarise at the end of the subsection how the results on the integrability of the $\mathbb{Z}_2$-coset model generalises to $\mathbb{Z}_T$-cosets for arbitrary T. When $T = 2$, σ is an involution (*i.e.* $\sigma^2 = \mathrm{Id}$), $\omega = -1$ and the eigenspaces

$$\mathfrak{g}^{(0)} = \big\{X \in \mathfrak{g} \;\big|\; \sigma(X) = X\big\} \qquad \text{and} \qquad \mathfrak{g}^{(1)} = \big\{X \in \mathfrak{g} \;\big|\; \sigma(X) = -X\big\}$$

satisfy

$$\big[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}\big] \subset \mathfrak{g}^{(0)}, \quad \big[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}\big] \subset \mathfrak{g}^{(1)} \quad \text{and} \quad \big[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\big] \subset \mathfrak{g}^{(0)}. \tag{3.2.10}$$

In this case, the Lie algebra $\mathfrak{h}$ of the subgroup H is identified with the subalgebra $\mathfrak{g}^{(0)} \cap \mathfrak{g}_0$ of elements of the real form $\mathfrak{g}_0$ stabilised by σ. In this case $T = 2$, the coset space G_0/H is called a symmetric-space.

Action and equations of motion. As explained briefly in Subsection 3.1.1, we will describe the σ-model on G_0/H as a model on a G_0-valued field g together with a gauge symmetry in H to eliminate the redundant degrees of freedom. More explicitly, the symmetric-space σ-model on G_0/H is given by the action

$$S_{\mathbb{Z}_2}[g] = \frac{K}{2} \int_\Sigma \mathrm{d}x\, \mathrm{d}t \;\Big(\kappa\Big(\big(g^{-1}\partial_t g\big)^{(1)}, \big(g^{-1}\partial_t g\big)^{(1)}\Big) - \kappa\Big(\big(g^{-1}\partial_x g\big)^{(1)}, \big(g^{-1}\partial_x g\big)^{(1)}\Big)\Big).$$

One can reexpress this action in terms of left currents j_μ^L as

$$S_{\mathbb{Z}_2}[g] = \frac{K}{2} \int_\Sigma \mathrm{d}x\, \mathrm{d}t \;\Big(\kappa\Big(j_0^{L\,(1)}, j_0^{L\,(1)}\Big) - \kappa\Big(j_1^{L\,(1)}, j_1^{L\,(1)}\Big)\Big).$$

In the same way, one can also use the light-cone left currents $j_\pm^L$ and the light-cone coordinates $x^\pm$:

$$S_{\mathbb{Z}_2}[g] = K \int_\Sigma \mathrm{d}x^+\, \mathrm{d}x^-\; \kappa\Big(j_+^{L\,(1)}, j_-^{L\,(1)}\Big). \tag{3.2.11}$$

Let us consider the local right multiplication

$$g(x,t) \longmapsto g(x,t)h(x,t), \quad h(x,t) \in H, \tag{3.2.12}$$

by a field h valued in $H = G^{(0)} \cap G_0$. Under this transformation, the left-current transforms as

$$j_\pm^L \longmapsto h^{-1} j_\pm^L h + h^{-1}\partial_\pm h.$$

By construction, h is valued in $G^{(0)}$ so $h^{-1}\partial_\pm h$ belongs to $\mathfrak{g}^{(0)}$. Moreover, according to the grading relations (3.2.10), the conjugacy transformation by $h \in G^{(0)}$ preserves the graded subspaces $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$. Thus, we deduce that the graded components of $j_\pm^L$ transform simply as

$$j_\pm^{L\,(0)} \longmapsto h^{-1} j_\pm^{L\,(0)} h + h^{-1}\partial_\pm h \qquad \text{and} \qquad j_\pm^{L\,(1)} \longmapsto h^{-1} j_\pm^{L\,(1)} h. \tag{3.2.13}$$

In particular, as the Killing form κ is invariant under conjugacy transformations, we see that the action (3.2.11) is invariant under the local transformation (3.2.12). Thus, the model possesses a gauge symmetry under the right multiplication by elements of H. The physical degrees of freedom of the model are then in the quotient G_0/H. One can then recover the usual σ-model on the symmetric-space G_0/H by gauge fixing the theory.

Let us end this paragraph by expressing the equations of motion of the model, obtained by varying the field g in the action (3.2.11). They read

$$D_+ j_-^{L\,(1)} + D_- j_+^{L\,(1)} = 0, \tag{3.2.14}$$

where $D_\pm$ denotes the covariant derivative

$$D_\pm = \partial_\pm + \left[j_\pm^{L\,(0)}, \cdot\right].$$

As a consistency check, we can verify that this equation of motion is invariant under the gauge transformation (3.2.12). According to equation (3.2.13), the currents $j_\pm^{L\,(1)}$ transform covariantly under this transformation and the currents $j_\pm^{L\,(0)}$ transform as gauge fields (which justifies the name of covariant derivative for $D_\pm$). Thus, the left-hand side of the equation of motion (3.2.14) is covariant under the gauge transformation, hence the gauge symmetry at the level of the equations of motion.

Global left symmetry. In addition to the right gauge symmetry described above, the $\mathbb{Z}_2$-coset model possesses a global left symmetry. Indeed, the action (3.2.11) is expressed only in terms of the left-invariant currents $j_\pm^L$. Thus, the action is invariant under the left multiplication $L_h : g \mapsto hg$ by a constant element h of the group G_0. By the Noether theorem, this global symmetry is associated with the following equation of conservation:

$$\partial_+\left(g j_-^{L\,(1)} g^{-1}\right) + \partial_-\left(g j_+^{L\,(1)} g^{-1}\right) = 0,$$

which is equivalent to the equation of motion (3.2.14). Thus, the charge

$$\int \mathrm{d}x\; g(x,t) j_0^{L\,(1)}(x,t) g^{-1}(x,t)$$

is conserved, *i.e.* does not depend on t.

Lax equation. Let us show that the equations of motion of the $\mathbb{Z}_2$-coset σ-model can be recast as a Lax equation (2.1.4) (in light-cone coordinates), as proved originally by Eichenherr and Forger in [126] (first results on particular examples were found by Pohlmeyer in [125]). As in the case of the PCM (subsection 3.2.1), we will need the flatness of the left-current $j_\pm^L$, *i.e.* the Maurer-Cartan equation (3.1.4). As the $\mathbb{Z}_2$-coset model is expressed in terms of the graded components $j_\pm^{L\,(0)}$ and $j_\pm^{L\,(1)}$, let us decompose the Maurer-Cartan equation along this grading. Using the grading relation (3.2.10), one finds the two following equations:

$$\partial_+ j_-^{L\,(0)} - \partial_- j_+^{L\,(0)} + \left[j_+^{L\,(0)}, j_-^{L\,(0)}\right] + \left[j_+^{L\,(1)}, j_-^{L\,(1)}\right] = 0, \tag{3.2.15a}$$

$$\partial_+ j_-^{L\,(1)} - \partial_- j_+^{L\,(1)} + \left[j_+^{L\,(0)}, j_-^{L\,(1)}\right] + \left[j_+^{L\,(1)}, j_-^{L\,(0)}\right] = 0. \tag{3.2.15b}$$

We define the light-cone Lax pair

$$\mathcal{L}_\pm^{\mathbb{Z}_2}(\lambda) = j_\pm^{L\,(0)} + \lambda^{\pm 1} j_\pm^{L\,(1)}, \tag{3.2.16}$$

depending on the spectral parameter $\lambda \in \mathbb{C}$. One has

$$\begin{aligned} &\partial_+\mathcal{L}_-^{\mathbb{Z}_2}(\lambda) - \partial_-\mathcal{L}_+^{\mathbb{Z}_2}(\lambda) + \left[\mathcal{L}_+^{\mathbb{Z}_2}(\lambda), \mathcal{L}_-^{\mathbb{Z}_2}(\lambda)\right] \\ &\quad= \partial_+ j_-^{L\,(0)} - \partial_- j_+^{L\,(0)} + \left[j_+^{L\,(0)}, j_-^{L\,(0)}\right] + \left[j_+^{L\,(1)}, j_-^{L\,(1)}\right] \\ &\qquad + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)\left(\partial_+ j_-^{L\,(1)} - \partial_- j_+^{L\,(1)} + \left[j_+^{L\,(0)}, j_-^{L\,(1)}\right] + \left[j_+^{L\,(1)}, j_-^{L\,(0)}\right]\right) \\ &\qquad + \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)\left(D_+ j_-^{L\,(1)} + D_- j_+^{L\,(1)}\right). \end{aligned} \tag{3.2.17}$$

The first two lines vanish off-shell (without the equations of motion) due to the Maurer-Cartan equations (3.2.15). Thus, we get

$$\partial_+\mathcal{L}_-^{\mathbb{Z}_2}(\lambda) - \partial_-\mathcal{L}_+^{\mathbb{Z}_2}(\lambda) + \left[\mathcal{L}_+^{\mathbb{Z}_2}(\lambda), \mathcal{L}_-^{\mathbb{Z}_2}(\lambda)\right] = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)\left(D_+ j_-^{L\,(1)} + D_- j_+^{L\,(1)}\right).$$

The equation of motion (3.2.14) of the $\mathbb{Z}_2$-coset is then equivalent to the Lax equation (2.1.4), as expected. To end this paragraph, let us give the corresponding Lax pair $(\mathcal{L}_{\mathbb{Z}_2}, \mathcal{M}_{\mathbb{Z}_2})$ in space-time coordinates:

$$\mathcal{L}_{\mathbb{Z}_2}(\lambda) = j_1^{L\,(0)} + \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) j_0^{L\,(1)} + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) j_1^{L\,(1)} \tag{3.2.18a}$$

$$\mathcal{M}_{\mathbb{Z}_2}(\lambda) = j_0^{L\,(0)} + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) j_0^{L\,(1)} + \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) j_1^{L\,(1)}. \tag{3.2.18b}$$

Using the fact that $\sigma\left(j_\mu^{L\,(0)}\right) = j_\mu^{L\,(0)}$ and $\sigma\left(j_\mu^{L\,(1)}\right) = -j_\mu^{L\,(1)}$, we can note that

$$\sigma\big(\mathcal{L}_{\mathbb{Z}_2}(\lambda)\big) = \mathcal{L}_{\mathbb{Z}_2}(-\lambda) \quad \text{and} \quad \sigma\big(\mathcal{M}_{\mathbb{Z}_2}(\lambda)\big) = \mathcal{M}_{\mathbb{Z}_2}(-\lambda).$$

Thus, the Lax matrix $\mathcal{L}_{\mathbb{Z}_2}(\lambda)$ satisfies the equivariance condition (2.4.3) (as $\omega = -1$ for $T = 2$).

Hamiltonian analysis. Let us discuss the Hamiltonian formulation of the $\mathbb{Z}_2$-coset model. The phase space of the model is described in Subsection 3.1.2. It contains the G_0-valued field $g(x)$ and the $\mathfrak{g}_0$-valued field $X(x)$. Choosing coordinate fields ϕ^i on G_0, one can compute the corresponding conjugate momenta π_i and deduce the Lagrangian expression (3.1.6) of X. One finds

$$X \approx K\big(g^{-1}\partial_t g\big)^{(1)} \approx K\, j_0^{L\,(1)}, \tag{3.2.19}$$

where the notation $\approx$ instead of $=$ will be justified in what follows. A direct consequence of this expression is that

$$X^{(0)} \approx 0.$$

This can seem surprising at first, as $X^{(0)} = 0$ is for example incompatible with the Poisson bracket (3.1.7b). This is a consequence of the gauge symmetry of the $\mathbb{Z}_2$-coset model. Indeed, according to Dirac theory [127], when passing from a Lagrangian system with gauge symmetry to the Hamiltonian formalism, one encounters constraints on the phase space. These are relations between the coordinates ϕ^i and the conjugate momentum π_i. In the case of the $\mathbb{Z}_2$-coset model, these relations are encoded in the fact that $X^{(0)}$ vanishes.

We shall use here the Dirac terminology and say that an equation is weak when it is true only with the constraint (we then use the notation $\approx$ instead of $=$). At the contrary, we will say that an equation is strong when it is true without imposing the constraint (we shall then keep the usual equality sign $=$ for those). The full treatment of the $\mathbb{Z}_2$-coset model as a constrained Hamiltonian system would require the application of the Dirac procedure. We shall not enter into too much details here and will just refer to the textbooks [127, 128] when we have to use results on constrained systems.

Let us finish this discussion about the constraint by a quick consistency check. For constraints coming from a gauge symmetry, it is a standard result that the Hamiltonian flow of this constraint generates the corresponding gauge transformation. Let us check this fact here. For $\epsilon(x)$ an infinitesimal field valued in $\mathfrak{h} = \mathfrak{g}^{(0)} \cap \mathfrak{g}_0$, one can check from the Poisson bracket (3.1.7b) that

$$\delta_\epsilon g(x) = \left\{ \int \mathrm{d}y\, \kappa\Big(X^{(0)}(y), \epsilon(y)\Big), g(x) \right\} = g(x)\epsilon(x). \tag{3.2.20}$$

This is the infinitesimal version of the gauge transformation

$$g(x) \mapsto g(x)h(x), \quad h(x) \in H.$$

Thus, we recover that the constraint $X^{(0)}$ generates the gauge symmetry of the model, as expected.

Let us discuss the Hamiltonian of the model, using the formula (3.1.12) obtained in the general discussion. As we are considering a constrained system, we have the freedom of adding to the Hamiltonian of equation (3.1.12) any term proportional to the constraint. Doing the explicit computation for the case of the $\mathbb{Z}_2$-model, one finds

$$\mathcal{H}_{\mathbb{Z}_2} = \frac{1}{2}\int \mathrm{d}x\, \left(\frac{1}{K}\kappa(X,X) - K\kappa\big(j^{L\,(1)}, j^{L\,(1)}\big) + \kappa\big(X^{(0)}(x), \mu(x)\big)\right), \tag{3.2.21}$$

where $j^L = j_1^L$ and μ is an arbitrary $\mathfrak{h}$-valued field, called the Lagrange multiplier. According to equation (3.2.20), the Hamiltonian flow generated by the term containing μ is purely a gauge term and thus does not change the physics of the model.

Maillet bracket and twist function. Let us now determine the Poisson bracket of the Lax matrix with itself and show that the $\mathbb{Z}_2$-coset model possesses a twist function [129, 122]. We first need to reexpress the Lax matrix (3.2.18a) in terms of the Hamiltonian fields g and X. Moreover, as $X^{(0)}$ is a constraint, one has the freedom to add to the Lax matrix a term proportional to $X^{(0)}$. Using the expression (3.2.19) of X, one then gets

$$\mathcal{L}_{\mathbb{Z}_2}(\lambda) = j^{L\,(0)} + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) j^{L\,(1)} + \frac{1}{2K}\left(\lambda - \frac{1}{\lambda}\right) X^{(1)} + f(\lambda) X^{(0)}, \tag{3.2.22}$$

where $f(\lambda)$ is an arbitrary function of the spectral parameter. Starting from the Poisson brackets (3.1.7) and (3.1.11), we compute the bracket of the Lax matrix (3.2.22) with itself. We find that it takes the form of a non-ultralocal Maillet bracket, strongly (*i.e.* without using the constraint $X^{(0)} \approx 0$), if we choose the function f to be

$$f(\lambda) = \frac{\lambda^2 - 1}{2K}. \tag{3.2.23}$$

In this case, the $\mathcal{R}$-matrix governing the Maillet bracket takes the form

$$\mathcal{R}^{\mathbb{Z}_2}_{\underline{\mathbf{12}}}(\lambda, \mu) = \mathcal{R}^{0}_{\underline{\mathbf{12}}}(\lambda, \mu) \varphi_{\mathbb{Z}_2}(\mu)^{-1}.$$

The matrix $\mathcal{R}^0$ is given by

$$\mathcal{R}^{0}_{\underline{\mathbf{12}}}(\lambda, \mu) = \frac{\mu C^{(0)}_{\underline{\mathbf{12}}}}{\mu^2 - \lambda^2} + \frac{\lambda C^{(1)}_{\underline{\mathbf{12}}}}{\mu^2 - \lambda^2},$$

with $C^{(p)}_{\underline{\mathbf{12}}} = \pi^{(p)}_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}$ the partial quadratic Casimirs. We recognize here the twisted standard $\mathcal{R}$-matrix (2.3.9), for $T = 2$. The twist function takes the form

$$\varphi_{\mathbb{Z}_2}(\lambda) = \frac{2K\lambda}{(\lambda^2 - 1)^2}. \tag{3.2.24}$$

With the function f chosen as in (3.2.23), the Lax matrix then becomes

$$\mathcal{L}_{\mathbb{Z}_2}(\lambda) = j^{L\,(0)} + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) j^{L\,(1)} + \frac{1}{2K}\left(\lambda^2 - 1\right) X^{(0)} + \frac{1}{2K}\left(\lambda - \frac{1}{\lambda}\right) X^{(1)}. \tag{3.2.25}$$

This Lax matrix satisfies the equivariance condition (2.4.3) and the reality condition (2.4.7). Moreover, the twist function also satisfies the equivariance property (2.4.5) and the reality condition (2.4.10), as expected from the consistency of the Maillet bracket. This proves that the $\mathbb{Z}_2$-coset σ-model belongs to the class of non-ultralocal models with twist function. Following the nomenclature of Section 2.4, we say that it is a cyclotomic model of order $T = 2$, as it is associated with the standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ twisted by the involution σ.

As for the PCM, let us study the analytical properties of the twist function and more precisely, the poles and zeros of the 1-form $\varphi_{\mathbb{Z}_2}(\lambda)\mathrm{d}\lambda$. One sees that it has two double poles at $+1$ and -1 and two simple zeros at 0 and infinity. These poles and zeros are shown in figure 3.2.

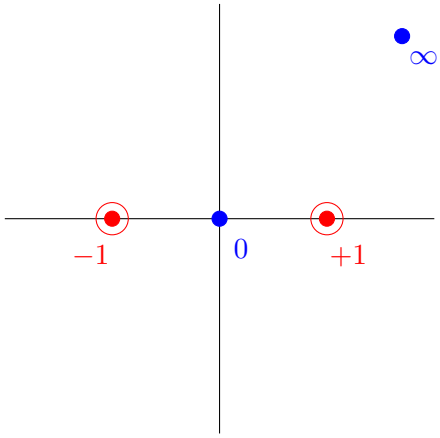


Figure 3.2: Poles and zeros of the twist function of the $\mathbb{Z}_2$-coset σ-model.

$\mathbb{Z}_T$-cosets. We close this subsection by discussing briefly general $\mathbb{Z}_T$-coset σ-models, for arbitrary T (we will use the notations introduced at the beginning of this subsection, with σ an automorphism of order T). These models were introduced by Young in [130], together with their Lax pair representation (see also [131] and [132]). The Hamiltonian analysis of these models and the computation of the Poisson bracket of their Lax matrix was done by Ke, Li, Wang and Yue in [133]. In particular, they showed that the $\mathbb{Z}_T$-coset models are non-ultralocal models with twist function, cyclotomic of order T. The Lax matrix of these models reads

$$\mathcal{L}_{\mathbb{Z}_T}(\lambda,x)=\sum_{k=1}^{T}\frac{(T-k)+k\lambda^{-T}}{T}\lambda^k j^{L\,(k)}(x)+\sum_{k=1}^{T}\frac{1-\lambda^{-T}}{T}\lambda^k X^{(k)}(x). \tag{3.2.26}$$

Note that in this equation, and in general, we consider the exponents (k) only modulo T, so that $X^{(T)}=X^{(0)}$ for example. All $\mathbb{Z}_T$-coset models possess a gauge symmetry under the action of the subgroup $H=G^{(0)}\cap G_0$ of G_0. As in the case of the $\mathbb{Z}_2$-coset presented above, the field $X^{(0)}$ is the constraint associated with this gauge symmetry. It is clear that equation (3.2.26) reduces to the Lax matrix (3.2.25) of the $\mathbb{Z}_2$-coset model for $T=2$ (and for the choice of the global factor $K=1$). Note that the Lax matrix (3.2.26) satisfies the equivariance condition (2.4.3) for the automorphism σ of order T introduced above.

The Poisson bracket of the Lax matrix (3.2.26) takes the form of a Maillet non-ultralocal bracket (2.2.6). The associated $\mathcal{R}$-matrix is given by a twist function, as in (2.4.1). More precisely, the matrix $\mathcal{R}^0$ is the standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ twisted by the automorphism σ of order T and the twist function is given by

$$\varphi_{\mathbb{Z}_T}(\lambda)=\frac{T\lambda^{T-1}}{(1-\lambda^T)^2}. \tag{3.2.27}$$

As for the Lax matrix, the twist function (3.2.27) reduces to the one (3.2.24) of the $\mathbb{Z}_2$-coset model when $T=2$ (and $K=1$). As expected, this twist function satisfies the equivariance condition (2.4.5). The 1-form $\varphi_{\mathbb{Z}_T}(\lambda)\mathrm{d}\lambda$ has two zeros in 0 and infinity and T double poles, located at each Tth-root of the unity ω^p.

Let us say a few words about supercoset models. Instead of a Lie algebra, one can consider a super-Lie algebra, equipped with a $\mathbb{Z}_{2T}$-grading (for a super-Lie algebra, we only consider even gradings, as the definition of a super-Lie algebra already contains a $\mathbb{Z}_2$-grading). The construction mentioned above for $\mathbb{Z}_T$-coset then generalises and one can write a $\mathbb{Z}_{2T}$-supercoset σ-model (this was also considered by Young in [130]). The Hamiltonian integrable structure of these models (Lax matrix, Maillet bracket, twist function) is the same as the one of the $\mathbb{Z}_{2T}$-coset models described above, when replacing Lie algebras by super-Lie algebras (this was also shown in [133]).

A slightly different construction allows to consider the Green-Schwarz superstring on $AdS_5\times S^5$: in particular, one has to take into account the worldsheet diffeomorphism gauge symmetry of the model,

as well as the κ-symmetry (the latter comes from the fact that the Green-Schwarz model, contrarily to the supercoset model mentioned above, has no kinetic terms involving fermions). The action of this model was worked out by Metsaev and Tseytlin in [4]. Its interpretation as a $\mathbb{Z}_4$-supercoset model may be seen from [134] (see also the review [63]). This model was extensively studied in the context of the AdS/CFT correspondence, as its holographic dual is the $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions (see for example [8]). It was shown to admit a Lax pair by Bena, Polchinski and Roiban in [5]. Its hamiltonian integrability and its Maillet structure were discovered by Magro in [6]. The underlying algebraic structure was understood in [7, 123] and the twist function determined in [123]. This twist function coincides with the one of a $\mathbb{Z}_4$-coset model, given by (3.2.27) with $T=4$.

3.3 Deformed integrable σ-models

In the past decades, several integrable deformations of the PCM and of the $\mathbb{Z}_T$-coset σ-models were discovered. These are models which depend on one or more deformation parameters, such that they reduce to the undeformed model when these parameters are equal to zero and such that they still admit a Lax pair formulation. Moreover, for all models for which the hamiltonian analysis of the Lax matrix has been carried out, it was found that they also belong to the class of models with twist function, as the undeformed one. Interestingly, as we shall see, the effect of the deformations is to deform the poles of the twist function.

We will not detail the construction and the analysis of all these models here. In the first subsection, we will develop the first known example of such a deformed model, the so-called Yang-Baxter model, which is a one-parameter deformation of the PCM that we will often use as an example in this thesis. In the second subsection, we will review briefly the whole panorama of deformations of the PCM, focusing in particular on their twist functions. Finally, in the third section, we shall discuss the deformations of $\mathbb{Z}_2$-coset models.

3.3.1 The Yang-Baxter model

Action. The Yang-Baxter model is a one-parameter integrable deformation of the PCM, introduced by Klimčik in [27]. We will use the notation of the subsection 3.2.1 on the PCM. We will suppose moreover that we are given a linear map $R : \mathfrak{g} \to \mathfrak{g}$, stabilising the real form $\mathfrak{g}_0$ and skew-symmetric with respect to the Killing form κ, which satisfies the so-called modified Classical Yang-Baxter Equation (mCYBE):

$$\forall\, X, Y \in \mathfrak{g},\ \ [RX, RY] - R\big([RX, Y] + [X, RY]\big) = -c^2[X, Y], \tag{3.3.1}$$

with $c=1$ (split case) or $c=i$ (non-split case). Although this equation is called the modified CYBE, it should not be confused with the CYBE (2.3.3) (note in particular that there is no dependence on some spectral parameter in the above equation). The link between these equations and their algebraic interpretation are explained in Appendix C. In most examples and applications, we will consider the so-called standard (split or non-split) matrix R, as described in Appendix C, although the results of the present subsection hold for any solution of the mCYBE.

The Yang-Baxter model is then defined by the following action, depending on the G_0-valued field g, a real parameter $\eta \in \mathbb{R}$ and a global factor K:

$$S_\eta[g] = K \int_\Sigma \mathrm{d}x^+\,\mathrm{d}x^-\ \kappa\left(j_+^R, \frac{1}{1-\eta R} j_-^R\right) = K \int_\Sigma \mathrm{d}x^+\,\mathrm{d}x^-\ \kappa\left(j_-^R, \frac{1}{1+\eta R} j_+^R\right), \tag{3.3.2}$$

where the second equality is obtained using the skew-symmetry of R with respect to κ. Note here that we use the notation η for the deformation parameter, following the conventions of [29], although this parameter is often denoted $\varkappa$ in the literature. Comparing this equation to equation (3.2.2), it is clear that

$$S_{\eta=0}[g] = S_{\mathrm{PCM}}[g].$$

Thus, this action defines a one-parameter deformation of the PCM.

Let us discuss briefly the invertibility of the operators $1 \pm \eta R$, which is necessary for the good definition of the action (3.3.2). This invertibility is always ensured for η close to 0. In the case of a standard matrix R, the eigenvalue of R are 0 and ± 1 for the split case and 0 and $\pm i$ for the non-split one. Thus, $1 \pm \eta R$ is invertible for any value of $\eta \in \mathbb{R}$ in the non-split case and invertible for $\eta \neq \pm 1$ for the split case.

The above action is expressed in terms of the right currents $j_{\pm}^R$. Using the invariance of κ under conjugacy transformation and the relation (3.1.1) between the right currents and the left ones, one can re-express this action as

$$S_\eta[g] = K \int_\Sigma \mathrm{d}x^+ \, \mathrm{d}x^- \, \kappa \left(j_+^L, \frac{1}{1 - \eta R_g} j_-^L \right) = K \int_\Sigma \mathrm{d}x^+ \, \mathrm{d}x^- \, \kappa \left(j_-^L, \frac{1}{1 + \eta R_g} j_+^L \right), \tag{3.3.3}$$

where

$$R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g.$$

Later in this subsection, when discussing the Hamiltonian formalism of the model, we will need the expression of the action in terms of space-time coordinates (x, t) instead of light-cone ones. This is given by

$$\begin{aligned} S_\eta[g] = \frac{K}{2} \int_\Sigma \mathrm{d}x \, \mathrm{d}t \Bigg[& \kappa \left(g^{-1} \partial_t g, \frac{1}{1 - \eta^2 R_g^2} g^{-1} \partial_t g \right) - \kappa \left(g^{-1} \partial_x g, \frac{1}{1 - \eta^2 R_g^2} g^{-1} \partial_x g \right) \\ & - \kappa \left(g^{-1} \partial_t g, \frac{2\eta R_g}{1 - \eta^2 R_g^2} g^{-1} \partial_x g \right) \Bigg]. \end{aligned} \tag{3.3.4}$$

(Preserved and broken) Symmetries. As already observed, the Yang-Baxter action (3.3.2) is expressed only in terms of the right currents $j_{\pm}^R$. As these currents are invariant under the right multiplication $g \mapsto gh$ by a constant element of G_0, it is clear that the Yang-Baxter model is invariant under this transformation, as the PCM is (we will come back on this fact later).

As explained in Subsection 3.2.1, the PCM is also invariant under the left multiplication $g \mapsto hg$. This left symmetry is broken by the Yang-Baxter deformation. Indeed, under the transformation $g \mapsto hg$, the action (3.3.2) transforms as

$$S_\eta[hg] = K \int_\Sigma \mathrm{d}x^+ \, \mathrm{d}x^- \, \kappa \left(j_+^R, \mathrm{Ad}_h^{-1} \circ \frac{1}{1 - \eta R} \circ \mathrm{Ad}_h \, j_-^R \right).$$

As R does not commute with Ad_h, this is different from $S_\eta[g]$ when $\eta \neq 0$. We shall discuss again this breaking of symmetry in Chapter 5.

Let us now come back to the right symmetry, which is not broken by the Yang-Baxter deformation. By the Noether theorem, it is associated with the conservation equation

$$\partial_+ K_- + \partial_- K_+ = 0 \tag{3.3.5}$$

of a $\mathfrak{g}_0$-valued current $K_\pm$. One finds that

$$K_\pm = \frac{1 - c^2 \eta^2}{1 \pm \eta R_g} j_\pm^L = \mathcal{O}_\pm j_\pm^L. \tag{3.3.6}$$

This conservation equation is one way to write the whole set of equations of motion of the Yang-Baxter model. A conserved current is only defined up to a global constant factor: here, we introduced the factor $1 - c^2\eta^2$ for future convenience. We note that in the non-split case $c = i$, this factor is equal to $1 + \eta^2$ and thus is non-zero for any value of η. In the split case $c = 1$, we see that this factor is not zero, except for the values $\eta = \pm 1$: as discussed above, these correspond also to the values where the the operator $1 \pm \eta R$ is non-invertible, so we shall exclude them for the present discussion. One easily checks that the current $K_\pm$ reduces to the left-current $j_\pm^L$ in the undeformed case $\eta = 0$.

As we have just seen, the Yang-Baxter deformation of the PCM preserves the right symmetry of the model but breaks the left symmetry. One could have also introduced a Yang-Baxter deformation which preserves the left symmetry and breaks the right one, by simply replacing the right current $j^R_\pm$ by the left current $j^L_\pm$ in the action (3.3.2). This model is equivalent to the one studied here and all the computations we present also apply to it by exchanging "left and right objects". Thus, we shall focus only on the model defined by (3.3.2).

Lax equation. Let us discuss the Lax representation of the equations of motion of the Yang-Baxter model, as first introduced by Klimčik in [28]. Recall that in the undeformed (PCM) case, we constructed a Lax pair following the scheme of Zakharov and Mikhailov, based on the existence of a conserved and flat current. In the case of the Yang-Baxter model, we already found a conserved current $K_\pm$, associated with the global right symmetry. It is thus natural to wonder whether $K_\pm$ is flat, *i.e.* whether we have or not

$$\partial_+ K_- - \partial_- K_+ + [K_+, K_-] = 0. \tag{3.3.7}$$

In the undeformed case, this flatness condition reduces to the Maurer-Cartan equation (3.1.4) of $j^L_\pm$. Thus, we should expect to also use the Maurer-Cartan equation in the deformed case. To compute the derivatives $\partial_\pm K_\mp$, one needs to take into account the fact that the operators $\mathcal{O}_\pm$ in the definition (3.3.6) of $K_\pm$ also depend on the field g. In general, for a g-dependent field Y and an infinitesimal variation δg of g, one finds that the induced variation of $\mathcal{O}_\pm Y$ is

$$\delta\big(\mathcal{O}_\pm Y\big) = \mathcal{O}_\pm\big(\delta Y\big) + \mathcal{O}_\pm\Big(\big[g^{-1}\delta g, Y\big]\Big) - \big[g^{-1}\delta g, \mathcal{O}_\pm Y\big]\,.$$

This allows to compute the derivatives $\partial_+ K_-$ and $\partial_- K_+$. Combining these with the Maurer-Cartan equation of $j^L_\pm$ and the conservation equation (3.3.5) of $K_\pm$, one checks that the flatness equation (3.3.7) indeed holds. In the derivation of this equation, one has to use extensively the mCYBE equation (3.3.1) (or more precisely the fact that R_g also satisfies the mCYBE). Thus, it is the fact that R is a solution of the mCYBE that ensures the flatness of $K_\pm$ and hence the integrability of the model. Note also that the zero curvature equation (3.3.7) is not invariant under the multiplication of $K_\pm$ by a global constant factor: this is the origin of the factor $1 - c^2\eta^2$ that we introduced in the definition (3.3.6) of $K_\pm$ (another factor would spoil the flatness of $K_\pm$).

Following Zakharov and Mikhailov, the Lax pair of the Yang-Baxter model is given by

$$\mathcal{L}_\eta(\lambda) = \frac{K_1 + \lambda K_0}{1-\lambda^2} \qquad \text{and} \qquad \mathcal{M}_\eta(\lambda) = \frac{K_0 + \lambda K_1}{1-\lambda^2}, \tag{3.3.8}$$

with $K_0 = \frac{1}{2}(K_+ + K_-)$ and $K_1 = \frac{1}{2}(K_+ - K_-)$. Although it is not equal to the Lax pair found in the original paper [28] of Klimčik, it is related to it by a formal gauge transformation by g. It is clear that this Lax pair reduces to the one (3.2.6) we considered for the PCM in the undeformed limit:

$$\mathcal{L}_{\eta=0}(\lambda) = \mathcal{L}_{\mathrm{PCM}}(\lambda) \qquad \text{and} \qquad \mathcal{M}_{\eta=0}(\lambda) = \mathcal{M}_{\mathrm{PCM}}(\lambda).$$

Hamiltonian analysis. Let us discuss the Hamitlonian formalism of the Yang-Baxter model. The phase-space of the model is described by the G_0-valued field g and the $\mathfrak{g}_0$-valued field X, as explained in Subsection 3.1.2. Choosing local coordinates ϕ^i on G_0, one can compute the corresponding conjugate momenta π_i from the action (3.3.4) and deduce the Lagrangian expression of X. One then finds

$$X = \frac{K}{1-\eta^2 R_g^2} j_0^L - \frac{K\eta R_g}{1-\eta^2 R_g^2} j_1^L, \tag{3.3.9}$$

which reduces to $X = K j_0^L$ in the undeformed limit, as expected from (3.2.7). From the action (3.3.4) and the equation (3.1.12), we compute the Hamiltonian of the Yang-Baxter model and find

$$\mathcal{H}_\eta = \int \mathrm{d}x \left(\frac{1}{2K} \kappa\left((1-\eta^2 R_g^2)X, X\right) + \frac{K}{2}\kappa\left(j^L, j^L\right) + \kappa\left(j^L, \eta R_g X\right)\right), \tag{3.3.10}$$

with $j^L = j_1^L$ as usual. Using the relation (3.3.9) between X and j_0^L, we can express the current K_0 and K_1 in terms of X and j^L:

$$K_0 = \frac{1-c^2\eta^2}{K} X \quad \text{and} \quad K_1 = (1-c^2\eta^2)\left(j^L - \frac{\eta}{K} R_g X\right). \tag{3.3.11}$$

We then note that the Hamiltonian (3.3.10) can be rewritten as

$$\mathcal{H}_\eta = \frac{K}{2(1-\eta^2c^2)^2} \int \mathrm{d}x \left(\kappa(K_0, K_0) + \kappa(K_1, K_1)\right).$$

Maillet non-ultralocal bracket. Using the Hamiltonian expression (3.3.11) of the current K_μ, we rewrite the Lax matrix (3.3.8) as

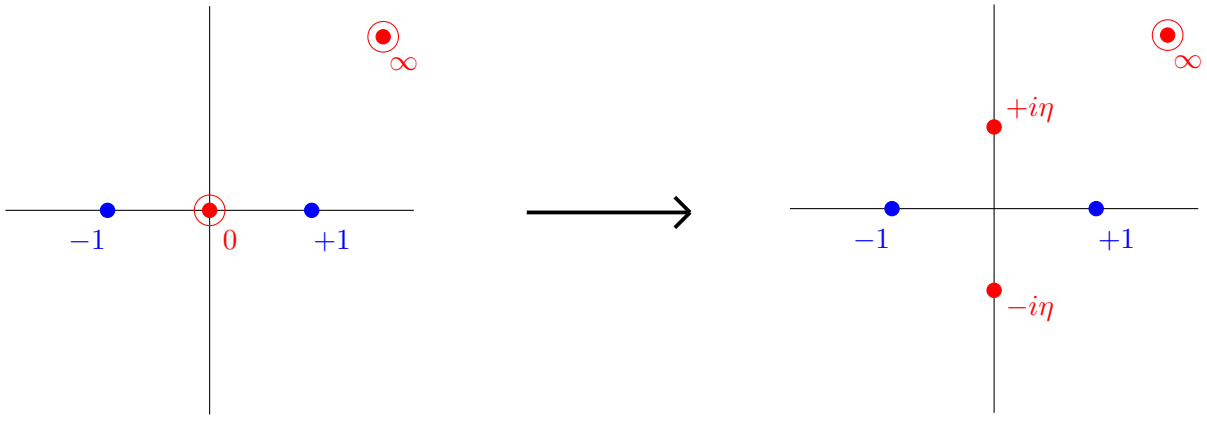
$$\mathcal{L}_\eta(\lambda) = \frac{1-c^2\eta^2}{1-\lambda^2}\left(j^L - \frac{\eta}{K} R_g X + \frac{\lambda}{K} X\right) \tag{3.3.12}$$

We now want the Poisson bracket of the Lax matrix with itself, as computed by Delduc, Magro and Vicedo in [29]. This necessitates the computation of some intermediate Poisson brackets involving the field $R_g X$. We will not enter into more details here, as some similar computations will be presented in the section 3.4 on the Bi-Yang-Baxter model. In the end, using the mCYBE (3.3.1) (or more precisely the mCYBE for R_g), one finds that the Poisson bracket of the Lax matrix takes a Maillet non-ultralocal form (2.2.6).

More precisely, the $\mathcal{R}$-matrix underlying this Maillet bracket is given by the standard non-twisted $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ and the twist function:

$$\varphi_\eta(\lambda) = \frac{K}{1-c^2\eta^2} \frac{1-\lambda^2}{\lambda^2 - c^2\eta^2}. \tag{3.3.13}$$

It is clear that this twist function reduces to the one (3.2.9) of the PCM in the undeformed limit $\eta = 0$. Let us now study the analytical properties of the 1-form $\varphi_\eta(\lambda)\mathrm{d}\lambda$, summarised on Figure 3.3. As for the PCM, it possesses two simple zeros at $\lambda = +1$ and $\lambda = -1$. It also possesses a double pole at infinity and two simple poles at $\pm c\eta$ (*i.e.* at $\pm\eta$ for a split matrix R and at $\pm i\eta$ for a non-split one). This is in contrast with the PCM case, for which there was a double pole at infinity and a double pole at 0. The effect of the Yang-Baxter deformation is thus to split the double pole at 0 into two simple poles at $\pm c\eta$, without deforming the zeros. We shall see that this is a common feature of all integrable deformations of σ-models.



Principal Chiral Model — Yang-Baxter Model (non-split R)

Figure 3.3: Effect of the Yang-Baxter deformation on the poles and zeros of the twist function.

Lax matrix at the pole of the twist function. As we have seen above, the twist function of the Yang-Baxter model possesses two simple poles at $\lambda_\pm = \pm c\eta$. We shall come back on the interpretation of these poles later, in the second part of this thesis. At this stage, let us just make the following observation, which will be useful for Chapter 5.

It is a classical result that a Lax pair of a model is not defined uniquely. In particular, one always has a freedom of performing a formal gauge transformation on the Lax pair $(\mathcal{L}, \mathcal{M})$ by a G-valued field $u(x,t)$ (this comes from the fact that the Lax equation is a zero curvature equation):

$$\mathcal{L}^u(\lambda) = u\mathcal{L}(\lambda)u^{-1} + u\partial_x u^{-1} \quad \text{and} \quad \mathcal{M}^u(\lambda) = u\mathcal{M}(\lambda)u^{-1} + u\partial_t u^{-1}. \tag{3.3.14}$$

The Lax equation (2.1.1) for the pair $(\mathcal{L}, \mathcal{M})$ is then equivalent to the one for the Lax pair $(\mathcal{L}^u, \mathcal{M}^u)$. For the Yang-Baxter model, we can then consider the Lax pair $(\mathcal{L}^g_\eta, \mathcal{M}^g_\eta)$, obtained by a gauge transformation with $u = g$.

We will be interested in the evaluation of this new Lax matrix $\mathcal{L}^g_\eta$ at the poles of the twist function. Starting from the expression (3.3.12), we get

$$\mathcal{L}^g_\eta(\lambda_\pm) = -\gamma R^\mp\big(gXg^{-1}\big), \tag{3.3.15}$$

with (see Appendix C for more informations about the operators $R^\pm$)

$$R^\pm = R \pm c\,\mathrm{Id} \quad \text{and} \quad \gamma = \frac{\eta}{K}.$$

The equation (3.3.15) will be the starting point for the studies of Poisson-Lie symmetries of the Yang-Baxter model in Chapter 5. To conclude this paragraph, let us also note that the factor γ satisfies

$$\frac{1}{\gamma} = \pm 2c \operatorname*{res}_{\lambda=\lambda_\pm} \varphi_\eta(\lambda)\,\mathrm{d}\lambda. \tag{3.3.16}$$

3.3.2 Deformations of the PCM

In the previous subsection, we presented in details the Yang-Baxter model, which is a particular example of an integrable deformation of the PCM. In this subsection, we will review briefly the whole landscape of these deformations.

As we have seen in the previous subsection, the Yang-Baxter model consists in breaking the left symmetry of the PCM, while preserving the right one. In [28], Klimčik proposed a deformation of the Yang-Baxter model itself, depending on another parameter $\widetilde{\eta}$, which also breaks the right symmetry while keeping the existence of a Lax pair. This model is called the Bi-Yang-Baxter model. We shall not discuss it further here: it will be the subject of the following section, as part of the presentation of my PhD works.

Another possibility to deform the PCM is to add [44] to its action a so-called Wess-Zumino term (multiplied by a constant parameter k). We will not explain here how this term is constructed and refer to [44,42,43] for more details on this matter. It was shown in [45] (see also [135]) that this model admits a Lax pair for any value of k. Its twist function has been studied in [46] and proved to be

$$\varphi_k(\lambda) = \frac{1-\lambda^2}{(\lambda-k)^2}. \tag{3.3.17}$$

In the undeformed limit k goes to 0, one recovers the twist function (3.2.9) of the PCM (for the global factor $K = 1$). An interesting remark is that, as for the Yang-Baxter case, the deformation modifies the poles of the 1-form $\varphi_k(\lambda)\mathrm{d}\lambda$, without modifying its zeros. More precisely, the deformation moved the double pole at $\lambda = 0$ of the PCM to a double pole at $\lambda = k$. The double pole at infinity and the two simple zeros at ± 1 of the PCM are not affected by this deformation.

Some deformations of the PCM can be combined together to form multi-parameter deformations. In fact, it was shown recently by Delduc, Hoare, Kameyama and Magro [47] that there exists a three-parameter integrable deformation of the PCM combining the left and right deformations η and $\widetilde{\eta}$ of the Bi-Yang-Baxter model (see above) and the addition of a Wess-Zumino term (with factor k). More precisely, they have constructed such a model which admits a Lax pair representation. The Hamiltonian integrability of this model, however, has not been studied yet although it is expected that it enters the class of models with a twist function. Although this might not be a simple thing to prove, this three-parameter deformation is expected to be the maximal way of deforming the twist function of the PCM.

As the twist function of the three-parameter deformation of the PCM has not been computed yet, we will restrict ourselves to its two-parameter limits. One of them is the Bi-Yang-Baxter model mentioned above. The other one is a combination of the Yang-Baxter deformation (with parameter η) and of the addition of a Wess-Zumino term (with parameter k). It was first proposed in [23] and further studied in [136,137] for the group $G_0 = SU(2)$. The model for a general group was constructed and shown to be integrable by Delduc, Magro and Vicedo in [46]. Its twist function takes the form

$$\varphi_{\mathrm{dPCM}}(\lambda) = \frac{1-\lambda^2}{(\lambda-k)^2+\mathcal{A}^2}, \tag{3.3.18}$$

where the subscript dPCM stands for deformed PCM and

$$\mathcal{A} = \eta\sqrt{1-\frac{k^2}{1+\eta^2}}.$$

Note that this model was considered in the litterature only for a Yang-Baxter deformation associated with a non-split solution R of the mCYBE (*i.e.* $c = i$ in (3.3.1)). One easily checks that $\mathcal{A} = 0$ for $\eta = 0$, so that the twist function (3.3.18) coincides with (3.3.17) in this limit. In the same way, $\mathcal{A} = \eta$ for $k = 0$, so (3.3.18) reduces to the twist function (3.3.13) of the Yang-Baxter model (for $c = i$ and an appropriate choice of K) in this case.

The integrable structure of the general deformed PCM shares some similarities with the one of the PCM. In particular, its Lax matrix takes the "model-independent" form

$$\mathcal{L}_{\mathrm{dPCM}}(\lambda, x) = \frac{A(x) + \lambda\,\Pi(x)}{1-\lambda^2}, \tag{3.3.19}$$

where A and Π are $\mathfrak{g}_0$-valued fields. The deformation is contained in the (model-dependent) expression of A and Π in terms of the fields g and X. In particular, according to Subsection 3.2.1, one has $A = j^L$ and $\Pi = X$ in the undeformed case $k = \eta = 0$. In the same way, in the Yang-Baxter case $k = 0$ and $\eta \neq 0$, A and Π coincide with the currents K_1 and K_0 of Subsection 3.3.1.

The expression of the currents A and Π for the general case $k \neq 0$ is more involved: it is easier to express them in terms of g and another field Y instead of X. The relation between (g, X) and (g, Y) is difficult to state explicitly due to some non-locality issues coming with Wess-Zumino terms. We shall not enter into this matter here as we will not need the explicit definition of A and Π in this PhD (see [46] for details).

The form of the Lax matrix (3.3.19), specific to a Zakharov-Mikhailov scheme, reflects the existence for all these models of a flat and conserved current. This current is associated by the Noether theorem with the right multiplication symmetry of the model, which is conserved both by the Yang-Baxter deformation and the addition of the Wess-Zumino term.

Another common feature shared by the two-parameter dPCM and its (less deformed) limits is the expression of the Hamiltonian and total momentum of the model. They are given in terms of the fields

A and Π by

$$\mathcal{H}_{\text{dPCM}} = \frac{\mathcal{B}}{2}\int \mathrm{d}x\, \Big((\mathcal{A}^2 + k^2 + 1)\,(\kappa(\Pi,\Pi) + \kappa(A,A)) + 4k\,\kappa(\Pi,A)\Big), \tag{3.3.20a}$$

$$\mathcal{P}_{\text{dPCM}} = \mathcal{B}\int \mathrm{d}x\, \Big(k\,(\kappa(\Pi,\Pi) + \kappa(A,A)) + (\mathcal{A}^2 + k^2 + 1)\kappa(\Pi,A)\Big), \tag{3.3.20b}$$

with $\mathcal{B}$ a global factor depending on $\mathcal{A}$ and k *via* the relation $\mathcal{B} = -\frac{1}{4}\varphi'_{\text{dPCM}}(1)\varphi'_{\text{dPCM}}(-1)$. Interestingly, one can always rewrite the above expressions as

$$\mathcal{H}_{\text{dPCM}} = \mathcal{H}_+ - \mathcal{H}_- \qquad \text{and} \qquad \mathcal{P}_{\text{dPCM}} = \mathcal{H}_+ + \mathcal{H}_-, \tag{3.3.21}$$

with

$$\mathcal{H}_\pm = -\frac{1}{2}\operatorname*{res}_{\lambda=\pm1}\varphi_{\text{dPCM}}(\lambda)\int \mathrm{d}x\, \kappa\big(\mathcal{L}_{\text{dPCM}}(\lambda,x), \mathcal{L}_{\text{dPCM}}(\lambda,x)\big).$$

We shall use this fact in Chapter 4.

There exists another type of integrable deformation of the PCM, called the λ-deformation [39]. It consists more of a deformation of the non-abelian T-dual of the PCM, which is a model equivalent to the PCM. We shall not present here the results on this model. The twist function of this model turns out to be of the same form than the one of a split Yang-Baxter deformation [40, 138]. This is due to a deeper relation between the Yang-Baxter model and the λ-deformation [138–140], involving a Poisson-Lie duality [141–143, 27].

3.3.3 Deformations of $\mathbb{Z}_2$-coset models

We shall now discuss the integrable deformations of the $\mathbb{Z}_2$-coset σ-model. We will use the notations of Subsection 3.2.2, in which we described the undeformed theory. There are two types of deformations of the $\mathbb{Z}_2$-coset model: the Yang-Baxter deformation and λ-deformation. As for the PCM, the two give a similar integrable structure and in particular a twist function of the same form. We shall focus on the Yang-Baxter deformation and refer to [40] for the λ-deformation.

The deformed model we are going to study here is part of a general scheme of integrable deformations called Yang-Baxter type deformations [138], or η-deformation, which applies to a certain class of models with twist function. In particular, the η-deformation of the PCM is the Yang-Baxter model, that we studied in Subsection 3.3.1. In this subsection, we shall skip most technical details and insist on the common features shared by this model and the Yang-Baxter one. They actually are common features of general Yang-Baxter type deformations and will be the starting key points for the chapter 5 of this thesis. Moreover, in this subsection, we will present a common formalism describing the integrable structure of both the undeformed $\mathbb{Z}_2$-coset and its η-deformation, as we did for the PCM and its deformations in the previous subsection.

Action and symmetries. The $\mathbb{Z}_2$-coset η-deformation has been constructed by Delduc, Magro and Vicedo in [29]. As for the Yang-Baxter model, it is based on a skew-symmetric solution R of the mCYBE (3.3.1). Its action is given by

$$S_{\mathbb{Z}_2,\eta}[g] = K\int_\Sigma \mathrm{d}x^+\,\mathrm{d}x^-\,\kappa\left(j_+^{L\,(1)}, \frac{1}{1-\eta R_g\circ\pi^{(1)}}j_-^{L\,(1)}\right), \tag{3.3.22}$$

where $R_g = \mathrm{Ad}_g^{-1}\circ R\circ \mathrm{Ad}_g$, as in the PCM case, and $\pi^{(1)}$ is the projection on the eigenspace $\mathfrak{g}^{(1)}$ of σ. Note the similarity between this action and the one of the Yang-Baxter model, written as (3.3.3).

We expect a deformation of the $\mathbb{Z}_2$-coset model to possess the same degrees of freedom as the undeformed model. Thus, we expect the gauge symmetry (3.2.12) of the undeformed model to also leave invariant the action (3.3.22). This can be easily verified using the expression (3.2.13) of the transformation of $j_\pm^{L\,(1)}$ under a gauge transformation and the fact that R_g transforms as $\mathrm{Ad}_h^{-1}\circ R\circ\mathrm{Ad}_h$. Thus, the η-deformation preserves the right gauge symmetry of the $\mathbb{Z}_2$-coset model.

Recall that the undeformed model also possesses a global left symmetry (see Subsection 3.2.2). Due to the presence of the operator R_g in the action (3.3.22), this symmetry is broken by the η-deformation. This is similar to what happens for the Yang-Baxter model, *i.e.* the η-deformation of the PCM, which preserves the right symmetry of the PCM but breaks the left one. This symmetry breaking is one of the common feature shared by the Yang-Baxter type deformations.

Lax matrix. It was found in [29] that the model (3.3.22) admits a Lax pair. In particular, the Lax matrix can be expressed as

$$\mathcal{L}_{\mathrm{d}\mathbb{Z}_2}(\lambda) = A^{(0)} + \frac{1}{2}\left(\frac{1}{\lambda}+\lambda\right)A^{(1)} + \frac{1}{2}(\lambda^2-1)\Pi^{(0)} + \frac{1}{2}\left(\lambda-\frac{1}{\lambda}\right)\Pi^{(1)}, \tag{3.3.23}$$

where A and Π are $\mathfrak{g}_0$-valued fields. Similarly to the deformations of the PCM in Subsection 3.3.2, this structure is common to the deformed and undeformed $\mathbb{Z}_2$-models. The deformation is purely contained in the dependence of A and Π in terms of g and X. According to equation (3.2.25), we have in the undeformed limit $\eta = 0$ that $\Pi = K^{-1}X$ and $A = j^L$. We refer to [29] for the expressions of A and Π in the deformed case. In both the undeformed and the deformed models, the field $\Pi^{(0)}$ is the constraint associated with the gauge symmetry (3.2.12). Finally, let us note that the Lax matrix satisfies the equivariance and reality conditions (2.4.3) and (2.4.7).

Twist function. The computation of the Poisson bracket of the Lax matrix is also carried out in [29]. It takes the form of a Maillet non-ultralocal bracket. The $\mathcal{R}$-matrix of the latter is given by the standard matrix $\mathcal{R}^0$, twisted by the automorphism σ, and the twist function:

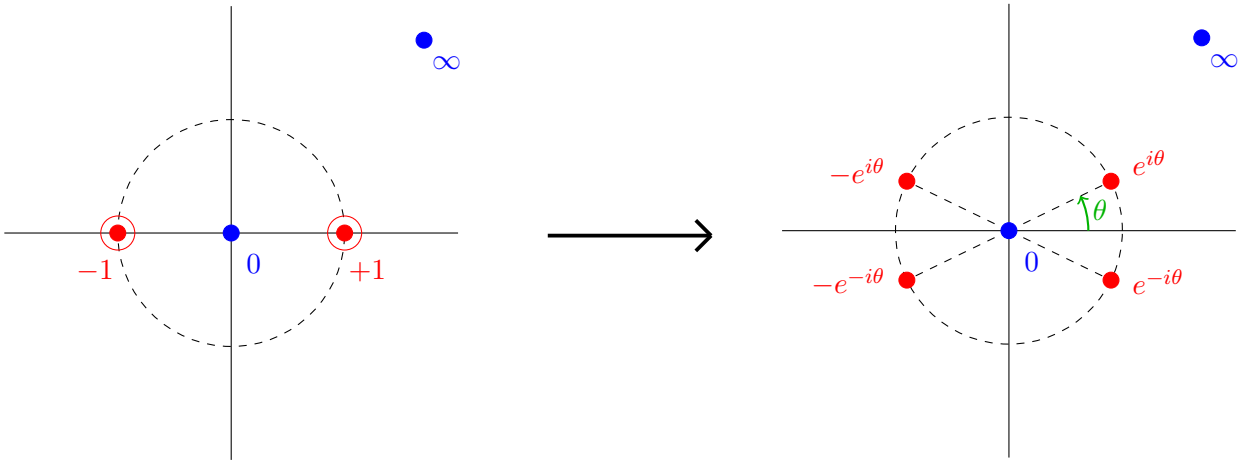
$$\varphi_{\mathrm{d}\mathbb{Z}_2}(\lambda) = \frac{2K\lambda}{(\lambda^2-1)^2 - c^2\eta^2(\lambda^2+1)^2}. \tag{3.3.24}$$

This twist function clearly reduces to the one (3.2.24) of the undeformed $\mathbb{Z}_2$-coset model when $\eta = 0$. Note that it also satisfies the equivariance and reality conditions (2.4.5) and (2.4.10).

The 1-form $\varphi_{\mathrm{d}\mathbb{Z}_2}(\lambda)\mathrm{d}\lambda$ possesses two simple zeros at 0 and infinity, as for the undeformed $\mathbb{Z}_2$-coset model. However, the two double poles $+1$ and -1 of the undeformed model have been split into simple poles $\lambda_\pm$ and $-\lambda_\pm$, given by

$$\lambda_\pm = \sqrt{\frac{1\pm c\eta}{1\mp c\eta}} = \frac{1}{\lambda_\mp}.$$

For a split matrix R ($c = 1$), these poles are situated on the real axis. For a non-split R ($c = i$), they belong to the unit circle of the complex plane. We can rewrite them as $\lambda_\pm = e^{\pm i\theta}$, with the angle θ defined by $\eta = \tan\theta$, as represented in Figure 3.4.



$\mathbb{Z}_2$-coset σ-model η-deformation (non-split R)

Figure 3.4: Effect of the Yang-Baxter deformation on the poles and zeros of the twist function.

We observe that the two poles $+1$ and -1 of the undeformed model split in a symmetric way under the reflection with respect to the origin. This is forced by the equivariance property (2.4.5) of the twist function, which in this case implies that

$$\varphi_{\mathrm{d}\mathbb{Z}_2}(-\lambda) = -\varphi_{\mathrm{d}\mathbb{Z}_2}(\lambda).$$

Indeed, for every pole λ_∞, there must then also be a pole at $-\lambda_\infty$. In some sense (which will be made clearer in the second part of this thesis), one can consider these two poles as a unique set of poles related by the transformation $\lambda \mapsto -\lambda$. Thus, we see that the effect of the η-deformation is to split a (set of) double pole(s) into two (sets of) simple poles. This is similar to the case of the Yang-Baxter model and is one of the common feature of the Yang-Baxter type deformations.

We end this paragraph by a last observation. Performing the gauge transformation of the Lax matrix (3.3.19) by g and evaluating it at $\lambda_\pm$, one finds

$$\mathcal{L}^g_{\mathrm{d}\mathbb{Z}_2}(\lambda_\pm) = -\gamma\, R^{\mp}\big(gXg^{-1}\big), \qquad \text{with} \qquad \frac{1}{\gamma} = \pm 4c \operatorname*{res}_{\lambda=\lambda_\pm} \varphi_{\mathrm{d}\mathbb{Z}_2}(\lambda)\, \mathrm{d}\lambda, \tag{3.3.25}$$

and $R^\pm = R \pm c\,\mathrm{Id}$. This is to be compared with equation (3.3.15), derived for the Yang-Baxter model. This is the main common feature of Yang-Baxter type models, that we shall use extensively in Chapter 5.

Deformed $\mathbb{Z}_T$-coset models. Let us end this section by discussing the deformations of $\mathbb{Z}_T$-coset σ-models. It is expected that the $\mathbb{Z}_T$-coset models admit an integrable deformation of Yang-Baxter type. The effect of this deformation would be to break the left symmetry of the model and to split the double poles of the twist function (3.2.27) into simple poles. However, this deformed model has never been constructed explicitly. This construction would be an interesting project, to complete the study of the whole landscape of deformations of σ-models. We expect the Lax matrix of this deformed model to be of the form

$$\mathcal{L}_{\mathrm{d}\mathbb{Z}_T}(\lambda, x) = \sum_{k=1}^{T} \frac{(T-k) + k\lambda^{-T}}{T} \lambda^k A^{(k)}(x) + \sum_{k=1}^{T} \frac{1-\lambda^{-T}}{T} \lambda^k \Pi^{(k)}(x), \tag{3.3.26}$$

with A and Π some $\mathfrak{g}_0$-valued field depending on g, X and η.

It is worth mentioning that there exists a Yang-Baxter type deformation of the Green-Schwarz superstring on $AdS_5 \times S^5$, which was discovered by Delduc, Magro and Vicedo in [31, 32]. Its Lax matrix is of the form (3.3.26) for $T = 4$. The undeformed model possesses the same twist function as the $\mathbb{Z}_4$-coset σ-model, *i.e.* equation (3.2.27) with $T = 4$. The deformation modifies this twist function as

$$\varphi_{\mathrm{dGS}}(\lambda) = \frac{4\alpha\lambda^3}{(\lambda^4-1)^2 + \beta(\lambda^4+1)^2},$$

where α and β are constants depending on the deformation parameter η, satisfying $\alpha = 1$ and $\beta = 0$ when $\eta = 0$ (thus, for $\eta = 0$, we recover the twist function (3.2.27) for $T = 4$). This twist function has 8 simple poles at $i^k e^{\pm i\theta}$, $k = 0, 1, 2, 3$, with $\eta = \tan\theta$. Thus, the effect of the deformation is to split the double poles $\{1, i, -1, -i\}$ into pairs of simple poles, as expected for a Yang-Baxter type deformation. The deformation of the Green-Schwarz model also exhibits other properties of Yang-Baxter type deformations: the left symmetry of the model is broken by the deformation and the Lax matrix of the model satisfies the relation (3.3.25) for the simple poles $\lambda_\pm = e^{\pm i\theta}$ (see [32]).

3.4 Bi-Yang-Baxter model

In this section, I will present some results I obtained during my PhD about the Bi-Yang-Baxter model. These results are part of the publication [P1]: here, I will summarize the main ideas of [P1] (and a

few subsequent results) without entering into too much technical details. The whole content of the article [P1], including these details, can be found at the end of this PhD thesis. Note that some conventions and notations (in particular the sign of the Lax matrix $\mathcal{L}$) are different in the core text of this thesis and in the article [P1]. In the present section, we will keep the conventions that we used in the previous chapter and sections and will transpose the results of [P1] in those.

The Bi-Yang-Baxter model (BYBM) is a double deformation of the PCM, introduced by Klimčik in [28], which depends on two deformation parameters η and $\widetilde{\eta}$. The limits $\eta = 0$ and $\widetilde{\eta} = 0$ respectively correspond to the right and left Yang-Baxter model, *i.e.* the one-parameter deformations of the PCM which break respectively the right and left symmetries of the model. The BYBM is then a combination of these two Yang-Baxter deformations, which breaks both the left and the right symmetries.

The BYBM was shown to possess a Lax pair representation by Klimčik in [33], hence proving the existence of an infinite number of conserved charges. However, the Hamiltonian integrability of the model, *i.e.* the fact that these conserved charges are in involution one with another, was not proved. This was the subject of my first PhD project, which resulted in the article [P1]. We showed that the BYBM enters the class of non-ultralocal models with twist function, thus proving its Hamiltonian integrability. More precisely, as we shall see, we have shown this for a reformulation of the BYBM as a $\mathbb{Z}_2$-coset model on $G_0 \times G_0/G_0^{\text{diag}}$. We shall start by introducing and studying this formulation and will end the section by discussing the more usual formulation as a deformation of the PCM.

3.4.1 The Bi-Yang-Baxter model in Lagrangian formulation

Action and symmetries. We shall use the notations of the previous chapter and sections. In particular, we consider a real Lie group G_0 with Lie algebra $\mathfrak{g}_0$. Let R and $\widetilde{R}$ be two skew-symmetric solutions of the mCYBE (3.3.1) on $\mathfrak{g}_0$ (we shall restrict here to the case of the non-split mCYBE, *i.e.* with $c = i$). We define the Bi-Yang-Baxter model (BYBM) by the following action, for a field $(g, \widetilde{g})$ valued in the double group $G_0 \times G_0$ [144]

$$S_{\eta,\widetilde{\eta}}[g,\widetilde{g}] = \frac{K}{2}\int \mathrm{d}x^+\mathrm{d}x^-\ \kappa\left(j_+^L - \widetilde{j}_+^L, \left(1 - \frac{\eta}{2}R_g - \frac{\widetilde{\eta}}{2}\widetilde{R}_{\widetilde{g}}\right)^{-1}(j_-^L - \widetilde{j}_-^L)\right). \tag{3.4.1}$$

K, η and $\widetilde{\eta}$ are real parameters, the currents $j_\pm^L = g^{-1}\partial_\pm g$ and $\widetilde{j}_\pm^L = \widetilde{g}^{-1}\partial_\pm\widetilde{g}$ are the left currents of g and $\widetilde{g}$, as introduced in Subsection 3.1.1 and finally, we have

$$R_g = \mathrm{Ad}_g^{-1}\circ R\circ \mathrm{Ad}_g \quad \text{and} \quad \widetilde{R}_{\widetilde{g}} = \mathrm{Ad}_{\widetilde{g}}^{-1}\circ \widetilde{R}\circ \mathrm{Ad}_{\widetilde{g}}.$$

Let us notice here that R_g and $\widetilde{R}_{\widetilde{g}}$ are also skew-symmetric solutions of the mCYBE (3.3.1).

Let us consider the subgroup G_0^{diag} of $G_0 \times G_0$ composed by elements of the form (h, h), $h \in G_0$. It acts on $(g, \widetilde{g}) \in G_0 \times G_0$ by the right multiplication

$$g \longmapsto gh \quad \text{and} \quad \widetilde{g} \longmapsto \widetilde{g}h. \tag{3.4.2}$$

One easily checks that the action (3.4.1) is invariant under the transformation (3.4.2), for h an arbitrary field valued in G_0. Thus, the action (3.4.1) possesses a gauge symmetry under the right action of G_0^{diag}.

Let us also consider the action on $(g, \widetilde{g})$ of the right multiplication by constant elements $(h, \widetilde{h})$ in $G_0 \times G_0$:

$$g \longmapsto hg \quad \text{and} \quad \widetilde{g} \longmapsto \widetilde{h}\widetilde{g}. \tag{3.4.3}$$

One checks that the action (3.4.1) is invariant under this transformation if and only if $\eta = \widetilde{\eta} = 0$. We shall come back on this remark later.

Gauge fixing and deformed "PCM" formulation. As we observed above, the action (3.4.1) possesses a gauge symmetry under the right multiplication by G_0^{diag}. The real degrees of freedom of this model then belong to the quotient $G_0 \times G_0/G_0^{\text{diag}}$. One can identify this quotient with the group

G_0, by observing that each orbit contains a unique element of the form (g', Id). Thus, one should be able to reformulate the model in terms of a G_0-valued field. This is done at the level of the action by performing a gauge transformation (3.4.2) with $h = \widetilde{g}^{-1}$. Doing so, one finds a new action on the gauge-invariant field $g' = g\widetilde{g}^{-1}$, which reads

$$S_{\eta,\widetilde{\eta}}[g'] = \frac{K}{2}\int \mathrm{d}x^+\mathrm{d}x^-\ \kappa\left(j'^L_+, \left(1 - \frac{\eta}{2}R_{g'} - \frac{\widetilde{\eta}}{2}\widetilde{R}\right)^{-1} j'^L_-\right). \tag{3.4.4}$$

This is the original action proposed by Klimčik in [28]. In this formulation, it is clear that the BYBM is a two-parameter deformation of the PCM (3.2.2).

Let us note that the action (3.4.3) of $G_0 \times G_0$ acts on the field g' by the following combination of the left and right multiplications

$$g' \mapsto hg\widetilde{h}^{-1}.$$

In particular, both the left and right multiplication are not symmetries of the action (3.4.4) if η and $\widetilde{\eta}$ are non zero. When $\eta = \widetilde{\eta} = 0$, these symmetries are restored, as the model then reduces to the undeformed PCM model.

It is clear in this gauge-fixed formulation that the BYBM (3.4.4) reduces to the Yang-Baxter model (3.3.3) when $\widetilde{\eta} = 0$ (with a change from η to $\frac{\eta}{2}$ and from K to $\frac{K}{2}$). In this case, the global right symmetry $g' \mapsto g'h$ is restored and the left symmetry $g' \mapsto hg'$ stays broken. In the same way, the limit $\eta = 0$ is also the Yang-Baxter model, but with the left symmetry preserved and the right symmetry broken. The BYBM is thus a combination of both the left and the right Yang-Baxter deformations. According to Subsection 3.3.1, the one-parameter limits $\eta = 0$ and $\widetilde{\eta} = 0$ are thus both integrable models with twist function.

$\mathbb{Z}_2$-coset Yang-Baxter limit. As we shall see now, there exist two other one-parameter limits of the model which also admit a twist function. They arise naturally from the non-gauge fixed formulation (3.4.1), once we identify it with a deformation of a $\mathbb{Z}_2$-coset σ-model.

Let us consider the double group $DG_0 = G_0 \times G_0$. It is equipped with an involutive automorphism δ defined as the exchange of the two G_0 factors in DG_0:

$$\delta: \begin{array}{ccc} DG_0 = G_0 \times G_0 & \longrightarrow & DG_0 = G_0 \times G_0 \\ (g, \widetilde{g}) & \longmapsto & (\widetilde{g}, g) \end{array}.$$

Let us note here that the subgroup $DG_0^{(0)}$ of δ-fixed-points is equal to G_0^{diag}: thus, we can identify the coset $G_0 \times G_0/G_0^{\mathrm{diag}}$ with the $\mathbb{Z}_2$-coset $DG_0/DG_0^{(0)}$. The automorphism δ induces an involutive automorphism of the Lie algebra $D\mathfrak{g}_0 = \mathfrak{g}_0 \times \mathfrak{g}_0$, that we shall still denote δ. We will use the notations of Subsection 3.2.2 and Appendix A.4 for finite order automorphisms. In particular, the eigenspaces of δ are

$$D\mathfrak{g}_0^{(0)} = \{(X, X),\ X \in \mathfrak{g}_0\} = \mathfrak{g}_0^{\mathrm{diag}} \quad \text{and} \quad D\mathfrak{g}_0^{(1)} = \{(X, -X),\ X \in \mathfrak{g}_0\}.$$

The projectors $\pi^{(p)}$ of the decomposition $D\mathfrak{g}_0 = D\mathfrak{g}_0^{(0)} \oplus D\mathfrak{g}_0^{(1)}$ are then given by

$$\pi^{(0)}: \begin{array}{ccc} D\mathfrak{g}_0 & \longrightarrow & D\mathfrak{g}_0^{(0)} \\ (X, Y) & \longmapsto & \frac{1}{2}(X + Y, X + Y) \end{array} \quad \text{and} \quad \pi^{(1)}: \begin{array}{ccc} D\mathfrak{g}_0 & \longrightarrow & D\mathfrak{g}_0^{(1)} \\ (X, Y) & \longmapsto & \frac{1}{2}(X - Y, Y - X) \end{array}.$$

We will encode the two fields g and $\widetilde{g}$ in the DG_0-valued field $f = (g, \widetilde{g})$. Note that the grade (1) part of the left current $f^{-1}\partial_\pm f$ is given by

$$\left(f^{-1}\partial_\pm f\right)^{(1)} = \frac{1}{2}\left(j^L_\pm - \widetilde{j}^L_\pm, \widetilde{j}^L_\pm - j^L_\pm\right).$$

The Killing form on the double algebra $D\mathfrak{g}_0$ is given by $\kappa_D\big((X, Y), (X', Y')\big) = \kappa(X, X') + \kappa(Y, Y')$, where κ is the Killing form on $\mathfrak{g}_0$. Let us define $\mathfrak{R} = \big(R, \widetilde{R}\big)$. It is a skew-symmetric solution of the mCYBE (3.3.1) on $D\mathfrak{g}_0$. In the limit $\eta = \widetilde{\eta}$, we can rewrite the action (3.4.1) as

$$S_{\eta=\widetilde{\eta}}[f] = K\int \mathrm{d}x^+\mathrm{d}x^-\ \kappa_D\left(\left(f^{-1}\partial_+ f\right)^{(1)}, \frac{1}{1 - \eta\,\mathfrak{R}_f \circ \pi^{(1)}}\left(f^{-1}\partial_- f\right)^{(1)}\right).$$

This is the Yang-Baxter type deformation of the $\mathbb{Z}_2$-coset σ-model on $DG_0/DG_0^{(0)}$, as described in Subsection 3.3.3. Thus, the one-parameter limit $\eta = \widetilde{\eta}$ is also an integrable model with twist function.

In the same way, we check that the limit $\eta = -\widetilde{\eta}$ is also a Yang-Baxter type deformation of the $\mathbb{Z}_2$-coset σ-model on $DG_0/DG_0^{(0)}$, but with the solution $\mathfrak{R} = \big(R, -\widetilde{R}\big)$ of the mCYBE on $D\mathfrak{g}_0$. Thus, this is also a one-parameter limit which admits a twist function. Moreover, let us note that the twist function of the limit $\eta = \widetilde{\eta}$ is the same as the one of the limit $\eta = -\widetilde{\eta}$, as the twist function of a Yang-Baxter type deformation does not depend on the choice of the matrix $\mathfrak{R}$.

3.4.2 Lax pair of the BYBM

In this section, we shall show that the BYBM equations of motion can be recast into a Lax equation. This was first done by Klimčik in [33], using the gauge-fixed action (3.4.4). Here, we shall start from the non-gauge-fixed one (3.4.1) and follow a construction close to the one for the Lax pair of the $\mathbb{Z}_2$-coset model (see Subsection 3.2.2). We will explain the main step of this construction and refer to our article [P1] (given at the end of this thesis) for more details.

Equations of motion. Varying the action (3.4.1) with respect to the field g, one finds the following equations of motion:

$$EOM = \partial_+ J_- + [a_+, J_-] + \partial_- J_+ + [a_-, J_+] = 0, \tag{3.4.5}$$

where we introduced

$$J_\pm = \left(1 \pm \frac{\eta}{2} R_g \pm \frac{\widetilde{\eta}}{2} \widetilde{R}_{\widetilde{g}}\right)^{-1} (j^L_\pm - \widetilde{j}^L_\pm) \tag{3.4.6}$$

and a "gauge field"

$$a_\pm = j^L_\pm \mp \frac{\eta}{2} R_g J_\pm = \left(1 \pm \frac{\widetilde{\eta}}{2} \widetilde{R}_{\widetilde{g}}\right) J_\pm + \widetilde{j}^L_\pm. \tag{3.4.7}$$

We note that the equation of motion is invariant under the redefinition $a_\pm \mapsto a_\pm + \rho J_\pm$ of $a_\pm$, where ρ is a constant. Varying the field $\widetilde{g}$ in the action (3.4.1), one finds some other equations of motion. Using the freedom in the definition of the gauge field $a_\pm$ mentioned above, one checks that this equation of motion is equivalent to the one (3.4.5) on g. This redundancy in the equations of motion is due to the gauge symmetry (3.4.2) of the model.

Search for a Lax pair. The equation of motion (3.4.5) takes the form of a covariant conservation equation, for the current $J_\pm$, with the covariant derivative $\partial_\pm + [a_\pm, \cdot]$. This was also the form of the equations of motion of the $\mathbb{Z}_2$-coset σ-model in Subsection 3.2.2. We shall use the results about the Lax representation of the $\mathbb{Z}_2$-coset model as a guide for the search of a Lax pair for the BYBM. In particular, following equation (3.2.16), one could try to define the Lax pair $\mathcal{L}^{\mathrm{BYB}}_\pm(\lambda) = a_\pm + \lambda^{\pm 1} J_\pm$.

However, let us note that the covariant conservation equation on $J_\pm$ is also verified by the current $\frac{\zeta}{2} J_\pm$, for any constant ζ. Moreover, recall that the "gauge-field" $a_\pm$ can be redefined by adding to it a term proportional to $J_\pm$. For now on, we shall then consider the most general gauge-field

$$A_\pm = a_\pm + \rho J_\pm, \tag{3.4.8}$$

with ρ an arbitrary constant, and define the Lax pair to be

$$\mathcal{L}^{\mathrm{BYB}}_\pm(\lambda) = A_\pm + \frac{\zeta}{2} \lambda^{\pm 1} J_\pm. \tag{3.4.9}$$

Following equation (3.2.17), we then find

$$\begin{aligned} &\partial_+ \mathcal{L}^{\mathrm{BYB}}_-(\lambda) - \partial_- \mathcal{L}^{\mathrm{BYB}}_+(\lambda) + \big[\mathcal{L}^{\mathrm{BYB}}_+(\lambda), \mathcal{L}^{\mathrm{BYB}}_-(\lambda)\big] \\ &\quad = \frac{\zeta}{4}\left(\lambda + \frac{1}{\lambda}\right) EOM + \left(\partial_+ A_- - \partial_- A_+ + \big[A_+, A_-\big] + \frac{\zeta^2}{4}\big[J_+, J_-\big]\right) \\ &\qquad + \frac{\zeta}{4}\left(\lambda + \frac{1}{\lambda}\right)\Big(\partial_+ J_- - \partial_- J_+ + \big[A_+, J_-\big] + \big[J_+, A_-\big]\Big). \end{aligned} \tag{3.4.10}$$

The first term on the right-hand side obviously vanishes on-shell, *i.e.* when the equations of motion $EOM = 0$ are satisfied. Using the fact that R_g and $\widetilde{R}_{\widetilde{g}}$ are solutions of the mCYBE (3.3.1), combined with the Maurer-Cartan equation (3.1.4) on $j_\pm^L$ (and the one for $\widetilde{j}_\pm^L$), one finds that

$$\partial_+ J_- - \partial_- J_+ + \big[A_+, J_-\big] + \big[J_+, A_-\big] = \left(2\rho + 1 - \frac{\eta^2 - \widetilde{\eta}^2}{2}\right)\big[J_+, J_-\big] + \frac{1}{2}\big(\eta R_g + \widetilde{\eta}\widetilde{R}_{\widetilde{g}}\big)(EOM).$$

Thus, if we choose

$$\rho = -\frac{1}{2}\left(1 - \frac{\eta^2 - \widetilde{\eta}^2}{4}\right), \tag{3.4.11}$$

we see that the last term in equation (3.4.10) vanishes on shell. In the same way, using the mCYBE and the Maurer-Cartan equations, one gets

$$\partial_+ A_- - \partial_- A_+ + \big[A_+, A_-\big] + \frac{\zeta^2}{4}\big[J_+, J_-\big] = -\frac{1}{4}\left(1 - \frac{\eta^2 - \widetilde{\eta}^2}{2}\right)\big(\eta R_g + \widetilde{\eta}\widetilde{R}_{\widetilde{g}}\big)(EOM) + \eta R_g(EOM),$$

for

$$\zeta = \sqrt{\left(1 + \frac{(\eta + \widetilde{\eta})^2}{4}\right)\left(1 + \frac{(\eta - \widetilde{\eta})^2}{4}\right)}. \tag{3.4.12}$$

Thus, for the choice (3.4.11) and (3.4.12) of ρ and ζ, one finds that the vanishing of (3.4.10) is equivalent to the one of EOM. In other words, we found a Lax pair formulation of the equations of motion of the BYBM.

Lax pair in the double algebra. As explained above, the action (3.4.1) is interpreted as a deformation of the $\mathbb{Z}_2$-coset σ-model on $DG_0/DG_0^{(0)}$. Thus, the Lax pair of the model should naturally belong to the double algebra $D\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ (where $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$). However, in the previous paragraph, we constructed a Lax pair $\mathcal{L}_\pm^{\mathrm{BYB}}(\lambda)$ in a single copy of $\mathfrak{g}$. We shall now see how this Lax pair can be "extended" to a Lax pair $\mathcal{L}_\pm^D$ valued in the double algebra $\mathfrak{g} \times \mathfrak{g}$.

It is clear that the Lax equation (2.1.4) for $\mathcal{L}_\pm^D$ is equivalent to the Lax equations for the two $\mathfrak{g}$-valued factors of $\mathcal{L}_\pm^D$. Thus, we shall choose the left factor to be $\mathcal{L}_\pm^{\mathrm{BYB}}$. Moreover, as we are considering a deformation of the $\mathbb{Z}_2$-coset model with involution δ (the exchange automorphism), it is natural to ask the Lax pair $\mathcal{L}_\pm^D$ to satisfy the equivariance relation

$$\delta\left(\mathcal{L}_\pm^D(\lambda)\right) = \mathcal{L}_\pm^D(-\lambda).$$

This fixes the right factor of $\mathcal{L}_\pm^D(\lambda)$ to be $\mathcal{L}_\pm^{\mathrm{BYB}}(-\lambda)$. We then define

$$\mathcal{L}_\pm^D(\lambda) = \big(\mathcal{L}_\pm^{\mathrm{BYB}}(\lambda), \mathcal{L}_\pm^{\mathrm{BYB}}(-\lambda)\big) \in D\mathfrak{g}.$$

It is clear that the Lax equation for $\mathcal{L}_\pm^D$ is satisfied, as it is satisfied by $\mathcal{L}_\pm^{\mathrm{BYB}}$.

In the undeformed limit $\eta = \widetilde{\eta} = 0$, we have $J_\pm = j_\pm^L - \widetilde{j}_\pm^L$, $A_\pm = \frac{1}{2}(j_\pm^L + \widetilde{j}_\pm^L)$ and $\zeta = 1$. Thus, in this limit, one gets

$$\mathcal{L}_\pm^D(\lambda) = \frac{1}{2}\big(j_\pm^L + \widetilde{j}_\pm^L, j_\pm^L + \widetilde{j}_\pm^L\big) + \frac{\lambda^{\pm 1}}{2}\big(j_\pm^L - \widetilde{j}_\pm^L, j_\pm^L - \widetilde{j}_\pm^L\big) = \big(f^{-1}\partial_\pm f\big)^{(0)} + \lambda^{\pm 1}\big(f^{-1}\partial_\pm f\big)^{(1)},$$

with the DG_0-valued field $f = (g, \widetilde{g})$. Thus, for $\eta = \widetilde{\eta} = 0$, $\mathcal{L}_\pm^D$ reduces to the Lax matrix of the $\mathbb{Z}_2$-coset model on $DG_0/DG_0^{(0)}$, as constructed in equation (3.2.16).

As we explained in the last paragraph of Subsection 3.4.1, the one-parameter limits $\eta = \pm\widetilde{\eta}$ coincide with the Yang-Baxter type deformation of the $\mathbb{Z}_2$-coset on $DG_0/DG_0^{(0)}$. One can also check that in this limit, the Lax pair $\mathcal{L}_\pm^D$ reduces to the Lax pair considered in Subsection 3.3.3 for the deformed $\mathbb{Z}_2$-coset model.

Lax matrix. In this subsection, we have constructed light-cone Lax pairs $\mathcal{L}^{\mathrm{BYB}}_{\pm}$ and $\mathcal{L}^{D}_{\pm}$ for the BYBM, valued respectively in $\mathfrak{g}$ and $D\mathfrak{g}$. From these pairs, one can construct the corresponding Lax matrices

$$\mathcal{L}_{\mathrm{BYB}}(\lambda) = \frac{1}{2}\left(\mathcal{L}^{\mathrm{BYB}}_{+}(\lambda) - \mathcal{L}^{\mathrm{BYB}}_{-}(\lambda)\right) \qquad \text{and} \qquad \mathcal{L}_{D}(\lambda) = \frac{1}{2}\left(\mathcal{L}^{D}_{+}(\lambda) - \mathcal{L}^{D}_{-}(\lambda)\right). \tag{3.4.13}$$

One then has

$$\mathcal{L}_{\mathrm{BYB}}(\lambda) = \frac{A_+ - A_-}{2} + \frac{\zeta\lambda}{4}J_+ - \frac{\zeta}{4\lambda}J_-$$

and

$$\mathcal{L}_{D}(\lambda) = \left(\mathcal{L}_{\mathrm{BYB}}(\lambda), \mathcal{L}_{\mathrm{BYB}}(-\lambda)\right). \tag{3.4.14}$$

3.4.3 $\mathcal{R}$-matrices on the double algebra

As explained in the previous subsection, the Lax matrix $\mathcal{L}_D$ of the BYBM in its non-gauged-fixed formulation is naturally valued in the double algebra $D\mathfrak{g}$. As we will prove later, the Hamiltonian integrability of the BYBM is ensured by the fact that this Lax matrix satisfies a Maillet non-ultralocal bracket (2.2.6), with a $\mathcal{R}$-matrix $\mathcal{R}^{D}_{\underline{\mathbf{12}}}$ valued in $D\mathfrak{g}\otimes D\mathfrak{g}$. As the BYBM is a deformation of the $\mathbb{Z}_2$-coset model on $DG_0/DG_0^{(0)}$, one could expect $\mathcal{R}^{D}_{\underline{\mathbf{12}}}$ to be of the form (2.4.1), with a twist function φ and $\mathcal{R}^0$ the standard $\mathcal{R}$-matrix on $\mathcal{L}(D\mathfrak{g})$ twisted by the automorphism δ. However, in the case of the BYBM, the matrix $\mathcal{R}^0$ takes a slightly more general form, that we shall explain here.

The standard construction. Let us recall the main steps of the construction of the standard twisted $\mathcal{R}$-matrix on $\mathcal{L}(D\mathfrak{g})$ (see the Appendix C for the general theory). Let us consider the loop algebra $\mathcal{L}(D\mathfrak{g}) = D\mathfrak{g}(\!(\lambda)\!) = D\mathfrak{g}\otimes\mathbb{C}(\!(\lambda)\!)$ of Laurent series in a complex parameter λ valued in the double algebra $D\mathfrak{g}$ and equipped with the natural Lie bracket. The exchange automorphism δ on $D\mathfrak{g}$ induces an automorphism $\hat{\delta}$ on $D\mathfrak{g}(\!(\lambda)\!)$ defined for all $M\in D\mathfrak{g}(\!(\lambda)\!)$ by

$$\hat{\delta}(M)(\lambda) = \delta\big(M(-\lambda)\big).$$

Denote by $D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}}$ the twisted loop algebra, *i.e.* the subalgebra of $D\mathfrak{g}(\!(\lambda)\!)$ formed by the fixed points of $\hat{\delta}$. It admits a natural vector space decomposition

$$D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}} = D\mathfrak{g}[[\lambda]]^{\hat{\delta}} \oplus \left(\lambda^{-1}D\mathfrak{g}[\lambda^{-1}]\right)^{\hat{\delta}} \tag{3.4.15}$$

into subalgebras of positive and strictly negative powers of the loop parameter λ, respectively. Let π_+ and π_- denote the projection operators relative to this decomposition. The operator

$$\mathcal{R}^D = \pi_+ - \pi_- \tag{3.4.16}$$

defines a solution of the mCYBE on $D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}}$. Suppose now that we are given an invariant inner product $\langle\cdot,\cdot\rangle$ on the twisted loop algebra $D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}}$. We define the kernel $\mathcal{R}^{D}_{\underline{\mathbf{12}}}(\lambda,\mu)$ of the operator $\mathcal{R}^D$ in (3.4.16), with respect to $\langle\cdot,\cdot\rangle$, as the rational function $\mathcal{R}^{D}_{\underline{\mathbf{12}}}(\lambda,\mu)$ of two complex variables and valued in $D\mathfrak{g}\otimes D\mathfrak{g}$, such that for all $M\in D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}}$ we have

$$\langle \mathcal{R}^{D}_{\underline{\mathbf{12}}}(\lambda,\mu), M_{\underline{\mathbf{2}}}(\mu)\rangle_{\underline{\mathbf{2}}} = (\mathcal{R}^D M)(\lambda).$$

This matrix is then a solution of the CYBE (2.3.3) (see Appendix C for more details about this construction). The usual construction for $\mathcal{R}$-matrices with twist function is based on the inner product

$$\langle M, N\rangle_{\varphi} = 2\operatorname*{res}_{\lambda=0}\kappa_D\big(M(\lambda), N(\lambda)\big)\varphi(\lambda)\mathrm{d}\lambda, \tag{3.4.17}$$

for any $M,N\in D\mathfrak{g}(\!(\lambda)\!)$. It is easy to check that this inner product is invariant under $\hat{\delta}$ and thus induces an inner product on the twisted loop algebra $D\mathfrak{g}(\!(\lambda)\!)^{\hat{\delta}}$, if and only if φ is an odd function. This yields a kernel $\mathcal{R}^{D}_{\underline{\mathbf{12}}}(\lambda,\mu)$ of the form (2.4.1), with twist function φ and $\mathcal{R}^0$ the standard matrix on $\mathcal{L}(D\mathfrak{g})$ twisted by δ. The parity condition on φ is then the equivariance condition (2.4.5).

Inner product for the BYBM. We shall now present a more general construction, which allows for a non-odd twist function φ. As we are considering the double Lie algebra $D\mathfrak{g}$, one can define an even more general inner product invariant under $\hat{\delta}$, by separating explicitly the left and right part of $D\mathfrak{g}$. That is, for any $M=(m,\tilde{m})$ and $N=(n,\tilde{n})$ in $D\mathfrak{g}(\!(\lambda)\!)$ we define

$$\langle M,N\rangle_{\varphi}=2\operatorname*{res}_{\lambda=0}\kappa\big(m(\lambda),n(\lambda)\big)\varphi_{\mathrm{BYB}}(\lambda)\mathrm{d}\lambda-2\operatorname*{res}_{\lambda=0}\kappa\big(\tilde{m}(\lambda),\tilde{n}(\lambda)\big)\varphi_{\mathrm{BYB}}(-\lambda)\mathrm{d}\lambda, \tag{3.4.18}$$

where κ is the Killing form on $\mathfrak{g}$. When φ_{BYB} is odd, we recover the twisted inner product (3.4.17). This construction allows to consider twist functions without parity constraint. The kernel of $\mathcal{R}^D$ with respect to the inner product (3.4.18) is given by

$$\mathcal{R}^D_{\underline{\mathbf{12}}}(\lambda,\mu)=\frac{1}{2}\left(\frac{C^{LL}_{\underline{\mathbf{12}}}}{\mu-\lambda}+\frac{C^{RL}_{\underline{\mathbf{12}}}}{\mu+\lambda}\right)\varphi_{\mathrm{BYB}}(\mu)^{-1}-\frac{1}{2}\left(\frac{C^{RR}_{\underline{\mathbf{12}}}}{\mu-\lambda}+\frac{C^{LR}_{\underline{\mathbf{12}}}}{\mu+\lambda}\right)\varphi_{\mathrm{BYB}}(-\mu)^{-1}, \tag{3.4.19}$$

where we defined the partial split Casimirs (κ^{ab} is the Killing form of $\mathfrak{g}$ in the basis $\{I^a\}$)

$$\begin{aligned}C^{LL}_{\underline{\mathbf{12}}}&=\kappa^{ab}(I_a,0)\otimes(I_b,0),\qquad & C^{RR}_{\underline{\mathbf{12}}}&=\kappa^{ab}(0,I_a)\otimes(0,I_b),\\ C^{LR}_{\underline{\mathbf{12}}}&=\kappa^{ab}(I_a,0)\otimes(0,I_b),\qquad & C^{RL}_{\underline{\mathbf{12}}}&=\kappa^{ab}(0,I_a)\otimes(I_b,0).\end{aligned}$$

If φ_{BYB} were an odd function, we would get

$$\mathcal{R}^D_{\underline{\mathbf{12}}}(\lambda,\mu)=\mathcal{R}^{0,D}_{\underline{\mathbf{12}}}(\lambda,\mu)\varphi_{\mathrm{BYB}}(\mu)^{-1},\qquad\text{with}\qquad\mathcal{R}^{0,D}_{\underline{\mathbf{12}}}(\lambda,\mu)=\frac{1}{2}\left(\frac{C^D_{\underline{\mathbf{12}}}}{\mu-\lambda}+\frac{\delta_{\underline{\mathbf{1}}}C^D_{\underline{\mathbf{12}}}}{\mu+\lambda}\right),$$

where $C^D_{\underline{\mathbf{12}}}=C^{LL}_{\underline{\mathbf{12}}}+C^{RR}_{\underline{\mathbf{12}}}$ is the split Casimir of the double algebra $D\mathfrak{g}$. $\mathcal{R}^{0,D}$ is the standard $\mathcal{R}$-matrix on $\mathcal{L}(D\mathfrak{g})$ twisted by δ. We thus recover the usual notion of $\mathcal{R}$-matrix with twist function φ_{BYB} in this case. The twist function φ_{BYB} of the BYBM that we will find is not an odd function in general, so we will need the matrix (3.4.19). However, in the limit $\eta=\pm\widetilde{\eta}$, the twist function φ_{BYB} will be odd and we will recover the usual setting, as expected from the fact that this limit is simply the η-deformation of the $\mathbb{Z}_2$-coset model on $DG_0/DG_0^{(0)}$.

Projection on $\mathfrak{g}$. Let us suppose that the $D\mathfrak{g}$-valued Lax matrix $\mathcal{L}_D$ (3.4.14) satisfies a Maillet bracket (2.2.6) with $\mathcal{R}$-matrix (3.4.19). Projecting this Poisson bracket on the left factor of $D\mathfrak{g}=\mathfrak{g}\times\mathfrak{g}$, one finds that the Lax matrix $\mathcal{L}_{\mathrm{BYB}}$ also satisfies a Maillet bracket with the $\mathcal{R}$-matrix:

$$\mathcal{R}^{\mathrm{BYB}}_{\underline{\mathbf{12}}}=\frac{1}{2}\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}\varphi_{\mathrm{BYB}}(\mu)^{-1}, \tag{3.4.20}$$

with $C_{\underline{\mathbf{12}}}=\kappa^{ab}I_a\otimes I_b$ the split Casimir of $\mathfrak{g}$. Conversely, if $\mathcal{L}_{\mathrm{BYB}}$ satisfies the Maillet bracket with $\mathcal{R}$-matrix (3.4.20), one checks that the matrix $\mathcal{L}^D$ in the double algebra satisfies the bracket with $\mathcal{R}$-matrix (3.4.19). For the rest of this section, we shall then focus on the Poisson bracket of $\mathcal{L}_{\mathrm{BYB}}$ and find the corresponding twist function φ_{BYB}.

3.4.4 Hamiltonian analysis and twist function of the BYBM

In Subsection 3.4.2, we have presented the Lax pair of the BYBM. We will now compute the Poisson bracket of the Lax matrix with itself. For that, we first need to go from the Lagrangian formulation (3.4.1) of the BYBM to its Hamiltonian formulation.

Phase space, constraint and Hamiltonian. The BYBM in its non-gauge-fixed formulation (3.4.1) is defined on two G_0-valued fields g and $\widetilde{g}$. Following Subsection 3.1.2, the phase space of the BYBM is thus parametrised in terms of the G_0-valued fields g and $\widetilde{g}$ and the $\mathfrak{g}_0$-valued fields X and $\widetilde{X}$. The Poisson bracket of g and X is then (3.1.7). The fields $\widetilde{g}$ and $\widetilde{X}$ satisfy exactly the same brackets as g and X. Moreover, the fields g and X have zero Poisson brackets with the fields $\widetilde{g}$ and $\widetilde{X}$.

Starting from the action (3.4.1), one can compute the conjugate momenta of the BYBM and deduce the Lagrangian expression of X and $\widetilde{X}$. One then finds

$$X \approx -\widetilde{X} \approx \frac{K}{4}(J_+ + J_-), \tag{3.4.21}$$

where the current $J_\pm$ was defined in (3.4.6) and the use of the symbol $\approx$ instead of $=$ will be explained in what follows. We deduce from equation (3.4.21) that

$$X + \widetilde{X} \approx 0. \tag{3.4.22}$$

This is a constraint on the phase space, which is due to the existence of a gauge symmetry in the model. We refer to the discussion of constraints in coset models in Subsection 3.2.2 and will use the notations introduced there in the rest of the section (in particular, the symbol $\approx$ holds for weak equations, true only on the constrained phase space).

In Subsection 3.4.1, we explained that we can see the BYBM as a deformation of a $\mathbb{Z}_2$-coset model, with dynamical field $f = (g, \widetilde{g})$ in the double group DG_0. Thus, one can also parametrise the phase space of the model with the DG_0-valued field f and a $D\mathfrak{g}_0$-valued field Z (equivalent of the field X for f). One then finds

$$Z = (X, \widetilde{X}) \approx K\big(J_0^D\big)^{(1)},$$

with

$$J_0^D = \frac{J_+^D + J_-^D}{2} \quad \text{and} \quad J_\pm^D = \left(\left(1 \pm \frac{\eta}{2} R_g \pm \frac{\widetilde{\eta}}{2}\widetilde{R}_{\widetilde{g}}\right)^{-1} j_\pm^L, \left(1 \pm \frac{\eta}{2} R_g \pm \frac{\widetilde{\eta}}{2}\widetilde{R}_{\widetilde{g}}\right)^{-1} \widetilde{j}_\pm^L\right).$$

We thus re-express the constraint (3.4.22) as $Z^{(0)} \approx 0$ (as we would have in the undeformed limit). Moreover, in the undeformed limit $\eta = \widetilde{\eta} = 0$, we have $J_0^D = f^{-1}\partial_t f$. Thus, we recover $Z \approx K\big(f^{-1}\partial_t f\big)^{(1)}$, as we expect from equation (3.2.19).

Let us end this subsection by expressing the Hamiltonian of the BYBM. Starting from equation (3.4.1) and performing the Legendre transformation, we express the Hamiltonian in terms of the current $J_\pm$ as

$$\mathcal{H}_{\mathrm{BYB}} = \frac{K}{8}\int \mathrm{d}x \, \Big(\kappa(J_+, J_+) + \kappa(J_-, J_-) + \kappa(\Lambda, X + \widetilde{X})\Big),$$

where Λ is a Lagrange multiplier associated with the gauge constraint $X + \widetilde{X}$. In terms of the $D\mathfrak{g}_0$-valued field currents $Z = K\big(J_0^D\big)^{(1)}$ and $J_1^D = \frac{1}{2}(J_+^D - J_-^D)$, one finds

$$\mathcal{H}_{\mathrm{BYB}} = \frac{1}{2}\int \mathrm{d}x \, \left(\frac{1}{K}\kappa_D(Z, Z) + K\,\kappa_D\Big(J_1^{D\,(1)}, J_1^{D\,(1)}\Big) + \kappa_D\Big(\mu, Z^{(0)}\Big)\right),$$

with μ a $D\mathfrak{g}_0^{(0)}$-valued Lagrange multiplier. In the undeformed limit $\eta = \widetilde{\eta} = 0$, one has $J_1^D = (j_1^L, \widetilde{j}_1^L) = f^{-1}\partial_x f$ and we then recover the Hamiltonian (3.2.21) for the $\mathbb{Z}_2$-coset $DG_0/DG_0^{(0)}$.

Hamiltonian Lax matrix. Let us consider the Lax matrix (3.4.13). Using the expression (3.4.21) of X and $\widetilde{X}$, one can re-express the currents $J_\pm$ and $A_\pm$ in terms of g, $\widetilde{g}$, $j = j_1^L$, $\widetilde{j} = \widetilde{j}_1^L$, X and $\widetilde{X}$. Thus, we can express the Lax matrix $\mathcal{L}_{\mathrm{BYB}}(\lambda)$ in terms of these fields.

Moreover, because of the constraint (3.4.22), one can add to the expression obtained this way a term $f(\lambda)(X + \widetilde{X})$, with f an arbitrary function of the spectral parameter. One could potentially add other extra terms, for instance proportional to $R_g(X + \widetilde{X})$ and $\widetilde{R}_{\widetilde{g}}(X + \widetilde{X})$. These allow to change terms like $R_g\widetilde{X}$ or $\widetilde{R}_{\widetilde{g}}X$ into R_gX and $\widetilde{R}_{\widetilde{g}}\widetilde{X}$. For reasons of symmetry and simplicity, we will use this freedom to keep only terms proportional to R_gX and $\widetilde{R}_{\widetilde{g}}\widetilde{X}$. Similarly, when dealing with terms proportional to

X or $\widetilde{X}$ (without operators R or $\widetilde{R}$), we shall always transform those in terms proportional to $X - \widetilde{X}$ (this can always be done by redefining the function $f(\lambda)$ introduced above).

Following this prescription, we obtain an expression for the Lax matrix $\mathcal{L}_{\mathrm{BYB}}(\lambda)$ which is linear in the set of fields $\mathcal{O} = \left\{j, \widetilde{j}, X, \widetilde{X}, R_g X, \widetilde{R}_{\widetilde{g}}\widetilde{X}\right\}$:

$$\mathcal{L}_{\mathrm{BYB}}(\lambda, x) = \sum_{Q \in \mathcal{O}} C_Q(\lambda) Q(x), \tag{3.4.23}$$

where the $C_Q(\lambda)$'s are non-dynamical functions of the spectral parameter λ. We shall not detail the computation and refer to [P1], Section 2.2.3 for more details. The expressions we obtained for the coefficients C_Q's are

$$C_j(\lambda) = \frac{1}{2}\left(1 + \frac{\eta^2 - \widetilde{\eta}^2}{4} + \frac{\zeta}{2}\left(\lambda + \frac{1}{\lambda}\right)\right), \tag{3.4.24a}$$

$$C_X(\lambda) = \frac{\zeta}{4K}\left(\lambda - \frac{1}{\lambda}\right) + f(\lambda), \tag{3.4.24b}$$

$$C_{R_g X}(\lambda) = -\frac{\eta}{2K}\left(1 + \frac{\eta^2 - \widetilde{\eta}^2}{4} + \frac{\zeta}{2}\left(\lambda + \frac{1}{\lambda}\right)\right), \tag{3.4.24c}$$

$$C_{\widetilde{j}}(\lambda) = \frac{1}{2}\left(1 + \frac{\widetilde{\eta}^2 - \eta^2}{4} - \frac{\zeta}{2}\left(\lambda + \frac{1}{\lambda}\right)\right), \tag{3.4.24d}$$

$$C_{\widetilde{X}}(\lambda) = -\frac{\zeta}{4K}\left(\lambda - \frac{1}{\lambda}\right) + f(\lambda), \tag{3.4.24e}$$

$$C_{\widetilde{R}_{\widetilde{g}}\widetilde{X}}(\lambda) = -\frac{\widetilde{\eta}}{2K}\left(1 + \frac{\widetilde{\eta}^2 - \eta^2}{4} - \frac{\zeta}{2}\left(\lambda + \frac{1}{\lambda}\right)\right). \tag{3.4.24f}$$

Maillet bracket and twist function. Starting from the expressions (3.4.23), we can compute the Poisson bracket of the Lax matrix $\mathcal{L}_{\mathrm{BYB}}$ with itself. For that, we need some intermediary Poisson brackets, between the different fields $Q \in \mathcal{O}$. The brackets between $j, X, R_g X$ and $\widetilde{j}, \widetilde{X}, \widetilde{R}_{\widetilde{g}}\widetilde{X}$ are all zero. The brackets of j and X are given by (3.1.7c) and (3.1.11) and $\widetilde{j}$ and $\widetilde{X}$ satisfy similar ones.

The brackets involving the fields $R_g X$ and $\widetilde{R}_{\widetilde{g}}\widetilde{X}$ can be computed from the brackets (3.1.7) and (3.1.11), using the fact that the action of any derivation δ on $R_g X$ is given by

$$\delta\big(R_g X\big) = R_g\big(\delta X\big) + R_g\big(\big[g^{-1}\delta g, X\big]\big) - \big[g^{-1}\delta g, R_g X\big]. \tag{3.4.25}$$

In particular, let us mention the following bracket

$$\big\{(R_g X)_{\underline{\mathbf{1}}}(x), (R_g X)_{\underline{\mathbf{2}}}(y)\big\} = \big[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}(x)\big]\,\delta_{xy},$$

which is derived using equation (3.4.25) and the mCYBE (3.3.1) on R_g (with $c = i$).

We will not show the details of the computation of the Poisson bracket of the Lax matrix (3.4.23) with itself here. The key steps can be found in the Sections 3.2 and 3.3 of [P1]. One then finds that this Poisson bracket takes the form of a Maillet bracket (2.2.6) if we choose the function f to be

$$f(\lambda) = \frac{\zeta^2}{4K}(1 + \lambda^2) - \frac{1}{2K}\left(1 - \frac{(\eta^2 - \widetilde{\eta}^2)^2}{16}\right) + \frac{\zeta(\eta^2 - \widetilde{\eta}^2)}{16K}\left(3\lambda + \frac{1}{\lambda}\right).$$

Moreover, the $\mathcal{R}$-matrix $\mathcal{R}^{\mathrm{BYB}}_{\underline{\mathbf{12}}}(\lambda, \mu)$ of this Maillet bracket is of the form (3.4.20), with the twist function

$$\varphi_{\mathrm{BYB}}(\lambda) = \frac{1}{\zeta^2}\frac{2K\lambda}{\lambda^4 + \dfrac{\eta^2 - \widetilde{\eta}^2}{\zeta}\lambda^3 + \left(2 + \dfrac{(\eta^2 - \widetilde{\eta}^2)^2 - 16}{4\zeta^2}\right)\lambda^2 + \dfrac{\eta^2 - \widetilde{\eta}^2}{\zeta}\lambda + 1}. \tag{3.4.26}$$

Analysis of the twist function. Let us first note that we can rewrite (3.4.26) as

$$\varphi_{\mathrm{BYB}}(\lambda) = \frac{2K\lambda}{(\lambda^2-1)^2 + (\zeta^2-1)(\lambda^2+1)^2 + (\eta^2-\widetilde{\eta}^2)\zeta\lambda\left(\lambda^2 + \dfrac{\eta^2-\widetilde{\eta}^2}{4\zeta}\lambda + 1\right)}. \tag{3.4.27}$$

This expression is appropriate to discuss the one-parameter limits $\eta = \pm\widetilde{\eta}$. Indeed, in this case, we have $\zeta = \sqrt{1+\eta^2}$ so it is clear that the twist function (3.4.27) reduces to the one (3.3.24) of the Yang-Baxter deformation of a $\mathbb{Z}_2$-coset model (with $c = i$, as here).

Let us now study the analytical properties of the 1-form $\varphi_{\mathrm{BYB}}(\lambda)\,\mathrm{d}\lambda$. It possesses two simple zeros at 0 and infinity, similarly to the undeformed model. The double poles $+1$ and -1 of the undeformed model have been splited into two pairs (λ_+, λ_-) and $(\widetilde{\lambda}_+, \widetilde{\lambda}_-)$ of simple poles by the Bi-Yang-Baxter deformation. The expressions of these poles in terms of the parameters η and $\widetilde{\eta}$ are given by

$$\lambda_\pm = \frac{1 - \frac{1}{4}(\eta^2-\widetilde{\eta}^2) \pm i\eta}{\zeta} = \lambda_\mp^* \tag{3.4.28a}$$

$$\widetilde{\lambda}_\pm = -\frac{1 + \frac{1}{4}(\eta^2-\widetilde{\eta}^2) \pm i\widetilde{\eta}}{\zeta} = \widetilde{\lambda}_\mp^*. \tag{3.4.28b}$$

These poles belong to the unit circle. They can be written in a trigonometric form $\lambda_\pm = e^{\pm i\theta}$ and $\widetilde{\lambda}_\pm = -e^{\pm i\widetilde{\theta}}$, with the angles θ and $\widetilde{\theta}$ defined by

$$\tan\theta = \frac{\eta}{1 - \frac{1}{4}(\eta^2-\widetilde{\eta}^2)} \quad \text{and} \quad \tan\widetilde{\theta} = \frac{\widetilde{\eta}}{1 + \frac{1}{4}(\eta^2-\widetilde{\eta}^2)}.$$

The zeros and poles of the 1-form $\varphi_{\mathrm{BYB}}(\lambda)\,\mathrm{d}\lambda$ are represented in Figure 3.5.

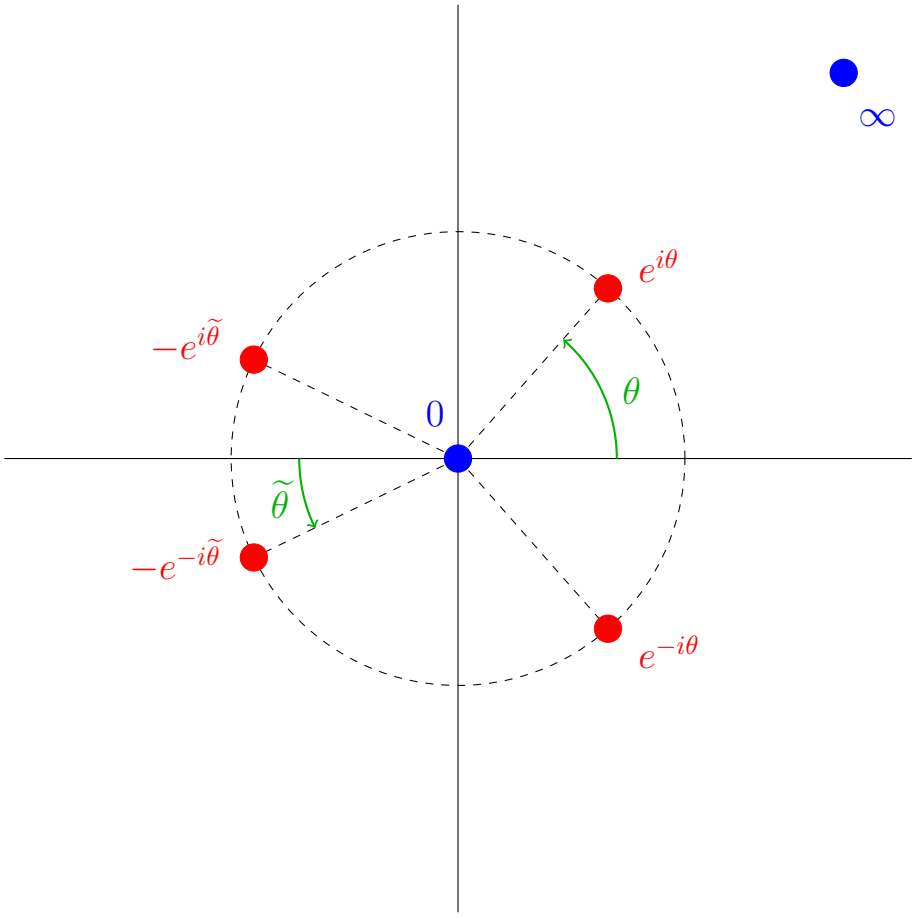


Figure 3.5: Poles and zeros of the twist function of the BYBM.

In the one-parameter limit $\eta = \widetilde{\eta}$, we recover $\theta = \widetilde{\theta} = \arctan \eta$. The poles of the twist function are then as in Figure 3.4. In particular, in this limit we recover the symmetry of the poles with respect to the symmetry $\lambda \mapsto -\lambda$, as the twist function reduces to the odd one in (3.3.24). The Bi-Yang-Baxter deformation allows to split the double poles ± 1 with different angles, by breaking the equivariance relation $\varphi(-\lambda) = -\varphi(\lambda)$. In the gauged-fixed formulation (3.4.4), this is related to the fact that η and $\widetilde{\eta}$ control the breaking of respectively the left and the right symmetry of the PCM. We shall explain this in Chapter 5.

Yang-Baxter type deformations. As we just observed, the Bi-Yang-Baxter deformation has for effect to split the double poles ± 1 into pairs of simple poles and to break the left and right symmetries of the undeformed model. In the previous Section 3.3, we have observed that these effects were some of the features shared by the Yang-Baxter type deformations. Let us consider the gauge transformation of the Lax matrix $\mathcal{L}_{\mathrm{BYB}}(\lambda)$ by the G_0-valued field g and evaluate it at the poles $\lambda = \lambda_\pm$ of the twist function. One then finds

$$\mathcal{L}^g_{\mathrm{BYB}}(\lambda_\pm) = -\gamma R^{\mp}\big(gXg^{-1}\big), \qquad \text{with} \qquad \frac{1}{\gamma} = \frac{K}{\eta} = \pm 4i \operatorname*{res}_{\lambda=\lambda_\pm} \varphi_{\mathrm{BYB}}(\lambda)\, \mathrm{d}\lambda, \tag{3.4.29}$$

with $R^{\pm} = R \pm i\,\mathrm{Id}$. Similarly, one finds

$$\mathcal{L}^{\widetilde{g}}_{\mathrm{BYB}}(\widetilde{\lambda}_\pm) = -\widetilde{\gamma} \widetilde{R}^{\mp}\big(\widetilde{g}\widetilde{X}\widetilde{g}^{-1}\big), \qquad \text{with} \qquad \frac{1}{\widetilde{\gamma}} = \frac{K}{\widetilde{\eta}} = \pm 4i \operatorname*{res}_{\lambda=\widetilde{\lambda}_\pm} \varphi_{\mathrm{BYB}}(\lambda)\, \mathrm{d}\lambda. \tag{3.4.30}$$

This is to be compared with the expressions (3.3.15) and (3.3.25). It is also one of the characteristic feature of Yang-Baxter type deformations and will allow us to apply the results of Chapter 5 to the BYBM.

3.4.5 Gauge-fixing and Lax matrix

As explained in Subsection 3.4.1, the gauge-fixed BYBM is given by the action (3.4.4) and is obtained by performing a gauge transformation (3.4.2) with $h = \widetilde{g}^{-1}$. We shall use the results for the non-gauge-fixed model to describe the Lax matrix of the gauge-fixed one and the corresponding Maillet bracket.

Lax pair and gauge transformation. We first need to understand how a gauge transformation (3.4.2) acts on the Lax pair (3.4.9) of the non-gauge-fixed model. From equation (3.4.6), one finds that the currents $J_\pm$ transform covariantly under the gauge transformation (3.4.2):

$$J_\pm \longmapsto h^{-1} J_\pm h.$$

In the same way, using the expression (3.4.7) and (3.4.8) of $a_\pm$ and $A_\pm$, one shows that they transform as gauge fields:

$$a_\pm \longmapsto h^{-1} a_\pm h + h^{-1}\partial_\pm h \qquad \text{and} \qquad A_\pm \longmapsto h^{-1} A_\pm h + h^{-1}\partial_\pm h.$$

We note in passing that these two properties ensure the invariance of the equation of motion (3.4.5) under the gauge transformation, as expected.

A consequence of these transformation properties is that the Lax pair (3.4.9) of the non-gauged-fixed BYBM transforms under a gauge transformation as

$$\mathcal{L}^{\mathrm{BYB}}_\pm(\lambda) \mapsto h^{-1}\mathcal{L}^{\mathrm{BYB}}_\pm(\lambda) h + h^{-1}\partial_\pm h = \big(\mathcal{L}^{\mathrm{BYB}}_\pm\big)^{h^{-1}}(\lambda).$$

Thus, the action of a gauge transformation (3.4.2) on the Lax pair coincides with the formal gauge transformation by h^{-1}, as defined in equation (3.3.14).

As a consequence, one performs the gauge-fixing of the model at the level of the Lax pair by considering the formal gauge transformation by $\widetilde{g}$. Moreover, we shall consider the following change of spectral parameter:

$$\lambda = \frac{1+\xi}{1-\xi}. \tag{3.4.31}$$

We then define the gauge-fixed Lax pair of the BYBM as

$$\mathcal{L}^{\mathrm{GF}}_{\pm}(\xi) = \left(\mathcal{L}^{\mathrm{BYB}}_{\pm}\right)^{\widetilde{g}}\left(\lambda = \frac{1+\xi}{1-\xi}\right).$$

This gauge-fixed Lax pair can be expressed in terms of the gauge-invariant field $g' = g\widetilde{g}^{-1}$ introduced in Subsection 3.4.1. Using

$$j'^{L}_{\pm} = g'^{-1}\partial_{\pm}g' = \widetilde{g}\left(j^{L}_{\pm} - \widetilde{j}^{L}_{\pm}\right)\widetilde{g}^{-1},$$

one finds

$$\mathcal{L}^{\mathrm{GF}}_{\pm}(\xi) = \frac{K_{\pm}}{1\mp\xi} + B_{\pm}, \tag{3.4.32}$$

with

$$K_{\pm} = \zeta\left(1 \pm \frac{\eta}{2}R_{g'} \pm \frac{\widetilde{\eta}}{2}\widetilde{R}\right)^{-1} j'^{L}_{\pm} \tag{3.4.33a}$$

$$B_{\pm} = \frac{1}{2\zeta}\left(1 + \frac{\eta^2 - \widetilde{\eta}^2}{4} - \zeta \pm \widetilde{\eta}\widetilde{R}\right)K_{\pm}. \tag{3.4.33b}$$

The Lax pair (3.4.32) is the one found by Klimčik in [33], up to a change of spectral parameter. It is to be compared with the one (3.2.5) of the PCM. In particular, due to the presence of the field $B_{\pm}$, this Lax pair is not of Zakharov-Mikhailov type. This is natural as the Bi-Yang-Baxter deformation breaks both left and right symmetries of the PCM: indeed, one then cannot find a flat and conserved current to construct a Zakharov-Mikhailov Lax pair, as we did for the PCM and the Yang-Baxter model.

Let us finish by discussing the one-parameter limit $\widetilde{\eta} = 0$. Recall from Subsection 3.4.1, equation (3.4.4), that this limit corresponds to the Yang-Baxter model (3.3.3) (where the deformation parameter η is replaced by $\frac{\eta}{2}$). In this case, one finds $\zeta = 1 + \frac{\eta^2}{4}$, so in particular, the field $B_{\pm}$ in (3.4.33b) vanishes. We are then left with a Lax pair of Zakharov-Mikhailov type. Moreover, the current $K_{\pm}$ in (3.4.33a) then coincides with the current $K_{\pm}$ given for the Yang-Baxter model in (3.3.6) (with $c = i$ and η replaced by $\frac{\eta}{2}$). Thus, the gauge-fixed Lax pair of the BYBM reduces to the one of the Yang-Baxter model in the limit $\widetilde{\eta} = 0$.

One can also take the limit $\eta = 0$. In this case, one finds the Yang-Baxter model obtained by breaking the right symmetry instead of the left. As explained in Subsection 3.3.1, this model is equivalent to the one deformed on the left by considering g'^{-1} instead of g', so we shall not discuss it further here.

Maillet bracket and gauge transformation. As we have explained above, one performs the gauge-fixing of the BYBM on the Lax pair by considering a formal gauge transformation by $\widetilde{g}$. This can also be done on the Hamiltonian Lax matrix (3.4.23) and one can then express the result in terms of the gauge-invariant fields

$$g' = g\widetilde{g}^{-1}, \quad j' = \widetilde{g}\left(j - \widetilde{j}\right) \quad \text{and} \quad X' = \widetilde{g}X\widetilde{g}^{-1},$$

using the gauge constraint $X + \widetilde{X} = 0$. One then finds

$$\mathcal{L}^{\widetilde{g}}_{\mathrm{BYB}}(\lambda) = C_j(\lambda)j' + \left(C_X(\lambda) - C_{\widetilde{X}}(\lambda)\right)X' + C_{R_gX}(\lambda)R_{g'}X' - C_{\tilde{R}_{\tilde{g}}\tilde{X}}(\lambda)\widetilde{R}X'.$$

Note here that the spectral parameter λ is the same as in the non-gauge-fixed Lax matrix (3.4.23). One can also perform the change of spectral parameter (3.4.31) and define

$$\mathcal{L}_{\mathrm{GF}}(\xi) = \mathcal{L}^{\widetilde{g}}_{\mathrm{BYB}}\left(\lambda = \frac{1+\xi}{1-\xi}\right).$$

This coincides with the Lax matrix one would obtain starting from the Lax pair (3.4.32), and expressing the corresponding Lax matrix in terms of the phase space fields (g' and X') of the gauge-fixed model. One could then compute the Poisson bracket of the Lax matrix with itself using the canonical Poisson brackets in terms of g' and X'. Here, we shall use a different method to obtain this result.

In the Hamiltonian formalism, one performs the gauge fixing at the level of the Poisson bracket using the Dirac bracket [127]. The quantities g' and X' being gauge-invariant, their Dirac bracket coincides with their non-gauged-fixed bracket. Thus, one can compute the Poisson bracket of the gauge-fixed Lax matrix $\mathcal{L}_{\mathrm{GF}}$ simply by computing the one of the non-gauged-fixed matrix $\mathcal{L}^{\widetilde{g}}_{\mathrm{BYB}}$.

For that, we shall use a general result on the transformation of Maillet brackets under a formal gauge transformation of the Lax matrix. We will not recall this general result here and refer to our article [P1] for details. Let us summarise the outcome of this procedure: starting form the Maillet bracket of $\mathcal{L}_{\mathrm{BYB}}$ with the $\mathcal{R}$-matrix (3.4.20), one finds that the gauge transformed matrix $\mathcal{L}^{\widetilde{g}}_{\mathrm{BYB}}$ also satisfies a Maillet bracket with $\mathcal{R}$-matrix

$$\mathcal{R}^{\widetilde{g}}_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{2}\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}\varphi_{\mathrm{BYB}}(\mu)^{-1} - C_{\widetilde{X}}(\mu)C_{\underline{\mathbf{12}}} + C_{\widetilde{R}_{\widetilde{g}}\widetilde{X}}(\mu)\widetilde{R}_{\underline{\mathbf{12}}}, \tag{3.4.34}$$

where

$$\widetilde{R}_{\underline{\mathbf{12}}} = \widetilde{R}_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}} = -\widetilde{R}_{\underline{\mathbf{21}}}$$

is the kernel of the operator $\widetilde{R}$ (the last equality is a consequence of the skew-symmetry of $\widetilde{R}$).

As a consequence, the Lax matrix $\mathcal{L}_{\mathrm{GF}}(\xi)$ also satisfies a Maillet non-ultralocal bracket. The corresponding $\mathcal{R}$-matrix is simply obtained from (3.4.34) by the change of spectral parameter (3.4.31):

$$\mathcal{R}^{\mathrm{GF}}_{\underline{\mathbf{12}}}(\xi,\upsilon) = \mathcal{R}^{\widetilde{g}}_{\underline{\mathbf{12}}}\left(\frac{1+\xi}{1-\xi},\frac{1+\upsilon}{1-\upsilon}\right).$$

One then finds

$$\mathcal{R}^{\mathrm{GF}}_{\underline{\mathbf{12}}}(\xi,\upsilon) = \mathcal{R}^{0,\mathrm{GF}}_{\underline{\mathbf{12}}}(\xi,\upsilon)\,\varphi_{\mathrm{GF}}(\upsilon)^{-1}$$

with

$$\mathcal{R}^{0,\mathrm{GF}}_{\underline{\mathbf{12}}}(\xi,\upsilon) = \frac{C_{\underline{\mathbf{12}}}}{\upsilon-\xi} - \frac{\alpha_4\upsilon}{\alpha_3+\alpha_4\upsilon^2}C_{\underline{\mathbf{12}}} + \frac{1}{2}\frac{\widetilde{\eta}}{\alpha_3+\alpha_4\upsilon^2}\widetilde{R}_{\underline{\mathbf{12}}}, \tag{3.4.35}$$

$$\varphi_{\mathrm{GF}}(\xi) = \frac{K}{2}\frac{1-\xi^2}{(\alpha_1+\alpha_2\xi^2)(\alpha_3+\alpha_4\xi^2)}, \tag{3.4.36}$$

and

$$\alpha_1 = \frac{1}{2}\left(\zeta - 1 + \frac{\eta^2-\widetilde{\eta}^2}{4}\right), \qquad \alpha_2 = \frac{1}{2}\left(\zeta + 1 - \frac{\eta^2-\widetilde{\eta}^2}{4}\right), \tag{3.4.37a}$$

$$\alpha_3 = \frac{1}{2}\left(\zeta + 1 + \frac{\eta^2-\widetilde{\eta}^2}{4}\right), \qquad \alpha_4 = \frac{1}{2}\left(\zeta - 1 - \frac{\eta^2-\widetilde{\eta}^2}{4}\right). \tag{3.4.37b}$$

The gauge-fixed BYBM does not exactly enter the class of models with twist function, as the matrix $\mathcal{R}^{0,GF}$ is not a standard $\mathcal{R}$-matrix on $\mathfrak{g}$. However, it shares some similar structures with these models. In particular, the matrix $\mathcal{R}^{0,GF}$ satisfies the CYBE (2.3.3) and the asymptotic condition

$$\mathcal{R}^{0,\mathrm{GF}}_{\underline{\mathbf{12}}}(\xi,\upsilon) = \frac{C_{\underline{\mathbf{12}}}}{\upsilon-\xi} + O\big((\xi-\upsilon)^0\big), \tag{3.4.38}$$

as the standard matrix $\mathcal{R}^0$. This will allow us to apply the results of Chapter 4 to the gauge-fixed BYBM. With a slight abuse of notation, we will call φ_{GF} the twist function of the gauge-fixed BYBM. We note the following relation between $\varphi_{\mathrm{BYB}}(\lambda)$ and $\varphi_{\mathrm{GF}}(\xi)$:

$$2\varphi_{\mathrm{BYB}}(\lambda)\,\mathrm{d}\lambda = \varphi_{\mathrm{GF}}(\xi)\,\mathrm{d}\xi.$$

Let us consider the one-parameter limit $\widetilde{\eta}=0$, which coincides with the Yang-Baxter model. In this case, we find $\zeta=\alpha_3=1+\frac{\eta^2}{4}$, $\alpha_1=\frac{\eta^2}{4}$, $\alpha_2=1$ and $\alpha_4=0$. Thus, we see that the matrix $\mathcal{R}^{0,\mathrm{GF}}$ reduces to the standard matrix on $\mathfrak{g}$ and φ_{GF} reduces to the twist function (3.3.13) of the Yang-Baxter model, with η and K replaced by $\frac{\eta}{2}$ and $\frac{K}{2}$.

Poles and zeros of the twist function. Finally, let us study the analytical properties of the 1-form $\varphi_{\mathrm{GF}}(\xi)\,\mathrm{d}\xi$. It possesses two simple zeros at $+1$ and -1, as for the undeformed PCM or the Yang-Baxter model. Recall that the Yang-Baxter deformation had for effect to split the double pole at 0 of the PCM into two simples poles on the imaginary axis, without modifying the double pole at infinity. In the case of the BYBM, both the double poles at 0 and infinity are split into pairs of simple poles, given by

$$\xi_\pm = \pm i\sqrt{\frac{\alpha_1}{\alpha_2}} \qquad \text{and} \qquad \widetilde{\xi}_\pm = \pm i\sqrt{\frac{\alpha_3}{\alpha_4}}.$$

They belong to the imaginary axis as one can show that the α_i's are all positive numbers. Note that in the one-parameter limit $\widetilde{\eta} = 0$, we get $\alpha_4 = 0$: we thus see that in this limit, the two poles $\widetilde{\xi}_\pm$ recombine as a double pole at infinity, as expected from the Yang-Baxter model case. Note that these poles are related to the ones of φ_{BYB} by the relation

$$\lambda_\pm = \frac{1+\xi_\mp}{1-\xi_\mp} \qquad \text{and} \qquad \widetilde{\lambda}_\pm = \frac{1+\widetilde{\xi}_\mp}{1-\widetilde{\xi}_\mp}.$$

As for the Yang-Baxter model, it is useful to evaluate the Lax matrix or its gauge transformation by g' at the poles of the twist function. Indeed, one then finds

$$\mathcal{L}^{g'}_{\mathrm{GF}}(\xi_\pm) = -\gamma R^\mp\big(g' X' g'^{-1}\big) \qquad \text{with} \quad \frac{1}{\gamma} = \frac{K}{\eta} = \pm 2i \operatorname*{res}_{\xi=\xi_\pm} \varphi_{\mathrm{GF}}(\xi)\,\mathrm{d}\xi, \tag{3.4.39a}$$

$$\mathcal{L}_{\mathrm{GF}}(\widetilde{\xi}_\pm) = -\widetilde{\gamma}\widetilde{R}^\mp X' \qquad \text{with} \quad \frac{1}{\widetilde{\gamma}} = \frac{K}{\widetilde{\eta}} = \pm 2i \operatorname*{res}_{\xi=\widetilde{\xi}_\pm} \varphi_{\mathrm{GF}}(\xi)\,\mathrm{d}\xi. \tag{3.4.39b}$$

We shall use these relations in Chapter 5.

One can also extract informations on the model from the zeros $+1$ and -1 of the twist function $\varphi_{\mathrm{GF}}(\xi)$. Indeed, one can reconstruct the Hamiltonian $\mathcal{H}_{\mathrm{GF}}$ and total momentum $\mathcal{P}_{\mathrm{GF}}$ of the gauge-fixed model from the Lax matrix:

$$\mathcal{H}_{\mathrm{GF}} = \mathcal{H}_+ - \mathcal{H}_- \qquad \text{and} \qquad \mathcal{P}_{\mathrm{GF}} = \mathcal{H}_+ + \mathcal{H}_-,$$

with

$$\mathcal{H}_\pm = -\frac{1}{2} \operatorname*{res}_{\xi=\pm 1} \varphi_{\mathrm{GF}}(\xi) \int \mathrm{d}x\, \kappa\big(\mathcal{L}_{\mathrm{GF}}(\xi,x), \mathcal{L}_{\mathrm{GF}}(\xi,x)\big).$$

Note that the same equation was already satisfied by the other deformations of the PCM, as seen in equation (3.3.21). We will use this fact in Chapter 4.

Chapter 4

Local charges in involution and integrable hierarchies

This chapter is based on the article [P3], that I wrote during my PhD with M. Magro and B. Vicedo. The content of this chapter is the same as the one of [P3] and is made to be read independently (the publication is thus not included at the end of this thesis).

In Chapter 2, we introduced the general Lax formalism for integrable fields theories. This formalism ensures the existence for these theories of an infinite number of conserved charges in involution, which are constructed from the monodromy matrix of the theory. In particular, these charges are said to be non-local. Before going further, let us briefly recall what we mean by local and non-local quantities.

We consider a Hamiltonian field theory which describes the time evolution of some fundamental fields $\phi_i(x)$'s, depending on the spatial coordinate x. A quantity $\mathscr{K}(x)$ is said to be a local field of the theory if it is a function of the evaluation of the ϕ_i's and their derivatives at a unique point x of space. For example, $\phi_1(x)^2$ and $\phi_1(x) + 2\phi_2'(x)$ are local fields but $\int_0^x \phi_1(y)\,\mathrm{d}y$ or $\phi_1(x) - \phi_2(0)$ are not. A local charge is the integral

$$\int \mathrm{d}x\, \mathscr{K}(x),$$

of a local field $\mathscr{K}(x)$ on the whole space (the real line $\mathbb{R}$ or the circle $\mathbb{S}^1$). The field $\mathscr{K}(x)$ is then called the density of the charge. In particular, we always consider here local field theories, for which the Hamiltonian is a local charge, ensuring that the time evolution of the ϕ_i's is encoded in local partial differential equations.

The charges extracted from the monodromy matrix are non-local, in the sense that they cannot be written as integrals of some local fields (more precisely, the monodromy matrix is expressed as a series of nested integrals). This makes the computation and the manipulation of these charges quite involved. Let us illustrate that on an example. Let $\mathcal{Q}$ be a conserved charge extracted from the monodromy and consider its Hamiltonian flow $\partial_{\mathcal{Q}} = \{\mathcal{Q}, \cdot\}$ on the phase space of the model: it is an interesting object for the study of the model as it commutes with the time flow ∂_t (because the charge $\mathcal{Q}$ is conserved). However, as $\mathcal{Q}$ is non-local, this flow $\partial_{\mathcal{Q}}$ does not take the form of partial differential equations on the fields ϕ_i's and is thus quite difficult to study.

It is thus interesting to find conserved charges in involution which are local (another motivation for that will be explained in the second part of this thesis). For integrable σ-models (see Chapter 3), an infinity of such charges were found in two particular examples. They were constructed for the Principal Chiral Model by Evans, Hassan, MacKay and Mountain in [79] and for the $\mathbb{Z}_2$-coset σ-model by Evans and Moutain in [80]. For completeness, let us also mention the papers [145] by Evans, Hassan, MacKay and Mountain and [146] by Evans and Young for super-symmetric generalisations of the above-mentioned publications.

However, the existence of such charges for other integrable σ-models, such as deformed models or $\mathbb{Z}_T$-coset models for $T > 2$, was not proved. In [P3], I have shown, together with my collaborators,

that all integrable σ-models, including integrable deformations, possess an infinite number of local conserved charges in involution. One of the main characteristic of our approach in [P3] is that we do not treat each model individually: at the contrary, we develop a general construction which applies to the whole family of integrable σ-models at once. This construction relies on the common mathematical structure shared by these models: the existence of a twist function (see Chapters 2 and 3). As such, it applies to all models with a twist function, provided the latter possesses what we call a regular zero (see the precise definition below).

Given the length of this chapter, and to avoid going directly into technicalities, we start by a summary of the results presented in more details in the subsequent sections.

4.1 Summary

The purpose of the present chapter is to provide another application of the general formalism of Maillet brackets with twist function (see Chapter 2). Specifically, we will describe how, in this general framework, infinite towers of local charges can be associated with certain zeros of the twist function, all of which are in pairwise involution. Following the same spirit as recalled above, the starting point of our approach was to reinterpret the construction of local charges in the principal chiral model due to Evans, Hassan, MacKay and Mountain [79] in the present language of twist functions. In fact, this construction had soon been generalised to include also the (supersymmetric) principal chiral model with a Wess-Zumino term in [145], symmetric space σ-models in [80] as well as supersymmetric coset σ-models in [146]. Each of these generalisations can be regarded as further evidence that such a construction should hold for any integrable field theory with twist function, while at the same time providing indications on how to do so. In this section, we will briefly summarise the main results of the chapter.

Let us first note that in all of the integrable σ-models with twist function described above, every zero of $\varphi(\lambda)$ is such that $\varphi(\lambda)\mathcal{L}(\lambda, x)$ is regular there. In a general integrable field theory with twist function $\varphi(\lambda)$ we shall say that any **zero** of $\varphi(\lambda)$ with this property is **regular**. We denote by $\mathcal{Z}$ the set of regular zeros of $\varphi(\lambda)$ in $\mathbb{C}$. We shall further distinguish between two types of zeros: cyclotomic ones and non-cyclotomic ones. This notion depends on the order T of the automorphism σ appearing in the $\mathcal{R}$-matrix of the system (see Chapter 2). In a model with $T = 1$, every point is by definition non-cyclotomic, whereas in a model with $T > 1$, every point is non-cyclotomic except for the origin and infinity. As explained in subsection 4.2.2, throughout our analysis the point at infinity will be treated in much the same way as the origin by using an inversion of the spectral parameter.

With every $\lambda_0 \in \mathcal{Z}$, or every $\lambda_0 \in \mathcal{Z} \cup \{\infty\}$ if infinity is also a regular zero, we will associate a subset of integers $\mathcal{E}_{\lambda_0} \subset \mathbb{Z}_{\geq 2}$ and a corresponding tower of local charges $\mathcal{Q}_n^{\lambda_0}$ labelled by $n \in \mathcal{E}_{\lambda_0}$. The first main property of these charges which we will establish is that any two such charges $\mathcal{Q}_n^{\lambda_0}$ and $\mathcal{Q}_m^{\mu_0}$ for any $\lambda_0, \mu_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$, $m \in \mathcal{E}_{\mu_0}$ are in involution. Moreover, if infinity is a regular zero and either λ_0 or μ_0 is taken to be the point at infinity, the corresponding local charges will only Poisson commute up to a certain field $\mathcal{C}(x)$ which will coincide with the coset constraint in $\mathbb{Z}_T$-coset σ-models. Following the standard terminology from the theory of constrained Hamiltonian systems, we will refer to equalities as being *weak* when they hold only after setting this particular field to zero, see subsection 4.5.1. Furthermore, we show that in every example of integrable σ-model considered, the Hamiltonian can be expressed as a particular linear combination of the collection of quadratic local charges $\mathcal{Q}_2^{\lambda_0}$ for $\lambda_0 \in \mathcal{Z} \cup \{\infty\}$ and the momentum of the model. It then follows that all of the local charges are conserved.

Let us briefly outline the construction of the local charges by considering first the case when $\lambda_0 \in \mathcal{Z}$ is non-cyclotomic. If the Lie algebra $\mathfrak{g}$ is of type B, C or D then the density of the local charge $\mathcal{Q}_n^{\lambda_0}$ is obtained simply by evaluating

$$\mathrm{Tr}\big(\varphi(\lambda)^n \mathcal{L}(\lambda, x)^n\big) \tag{4.1.1}$$

at the regular zero λ_0. When $\mathfrak{g}$ is of type A, on the other hand, the density of the local charge $\mathcal{Q}_n^{\lambda_0}$ is given instead by a certain polynomial in the above expressions, determined as in [79] with the help

of a generating function. In either case, $\mathcal{E}_{\lambda_0}$ is given here by the set of **exponents of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$** associated with $\mathfrak{g}$, shifted by one (we do not treat the case of the Pfaffian in type D). In the example of the principal chiral model on a real Lie group G_0 (see Subsection 4.7.1) treated in [79], the twist function has simple zeros at ± 1 and the evaluation of $\varphi(\lambda)\mathcal{L}(\lambda, x)$ at $\lambda = \pm 1$ produces the light-cone currents $j^L_\pm = g^{-1}\partial_\pm g$ of the theory (see Subsection 3.1.1). We recover in this way the higher spin local charges in involution of the principal chiral model constructed in [79].

When the regular zero $\lambda_0 \in \mathcal{Z}$ is cyclotomic, *i.e.* $\lambda_0 = 0$, it may happen, as a result of the equivariance properties of both the Lax matrix and twist function, that the evaluation of (4.1.1) at the point λ_0 vanishes identically. More precisely, the first non-vanishing term in the power series expansion of (4.1.1) around $\lambda = 0$ is of order λ^{r_n} for some $0 \leq r_n \leq T-1$. If the Lie algebra $\mathfrak{g}$ is of type B, C or D, or also of type A with an inner automorphism σ, then we define the density of the local charge $\mathcal{Q}^0_n$ as the coefficient of this leading term. The case when $\mathfrak{g}$ is of type A and the automorphism σ is not inner is treated in a similar fashion to the case of a non-cyclotomic point in type A, with the densities of the local charges $\mathcal{Q}^0_n$ being obtained by means of a generating function. In each case it turns out that we need to restrict attention to indices n such that $0 \leq r_n < T-1$. As a result, and in contrast to the case of a non-cyclotomic regular zero, some exponents of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ are 'dropped' in the construction of the subset $\mathcal{E}_0$, specifically those such that $r_n = T-1$. In the case of a symmetric space σ-model, for which $T = 2$ so that only charges for which $r_n = 0$ are kept, we recover in this way the local charges found in [80].

The collection of local charges $\mathcal{Q}^{\lambda_0}_n$, $\lambda_0 \in \mathcal{Z}$, $n \in \mathcal{E}_{\lambda_0}$ in involution generates an infinite set of Poisson commuting Hamiltonian flows $\{\mathcal{Q}^{\lambda_0}_n, \cdot\}$ on the phase space of the model. With every such flow we then associate a corresponding $\mathfrak{g}$-valued connection $\nabla^{\lambda_0}_n = \{\mathcal{Q}^{\lambda_0}_n, \cdot\} + \mathcal{M}^{\lambda_0}_n(\lambda, x)$ for some $\mathfrak{g}$-valued matrix $\mathcal{M}^{\lambda_0}_n(\lambda, x)$ depending on the spectral parameter λ. The second main property of the local charges $\mathcal{Q}^{\lambda_0}_n$, $\lambda_0 \in \mathcal{Z}$, $n \in \mathcal{E}_{\lambda_0}$ which we establish is that the connection $\nabla^{\lambda_0}_n$ for any $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$ commutes with the connection $\nabla_x = \partial_x + \mathcal{L}(\lambda, x)$. In this sense, the local charges generate a **hierarchy of integrable equations**. We use this result to deduce that the local charges $\mathcal{Q}^{\lambda_0}_n$, $\lambda_0 \in \mathcal{Z}$, $n \in \mathcal{E}_{\lambda_0}$ are in involution with the non-local charges extracted from the monodromy of $\mathcal{L}(\lambda, x)$. Moreover, we go on to show that when $\mathfrak{g}$ is of type B, C or D, any two such connections $\nabla^{\lambda_0}_n$ and $\nabla^{\mu_0}_m$ for $\lambda_0, \mu_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$, $m \in \mathcal{E}_{\mu_0}$ also commute with one another. Finally, we have also checked these results in the case of type A for low values of n and m and on this basis we conjecture it to hold in general. If infinity is a regular zero then the majority of these results still hold in the weak sense when we consider also the local charges associated with infinity.

This chapter is organised as follows. The general framework which we employ throughout the chapter is introduced in Section 4.2. In particular, we introduce the notion of a regular zero in the complex plane which plays a central role in our analysis. In subsection 4.2.2, we define the notion of a regular zero at infinity and relate it to that of a regular zero at the origin by inversion of the spectral parameter. Finally, we establish some general results in subsection 4.2.3. Section 4.3 is devoted to the procedure for extracting local charges in involution in the case of a non-cyclotomic regular zero. In particular, we present in subsection 4.3.4 an explicit construction of the currents $\mathscr{K}^{\lambda_0}_n$ for type A algebras using generating functions in the spirit of [79]. Section 4.4 deals with charges at cyclotomic zeros. We explain how the equivariance properties of the various objects affect the construction of local conserved charges in involution. Here the Lie algebras of type B, C and D can still be treated uniformly but in type A we need to consider separately the cases when the automorphism σ is inner or not. The generating function for Lie algebras of type A with non-inner automorphism is presented in subsection 4.4.6. A list of properties of these local charges is collated in section 4.5, including the fact that the local charges extracted from different regular zeros Poisson commute (weakly when the point at infinity is involved). Moreover, we show that all the local charges commute with the field $\mathcal{C}(x)$ which will play the role of the constraint in $\mathbb{Z}_T$-coset σ-models, therefore showing that they are gauge invariant. We also discuss the reality conditions of all the local charges. The Hamiltonian flows of the local charges $\mathcal{Q}^{\lambda_0}_n$ are studied in detail in section 4.6. The main result that any two of the $\mathfrak{g}$-valued

connections $\nabla_n^{\lambda_0}$ and $\nabla_m^{\mu_0}$ satisfy a zero curvature equation is established in subsection 4.6.3. Finally, in section 4.7 we apply all these results to the whole family of integrable σ-models described in Chapter 3. Some technical appendices, specific to the present chapter, are given here as sections 4.A and 4.B.

4.2 Framework and general results

4.2.1 Framework: regular zeros of a model with twist function

In this chapter, we consider an integrable model with twist function, as described in Section 2.4. We will use the notations of this section and more generally of Chapter 2. In particular, let us consider the twist function $\varphi(\lambda)$ and the Lax matrix $\mathcal{L}(\lambda, x)$ of the model, which are rational functions of the spectral parameter λ. Let $\lambda_0 \in \mathbb{C}$ be a **zero** of the twist function, so that $\varphi(\lambda_0) = 0$. We will say that this zero is **regular** if $\varphi(\lambda)\mathcal{L}(\lambda, x)$ is holomorphic at $\lambda = \lambda_0$.

Let us suppose that the model we are considering is cyclotomic with respect to an automorphism σ of order T (note that a non-cyclotomic model can be considered as a cyclotomic one with $\sigma = \mathrm{Id}$ and $T = 1$). As in the Section 2.4, we consider the action of the cyclic group $\mathbb{Z}_T$ on the complex plane $\mathbb{C}$ by the multiplication by ω, a Tth-root of unit. The Lax matrix and the twist function then satisfy the equivariance properties (2.4.3) and (2.4.5). Due to these properties, if $\lambda_0 \in \mathbb{C}$ is a regular zero, all points of the orbit $\mathbb{Z}_T\lambda_0$ are also regular zeros. Let us pick (arbitrarily) one of them. We then form a set $\mathcal{Z}$ of regular zeros of φ such that for every pair of distinct points λ_0 and μ_0 in $\mathcal{Z}$, the orbits $\mathbb{Z}_T\lambda_0$ and $\mathbb{Z}_T\mu_0$ are disjoint.

As explained in subsection 4.2.2, we will also be interested in the case where the differential form $\varphi(\lambda)\mathrm{d}\lambda$ has a zero at infinity, *i.e.* where

$$\psi(\alpha) = -\frac{1}{\alpha^2}\varphi\left(\frac{1}{\alpha}\right) \tag{4.2.1}$$

has a zero at $\alpha = 0$. We will also see that the appropriate notion of a regular zero at infinity corresponds to requiring that $\frac{1}{\alpha}\varphi\left(\frac{1}{\alpha}\right)\mathcal{L}\left(\frac{1}{\alpha}, x\right)$ be holomorphic at $\alpha = 0$.

As an example, let us determine the regular zeros of the PCM. Its Lax matrix and twist function are respectively given by (3.2.8) and (3.2.9). As mentioned in Subsection 3.2.1, the twist function admits two simple zeros, at $+1$ and -1. Moreover, the Lax matrix has simple poles at $\lambda = \pm 1$, *i.e.* at these zeros. Thus, the product $\varphi_{\mathrm{PCM}}(\lambda)\mathcal{L}_{\mathrm{PCM}}(\lambda, x)$ is regular at $\lambda = \pm 1$, which are then regular zeros. Let us note here that the evaluation of $\varphi_{\mathrm{PCM}}(\lambda)\mathcal{L}_{\mathrm{PCM}}(\lambda, x)$ at these zeros gives the light-cone currents $j_{\pm}^L$ (see Subsection 3.1.1).

4.2.2 Infinity and inversion of the spectral parameter

In this chapter we will construct a tower of local charges associated with each regular zero of the twist function. As mentioned in the previous subsection, the set of regular zeros can include the point at infinity, although the sense in which infinity can be a regular zero is slightly different from the definition of finite regular zeros. In this subsection, we show how the notion of a regular zero at infinity is related to that of a regular zero at the origin through inversion of the spectral parameter, *i.e.* by the change of parameter $\lambda \mapsto \alpha = \lambda^{-1}$. Under such a change of spectral parameter we have

$$\varphi(\lambda)\mathrm{d}\lambda = \psi(\alpha)\mathrm{d}\alpha,$$

where $\psi(\alpha)$ is defined in equation (4.2.1). Suppose that infinity is a zero of the twist function, *i.e.* that $\psi(0) = 0$, and define

$$P(\alpha, x) = \frac{1}{\alpha}\varphi\left(\frac{1}{\alpha}\right)\mathcal{L}\left(\frac{1}{\alpha}, x\right). \tag{4.2.2}$$

We will say that infinity is a **regular zero** if $P(\alpha, x)$ is regular at $\alpha = 0$. In the remainder of this subsection we will assume this to be the case. We then set

$$\mathcal{C}(x) = P(0, x). \tag{4.2.3}$$

From the equivariance properties (2.4.3) and (2.4.5) of $\mathcal{L}$ and φ, we deduce that $\mathcal{C}$ is valued in the grading $\mathfrak{g}^{(0)}$. Let us note here that in the $\mathbb{Z}_T$-coset models, described in Subsection 3.2.2, this field $\mathcal{C}$ coincides with the gauge constraint $X^{(0)}$.

Starting from the Poisson bracket (2.2.6) and using the form (2.4.1) of the $\mathcal{R}$-matrix, we find

$$\begin{aligned}\left\{\mathcal{L}(\lambda,x)_{\underline{1}},P(\alpha,y)_{\underline{2}}\right\} = &\left[\alpha^{-1}\mathcal{R}^0_{\underline{12}}\left(\lambda,\alpha^{-1}\right),\mathcal{L}(\lambda,x)_{\underline{1}}\right]\delta_{xy} - \left[\mathcal{R}^0_{\underline{21}}\left(\alpha^{-1},\lambda\right)\varphi(\lambda)^{-1},P(\alpha,x)_{\underline{2}}\right]\delta_{xy}\\ &- \Big(\alpha^{-1}\mathcal{R}^0_{\underline{12}}\left(\lambda,\alpha^{-1}\right) - \alpha\psi(\alpha)\mathcal{R}^0_{\underline{21}}\left(\alpha^{-1},\lambda\right)\varphi(\lambda)^{-1}\Big)\delta'_{xy}\end{aligned}$$

Using the expression (2.3.8) of $\mathcal{R}^0$, we have

$$\alpha^{-1}\mathcal{R}^0_{\underline{12}}\left(\lambda,\alpha^{-1}\right) \xrightarrow{\alpha\to 0} \frac{1}{T}\sum_{k=0}^{T-1}\sigma^k_{\underline{1}}C_{\underline{12}} = C^{(0)}_{\underline{12}}, \qquad \mathcal{R}^0_{\underline{21}}\left(\alpha^{-1},\lambda\right) \xrightarrow{\alpha\to 0} 0. \tag{4.2.4}$$

As $P(\alpha,x)$ and $\alpha\psi(\alpha)$ are regular at 0, taking the limit $\alpha\to 0$ in the above Poisson bracket, we then obtain

$$\left\{\mathcal{L}(\lambda,x)_{\underline{1}},\mathcal{C}(y)_{\underline{2}}\right\} = \left[C^{(0)}_{\underline{12}},\mathcal{L}(\lambda,x)_{\underline{1}}\right]\delta_{xy} - C^{(0)}_{\underline{12}}\delta'_{xy}. \tag{4.2.5}$$

Applying the same kind of reasoning we also find

$$\left\{\mathcal{C}(x)_{\underline{1}},\mathcal{C}(y)_{\underline{2}}\right\} = \left[C^{(0)}_{\underline{12}},\mathcal{C}(x)_{\underline{1}}\right]\delta_{xy}. \tag{4.2.6}$$

Let us define a new Lax matrix

$$\widetilde{\mathcal{L}}(\lambda,x) = \mathcal{L}(\lambda,x) - \lambda^{-1}\varphi(\lambda)^{-1}\mathcal{C}(x).$$

From the fact that $[C^{(k)}_{\underline{12}},Z_{\underline{1}}] = -[C^{(k)}_{\underline{12}},Z_{\underline{2}}]$ for any $Z\in\mathfrak{g}^{(0)}$, we find that

$$\lambda\left[\mathcal{R}^0_{\underline{21}}(\mu,\lambda),Z_{\underline{2}}\right] - \mu\left[\mathcal{R}^0_{\underline{12}}(\lambda,\mu),Z_{\underline{1}}\right] = \left[C^{(0)}_{\underline{12}},Z_{\underline{2}}\right].$$

Using this identity and the Poisson brackets (2.2.6), (4.2.5) and (4.2.6), we prove that the Poisson bracket of $\widetilde{\mathcal{L}}$ with itself is also of the Maillet form, namely

$$\begin{aligned}\left\{\widetilde{\mathcal{L}}(\lambda,x)_{\underline{1}},\widetilde{\mathcal{L}}(\mu,y)_{\underline{2}}\right\} = &\left[\widetilde{\mathcal{R}}_{\underline{12}}(\lambda,\mu),\widetilde{\mathcal{L}}(\lambda,x)_{\underline{1}}\right]\delta_{xy} - \left[\widetilde{\mathcal{R}}_{\underline{21}}(\mu,\lambda),\widetilde{\mathcal{L}}(\mu,y)_{\underline{2}}\right]\delta_{xy}\\ &- \Big(\widetilde{\mathcal{R}}_{\underline{12}}(\lambda,\mu)+\widetilde{\mathcal{R}}_{\underline{21}}(\mu,\lambda)\Big)\delta'_{xy},\end{aligned} \tag{4.2.7}$$

where $\widetilde{\mathcal{R}}_{\underline{12}}(\lambda,\mu) = \widetilde{\mathcal{R}}^0_{\underline{12}}(\lambda,\mu)\varphi(\mu)^{-1}$ and

$$\widetilde{\mathcal{R}}^0_{\underline{12}}(\lambda,\mu) = \mathcal{R}^0_{\underline{12}}(\lambda,\mu) - \mu^{-1}C^{(0)}_{\underline{12}}. \tag{4.2.8}$$

We now define

$$\mathcal{L}^\infty(\alpha,x) = \widetilde{\mathcal{L}}\left(\frac{1}{\alpha},x\right).$$

The following theorem is the main result of this subsection.

Theorem 4.2.1. *The Poisson bracket of $\mathcal{L}^\infty$ with itself reads*

$$\begin{aligned}\left\{\mathcal{L}^\infty(\alpha,x)_{\underline{1}},\mathcal{L}^\infty(\beta,y)_{\underline{2}}\right\} = &\left[\mathcal{R}^\infty_{\underline{12}}(\alpha,\beta),\mathcal{L}^\infty(\alpha,x)_{\underline{1}}\right]\delta_{xy} - \left[\mathcal{R}^\infty_{\underline{21}}(\beta,\alpha),\mathcal{L}^\infty(\beta,y)_{\underline{2}}\right]\delta_{xy}\\ &- \Big(\mathcal{R}^\infty_{\underline{12}}(\alpha,\beta)+\mathcal{R}^\infty_{\underline{21}}(\beta,\alpha)\Big)\delta'_{xy},\end{aligned} \tag{4.2.9}$$

where

$$\mathcal{R}^\infty_{\underline{12}}(\alpha,\beta) = \mathcal{R}^0_{\underline{21}}(\alpha,\beta)\psi(\beta)^{-1}$$

satisfies the classical Yang-Baxter equation (2.3.3).

Proof. Using equation (2.3.9), we find that

$$\widetilde{\mathcal{R}}^0_{\underline{\mathbf{12}}}(\lambda,\mu)=\sum_{k=1}^{T}\frac{\lambda^k\mu^{T-1-k}}{\mu^T-\lambda^T}C^{(p)}_{\underline{\mathbf{12}}}. \tag{4.2.10}$$

The theorem follows from the Poisson bracket (4.2.7) and the identity

$$\widetilde{\mathcal{R}}^0_{\underline{\mathbf{12}}}\left(\frac{1}{\alpha},\frac{1}{\beta}\right)=-\beta^2\mathcal{R}^0_{\underline{\mathbf{21}}}(\alpha,\beta), \tag{4.2.11}$$

which is a consequence of equation (4.2.10). □

To interpret Theorem 4.2.1, let us note that the matrix $\mathcal{R}^0_{\underline{\mathbf{21}}}$ is nothing but the matrix $\mathcal{R}^0_{\underline{\mathbf{12}}}$ for the automorphism σ^{-1}. Moreover, from the equivariance properties (2.4.3) and (2.4.5), we find that the corresponding properties of $\mathcal{L}^\infty$ and ψ are

$$\sigma^{-1}\big(\mathcal{L}^\infty(\alpha,x)\big)=\mathcal{L}^\infty(\omega\alpha,x)\qquad\text{and}\qquad\psi(\omega\alpha)=\omega^{-1}\psi(\alpha). \tag{4.2.12}$$

The Poisson bracket of $\mathcal{L}^\infty$ is thus the one of a model with twist function ψ, automorphism σ^{-1} and spectral parameter $\alpha=\lambda^{-1}$. Moreover, the point $\alpha=0$ is a regular zero of this model. Indeed, we supposed that α was a zero of $\psi(\alpha)$ and one can check explicitly that $\psi(\alpha)\mathcal{L}^\infty(\alpha,x)$ is regular at $\alpha=0$.

It is worth noting that the procedure just described is involutive, in the following sense. If $\varphi(\lambda)\mathcal{L}(\lambda,x)$ is regular at $\lambda=0$, one can check that $\alpha=\infty$ (which corresponds to $\lambda=0$) is a regular zero of the model with Lax matrix $\mathcal{L}^\infty$ and, moreover, that the corresponding field $\mathcal{C}^\infty$ obtained by evaluating $\lambda^{-1}\psi(\lambda^{-1})\mathcal{L}^\infty(\lambda^{-1},x)$ at $\lambda=0$ is equal to $\mathcal{C}$. Re-inverting the spectral parameter α to $\lambda=\alpha^{-1}$, we can thus construct a "new" Lax matrix $\mathcal{L}^\infty(\lambda^{-1},x)-\lambda\psi(\lambda^{-1})^{-1}\mathcal{C}(x)$. According to Theorem 4.2.1, this Lax matrix should satisfy a Maillet bracket with twist function φ and automorphism σ. A direct computation reveals that this Lax matrix is actually equal to the initial Lax matrix $\mathcal{L}$.

Let us end this subsection by illustrating the inversion of spectral parameter on the example of $\mathbb{Z}_T$-coset models. As noted above, for these models the field $\mathcal{C}$ coincides with the constraint $X^{(0)}$. After performing the change of spectral parameter $\lambda\mapsto\alpha=\lambda^{-1}$, we find a twist function

$$\psi_{\mathbb{Z}_T}(\alpha)=-\frac{T\alpha^{T-1}}{(1-\alpha^T)^2}=-\varphi_{\mathbb{Z}_T}(\alpha).$$

Note that the property $\psi(\alpha)=-\varphi(\alpha)$ is also true for the twist function (3.2.24) of the η-deformed $\mathbb{Z}_2$-model. The new Lax matrix is

$$\mathcal{L}^\infty_{\mathbb{Z}_T}(\alpha,x)=\sum_{k=1}^{T}\frac{(T-k)+k\alpha^{-T}}{T}\alpha^k j^{L\,(T-k)}(x)-\sum_{k=1}^{T}\frac{1-\alpha^{-T}}{T}\alpha^k X^{(T-k)}(x).$$

Comparing this to the initial Lax matrix (3.2.26), we see that it simply corresponds (up to a minus sign on terms involving $X^{(k)}$) to changing every grading (k) to $(T-k)$, which is equivalent to considering the automorphism σ^{-1} instead of σ.

4.2.3 Poisson brackets of traces of powers of $\mathcal{L}$

In this chapter, we will focus on the case where the Lie algebra $\mathfrak{g}$ is simple. More precisely, we will restrict to the classical types A, B, C and D of the Cartan classification (see Appendix A.2), seen in their defining representations[1]:

[1] Here J_n is the standard symplectic structure on $\mathbb{C}^{2n}$ given by $J_n=\begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}$ and ${}^tM$ denotes the transpose of M.

Type	Algebra
A	$\mathfrak{sl}(n,\mathbb{C}) = \{M \in M_n(\mathbb{C}) \mid \mathrm{Tr}(M) = 0\}$
B,D	$\mathfrak{so}(n,\mathbb{C}) = \{M \in M_n(\mathbb{C}) \mid {}^tM + M = 0\}$
C	$\mathfrak{sp}(2n,\mathbb{C}) = \{M \in M_{2n}(\mathbb{C}) \mid {}^tMJ_n + J_nM = 0\}$

Table 4.1: Defining representations of classical Lie algebras.

We may therefore take powers of elements of $\mathfrak{g}$ and traces of these matrices. In the following sections, we will extract local charges in involution from the traces of powers of the Lax matrix $\mathcal{L}$. In this subsection, we will establish general results on the Poisson brackets of powers of $\mathcal{L}$ and their traces.

For simplicity, we will change the definition of the Casimir $C_{\underline{12}}$ for this chapter, by considering the bilinear form $\mathrm{Tr}(XY)$ in the defining representation $\mathfrak{g}$, instead of $\mathrm{Tr}(\mathrm{ad}_X\mathrm{ad}_Y)$ in the adjoint representation (see Appendix A.2.4). This only changes the normalisation of the Casimir, as these two bilinear forms are proportional. Thus it does not affect the results described here but simplifies greatly the presentation, as all computations of this chapter are done in the defining representation. In particular, the completeness relation (A.2.7) has to be changed into

$$\mathrm{Tr}_{\underline{2}}(C_{\underline{12}}X_{\underline{2}}) = X, \qquad \forall\, X \in \mathfrak{g}. \tag{4.2.13}$$

Lemma 4.2.2. *Suppose that X and Y are $\mathfrak{g}$-valued quantities such that*

$$\{X_{\underline{1}}, Y_{\underline{2}}\} = [a_{\underline{12}}, X_{\underline{1}}] + [b_{\underline{12}}, Y_{\underline{2}}] + c_{\underline{12}}.$$

Then the Poisson brackets of powers of X and Y are

$$\{X^n_{\underline{1}}, Y^m_{\underline{2}}\} = \left[a^{(nm)}_{\underline{12}}, X_{\underline{1}}\right] + \left[b^{(nm)}_{\underline{12}}, Y_{\underline{2}}\right] + c^{(nm)}_{\underline{12}},$$

where, for $t = a, b, c$, we defined

$$t^{(nm)}_{\underline{12}} = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X^k_{\underline{1}}Y^l_{\underline{2}}\, t_{\underline{12}}\, X^{n-1-k}_{\underline{1}}Y^{m-1-l}_{\underline{2}}.$$

Proof. The Poisson bracket being a derivation, we can use the Leibniz rule yielding

$$\{X^n_{\underline{1}}, Y^m_{\underline{2}}\} = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X^k_{\underline{1}}Y^l_{\underline{2}}\, \{X_{\underline{1}}, Y_{\underline{2}}\}\, X^{n-1-k}_{\underline{1}}Y^{m-1-l}_{\underline{2}}.$$

We conclude observing that $X^k_{\underline{1}}Y^l_{\underline{2}}$ and $X^{n-1-k}_{\underline{1}}Y^{m-1-l}_{\underline{2}}$ commute with $X_{\underline{1}}$ and $Y_{\underline{2}}$ and using the identity

$$M_1[M_2, N]M_3 = [M_1M_2M_3, N],$$

true for any matrices M_1, M_2, M_3 and N such that $[M_1, N] = [M_3, N] = 0$. □

Corollary 4.2.3. *Suppose that X and Y are $\mathfrak{g}$-valued quantities such that*

$$\{X_{\underline{1}}, Y_{\underline{2}}\} = [a_{\underline{12}}, X_{\underline{1}}] + [b_{\underline{12}}, Y_{\underline{2}}] + c_{\underline{12}}.$$

Then we have

$$\begin{aligned}\{\mathrm{Tr}(X^n), \mathrm{Tr}(Y^m)\} &= nm\,\mathrm{Tr}_{\underline{12}}\big(c_{\underline{12}}X^{n-1}_{\underline{1}}Y^{m-1}_{\underline{2}}\big),\\ \{X, \mathrm{Tr}(Y^m)\} &= m\left[\mathrm{Tr}_{\underline{2}}\big(a_{\underline{12}}Y^{m-1}_{\underline{2}}\big), X\right] + m\,\mathrm{Tr}_{\underline{2}}\big(c_{\underline{12}}Y^{m-1}_{\underline{2}}\big).\end{aligned}$$

Proof. Starting with Lemma 4.2.2, the corollary follows from the cyclicity of the trace and the vanishing of traces of commutators. □

Let us now apply these results to the Lax matrix $\mathcal{L}$. We work in the framework described in Chapter 2. We define

$$S_n(\lambda, x) = \varphi(\lambda)^n \mathcal{L}(\lambda, x)^n \tag{4.2.15}$$

and

$$\mathscr{T}_n(\lambda, x) = \mathrm{Tr}\big(S_n(\lambda, x)\big). \tag{4.2.16}$$

Starting with the Poisson bracket (2.2.6) and the expression (2.4.1) of the $\mathcal{R}$-matrix, we apply Corollary 4.2.3. We find that

$$\{\mathscr{T}_n(\lambda, x), \mathscr{T}_m(\mu, y)\} = -nm\,\mathrm{Tr}_{\underline{\mathbf{12}}}\Big(U_{\underline{\mathbf{12}}}(\lambda, \mu) S_{n-1}(\lambda, x)_{\underline{\mathbf{1}}} S_{m-1}(\mu, y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}, \tag{4.2.17}$$

with

$$U_{\underline{\mathbf{12}}}(\lambda, \mu) = \varphi(\lambda)\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) + \varphi(\mu)\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu, \lambda). \tag{4.2.18}$$

4.3 Charges at non-cyclotomic zeros

The purpose of this section is to describe the procedure for extracting local charges in involution from non-cyclotomic regular zeros of the twist function φ. Let us first explain what we mean here by a *non-cyclotomic* point. If $T = 1$, *i.e.* if $\sigma = \mathrm{Id}$ and there is no cyclotomic invariance, we define any point as being non-cyclotomic. If $T \in \mathbb{Z}_{>1}$, a non-cyclotomic point is a point which is not fixed by the action of the cyclic group $\mathbb{Z}_T$, *i.e.* which is not the origin or infinity.

Throughout this section we fix a non-cyclotomic regular zero λ_0. We will focus here on the case where λ_0 is different from infinity. The case $\lambda_0 = \infty$ is treated by the same method, just replacing $\mathcal{L}$ by $\mathcal{L}^\infty$ and φ by ψ (cf. subsection 4.2.2). The fact that λ_0 is a regular zero implies that $S_n(\lambda, x)$ and $\mathscr{T}_n(\lambda, x)$, defined in equations (4.2.15) and (4.2.16), are both holomorphic at $\lambda = \lambda_0$. Thus, we can define the current

$$\mathscr{J}_n^{\lambda_0}(x) = \mathscr{T}_n(\lambda_0, x). \tag{4.3.1}$$

Let us briefly comment on the explicit expression of these currents in the case of the PCM. As explained in paragraph 4.2.1, the PCM has two regular zeros at $+1$ and -1. The corresponding currents are

$$\mathscr{J}^{\pm 1}_{n,\mathrm{PCM}}(x) = \mathrm{Tr}\big(j^L_\pm(x)^n\big).$$

These currents are the one investigated in [79], from which local charges in involution for the PCM are constructed. In this section, we will follow the method developed in [79], generalising it to any current (4.3.1) associated with a non-cyclotomic regular zero λ_0 of the model.

4.3.1 Poisson algebra of the currents

We begin by computing the Poisson bracket of the currents $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\lambda_0}(y)$. Specifically, we would like to evaluate equation (4.2.17) at $\lambda = \mu = \lambda_0$. Since λ_0 is a regular zero, $S_{n-1}(\lambda_0, x)$ and $S_{m-1}(\lambda_0, y)$ are well defined. Thus, it remains to determine $U_{\underline{\mathbf{12}}}(\lambda_0, \lambda_0)$. Starting with the definition (4.2.18) of U and using $\varphi(\lambda_0) = 0$, one has

$$U_{\underline{\mathbf{12}}}(\lambda, \lambda_0) = \varphi(\lambda)\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \lambda_0).$$

Recall from equation (2.3.8) that $\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \lambda_0)$ is not regular at $\lambda = \lambda_0$, so that we cannot simply evaluate the above equation at $\lambda = \lambda_0$. However, as λ_0 is a non-cyclotomic point, the matrix $\mathcal{R}^0$ has the following local behaviour

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \lambda_0) = -\frac{1}{T}\frac{C_{\underline{\mathbf{12}}}}{\lambda - \lambda_0} + A^{\lambda_0}_{\underline{\mathbf{12}}}(\lambda), \tag{4.3.2}$$

where $A^{\lambda_0}_{\underline{\mathbf{12}}}(\lambda)$ is regular at $\lambda = \lambda_0$. Using again $\varphi(\lambda_0) = 0$, we then obtain

$$U_{\underline{\mathbf{12}}}(\lambda_0, \lambda_0) = -\frac{\varphi'(\lambda_0)}{T} C_{\underline{\mathbf{12}}},$$

where φ' denotes the derivative of φ with respect to the spectral parameter λ. Thus, one has

$$\Big\{ \mathscr{J}^{\lambda_0}_n(x), \mathscr{J}^{\lambda_0}_m(y) \Big\} = \frac{nm}{T} \varphi'(\lambda_0) \operatorname{Tr}_{\underline{\mathbf{12}}} \Big(C_{\underline{\mathbf{12}}} S_{n-1}(\lambda_0, x)_{\underline{\mathbf{1}}} S_{m-1}(\lambda_0, y)_{\underline{\mathbf{2}}} \Big) \delta'_{xy}. \tag{4.3.3}$$

Recall the completeness relation (4.2.13). We cannot directly apply this identity to equation (4.3.3) as $S_{m-1}(\lambda_0, y)$ does not belong to $\mathfrak{g}$ in general (recall that S_{m-1} is defined as the $(m-1)^{\text{st}}$ power of a matrix in $\mathfrak{g}$).

Following [79], we will show in the next subsections how to circumvent this difficulty. We will treat separately the case where $\mathfrak{g}$ is of type B, C or D and the case where $\mathfrak{g}$ is of type A.

4.3.2 Type B, C and D algebras

Let us first consider the case where $\mathfrak{g}$ is of type B, C or D, *i.e.* where $\mathfrak{g}$ is an orthogonal or a symplectic algebra (cf. Table 4.1). One can check that, for these algebras, if X belongs to $\mathfrak{g}$, X^n also belongs to $\mathfrak{g}$ if n is odd. Moreover, all matrices in $\mathfrak{g}$ are traceless. We then deduce that the currents $\mathscr{J}^{\lambda_0}_n$ are zero for n odd. Thus, we will only extract local charges from the traces of even powers of $\mathcal{L}$, *i.e.* from the currents $\mathscr{J}^{\lambda_0}_{2n}$.

The Poisson bracket of such currents is given by equation (4.3.3). The right hand side contains $\operatorname{Tr}_{\underline{\mathbf{2}}}\big(C_{\underline{\mathbf{12}}} S_{2m-1}(\lambda_0, y)_{\underline{\mathbf{2}}}\big)$, and since $2m-1$ is odd we have $S_{2m-1}(\lambda_0, y) \in \mathfrak{g}$. Hence, we can apply the completeness relation (4.2.13), which yields

$$\Big\{ \mathscr{J}^{\lambda_0}_{2n}(x), \mathscr{J}^{\lambda_0}_{2m}(y) \Big\} = 4nm \frac{\varphi'(\lambda_0)}{T} \operatorname{Tr}\Big(S_{2n-1}(\lambda_0, x) S_{2m-1}(\lambda_0, y) \Big) \delta'_{xy}.$$

Using the definition (4.2.15) of S, one has

$$\operatorname{Tr}\big(S_p(\lambda, x) \partial_x S_q(\lambda, x) \big) = \frac{q}{p+q} \partial_x \mathscr{T}_{p+q}(\lambda, x). \tag{4.3.4}$$

Using the identities $f(y)\delta'_{xy} = \partial_x\big(f(x)\big)\delta_{xy} + f(x)\delta'_{xy}$ and (4.3.4), we obtain

$$\Big\{ \mathscr{J}^{\lambda_0}_{2n}(x), \mathscr{J}^{\lambda_0}_{2m}(y) \Big\} = 4nm \frac{\varphi'(\lambda_0)}{T} \left(\mathscr{J}^{\lambda_0}_{2n+2m-2}(x) \delta'_{xy} + \frac{2m-1}{2n+2m-2} \partial_x\big(\mathscr{J}^{\lambda_0}_{2n+2m-2}(x) \big) \delta_{xy} \right). \tag{4.3.5}$$

Define the local charges

$$\mathcal{Q}^{\lambda_0}_{2n} = \int \mathrm{d}x \; \mathscr{J}^{\lambda_0}_{2n}(x), \tag{4.3.6}$$

where the integration is over the whole domain of the spatial coordinate x (*i.e.* the real line $\mathbb{R}$ or the circle $\mathbb{S}^1$). Once integrated over y, the right hand side of (4.3.5) is a total derivative with respect to x. Assuming the periodicity of the fields if $x \in \mathbb{S}^1$ or that they decrease at infinity if $x \in \mathbb{R}$, we then conclude that

$$\Big\{ \mathcal{Q}^{\lambda_0}_{2n}, \mathcal{Q}^{\lambda_0}_{2m} \Big\} = 0.$$

In conclusion, we have constructed a tower of local charges $\mathcal{Q}^{\lambda_0}_{2n}$ in involution, as integrals of the currents $\mathscr{J}^{\lambda_0}_{2n}(x)$. These currents are polynomials in the fields appearing in the Lax matrix $\mathcal{L}(\lambda, x)$. More precisely, the current $\mathscr{J}^{\lambda_0}_{2n}$ is a homogeneous polynomial of degree $2n$.

Up to a global factor, the Poisson bracket (4.3.5) is the same as the bracket (4.16) of [79]. Thus, we can apply the methods developed in [79]. In particular, this allows to construct a more general tower

of local charges $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$ in involution, depending on a free parameter $\xi \in \mathbb{R}$. These charges are defined as integrals

$$\mathcal{Q}_{2n}^{\lambda_0}(\xi) = \int \mathrm{d}x \, \mathscr{K}_{2n}^{\lambda_0}(\xi, x)$$

of some currents $\mathscr{K}_{2n}^{\lambda_0}(\xi)$. These currents are given by homogeneous polynomials in the $\mathscr{J}_{2k}^{\lambda_0}$'s, depending on the free parameter $\xi \in \mathbb{R}$. In particular, the first currents $\mathscr{K}_{2n}^{\lambda_0}(\xi)$ are given by:

$$\begin{aligned} \mathscr{K}_2^{\lambda_0}(\xi) &= \mathscr{J}_2^{\lambda_0}, \qquad \mathscr{K}_4^{\lambda_0}(\xi) = \mathscr{J}_4^{\lambda_0} - \frac{3\xi}{2}(\mathscr{J}_2^{\lambda_0})^2, \\ \mathscr{K}_6^{\lambda_0}(\xi) &= \mathscr{J}_6^{\lambda_0} - \frac{15\xi}{4}\mathscr{J}_2^{\lambda_0}\mathscr{J}_4^{\lambda_0} + \frac{25\xi^2}{8}(\mathscr{J}_2^{\lambda_0})^3. \end{aligned} \tag{4.3.7}$$

The expression of the current $\mathscr{K}_{2n}^{\lambda_0}(\xi)$ is determined (up to a global factor) recursively from equation (4.3.5) by demanding that the charge $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$ be in involution with all the charges $\mathcal{Q}_{2m}^{\lambda_0}(\xi)$ ($m = 2, \dots, n-1$) constructed thus far. It can also be found without recursion with the help of a generating function, which allows a general proof of the involution of the charges $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$: we refer the reader to the subsection 4.3.4 for more details.

Taking $\xi = 0$ in equation (4.3.7), we get $\mathscr{K}_{2n}^{\lambda_0}(\xi = 0) = \mathscr{J}_{2n}^{\lambda_0}$. Hence, we recover the local charges $\mathcal{Q}_{2n}^{\lambda_0}$ introduced in equation (4.3.6) as a special case of this one-parameter family of local charges. For different parameters ξ and ξ', the towers of charges $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$ and $\mathcal{Q}_{2n}^{\lambda_0}(\xi')$ do not commute with one another. We thus have to work with a fixed value of ξ: in the rest of this chapter, we will mainly focus on the simplest case $\xi = 0$. This choice is justified first by simplicity, but also because the proof of the existence of an integrable hierarchy associated to the charges $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$, presented in section 4.6, works only for the case $\xi = 0$.

4.3.3 Type A algebras

Let us now consider the case where $\mathfrak{g}$ is of type A, *i.e.* where $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$ for some $d \in \mathbb{Z}_{\geq 2}$ (see Table 4.1). If $X \in \mathfrak{g}$, we have $\mathrm{Tr}(X) = 0$ by definition, but in general $X^n \notin \mathfrak{g}$ and $\mathrm{Tr}(X^n) \neq 0$ for $n \geq 2$. Thus, we consider the currents $\mathscr{J}_n^{\lambda_0}$ for $n \geq 2$. The Poisson bracket between two such currents is given by equation (4.3.3). Since in general $S_{m-1}(\lambda_0, y)$ does not belong to $\mathfrak{g}$, we cannot use the completeness relation (4.2.13) to simplify this equation. However, a variant of the identity (4.2.13) exists for any matrix $Z \in M_d(\mathbb{C})$. Indeed, using the facts that $Z - \frac{1}{d}\mathrm{Tr}(Z)\mathrm{Id}$ belongs to $\mathfrak{g}$ and that $\mathrm{Tr}_{\underline{\mathbf{2}}}(C_{\underline{\mathbf{12}}}) = 0$, we find that

$$\mathrm{Tr}_{\underline{\mathbf{2}}}\big(C_{\underline{\mathbf{12}}} Z_{\underline{\mathbf{2}}}\big) = Z - \frac{1}{d}\mathrm{Tr}(Z)\mathrm{Id}. \tag{4.3.8}$$

Applying this relation to equation (4.3.3) and using the identities $f(y)\delta'_{xy} = \partial_x\big(f(x)\big)\delta_{xy} + f(x)\delta'_{xy}$ and (4.3.4), we obtain

$$\begin{aligned} \Big\{\mathscr{J}_n^{\lambda_0}(x), \mathscr{J}_m^{\lambda_0}(y)\Big\} = nm\frac{\varphi'(\lambda_0)}{T}\Bigg(& \mathscr{J}_{n+m-2}^{\lambda_0}(x)\delta'_{xy} - \frac{1}{d}\mathscr{J}_{n-1}^{\lambda_0}(x)\mathscr{J}_{m-1}^{\lambda_0}(x)\delta'_{xy} \\ & + \frac{m-1}{n+m-2}\partial_x\Big(\mathscr{J}_{n+m-2}^{\lambda_0}(x)\Big)\delta_{xy} - \frac{1}{d}\mathscr{J}_{n-1}^{\lambda_0}(x)\partial_x\Big(\mathscr{J}_{m-1}^{\lambda_0}(x)\Big)\delta_{xy}\Bigg). \end{aligned} \tag{4.3.9}$$

Integrating both sides over x and y, we see that the right hand side does not vanish identically as it did in subsection 4.3.2. Nevertheless, following the method of [79] we will be able to construct new currents $\mathscr{K}_n^{\lambda_0}$ such that the charges

$$\mathcal{Q}_n^{\lambda_0} = \int \mathrm{d}x \, \mathscr{K}_n^{\lambda_0}(x) \tag{4.3.10}$$

Poisson commute with one another.

The Poisson bracket (4.3.9) is to be compared to equation (4.5) of [79], from which it differs only by an overall factor. We can therefore directly apply the procedure developed in [79] to the present case so as to construct the desired currents $\mathscr{K}_n^{\lambda_0}$'s. The expression for the first $\mathscr{K}_n^{\lambda_0}$'s read

$$\mathscr{K}_2^{\lambda_0} = \mathscr{J}_2^{\lambda_0}, \quad \mathscr{K}_3^{\lambda_0} = \mathscr{J}_3^{\lambda_0}, \quad \mathscr{K}_4^{\lambda_0} = \mathscr{J}_4^{\lambda_0} - \frac{3}{2d}(\mathscr{J}_2^{\lambda_0})^2, \quad \mathscr{K}_5^{\lambda_0} = \mathscr{J}_5^{\lambda_0} - \frac{10}{3d}\mathscr{J}_2^{\lambda_0}\mathscr{J}_3^{\lambda_0},$$
$$\mathscr{K}_6^{\lambda_0} = \mathscr{J}_6^{\lambda_0} - \frac{5}{3d}(\mathscr{J}_3^{\lambda_0})^2 - \frac{15}{4d}\mathscr{J}_2^{\lambda_0}\mathscr{J}_4^{\lambda_0} + \frac{25}{8d^2}(\mathscr{J}_2^{\lambda_0})^3. \tag{4.3.11}$$

These currents are similar to the currents $\mathscr{K}_n^{\lambda_0}(\xi)$ described in (4.3.7) for $\mathfrak{g}$ of type B, C or D. More precisely, the current (4.3.11) coincide with the currents $\mathscr{K}_n^{\lambda_0}\left(\frac{1}{d}\right)$, recalling that for type B, C and D, the $\mathscr{J}_{2k+1}^{\lambda_0}$'s vanish. As for $\mathscr{K}_n^{\lambda_0}(\xi)$ in type B, C and D, the expression of the current $\mathscr{K}_n^{\lambda_0}$ for type A is determined (up to a global factor) recursively from equation (4.3.9) by demanding that the charge $\mathcal{Q}_n^{\lambda_0}$ be in involution with all the charges $\mathcal{Q}_m^{\lambda_0}$ $(m = 2, \dots, n-1)$ constructed thus far. However, in the present case, one does not have the freedom of a free parameter ξ in the definition of $\mathscr{K}_n^{\lambda_0}$: there is a unique tower of charges in involution $\mathcal{Q}_n^{\lambda_0}$.

As in the case of type B, C and D algebras, the current $\mathscr{K}_n^{\lambda_0}(x)$ is a homogeneous polynomial of degree n in the fields appearing in the Lax matrix $\mathcal{L}(\lambda, x)$. And as explained in [79], the degrees n for which the current $\mathscr{K}_n^{\lambda_0}(x)$ is non-zero are the exponents of the untwisted affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ plus one.

At this stage, we do not have a proof that the recursive algorithm described above can be applied indefinitely. We shall now recall from [79] how to construct explicitly the current $\mathscr{K}_n^{\lambda_0}$ without a recursive algorithm, using generating functions.

4.3.4 Generating functions

In the previous subsections 4.3.2 and 4.3.3, we introduced currents $\mathscr{K}_n^{\lambda_0}(\xi)$ (for types B, C and D) and $\mathscr{K}_n^{\lambda_0}$ (for type A), constructed recursively from the currents $\mathscr{J}_n^{\lambda_0}$ (and which depended on a free parameter ξ for types B, C and D). In this subsection, we will show how to construct these currents using generating functions.

We will mainly focus on the case where $\mathfrak{g}$ is of type A and will briefly comment on types B, C and D at the end of the subsection. Let us then suppose that $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$, so that we can use the notations and results of subsection 4.3.3. We introduce

$$F(\lambda, \mu, x) = \operatorname{Tr} \log\big(\mathrm{Id} - \mu\varphi(\lambda)\mathcal{L}(\lambda, x)\big) \tag{4.3.12}$$

and

$$A(\lambda, \mu, x) = \det\big(\mathrm{Id} - \mu\varphi(\lambda)\mathcal{L}(\lambda, x)\big), \tag{4.3.13}$$

so that $A(\lambda, \mu, x) = \exp\big(F(\lambda, \mu, x)\big)$. By expanding the matricial logarithm in (4.3.12) as a power series in μ one finds

$$F(\lambda, \mu, x) = -\sum_{k=2}^{\infty} \frac{\mu^k}{k} \mathscr{T}_k(\lambda, x), \tag{4.3.14}$$

with $\mathscr{T}_n(\lambda, x)$ defined in equation (4.2.16). We are interested in the evaluations of $F(\lambda, \mu, x)$ and $A(\lambda, \mu, x)$ at $\lambda = \lambda_0$, which are well defined as λ_0 is a regular zero. Following [79], we look for $\mathscr{K}_n^{\lambda_0}(x)$ in the form of

$$\mathscr{K}_n^{\lambda_0}(x) = A(\lambda_0, \mu, x)^{p_n}\Big|_{\mu^n} \tag{4.3.15}$$

for some rational number p_n, where $f(\mu)|_{\mu^n}$ denotes the coefficient of μ^n in the power series expansion of $f(\mu)$.

The Poisson brackets of the currents $\mathscr{T}_n(\lambda_0, x) = \mathscr{J}_n^{\lambda_0}(x)$ are given by equation (4.3.9). This allows one to compute $\{F(\lambda_0, \mu, x), F(\lambda_0, \nu, y)\}$ and $\{A(\lambda_0, \mu, x), A(\lambda_0, \nu, y)\}$. As equation (4.3.9) coincides with the equation (4.5) of [79] up to a global factor, these Poisson brackets are the same as in [79] (equations (4.13) and (4.14)), still up to the global factor. Thus, the procedure of [79] applies and we

conclude that the Poisson bracket of the local charges (4.3.10) defined in terms of the currents (4.3.15) is

$$\left\{\mathcal{Q}_n^{\lambda_0}, \mathcal{Q}_m^{\lambda_0}\right\} = p_n p_m \mu\nu \frac{\varphi'(\lambda_0)}{T} \int \mathrm{d}x\; A(\lambda_0,\mu,x)^{p_n} \partial_x \big(A(\lambda_0,\nu,x)^{p_m}\big) h_{mn}(\mu,\nu)\Big|_{\mu^n\nu^m},$$

where

$$h_{nm}(\mu,\nu) = \left[\left(\frac{n-1}{p_n}\nu - \frac{m-1}{p_m}\mu\right)\frac{1}{\mu-\nu} + \frac{1}{d}\frac{(n-1)(m-1)}{p_n p_m}\right]. \tag{4.3.16}$$

It follows that the charges $\mathcal{Q}_n^{\lambda_0}$ are in involution if we choose, for any $k \in \mathbb{Z}_{\geq 2}$, $p_k = \frac{k-1}{d}$. The corresponding currents are given by

$$\mathscr{K}_n^{\lambda_0}(x) = \exp\left(-\frac{n-1}{d}\sum_{k=2}^{\infty}\frac{\mu^k}{k}\mathscr{J}_k^{\lambda_0}(x)\right)\Big|_{\mu^n}. \tag{4.3.17}$$

One can check that the first currents defined by this generating function are given by equation (4.3.11), up to overall global factors. The current $\mathscr{K}_n^{\lambda_0}(x)$ is the evaluation at $\lambda = \lambda_0$ of the more general current

$$\mathcal{W}_n(\lambda,x) = A(\lambda,\mu,x)^{(n-1)/d}\Big|_{\mu^n}, \tag{4.3.18}$$

which we will need later. The equation

$$\mathcal{W}_n(\lambda,x) = \exp\left(-\frac{n-1}{d}\sum_{k=2}^{\infty}\frac{\mu^k}{k}\mathscr{T}_k(\lambda,x)\right)\Big|_{\mu^n} \tag{4.3.19}$$

allows one to compute $\mathcal{W}_n(\lambda,x)$ as a polynomial in the $\mathscr{T}_k(\lambda,x)$. More precisely, $\mathcal{W}_n$ is related to the $\mathscr{T}_k$'s in the same way that $\mathscr{K}_n^{\lambda_0}$ is related to the $\mathscr{J}_k^{\lambda_0}$'s.

We end this subsection by saying a few words on Lie algebras $\mathfrak{g}$ of type B, C or D. In this case, we saw in subsection 4.3.2 that the local charges in involution can be taken as integrals of currents $\mathscr{K}_{2n}^{\lambda_0}(\xi)$, depending on a free parameter ξ (see equation (4.3.7)). These currents can be obtained from the $\mathscr{J}_{2k}^{\lambda_0}$'s using a generating function, similar to the one presented above for type A. We will not enter into details here and will just present the final result, based on reference [79]. The current $\mathscr{K}_{2n}^{\lambda_0}(\xi)$ can be computed as:

$$\mathscr{K}_{2n}^{\lambda_0}(\xi,x) = \exp\left(-\frac{\xi(2n-1)}{2}\sum_{k=1}^{\infty}\frac{\mu^k}{k}\mathscr{J}_{2k}^{\lambda_0}(x)\right)\Big|_{\mu^n}. \tag{4.3.20}$$

Starting from the Poisson bracket (4.3.5), one can show that the corresponding charges $\mathcal{Q}_{2n}^{\lambda_0}(\xi)$ are in involution, using similar techniques as above for type A. We refer the interested reader to reference [79] for details on the proof. An explicit computation shows that the first currents $\mathscr{K}_{2n}^{\lambda_0}(\xi)$ obtained from the above equation are given by equation (4.3.7), up to overall global factors.

4.3.5 Summary

To conclude this section, let us summarise the results that we obtained. In particular, we will use this as an opportunity to extend the notations $\mathscr{K}_n^{\lambda_0}$ and $\mathcal{W}_n$, defined for a type A algebra in the previous subsections, to other types. This will serve to uniformise the notation in the rest of the chapter.

When $\mathfrak{g}$ is of type A, the currents $\mathscr{K}_n^{\lambda_0}(x)$ are given in subsection 4.3.4 through equation (4.3.17). We also defined a current $\mathcal{W}_n(\lambda,x)$ depending on the spectral parameter λ in equation (4.3.19). For a Lie algebra $\mathfrak{g}$ of type B, C or D (as treated in subsection 4.3.2), we introduced currents $\mathscr{K}_n^{\lambda_0}(\xi)$, depending on a free parameter ξ. However, as explained at the end of susbection 4.3.2, we will only use the currents $\mathscr{J}_n^{\lambda_0}(x) = \mathscr{K}_n^{\lambda_0}(\xi=0,x)$ in the rest of this chapter. In order to employ uniform notations throughout the chapter, we shall define in this case $\mathscr{K}_n^{\lambda_0}(x) = \mathscr{J}_n^{\lambda_0}(x)$ and $\mathcal{W}_n(\lambda,x) = \mathscr{T}_n(\lambda,x)$.

With these conventions, independently of the type of $\mathfrak{g}$, the current $\mathscr{K}_n^{\lambda_0}(x)$ is the evaluation of $\mathcal{W}_n(\lambda, x)$ at $\lambda = \lambda_0$ and the charge $\mathcal{Q}_n^{\lambda_0}$ is given by

$$\mathcal{Q}_n^{\lambda_0} = \int \mathrm{d}x \, \mathscr{K}_n^{\lambda_0}(x). \tag{4.3.21}$$

Recall also that we restrict the degrees n of the currents $\mathscr{K}_n^{\lambda_0}$ to some subset $\mathcal{E}_{\lambda_0}$ of $\mathbb{Z}_{\geq 2}$. In fact, independently of the type of $\mathfrak{g}$, $\mathcal{E}_{\lambda_0}$ can (almost) be seen as the set of exponents of the affine algebra $\widehat{\mathfrak{g}}$ plus one. This was already observed for type A in subsection 4.3.3, based on the results of [79]. For types B, C and D, we saw in subsection 4.3.2 that $\mathcal{E}_{\lambda_0}$ is the set of all even numbers, which turns out to coincide with the exponents of $\widehat{\mathfrak{g}}$ plus one for types B and C [79]. For type D, there are some exponents missing in this construction (the rank modulo the Coxeter number), which are related to the Pfaffian (see [79]). Although we do not consider the Pfaffian here, we expect that it should be possible to construct a corresponding local charge in the present framework too.

Having introduced these type-independent notations, we can summarise the results of this section by the following theorem.

Theorem 4.3.1. *Let λ_0 be a non-cyclotomic regular zero of the model. Then, for any m and n in $\mathcal{E}_{\lambda_0}$, the charges $\mathcal{Q}_n^{\lambda_0}$ and $\mathcal{Q}_m^{\lambda_0}$ are in involution, i.e. we have*

$$\{\mathcal{Q}_n^{\lambda_0}, \mathcal{Q}_m^{\lambda_0}\} = 0.$$

The notations and results summarised above will be generalised to the case of cyclotomic zeros in the following section. Let us note here that there will be some subtlety in the definition of the current $\mathcal{W}_n$ for an algebra of type A in the case when the automorphism σ is inner, compared to the definition given above. We shall discuss this in subsection 4.4.7.

4.4 Charges at cyclotomic zeros

In this section, we explain how to construct towers of local charges in involution attached to cyclotomic regular zeros of the twist function φ. Recall that a cyclotomic point is a point fixed by the action of the cyclic group $\mathbb{Z}_T$, *i.e.* the origin or infinity. Suppose we are considering a model with a regular zero at infinity. As explained in subsection 4.2.2, working in the new spectral parameter $\alpha = \lambda^{-1}$ and with the new Lax matrix $\mathcal{L}^\infty$ amounts to treating, instead, a model with a regular zero at $\alpha = 0$ and automorphism σ^{-1}. Hence it is sufficient to describe the extraction of local charges at the origin.

Throughout this section we therefore consider a model with $T > 1$ and a regular zero at the origin. We thus have $\varphi(0) = 0$ and $\varphi(\lambda)\mathcal{L}(\lambda, x)$ regular at 0. Using the equivariance property (2.4.5), we see that the smallest power of λ in φ is of the form $\alpha T - 1$, for some $\alpha \in \mathbb{Z}_{\geq 1}$. In terms of the function ζ, defined in equation (2.4.6), this implies that $\zeta(\lambda^T) = O(\lambda^{\alpha T})$. We will mostly need the fact that $\zeta(\lambda^T) = O(\lambda^T)$, *i.e.* that $\varphi(\lambda) = O(\lambda^{T-1})$, and more precisely the asymptotic property

$$\zeta(\lambda^T) = \zeta'(0)\lambda^T + O(\lambda^{2T}). \tag{4.4.1}$$

Recall that in the previous section we extracted local charges by evaluating the traces of powers of $\varphi(\lambda)\mathcal{L}(\lambda, x)$ at the regular zeros. In the case of a cyclotomic point, this method is not sufficient to extract all charges, as such traces can vanish. To understand how to construct the whole algebra of local charges, we will first need to establish equivariance properties of $S_n(\lambda, x)$.

4.4.1 Equivariance properties

Recall the equivariance properties (2.4.5) and (2.4.3) of φ and $\mathcal{L}$. In this subsection, we look for a similar relation for $S_n(\lambda, x)$. In general, $S_n(\lambda, x)$ does not belong to the Lie algebra $\mathfrak{g}$ since it is defined as the power of an element of $\mathfrak{g}$ seen in the fundamental representation. Thus, one cannot consider directly the action of σ on $S_n(\lambda, x)$.

We refer here to the discussion of appendix 4.A. We will restrict to the case where σ is not one of the special automorphisms of $D_4 = \mathfrak{so}(8,\mathbb{C})$. In this case, we can extend σ to a linear endomorphism on the space F of all matrices acting on the fundamental representation, that we shall still denote σ (see details in appendix 4.A). Note that this new endomorphism σ of F is still of order T. We will also need the following properties of σ. For any $Z \in F$ we have

$$\sigma(Z^n) = \epsilon^{n-1}\sigma(Z)^n, \tag{4.4.2a}$$
$$\mathrm{Tr}\big(\sigma(Z)\big) = \epsilon \mathrm{Tr}(Z), \tag{4.4.2b}$$

for some ϵ in $\{1,-1\}$. Note that ϵ is always 1 except when $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{C})$ and σ has a non-trivial outer part, in which case T is even. We shall write $\epsilon = \omega^{\frac{\eta T}{2}}$, with η in $\{0,1\}$.

From equations (2.4.5) and (2.4.3) and the identity (4.4.2a), we deduce that S_n satisfies the equivariance property

$$\sigma\big(S_n(\lambda,x)\big) = \omega^{\kappa(n-1)+1} S_n(\omega\lambda, x), \tag{4.4.3}$$

with $\kappa = 1 + \frac{\eta T}{2}$. Let us consider the power series expansion

$$S_n(\lambda,x) = \sum_{r=0}^{\infty} A_{n,r}(x)\lambda^r. \tag{4.4.4}$$

We then find

$$\sigma(A_{n,r}) = \omega^{r+\kappa(n-1)+1} A_{n,r}. \tag{4.4.5}$$

Taking the trace and using equation (4.4.2b), we find

$$\mathrm{Tr}(A_{n,r}) = \omega^{r+n\kappa}\mathrm{Tr}(A_{n,r}). \tag{4.4.6}$$

Thus, $\mathrm{Tr}(A_{n,r})$ vanishes except if $r \equiv r_n\,[T]$, where r_n is the remainder of the euclidian division of $-n\kappa$ by T. We define

$$\mathscr{J}^0_n(x) = \mathrm{Tr}\big(A_{n,r_n}(x)\big).$$

In particular, the first term in the power series expansion of $\mathscr{T}_n(\lambda,x)$ is $\lambda^{r_n}\mathscr{J}^0_n(x)$. Note that $\mathscr{J}^0_n(x)$ is the evaluation of $\mathscr{T}_n(\lambda,x)$ at $\lambda = 0$ if and only if $r_n = 0$, *i.e.* if T divides $n\kappa$. Note also, as $-2\kappa \equiv -2\,[T]$, that $r_2 = T-2$. Thus, we find

$$\mathscr{J}^0_2(x) = \zeta'(0) \operatorname*{res}_{\lambda=0} \varphi(\lambda)\mathrm{Tr}\big(\mathcal{L}(\lambda,x)^2\big),$$

where ζ was defined in equation (2.4.6). Finally, let us remark that equation (4.4.6) implies that $\mathscr{T}_n(\lambda,x)$ has the following equivariance property

$$\mathscr{T}_n(\omega\lambda,x) = \omega^{r_n}\mathscr{T}_n(\lambda,x). \tag{4.4.7}$$

4.4.2 Poisson algebra of the currents

One can extract the Poisson brackets of the currents $\mathscr{J}^0_n(x)$ and $\mathscr{J}^0_m(y)$ as the coefficient of $\lambda^{r_n+r_m}$ in the power series expansion of $\{\mathscr{T}_n(\lambda,x),\mathscr{T}_m(\lambda,y)\}$. The latter can be computed from equation (4.2.17). Specifically, using the identity (2.3.9) we find

$$\lambda\mu U_{\underline{\mathbf{12}}}(\lambda,\mu) = -\frac{\zeta(\lambda^T)-\zeta(\mu^T)}{\lambda^T-\mu^T}\sum_{k=0}^{T-1}\lambda^k\mu^{T-k}C^{(k)}_{\underline{\mathbf{12}}} + \zeta(\mu^T)C^{(0)}_{\underline{\mathbf{12}}},$$

with ζ defined in equation (2.4.6). Taking the limit $\mu \to \lambda$ we obtain

$$U_{\underline{\mathbf{12}}}(\lambda,\lambda) = -\lambda^{T-2}\zeta'(\lambda^T)C_{\underline{\mathbf{12}}} + \lambda^{-2}\zeta(\lambda^T)C^{(0)}_{\underline{\mathbf{12}}}, \tag{4.4.8}$$

so that

$$\{\mathscr{T}_n(\lambda,x),\mathscr{T}_m(\lambda,y)\} = nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}_{\underline{\mathbf{12}}}\Big(C_{\underline{\mathbf{12}}}S_{n-1}(\lambda,x)_{\underline{\mathbf{1}}}S_{m-1}(\lambda,y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy} - nm\lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}_{\underline{\mathbf{12}}}\Big(C^{(0)}_{\underline{\mathbf{12}}}S_{n-1}(\lambda,x)_{\underline{\mathbf{1}}}S_{m-1}(\lambda,y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}. \tag{4.4.9}$$

The first term of this Poisson bracket has the same structure as the Poisson bracket (4.3.3). The main difference coming from cyclotomy is thus the second term, which involves the partial Casimir $C^{(0)}_{\underline{\mathbf{12}}}$. We recall that we have the partial completeness relation

$$\mathrm{Tr}_{\underline{\mathbf{2}}}(C^{(0)}_{\underline{\mathbf{12}}}Z_{\underline{\mathbf{2}}}) = \pi^{(0)}(Z), \tag{4.4.10}$$

for any $Z\in\mathfrak{g}$.

The second term in (4.4.9) will therefore involve the projection $S^{(0)}_{n-1}(\lambda,x)$ of $S_{n-1}(\lambda,x)$ onto the grading zero $F^{(0)} = \{Z\in F\,|\,\sigma(Z)=Z\}$. To determine these projections, we can make use of the power series expansion (4.4.4) and equation (4.4.5). In particular, one finds that $A_{n-1,r+T}$ is in $F^{(0)}$ if and only if $A_{n-1,r}$ also belongs to $F^{(0)}$. Let us then define q_n to be the unique integer between 0 and $T-1$ such that $A_{n-1,r}$ belongs to $F^{(0)}$ if and only if $r\equiv q_n\,[T]$. Using equation (4.4.5) we find $q_n\equiv r_n+1\,[T]$. So $q_n = r_n+1$ if $r_n\leq T-2$ and $q_n=0$ if $r_n = T-1$.

To simplify the Poisson bracket (4.4.9), we will need to distinguish between three cases:

- $\mathfrak{g}$ is of type B, C or D,
- $\mathfrak{g}$ is of type A and σ is inner,
- $\mathfrak{g}$ is of type A and σ is not inner.

4.4.3 Algebra of type B, C or D

We first consider $\mathfrak{g}$ to be of type B, C or D. Recall that in this case $S_{2n-1}(\lambda,x)$ belongs to the Lie algebra so that $\mathscr{T}_{2n-1}(\lambda,x)$ is zero and hence we consider only the currents $\mathscr{J}^0_{2n}(x)$. Moreover, we can use the completeness relations (4.2.13) and (4.4.10) in (4.4.9). We then find

$$\{\mathscr{T}_{2n}(\lambda,x),\mathscr{T}_{2m}(\lambda,y)\} = 4nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}\Big(S_{2n-1}(\lambda,x)S_{2m-1}(\lambda,y)\Big)\delta'_{xy} - 4nm\lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}\Big(S^{(0)}_{2n-1}(\lambda,x)S^{(0)}_{2m-1}(\lambda,y)\Big)\delta'_{xy}. \tag{4.4.11}$$

After integration over y, the first term becomes a total derivative with respect to x by virtue of equation (4.3.4) and thus vanishes when integrated over x.

Recall moreover that the Poisson bracket of $\mathscr{J}^0_{2n}(x)$ with $\mathscr{J}^0_{2m}(y)$ is obtained from (4.4.11) by keeping only the term $\lambda^{r_{2n}+r_{2m}}$ in the power series expansion. We note that the smallest power of λ in the second term of (4.4.11) is $\alpha T-2+q_{2n}+q_{2m}$ (cf. equation (4.4.1) and above). As we saw in the previous subsection, $q_k = r_k+1$ if $r_k\leq T-2$ and $q_k=0$ if $r_k=T-1$. In the case where r_{2n} and r_{2m} are different from $T-1$, the smallest power of λ is then $r_{2n}+r_{2m}+\alpha T$ so the second term of (4.4.11) does not contribute to the Poisson bracket of $\mathscr{J}^0_{2n}(x)$ with $\mathscr{J}^0_{2m}(y)$, as $\alpha\geq 1$.

If r_{2n} or r_{2m} is equal to $T-1$ then there will be a contribution from this term involving other objects than only the $\mathscr{J}^0_k$'s, preventing us from constructing charges in involution. Thus, we will only consider the currents $\mathscr{J}^0_{2k}(x)$ such that $r_{2k}\neq T-1$. We then have

$$\{\mathcal{Q}^0_{2n},\mathcal{Q}^0_{2m}\} = 0,$$

where $\mathcal{Q}^0_{2k}$ is the integral of the current $\mathscr{J}^0_{2k}(x)$.

We have thus extracted a tower of local charges in involution from the Lax matrix around the origin. Just as in the non-cyclotomic case, these charges are integrals of some polynomials of even degrees in the fields appearing in the Lax matrix. The main difference with the non-cyclotomic case is the fact that, in general, we do not have a current of any even degree. More precisely, we 'dropped' the currents of degree $2n$, for all n such that $r_{2n} = T-1$. Recall from appendix 4.A that in the case of an algebra of type B, C or D, we have $\epsilon = 1$ and $\kappa = 1$. Thus r_{2n} is the remainder of the euclidian division of $-2n$ by T, which means that $r_{2n} = T-1$ if and only if $2n \equiv 1\,[T]$. In particular, we see that there is no drop of any degrees if T is even.

4.4.4 Algebra of type A and σ inner

Let us now suppose that $\mathfrak{g}$ is $\mathfrak{sl}(d,\mathbb{C})$ and σ is inner. In this case, we have the generalised completeness relation (4.3.8). Moreover, we also have a similar identity for the partial Casimir $C^{(0)}_{\underline{\mathbf{12}}}$, derived as follows. Recall that for any $Z \in F$, $Z - \frac{1}{d}\mathrm{Tr}(Z)\mathrm{Id}$ belongs to $\mathfrak{g}$. Moreover, we note that the identity Id is in the grading zero $F^{(0)}$ for σ inner (cf. appendix 4.A). Using equation (4.4.10), we then have

$$\mathrm{Tr}_{\underline{\mathbf{2}}}\left(C^{(0)}_{\underline{\mathbf{12}}} Z_{\underline{\mathbf{2}}}\right) = \pi^{(0)}(Z) - \frac{1}{d}\mathrm{Tr}(Z)\mathrm{Id}. \tag{4.4.12}$$

Using equations (4.3.8) and (4.4.12) in the Poisson bracket (4.4.9), we obtain

$$\begin{aligned}\{\mathscr{T}_n(\lambda,x),\mathscr{T}_m(\lambda,y)\} &= nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}\Big(S_{n-1}(\lambda,x)S_{m-1}(\lambda,y)\Big)\delta'_{xy} \\ &- nm\lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}\Big(S^{(0)}_{n-1}(\lambda,x)S^{(0)}_{m-1}(\lambda,y)\Big)\delta'_{xy} + \frac{nm}{d}\frac{\zeta(\lambda^T)-\lambda^T\zeta'(\lambda^T)}{\lambda^2}\mathscr{T}_{n-1}(\lambda,x)\mathscr{T}_{m-1}(\lambda,y)\delta'_{xy}\end{aligned} \tag{4.4.13}$$

The Poisson bracket of $\mathscr{J}^0_n(x)$ with $\mathscr{J}^0_m(y)$ is obtained by extracting the coefficient of $\lambda^{r_n+r_m}$ in the above equation. To treat the second term on the right hand side of this equation, we follow the discussion of the previous subsection 4.4.3. The smallest power of λ appearing in this term is $\alpha T - 2 + q_n + q_m$ and if we restrict to n and m such that r_n and r_m are different from $T-1$, this power is strictly greater than $r_n + r_m$. The term then does not contribute to the Poisson bracket $\{\mathscr{J}^0_n(x),\mathscr{J}^0_m(y)\}$. Let us turn to the third term on the right hand side of equation (4.4.13). It can be seen from equation (4.4.1) that $\zeta(\lambda^T) - \lambda^T\zeta'(\lambda^T) = O(\lambda^{2T})$. The smallest power of λ that can appear in this term is thus $2T - 2 + r_{n-1} + r_{m-1}$, which is always greater than $2T-2$ and therefore strictly greater than $r_n + r_m$ if r_n and r_m are different from $T-1$.

In conclusion, only the first term of the right hand side of (4.4.13) contributes to the Poisson bracket $\{\mathscr{J}^0_n(x),\mathscr{J}^0_m(y)\}$, which then has the same structure as in the previous subsection. Integrating this bracket over x and y, we recognise the integral of a total derivative proportional to $\partial_x\mathscr{T}_{n+m-2}(\lambda,x)$, which then vanishes, assuming appropriate boundary conditions. Thus, for any n and m such that r_n and r_m are different from $T-1$, we have

$$\{\mathcal{Q}^0_n,\mathcal{Q}^0_m\} = 0$$

with $\mathcal{Q}^0_k$ the integral of the current $\mathscr{J}^0_k(x)$. As in the subsection 4.4.3, we have $\epsilon = 1$ and $\kappa = 1$ for σ inner. It follows that the integers n such that $r_n = T-1$ (for which we do not consider the charge $\mathcal{Q}^0_n$) are the ones equal to 1 modulo T.

4.4.5 Algebra of type A and σ not inner

Finally, let us treat the case where $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{C})$ and σ not inner. In particular, this implies that T is even and we shall write $T = 2S$ in this subsection. We still have the generalised completeness relation (4.3.8). As σ is not inner, we have $\sigma(\mathrm{Id}) = -\mathrm{Id}$ and hence $\pi^{(0)}(\mathrm{Id}) = 0$. We deduce that in this case, the partial completeness relation (4.4.10) actually holds for any $Z \in F$. Equation (4.4.9) then gives

$$\begin{aligned}\{\mathscr{T}_n(\lambda,x),\mathscr{T}_m(\lambda,y)\} &= nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}\Big(S_{n-1}(\lambda,x)S_{m-1}(\lambda,y)\Big)\delta'_{xy} \\ &- nm\lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}\Big(S^{(0)}_{n-1}(\lambda,x)S^{(0)}_{m-1}(\lambda,y)\Big)\delta'_{xy} - \frac{nm}{d}\lambda^{T-2}\zeta'(\lambda^T)\mathscr{T}_{n-1}(\lambda,x)\mathscr{T}_{m-1}(\lambda,y)\delta'_{xy}.\end{aligned} \tag{4.4.14}$$

We follow the method of the previous subsections and look for the power $r_n + r_m$ of λ in the right hand side of this bracket. As explained in subsection 4.4.3, the second term does not contribute when we restrict to r_n and r_m different from $T-1$.

The first term is treated as in the case of a non-cyclotomic point: using the identity $f(y)\delta'_{xy} = f(x)\delta'_{xy} + \partial_x\big(f(x)\big)\delta_{xy}$ and the equation (4.3.4), we find

$$\mathrm{Tr}\Big(S_{n-1}(\lambda,x)S_{m-1}(\lambda,y)\Big)\delta'_{xy} = \mathscr{T}_{n+m-2}(\lambda,x)\delta'_{xy} + \frac{m-1}{n+m-2}\partial_x\big(\mathscr{T}_{n+m-2}(\lambda,x)\big)\delta_{xy}. \tag{4.4.15}$$

The powers of λ appearing in the power series expansion of the first term are then of the form $r_{n+m-2} - 2 + aT$, with $a \in \mathbb{Z}_{\geq 1}$. One has $r_{n+m-2} \equiv r_n + r_m + 2\,[T]$ and $0 \leq r_{n+m-2} \leq T-1$. Moreover, $r_n + r_m + 2$ is always between 0 and $2T-2$ if we suppose r_n and r_n different from $T-1$. If $0 \leq r_n + r_m + 2 < T$, we have $r_{n+m-2} = r_n + r_m + 2$ and the powers $r_{n+m+2} - 2 + aT$ are then all strictly greater than $r_n + r_m$. If $T \leq r_n + r_m + 2 \leq 2T-2$ then $r_{n+m-2} = r_n + r_m + 2 - T$ and the power $r_{n+m-2} - 2 + aT$ is equal to $r_n + r_m$ if and only if $a = 1$.

Finally, let us consider the third term on the right hand side of (4.4.14). The powers of λ in its power series expansion are of the form $r_{n-1} + r_{m-1} - 2 + aT$, with $a \in \mathbb{Z}_{\geq 1}$. Note that $r_{k-1} \equiv r_k + 1 + S \equiv r_k + 1 - S\,[T]$. Thus, $r_{k-1} = r_k + 1 + S$ if $0 \leq r_k < S-1$ and $r_{k-1} = r_k + 1 - S$ if $S-1 \leq r_k \leq T-1$. We then conclude that the power $r_{n-1} + r_{m-1} - 2 + aT$ is equal to $r_n + r_m$ if and only if $r_n + 1 - S \geq 0$, $r_m + 1 - S \geq 0$ and $a = 1$.

Combining all the above results, we find a closed expression for the Poisson bracket of the currents $\mathscr{J}^0_n(x)$ and $\mathscr{J}^0_m(y)$ when r_n and r_m are different form $T-1$, specifically

$$\begin{aligned}\big\{\mathscr{J}^0_n(x), \mathscr{J}^0_m(y)\big\} &= \theta_{r_n+r_m+2-T}\, nm\zeta'(0)\left(\mathscr{J}^0_{n+m-2}(x)\delta'_{xy} + \frac{m-1}{n+m-2}\partial_x\left(\mathscr{J}^0_{n+m-2}(x)\right)\delta_{xy}\right)\\ &\quad - \theta_{r_n+1-S}\,\theta_{r_m+1-S}\,\frac{nm}{d}\zeta'(0)\Big(\mathscr{J}^0_{n-1}(x)\mathscr{J}^0_{m-1}(x)\delta'_{xy} + \mathscr{J}^0_{n-1}(x)\partial_x\left(\mathscr{J}^0_{m-1}(x)\right)\delta_{xy}\Big),\end{aligned} \tag{4.4.16}$$

where $\theta_k = 1$ if $k \in \mathbb{Z}_{\geq 0}$ and $\theta_k = 0$ if $k \in \mathbb{Z}_{<0}$.

As in the case of a non-cyclotomic zero, one can construct charges in involution by taking integrals of new currents $\mathscr{K}^0_n(x)$, constructed as polynomials of the currents $\mathscr{J}^0_m(x)$ for $m \leq n$. The method in the present case is similar: one can construct the currents $\mathscr{K}^0_n$ recursively by asking that the corresponding charge $\mathcal{Q}^0_n$ Poisson commutes with $\mathcal{Q}^0_m$ for all $m < n$. One of the main difference with the non-cyclotomic case is the fact that we do not consider a current $\mathscr{K}^0_n$ when $r_n = T-1$ (we say that such a degree n drops from the construction). The second difference is the presence of the terms θ_k in the Poisson bracket (4.4.16) compared to (4.3.9). Since these terms depend on the numbers r_k, the construction of the $\mathscr{K}^0_n$'s will depend on T. As an illustration, we give here the expression of the first $\mathscr{K}^0_n$'s for T equal to 2 and 4.

In the case $T = 2$, we have $\kappa = 2$ hence $r_n = 0$ for all $n \geq 2$. Thus, there is no drop of any current due to the condition $r_n \neq T-1$ and the current $\mathscr{J}^0_n(x)$ is simply the evaluation of $\mathscr{T}_n(\lambda,x)$ at $\lambda = 0$. Moreover, since $2-T$ and $1-S$ are both zero for $T = 2$, we note that all θ_k terms in the Poisson bracket (4.4.16) are equal to 1. The construction for the $\mathscr{K}^0_n$'s is therefore the same as in the non-cyclotomic case and their expression is given by (4.3.11).

Let us now consider the case $T = 4$. We find $\kappa = 3$ and $r_n \equiv n\,[4]$ and therefore drop the currents $\mathscr{J}^0_{4k+3}(x)$. Constructing the $\mathscr{K}^0_n$'s recursively we find

$$\begin{aligned}&\mathscr{K}^0_2 = \mathscr{J}^0_2, \quad \mathscr{K}^0_4 = \mathscr{J}^0_4, \quad \mathscr{K}^0_5 = \mathscr{J}^0_5,\\ &\mathscr{K}^0_6 = \mathscr{J}^0_6 - \frac{15}{4d}\mathscr{J}^0_2\mathscr{J}^0_4, \quad \mathscr{K}^0_8 = \mathscr{J}^0_8 - \frac{7}{4d}\big(\mathscr{J}^0_4\big)^2\end{aligned} \tag{4.4.17}$$

where we dropped the currents $\mathscr{K}^0_3$ and $\mathscr{K}^0_7$ and more generally all currents $\mathscr{K}^0_{4k+3}$.

Comparing these currents to the ones constructed in the non-cyclotomic case (4.3.11), one can observe a pattern in the cyclotomic procedure. Here also, the current $\mathscr{K}_n^0$ is constructed by correcting $\mathscr{J}_n^0$ with monomials $\mathscr{J}_{m_1}^0 \dots \mathscr{J}_{m_p}^0$ such that $m_1+\ldots+m_p = n$. We observe that not all the monomials in the non-cyclotomic corrections appear among the cyclotomic ones but the ones that do have the same coefficients (for example $15/4d$ for the $\mathscr{J}_2\mathscr{J}_4$ correction of $\mathscr{J}_6$). Moreover, we note that a monomial $\mathscr{J}_{m_1}^0 \dots \mathscr{J}_{m_p}^0$ appearing in the non-cyclotomic procedure is also present in the cyclotomic expression if and only if $r_{m_1} + \ldots + r_{m_p} = r_n$.

The above observations are still found to hold for larger values of T and n (although we do not include the corresponding expression of $\mathscr{K}_n^0$ for conciseness). This allows one to find the currents $\mathscr{K}_n^0(x)$ without going through the recursive procedure if one already knows the result for the non-cyclotomic case. A more systematic approach to constructing higher conserved charges in involution in the cyclotomic case would be to find a generating function for the $\mathscr{K}_n^0$'s, generalising the results of subsection 4.3.4. This will be the subject of the next subsection.

4.4.6 Generating function for type A with σ not inner

In subsection 4.3.4 we presented, following [79], the generating function for constructing the currents $\mathscr{K}_n(x)$ in the non-cyclotomic setting. In particular, we found that the relation between the $\mathscr{K}_n(x)$'s and the $\mathscr{J}_m(x)$'s is given by equation (4.3.17). In the previous subsection we showed how the currents $\mathscr{K}_n^0(x)$ could be constructed in the cyclotomic case from the knowledge of the corresponding result in the non-cyclotomic case. In particular, starting from the expression of $\mathscr{K}_n$ as a polynomial of the $\mathscr{J}_m$'s, we observed that $\mathscr{K}_n^0$ can be constructed in the same way by keeping monomials $\mathscr{J}_{m_1}^0 \dots \mathscr{J}_{m_p}^0$ with the same coefficient if and only if $r_n = r_{m_1} + \ldots + r_{m_p}$. This procedure for going from the non-cyclotomic to the cyclotomic setting has a natural interpretation in terms of equation (4.3.17). Indeed, the current $\mathscr{K}_n^0$ constructed above is equal to

$$\mathscr{K}_n^0(x) = \exp\left(-\frac{n-1}{d}\sum_{k=2}^{\infty}\frac{\mu^k}{k}\lambda^{r_k}\mathscr{J}_k^0(x)\right)\Bigg|_{\mu^n\,\lambda^{r_n}}, \tag{4.4.18}$$

where the projection onto the term λ^{r_n} ensures that we keep only the monomials satisfying the condition $r_n = r_{m_1} + \ldots + r_{m_p}$.

Recall that $\lambda^{r_k}\mathscr{J}_k^0(x)$ is the first term in the power series expansion of $\mathscr{T}_k(\lambda, x)$. Moreover, the next terms start with a $(r_k+T)^{\text{th}}$ power of λ. Since $r_n < T$, such terms can be added to the exponent in equation (4.4.18) without changing the left hand side as they cannot contribute to a λ^{r_n}-term. We may therefore also write

$$\mathscr{K}_n^0(x) = \exp\left(-\frac{n-1}{d}\sum_{k=2}^{\infty}\frac{\mu^k}{k}\mathscr{T}_k(\lambda, x)\right)\Bigg|_{\mu^n\,\lambda^{r_n}}. \tag{4.4.19}$$

In terms of the definitions (4.3.12) and (4.3.13) of $F(\lambda,\mu,x)$ and $A(\lambda,\mu,x)$ and equation (4.3.14), we can further re-express equation (4.4.19) as

$$\mathscr{K}_n^0(x) = A(\lambda,\mu,x)^{(n-1)/d}\Big|_{\mu^n\,\lambda^{r_n}}, \tag{4.4.20}$$

or again

$$\mathscr{K}_n^0(x) = \exp\left(\frac{n-1}{d}F(\lambda,\mu,x)\right)\Bigg|_{\mu^n\,\lambda^{r_n}}.$$

Starting from equation (4.4.14), using equation (4.4.15) and the identity $f(y)\delta'_{xy} = f(x)\delta'_{xy} + \partial_x\big(f(x)\big)\delta_{xy}$, we get

$$\{\mathscr{T}_n(\lambda,x), \mathscr{T}_m(\lambda,y)\} = nm\,\Omega_{nm}(\lambda,x,y) + nm\,\Delta_{nm}(\lambda,x,y) \tag{4.4.21}$$

where

$$\Omega_{nm}(\lambda,x,y) = \lambda^{T-2}\zeta'(\lambda)\Big(\mathscr{T}_{n+m-2}(\lambda,x)\delta'_{xy} - \frac{1}{d}\mathscr{T}_{n-1}(\lambda,x)\mathscr{T}_{m-1}(\lambda,x)\delta'_{xy} \tag{4.4.22a}$$
$$+\frac{m-1}{n+m-2}\partial_x\left(\mathscr{T}_{n+m-2}(\lambda,x)\right)\delta_{xy} - \frac{1}{d}\mathscr{T}_{n-1}(\lambda,x)\partial_x\left(\mathscr{T}_{m-1}(\lambda,x)\right)\delta_{xy}\Big),$$

$$\Delta_{nm}(\lambda,x,y) = \lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}\Big(S^{(0)}_{n-1}(\lambda,x)S^{(0)}_{m-1}(\lambda,y)\Big)\delta'_{xy}. \tag{4.4.22b}$$

We want to compute the Poisson brackets between the charges $\mathcal{Q}^0_n$ defined as integrals of the currents (4.4.20). To begin with, note that the Poisson bracket between $F(\lambda,\mu,x)$ and $F(\lambda,\nu,y)$ can be obtained from equations (4.3.14) and (4.4.21). We then find that

$$\{A(\lambda,\mu,x)^p, A(\lambda,\nu,y)^q\} = pq\,A(\lambda,\mu,x)^pA(\lambda,\nu,y)^q\sum_{k,l=2}^{\infty}\big(\Omega_{kl}(\lambda,x,y)+\Delta_{kl}(\lambda,x,y)\big)\mu^k\nu^l. \tag{4.4.23}$$

Up to a global factor and treating the spectral parameter λ as an external parameter, $\Omega_{nm}(\lambda,x,y)$ has the same structure as the right hand side of equation (4.3.9), which as we saw already had the same structure as equation (4.5) of [79]. This equation (4.5) is used in [79] to compute the Poisson brackets of the generating functions (equations (4.13) to (4.15)). The method developed in [79] for computing these Poisson brackets then applies to the terms Ω_{kl} in equation (4.4.23) and gives a similar result, up to the global factor and the dependence on λ that we keep. Specifically, we find

$$\int \mathrm{d}x\int \mathrm{d}y\left\{A(\lambda,\mu,x)^{p_n}\Big|_{\mu^n}, A(\lambda,\nu,y)^{p_m}\Big|_{\nu^m}\right\} \tag{4.4.24}$$
$$= p_np_m\,\lambda^{T-2}\zeta'(\lambda)\,\mu\nu\int \mathrm{d}x\;A(\lambda,\mu,x)^{p_n}\partial_x\big(A(\lambda,\nu,x)^{p_m}\big)h_{nm}(\mu,\nu)\;\Big|_{\mu^n\nu^m}$$
$$+\,p_np_m\int\mathrm{d}x\int\mathrm{d}y\sum_{k=2}^{n}\sum_{l=2}^{m}\Delta_{kl}(\lambda,x,y)\,A(\lambda,\mu,x)^{p_n}\Big|_{\mu^{n-k}}A(\lambda,\nu,y)^{p_m}\;\Big|_{\nu^{n-l}},$$

where $h_{nm}(\mu,\nu)$ was defined in equation (4.3.16).

The first term on the right hand side of (4.4.24), proportional to $h_{nm}(\mu,\nu)$, vanishes when $p_k = \frac{k-1}{d}$ for all $k\in\mathbb{Z}_{\geq 2}$, as expected. It therefore remains to show that the second term also does not contribute when we restrict to the (r_n+r_m)-th power of λ. From equation (4.4.22b), we see that the powers of λ appearing in the power series expansion of $\Delta_{kl}(\lambda,x,y)$ are of the form q_k+q_l-2+aT, with $a\geq\alpha\geq 1$ and q_n defined in subsection 4.4.2.

The equivariance property (4.4.7) can be rewritten as $\mathscr{T}_n(\omega\lambda,x) = \omega^{-n\kappa}\mathscr{T}_n(\lambda,x)$. In terms of the generating function $F(\lambda,\mu,x)$, this can be re-expressed as $F(\omega\lambda,\mu,x) = F(\lambda,\omega^{-\kappa}\mu,x)$. Thus, we also have $A(\omega\lambda,\mu,x)^p = A(\lambda,\omega^{-\kappa}\mu,x)^p$. Finally, we deduce that

$$A(\omega\lambda,\mu,x)^p\Big|_{\mu^k} = \omega^{r_k}A(\lambda,\mu,x)^p\Big|_{\mu^k} \tag{4.4.25}$$

In particular, this implies that the powers of λ appearing in $A(\lambda,\mu,x)^p\big|_{\mu^k}$ are of the form r_k+bT with $b\geq 0$. In conclusion, the powers of λ in

$$\Delta_{kl}(\lambda,x,y)\,A(\lambda,\mu,x)^p\Big|_{\mu^{n-k}}A(\lambda,\nu,y)^q\Big|_{\nu^{n-l}} \tag{4.4.26}$$

are of the form $q_k+r_{n-k}+q_l+r_{m-l}-2+cT$, with $c\geq 1$.

Recall from subsection 4.4.2 that $q_k\equiv r_k+1\,[T]$, and therefore $q_k+r_{n-k}\equiv r_n+1\,[T]$. Suppose now that r_n and r_m are different from, and so in particular strictly less than, $T-1$. As q_k+r_{n-k} is always positive and congruent to r_n+1 modulo T, which is strictly less than T, it then follows that $q_k+r_{n-k}\geq r_n+1$. Similarly, we have $q_l+r_{m-l}\geq r_m+1$ and we thus deduce that

$q_k + r_{n-k} + q_l + r_{m-l} - 2 + cT$ is greater than or equal to $r_n + r_m + T$. We deduce that the term (4.4.26) cannot contribute to the $(r_n + r_m)^{\text{th}}$ power of λ, as required.

In conclusion, we have found that, for any n and m such that r_n and r_m are different from $T-1$, one has $\{\mathcal{Q}_n^0, \mathcal{Q}_m^0\} = 0$, where the charge $\mathcal{Q}_k^0$ is defined as the integral of the current $\mathscr{K}_k^0(x)$ given by (4.4.20). Recall that the current $\mathscr{J}_n^0(x)$ is constructed as the coefficient of λ^{r_n} in the power series expansion of $\mathscr{T}_n(\lambda, x)$. Similarly, one can rewrite equation (4.4.20) as

$$\mathscr{K}_n^0(x) = \mathcal{W}_n(\lambda, x)\Big|_{\lambda^{r_n}}, \tag{4.4.27}$$

with $\mathcal{W}_n$ defined in equation (4.3.18).

4.4.7 Summary

Let us summarise the results of this section, as we did for non-cyclotomic zeros in subsection 4.3.5. In general, we define a charge $\mathcal{Q}_n^0$, associated with a current $\mathscr{K}_n^0(x)$, by

$$\mathcal{Q}_n^0 = \int \mathrm{d}x\, \mathscr{K}_n^0(x), \qquad \text{with} \qquad \mathscr{K}_n^0(x) = \mathcal{W}_n(\lambda, x)\Big|_{\lambda^{r_n}}, \tag{4.4.28}$$

where the definition of $\mathcal{W}_n(\lambda, x)$ and r_n depends on the type of $\mathfrak{g}$ and whether σ is inner or not.

For $\mathfrak{g}$ of type B, C or D (see subsection 4.4.3) and for $\mathfrak{g}$ of type A and σ inner (see subsection 4.4.4), we simply choose $\mathcal{W}_n(\lambda, x) = \mathscr{T}_n(\lambda, x)$, so that $\mathscr{K}_n^0(x) = \mathscr{J}_n^0(x)$. In this case, we consider r_n as the remainder of the euclidian division of $-n$ by T. When $\mathfrak{g}$ is of type A and σ is not inner (the case discussed in subsections 4.4.5 and 4.4.6), we choose $\mathcal{W}_n(\lambda, x)$ as given by equation (4.3.19). In this case, r_n is defined as the remainder of the euclidian division of $-n\big(1 + \frac{T}{2}\big)$ by T.

The equivariance properties (4.4.7) and (4.4.25) imply that, independently of the type of $\mathfrak{g}$ and of σ being inner or not, we have

$$\mathcal{W}_n(\omega\lambda, x) = \omega^{r_n} \mathcal{W}_n(\lambda, x), \tag{4.4.29}$$

for $\mathcal{W}_n$ defined as above. It therefore follows that the powers of λ appearing in $\mathcal{W}_n(\lambda, x)$ are of the form $r_n + kT$, with $k \in \mathbb{Z}_{\geq 0}$. In particular, the current $\mathscr{K}_n^0(x)$ of equation (4.4.28) is the coefficient of the smallest power of λ in $\mathcal{W}_n(\lambda, x)$.

As in the non-cyclotomic case, we restrict the degree n of the currents $\mathscr{K}_n^0(x)$ to some subset $\mathcal{E}_0$ of $\mathbb{Z}_{\geq 2}$. More precisely, n belongs to $\mathcal{E}_0$ if $n-1$ is an exponent of the affine algebra $\widehat{\mathfrak{g}}$ and r_n is different from $T-1$ (with the exception of the exponents related to the Pfaffian in type D, as in subsection 4.3.5). The results of this section can be summarised as the following theorem.

Theorem 4.4.1. *Let $n, m \in \mathcal{E}_0$. Then the charges $\mathcal{Q}_n^0$ and $\mathcal{Q}_m^0$ are in involution, i.e. we have*

$$\{\mathcal{Q}_n^0, \mathcal{Q}_m^0\} = 0.$$

There is a subtlety in the definition of $\mathcal{W}_n(\lambda, x)$ for $\mathfrak{g}$ of type A and σ inner. Indeed, in this case the current $\mathscr{K}_n^0(x)$ is extracted just from $\mathscr{T}_n(\lambda, x)$ as recalled above. Yet in section 4.3, the current $\mathscr{K}_n^{\lambda_0}$ at a non-cyclotomic point was extracted instead from $\mathcal{W}_n(\lambda, x)$ which differs from $\mathscr{T}_n(\lambda, x)$. Thus, for this case, we choose the appropriate definition of $\mathcal{W}_n(\lambda, x)$ depending on whether the regular zeros of the considered model are cyclotomic or not.

We end this section by an open question. For a non-cyclotomic regular zero λ_0 and $\mathfrak{g}$ of type B, C or D, we considered local charges in involution $\mathcal{Q}_n^{\lambda_0}$ built as the integral of the currents $\mathscr{J}_n^{\lambda_0}(x)$ (see subsection 4.3.2). However, based on the results of [79], we also exhibited a more general family of local charges in involution $\mathcal{Q}_n^{\lambda_0}(\xi)$, depending on a free parameter $\xi \in \mathbb{R}$ and whose corresponding currents $\mathscr{K}_n^{\lambda_0}(\xi, x)$ are constructed as polynomials in the $\mathscr{J}_k^{\lambda_0}$'s.

In the subsection 4.4.3 of the present section, where we deal with a cyclotomic regular zero at the origin for $\mathfrak{g}$ of type B, C or D, the charges $\mathcal{Q}_n^0$ are also constructed simply as integrals of the currents $\mathscr{J}_n^0(x)$. It is thus natural to ask if there exists in this case a more general family of charges $\mathcal{Q}_n^0(\xi)$, depending on a free parameter ξ, as for the non-cyclotomic case. Moreover, for $\mathfrak{g}$ of type A and σ inner (as treated in subsection 4.4.4), the charges $\mathcal{Q}_n^0$ are also integrals of the currents $\mathscr{J}_n^0(x)$ (we do not need to construct more complicated currents to obtain charges in involution). It is thus also natural to look for a more general family of charges $\mathcal{Q}_n^0(\xi)$. This would be an interesting result, as it would exhibit an important qualitative difference between the non-cyclotomic case and the cyclotomic one (with σ inner), for $\mathfrak{g}$ of type A.

We expect these one-parameter families (for $\mathfrak{g}$ of type B, C and D or for $\mathfrak{g}$ of type A with σ inner) to exist. More precisely, we expect them to be given by the first non-zero coefficient in the power series expansions of a suitable generating function, depending on ξ, around the cyclotomic regular zero $\lambda = 0$. As for the non-cyclotomic case, we will focus in this chapter on the local charges which do not depend on a free parameter ξ (as described at the beginning of this subsection), for the same reasons as the ones discussed at the end of subsection 4.3.2 for a non-cyclotomic zero.

4.5 Properties of the local charges

4.5.1 Algebra of local charges in involution

In the previous sections, we constructed a tower of local charges in involution at every regular zero $\lambda_0 \in \mathcal{Z}$. More precisely, we constructed currents $\mathscr{K}_n^{\lambda_0}(x)$, with degrees n in some subset $\mathcal{E}_{\lambda_0}$ of $\mathbb{Z}_{\geq 2}$, such that the charges $\mathcal{Q}_n^{\lambda_0}$ defined as the integral of $\mathscr{K}_n^{\lambda_0}(x)$ are in involution with one another. In this subsection, we study the whole algebra of local charges in involution, formed by all the $\mathcal{Q}_n^{\lambda_0}$ for $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$. More precisely, we prove that currents $\mathscr{K}_n^{\lambda_0}(x)$ and $\mathscr{K}_m^{\mu_0}(y)$ extracted at different regular zeros are in involution. We establish this result for regular zeros in $\mathcal{Z}$, excluding the point at infinity. If infinity is a regular zero then one can also extract charges in involution $\mathcal{Q}_n^{\infty}$, using the results of subsection 4.2.2. The Poisson brackets of the corresponding currents with the currents at finite regular zeros involve some subtleties and will be treated separately at the end of the subsection.

In general, the currents $\mathscr{K}_n^{\lambda_0}(x)$ and $\mathscr{K}_m^{\mu_0}(y)$ are constructed as polynomials of the currents $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\mu_0}(y)$. It is therefore sufficient to prove that the Poisson bracket of $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\mu_0}(y)$ is zero. The currents $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\mu_0}(y)$ are extracted from $\mathscr{T}_n(\lambda, x)$ and $\mathscr{T}_m(\mu, y)$, whose Poisson bracket is given by equation (4.2.17). We can suppose that μ_0 is different from 0 and thus is a non-cyclotomic point, so that $\mathscr{J}_m^{\mu_0}(y) = \mathscr{T}_m(\mu_0, y)$. Using the fact that $U_{\underline{\mathbf{12}}}(\lambda, \mu_0) = \varphi(\lambda)\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu_0)$ since $\varphi(\mu_0) = 0$, we can evaluate equation (4.2.17) at $\mu = \mu_0$ to find

$$\{\mathscr{T}_n(\lambda, x), \mathscr{J}_m^{\mu_0}(y)\} = -nm\,\varphi(\lambda)\mathrm{Tr}_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu_0)S_{n-1}(\lambda, x)_{\underline{\mathbf{1}}}S_{m-1}(\mu_0, y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}. \tag{4.5.1}$$

We will now treat separately the cases λ_0 cyclotomic or λ_0 non-cyclotomic.

Suppose that λ_0 is non-cyclotomic so that $\mathscr{J}_n^{\lambda_0}(x)$ is simply the evaluation of $\mathscr{T}_n(\lambda, x)$ at $\lambda = \lambda_0$. Recall from paragraph 4.2.1 that, by construction of the set $\mathcal{Z}$, as λ_0 and μ_0 are different elements of $\mathcal{Z}$, the cyclotomic orbits $\mathbb{Z}_T\lambda_0$ and $\mathbb{Z}_T\mu_0$ are disjoint. In particular, this means that $\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu_0)$ is regular at $\lambda = \lambda_0$. Indeed, by equation (2.3.8) the poles of $\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu_0)$ are the points of the orbit $\mathbb{Z}_T\mu_0$. Moreover, $S_{n-1}(\lambda, x)$ is regular at $\lambda = \lambda_0$ and we have $\varphi(\lambda_0) = 0$. Thus, evaluating equation (4.5.1) at $\lambda = \lambda_0$ we find that the currents $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\mu_0}(y)$ are in involution, as expected.

Let us now treat the cyclotomic case where $\lambda_0 = 0$. In this case, $\mathscr{J}_n^0(x)$ is the coefficient of λ^{r_n} in the power series expansion of $\mathscr{T}_n(\lambda, x)$ (cf. subsection 4.4.1). The Poisson bracket of $\mathscr{J}_n^0(x)$ with $\mathscr{J}_m^{\mu_0}(y)$ is then the coefficient of λ^{r_n} in equation (4.5.1). Recall from section 4.4 that for $n \in \mathcal{E}_0$, we have $r_n < T - 1$. Yet, in equation (4.5.1), $\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu_0)$ and $S_{n-1}(\lambda, x)$ are regular at $\lambda = 0$ and $\varphi(\lambda) = O(\lambda^{T-1})$, hence the involution of $\mathscr{J}_n^0(x)$ and $\mathscr{J}_m^{\mu_0}(y)$. In conclusion, we have proved

Theorem 4.5.1. *Let $\lambda_0, \mu_0 \in \mathcal{Z}$ and let $n \in \mathcal{E}_{\lambda_0}$ and $m \in \mathcal{E}_{\mu_0}$. Then, if $\lambda_0 \neq \mu_0$, we have*

$$\Big\{ \mathscr{J}_n^{\lambda_0}(x), \mathscr{J}_m^{\mu_0}(y) \Big\} = 0.$$

Combining this theorem with the results of previous sections, we conclude that the local charges $\mathcal{Q}_n^{\lambda_0}$, for all $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$, are in involution with one another.

We now turn to the case where one of the regular zeros is the point at infinity. In this case, the current $\mathscr{J}_m^{\infty}(y)$ is extracted from the Lax matrix $\mathcal{L}^{\infty}(\alpha, y)$. From the Poisson brackets (2.2.6) and (4.2.5), we get

$$\begin{aligned} \big\{ \mathcal{L}(\lambda,x)_{\underline{\mathbf{1}}}, \mathcal{L}^{\infty}(\alpha,y)_{\underline{\mathbf{2}}} \big\} = & \Big[\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda,\alpha^{-1}), \mathcal{L}(\lambda,x)_{\underline{\mathbf{1}}} \Big] \delta_{xy} - \big[\mathcal{R}_{\underline{\mathbf{21}}}(\alpha^{-1},\lambda), \mathcal{L}^{\infty}(\alpha,y)_{\underline{\mathbf{2}}} \big] \delta_{xy} \\ & - \big(\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda,\alpha^{-1}) + \mathcal{R}_{\underline{\mathbf{21}}}(\alpha^{-1},\lambda) \big) \delta'_{xy} + \alpha^{-1} \psi(\alpha)^{-1} \big[\mathcal{R}_{\underline{\mathbf{21}}}(\alpha^{-1},\lambda), \mathcal{C}(x)_{\underline{\mathbf{2}}} \big] \delta_{xy}, \end{aligned} \tag{4.5.2}$$

with the matrix $\widetilde{\mathcal{R}}$ defined in (4.2.8).

In the following, we will say that an equation is true *weakly*, and we will then use the symbol $\approx$ instead of $=$, if the equation holds when one puts the field $\mathcal{C}(x)$ to zero. This denomination and its interest will be made clear when studying $\mathbb{Z}_T$-coset models, in which case the field $\mathcal{C}(x)$ is interpreted as a gauge constraint. Note, in particular, that the last term of equation (4.5.2) vanishes weakly. Using Corollary 4.2.3, we may then compute the Poisson brackets of $\mathscr{T}_n(\lambda, x)$ with

$$\mathscr{T}_m^{\infty}(\alpha, y) = \mathrm{Tr}\big(S_n^{\infty}(\alpha, x)\big) = \mathrm{Tr}\big(\psi(\alpha)^n \mathcal{L}^{\infty}(\alpha, x)^n\big)$$

weakly, to find

$$\{ \mathscr{T}_n(\lambda,x), \mathscr{T}_m^{\infty}(\alpha,y) \} \approx -nm\, \mathrm{Tr}_{\underline{\mathbf{12}}}\Big(V_{\underline{\mathbf{12}}}(\lambda,\alpha) S_{n-1}(\lambda,x)_{\underline{\mathbf{1}}} S_{m-1}^{\infty}(\alpha,y)_{\underline{\mathbf{2}}} \Big) \delta'_{xy}, \tag{4.5.3}$$

where

$$V_{\underline{\mathbf{12}}}(\lambda,\alpha) = -\alpha^{-2} \varphi(\lambda) \widetilde{\mathcal{R}}^0_{\underline{\mathbf{12}}}(\lambda,\alpha^{-1}) + \psi(\alpha) \mathcal{R}^0_{\underline{\mathbf{21}}}(\alpha^{-1},\lambda).$$

Suppose first that $\lambda_0 \in \mathcal{Z}$ is non-cyclotomic. We have $\varphi(\lambda_0) = 0$, and hence

$$\Big\{ \mathscr{J}_n^{\lambda_0}(x), \mathscr{T}_m^{\infty}(\alpha,y) \Big\} \approx -nm\, \psi(\alpha) \mathrm{Tr}_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{21}}}(\alpha^{-1},\lambda_0) S_{n-1}(\lambda_0,x)_{\underline{\mathbf{1}}} S_{m-1}^{\infty}(\alpha,y)_{\underline{\mathbf{2}}} \Big) \delta'_{xy}.$$

The Poisson bracket between $\mathscr{J}_m^{\lambda_0}(x)$ and $\mathscr{J}_m^{\infty}(y)$ is then obtained, weakly, by extracting the coefficient of α^{r_m} in the equation above. For $m \in \mathcal{E}_{\infty}$, we have $r_m < T-1$. Yet $\psi(\alpha) = O(\alpha^{T-1})$ and $\mathcal{R}^0_{\underline{\mathbf{21}}}(\alpha^{-1}, \lambda_0)$ and $S_{m-1}^{\infty}(\alpha, y)$ are regular at $\alpha = 0$. Thus $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_m^{\infty}(y)$ are weakly in involution.

It remains to consider the case where $\lambda_0 = 0$. In this case, $\mathscr{J}_n^0(x)$ is the coefficient of λ^{r_n} in $\mathscr{T}_n(\lambda, x)$ and we restrict to n such that $r_n < T-1$. We have to find the coefficient of $\lambda^{r_n}\alpha^{r_m}$ in equation (4.5.3). Due to the presence of $\varphi(\lambda)$ or $\psi(\alpha)$ in the two terms appearing in $V_{\underline{\mathbf{12}}}(\lambda,\alpha)$, we see that either the power of λ or the power of α is greater than $T-1$ and thus cannot contribute to the term $\lambda^{r_n}\alpha^{r_m}$. In conclusion, we have the following theorem.

Theorem 4.5.2. *Suppose that infinity is a regular zero of the model. Let $\lambda_0 \in \mathcal{Z}$, $n \in \mathcal{E}_{\lambda_0}$ and $m \in \mathcal{E}_{\infty}$. Then we have*

$$\Big\{ \mathscr{J}_n^{\lambda_0}(x), \mathscr{J}_m^{\infty}(y) \Big\} \approx 0.$$

Combining this theorem with the results of previous sections, we see that the local charges $\mathcal{Q}_n^{\lambda_0}$, for all $\lambda_0 \in \mathcal{Z} \cup \{\infty\}$ and $n \in \mathcal{E}_{\lambda_0}$, are (at least weakly) in involution with one another.

We thus constructed a large algebra of local charges in involution, composed of the charges $\mathcal{Q}_n^{\lambda_0}$, with λ_0 regular zeros and $n \in \mathcal{E}_{\lambda_0}$. Since these charges are local, they are also in involution with the momentum $\mathcal{P}$ of the theory, whose Poisson bracket generates the derivative with respect to the spatial coordinate x. We have not yet discussed the conservation properties of these charges. For the models that we will consider as examples in this chapter, we will see in section 4.7 that the Hamiltonian $\mathcal{H}$ of the theory always belongs to the algebra of local charges described above. It therefore follows that all these charges are conserved. More precisely, we will see that $\mathcal{H}$ is always a linear combination of the quadratic charges $\mathcal{Q}_2^{\lambda_0}$ and the momentum $\mathcal{P}$.

4.5.2 Gauge invariance

In this subsection, we anticipate the application of the construction developed in the previous sections to integrable σ-models on $\mathbb{Z}_T$-coset spaces. In those models, infinity is a regular zero and the corresponding field $\mathcal{C}(x)$ (cf. subsection 4.2.2) is a gauge constraint. We prove here that the currents $\mathscr{K}_n^{\lambda_0}(x)$ constructed at regular zeros λ_0 in the previous sections are gauge invariant, in the sense that they Poisson commute with the constraint $\mathcal{C}(y)$. As the $\mathscr{K}_n^{\lambda_0}$'s are polynomials of the $\mathscr{J}_n^{\lambda_0}$'s, it is enough to prove the following theorem.

Theorem 4.5.3. *Suppose that infinity is a regular zero. Let $\lambda_0 \in \mathcal{Z} \cup \{\infty\}$ and $n \in \mathcal{E}_{\lambda_0}$. Then, we have*

$$\Big\{ \mathscr{J}_n^{\lambda_0}(x), \mathcal{C}(y) \Big\} = 0.$$

Proof. Let us first suppose that λ_0 is different from infinity. The current $\mathscr{J}_n^{\lambda_0}(x)$ is extracted from $\mathscr{T}_n(\lambda, x) = \varphi(\lambda)^n \mathcal{L}(\lambda, x)^n$. Recall the Poisson bracket between the Lax matrix $\mathcal{L}(\lambda, x)$ and $\mathcal{C}(y)$, given by equation (4.2.5). By Corollary 4.2.3 we then have

$$\{\mathscr{T}_n(\lambda, x), \mathcal{C}(y)\} = -n\,\varphi(\lambda)\,\mathrm{Tr}_{\underline{\mathbf{1}}}\big(C^{(0)}_{\underline{\mathbf{12}}} S_{n-1}(\lambda, x)_{\underline{\mathbf{1}}}\big)\delta'_{xy}.$$

If $\lambda_0 \neq 0$, $\mathscr{J}_n^{\lambda_0}(x)$ is simply $\mathscr{T}_n(\lambda_0, x)$. And since λ_0 is a regular zero, $S_{n-1}(\lambda, x)$ is regular at $\lambda = \lambda_0$ and $\varphi(\lambda_0) = 0$. Evaluating the above Poisson bracket at $\lambda = \lambda_0$, we get the involution of $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathcal{C}(y)$. If $\lambda_0 = 0$, $\mathscr{J}_n^0(x)$ is the coefficient of λ^{r_n} in $\mathscr{T}_n(\lambda, x)$ and, since $n \in \mathcal{E}_0$, we have $r_n < T - 1$. Moreover, since $S_{n-1}(\lambda, x)$ is regular at $\lambda = 0$ and $\varphi(\lambda) = O(\lambda^{T-1})$, the λ^{r_n}-term in the Poisson bracket above is then zero, as required.

Finally, let us treat the case $\lambda_0 = \infty$, for which $\mathscr{J}_n^{\infty}(x)$ is given by the coefficient of α^{r_n} in $\mathscr{T}_n^{\infty}(\alpha, x) = \mathrm{Tr}\big(\psi(\alpha)^n \mathcal{L}^{\infty}(\alpha, x)^n\big)$. Using the definition of $\mathcal{L}^{\infty}$ and the Poisson brackets (4.2.5) and (4.2.6), we find

$$\big\{\mathcal{L}^{\infty}(\alpha, x)_{\underline{\mathbf{1}}}, \mathcal{C}(y)_{\underline{\mathbf{2}}}\big\} = \big[C^{(0)}_{\underline{\mathbf{12}}}, \mathcal{L}^{\infty}(\alpha, x)_{\underline{\mathbf{1}}}\big]\delta_{xy} - C^{(0)}_{\underline{\mathbf{12}}}\delta'_{xy}.$$

This bracket has the same structure as equation (4.2.5). Therefore, the case $\lambda_0 = \infty$ is treated exactly in the same way than the case $\lambda_0 = 0$, which ends the proof. □

4.5.3 Reality conditions

To close this section let us discuss the reality conditions on the charges $\mathcal{Q}_n^{\lambda_0}$ extracted at regular zeros in the previous sections. In Chapter 3, we considered integrable σ-models with target space G_0 or a quotient of G_0, where G_0 is a real Lie group. If $\mathfrak{g}_0$ is the Lie algebra of G_0, then the Lax matrix of the model is a $\mathfrak{g}$-valued field, where $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$. In other words, $\mathfrak{g}_0$ is a real form of the complex Lie algebra $\mathfrak{g}$: it is thus characterised by a antilinear involutive automorphism τ (see Appendix A.3). The fact that the σ-models we consider are on the real form G_0 (or one of its quotient) is encoded in the reality conditions (2.4.7) and (2.4.10) of the Lax matrix and of the twist function.

In particular, by equation (2.4.10), if λ_0 is a zero of φ, its conjugate $\bar{\lambda}_0$ is also a zero of φ. Combining the reality conditions (2.4.7) and (2.4.10), we also see that if λ_0 is a regular zero (see paragraph 4.2.1), $\bar{\lambda}_0$ is also a regular zero. Thus, the regular zeros can be of two types: real ones $\lambda_0 \in \mathbb{R}$ and conjugate pairs λ_0, $\bar{\lambda}_0$ in $\mathbb{C} \setminus \mathbb{R}$.

We will use the reality condition (2.4.7) in a similar way to the way we used the equivariance property (2.4.3) in subsection 4.4.1. In particular, as we consider powers of the Lax matrix, which are not in the Lie algebra $\mathfrak{g}$ in general, we will need to extend "naturally" the automorphism τ to the whole algebra F of matrices acting on the defining representation of $\mathfrak{g}$. This was done for the automorphism σ in subsection 4.4.1 and appendix 4.A. One can apply similar ideas to τ, using the classification of real forms of the classical Lie algebras A, B, C and D. We do not present the details here and just summarise the results.

There exists an extension of τ on the whole algebra of matrices F, which coincides with τ when restricted to the Lie algebra $\mathfrak{g}$, and that we shall still denote τ. This extension is still an involutive

semi-linear map of F to itself. However, it is not in general an algebra homomorphism. The main properties of the extension τ that we will need are the following. There exists $\gamma \in \{1, -1\}$ such that

$$\tau(Z^n) = \gamma^{n-1}\,\tau(Z)^n, \tag{4.5.4a}$$

$$\mathrm{Tr}\big(\tau(Z)\big) = \gamma\,\overline{\mathrm{Tr}(Z)}, \tag{4.5.4b}$$

for any $Z \in F$. For every real form $\mathfrak{g}_0$ of a classical algebra $\mathfrak{g}$ we have $\gamma = 1$, except for the real forms $\mathfrak{su}(p,q,\mathbb{R})$ of $\mathfrak{sl}(d,\mathbb{C})$ (with $p+q=d$), for which $\gamma = -1$. Using the properties (4.5.4) with the reality conditions (2.4.7) and (2.4.10), one finds that $\tau\big(S_n(\lambda,x)\big) = \gamma^{n-1}S_n(\bar\lambda,x)$ and that

$$\overline{\mathscr{T}_n(\lambda,x)} = \gamma^n\,\mathscr{T}_n(\bar\lambda,x). \tag{4.5.5}$$

Consider a regular zero λ_0. Suppose first that λ_0 is complex: its conjugate $\bar\lambda_0$ is then also a regular zero. According to the previous sections, we can extract two towers of (possibly complex) currents $\mathscr{J}_n^{\lambda_0}(x)$ and $\mathscr{J}_n^{\bar\lambda_0}(x)$ by evaluating $\mathscr{T}_n(\lambda,x)$ at $\lambda=\lambda_0$ or $\lambda=\bar\lambda_0$ (note that λ_0 cannot be a cyclotomic point as it is complex). However, according to equation (4.5.5), these currents are not independent. Indeed, they are related by the reality condition

$$\mathscr{J}_n^{\bar\lambda_0}(x) = \gamma^n\,\overline{\mathscr{J}_n^{\lambda_0}(x)}.$$

Thus, considering linear combination of $\mathcal{Q}_n^{\lambda_0}$ and $\mathcal{Q}_n^{\bar\lambda_0}$, we extract from each pair λ_0, $\bar\lambda_0$ of complex regular zeros two towers of real charges in involution: $\mathcal{Q}_n^{\lambda_0} + \gamma^n\mathcal{Q}_n^{\bar\lambda_0}$ and $i\big(\mathcal{Q}_n^{\lambda_0} - \gamma^n\mathcal{Q}_n^{\bar\lambda_0}\big)$.

Suppose now that λ_0 is a real and non-cyclotomic regular zero. Equation (4.5.5) then imposes the reality condition

$$\mathscr{J}_n^{\lambda_0}(x) = \gamma^n\,\overline{\mathscr{J}_n^{\lambda_0}(x)}. \tag{4.5.6}$$

Thus, the current $\mathscr{J}_n^{\lambda_0}$ is either real or pure imaginary. In each case, we can extract only one tower of real local charges. Consider now the case where λ_0 is the origin and thus a cyclotomic real point. The current $\mathscr{J}_n^0(x)$ is then the coefficient of λ^{r_n} in the power series expansion of $\mathscr{T}_n(\lambda,x)$. Yet, this coefficient is also the one of $\bar\lambda^{r_n}$ in the power series expansion of $\mathscr{T}_n(\bar\lambda,x)$. The reality condition (4.5.5) then implies that equation (4.5.6) also holds for $\lambda_0 = 0$.

Finally, let us discuss the case where λ_0 is infinity, which we consider as a real point. From the reality conditions (2.4.7) and (2.4.10), we find that the field $\mathcal{C}(x)$ defined in subsection 4.2.2 is real, in the sense that $\tau\big(\mathcal{C}(x)\big) = \mathcal{C}(x)$. We then obtain reality conditions on the Lax matrix $\mathcal{L}^\infty(\alpha,x)$ and the twist function $\psi(\alpha)$ similar to equations (2.4.7) and (2.4.10). As a result we can apply the above discussion, since the point at infinity in the variable λ corresponds to the origin in the variable α, and conclude that equation (4.5.6) also holds for $\lambda_0 = \infty$.

To summarise this subsection, we have shown that one can extract:

- one tower of real local charges for each real regular zero λ_0,
- two towers of real local charges for each pair λ_0, $\bar\lambda_0$ of complex regular zeros.

In other words, one can extract as many towers of real charges as there are regular zeros.

4.6 Integrable hierarchies and zero curvature equations

In the previous sections, we constructed a infinite set of local charges $\mathcal{Q}_n^{\lambda_0}$ in involution, with λ_0 regular zeros. It induces an infinite set of commuting Hamiltonian flows, defined by $\big\{\mathcal{Q}_n^{\lambda_0},\cdot\big\}$. In this section, we show that these flows generate a hierarchy of integrable equations. More precisely, we associate with each charge $\mathcal{Q}_n^{\lambda_0}$ a connection

$$\nabla_n^{\lambda_0} = \Big\{\mathcal{Q}_n^{\lambda_0},\cdot\Big\} + \mathcal{M}_n^{\lambda_0}(\lambda,x)$$

which commutes with the connection $\nabla_x = \partial_x + \mathcal{L}(\lambda, x)$. We show that the connections $\nabla_n^{\lambda_0}$ also commute with one another for finite regular zeros λ_0. The commutativity of these connections takes the form of zero curvature equations. In particular, we will use the zero curvature equations involving $\mathcal{L}(\lambda, x)$ and the $\mathcal{M}_n^{\lambda_0}(\lambda, x)$'s to prove that the local charges $\mathcal{Q}_n^{\lambda_0}$ are in involution with the non-local charges extracted from the monodromy of $\mathcal{L}(\lambda, x)$.

4.6.1 Zero curvature equations with $\mathcal{L}$

The starting point of this chapter is an integrable system with Lax matrix $\mathcal{L}(\lambda, x)$ and Hamiltonian $\mathcal{H}$. The dynamical equations of this system are generated by the Poisson bracket with $\mathcal{H}$. They are encoded in the form of the zero curvature equation (2.2.2). In this subsection, we study the dynamics of the Lax matrix under the Hamiltonian flows generated by the local charges $\mathcal{Q}_n^{\lambda_0}$ constructed in the previous sections. More precisely, we show that these dynamics also take the form of a zero curvature equation on $\mathcal{L}(\lambda, x)$:

Theorem 4.6.1. *Let $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$. There exists a matrix $\mathcal{M}_n^{\lambda_0}(\lambda, x)$ such that we have the zero curvature equation*

$$\Big\{\mathcal{Q}_n^{\lambda_0}, \mathcal{L}(\lambda, x)\Big\} - \partial_x \mathcal{M}_n^{\lambda_0}(\lambda, x) + \Big[\mathcal{M}_n^{\lambda_0}(\lambda, x), \mathcal{L}(\lambda, x)\Big] = 0.$$

Proof. Let us apply the second result of Corollary 4.2.3 to the Maillet bracket (2.2.6). Using the form (2.4.1) of the $\mathcal{R}$-matrix, we find

$$\begin{aligned}\{\mathcal{L}(\lambda, x), \mathscr{T}_n(\mu, y)\} = n\left[\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) S_{n-1}(\mu, y)_{\underline{\mathbf{2}}}\Big), \mathcal{L}(\lambda, x)\right]\delta_{xy} \\ - n\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) S_{n-1}(\mu, y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy} - n\frac{\varphi(\mu)}{\varphi(\lambda)}\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu, \lambda) S_{n-1}(\mu, y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}.\end{aligned} \tag{4.6.1}$$

Consider first the case where λ_0 is a non-cylotomic regular zero. Evaluating the equation above at $\mu = \lambda_0$ and using $\varphi(\lambda_0) = 0$, we have

$$\Big\{\mathcal{L}(\lambda, x), \mathscr{J}_n^{\lambda_0}(y)\Big\} = \Big[\mathcal{N}_n^{\lambda_0}(\lambda, y), \mathcal{L}(\lambda, x)\Big]\,\delta_{xy} - \mathcal{N}_n^{\lambda_0}(\lambda, y)\delta'_{xy}, \tag{4.6.2}$$

where

$$\mathcal{N}_n^{\lambda_0}(\lambda, x) = n\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \lambda_0) S_{n-1}(\lambda_0, x)_{\underline{\mathbf{2}}}\Big). \tag{4.6.3}$$

Suppose now that λ_0 is the origin, which is a cyclotomic point, in which case $\mathscr{J}_n^0(y)$ is constructed as the coefficient of μ^{r_n} in the power series expansion of $\mathscr{T}_n(\mu, y)$. Moreover, as $n \in \mathcal{E}_0$, we have $r_n < T - 1$ (see section 4.4). The Poisson bracket $\big\{\mathcal{L}(\lambda, x), \mathscr{J}_n^0(y)\big\}$ is thus the μ^{r_n}-term in equation (4.6.1). We have $\varphi(\mu) = O(\mu^{T-1})$ and $r_n < T - 1$, thus the last term of equation (4.6.1) cannot contribute to μ^{r_n}. Thus, we also have equation (4.6.2) for $\lambda_0 = 0$, with

$$\mathcal{N}_n^0(\lambda, x) = n\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) S_{n-1}(\mu, x)_{\underline{\mathbf{2}}}\Big)\Big|_{\mu^{r_n}}.$$

We will say that $\mathcal{N}_n^{\lambda_0}$ is the Lax matrix associated with the charge defined as the integral of the current $\mathscr{J}_n^{\lambda_0}$. Equation (4.6.2) implies a zero curvature equation for the evolution of $\mathcal{L}(\lambda, x)$ under the Hamiltonian flow of this charge. In general, the charge $\mathcal{Q}_n^{\lambda_0}$ is not the integral of $\mathscr{J}_n^{\lambda_0}$ but of $\mathscr{K}_n^{\lambda_0}$ (see previous sections). Recall that $\mathscr{K}_n^{\lambda_0}$ is a polynomial in the $\mathscr{J}_m^{\lambda_0}$'s. We construct the Lax matrix $\mathcal{M}_n^{\lambda_0}(\lambda, x)$ associated with $\mathcal{Q}_n^{\lambda_0}$ by assigning any monomial $\mathscr{J}_{m_1}^{\lambda_0} \dots \mathscr{J}_{m_p}^{\lambda_0}$ in this polynomial to the matrix

$$\sum_{k=1}^{p}\Big(\prod_{j\neq k} \mathscr{J}_{m_j}^{\lambda_0}(x)\Big)\mathcal{N}_{m_k}^{\lambda_0}(\lambda, x).$$

Using the fact that the Poisson bracket is a derivation, we find from equation (4.6.2) that

$$\Big\{\mathcal{L}(\lambda, x), \mathscr{K}_n^{\lambda_0}(y)\Big\} = \Big[\mathcal{M}_n^{\lambda_0}(\lambda, y), \mathcal{L}(\lambda, x)\Big]\,\delta_{xy} - \mathcal{M}_n^{\lambda_0}(\lambda, y)\delta'_{xy}. \tag{4.6.4}$$

After integration over y, we get the required zero curvature equation. □

Thus, the Hamiltonian flows of the charges $\mathcal{Q}_n^{\lambda_0}$ generate dynamical equations that can be recast in the form of zero curvature equations. In conclusion, we have constructed a hierarchy of integrable systems with Lax matrix $\mathcal{L}(\lambda,x)$ and Hamiltonians $\mathcal{Q}_n^{\lambda_0}$. The zero curvature equations of Theorem 4.6.1 can be seen as the commutativity of the connections

$$\nabla_n^{\lambda_0} = \Big\{\mathcal{Q}_n^{\lambda_0},\cdot\Big\} + \mathcal{M}_n^{\lambda_0}(\lambda,x) \tag{4.6.5}$$

with $\nabla_x = \partial_x + \mathcal{L}(\lambda,x)$. This connection ∇_x can be thought of as the connection associated with the local momentum $\mathcal{P}$ of the theory. As already mentioned, we will see in section 4.7 that for the models we consider, the Hamiltonian is given by a linear combination $\mathcal{H} = \sum_{\lambda_0\in\mathcal{Z}} a_{\lambda_0}\mathcal{Q}_2^{\lambda_0} + b\mathcal{P}$ of the quadratic charges $\mathcal{Q}_2^{\lambda_0}$ and the momentum $\mathcal{P}$. Therefore, the matrix $\mathcal{M}(\lambda,x)$ of equation (2.2.2) can be constructed as $\sum_{\lambda_0\in\mathcal{Z}} a_{\lambda_0}\mathcal{M}_2^{\lambda_0}(\lambda,x) + b\mathcal{L}(\lambda,x)$.

Theorem 4.6.1 only treats the case of finite regular zeros λ_0. Let us also briefly discuss what happens when $\lambda_0=\infty$. In this case, $\mathscr{J}_n^{\infty}(x)$ is extracted from the Lax matrix $\mathcal{L}^{\infty}(\alpha,x)$. Since this matrix satisfies a Maillet bracket with twist function $\psi(\alpha)$, one can apply the method developed here. Doing so we find that the dynamics of $\mathcal{L}^{\infty}(\alpha,x)$ under the Hamiltonian flow of $\mathcal{Q}_n^{\infty}$ takes the form of a zero curvature equation. Moreover, starting with the Poisson bracket (4.5.2) and working weakly, we also find a weak curvature equation

$$\{\mathcal{Q}_n^{\infty},\mathcal{L}(\lambda,x)\} - \partial_x\mathcal{M}_n^{\infty}(\lambda,x) + [\mathcal{M}_n^{\infty}(\lambda,x),\mathcal{L}(\lambda,x)] \approx 0,$$

where the matrix $\mathcal{M}_n^{\infty}(\lambda,x)$ is constructed from

$$\mathcal{N}_n^{\infty}(\lambda,x) = -n\,\alpha^{-2}\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\widetilde{\mathcal{R}}^0_{\underline{\mathbf{12}}}(\lambda,\alpha^{-1})S^{\infty}_{n-1}(\alpha,x)_{\underline{\mathbf{2}}}\Big)\Big|_{\alpha^{r_n}}$$

in the same way as $\mathcal{M}_n^{\lambda_0}(\lambda,x)$ was built from $\mathcal{N}_n^{\lambda_0}(\lambda,x)$ for a finite regular zero λ_0. In other words, Theorem 4.6.1 also applies for $\lambda_0=\infty$ when Poisson brackets are considered weakly.

Let us end this subsection by stating a few properties of the Lax matrix $\mathcal{M}_n^{\lambda_0}(\lambda,x)$. Using the equivariance property (2.4.2), we find that

$$\sigma\Big(\mathcal{M}_n^{\lambda_0}(\lambda,x)\Big) = \mathcal{M}_n^{\lambda_0}(\omega\lambda,x).$$

The Lax matrix $\mathcal{M}_n^{\lambda_0}$ thus satisfies the same equivariance property (2.4.3) as $\mathcal{L}$. Recall that the Lax matrix $\mathcal{N}_n^0(\lambda,x)$ is extracted as the μ^{r_n}-term in

$$\mathcal{N}_n(\mu\,;\lambda,x) = n\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)S_{n-1}(\mu,x)_{\underline{\mathbf{2}}}\Big). \tag{4.6.6}$$

Consider the equivariance properties (4.4.3) and

$$\sigma_{\mathbf{2}}\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \omega\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\omega\mu).$$

Combining it with the fact that $\mathrm{Tr}\big(\sigma(Y)\sigma(Z)\big) = \mathrm{Tr}(YZ)$ for any matrices $Y,Z\in F$ (see appendix 4.A), we find that

$$\mathcal{N}_n(\omega\mu\,;\lambda,x) = \omega^{r_n}\mathcal{N}_n(\mu\,;\lambda,x). \tag{4.6.7}$$

Therefore, the power series expansion of $\mathcal{N}_n(\mu\,;\lambda,x)$ in μ contains powers of the form r_n+kT, with $k\in\mathbb{Z}_{\geq 0}$. In particular, $\mathcal{N}_n^0(\lambda,x)$ is the coefficient of the smallest power in this expansion, in the same way as $\mathscr{J}_n^0(x)$ is in the expansion of $\mathscr{T}_n(\mu,x)$.

Let us define $\mathcal{M}_n(\mu\,;\lambda,x)$ from $\mathcal{N}_n(\mu\,;\lambda,x)$ and $\mathscr{T}_n(\mu,x)$ in the same way we constructed $\mathcal{M}_n^{\lambda_0}(\lambda,x)$ from $\mathcal{N}_n^{\lambda_0}(\lambda,x)$ and $\mathscr{J}_n^{\lambda_0}(x)$. In particular, $\mathcal{M}_n^{\lambda_0}(\lambda,x)$ is the evaluation of $\mathcal{M}_n(\mu\,;\lambda,x)$ at $\mu=\lambda_0$. From equations (4.4.6) and (4.6.7), we find the following equivariance property

$$\mathcal{M}_n(\omega\mu\,;\lambda,x) = \omega^{r_n}\mathcal{M}_n(\mu\,;\lambda,x). \tag{4.6.8}$$

So $\mathcal{M}_n^0(\lambda,x)$ is the coefficient of the first term μ^{r_n} in the power series expansion of $\mathcal{M}_n(\mu\,;\lambda,x)$.

4.6.2 Involution with non-local charges

In this subsection, we use the result of the previous one to prove that the local charges $\mathcal{Q}_n^{\lambda_0}$ are in involution with the non-local charges extracted from the monodromy of the Lax matrix $\mathcal{L}(\lambda, x)$. This monodromy is defined as the path-ordered exponential (see Appendix A.5)

$$T(\lambda) = \overleftarrow{\mathrm{Pexp}}\left(-\int \mathrm{d}z\, \mathcal{L}(\lambda, z)\right),$$

where the integral is taken on the real line $\mathbb{R}$ or the circle S^1, depending on the coordinate space of the model. Consider also the partial transfer matrices

$$T(\lambda\,; x, y) = \overleftarrow{\mathrm{Pexp}}\left(-\int_y^x \mathrm{d}z\, \mathcal{L}(\lambda, z)\right).$$

The properties of these matrices are described in Appendix A.5. In particular, their variation under an infinitesimal variation of $\mathcal{L}$ is given by equation (A.5.2).

This formula allows one to compute derivatives of T and in particular its Poisson bracket with the local charge $\mathcal{Q}_n^{\lambda_0}$, for $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$. Specifically, we have

$$\left\{\mathcal{Q}_n^{\lambda_0}, T(\lambda\,; x, y)\right\} = -\int_y^x \mathrm{d}z\, T(\lambda\,; x, z)\left\{\mathcal{Q}_n^{\lambda_0}, \mathcal{L}(\lambda, z)\right\} T(\lambda\,; z, y). \tag{4.6.9}$$

The Poisson bracket of $\mathcal{Q}_n^{\lambda_0}$ and $\mathcal{L}(\lambda, z)$ is given by Theorem 4.6.1. Using this together with the equations (A.5.1) and (4.6.9), we find

$$\left\{\mathcal{Q}_n^{\lambda_0}, T(\lambda\,; x, y)\right\} = T(\lambda\,; x, y)\mathcal{M}_n^{\lambda_0}(\lambda, y) - \mathcal{M}_n^{\lambda_0}(\lambda, x)T(\lambda\,; x, y).$$

If the spatial coordinate is taken on the real line (from $-\infty$ to ∞) and the fields are assumed to be decreasing at infinity fast enough, we get

$$\left\{\mathcal{Q}_n^{\lambda_0}, T(\lambda)\right\} = 0,$$

i.e. the whole monodromy $T(\lambda)$ is in involution with $\mathcal{Q}_n^{\lambda_0}$. If the spatial coordinate is taken on the circle (from 0 to 2π) and the fields are assumed to be periodic, we get

$$\left\{\mathcal{Q}_n^{\lambda_0}, T(\lambda)\right\} = \left[T(\lambda), \mathcal{M}_n^{\lambda_0}(\lambda, 0)\right].$$

In this case, $\mathcal{Q}_n^{\lambda_0}$ Poisson commutes with any central function of $T(\lambda)$, *e.g.* the traces $\mathrm{Tr}\big(T(\lambda)^k\big)$ and the determinant $\det\big(T(\lambda)\big)$. Thus, we have

Theorem 4.6.2. *The monodromy $T(\lambda)$ (resp. the central functions of $T(\lambda)$) is in involution with the local charges $\mathcal{Q}_n^{\lambda_0}$ for $\lambda_0 \in \mathcal{Z}$ and $n \in \mathcal{E}_{\lambda_0}$, if the spatial coordinate is taken on the real line (resp. the circle). In particular, it is conserved.*

Proof. It just remains to prove the conservation of the non-local charges. This follows from the fact that the Hamiltonian $\mathcal{H}$ can be expressed as a linear combination of the quadratic charges $\mathcal{Q}_2^{\lambda_0}$ and the momentum $\mathcal{P}$. □

Once again, this theorem applies only for finite regular zeros λ_0. Following a similar argument to the one given in the previous subsection, it also holds for the charges $\mathcal{Q}_n^{\infty}$ if we consider Poisson brackets only weakly.

4.6.3 Zero curvature equations between the $\mathcal{M}_n^{\lambda_0}$'s

In subsection 4.6.1, we showed that the dynamics of the Lax matrix $\mathcal{L}(\lambda, x)$ under the Hamiltonian flow of the local charge $\mathcal{Q}_n^{\lambda_0}$ takes the form of a zero curvature equation with a matrix $\mathcal{M}_n^{\lambda_0}(\lambda, x)$. We thus exhibited a hierarchy of integrable equations, corresponding to the commutativity of the connections $\nabla_n^{\lambda_0}$ with ∇_x. This can be seen as the compatibility condition of the two auxiliary linear problems $\nabla_x \Psi = 0$ and $\nabla_n^{\lambda_0}\Psi = 0$, with Ψ a function on the phase space, valued in the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. In this subsection, we prove that the connections $\nabla_n^{\lambda_0}$ and $\nabla_m^{\mu_0}$ also commute with one another (except when λ_0 is finite and $\mu_0 = \infty$). This can be seen as the simultaneous compatibility of all auxiliary linear problems $\nabla_n^{\lambda_0}\Psi = 0$ and it takes the form of zero curvature equations:

Theorem 4.6.3. *Let $\lambda_0, \mu_0 \in \mathcal{Z}$, $n \in \mathcal{E}_{\lambda_0}$ and $m \in \mathcal{E}_{\mu_0}$. We have the zero curvature equation*

$$\Big\{\mathcal{Q}_n^{\lambda_0}, \mathcal{M}_m^{\mu_0}(\lambda, x)\Big\} - \Big\{\mathcal{Q}_m^{\mu_0}, \mathcal{M}_n^{\lambda_0}(\lambda, x)\Big\} + \Big[\mathcal{M}_n^{\lambda_0}(\lambda, x), \mathcal{M}_m^{\mu_0}(\lambda, x)\Big] = 0.$$

This subsection is entirely devoted to the proof of Theorem 4.6.3. After stating some general results, we will treat separately the cases $\lambda_0 \neq \mu_0$ and $\lambda_0 = \mu_0$. Note that for the latter, we only have a complete proof for an algebra $\mathfrak{g}$ of type B, C and D. For $\mathfrak{g}$ of type A, we verified Theorem 4.6.3 for the first degrees n and m and conjecture that it holds more generally for any n and m. To improve the clarity of the subsection, some technical details of the proof are presented in appendix 4.B.

Here also the theorem concerns the finite regular zeros λ_0 and μ_0. The method presented in this subsection also applies for $\lambda_0 = \mu_0 = \infty$ as $\mathcal{L}^\infty(\lambda, x)$ also satisfies an r/s-system with twist function (Theorem 4.2.1). However, the theorem does not hold when λ_0 is finite and $\mu_0 = \infty$, even if Poisson brackets are considered only weakly.

Some general results

Let us consider the Poisson bracket (4.6.1). It can be rewritten

$$\begin{aligned}\{S(\lambda, x), \mathscr{T}_m(\mu, y)\} = &-m\,\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(U_{\underline{\mathbf{12}}}(\lambda, \mu) S_{m-1}(\mu, y)_{\underline{\mathbf{2}}}\Big)\delta'_{xy} \\ &+ m\left[\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) S_{m-1}(\mu, x)_{\underline{\mathbf{2}}}\Big), S(\lambda, x)\right]\delta_{xy},\end{aligned}$$

with $S(\lambda, x) = S_1(\lambda, x) = \varphi(\lambda)\mathcal{L}(\lambda, x)$. Starting from this Poisson bracket, we elevate $S(\lambda, x)$ to the power $n-1$ and find, using the fact that the Poisson bracket and the commutator are derivations, that

$$\begin{aligned}\{S_{n-1}(\lambda, x), \mathscr{T}_m(\mu, y)\} = &\; m\left[\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) S_{m-1}(\mu, x)_{\underline{\mathbf{2}}}\Big), S_{n-1}(\lambda, x)\right]\delta_{xy} \\ &- m\sum_{k=0}^{n-2} S_k(\lambda, x)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(U_{\underline{\mathbf{12}}}(\lambda, \mu) S_{m-1}(\mu, y)_{\underline{\mathbf{2}}}\Big) S_{n-2-k}(\lambda, x)\delta'_{xy}.\end{aligned} \tag{4.6.10}$$

Recall the definition (4.6.6) of $\mathcal{N}_n(\lambda\,;\rho, x)$. From the Poisson bracket (4.6.10), using the cyclicity of the trace, we find

$$\{\mathcal{N}_n(\lambda\,;\rho, x), \mathscr{T}_m(\mu, y)\} = \Gamma^{\lambda\mu}_{nm}(\rho, x)\delta_{xy} + \Xi^{\lambda\mu}_{nm}(\rho, x, y),$$

where

$$\Gamma^{\lambda\mu}_{nm}(\rho, x) = nm\mathrm{Tr}_{\underline{\mathbf{23}}}\Big(\big[\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho, \lambda), \mathcal{R}^0_{\underline{\mathbf{23}}}(\lambda, \mu)\big] S_{n-1}(\lambda, x)_{\underline{\mathbf{2}}} S_{m-1}(\mu, y)_{\underline{\mathbf{3}}}\Big), \tag{4.6.11}$$

$$\Xi^{\lambda\mu}_{nm}(\rho, x, y) = nm\mathrm{Tr}_{\underline{\mathbf{23}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho, \lambda) S_{m-1}(\mu, y)_{\underline{\mathbf{3}}} \sum_{k=0}^{n-2} S_k(\lambda, x)_{\underline{\mathbf{2}}} U_{\underline{\mathbf{23}}}(\lambda, \mu) S_{n-2-k}(\lambda, x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}, \tag{4.6.12}$$

with U defined in equation (4.2.18). Let us introduce

$$\begin{aligned}\mathcal{Y}^{\lambda\mu}_{nm}(\rho, x, y) = &\big[\mathcal{N}_n(\lambda\,;\rho, x), \mathcal{N}_m(\mu\,;\rho, x)\big]\,\delta_{xy} \\ &+ \{\mathscr{T}_n(\lambda, y), \mathcal{N}_m(\mu\,;\rho, x)\} - \{\mathscr{T}_m(\mu, y), \mathcal{N}_n(\lambda\,;\rho, x)\}\,.\end{aligned}$$

It contains a term equal to δ_{xy} times

$$[\mathcal{N}_n(\lambda\,;\rho,x),\mathcal{N}_m(\mu\,;\rho,x)]+\Gamma^{\lambda\mu}_{nm}(\rho,x)-\Gamma^{\mu\lambda}_{mn}(\rho,x).$$

One can show from equations (4.6.6) and (4.6.11) that this is equal to

$$\mathrm{Tr}_{\underline{\mathbf{23}}}\Big(\Upsilon_{\underline{\mathbf{123}}}(\rho,\lambda,\mu)S_{n-1}(\lambda,x)_{\underline{\mathbf{2}}}S_{m-1}(\mu,x)_{\underline{\mathbf{3}}}\Big),$$

with

$$\Upsilon_{\underline{\mathbf{123}}}(\rho,\lambda,\mu)=\left[\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda),\mathcal{R}^0_{\underline{\mathbf{13}}}(\rho,\mu)\right]+\left[\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda),\mathcal{R}^0_{\underline{\mathbf{23}}}(\lambda,\mu)\right]+\left[\mathcal{R}^0_{\underline{\mathbf{32}}}(\mu,\lambda),\mathcal{R}^0_{\underline{\mathbf{13}}}(\rho,\mu)\right].$$

This terms vanishes as $\mathcal{R}^0$ is a solution of the classical Yang-Baxter equation (2.3.3). We are therefore simply left with

$$\mathcal{Y}^{\lambda\mu}_{nm}(\rho,x,y)=\Xi^{\lambda\mu}_{nm}(\rho,x,y)-\Xi^{\mu\lambda}_{mn}(\rho,x,y). \tag{4.6.13}$$

The currents $\mathscr{J}_k$ are extracted from $\mathscr{T}_k$. But in general, the charges are constructed from currents $\mathscr{K}_k$ which are extracted from $\mathcal{W}_k$, where the definition of $\mathcal{W}_k$ depends on $\mathfrak{g}$ and σ (see subsections 4.3.5 and 4.4.7). In particular, we have $\mathscr{K}_k^{\lambda_0}(x)=\mathcal{W}_k(\lambda_0,x)$ for a non-cyclotomic regular zero λ_0. For the origin, which is cyclotomic, $\mathscr{K}_k^0(x)$ is the coefficient of λ^{r_k} in $\mathcal{W}_k(\lambda,x)$. Let us define

$$\begin{aligned}\mathcal{Z}^{\lambda\mu}_{nm}(\rho,x,y)=&\left[\mathcal{M}_n(\lambda\,;\rho,x),\mathcal{M}_m(\mu\,;\rho,x)\right]\delta_{xy}\\&+\left\{\mathcal{W}_n(\lambda,y),\mathcal{M}_m(\mu\,;\rho,x)\right\}-\left\{\mathcal{W}_m(\mu,y),\mathcal{M}_n(\lambda\,;\rho,x)\right\}.\end{aligned} \tag{4.6.14}$$

Using the expression of $\mathcal{W}_k$ and $\mathcal{M}_k$ in terms of $\mathscr{T}_k$ and $\mathcal{N}_k$, we see that $\mathcal{Z}^{\lambda\mu}_{nm}(\rho,x,y)$ contains several types of terms:

1. commutators $[\mathcal{N}_k(\lambda\,;\rho,x),\mathcal{N}_l(\mu\,;\rho,x)]$, multiplied by polynomials in the $\mathscr{T}_j$'s,
2. $\Gamma^{\lambda\mu}_{kl}(\rho,x)$ and $\Gamma^{\mu\lambda}_{lk}(\rho,x)$, multiplied by polynomials in the $\mathscr{T}_j$'s,
3. $\Xi^{\lambda\mu}_{kl}(\rho,x,y)$ and $\Xi^{\mu\lambda}_{lk}(\rho,x,y)$, multiplied by polynomials in the $\mathscr{T}_j$'s,
4. $\{\mathscr{T}_k(\lambda,x),\mathscr{T}_l(\mu,y)\}$ and $\{\mathscr{T}_k(\mu,x),\mathscr{T}_l(\lambda,y)\}$, multiplied by polynomials in $\mathscr{T}_j$'s and $\mathcal{N}_j$'s.

Moreover, the terms of type 1 and 2 are always ultralocal, *i.e.* proportional to δ_{xy}. It can be seen that these terms always combine into polynomials of $\mathscr{T}_j$ multiplied by

$$[\mathcal{N}_k(\lambda\,;\rho,x),\mathcal{N}_l(\mu\,;\rho,x)]+\Gamma^{\lambda\mu}_{kl}(\rho,x)-\Gamma^{\mu\lambda}_{lk}(\rho,x).$$

As explained above, this vanishes by virtue of the classical Yang-Baxter equation. Therefore, $\mathcal{Z}^{\lambda\mu}_{nm}(\rho,x,y)$ is composed only of terms of type 3 and 4.

Zero curvature equation at different regular zeros

Let us now prove Theorem 4.6.3 when λ_0 and μ_0 are different regular zeros. Since we are not considering here the point at infinity (see discussion after Theorem 4.6.3), at least one of them is non-cyclotomic, say μ_0. Recall that $U_{\underline{\mathbf{23}}}(\lambda,\mu_0)=\varphi(\lambda)\mathcal{R}^0(\lambda,\mu_0)$, as $\varphi(\mu_0)=0$.

Consider first the case where λ_0 is also non-cyclotomic. We will prove the that zero curvature equation of Theorem 4.6.3 holds by showing that $\mathcal{Z}^{\lambda_0\mu_0}_{nm}(\rho,x,y)$ vanishes. As explained above, it contains two types of terms. The ones of types 4 contain Poisson brackets between currents $\mathscr{J}_k^{\lambda_0}$ and $\mathscr{J}_l^{\mu_0}$. According to Theorem 4.5.1, these brackets are all zeros. As λ_0 and μ_0 are two distinct elements of $\mathcal{Z}$, the cyclotomic orbits $\mathbb{Z}_T\lambda_0$ and $\mathbb{Z}_T\mu_0$ are disjoint and thus $\mathcal{R}^0(\lambda,\mu_0)$ is regular at $\lambda=\lambda_0$. We then have $U_{\underline{\mathbf{23}}}(\lambda_0,\mu_0)=0$, as $\varphi(\lambda_0)=0$. We deduce from this that $\Xi^{\lambda_0\mu_0}_{kl}(\rho,x,y)=0$ and similarly $\Xi^{\mu_0\lambda_0}_{lk}(\rho,x,y)=0$, *i.e.* the terms of type 3 also vanish. Thus $\mathcal{Z}^{\lambda_0\mu_0}_{nm}(\rho,x,y)=0$, as required.

Suppose now that λ_0 is the origin and hence a cyclotomic point. Recall that $\mathscr{K}_n^0(x)$ and $\mathcal{M}_n^0(\rho,x)$ are the coefficients of λ^{r_n} in respectively $\mathcal{W}_n(\lambda,x)$ and $\mathcal{M}_n(\lambda\,;\rho,x)$. Thus, it is enough to show that there is no term λ^{r_n} in $\mathcal{Z}_{nm}^{\lambda\mu_0}(\rho,x,y)$ to prove Theorem 4.6.3 in this case. Recall that $\mathscr{T}_k(\lambda,x)$ and $\mathcal{N}_k(\lambda\,;\rho,x)$ contain powers of λ of the form r_k+aT with $a\in\mathbb{Z}_{\geq 0}$. As $r_n\leq T-2<T$ for $n\in\mathcal{E}_{\lambda_0}$, the powers with $a\geq 1$ cannot contribute to the λ^{r_n}-term. Following the discussion at the end of paragraph 4.6.3, the term λ^{r_n} of $\mathcal{Z}_{nm}^{\lambda\mu_0}(\rho,x,y)$ is thus composed of polynomials in the $\mathscr{J}_k^0$'s and $\mathcal{N}_k^0$'s times Poisson brackets of $\mathscr{J}_k^0$ with $\mathscr{J}_l^{\mu_0}$ or terms of the form

$$\Xi_{kl}^{\lambda\mu_0}(\rho,x,y)\Big|_{\lambda^{r_k}}\quad\text{or}\quad\Xi_{lk}^{\mu_0\lambda}(\rho,x,y)\Big|_{\lambda^{r_k}},$$

for k such that $r_k<T-1$. According to Theorem 4.5.1, the Poisson brackets of such $\mathscr{J}_k^0$ with $\mathscr{J}_l^{\mu_0}$ vanish. Moreover, $\Xi_{kl}^{\lambda\mu_0}(\rho,x,y)$ is proportional to $\varphi(\lambda)\mathcal{R}^0(\lambda,\mu_0)$. Yet, $\varphi(\lambda)=O(\lambda^{T-1})$ and $r_k<T-1$, hence $\Xi_{kl}^{\lambda\mu_0}(\rho,x,y)\Big|_{\lambda^{r_k}}=0$. Similarly $\Xi_{lk}^{\mu_0\lambda}(\rho,x,y)\Big|_{\lambda^{r_k}}=0$. Thus, the coefficient of λ^{r_n} in $\mathcal{Z}_{nm}^{\lambda\mu_0}(\rho,x,y)$ vanishes, as required. This ends the proof of Theorem 4.6.3 for different regular zeros λ_0 and μ_0.

Zero curvature equations at a non-cyclotomic regular zero

Let us now prove Theorem 4.6.3 for $\lambda_0=\mu_0$. We start with the case where λ_0 is a non-cyclotomic point. We then want to show that $\displaystyle\int \mathrm{d}y\,\mathcal{Z}_{nm}^{\lambda_0\lambda_0}(\rho,x,y)=0$.

As in section 4.3, we treat separately the Lie algebras of type B, C and D and the Lie algebras of type A. Suppose first that $\mathfrak{g}$ is of type B, C or D. In this case, the currents $\mathscr{K}_{2n}^{\lambda_0}$ are equal to the currents $\mathscr{J}_{2n}^{\lambda_0}$ (see subsections 4.3.2 and 4.3.5) and the corresponding Lax matrices $\mathcal{M}_{2n}^{\lambda_0}$ are equal to the matrices $\mathcal{N}_{2n}^{\lambda_0}$. Thus, $\mathcal{Z}_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)$ is simply equal to $\mathcal{Y}_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)$ (see paragraph 4.6.3). According to equation (4.6.13), we have

$$\mathcal{Y}_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)=\Xi_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)-\Xi_{2m\,2n}^{\lambda_0\lambda_0}(\rho,x,y),$$

where Ξ was defined in equation (4.6.12). To avoid cluttering the argument in the present paragraph with too many technicalities, we postpone the details of the computation of $\Xi_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)$ in appendix 4.B.1. We find

$$\Xi_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)=\frac{\varphi'(\lambda_0)}{T}\frac{4nm(1-2n)}{2n+2m-2}\mathcal{N}_{2n+2m-2}^{\lambda_0}(\rho,x)\delta'_{xy}+f_{2n\,2m}^{\lambda_0}(\rho,x)\delta_{xy},$$

where the function $f_{2n\,2m}^{\lambda_0}$ satisfies $f_{2n\,2m}^{\lambda_0}=f_{2m\,2n}^{\lambda_0}$ (cf. appendix 4.B.1). It then follows that

$$\mathcal{Y}_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)=\frac{\varphi'(\lambda_0)}{T}\frac{8nm(m-n)}{2n+2m-2}\mathcal{N}_{2n+2m-2}^{\lambda_0}(\rho,x)\delta'_{xy},$$

from which we deduce that $\displaystyle\int \mathrm{d}y\,\mathcal{Y}_{2n\,2m}^{\lambda_0\lambda_0}(\rho,x,y)=0$, as required.

Suppose now that $\mathfrak{g}$ is of type A. In this case, the currents $\mathscr{K}_n^{\lambda_0}$ are different from the currents $\mathscr{J}_n^{\lambda_0}$ and we therefore have to consider $\mathcal{Z}_{nm}^{\lambda_0\lambda_0}$ rather than simply $\mathcal{Y}_{nm}^{\lambda_0\lambda_0}$. According to the discussion at the end of paragraph 4.6.3, it contains polynomials in the $\mathscr{J}_p^{\lambda_0}$'s and $\mathcal{N}_p^{\lambda_0}$'s, multiplied by either $\Xi_{kl}^{\lambda_0\lambda_0}(\rho,x,y)$ or $\{\mathscr{J}_k^{\lambda_0}(x),\mathscr{J}_l^{\lambda_0}(y)\}$. This last Poisson bracket is given by equation (4.3.9) and is expressed in terms of the $\mathscr{J}_p^{\lambda_0}$'s. As for type B, C and D, we compute the expression of $\Xi_{kl}^{\lambda_0\lambda_0}$ in appendix 4.B.1. We find

$$\begin{aligned}\Xi_{k\,l}^{\lambda_0\lambda_0}(\rho,x,y)=&-\frac{\varphi'(\lambda_0)}{T}\frac{kl(k-1)}{k+l-2}\mathcal{N}_{k+l-2}^{\lambda_0}(\rho,x)\delta'_{xy}\\&+\frac{\varphi'(\lambda_0)}{dT}kl\,\mathscr{J}_{l-1}^{\lambda_0}(y)\mathcal{N}_{k-1}^{\lambda_0}(\rho,x)\delta'_{xy}+f_{kl}^{\lambda_0}(\rho,x)\delta_{xy},\end{aligned}$$

for some function $f_{kl}^{\lambda_0}$ such that $f_{kl}^{\lambda_0} = f_{lk}^{\lambda_0}$.

Hence $\mathcal{Z}_{nm}^{\lambda_0\lambda_0}$ can be expressed in terms of the $\mathscr{J}_p^{\lambda_0}$'s and $\mathcal{N}_p^{\lambda_0}$'s, up to terms involving $f_{kl}^{\lambda_0}$. The latter are always of the form

$$\alpha \mathscr{J}_{p_1}^{\lambda_0}(x) \dots \mathscr{J}_{p_q}^{\lambda_0}(x) f_{kl}^{\lambda_0}(\rho, x)\delta_{xy},$$

with α a constant. Moreover, one can check that for any such term, there is also a similar one but with an opposite sign and k and l interchanged. Using the symmetry property $f_{kl}^{\lambda_0} = f_{lk}^{\lambda_0}$, one can then conclude that these terms always vanish.

Therefore $\mathcal{Z}_{nm}^{\lambda_0\lambda_0}$ can be expressed in terms of the $\mathscr{J}_p^{\lambda_0}$'s and $\mathcal{N}_p^{\lambda_0}$'s. Using the first explicit expressions (4.3.11) for the current $\mathscr{K}_n^{\lambda_0}$ and the corresponding expressions for the matrices $\mathcal{M}_n^{\lambda_0}$, it can be check directly that $\int \mathrm{d}y\, \mathcal{Z}_{nm}^{\lambda_0\lambda_0}(\rho, x, y) = 0$ for the first few degrees n, m. Specifically, we have checked this for degrees n and m up to 7. In particular, we observed that we could not have chosen different coefficients in equation (4.3.11) for these zero curvature equations to hold (in the same way that these coefficients were uniquely fixed by requiring the involution of $\mathcal{Q}_n^{\lambda_0}$ and $\mathcal{Q}_m^{\lambda_0}$). Based on these strong observations, we conjecture that it holds for any $n, m \in \mathcal{E}_{\lambda_0}$.

Zero curvature equations at a cyclotomic regular zero

Finally, let us prove Theorem 4.6.3 for $\lambda_0 = \mu_0 = 0$, which is a cyclotomic point. Remember that $\mathscr{K}_n^0(x)$ and $\mathcal{M}_n^0(\rho, x)$ are extracted as the coefficient of λ^{r_n} in the power series expansion of $\mathcal{W}_n(\lambda, x)$ and $\mathcal{M}_n(\lambda\,; \rho, x)$ where r_n is the smallest power appearing in these expansions. That is, Theorem 4.6.3 for $\lambda_0 = \mu_0 = 0$ is equivalent to the statement that

$$\int \mathrm{d}y\, \mathcal{Z}_{nm}^{\lambda\lambda}(\rho, x, y)\Big|_{\lambda^{r_n+r_m}} = 0.$$

Let us start with the case of a Lie algebra $\mathfrak{g}$ of type B, C or D, for which $\mathcal{Z}_{2n\,2m}^{\lambda\lambda} = \mathcal{Y}_{2n\,2m}^{\lambda\lambda}$. According to equation (4.6.13), we have

$$\mathcal{Y}_{2n\,2m}^{\lambda\lambda}(\rho, x, y) = \Xi_{2n\,2m}^{\lambda\lambda}(\rho, x, y) - \Xi_{2m\,2n}^{\lambda\lambda}(\rho, x, y).$$

The computation of $\Xi_{2n\,2m}^{\lambda\lambda}\big|_{\lambda^{r_n+r_m}}$ is performed in appendix 4.B.2. The final result is

$$\Xi_{2n\,2m}^{\lambda\lambda}(\rho, x, y)\Big|_{\lambda^{r_{2n}+r_{2m}}} = f_{2n\,2m}^{(0)}(\rho, x)\delta_{xy} - \theta_{r_{2n}+r_{2m}+2-T}\,\zeta'(0)\frac{4nm(2n-1)}{2n+2m-2}\mathcal{N}_{2n+2m-2}^0(\rho, x)\delta'_{xy},$$

with $f_{2n\,2m}^{(0)}$ a function symmetric under the exchange of n and m. By virtue of this symmetry we find that the terms involving f disappear in $\mathcal{Y}_{2n\,2m}^{\lambda\lambda}(\rho, x, y)\big|_{\lambda^{r_n+r_m}}$, while the other terms vanish when integrated over y, as required.

Consider now $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$ of type A. The construction of the currents $\mathscr{K}_k^0$ depends on σ being inner or not (see subsections 4.4.4, 4.4.5 and 4.4.7). If σ is inner, then the currents $\mathscr{K}_k^0$ and $\mathcal{W}_k$ are equal to the currents $\mathscr{J}_k^0$ and $\mathscr{T}_k$. In this case, we have

$$\mathcal{Z}_{nm}^{\lambda\lambda}(\rho, x, y) = \mathcal{Y}_{nm}^{\lambda\lambda}(\rho, x, y) = \Xi_{nm}^{\lambda\lambda}(\rho, x, y) - \Xi_{mn}^{\lambda\lambda}(\rho, x, y).$$

The expression for $\Xi_{nm}^{\lambda\lambda}(\rho, x, y)\big|_{\lambda^{r_n+r_m}}$ is given by equation (4.B.7) of appendix 4.B.2. It has the same structure as in the case of types B, C and D: the same arguments then apply and we conclude that the integration of $\mathcal{Y}_{nm}^{\lambda\lambda}(\rho, x, y)\big|_{\lambda^{r_n+r_m}}$ over y vanishes.

Finally, consider $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$ of type A with σ not inner. In this case, the currents $\mathscr{K}_k^0(x)$ and $\mathcal{W}_k(\lambda, x)$ are constructed as polynomials of respectively $\mathscr{J}_k^0(x)$ and $\mathscr{T}_k(\lambda, x)$. The corresponding structure of $\mathcal{Z}_{nm}^{\lambda\mu}$ is discussed at the end of paragraph 4.6.3. In particular, $\mathcal{Z}_{nm}^{\lambda\lambda}(\rho, x, y)$ is composed of two types of terms:

- $\Xi^{\lambda\lambda}_{kl}(\rho,x,y)$, multiplied by polynomials in the $\mathscr{T}_j(\lambda,\cdot)$'s,
- $\{\mathscr{T}_k(\lambda,x),\mathscr{T}_l(\lambda,y)\}$, multiplied by polynomials in the $\mathscr{T}_j(\lambda,\cdot)$'s and $\mathcal{N}_j(\lambda\,;\rho,\cdot)$'s.

We want to extract the coefficient of $\lambda^{r_n+r_m}$ in $\mathcal{Z}^{\lambda\lambda}_{nm}(\rho,x,y)$. Recall that the powers of λ appearing in $\mathscr{T}_j(\lambda,\cdot)$ and $\mathcal{N}_j(\lambda\,;\rho,\cdot)$ are of the form r_j+aT, with $a\in\mathbb{Z}_{\geq 0}$, and that the coefficients corresponding to $a=0$ are $\mathscr{J}^0_j(\cdot)$ and $\mathcal{N}^0_j(\rho,\cdot)$. In the two types of terms mentioned above, one can check that the terms with $a>0$ will not contribute to the coefficient of $\lambda^{r_n+r_m}$. More precisely, $\mathcal{Z}^{\lambda\lambda}_{nm}(\rho,x,y)\big|_{\lambda^{r_n+r_m}}$ is composed of two types of terms:

- $\Xi^{\lambda\lambda}_{kl}(\rho,x,y)\big|_{\lambda^{r_k+r_l}}$, multiplied by polynomials in the $\mathscr{J}^0_j(\cdot)$'s,
- $\{\mathscr{J}^0_k(x),\mathscr{J}^0_l(y)\}$, multiplied by polynomials in the $\mathscr{J}^0_j(\cdot)$'s and $\mathcal{N}^0_j(\rho,\cdot)$'s.

The Poisson brackets $\{\mathscr{J}^0_k(x),\mathscr{J}^0_l(y)\}$ are given by equation (4.4.16). The expression for $\Xi^{\lambda\lambda}_{kl}(\rho,x,y)\big|_{\lambda^{r_k+r_l}}$ is worked out in appendix 4.B.2 and reads

$$\begin{aligned}\Xi^{\lambda\lambda}_{kl}(\rho,x,y)\Big|_{\lambda^{r_k+r_l}} = f^{(0)}_{kl}(\rho,x)\delta_{xy} - \theta_{r_k+r_l+2-T}\,\zeta'(0)\frac{kl(k-1)}{k+l-2}\mathcal{N}^0_{k+l-2}(\rho,x)\delta'_{xy}\\ - \theta_{r_k+1-S}\theta_{r_l+1-S}\,\frac{\zeta'(0)}{d}\,kl\,\mathscr{J}^0_{l-1}(y)\mathcal{N}^0_{k-1}(\rho,x)\delta'_{xy},\end{aligned}$$

where $f^{(0)}_{kl}$ is a function invariant under the interchange of k and l.

The rest of the argument follows closely that given in the non-cyclotomic case. Specifically, the terms containing $f^{(0)}_{kl}$ are seen to vanish by virtue of this symmetry property. We can thus express $\mathcal{Z}^{\lambda\lambda}_{nm}(\rho,x,y)\big|_{\lambda^{r_n+r_m}}$ in terms of the $\mathscr{J}^0_k$'s and $\mathcal{N}^0_k$'s only. One then can check explicitly that this expression vanishes when integrated over y, as required. We verified this for the first few degrees n and m (up to 8) and different values of T (from 2 to 6). We therefore conjecture that this is also true for any $n,m\in\mathcal{E}_0$ and any T.

4.7 Applications

In Chapter 3, we gave a list of integrable σ-models which fit in the framework of models with twist function. In this section, we apply the methods developed in the previous sections to these particular examples, analyse the results and compare them to some existing work in the literature. These models were recently re-interpreted as particular examples of so-called dihedral affine Gaudin models [78]. We will explain in second part of this thesis how the framework of dihedral affine Gaudin models is particularly suited to apply the methods of the present chapter.

4.7.1 Principal chiral model and its deformations

Let us start with the simplest integrable σ-model, the Principal Chiral Model (PCM). The study of local charges of the PCM is already well known and was treated in the reference [79]: these results were the principal motivation and guideline for the present chapter. In particular, one of the aims was to generalise the construction of [79] to a wider class of models, among which are the integrable two-parameters deformations of the PCM (dPCM). We shall discuss the latter in this subsection.

The integrable structure of the dPCM was discussed in subsection 3.3.2. In this case, $\sigma=\mathrm{Id}$ so that $T=1$. Their Lax matrix and twist function are given by equations (3.3.19) and (3.3.18) respectively. In the language of this chapter, the regular zeros of these deformed models are $+1$ and -1. The evaluation of $\varphi(\lambda)\mathcal{L}(\lambda,x)$ at these zeros gives the fields

$$J_\pm(x)=\frac{A(x)\pm\Pi(x)}{(k\pm 1)^2+\mathcal{A}^2}, \tag{4.7.1}$$

which reduce to the light-cone currents $j^L_\pm$ in the undeformed case.

The local charges $\mathcal{Q}_n^{\pm 1}$ constructed in the present chapter are related to the traces of powers of these fields. In the undeformed case, *i.e.* for the PCM, these fields coincide with the fields $j_\pm^L$ used in reference [79] to construct the local charges. Thus, the method presented in this chapter gives back the results of [79] for the PCM, as expected. In the deformed case, it generalises these results, while keeping a similar structure in the construction: in particular, we obtain two towers of local charges in involution, corresponding to the two chiralities of the model, and the spin of these charges is still related to the exponents of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, as it was in the PCM case [79].

The Hamiltonian and momentum of the deformed PCM are given by equations (3.3.20). One can check that these are related to the quadratic charges $\mathcal{Q}_2^{\pm 1}$ as follows

$$\mathcal{H}_{\text{dPCM}} = -\frac{\mathcal{Q}_2^{+1}}{2\varphi'_{\text{dPCM}}(+1)} + \frac{\mathcal{Q}_2^{-1}}{2\varphi'_{\text{dPCM}}(-1)},$$
$$\mathcal{P}_{\text{dPCM}} = -\frac{\mathcal{Q}_2^{+1}}{2\varphi'_{\text{dPCM}}(+1)} - \frac{\mathcal{Q}_2^{-1}}{2\varphi'_{\text{dPCM}}(-1)}.$$

In particular, the Hamiltonian belongs to the algebra of local charges in involution so that these charges are conserved (see also the discussion at the end of subsection 4.5.1).

This observation also allows one to recover the temporal component $\mathcal{M}(\lambda, x)$ of the Lax pair of the model (see the paragraph below equation (4.6.5)). More precisely, the equation of motion of the dPCM can be recast as the Lax equation (2.2.2), where

$$\mathcal{M}_{\text{dPCM}}(\lambda,x) = -\frac{\mathcal{M}_2^{+1}(\lambda, x)}{2\varphi'_{\text{dPCM}}(+1)} + \frac{\mathcal{M}_2^{-1}(\lambda, x)}{2\varphi'_{\text{dPCM}}(-1)} = \frac{\Pi(x) + \lambda A(x)}{1-\lambda^2}.$$

This zero curvature equation (2.2.2) is the first among a whole hierarchy of integrable equations generated by the local charges $\mathcal{Q}_n^{\pm 1}$ (cf. subsection 4.6.1).

In particular, this result was used in subsection 4.6.2 to show that the local charges $\mathcal{Q}_n^{\pm 1}$ are in involution with the non-local charges extracted from the monodromy of the Lax matrix $\mathcal{L}(\lambda, x)$ (see Theorem 4.6.2). In [79], it was shown that the local charges of the undeformed PCM Poisson commute with the non-local charges generating the classical Yangian symmetry of the model. In the framework of this chapter, if we consider the model on the real line $\mathbb{R}$, we expect these non-local charges to be extracted from the expansion of (a gauge transformation of) the monodromy around the pole $\lambda = 0$ of the twist function of the PCM.

For the Yang-Baxter model ($k = 0$ and $\eta \neq 0$, see paragraph 3.3.1), this Yangian symmetry gets deformed to a quantum affine symmetry [36] (see also Chapter 5 of this thesis for the finite part of the deformed symmetry). In particular, studying the monodromy around the poles $\pm i\eta$ of the twist function of the Yang-Baxter model, one can extract a q-deformed affine Poisson-Hopf algebra $\mathscr{U}_q(\widehat{\mathfrak{g}})$. We have therefore proved that this algebra of non-local charges is in involution with the algebra of local charges consisting of the $\mathcal{Q}_n^{\pm 1}$'s.

As explained in the chapter 3, the PCM and its deformation are defined on a real Lie group G_0, whose Lie algebra $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. This real form is characterised by a semi-linear involutive automorphism τ. The Lax matrix (3.3.19) of these models satisfies the reality condition (2.4.7). Moreover, the twist function (3.3.18) verifies the reality condition (2.4.10) and the regular zeros of the model ($+1$ and -1) are real. Thus, the discussion of the subsection 4.5.3 applies to these models and the charges $\mathcal{Q}_n^{\pm 1}$ are real (possibly up to a redefinition of some $\mathcal{Q}_n^{\pm 1}$ by a factor of i, depending on τ).

4.7.2 Bi-Yang-Baxter model

There exists another two-parameter deformation of the PCM, the Bi-Yang-Baxter (BYB) σ-model [28, 33], which is the subject of the Section 3.4 of this thesis. Here, we choose to treat the BYB model in its gauge fixed formulation, as described in Subsection 3.4.5. As we saw, this formulation does not fit

exactly within the framework of models with twist function, but we will explain how one can overcome this difficulty by further relaxing the general assumptions we made. Even though this generalisation could have been done throughout the entire chapter, we chose here to work in a more restricting but more common framework for clarity and simplicity. In this regard, the present subsection is also used to illustrate, *a posteriori*, how the methods and results we found apply under the generalised conditions.

The Lax pair of the gauge-fixed BYB model takes the form (3.4.32). The corresponding Lax matrix then reads

$$\mathcal{L}_{\mathrm{GF}}(\xi, x) = \frac{K_1(x) + \lambda K_0(x)}{1 - \xi^2} + K_\infty(x), \tag{4.7.3}$$

for some $\mathfrak{g}$-valued fields K_0, K_1 and K_∞. The Hamiltonian analysis of the BYB model was done in Section 3.4. In particular, the Poisson bracket of the Lax matrix (4.7.3) was computed in Subsection 3.4.5. It takes the form of a Maillet bracket with twist function φ_{GF} defined in (3.4.36) but with the standard $\mathcal{R}$-matrix $\mathcal{R}^0$ replaced by the matrix $\mathcal{R}^{0,\mathrm{GF}}$ defined in (3.4.35).

As $\mathcal{R}^{0,\mathrm{GF}}$ is different from $\mathcal{R}^0$, we cannot directly apply the results of this chapter. However, going through the details of the proofs of these results for a non-constrained model with $T = 1$, we see that the only properties of the matrix $\mathcal{R}^0$ that we used are the CYBE (for the zero curvature equations), the fact that $\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu)$ is holomorphic at pairs (λ_0, μ_0) of distinct regular zeros and the asymptotic property (4.3.2) near a regular zero $\mu = \lambda_0$. The matrix $\mathcal{R}^{0,\mathrm{GF}}$ also satisfies the CYBE, as explained in Subsection 3.4.5. Moreover, one easily checks that it also verifies the holomorphy condition and the asymptotic property mentioned above. Thus, the results we found in this chapter also apply to the BYB model.

This is a general observation: we can also treat the models where the matrix $\mathcal{R}^0$ is replaced by a matrix $\mathcal{R}'$ satisfying some similar properties. More precisely, we require that $\mathcal{R}'$

- obeys the CYBE (2.3.3),
- is holomorphic at (λ_0, μ_0) with λ_0 and μ_0 different regular zeros in $\mathcal{Z}$,
- verifies the asymptotic property (4.3.2) around non-cyclotomic regular zeros,
- satisfies the equation (4.4.8) for $U_{\underline{\mathbf{12}}}(\lambda, \lambda)$ around a cyclotomic regular zero, up to a term $O(\lambda^{2T-3})$ (which would not contribute to some $(r_n + r_m)^{\mathrm{th}}$ power of λ in (4.4.9)).

In particular, let us consider a matrix $\mathcal{R}'$ of the form

$$\mathcal{R}'_{\underline{\mathbf{12}}}(\lambda, \mu) = \mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda, \mu) + \mathcal{D}_{\underline{\mathbf{12}}}(\mu), \tag{4.7.4}$$

like the matrix $\mathcal{R}^{0,\mathrm{GF}}$. Then $\mathcal{R}'_{\underline{\mathbf{12}}}(\lambda, \mu)$ is holomorphic for λ and μ going to different regular zeros if $\mathcal{D}$ is holomorphic at any regular zero (this is for example the case for the BYB model). This condition also ensures that the asymptotic property (4.3.2) is satisfied by $\mathcal{R}'$. In the same way the condition on $U_{\underline{\mathbf{12}}}(\lambda, \lambda)$ is satisfied by $\mathcal{R}'$ if $\mathcal{D}_{\underline{\mathbf{12}}}(\lambda) + \mathcal{D}_{\underline{\mathbf{21}}}(\lambda) = O(\lambda^{T-2})$.

Let us note, however, that these conditions do not allow to treat the case where infinity is a regular zero in the same way that we did in this chapter (subsections 4.2.2 and 4.4). This would require, among other conditions, that the asymptotic properties (4.2.4) at infinity are also satisfied by the matrix $\mathcal{R}'$. One can check that a matrix $\mathcal{R}'$ of the form (4.7.4) can never satisfy the second property of equation (4.2.4).

As explained above, we can apply the construction of local charges in involution to the BYB model. These local charges will be very similar to the ones of the PCM and its deformations, described in the previous subsection, so we shall not enter into much details here. Let us note that these charges are related to traces of powers of $K_0(x) \pm K_1(x)$, where K_0 and K_1 are the fields appearing in the Lax matrix (4.7.3). As in the case of the PCM (see previous subsection), the Hamiltonian and the momentum of the BYB model are related to the quadratic charges $\mathcal{Q}_2^{\pm 1}$ by the relation

$$\mathcal{H}_{\mathrm{GF}} = -\frac{\mathcal{Q}_2^{+1}}{2\varphi'_{\mathrm{GF}}(+1)} + \frac{\mathcal{Q}_2^{-1}}{2\varphi'_{\mathrm{GF}}(-1)},$$
$$\mathcal{P}_{\mathrm{GF}} = -\frac{\mathcal{Q}_2^{+1}}{2\varphi'_{\mathrm{GF}}(+1)} - \frac{\mathcal{Q}_2^{-1}}{2\varphi'_{\mathrm{GF}}(-1)}.$$

In particular, the local charges constructed above are all conserved.

4.7.3 $\mathbb{Z}_T$-coset models and their deformations

In this subsection, we discuss the construction of local charges in involution for $\mathbb{Z}_T$-coset models (and the deformations of $\mathbb{Z}_2$-coset models). These models were described in Subsections 3.2.2 and 3.3.3. Their twist function and Lax matrix are given by equations (3.2.27) and (3.2.26) for undeformed $\mathbb{Z}_T$-coset and by (3.3.24) and (3.3.23) for the deformed $\mathbb{Z}_2$-coset. Local charges in involution were constructed for symmetric spaces, *i.e.* $\mathbb{Z}_2$-cosets, in references [80] and [146]: we shall compare these results with the ones of this chapter.

The regular zeros of these models are the origin and infinity, which are both cyclotomic points. We shall therefore apply here the construction of section 4.4. Moreover, all these models possess a gauge constraint $X^{(0)}$ (see Chapter 3), which is identified with the field at infinity $\mathcal{C}(x)$ described in subsection 4.2.2. The results of subsection 4.5.2 ensure that the densities of the local charges that we construct here are gauge invariant. Indeed, by Theorem 4.5.3, these densities Poisson commute with the constraint $\mathcal{C}$.

As in the case of the PCM (see subsection 4.7.1), the degrees of the local charges are related to the exponents of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ plus one (here also, we do not consider the exponents corresponding to the Pfaffian for type D). However, as explained in section 4.4, the fact that the regular zeros of the model are cyclotomic makes some of the exponents 'drop out', in the sense that we cannot construct a charge of the corresponding degree. Recall that a degree n (corresponding to an exponent $n-1$) drops out if r_n is equal to $T-1$ (where r_n was defined in subsection 4.4.1).

Let us study this in more detail for the case of $\mathbb{Z}_2$-cosets. In particular, we shall compare this phenomenon of exponents dropping out with some results of reference [80]. Indeed, in this reference, some local charges in involution were constructed for symmetric spaces (*i.e.* $\mathbb{Z}_2$-cosets). These symmetric spaces correspond to quotients G_0/G_0^σ of the real Lie group G_0 by the subgroup of fixed points under the involutive automorphism σ. Such spaces were classified, up to isomorphism, for classical compact groups G_0.

In particular, the possible exponents (*i.e.* the degrees minus one) of the local charges for each symmetric space of this classification were listed in Table 1 of [80]: they form a (potentially proper) subset of the exponents of $\widehat{\mathfrak{g}}$. A simple case by case computation of the integers r_n for these symmetric spaces, and thus these automorphisms σ, shows that the exponents of $\widehat{\mathfrak{g}}$ which do not appear in this list are exactly the exponents that drop out in the formalism of the present chapter. We therefore recover the structure of the degrees of local charges found in [80] (except for the integer h of [80], which we could not interpret in the present formalism).

An explicit computation of the traces of powers of $\varphi_{\mathbb{Z}_2}(\lambda)\mathcal{L}_{\mathbb{Z}_2}(\lambda, x)$ around the origin $\lambda = 0$ shows that the charges constructed in this chapter coincide, up to some factors, with the ones constructed in reference [80]. The two regular zeros 0 and ∞ correspond to the two chiralities of the model. The article [80] focused on one particular chirality. Here, we also have the Poisson brackets between the two towers of local charges constructed in this way. Indeed, according to Theorem 4.5.2, we show that these two towers of charges Poisson commute weakly.

This chapter also generalises the results of [80] in different directions. First of all, the present formalism also allows to treat the integrable deformations of the $\mathbb{Z}_2$-coset model. Indeed, as explained in Subsection 3.3.3, the regular zeros of these models are still 0 and ∞ and so the methods developed here still apply. The main generalisation is that this chapter does not restrict to (compact) symmetric

spaces and also generalises the construction to any $\mathbb{Z}_T$-coset model. Finally, in this chapter we have also studied the hierarchy of equations induced by the flow of these local charges.

Recall the supercoset models discussed at the end of Subsection 3.2.2, which have a similar structure than the $\mathbb{Z}_T$-coset models but with $\mathfrak{g}$ a super-Lie algebra. Although we did not consider such models in this chapter, we expect the construction to extend to these theories, by working with the Grassmann envelope of $\mathfrak{g}$ and replacing all traces by supertraces. One should however be careful about how the automorphism σ is extented to the whole matrix algebra (see appendix 4.A) in these supercoset cases. Such considerations could allow the construction of local charges in involution for supercoset σ-models whose target space includes AdS manifolds, with possible applications to the Green-Schwarz formulation of string theory.

We end this subsection by observing that the Hamiltonian of the $\mathbb{Z}_T$-coset model is related to the quadratic charge $\mathcal{Q}_2^0$ at the origin and the momentum $\mathcal{P}_{\mathbb{Z}_T}$ of the theory by

$$\mathcal{H}_{\mathbb{Z}_T} = \frac{\mathcal{Q}_2^0}{\zeta'(0)} + \mathcal{P}_{\mathbb{Z}_T},$$

where ζ was defined in equation (2.4.6) (note that this expression also holds for the deformed $\mathbb{Z}_2$ model). Thus, we conclude that the local charges constructed above are conserved, as they commute (at least weakly) with the Hamiltonian.

As the $\mathbb{Z}_T$-coset models are constrained models, their Hamiltonian is defined up to a term $\mathrm{Tr}\big(\mu(x)\mathcal{C}(x)\big)$, where μ is a $\mathfrak{g}$-valued Lagrange multiplier. In this sense, $\mathcal{H}_{\mathbb{Z}_T}$ defined above is a particular choice of such a Hamiltonian, which generates a strong zero curvature equation (2.2.2) on the Lax matrix $\mathcal{L}$. Another choice of Hamiltonian involves the quadratic charge $\mathcal{Q}_2^\infty$ extracted at infinity, namely

$$\widetilde{\mathcal{H}}_{\mathbb{Z}_T} = -\frac{\mathcal{Q}_2^\infty}{\zeta'_\infty(0)} - \mathcal{P}_{\mathbb{Z}_T},$$

where ζ_∞ is defined in the same way than ζ by $\zeta_\infty(\alpha^T) = \alpha\psi(\alpha)$. This Hamiltonian is weakly equal to $\mathcal{H}_{\mathbb{Z}_T}$ and generates a strong curvature equation on the Lax matrix $\mathcal{L}^\infty$.

4.A Extension of the automorphism to the whole space of matrices

We consider a Lie algebra $\mathfrak{g}$ of classical type A, B, C or D, in its defining matricial representation. We therefore regard elements of $\mathfrak{g}$ as acting linearly on a vector space V, *i.e.* as element of the space F of endomorphisms of V. Note that we can consider some connected matrix group $G \subset F$ whose Lie algebra is $\mathfrak{g}$. Let σ be an automorphism of $\mathfrak{g}$, of finite order T. In this chapter, we are considering powers of elements of $\mathfrak{g}$, which do not belong to $\mathfrak{g}$ in general but are elements of F. Thus, we want to extend the automorphism σ to the whole space of matrices F, in a "natural way".

4.A.1 The conjugacy case

Let us begin with the case where σ is inner, *i.e.* when $\sigma : X \in \mathfrak{g} \mapsto QXQ^{-1}$ for some $Q \in G$. Then the extension of σ to F, which by a slight abuse of notation we still denote as σ, can be naturally defined as

$$\begin{array}{rccc} \sigma : & F & \longmapsto & F \\ & X & \longrightarrow & QXQ^{-1} \end{array} \tag{4.A.1}$$

This covers the case of types B and C, as they do not have any non trivial diagram automorphism.

Let us now consider the algebra D_n, *i.e.* $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, for $n \geq 5$. In this case, there always exists one non trivial diagram automorphism. However, this automorphism can be realised on the defining representation as an external conjugation: $\sigma : X \in \mathfrak{g} \mapsto QXQ^{-1}$ where Q is not in the group $SO(2n, \mathbb{C})$ but belongs to $O(2n, \mathbb{C})$. In this case, the endomorphism σ as defined in equation (4.A.1) still naturally extends σ on F.

Let us say a few words on the algebra $\mathrm{D}_4 = \mathfrak{so}(8,\mathbb{C})$. It is known to have 6 diagram automorphisms, forming the *triality*, isomorphic to the symmetric group S_3. One of them, of order 2, can also be realised as conjugation by a matrix $Q \in O(8,\mathbb{C})$ and so extends to $F = M_8(\mathbb{C})$ by equation (4.A.1). The other non-trivial ones cannot be realised in any "natural way" on the defining representation and thus cannot easily be extended to the whole space $M_8(\mathbb{C})$. We shall not consider them in this chapter.

We now describe the properties of the extension σ defined in (4.A.1). The fact that the automorphism σ of $\mathfrak{g}$ is of order T is equivalent to the fact that Q^T belongs to the centraliser of $\mathfrak{g}$ in F,

$$Z_F(\mathfrak{g}) = \{X \in F \text{ s.t. } [X,Y] = 0,\ \forall Y \in \mathfrak{g}\}\,.$$

By Schur's lemma, this implies that $Q^T = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Therefore σ on F defined by (4.A.1) is also of order T. We shall make extensive use of the following five obvious properties of σ as defined in (4.A.1):

$$\sigma(XY) = \sigma(X)\sigma(Y), \qquad \sigma(X^n) = \sigma(X)^n, \qquad \sigma(\mathrm{Id}) = \mathrm{Id},$$
$$\mathrm{Tr}\big(\sigma(X)\big) = \mathrm{Tr}(X), \qquad \mathrm{Tr}\big(\sigma(X)\sigma(Y)\big) = \mathrm{Tr}(XY),$$

for any X, Y in F.

4.A.2 The transpose case

The last case that we have to treat is the one of a type A algebra, with σ being not inner. We thus consider the defining representation $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{C})$. The action of σ on $\mathfrak{g}$ can then always be expressed as $\sigma : X \in \mathfrak{g} \mapsto -QX^{\mathsf{T}}Q^{-1}$, where X^{T} is the transpose of X and Q is a matrix in $SL(d,\mathbb{C})$. Here also we can naturally extend σ to an endomorphism of $F = M_d(\mathbb{C})$, which we still denote σ, by letting

$$\sigma : \begin{array}{ccc} F & \longmapsto & F \\ X & \longrightarrow & -Q\,X^{\mathsf{T}}Q^{-1} \end{array}\,. \tag{4.A.2}$$

Once again, let us investigate the properties of σ. As the automorphism σ of $\mathfrak{g}$ is not inner, its order T must be even, and we shall write $T = 2S$. We note that σ^2 acts as conjugation by $R = Q(Q^{\mathsf{T}})^{-1}$. The fact that $\sigma^T = (\sigma^2)^S = \mathrm{Id}|_{\mathfrak{g}}$ is thus equivalent to the fact that R^S belongs to the centraliser $Z_F(\mathfrak{g})$. Thus $R^S = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$ and so σ defined in (4.A.2) is also of order T. We end the subsection by noting the following five properties of σ:

$$\sigma(XY) = -\sigma(Y)\sigma(X), \qquad \sigma(X^n) = (-1)^{n-1}\sigma(X)^n, \qquad \sigma(\mathrm{Id}) = -\mathrm{Id},$$
$$\mathrm{Tr}\big(\sigma(X)\big) = -\mathrm{Tr}(X), \qquad \mathrm{Tr}\big(\sigma(X)\sigma(Y)\big) = \mathrm{Tr}(XY),$$

for any X, Y in F.

4.B Computation of Ξ

In this appendix, we give the details of the computation in some particular cases of the term $\Xi^{\lambda\mu}_{nm}(\rho,x,y)$, defined by (4.6.12).

4.B.1 At a non-cyclotomic regular zero

We first suppose that λ_0 is a non-cyclotomic regular zero and that $\mathfrak{g}$ is of type B, C or D. Recall that in this case, we constructed currents $\mathscr{K}^{\lambda_0}_{2n} = \mathscr{J}^{\lambda_0}_{2n}$ and the associated Lax matrices $\mathcal{M}^{\lambda_0}_{2n} = \mathcal{N}^{\lambda_0}_{2n}$. We want to compute $\Xi^{\lambda_0\lambda_0}_{2n\,2m}(\rho,x,y)$, starting from equation (4.6.12). Recall from section 4.3 that for a non-cyclotomic zero λ_0, one has $U_{\underline{\mathbf{23}}}(\lambda_0,\lambda_0) = -\frac{1}{T}\varphi'(\lambda_0)C_{\underline{\mathbf{12}}}$. Recall also from subsection 4.3.2 that

$S_{2m-1}(\lambda_0,x)$ belongs to the Lie algebra $\mathfrak{g}$, as it is an odd power of a matrix in $\mathfrak{g}$. Using the completeness relation (4.2.13), we find

$$\Xi^{\lambda_0\lambda_0}_{2n\,2m}(\rho,x,y)=-\frac{4nm\varphi'(\lambda_0)}{T}\mathrm{Tr}_{\underline{\mathbf{2}}}\left(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)\sum_{k=0}^{2n-2}S_k(\lambda_0,x)_{\underline{\mathbf{2}}}S_{2m-1}(\lambda_0,y)_{\underline{\mathbf{2}}}S_{2n-2-k}(\lambda_0,x)_{\underline{\mathbf{2}}}\right)\delta'_{xy}, \tag{4.B.1}$$

Using the identity $f(y)\delta'_{xy}=f(x)\delta'_{xy}+\big(\partial_x f(x)\big)\delta_{xy}$ and the fact that $S_p(\lambda,x)S_q(\lambda,x)=S_{p+q}(\lambda,x)$, we get

$$\Xi^{\lambda_0\lambda_0}_{2n\,2m}(\rho,x,y)=f^{\lambda_0}_{2n\,2m}(\rho,x)\delta_{xy}-\frac{4nm\varphi'(\lambda_0)(2n-1)}{T}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)S_{2n+2m-3}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy},$$

where

$$f^{\lambda_0}_{2n\,2m}(\rho,x)=-\frac{4nm\varphi'(\lambda_0)}{T}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)\sum_{k=0}^{2n-2}S_k(\lambda_0,x)_{\underline{\mathbf{2}}}\partial_x\big(S_{2m-1}(\lambda_0,x)_{\underline{\mathbf{2}}}\big)S_{2n-2-k}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big).$$

Recalling the definition (4.6.3) of $\mathcal{N}^{\lambda_0}_p$, we obtain

$$\Xi^{\lambda_0\lambda_0}_{2n\,2m}(\rho,x,y)=\frac{\varphi'(\lambda_0)}{T}\frac{4nm(1-2n)}{2n+2m-2}\mathcal{N}^{\lambda_0}_{2n+2m-2}(\rho,x)\delta'_{xy}+f^{\lambda_0}_{2n\,2m}(\rho,x)\delta_{xy}. \tag{4.B.2}$$

As $\partial_x S_p(\lambda,x)=\sum_{l=0}^{p-1}S_l(\lambda,x)\partial_x\big(S(\lambda,x)\big)S_{p-1-l}(\lambda,x)$, one can rewrite the function $f^{\lambda_0}_{2n\,2m}$ as

$$f^{\lambda_0}_{2n\,2m}(\rho,x)=-\frac{4nm\varphi'(\lambda_0)}{T}\sum_{k=0}^{2n-2}\sum_{l=0}^{2m-2}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)\\ S_{k+l}(\lambda_0,x)_{\underline{\mathbf{2}}}\partial_x\big(S(\lambda_0,x)_{\underline{\mathbf{2}}}\big)S_{2n+2m-4-k-l}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big).$$

In particular, note that $f^{\lambda_0}_{2n\,2m}=f^{\lambda_0}_{2m\,2n}$.

Let us now compute $\Xi^{\lambda_0\lambda_0}_{nm}$ for a non-cyclotomic regular zero λ_0 and an algebra $\mathfrak{g}$ of type A. As in the case of type B, C or D, we have $U_{\underline{\mathbf{23}}}(\lambda_0,\lambda_0)=-\frac{1}{T}\varphi'(\lambda_0)C_{\underline{\mathbf{23}}}$. Using the generalised completeness relation (4.3.8) and the fact that $\mathscr{J}^{\lambda_0}_p(x)=\mathrm{Tr}\big(S_p(\lambda_0,x)\big)$, we find from equation (4.6.12) that

$$\Xi^{\lambda_0\lambda_0}_{n\,m}(\rho,x,y)=-\frac{nm\varphi'(\lambda_0)}{T}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)\sum_{k=0}^{n-2}S_k(\lambda_0,x)_{\underline{\mathbf{2}}}S_{m-1}(\lambda_0,y)_{\underline{\mathbf{2}}}S_{n-2-k}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}\\ +\frac{nm(n-1)\varphi'(\lambda_0)}{dT}\mathscr{J}^{\lambda_0}_{m-1}(y)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)S_{n-2}(\lambda_0,x)\Big)\delta'_{xy}.$$

From the identity $f(y)\delta'_{xy}=f(x)\delta'_{xy}+\big(\partial_x f(x)\big)\delta_{xy}$ and equation (4.6.3), we find

$$\Xi^{\lambda_0\lambda_0}_{n\,m}(\rho,x,y)=-\frac{\varphi'(\lambda_0)}{T}\frac{nm(n-1)}{n+m-2}\mathcal{N}^{\lambda_0}_{n+m-2}(\rho,x)\delta'_{xy}\\ +\frac{\varphi'(\lambda_0)}{dT}nm\,\mathscr{J}^{\lambda_0}_{m-1}(y)\mathcal{N}^{\lambda_0}_{n-1}(\rho,x)\delta'_{xy}+f^{\lambda_0}_{nm}(\rho,x)\delta_{xy},$$

with

$$f^{\lambda_0}_{nm}(\rho,x)=-\frac{nm\varphi'(\lambda_0)}{T}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)\sum_{k=0}^{n-2}S_k(\lambda_0,x)_{\underline{\mathbf{2}}}\partial_x\big(S_{m-1}(\lambda_0,x)_{\underline{\mathbf{2}}}\big)S_{n-2-k}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big).$$

As in the case of type B, C or D, we can re-express $f^{\lambda_0}_{nm}$ as

$$f^{\lambda_0}_{nm}(\rho,x)=-\frac{nm\varphi'(\lambda_0)}{T}\sum_{k=0}^{n-2}\sum_{l=0}^{m-2}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda_0)S_{k+l}(\lambda_0,x)_{\underline{\mathbf{2}}}\partial_x\big(S(\lambda_0,x)_{\underline{\mathbf{2}}}\big)S_{n+m-4-k-l}(\lambda_0,x)_{\underline{\mathbf{2}}}\Big).$$

In particular, note that $f^{\lambda_0}_{nm}=f^{\lambda_0}_{mn}$.

4.B.2 Around a cyclotomic regular zero

This subsection is devoted to the computation of $\Xi^{\lambda\lambda}_{nm}(\rho,x,y)$ around the origin $\lambda=0$ and more precisely to the computation of the coefficient of $\lambda^{r_n+r_m}$ in its series expansion. Our starting point is the definition (4.6.12) of $\Xi^{\lambda\mu}_{nm}$. To evaluate this equation at $\mu=\lambda$, we will need the expression of $U_{\underline{\mathbf{12}}}(\lambda,\lambda)$. We saw in section 4.4 that this is given by equation (4.4.8).

The presence of the partial Casimir $C^{(0)}_{\underline{\mathbf{12}}}$ in this equation will gives rise to projections of $S_{m-1}(\lambda,y)$ onto the grading $F^{(0)}=\{Z\in F\,|\,\sigma(Z)=Z\}$ of the matrix algebra. More precisely, the calculations will involve

$$g^{\lambda}_{mn}(\rho,x,y)=nm\lambda^{-2}\zeta(\lambda^T)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda)\sum_{k=0}^{n-2}S_k(\lambda,x)_{\underline{\mathbf{2}}}S^{(0)}_{m-1}(\lambda,y)_{\underline{\mathbf{2}}}S_{n-2-k}(\lambda,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy},\tag{4.B.3}$$

where $S^{(0)}_p$ denotes the projection of S_p on $F^{(0)}$. As we are computing the coefficient of $\lambda^{r_n+r_m}$ in $\Xi^{\lambda\lambda}_{nm}$, we will consider the $\lambda^{r_n+r_m}$-term of g^{λ}_{nm}. Let us show that this term is actually always zero. Using the conventions and results of the subsections 4.4.1 and 4.4.2, in particular the integers α and q_m, we see that the smallest power of λ appearing in g^{λ}_{nm} is $a=\alpha T-2+q_m$, as the $S_p(\lambda,x)$'s are regular at $\lambda=0$. Recall that r_n and r_m are both strictly less than $T-1$ when $n,m\in\mathcal{E}_0$ and that in this case, we have $q_m=r_m+1$ (see subsection 4.4.2). Thus $a=\alpha T-1+r_m$ and hence $a>r_n+r_m$ since $\alpha\geq 1$ and $T-1>r_n$. We can then conclude that the coefficient of $\lambda^{r_n+r_m}$ in g^{λ}_{nm} vanishes, as announced.

We will also need the function

$$f^{\lambda}_{nm}(\rho,x)=-nm\lambda^{T-2}\zeta'(\lambda)\sum_{k=0}^{n-2}\sum_{l=0}^{m-2}\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda)S_{k+l}(\lambda,x)_{\underline{\mathbf{2}}}\partial_x\big(S(\lambda,x)_{\underline{\mathbf{2}}}\big)S_{n+m-4-k-l}(\lambda,x)_{\underline{\mathbf{2}}}\Big),$$

similar to the function $f^{\lambda_0}_{nm}$ defined in the non-cyclotomic case (see previous subsection) and which possesses the same symmetry property $f^{\lambda}_{nm}=f^{\lambda}_{mn}$. As for g^{λ}_{nm}, we will use more precisely the function $f^{(0)}_{nm}=f^{\lambda}_{nm}\big|_{\lambda^{r_n+r_m}}$, which is also symmetric under the exchange of n and m.

To go further in the computation, we will need to distinguish between the algebras of type B, C and D and the ones of type A. Let us start with types B, C and D. In this case, we restrict to degrees $2n$ and $2m$ (see subsections 4.3.2 and 4.4.3) and thus compute $\Xi^{\lambda\lambda}_{2n\,2m}$. Recall that $S_{2m-1}(\lambda,y)$ belongs to the Lie algebra $\mathfrak{g}$, so that we can apply the completeness relations (4.2.13) and (4.4.10) to it. One then gets

$$\Xi^{\lambda\lambda}_{2n\,2m}(\rho,x,y)=\widetilde{\Xi}^{\lambda\lambda}_{2n\,2m}(\rho,x,y)+g^{\lambda}_{2n\,2m}(\rho,x,y),$$

with $g^{\lambda}_{2n\,2m}$ defined in equation (4.B.3) and

$$\widetilde{\Xi}^{\lambda\lambda}_{2n\,2m}(\rho,x,y)=-4nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda)\sum_{k=0}^{2n-2}S_k(\lambda,x)_{\underline{\mathbf{2}}}S_{2m-1}(\lambda,y)_{\underline{\mathbf{2}}}S_{2n-2-k}(\lambda,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}.$$

The first term $\widetilde{\Xi}^{\lambda\lambda}_{2n\,2m}$ has the same structure as $\Xi^{\lambda_0\lambda_0}_{2n\,2m}$ studied in the previous subsection (see equation (4.B.1)). Thus, the calculations of that subsection apply here and we get to an equation similar to (4.B.2) for $\widetilde{\Xi}^{\lambda\lambda}_{2n\,2m}$. Namely, we have

$$\Xi^{\lambda\lambda}_{2n\,2m}(\rho,x,y)=f^{\lambda}_{2n\,2m}(\rho,x)\delta_{xy}+g^{\lambda}_{2n\,2m}(\rho,x,y)-\lambda^{T-2}\zeta'(\lambda^T)\frac{4nm(2n-1)}{2n+2m-2}\mathcal{N}_{2n+2m-2}(\lambda\,;\rho,x)\delta'_{xy},\tag{4.B.4}$$

where $f^{\lambda}_{2n\,2m}$ is defined above.

We now compute the coefficient of $\lambda^{r_{2n}+r_{2m}}$ in this expression. We showed above that $g^{\lambda}_{2n\,2m}$ does not contribute to this term and we have defined $f^{(0)}_{2n\,2m}$ as its contribution from $f^{\lambda}_{2n\,2m}$. Recall also

that $\mathcal{N}_k(\lambda\,;\rho,x)$ has the same equivariance property as $\mathscr{T}_k(\lambda,x)$ (equation (4.6.7)) so that its power series expansion starts with λ^{r_k}. Thus, the smallest power of λ in the second line of equation (4.B.4) is greater than or equal to $T-2+r_{2n+2m-2}$. We have shown in subsection 4.4.5 that this is equal to $r_{2n}+r_{2m}$ or $r_{2n}+r_{2m}+T$, depending on whether $r_{2n}+r_{2m}$ is greater than or strictly less than $T-2$. We find

$$\Xi^{\lambda\lambda}_{2n\,2m}(\rho,x,y)\Big|_{\lambda^{r_{2n}+r_{2m}}} = f^{(0)}_{2n\,2m}(\rho,x)\delta_{xy} - \theta_{r_{2n}+r_{2m}+2-T}\,\zeta'(0)\frac{4nm(2n-1)}{2n+2m-2}\mathcal{N}^0_{2n+2m-2}(\rho,x)\delta'_{xy}. \tag{4.B.5}$$

Finally, let us study the case of a Lie algebra of type A, *i.e.* of $\mathfrak{g}=\mathfrak{sl}(d,\mathbb{C})$. The term in $\Xi^{\lambda\lambda}_{nm}$ involving the Casimir $C_{\underline{\mathbf{12}}}$ is treated with the generalised completeness relation (4.3.8). In the same way, one has a generalised completeness relation for the partial Casimir $C^{(0)}_{\underline{\mathbf{12}}}$. This relation depends on whether the extension of σ to the whole algebra of matrices fixes the identity Id or not, and thus whether σ is inner or not (see appendix 4.A and subsections 4.4.4 and 4.4.5). In general, one can write

$$\mathrm{Tr}_{\underline{\mathbf{12}}}\left(C^{(0)}_{\underline{\mathbf{12}}}Z_{\underline{\mathbf{2}}}\right) = \pi^{(0)}(Z) - \frac{a}{d}\mathrm{Tr}(Z),$$

for any $Z\in M_d(\mathbb{C})$, with $a=1$ if σ is inner and $a=0$ if not. Using this relation and the relation (4.3.8), one finds

$$\Xi^{\lambda\lambda}_{nm}(\rho,x,y) = \widetilde{\Xi}^{\lambda\lambda}_{nm}(\rho,x,y) + g^{\lambda}_{nm}(\rho,x,y),$$

with g^{λ}_{nm} defined in equation (4.B.3) and

$$\begin{aligned}\widetilde{\Xi}^{\lambda\lambda}_{nm}(\rho,x,y) = -nm\lambda^{T-2}\zeta'(\lambda^T)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda)\sum_{k=0}^{n-2}S_k(\lambda,x)_{\underline{\mathbf{2}}}S_{m-1}(\lambda,y)_{\underline{\mathbf{2}}}S_{n-2-k}(\lambda,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}\\ + \frac{nm(n-1)}{d}\frac{\lambda^T\zeta'(\lambda^T)-a\zeta(\lambda^T)}{\lambda^2}\mathscr{T}_{m-1}(\lambda,y)\mathrm{Tr}_{\underline{\mathbf{2}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(\rho,\lambda)S_{n-2}(\lambda,x)_{\underline{\mathbf{2}}}\Big)\delta'_{xy}.\end{aligned}$$

The first term in this expression is treated in the same way as in the case of types B, C and D. Moreover, we recognise in the second term the definition of $\mathcal{N}_{n-1}(\lambda\,;\rho,x)$. Finally, we obtain

$$\begin{aligned}\Xi^{\lambda\lambda}_{nm}(\rho,x,y) = f^{\lambda}_{nm}(\rho,x)\delta_{xy} + g^{\lambda}_{nm}(\rho,x,y) - \lambda^{T-2}\zeta'(\lambda^T)\frac{nm(n-1)}{n+m-2}\mathcal{N}_{n+m-2}(\lambda\,;\rho,x)\delta'_{xy},\\ + \frac{nm}{d}\frac{\lambda^T\zeta'(\lambda^T)-a\zeta(\lambda^T)}{\lambda^2}\mathscr{T}_{m-1}(\lambda,y)\mathcal{N}_{n-1}(\lambda\,;\rho,x)\delta'_{xy}.\end{aligned} \tag{4.B.6}$$

We now compute the coefficient of $\lambda^{r_n+r_m}$ in $\Xi^{\lambda\lambda}_{nm}$. As explained at the beginning of this subsection, g^{λ}_{nm} does not contribute to this term and the contribution of f^{λ}_{nm} is defined as $f^{(0)}_{nm}$. The contribution from the third term is calculated as in the case of types B, C and D. In particular, it vanishes when r_n+r_m is strictly less than $T-2$.

Finally, let us discuss the contribution of the last term. First of all, we note that if $a=1$, $\lambda^T\zeta'(\lambda^T)-a\zeta(\lambda^T)=O(\lambda^{2T})$. Thus the powers of λ in this term are greater than $2T-2$. Yet, we have $r_n+r_m<2T-2$ for $n,m\in\mathcal{E}_0$, so this term does not contribute to the $\lambda^{r_n+r_m}$-term in this case. Hence for σ inner, we have

$$\Xi^{\lambda\lambda}_{nm}(\rho,x,y)\Big|_{\lambda^{r_n+r_m}} = f^{(0)}_{nm}(\rho,x)\delta_{xy} - \theta_{r_n+r_m+2-T}\,\zeta'(0)\frac{nm(n-1)}{n+m-2}\mathcal{N}^0_{n+m-2}(\rho,x)\delta'_{xy}, \tag{4.B.7}$$

as in the case of types B, C and D.

Suppose now that σ is not inner, so that $a=0$. Then the smallest power of λ in the last term of equation (4.B.6) is greater than or equal to $T-2+r_{n-1}+r_{m-1}$. In subsection 4.4.5, we have shown that this is equal to r_n+r_m if both r_n and r_m are greater than $S-1$ and that it is strictly greater than r_n+r_m otherwise. In conclusion, we find

$$\begin{aligned}\Xi^{\lambda\lambda}_{nm}(\rho,x,y)\Big|_{\lambda^{r_n+r_m}} = f^{(0)}_{nm}(\rho,x)\delta_{xy} - \theta_{r_n+r_m+2-T}\,\zeta'(0)\frac{nm(n-1)}{n+m-2}\mathcal{N}^0_{n+m-2}(\rho,x)\delta'_{xy}\\ + \theta_{r_n+1-S}\theta_{r_m+1-S}\,\zeta'(0)\,\frac{nm}{d}\,\mathscr{J}^0_{m-1}(y)\mathcal{N}^0_{n-1}(\rho,x)\delta'_{xy}.\end{aligned} \tag{4.B.8}$$

Chapter 5

Deformed symmetries of Yang-Baxter deformations as Poisson-Lie symmetries

This chapter is based on the article [P2], that I wrote during my PhD with F. Delduc, M. Magro and B. Vicedo. The content of this chapter is the same as the one of [P2] and is made to be read independently (the publication is thus not included at the end of this thesis).

In the chapter 3, we introduced the Yang-Baxter type deformations of σ-models, which include the (one-parameter) η-deformations of the PCM (Subsection 3.3.1) and of the $\mathbb{Z}_2$-coset σ-model (Subsection 3.3.3), as well as the BYB model (Section 3.4), which is a combination of two Yang-Baxter deformations. One of the characteristics of these deformations is that it breaks a global symmetry of the model: the global left multiplication symmetry for the one-parameter deformations of the PCM and $\mathbb{Z}_2$-coset model and both the left and right multiplication symmetries for the BYB model.

This global symmetry of the undeformed model is associated with a set of conserved charges (see Section 3.2), which satisfies the Kirillov-Kostant bracket of the underlying Lie algebra $\mathfrak{g}_0$ (see appendices B.2 and B.4). It was already observed in previous articles about Yang-Baxter type deformations [29, 32, P1, 34, 35, 144] that the deformed models admit a set of deformed conserved charges, which satisfy a q-deformed algebra (technically the semi-classical limit $\mathscr{U}_q(\mathfrak{g}_0)$ of a quantum group). In the article [P2], I have shown, together with my collaborators, that this is a general feature of all Yang-Baxter type deformations. The originality of the approach we developed in [P2] is that it is model independent: it relies mostly on the effect of the deformation on the twist function of the model.

A natural question emerging from the existence of these deformed conserved charges is whether they are associated with a symmetry of the model. This question is made difficult by the fact that the charges do not satisfy a Kirillov-Kostant algebra but rather a q-deformed algebra $\mathscr{U}_q(\mathfrak{g}_0)$. Indeed, charges satisfying the Kirillov-Kostant bracket of the Lie algebra $\mathfrak{g}_0$ form the moment map of an infinitesimal Hamiltonian action of $\mathfrak{g}_0$ (see Appendix B.4). However, this is not the case for charges satisfying a q-deformed algebra $\mathscr{U}_q(\mathfrak{g}_0)$. Instead, these charges form the so-called non-abelian moment map of a Poisson-Lie action of $\mathfrak{g}_0$, which is a generalisation of a Hamiltonian action.

Using the theory of Poisson-Lie actions, we constructed in [P2] the symmetry associated with the deformed conserved charges of Yang-Baxter type deformations, in a model-independent way. It is a transformation of the fields of the theory, depending on the deformation parameter, which is a symmetry of the deformed model and which reduces to the global symmetry of the undeformed one when the deformation parameter goes to zero. This transformation is the main subject of this chapter.

The chapter is constructed as follows. Sections 5.1 to 5.3 are a general review of the theory of Poisson-Lie groups and their actions. Section 5.4 establishes the link between Poisson-Lie actions and the q-deformed algebra $\mathscr{U}_q(\mathfrak{g}_0)$ (which is a generalisation of the link between Hamiltonian actions and the Kirillov-Kostant bracket). Finally, Section 5.5 is the application of the Poisson-Lie formalism to the study of deformed symmetries in Yang-Baxter type deformations. Some technical results, specific to this chapter, are gathered in Appendix 5.A.

5.1 Poisson-Lie groups and Drinfel'd doubles

In this section, we recall the main points of the general theory of Poisson-Lie groups[1] and their link to Lie bialgebras, including the formulation in terms of Drinfel'd doubles.

5.1.1 Poisson-Lie groups and Lie bialgebras

A Poisson-Lie group is a real Lie group G_0 equipped with a Poisson bracket $\{\cdot,\cdot\}_{G_0}$ which is compatible with the multiplication $G_0 \times G_0 \to G_0$ in the sense that the latter is a Poisson map.

Consider the dual space $\mathfrak{g}_0^*$ of the Lie algebra $\mathfrak{g}_0$. As $\mathfrak{g}_0 \simeq T_eG_0$, any element in $\mathfrak{g}_0^*$ can be realised as the differential $\mathrm{d}_e f : T_eG_0 \to \mathbb{R}$ of a smooth function $f : G_0 \to \mathbb{R}$, taken at the identity e. Using this, we define a skew-symmetric product on $\mathfrak{g}_0^*$ by

$$[d_e f, d_e g]_* = d_e \{f, g\}_{G_0} . \tag{5.1.1}$$

One can show that this product is well defined, *i.e.* that the results only depend on $d_e f$ and $d_e g$ and not on the choice of f and g. Using the Jacobi identity of the Poisson bracket, one finds that $[\cdot,\cdot]_*$ also satisfies the Jacobi identity, so that

$$(\mathfrak{g}_0^*, [\cdot,\cdot]_*)$$

is a Lie algebra. The Lie bracket $[\cdot,\cdot]_*$ can be seen as a skew-symmetric map $\delta^* : \mathfrak{g}_0^* \otimes \mathfrak{g}_0^* \to \mathfrak{g}_0^*$. Using the compatibility of the Poisson bracket $\{\cdot,\cdot\}_{G_0}$ with the multiplication on G_0, one can show that the dual map $\delta : \mathfrak{g}_0 \to \mathfrak{g}_0 \otimes \mathfrak{g}_0$ is a 1-cocycle, *i.e.* that it verifies

$$\delta([X,Y]) = (\mathrm{ad}_X \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{ad}_X)\,\delta(Y) - (\mathrm{ad}_Y \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{ad}_Y)\,\delta(X), \tag{5.1.2}$$

with the adjoint actions ad_X and ad_Y as defined in Appendix A.1. This proves that $(\mathfrak{g}_0, \mathfrak{g}_0^*)$ is a Lie bialgebra (see for instance [156]). Conversely, from any Lie bialgebra, one can define a unique connected and simply connected Poisson-Lie group.

5.1.2 Drinfel'd doubles

Let G_0 be a Poisson-Lie group, with Lie bialgebra $(\mathfrak{g}_0, \mathfrak{g}_0^*)$. We define the Drinfel'd double of $\mathfrak{g}_0$ as the vector space direct sum

$$D\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_0^*.$$

We will write ι and ι^* for the natural embeddings of $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ into $D\mathfrak{g}_0$ and we will denote elements of $D\mathfrak{g}_0$ as (X, λ), where X is in $\mathfrak{g}_0$ and λ is a linear form in $\mathfrak{g}_0^*$. One can define a non-degenerate bilinear form on the double $D\mathfrak{g}_0$ by

$$\big\langle (X,\lambda) | (Y,\mu) \big\rangle = \langle X, \mu \rangle + \langle Y, \lambda \rangle = \mu(X) + \lambda(Y) \tag{5.1.3}$$

for any $X, Y \in \mathfrak{g}_0$ and $\lambda, \mu \in \mathfrak{g}_0^*$, where $\langle \cdot,\cdot \rangle$ denotes the canonical pairing between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$. One then has the following result [156]:

Theorem 5.1.1. *There exists a unique Lie bracket $[\cdot,\cdot]_D$ on $D\mathfrak{g}_0$ such that ι and ι^* are Lie homomorphisms from $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ to $D\mathfrak{g}_0$, and such that the bilinear form $\langle \cdot | \cdot \rangle$ is ad-invariant.*

The decomposition $D\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_0^*$ satisfies the conditions for the application of the Adler-Kostant-Symes (AKS) scheme, described in Appendix C.2. Thus, if $\pi_{\mathfrak{g}_0}$ and $\pi_{\mathfrak{g}_0^*}$ denote the projections along this decomposition, the operator $\mathcal{R}^D = \pi_{\mathfrak{g}_0} - \pi_{\mathfrak{g}_0^*}$ of $D\mathfrak{g}_0$ is a solution of the split mCYBE on $D\mathfrak{g}_0$ (Theorem C.2.1). As $\langle \cdot | \cdot \rangle$ defines a non-degenerate bilinear form on $D\mathfrak{g}_0$, one can consider the kernel $\mathcal{R}^D_{\underline{\mathbf{12}}}$ of $\mathcal{R}^D$ with respect to this form (see Appendix C.2). It is clear from the definition (5.1.3) that the subspaces $\iota(\mathfrak{g}_0)$ and $\iota^*(\mathfrak{g}_0^*)$ of $D\mathfrak{g}_0$ are both isotropic with respect to $\langle \cdot | \cdot \rangle$: $\mathcal{R}^D_{\underline{\mathbf{12}}}$ is thus skew-symmetric

[1] There are many references on Poisson-Lie groups. For the aspects reviewed in the present article, we mainly refer to the articles [147–155] and to the books [15, 156]. Further references may be found in [157].

(see Proposition C.2.3). By Proposition C.1.2, $\mathcal{R}^D_{\underline{\mathbf{12}}}$ is a solution of the matricial split mCYBE on $D\mathfrak{g}_0$, which reads

$$\left[\mathcal{R}^D_{\underline{\mathbf{12}}},\mathcal{R}^D_{\underline{\mathbf{13}}}\right]_D+\left[\mathcal{R}^D_{\underline{\mathbf{12}}},\mathcal{R}^D_{\underline{\mathbf{23}}}\right]_D+\left[\mathcal{R}^D_{\underline{\mathbf{13}}},\mathcal{R}^D_{\underline{\mathbf{23}}}\right]_D=\left[C^D_{\underline{\mathbf{12}}},C^D_{\underline{\mathbf{13}}}\right]_D,$$

with $C^D_{\underline{\mathbf{12}}}$ the quadratic Casimir in $D\mathfrak{g}_0\otimes D\mathfrak{g}_0$. Let us consider a basis $\{I^a\}$ of $\mathfrak{g}_0$ and the dual basis $\{I_a\}$ of $\mathfrak{g}_0^*$. Then by Proposition C.2.3, $\mathcal{R}^D_{\underline{\mathbf{12}}}$ reads

$$\mathcal{R}^D_{\underline{\mathbf{12}}}=\iota(I^a)\otimes\iota^*(I_a)-\iota^*(I_a)\otimes\iota(I^a). \tag{5.1.4}$$

In the same way, the quadratic Casimir $C^D_{\underline{\mathbf{12}}}$ is given by

$$C^D_{\underline{\mathbf{12}}}=\iota(I^a)\otimes\iota^*(I_a)+\iota^*(I_a)\otimes\iota(I^a).$$

5.2 Poisson-Lie actions

In this section we study actions of Poisson-Lie groups on Poisson manifolds *via* the non-abelian moment map formulation.

5.2.1 Non-abelian moment map

Let G_0 be a Poisson-Lie group and M be a Poisson manifold, with Poisson bracket $\{\cdot,\cdot\}$. Let

$$\rho : G_0\times M\longrightarrow M$$

be a smooth group action of G_0 on M. We say that ρ is a Poisson-Lie action if it is a Poisson map, from $G_0\times M$, with the direct product Poisson structure, to M. The map ρ can alternatively be seen as a group homomorphism from G_0 to $\mathrm{Diff}(M)$, the group of diffeomorphisms of M. Its differential at the identity induces a Lie algebra action

$$\delta : \mathfrak{g}_0\longrightarrow T\big(\mathrm{Diff}(M)\big)=\mathcal{X}[M],$$

on the space $\mathcal{X}[M]$ of vector fields on M (see Appendix B.4). For $\epsilon\in\mathfrak{g}_0$, the vector field δ_ϵ acts naturally on any smooth function $f : M\to\mathbb{R}$. We consider the case where there exists a map

$$\Gamma : M\longrightarrow G_0^*,$$

where the dual group G_0^* is the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_0^*$, such that

$$\delta_\epsilon f=-\left\langle\epsilon,\Gamma^{-1}\left\{\Gamma,f\right\}\right\rangle. \tag{5.2.1}$$

The map Γ is called the non-abelian moment map of the action of G on M. If M is symplectic and simply connected, then such a map always exists. We can note here that Γ is defined up to a left multiplication by a constant element in G_0^*. Conversely, every transformation of the form (5.2.1) preserves the Poisson bracket if the parameter ϵ has a non-zero bracket with itself, coming from the Poisson-Lie structure on G_0.

To illustrate this concept, let us investigate here the case of a usual Hamiltonian action of G_0 on M. For any fixed g, the action $\rho(g,\cdot)$ is then a canonical transformation on M. In other words, ρ is a Poisson map for the trivial Poisson structure on G_0. The induced Lie bracket on $\mathfrak{g}_0^*$ defined by (5.1.1) is then trivial, so that the dual group G_0^* is abelian. We write $\Gamma=\exp(-Q)$, with $Q : M\to\mathfrak{g}_0^*$. As G_0^* is abelian, the transformation (5.2.1) simply becomes

$$\delta_\epsilon f=\langle\epsilon,\{Q,f\}\rangle. \tag{5.2.2}$$

We recognize here the usual expression (B.4.3) for a Hamiltonian action of G_0 on M, with Q the moment map (see Appendix B.4). When this action is a symmetry of a Hamiltonian system, decomposing Q with respect to the dual basis of $\mathfrak{g}_0^*$ as $Q=Q^aI_a$, we obtain $\dim G$ conserved charges Q^a.

5.2.2 Poisson brackets of the non-abelian moment map

Let us recall that since ρ is a Lie group action, δ is a Lie algebra action. In other words, δ is a Lie homomorphism which is to say that

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon,\epsilon']}, \tag{5.2.3}$$

for any $\epsilon, \epsilon' \in \mathfrak{g}_0$. In the case of a usual Hamiltonian action of G_0 on M, where δ_ϵ is given by equation (5.2.2), it is a well-known fact that the homomorphism condition (5.2.3) implies that the Poisson algebra of the charges Q^a takes the form

$$\{Q^a, Q^b\} = f^{ab}{}_c\, Q^c + N^{ab}$$

of the Lie algebra relations in $\mathfrak{g}_0$ up to central charges N^{ab}, where $f^{ab}{}_c$ are structure constants of $\mathfrak{g}_0$ with respect to the basis $\{I^a\}$, *i.e.*

$$[I^a, I^b] = f^{ab}{}_c\, I^c.$$

It is therefore natural to ask whether we can extract from equation (5.2.3) some informations on the Poisson bracket of Γ with itself. One important step is to note that, from equation (5.2.1), using the Jacobi and Leibniz identities on $\{\cdot,\cdot\}$, the action of $[\delta_\epsilon, \delta_{\epsilon'}]$ on any function f takes the following rather simple form

$$[\delta_\epsilon, \delta_{\epsilon'}]\, f = \left\langle \epsilon_{\underline{1}}\epsilon'_{\underline{2}}, \Gamma_{\underline{1}}^{-1}\Gamma_{\underline{2}}^{-1}\left\{\{\Gamma_{\underline{1}}, \Gamma_{\underline{2}}\}\Gamma_{\underline{1}}^{-1}\Gamma_{\underline{2}}^{-1}, f\right\}\Gamma_{\underline{1}}\Gamma_{\underline{2}}\right\rangle_{\underline{12}}. \tag{5.2.4}$$

In order to treat the right hand side of equation (5.2.3), we pass to the Drinfel'd double formulation recalled in section 5.1.2. Indeed, we can write

$$\delta_{[\epsilon,\epsilon']} f = -\left\langle [\iota(\epsilon), \iota(\epsilon')]_D \,\middle|\, \iota^*(\Gamma)^{-1}\left\{\iota^*(\Gamma), f\right\}\right\rangle,$$

where, by abuse of notation, we still denote by ι^* the lift of $\iota^* : \mathfrak{g}_0^* \hookrightarrow D\mathfrak{g}_0$ to the group embedding $G_0^* \hookrightarrow DG_0$. Using the expression (5.1.4) of $\mathcal{R}^D_{\underline{12}}$ we note that

$$\iota(\epsilon')_{\underline{1}} = \langle \iota(\epsilon')_{\underline{2}} | \mathcal{R}^D_{\underline{12}} \rangle_{\underline{2}} \quad \text{and} \quad \iota(\epsilon)_{\underline{2}} = -\langle \iota(\epsilon)_{\underline{1}} | \mathcal{R}^D_{\underline{12}} \rangle_{\underline{1}}.$$

From these two equations and the fact that the pairing $\langle\cdot|\cdot\rangle$ is invariant with respect to the $[\cdot,\cdot]_D$ bracket, we obtain

$$\delta_{[\epsilon,\epsilon']} f = -\frac{1}{2}\left\langle \iota(\epsilon)_{\underline{1}}\iota(\epsilon')_{\underline{2}} \,\middle|\, \left[\mathcal{R}^D_{\underline{12}}, \iota^*(\Gamma)_{\underline{1}}^{-1}\left\{\iota^*(\Gamma)_{\underline{1}}, f\right\} + \iota^*(\Gamma)_{\underline{2}}^{-1}\left\{\iota^*(\Gamma)_{\underline{2}}, f\right\}\right]_D\right\rangle_{\underline{12}}.$$

This expression can be rewritten as

$$\delta_{[\epsilon,\epsilon']} f = \frac{1}{2}\left\langle \iota(\epsilon)_{\underline{1}}\iota(\epsilon')_{\underline{2}} \,\middle|\, \iota^*(\Gamma)_{\underline{1}}^{-1}\iota^*(\Gamma)_{\underline{2}}^{-1}\left\{\iota^*(\Gamma)_{\underline{1}}\iota^*(\Gamma)_{\underline{2}}\mathcal{R}^D_{\underline{12}}\iota^*(\Gamma)_{\underline{1}}^{-1}\iota^*(\Gamma)_{\underline{2}}^{-1}, f\right\}\iota^*(\Gamma)_{\underline{1}}\iota^*(\Gamma)_{\underline{2}}\right\rangle_{\underline{12}}. \tag{5.2.5}$$

Using the fact that the pairing $\langle\cdot|\cdot\rangle$ is non-degenerate between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, by equating (5.2.4) and (5.2.5) we arrive at

$$\left\{\iota^*(\Gamma)_{\underline{1}}, \iota^*(\Gamma)_{\underline{2}}\right\}\iota^*(\Gamma)_{\underline{1}}^{-1}\iota^*(\Gamma)_{\underline{2}}^{-1} = \frac{1}{2}\iota^*(\Gamma)_{\underline{1}}\iota^*(\Gamma)_{\underline{2}}\mathcal{R}^D_{\underline{12}}\iota^*(\Gamma)_{\underline{1}}^{-1}\iota^*(\Gamma)_{\underline{2}}^{-1} + P_{\underline{12}} \tag{5.2.6}$$

where the element $P_{\underline{12}} \in D\mathfrak{g}_0 \otimes D\mathfrak{g}_0$ is a central charge of the Poisson bracket $\{\cdot,\cdot\}$. Let us study the properties that must fulfil $P_{\underline{12}}$. First of all, it must be skew-symmetric. Moreover, it should be such that the right hand side of (5.2.6) lives in $\iota^*(\mathfrak{g}_0^*) \otimes \iota^*(\mathfrak{g}_0^*)$. It is a well-known consequence of the Adler-Kostant-Symes construction that, for any $y \in \iota^*(G_0^*)$, we have $y_{\underline{1}}y_{\underline{2}}\mathcal{R}^D_{\underline{12}}y_{\underline{1}}^{-1}y_{\underline{2}}^{-1} - \mathcal{R}^D_{\underline{12}} \in \iota^*(\mathfrak{g}_0^*) \otimes \iota^*(\mathfrak{g}_0^*)$. Thus, defining $N_{\underline{12}} = P_{\underline{12}} + \frac{1}{2}\mathcal{R}^D_{\underline{12}}$, we can write

$$\left\{\iota^*(\Gamma)_{\underline{1}}, \iota^*(\Gamma)_{\underline{2}}\right\} = -\frac{1}{2}\left[\mathcal{R}^D_{\underline{12}}, \iota^*(\Gamma)_{\underline{1}}\iota^*(\Gamma)_{\underline{2}}\right] + N_{\underline{12}}\iota^*(\Gamma)_{\underline{1}}\iota^*(\Gamma)_{\underline{2}}, \tag{5.2.7}$$

with $N_{\underline{\mathbf{12}}} \in \iota^*(\mathfrak{g}_0^*) \otimes \iota^*(\mathfrak{g}_0^*)$ skew-symmetric. The last requirement on $N_{\underline{\mathbf{12}}}$ is that the Poisson bracket (5.2.7) must satisfy the Jacobi identity. Let us first remark that this is the case when $N_{\underline{\mathbf{12}}} = 0$, as $\mathcal{R}^D_{\underline{\mathbf{12}}}$ verifies the mCYBE. We will see in the next sections why this case is of particular interest.

More generally, we recognise in (5.2.7) a quadratic algebra of *ad*-type, in the nomenclature of [85,86]. In this case, a necessary and sufficient condition for the Jacobi identity to hold is that $N_{\underline{\mathbf{12}}} - \frac{1}{2}\mathcal{R}^D_{\underline{\mathbf{12}}}$ satisfies the mCYBE. In particular, this is the case if

$$N_{\underline{\mathbf{12}}} = \frac{1}{2}\mathcal{R}^D_{\underline{\mathbf{12}}} - \frac{1}{2}\iota^*(C)^{-1}_{\underline{\mathbf{1}}}\iota^*(C)^{-1}_{\underline{\mathbf{2}}}\mathcal{R}^D_{\underline{\mathbf{12}}}\iota^*(C)_{\underline{\mathbf{1}}}\iota^*(C)_{\underline{\mathbf{2}}},$$

for some constant $C \in G_0^*$. For this $N_{\underline{\mathbf{12}}}$ we define $\tilde{\Gamma} = C\Gamma$. As noted in subsection 5.2.1, $\tilde{\Gamma}$ is still a good non-abelian moment map. Moreover, the Poisson bracket of $\tilde{\Gamma}$ becomes

$$\{\iota^*(\tilde{\Gamma})_{\underline{\mathbf{1}}}, \iota^*(\tilde{\Gamma})_{\underline{\mathbf{2}}}\} = -\frac{1}{2}[\mathcal{R}^D_{\underline{\mathbf{12}}}, \iota^*(\tilde{\Gamma})_{\underline{\mathbf{1}}}\iota^*(\tilde{\Gamma})_{\underline{\mathbf{2}}}].$$

Conversely, if the Poisson brackets of Γ are of the form (5.2.7), then the transformation (5.2.1) is a Lie algebra action of $\mathfrak{g}_0$, *i.e.* it satisfies (5.2.3).

5.3 Coboundary Poisson-Lie groups and R-matrices

One important class of Lie bialgebras are the so-called coboundary ones, which are given by R-matrices, solutions of the mCYBE (see Appendix C.1). In this section, we recall their properties and apply the abstract result (5.2.7) of the previous section to this particular case.

5.3.1 R-matrices, Sklyanin bracket and $\mathfrak{g}_R$ dual algebra

Let $\mathfrak{g}_0$ be a Lie algebra and $R : \mathfrak{g}_0 \to \mathfrak{g}_0$ a linear map solution of the mCYBE on $\mathfrak{g}_0$, namely

$$[RX, RY] - R\big([RX, Y] + [X, RY]\big) = -c^2[X, Y], \tag{5.3.1}$$

for all $X, Y \in \mathfrak{g}_0$ with $c = 1$ (split case) or $c = i$ (non-split case). We define $R^{\pm} = R \pm c\,\mathrm{Id}$, and introduce the R-bracket

$$[X, Y]_R = \gamma\big([RX, Y] + [X, RY]\big) = \gamma\big([R^{\pm}X, Y] + [X, R^{\mp}Y]\big),$$

with γ a real constant. An important consequence of the mCYBE is that the vector space $\mathfrak{g}_0$ equipped with the R-bracket is also a Lie algebra. We therefore have two Lie algebra structures on the vector space $\mathfrak{g}_0$: the usual one $(\mathfrak{g}_0, [\cdot,\cdot])$, that we shall still note $\mathfrak{g}_0$ and

$$\mathfrak{g}_R = (\mathfrak{g}_0, [\cdot,\cdot]_R).$$

This construction is related to Poisson-Lie groups. Suppose now that $\mathfrak{g}_0$ is semi-simple and let κ denote its Killing form. Let R be a skew-symmetric solution of the mCYBE on $\mathfrak{g}_0$. We denote by $R_{\underline{\mathbf{12}}} \in \mathfrak{g}_0 \otimes \mathfrak{g}_0$ its kernel with respect to κ. One can then define a Poisson-Lie structure on G_0 with the Sklyanin bracket

$$\{x_{\underline{\mathbf{1}}}, x_{\underline{\mathbf{2}}}\}_{G_0} = \gamma\,[R_{\underline{\mathbf{12}}}, x_{\underline{\mathbf{1}}}x_{\underline{\mathbf{2}}}]\,.$$

Since $\mathfrak{g}_0$ is semi-simple, its Killing form κ is non-degenerate. This allows us to define a natural pairing π between $\mathfrak{g}_0$ and its dual $\mathfrak{g}_0^*$ by considering, for any $X \in \mathfrak{g}_0$, the linear form

$$\begin{array}{rccc} \pi(X): & \mathfrak{g}_0 & \longrightarrow & \mathbb{R} \\ & Y & \longmapsto & \kappa(X,Y) \end{array}$$

As a vector space, $\mathfrak{g}_R$ is equal to $\mathfrak{g}_0$, so π can be seen as a linear isomorphism from $\mathfrak{g}_R$ to $\mathfrak{g}_0^*$. The following lemma then gives a concrete realisation of the dual Lie algebra $\mathfrak{g}_0^*$.

Lemma 5.3.1. *Let G_0 be a Poisson-Lie group, with the Sklyanin bracket associated with a solution R of the mCYBE on $\mathfrak{g}_0$. Equip $\mathfrak{g}_0^*$ with the Lie bracket (5.1.1). Then the map*

$$\pi : \mathfrak{g}_R \longrightarrow \mathfrak{g}_0^*$$

is a Lie algebra isomorphism.

5.3.2 Real and complex doubles and the dual Lie algebras $\mathfrak{g}_{DR}$ and $\mathfrak{g}_{\pm}$

In this section, we study separately the split and non-split cases.

Split case. Define the real double of $\mathfrak{g}_0$ as the Lie algebra direct sum $\mathfrak{g}_0 \oplus \mathfrak{g}_0$. We introduce the subspaces

$$\mathfrak{g}_{0,\mathrm{diag}} = \big\{(X, X),\, X \in \mathfrak{g}_0\big\}, \qquad \mathfrak{g}_{DR} = \big\{(R^+X, R^-X),\, X \in \mathfrak{g}_0\big\}.$$

It is clear that, for any endomorphism R, $\mathfrak{g}_{0,\mathrm{diag}}$ and $\mathfrak{g}_{DR}$ form a direct sum decomposition of $\mathfrak{g}_0 \oplus \mathfrak{g}_0$. Moreover, $\mathfrak{g}_{0,\mathrm{diag}}$ is a Lie subalgebra of $\mathfrak{g}_0 \oplus \mathfrak{g}_0$. One shows that, when R is a split solution of the mCYBE, $\mathfrak{g}_{DR}$ is also a Lie subalgebra of $\mathfrak{g}_0 \oplus \mathfrak{g}_0$ isomorphic to $\mathfrak{g}_R$. More precisely,

Lemma 5.3.2. *If R is a solution of the split mCYBE on $\mathfrak{g}_0$, then*

$$\begin{array}{rccc} \Delta : & \mathfrak{g}_R & \longrightarrow & \mathfrak{g}_{DR} \\ & X & \longmapsto & \gamma(R^+X, R^-X) \end{array}$$

is a Lie algebra isomorphism, whose inverse is given for all $(X, Y) \in \mathfrak{g}_{DR} \subset \mathfrak{g}_0 \oplus \mathfrak{g}_0$ by

$$\Delta^{-1}(X, Y) = \frac{1}{2\gamma}(X - Y). \tag{5.3.2}$$

We have obtained yet another realisation of $\mathfrak{g}_0^*$, this time in the real double. Moreover the subalgebra $\mathfrak{g}_{0,\mathrm{diag}}$ is isomorphic to $\mathfrak{g}_0$. Hence we have realised both $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ as subalgebras of the real double $\mathfrak{g}_0 \oplus \mathfrak{g}_0$. In fact, by the following lemma the real double $\mathfrak{g}_0 \oplus \mathfrak{g}_0$ itself is a realisation of the abstract Drinfel'd double $D\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_0^*$ (cf. section 5.1.2).

Lemma 5.3.3. *If R is a skew-symmetric solution of the split mCYBE on $\mathfrak{g}_0$, then*

$$\begin{array}{rccc} \Phi : & D\mathfrak{g}_0 & \longrightarrow & \mathfrak{g}_0 \oplus \mathfrak{g}_0 \\ & (X, \lambda) & \longmapsto & (X, X) + \Delta \circ \pi^{-1}(\lambda) \end{array}$$

is a Lie algebra isomorphism, such that $\Phi\big(\iota(\mathfrak{g}_0)\big) = \mathfrak{g}_{0,\mathrm{diag}}$ and $\Phi\big(\iota^*(\mathfrak{g}_0^*)\big) = \mathfrak{g}_{DR}$. Its inverse is given for every $(X, Y) \in \mathfrak{g}_0 \oplus \mathfrak{g}_0$ by

$$\Phi^{-1}(X, Y) = \frac{1}{2}\left(R^+Y - R^-X, \frac{1}{\gamma}\pi(X - Y)\right).$$

Moreover, Φ sends the pairing (5.1.3) on $D\mathfrak{g}_0$ to the non-degenerate ad-invariant bilinear form on $\mathfrak{g}_0 \oplus \mathfrak{g}_0$ defined, for all $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}_0$, by

$$\big\langle (X_1, Y_1) | (X_2, Y_2) \big\rangle = \frac{1}{2\gamma}\big(\kappa(X_1, X_2) - \kappa(Y_1, Y_2)\big).$$

Non-split case. One can perform a similar analysis in the case of a non-split solution of the mCYBE ($c = i$). Here we introduce the complex double $\mathfrak{g}_0^{\mathbb{C}}$ as the complexification of $\mathfrak{g}_0$ (see Appendix A.3), namely

$$\mathfrak{g}_0^{\mathbb{C}} = \{X + iY,\, X, Y \in \mathfrak{g}_0\}$$

We define the complex conjugation relative to the real form $\mathfrak{g}_0$ as

$$\begin{array}{rccc} \tau : & \mathfrak{g}_0^{\mathbb{C}} & \longrightarrow & \mathfrak{g}_0^{\mathbb{C}} \\ & X + iY & \longmapsto & X - iY \end{array}$$

This is a semi-linear involutive automorphism of $\mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{g}_0$ itself can be seen as a Lie subalgebra of $\mathfrak{g}_0^{\mathbb{C}}$, viewed as a real Lie algebra. More precisely, $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{g}_0^{\mathbb{C}}$ fixed by τ (see appendix A.3).

We introduce the subspaces

$$\mathfrak{g}_{\pm} = \{R^{\pm}X,\, X \in \mathfrak{g}_0\}$$

of $\mathfrak{g}_0^{\mathbb{C}}$. Note that $\mathfrak{g}_{\pm} = \tau(\mathfrak{g}_{\mp})$. We have the vector space decompositions $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_-$. Moreover, as a consequence of the mCYBE, $\mathfrak{g}_{\pm}$ are Lie subalgebras of $\mathfrak{g}_0^{\mathbb{C}}$ isomorphic to $\mathfrak{g}_R$.

Lemma 5.3.4. *If R is a solution of the non-split mCYBE on $\mathfrak{g}_0$, then*

$$\Delta_\pm = \gamma R^\pm : \mathfrak{g}_R \longrightarrow \mathfrak{g}_\pm$$

is a Lie algebra isomorphism, whose inverse is given for each $X \in \mathfrak{g}_\pm \subset \mathfrak{g}_0^{\mathbb{C}}$ by

$$\Delta_\pm^{-1}(X) = \pm \frac{X - \tau(X)}{2i\gamma}. \tag{5.3.3}$$

As in the split case, we realised $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ as subalgebras of $\mathfrak{g}_0^{\mathbb{C}}$. Moreover, the complex double $\mathfrak{g}_0^{\mathbb{C}}$ provides another realisation of the abstract Drinfel'd double $D\mathfrak{g}_0$ by the following result.

Lemma 5.3.5. *If R is a skew-symmetric solution of the non-split mCYBE on $\mathfrak{g}_0$, then*

$$\begin{array}{rccc} \Phi_\pm : & D\mathfrak{g}_0 & \longrightarrow & \mathfrak{g}_0^{\mathbb{C}} \\ & (X, \lambda) & \longmapsto & X + \gamma R^\pm \circ \pi^{-1}(\lambda) \end{array}$$

is a Lie algebra isomorphism, such that $\Phi_\pm\big(\iota(\mathfrak{g}_0)\big) = \mathfrak{g}_0$ and $\Phi_\pm\big(\iota^*(\mathfrak{g}_0^*)\big) = \mathfrak{g}_\pm$. Its inverse is given for any $X \in \mathfrak{g}_0^{\mathbb{C}}$ by

$$\Phi_\pm^{-1}(X) = \frac{1}{2i}\left(R^+\big(\tau(X)\big) - R^-(X), \pm\frac{1}{\gamma}\pi\big(X - \tau(X)\big)\right).$$

Moreover, $\Phi_\pm$ sends the pairing (5.1.3) on $D\mathfrak{g}_0$ to the non-degenerate ad-invariant bilinear form on $\mathfrak{g}_0^{\mathbb{C}}$ defined, for all $X, Y \in \mathfrak{g}_0^{\mathbb{C}}$, by

$$\langle X | Y \rangle = \pm\frac{1}{\gamma}\,\mathrm{Im}\big(\kappa(X, Y)\big).$$

5.3.3 Poisson-Lie action of G_0: Semenov-Tian-Shansky brackets

In the previous subsections, we provided concrete realisations of both the dual Lie algebra $\mathfrak{g}_0^*$ and the Drinfel'd double $D\mathfrak{g}_0$ for (split and non-split) coboundary Poisson-Lie groups. By abuse of notation, we will denote by the same symbols the lift of these realisations to the dual group G_0^* and the Drinfel'd double group DG_0. In section 5.2.2, we found an abstract expression (5.2.7) for the Poisson bracket of the non-abelian moment map viewed in the Drinfel'd double. We will now investigate what this Poisson bracket becomes in the concrete realisations of DG_0.

Split case. The non-abelian moment map Γ can be seen as a map to the group G_R *via* the Killing pairing π, namely

$$\Gamma_R = \pi^{-1}(\Gamma) \in G_R,$$

and in turn as an element of the group G_{DR} *via* the morphism Δ,

$$(\Gamma^+, \Gamma^-) = \Delta(\Gamma_R) = \Delta \circ \pi^{-1}(\Gamma) \in G_{DR} \subset G \times G.$$

The real double $G_0 \times G_0$ is related to the Drinfel'd double DG_0 by the morphism Φ (cf lemma 5.3.3). Let us remark here that:

$$\Phi\big(\iota^*(\Gamma)\big) = (\Gamma^+, \Gamma^-).$$

The Poisson bracket (5.2.7) then becomes, under the action of $\Phi_{\underline{\mathbf{1}}}\Phi_{\underline{\mathbf{2}}}$:

$$\big\{(\Gamma^+, \Gamma^-)_{\underline{\mathbf{1}}}, (\Gamma^+, \Gamma^-)_{\underline{\mathbf{2}}}\big\} = -\frac{1}{2}\left[\Phi_{\underline{\mathbf{1}}}\Phi_{\underline{\mathbf{2}}}\mathcal{R}^D_{\underline{\mathbf{12}}}, (\Gamma^+, \Gamma^-)_{\underline{\mathbf{1}}}(\Gamma^+, \Gamma^-)_{\underline{\mathbf{2}}}\right] + \tilde{N}_{\underline{\mathbf{12}}}(\Gamma^+, \Gamma^-)_{\underline{\mathbf{1}}}(\Gamma^+, \Gamma^-)_{\underline{\mathbf{2}}} \tag{5.3.4}$$

with $\widetilde{N}_{\underline{\mathbf{12}}} = \Phi_{\underline{\mathbf{1}}}\Phi_{\underline{\mathbf{2}}}N_{\underline{\mathbf{12}}}$ the central charge.

The objects in the above formula belongs to $(\mathfrak{g}_0 \oplus \mathfrak{g}_0) \otimes (\mathfrak{g}_0 \oplus \mathfrak{g}_0)$. Such objects can be written as vectors with four $\mathfrak{g}_0 \otimes \mathfrak{g}_0$-valued components as

$$(X, Y) \otimes (X', Y') = \begin{pmatrix} X \otimes X' \\ X \otimes Y' \\ Y \otimes X' \\ Y \otimes Y' \end{pmatrix}.$$

Let us now compute $\Phi_{\underline{\mathbf{1}}}\Phi_{\underline{\mathbf{2}}}\mathcal{R}^D_{\underline{\mathbf{12}}}$. We have

$$\Phi \circ \iota(I^a) = (I^a, I^a)$$

and

$$\Phi \circ \iota^*(I_a) = \Delta \circ \pi^{-1}(I_a) = \kappa_{ab}\Delta(I^b) = \gamma\kappa_{ab}(R^+I^b, R^-I^b)$$

where κ_{ab} is the inverse of the Killing form written in the basis $\{I^a\}$. We therefore find that

$$\Phi_{\underline{\mathbf{1}}}\Phi_{\underline{\mathbf{2}}}\mathcal{R}^D_{\underline{\mathbf{12}}} = -2\gamma \begin{pmatrix} R_{\underline{\mathbf{12}}} \\ R^+_{\underline{\mathbf{12}}} \\ R^-_{\underline{\mathbf{12}}} \\ R_{\underline{\mathbf{12}}} \end{pmatrix}. \tag{5.3.5}$$

Let us write

$$\widetilde{N}_{\underline{\mathbf{12}}} = \begin{pmatrix} N^{++}_{\underline{\mathbf{12}}} \\ N^{+-}_{\underline{\mathbf{12}}} \\ N^{-+}_{\underline{\mathbf{12}}} \\ N^{--}_{\underline{\mathbf{12}}} \end{pmatrix}.$$

The four components of the Poisson bracket (5.3.4) then read

$$\left\{\Gamma^+_{\underline{\mathbf{1}}}, \Gamma^+_{\underline{\mathbf{2}}}\right\} = \gamma\left[R_{\underline{\mathbf{12}}}, \Gamma^+_{\underline{\mathbf{1}}}\Gamma^+_{\underline{\mathbf{2}}}\right] + N^{++}_{\underline{\mathbf{12}}}\Gamma^+_{\underline{\mathbf{1}}}\Gamma^+_{\underline{\mathbf{2}}}, \tag{5.3.6a}$$

$$\left\{\Gamma^+_{\underline{\mathbf{1}}}, \Gamma^-_{\underline{\mathbf{2}}}\right\} = \gamma\left[R^+_{\underline{\mathbf{12}}}, \Gamma^+_{\underline{\mathbf{1}}}\Gamma^-_{\underline{\mathbf{2}}}\right] + N^{+-}_{\underline{\mathbf{12}}}\Gamma^+_{\underline{\mathbf{1}}}\Gamma^-_{\underline{\mathbf{2}}}, \tag{5.3.6b}$$

$$\left\{\Gamma^-_{\underline{\mathbf{1}}}, \Gamma^+_{\underline{\mathbf{2}}}\right\} = \gamma\left[R^-_{\underline{\mathbf{12}}}, \Gamma^-_{\underline{\mathbf{1}}}\Gamma^+_{\underline{\mathbf{2}}}\right] + N^{-+}_{\underline{\mathbf{12}}}\Gamma^-_{\underline{\mathbf{1}}}\Gamma^+_{\underline{\mathbf{2}}}, \tag{5.3.6c}$$

$$\left\{\Gamma^-_{\underline{\mathbf{1}}}, \Gamma^-_{\underline{\mathbf{2}}}\right\} = \gamma\left[R_{\underline{\mathbf{12}}}, \Gamma^-_{\underline{\mathbf{1}}}\Gamma^-_{\underline{\mathbf{2}}}\right] + N^{--}_{\underline{\mathbf{12}}}\Gamma^-_{\underline{\mathbf{1}}}\Gamma^-_{\underline{\mathbf{2}}}. \tag{5.3.6d}$$

When the central charges $N^{\pm\pm}_{\underline{\mathbf{12}}}$ vanish, these are the Semenov-Tian-Shansky brackets

$$\left\{\Gamma^\pm_{\underline{\mathbf{1}}}, \Gamma^\pm_{\underline{\mathbf{2}}}\right\} = \gamma\left[R_{\underline{\mathbf{12}}}, \Gamma^\pm_{\underline{\mathbf{1}}}\Gamma^\pm_{\underline{\mathbf{2}}}\right], \tag{5.3.7a}$$

$$\left\{\Gamma^\pm_{\underline{\mathbf{1}}}, \Gamma^\mp_{\underline{\mathbf{2}}}\right\} = \gamma\left[R^\pm_{\underline{\mathbf{12}}}, \Gamma^\pm_{\underline{\mathbf{1}}}\Gamma^\mp_{\underline{\mathbf{2}}}\right]. \tag{5.3.7b}$$

Finally, let us emphasise that the transformation law (5.2.1) can be re-expressed in terms of the non-abelian moment map (Γ^+, Γ^-) *via* the morphism $\Delta \circ \pi^{-1}$, giving explicitly

$$\delta_\epsilon f = -\frac{1}{2\gamma}\kappa\Big(\epsilon, (\Gamma^+)^{-1}\left\{\Gamma^+, f\right\} - (\Gamma^-)^{-1}\left\{\Gamma^-, f\right\}\Big). \tag{5.3.8}$$

Non-split case. From the non-abelian moment map $\Gamma_R = \pi^{-1}(\Gamma)$ seen in the group G_R, we can construct two different realisations of Γ in the complex double $G_0^{\mathbb{C}}$:

$$\Gamma^+ = \Delta_+(\Gamma_R) \in G_+ \quad \text{and} \quad \Gamma^- = \Delta_-(\Gamma_R) \in G_-.$$

These are not independent. They are related by the semi-linear automorphism τ as $\Gamma^{\pm} = \tau(\Gamma^{\mp})$. Note that

$$\Phi_{\pm} \circ \iota^*(\Gamma) = \Gamma^{\pm}.$$

For any $\eta, \varepsilon \in \{+,-\}$, applying $\Phi_{\eta\underline{1}}\Phi_{\varepsilon\underline{2}}$ to the bracket (5.2.7), we obtain

$$\left\{\Gamma^{\eta}_{\underline{1}}, \Gamma^{\varepsilon}_{\underline{2}}\right\} = -\frac{1}{2}\left[\Phi_{\eta\underline{1}}\Phi_{\varepsilon\underline{2}}\mathcal{R}^{D}_{\underline{12}}, \Gamma^{\eta}_{\underline{1}}\Gamma^{\varepsilon}_{\underline{2}}\right] + N^{\eta\varepsilon}_{\underline{12}}\Gamma^{\eta}_{\underline{1}}\Gamma^{\varepsilon}_{\underline{2}},$$

with the central charges $N^{\eta\varepsilon}_{\underline{12}} = \Phi_{\eta\underline{1}}\Phi_{\varepsilon\underline{2}}\tilde{N}_{\underline{12}}$. We have

$$\Phi_{\pm} \circ \iota(I^a) = I^a \quad \text{and} \quad \Phi_{\pm} \circ \iota^*(I_a) = \gamma R^{\pm}\big(\pi^{-1}(I_a)\big) = \gamma \sum_b \kappa_{ab} R^{\pm} I^b,$$

so that

$$\Phi_{\eta\underline{1}}\Phi_{\varepsilon\underline{2}}\mathcal{R}^{D}_{\underline{12}} = \gamma\big(R^{\varepsilon}_{\underline{21}} - R^{\eta}_{\underline{12}}\big) = -\gamma\big(R^{\eta}_{\underline{12}} + R^{-\varepsilon}_{\underline{12}}\big).$$

Thus, in the non-split case, the non-abelian moment maps Γ^+ and Γ^- also satisfy, up to central charges, the Semenov-Tian-Shansky Poisson brackets (5.3.7).

Applying $\Phi_{\pm}$ to equation (5.2.1) yields the transformation law in terms of the non-abelian moment map $\Gamma^{\pm}$ which reads

$$\delta_{\epsilon} f = \mp\frac{1}{\gamma}\,\mathrm{Im}\left(\kappa\big(\epsilon, (\Gamma^{\pm})^{-1}\left\{\Gamma^{\pm}, f\right\}\big)\right) = -\frac{1}{2i\gamma}\kappa\Big(\epsilon, (\Gamma^{+})^{-1}\left\{\Gamma^{+}, f\right\} - (\Gamma^{-})^{-1}\left\{\Gamma^{-}, f\right\}\Big). \tag{5.3.9}$$

5.3.4 Poisson-Lie action of G_0^*: Sklyanin bracket

In this section we consider the case of a coboundary Lie bialgebra specified by a split R-matrix. We have canonical isomorphisms $\mathfrak{g}^*_{DR} \simeq \mathfrak{g}_0^{**} \simeq \mathfrak{g}_0$ of vector spaces. Moreover, the dual Lie group $G_0^* \simeq G_{DR}$ is a Poisson-Lie group when equipped with the Semenov-Tian-Shansky bracket

$$\left\{x^{\pm}_{\underline{1}}, x^{\pm}_{\underline{2}}\right\} = \gamma\left[R_{\underline{12}}, x^{\pm}_{\underline{1}}x^{\pm}_{\underline{2}}\right], \tag{5.3.10a}$$

$$\left\{x^{\pm}_{\underline{1}}, x^{\mp}_{\underline{2}}\right\} = \gamma\left[R^{\pm}_{\underline{12}}, x^{\pm}_{\underline{1}}x^{\mp}_{\underline{2}}\right]. \tag{5.3.10b}$$

The induced Lie structure on $\mathfrak{g}^*_{DR}$ is isomorphic to $\mathfrak{g}_0$, so that the isomorphism $\mathfrak{g}_0^{**} \simeq \mathfrak{g}^*_{DR} \simeq \mathfrak{g}_0$ also holds at the level of Lie algebras. Thus the Drinfel'd double $D\mathfrak{g}_0^*$ of the dual Lie algebra $\mathfrak{g}_0^*$ is isomorphic to the Drinfel'd double $D\mathfrak{g}_0$ of the original Lie algebra $\mathfrak{g}_0$.

As a consequence, the formalism developed in the previous sections can also be used to treat a Poisson-Lie action of the dual group $G_0^* \simeq G_{DR}$ on a Poisson manifold M. In this case, the non-abelian moment map is an application

$$U : M \to G_0.$$

We can see it as an element $\iota(U)$ of the Drinfel'd double DG_0. The results presented in section 5.2.2 still apply in this case and the Poisson bracket of U is then given by

$$\left\{\iota(U)_{\underline{1}}, \iota(U)_{\underline{2}}\right\} = -\frac{1}{2}\left[\mathcal{R}^{D}_{\underline{12}}, \iota(U)_{\underline{1}}\iota(U)_{\underline{2}}\right] + M_{\underline{12}}\iota(U)_{\underline{1}}\iota(U)_{\underline{2}},$$

where $M_{\underline{12}}$ is a central charge valued in $\iota(\mathfrak{g}_0) \otimes \iota(\mathfrak{g}_0)$. Applying the morphism Φ to this equation, noting that $\Phi\big(\iota(U)\big) = (U, U)$ and recalling equation (5.3.5), we obtain

$$\left\{U_{\underline{1}}, U_{\underline{2}}\right\} = \gamma\left[R_{\underline{12}}, U_{\underline{1}}U_{\underline{2}}\right] + \widetilde{M}_{\underline{12}}U_{\underline{1}}U_{\underline{2}}, \tag{5.3.11}$$

with $\widetilde{M}_{\underline{12}} = \Phi_{\underline{1}}\Phi_{\underline{2}}\big(M_{\underline{12}}\big)$. This is, up to central charges, the Sklyanin bracket.

5.4 Link with q-deformed algebras

In this section we suppose that $\mathfrak{g}_0$ is either the split real form or a non-split real form of the semi-simple complexification $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$. The definitions and basic properties of semi-simple complex Lie algebras are recalled in appendix A.2 and those of (non-)split real forms in appendix A.3.2.

5.4.1 Real forms and standard R-matrices

Recall the construction of the standard R-matrix on $\mathfrak{g}$, explained in Appendix C.3. If $\pi_\pm$ and $\pi_\mathfrak{h}$ denotes the projections on $\mathfrak{n}_\pm$ and $\mathfrak{h}$ in the Cartan-Weyl decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+$, the standard R-matrix on $\mathfrak{g}$ reads

$$R = c(\pi_+ - \pi_-), \tag{5.4.1}$$

with $c = 1$ (split case) or $c = i$ (non-split case). It is a solution of the operator mCYBE (5.3.1) on $\mathfrak{g}$. Moreover, as explained in Appendix C.3, it defines a R-matrix on the real form $\mathfrak{g}_0$ (for $c = 1$ for the split real form and for $c = i$ for the non-split real form).

As shown in Appendix C.3, the kernel of R with respect to the Killing form is

$$R_{\underline{\mathbf{12}}} = c \sum_{\alpha>0} (E_\alpha \otimes F_\alpha - F_\alpha \otimes E_\alpha)$$

Likewise, the kernel of $R^\pm$ is $R^\pm_{\underline{\mathbf{12}}} = R_{\underline{\mathbf{12}}} \pm c\, C_{\underline{\mathbf{12}}}$, where the quadratic Casimir is given

$$C_{\underline{\mathbf{12}}} = H_{\underline{\mathbf{12}}} + \sum_{\alpha>0} (E_\alpha \otimes F_\alpha + F_\alpha \otimes E_\alpha)\,,$$

with $H_{\underline{\mathbf{12}}} \in \mathfrak{g} \otimes \mathfrak{g}$ as introduced in Appendix A.2.4.

5.4.2 Extraction of charges

We saw in section 5.3.3 that, in the split case, the non-abelian moment map can be regarded as an element (Γ^+, Γ^-) of $G_{DR} \subset G_0 \times G_0$. In the non-split case, it can be represented as either $\Gamma^+ \in G_+ \subset G$ or $\Gamma^- \in G_- \subset G$ (with Γ^+ and Γ^- related by $\Gamma^\pm = \tau(\Gamma^\mp)$, where τ is the semi-linear involutive automorphism of G defining the real subgroup G_0). Here we will treat the two cases together.

In the split (resp. non-split) case, the Lie algebras $R^\pm(\mathfrak{g}_0)$ are the positive and negative Borel subalgebras of $\mathfrak{g}_0$ (resp. $\mathfrak{g}$), with opposite Cartan parts. Therefore Γ^+ and Γ^- are elements of the positive and negative Borel subgroups of G_0 (resp. G), with Cartan parts inverses of one another. We choose to parametrise them as follows

$$\Gamma^+ = M^+ D, \quad \Gamma^- = D^{-1} M^-, \quad M^\pm \in N_\pm, \quad D \in H. \tag{5.4.2}$$

We now extract scalar charges from D and $M^\pm$. Starting with the Cartan part D, we choose a decomposition with respect to the basis of fundamental weight, recalled in appendix A.2.2,

$$D = \exp\left(ic\gamma \sum_{i=1}^{\ell} Q_i^H P_i \right) = \prod_{i=1}^{\ell} \exp\left(ic\gamma Q_i^H P_i \right). \tag{5.4.3}$$

The order of the product has no importance since the Cartan subgroup is abelian. We define

$$Z = ic\gamma \sum_{i=1}^{\ell} Q_i^H P_i \in \mathfrak{h}, \tag{5.4.4}$$

so that $D = \exp(Z)$.

The extraction of suitable charges from M^+ and M^- is more involved. Let us fix a labelling $\beta_1, \dots, \beta_n$ of the positive roots, where n is the number of positive roots. We can parametrise $M^\pm$ by scalar charges Q^E_β as

$$M^\pm = \prod_{i=1}^{n} \exp\left(\pm ic\gamma A_{\pm\beta_i} Q^E_{\pm\beta_i} E_{\pm\beta_i}\right), \tag{5.4.5}$$

where the A_β's are normalisation constants to be fixed later. Define

$$u_{(i)} = \exp\left(ic\gamma A_{\beta_i} Q^E_{\beta_i} E_{\beta_i}\right) \quad \text{and} \quad v_{(i)} = \exp\left(-ic\gamma A_{-\beta_i} Q^E_{-\beta_i} E_{-\beta_i}\right) \tag{5.4.6}$$

so that we can write

$$M^+ = u_{(1)} \dots u_{(n)} \quad \text{and} \quad M^- = v_{(1)} \dots v_{(n)}. \tag{5.4.7}$$

Since the Lie groups $N_\pm$ are not abelian, these products depend on the choice of the ordering $\beta_1, \dots, \beta_n$ of the positive roots. We choose an ordering such that

$$\text{if } i < j \text{ and } \beta_i + \beta_j \text{ is a root, then } \beta_i + \beta_j = \beta_k \text{ with } i < k < j. \tag{5.4.8}$$

Such an ordering can be constructed from the (partial) normal order described in [29]. We label the simple roots $\alpha_1, \dots, \alpha_\ell$ in a way which is compatible with the ordering $\beta_1, \dots, \beta_n$, *i.e.* such that $\alpha_i = \beta_{k_i}$ with $1 = k_1 \leq \dots \leq k_\ell = n$.

5.4.3 Semenov-Tian-Shansky brackets and $\mathscr{U}_q(\mathfrak{g}_0)$ algebra

We will now start from the Semenov-Tian-Shansky brackets (5.3.7) for the non-abelian moment map $\Gamma^\pm$ and extract from it the corresponding Poisson brackets between the charges Q^H_i and Q^E_β, as defined above. We will make extensive use of two theorems for the extraction of Poisson brackets that we present in appendix 5.A. We shall denote by π_k and π_{-k} the projections onto $\mathbb{C}E_{\beta_k}$ and $\mathbb{C}E_{-\beta_k}$ with respect to the Cartan-Weyl decomposition

$$\mathfrak{g} = \bigoplus_{k=1}^{n} (\mathbb{C}E_{\beta_k} \oplus \mathbb{C}E_{-\beta_k}) \oplus \mathfrak{h}.$$

Poisson brackets of D and $M^\pm$

Consider the decomposition (5.4.2) of Γ^+ and Γ^-. Using Theorem 5.A.1, we obtain

$$\left\{D_{\underline{1}}, D_{\underline{2}}\right\} = 0, \tag{5.4.9a}$$

$$\left\{D_{\underline{1}}, M^\pm_{\underline{2}}\right\} = c\gamma D_{\underline{1}} \left[H_{\underline{12}}, M^\pm_{\underline{2}}\right], \tag{5.4.9b}$$

$$\left\{M^+_{\underline{1}}, M^-_{\underline{2}}\right\} = \gamma\Big(D_{\underline{2}} R^{++}_{\underline{12}} D^{-1}_{\underline{2}} M^+_{\underline{1}} M^-_{\underline{2}} - M^+_{\underline{1}} M^-_{\underline{2}} D^{-1}_{\underline{2}} R^{++}_{\underline{12}} D_{\underline{2}}\Big), \tag{5.4.9c}$$

$$\left\{M^\pm_{\underline{1}}, M^\pm_{\underline{2}}\right\} = \gamma\Big(\left[R_{\underline{12}}, M^\pm_{\underline{1}} M^\pm_{\underline{2}}\right] \mp c\, M^\pm_{\underline{1}} H_{\underline{12}} M^\pm_{\underline{2}} \pm c\, M^\pm_{\underline{2}} H_{\underline{12}} M^\pm_{\underline{1}}\Big). \tag{5.4.9d}$$

where we have introduced (see Appendix C.3)

$$R^{++}_{\underline{12}} = R^+_{\underline{12}} - c\, H_{\underline{12}} = 2c \sum_{\alpha>0} E_\alpha \otimes F_\alpha, \tag{5.4.10a}$$

$$R^{--}_{\underline{12}} = R^-_{\underline{12}} + c\, H_{\underline{12}} = -2c \sum_{\alpha>0} F_\alpha \otimes E_\alpha. \tag{5.4.10b}$$

We also made use of the following identity, valid for any $h \in H$ and $\epsilon \in \{\emptyset, +, ++, -, --\}$,

$$h_{\underline{1}} h_{\underline{2}} R^\epsilon_{\underline{12}} h^{-1}_{\underline{1}} h^{-1}_{\underline{2}} = R^\epsilon_{\underline{12}}.$$

From the Poisson bracket (5.4.9a), one simply finds

$$\left\{Q^H_i, Q^H_j\right\} = 0. \tag{5.4.11}$$

Poisson bracket between Q_i^H and Q_β^E

The partial Casimir tensor $H_{\underline{12}}$ on the Cartan subalgebra can be expressed in terms of the dual bases of weights P_i and co-roots $\check{\alpha}_i$ (cf. Appendices A.2.2 and A.2.4) as

$$H_{\underline{12}} = \sum_{i=1}^{\ell} P_i \otimes \check{\alpha}_i.$$

This allows us to extract the Poisson bracket between Q_i^H and $M^\pm$ by projecting equation (5.4.9b) onto P_i in the first tensor factor, namely

$$i\{Q_i^H, M^\pm\} = \left[\check{\alpha}_i, M^\pm\right].$$

We will now treat the bracket with M^+, the case of M^- being similar. We introduce

$$w_{(k)} = u_{(k)} \dots u_{(n)},$$

such that $M^+ = w_{(1)}$ and $w_{(k)} = u_{(k)} w_{(k+1)}$. Using this decomposition and Theorem 5.A.1, one shows by induction on k that, for every $k \in \{1, \dots, n\}$, we have

$$iu_{(k)}^{-1}\{Q_i^H, u_{(k)}\} = u_{(k)}^{-1}\check{\alpha}_i u_{(k)} - \check{\alpha}_i, \tag{5.4.12a}$$

$$i\{Q_i^H, w_{(k)}\}w_{(k)}^{-1} = \check{\alpha}_i - w_{(k)}\check{\alpha}_i w_{(k)}^{-1}. \tag{5.4.12b}$$

This induction relies on the fact that for any k, the adjoint action of $w_{(k+1)}$ on $\check{\alpha}_i$ only creates nilpotent generators E_γ coresponding to roots of the form $\gamma = a_{k+1}\beta_{k+1} + \dots + a_n\beta_n$, with $a_{k+1}, \dots, a_n \in \mathbb{N}$. One can show from the ordering condition (5.4.8) that these roots are always strictly superior to the root β_k, which allows to perform the projection needed in Theorem 5.A.1. Using the definition (5.4.6) of $u_{(k)}$, equation (5.4.12a) becomes

$$i\{Q_i^H, Q_{\beta_k}^E\} = \beta_k(\check{\alpha}_i) Q_{\beta_k}^E.$$

Applying the same method to the Poisson bracket with M^-, we find that this equation holds for any root β, positive or negative. In the case of a simple root (or its opposite) $\beta = \pm\alpha_j$, we have $\beta(\check{\alpha}_i) = \pm\alpha_j(\check{\alpha}_i) = \pm a_{ij}$ (cf. appendix A.2.2). We therefore obtain

$$i\{Q_i^H, Q_{\pm\alpha_j}^E\} = \pm a_{ij} Q_{\pm\alpha_j}^E. \tag{5.4.13}$$

Poisson bracket between $Q_{\alpha_i}^E$ and $Q_{-\alpha_j}^E$

Fixing two simple roots α_i and α_j, we want to compute the Poisson bracket between $Q_{\alpha_i}^E$ and $Q_{-\alpha_j}^E$. Recall that $\alpha_i = \beta_{k_i}$ and $\alpha_j = \beta_{k_j}$. Considering the decomposition (5.4.7) of $M^\pm$, we need to extract the Poisson bracket of $u_{(k_i)}$ with $v_{(k_j)}$. Define

$$x = u_{(1)} \dots u_{(k_i)}, \qquad \tilde{x} = v_{(1)} \dots v_{(k_j)},$$
$$y = u_{(k_i+1)} \dots u_{(n)}, \qquad \tilde{y} = v_{(k_j+1)} \dots v_{(n)}.$$

By Theorem 5.A.2, applied on both tensor factors, we may write

$$u_{(k_i)\,\underline{1}}^{-1} v_{(k_j)\,\underline{2}}^{-1} \left\{u_{(k_i)\,\underline{1}}, v_{(k_j)\,\underline{2}}\right\} = \pi_{k_i} \otimes \pi_{-k_j}\left(\mathcal{P}_{\underline{12}}\right),$$

where $\mathcal{P}_{\underline{12}} = x_{\underline{1}}^{-1}\tilde{x}_{\underline{2}}^{-1}\left\{M_{\underline{1}}^+, M_{\underline{2}}^-\right\} y_{\underline{1}}^{-1}\tilde{y}_{\underline{2}}^{-1}$. On the other hand, from equation (5.4.9c) we find

$$\mathcal{P}_{\underline{12}} = \gamma\Big(x_{\underline{1}}^{-1}\tilde{x}_{\underline{2}}^{-1} D_{\underline{2}} R_{\underline{12}}^{++} D_{\underline{2}}^{-1} x_{\underline{1}}\tilde{x}_{\underline{2}} - y_{\underline{1}}\tilde{y}_{\underline{2}} D_{\underline{2}}^{-1} R_{\underline{12}}^{++} D_{\underline{2}} y_{\underline{1}}^{-1}\tilde{y}_{\underline{2}}^{-1}\Big).$$

Recalling from (5.4.4) that $D = \exp(Z)$ with $Z \in \mathfrak{h}$, we have

$$D_{\underline{\mathbf{2}}}^{\pm 1} R_{\underline{\mathbf{12}}}^{++} D_{\underline{\mathbf{2}}}^{\mp 1} = 2c \sum_{\alpha>0} \exp\big(\mp\alpha(Z)\big) E_\alpha \otimes E_{-\alpha},$$

so that

$$\mathcal{P}_{\underline{\mathbf{12}}} = 2c\gamma \sum_{\alpha>0} \Big(\exp\big(-\alpha(Z)\big) \left(x^{-1} E_\alpha x\right) \otimes \left(\tilde{x}^{-1} E_{-\alpha} \tilde{x}\right) - \exp\big(\alpha(Z)\big) \left(y E_\alpha y^{-1}\right) \otimes \left(\tilde{y} E_{-\alpha} \tilde{y}^{-1}\right) \Big).$$

The adjoint action of any E_β (appearing in x or y) on E_α cannot create the simple root generator E_{α_i} and similarly for $E_{-\alpha_j}$ on the second space. It follows that

$$\pi_{k_i} \otimes \pi_{-k_j}\big(\mathcal{P}_{\underline{\mathbf{12}}}\big) = 2c\gamma\delta_{ij} \Big(\exp\big(-\alpha_i(Z)\big) - \exp\big(\alpha_i(Z)\big) \Big) E_{\alpha_i} \otimes E_{-\alpha_j}.$$

Yet, by definition (5.4.6) of the $u_{(k)}$'s and $v_{(k)}$'s, we find

$$\pi_{k_i} \otimes \pi_{-k_j}\big(\mathcal{P}_{\underline{\mathbf{12}}}\big) = u^{-1}_{(k_i)\,\underline{\mathbf{1}}} v^{-1}_{(k_j)\,\underline{\mathbf{2}}} \Big\{ u_{(k_i)\,\underline{\mathbf{1}}}, v_{(k_j)\,\underline{\mathbf{2}}} \Big\} = c^2\gamma^2 A_{\alpha_i} A_{-\alpha_j} \Big\{ Q^E_{\alpha_i}, Q^E_{-\alpha_j} \Big\} E_{\alpha_i} \otimes E_{-\alpha_j},$$

so that

$$i\big\{Q^E_{\alpha_i}, Q^E_{-\alpha_j}\big\} = \frac{2i}{c\gamma A_{\alpha_i} A_{-\alpha_j}} \delta_{ij} \Big(\exp\big(-\alpha_i(Z)\big) - \exp\big(\alpha_i(Z)\big) \Big).$$

From equation (5.4.4), one has (cf. appendix A.2.2)

$$\alpha_i(Z) = ic\gamma \sum_{k=1}^{\ell} Q^H_k \alpha_i(P_k) = ic\gamma \sum_{k=1}^{\ell} Q^H_k d_i \delta_{ik} = ic\gamma d_i Q^H_i.$$

Introducing the deformation parameter

$$q = e^{-ic\gamma}, \tag{5.4.14}$$

we therefore have

$$i\big\{Q^E_{\alpha_i}, Q^E_{-\alpha_j}\big\} = \frac{2i}{\gamma c A_{\alpha_i} A_{-\alpha_j}} \delta_{ij} \Big(q^{d_i Q^H_i} - q^{-d_i Q^H_i} \Big).$$

Finally, if we fix the normalisation $A_{\pm\alpha}$ for simple roots as

$$A_{\pm\alpha_i} = \left(\frac{4 \sinh(ic\gamma d_i)}{ic\gamma} \right)^{\frac{1}{2}}, \tag{5.4.15}$$

then we may rewrite the above Poisson brackets as

$$i\big\{Q^E_{\alpha_i}, Q^E_{-\alpha_j}\big\} = \delta_{ij} \frac{q^{d_i Q^H_i} - q^{-d_i Q^H_i}}{q^{d_i} - q^{-d_i}}. \tag{5.4.16}$$

q-Poisson-Serre relations

The Poisson brackets (5.4.11), (5.4.13) and (5.4.16) obtained so far are part of the defining relations of the semiclassical limit $\mathscr{U}_q(\mathfrak{g}_0)$ of the quantum group $U_{\widehat{q}}(\mathfrak{g}_0)$ with $\widehat{q} = q^\hbar$, as given in [29] (see also [158]). The complete set of relations characterising $\mathscr{U}_q(\mathfrak{g}_0)$ also includes the so-called q-Poisson-Serre relations. The purpose of the present subsection is to derive these from the Poisson bracket (5.4.9d). We will only treat the case of positive roots, the negative one being handled similarly.

Poisson brackets of $Q^E_{\alpha_i}$ with M^+. Let us fix a simple root α_i. We recall that $\alpha_i = \beta_{k_i}$, so that $Q^E_{\alpha_i}$ is to be extracted from $u_{(k_i)}$. Introduce

$$x = u_{(1)} \dots u_{(k_i)},$$
$$y = u_{(k_i+1)} \dots u_{(n)},$$

so that $M^+ = xy$. By Theorem 5.A.2 we have

$$\big(u_{(k_i)}\big)^{-1}_{\underline{\mathbf{1}}} \Big\{ u_{(k_i)\underline{\mathbf{1}}}, M^+_{\underline{\mathbf{2}}} \Big\} = (\pi_{k_i})_{\underline{\mathbf{1}}} \big(\mathcal{P}_{\underline{\mathbf{12}}}\big),$$

where $\mathcal{P}_{\underline{\mathbf{12}}} = x^{-1}_{\underline{\mathbf{1}}} \Big\{ M^+_{\underline{\mathbf{1}}}, M^+_{\underline{\mathbf{2}}} \Big\} y^{-1}_{\underline{\mathbf{1}}}$. On the other hand, from (5.4.9d) we have

$$\mathcal{P}_{\underline{\mathbf{12}}} = \gamma \mathcal{P}^R_{\underline{\mathbf{12}}} + \gamma \mathcal{P}^H_{\underline{\mathbf{12}}}$$

with

$$\mathcal{P}^R_{\underline{\mathbf{12}}} = x^{-1}_{\underline{\mathbf{1}}} R^{++}_{\underline{\mathbf{12}}} x_{\underline{\mathbf{1}}} M^+_{\underline{\mathbf{2}}} - M^+_{\underline{\mathbf{2}}} y_{\underline{\mathbf{1}}} R^{++}_{\underline{\mathbf{12}}} y^{-1}_{\underline{\mathbf{1}}},$$
$$\mathcal{P}^H_{\underline{\mathbf{12}}} = c \Big(x^{-1}_{\underline{\mathbf{1}}} H_{\underline{\mathbf{12}}} x_{\underline{\mathbf{1}}} - y_{\underline{\mathbf{1}}} H_{\underline{\mathbf{12}}} y^{-1}_{\underline{\mathbf{1}}} \Big) M^+_{\underline{\mathbf{2}}} + c\, M^+_{\underline{\mathbf{2}}} \Big(x^{-1}_{\underline{\mathbf{1}}} H_{\underline{\mathbf{12}}} x_{\underline{\mathbf{1}}} - y_{\underline{\mathbf{1}}} H_{\underline{\mathbf{12}}} y^{-1}_{\underline{\mathbf{1}}} \Big).$$

By writing $H_{\underline{\mathbf{12}}} = \sum_{j=1}^{\ell} \check{\omega}_j \otimes H_j$ (cf. appendices A.2.2 and A.2.4), these can be rewritten

$$\mathcal{P}^R_{\underline{\mathbf{12}}} = 2c \sum_{\alpha>0} \big(x^{-1} E_\alpha x\big) \otimes \big(E_{-\alpha} M^+\big) - 2c \sum_{\alpha>0} \big(y E_\alpha y^{-1}\big) \otimes \big(M^+ E_{-\alpha}\big),$$
$$\mathcal{P}^H_{\underline{\mathbf{12}}} = c \sum_{j=1}^{\ell} \big(x^{-1} \check{\omega}_j x - y \check{\omega}_j y^{-1}\big) \otimes \big(H_j M^+ + M^+ H_j\big).$$

The adjoint action of any E_β (appearing in x or y) on E_α cannot create the simple root generator E_{α_i}. Thus, we have

$$(\pi_{k_i})_{\underline{\mathbf{1}}} \big(\mathcal{P}^R_{\underline{\mathbf{12}}}\big) = 2c\, E_{\alpha_i} \otimes \big(E_{-\alpha_i} M^+ - M^+ E_{-\alpha_i}\big).$$

In the same way, in the adjoint actions of E_β's from x or y on $\check{\omega}_j$, only a unique adjoint action of E_{α_i}, coming from $u_{(k_i)}$ in x, can create the simple root generator E_{α_i}. Therefore

$$\pi_{k_i}\big(x^{-1} \check{\omega}_j x - y \check{\omega}_j y^{-1}\big) = -ic\gamma A_{\alpha_i} Q^E_{\alpha_i} \mathrm{ad}_{E_{\alpha_i}}\big(\check{\omega}_j\big) = ic\gamma A_{\alpha_i} Q^E_{\alpha_i} \delta_{ij} E_{\alpha_i},$$

and hence

$$(\pi_{k_i})_{\underline{\mathbf{1}}} \big(\mathcal{P}^H_{\underline{\mathbf{12}}}\big) = ic^2 \gamma A_{\alpha_i} Q^E_{\alpha_i} E_{\alpha_i} \otimes \big(H_i M^+ + M^+ H_i\big).$$

Putting together all the above we arrive at

$$\big(u_{(k_i)}\big)^{-1}_{\underline{\mathbf{1}}} \Big\{ u_{(k_i)\underline{\mathbf{1}}}, M^+_{\underline{\mathbf{2}}} \Big\} = c\gamma E_{\alpha_i} \otimes \Big(2\big[E_{-\alpha_i}, M^+\big] + ic\gamma A_{\alpha_i} Q^E_{\alpha_i} \big(H_i M^+ + M^+ H_i\big) \Big).$$

Yet, by definition of $u_{(k_i)}$ in (5.4.6) we have

$$\big(u_{(k_i)}\big)^{-1}_{\underline{\mathbf{1}}} \Big\{ u_{(k_i)\underline{\mathbf{1}}}, M^+_{\underline{\mathbf{2}}} \Big\} = ic\gamma A_{\alpha_i} E_{\alpha_i} \otimes \big\{ Q^E_{\alpha_i}, M^+ \big\},$$

and hence

$$iA_{\alpha_i} \big\{ Q^E_{\alpha_i}, M^+ \big\} = 2\big[E_{-\alpha_i}, M^+\big] + ic\gamma A_{\alpha_i} Q^E_{\alpha_i} \big(H_i M^+ + M^+ H_i\big) \tag{5.4.17}$$

α_i-string through α_j. Let us now consider another simple root α_j. We suppose here that $\alpha_i > \alpha_j$. The α_i-string through α_j is then contained between α_j and α_i. Specifically, we have

$$\alpha_j < \alpha_j + \alpha_i < \ldots < \alpha_j - a_{ij}\alpha_i < \alpha_i,$$

with a the Cartan matrix of $\mathfrak{g}$ (cf. appendix A.2.2). Let $r \in \{0, \ldots, -a_{ij}\}$ and $p \in \{1, \ldots, n\}$ be such that

$$\beta_p = \alpha_j + r\alpha_i,$$

We define

$$x = u_{(1)} \ldots u_{(p)},$$
$$y = u_{(p+1)} \ldots u_{(n)},$$

and $\mathcal{Q} = x^{-1}\left\{Q^E_{\alpha_i}, M^+\right\} y^{-1}$. By Theorem 5.A.2, we have

$$u^{-1}_{(p)}\left\{Q^E_{\alpha_i}, u_{(p)}\right\} = \pi_p(\mathcal{Q}). \tag{5.4.18}$$

On the other hand, from the Poisson bracket (5.4.17) we get

$$iA_{\alpha_i}\mathcal{Q} = 2\big(x^{-1}E_{-\alpha_i}x - yE_{-\alpha_i}y^{-1}\big) + ic\gamma A_{\alpha_i}Q^E_{\alpha_i}\big(x^{-1}H_i x + yH_i y^{-1}\big). \tag{5.4.19}$$

The projection onto E_{β_p} of the terms involving H_i on the right hand side of (5.4.19) can be computed as follows. We note that yH_iy^{-1} is composed of nilpotent generators E_β with β a sum of roots superior to β_p, which therefore cannot be β_p. In the same way, the adjoint action of x on H_i creates nilpotent generators E_β with β a sum of roots inferior or equal to β_p. Such β can be either strictly inferior to β_p or β_p itself. Therefore the only way to have E_{β_p} in $x^{-1}H_ix$ is by the simple adjoint action on H_i of the generator E_{β_p} (appearing in $u_{(p)}$). Thus, we have

$$\pi_p\big(x^{-1}H_ix + yH_iy^{-1}\big) = -ic\gamma A_{\beta_p}Q^E_{\beta_p}\mathrm{ad}_{E_{\beta_p}}(H_i) = ic\gamma A_{\beta_p}Q^E_{\beta_p}(\alpha_i, \beta_p)E_{\beta_p}. \tag{5.4.20}$$

Next, consider the term $x^{-1}E_{-\alpha_i}x - yE_{-\alpha_i}y^{-1}$ on the right hand side of (5.4.19). It is composed of generators E_β, with $\beta = \gamma - \alpha_i$ and γ a sum of roots either all inferior or equal to β_p (for x) or all superior (for y). We want to project this on E_{β_p}. Yet, having $\beta = \beta_p$ requires $\gamma = \beta_p + \alpha_i$. As $\beta_p < \alpha_i$, this means that $\beta_p < \gamma$, hence γ comes from the adjoint action of y. To be more precise, $yE_{-\alpha_i}y^{-1}$ is composed of elements of the form (up to prefactors)

$$\mathrm{ad}^{a_{p+1}}_{E_{\beta_{p+1}}} \ldots \mathrm{ad}^{a_n}_{E_{\beta_n}}\big(E_{-\alpha_i}\big), \quad \text{with } a_{p+1}, \ldots, a_n \in \mathbb{N}.$$

Such a term is proportional to $E_{\gamma-\alpha_i}$, with $\gamma = a_{p+1}\beta_{p+1} + \ldots + a_n\beta_n$. In order to get E_{β_p}, one must have $\gamma = \alpha_i + \beta_p = \alpha_j + (r+1)\alpha_i$. Therefore, we want to solve

$$a_{p+1}\beta_{p+1} + \ldots + a_n\beta_n = \alpha_j + (r+1)\alpha_i,$$

with $a_{p+1}, \ldots, a_n$ non-negative integers. If a root β_q, for $q > p$, contains a simple root α_k different from α_i and α_j, it is clear from the equation above that a_q must be zero, as α_k does not appear in the right hand side of the equation.

Moreover, the only roots superior to β_p and containing only α_i and α_j as simple roots are $\alpha_j + (r+1)\alpha_i, \alpha_j + (r+2)\alpha_i, \ldots, \alpha_j - A_{ij}\alpha_i$ and α_i. The only way that a non-negative integer linear combination of these roots can give $\alpha_j + (r+1)\alpha_i$ is if all the coefficients are zero except for that of the root $\alpha_j + (r+1)\alpha_i$ itself. Thus, the projection of $x^{-1}E_{-\alpha_i}x - yE_{-\alpha_i}y^{-1}$ onto E_{β_p} comes from the simple adjoint action of $E_{\alpha_j+(r+1)\alpha_i}$ on $E_{-\alpha_i}$ (if $\alpha_j + (r+1)\alpha_i$ is a root). Hence

$$\begin{aligned}\pi_p\big(x^{-1}E_{-\alpha_i}x - yE_{-\alpha_i}y^{-1}\big) &= -ic\gamma A_{\alpha_j+(r+1)\alpha_i}Q^E_{\alpha_j+(r+1)\alpha_i}\big[E_{\alpha_j+(r+1)\alpha_i}, E_{-\alpha_i}\big] \\ &= -ic\gamma A_{\alpha_j+(r+1)\alpha_i}Q^E_{\alpha_j+(r+1)\alpha_i}N_{\beta_p,\alpha_i}E_{\beta_p},\end{aligned} \tag{5.4.21}$$

if $\alpha_j + (r+1)\alpha_i$ is a root, and is zero otherwise.

Applying π_p to (5.4.19) and using the results (5.4.20) and (5.4.21) gives

$$iA_{\alpha_i}\pi_p(\mathcal{Q}) = -2ic\gamma A_{\alpha_j+(r+1)\alpha_i} N_{\beta_p,\alpha_i} Q^E_{\alpha_j+(r+1)\alpha_i} E_{\beta_p} + (ic\gamma)^2 A_{\alpha_i} A_{\beta_p}(\alpha_i,\beta_p) Q^E_{\alpha_i} Q^E_{\beta_p} E_{\beta_p}.$$

Yet from (5.4.18) together with the definition of $u_{(p)}$ in (5.4.6) we have

$$\pi_p(\mathcal{Q}) = u^{-1}_{(p)}\{Q^E_{\alpha_i}, u_{(p)}\} = ic\gamma A_{\beta_p}\{Q^E_{\alpha_i}, Q^E_{\beta_p}\}E_{\beta_p},$$

and hence

$$\{Q^E_{\alpha_i}, Q^E_{\beta_p}\} = \frac{A_{\alpha_j+(r+1)\alpha_i}}{A_{\alpha_i}A_{\beta_p}} 2iN_{\beta_p,\alpha_i} Q^E_{\alpha_j+(r+1)\alpha_i} + c\gamma(\alpha_i,\beta_p) Q^E_{\alpha_i} Q^E_{\beta_p}.$$

We define the q-bracket of two charges associated with the positive roots α and β as

$$\{Q^E_\alpha, Q^E_\beta\}_q = \{Q^E_\alpha, Q^E_\beta\} + c\gamma(\alpha,\beta) Q^E_\alpha Q^E_\beta.$$

Moreover, if we fix the normalisation constant A_α for α in the α_i-string through α_j as

$$A_{\alpha_j+r\alpha_i} = A_{\alpha_j} A^r_{\alpha_i},$$

then we deduce that

$$\{Q^E_{\alpha_j+r\alpha_i}, Q^E_{\alpha_i}\}_q = 2iN_{\alpha_i,\alpha_j+r\alpha_i} Q^E_{\alpha_j+(r+1)\alpha_i},$$

if $\alpha_j + (r+1)\alpha_i$ is a root and is zero otherwise.

By induction, we get the q-Poisson-Serre relation

$$\{\{\dots\{Q^E_{\alpha_j}, \underbrace{Q^E_{\alpha_i}\}_q, \dots Q^E_{\alpha_i}\}_q, Q^E_{\alpha_i}\}_q}_{1-a_{ij} \text{ times}} = 0. \tag{5.4.22}$$

One can treat the case $\alpha_i < \alpha_j$ in a similar way. For that, one needs to use a slightly different version of Theorem 5.A.2, involving the quantity $\left(u_{(1)}\dots u_{(p-1)}\right)^{-1}\left\{Q^E_{\alpha_i}, M^+\right\}\left(u_{(p)}\dots u_{(n)}\right)^{-1}$ instead of $\left(u_{(1)}\dots u_{(p)}\right)^{-1}\left\{Q^E_{\alpha_i}, M^+\right\}\left(u_{(p+1)}\dots u_{(n)}\right)^{-1}$. This yields the q-Poisson-Serre relation

$$\{\underbrace{Q^E_{\alpha_i}, \{Q^E_{\alpha_i}, \dots, \{Q^E_{\alpha_i}}_{1-a_{ij} \text{ times}}, Q^E_{\alpha_j}\}_q \dots\}_q\}_q = 0. \tag{5.4.23}$$

Applying the same method as above to the Poisson bracket in (5.4.9d) involving M^-, one finds that the charges $Q^E_{-\alpha_i}$ also verifiy q-Poisson-Serre relations, but with respect to the deformed bracket $\{\cdot,\cdot\}_{q^{-1}}$, defined for two negative roots α and β as

$$\{Q^E_\alpha, Q^E_\beta\}_{q^{-1}} = \{Q^E_\alpha, Q^E_\beta\} - c\gamma(\alpha,\beta) Q^E_\alpha Q^E_\beta.$$

Reality conditions

The Poisson brackets (5.4.11), (5.4.13) and (5.4.16), together with the q-Poisson-Serre relations stated above are the defining Poisson bracket relations of the semiclassical limit $\mathscr{U}_q(\mathfrak{g}_0)$ of the quantum group $U_{\widehat{q}}(\mathfrak{g}_0)$ with $\widehat{q} = q^\hbar$. It only remains to check that the required reality conditions are verified by the charges Q^H_i and Q^E_α. We shall address this question in the present subsection, first in the split case and then in the non-split one.

Split case. When $c = 1$, the deformation parameter (5.4.14) becomes

$$q = e^{-i\gamma},$$

so that $|q| = 1$, *i.e.* q is a phase. Now the moment map (Γ^+, Γ^-) takes values in the real double $G_0 \times G_0$, therefore the reality condition is simply

$$\tau(\Gamma^\pm) = \Gamma^\pm,$$

with τ the split semi-linear automorphism described in appendix A.3.2 and lifted to the complexified group G. As τ stabilises the Cartan and the nilpotent subgroups and since the decomposition (5.4.2) is unique, one has

$$\tau(D) = D \quad \text{and} \quad \tau(M^\pm) = M^\pm.$$

We recall (cf. appendix A.3.2) that in the split case $\tau(P_i) = P_i$ for $i \in \{1, \dots, \ell\}$ (since P_i is a real linear combination of the H_j) and $\tau(E_\alpha) = E_\alpha$ for any root α. Considering the extraction of charges (5.4.3) and (5.4.5) with $c = 1$, the above reality condition gives

$$\begin{aligned} \overline{Q_i^H} &= -Q_i^H, \\ \overline{A_{\pm\alpha_i} Q^E_{\pm\alpha_i}} &= -A_{\pm\alpha_i} Q^E_{\pm\alpha_i}. \end{aligned}$$

The normalisation constants $A_{\pm\alpha_i}$ are given by equation (5.4.15), which in the split case reads

$$A_{\pm\alpha_i} = \left(\frac{4 \sin(\gamma d_i)}{\gamma} \right)^{\frac{1}{2}}.$$

We will restrict attention to the case where

$$-\pi \leq \gamma d_i \leq \pi,$$

for any i, so that the $A_{\pm\alpha_i}$ are real numbers. As a result, the reality conditions are simply

$$|q| = 1, \quad \overline{Q_i^H} = -Q_i^H \quad \text{and} \quad \overline{Q^E_{\pm\alpha_i}} = -Q^E_{\pm\alpha_i}. \tag{5.4.24}$$

These are the reality conditions for the split real form $\mathscr{U}_q(\mathfrak{g})$, which correspond precisely to the semi-classical counterpart of the reality conditions on $U_{\widehat{q}}(\mathfrak{g}_0)$ as given in [159].

Non-split case. For $c = i$, the deformation parameter (5.4.14) now reads

$$q = e^{\gamma},$$

which is a real number. As explained in subsection 5.3.3, the two moment maps Γ^+ and Γ^- are not independent. They are related by the reality condition

$$\tau(\Gamma^\pm) = \Gamma^\mp,$$

where τ is the non-split semi-linear automorphism described in appendix A.3.2, lifted to the complexified group G. Since τ stabilises the Cartan subgroup H but exchanges the unipotent subgroups $N_\pm$, applying τ to the decomposition (5.4.2) we get

$$\begin{aligned} \tau(\Gamma^+) &= \tau(M^+)\tau(D) = \underbrace{\tau(D)}_{\in H} \underbrace{\tau(D)^{-1}\tau(M^+)\tau(D)}_{\in N_-}, \\ \tau(\Gamma^-) &= \tau(D)^{-1}\tau(M^-) = \underbrace{\tau(D)^{-1}\tau(M^-)\tau(D)}_{\in N_+} \underbrace{\tau(D)^{-1}}_{\in H}, \end{aligned}$$

where we used the fact that an adjoint action of a Cartan element on a element of $N_\pm$ is still in $N_\pm$. Equating $\tau(\Gamma^+)$ with $\Gamma^- = D^{-1}M^-$ and $\tau(\Gamma^-)$ with $\Gamma^+ = M^+D$ we obtain

$$\tau(D) = D^{-1} \quad \text{and} \quad \tau(M^\pm) = D^{-1}M^\mp D. \tag{5.4.25}$$

Recall (cf. appendix A.3.2) that $\tau(P_i) = -P_i$. Using the extraction of Cartan charges (5.4.3), with $c = i$, we find

$$\overline{Q_i^H} = Q_i^H.$$

From the decomposition (5.4.5), we have

$$D^{-1}M^\pm D = \prod_{k=1}^{n} \exp\left(\mp\gamma A_{\pm\beta_k} Q^E_{\pm\beta_k} D^{-1} E_{\pm\beta_k} D\right).$$

Moreover, since $D = \exp(Z)$ since Z defined in (5.4.4),

$$D^{-1}E_{\pm\beta_k}D = \exp\big(\mp\beta_k(Z)\big)E_{\pm\beta_k} = \exp\big(\pm\gamma\sum_{j=1}^{\ell} Q_j^H \beta_k(P_j)\big)E_{\pm\beta_k}.$$

In particular, for $k = k_i$, *i.e.* for β_k the simple root α_i, using (A.2.5) we get

$$D^{-1}E_{\pm\alpha_i}D = q^{\pm d_i Q_i^H} E_{\pm\alpha_i}.$$

The term corresponding to the simple root α_i in $D^{-1}M^\mp D$ therefore reads

$$\exp\Big(\pm\gamma A_{\mp\alpha_i} q^{\mp d_i Q_i^H} Q^E_{\mp\alpha_i} E_{\mp\alpha_i}\Big).$$

Yet we have $\tau(E_{\pm\alpha_i}) = -\lambda_i E_{\mp\alpha_i}$, so that the corresponding term in $\tau(M^\pm)$ is

$$\exp\Big(\pm\gamma\overline{A_{\pm\alpha_i}Q^E_{\pm\alpha_i}}\lambda_i E_{\mp\alpha_i}\Big).$$

It now follows from the second equality in (5.4.25) that

$$\overline{A_{\pm\alpha_i}Q^E_{\pm\alpha_i}} = \lambda_i q^{\mp d_i Q_i^H} A_{\mp\alpha_i} Q^E_{\mp\alpha_i}.$$

The normalisation constants $A_{\pm\alpha_i}$ are given by (5.4.15), which for $c = i$ take the form

$$A_{\pm\alpha_i} = \left(\frac{4\sinh(\gamma d_i)}{\gamma}\right)^{\frac{1}{2}}$$

and are therefore real numbers. Hence, the reality conditions are

$$q \in \mathbb{R}, \quad \overline{Q_i^H} = Q_i^H \quad \text{and} \quad \overline{Q^E_{\pm\alpha_i}} = \lambda_i q^{\mp d_i Q_i^H} Q^E_{\mp\alpha_i}. \tag{5.4.26}$$

According to [159], these are the reality conditions of the non-split real form $\mathscr{U}_q(\mathfrak{g}_0)$.

5.4.4 Sklyanin bracket and $\mathscr{U}_q(\mathfrak{g}_0^*)$ algebra

As in subsection 5.3.4, in what follows we consider only the split case. We start from the Poisson bracket (5.3.11) with the central quantity $\widetilde{M_{\underline{12}}}$ set to zero. In other words, U satisfies the Sklyanin Poisson bracket

$$\{U_{\underline{1}}, U_{\underline{2}}\} = \gamma\left[R_{\underline{12}}, U_{\underline{1}}U_{\underline{2}}\right]. \tag{5.4.27}$$

Let us decompose U as

$$U = M^- D M^+,$$

with $D \in H$ and $M^{\pm} \in N_{\pm}$. Using Theorem 5.A.1, we can extract the Poisson brackets between D, M^+ and M^- from equation (5.4.27) to find

$$\{D_{\underline{1}}, D_{\underline{2}}\} = 0, \tag{5.4.28a}$$

$$\left\{D_{\underline{1}}, M^{\pm}_{\underline{2}}\right\} = \pm c\gamma D_{\underline{1}} \left[H_{\underline{12}}, M^{\pm}_{\underline{2}}\right], \tag{5.4.28b}$$

$$\left\{M^{+}_{\underline{1}}, M^{-}_{\underline{2}}\right\} = 0, \tag{5.4.28c}$$

$$\left\{M^{\pm}_{\underline{1}}, M^{\pm}_{\underline{2}}\right\} = \gamma\Big(\left[R_{\underline{12}}, M^{\pm}_{\underline{1}} M^{\pm}_{\underline{2}}\right] \mp c\, M^{\pm}_{\underline{1}} H_{\underline{12}} M^{\pm}_{\underline{2}} \pm c\, M^{\pm}_{\underline{2}} H_{\underline{12}} M^{\pm}_{\underline{1}}\Big). \tag{5.4.28d}$$

We notice that these Poisson brackets are very similar to (5.4.9), the main difference being that M^+ and M^- Poisson commute in the present case.

The methods of subsection 5.4.3 can be applied to this case. For each positive root α, we extract the positive nilpotent charge Q^E_α from M^+ and negative nilpotent charge $Q^E_{-\alpha}$ from M^- as we did before using the decomposition (5.4.5). Likewise, we extract Cartan charges Q^H_i from D as we did in equation (5.4.3).

From equation (5.4.28d), following same the procedure outline in subsection 5.4.3, we find that the nilpotent charges Q^E_α and $Q^E_{-\alpha}$ satisfy the q-Poisson-Serre relations. In other words, these charges span nilpotent q-deformed Poisson algebras $\mathscr{U}_q(\mathfrak{n}_+)$ and $\mathscr{U}_q(\mathfrak{n}_-)$. Moreover, it is clear from (5.4.28c) that elements from these two algebras Poisson commute.

Similarly, we can apply the methods of subsection 5.4.3 to the Poisson bracket (5.4.28b). Doing so, we find that, for any positive root α,

$$i\{Q^H_i, Q^E_\alpha\} = \alpha(\check{\alpha}_i) Q^E_\alpha \quad \text{and} \quad i\{Q^H_i, Q^E_{-\alpha}\} = \alpha(\check{\alpha}_i) Q^E_{-\alpha}.$$

This implies that the Cartan charges Q^H_i for $i = 1, \dots, n$ together with the charges Q^E_α for $\alpha > 0$ span a q-deformed positive Borel algebra $\mathscr{U}_q(\mathfrak{b}_+)$. In the same way, the charges $-Q^H_i$ for $i = 1, \dots, n$ together with the charges $Q^E_{-\alpha}$ for $\alpha > 0$ span a q-deformed negative Borel algebra $\mathscr{U}_q(\mathfrak{b}_-)$. The combination of all the charges Q^H_i for $i = 1, \dots, n$ and Q^E_α for all roots α therefore span a q-deformed Poisson algebra which we could call $\mathscr{U}_q(\mathfrak{g}_{DR})$.

Since U takes value in the split real form G_0 we have $\tau(U) = U$, with τ the split semi-linear automorphism of appendix A.3.2. Moreover, since τ stabilises the subgroups H and $N_\pm$, we deduce that $\tau(D) = D$ and $\tau(M^\pm) = M^\pm$. The reality conditions are then identical to those of the split case in the subsection 5.4.3, so that

$$|q| = 1, \quad \overline{Q^H_i} = -Q^H_i \quad \text{and} \quad \overline{Q^E_{\pm\alpha}} = -Q^E_{\pm\alpha}. \tag{5.4.29}$$

5.5 Application to Yang-Baxter type models

5.5.1 Yang-Baxter type models

In this section we will apply the formalism of Poisson-Lie groups and non-abelian moment maps to discuss the symmetries of Yang-Baxter type models. The latter can be defined as the result of applying a general procedure for constructing integrable deformations of a broad family of integrable models. We refer to [138] for details on the general construction. To start this section, we will recall the main features of the construction, mostly based on the examples of Yang-Baxter type deformations presented in Chapter 3: the Yang-Baxter model (deformation of the PCM, Subsection 3.3.1), the η-deformation of the $\mathbb{Z}_2$-coset model (Subsection 3.3.3) and the Bi-Yang-Baxter model (Section 3.4).

Yang-Baxter type deformations apply to models which possesses a twist function (see Chapter 2) with a double pole $\lambda_0 \in \mathbb{R}$. To construct a Yang-Baxter type model we begin by modifying the twist function $\varphi(\lambda)$ of the undeformed model, by deforming its double pole at λ_0 to a pair of simple poles which we denote $\lambda_\pm$, while keeping all other poles and zeroes fixed. In order to preserve the reality condition (2.4.10) of the twist function, we should take either $\lambda_\pm$ both real or λ_+ and λ_- complex conjugate of one another. We refer to these two cases as the real and complex branches respectively.

Finally, we require also that the deformed twist function $\varphi(\lambda)$ has opposite residues at the simple poles λ_+ and λ_-, which allows us to define a single real deformation parameter $\gamma \in \mathbb{R}$ by

$$\frac{1}{\gamma} = 2cT \operatorname*{res}_{\lambda=\lambda_+} \varphi(\lambda)\mathrm{d}\lambda = -2cT \operatorname*{res}_{\lambda=\lambda_-} \varphi(\lambda)\mathrm{d}\lambda,$$

with $c = 1$ in the real branch and $c = i$ in the complex branch, and with T the order of cyclotomy of the model (see Section 2.4). This definition of γ agrees with the ones given for examples of Yang-Baxter type models, namely in equations (3.3.16), (3.3.25) and (3.4.39a).

The phase space of Yang-Baxter type models contains a G_0-valued field g and a $\mathfrak{g}_0$-valued field X satisfying the Poisson brackets (3.1.7). In particular, the evaluation of the (gauge transformed) Lax matrix of the model at the poles $\lambda_\pm$ is given by

$$\mathcal{L}^g(\lambda_\pm) = -\gamma R^{\mp}\big(gXg^{-1}\big), \tag{5.5.1}$$

where $R^\pm = R \pm c\,\mathrm{Id}$, for a certain solution R of the mCYBE on $\mathfrak{g}$ (which is part of the definition of the Yang-Baxter deformation). This general statement was verified for the examples mentioned above in equations (3.3.15), (3.3.25), (3.4.39a).

In this section, we will restrict to models whose space coordinate x belongs to the real line $\mathbb{R}$ (and not the circle $\mathbb{S}^1$). Let us then introduce the transfer matrices of the $\mathfrak{g}_0$-valued fields $\mathcal{L}^g(\lambda_\pm)$, given as the following path-order exponential (see Appendix A.5):

$$T^g(\lambda_\pm\,;\,x,y) = \mathrm{P}\overleftarrow{\exp}\left(-\int_x^y \mathcal{L}^g(\lambda_\pm,z)\,\mathrm{d}z\right).$$

In particular, we will be interested in the monodromy matrices

$$T^g(\lambda_\pm) = T^g(\lambda_\pm\,;\,-\infty,+\infty).$$

As $\mathcal{L}^g(\lambda_\pm)$ satisfies a zero curvature equation (because $\mathcal{L}$ satisfies the Lax equation (2.1.1)) and as the space coordinate is taken on the real line, the matrices $T^g(\lambda_\pm)$ are conserved quantities of the model (see Appendix A.5).

5.5.2 Poisson brackets of $T^g(\lambda_\pm)$

In this subsection, we will compute the Poisson bracket of $T^g(\lambda_\pm)$ with itself and with $T^g(\lambda_\mp)$. In general, when a Lax matrix obeys Poisson brackets of Maillet type, the Poisson bracket of its path-ordered exponential is ill-defined, due to the presence of non-ultralocal terms (the calculation then requires a regularisation procedure [20]). As we will see, this issue will not appear for the particular case of $T^g(\lambda_\pm)$.

Recall that g and X satisfy the Poisson brackets (3.1.7). For this section, we will use another parametrisation of the phase space of the model, by defining the fields

$$h = g^{-1} \quad \text{and} \quad Y = -gXg^{-1}$$

Let us motivate this choice. Considering Y instead of X is natural given the expression (5.5.1) of $\mathcal{L}^g(\lambda_\pm)$. Indeed, one then have

$$\mathcal{L}^g(\lambda_\pm) = \gamma R^{\mp} Y. \tag{5.5.2}$$

The Poisson brackets of g and Y were considered in equation (3.1.9). Our motivation to consider the field h instead of g is a matter of convention (more precisely it simplifies some changes of conventions between our article [P2] and this thesis): indeed, the Poisson brackets of h and Y are exactly the same as the Poisson brackets of g and X:

$$\big\{h_{\underline{\mathbf{1}}}(x), h_{\underline{\mathbf{2}}}(y)\big\} = 0, \tag{5.5.3a}$$
$$\big\{Y_{\underline{\mathbf{1}}}(x), h_{\underline{\mathbf{2}}}(y)\big\} = h_{\underline{\mathbf{2}}}(x) C_{\underline{\mathbf{12}}}\,\delta_{xy}, \tag{5.5.3b}$$
$$\big\{Y_{\underline{\mathbf{1}}}(x), Y_{\underline{\mathbf{2}}}(y)\big\} = -\big[C_{\underline{\mathbf{12}}}, Y_{\underline{\mathbf{2}}}(x)\big]\,\delta_{xy}. \tag{5.5.3c}$$

The variation of the path-ordered exponential of some field under an infinitesimal variation of this field is given by equation (A.5.2). In particular, for $\varepsilon, \varkappa \in \{+,-\}$, one has

$$\begin{aligned}\left\{T^g(\lambda_\varepsilon)_{\underline{\mathbf{1}}}, T^g(\lambda_\varkappa)_{\underline{\mathbf{2}}}\right\} = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \; & T^g(\lambda_\varepsilon\,;\, x, +\infty)_{\underline{\mathbf{1}}} T^g(\lambda_\varkappa\,;\, y, +\infty)_{\underline{\mathbf{2}}} \\ & \left\{\mathcal{L}^g(\lambda_\varepsilon, x)_{\underline{\mathbf{1}}}, \mathcal{L}^g(\lambda_\varkappa, y)_{\underline{\mathbf{2}}}\right\} T^g(\lambda_\varepsilon\,;\, -\infty, x)_{\underline{\mathbf{1}}} T^g(\lambda_\varkappa\,;\, -\infty, y)_{\underline{\mathbf{2}}}.\end{aligned} \tag{5.5.4}$$

One then needs the bracket $\left\{\mathcal{L}^g(\lambda_\varepsilon, x)_{\underline{\mathbf{1}}}, \mathcal{L}^g(\lambda_\varkappa, y)_{\underline{\mathbf{2}}}\right\}$. Starting from equation (5.5.2) and using the bracket (5.5.3c), we get

$$\left\{\mathcal{L}^g(\lambda_\varepsilon, x)_{\underline{\mathbf{1}}}, \mathcal{L}^g(\lambda_\varkappa, y)_{\underline{\mathbf{2}}}\right\} = \gamma \left[\mathcal{L}^g(\lambda_\varepsilon, x)_{\underline{\mathbf{1}}} + \mathcal{L}^g(\lambda_\varkappa, x)_{\underline{\mathbf{2}}}, R^\varkappa_{\underline{\mathbf{12}}}\right] \delta_{xy}. \tag{5.5.5}$$

To obtain this Poisson bracket, we used the mCYBE (3.3.1) on R and the skew-symmetry of R, in the form

$$R^\varepsilon_{\underline{\mathbf{12}}} = R^\varepsilon_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}} = -R^{-\varepsilon}_{\underline{\mathbf{2}}} C_{\underline{\mathbf{12}}} = -R^{-\varepsilon}_{\underline{\mathbf{21}}}.$$

In particular, the Poisson bracket (5.5.5) is ultralocal. Thus, as mentioned above, the computation of the Poisson brackets of the monodromies $T^g(\lambda_\pm)$ does not suffer of non-ultralocality issues. Combining equations (5.5.4) and (5.5.5) with the property (A.5.1) of path-order exponentials, one gets

$$\left\{T^g(\lambda_\pm)_{\underline{\mathbf{1}}}, T^g(\lambda_\pm)_{\underline{\mathbf{2}}}\right\} = \gamma \left[R_{\underline{\mathbf{12}}}, T^g(\lambda_\pm)_{\underline{\mathbf{1}}} T^g(\lambda_\pm)_{\underline{\mathbf{2}}}\right]. \tag{5.5.6}$$

$$\left\{T^g(\lambda_\mp)_{\underline{\mathbf{1}}}, T^g(\lambda_\pm)_{\underline{\mathbf{2}}}\right\} = \gamma \left[R^\pm_{\underline{\mathbf{12}}}, T^g(\lambda_\mp)_{\underline{\mathbf{1}}} T^g(\lambda_\pm)_{\underline{\mathbf{2}}}\right]. \tag{5.5.7}$$

In conclusion, $T^g(\lambda_-)$ and $T^g(\lambda_+)$ satisfy the Semenov-Tian-Shansky Poisson brackets (5.3.7).

In particular, let us suppose that R is a standard R-matrix on $\mathfrak{g}$ as described in Subsection 5.4.1 and Appendix C.3. Then, using the methods of Subsection 5.4.3, one can extract from $T^g(\lambda_\pm)$ conserved charges satisfying the q-deformed Poisson-Hopf algebra $\mathscr{U}_q(\mathfrak{g}_0)$ (note that the reality conditions on $T^g(\lambda_\pm)$ necessary to apply the results of Subsection 5.4.3 can be proven from the reality condition (2.4.7) of the Lax matrix). These deformed conserved charges were already found in [29] for the η-deformations of the PCM and the $\mathbb{Z}_2$-coset σ-model. The result presented above then generalises this to all Yang-Baxter type models, in a model independent way. The rest of this section is devoted to the study of the (Poisson-Lie) symmetry associated with these deformed charges (note however that this study does not require R to be standard).

Before pursuing, let us make a brief parenthesis. For the undeformed PCM, the conserved charges mentioned above are part of a bigger (actually infinite) algebra, the classical analogue of the Yangian $Y(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ [160, 161]. It was proved recently [36] (after the publication of the results presented here) that for the Yang-Baxter model, this Yangian is deformed in the algebra $\mathscr{U}_q(\widehat{\mathfrak{g}}_0)$, where $\widehat{\mathfrak{g}}_0$ is the affine algebra associated with $\mathfrak{g}_0$.

5.5.3 Poisson-Lie G_0-symmetry

For the remainder of this section we shall restrict attention to the non-split case. The treatment of the split case is completely analogous.

The non-abelian moment map

According to (5.5.2), $\mathcal{L}^g(\lambda_\mp, \sigma)$ take values in the subalgebras $\mathfrak{g}_\pm$ of the complex double $\mathfrak{g}$ (cf. subsection 5.3.2). Hence the path-ordered exponentials $T^g(\lambda_\mp\,; x, y)$ belong to the subgroups $G_\pm$, which are realisations of the dual group G_0^*. Moreover, we proved in the previous subsection that $T^g(\lambda_\mp)$ satisfies the Semenov-Tian-Shansky bracket. It follows from subsection 5.2.2 that $T^g(\lambda_\mp)$ has the right Poisson brackets for being the non-abelian moment map of a Poisson-Lie action of G_0. In the notations of the previous sections, we therefore consider

$$\Gamma^\pm = T^g(\lambda_\mp). \tag{5.5.8}$$

as the two realisations of a non-abelian moment map in $G_\pm$, embedded in the complex double G. It is natural to also look for the expression of this non-abelian moment map in the other realisation of the dual group G_0^*, namely the group G_R described in subsection 5.3.1. This is given by $\Gamma_R = \Delta_\pm^{-1}\big(T^g(\lambda_\mp)\big)$, with $\Delta_\pm : G_R \to G_\pm$ the automorphisms described in subsection 5.3.2. In order to evaluate this explicitly we note that (5.5.2) can be written as $\mathcal{L}^g(\lambda_\mp, x) = \Delta_\pm Y(x)$. Therefore, according to equation (A.5.3), the non-abelian moment map seen in G_R simply reads

$$\Gamma_R = \overleftarrow{\mathrm{P}\exp}_{G_R}\left(-\int_{-\infty}^{+\infty} \mathrm{d}x\, Y(x)\right). \tag{5.5.9}$$

In this expression, $Y(x)$ is seen as an element of $\mathfrak{g}_R$ and the path-ordered exponential is taken in the group G_R.

Transformation law of g, h and Y

As motivated in the previous subsection, we consider the Poisson-Lie action of G_0 generated by the non-abelian moment map $\Gamma^\pm = T^g(\lambda_\mp) \in G_\pm$. According to equation (5.3.9), the infinitesimal form of this action is given by

$$\delta_\epsilon f = \frac{1}{2i\gamma}\kappa\Big(\epsilon, T^g(\lambda_+)^{-1}\{T^g(\lambda_+), f\} - T^g(\lambda_-)^{-1}\{T^g(\lambda_-), f\}\Big) = -\kappa\big(\epsilon, \Gamma_R^{-1}\{\Gamma_R, f\}\big). \tag{5.5.10}$$

In the undeformed case $\gamma = 0$, the group G_R is abelian and, from the expression (5.5.9) of the non-abelian moment map Γ_R, the transformation (5.5.10) becomes the usual Hamiltonian action with moment map $\int_{-\infty}^{+\infty}\mathrm{d}x\, Y(x)$. From the brackets (5.5.3b) and (5.5.3c), one checks that this is the action of $\mathfrak{g}_0$ by right multiplication on h:

$$\delta_\epsilon h(x) = h(x)\epsilon \quad \text{and} \quad \delta_\epsilon Y(x) = [Y(x), \epsilon]. \tag{5.5.11}$$

We will see in the rest of this subsection that, for $\gamma \neq 0$, the Poisson-Lie action generated by $T^g(\lambda_\mp)$ is still a right multiplication of h, but with a more complicated parameter. Note that in terms of the initial field $g = h^{-1}$, the transformation (5.5.11) becomes the left multiplication on g.

Since the Poisson bracket of $\mathcal{L}^g(\lambda_\mp, x)$ with the fields h and Y is ultralocal, we can compute the Poisson brackets of $T^g(\lambda_\pm)$ with h and Y using equation (A.5.2), without the need for any regularisation. We find

$$\begin{aligned}
\big\{T^g(\lambda_\mp)_{\underline{1}}, h(x)_{\underline{2}}\big\} &= -\gamma\, T^g(\lambda_\mp\,;\, x, +\infty)_{\underline{1}}\, h(x)_{\underline{2}}\, R^\pm_{\underline{12}}\, T^g(\lambda_\mp\,;\, -\infty, x)_{\underline{1}},\\
\big\{T^g(\lambda_\mp)_{\underline{1}}, Y(x)_{\underline{2}}\big\} &= -\gamma\, T^g(\lambda_\mp\,;\, x, +\infty)_{\underline{1}}\, \Big[Y_{\underline{2}}(x), R^\pm_{\underline{12}}\Big]\, T^g(\lambda_\mp\,;\, -\infty, x)_{\underline{1}}.
\end{aligned}$$

Inserting these expressions into (5.5.10), we obtain the transformation law of h and Y,

$$\delta_\epsilon h(x) = h(x)K(x) \quad \text{and} \quad \delta_\epsilon Y(x) = [Y(x), K(x)], \tag{5.5.12}$$

where we have defined

$$\begin{aligned}
K(x) = \frac{1}{2i}R^+\Big(T^g(\lambda_+\,;\, -\infty, x)\,\epsilon\, T^g(\lambda_+\,;\, -\infty, x)^{-1}\Big)\\
- \frac{1}{2i}R^-\Big(T^g(\lambda_-\,;\, -\infty, x)\,\epsilon\, T^g(\lambda_-\,;\, -\infty, x)^{-1}\Big).
\end{aligned} \tag{5.5.13}$$

We note that this transformation law has the same structure as the undeformed one (5.5.11) but with ϵ replaced by a more complicated (and non-constant) expression $K(x)$. In particular, this field is non-local, as it contains $T^g(\lambda_\pm\,;\, -\infty, x)$. Since $T^g(\lambda_\pm\,;\, -\infty, x)$ becomes equal to the identity when $\gamma = 0$, we see that K turns back into ϵ in the undeformed case. As for the undeformed transformation, if one goes back to the initial field $g = h^{-1}$, one finds that the transformation (5.5.12) acts as a left multiplication

$$\delta_\epsilon g(x) = -K(x)g(x). \tag{5.5.14}$$

According to the paragraph following equation (5.2.1), the transformation (5.5.12) must preserve the Poisson brackets on (h, Y) if ϵ possesses a Poisson bracket with itself, coming from the linearisation of the Sklyanin Poisson bracket on G_0:

$$\{\epsilon_{\underline{\mathbf{1}}}, \epsilon_{\underline{\mathbf{2}}}\} = \gamma\big[R_{\underline{\mathbf{12}}}, \epsilon_{\underline{\mathbf{1}}} + \epsilon_{\underline{\mathbf{2}}}\big]. \tag{5.5.15}$$

For coherence, one can check this directly from the expression (5.5.13) of $K(x)$. This (slightly long) computation involves some algebraic manipulations to simplify the expressions, in particular the identity

$$\mathrm{Ad}_{T^g(\lambda_\mp)} \circ R^\pm = R^\pm \circ \mathrm{Ad}^{G_R}_{\Gamma_R},$$

which is a consequence of equation (A.5.3), applied to the automorphism $\Delta_\pm$.

The transformation law (5.5.12) may seem complicated because of the non-local expression (5.5.13) for $K(x)$. However, it can be re-interpreted in a simpler way by introducing the more adapted variables [27, 29, 138]

$$\Psi_\pm(x) = h(x)\xi_\pm(x), \qquad \text{with} \qquad \xi_\pm(x) = T^g(\lambda_\mp\,; -\infty, x).$$

In terms of these, the quantity (5.5.13) may be then rewritten as

$$K(x) = \frac{1}{2i}R^+\Big(\xi_-(x)\,\epsilon\,\xi_-(x)^{-1}\Big) - \frac{1}{2i}R^-\Big(\xi_+(x)\,\epsilon\,\xi_+(x)^{-1}\Big). \tag{5.5.16}$$

If we also introduce

$$Z(x) = \frac{1}{2i}\Big(\xi_+(x)\,\epsilon\,\xi_+(x)^{-1} - \xi_-(x)\,\epsilon\,\xi_-(x)^{-1}\Big),$$

then one checks that

$$\delta_\epsilon Y(x) = \big[Y(x), K(x)\big] = -\frac{1}{\gamma}\Big(\partial_x Z(x) + \big[Y(x), Z(x)\big]_R\Big).$$

Using this identity and equation (A.5.2), we find that the transformation law of $\xi_\pm$ reads

$$\delta_\epsilon \xi_\pm(x) = R^\pm Z(x)\,\xi_\pm(x).$$

Finally, it follows that the pair of fields $\Psi_\pm$ simply transform as

$$\delta_\epsilon \Psi_\pm(x) = \Psi_\pm(x)\epsilon.$$

It was observed in [29] that for Yang-Baxter type deformations with standard R-matrices, the Cartan part of the G_0-symmetry is preserved. This can be checked here explicitly: indeed, for $\epsilon \in \mathfrak{h}$ we find that the definition (5.5.13) of K reduces to ϵ, so that the infinitesimal transformation in the Cartan direction remains undeformed, as in (5.5.11). This fact can also be seen in terms of Poisson-Lie actions. For standard R-matrices, the Sklyanin bracket (5.5.15) vanishes when restricted to the Cartan subalgebra $\mathfrak{h}$. The corresponding action is then a usual Hamiltonian symmetry.

Let us end this paragraph by saying a few words about the BYB model. As explained in Section 3.4, it is the combination of two Yang-Baxter type deformations, breaking both the left and the right multiplication symmetries. The discussion above applies to the BYB model, as it satisfies equation (5.5.2), by equation (3.4.39a). The BYB model then admits a deformed left multiplication symmetry (5.5.14) (on g', or equivalently the right symmetry (5.5.12) on $h' = g'^{-1}$). This deformed symmetry is associated with conserved charges forming a deformed algebra $\mathscr{U}_q(\mathfrak{g}_0)$.

Recall however that the BYB model aslo satisfies equation (3.4.39b), which is similar to equation (3.4.39a) but with Y replaced by X'. This can be understood as the result of the Yang-Baxter type deformation on the right (recall from Subsection 3.1.2 that the integral of X' generates the right multiplication on g'). All the computation done above can be done with Y replaced by X' and h by g' and we would then find that the BYB model also admits a deformed right multiplication symmetry on g', associated with a deformed algebra $\mathscr{U}_{\tilde{q}}(\mathfrak{g}_0)$, where $\tilde{q} = e^{-ic\tilde{\gamma}}$.

Recall also that the BYB model can be seen as a deformation of a $\mathbb{Z}_2$-coset σ-model (this was the approach followed in Section 3.4). In particular, in this formulation, we obtained two equations of the form (5.5.2), respectively in (3.4.29) and (3.4.30). Thus, one can apply the method presented above and find two deformed left multiplication symmetries of the BYB model, acting respectively on g and $\widetilde{g}$. After gauge fixing, these transformations will reduce to the deformed left and right multiplication on the gauge-invariant field $g' = g\widetilde{g}^{-1}$.

Poisson-Lie symmetry: variation of the Hamiltonian and the action

In this section, we consider the case of the Yang-Baxter σ-model. The conservation of $T^g(\lambda_\pm)$ can be seen as the fact that it has a vanishing Poisson bracket with the Hamiltonian $\mathcal{H}$ of the model. This implies that the Hamiltonian is invariant under the Poisson-Lie action generated by $T^g(\lambda_\pm)$, namely

$$\delta_\epsilon \mathcal{H} = 0.$$

Thus, the transformation (5.5.12) is a symmetry of the Hamiltonian.

Let us now compute the variation of the action under the transformation. In the case of a Hamiltonian action (G_0^* abelian), the transformation is canonical and the invariance of the Hamiltonian is then equivalent to the invariance of the action. The situation is slightly more involved in the case of a Poisson-Lie action. The first order action is given

$$S = \int \mathrm{d}t\, \mathrm{d}x\, \kappa\big(h^{-1}\partial_t h, Y\big) - \int \mathrm{d}t\, \mathcal{H}.$$

Consider the transformation (5.5.12) of h and Y and, at first, let us allow the parameter ϵ to be a function of the time parameter t. Since $\mathcal{H}$ is invariant under this transformation, the variation of the action becomes

$$\delta_\epsilon S = \int \mathrm{d}t\, \mathrm{d}x\, \delta_\epsilon\Big(\kappa\big(h^{-1}\partial_t h, Y\big)\Big).$$

We have $\delta_\epsilon\big(h^{-1}\partial_t h\big) = \partial_t K + \big[h^{-1}\partial_t h, K\big]$, so that

$$\delta_\epsilon\Big(\kappa\big(h^{-1}(\partial_t h), Y\big)\Big) = \kappa\big(\partial_t K, Y\big) = \partial_t\big(\kappa(K,Y)\big) - \kappa\big(K, \partial_t Y\big).$$

Using expression (5.5.16) for K, the skew-symmetry of R and the invariance of κ under adjoint action, one finds

$$\kappa\big(K, \partial_t Y\big) = \frac{1}{2i}\kappa\big(\epsilon, \xi_+^{-1}\partial_t(R^+Y)\xi_+ - \xi_-^{-1}\partial_t(R^-Y)\xi_+\big).$$

Discarding the boundary terms at initial and final times, we get

$$\begin{aligned}\delta_\epsilon S = &\int \mathrm{d}t\, \frac{1}{2i\gamma}\kappa\left(\epsilon, T^g(\lambda_+)^{-1}\int_{-\infty}^{\infty}\mathrm{d}x\, T^g(\lambda_+\,;\,x,+\infty)\partial_t\mathcal{L}^g(\lambda_+,x)T^g(\lambda_+\,;\,-\infty,x)\right)\\ &- \int \mathrm{d}t\, \frac{1}{2i\gamma}\kappa\left(\epsilon, T^g(\lambda_-)^{-1}\int_{-\infty}^{\infty}\mathrm{d}x\, T^g(\lambda_-\,;\,x,+\infty)\partial_t\mathcal{L}^g(\lambda_-,x)T^g(\lambda_-\,;\,-\infty,x)\right).\end{aligned}$$

Using equation (A.5.2), this may be rewritten as

$$\delta_\epsilon S = \frac{1}{2i\gamma}\int \mathrm{d}\tau\, \kappa\big(\epsilon, T^g(\lambda_+)^{-1}\partial_t T^g(\lambda_+) - T^g(\lambda_-)^{-1}\partial_t T^g(\lambda_-)\big).$$

In terms of the "abstract" non-abelian moment map Γ, seen as a G_0^*-valued map, this is simply

$$\delta_\epsilon S = -\int \mathrm{d}t\, \big\langle \epsilon(t), \Gamma^{-1}\partial_t\Gamma\big\rangle.$$

By the principle of least action, $\delta_\epsilon S$ must be zero for any function $\epsilon(t)$, as long as the fields are on-shell. Thus, we recover the fact that Γ is conserved.

It is worth noticing that, if G_0^* is non-abelian, $\Gamma^{-1}\partial_t\Gamma$ is not a total time derivative. Thus, when we choose a constant parameter ϵ, we cannot conclude that $\delta_\epsilon S = 0$. That is to say, the action is not invariant under the Poisson-Lie symmetry. However, the latter is still a symmetry of the model since the Hamiltonian is invariant.

5.A Poisson brackets extraction theorems

In this appendix, we state and prove two theorems allowing to extract the Poisson brackets of the factors of a Lie-group-valued quantity.

Theorem 5.A.1. *Let F_1 and F_2 be two Lie groups, that are decomposable into two subgroups: $F_i = G_i H_i$. This group factorisation corresponds to a direct sum of Lie algebras $\mathfrak{f}_i = \mathfrak{g}_i \oplus \mathfrak{h}_i$. We denote by $\pi_{\mathfrak{g}_i}$ and $\pi_{\mathfrak{h}_i}$ the associated projections. We consider $A \in F_1$ and $B \in F_2$ that we factorise as*

$$A = uv, \quad \textit{and} \quad B = xy,$$

for $(u, v) \in G_1 \times H_1$ and $(x, y) \in G_2 \times H_2$. We define

$$\mathcal{P}_{\underline{12}} = u_{\underline{1}}^{-1} x_{\underline{2}}^{-1} \{A_{\underline{1}}, B_{\underline{2}}\} v_{\underline{1}}^{-1} y_{\underline{2}}^{-1} \ \in \mathfrak{f}_1 \otimes \mathfrak{f}_2.$$

Then, we have

$$\begin{aligned}
\{u_{\underline{1}}, x_{\underline{2}}\} &= u_{\underline{1}} x_{\underline{2}}\ \pi_{\mathfrak{g}_1} \otimes \pi_{\mathfrak{g}_2}\big(\mathcal{P}_{\underline{12}}\big), \\
\{u_{\underline{1}}, y_{\underline{2}}\} &= u_{\underline{1}}\ \pi_{\mathfrak{g}_1} \otimes \pi_{\mathfrak{h}_2}\big(\mathcal{P}_{\underline{12}}\big)\ y_{\underline{2}}, \\
\{v_{\underline{1}}, x_{\underline{2}}\} &= x_{\underline{2}}\ \pi_{\mathfrak{h}_1} \otimes \pi_{\mathfrak{g}_2}\big(\mathcal{P}_{\underline{12}}\big)\ v_{\underline{1}}, \\
\{v_{\underline{1}}, y_{\underline{2}}\} &= \pi_{\mathfrak{h}_1} \otimes \pi_{\mathfrak{h}_2}\big(\mathcal{P}_{\underline{12}}\big)\ v_{\underline{1}} y_{\underline{2}},
\end{aligned}$$

Proof. Using the Leibniz rule, we have

$$\{A_{\underline{1}}, B_{\underline{2}}\} = \{u_{\underline{1}}, x_{\underline{2}}\} v_{\underline{1}} y_{\underline{2}} + x_{\underline{2}} \{u_{\underline{1}}, y_{\underline{2}}\} v_{\underline{1}} + u_{\underline{1}} \{v_{\underline{1}}, x_{\underline{2}}\} y_{\underline{2}} + u_{\underline{1}} x_{\underline{2}} \{v_{\underline{1}}, y_{\underline{2}}\},$$

so that

$$\mathcal{P}_{\underline{12}} = \underbrace{u_{\underline{1}}^{-1} x_{\underline{2}}^{-1} \{u_{\underline{1}}, x_{\underline{2}}\}}_{\in \mathfrak{g}_1 \otimes \mathfrak{g}_2} + \underbrace{u_{\underline{1}}^{-1} \{u_{\underline{1}}, y_{\underline{2}}\} y_{\underline{2}}^{-1}}_{\in \mathfrak{g}_1 \otimes \mathfrak{h}_2} + \underbrace{x_{\underline{2}}^{-1} \{v_{\underline{1}}, x_{\underline{2}}\} v_{\underline{1}}^{-1}}_{\in \mathfrak{h}_1 \otimes \mathfrak{g}_2} + \underbrace{\{v_{\underline{1}}, y_{\underline{2}}\} v_{\underline{1}}^{-1} y_{\underline{2}}^{-1}}_{\in \mathfrak{h}_1 \otimes \mathfrak{h}_2},$$

and hence the theorem. □

Theorem 5.A.2. *Let G be a Lie group that factorises into subgroups as $G = G_1 \dots G_p$. This group factorisation corresponds to a direct sum of Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_p$, with associated projections π_i. Suppose this decomposition is such that, for all $i \in \{1, \dots, p\}$,*

$$\mathfrak{g}_{<i} = \bigoplus_{k=1}^{i-1} \mathfrak{g}_k \qquad \textit{and} \qquad \mathfrak{g}_{>i} = \bigoplus_{k=i+1}^{p} \mathfrak{g}_k$$

are subalgebras of $\mathfrak{g}$ and $[\mathfrak{g}_i, \mathfrak{g}_{<i}] \subseteq \mathfrak{g}_{<i}$.

Let f be a $\mathbb{R}$-valued function and A a G-valued function, that we factorise as

$$A = A^{(1)} \dots A^{(p)}, \quad \textit{with } \left(A^{(1)}, \dots, A^{(p)}\right) \in G_1 \times \dots \times G_p.$$

If we define

$$\mathcal{P}^{(i)} = \left(A^{(1)} \dots A^{(i)}\right)^{-1} \{A, f\} \left(A^{(i+1)} \dots A^{(p)}\right)^{-1} \in \mathfrak{g},$$

then we have

$$\left(A^{(i)}\right)^{-1} \left\{A^{(i)}, f\right\} = \pi_i\big(\mathcal{P}^{(i)}\big).$$

Proof. Let $B = A^{(1)} \dots A^{(i-1)}$ and $C = A^{(i+1)} \dots A^{(p)}$. Using the Leibniz rule, we have

$$\mathcal{P}^{(i)} = \underbrace{\left(A^{(i)}\right)^{-1} \left\{A^{(i)}, f\right\}}_{\in \mathfrak{g}_i} + \left(A^{(i)}\right)^{-1} \underbrace{\left(B^{-1} \{B, f\}\right)}_{\in \mathfrak{g}_{<i}} A^{(i)} + \underbrace{\{C, f\} C^{-1}}_{\in \mathfrak{g}_{>i}}.$$

Since $B^{-1}\{B, f\}$ belongs to $\mathfrak{g}_{<i}$, the assumption on the Lie subalgebras $\mathfrak{g}_k$ tells us that the adjoint action

$$\left(A^{(i)}\right)^{-1} \left(B^{-1} \{B, f\}\right) A^{(i)}$$

still belongs to $\mathfrak{g}_{<i}$, hence the theorem. □

Part II

Gaudin models

Chapter 6

Classical Gaudin models

The first part of this thesis concerned integrable models with twist function and in particular integrable σ-models. In the recent publication [78], Vicedo reinterpreted these field theories as so-called classical Dihedral Affine Gaudin Models (DAGM). Gaudin models, which are the subject of the second part of this thesis, are a particular class of integrable systems, associated with quadratic Lie algebras (Lie algebras with a non-degenerate invariant bilinear form).

In particular, one can construct Gaudin models associated with Kac-Moody algebras. These are a particular class of quadratic Lie algebras (see for example [162]). They are classified in three types: finite, affine and indefinite. The simplest Kac-Moody algebras are the ones of finite type: they exactly coincide with finite-dimensional semi-simple Lie algebras, as described in Appendix A.2. The associated Gaudin models, that we shall call finite Gaudin models, are thus integrable systems with a finite number of degrees of freedom. They have been extensively studied in the literature and are used to describe mechanical systems and spin chains.

The Kac-Moody algebras of non-finite type are much more complicated: in particular, they are all infinite dimensional. Among them, the affine Kac-Moody algebras are the simplest, as they are related to loop algebras of finite ones. Using this relation, Vicedo has shown in [78] that classical affine Gaudin models (associated with affine Kac-Moody algebras) can be viewed as two dimensional integrable field theories. More precisely, they are models with twist function, as described in the first part of this thesis.

In this chapter, we will focus on classical Gaudin models. In Section 6.1, we will develop the general theory of classical Gaudin models for an arbitrary quadratic Lie algebra. We will also describe briefly finite Gaudin models (illustrated with an example, the unreduced Neumann model). In Section 6.2, we will discuss classical affine Gaudin models and their relations with integrable fields theories with twist function. Most of this chapter is a review of the literature on classical Gaudin models, in particular based on the article [78]. However, we will end Section 6.2 by a new result about integrable hierarchies of affine Gaudin models, which is part of my article [P3] with M. Magro and B. Vicedo.

6.1 Classical Gaudin models

Let $\mathfrak{g}$ be a quadratic Lie algebra, *i.e.* a Lie algebra possessing a non-degenerate invariant bilinear form κ. For now, we will consider only complex Lie algebras. Real Gaudin models will be discussed later in Subsection 6.1.3.

Let us consider a basis $\{I^a\}$ of $\mathfrak{g}$, the bilinear form $\kappa^{ab} = \kappa(I^a, I^b)$ written in this basis, its inverse κ_{ab} and the dual basis $\{I_a = \kappa_{ab} I^b\}$. One can then consider the quadratic Casimir

$$C_{\underline{\mathbf{12}}} = I^a \otimes I_a = \kappa_{ab}\, I^a \otimes I^b \in \mathfrak{g} \otimes \mathfrak{g}. \tag{6.1.1}$$

It is defined in Appendix A.2.4 in the particular case of a semi-simple finite dimensional algebra equipped with the Killing form. However, its main properties (A.2.6) and (A.2.7) generalise easily to any quadratic Lie algebra.

Let us mention here a technical subtlety. If the Lie algebra $\mathfrak{g}$ is infinite dimensional, the sum defining $C_{\underline{\mathbf{12}}}$ in equation (6.1.1) is infinite. A rigorous definition of $C_{\underline{\mathbf{12}}}$ then requires to give a sense to this infinite sum, by considering an appropriate topology on $\mathfrak{g}\otimes\mathfrak{g}$. The quadratic Casimir then belongs to the completion $\mathfrak{g}\hat{\otimes}\mathfrak{g}$ with respect to this topology. We shall not enter into these technicalities here: when manipulating infinite sums, we will suppose that these can be well defined in an appropriate completion. These questions of convergence will appear in Section 6.2, as we will consider affine Kac-Moody algebras, which are infinite dimensional. We will then give a brief summary of how to treat these infinite sums in this case.

In addition to the quadratic Lie algebra $\mathfrak{g}$, a Gaudin model is defined from the data of N points $\lambda_1,\cdots,\lambda_N$ of the complex plane $\mathbb{C}$, given with multiplicities $m_1,\cdots,m_N\in\mathbb{Z}_{\geq 1}$. These points are called the sites of the model. For simplicity, we will restrict in this section to the case where all sites have multiplicity one. In the same way, for this first presentation of Gaudin models, we will not consider the so-called cyclotomic and dihedral Gaudin models. For completeness, and as it is important for the link between affine Gaudin models and integrable field theories, we will say a few words about the general formalism with arbitrary multiplicities and dihedrality at the end of this section. We refer to [78] for an exhaustive and rigorous presentation.

6.1.1 Phase space and Lax matrix

Phase space. Consider the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Recall that it can be made a Poisson manifold by the so-called Kirillov-Kostant bracket, as described in Appendix B.2 (technically the Appendix B.2 treats the case of a real Lie algebra but the discussion generalises easily to complex ones, the Poisson bracket then being a $\mathbb{C}$-linear derivation of the complex algebra of functions on $\mathfrak{g}^*$). The phase space M of the Gaudin model with sites $\lambda_1,\cdots,\lambda_N$ is the Cartesian product of N copies of the Poisson manifold $\mathfrak{g}^*$. Concretely, this means that the algebra $\mathcal{F}[M]$ of functions on M is generated by elements $X^a_{(r)}$ $(r=1,\cdots,N)$, satisfying the Poisson bracket

$$\left\{X^a_{(r)},X^b_{(s)}\right\}=\delta_{rs}\,f^{ab}{}_c\,X^c_{(r)},$$

where the $f^{ab}{}_c$'s are the structure constants of $\mathfrak{g}$.

Using the non-degenerate form κ, one can encode all the fundamental functions $X^a_{(r)}$, for a fixed $r\in\{1,\cdots,N\}$, in a unique object

$$X^{(r)}=\kappa_{ab}\,I^a\otimes X^b_{(r)}\in\mathfrak{g}\otimes\mathcal{F}[M]. \tag{6.1.2}$$

Here also, one would have to consider a completion of $\mathfrak{g}\otimes\mathcal{F}[M]$ to consider $X^{(r)}$ if the algebra $\mathfrak{g}$ is infinite. As an element of $\mathfrak{g}\otimes\mathcal{F}[M]$, $X^{(r)}$ is a $\mathfrak{g}$-valued function on M. As explained in Appendix B.2, the Poisson bracket of the $X^{(r)}$'s can be written in tensorial notations as

$$\left\{X^{(r)}_{\underline{\mathbf{1}}},X^{(s)}_{\underline{\mathbf{2}}}\right\}=\delta_{rs}\left[C_{\underline{\mathbf{12}}},X^{(r)}_{\underline{\mathbf{1}}}\right]=-\delta_{rs}\left[C_{\underline{\mathbf{12}}},X^{(r)}_{\underline{\mathbf{2}}}\right].$$

Lax matrix. Let us now define the Lax matrix of the Gaudin model. It is a rational function of a complex parameter λ (the spectral parameter) which depends on the positions λ_r's of the sites and contains all the $\mathfrak{g}$-valued functions $X^{(r)}$:

$$\mathscr{L}(\lambda)=\sum_{r=1}^{N}\frac{X^{(r)}}{\lambda-\lambda_r}+\Omega, \tag{6.1.3}$$

where Ω is a constant element in $\mathfrak{g}$, in the sense that it has a vanishing Poisson bracket with all functions in $\mathcal{F}[M]$ (this constant element can be seen as a site with multiplicity two at infinity, but we shall not develop further this interpretation here, cf. [78]). Using the circle lemma (2.3.5), one checks that the Poisson bracket of the Lax matrix reads

$$\left\{\mathscr{L}_{\underline{\mathbf{1}}}(\lambda),\mathscr{L}_{\underline{\mathbf{2}}}(\mu)\right\}=\left[r_{\underline{\mathbf{12}}}(\lambda,\mu),\mathscr{L}_{\underline{\mathbf{1}}}(\lambda)+\mathscr{L}_{\underline{\mathbf{2}}}(\mu)\right], \tag{6.1.4}$$

with

$$\pi_{\underline{\mathbf{12}}}(\lambda,\mu)=\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}.$$

Note that this Poisson bracket is independent of the constant element $\Omega\in\mathfrak{g}$ (to get this result, one needs to use the identity (A.2.6)).

In particular, if $\mathfrak{g}$ is taken to be a finite dimensional semi-simple algebra, the matrix π coincides with $\mathcal{R}^0$, the standard non-twisted $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$, as introduced in Section 2.3. As explained in this section, this matrix satisfies the CYBE (2.3.3). Although we considered the case of a semi-simple finite algebra $\mathfrak{g}$, the proof generalises to any quadratic Lie algebra $\mathfrak{g}$ and in general the matrix $\pi_{\underline{\mathbf{12}}}(\lambda,\mu)$ satisfies the CYBE.

Note that π is skew-symmetric, in the sense that $\pi_{\underline{\mathbf{12}}}(\lambda,\mu)=-\pi_{\underline{\mathbf{21}}}(\mu,\lambda)$. This property ensures the skew-symmetry of the bracket (6.1.4). Note that one can rewrite the Poisson bracket (6.1.4) in a form where the skew-symmetry is obvious:

$$\big\{\mathscr{L}_{\underline{\mathbf{1}}}(\lambda),\mathscr{L}_{\underline{\mathbf{2}}}(\mu)\big\}=\big[\pi_{\underline{\mathbf{12}}}(\lambda,\mu),\mathscr{L}_{\underline{\mathbf{1}}}(\lambda)\big]-\big[\pi_{\underline{\mathbf{21}}}(\mu,\lambda),\mathscr{L}_{\underline{\mathbf{2}}}(\mu)\big]\,.$$

6.1.2 Hamiltonians and Lax equation

Quadratic Hamiltonians. So far, we defined the phase space M of the Gaudin model, whose fundamental coordinates are encoded in the Lax matrix (6.1.3). To define entirely the model, one has to specify a time evolution on this phase space, in the form of a Hamiltonian. Recall the bilinear form κ on $\mathfrak{g}$. We define the so-called quadratic Hamiltonian of the model as

$$\mathscr{H}(\lambda)=\frac{1}{2}\kappa\big(\mathscr{L}(\lambda),\mathscr{L}(\lambda)\big). \tag{6.1.5}$$

As $\mathscr{L}(\lambda)$ is a $\mathfrak{g}$-valued function on M, $\mathscr{H}(\lambda)$ is simply a function on M. Using the expression (6.1.3), one finds the following partial fraction decomposition of $\mathscr{H}(\lambda)$:

$$\mathscr{H}(\lambda)=\Delta_\infty+\sum_{r=1}^{N}\left(\frac{\Delta_r}{(\lambda-\lambda_r)^2}+\frac{\mathscr{H}_r}{\lambda-\lambda_r}\right), \tag{6.1.6}$$

with

$$\Delta_r=\frac{1}{2}\kappa\big(X^{(r)},X^{(r)}\big),\qquad \Delta_\infty=\frac{1}{2}\kappa(\Omega,\Omega) \tag{6.1.7}$$

and

$$\mathscr{H}_r=\sum_{s\neq r}\frac{\kappa\big(X^{(r)},X^{(s)}\big)}{\lambda_r-\lambda_s}+\kappa\big(X^{(r)},\Omega\big). \tag{6.1.8}$$

Using the Poisson bracket (6.1.4) and the fact that κ is invariant (*i.e.* that $\kappa([X,Y],Z)=\kappa(X,[Y,Z])$ for all $X,Y,Z\in\mathfrak{g}$), one finds that

$$\{\mathscr{H}(\lambda),\mathscr{H}(\mu)\}=0,\qquad\forall\,\lambda,\mu\in\mathbb{C}. \tag{6.1.9}$$

Thus the quadratic Hamiltonian is in involution with itself for any values of the spectral parameter. In particular, one finds that the $\mathscr{H}_r$'s, the Δ_r's and Δ_∞ are all in involution with one another.

As it turns out, one can show that the Δ_r's and Δ_∞ are Poisson Casimirs of the model, *i.e.* that they have a vanishing Poisson bracket with any function in $\mathcal{F}[M]$. However, the $\mathscr{H}_r$'s are not Poisson Casimirs in general and their involution is thus a non trivial result. We will define the Hamiltonian of the model as a linear combination

$$\mathscr{H}=\sum_{r=1}^{N}c_r\mathscr{H}_r \tag{6.1.10}$$

of the $\mathscr{H}_r$. In particular, the $\mathscr{H}_r$'s are conserved charges in involution of the model.

Lax equation. It is natural to study the time flow of the Gaudin model, defined as the Hamiltonian flow of $\mathscr{H}$ on the phase space: $\partial_t = \{\mathscr{H}, \cdot\}$. For that, let us compute the Hamiltonian flow of the quadratic Hamiltonian $\mathscr{H}(\mu)$ on the Lax matrix $\mathscr{L}(\lambda)$. Starting from the bracket (6.1.4), using the invariance of κ and the completeness relation (A.2.7), one finds that

$$\big\{\mathscr{H}(\mu), \mathscr{L}(\lambda)\big\} = \big[\mathscr{M}(\mu,\lambda), \mathscr{L}(\lambda)\big], \qquad \text{with} \qquad \mathscr{M}(\mu,\lambda) = \frac{\mathscr{L}(\mu)}{\lambda-\mu}. \tag{6.1.11}$$

In particular, we find

$$\{\mathscr{H}_r, \mathscr{L}(\lambda)\} = \left[\frac{X^{(r)}}{\lambda-\lambda_r}, \mathscr{L}(\lambda)\right].$$

Thus, the time evolution of $\mathscr{L}(\lambda)$ takes the form of the Lax equation

$$\partial_t \mathscr{L}(\lambda) = \big\{\mathscr{H}, \mathscr{L}(\lambda)\big\} = \big[\mathscr{M}(\lambda), \mathscr{L}(\lambda)\big], \tag{6.1.12}$$

with

$$\mathscr{M}(\lambda) = \sum_{r=1}^{N} c_r \frac{X^{(r)}}{\lambda-\lambda_r}.$$

We will explain in Section 6.2 the relation between this Lax equation when $\mathfrak{g}$ is an affine Kac-Moody algebra and the Lax equation (2.1.1) for a field theory.

Higher degree Hamiltonians. In this subsection, we defined the quadratic Hamiltonians of the Gaudin model, using the bilinear form κ. These quadratic Hamiltonians were then in involution, as a consequence of the invariance of κ. This construction generalises for any invariant polynomial on $\mathfrak{g}$, as we shall explain now.

Let Φ be a polynomial of degree d on $\mathfrak{g}$. One can see Φ as a totally symmetric d-linear form on $\mathfrak{g}$, *i.e.* such that, for all permutation σ of $\{1,\cdots,d\}$, we have

$$\Phi\big(Y_1,\cdots,Y_d\big) = \Phi\big(Y_{\sigma(1)},\cdots,Y_{\sigma(d)}\big), \quad \forall\, Y_1,\cdots,Y_d \in \mathfrak{g}.$$

We say that Φ is invariant if for any $Y_1,\cdots,Y_d,Z \in \mathfrak{g}$ we have

$$\Phi\big([Z,Y_1],Y_2,\cdots,Y_d\big) + \cdots + \Phi\big(Y_1,Y_2,\cdots,[Z,Y_d]\big) = 0.$$

In particular, the bilinear form defines an invariant polynomial of degree two on $\mathfrak{g}$.

Let Φ be an invariant polynomial of degree d on $\mathfrak{g}$. Evaluating this polynomial on the Lax matrix of the Gaudin model, we define the charge

$$\mathscr{Q}_\Phi(\lambda) = \frac{1}{d}\Phi\big(\mathscr{L}(\lambda),\cdots,\mathscr{L}(\lambda)\big). \tag{6.1.13}$$

In particular, the quadratic Hamiltonian is then $\mathscr{H}(\lambda) = \mathscr{Q}_\kappa(\lambda)$.

Let Φ and Ψ be two invariant polynomials on $\mathfrak{g}$. Starting from the Poisson bracket (6.1.4) of the Lax matrix, one can compute the Poisson bracket of the charges $\mathscr{Q}_\Phi(\lambda)$ and $\mathscr{Q}_\Psi(\mu)$. Using the invariance of Φ and Ψ, one finds that

$$\big\{\mathscr{Q}_\Phi(\lambda), \mathscr{Q}_\Psi(\mu)\big\} = 0.$$

Thus, charges constructed from invariant polynomials are in involution one with another. In particular, they are in involution with $\mathscr{H}(\lambda)$ and thus with the Hamiltonian $\mathscr{H}$, hence they are conserved. This is a general method for constructing a large number of commuting conserved charges in Gaudin models.

6.1.3 Real classical Gaudin models

So far, we discussed complex Gaudin models, defined on complex Lie algebras. Let us now consider real Gaudin models. Let $\mathfrak{g}_0$ be a real quadratic Lie algebra. If we choose points $\lambda_1, \cdots, \lambda_N$ on the real line $\mathbb{R}$, the construction presented above for a complex Lie algebra easily applies to the real algebra (by also considering the spectral parameter λ to be real). We then get a real Gaudin model.

However, one can consider a larger class of real Gaudin models, by allowing the sites λ_k's to be complex. In this case, one has to change the construction of the Gaudin model. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$: $\mathfrak{g}_0$ can then be seen as the real form $\mathfrak{g}_0 = \mathfrak{g}^\tau$ for some antilinear involutive automorphism τ of $\mathfrak{g}$ (see Appendix A.3). Note that the quadratic form κ on $\mathfrak{g}_0$ extends naturally in a non-degenerate form on $\mathfrak{g}$ invariant under τ.

As complex ones, a real Gaudin model is defined by the data of sites $\lambda_1, \cdots, \lambda_N$ in $\mathbb{C}$. However, for the real model we shall suppose that if λ_k is a site, $\overline{\lambda}_k$ is not. The phase space of the real model is then defined as follows. If λ_k is a site of the model in $\mathbb{R}$, then we consider the real dual $\mathfrak{g}_0^*$ of $\mathfrak{g}_0$, equipped with the Kirillov-Kostant bracket. This space is described by a $\mathfrak{g}_0$-valued observable $X^{(k)}$ as defined in equation (6.1.2) (bur for $\mathfrak{g}_0$ instead of $\mathfrak{g}$). If λ_k is a site in $\mathbb{C} \setminus \mathbb{R}$, then we consider the complex dual $\mathfrak{g}^*$ of the complexification $\mathfrak{g}$, but as a real vector space (which is then of dimension $2\dim(\mathfrak{g}_0)$). This space is described by an observable $X^{(k)}$ which is $\mathfrak{g}$-valued and not $\mathfrak{g}_0$-valued. The phase space M of the real Gaudin model is then the Cartesian product of these vectors spaces associated with each site.

We define the Lax matrix of the model as the following $\mathfrak{g}$-valued function of the spectral parameter $\lambda \in \mathbb{C}$:

$$\mathscr{L}(\lambda) = \sum_{\substack{k=1 \\ \lambda_k \in \mathbb{R}}}^{N} \frac{X^{(k)}}{\lambda - \lambda_k} + \sum_{\substack{k=1 \\ \lambda_k \in \mathbb{C}\setminus\mathbb{R}}}^{N} \left(\frac{X^{(k)}}{\lambda - \lambda_k} + \frac{\tau\big(X^{(k)}\big)}{\lambda - \overline{\lambda}_k} \right) + \Omega, \tag{6.1.14}$$

where Ω is a constant element of $\mathfrak{g}_0$. Using the definitions of $X^{(k)}$ as being either $\mathfrak{g}_0$-valued or $\mathfrak{g}$-valued, depending on whether λ_k is real or not, one checks that this Lax matrix satisfies the reality condition

$$\tau\big(\mathscr{L}(\lambda)\big) = \mathscr{L}(\overline{\lambda}). \tag{6.1.15}$$

The rest of the construction of the real Gaudin model is then similar to the complex case. The Poisson bracket of the Lax matrix (6.1.14) of the real model is exactly the same as the complex one, *i.e.* the bracket (6.1.4).

One defines the quadratic Hamiltonian $\mathscr{H}(\lambda)$ as in (6.1.5), which also satisfies the involution equation (6.1.9). Using the fact that the extended form κ on $\mathfrak{g}$ is τ-invariant, one finds that this Hamiltonian satisfies the reality condition

$$\overline{\mathscr{H}(\lambda)} = \mathscr{H}(\overline{\lambda}).$$

One can then write a partial fraction decomposition of $\mathscr{H}(\lambda)$ as in (6.1.6) but with also double and simple poles at $\overline{\lambda}_k$ if $\lambda_k \notin \mathbb{R}$, with coefficients $\overline{\Delta_k}$ and $\overline{\mathscr{H}_k}$. One can then define a real Hamiltonian $\mathscr{H}$ as in (6.1.10) by a linear combination of the $\mathscr{H}_k$'s and $\overline{\mathscr{H}_k}$'s with appropriate reality conditions on the coefficients.

If one considers only real sites λ_k's, then this definition of the real Gaudin model agrees with the simplest one presented at the beginning of this subsection, as one then considers only $\mathfrak{g}_0$-valued $X^{(k)}$'s.

6.1.4 Finite Gaudin models and an example

Classical finite Gaudin models. In this subsection, we will suppose that $\mathfrak{g}$ is a Kac-Moody algebra of finite type, *i.e.* a finite dimensional semi-simple Lie algebra (see Appendix A.2). The associated Gaudin model then possesses a finite number of degrees of freedom and is thus a system of classical mechanics. In this case, the equation (6.1.12) is what is generally called a Lax equation for a mechanical system (see for example [15]).

Finite algebras admit many invariant polynomials. For example, let us consider a representation of $\mathfrak{g}$ (we can then consider the elements of $\mathfrak{g}$ as matrices): the polynomial $X \in \mathfrak{g} \mapsto \mathrm{Tr}(X^d)$ is then

an invariant polynomial of degree d on $\mathfrak{g}$. This way, one can construct a large number of commuting conserved charges in involution for finite Gaudin models.

Following the construction explained above, it appears that there are an infinite number of such charges (for example considering all powers $d \in \mathbb{N}$). However, these charges are not all independent: for example, for matrices of size n, the traces $\mathrm{Tr}(X^d)$'s for $d > n$ can be expressed in terms of the ones for $d \leq n$. The structure of the algebra of invariant polynomials on a finite Lie algebra $\mathfrak{g}$ has been completely described in the literature [163]. In particular, it is generated by ℓ independent polynomials Φ_i's, where ℓ is the rank of $\mathfrak{g}$. These polynomials have degrees $d_i + 1$, where the d_i's are the so-called exponents of the Lie algebra $\mathfrak{g}$ (we shall come back on that fact in Chapter 7). One then finds all independent charges in involution of the Gaudin model by considering the charges (6.1.13) associated with the fundamental invariant polynomials Φ_i.

An example: the unreduced Neumann model. Let us consider the so-called Neumann model [164]. This model describes the movement of a particle in an (anisotropic) harmonic potential and constrained on a $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$. We describe the particle by its position $(x_1, \cdots, x_N)$ by embedding the sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. We then have the constraint $\sum_{i=1}^N x_i^2 = 1$. This model was shown to be integrable by Uhlenbeck [165].

We will consider here a slightly different model, that we shall call the unreduced Neumann model. It is a model on the whole space $\mathbb{R}^N$, which once reduced to the sphere $\mathbb{S}^{N-1}$ coincides with the Neumann model. The phase space of the model is given by the positions x_i's and the corresponding momenta p_i's $(i = 1, \cdots, N)$. The Hamiltonian of the model is

$$\mathcal{H}_N = \frac{1}{2}\sum_{i=1}^N \omega_i^2 x_i^2 + \frac{1}{4}\sum_{i\neq j} J_{ij}^2,$$

with ω_i the pulsation of the harmonic oscillator in the axis i and

$$J_{ij} = x_i p_j - x_j p_i.$$

Let us define the so-called Ulhenbeck quantities

$$\mathcal{F}_i = x_i^2 + \sum_{\substack{j=1\\ j\neq i}}^N \frac{J_{ij}^2}{\omega_i^2 - \omega_j^2}, \tag{6.1.16}$$

for $i \in \{1, \cdots, N\}$. One shows that these are independent conserved charges in involution of the unreduced Neumann model (hence proving its Liouville integrability [15]). They are related to the Hamiltonian by

$$\mathcal{H}_N = \sum_{i=1}^N \omega_i^2 \mathcal{F}_i. \tag{6.1.17}$$

Let us consider the two by two matrices [166] (see also [167])

$$\mathcal{L}_N(\lambda) = \sum_{k=1}^N \frac{1}{\lambda - \omega_k^2}\begin{pmatrix} x_k p_k & x_k^2 \\ -p_k^2 & -x_k p_k \end{pmatrix} + \Omega, \quad \mathcal{M}_N(\lambda) = \sum_{k=1}^N \begin{pmatrix} x_k p_k & x_k^2 \\ -p_k^2 & -x_k p_k \end{pmatrix} + \lambda\Omega, \tag{6.1.18}$$

depending on a spectral parameter λ, with $\Omega = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. The equations of motion of the unreduced Neumann model can be recast in the form of the Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_N(\lambda) = \{\mathcal{H}_N, \mathcal{L}_N(\lambda)\} = [\mathcal{M}_N(\lambda), \mathcal{L}_N(\lambda)]\,.$$

During my PhD, I have co-supervised a master thesis about the Neumann model. The main subject of this project was the search of a two by two Lax pair for an integrable deformation of the Neumann model (for $N = 3$), introduced recently by Arutyunov and Medina-Rincon in [168]. Another part of the project was to understand the reinterpretation of the unreduced Neumann model as a Gaudin model on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, as stated in [169]. We will present this reinterpretation here, as an example of what we mean exactly when saying that a model is realised as a Gaudin model.

We consider the simple real Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$, equipped with the invariant bilinear form $\kappa(X,Y) = \frac{1}{2}\mathrm{Tr}(XY)$, making it a quadratic Lie algebra. Let us fix a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{g}_0$. We consider a classical real Gaudin model associated with $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ and with N sites $\lambda_r = \omega_r^2$ $(r = 1, \cdots, N)$. As these are real sites, the construction of this real Gaudin model is similar to the complex one (see Subsection 6.1.3). The phase space M of this model (see subsections 6.1.1 and 6.1.3) is described by $3N$ fundamental functions $X^H_{(r)}$, $X^E_{(r)}$ and $X^F_{(r)}$. The non-vanishing Poisson brackets of these functions are

$$\{X^H_{(r)}, X^E_{(s)}\} = 2\delta_{rs}X^E_{(r)}, \quad \{X^H_{(r)}, X^F_{(s)}\} = -2\delta_{rs}X^F_{(r)} \quad \text{and} \quad \{X^E_{(r)}, X^F_{(s)}\} = \delta_{rs}X^H_{(r)}.$$

The Lax matrix of the Gaudin model is then

$$\mathscr{L}(\lambda) = \sum_{k=1}^{N} \frac{1}{\lambda - \omega_k^2} \begin{pmatrix} X^H_{(k)} & 2X^F_{(k)} \\ 2X^E_{(k)} & -X^H_{(k)} \end{pmatrix} + \Omega,$$

where by anticipation we already choose the constant matrix Ω to be the one appearing in (6.1.18). The presence of factors 2 in the above equation is due to the expression of the form κ in the basis $\{H, E, F\}$: $\kappa(H,H) = 1$, $\kappa(E,F) = \kappa(F,E) = \frac{1}{2}$.

The phase space M of the Gaudin model can be seen as the vector space $\mathbb{R}^{3N}$ with coordinates $\big(X^H_{(k)}, X^E_{(k)}, X^F_{(k)}\big)_{k=1,\cdots,N}$. In the same way, the phase space of the unreduced Neumann model is simply the vector space $\mathbb{R}^{2N}$ with canonical coordinates $(x_k, p_k)_{k=1,\cdots,N}$. We define the following map from $\mathbb{R}^{2N}$ to M:

$$\begin{array}{rccc} \pi : & \mathbb{R}^{2N} & \longrightarrow & M \simeq \mathbb{R}^{3N} \\ & (x_k, p_k)_{k=1,\cdots,N} & \longmapsto & \big(x_k p_k, -\frac{1}{2}p_k^2, \frac{1}{2}x_k^2\big)_{k=1,\cdots,N} \end{array}.$$

One checks that this is a Poisson map, *i.e.* that $x_k p_k$, $-\frac{1}{2}p_k^2$ and $\frac{1}{2}x_k^2$ satisfy the same Poisson bracket as $X^H_{(k)}$, $X^E_{(k)}$ and $X^F_{(k)}$. The map π induces a morphism of algebras π^* from $\mathcal{F}[M]$ to $\mathcal{F}[\mathbb{R}^{2N}]$ by pullback:

$$\begin{array}{rccc} \pi^* : & \mathcal{F}[M] & \longrightarrow & \mathcal{F}[\mathbb{R}^{2N}] \\ & f & \longmapsto & f \circ \pi \end{array}.$$

As π is a Poisson map, π^* sends the Poisson bracket on $\mathcal{F}[M]$ to the one on $\mathcal{F}[\mathbb{R}^{2N}]$. We will say that π is a realisation of the Gaudin phase space M.

As we will see, π maps the unreduced Neumann model on the $\mathfrak{sl}(2,\mathbb{R})$-Gaudin model. Indeed, it is clear that

$$\pi^*\big(\mathscr{L}(\lambda)\big) = \mathcal{L}_N(\lambda),$$

hence the Lax matrix of the unreduced Neumann model coincides with the one of the Gaudin model through the map π^*. In the same way, one checks that the Ulhenbeck quantities (6.1.16) coincide with the quadratic Hamiltonians (6.1.8) of the Gaudin model:

$$\pi^*\big(\mathscr{H}_k\big) = \mathcal{F}_k = \frac{1}{2}\operatorname*{res}_{\lambda=\omega_k^2} \kappa\big(\mathcal{L}_N(\lambda), \mathcal{L}_N(\lambda)\big).$$

By equation (6.1.17), we then have

$$\pi^*(\mathscr{H}) = \mathcal{H}_N, \qquad \text{with} \qquad \mathscr{H} = \sum_{k=1}^{N} \omega_k^2 \mathscr{H}_k$$

of the form (6.1.10). The quadratic Hamiltonians $\mathscr{H}_k$ are part of the quadratic charge $\mathscr{H}(\lambda)$. Recall from (6.1.6) that $\mathscr{H}(\lambda)$ also contains the Casimirs Δ_k, defined in (6.1.7). It is easy to check that these Casimirs vanishes under the realisation π:

$$\pi^*(\Delta_k) = 0.$$

This is to be partially expected: indeed, the canonical phase space $\mathbb{R}^{2N}$ of the model is symplectic, *i.e.* its Poisson bracket does not possess non constant Casimirs. The image of Δ_k under π^* should then be a constant (and in this case is actually equal to zero).

This illustrates more generally what we mean when saying that a model is identified as a Gaudin model: there exists a Poisson map from the phase space of the model to the phase space of the Gaudin model, which maps the Hamiltonian of the model to the Gaudin Hamiltonian. Note however that this map is in general not an isomorphism, so that the phase space of the initial model is not isomorphic to the phase space of the Gaudin model. This is the case for the map π in the case of the unreduced Neumann model: it clearly cannot be surjective on dimensional grounds (and the phase spaces $\mathbb{R}^{2N}$ and M cannot be isomorphic as $\mathbb{R}^{2N}$ is symplectic and M is not).

6.1.5 Cyclotomic and dihedral Gaudin models with arbitrary multiplicities

In this subsection, we say a few words about various generalisations of the classical Gaudin model described previously in this section. The most general formalism of a dihedral Gaudin model with arbitrary singularities is described in the article [78] (although this article treats the case of an affine Lie algebra, the transcription of the formalism to an arbitrary quadratic Lie algebra is straightforward). These generalisations are necessary to understand entirely the reinterpretation of integrable σ-models as (dihedral) affine Gaudin models.

Gaudin models with multiplicities. For this section, we restricted to Gaudin models whose sites $\lambda_1, \cdots, \lambda_N$ have multiplicity one. Let us say a few words about Gaudin models with arbitrary multiplicities $m_1, \cdots, m_N \in \mathbb{Z}_{\geq 1}$ [109]. We will focus on complex Gaudin models, the generalisation to real ones being quite similar to the case with multiplicity one (as discussed in Subsection 6.1.3).

The phase space of the model is the Cartesian product of N Poisson vector spaces associated with the N sites. The space associated with λ_r is the Kirillov-Kostant space $\mathsf{T}^{m_r}\mathfrak{g}^*$ associated with the so-called Takiff algebra $\mathsf{T}^{m_r}\mathfrak{g}$ of multiplicity m_r [170]. The Takiff algebra $\mathsf{T}^m\mathfrak{g}$ is constructed from the Lie algebra $\mathfrak{g}$ as follows. We denote by $\mathbb{C}[\varepsilon]$ the algebra of complex polynomials of a formal variable ε. The subspace $\varepsilon^m\mathbb{C}[\varepsilon]$ is then an ideal of $\mathbb{C}[\varepsilon]$. The Takiff algebra associated with $\mathfrak{g}$ and of multiplicity m is then defined as

$$\mathsf{T}^m\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[\epsilon]/\epsilon^m\mathbb{C}[\epsilon].$$

If $m = 1$, one has $\mathbb{C}[\epsilon]/\epsilon\mathbb{C}[\epsilon] \simeq \mathbb{C}$, so that the Takiff algebra $\mathsf{T}^1\mathfrak{g}$ coincide with $\mathfrak{g}$. Let us describe more concretely the Takiff algebra $\mathsf{T}^m\mathfrak{g}$. We consider a basis $\{I^a\}$ of $\mathfrak{g}$, with associated structure constants $f^{ab}{}_c$. Then the Takiff algebra has a basis $I^a_{[p]}$, with $p = 0, \cdots, m-1$, with bracket

$$\left[I^a_{[p]}, I^b_{[q]}\right] = \begin{cases} f^{ab}{}_c\, I^c_{[p+q]} & \text{if } p+q \leq m-1, \\ 0 & \text{if } p+q > m-1. \end{cases}$$

The phase space of the Gaudin model is defined as

$$M = \mathsf{T}^{m_1}\mathfrak{g}^* \times \cdots \times \mathsf{T}^{m_N}\mathfrak{g}^*,$$

where each dual $\mathsf{T}^m\mathfrak{g}^*$ is equipped with the Kirillov-Kostant bracket. This phase space is then parametrized by $\mathfrak{g}$-valued observables $X^{(r)}_{[p]}$, with $r = 1, \cdots, N$ and $p = 0, \cdots, m_r - 1$, satisfying the brackets

$$\left\{X^{(r)}_{[p]\,\underline{\mathbf{1}}}, X^{(s)}_{[q]\,\underline{\mathbf{2}}}\right\} = \begin{cases} \delta_{rs}\left[C_{\underline{\mathbf{12}}}, X^{(r)}_{[p+q]\underline{\mathbf{1}}}\right] & \text{if } p+q \leq m_r - 1, \\ 0 & \text{if } p+q > m_r - 1. \end{cases} \tag{6.1.19}$$

The Lax matrix of the model is defined as [109]

$$\mathscr{L}(\lambda) = \sum_{r=1}^{N} \sum_{p=0}^{m_r-1} \frac{X_{[p]}^{(r)}}{(\lambda-\lambda_r)^{p+1}} + \Omega, \tag{6.1.20}$$

with Ω a constant element of $\mathfrak{g}$. As the multiplicity m_r appears as the order of the pole at λ_r in the Lax matrix, we will sometimes speak of a Gaudin model with simple, double, triple ... poles instead of a model with sites and multiplicities.

The important result is that this Lax matrix satisfies the same Poisson bracket (6.1.4) as the Lax matrix for simple poles, with the same matrix $r_{\underline{\mathbf{12}}}(\lambda,\mu)$. One can then construct the Gaudin model in a similar fashion as in the multiplicity one case. In particular, we can consider the quadratic Hamiltonian (6.1.5), which satisfies the involution equation (6.1.9). This Hamiltonian can then be written as a partial fraction decomposition similar to (6.1.6) but with higher order poles (which gives more conserved charges in involution). The Lax equation (6.1.11) still holds without corrections.

The most general Gaudin model can also have a site at infinity, with an arbitrary multiplicity. As this requires a somehow special treatment, we will not consider this case here and refer to [78] (note however that the constant element Ω in (6.1.20) corresponds to a double pole at infinity).

Cyclotomic Gaudin models. Let us now say a few words about cyclotomic Gaudin models. These are a generalisation of the complex Gaudin model presented in this section (which appeared first in [171] for the case with no multiplicities and in [115] for the case with arbitrary multiplicities), for Lie algebras $\mathfrak{g}$ which possess an automorphism σ of finite order T (see appendix A.4). We will use here a formalism close to the one we used in subsection 3.2.2 to described $\mathbb{Z}_T$-coset σ-models. The action of σ on $\mathfrak{g}$ defines an action of the cyclic group $\mathbb{Z}_T$. One can also define an action of $\mathbb{Z}_T$ on the complex plane *via* the multiplication by ω a primitive T^{th}-root of unity. Cyclotomic Gaudin models are models with a Lax matrix of a form similar to (6.1.20) but which satisfies the equivariance condition

$$\sigma\big(\mathscr{L}(\lambda)\big) = \omega \mathscr{L}(\omega\lambda). \tag{6.1.21}$$

Before explaining the construction *ex nihilo* of cyclotomic Gaudin models, let us first gain some intuitions about what these models should satisfy. Indeed, the equivariance condition (6.1.21) imposes several restrictions on the theory.

Let λ_r be a pole of $\mathscr{L}(\lambda)$ different from the origin (which is then not fixed under the multiplication by ω), then all the points of the orbit $\mathbb{Z}_T.\lambda_r = \{\omega^k\lambda_r, k=0,\cdots,T-1\}$ must be poles of $\mathscr{L}(\lambda)$, with same multiplicity m_r. Moreover, the coefficients over the poles at these points are related to the $X_{[p]}^{(r)}$'s by action of powers of σ.

In the same way, let us consider the point 0, which is fixed under the multiplication by ω. If it is a pole of $\mathscr{L}(\lambda)$, then one has

$$\sigma\left(X_{[p]}^{(0)}\right) = \omega^{-p} X_{[p]}^{(0)},$$

i.e. $X_{[p]}^{(0)}$ is in the grading $\mathfrak{g}^{(-p)}$ of $\mathfrak{g}$ (see $\mathbb{Z}_T$-gradings in Appendix A.4).

To construct a cyclotomic Gaudin model, one then considers sites $\lambda_0=0,\lambda_1,\cdots,\lambda_N$, with multiplicities $m_0,m_1,\cdots,m_N$, and associate with them some observables $X_{[p]}^{(r)}$ $(p=0,\cdots,m_r-1)$, which are $\mathfrak{g}$-valued for $r=1,\cdots,N$ and which are in gradings $\mathfrak{g}^{(-p)}$ for $r=0$. We suppose that the λ_r are not in the same $\mathbb{Z}_T$-orbits. We then define the Lax matrix to be (with $\Omega\in\mathfrak{g}^{(1)}$ constant) [115, 78]

$$\mathscr{L}(\lambda) = \frac{1}{T}\sum_{k=0}^{T-1}\sum_{r=1}^{N}\sum_{p=0}^{m_r-1} \frac{\omega^{kp}\sigma^k X_{[p]}^{(r)}}{(\lambda-\omega^k\lambda_r)^{p+1}} + \sum_{p=0}^{m_0-1}\frac{X_{[p]}^{(0)}}{\lambda^{p+1}} + \Omega. \tag{6.1.22}$$

One then checks that this Lax matrix satisfies the equivariance condition (6.1.21). Note that for all sites λ_r $(r=1,\cdots,N)$, this Lax matrix possesses poles at all $\omega^k\lambda_r$ in the orbit $\mathbb{Z}_T.\lambda_r$, as expected

above. However, the coefficients of these poles are not independent of the ones at λ_r. Thus, in the cyclotomic Gaudin model, all these poles have to be regarded as attached to one unique site.

So far, we specified a Lax matrix but not the phase space of the model. This phase space is parametrised by the observables $X^{(r)}_{[p]}$. For $r = 1, \cdots, N$, we will suppose that $X^{(r)}_{[p]}$ satisfies the Kirillov-Kostant bracket (6.1.19) of the Takiff algebra $\mathsf{T}^{m_r}\mathfrak{g}$. The observables $X^{(0)}_{[p]}$ satisfy more complicated brackets, which take into account the fact that these are restricted to particular gradings. We will not enter into details here and refer to [78]. These brackets on $X^{(r)}_{[p]}$ are made for the Lax matrix $\mathscr{L}(\lambda)$ to satisfy the bracket

$$\big\{\mathscr{L}_{\underline{\mathbf{1}}}(\lambda), \mathscr{L}_{\underline{\mathbf{2}}}(\mu)\big\} = \big[\hbar_{\underline{\mathbf{12}}}(\lambda,\mu), \mathscr{L}_{\underline{\mathbf{1}}}(\lambda)\big] - \big[\hbar_{\underline{\mathbf{21}}}(\mu,\lambda), \mathscr{L}_{\underline{\mathbf{2}}}(\mu)\big]\,, \tag{6.1.23}$$

with

$$\hbar_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\sum_{k=0}^{T-1} \frac{\sigma^k_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}}{\mu - \omega^{-k}\lambda}. \tag{6.1.24}$$

In particular, for a finite algebra $\mathfrak{g}$ with an automorphism σ, $\hbar$ coincides with the standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ twisted by σ, as introduced in (2.3.8). The construction of quadratic Hamiltonians in involution and the associated Lax equations developed in the non-cyclotomic case then generalise to the cyclotomic case.

Dihedral Gaudin models. Let us finally introduce dihedral Gaudin models [78]. These are the real equivalents of the cyclotomic Gaudin models introduced above. Let $\mathfrak{g}_0$ be a real Lie algebra with a $\mathbb{Z}_T$-grading $\mathfrak{g}_0 = \bigoplus_{p=0}^{T-1} \mathfrak{g}_0^{(p)}$. According to Corollary A.4.4, if $\mathfrak{g}_0$ is the real form $\mathfrak{g}^\tau$ of $\mathfrak{g}$, these gradings are in one-to-one correspondence with automorphisms σ of $\mathfrak{g}$ of order T, which satisfy the **dihedrality condition**

$$\sigma \circ \tau = \tau \circ \sigma^{-1}.$$

A dihedral Gaudin model is then a model whose Lax matrix $\mathscr{L}(\lambda)$ satisfies both the reality condition (6.1.15) and the equivariance condition (6.1.21):

$$\tau\big(\mathscr{L}(\lambda)\big) = \mathscr{L}(\overline{\lambda}) \qquad \text{and} \qquad \sigma\big(\mathscr{L}(\lambda)\big) = \omega\mathscr{L}(\omega\lambda). \tag{6.1.25}$$

Such a model is constructed by combining the constructions of real Gaudin models (in Subsection 6.1.3) and the one of cyclotomic Gaudin models above. Let Γ_T be the dihedral group of order $2T$, as defined in (2.4.12), and consider its action (2.4.13) on the complex plane *via* multiplication by ω and complex conjugation. We fix a certain number of sites $\lambda_0 = 0, \lambda_1, \cdots, \lambda_N$ in $\mathbb{C}$, whose orbits under Γ_T are disjoint, with multiplicity $m_0, m_1, \cdots, m_N$. We then attach to these sites certain objects $X^{(r)}_{[p]}$ ($p = 0, \cdots, m_r - 1$), whose properties depend on the orbit $\Gamma_T.\lambda_r$:

(i) if $\Gamma_T.\lambda_r$ is of size one (fixed point), *i.e.* $r = 0$, then $X^{(0)}_{[p]} \in \mathfrak{g}_0^{(-p)}$,

(ii) if $\Gamma_T.\lambda_r$ is of size T, *i.e.* (at least) one element of $\Gamma_T.\lambda_r$ is real and non-zero, then $X^{(r)}_{[p]} \in \mathfrak{g}_0$,

(iii) if $\Gamma_T.\lambda_r$ is of maximal size $2T$, *i.e.* all elements $\Gamma_T.\lambda_r$ are in $\mathbb{C} \setminus \mathbb{R}$, then $X^{(r)}_{[p]} \in \mathfrak{g}$.

The Lax matrix is then constructed from these objects to satisfy the conditions (6.1.25). In particular, it has poles at all points of the orbits $\Gamma_T.\lambda_r$, but the coefficients at these poles are related to the ones at λ_r by actions of τ and powers of σ. Hence, we see all these poles as a unique site of the model.

The Poisson brackets satisfied by the $X^{(r)}_{[p]}$ are made so that the Lax matrix satisfies the bracket (6.1.23) with $\hbar$ as in (6.1.24). In the cases (ii) and (iii) above, the $X^{(r)}_{[p]}$'s obey the Takiff Kirillov-Kostant brackets (6.1.19) of respectively $\mathfrak{g}_0$ and $\mathfrak{g}$. The Poisson brackets of the $X^{(0)}_{[p]}$'s are more complicated and we simply refer to [78] for their expression.

One could also consider a site at infinity. In this case, the associated $X^{(\infty)}_{[p]}$ would have properties similar to the ones attached to the origin, as ∞ is a fixed point under the action of Γ_T (we refer to [78] for the full construction).

Let us end this paragraph by an example. The unreduced Neumann model presented in Subsection 6.1.4 also possesses another Lax matrix, which is of size $N \times N$. This alternative formulation is a realisation of a Γ_2-dihedral Gaudin model on $\mathfrak{sl}(N, \mathbb{C})$, with automorphism $\sigma : X \mapsto -{}^tX$.

6.2 Classical AGM as field theories with twist function

In this section, we will focus on classical Affine Gaudin Models (AGM). In particular, using the description of affine Kac-Moody algebras in terms of loop algebras, we will explain how AGM can be seen as integrable field theories with twist function and with space coordinate x on the circle $\mathbb{S}^1$. This section is mostly a review of the article [78] of Vicedo.

6.2.1 Models with twist function and the Lie algebra of $\mathfrak{g}$-connections on the circle

Before entering into the formalism of affine algebras, let us try to get intuition on how a Gaudin model can be understood as a model with twist function. Recall that the Lax matrix of a Gaudin model satisfies a Poisson bracket of the form (6.1.23). The Lax matrix of a model with twist function, on the other hand, satisfies a Poisson bracket of the form (2.2.6). At first sight, these two brackets seem quite dissimilar. In this section, we explain how the bracket (2.2.6) can be put in a form similar to the one of Gaudin models, using the Lie algebra of $\mathfrak{g}$-connections on the circle.

The fundamental dynamical fields of a field theory on the circle are observables-valued distributions on $\mathbb{S}^1$. They can be seen as infinite Fourier series

$$\phi(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad x \in \mathbb{S}^1 \simeq [0, 2\pi], \tag{6.2.1}$$

where the coefficients c_n belong to $\mathcal{F}[M]$, *i.e.* are functions on the phase space M of the model (here, we consider complex fields: real ones are treated in a similar fashion requiring $\bar{c}_n = c_{-n}$). The kth-derivative of the field ϕ with respect to x is then

$$\partial_x^k \phi(x) = \sum_{n\in\mathbb{Z}} (in)^k c_n e^{inx}.$$

We will denote by $\mathcal{T}(\mathbb{S}^1)$ the space of trigonometric polynomials on $\mathbb{S}^1$, *i.e.* the vector space with basis

$$\begin{array}{rccc} e_n : & \mathbb{S}^1 & \longrightarrow & \mathbb{C} \\ & x & \longmapsto & e^{inx} \end{array}.$$

A field $\phi(x)$ of the form (6.2.1) is then in the tensor product $\mathcal{F}[M] \otimes \mathcal{T}(\mathbb{S}^1)$ (more precisely, in a completion of this tensor product, as the sum in (6.2.1) is infinite).

Let $\mathfrak{g}$ be the finite Lie algebra underlying the considered model with twist function. We define the space of $\mathfrak{g}$-connections on $\mathbb{S}^1$ as

$$\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1) = \mathbb{C}\partial \oplus \mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1).$$

Elements of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ are represented by their evaluation at $x \in \mathbb{S}^1$, which sends ∂ to ∂_x and e_n to $e_n(x) = e^{inx}$. An element $\nabla = \hbar\partial + \mathcal{J}$ in $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$, with $\hbar \in \mathbb{C}$ and $\mathcal{J} \in \mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1)$ a $\mathfrak{g}$-valued trigonometric polynomial, can then be seen as the $\mathfrak{g}$-connection

$$\nabla(x) = \hbar\partial_x + \mathcal{J}(x).$$

One defines a bracket $[\cdot, \cdot]$ of two $\mathfrak{g}$-valued trigonometric polynomials by considering the point-wise bracket on $\mathfrak{g}$. We then extend it to $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ as

$$\big[\hbar_1\partial + \mathcal{J}_1, \hbar_2\partial + \mathcal{J}_2\big](x) = [\mathcal{J}_1(x), \mathcal{J}_2(x)] + \hbar_1\partial_x\mathcal{J}_2(x) - \hbar_2(x)\partial_x\mathcal{J}_1(x).$$

One checks that $[\cdot,\cdot]$ is a Lie bracket on $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$.

Let us come back to the considered field theory with twist function. From its twist function $\varphi(\lambda)$ and its Lax matrix $\mathcal{L}(\lambda,x)$, we define the following $\mathfrak{g}$-connection on $\mathbb{S}^1$, depending on the spectral parameter λ:

$$\nabla(\lambda,x) = \varphi(\lambda)\big(\partial_x + \mathcal{L}(\lambda,x)\big) \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1). \tag{6.2.2}$$

As $\mathcal{L}(\lambda,\cdot)$ is a $\mathfrak{g}$-valued field, its coefficients in a basis of $\mathfrak{g}$ are thus elements of (the completion of) the tensor factor $\mathcal{F}[M]\otimes\mathcal{T}(\mathbb{S}^1)$: thus, $\nabla(\lambda,\cdot)$ is an element of (the completion of) the tensor product $\mathcal{F}[M]\otimes\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. The main result of this subsection is the fact that the Poisson bracket (2.2.6) of $\mathcal{L}$, with the $\mathcal{R}$-matrix (2.4.1), can be rewritten as

$$\big\{\nabla_{\underline{\mathbf{1}}}(\lambda,x),\nabla_{\underline{\mathbf{2}}}(\mu,y)\big\} = \big[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\delta_{xy},\nabla_{\underline{\mathbf{1}}}(\lambda,x)\big] - \big[\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda)\delta_{yx},\nabla_{\underline{\mathbf{2}}}(\mu,y)\big]\,. \tag{6.2.3}$$

This bracket is valued in $\mathcal{F}[M]\otimes\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)\otimes\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ (or more technically in its completion with respect to an appropriate topology). An element of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ is a $\mathfrak{g}$-connection on $\mathbb{S}^1$ and can thus be evaluated at any point of $\mathbb{S}^1$. In this bracket, the points x and y are thus to be understood as attached to the first and second tensor factors of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)\otimes\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. Note that the dependence of the bracket in the twist function is now only contained in the connection ∇.

Before interpreting this bracket, let us briefly prove the equation (6.2.3). From the Poisson bracket (2.2.6) of $\mathcal{L}(\lambda,x)$ and the expression (2.4.1) of $\mathcal{R}$, one gets

$$\begin{aligned}
\big\{\nabla_{\underline{\mathbf{1}}}(\lambda,x),\nabla_{\underline{\mathbf{2}}}(\mu,y)\big\} &= \varphi(\lambda)\varphi(\mu)\left\{\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x),\mathcal{L}_{\underline{\mathbf{2}}}(\mu,y)\right\}\\
&= \left[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu),\varphi(\lambda)\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x)\right]\delta_{xy} - \left[\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda),\varphi(\mu)\mathcal{L}_{\underline{\mathbf{2}}}(\mu,x)\right]\delta_{xy}\\
&\qquad - \big(\varphi(\lambda)\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) + \varphi(\mu)\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda)\big)\,\delta'_{xy}\\
&= \left[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\delta_{xy},\varphi(\lambda)\mathcal{L}_{\underline{\mathbf{1}}}(\lambda,x)\right] - \left[\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda)\delta_{yx},\varphi(\mu)\mathcal{L}_{\underline{\mathbf{2}}}(\mu,x)\right]\\
&\qquad - \left[\varphi(\lambda)\partial_x,\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\delta_{xy}\right] + \left[\varphi(\mu)\partial_y,\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda)\delta_{yx}\right]\\
&= \left[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\delta_{xy},\nabla_{\underline{\mathbf{1}}}(\lambda,x)\right] - \left[\mathcal{R}^0_{\underline{\mathbf{21}}}(\mu,\lambda)\delta_{yx},\nabla_{\underline{\mathbf{2}}}(\mu,y)\right]\,.
\end{aligned}$$

The bracket (6.2.3) now has a form similar to the bracket (6.1.23) of a Gaudin model. One could then hope that models with twist function are related to Gaudin models on the Lie algebra $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. However, this turns out to be impossible. Indeed, there is no invariant non-degenerate bilinear form on the Lie algebra $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. As we will see in the next subsection, we will overcome this difficulty by considering affine Kac-Moody algebras.

6.2.2 Affine Kac-Moody algebras from loop algebras

Loop algebra. In this section we present the construction of affine Kac-Moody algebras from loop algebras. We will not use here the general abstract theory of Kac-Moody algebras, defined from generalised Cartan matrices. We refer the reader to [162] for a general treatment of Kac-Moody algebras and the link between the two descriptions of affine ones.

Let us consider a finite dimensional complex semi-simple Lie algebra $\mathfrak{g}$, with Killing form κ. The loop algebra of $\mathfrak{g}$ is the algebra

$$\mathfrak{g}[t,t^{-1}] = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$$

of $\mathfrak{g}$-valued Laurent polynomials in a formal variable t (note that this definition of the loop algebra is slightly different from the one used in Appendix C.4, where we considered Laurent series and not Laurent polynomials). The space $\mathfrak{g}[t,t^{-1}]$ is a Lie algebra when equipped with the point-wise bracket

$$[X\otimes t^p, Y\otimes t^q] = [X,Y]\otimes t^{p+q},\quad \forall\, X,Y\in\mathfrak{g},\ \forall\, p,q\in\mathbb{Z}.$$

We define the (untwisted) affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ as a central and cocentral extension of $\mathfrak{g}[t,t^{-1}]$:

$$\widetilde{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}\mathsf{D}\oplus\mathbb{C}\mathsf{K},$$

with the Lie bracket

$$\begin{aligned}\big[X\otimes t^p + a\mathsf{D} + b\mathsf{K}, Y\otimes t^q + c\mathsf{D} + d\mathsf{K}\big] = \\ [X,Y]\otimes t^{p+q} + aqY\otimes t^q - cpX\otimes t^p + p\,\kappa(X,Y)\delta_{p+q,0}\mathsf{K}.\end{aligned} \tag{6.2.4}$$

We define a bilinear form $(\cdot,\cdot)$ on $\widetilde{\mathfrak{g}}$ by letting

$$\big(X\otimes t^p + a\mathsf{D} + b\mathsf{K}, Y\otimes t^q + c\mathsf{D} + d\mathsf{K}\big) = \delta_{p+q,0}\kappa(X,Y) + ad + bc. \tag{6.2.5}$$

One checks that $(\cdot,\cdot)$ defines an invariant and non-degenerate bilinear form on $\widetilde{\mathfrak{g}}$.

As the affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ is quadratic, one can consider the split quadratic Casimir $\widetilde{C}_{\underline{\mathbf{12}}}$ of $\widetilde{\mathfrak{g}}$, as defined in equation (6.1.1). If $\{I^a\}$ is a basis of $\mathfrak{g}$ and $\{I_a\}$ is the associated dual basis with respect to κ, then $\{I^a\otimes t^n, \mathsf{K}, \mathsf{D}\}$ is a basis of $\widetilde{\mathfrak{g}}$, whose dual basis with respect to $(\cdot,\cdot)$ is $\{I_a\otimes t^{-n}, \mathsf{D}, \mathsf{K}\}$. The split quadratric Casimir of $\widetilde{\mathfrak{g}}$ is then

$$\widetilde{C}_{\underline{\mathbf{12}}} = \mathsf{K}\otimes\mathsf{D} + \mathsf{D}\otimes\mathsf{K} + \sum_{n\in\mathbb{Z}}\big(I^a\otimes t^n\big)\otimes\big(I_a\otimes t^{-n}\big). \tag{6.2.6}$$

One issue regarding this construction is the fact that $\widetilde{\mathfrak{g}}$ is infinite dimensional, which makes the sum in equation (6.2.6) infinite and thus ill-defined in $\widetilde{\mathfrak{g}}\otimes\widetilde{\mathfrak{g}}$. To make sense of this sum, one needs to consider a completion $\widetilde{\mathfrak{g}}\hat{\otimes}\widetilde{\mathfrak{g}}$ of this tensor product. For completeness, we will say a few words about completions in the following paragraph (note that this paragraph is not necessary for the comprehension of the main ideas introduced in the rest of this section).

Infinite sums and completion. Issues of convergence and completion appear frequently when dealing with AGM. In this thesis, we will not treat these in details, in order to focus on the general ideas behind AGM. We refer to [78] for a thorough rigorous treatment. However, we will illustrate the kind of completions used when manipulating affine algebras and related vector spaces by a simple example, a completion of $\widetilde{\mathfrak{g}}$ itself. Consider the subspaces

$$\mathsf{F}_n\widetilde{\mathfrak{g}} = \mathfrak{g}\otimes t^n\mathbb{C}[t], \quad n\in\mathbb{Z}_{\geq 0},$$

of $\widetilde{\mathfrak{g}}$. They form a descending $\mathbb{Z}_{\geq 0}$-filtration of $\widetilde{\mathfrak{g}}$:

$$\widetilde{\mathfrak{g}} \supset \mathsf{F}_0\widetilde{\mathfrak{g}} \supset \mathsf{F}_1\widetilde{\mathfrak{g}} \supset \cdots \supset \mathsf{F}_n\widetilde{\mathfrak{g}} \supset \cdots .$$

Let us consider a sequence $U = (u_k)_{k\in\mathbb{Z}_{\geq 0}}$ in $\widetilde{\mathfrak{g}}$. We say that U is a Cauchy sequence with respect to this filtration if

$$\forall\, n\in\mathbb{Z}_{\geq 0}, \quad \exists\, k\in\mathbb{Z}_{\geq 0} \quad \text{such that} \quad u_p - u_q \in \mathsf{F}_n\widetilde{\mathfrak{g}} \quad \text{if} \quad p,q\geq k.$$

There exists a general construction of completions of vector spaces with descending $\mathbb{Z}_{\geq 0}$-filtration such that all Cauchy sequences with respect to this filtration converge in the completion (see more details in [78]). In particular, this construction allows to make sense of infinite series

$$\sum_{n\in\mathbb{Z}_{\geq 0}} v_n, \qquad \text{with} \quad v_n\in\mathsf{F}_n\widetilde{\mathfrak{g}}.$$

Note that the $\mathbb{Z}_{\geq 0}$-filtration is a Lie algebra filtration, *i.e.* satisfies

$$[\mathsf{F}_n\widetilde{\mathfrak{g}}, \mathsf{F}_m\widetilde{\mathfrak{g}}] \subset \mathsf{F}_{n+m}\widetilde{\mathfrak{g}}.$$

This ensures that the Lie bracket $\widetilde{\mathfrak{g}}$ extends to the associated completion, making it a Lie algebra. In the present case, the completion is simply

$$\mathfrak{g}(\!(t)\!)\oplus\mathbb{C}\mathsf{D}\oplus\mathbb{C}\mathsf{K},$$

constructed as $\widetilde{\mathfrak{g}}$ but where we consider $\mathfrak{g}$-valued Laurent series in t instead of Laurent polynomials.

The rigorous construction of AGM requires to consider similar completions of various vector spaces, with respect to some filtrations constructed from the one above and the "conjugate" one (with powers of t^{-1} instead of t). In particular, this allows to make sense of the split quadratic Casimir (6.2.6).

Affine algebras and connections on the circle. Recall the trigonometric functions on the circle, $e_n : x \in \mathbb{S}^1 \mapsto e^{inx}$ $(n \in \mathbb{Z})$, introduced in the previous subsection. We define a map

$$\rho : \widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\mathsf{D} \oplus \mathbb{C}\mathsf{K} \longrightarrow \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1) = \mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1) \oplus \mathbb{C}\partial$$

by

$$\rho(X \otimes t^n) = X \otimes e_n, \quad \rho(\mathsf{D}) = -i\partial \quad \rho(\mathsf{K}) = 0,$$

for $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. One checks that ρ is a Lie algebra morphism from the affine algebra $\widetilde{\mathfrak{g}}$ to the Lie algebra $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ of $\mathfrak{g}$-connections on the circle.

Let us consider the quadratic split Casimir (6.2.6) of $\widetilde{\mathfrak{g}}$. It is clear that its image under the Lie algebra morphism ρ is given by

$$(\rho \otimes \rho)\widetilde{C}_{\underline{\mathbf{12}}} = \sum_{n \in \mathbb{Z}} \big(I^a \otimes e_n\big) \otimes \big(I_a \otimes e_{-n}\big)$$

and belongs to (a completion of) $\big(\mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1)\big) \otimes \big(\mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1)\big) \subset \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1) \otimes \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. Recall that in Subsection 6.2.1, we considered elements of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ by evaluating them at points of $\mathbb{S}^1$. Let us evaluate the left factor of the tensor product $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1) \otimes \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ at $x \in \mathbb{S}^1$ and the second at $y \in \mathbb{S}^1$ (as we did to consider the Poisson bracket (6.2.3) in Subsection 6.2.1). We then get

$$(\rho \otimes \rho)\widetilde{C}_{\underline{\mathbf{12}}}(x, y) = C_{\underline{\mathbf{12}}}\mathcal{D}(x, y),$$

where $C_{\underline{\mathbf{12}}} = I^a \otimes I_a \in \mathfrak{g} \otimes \mathfrak{g}$ is the split Casimir of the finite algebra $\mathfrak{g}$ and

$$\mathcal{D}(x, y) = \sum_{n \in \mathbb{Z}} e_n(x) e_{-n}(y) = \sum_{n \in \mathbb{Z}} e^{in(x-y)}.$$

The latter cannot be understood as a function of $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ because of the infinite sum on n. In fact, such infinite Fourier series where we do not require additional convergence conditions can be seen as distributions on $\mathcal{T}(\mathbb{S}^1)$. Let

$$f = \sum_{c_n \in \mathbb{Z}} c_n e_n,$$

be an element of $\mathcal{T}(\mathbb{S}^1)$ (we then have a finite number of non-vanishing c_n's). The action of the distribution $\mathcal{D}(x, \cdot)$ on the trigonometric polynomial f is given by

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}y\, \mathcal{D}(x, y) f(y) = \sum_{n,m \in \mathbb{Z}} c_m e^{inx} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}y\, e^{i(m-n)y}}_{\delta_{nm}} = \sum_{n \in \mathbb{Z}} c_n e^{inx} = f(x).$$

Thus the distribution $\mathcal{D}(x, y)$ is in fact the Dirac δ-distribution δ_{xy} (note that, as we are on the circle, we consider integrals with a measure normalised by 2π). We then have

$$(\rho \otimes \rho)\widetilde{C}_{\underline{\mathbf{12}}}(x, y) = C_{\underline{\mathbf{12}}}\delta_{xy}. \tag{6.2.7}$$

Note that the morphism ρ is surjective but not injective, as it sends K to zero. Thus, $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ is not isomorphic to $\widetilde{\mathfrak{g}}$. In fact, it is isomorphic to its quotient $\widetilde{\mathfrak{g}}/\mathbb{C}\mathsf{K}$ by the central ideal $\mathbb{C}\mathsf{K}$. This prevents us to translate all properties of $\widetilde{\mathfrak{g}}$ on $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. In particular, one cannot "push" the invariant non-degenerate bilinear form $(\cdot, \cdot)$ to an invariant non-degenerate form on $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. We already stated the non-existence of such a form on $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ at the end of subsection 6.2.1. In particular, this prevented us to define a Gaudin model on $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. However, one can define a Gaudin model on $\widetilde{\mathfrak{g}}$, whose Lax matrix would then be $\widetilde{\mathfrak{g}}$-valued, and apply the morphism ρ to it to get a Lax matrix valued in $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$. We will apply this idea in the next subsection. First, let us remark that although one cannot "push" the bilinear form $(\cdot, \cdot)$ from $\widetilde{\mathfrak{g}}$ to $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$, one can do it when restricting to $\mathfrak{g}[t, t^{-1}] \subset \widetilde{\mathfrak{g}}$:

$$\forall\, J_1, J_2 \in \mathfrak{g}[t, t^{-1}] \subset \widetilde{\mathfrak{g}}, \quad (J_1, J_2) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \kappa\big(\mathcal{J}_1(x), \mathcal{J}_2(x)\big), \quad \text{with} \quad \mathcal{J}_i = \rho(J_i). \tag{6.2.8}$$

6.2.3 Classical Affine Gaudin models without dihedrality and multiple poles

Lax matrix and change of auxiliary space. For simplicity, we will start with the simplest AGM on $\widetilde{\mathfrak{g}}$: complex, without cyclotomy and with sites of multiplicity one. Its Lax matrix $\mathscr{L}(\lambda)$ then satisfies the Poisson bracket

$$\{\mathscr{L}_{\underline{\mathbf{1}}}(\lambda), \mathscr{L}_{\underline{\mathbf{2}}}(\mu)\} = \left[\frac{\widetilde{C}_{\underline{\mathbf{12}}}}{\mu-\lambda}, \mathscr{L}_{\underline{\mathbf{1}}}(\lambda) + \mathscr{L}_{\underline{\mathbf{2}}}(\mu)\right], \tag{6.2.9}$$

according to equation (6.1.4).

The Lax matrix $\mathscr{L}(\lambda)$ is valued in (a completion of) $\widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$, where M is the phase space of the Gaudin model. We call the space $\mathcal{F}[M]$ the algebra of observables of the model and $\widetilde{\mathfrak{g}}$ the auxiliary space. Let us modify this auxiliary space by applying the morphism $\rho : \widetilde{\mathfrak{g}} \to \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$:

$$\widetilde{\nabla}(\lambda) = (\rho \otimes \mathrm{Id})\mathscr{L}(\lambda) \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1) \otimes \mathcal{F}[M],$$

where ρ and Id acts respectively on the left and right factor of the tensor product $\widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$. The bracket (6.2.9) is valued in $\widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$. We get the Poisson bracket of $\widetilde{\nabla}_{\underline{\mathbf{1}}}(\lambda)$ with $\widetilde{\nabla}_{\underline{\mathbf{2}}}(\mu)$ by applying $\rho \otimes \rho \otimes \mathrm{Id}$ on equation (6.2.9). As ρ is a Lie algebra morphism, we get

$$\left\{\widetilde{\nabla}_{\underline{\mathbf{1}}}(\lambda), \widetilde{\nabla}_{\underline{\mathbf{2}}}(\mu)\right\} = \left[\frac{(\rho \otimes \rho)\widetilde{C}_{\underline{\mathbf{12}}}}{\mu-\lambda}, \widetilde{\nabla}_{\underline{\mathbf{1}}}(\lambda) + \widetilde{\nabla}_{\underline{\mathbf{2}}}(\mu)\right].$$

Recall that to write the Poisson bracket (6.2.3) of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$-valued objects we evaluated the connections at points x and y of $\mathbb{S}^1$. Using equation (6.2.7), one then gets

$$\left\{\widetilde{\nabla}_{\underline{\mathbf{1}}}(\lambda, x), \widetilde{\nabla}_{\underline{\mathbf{2}}}(\mu, y)\right\} = \left[\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}\delta_{xy}, \widetilde{\nabla}_{\underline{\mathbf{1}}}(\lambda, x) + \widetilde{\nabla}_{\underline{\mathbf{2}}}(\mu, x)\right]. \tag{6.2.10}$$

This has exactly the form of the Lax bracket (6.2.3) (for $\mathcal{R}^0$ the non-twisted standard $\mathcal{R}$-matrix $C_{\underline{\mathbf{12}}}/(\mu-\lambda)$). The Lax matrix of the AGM, once the auxiliary space changed, then has the same bracket as the Lax connection (6.2.2) of a model with twist function. However, to complete the interpretation of the AGM as a field theory, one also has to understand its phase space and in particular the components of $\widetilde{\nabla}(\lambda)$.

Phase space and dynamical fields. As we consider a non-cyclotomic complex AGM with only simple poles, its phase space M is the product of N copies of the dual space $\widetilde{\mathfrak{g}}^*$, equipped with the Kirillov-Kostant bracket. This phase space is then encoded in N quantities $X^{(r)}$ in (a completion of) $\widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$ satisfying the bracket

$$\left\{X^{(r)}_{\underline{\mathbf{1}}}, X^{(s)}_{\underline{\mathbf{2}}}\right\} = \delta_{rs}\left[\widetilde{C}_{\underline{\mathbf{12}}}, X^{(r)}_{\underline{\mathbf{1}}}\right]. \tag{6.2.11}$$

We will write $X^{(r)}$ as

$$X^{(r)} = D^{(r)}\mathsf{K} + iK^{(r)}\mathsf{D} + J^{(r)}, \tag{6.2.12}$$

with $D^{(r)}, K^{(r)} \in \mathcal{F}[M]$ and $J^{(r)} \in \mathfrak{g}[t, t^{-1}] \otimes \mathcal{F}[M]$. We will focus first on $K^{(r)}$ and $J^{(r)}$ by considering the image of $X^{(r)}$ through the morphism ρ:

$$\big(\rho \otimes \mathrm{Id}\big)X^{(r)}(x) = K^{(r)}\partial_x + \mathcal{J}^{(r)}(x), \tag{6.2.13}$$

where $\mathcal{J}^{(r)} = \rho\big(J^{(r)}\big)$ belongs to (a completion of) $\mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1) \otimes \mathcal{F}[M]$. In particular, the components of $\mathcal{J}^{(r)}$ in a basis $\{I^a\}$ of $\mathfrak{g}$ belong to the completion of $\mathcal{T}(\mathbb{S}^1) \otimes \mathcal{F}[M]$:

$$\mathcal{J}^{(r)}_a(x) = \sum_{n \in \mathbb{Z}} c^{(r)}_{a,n} e^{inx},$$

with $c^{(r)}_{a,n} \in \mathcal{F}[M]$. Thus, $\mathcal{J}^{(r)}_a$ is an observable-valued distribution on $\mathbb{S}^1$. As explained at the beginning of subsection 6.2.1, such an object is the dynamical field of an Hamiltonian theory on $\mathbb{S}^1$.

Levels and Kac-Moody currents. Now that we interpreted the observable $\mathcal{J}^{(r)} = \rho\big(J^{(r)}\big)$ as dynamical fields, let us consider the other observables. In particular, the observable $K^{(r)}$ appears as the coefficient of $i\mathsf{D}$ in $X^{(r)}$. However, one sees in equation (6.2.4) that the element D can never be created by a commutator in $\widetilde{\mathfrak{g}}$. Thus, it cannot appear on the left factor of the bracket (6.2.11). Therefore, one gets

$$\Big\{K^{(r)}, X^{(s)}\Big\} = 0,$$

for all $r, s = 1, \cdots, N$. This means that the observable $K^{(r)}$ is a Poisson Casimir of $\mathcal{F}[M]$ (it Poisson commutes with all functions in $\mathcal{F}[M]$). The elements $K^{(r)}$ are the classical equivalent for the Kostant-Kirillov bracket of the central element K of $\widetilde{\mathfrak{g}}$.

In a Hamiltonian field theory, there should not be Poisson Casimirs which are not constant functions. In order to interpret the AGM as a field theory, we thus fix the quantities $K^{(r)}$ to constants. For that, we will consider a Poisson map $\pi_{\boldsymbol{k}}$ from $\mathcal{F}[M]$ to a new algebra of observables Obs, such that

$$\pi_{\boldsymbol{k}}\left(K^{(r)}\right) = k_r, \qquad \text{with} \qquad \boldsymbol{k} = (k_1, \cdots, k_N) \in \mathbb{C}^N. \tag{6.2.14}$$

We will explain more precisely how we construct this map later. The numbers in $\boldsymbol{k}$ are called the **levels** of the AGM. As the quantities $\mathcal{J}^{(r)}$ have already been interpreted as dynamical fields on $\mathbb{S}^1$, we shall consider that $\pi_{\boldsymbol{k}}$ does not affect these quantities (again, see the precise statement after), so we still write $\mathcal{J}^{(r)}$ their image under $\pi_{\boldsymbol{k}}$. We then have

$$(\rho \otimes \pi_{\boldsymbol{k}}) X^{(r)}(x) = k_r \partial_x + \mathcal{J}^{(r)}(x).$$

The bracket (6.2.11) is valued in $\widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$. As $\pi_{\boldsymbol{k}}$ is a Poisson map, we get the Poisson brackets of $(\rho \otimes \pi_{\boldsymbol{k}}) X^{(r)}_{\underline{\mathbf{1}}}(x)$ and $(\rho \otimes \pi_{\boldsymbol{k}}) X^{(s)}_{\underline{\mathbf{2}}}(y)$ by applying the map $\rho \otimes \rho \otimes \pi_{\boldsymbol{k}}$ to (6.2.11) (and evaluate at x and y in $\mathbb{S}^1$). Doing this, and using the expression (6.2.7) of $(\rho \otimes \rho)\widetilde{C}_{\underline{\mathbf{12}}}(x, y)$, we obtain

$$\Big\{\mathcal{J}^{(r)}_{\underline{\mathbf{1}}}(x), \mathcal{J}^{(r)}_{\underline{\mathbf{2}}}(y)\Big\} = \Big[C_{\underline{\mathbf{12}}}, \mathcal{J}^{(r)}_{\underline{\mathbf{1}}}(x)\Big]\, \delta_{xy} - k_r C_{\underline{\mathbf{12}}} \delta'_{xy} \tag{6.2.15}$$

and

$$\Big\{\mathcal{J}^{(r)}_{\underline{\mathbf{1}}}(x), \mathcal{J}^{(s)}_{\underline{\mathbf{2}}}(y)\Big\} = 0 \qquad \text{if} \quad k \neq s.$$

The bracket (6.2.15) is then the one of a so-called **Kac-Moody current** of level k_r. The algebra of observables Obs of the model then contains N commuting Kac-Moody currents.

Momentum of the theory. So far, we understood the interpretation of the components $\mathcal{J}^{(r)}$ and $K^{(r)}$ of $X^{(r)}$. However, there is still a set of observables that we did not study. Indeed, recall that the morphism ρ is not injective, as it sends K to zero. Thus there is still to interpret the coefficient $D^{(r)}$ of K in (6.2.12).

Let us compute its Poisson bracket with the Kac-Moody current $\mathcal{J}^{(r)}$. For that, we project the bracket (6.2.11) on K on the second tensor factor. Given the expression (6.2.6) of $\widetilde{C}_{\underline{\mathbf{12}}}$, one gets

$$\Big\{X^{(r)}, D^{(r)}\Big\} = \Big[\mathsf{D}, X^{(r)}\Big].$$

Applying the morphism ρ to this bracket, we get

$$\Big\{\mathcal{J}^{(r)}(x), D^{(r)}\Big\} = \Big[-i\partial_x, K^{(r)}\partial_x + \mathcal{J}^{(r)}(x)\Big],$$

hence

$$-i\big\{D^{(r)}, \mathcal{J}^{(r)}(x)\big\} = \partial_x \mathcal{J}^{(r)}(x).$$

Thus, $-iD^{(r)}$ generates the space derivative on the current $\mathcal{J}^{(r)}$. Its Hamiltonian flow then coincides with the one of the momentum of the theory. More precisely, it is clear from (6.2.11) that

$$\big\{D^{(r)}, \mathcal{J}^{(s)}(x)\big\} = 0 \qquad \text{if} \quad r \neq s,$$

hence the Hamiltonian flow of $-iD^{(r)}$ coincides with the momentum of the Kac-Moody current $\mathcal{J}^{(r)}_{[p]}$ associated with the site λ_r.

Here, $D^{(r)}$ is an abstract generator in $\mathcal{F}[M]$. Yet, in general in a Hamiltonian field theory, the momentum of the model is not an independent quantity: it is expressed in terms of the dynamical fields so that it generates the space derivative. For the algebra Obs to describe the observables of a field theory, one should then realise the abstract generator $-iD^{(r)}$ as the momentum $\mathcal{P}^{(r)}$ of the Kac-Moody current $\mathcal{J}^{(r)}$. We thus choose the map $\pi_{\boldsymbol{k}}$ to satisfy

$$\pi_{\boldsymbol{k}}\left(D^{(r)}\right) = i\mathcal{P}^{(r)}. \tag{6.2.16}$$

As for the treatment of the observables $K^{(r)}$'s, the construction of the map $\pi_{\boldsymbol{k}}$ satisfying (6.2.16) will be made more precise in the next paragraph.

Above, we introduced the observable $\mathcal{P}^{(r)}$ to be the momentum generating the space derivative on the Kac-Moody current $\mathcal{J}^{(r)}$. Let us be more concrete. Indeed, one can find an explicit expression for $\mathcal{P}^{(r)}$. From the Poisson bracket (6.2.15), one finds that

$$\mathcal{P}^{(r)} = \frac{1}{4\pi k_r}\int_0^{2\pi} \mathrm{d}x\ \kappa\left(\mathcal{J}^{(r)}(x), \mathcal{J}^{(r)}(x)\right) \tag{6.2.17}$$

generates the space derivative on $\mathcal{J}^{(r)}$. Note that in order to define $\mathcal{P}^{(r)}$, we supposed that the level k_r is non-zero. This expression for $\mathcal{P}^{(r)}$ is the so-called classical Segal-Sugawara construction (see for instance [172]). Its algebraic origin will be explained in the next paragraph, together with the precise construction of $\pi_{\boldsymbol{k}}$.

The total momentum $\mathcal{P} \in \mathrm{Obs}$ of the resulting field theory is then simply given by

$$\mathcal{P} = \sum_{r=1}^{N} \mathcal{P}^{(r)}. \tag{6.2.18}$$

Local AGM and the Segal-Sugawara map $\pi_{\boldsymbol{k}}$. In this section, we will explain how to construct rigorously the map $\pi_{\boldsymbol{k}}$ such that it satisfies (6.2.14) and (6.2.16). In particular, this map sends the element $K^{(r)} - k_r$ of $\mathcal{F}[M]$ (where we see k_r as a constant function on M) to zero. Such a map is easily constructed by taking the quotient by the ideal $(K^{(r)} - k_r)\mathcal{F}[M]$ generated by $K^{(r)} - k_r$. By construction, the canonical map from $\mathcal{F}[M]$ to $\mathcal{F}[M]/(K^{(r)} - k_r)\mathcal{F}[M]$ is thus a morphism of algebra. However, we also want this map to induce a Poisson bracket on the quotient. For that, we need the algebra ideal $(K^{(r)} - k_r)\mathcal{F}[M]$ to also be a Poisson ideal, *i.e.* to be stabilised under the Poisson bracket with any element of $\mathcal{F}[M]$. This is simply ensured by the fact that $K^{(r)} - k_r$ is a Poisson Casimir and thus has vanishing Poisson bracket with all elements of $\mathcal{F}[M]$.

Recall from Subsection 6.1.2 that $\mathcal{F}[M]$ contains other Poisson Casimirs than the $K^{(r)}$'s, namely the quantities Δ_r's defined as

$$\Delta_r = \frac{1}{2}\left(X^{(r)}, X^{(r)}\right).$$

Given the expression (6.2.12) of $X^{(r)}$, the definition (6.2.5) of $(\cdot,\cdot)$ and equation (6.2.8), we get

$$\Delta_r = iK^{(r)}D^{(r)} + \frac{1}{2}\left(J^{(r)}, J^{(r)}\right) = iK^{(r)}D^{(r)} + \frac{1}{4\pi}\int_0^{2\pi} \kappa\left(\mathcal{J}^{(r)}(x), \mathcal{J}^{(r)}(x)\right) = iK^{(r)}D^{(r)} + k_r\mathcal{P}^{(r)}.$$

One then sees that a map $\pi_{\boldsymbol{k}}$ which would verify (6.2.14) and (6.2.16) should send the Casimir Δ_r to zero. As Δ_r is a Poisson Casimir, the algebra ideal $\Delta_r\mathcal{F}[M]$ is also a Poisson ideal of $\mathcal{F}[M]$. Thus the canonical surjection from $\mathcal{F}[M]$ to $\mathcal{F}[M]/\Delta_r\mathcal{F}[M]$ is an algebra morphism preserving the Poisson bracket.

The map $\pi_{\boldsymbol{k}}$ is obtained by considering the quotient of $\mathcal{F}[M]$ by all Poisson ideals $(K^{(r)} - k_r)\mathcal{F}[M]$'s and $\Delta_r\mathcal{F}[M]$'s. More precisely, let us define

$$\mathcal{I}_{\boldsymbol{k}} = \bigoplus_{r=1}^{N}\Big((K^{(r)} - k_r)\mathcal{F}[M] \oplus \Delta_r\mathcal{F}[M]\Big).$$

It is a Poisson ideal of $\mathcal{F}[M]$ as all $(K^{(r)} - k_r)$'s and Δ_r's are Poisson Casimirs. We then define the algebra of observables of the field theory as the quotient

$$\mathrm{Obs} = \mathcal{F}[M]/\mathcal{I}_{\boldsymbol{k}}$$

and the map $\pi_{\boldsymbol{k}}$ as the canonical surjection

$$\pi_{\boldsymbol{k}} : \mathcal{F}[M] \longrightarrow \mathrm{Obs}.$$

This map is called the (classical) Segal-Sugawara map.

We shall call the resulting Hamiltonian theory with observables Obs the local AGM. We will often speak of the formal AGM for the model before quotienting. Recall that the $X^{(r)}$'s generate the algebra $\mathcal{F}[M]$. Thus, as $\pi_{\boldsymbol{k}}$ is surjective, Obs is generated by the images $\pi_{\boldsymbol{k}}(X^{(r)})$'s. Recall the expression (6.2.12) of $X^{(r)}$. As $\pi_{\boldsymbol{k}}$ sends $K^{(r)}$ to a constant k_r and expresses $D^{(r)}$ in terms of $\mathcal{J}^{(r)} = (\rho \otimes \pi_k)J^{(r)}$, we find that Obs is generated by the $\mathcal{J}^{(r)}$. Moreover, as $\pi_{\boldsymbol{k}}$ does not impose any additional relations, these generators $\mathcal{J}^{(r)}$ are independent in Obs (this is what we meant after equation (6.2.14) when saying that the map $\pi_{\boldsymbol{k}}$ does not affect these quantities).

As explained in this subsection, the quantities $\mathcal{J}^{(r)}$ are elements of the completion of $\mathfrak{g} \otimes \mathcal{T}(\mathbb{S}^1) \otimes \mathcal{F}[M]$, *i.e.* observables-valued distributions on $\mathbb{S}^1$, living in $\mathfrak{g}$. Thus, they are $\mathfrak{g}$-valued dynamical fields of an Hamiltonian field theory. As they are independent generators of Obs, they form the fundamental fields of the theory. This shows that the local AGM is an Hamiltonian field theory. Note that to define this field theory, we required that the levels k_r's are different from 0.

Lax matrix and twist function. Let us now turn our attention to the Lax matrix of the model. According to equation (6.1.3), the Lax matrix of the formal AGM is given by

$$\mathscr{L}(\lambda) = \sum_{i=1}^{N} \frac{X^{(r)}}{\lambda - \lambda_r} + \Omega,$$

where Ω is a constant in $\widetilde{\mathfrak{g}}$. This Lax matrix is valued in (a completion of) $\widetilde{\mathfrak{g}} \otimes \mathcal{F}[M]$. We will write the constant Ω as

$$\Omega = ip_\infty \mathsf{K} + ik_\infty \mathsf{D} + B,$$

with k_∞ and p_∞ complex numbers and $B \in \mathfrak{g}[t, t^{-1}]$.

As explained in the first paragraph of this subsection, to interpret this Lax matrix as the Lax connection of a model with twist function, we made a change of auxiliary space and considered the image $\widetilde{\nabla}(\lambda)$ of $\mathscr{L}(\lambda)$ under the morphism $(\rho \otimes \mathrm{Id})$ (where Id acts on $\mathcal{F}[M]$). Let us now consider the equivalent object for the local AGM and define

$$\nabla(\lambda) = (\rho \otimes \pi_{\boldsymbol{k}})\, \mathscr{L}(\lambda). \tag{6.2.19}$$

Evaluating at $x \in \mathbb{S}^1$, we then find

$$\nabla(\lambda, x) = \varphi(\lambda)\partial_x + \mathcal{S}(\lambda, x) \tag{6.2.20}$$

with

$$\varphi(\lambda) = \sum_{r=1}^{N} \frac{k_r}{\lambda - \lambda_r} + k_\infty \tag{6.2.21}$$

and

$$\mathcal{S}(\lambda, x) = \sum_{r=1}^{N} \frac{\mathcal{J}^{(r)}(x)}{\lambda - \lambda_r} + \mathcal{B}(x), \qquad \text{where} \qquad \mathcal{B} = (\rho \otimes \pi_{\boldsymbol{k}})B. \tag{6.2.22}$$

As $\pi_{\boldsymbol{k}}$ is a Poisson map, the $\mathfrak{g}$-connection ∇ satisfies the same Poisson bracket (6.2.10) than $\widetilde{\nabla}$. According to equation (6.2.2), the function $\varphi(\lambda)$ is thus the **twist function** of the model. Moreover, one recovers the usual **Lax matrix** of the model as

$$\mathcal{L}(\lambda, x) = \varphi(\lambda)^{-1}\mathcal{S}(\lambda, x).$$

Considering the inverse reasoning of Subsection 6.2.1, this Lax matrix then satisfies the Maillet bracket (2.2.6) with $\mathcal{R}$-matrix

$$\frac{C_{\mathbf{12}}}{\mu-\lambda}\varphi(\mu)^{-1}.$$

Note that the twist function appears first in the formal AGM as the coefficient of $i\mathsf{D}$ in the formal Lax matrix $\mathscr{L}(\lambda)$: it is then a fundamental observable of the formal theory and is later realised as a rational function through the Segal-Sugawara map $\pi_{\boldsymbol{k}}$. Note also that as $\varphi(\lambda)$ and $\mathcal{S}(\lambda,x)$ are extracted from the same object $\mathscr{L}(\lambda)$, they possess similar partial fraction decompositions.

Hamiltonian. So far, we did not discuss the dynamic of the AGM. At the level of the formal AGM, we define the quadratic Hamiltonian (still depending on the spectral parameter) as in (6.1.5):

$$\mathscr{H}(\lambda)=\frac{1}{2}\big(\mathscr{L}(\lambda),\mathscr{L}(\lambda)\big)=\sum_{r=1}^{N}\left(\frac{\Delta_r}{(\lambda-\lambda_r)^2}+\frac{\mathscr{H}_r}{\lambda-\lambda_r}\right)+\frac{1}{2}(\Omega,\Omega).$$

This Hamiltonian is in involution with itself for any values of the spectral parameter, according to equation (6.1.9).

We then define the Hamiltonian of the local AGM from the formal one *via* the Segal-Sugawara map $\pi_{\boldsymbol{k}}$:

$$\mathcal{H}(\lambda)=\pi_{\boldsymbol{k}}\big(\mathscr{H}(\lambda)\big)\in\mathrm{Obs}.$$

As $\pi_{\boldsymbol{k}}$ is a Poisson map, we have

$$\{\mathcal{H}(\lambda),\mathcal{H}(\mu)\}=0,\quad\forall\,\lambda,\mu\in\mathbb{C}.$$

As explained above, $\pi_{\boldsymbol{k}}$ sends the Poisson Casimirs Δ_r's to zero. Thus, the local Gaudin Hamiltonian has only simple poles:

$$\mathcal{H}(\lambda)=\sum_{r=1}^{N}\frac{\mathcal{H}_r}{\lambda-\lambda_r}+\frac{1}{2}(\Omega,\Omega). \tag{6.2.23}$$

Note that equation (6.2.12) implies

$$(\mathrm{Id}\otimes\pi_{\boldsymbol{k}})\big(\mathscr{L}(\lambda)\big)=i\mathcal{P}(\lambda)\mathsf{K}+i\varphi(\lambda)\mathsf{D}+S(\lambda),$$

where $S(\lambda)\in\mathfrak{g}[t,t^{-1}]$ is such that $\rho\big(S(\lambda)\big)=\mathcal{S}(\lambda)$ and

$$\mathcal{P}(\lambda)=\sum_{r=1}^{N}\frac{\mathcal{P}^{(r)}}{\lambda-\lambda_r}+p_\infty.$$

Note that the total momentum (6.2.18) of the local AGM can be seen as

$$\mathcal{P}=-\operatorname*{res}_{\lambda=\infty}\mathcal{P}(\lambda)\,\mathrm{d}\lambda.$$

We then express the local Hamiltonian, using the identity (6.2.8), as

$$\mathcal{H}(\lambda)=\frac{1}{2}\big(S(\lambda),S(\lambda)\big)-\varphi(\lambda)\mathcal{P}(\lambda)=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}x\;\kappa\big(\mathcal{S}(\lambda,x),\mathcal{S}(\lambda,x)\big)-\varphi(\lambda)\mathcal{P}(\lambda). \tag{6.2.24}$$

One easily checks that the double pole of $\mathcal{H}(\lambda)$ at $\lambda=\lambda_r$ is equal to

$$\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}x\;\kappa\big(\mathcal{J}^{(r)}(x),\mathcal{J}^{(r)}(x)\big)-k_r\mathcal{P}^{(r)}=0,$$

by equation (6.2.17), as expected from equation (6.2.23). In general, we define the Hamiltonian of the model as a linear combination of the $\mathcal{H}_r$'s.

Lax equation. Let us end this subsection by discussing the dynamic of the Lax matrix. Recall the formal Lax equation (6.1.3) describing the evolution of the formal Lax matrix $\mathscr{L}(\lambda)$ under the formal Hamiltonian $\mathscr{H}(\mu)$. As usual, we will apply the map $\rho\otimes\pi_{\boldsymbol{k}}$ to this equation to get informations on the local AGM. We define $\mathcal{N}$ and χ by

$$(\rho\otimes\pi_{\boldsymbol{k}})\mathscr{M}(\mu,\lambda,x)=\chi(\mu,\lambda)\partial_x+\mathcal{N}(\mu,\lambda,x).$$

Starting from the formal Lax equation (6.1.3) and using the fact that ρ is a Lie morphism on $\widetilde{\mathfrak{g}}$, we get

$$\begin{aligned}
(\rho\otimes\pi_{\boldsymbol{k}})\left\{\mathscr{H}(\mu),\mathscr{L}(\lambda)\right\}(x) &= (\rho\otimes\pi_{\boldsymbol{k}})\left[\mathscr{M}(\mu,\lambda),\mathscr{L}(\lambda)\right](x)\\
&= \left[(\rho\otimes\pi_{\boldsymbol{k}})\mathscr{M}(\mu,\lambda,x),(\rho\otimes\pi_{\boldsymbol{k}})\mathscr{L}(\lambda,x)\right]\\
&= \varphi(\lambda)\left[\chi(\mu,\lambda)\partial_x+\mathcal{N}(\mu,\lambda,x),\partial_x+\mathcal{L}(\lambda,x)\right]\right]\\
&= \varphi(\lambda)\Big(\partial_x\big(\chi(\mu,\lambda)\mathcal{L}(\lambda,x)-\mathcal{N}(\mu,\lambda,x)\big)+\left[\mathcal{N}(\mu,\lambda,x),\mathcal{L}(\lambda,x)\right]\Big)\\
&= \varphi(\lambda)\Big(\partial_x\mathcal{M}(\mu,\lambda,x)-\left[\mathcal{M}(\mu,\lambda,x),\mathcal{L}(\lambda,x)\right]\Big),
\end{aligned}$$

with

$$\mathcal{M}(\mu,\lambda,x)=\chi(\mu,\lambda)\mathcal{L}(\lambda,x)-\mathcal{N}(\mu,\lambda,x).$$

On the other hand, using the fact that $\pi_{\boldsymbol{k}}$ is a Poisson map on $\mathcal{F}[M]$, we have

$$\begin{aligned}
(\rho\otimes\pi_{\boldsymbol{k}})\left\{\mathscr{H}(\mu),\mathscr{L}(\lambda)\right\}(x) &= \left\{\pi_{\boldsymbol{k}}\big(\mathscr{H}(\mu)\big),(\rho\otimes\pi_{\boldsymbol{k}})\mathscr{L}(\lambda)(x)\right\}\\
&= \varphi(\lambda)\left\{\mathcal{H}(\mu),\partial_x+\mathcal{L}(\lambda,x)\right\}\\
&= \varphi(\lambda)\left\{\mathcal{H}(\mu),\mathcal{L}(\lambda,x)\right\}.
\end{aligned}$$

Thus, one gets

$$\left\{\mathcal{H}(\mu),\mathcal{L}(\lambda,x)\right\}-\partial_x\mathcal{M}(\mu,\lambda,x)+\left[\mathcal{M}(\mu,\lambda,x),\mathcal{L}(\lambda,x)\right]=0.$$

This is a zero curvature equation, hence proving that the dynamic of the Lax matrix $\mathcal{L}(\lambda)$ under the local quadratic Gaudin Hamiltonian $\mathcal{H}(\mu)$ takes the form of a Lax equation (2.2.2).

Summary. Let us summarise what we have done in this subsection by recalling the main steps of the construction of the local AGM.

1. We start with the formal AGM on the affine algebra $\widetilde{\mathfrak{g}}$.
2. Using the definition of $\widetilde{\mathfrak{g}}$ in terms of loop algebra, we change the auxiliary space $\widetilde{\mathfrak{g}}$ in the space of $\mathfrak{g}$-connections on $\mathbb{S}^1$, *via* the morphism ρ.
3. Using the Segal-Sugawara map $\pi_{\boldsymbol{k}}$, we change the algebra of observables of the model so that the new algebra Obs describes the phase space of a Hamiltonian field theory .
4. We extract from the formal Gaudin Lax matrix $\mathscr{L}(\lambda)$ the twist function $\varphi(\lambda)$ and the usual Lax matrix $\mathcal{L}(\lambda)$ of the local AGM.
5. The Poisson bracket of the formal Gaudin Lax matrix translates to the Maillet bracket with twist function φ of the matrix $\mathcal{L}$.
6. We define the Hamiltonian of the local AGM as the image of the formal one under $\pi_{\boldsymbol{k}}$.
7. The Lax equation on $\mathscr{L}$ gives the zero curvature equation on $\mathcal{L}$ (*i.e.* a Lax equation for a field theory).

As a conclusion, the local AGM is an integrable field theory with twist function.

6.2.4 Dihedral Affine Gaudin models with arbitrary multiplicities

In this subsection, we will discuss generalisations of the AGM presented above by adding successively multiplicities of the sites, cyclotomy and reality conditions, hence ending with the most general DAGM. We will follow the general discussion of these generalisations for an arbitrary Gaudin model in Subsection 6.1.5, but adapting to the particular case of an affine algebra. In particular, we shall explain how these formal AGM can be mapped to local AGM, as in the previous subsection, and how to interpret local AGM as integrable field theories with twist function.

AGM with multiplicities. Let us start by considering sites $\lambda_1, \cdots, \lambda_N$ with corresponding multiplicities $m_1, \cdots, m_N \in \mathbb{Z}_{\geq 1}$. In this case, recall from Subsection 6.1.5 that the formal Lax matrix $\mathscr{L}(\lambda)$ satisfies the same Poisson bracket (6.2.9) than in the case with multiplicities equal to one. Therefore, applying the Lie morphism $\rho : \widetilde{\mathfrak{g}} \to \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ to $\mathscr{L}(\lambda)$ to change the auxiliary space of the model, one gets a connection satisfying the bracket (6.2.10), characteristic of a model with twist function.

The main change in the study of AGM with multiplicities is then the interpretation of the observables of the model as the observables of a Hamiltonian field theory. Recall from Subsection 6.1.5 that the phase space of the formal AGM with multiplicities is the Cartesian product

$$M = \mathsf{T}^{m_1}\widetilde{\mathfrak{g}}^* \otimes \cdots \otimes \mathsf{T}^{m_N}\widetilde{\mathfrak{g}}^*$$

of Kirillov-Kostant spaces dual to Takiff algebras $\mathsf{T}^m\widetilde{\mathfrak{g}}$. This phase space is encoded by $\widetilde{\mathfrak{g}}$-valued observables $X^{(r)}_{[p]}$ $(p = 0, \cdots, m-1)$ satisfying the bracket

$$\left\{X^{(r)}_{[p]}\,\underline{\mathbf{1}}, X^{(s)}_{[q]}\,\underline{\mathbf{2}}\right\} = \begin{cases} \delta_{rs}\left[\widetilde{C}_{\underline{\mathbf{12}}}, X^{(r)}_{[p+q]}\underline{\mathbf{1}}\right] & \text{if } p+q \leq m_r - 1, \\ 0 & \text{if } p+q > m_r - 1. \end{cases} \tag{6.2.25}$$

We will follow the ideas developed in Subsection 6.2.3 for the AGM with multiplicities one. In particular, we want to find a Poisson map

$$\pi_{\boldsymbol{k}} : \mathcal{F}[M] \longrightarrow \mathrm{Obs},$$

from the algebra of observables $\mathcal{F}[M]$ of the formal AGM to the one Obs of a field theory, which we will call the local AGM. We define observables $K^{(r)}_{[p]}$, $D^{(r)}_{[p]}$ (scalars) and $J^{(r)}_{[p]}$ (in $\mathfrak{g}[t, t^{-1}]$) by

$$X^{(r)}_{[p]} = D^{(r)}_{[p]}\mathsf{K} + iK^{(r)}_{[p]}\mathsf{D} + J^{(r)}_{[p]},$$

as in (6.2.12). As in the case with no multiplicities, we find that the $K^{(r)}_{[p]}$ are Poisson Casimirs of the algebra $\mathcal{F}[M]$ and choose $\pi_{\boldsymbol{k}}$ to send them to some constants:

$$\pi_{\boldsymbol{k}}\left(K^{(r)}_{[p]}\right) = k_{r,p}, \qquad \text{with} \qquad \boldsymbol{k} = (k_{1,0}, \cdots, k_{r,p}, \cdots, k_{N,m_N-1}) \in \mathbb{C}^{m_1+\cdots+m_N},$$

that we also call the levels of the local AGM. We also define

$$\mathcal{J}^{(r)}_{[p]} = (\rho \otimes \pi_{\boldsymbol{k}})J^{(r)}_{[p]},$$

which belong to (a completion of) $\mathfrak{g} \otimes \mathrm{Obs} \otimes \mathcal{T}(\mathbb{S}^1)$ and are thus $\mathfrak{g}$-valued dynamical fields. Applying $(\rho \otimes \pi_{\boldsymbol{k}})$ to the Poisson bracket (6.2.25) and using identity (6.2.7), we get

$$\left\{\mathcal{J}^{(r)}_{[p]}\,\underline{\mathbf{1}}(x), \mathcal{J}^{(s)}_{[q]}\,\underline{\mathbf{2}}(y)\right\} = \delta_{rs}\begin{cases} \left[C_{\underline{\mathbf{12}}}, \mathcal{J}^{(r)}_{[p+q]}\underline{\mathbf{1}}(x)\right]\delta_{xy} - k_{r,p+q}C_{\underline{\mathbf{12}}}\delta'_{xy} & \text{if } p+q \leq m_r - 1, \\ 0 & \text{if } p+q > m_r - 1. \end{cases}$$

The treatment of the $D^{(r)}_{[p]}$'s is similar to the one of the $D^{(r)}$'s in the case of multiplicities one. We find that $D^{(r)}_{[p]}$ acts on the fields $\mathcal{J}^{(r)}_{[p]}$ (here also, we denote by the same notations the fields before and after taking the realisation $\pi_{\boldsymbol{k}}$) as

$$-i\left\{D^{(r)}_{[p]}, \mathcal{J}^{(s)}_{[q]}(x)\right\} = \delta_{rs}\begin{cases} \partial_x\mathcal{J}^{(r)}_{[p+q]}(x) & \text{if } p+q \leq m_r - 1, \\ 0 & \text{if } p+q > m_r - 1. \end{cases}$$

Moreover, for all r, s, p and q, we have

$$\left\{D^{(r)}_{[p]}, D^{(s)}_{[q]}\right\} = 0.$$

It is possible to construct quantities $\mathcal{P}^{(r)}_{[p]}$ from the $\mathcal{J}^{(r)}_{[q]}$'s that satisfy the same Poisson brackets as the $D^{(r)}_{[p]}$'s. We then consider a generalised Segal-Sugawara map

$$\pi_{\boldsymbol{k}}\left(D^{(r)}_{[p]}\right) = \mathcal{P}^{(r)}_{[p]}.$$

For a Takiff algebra of multuplicity two, this generalised Segal-Sugawara construction is the classical analogue of a quantum one, found in [173]. For the most general (classical) construction with arbitrary multiplicity, we refer to the article [78]. Note that this construction requires that the "highest" levels k_{r,m_r-1}'s (in the sense of the levels associated with the highest Takiff modes $[p] = [m_r - 1]$) are non-zero.

As the map $\pi_{\boldsymbol{k}}$ realises the observables $K^{(r)}_{[p]}$'s and $D^{(r)}_{[p]}$'s in terms of the fields $\mathcal{J}^{(r)}_{[p]}$'s, the resulting Poisson algebra Obs describes the observables of a field theory, that we shall call the local AGM. Once this algebra constructed, the rest of the construction of the local AGM presented in Subsection 6.2.3 for multiplicities one generalises easily to the case of arbitrary multiplicities. One extracts the twist function φ and the Lax matrix $\mathcal{L}$ of the local AGM from the formal Gaudin Lax matrix:

$$\varphi(\lambda) = \sum_{r=1}^{N}\sum_{p=0}^{m_r-1}\frac{k_{r,p}}{(\lambda-\lambda_r)^{p+1}} + k_\infty \quad \text{and} \quad \varphi(\lambda)\mathcal{L}(\lambda,x) = \sum_{r=1}^{N}\sum_{p=0}^{m_r-1}\frac{\mathcal{J}^{(r)}_{[p]}(x)}{(\lambda-\lambda_r)^{p+1}} + \mathcal{B}(x). \tag{6.2.26}$$

This Lax matrix $\mathcal{L}$ satisfies a Maillet bracket (2.2.6) with twist function (and non twisted standard $\mathcal{R}$-matrix) and a zero curvature equation, hence proving that the local AGM with arbitrary multiplicities is an integrable field theory with twist function.

As explained already for the Gaudin model on an arbitrary quadratic Lie algebra, one can also consider a site at infinity with arbitrary multiplicity (in fact, the level k_∞ and the non-dynamical field $\mathcal{B}(x)$ in (6.2.26) already correspond to a double pole at infinity). The construction of the local AGM as a field theory with twist function also generalises to this case (see [78]): the effect on the twist function and Lax matrix (6.2.26) would then be to add polynomials in the spectral parameter λ.

Cyclotomic AGM. Let us now consider a cyclotomic affine Gaudin model, associated with an automorphism σ of $\widetilde{\mathfrak{g}}$ of order T. We shall restrict to a particular type of automorphism, which is the lift on $\widetilde{\mathfrak{g}}$ of an automorphism of $\mathfrak{g}$. Let σ be an automorphism of $\mathfrak{g}$ of order T. We define an automorphism of $\widetilde{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathsf{K} \oplus \mathbb{C}\mathsf{D}$, that we still denote σ, by

$$\sigma(X \otimes t^n) = \sigma(X) \otimes t^n, \quad \sigma(\mathsf{K}) = \mathsf{K}, \quad \text{and} \quad \sigma(\mathsf{D}) = \mathsf{D}.$$

The formal Gaudin Lax matrix $\mathscr{L}(\lambda)$ then satisfies the equivariance condition (6.1.21).

The phase space M of the formal cyclotomic AGM is similar to the one of a non-cyclotomic model, taking into account that the observables $X^{(0)}_{[p]}$ attached to the site $\lambda_0 = 0$ at the origin belong to appropriate gradings $\widetilde{\mathfrak{g}}^{(-p)}$. One can also construct a Segal-Sugawara map $\pi_{\boldsymbol{k}}$ from $\mathcal{F}[M]$ to the algebra Obs of observables of a field theory. In particular, it sends the observables $K^{(r)}_{[p]}$ (the coefficient of $i\mathsf{D}$ in $X^{(r)}_{[p]}$) to complex numbers $k_{r,p}$, the levels of the theory. We will not enter into the details of this construction here, as it is quite similar to the one presented above, with a particular treatment of the observables $X^{(0)}_{[p]}$, and refer to [78] for details.

To generalise the construction of the local AGM to the cyclotomic case, we will focus more on the auxiliary space $\widetilde{\mathfrak{g}}$ of the model. One easily checks that the Lie morphism $\rho : \widetilde{\mathfrak{g}} \to \mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ satisfies:

$$\rho \circ \sigma = \sigma_c \circ \rho, \tag{6.2.27}$$

where σ_c is an automorphism of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ defined by

$$\sigma_c(X \otimes e_n) = \sigma(X) \otimes e_n \qquad \text{and} \qquad \sigma_c(\partial) = \partial. \tag{6.2.28}$$

Recall that the formal Gaudin Lax matrix $\mathscr{L}(\lambda)$ of the AGM satisfies the Poisson bracket (6.1.23), with the matrix r given as in (6.1.24) by

$$\mathit{r}_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\sum_{k=0}^{T-1} \frac{\sigma_{\underline{\mathbf{1}}}^k \widetilde{C}_{\underline{\mathbf{12}}}}{\mu - \omega^{-k}\lambda}.$$

Applying $\rho \otimes \rho$ to this matrix and using the equivariance property (6.2.27), one finds

$$(\rho\otimes\rho)\mathit{r}_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\sum_{k=0}^{T-1} \frac{\sigma_{c\,\underline{\mathbf{1}}}^k}{\mu - \omega^{-k}\lambda}(\rho\otimes\rho)\widetilde{C}_{\underline{\mathbf{12}}}.$$

Evaluating this $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)\otimes\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$-valued equation at x and y in $\mathbb{S}^1$ (on the left and right factors) and using the definition (6.2.28) of σ_c and the identity (6.2.7), one finds

$$(\rho\otimes\rho)\mathit{r}_{\underline{\mathbf{12}}}(\lambda,\mu)(x,y) = \frac{1}{T}\sum_{k=0}^{T-1} \frac{\sigma_{\underline{\mathbf{1}}}^k C_{\underline{\mathbf{12}}}}{\mu - \omega^{-k}\lambda}\delta_{xy} = \mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\delta_{xy},$$

where $\mathcal{R}^0$ is the standard $\mathcal{R}$-matrix (2.3.8) on $\mathcal{L}(\mathfrak{g})$ twisted by σ. Defining the Lax connection ∇ of the local AGM as in (6.2.19), one then finds that it satisfies the Poisson bracket (6.2.3), characteristic of a model with twist function. We then extract the twist function $\varphi(\lambda)$ and the Lax matrix $\mathcal{L}(\lambda,x)$ of the local AGM as in equation (6.2.2). By equation (6.2.28), the equivariance condition (6.1.21) translates to

$$\sigma_c\big(\nabla(\lambda)\big) = \omega\nabla(\omega\lambda).$$

We thus get that the Lax matrix and the twist function of the local AGM satisfy the equivariance conditions (2.4.5) and (2.4.3).

The construction of the local Hamiltonian and of the zero curvature equation under its flow generalises easily from the non-cyclotomic case to the cyclotomic one. The local cyclotomic AGM is thus a cyclotomic integrable field theory with twist function (as defined in Section 2.4, explaining the denomination cyclotomic used at the time).

Dihedral AGM. Let us end this subsection by discussing briefly the Dihedral AGM (DAGM), *i.e.* cyclotomic AGM with an additional reality condition. As for the automorphism σ in the cyclotomic AGM, we start with an antilinear involutive automorphism τ of the finite algebra $\mathfrak{g}$. We extend it to $\widetilde{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}\mathsf{K} \oplus \mathbb{C}\mathsf{D}$ as

$$\tau(X\otimes t^n) = \tau(X)\otimes t^{-n}, \quad \tau(\mathsf{K}) = -\mathsf{K} \text{ and } \quad \tau(\mathsf{D}) = -\mathsf{D}. \tag{6.2.29}$$

One checks that this defines an antilinear involutive automorphism of $\widetilde{\mathfrak{g}}$.

We then define a formal DAGM as explained generally in Subsection 6.1.5. We shall not discuss in detail the algebra of observables $\mathcal{F}[M]$ of the formal model as the discussion is quite similar to what we did previously. One constructs a generalised Segal-Sugawara map $\pi_{\boldsymbol{k}}$ from $\mathcal{F}[M]$ to Obs, which is the algebra of observables of a field theory, the local DAGM. The important property of this map is that it is compatible with the conjugacy on the complex algebra $\mathcal{F}[M]$ and Obs, *i.e.*

$$\forall\, f \in \mathcal{F}[M], \quad \pi_{\boldsymbol{k}}(\overline{f}) = \overline{\pi_{\boldsymbol{k}}(f)}.$$

The treatment of the auxiliary space of the DAGM is also similar to what was done above. We find that

$$\rho\circ\tau = \tau_c\circ\rho,$$

where τ_c is the antilinear involutive automorphism of $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{S}^1)$ defined by

$$\tau_c(X \otimes e_n) = X \otimes e_{-n} \qquad \text{and} \qquad \tau_c(\partial) = \partial.$$

Note that this definition is compatible with the conjugacy *via* the evaluation at $x \in \mathbb{S}^1$, as

$$\overline{e_n(x)} = e^{-inx} = e_{-n}(x) \qquad \text{and} \qquad \overline{\partial_x f(x)} = \partial_x \overline{f(x)}.$$

Using these properties and equation (6.2.29) on $\pi_{\boldsymbol{k}}$, one finds that the Lax matrix $\mathcal{L}(\lambda, x)$ and twist function $\varphi(\lambda)$ extracted from the formal Gaudin Lax matrix $\mathscr{L}(\lambda)$ satisfy the reality conditions (2.4.7) and (2.4.10) (using the reality condition (6.1.15) on $\mathscr{L}$).

One constructs a local quadratic Hamiltonian of the theory as in the non-dihedral case and finds that its flow on the Lax matrix takes the form of a zero curvature equation. As in equation (6.2.24) for the non-dihedral case, this Hamiltonian reads

$$\mathcal{H}(\lambda) = \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}x \; \kappa\big(\mathcal{S}(\lambda, x), \mathcal{S}(\lambda, x)\big) - \varphi(\lambda)\mathcal{P}(\lambda), \tag{6.2.30}$$

where

$$\mathcal{S}(\lambda, x) = \varphi(\lambda)\mathcal{L}(\lambda, x)$$

and $\mathcal{P}$ is the image under $\pi_{\boldsymbol{k}}$ of the coefficient of $i\mathsf{K}$ in $\mathscr{L}(\lambda)$. This Hamiltonian satisfies the reality condition:

$$\overline{\mathcal{H}(\lambda)} = \mathcal{H}(\overline{\lambda}).$$

In general, we can choose the Hamiltonian $\mathcal{H}$ of the theory as any quantity extracted linearly from $\mathcal{H}(\lambda)$ (evaluation at particular points, residues, coefficients of higher order poles, integral over λ, any linear combination of these ...). Such an Hamiltonian is then in involution with $\mathcal{H}(\lambda)$ for any value of λ and generates a zero curvature equation on $\mathcal{L}(\lambda, x)$.

6.2.5 Integrable field theories with twist function as DAGM

In the previous subsections, we proved that a formal DAGM can be realised as an integrable field theory with twist function, the local DAGM. Given the generality of this construction, it is natural to ask whether the known models with twist function can be interpreted as DAGM. The answer is yes for all integrable σ-models and their deformations, as shown by B. Vicedo in [78].

Let us be more precise about what we mean by that. We consider a local DAGM with algebra of observables Obs, as constructed above, and an integrable field theory $\mathcal{T}$ with twist function, whose algebra of observables we denote $\mathrm{Obs}_{\mathcal{T}}$. The local DAGM possesses a twist function $\varphi(\lambda)$, a Lax matrix $\mathcal{L}(\lambda, x)$ and a Hamiltonian $\mathcal{H}$, as constructed in the previous subsections. In the same way, by definition, the model with twist function is described by a twist function $\varphi_{\mathcal{T}}(\lambda)$, a Lax matrix $\mathcal{L}_{\mathcal{T}}(\lambda, x)$ and a Hamitlonian $\mathcal{H}_{\mathcal{T}}$. We will say that $\mathcal{T}$ is a realisation of the local DAGM if:

- there exists a Poisson map $\pi : \mathrm{Obs} \mapsto \mathrm{Obs}_{\mathcal{T}}$, which realises the Gaudin observables in terms of the observables of the model $\mathcal{T}$;
- the twist functions $\varphi(\lambda)$ and $\varphi_{\mathcal{T}}(\lambda)$ are equal ;
- the Lax matrices and Hamiltonians of the two models are such that:

$$\pi\big(\mathcal{L}(\lambda, x)\big) = \mathcal{L}_{\mathcal{T}}(\lambda, x) \qquad \text{and} \qquad \pi\big(\mathcal{H}\big) = \mathcal{H}_{\mathcal{T}}.$$

Yang-Baxter model as a DAGM. Let us illustrate this on a simple example, the Yang-Baxter model. It could seem surprising to take the Yang-Baxter model as a first example, as the undeformed PCM seems simpler. As we will see, in terms of their Gaudin representations, the Yang-Baxter model is actually simpler than the PCM, as it is a DAGM with simple poles, whereas the PCM is not.

Recall then the Yang-Baxter model on a real Lie group G_0, described in Subsection 3.3.1 of this thesis. For simplicity, we will restrict to the split case $c = 1$. We consider the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$

and the antilinear involutive automorphism τ such that $\mathfrak{g}_0 = \mathfrak{g}^\tau$. Let us check what are the properties of the Yang-Baxter model as a theory with twist function and deduce what would be the characteristic of an affine Gaudin model that would be realised by the Yang-Baxter model.

The Yang-Baxter model possesses a Lax matrix $\mathcal{L}(\lambda, x)$ valued in the Lie algebra $\mathfrak{g}$ and satisfying the reality condition (2.4.7). Thus, the corresponding AGM should be constructed from the Lie algebra $\mathfrak{g}$ and be real with respect to the antilinear automorpism τ. The Lax matrix does not satisfy an equivariance condition of the form (2.4.3) and its Maillet bracket is described by a non-twisted standard $\mathcal{R}$-matrix (2.3.6). Thus, the AGM should not be cyclotomic, *i.e.* $\sigma = \mathrm{Id}$.

One still has to find what would be the sites and the levels of the AGM. For that, let us consider the twist function $\varphi_\eta(\lambda)$ of the Yang-Baxter model, given by (3.3.13). We find that its partial fraction decomposition is

$$\varphi_\eta(\lambda) = \frac{1}{\lambda - \eta}\frac{K}{2\eta} - \frac{1}{\lambda + \eta}\frac{K}{2\eta} - \frac{K}{1-\eta^2}.$$

Comparing to equation (6.2.21), one then consider an AGM with sites $\lambda_1 = \eta$ and $\lambda_2 = -\eta$ and fix the levels to be

$$k_1 = \frac{K}{2\eta}, \quad k_2 = -\frac{K}{2\eta}, \quad k_\infty = -\frac{K}{1-\eta^2}.$$

This way, the Gaudin twist function (6.2.21) agrees with the twist function $\varphi_\eta(\lambda)$.

Let us now turn to the phase space of the model, as we would like to construct a map π from the observables Obs of the local real AGM (with sites and levels described above) to the observables $\mathrm{Obs}_{\mathrm{YB}}$ of the Yang-Baxter model. For that, we consider the Lax matrix $\mathcal{L}_\eta(\lambda, x)$ of the Yang-Baxter model, given by (3.3.12), and construct the matrix

$$\mathcal{S}_\eta(\lambda, x) = \varphi_\eta(\lambda)\mathcal{L}_\eta(\lambda, x).$$

We find that the partial fraction decomposition of this matrix is

$$\mathcal{S}_\eta(\lambda) = \frac{\mathcal{K}_+}{\lambda - \eta} + \frac{\mathcal{K}_-}{\lambda + \eta},$$

where

$$\mathcal{K}_+ = \frac{1}{2}\left(X - R_g X + \frac{K}{\eta} j^L\right) \quad \text{and} \quad \mathcal{K}_- = \frac{1}{2}\left(X + R_g X - \frac{K}{\eta} j^L\right).$$

The fundamental observables of the local AGM (which generate Obs) are the $\mathfrak{g}_0$-valued Kac-Moody current $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ in equation (6.2.22) (for two sites). Comparing this equation to $\mathcal{S}_\eta$ above, one then defines the map

$$\pi : \mathrm{Obs} \longrightarrow \mathrm{Obs}_{\mathrm{YB}}$$

by

$$\pi\big(\mathcal{J}^{(1)}\big) = \mathcal{K}_+ \quad \text{and} \quad \pi\big(\mathcal{J}^{(2)}\big) = \mathcal{K}_-.$$

By construction, this map satisfies $\pi\big(\mathcal{S}(\lambda, x)\big) = \mathcal{S}_\eta(\lambda, x)$, hence also $\pi\big(\mathcal{L}(\lambda, x)\big) = \mathcal{L}_\eta(\lambda, x)$ as the twist functions of the models are equal (note that we choose the free non-dynamical field $\mathcal{B}$ in (6.2.22) to be zero). Moreover, one checks explicitly that π is a Poisson map, *i.e.* that the currents $\mathcal{K}_\pm$ are commuting Kac-Moody currents with levels k_1 and k_2 defined above. The fact that Yang-Baxter deformations possess such Kac-Moody currents was already known before their reinterpretation in terms of Gaudin models (see [24, 40, 138]).

To prove that the Yang-Baxter model is a realisation of the local DAGM constructed here, one still has to find that the Hamiltonian $\mathcal{H}_\eta$ of the Yang-Baxter model is realised as the image under π of the Hamiltonian $\mathcal{H}$ of the local DAGM. The Hamiltonian $\mathcal{H}$ can be defined as any linear combination

$$\mathcal{H} = c_1\mathcal{H}_1 + c_2\mathcal{H}_2, \qquad \text{with} \qquad \mathcal{H}_r = \operatorname*{res}_{\lambda=\lambda_r} \mathcal{H}(\lambda)\mathrm{d}\lambda,$$

with the local quadratic Hamiltonian $\mathcal{H}(\lambda)$ defined in (6.2.24). Starting from the expression (3.3.10) of the Yang-Baxter Hamiltonian $\mathcal{H}_\eta$, one finds that there is a choice of c_1 and c_2 such that $\pi(\mathcal{H}) = \mathcal{H}_\eta$. Thus, we successfully interpreted the Yang-Baxter model as the realisation of a local DAGM.

Integrable σ-models as DAGM. More generally, it was proved in [78] that all integrable σ-models described in Chapter 3 are realisations of local DAGM, following a method close to the one above.

The PCM and its deformations are realisations of DAGM without cyclotomy (*i.e.* $\sigma = \mathrm{Id}$). For example, the PCM corresponds to a real AGM with a site at 0 of multiplicity two. For the Yang-Baxter model in the non-split case ($c = i$), there is only one site at $i\eta$: the associated Kac-Moody current is then complex (*i.e.* $\mathfrak{g}$-valued and not $\mathfrak{g}_0$-valued) and the matrix $\mathcal{S}$ then contains this current at the pole $i\eta$ and its image under τ at the pole $-i\eta$.

The $\mathbb{Z}_T$-cosets and their deformations are realisations of diheral AGM, with the automorphism σ of order T appearing in the definition of the considered coset space. For these models, it is also explained in [78] how to treat the gauge constraint at the level of the local DAGM.

For completeness, let us also mention the fact that affine Toda field theories have been realised as DAGM in [78]. However, for these models, it is a realisation of a formal DAGM and not of a local one. This is due to the fact that in this case, one of the highest level k_{r,m_r-1} is zero, which prevents the construction of a Segal-Sugawara map and thus of a local DAGM (see Subsections 6.2.3 and 6.2.4).

6.2.6 Integrable hierarchies in AGM

In this subsection, we discuss integrable hierarchies of (local) AGM. Recall indeed from Subsections 6.2.3 and 6.2.4 that formal AGM (and DAGM) can be mapped to integrable field theories with twist function (the local models). Thus, the construction of integrable hierarchies for models with twist function, as presented in Chapter 4 of this thesis, applies to local AGM.

Regular zeros. Recall from Chapter 4 that this construction necessitates the existence of a regular zero of the model, *i.e.* a zero λ_0 of $\varphi(\lambda)$ such that $\mathcal{S}(\lambda, x) = \varphi(\lambda)\mathcal{L}(\lambda, x)$ is regular at $\lambda = \lambda_0$. In our construction of local AGM, the matrix $\mathcal{S}(\lambda, x)$ appears quite naturally as a part of the Lax connection (6.2.20). Moreover, comparing the expressions (6.2.26) of the twist function and the matrix $\mathcal{S}(\lambda, x)$, one sees that the poles of $\mathcal{S}(\lambda, x)$ are exactly the poles of the twist function. Thus, if λ_0 is a zero of the twist function (and so is different from a pole), the matrix $\mathcal{S}(\lambda, x)$ is regular at λ_0.

The expressions used here for φ and $\mathcal{S}$ were constructed for the case of a non dihedral AGM. However, the observation that the poles of φ and $\mathcal{S}$ exactly coincide stays true for the most general local DAGM. Therefore, for a local DAGM, **every zero of $\boldsymbol{\varphi(\lambda)}$ is regular**. Thus, the construction of local charges in involution presented in Chapter 4 applies naturally to local DAGM.

Hamiltonian and conservation. So far, we constructed an infinite number of local charges in involution. However, we do not know yet if these charges are conserved. The Hamiltonian of the local AGM is extracted from the charge $\mathcal{H}(\lambda)$, given by (6.2.30). Note that the density

$$\frac{1}{4\pi}\kappa\big(\mathcal{S}(\lambda, x), \mathcal{S}(\lambda, x)\big)$$

appearing in $\mathcal{H}(\lambda)$ coincides, up to a global factor a, with the density

$$\mathcal{W}_2(\lambda, x) = \varphi(\lambda)^2 \mathrm{Tr}\big(\mathcal{L}(\lambda, x)^2\big)$$

introduced in Chapter 4. Recall that the quadratic charge $\mathcal{Q}_2^{\lambda_0}$ at a regular zero is defined as

$$\mathcal{Q}_2^{\lambda_0} = \int_0^{2\pi} \mathcal{W}_2(\lambda_0, x)\mathrm{d}x.$$

As λ_0 is a zero of the twist function $\varphi(\lambda)$, we then find that

$$\mathcal{Q}_2^{\lambda_0} = a\mathcal{H}(\lambda_0).$$

Thus, the quadratic charge at λ_0 is extracted from the local quadratic Hamiltonian $\mathcal{H}(\lambda)$. At the end of paragraph 6.2.4, we said that the Hamiltonian $\mathcal{H}$ of the local AGM can be chosen as any quantity

linearly extracted from $\mathcal{H}(\lambda)$. In particular, motivated by the discussion above, one can define $\mathcal{H}$ as a linear combination

$$\mathcal{H} = \sum_{\lambda_0 \text{ zero of } \varphi(\lambda)} b_{\lambda_0} \mathcal{H}(\lambda_0). \tag{6.2.31}$$

By construction, such an Hamiltonian is thus part of the infinite algebra of local charges in involution, which are then all conserved.

The question whether any choice of local Hamiltonian extracted from $\mathcal{H}(\lambda)$ can be written in the form (6.2.31) is not easy to answer in general. However, let us explain why this is the case for the simplest AGM: complex, non-cyclotomic and with multiplicities one. In Subsection 6.2.3, we observed that in this simple case, the Hamiltonian is expressed in (6.2.23) as a sum of simple poles (and a constant non-dynamical term). The most general Hamiltonian linearly extracted from $\mathcal{H}(\lambda)$ is thus a linear combination

$$\mathcal{H} = \sum_{r=1}^{N} c_r \mathcal{H}_r, \tag{6.2.32}$$

of the residues $\mathcal{H}_r$'s of $\mathcal{H}(\lambda)$ at the sites $\lambda = \lambda_r$ (recall that the symbol λ_0 above did not stand for a site of the model but for a zero of the twist function). It is clear that any Hamiltonian of the form (6.2.31) is of the form (6.2.32) (up to the non-dynamical term (Ω, Ω) in (6.2.23)).

On the other hand, the twist function (6.2.21) of the local DAGM can be written as

$$\varphi(\lambda) = \frac{P(\lambda)}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_N)},$$

where $P(\lambda)$ is a polynomial of order N. Thus, the twist function possesses N zeros $\mu_1, \cdots, \mu_N$. Except maybe for degenerate cases (multiples zeros, ...), we then expect the N residues $\mathcal{H}_r$'s of $\mathcal{H}(\lambda)$ to be linear combinations of the N evaluations $\mathcal{H}(\mu_r)$'s. Therefore, an Hamiltonian of the form (6.2.32) is also of the form (6.2.31), ensuring that the local charges in involution constructed above are conserved.

Chapter 7

Quantum finite Gaudin models

In this chapter, we discuss quantum finite Gaudin models. These were introduced by Gaudin, before the classical Gaudin models, in [89] for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and in [90] for a general semi-simple Lie algebra $\mathfrak{g}$. As physical models, they are interesting as they describe quantum spin chains with long range interaction, while being integrable. For simplicity, we will restrict here to complex quantum Gaudin models with sites of multiplicities one. Moreover, for most of this chapter, we will consider non-cyclotomic Gaudin models.

As explained in Chapter 6, Subsection 6.1.4, classical finite Gaudin models possess a large number of conserved charges in involution, given by the evaluation of invariant polynomials of $\mathfrak{g}$ on the Lax matrix on the model. There exist quantum analogues of these conserved charges [95, 97, 98, 100, 101]: they form a large commutative subalgebra of the algebra of operators of the quantum Gaudin model, called the Gaudin [95] (or Bethe [96]) subalgebra. In particular, this subalgebra contains the quantum analogue of the quadratic Hamiltonians $\mathscr{H}_r$ defined for classical Gaudin models in Chapter 6.

A natural question coming with the existence of conserved commuting charges is their diagonalisation. Indeed, as these operators commute, they can be simultaneously diagonalised. A first step in the resolution of quantum finite Gaudin models is thus to find the common eigenvectors and the spectrum (eigenvalues) of these commuting conserved charges.

This goal is partially achieved by the so-called Bethe ansatz for Gaudin models [92]. The Bethe ansatz is a general method to solve quantum integrable systems by finding common eigenvectors of their commuting charges. The Bethe ansatz for Gaudin models has proven to be quite successful and has led to very rich developments in the study of these models. However, it has some limits, principally in two directions. The first is the fact that it can be applied only for models where the Hilbert space is a tensor product of highest weight representations of the underlying Lie algebra $\mathfrak{g}$. In particular, this excludes models which do not possess a vacuum. The second limitation of the Bethe ansatz is its non-completeness: indeed, in some (rather degenerate cases), the family of common eigenvectors obtained by the Bethe ansatz does not form a basis of the Hilbert space of the model [174].

An alternative but more abstract approach for the description of the spectrum of Gaudin models is the so-called Feigin-Frenkel-Reshetikhin (FFR) approach [95]. It describes the eigenvalues of the conserved commuting charges of the Gaudin model in terms of some differential operators, called opers, which are associated with the underlying Lie algebra. One of the great assets of this approach is that it gives a theoretical description of all eigenvalues of the Gaudin charges, for any eigenvector, in any Hilbert space. Moreover, it is related [105] to a very deep mathematical result, the Geometric Langlands Correspondence, and is thus of interest in pure mathematics.

This chapter is mostly an introductory review of known results on quantum finite Gaudin models. Its plan is the following. In Section 7.1, we will explain the construction of quantum Gaudin models: the algebra of operators, the Hilbert space, the quadratic Hamiltonians and the Gaudin subalgebra. In Section 7.2, we will recall the basics of the Bethe ansatz for Gaudin models. Finally, in Section 7.3, we give an introduction to the FFR approach of Gaudin models. We will end this section with some new results concerning the generalisation of the FFR approach to quantum cyclotomic Gaudin models, based on my PhD work [P4] with B. Vicedo.

7.1 Constructing quantum finite Gaudin models

In this section, we explain how to quantise classical Gaudin models associated with a finite Lie algebra $\mathfrak{g}$ [90]. As this chapter concerns only finite Gaudin models, we will simply refer to them as Gaudin models for simplicity: one has to bear in mind that the results presented here do not hold in general for more general Gaudin models (see Chapter 8 for first results on quantum affine Gaudin models). As explained in the introduction, we will also restrict here to the simplest Gaudin model: complex, without cyclotomy and with simple poles. In this section, we will align our notations to the ones more currently used in the literature about Gaudin models and will thus denote by z the spectral parameter instead of λ. In particular, we will write $\boldsymbol{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$ the sites of the considered Gaudin model.

7.1.1 Algebra of quantum operators and Hilbert spaces

Quantizing the Kirillov-Kostant bracket. Recall that the phase space of the classical Gaudin model is a product of N copies of the dual space $\mathfrak{g}^*$, equipped with the Kirillov-Kostant bracket. We fix a basis $\{I^a\}$ of $\mathfrak{g}$, with structure constants $f^{ab}{}_c$. The algebra of observables on the space $\mathfrak{g}^*$ is then generated by $\dim \mathfrak{g}$ observables X^a's, satisfying the bracket

$$\{X^a, X^b\} = f^{ab}{}_c \, X^c.$$

A quantisation of this bracket then consists of a (non-commutative) algebra, generated by operators $\widehat{X}^a$, satisfying the commutation relations

$$\left[\widehat{X}^a, \widehat{X}^b\right] = \hbar f^{ab}{}_c \, \widehat{X}^c. \tag{7.1.1}$$

These commutation relations are thus the same as the Lie relations of the Lie algebra $\mathfrak{g}$ itself (up to the factor $\hbar$). In particular, if V is a vector space and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$, the operators

$$\widehat{X}^a = \hbar \, \rho(I^a)$$

satisfy the commutation relations (7.1.1).

The most general (in a sense made precise below) algebra with generators satisfying the commutation relations (7.1.1) is the universal enveloping algebra $U_\hbar(\mathfrak{g})$ of $\mathfrak{g}$ [163]. This algebra is defined in the following way. Consider the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \oplus \mathfrak{g}^{\otimes n} \oplus \cdots,$$

equipped with the tensor product $\otimes$. It is an associative and unital algebra (the identity element being the element 1 of $\mathbb{C} = \mathfrak{g}^{\otimes 0} \subset T(\mathfrak{g})$). We denote by $I_\hbar(\mathfrak{g})$ the ideal of $T(\mathfrak{g})$ generated by all

$$X \otimes Y - Y \otimes X - \hbar[X, Y],$$

for $X, Y \in \mathfrak{g}$. The universal enveloping algebra is then defined as the quotient

$$U_\hbar(\mathfrak{g}) = T(\mathfrak{g}) / I_\hbar(\mathfrak{g}).$$

This is also an associative and unital algebra. Moreover, it contains a copy of $\mathfrak{g}$ (the image of $\mathfrak{g}^{\otimes 1}$ under the quotient by $I_\hbar(\mathfrak{g})$) such that for any X, Y in $\mathfrak{g} \subset U_\hbar(\mathfrak{g})$, we have

$$XY - YX = \hbar[X, Y],$$

where XY denotes the product of X and Y in $U_\hbar(\mathfrak{g})$. The elements of the basis $\{I^a\}$, when seen as elements of $U_\hbar(\mathfrak{g})$, satisfy the commutation relation (7.1.1).

Moreover, one can show [163] that if $\mathcal{A}$ is any associative algebra generated by elements $\widehat{X}_a$ which satisfy these relations, there exists a surjective morphism of algebra from $U_\hbar(\mathfrak{g})$ to $\mathcal{A}$ which sends I^a to $\widehat{X}^a$. In particular, there is a one-to-one correspondence between representations of the Lie algebra $\mathfrak{g}$ and modules over the algebra $U_\hbar(\mathfrak{g})$. In this sense, $U_\hbar(\mathfrak{g})$ is the most general quantisation of the Kirillov-Kostant bracket.

A simple rescaling of the generators by $\hbar$ shows that $U_\hbar(\mathfrak{g})$ and $U(\mathfrak{g}) = U_1(\mathfrak{g})$ are isomorphic. For simplicity, we will then consider the universal enveloping algebra $U(\mathfrak{g})$ and forget about the Planck constant $\hbar$. When we will talk about classical limits, we will then mean re-introducing the constant $\hbar$ and take the limit $\hbar$ goes to zero.

The algebra of operators of the Gaudin model. Let us come back to the Gaudin model with sites $\boldsymbol{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$. The phase space of the model is the Cartesian products of N copies of the dual space $\mathfrak{g}^*$. Thus, the algebra of observables $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ of the quantum model is the Nth-tensor product of the universal enveloping algebra:

$$\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}) = U(\mathfrak{g})^{\otimes N}. \tag{7.1.2}$$

For any $X \in \mathfrak{g}$, one can consider an operator $X_{(r)} \in \mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, which belongs to the copy of $\mathfrak{g}$ in the rth-factor of the tensor product (7.1.2). The algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ is then generated by the $N \dim \mathfrak{g}$ operators $I^a_{(k)}$, satisfying the commutation relations

$$\left[I^a_{(i)}, I^b_{(j)}\right] = \delta_{ij}\, f^{ab}{}_c\, I^c_{(i)}.$$

We will encode these generators in operators depending on the spectral parameter z, by defining, for any $X \in \mathfrak{g}$:

$$X(z) = \sum_{k=1}^{N} \frac{X_{(k)}}{z - z_k}.$$

From the circle lemma (2.3.5), one finds the following commutation relation:

$$\big[X(z), Y(z')\big] = -\frac{[X,Y](z) - [X,Y](z')}{z - z'}, \quad \forall\, X, Y \in \mathfrak{g}, \ \ \forall\, z, z' \in \mathbb{C}. \tag{7.1.3}$$

In particular, we have

$$\big[X(z), Y(z)\big] = -[X,Y]'(z),$$

where on the right-hand side, the prime denotes the derivative with respect to z.

In the classical limit, $I^a_{(i)}$ becomes the observable $X^a_{(i)}$ of the classical Gaudin model. Thus, the classical limit of the operators $I^a(z)$ are

$$\sum_{i=1}^{N} \frac{X^a_{(i)}}{z - z_i}.$$

These are the coefficients of the Lax matrix (6.1.3) in the dual basis I_a (with the constant element Ω chosen to be zero).

Hilbert space. So far, we found an algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ which is a quantisation of the algebra of observables of the Gaudin model. Let us define the Hilbert space H of the theory, on which the algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ acts linearly. We choose N representations $R_1, \cdots, R_N$ of the Lie algebra $\mathfrak{g}$. As explained in the first paragraph of this subsection, these representations are also modules of the universal enveloping algebra $U(\mathfrak{g})$. We define the Hilbert space to be

$$H = R_1 \otimes \cdots \otimes R_N. \tag{7.1.4}$$

The algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ acts on H by letting the kth-factor $U(\mathfrak{g})$ of the tensor product (7.1.2) act on the kth-factor R_k in H.

For most of our purposes, we will consider the R_k's to be highest weight representations of $\mathfrak{g}$ [175]. For $\lambda \in \mathfrak{h}^*$ a weight of $\mathfrak{g}$ (where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ its dual), we will denote by V_λ the Verma module of $\mathfrak{g}$. If λ is an integral and dominant weight, we can also consider F_λ, the finite dimensional irreducible representation of $\mathfrak{g}$, obtained as a quotient of V_λ. Let us fix a collection $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N) \in {\mathfrak{h}^*}^N$ of weights. We will often consider the Hilbert space

$$H_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}. \tag{7.1.5}$$

The particularity of the Verma module V_λ is that it possesses a highest-weight vector v_λ. Recall from Appendix A.2.1 the definition of the nilpotent elements E_α of $\mathfrak{g}$, associated with roots $\alpha \in \Delta$. In particular, we distinguish the positive nilpotent elements E_α, for $\alpha \in \Delta_+$ a positive root, and the negative elements $F_\alpha = E_{-\alpha}$. The highest-weight vector satisfies

$$E_\alpha.v_\lambda = 0, \quad \forall\, \alpha \in \Delta_+. \tag{7.1.6}$$

Moreover, the action of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ on v_λ is given by

$$X.v_\lambda = \lambda(X)v_\lambda = \langle X, \lambda\rangle v_\lambda, \quad \forall\, X \in \mathfrak{h}, \tag{7.1.7}$$

where $\langle \cdot, \cdot \rangle$ represents the canonical pairing on $\mathfrak{h} \times \mathfrak{h}^*$.

We define the vacuum state of the Hilbert space $H_{\boldsymbol{\lambda}}$ as the tensor product of the highest-weight vectors:

$$v_{\boldsymbol{\lambda}} = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_N}.$$

This vacuum state satisfies

$$E_\alpha(z).v_{\boldsymbol{\lambda}} = 0, \quad \forall\, \alpha \in \Delta_+$$

and

$$X(z).v_{\boldsymbol{\lambda}} = \big\langle X, \lambda(z)\big\rangle v_{\boldsymbol{\lambda}},$$

with

$$\lambda(z) = \sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} \in \mathfrak{h}^*. \tag{7.1.8}$$

7.1.2 Hamiltonians and Gaudin subalgebra

Quadratic Casimir of $U(\mathfrak{g})$. Recall the Killing form κ (see Appendix A.2), which is a non-degenerate invariant bilinear form on $\mathfrak{g}$. We denote by $\kappa^{ab} = \kappa(I^a, I^b)$ its evaluation in the basis $\{I^a\}$. As κ is non-degenerate, we can define the inverse κ_{ab} of κ^{ab} and the dual basis $I_a = \kappa_{ab} I^b$. We define the following element of $U(\mathfrak{g})$:

$$\Delta = \kappa_{ab} I^a I^b = I^a I_a.$$

It is independent of the choice of basis $\{I^a\}$. Moreover, by the invariance of κ, one finds that Δ is a Casimir of $U(\mathfrak{g})$, *i.e.* that

$$[\Delta, X] = 0, \quad \forall\, X \in U(\mathfrak{g}).$$

This Casimir Δ is the quantum equivalent of the quadratic Poisson Casimir $\kappa_{ab} X^a X^b$ of the Kirillov-Kostant bracket, as described in Subsection 6.1.2 on classical Gaudin Hamiltonians.

Quadratic Hamiltonians. We define the quadratic quantum Gaudin hamiltonian as [90]

$$\mathscr{H}(z) = \frac{1}{2}\kappa_{ab} I^a(z) I^b(z) \in \mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}). \tag{7.1.9}$$

As explained in subsection 7.1.1, in the classical limit, $I^a(z)$ becomes the coefficient of I_a in the classical Lax matrix $\mathscr{L}(z)$ (without the constant element Ω). Thus, in this limit, the quantum Hamiltonian $\mathscr{H}(z)$ becomes the classical quadratic Hamiltonian (6.1.5).

We will denote by $\Delta_{(k)} = \kappa_{ab} I^a_{(k)} I^b_{(k)}$ the Casimir of the algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}) = U(\mathfrak{g})^{\otimes N}$, constructed as Δ in the kth-tensor factor of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$. The partial fraction decomposition of $\mathscr{H}(z)$ is then

$$\mathscr{H}(z) = \sum_{i=1}^{N} \left(\frac{1}{2} \frac{\Delta_{(i)}}{(z-z_i)^2} + \frac{\mathscr{H}_i}{z-z_i} \right), \tag{7.1.10}$$

with

$$\mathscr{H}_i = \sum_{j \neq i} \kappa_{ab} \frac{I^a_{(i)} I^b_{(j)}}{z_i - z_j}. \tag{7.1.11}$$

Recall the involution property (6.1.9) of the classical Gaudin Hamiltonian. Using the commutation relation (7.1.3) and the invariance of κ, one checks that the Hamiltonian (7.1.9) satisfies a similar property at the quantum level:

$$\big[\mathscr{H}(z), \mathscr{H}(z')\big] = 0, \quad \forall\, z, z' \in \mathbb{C}.$$

As the $\Delta_{(k)}$'s are Casimirs of the algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, this is equivalent to

$$[\mathscr{H}_i, \mathscr{H}_j] = 0, \quad \forall\, i, j \in \{1, \cdots, N\}.$$

In general, we will define the Hamiltonian $\mathscr{H}$ of the Gaudin model to be a linear combination

$$\mathscr{H} = \sum_{k=1}^{N} c_k \mathscr{H}_k.$$

Global $\mathfrak{g}_{(\infty)}$-symmetry. For $X \in \mathfrak{g}$, we define the operator

$$X_{(\infty)} = \sum_{i=1}^{N} X_{(i)} \in \mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}).$$

One checks that the map $X \mapsto X_{(\infty)}$ is a Lie algebra homomorphism, *i.e.* that

$$\big[X_{(\infty)}, Y_{(\infty)}\big] = [X, Y]_{(\infty)}, \quad \forall\, X, Y \in \mathfrak{g}.$$

Thus, the $X_{(\infty)}$'s form a Lie subalgebra $\mathfrak{g}_{(\infty)}$ of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, isomorphic to $\mathfrak{g}$. On the Hilbert space (7.1.4), the operator $X_{(\infty)}$ acts by X on each representation R_i:

$$\forall\, w = w_1 \otimes \cdots \otimes w_N \in H, \quad X_{(\infty)}.w = \Big((X.w_1) \otimes \cdots \otimes w_N\Big) + \cdots + \Big(w_1 \otimes \cdots \otimes (X.w_N)\Big). \tag{7.1.12}$$

This defines a representation of $\mathfrak{g}_{(\infty)} \simeq \mathfrak{g}$ on H that we call the diagonal action.

As we will see, this diagonal action is an infinitesimal symmetry of the Gaudin model. For that, we will need the following commutation relation, which is straightforward to prove:

$$\big[X_{(\infty)}, Y(z)\big] = \big[X(z), Y_{(\infty)}\big] = [X, Y](z), \quad \forall\, X, Y \in \mathfrak{g}, \ \ \forall\, z \in \mathbb{C}. \tag{7.1.13}$$

The commutator of the Gaudin hamiltonian $\mathscr{H}(z)$ with the diagonal operator $I^a_{(\infty)}$ is then given by

$$\begin{aligned}\big[I^a_{(\infty)},\mathscr{H}(z)\big] &= \frac{1}{2}\kappa_{bc}\Big(\big[I^a_{(\infty)},I^b(z)\big]I^c(z)+I^b(z)\big[I^a_{(\infty)},I^c(z)\big]\Big)\\ &= \frac{1}{2}\kappa_{bc}\Big(f^{ab}{}_d I^d(z)I^c(z)+I^b(z)f^{ac}{}_e I^e(z)\Big)\\ &= \frac{1}{2}\big(f^{ab}{}_d\,\kappa_{be}+f^{ac}{}_e\,\kappa_{dc}\big)I^d(z)I^e(z),\end{aligned}$$

where in the last equality, we relabelled some of the indices. The parenthesis in the last line vanishes due to the invariance of κ. We thus find that

$$\big[X_{(\infty)},\mathscr{H}(z)\big]=0,\quad \forall X\in\mathfrak{g},\ \ \forall z\in\mathbb{C}.$$

The diagonal action of $\mathfrak{g}$ then commutes with the Gaudin Hamiltonian $\mathscr{H}(z)$, so it is a symmetry of the Gaudin model.

Higher-degrees Hamiltonians and Gaudin subalgebra. In this subsection, we already found commuting operators in $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$: the Casimirs $\Delta_{(k)}$ and the Hamiltonians $\mathscr{H}_k$, which were parts of the quadratic Hamiltonian $\mathscr{H}(z)$.

We discussed the classical limit of the quantum (finite) Gaudin model in Subsection 6.1.4. In particular, we said that these classical models admit a large number of conserved charges in involution. These charges were constructed from invariant polynomials on $\mathfrak{g}$. These polynomials are generated by ℓ elementary polynomials $\Phi_1,\cdots,\Phi_\ell$, of degrees d_i+1, where the d_i's are the so-called exponents of $\mathfrak{g}$. We will denote by $E=\{d_1,\cdots,d_\ell\}$ the set[1] of exponents of $\mathfrak{g}$. There are then ℓ corresponding independent conserved charges in involution $\mathscr{H}^d(z)$, $d\in E$, depending on the spectral parameter z and constructed as polynomials of the Lax matrix of degrees $d+1$. The first exponent of a semi-simple Lie algebra is always 1, corresponding to the existence of a quadratic invariant polynomial on $\mathfrak{g}$, namely the Killing form. The corresponding Hamiltonian is simply the quadratic Hamiltonian $\mathscr{H}^1(z)=\mathscr{H}(z)$ constructed above. The other Hamiltonians $\mathscr{H}^d(z)$ for $d>1$ are then called the higher-degree Hamiltonians of the classical Gaudin model.

The existence of a large number of invariant polynomials on $\mathfrak{g}$ can be seen as a classical statement on Poisson Casimirs of the Kirillov-Kostant bracket. The quantum analogue of this statement is the existence of a large number of Casimirs of the universal enveloping algebra $U(\mathfrak{g})$. Together, these Casimirs form the center $\mathcal{Z}(g)$ of the algebra $U(\mathfrak{g})$. This center is generated by ℓ elements Δ^d, labelled by exponents $d\in E$, corresponding to the ℓ fundamental invariant polynomials. In particular, the Casimir associated with the first exponent $d=1$ is the quadratic Casimir $\Delta^1=\Delta$ constructed above.

As one can construct a quantum quadratic Hamiltonian $\mathscr{H}^1(z)=\mathscr{H}(z)$, one can also construct **higher-degree quantum Hamiltonians** $\mathscr{H}^d(z)$ for all exponents $d\in E$ strictly superior to one, whose classical limits are the higher-degree Hamiltonians mentioned above [95] (see also [97,98,100] for explicit constructions). The quantum analogue of the involution of these classical Hamiltonians is

$$\big[\mathscr{H}^p(z),\mathscr{H}^q(z')\big]=0,\quad \forall p,q\in E,\ \ \forall z,z'\in\mathbb{C}.$$

Moreover, one finds that all Hamiltonians $\mathscr{H}^d(z)$ are invariant under the global diagonal action of $\mathfrak{g}_{(\infty)}$, as described in the previous paragraph for the quadratic one. Concretely, this means that

$$\big[X_{(\infty)},\mathscr{H}^d(z)\big]=0,\quad \forall X\in\mathfrak{g},\ \ \forall d\in E,\ \ \forall z\in\mathbb{C}.$$

The Hamiltonian $\mathscr{H}^d(z)$ is a rational function of the spectral parameter z, which possesses poles at the N sites z_i's of the Gaudin model. These poles are of order inferior or equal to $d+1$ and the coefficient

[1] Technically, E is a multiset, as some of the exponents can come with multiplicities, *i.e.* some d_i's can be identical for different i's. In this case, one has to consider the corresponding d_i's as different elements of E.

of the higher order $(d+1)$ pole at $z = z_k$ is (proportional to) the Casimir $\Delta^d_{(k)}$ in the kth-site of the algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}) = U(\mathfrak{g})^{\otimes N}$.

Let us consider the coefficients of all poles of all rational functions $\mathscr{H}^d(z)$ $(d \in E)$. These are commuting operators in $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$. We will denote by $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ the algebra generated by all these operators and will call it the **Gaudin subalgebra** of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ [95] (it is also sometimes called the **Bethe subalgebra** [96]). By construction, this is an abelian subalgebra of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, which contains the Gaudin Hamitlonian $\mathscr{H}$ and the center $\mathcal{Z}(\mathfrak{g})^{\otimes N}$ of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$.

7.1.3 Realisations of quantum Gaudin models

Let us consider a quantum system $\mathcal{S}$, with algebra of operators $\mathcal{A}_{\mathcal{S}}$ acting on a Hilbert space $H_{\mathcal{S}}$ and with an Hamiltonian $\mathcal{H}_{\mathcal{S}}$. We say that $\mathcal{S}$ is a realisation of the quantum Gaudin model if there exists an algebra morphism

$$\pi : \mathcal{A}_{\boldsymbol{z}}(\mathfrak{g}) \longrightarrow \mathcal{A}_{\mathcal{S}},$$

such that $\pi(\mathscr{H}) = \mathcal{H}_{\mathcal{S}}$. The image of the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ then contains a large number of commuting conserved charges of the system $\mathcal{S}$.

As the algebra $\mathcal{A}_{\mathcal{S}}$ acts on the Hilbert space $H_{\mathcal{S}}$, the map π makes $H_{\mathcal{S}}$ a representation of the algebra $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ of Gaudin operators. In particular, if one is able to solve the Gaudin model on any representation of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$ (in particular, find the spectrum of the Hamiltonian $\mathscr{H}$ on this representation), one can theoretically solve all realisations of the Gaudin model.

7.2 The Bethe ansatz for Gaudin models

In Section 7.1, we explained how to construct a quantum Gaudin model and its algebra of operators $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$. In particular, we found a large commutative subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ of $\mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, containing the Hamiltonians $\mathscr{H}^d(z)$. The next step in the study of the Gaudin model is to diagonalise these Hamiltonians and find their spectrum (eigenvalues). In particular, as all operators in $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ commute, they can be diagonalised simultaneously and thus possess a common basis of eigenvectors.

The Bethe ansatz is a general method to construct these common eigenvectors when the Hilbert space is a product of highest-weight representations of $\mathfrak{g}$ [92] (see also [93,94]). We will use the notations of paragraph 7.1.1, where we already discussed such Hilbert spaces. In particular, we fix a collection $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N) \in {\mathfrak{h}^*}^N$ of weights and consider the Hilbert space $H_{\boldsymbol{\lambda}}$ as defined in equation (7.1.5).

Although the Bethe ansatz provides common eigenvectors to all Hamiltonians $\mathscr{H}^d(z)$, the description of the eigenvalues of the higher-degree Hamiltonians is more difficult than the ones of the quadratic Hamiltonian. In particular there exists a common formalism for the description of the quadratic Hamiltonian, independent of the Lie algebra $\mathfrak{g}$, but such a formalism for higher-degree Hamiltonians is difficult to construct due to the fact that the exponents depend on the Lie algebra $\mathfrak{g}$. In this section, we will therefore mainly focus on the diagonalisation of the quadratic Hamiltonian $\mathscr{H}(z) = \mathscr{H}^1(z)$. However, note that the section 7.3, which concerns the more abstract Feigin-Frenkel-Reshetikhin approach, will use a formalism independent of $\mathfrak{g}$.

7.2.1 Preliminary: eigenvalues of the quadratic Casimirs

Recall from Paragraph 7.1.2 the quadratic Casimir Δ of $U(\mathfrak{g})$. Let us consider the Verma module V_λ of weight $\lambda \in \mathfrak{h}^*$ (or any representation with highest-weight λ). As we will see, the highest-weight vector v_λ of V_λ is an eigenvector under the action of Δ. A generating family of V_λ is given by successive applications of the simple negative generators $F_i = F_{\alpha_i}$'s (associated with simple roots α_i's). As Δ commutes with all the F_i's, all vectors of V_λ are eigenvectors of Δ, with the same eigenvalue as v_λ.

Let us compute this eigenvalue. Recall that a basis of $\mathfrak{g}$ is given by the E_α's and F_α's, $\alpha \in \Delta_+$, together with a basis K_i $(i = 1, \cdots, \ell)$ of the Cartan subalgebra $\mathfrak{h}$ (where $\ell = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$). We will suppose that K_i is orthonormal with respect to the Killing form κ (one can always find such

a basis as κ restricts non-degenerately to $\mathfrak{h}$). The dual basis of $\{E_\alpha, F_\alpha, K_i\}_{\alpha\in\Delta_+, i=1,\cdots,\ell}$ with respect to κ is then $\{F_\alpha, E_\alpha, K_i\}_{\alpha\in\Delta_+, i=1,\cdots,\ell}$. We thus find that

$$\Delta = \sum_{i=1}^{\ell} K_i K_i + \sum_{\alpha\in\Delta_+} \big(E_\alpha F_\alpha + F_\alpha E_\alpha\big).$$

Let $\zeta : \mathfrak{h}^* \to \mathfrak{h}$ be the canonical isomorphism induced by the Killing form (see Appendix A.2.2). It is known that the commutator of E_α with F_α is given by

$$[E_\alpha, F_\alpha] = \zeta(\alpha), \quad \forall\, \alpha \in \Delta_+.$$

We then find

$$\Delta = \sum_{i=1}^{\ell} K_i K_i + 2\zeta(\rho) + 2\sum_{\alpha\in\Delta_+} F_\alpha E_\alpha, \tag{7.2.1}$$

where ρ is the so-called Weyl weight of $\mathfrak{g}$:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha. \tag{7.2.2}$$

By equations (7.1.6) and (7.1.7), one then finds

$$\Delta.v_\lambda = \left(\sum_{i=1}^{\ell} \langle K_i, \lambda\rangle^2 + 2\langle\zeta(\rho), \lambda\rangle\right) v_\lambda.$$

The restriction of the Killing form on $\mathfrak{h}$ induces a bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ by (see also Appendix A.2.2)

$$(\alpha, \beta) = \kappa\big(\zeta(\alpha), \zeta(\beta)\big) = \big\langle\zeta(\alpha), \beta\big\rangle.$$

Using the fact that $\{K_i\}_{i=1,\cdots,\ell}$ is an orthonormal basis of $\mathfrak{h}$, one finds

$$\Delta.v_\lambda = (\lambda, \lambda + 2\rho)v_\lambda.$$

Thus, one has

$$\Delta.w = (\lambda, \lambda + 2\rho)w, \quad \forall\, w \in V_\lambda.$$

Let us consider the quadratic Hamiltonian $\mathscr{H}(z)$, acting on the Hilbert space $H_{\boldsymbol{\lambda}}$. Recall the partial fraction decomposition (7.1.10) of $\mathscr{H}(z)$. The Casimir $\Delta_{(k)}$, which is the coefficient of the double pole of $\mathscr{H}(z)$ at $z = z_k$, acts as the Casimir Δ on the kth-tensor fact of $H_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$. According to the discussion above, we then have

$$\Delta_{(k)}.w = (\lambda_k, \lambda_k + 2\rho)w, \quad \forall\, w \in H_{\boldsymbol{\lambda}}. \tag{7.2.3}$$

The diagonalisation of the quadratic Hamiltonian $\mathscr{H}(z)$ thus reduces to the diagonalisation of the residues $\mathscr{H}_k$ in (7.1.10).

7.2.2 Vacuum eigenvalues

Recall from the paragraph 7.1.1 the vacuum state $v_{\boldsymbol{\lambda}} = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_N}$ of $H_{\boldsymbol{\lambda}}$. Let us study the action of $\mathscr{H}_k$ on $v_{\boldsymbol{\lambda}}$. We will use the notations of the subsection 7.2.1. From equation (7.1.11), We get

$$\mathscr{H}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{z_i - z_j}\left(\sum_{k=1}^{\ell} K_{k\,(i)} K_{k\,(j)} + \sum_{\alpha\in\Delta_+} \big(F_{\alpha\,(i)} E_{\alpha\,(j)} + F_{\alpha\,(j)} E_{\alpha\,(i)}\big)\right)$$

Using equations (7.1.6) and (7.1.7), we get that the vacuum state $v_{\boldsymbol{\lambda}}$ is an eigenvector of the $\mathscr{H}_i$'s:

$$\mathscr{H}_i.v_{\boldsymbol{\lambda}} = \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{k=1}^{\ell} \frac{\langle K_k, \lambda_i \rangle \langle K_k, \lambda_j \rangle}{z_i - z_j} \right) v_{\boldsymbol{\lambda}}.$$

As in Subsection 7.2.1, by the orthonormality of the basis $\{K_k\}_{k=1,\cdots,\ell}$ of $\mathfrak{h}$, one gets

$$\mathscr{H}_i.v_{\boldsymbol{\lambda}} = \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(\lambda_i, \lambda_j)}{z_i - z_j} \right) v_{\boldsymbol{\lambda}}.$$

Combining this with equation (7.2.3), we get

$$\mathscr{H}(z).v_{\boldsymbol{\lambda}} = \left(\frac{1}{2} \sum_{i=1}^{N} \frac{(\lambda_i, \lambda_i + 2\rho)}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{1}{z - z_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(\lambda_i, \lambda_j)}{z_i - z_j} \right) v_{\boldsymbol{\lambda}}.$$

This can be rewritten in terms of $\lambda(z)$, defined in (7.1.8), as

$$\mathscr{H}(z).v_{\boldsymbol{\lambda}} = \left(\frac{1}{2} \big(\lambda(z), \lambda(z)\big) - \big(\lambda'(z), \rho\big) \right) v_{\boldsymbol{\lambda}} = \mathcal{E}_{\mathrm{vac}}(z) v_{\boldsymbol{\lambda}},$$

where $\lambda'(z)$ denotes the derivative of $\lambda(z)$ with respect to z. We call $\mathcal{E}_{\mathrm{vac}}(z)$ the vacuum eigenvalue of $\mathscr{H}(z)$.

7.2.3 Diagonal decomposition of $H_{\boldsymbol{\lambda}}$

$\mathfrak{h}_{(\infty)}$-weight decomposition. Recall from Paragraph 7.1.2 the Lie algebra $\mathfrak{g}_{(\infty)} \subset \mathcal{A}_{\boldsymbol{z}}(\mathfrak{g})$, which acts on $H_{\boldsymbol{\lambda}}$ by the diagonal elements $X_{(\infty)}$, $X \in \mathfrak{g}$, as in (7.1.12). Consider the Cartan subalgebra $\mathfrak{h}_{(\infty)}$ of $\mathfrak{g}_{(\infty)}$. One can decompose the Hilbert space $H_{\boldsymbol{\lambda}}$ in $\mathfrak{h}_{(\infty)}$-weight spaces:

$$H_{\boldsymbol{\lambda}} = \bigoplus_{\mu \in L_{\boldsymbol{\lambda}}} W_\mu,$$

where $L_{\boldsymbol{\lambda}} \subset \mathfrak{h}^*$ is composed of weights $\mu \in \mathfrak{h}^*$ such that the weight space

$$W_\mu = \big\{ w \in H_{\boldsymbol{\lambda}} \mid X_{(\infty)}.w = \langle X, \mu \rangle w, \ \forall\, X \in \mathfrak{h} \big\}$$

is non trivial.

If the w_i's are elements of V_{λ_i} of weights $\mu_i \in \mathfrak{h}^*$ under the $\mathfrak{h}$-weight decomposition of V_{λ_i}, the vector $w = w_1 \otimes \cdots \otimes w_N$ of $H_{\boldsymbol{\lambda}}$ belongs to $W_{\mu_1 + \cdots + \mu_N}$. The highest weight in $L_{\boldsymbol{\lambda}}$ is the sum $\lambda_\infty = \lambda_1 + \cdots + \lambda_N$ and the corresponding weight space is simply $W_{\lambda_\infty} = \mathbb{C} v_{\boldsymbol{\lambda}}$. More generally, if

$$\mu = \lambda_\infty - \sum_{i=1}^{\ell} p_i \alpha_i,$$

a generating family of the weight space W_μ is given by vectors of the form $\mathcal{F}.v_{\boldsymbol{\lambda}}$, where $\mathcal{F}$ is a product of simple negative generators $F_i = F_{\alpha_i}$, appearing p_i times, and acting on any tensor factor of $H_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$. For example, $W_{\lambda_\infty - \alpha_i}$ is of dimension N and admits the following basis:

$$\{ F_{i\,(k)}.v_{\boldsymbol{\lambda}} \}_{k=1,\cdots,N}.$$

The description of the weight space W_μ above implies in particular that all the W_μ's are finite dimensional.

Recall that the diagonal action $\mathfrak{g}_{(\infty)}$ is a symmetry of the Gaudin model, in the sense that all $X_{(\infty)}$ ($X \in \mathfrak{g}$) commute with the Gaudin Hamiltonians $\mathscr{H}^d(z)$. In particular, these Hamiltonians commute with the Cartan subalgebra $\mathfrak{h}_{(\infty)}$ of $\mathfrak{g}_{(\infty)}$. This implies that they stabilise the weight spaces W_μ's and that one can find a basis of common eigenvectors of the $\mathscr{H}^d(z)$'s and $\mathfrak{h}_{(\infty)}$.

Decomposition in highest-weight representations of $\mathfrak{g}_{(\infty)}$. Let us consider the Hilbert space $H_{\boldsymbol{\lambda}}$ as a representation of $\mathfrak{g}_{(\infty)}$. In particular, as it is a tensor product of Verma modules, one can decompose $H_{\boldsymbol{\lambda}}$ as a direct sum of Verma modules of $\mathfrak{g}_{(\infty)}$ [176]. In general, the weights and the multiplicities of the Verma modules appearing in this decomposition are difficult to describe. We shall then label these modules by abstract indices s in a set $I_{\boldsymbol{\lambda}}$ and write

$$H_{\boldsymbol{\lambda}} = \bigoplus_{s \in I_{\boldsymbol{\lambda}}} M_s, \tag{7.2.4}$$

where M_s is a $\mathfrak{g}_{(\infty)}$-submodule of $H_{\boldsymbol{\lambda}}$, isomorphic to the Verma module V_{μ_s}, for some weight $\mu_s \in \mathfrak{h}^*$. In particular, there is a unique (up to scalar multiplication) highest-weight vector B_s in M_s, which is then in W_{μ_s}. Note that one of these modules M_{s_∞} is a Verma module of weight $\mu_{s_\infty} = \lambda_\infty$, whose highest-weight vector is the vacuum state $B_{s_\infty} = v_{\boldsymbol{\lambda}}$. It is the unique module of highest weight λ_∞ in the decomposition (7.2.4).

Recall that the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ (*i.e.* the Gaudin Hamiltonians $\mathscr{H}^d(z)$) commute with the diagonal action $\mathfrak{g}_{(\infty)}$. One can then choose the labelling $I_{\boldsymbol{\lambda}}$ of the decomposition (7.2.4) such that the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ stabilises all M_s's ($s \in I_{\boldsymbol{\lambda}}$).

Let $\mathcal{Q}$ be an operator in $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$. As $\mathcal{Q}$ stabilises M_s and commutes with the action of $\mathfrak{h}_{(\infty)}$, the state $\mathcal{Q}.B_s$ is a vector of M_s of $\mathfrak{h}_{(\infty)}$-weight μ_s. Thus, it is proportional to B_s itself as μ_s is the highest-weight of M_s. This means that there exists $\chi_s(\mathcal{Q}) \in \mathbb{C}$ such that

$$\mathcal{Q}.B_s = \chi_s(\mathcal{Q}) B_s.$$

As B_s is the highest-weight vector of M_s, all other vectors are obtained from B_s by successive actions of the simple lowering operators $F_{i\,(\infty)}$ ($i = 1, \cdots, \ell$). As $\mathcal{Q}$ commutes with the diagonal operators $F_{i\,(\infty)}$, all vectors of M_s are also eigenvectors of $\mathcal{Q}$ of eigenvalue $\chi_s(\mathcal{Q})$. Thus, we have

$$\mathcal{Q}.v = \chi_s(\mathcal{Q}) v, \quad \forall\, v \in M_s.$$

To diagonalise the operators of the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ on $H_{\boldsymbol{\lambda}}$, it is then enough to find all highest-weight vectors B_s, $s \in I_{\boldsymbol{\lambda}}$. This is the goal of the Bethe ansatz.

As highest-weight vectors under the action of $\mathfrak{g}_{(\infty)}$, the states B_s's are $\mathfrak{g}_{(\infty)}$-singular, in the sense that

$$E_{\alpha\,(\infty)}.B_s = 0, \quad \forall\, \alpha \in \Delta_+, \ \ \forall\, s \in I_{\boldsymbol{\lambda}}.$$

There exist other $\mathfrak{g}_{(\infty)}$-singular vectors in $H_{\boldsymbol{\lambda}}$, which are not highest-weight vectors of one of the M_s's. In fact, the Bethe ansatz also generates these other singular vectors. We shall come back on this fact in Section 7.3.

7.2.4 Bethe ansatz at one excitation

Off-shell Bethe vectors. Let us illustrate the Bethe ansatz method by finding the simplest eigenvectors of the Hamiltonian $\mathscr{H}(z)$'s (after the vacuum state), the so-called Bethe vectors with one excitation. Let w be a complex number and $i \in \{1, \cdots, \ell\}$. We define the so-called off-shell Bethe vector

$$\Psi_i(w) = F_i(w).v_{\boldsymbol{\lambda}} = \sum_{k=1}^{N} \frac{F_{i\,(k)}}{w - z_k}.v_{\boldsymbol{\lambda}}.$$

It is clear that $\Psi_i(w)$ belongs to $W_{\lambda_\infty - \alpha_i}$, *i.e.* is of weight $\lambda_\infty - \alpha_i$ under the $\mathfrak{h}_{(\infty)}$-weight decomposition of $H_{\boldsymbol{\lambda}}$. We say that $\Psi_i(w)$ is a Bethe vector with one excitation in the direction α_i.

Action of $\mathscr{H}_k$ on the off-shell Bethe vector. Recall from subsection 7.2.2 that the vacuum state $v_{\boldsymbol{\lambda}}$ is an eigenvector of $\mathscr{H}(z)$ with eigenvalue $\mathcal{E}_{\mathrm{vac}}(z)$. We then have

$$\mathscr{H}(z).\Psi_i(w) = F_i(w)\mathscr{H}(z).v_{\boldsymbol{\lambda}} + \big[\mathscr{H}(z), F_i(w)\big].v_{\boldsymbol{\lambda}} = \mathcal{E}_{\mathrm{vac}}(z)\Psi_i(w) + \big[\mathscr{H}(z), F_i(w)\big].v_{\boldsymbol{\lambda}}.$$

In particular, taking the residue at $z = z_k$, one has

$$\mathscr{H}_k.\Psi_i(w) = \mathcal{E}_{\mathrm{vac},k}\Psi_i(w) + \big[\mathscr{H}_k, F_i(w)\big].v_{\boldsymbol{\lambda}}, \tag{7.2.5}$$

where

$$\mathcal{E}_{\mathrm{vac},k} = \operatorname*{res}_{z=z_k} \mathcal{E}_{\mathrm{vac}}(z) = \sum_{j\neq k} \frac{(\lambda_k, \lambda_j)}{z_k - z_j}.$$

Using the expression (7.1.11) of $\mathscr{H}_k$, one gets

$$\big[\mathscr{H}_k, F_i(w)\big] = \kappa_{ab} \sum_{j\neq k} \frac{\big[I^a_{(k)}, F_i(w)\big] I^b_{(j)} + I^a_{(k)} \big[I^b_{(j)}, F_i(w)\big]}{z_k - z_j}.$$

Note that

$$\big[X_{(k)}, Y(w)\big] = \frac{[X,Y]_{(k)}}{w - z_k}, \quad \forall\, X, Y \in \mathfrak{g}.$$

Thus, one gets

$$\big[\mathscr{H}_k, F_i(w)\big] = \sum_{j\neq k} \left(\frac{[I_a, F_i]_{(k)} I^a_{(j)}}{(w - z_k)(z_k - z_j)} + \frac{[I_a, F_i]_{(j)} I^a_{(k)}}{(w - z_j)(z_k - z_j)} \right). \tag{7.2.6}$$

Let us now compute the action of this operator on the vacuum state $v_{\boldsymbol{\lambda}}$. We let I^a run through the basis $\{E_\alpha\}_{\alpha\in\Delta} \cup \{K_m\}_{m=1,\cdots,\ell}$. If $I^a = E_\alpha$ with $\alpha \in \Delta_+$, then $I^a_{(j)}.v_{\boldsymbol{\lambda}} = I^a_{(k)}.v_{\boldsymbol{\lambda}} = 0$ so this choice does not contribute to the action of (7.2.6) on $v_{\boldsymbol{\lambda}}$. Let now $I^a = F_\alpha$ with $\alpha \in \Delta_+$ of height h: we then have $I_a = E_\alpha$ so $[I_a, F_i]$ is of height $h-1$. Thus, if $h > 1$, we get $[I_a, F_i]_{(j)}.v_{\boldsymbol{\lambda}} = [I_a, F_i]_{(k)}.v_{\boldsymbol{\lambda}} = 0$. Therefore, the only contributions to the action of (7.2.6) on $v_{\boldsymbol{\lambda}}$ are $I^a = F_m$ and $I^a = K_m$ ($m \in \{1, \cdots, \ell\}$). We then get

$$\begin{aligned}
\big[\mathscr{H}_k, F_i(w)\big].v_{\boldsymbol{\lambda}} \;=\; & \sum_{j\neq k} \sum_{m=1}^{\ell} \left(\frac{[E_m, F_i]_{(k)} F_{m\,(j)}}{(w - z_k)(z_k - z_j)} + \frac{[E_m, F_i]_{(j)} F_{m\,(k)}}{(w - z_j)(z_k - z_j)} \right).v_{\boldsymbol{\lambda}} \\
& + \sum_{j\neq k} \sum_{m=1}^{\ell} \left(\frac{[K_m, F_i]_{(k)} K_{m\,(j)}}{(w - z_k)(z_k - z_j)} + \frac{[K_m, F_i]_{(j)} K_{m\,(k)}}{(w - z_j)(z_k - z_j)} \right).v_{\boldsymbol{\lambda}}.
\end{aligned}$$

Recall the canonical isomorphism $\zeta : \mathfrak{h}^* \to \mathfrak{h}$ induced by the Killing form (see Appendix A.2.2 and Subsection 7.2.1). We then have $[E_m, F_i] = \delta_{mi}\, \zeta(\alpha_i)$. Thus, we get

$$\begin{aligned}
\big[\mathscr{H}_k, F_i(w)\big].v_{\boldsymbol{\lambda}} \;=\; & \sum_{j\neq k} \left(\frac{\zeta(\alpha_i)_{(k)} F_{i\,(j)}}{(w - z_k)(z_k - z_j)} + \frac{\zeta(\alpha_i)_{(j)} F_{i\,(k)}}{(w - z_j)(z_k - z_j)} \right).v_{\boldsymbol{\lambda}} \\
& - \sum_{j\neq k} \sum_{m=1}^{\ell} \langle K_m, \alpha_i \rangle \left(\frac{F_{i\,(k)} K_{m\,(j)}}{(w - z_k)(z_k - z_j)} + \frac{F_{i\,(j)} K_{m\,(k)}}{(w - z_j)(z_k - z_j)} \right).v_{\boldsymbol{\lambda}} \\
= \; & \frac{\langle \zeta(\alpha_i), \lambda_k \rangle}{(w - z_k)} \sum_{j\neq k} \frac{F_{i\,(j)}}{z_k - z_j}.v_{\boldsymbol{\lambda}} + \left(\sum_{j\neq k} \frac{\langle \zeta(\alpha_i), \lambda_j \rangle}{(w - z_j)(z_k - z_j)} \right) F_{i\,(k)}.v_{\boldsymbol{\lambda}} \\
& - \sum_{m=1}^{\ell} \langle K_m, \alpha_i \rangle \left(\sum_{j\neq k} \frac{\langle K_m, \lambda_j \rangle}{(z_k - z_j)} \right) \frac{F_{i\,(k)}}{(w - z_k)}.v_{\boldsymbol{\lambda}} \\
& - \left(\sum_{m=1}^{\ell} \langle K_m, \alpha_i \rangle \langle K_m, \lambda_k \rangle \right) \left(\sum_{j\neq k} \frac{F_{i\,(j)}}{(w - z_j)(z_k - z_j)} \right).v_{\boldsymbol{\lambda}}
\end{aligned}$$

Recall the bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ induced by κ (see Appendix A.2.2 and Subsection 7.2.1). As K_m is an orthonormal basis of $\mathfrak{h}$, we get

$$\begin{aligned}\big[\mathscr{H}_k, F_i(w)\big].v_{\boldsymbol{\lambda}} &= \frac{(\alpha_i,\lambda_k)}{(w-z_k)}\sum_{j\neq k}\frac{F_{i\,(j)}}{z_k-z_j}.v_{\boldsymbol{\lambda}} + \left(\sum_{j\neq k}\frac{(\alpha_i,\lambda_j)}{(w-z_j)(z_k-z_j)}\right)F_{i\,(k)}.v_{\boldsymbol{\lambda}} \\ &\quad - \left(\sum_{j\neq k}\frac{(\alpha_i,\lambda_j)}{(z_k-z_j)}\right)\frac{F_{i\,(k)}}{(w-z_k)}.v_{\boldsymbol{\lambda}} - (\alpha_i,\lambda_k)\left(\sum_{j\neq k}\frac{F_{i\,(j)}}{(w-z_j)(z_k-z_j)}\right).v_{\boldsymbol{\lambda}} \\ &= (\alpha_i,\lambda_k)\sum_{j\neq k}\left(\frac{1}{(w-z_k)(z_k-z_j)} - \frac{1}{(w-z_j)(z_k-z_j)}\right)F_{i\,(j)}.v_{\boldsymbol{\lambda}} \\ &\quad + \left(\sum_{j\neq k}(\alpha_i,\lambda_j)\left(\frac{1}{(w-z_j)(z_k-z_j)} - \frac{1}{(w-z_k)(z_k-z_j)}\right)\right)F_{i\,(k)}.v_{\boldsymbol{\lambda}}.\end{aligned}$$

Using the circle lemma (2.3.5), one gets

$$\begin{aligned}\big[\mathscr{H}_k, F_i(w)\big].v_{\boldsymbol{\lambda}} &= \frac{(\alpha_i,\lambda_k)}{w-z_k}\left(\sum_{j\neq k}\frac{F_{i\,(j)}}{w-z_j}\right).v_{\boldsymbol{\lambda}} - \left(\sum_{j\neq k}\frac{(\alpha_i,\lambda_j)}{w-z_j}\right)\frac{F_{i\,(k)}}{w-z_k}.v_{\boldsymbol{\lambda}} \\ &= \frac{(\alpha_i,\lambda_k)}{w-z_k}\left(\sum_{j=1}^{N}\frac{F_{i\,(j)}}{w-z_j}\right).v_{\boldsymbol{\lambda}} - \left(\sum_{j=1}^{N}\frac{(\alpha_i,\lambda_j)}{w-z_j}\right)\frac{F_{i\,(k)}}{w-z_k}.v_{\boldsymbol{\lambda}} \\ &= \frac{(\alpha_i,\lambda_k)}{w-z_k}\Psi_i(w) - \big(\alpha_i,\lambda(w)\big)\frac{F_{i\,(k)}}{w-z_k}.v_{\boldsymbol{\lambda}}.\end{aligned}$$

Combining this equation with (7.2.5), we find

$$\mathscr{H}_k.\Psi_i(w) = \mathcal{E}_{i,k}(w)\Psi_i(w) - \big(\alpha_i,\lambda(w)\big)\frac{F_{i\,(k)}}{w-z_k}.v_{\boldsymbol{\lambda}}, \tag{7.2.7}$$

with

$$\mathcal{E}_{i,k}(w) = \mathcal{E}_{\mathrm{vac},k} + \frac{(\alpha_i,\lambda_k)}{w-z_k}.$$

Bethe equation and eigenvalue with excitation. It is clear from equation (7.2.7) that $\Psi_i(w)$ is an eigenvector of all $\mathscr{H}_k$'s if w satisfies the **Bethe equation**:

$$\big(\alpha_i,\lambda(w)\big) = \sum_{k=1}^{N}\frac{(\alpha_i,\lambda_k)}{w-z_k} = 0. \tag{7.2.8}$$

In this case, we say that the Bethe vector $\Psi_i(w)$ is on-shell and write it as $\Psi_i^{\mathrm{on}}(w)$. The eigenvalue of $\mathscr{H}_k$ on $\Psi_i^{\mathrm{on}}(w)$ is then $\mathcal{E}_{i,k}^{\mathrm{on}}(w)$, defined as the value of $\mathcal{E}_{i,k}(w)$ for w solution of the Bethe equation. Such a solution is called a **Bethe root**.

Recall the partial fraction decomposition (7.1.10) of $\mathscr{H}(z)$. Recall also that the Casimirs $\Delta_{(i)}$ at the double poles of $\mathscr{H}(z)$ act as multiples of the identity on the whole Hilbert space $H_{\boldsymbol{\lambda}}$. We then get that the action of $\mathscr{H}(z)$ on $\Psi_i^{\mathrm{on}}(w)$ is

$$\mathscr{H}(z).\Psi_i^{\mathrm{on}}(w) = \mathcal{E}_i^{\mathrm{on}}(z,w)\Psi_i^{\mathrm{on}}(w),$$

with

$$\mathcal{E}_i^{\mathrm{on}}(z,w) = \mathcal{E}_{\mathrm{vac}}(z) + \sum_{k=1}^{N}\frac{\mathcal{E}_{i,k}^{\mathrm{on}}(w)}{z-z_k}.$$

Let us define

$$\lambda^i(z,w) = \sum_{i=1}^{N} \frac{\lambda_k}{z-z_k} - \frac{\alpha_i}{z-w} = \lambda(z) - \frac{\alpha_i}{z-w}.$$

One finds

$$\mathcal{E}_{\rm vac}(z) + \sum_{k=1}^{N} \frac{\mathcal{E}_{i,k}(w)}{z-z_k} = \frac{1}{2}\big(\lambda^i(z,w),\lambda^i(z,w)\big) - \frac{\partial}{\partial z}\big(\lambda^i(z,w),\rho\big) + \frac{\big(\alpha_i,\lambda(w)\big)}{z-w} - \frac{(\alpha_i,\alpha_i-2\rho)}{2(z-w)^2}.$$

It is a standard result that for a simple root α_i, the quantity $(\alpha_i,\alpha_i-2\rho)$ vanishes, so that the last term of the equation above disappears. Moreover, the third term cancels on-shell. Thus, we find

$$\mathcal{E}_i^{\rm on}(z,w) = \frac{1}{2}\big(\lambda^i(z,w),\lambda^i(z,w)\big) - \frac{\partial}{\partial z}\big(\lambda^i(z,w),\rho\big).$$

$\mathfrak{g}_{(\infty)}$-singularity of the on-shell Bethe vector. Recall the discussion of Subsection 7.2.3 about the diagonal action of $\mathfrak{g}_{(\infty)}$. For any $j\in\{1,\cdots,\ell\}$, using the commutation relation (7.1.13), we get

$$E_{j\,(\infty)}.\Psi_i(w) = \big[E_{j\,(\infty)},F_i(w)\big].v_{\boldsymbol{\lambda}} = [E_j,F_i](w).v_{\boldsymbol{\lambda}} = \delta_{ij}\,\zeta\big(\alpha_i(w)\big).v_{\boldsymbol{\lambda}} = \delta_{ij}\big(\alpha_i,\lambda(w)\big)v_{\boldsymbol{\lambda}}.$$

Given the form (7.2.8) of the Bethe equation, one sees that the on-shell Bethe vector is $\mathfrak{g}_{(\infty)}$-singular:

$$E_{j\,(\infty)}.\Psi_i^{\rm on}(w) = 0, \quad \forall\, j\in\{1,\cdots,\ell\}.$$

The on-shell Bethe vector $\Psi_i^{\rm on}(w)$ is the highest-weight vector B_s of one of the Verma module M_s in the decomposition (7.2.4). All vectors of M_s are then eigenvectors of $\mathscr{H}(z)$ with eigenvalue $\mathcal{E}_i^{\rm on}(z,w)$. The highest weight of M_s is

$$\mu_s = \lambda_\infty - \alpha_i = \sum_{k=1}^{N}\lambda_k - \alpha_i = -\operatorname*{res}_{z=\infty}\lambda^i(z,w)\mathrm{d}z.$$

Eigenvectors at one excitation. Let us describe all eigenvectors of $\mathscr{H}(z)$ with one excitation in the direction α_i. For that we need to find a basis of eigenvectors of the weight space $W_{\lambda_\infty-\alpha_i}$. As explained in Subsection 7.2.3, this weight space is of dimension N.

After a multiplication by $\prod_{k=1}^{N}(w-z_k)$, the Bethe equation (7.2.8) becomes a polynomial equation of degree $N-1$ on the Bethe root w. Thus, for generic values of the z_i's, there exist $N-1$ distinct solutions to this equation. One then finds $N-1$ linearly independent eigenvectors of $\mathscr{H}(z)$ as the corresponding on-shell Bethe vectors.

The last eigenvector in $W_{\lambda_\infty-\alpha_i}$ is not a Bethe vector, as it is not a singular vector under the diagonal action. It belongs to the $\mathfrak{g}_{(\infty)}$-module M_{s_∞} and is obtained from the vacuum state as

$$F_{i\,(\infty)}.v_{\boldsymbol{\lambda}}.$$

As a vector in M_{s_∞}, it is an eigenvector of $\mathscr{H}(z)$ of eigenvalue $\mathcal{E}_{\rm vac}(z)$.

7.2.5 Higher excited Bethe vectors

Let us end this section by saying a few words about the Bethe ansatz with M excitations, for $M>1$. Let us consider M complex numbers $\boldsymbol{w}=(w_1,\cdots,w_M)$, which will be the Bethe roots of the Bethe vector. We associate with each of these Bethe roots w_j $(j=1,\cdots,M)$ a "color" $c(j)\in\{1,\cdots,\ell\}$. The off-shell Bethe vector

$$\Psi_{\boldsymbol{c}}(\boldsymbol{w}) = \Psi_{c(1),\cdots,c(M)}(w_1,\cdots,w_M)$$

is then a function of the numbers w_j $(j=1,\cdots,M)$. More precisely, it is obtained from the vacuum state by actions of products of the lowering operators $F_{c(1)},\cdots,F_{c(m)}$, distributed on the N tensor

factors of $H_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$, with weightings depending rationally on the Bethe roots w_j's ($j = 1, \cdots, M$) and the sites z_k's ($k = 1, \cdots, N$). In particular, we have

$$\Psi_{\boldsymbol{c}}(\boldsymbol{w}) \in W_\mu, \qquad \text{with} \qquad \mu = \sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{M} \alpha_{c(j)}.$$

We refer to [93, 92] for the precise construction of $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$.

One then finds that $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ is an eigenvector of the quadratic Hamiltonian $\mathscr{H}(z)$ if the Bethe roots satisfy the M Bethe equations:

$$\sum_{i=1}^{N} \frac{(\alpha_{c(k)}, \lambda_i)}{w_k - z_i} - \sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{(\alpha_{c(k)}, \alpha_{c(j)})}{w_k - w_j} = 0, \tag{7.2.9}$$

for $k \in \{1, \cdots, M\}$. In this case, we get the on-shell Bethe vectors $\Psi^{\text{on}}_{\boldsymbol{c}}(\boldsymbol{w})$, which then satisfy

$$\mathscr{H}(z).\Psi^{\text{on}}_{\boldsymbol{c}}(\boldsymbol{w}) = \mathcal{E}^{\text{on}}_{\boldsymbol{c}}(z, \boldsymbol{w})\Psi^{\text{on}}_{\boldsymbol{c}}(\boldsymbol{w}).$$

To express the eigenvalue $\mathcal{E}^{\text{on}}_{\boldsymbol{c}}(z, \boldsymbol{w})$, let us introduce the z-dependent weight

$$\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) = \sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} - \sum_{j=1}^{M} \frac{\alpha_{c(j)}}{z - w_j}. \tag{7.2.10}$$

We then have

$$\mathcal{E}^{\text{on}}_{\boldsymbol{c}}(z, \boldsymbol{w}) = \frac{1}{2}\big(\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}), \lambda^{\boldsymbol{c}}(z, \boldsymbol{w})\big) - \frac{\partial}{\partial z}\big(\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}), \rho\big). \tag{7.2.11}$$

Note that the weight $\mu \in \mathfrak{h}^*$ such that $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ belongs to W_μ can be expressed as

$$\mu = -\operatorname*{res}_{\lambda=\infty} \lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) \mathrm{d}z.$$

Note also that if we write

$$\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) = \overline{\lambda}^{\boldsymbol{c}}_k(z, \boldsymbol{w}) - \frac{\alpha_{c(k)}}{z - w_k},$$

then $\overline{\lambda}^{\boldsymbol{c}}_k(z, \boldsymbol{w})$ is regular at $z = z_k$ and the Bethe equation (7.2.9) reads

$$\Big(\alpha_{c(k)}, \overline{\lambda}^{\boldsymbol{c}}_k(z_k, \boldsymbol{w})\Big) = 0$$

Finally, let us mention that because of the Bethe equations, the on-shell Bethe vector $\Psi^{\text{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ is $\mathfrak{g}_{(\infty)}$-singular.

7.3 The Feigin-Frenkel-Reshetikhin approach

This section is devoted to the Feigin-Frenkel-Reshetikhin (FFR) approach of Gaudin models. It consists of a description of the spectrum of Gaudin models in terms of differential operators called opers. We will give an introductory review about this approach and refer to [95, 102–108] for more details. We will end the section with some new results concerning the generalisation of this approach for cyclotomic Gaudin models. Before explaining the general theory, let us gain some intuition about it by considering the simplest Gaudin model, on the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

7.3.1 The $\mathfrak{sl}(2,\mathbb{C})$ case

Bethe ansatz for $\mathfrak{sl}(2,\mathbb{C})$. In this subsection, we consider a Gaudin model on the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, with sites $\boldsymbol{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(2,\mathbb{C})$ is one dimensional, hence $\mathfrak{sl}(2,\mathbb{C})$ is of rank $\ell = 1$. In particular, $\mathfrak{sl}(2,\mathbb{C})$ admits only one exponent, equal to 1. Thus, the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ is generated by the coefficients of the quadratic Hamiltonian $\mathscr{H}(z)$.

As $\mathfrak{h}^*$ is one dimensional, any weight $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = S\alpha$, with $S \in \mathbb{C}$ and $\alpha \in \mathfrak{h}^*$ the unique positive root of $\mathfrak{sl}(2,\mathbb{C})$. In particular, the Weyl vector is equal to $\rho = \frac{1}{2}\alpha$. The bilinear form $(\cdot,\cdot)$ induced on $\mathfrak{h}^*$ by the Killing form is given by $(\alpha,\alpha) = 4$.

Let us consider N weights $\lambda_i = S_i\alpha$, with $S_i \in \mathbb{C}$ and $i \in \{1, \cdots, N\}$, and the associated Hilbert space $H_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$, as in the previous sections. As $H_{\boldsymbol{\lambda}}$ is a tensor product of highest-weight vectors, we can study the spectrum of $\mathscr{H}(z)$ on $H_{\boldsymbol{\lambda}}$ by the Bethe ansatz, as described in Section 7.2. Let us then consider a Bethe vector with M excitations and Bethe roots $\boldsymbol{w} = (w_1, \cdots, w_M)$. As $\mathfrak{sl}(2,\mathbb{C})$ is of rank 1, there is only one possible color, *i.e.* each Bethe root is associated with an excitation in the direction α. The Bethe equations are given by equation (7.2.9) and read in this particualar case

$$B_k = \sum_{i=1}^{N} \frac{S_i}{w_k - z_i} - \sum_{\substack{j=1 \\ j\neq k}}^{M} \frac{1}{w_k - w_j} = 0, \quad \forall\, k \in \{1, \cdots, M\}. \tag{7.3.1}$$

The weight (7.2.10) associated with the Bethe vector is then

$$\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) = S(z, \boldsymbol{w})\alpha \qquad \text{with} \qquad S(z, \boldsymbol{w}) = \sum_{i=1}^{N} \frac{S_i}{z - z_i} - \sum_{j=1}^{M} \frac{1}{z - w_j}. \tag{7.3.2}$$

The eigenvalue of the on-shell Bethe vector with Bethe roots $\boldsymbol{w}$ is then given by (7.2.11)

$$\mathcal{E}_{\boldsymbol{c}}^{\text{on}}(z, \boldsymbol{w}) = 2\big(S(z, \boldsymbol{w})^2 - S'(z, \boldsymbol{w})\big), \tag{7.3.3}$$

where $S'(z, \boldsymbol{w}) = \dfrac{\partial S(z, \boldsymbol{w})}{\partial z}$.

$\mathfrak{sl}(2,\mathbb{C})$-opers. The goal of this subsection is to show a reformulation of the Bethe ansatz described above in terms of objects called $\mathfrak{sl}(2,\mathbb{C})$-opers. Let us start by defining these objects. We will be interested in the space $\text{Conn}_{\mathfrak{sl}(2,\mathbb{C})}\big(\mathbb{P}^1\big)$ of meromorphic $\mathfrak{sl}(2,\mathbb{C})$-connections on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. These are differential operators of the form

$$a\partial_z + A(z),$$

where a is a complex number and A is a $\mathfrak{sl}(2,\mathbb{C})$-valued meromorphic function of z. We will use the explicit representation of elements of $\mathfrak{sl}(2,\mathbb{C})$ as two by two traceless matrices. If $g(z)$ is a meromorphic function valued in the associated Lie group $SL(2,\mathbb{C})$ (composed of two by two matrices of determinant 1), we can consider the formal gauge transformation of the connection $a\partial_z + A(z)$ by g:

$$g(z)\big(a\partial_z + A(z)\big)g(z)^{-1} = a\partial_z + g(z)A(z)g(z)^{-1} - a\big(\partial_z g(z)\big)g(z)^{-1}.$$

Let us now consider a particular type of $\mathfrak{sl}(2,\mathbb{C})$-connections:

$$\text{op}_{\mathfrak{sl}(2,\mathbb{C})}\big(\mathbb{P}^1\big) = \left\{ \nabla = \partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix} \in \text{Conn}_{\mathfrak{sl}(2,\mathbb{C})}\big(\mathbb{P}^1\big) \right\}. \tag{7.3.4}$$

We also define a group

$$N = \left\{ g(z) = \begin{pmatrix} 1 & f(z) \\ 0 & 1 \end{pmatrix},\ f \text{ meromorphic} \right\} \tag{7.3.5}$$

of meromorphic functions valued in the group $SL(2,\mathbb{C})$. It is a straightforward computation that for ∇ and g of the form above, the gauge transformation of ∇ by g is given by

$$g\nabla g^{-1} = \partial_z + \begin{pmatrix} a(z)+f(z) & b(z)-f(z)^2-2f(z)a(z)-f'(z) \\ 1 & -a(z)-f(z), \end{pmatrix}. \tag{7.3.6}$$

In particular, the gauge transformation by elements of N stabilises the space $\mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$. This defines a group action of N on $\mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$. We can thus define the quotient

$$\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1) = \mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)/N. \tag{7.3.7}$$

An element of this quotient is called a **$\mathfrak{sl}(2,\mathbb{C})$-oper**. If ∇ is an element of $\mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$, we denote by $[\nabla] \in \mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ the corresponding oper, *i.e.* the equivalence class of ∇ under the gauge transformations in N. The connection ∇ is then a representative of the oper $[\nabla]$.

It is clear from equation (7.3.6) that there is a unique element in the equivalence class $[\nabla]$ with no diagonal coefficients: it is obtained by taking $f = -a$. Such an element is called a **canonical representative** of the oper $[\nabla]$. Through this representative, the space $\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ is thus parametrised by a unique meromorphic function c of z:

$$\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1) \simeq \left\{ \partial_z + \begin{pmatrix} 0 & c(z) \\ 1 & 0 \end{pmatrix} \in \mathrm{Conn}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1) \right\}. \tag{7.3.8}$$

Reformulation of the Bethe ansatz in terms of $\mathfrak{sl}(2,\mathbb{C})$-opers. Let us come back to the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model and the Bethe ansatz described above. We introduce the following $\mathfrak{sl}(2,\mathbb{C})$-connection:

$$\nabla_{\boldsymbol{w}} = \partial_z + \begin{pmatrix} -S(z,\boldsymbol{w}) & 0 \\ 1 & S(z,\boldsymbol{w}) \end{pmatrix}, \tag{7.3.9}$$

with $S(z,\boldsymbol{w})$ as in (7.3.2). We will call a connection of this form (with any function of z instead of S) a **Miura $\mathfrak{sl}(2,\mathbb{C})$-oper**. In particular, it belongs to the space $\mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ and we can thus consider the associated oper $[\nabla_{\boldsymbol{w}}]$ in $\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$. The canonical representative of $[\nabla_{\boldsymbol{w}}]$ is given explicitly by

$$[\nabla_{\boldsymbol{w}}]_{\mathrm{can}} = \partial_z + \begin{pmatrix} 0 & S(z,\boldsymbol{w})^2 - S'(z,\boldsymbol{w}) \\ 1 & 0 \end{pmatrix}.$$

We recognize in the coefficient of this canonical representative the eigenvalue (7.3.3) of $\mathscr{H}(z)$ on the Bethe vector with Bethe roots $\boldsymbol{w}$ (up to a global factor).

Bethe equations. Let us consider the coefficient

$$\mathcal{C}(z,\boldsymbol{w}) = S(z,\boldsymbol{w})^2 - S'(z,\boldsymbol{w}),$$

as a function of z and $\boldsymbol{w}$, without requiring that the Bethe roots $\boldsymbol{w}$ are on-shell. In particular, as $S(z,\boldsymbol{w})$ has simple poles at $z = w_j$ ($j \in \{1,\cdots,N\}$), $\mathcal{C}(z,\boldsymbol{w})$ might also have poles at $z = w_j$. One finds that due to a cancellation between the term $S(z,\boldsymbol{w})^2$ and the term $S'(z,\boldsymbol{w})$, the double pole of $\mathcal{C}(z,\boldsymbol{w})$ at $z = w_j$ vanishes. Moreover, an explicit computation shows that

$$\operatorname*{res}_{z=w_j} \mathcal{C}(z,\boldsymbol{w})\mathrm{d}z = -2B_j,$$

with B_j defined in equation (7.3.1). Thus, if the Bethe root w_j is on-shell, this residue vanishes and the coefficient $\mathcal{C}(z,\boldsymbol{w})$ is then regular at $z = w_j$.

Summary. As a conclusion of this subsection, let us summarise what we observed. We considered a Bethe vector $\Psi(\boldsymbol{w})$ for the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model and observed the following.

1. From the weight (7.2.10) associated with $\Psi(\boldsymbol{w})$, we constructed a Miura $\mathfrak{sl}(2,\mathbb{C})$-oper $\nabla_{\boldsymbol{w}}$.
2. We constructed the canonical representative $[\nabla_{\boldsymbol{w}}]_{\text{can}}$ of the corresponding oper $[\nabla_{\boldsymbol{w}}]$.
3. Although the Miura oper $\nabla_{\boldsymbol{w}}$ has simple poles at z equal to a Bethe root, the canonical representative $[\nabla_{\boldsymbol{w}}]_{\text{can}}$ is regular at this Bethe root if and only if this root satisfies the Bethe equation.
4. When all Bethe equations are satisfied, the coefficient of the canonical representative $[\nabla_{\boldsymbol{w}}]_{\text{can}}$ coincides (up to a factor) with the eigenvalue of $\mathscr{H}(z)$ on the Bethe vector.

These results are in fact part of the FFR approach for the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model, as we shall see in the rest of this section.

7.3.2 The FFR approach and the spectrum of the Gaudin model

Let us now state and motivate the main ideas of the FFR approach for a general $\mathfrak{g}$-Gaudin model, before going into more details.

The main theorem. The main result of the FFR approach can be stated as follows [105]:

Theorem 7.3.1. *There exists an algebraic variety* $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ *of so-called* ${}^L\mathfrak{g}$*-opers such that the Gaudin subalgebra* $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ *is isomorphic to the algebra of polynomial functions on* $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$*. We will write*

$$\Phi_{\boldsymbol{z},\mathfrak{g}} : \mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g}) \longrightarrow \mathrm{Fun}\left(\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)\right)$$

this isomorphism.

The definition of the space of opers $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ and of the map $\Phi_{\boldsymbol{z},\mathfrak{g}}$ will be explained later in this section, as it requires the introduction of a new mathematical formalism. For now, let us try to understand the consequence of this theorem for the study of quantum Gaudin models.

If $\mathcal{Z}$ is a commutative algebra over $\mathbb{C}$, we define the space $\Xi(\mathcal{Z})$ of algebra homomorphisms from $\mathcal{Z}$ to $\mathbb{C}$ and we call the elements of $\Xi(\mathcal{Z})$ the **characters** of $\mathcal{Z}$. Let V be an algebraic variety and $\mathrm{Fun}\,(V)$ the algebra of polynomial functions on V. For x a point of V, we define the evaluation at x as

$$\begin{array}{rccc} \mathrm{ev}_x : & \mathrm{Fun}\,(V) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(x) \end{array}.$$

It is clear that ev_x is an algebra homomorphism from $\mathrm{Fun}\,(V)$ to $\mathbb{C}$ and thus is a character of $\mathrm{Fun}\,(V)$. It is a fundamental result of algebraic geometry [177] that

$$\begin{array}{rccc} \mathrm{ev} : & V & \longrightarrow & \Xi\big(\mathrm{Fun}\,(V)\big) \\ & x & \longmapsto & \mathrm{ev}_x \end{array}$$

is a bijection. We then get the following consequence of Theorem 7.3.1:

Corollary 7.3.2. *The map*

$$\begin{array}{rccc} \eta_{\boldsymbol{z},\mathfrak{g}} : & \mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1) & \longrightarrow & \Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big) \\ & [\nabla] & \longmapsto & \mathrm{ev}_{[\nabla]} \circ \Phi_{\boldsymbol{z},\mathfrak{g}} \end{array}$$

is a bijection.

Proof. Theorem 7.3.1 states that the commutative $\mathbb{C}$-algebras $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ and $\mathrm{Fun}\left(\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)\right)$ are isomorphic (through $\Phi_{\boldsymbol{z},\mathfrak{g}}$). Thus the spaces of characters of these algebras should be isomorphic. Concretely, this isomorphism is given by the precomposition by $\Phi_{\boldsymbol{z},\mathfrak{g}}$:

$$\begin{array}{rccc} \widetilde{\eta}_{\boldsymbol{z},\mathfrak{g}} : & \Xi\left(\mathrm{Fun}\left(\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)\right)\right) & \longrightarrow & \Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big) \\ & \chi & \longmapsto & \chi \circ \Phi_{\boldsymbol{z},\mathfrak{g}} \end{array}.$$

We then get the bijection $\eta_{\boldsymbol{z},\mathfrak{g}}$ from $\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$ to $\Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big)$ by composing $\widetilde{\eta}_{\boldsymbol{z},\mathfrak{g}}$ on the right with the bijection $\mathrm{ev} : \mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1) \to \Xi\Big(\mathrm{Fun}\Big(\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)\Big)\Big)$. □

The main Theorem 7.3.1 of the FFR approach then allows to realise all characters of the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ as an oper in $\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$. Let us now explain why such characters are useful for the study of quantum integrable models.

Characters and quantum integrable models. Let us consider a quantum model, with an algebra of operators $\mathcal{A}$ and Hamiltonian $\mathcal{H}$. We suppose that this model is integrable, in the sense that it possesses a large number of conserved commuting charges. Let us then consider the subalgebra $\mathcal{Z}$ of $\mathcal{A}$ generated by these conserved charges (in particular, it contains the Hamiltonian $\mathcal{H}$ of the system). As the charges are commuting, the algebra $\mathcal{Z}$ is a commutative subalgebra of $\mathcal{A}$ (in the case of the Gaudin model, this is the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$).

Let us now consider the Hilbert space H of the model, on which $\mathcal{A}$ acts linearly. A first step towards the resolution of the model is to describe the spectrum of the Hamiltonian $\mathcal{H}$ on H, or even better, the joint spectrum of the commuting charges in $\mathcal{Z}$. Indeed, as these charges commute, they can be simultaneously diagonalised. Let then $v \in H$ be a common eigenvector of the operators in $\mathcal{Z}$. For $\mathcal{Q}$ in $\mathcal{Z}$, we denote by $\chi_v(\mathcal{Q}) \in \mathbb{C}$ the eigenvalue of $\mathcal{Q}$ on v, so that we have

$$\mathcal{Q}.v = \chi_v(\mathcal{Q})v.$$

If $\mathcal{Q}$ and $\mathcal{Q}'$ are two operators in $\mathcal{Z}$, we have

$$\chi_v(\mathcal{Q}\mathcal{Q}')v = (\mathcal{Q}\mathcal{Q}').v = \mathcal{Q}.\big(\mathcal{Q}'.v\big) = \chi_v(\mathcal{Q}')\,\mathcal{Q}.v = \chi_v(\mathcal{Q}')\chi_v(\mathcal{Q})v.$$

Thus, we have

$$\chi_v(\mathcal{Q}\mathcal{Q}') = \chi_v(\mathcal{Q})\chi_v(\mathcal{Q}'), \quad \forall\, \mathcal{Q}, \mathcal{Q}' \in \mathcal{Z},$$

i.e. $\boldsymbol{\chi_v}$ **is a character of** $\boldsymbol{\mathcal{Z}}$.

Therefore, if one knows all characters of $\mathcal{Z}$, one knows all possible eigenvalues of the charges $\mathcal{Q}$, *i.e.* the spectrum of the considered integrable model. The description of $\Xi(\mathcal{Z})$ is thus a first step towards the resolution of the model.

Spectrum of the Gaudin model. Let us come back to the particular case of the Gaudin model. The characters $\Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big)$ of the Gaudin subalgebra are realised in terms of opers in $\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$. Thus, one can describe the spectrum of the Gaudin model (*i.e.* the eigenvalues of the operators in $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$) in terms of opers. More precisely, we get

Theorem 7.3.3. *Let H be a Hilbert space of the Gaudin model, i.e. a linear representation of $\mathcal{A}$. Let $v \in H$ be a common eigenvector of the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$. Then, there exists an oper $[\nabla_v] \in \mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$ such that for any $\mathcal{Q} \in \mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$, the eigenvalue of $\mathcal{Q}$ on v is given by*

$$\big(\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})\big)\big([\nabla_v]\big). \tag{7.3.10}$$

Remark: Let us explain exactly what we mean by equation (7.3.10). Recall from Theorem 7.3.1 that $\Phi_{\boldsymbol{z},\mathfrak{g}}$ is a bijection from $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ to the algebra of functions on $\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$. Thus, $\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})$ is such a function and we get a complex number $\big(\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})\big)\big([\nabla_v]\big)$ when evaluating it on the oper $[\nabla_v] \in \mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$.

Proof. We use the notations of the previous paragraph and denote by $\chi_v(\mathcal{Q})$ the eigenvalue of $\mathcal{Q}$ on v: χ_v is then a character in $\Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big)$. Let us define

$$[\nabla_v] = \eta^{-1}_{\boldsymbol{z},\mathfrak{g}}(\chi_v) \in \mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1), \tag{7.3.11}$$

where $\eta_{\boldsymbol{z},\mathfrak{g}} : \mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1) \to \Xi\big(\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})\big)$ is the bijection of Corollary 7.3.2. Then, by definition of this bijection, we have

$$\chi_v(\mathcal{Q}) = \big(\eta_{\boldsymbol{z},\mathfrak{g}}\big([\nabla_v]\big)\big)(\mathcal{Q}) = \mathrm{ev}_{[\nabla_v]}\big(\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})\big) = \big(\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})\big)\big([\nabla_v]\big).$$ □

Theorem 7.3.3 gives a theoretical description of the spectrum of the Gaudin model. In particular, note that in this theorem, we did not put any restriction on the Hilbert space H of the model. Thus, the FFR approach describes the spectrum of the Gaudin model even on Hilbert spaces which are not a tensor product of highest-weight representations, *i.e.* on Hilbert spaces where the Bethe ansatz does not apply. Moreover, as we mentioned in the introduction of this chapter, it can happen in some degenerate cases that the Bethe ansatz is not complete [174], *i.e.* that it does not result in a basis of eigenvectors. In this case, there is at least one eigenvector $v \in H$ whose eigenvalues are not described by the Bethe ansatz. However, by Theorem 7.3.3, one knows that these eigenvalues are described by some oper $[\nabla_v]$. These are some of the advantages of the FFR approach.

However, this approach also has downsides. The main one is the fact that it is mainly an existence result, and not a constructive one. Indeed, the Theorem 7.3.3 only proves the existence of the oper $[\nabla_v]$ associated with the eigenvector v, it does not explain how to construct it. In the proof of the theorem, we defined $[\nabla_v]$ through equation (7.3.11). However, this is not a constructive result: indeed, this equation supposed that we already knew the eigenvector v and the corresponding eigenvalues χ_v. Similarly, given an oper $[\nabla]$, one can construct an associated character χ but the theorem 7.3.3 does not explain how to find an eigenvector v whose eigenvalues would be described by χ.

These downsides can be overcome when one considers a Hilbert space which is a tensor product of highest-weight representations of $\mathfrak{g}$, *i.e.* when the Bethe ansatz applies. In this case, one can re-interpret the Bethe ansatz in terms of opers and find the oper associated with each Bethe vector. The FFR approach then contains the Bethe ansatz.

The rest of this Section is mainly devoted to the description of the space of opers $\mathrm{Op}^{\mathrm{RS}}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$ and of the isomorphism $\Phi_{\boldsymbol{z},\mathfrak{g}}$ of Theorem 7.3.1, together with the reinterpretation of the Bethe ansatz in terms of these opers. Before that, let us come back to the example of the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model studied in Subsection 7.3.1 and illustrate the notions introduced above in this example.

Back to $\mathfrak{sl}(2,\mathbb{C})$. We will use the notations of Subsection 7.3.1. In particular, we introduced $\mathfrak{sl}(2,\mathbb{C})$-opers $\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ as gauge equivalence class of connections of the form

$$\nabla = \partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix}.$$

We will say that such a connection has regular singularities at $x \in \mathbb{C}$ if a possesses at most simple poles at $z = x$ and b at most double poles. We will also say that it is regular at x if a and b are regular at $z = x$.

For a $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model with sites $\boldsymbol{z} = (z_1, \cdots, z_n)$, we define the space of opers $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1)$ appearing in the FFR approach as opers in $\mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ which possess a representative that has regular singularities at the z_i's, is regular elsewhere and has no constant term. An oper $[\nabla] \in \mathrm{Op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ is in $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1)$ if and only if its canonical representative is of the form

$$[\nabla]_{\mathrm{can}} = \partial_z + \begin{pmatrix} 0 & c(z) \\ 1 & 0 \end{pmatrix} \qquad \text{with} \qquad c(z) = \sum_{i=1}^{N} \left(\frac{c_{i,0}}{2(z-z_i)} + \frac{c_{i,1}}{4(z-z_i)^2} \right). \tag{7.3.12}$$

We define $2N$ functions $\Gamma_{i,p}$ ($i = 1, \cdots, N$ and $p = 0, 1$) on $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1)$ by

$$\Gamma_{i,p} : [\nabla] \in \mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1) \to c_{i,p}.$$

As any oper in $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1)$ has a canonical representative of the form (7.3.12), it is clear that these functions generate all functions on $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}(\mathbb{P}^1)$.

For a Gaudin model on $\mathfrak{sl}(2,\mathbb{C})$, there is only the quadratic Hamiltonians $\mathscr{H}(z)$ and no higher-order ones (see Subsection 7.3.1). The Gaudin algebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{sl}(2,\mathbb{C}))$ of the model is then generated by the

Casimirs $\Delta_{(i)}$'s and the Hamiltonians $\mathscr{H}_i$'s appearing in the partial fraction decomposition (7.1.10) of $\mathscr{H}(z)$. In this case, the algebra isomorphism $\Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}$ of Theorem 7.3.1 is such that

$$\Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}\big(\mathscr{H}_i\big) = \Gamma_{i,0} \qquad \text{and} \qquad \Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}\big(\Delta_{(i)}\big) = \Gamma_{i,1} \tag{7.3.13}$$

and extends to the whole algebra $\mathcal{Z}_{\boldsymbol{z}}\big(\mathfrak{sl}(2,\mathbb{C})\big)$ by linearity and multiplication.

Let us now come back to the Bethe ansatz for this model. We consider $\Psi^{\mathrm{on}}(\boldsymbol{w})$ an on-shell Bethe vector, which is then an eigenvector of the Gaudin algebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{sl}(2,\mathbb{C}))$. Theorem 7.3.3 implies the theoretical existence of an oper $[\nabla_{\boldsymbol{w}}]$ in $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}\big(\mathbb{P}^1\big)$ encoding the eigenvalues of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{sl}(2,\mathbb{C}))$ on $\Psi^{\mathrm{on}}(\boldsymbol{w})$. Let us show that we can construct this oper explicitly.

We will use the notations of Subsection 7.3.1. Let us first consider the off-shell Bethe vector $\Psi(\boldsymbol{w})$. We define an associated Miura oper $\nabla_{\boldsymbol{w}}$ as in (7.3.9). As explained in Subsection 7.3.1, the canonical representative of the oper $[\nabla_{\boldsymbol{w}}]$ is

$$[\nabla_{\boldsymbol{w}}]_{\mathrm{can}} = \partial_z + \begin{pmatrix} 0 & \mathcal{C}(z,\boldsymbol{w}) \\ 1 & 0 \end{pmatrix}, \qquad \text{with} \qquad \mathcal{C}(z,\boldsymbol{w}) = S(z,\boldsymbol{w})^2 - S'(z,\boldsymbol{w}).$$

In particular, $\mathcal{C}(z,\boldsymbol{w})$ has simple and double poles at all $z = z_i$'s. Moreover, as remarked in Subsection 7.3.1, it is regular at $z = w_j$ if and only if the Bethe root w_j satisfies the Bethe equation (7.3.1). Thus, we have

$$[\nabla_{\boldsymbol{w}}] \in \mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}\big(\mathbb{P}^1\big) \iff \boldsymbol{w} \text{ satisfy the Bethe equations} \iff \Psi(\boldsymbol{w}) \text{ is on-shell.}$$

If this is the case, as explained in Subsection 7.3.1, the on-shell Bethe vector $\Psi^{\mathrm{on}}(w)$ is an eigenvector of $\mathscr{H}(z)$ with eigenvalue $2\mathcal{C}(z,\boldsymbol{w})$. In particular, given the expression (7.1.10) of $\mathscr{H}(z)$, the eigenvalues of $\mathscr{H}_i$ and $\Delta_{(i)}$ are

$$\chi_{\boldsymbol{w}}(\mathscr{H}_i) = 2 \operatorname*{res}_{z=z_i} \mathcal{C}(z,\boldsymbol{w}) \qquad \text{and} \qquad \chi_{\boldsymbol{w}}(\Delta_{(i)}) = 4 \operatorname*{res}_{z=z_i} (z-z_i)\mathcal{C}(z,\boldsymbol{w}).$$

Recall the functions $\Gamma_{i,p}$ on $\mathrm{Op}^{\mathrm{RS}}_{\mathfrak{sl}(2,\mathbb{C}),\boldsymbol{z}}\big(\mathbb{P}^1\big)$ defined above. As $\mathcal{C}(z,\boldsymbol{w})$ is the coefficient of the canonical representative $[\nabla_{\boldsymbol{w}}]_{\mathrm{can}}$, the equation above can be rewritten

$$\chi_{\boldsymbol{w}}(\mathscr{H}_i) = \Gamma_{i,0}\big([\nabla_{\boldsymbol{w}}]\big) \qquad \text{and} \qquad \chi_{\boldsymbol{w}}(\Delta_{(i)}) = \Gamma_{i,1}\big([\nabla_{\boldsymbol{w}}]\big).$$

Given the definition (7.3.13) of the FFR isomorphism $\Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}$, we find that for all $\mathcal{Q} \in \mathcal{Z}_{\boldsymbol{z}}\big(\mathfrak{sl}(2,\mathbb{C})\big)$, the eigenvalue of $\mathcal{Q}$ on $\Psi^{\mathrm{on}}(\boldsymbol{w})$ is

$$\chi_{\boldsymbol{w}}(\mathcal{Q}) = \big(\Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}(\mathcal{Q})\big)\big([\nabla_{\boldsymbol{w}}]\big).$$

This illustrates Theorem 7.3.3 about the spectrum of the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model. In particular, it shows that for the eigenvector $\Psi^{\mathrm{on}}(\boldsymbol{w})$ constructed from the Bethe ansatz, one can give an explicit construction of the associated oper $[\nabla_{\boldsymbol{w}}]$ (through a Miura oper).

7.3.3 Principal $\mathfrak{sl}(2,\mathbb{C})$ subalgebra and exponents

In this subsection, we introduce the notions of principal gradation, of principal $\mathfrak{sl}(2,\mathbb{C})$ subalgebra and of exponents of a semi-simple Lie algebra $\mathfrak{g}$, which are necessary for the construction of opers. Most of these notions were introduced by Kostant in [178–180]. We refer to these articles for details and proofs and mention just the results we will need in this section.

Principal gradation. Let us consider the root system Δ of $\mathfrak{g}$. A root α can be written in the basis of simple roots $\{\alpha_i\}_{i=1,\cdots,\ell} \subset \mathfrak{h}^*$:

$$\alpha = \sum_{i=1}^{\ell} m_i \alpha_i.$$

The numbers m_i's are either all non-negative integers or all non-positive integers (we then speak of a positive or negative root). We define the height of α as the integer

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{\ell} m_i.$$

Recall the Cartan-Weyl decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha \right).$$

We define the **principal gradation** of $\mathfrak{g}$ as

$$\mathfrak{g}_0 = \mathfrak{h} \qquad \text{and} \qquad \mathfrak{g}_d = \bigoplus_{\substack{\alpha \in \Delta \\ \mathrm{ht}(\alpha) = d}} \mathbb{C} E_\alpha, \quad \text{for } d \in \mathbb{Z} \setminus \{0\}.$$

If $X \in \mathfrak{g}_d$, we say that X is of degree d. This defines a $\mathbb{Z}$-gradation of $\mathfrak{g}$, *i.e.*

$$\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{g}_d, \qquad \text{and} \qquad \big[\mathfrak{g}_p, \mathfrak{g}_q\big] \subset \mathfrak{g}_{p+q}. \tag{7.3.14}$$

Note that the positive nilpotent subalgebra $\mathfrak{n}_+$ and Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ are given by

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathbb{C} E_\alpha = \bigoplus_{d \in \mathbb{Z}_{\geq 1}} \mathfrak{g}_d \qquad \text{and} \qquad \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+ = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_d. \tag{7.3.15}$$

The roots of $\mathfrak{g}$ have a maximal (resp. minimal) height $h-1$ (resp. $-h+1$), where h is the so-called Coxeter number of $\mathfrak{g}$. Thus, we have

$$\mathfrak{g}_{\pm d} = 0 \qquad \text{for} \qquad d \geq h.$$

Principal $\mathfrak{sl}(2,\mathbb{C})$ subalgebra. Let us consider the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and its basis of coweights $\{\check{\omega}_i\}_{i=1,\cdots,\ell}$ (see appendix A.2.2). This is the dual basis of the simple roots $\{\alpha_i\}_{i=1,\cdots,\ell} \subset \mathfrak{h}^*$:

$$\langle \check{\omega}_i, \alpha_j \rangle = \delta_{ij}.$$

Let us then introduce the Weyl co-vector

$$\check{\rho} = \sum_{i=1}^{\ell} \check{\omega}_i.$$

By definition, it satisfies

$$\langle \check{\rho}, \alpha \rangle = \mathrm{ht}(\alpha), \quad \forall\, \alpha \in \Delta.$$

One can interpret the principal gradation (7.3.14) as a decomposition of $\mathfrak{g}$ in eigenspaces of $\mathrm{ad}_{\check{\rho}}$:

$$\mathfrak{g}_d = \big\{ X \in \mathfrak{g} \;\big|\; [\check{\rho}, X] = d\, X \big\}, \qquad \forall\, d \in \mathbb{Z}.$$

Recall the isomorphism $\zeta : \mathfrak{h}^* \to \mathfrak{h}$ induced by the Killing form. We define numbers b_i's ($i = 1, \cdots, \ell$) by the decomposition of the Weyl co-weight in the basis $\{\zeta(\alpha_i)\}_{i=1,\cdots,\ell}$ of $\mathfrak{h}$:

$$\check{\rho} = \sum_{i=1}^{\ell} b_i \zeta(\alpha_i).$$

Let $E_i = E_{\alpha_i}$'s and $F_i = E_{-\alpha_i}$'s ($i \in \{1, \cdots, \ell\}$) be the Chevalley generators of $\mathfrak{g}$ (see Appendix A.2). We introduce the so-called negative and positive principal nilpotent elements as:

$$p_{-1} = \sum_{i=1}^{\ell} F_i \in \mathfrak{g}_{-1} \qquad \text{and} \qquad p_1 = \sum_{i=1}^{\ell} b_i E_i \in \mathfrak{g}_1.$$

The three elements $\{\check{\rho}, p_1, p_{-1}\}$ of $\mathfrak{g}$ satisfy the commutation relations:

$$\big[\check{\rho}, p_{\pm 1}\big] = \pm\, p_{\pm 1} \qquad \text{and} \qquad \big[p_1, p_{-1}\big] = \check{\rho}.$$

The subalgebra $\mathrm{Span}(\check{\rho}, p_1, p_{-1})$ of $\mathfrak{g}$ is thus isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We call it the **principal $\mathfrak{sl}(2, \mathbb{C})$ subalgebra** of $\mathfrak{g}$.

Exponents of $\mathfrak{g}$. Let us consider the centraliser of p_1:

$$\mathfrak{a} = \mathrm{Ker}(\mathrm{ad}_{p_1}) = \big\{ X \in \mathfrak{g} \;\big|\; [p_1, X] = 0 \big\}.$$

As shown in [178], one has $\dim \mathfrak{a} = \ell$, where ℓ is the rank of $\mathfrak{g}$. Moreover, for any $X \in \mathfrak{a}$, using the Jacoby identity, we have

$$\big[p_1, [\check{\rho}, X]\big] = \big[[p_1, \check{\rho}], X\big] + \big[\check{\rho}, [p_1, X]\big] = -[p_1, X] = 0.$$

Thus, $\mathrm{ad}_{\check{\rho}}$ stabilises the centraliser $\mathfrak{a}$. In particular, this implies that one can find a basis $\{q_1, \cdots, q_\ell\}$ of $\mathfrak{a}$ which is composed of eigenvectors of $\mathrm{ad}_{\check{\rho}}$. Note that we can choose the first element of this basis to be $q_1 = p_1$ itself, as it belongs to $\mathfrak{a}$. Thus, there exist numbers $d_i \in \mathbb{Z}$ such that

$$q_i \in \mathfrak{a} \cap \mathfrak{g}_{d_i}.$$

The first d_1 is the degree of p_1, hence $d_1 = 1$. One shows that the d_i's are positive integers and that the largest d_i is equal to the highest degree $h - 1$. These integers d_i's are the **exponents $\boldsymbol{E}$** of $\mathfrak{g}$. We can order the basis such that

$$1 = d_1 \leq d_2 \leq \cdots \leq d_\ell = h - 1.$$

We will now label the elements q_i's ($i = 1, \cdots, \ell$) by their corresponding exponent $d_i \in E$ and hence obtain a basis[2] $\{p_d\}_{d \in E}$ of $\mathfrak{a}$. Note that as $d_1 = 1$, the element $p_1 = q_1$ indeed coincides with p_1, justifying the notation. One shows [178] that for every $d \in \{1, \cdots, h-1\}$, we have

$$\mathfrak{g}_d = \begin{cases} [p_{-1}, \mathfrak{g}_{d+1}] & \text{if} \quad d \notin E, \\ [p_{-1}, \mathfrak{g}_{d+1}] \oplus \mathbb{C} p_d & \text{if} \quad d \in E. \end{cases} \tag{7.3.16}$$

7.3.4 Opers and Miura Opers associated with $\mathfrak{g}$

In this subsection, we define the opers and Miura opers associated with an arbitrary semi-simple Lie algebra $\mathfrak{g}$. These are generalisations of the opers and Miura opers defined in Subsection 7.3.1 for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. They were first introduced by Drinfeld and Sokolov in [181], for the study of the KdV equation and its generalisations. For a more complete introduction to opers, see for instance [182, 183].

Meromorphic $\mathfrak{g}$-connections. We will denote by $\mathcal{M}$ the algebra of meromorphic functions on the Riemann sphere $\mathbb{P}^1$. We define the space of $\mathfrak{g}$-connections on $\mathbb{P}^1$ as

$$\mathrm{Conn}_{\mathfrak{g}}\big(\mathbb{P}^1\big) = \{ a\partial_z + A(z),\ a \in \mathbb{C},\ A \in \mathfrak{g} \otimes \mathcal{M} \}\,.$$

In this definition, the element $A \in \mathfrak{g} \otimes \mathcal{M}$ is a $\mathfrak{g}$-valued meromorphic function of z.

[2] As in Subsection 7.1.2, we define E as the multiset of exponents. Therefore, if two exponents d_i's coincide, they appear both in E and one then has two corresponding elements p_{d_i}'s.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$[3]. Let us consider the group $G(\mathcal{M})$ of G-valued meromorphic functions. A rigorous definition of this group can become quite involved. In the article [P4], B. Vicedo and myself used a construction of $G(\mathcal{M})$ in terms of affine group schemes: as this publication is joined at the end of this thesis and is the subject of Subsection 7.3.6, we refer to it for a general rigorous definition of $G(\mathcal{M})$. For the purpose of this Subsection, one can consider a matricial representation of G and think of $G(\mathcal{M})$ as matrices in this representation with meromorphic entries. For $g \in G(\mathcal{M})$, we define the gauge transformation of $\nabla = a\partial_z + A(z)$ in $\mathrm{Conn}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$ as

$$g\nabla g^{-1} = a\partial_z + g(z)A(z)g(z)^{-1} - a\big(\partial_z g(z)\big)g(z)^{-1}.$$

This defines an action of $G(\mathcal{M})$ on $\mathrm{Conn}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$.

Opers and Miura opers. Recall the positive nilpotent subalgebra $\mathfrak{n}_+$ and Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ (see Appendix A.2.1 and Subsection 7.3.3). We define the following subspace of $\mathrm{Conn}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$:

$$\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big) = \big\{\partial_z + p_{-1} + A(z),\ A \in \mathfrak{b}_+ \otimes \mathcal{M}\big\}. \tag{7.3.17}$$

There is a unique connected subgroup N_+ of G with Lie algebra $\mathfrak{n}_+$. This subgroup can be seen as the exponentials of elements of $\mathfrak{n}_+$ in G:

$$N_+ = \{\exp(X),\ X \in \mathfrak{n}_+\} \subset G.$$

Let us consider the group $N_+(\mathcal{M})$ of N_+-valued meromorphic functions:

$$N_+(\mathcal{M}) = \big\{\exp\big(m(z)\big),\ m \in \mathfrak{n}_+ \otimes \mathcal{M}\big\}, \tag{7.3.18}$$

which is then a subgroup of $G(\mathcal{M})$.

Let $g = \exp(m)$ be an element of $N_+(\mathcal{M})$ and $\nabla = \partial_z + p_{-1} + A(z)$ be an element of $\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$. Then the gauge transformation of ∇ by g reads

$$g\nabla g^{-1} = \partial_z + p_{-1} + \big(g(z)p_{-1}g(z)^{-1} - p_{-1}\big) + g(z)A(z)g(z)^{-1} - \big(\partial_z g(z)\big)g(z)^{-1}.$$

As g is N_+-valued, $\big(\partial_z g(z)\big)g(z)^{-1}$ belongs to $\mathfrak{n}_+ \subset \mathfrak{b}_+$. In the same way, $A(z)$ belongs to $\mathfrak{b}_+$ and $g(z)$ to $N_+ \subset B_+$ (the subgroup of G with Lie algebra $\mathfrak{b}_+$), thus the conjugacy $g(z)A(z)g(z)^{-1}$ belongs to $\mathfrak{b}_+$. Finally, using the fact that $\mathrm{Ad}_{g(z)} = \exp\big(\mathrm{ad}_{m(z)}\big)$ (see Appendix A), we get

$$g(z)p_{-1}g(z)^{-1} - p_{-1} = \sum_{n=1}^{+\infty} \frac{1}{n!}\mathrm{ad}^n_{m(z)}\big(p_{-1}\big).$$

Note that this sum is finite as $m(z)$ is a nilpotent element of $\mathfrak{g}$. The element p_{-1} is of degree -1 in the principal gradation and m is composed of elements of degrees at least 1, thus $\mathrm{ad}_{m(z)}\big(p_{-1}\big)$ is composed of elements of degrees at least 0, $\mathrm{ad}^2_{m(z)}\big(p_{-1}\big)$ of elements of degrees at least 1, and so on. Thus, $g(z)p_{-1}g(z)^{-1} - p_{-1}$ belongs to $\mathfrak{b}_+$.

As a conclusion, we see that $g\nabla g^{-1}$ belongs to $\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$. Thus, the group $N_+(\mathcal{M})$ acts by gauge transformations on the space $\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$. Let us consider the quotient of this action

$$\mathrm{Op}_{\mathfrak{g}}\big(\mathbb{P}^1\big) = \mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)/N_+(\mathcal{M}). \tag{7.3.19}$$

We call en element of $\mathrm{Op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$ a **$\mathfrak{g}$-oper**. If ∇ belongs to $\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$, we denote by $[\nabla]$ the equivalence class of ∇ in the quotient $\mathrm{Op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$: ∇ is then a representative of $[\nabla]$.

We define the space of **$\mathfrak{g}$-Miura opers** as

$$\mathrm{mOp}_{\mathfrak{g}}\big(\mathbb{P}^1\big) = \big\{\partial_z + p_{-1} + A(z),\ A \in \mathfrak{h} \otimes \mathcal{M}\big\}, \tag{7.3.20}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. As $\mathfrak{h}$ is included in the Borel algebra $\mathfrak{b}_+$, $\mathrm{mOp}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$ is a subspace of $\mathrm{op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$. Thus, for any Miura oper $\nabla \in \mathrm{mOp}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$, one can define the associated oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}\big(\mathbb{P}^1\big)$ (note however that a Miura oper is not an oper, as it is not an equivalence class).

[3] The rest of this section will not depend on the choice of the group G. One can for example choose the adjoint group $\mathrm{Ad}(\mathfrak{g})$ or the unique connected simply-connected group with Lie algebra $\mathfrak{g}$.

Back to $\mathfrak{sl}(2,\mathbb{C})$. Let us come back to our first example $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, for which we defined opers and Miura opers in Subsection 7.3.1. The negative principal element p_{-1} and the Borel subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ are respectively

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix},\ a, b \in \mathbb{C} \right\}.$$

Thus, the definition (7.3.4) of the space $\mathrm{op}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{P}^1)$ agrees with the general definition (7.3.17) for an arbitrary Lie algebra $\mathfrak{g}$.

In subsection 7.3.1, we consider the group $SL(2,\mathbb{C})$, which is a connected (and simply-connected) Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The connected subgroup of $SL(2,\mathbb{C})$ with Lie algebra the positive nilpotent subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ reads matricially

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},\ a \in \mathbb{C} \right\}.$$

Thus, the group $N_+(\mathcal{M})$ defined in general by (7.3.18) coincides with the group N defined in (7.3.5) for $G = SL(2,\mathbb{C})$. As a consequence, the definition (7.3.7) of $\mathfrak{sl}(2,\mathbb{C})$-opers agrees with the general definition (7.3.19).

Finally, the Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ is composed by traceless diagonal matrices. Thus the general definition (7.3.20) of Miura oper reduces to the definition (7.3.9) for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.

Canonical representatives of $\mathfrak{g}$-opers. Recall that, for $\mathfrak{sl}(2,\mathbb{C})$-opers, we defined a notion of unique canonical representative. Such a notion exists also for opers associated with any semi-simple Lie algebra $\mathfrak{g}$. It involves the centraliser $\mathfrak{a}$ of p_1 introduced in Subsection 7.3.3.

Theorem 7.3.4. *Let $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$ be a $\mathfrak{g}$-oper. There exists a unique representative of $[\nabla]$ of the form*

$$\partial_z + p_{-1} + A(z), \qquad \text{with} \qquad A \in \mathfrak{a} \otimes \mathcal{M}. \tag{7.3.21}$$

*It is called the **canonical representative** of $[\nabla]$ and is written $[\nabla]_{can}$.*

Proof. We will not detail the construction of this canonical representative here. As for the construction of $G(\mathcal{M})$ above, we refer to the article [P4] joined at the end of this thesis for a detailed proof. Let us first note that the centraliser $\mathfrak{a}$ of p_1 is included in $\mathfrak{b}_+$: thus, a connection of the form (7.3.21) indeed belongs to the space $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$.

The proof rests on a constructive algorithm, which, from any representative $\nabla \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ of the oper, constructs $g \in G(\mathcal{M})$ such that the gauge transformation $g\nabla g^{-1}$ is of the form (7.3.21). The proof of this algorithm works recursively on the degrees (in the principal gradation) of the elements of ∇. It is mostly based on the property (7.3.16) of the basis $\{p_d\}_{d\in E}$ of $\mathfrak{a}$. One sees in the proof that the constructed element g is uniquely determined by the initial representative ∇, hence proving the uniqueness of the canonical representative.

An important point in the proof is the following. Although the gauge transformation $g\nabla g^{-1}$ involves derivatives of g, fixing the components of g such that $g\nabla g^{-1}$ is of the form (7.3.21) does not require any integration with respect to z. The construction is purely algebraic, which ensures that the element $A(z) \in \mathfrak{a}$ obtained in the end is a meromorphic function. In the same way, it ensures that the algorithm does not create new singularities: if the initial representative is regular at a point $x \in \mathbb{C}$, so is the canonical representative (the algorithm can however increase the order of existing poles of the initial representative). □

Using the basis $\{p_d\}_{d\in E}$ of $\mathfrak{a}$, we can express the canonical representative (7.3.21) as

$$[\nabla]_{\mathrm{can}} = \partial_z + p_{-1} + \sum_{d\in E} c^d(z) p_d, \qquad \text{with} \qquad c^d(z) \in \mathcal{M}.$$

The canonical element of a $\mathfrak{g}$-oper is thus uniquely described by ℓ meromorphic functions $c^d(z)$'s, labelled by exponents $d \in E$.

Let us end this paragraph by comparing this general notion of canonical element with the one for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, introduced in Subsection 7.3.1. The rank of $\mathfrak{sl}(2,\mathbb{C})$ is one, so there is only one element p_d, which is p_1. An explicit computation of p_1 shows that it is matricially given by

$$p_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, the notion of canonical representative of Theorem 7.3.4 coincides with the one (7.3.8) for $\mathfrak{sl}(2,\mathbb{C})$.

Regularities and singularities of opers. Let x be a point of $\mathbb{C}$. We say that a $\mathfrak{g}$-oper $[\nabla]$ is regular at x if it possesses a representative which is regular at x.

Lemma 7.3.5. *Let $[\nabla]$ be a $\mathfrak{g}$-oper. Then, $[\nabla]$ is regular at $x \in \mathbb{C}$ if and only if its canonical representative $[\nabla]_{can}$ is regular at x.*

Proof. If $[\nabla]_{\mathrm{can}}$ is regular at x, the oper $[\nabla]$ admits a representative which is regular at x and is thus by definition regular at x.

Conversely, let us suppose that $[\nabla]$ possesses a representative $\nabla \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ regular at x. Apply then the algorithm of Theorem 7.3.4 to find the canonical representative $[\nabla]_{\mathrm{can}}$. As explained in the proof of the theorem, this algorithm does not create singularities, hence $[\nabla]_{\mathrm{can}}$ is also regular at x. □

Let ∇ be a connection in $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$. Using the decomposition (7.3.15) of $\mathfrak{b}_+$ in terms of the principal gradation, one can write ∇ as

$$\nabla = \partial_z + p_{-1} + \sum_{d=0}^{h-1} A_d(z), \qquad \text{with} \qquad A_d \in \mathfrak{g}_d \otimes \mathcal{M}.$$

We say that ∇ has (at most) a regular singularity at $x \in \mathbb{C}$ if the functions $A_d(z)$ have poles of order (at most) $d+1$. An oper in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$ is said to have a regular singularity at x if it possesses a representative with a regular singularity at x. Considering the algorithm constructing the canonical representative of an oper one proves the following lemma [103], similar to the one above for regularity.

Lemma 7.3.6. *Let $[\nabla]$ be a $\mathfrak{g}$-oper. Then, $[\nabla]$ has a regular singularity at $x \in \mathbb{C}$ if and only if its canonical representative $[\nabla]_{can}$ has a regular singularity at x.*

One easily checks that these notions of regularity and regular singularity coincide with the ones defined in Paragraph 7.3.2 for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.

7.3.5 The FFR isomorphism and the case of Bethe vectors

We now have almost all tools to express the FFR isomorphism $\Phi_{\boldsymbol{z},\mathfrak{g}}$ of Theorem 7.3.1. We will then study how one can reformulate the Bethe ansatz in terms of the FFR approach and of opers.

The Langlands dual ${}^L\mathfrak{g}$. Let us consider the Cartan matrix A of the Lie algebra $\mathfrak{g}$. Its transpose ${}^tA$ is also a Cartan matrix. The simple roots $\alpha_i \in \mathfrak{h}^*$ and coroots $\check{\alpha}_i \in \mathfrak{h}$ associated with A are respectively the simple coroots and roots associated with ${}^tA$. We denote by ${}^L\mathfrak{g}$ the semi-simple Lie algebra with Cartan matrix ${}^tA$. The algebra ${}^L\mathfrak{g}$ is called the **Langlands dual** of $\mathfrak{g}$. For a simple Lie algebra $\mathfrak{g}$ of type A, D, E, F or G in Cartan's classification, the Langlands dual is isomorphic to $\mathfrak{g}$ itself (as A is symmetric or a permutation of its transpose). The Langlands duality exchanges Lie algebras of types B and C.

As the simple coroots of ${}^L\mathfrak{g}$ are the simple roots of $\mathfrak{g}$, the Cartan subalgebra of ${}^L\mathfrak{g}$ is naturally identified with the dual $\mathfrak{h}^*$ of the Cartan subalgebra of $\mathfrak{g}$. The algebras $\mathfrak{g}$ and ${}^L\mathfrak{g}$ then have the same

rank ℓ. One can define the Cartan-Weyl decomposition and Chevalley generators of ${}^L\mathfrak{g}$ as for $\mathfrak{g}$. One then also constructs the principal $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of ${}^L\mathfrak{g}$, generated by ρ, $\check{p}_1$ and $\check{p}_{-1}$. In particular, the Weyl coweight ρ of ${}^L\mathfrak{g}$ is defined as for $\mathfrak{g}$ and is thus an element of ${}^L\mathfrak{h} = \mathfrak{h}^*$. In fact, it coincides with the Weyl weight of $\mathfrak{g}$ [175], as defined in equation (7.2.2). One then defines the centraliser ${}^L\mathfrak{a}$ of $\check{p}_1$ and its basis $\{\check{p}_d\}$, with d running through the exponents of ${}^L\mathfrak{g}$. One can show [162] that the latter are equal to the exponents of $\mathfrak{g}$, so we can label the $\check{p}_d$'s by $d \in E$. This allows to define the space of ${}^L\mathfrak{g}$-opers $\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ and their canonical representative, as for $\mathfrak{g}$ in the previous Subsection.

The FFR isomorphism. Let us consider the Gaudin model on $\mathfrak{g}$ with sites $\boldsymbol{z} = (z_1, \cdots, z_N)$. Recall that the Gaudin Hamiltonian $\mathscr{H}^d(z)$ associated with the exponent $d \in E$ is a rational function of z, with poles of order $d+1$ at all sites z_i's. We will then write $\mathscr{H}^d(z)$ as

$$\mathscr{H}^d(z) = \sum_{i=1}^{N} \sum_{p=0}^{d} \frac{1}{(p+1)!} \frac{\mathcal{Q}^d_{i,p}}{(z-z_i)^{p+1}}. \tag{7.3.22}$$

In particular, we have $\mathcal{Q}^1_{i,0} = \mathscr{H}_i$ (residue at $z = z_i$ of the quadratic Hamiltonian $\mathscr{H}(z)$) and $\mathcal{Q}^1_{i,1} = \Delta_{(i)}$ (the quadratic Casimir). The Gaudin algebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ is then generated by the operators $\mathcal{Q}^d_{i,p}$, for $i \in \{1, \cdots, N\}$, $d \in E$ and $p \in \{0, \cdots, d\}$.

We define the space $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ as the space of ${}^L\mathfrak{g}$-opers with regular singularities at the z_i's and regular elsewhere. Following the results of Subsection 7.3.4, if $[\nabla]$ is such an oper, its canonical representative can be written as

$$[\nabla]_{\mathrm{can}} = \partial_z + \check{p}_{-1} + \sum_{d \in E} A_d \, c^d(z) \, \check{p}_d, \tag{7.3.23}$$

where the A_d's are some normalisation constants, independent of $[\nabla]$, and

$$c^d(z) = \sum_{i=1}^{N} \sum_{p=0}^{d} \frac{1}{(p+1)!} \frac{c^d_{i,p}}{(z-z_i)^{p+1}},$$

with $c^d_{i,p}$ complex numbers. We then define the following functions $\Gamma^d_{i,p}$ on $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$, for $i \in \{1, \cdots, N\}$, $d \in E$ and $p \in \{0, \cdots, d\}$:

$$\begin{array}{rccc} \Gamma^d_{i,p} : & \mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1) & \longrightarrow & \mathbb{C} \\ & [\nabla] & \longmapsto & c^d_{i,p} \end{array}. \tag{7.3.24}$$

These functions generate the algebra of functions on $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$.

We now finally have all the conventions necessary to express the FFR isomorphism $\Phi_{\boldsymbol{z},\mathfrak{g}}$ of Theorem 7.3.1. We characterize it by specifying its images on the generators $\mathcal{Q}^d_{i,p}$ of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ and extending it by linearity and multiplication:

$$\begin{array}{rccc} \Phi_{\boldsymbol{z},\mathfrak{g}} : & \mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g}) & \longrightarrow & \mathrm{Fun}\left(\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)\right) \\ & \mathcal{Q}^d_{i,p} & \longmapsto & \Gamma^d_{i,p} \end{array}. \tag{7.3.25}$$

This expression coincides with the one (7.3.13) of $\Phi_{\boldsymbol{z},\mathfrak{sl}(2,\mathbb{C})}$ for the $\mathfrak{sl}(2,\mathbb{C})$-Gaudin model, recalling that the Langlands dual of $\mathfrak{sl}(2,\mathbb{C})$ is $\mathfrak{sl}(2,\mathbb{C})$ itself.

The opers associated with Bethe vectors. Let us now consider the Gaudin model on the Hilbert space $H_{\boldsymbol{\lambda}}$, tensor product of highest-weight representations. One can then apply the Bethe ansatz. If $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ is an on-shell Bethe vector and thus an eigenvector of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$, Theorem 7.3.3 ensures the

existence of an oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ in $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ which encodes the eigenvalues of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ on $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$. In this paragraph, we will show how one constructs this oper explicitly.

Let us start with an off-shell Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$, with colors $\boldsymbol{c} = (c(1), \cdots, c(M))$ and Bethe roots $\boldsymbol{w} = (w_1, \cdots, w_M)$. Recall the weight $\lambda^{\boldsymbol{c}}(z, \boldsymbol{w})$ associated with the Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ by equation (7.2.10). It belongs to the dual $\mathfrak{h}^*$ of the Cartan subalgebra of $\mathfrak{g}$. Recall from the first paragraph of this subsection that $\mathfrak{h}^*$ is naturally identified with the Cartan subalgebra of ${}^L\mathfrak{g}$. We then define the following ${}^L\mathfrak{g}$-Miura oper

$$\nabla^{\boldsymbol{c}}_{\boldsymbol{w}} = \partial_z + \check{p}_{-1} - \lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) \ \in \ \mathrm{mOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$$

and the associated oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$. The FFR reformulation of the Bethe ansatz can be summarised as follows.

Theorem 7.3.7. *The following points are equivalent.*

(i) The oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ belongs to $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ (i.e. has regular singularities at the z_i's and is regular elsewhere).

(ii) The Bethe roots $\boldsymbol{w}$ satisfy the Bethe equations.

(iii) The Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ is on-shell and thus an eigenvector of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$.

In this case, the canonical representative of $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ can be written as

$$[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]_{\mathrm{can}} = \partial_z + \check{p}_{-1} + \sum_{d \in E} A_d \, \mathcal{C}^d(z, \boldsymbol{w}) \, \check{p}_d,$$

with A_d's the normalisation constant introduced in equation (7.3.23) and $\mathcal{C}^d(z, \boldsymbol{w})$'s some meromorphic functions. Then, the eigenvalue of $\mathscr{H}^d(z)$ on the on-shell Bethe vector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ is equal to $\mathcal{C}^d(z, \boldsymbol{w})$.

We will not prove this theorem here and refer to [103] for a demonstration. Let us analyse the different aspects and consequences of this theorem. The Miura oper $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$ is a representative of the oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$. Its component $\lambda^{\boldsymbol{c}}(z, \boldsymbol{w})$ is valued in the Cartan subalgebra $\mathfrak{h}^*$ of ${}^L\mathfrak{g}$. In terms of the principal gradation ${}^L\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} {}^L\mathfrak{g}_d$, this Cartan subalgebra corresponds to the grade zero, *i.e.* $\mathfrak{h}^* = {}^L\mathfrak{g}_0$ (see Subsection 7.3.3). Yet, the component $\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) \in {}^L\mathfrak{g}_0$ has simple poles at the z_i's. Thus, the representative $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$ has regular singularities at the z_i's and so has the corresponding oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$.

The oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ then belongs to $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ if and only if it is regular at points different from the z_i's. Given the expression (7.2.10) of $\lambda^{\boldsymbol{c}}(z, \boldsymbol{w})$, the oper can have only singularities (other than at the z_i's) at the Bethe roots w_j's. The first main point of Theorem 7.3.7 is that the oper is regular at w_j if and only the Bethe root w_j satisfies the Bethe equation. We already observed this phenomenon in Subsections 7.3.1 and 7.3.2 for the case where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In particular, the oper is regular at all w_j's if and only if all Bethe roots satisfy the Bethe equations, hence if and only if the Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ is on-shell.

Let us now consider the second part of Theorem 7.3.7, stating that the eigenvalue of $\mathscr{H}^d(z)$ on the $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ is $\mathcal{C}^d(z, w)$. Here also we observed it explicitly for the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ in Subsections 7.3.1 and 7.3.2. Given the decomposition (7.3.22) of $\mathscr{H}^d(z)$ and the definition (7.3.24) of the functions $\Gamma^d_{i,p}$ on $\mathrm{Op}^{\mathrm{RS}}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$, this can be reinterpreted as the fact that

$$\chi_{\boldsymbol{c},\boldsymbol{w}}\big(\mathcal{Q}^d_{i,p}\big) = \Gamma^d_{i,p}\big([\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]\big),$$

where $\chi_{\boldsymbol{c},\boldsymbol{w}}(\mathcal{Q})$ denotes the eigenvalue of $\mathcal{Q} \in \mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ on $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$. Thus, according to the definition (7.3.25) of the FFR isomorphism $\Phi_{\boldsymbol{z},\mathfrak{g}}$, we have

$$\chi_{\boldsymbol{c},\boldsymbol{w}}(\mathcal{Q}) = \big(\Phi_{\boldsymbol{z},\mathfrak{g}}(\mathcal{Q})\big)\big([\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]\big).$$

This result illustrates the Theorem 7.3.3 for the eigenvector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$. In particular, it shows that the oper associated with this eigenvector (whose existence is predicted by Theorem 7.3.3) can be constructed explicitly *via* the Miura oper $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$.

As a conclusion, we see that the FFR approach then contains the Bethe ansatz. As explained in Subsection 7.3.2, one of the assets of this approach is that it also works when the Bethe ansatz fails (for example on other types of Hilbert spaces). Another asset of the FFR approach that one can observe in the present paragraph is that it gives a general formalism to describe the eigenvalues of both the quadratic Hamiltonian and the higher-degree ones.

Reproduction procedure. Let us end this subsection by saying a few words about the so-called reproduction procedure. This procedure was first introduced and studied in [184] outside of the FFR formalism. Here, we present the reinterpretation of this procedure in the FFR approach, as understood in [103, 104]. More details about this can be found in the article [P4] joined with this thesis (indeed, part of the subject of this publication is the generalisation of this procedure to cyclotomic Gaudin models, as explained in the following subsection).

Let us consider an on-shell Bethe vector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ and the associated ${}^L\mathfrak{g}$-Miura oper $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$. The corresponding oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ is thus in $\mathrm{Op}^{\mathrm{RS}}_{L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$, as the Bethe roots $\boldsymbol{w}$ satisfy the Bethe equations. Let g be an element of the group $N_+(\mathcal{M})$ and let us consider the gauge transformation $\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}} = g\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}g^{-1}$. This is an other representative of the oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$. Suppose that one can choose g such that $\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}}$ is also a ${}^L\mathfrak{g}$-Miura oper, *i.e.* that

$$\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}} = \partial_z + \check{p}_{-1} - \widetilde{\mu}(z),$$

with $\widetilde{\mu}$ a $\mathfrak{h}^*$-valued meromorphic function. This condition takes the form of a differential equation on $g(z)$. It can be made into the form of a Ricatti equation and can then be solved formally (under some assumptions on the weights λ_i's defining the Hilbert space $H_{\boldsymbol{\lambda}}$). The Miura oper $\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}}$ is then called a reproduction of the initial one $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$.

For generic solutions of this reproduction procedure, we obtain a weight $\widetilde{\mu}(z)$ of the same form as the weight $\lambda^{\widetilde{\boldsymbol{c}}}(z,\widetilde{\boldsymbol{w}})$ associated with a Bethe vector $\Psi_{\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{w}})$, for other sets of Bethe roots $\widetilde{\boldsymbol{w}}$ and colors $\widetilde{\boldsymbol{c}}$. By construction, the oper $[\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}}] = [\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ belongs to $\mathrm{Op}^{\mathrm{RS}}_{L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$. Thus, by Theorem 7.3.7, the Bethe vector $\Psi_{\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{w}})$ is on-shell and is therefore an eigenvector of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$. As the two Miura opers $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$ and $\widetilde{\nabla}^{\boldsymbol{c}}_{\boldsymbol{w}}$ correspond to the same oper, they have the same canonical representative. Theorem 7.3.7 then implies that the eigenvalues of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ on $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ and $\Psi^{\mathrm{on}}_{\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{w}})$ are identical.

The generic solutions of the reproduction procedure then start from an on-shell Bethe vector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ and generate other on-shell Bethe vectors $\Psi^{\mathrm{on}}_{\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{w}})$ with the same eigenvalues. These different Bethe vectors have an interpretation in terms of the diagonal action of $\mathfrak{g}_{(\infty)}$. We discussed this action in Paragraph 7.1.2: we shall use the notation of this paragraph here. Recall in particular that we decomposed $H_{\boldsymbol{\lambda}}$ as a sum of Verma modules M_s of the diagonal action, on which elements of $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$ have a unique eigenvalue. These Verma modules contain several $\mathfrak{g}_{(\infty)}$-singular vectors (including the highest-weight ones B_s). The different Bethe vectors obtained above correspond in fact to different singular vectors with the same eigenvalue, including non highest-weight ones.

Outside of these generic solutions, the obtained Miura opers describe eigenvectors which are not Bethe vectors. We shall not enter into details about this here. An interesting result is that the set of all possibles solutions of the reproduction procedure possesses a nice geometric structure. Indeed, it is isomorphic to the flag manifold G/B_-, obtained as the quotient of the Lie group G by its Borel subgroup B_-.

7.3.6 Cyclotomic opers and a conjectural FFR approach for cyclotomic Gaudin models

This subsection is based on the article [P4], that I wrote during my PhD with B. Vicedo. The complete publication is joined at the end of this thesis. In this subsection, we will simply give a brief summary of its content.

In Chapter 6, Subsection 6.1.5, we introduced a generalisation of classical Gaudin models that are called cyclotomic models and which possess additional equivariance properties under an automorphism σ of the underlying Lie algebra $\mathfrak{g}$. Quantum cyclotomic Gaudin models for finite algebras can be seen as a particular example of generalised quantum Gaudin model introduced by Skrypnyk in [111]. Vicedo

and Young proved [113] that cyclotomic Gaudin models possess, similarly to non-cyclotomic ones, a large family of conserved commuting Hamiltonians, whose degrees are labelled by the exponents of the Lie algebra. These Hamiltonians generate a commutative algebra $\mathcal{Z}^{\sigma}_{\boldsymbol{z}}(\mathfrak{g})$, which is the cyclotomic equivalent of the Gaudin subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathfrak{g})$. The Bethe ansatz for cyclotomic Gaudin models has also been developed in [113] (see also [114] for particular cases) and allows to diagonalise $\mathcal{Z}^{\sigma}_{\boldsymbol{z}}(\mathfrak{g})$ on a tensor product of highest-weight representations.

The goal of one of my PhD project with B. Vicedo, which resulted in the article [P4], was to lay down the foundations for a generalisation of the FFR approach to cyclotomic models. We summarise briefly the main results of the article here and refer to the whole publication (at the end of this thesis) for details.

Cyclotomic opers. We introduce a notion of cyclotomic $\mathfrak{g}$-opers as gauge equivalence classes of particular $\mathfrak{g}$-connections. The main difference with non-cyclotomic opers is that these connections must satisfy some equivariance condition with respect to a certain automorphism τ of $\mathfrak{g}$. We generalise most definitions and results about non-cyclotomic opers to the cyclotomic setting (Miura opers, canonical representatives, singularities, ...).

Cyclotomic reproduction procedure. We study cyclotomic reproduction procedure (see previous Subsection for the non-cyclotomic case). More precisely we characterise the space of cyclotomic Miura opers which correspond to the same cyclotomic oper. The main difference with the non-cylcotomic case is that the possible reproduction transformations must satisfy some invariance condition under an automorphism θ of $\mathfrak{g}$ (which is related to the automorphism τ above but is not identical). In particular, we show that this space is isomorphic to a subspace of the flag variety G/B_-, invariant under the automorphism θ.

Conjectural cyclotomic FFR approach. We conjecture that the cyclotomic Gaudin algebra $\mathcal{Z}^{\sigma}_{\boldsymbol{z}}(\mathfrak{g})$ is isomorphic to the algebra of functions on a certain type of cyclotomic opers $\mathrm{Op}^{\mathrm{RS},\sigma}_{{}^L\mathfrak{g},\boldsymbol{z}}(\mathbb{P}^1)$ on the Langlands dual ${}^L\mathfrak{g}$ (the automorphism τ of ${}^L\mathfrak{g}$ which appears in the definition of these cyclotomic ${}^L\mathfrak{g}$-opers is constructed from the automorphism σ characterising the cyclotomic Gaudin model).

We motivate this conjecture by several observations coming from the cyclotomic Bethe ansatz. As in the non-cyclotomic case, we associate cyclotomic Bethe vectors with particular cyclotomic Miura opers. We then prove that the cyclotomic Bethe equations can be seen as some regularity conditions on the associated oper, as in the non-cyclotomic setting. We also prove that our conjecture gives the correct eigenvalue of the quadratic cyclotomic Gaudin Hamiltonian.

Assuming that the conjecture is true, we use the results on cyclotomic reproduction mentionned above to get some informations about reproduction of cyclotomic Bethe vectors. *Via* the constraints on these reproduction imposed by the automorphism θ, we recover (and to some extent generalise) some results already obtained in [185] using other techniques.

Chapter 8

Quantum affine Gaudin models: towards a quantum hierarchy

This chapter is based on the preprints [P5] and [P6], that I wrote during my PhD with B. Vicedo and C.A.S. Young. These are joined at the end of this thesis. Here, we will summarise the main ideas developed in them.

This chapter is devoted to quantum Affine Gaudin models. In general, for any Kac-Moody algebra $\mathfrak{g}$, one can define a quantum Gaudin model (complex, non cyclotomic and with simple poles) on $\mathfrak{g}$. The algebra of observables of this model is the tensor product $U(\mathfrak{g})^{\otimes N}$, where N is the number of sites of the model and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (more precisely the algebra of operators is a completion of this tensor product). We already introduced the universal enveloping algebra $U(\mathfrak{g})$ as a natural quantisation of the Kirillov-Kostant phase space $\mathfrak{g}^*$ for a finite Lie algebra in Subsection 7.1.1: this discussion generalises straightforwardly to Kac-Moody algebras (modulo the treatment of infinite sums by appropriate completions), justifying this choice of algebra of operators.

One then defines [93] a quadratic Hamiltonian $\mathscr{H}(z)$ in $U(\mathfrak{g})^{\otimes N}$ using the non-degenerate invariant bilinear form on the Kac-Moody algebra $\mathfrak{g}$ (recall that in this thesis, we consider Kac-Moody algebras to be associated with symmetrisable Cartan matrices, ensuring that these algebras possess such a form). When the Gaudin model is considered on a tensor product of highest-weight representations of $\mathfrak{g}$, this quadratic Hamiltonian has been diagonalised by Schechtman and Varchenko in [93], using the Bethe ansatz.

In particular, this construction applies to affine Kac-Moody algebras and one can thus consider quantum AGM. Classical AGM were discussed in Section 6.2 of this thesis. In particular, we explained how an AGM can be realised as an integrable field theory with twist function, that we called the local AGM. Using this fact, we constructed in Subsection 6.2.6 an infinite hierarchy of local conserved charges in involution for these models, which contains the quadratic Hamiltonian mentioned above. A natural question is then whether this hierarchy can be quantised, *i.e.* whether one can find commuting conserved operators in the quantum AGM which reduce to these local charges in the classical limit. This is the main question addressed in this chapter.

An important property of the classical hierarchy is that the charges are labelled by the positive exponents of the affine Kac-Moody algebra. This bears a striking resemblance with the finite case. Indeed, for finite Gaudin models, one constructs a hierarchy of conserved charges of degrees labelled by the exponents of the underlying finite algebra (see Subsection 6.1.4). Moreover, as explained in Chapter 7, this hierarchy can be quantised and its spectrum can be described by the Bethe ansatz and the FFR approach. The latter relates the spectrum of the conserved charges with opers associated with the finite algebra, whose description also involves the exponents of the finite algebra.

One can also define a notion of oper associated with an affine algebra. Moreover, these opers are described in terms of the positive exponents of the affine algebra. It is then natural to conjecture that there exists a quantum hierarchy of AGM and that its spectrum is described in terms of affine op-

ers. This was first proposed by Feigen and Frenkel in [116] for the study of the quantum KdV hierarchy.

As explained above, the first charge of the hierarchy of AGM, the quadratic Hamiltonian, has already been constructed at the quantum level by Schechtman and Varchenko in [93]. This construction was made possible by the existence of a quadratic Casimir of the affine algebra. In the same way, the existence of higher-degree quantum hamiltonians for finite Gaudin models was ensured by the existence of Casimirs of the corresponding degrees for the finite algebra. However, it is known [186] that affine algebras do not possess Casimirs of order greater than two. The construction of higher-order quantum Hamiltonians for AGM thus cannot be done in a similar way than the quadratic one.

This issue can already be seen at the classical level. As explained in Subsection 6.1.2, Hamiltonians in involution for a classical Gaudin model on a Lie algebra $\mathfrak{g}$ can be constructed by evaluating invariant polynomials of $\mathfrak{g}$ on the Lax matrix. One then obtains an Hamiltonian depending freely on the spectral parameter of the theory. For instance, the quadratic Hamiltonian of the model is constructed from an invariant polynomial of degree two, the invariant bilinear form. The classical analogue of the non-existence of higher-order Casimirs of the affine algebra is the non-existence of higher order invariant polynomials on this algebra.

In the article [P3] (see also Chapter 4 and Subsection 6.2.6 of this thesis), we overcame this difficulty and constructed higher-degree classical Hamiltonians by considering some observables depending on the Lax matrix and evaluating the spectral parameter at particular points, the zeros of the twist function. To find a quantisation of these charges, one then has to find an analogue of this procedure at the quantum level.

In the preprint [P5], B. Vicedo, C.A.S. Young and myself propose a conjecture for such a construction, guided by the idea that the spectrum of these quantum charges should be described by affine opers. We develop further the study of affine opers and construct functions on these opers which take the form of hypergeometric integrals in the spectral parameter. Based on the idea that such functions should describe the eigenvalues of quantum Hamiltonians of AGM, we conjecture that these Hamiltonians also take the form of such integrals. We support this conjecture by reasoning on the classical limit of the AGM: indeed in this limit, these hypergometric integrals in the spectral parameter become, by a saddle point approximation, evaluations of the spectral parameter at the zeros of the twist function. Considering these integrals is thus a quantum analogue of the procedure used to construct the classical hierarchy.

The plan of the chapter is the following. In Section 8.1, we recall some generalities about affine Lie algebras and quantum affine Gaudin models. In the rest of this chapter, we will summarise the main ideas developed in the preprint [P5] and state some of the important results without proving them. We refer to the complete preprint, joined to this thesis, for more details and proofs. In Section 8.2, we discuss the theory of affine opers and more precisely how to define functions on affine opers by hypergeometric integrals. In Section 8.3, we conjecture the existence of a quantum hierarchy for Gaudin models and the description of its spectrum by affine opers. The classical limit of this hierarchy will also be discussed in this section. Some first results on the cubic quantum Hamiltonian supporting these conjectures are the subject of our second preprint [P6] and are also mentioned in Section 8.3.

8.1 Affine algebras and quantum affine Gaudin models

8.1.1 Affine Kac-Moody algebras

Let us consider an affine algebra $\mathfrak{g}$. In Chapter 6, Subsection 6.2.2, we used a description of this algebra in terms of loop algebras, as it was useful for the reinterpretation of classical AGM as field theories. In this chapter, we will need the description of $\mathfrak{g}$ as a Kac-Moody algebra. This can be found for example in [162]. The structure theory of the affine algebra $\mathfrak{g}$ is similar to the one of a semi-simple Lie algebra: $\mathfrak{g}$ possesses a Cartan subalgebra $\mathfrak{h}$, a root system $\Delta \subset \mathfrak{h}^*$ and some Chevalley generators E_i's and F_i's associated with simple roots $\alpha_i \in \Delta$ and coroots $\check{\alpha}_i \in \mathfrak{h}$ $(i = 0, \cdots, \ell)$. A basis of $\mathfrak{h}$ is given by $\{\check{\alpha}_0, \cdots, \check{\alpha}_\ell, \mathsf{D}\}$, where D is a so-called derivation element of $\mathfrak{g}$. The algebra $\mathfrak{g}$ is equipped with a

non-degenerate invariant bilinear form $(\cdot,\cdot)$. In particular, this form restricts to a non-degenerate form on $\mathfrak{h}$, which induces a non-degenerate form on $\mathfrak{h}^*$, that we shall still write $(\cdot,\cdot)$.

Let a_i and $a_i^\vee$ be the Kac labels and dual labels of $\mathfrak{g}$ (these are coprime positive integers defined from the Cartan matrix of $\mathfrak{g}$, see [162]). The Coxeter and dual Coxeter numbers of $\mathfrak{g}$ are defined as

$$h = \sum_{i=0}^{\ell} a_i \qquad \text{and} \qquad h^\vee = \sum_{i=0}^{\ell} a_i^\vee. \tag{8.1.1}$$

We define the elements δ of $\mathfrak{h}^*$ and K of $\mathfrak{h}$ as

$$\delta = \sum_{i=0}^{\ell} a_i \alpha_i \qquad \text{and} \qquad \mathsf{K} = \sum_{i=0}^{\ell} a_i^\vee \check{\alpha}_i. \tag{8.1.2}$$

They satisfy

$$\langle \check{\alpha}_i, \delta \rangle = \langle \mathsf{K}, \alpha_i \rangle = 0, \qquad \forall\, i \in \{0, \cdots, \ell\}.$$

The element K is the central element of $\mathfrak{g}$ (which commutes with every $X \in \mathfrak{g}$). Let ρ be the unique element of $\mathfrak{h}^*$ such that $(\rho,\rho) = 0$ and

$$\langle \check{\alpha}_i, \rho \rangle = 1, \qquad \forall\, i \in \{0, \cdots, \ell\}. \tag{8.1.3}$$

Then a basis of $\mathfrak{h}^*$ is given by $\{\alpha_0, \cdots, \alpha_\ell, \rho\}$.

One constructs a quadratic Casimir Δ of $\mathfrak{g}$ using the invariant bilinear form $(\cdot,\cdot)$ [162]. This Casimir is an element of the center $\mathcal{Z}(\mathfrak{g})$ of the (completed) universal enveloping algebra $U(\mathfrak{g})$. Let $\lambda \in \mathfrak{h}^*$ be a weight of $\mathfrak{g}$ and V_λ be the Verma module with highest-weight λ. Then all elements of V_λ are eigenvectors of the Casimir Δ, with a unique eigenvalue:

$$\Delta.v = \big(\lambda, \lambda + 2\rho\big) v, \qquad \forall\, v \in V_\lambda.$$

8.1.2 Quantum affine Gaudin model

Generalities. Let us briefly introduce the quantum affine Gaudin model on $\mathfrak{g}$ with sites $\boldsymbol{z} = (z_1, \cdots, z_N)$. Its algebra of operators $\mathcal{A}_{\boldsymbol{z}}$ is given by a completion of the tensor product $U(\mathfrak{g})^{\otimes N}$ (this completion is constructed from the homogeneous gradation of $\mathfrak{g}$: cf. Paragraph 6.2.2 in the classical case). As for the finite case, if X is an element of $\mathfrak{g}$, we will write $X_{(k)}$ the copy of X in the kth-tensor factor of $\mathcal{A}_{\boldsymbol{z}}$.

Let I^a be a basis of $\mathfrak{g}$, with dual basis I_a with respect to the bilinear form $(\cdot,\cdot)$. We define the Gaudin Hamiltonian at site k to be

$$\mathscr{H}_k = \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{I^a_{(k)} I_{a\,(j)}}{z_k - z_j} \ \in \mathcal{A}_{\boldsymbol{z}}.$$

We define the spectral parameter dependent quadratic Hamiltonian to be

$$\mathscr{H}(z) = \sum_{k=1}^{N} \left(\frac{1}{2} \frac{\Delta_{(k)}}{(z - z_k)^2} + \frac{\mathscr{H}_k}{z - z_k} \right).$$

One then has the commutation relation

$$\big[\mathscr{H}(z), \mathscr{H}(z')\big] = 0, \qquad \forall\, z, z' \in \mathbb{C}.$$

Hilbert space and levels. As in the finite case, the tensor product $U(\mathfrak{g})^{\otimes N}$ acts naturally on any tensor product of N representations of $\mathfrak{g}$. However, as $\mathcal{A}_{\boldsymbol{z}}$ is a completion of $U(\mathfrak{g})^{\otimes N}$, the extension of this action to $\mathcal{A}_{\boldsymbol{z}}$ requires additional constraints on the considered representations. We will not enter into more details about which representations can be considered or not. We shall just need the fact that one can choose a tensor product of highest-weight representations of $\mathfrak{g}$. Let then $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N)$ be a collection of N weights of $\mathfrak{g}$. As in the finite case, we define the Hilbert space $H_{\boldsymbol{\lambda}}$ as the tensor product of the Verma modules V_{λ_i}, which is then a representation of the algebra of operators $\mathcal{A}_{\boldsymbol{z}}$.

Recall from Subsection 8.1.1 that a basis of $\mathfrak{h}^*$ is given by $\{\alpha_0, \cdots, \alpha_\ell, \rho\}$. Let us then decompose the weights λ_i's as

$$\lambda_i = \widetilde{\lambda}_i + \frac{k_i}{h^\vee}\rho, \qquad \text{with} \qquad \widetilde{\lambda}_i \in \mathrm{Span}(\alpha_0, \cdots, \alpha_\ell) \quad \text{and} \quad k_i \in \mathbb{C}, \tag{8.1.4}$$

where the dual Coxeter number $h^\vee$ has been introduced for later convenience.

The central element $\mathsf{K}_{(i)}$ acts on the Hilbert space $H_{\boldsymbol{\lambda}}$ by the multiplication by $\langle \mathsf{K}, \lambda_i \rangle$. Recall that $\langle \mathsf{K}, \alpha_j \rangle = 0$, for all $j \in \{0, \cdots, \ell\}$. Thus, we have $\langle \mathsf{K}, \widetilde{\lambda}_i \rangle = 0$. From equations (8.1.1), (8.1.2) and (8.1.3), we get

$$\langle \mathsf{K}, \rho \rangle = \sum_{i=0}^{\ell} a_i^\vee = h^\vee.$$

Thus, we get that $\langle \mathsf{K}, \lambda_i \rangle = k_i$. Therefore, the operator $\mathsf{K}_{(i)}$ acts on the Hilbert space $H_{\boldsymbol{\lambda}}$ as a complex number k_i. This is the quantum equivalent of the fact that for classical AGM, we realised the abstract observables $K_{(i)}$'s as complex numbers (see Section 6.2). The numbers k_i's are thus the **levels** of the quantum AGM.

Bethe ansatz. The Bethe ansatz allows the diagonalisation of the quadratic Hamiltonian $\mathscr{H}(z)$ on $H_{\boldsymbol{\lambda}}$. It can be derived from the results [93] of Schechtman and Varchenko, for any Kac-Moody algebra. In particular, it is quite similar to the Bethe ansatz for the finite Gaudin model presented in Section 7.2, as it is based on the common structure shared by all Kac-Moody algebras. Hence, we will not enter into more details about the Bethe ansatz here and just recall the result.

The off-shell Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$ is defined from Bethe roots $\boldsymbol{w} = (w_1, \cdots, w_M) \in \mathbb{C}^M$ and associated colors $\boldsymbol{c} = \big(c(1), \cdots, c(M)\big) \in \{0, \cdots, \ell\}^M$. It is on-shell (and thus an eigenvector of $\mathscr{H}(z)$) if the Bethe roots satisfy the Bethe equations

$$\sum_{i=1}^{N} \frac{(\alpha_{c(k)}, \lambda_i)}{w_k - z_i} - \sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{(\alpha_{c(k)}, \alpha_{c(j)})}{w_k - w_j} = 0,$$

for $k \in \{1, \cdots, M\}$. The eigenvalue of $\mathscr{H}(z)$ on $\Psi_{\boldsymbol{c}}^{\mathrm{on}}(\boldsymbol{w})$ is given by

$$\mathcal{E}_{\boldsymbol{c}}^{\mathrm{on}}(z, \boldsymbol{w}) = \frac{1}{2}\big(\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}), \lambda^{\boldsymbol{c}}(z, \boldsymbol{w})\big) - \frac{\partial}{\partial z}\big(\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}), \rho\big), \tag{8.1.5}$$

where we define $\lambda^{\boldsymbol{c}}(z, \boldsymbol{w})$ in $\mathfrak{h}^*$ by

$$\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) = \sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} - \sum_{j=1}^{M} \frac{\alpha_{c(j)}}{z - w_j}. \tag{8.1.6}$$

An important point for the rest of this section is the following. Recall the decomposition (8.1.4) of the weights λ_i's. The weight (8.1.6) can then be written as

$$\lambda^{\boldsymbol{c}}(z, \boldsymbol{w}) = \widetilde{\lambda}^{\boldsymbol{c}}(z, \boldsymbol{w}) + \frac{\varphi(z)}{h^\vee}\rho, \tag{8.1.7}$$

where $\widetilde{\lambda}^{\boldsymbol{c}}(z,\boldsymbol{w})$ belongs to $\mathrm{Span}(\alpha_0,\cdots,\alpha_\ell)$ and

$$\varphi(z)=\sum_{i=1}^{N}\frac{k_i}{z-z_i}. \tag{8.1.8}$$

As the k_i's are the levels of the AGM, the function $\varphi(z)$ is the **twist function** of the model (see equation (6.2.21) in the classical case). The twist function then appears as the coefficient of ρ in the weight $\lambda^{\boldsymbol{c}}(z,\boldsymbol{w})$ (note in particular that, although this weight depends on the considered Bethe vector $\Psi_{\boldsymbol{c}}(\boldsymbol{w})$, the coefficient of ρ does not and is always given by the twist function).

8.1.3 Langlands dual and exponents

Langlands dual. Our goal in this chapter is the generalisation of the FFR approach to affine Gaudin models. Recall from Section 7.3 that this approach for finite models involves opers on the Langlands dual of the underlying finite algebra. Thus, we shall need the Langlands dual ${}^L\mathfrak{g}$ of the affine algebra $\mathfrak{g}$. As in the finite case, it is the Kac-Moody algebra with Cartan matrix the transpose of the one of $\mathfrak{g}$. Thus, it is also an affine algebra. Its Cartan subalgebra ${}^L\mathfrak{h}$ is naturally identified with the dual $\mathfrak{h}^*$ of the one of $\mathfrak{g}$. In particular, the coroots of ${}^L\mathfrak{g}$ are the roots α_i's and its central element is the root δ defined in equation (8.1.2). As a Kac-Moody algebra, ${}^L\mathfrak{g}$ possesses a Cartan-Weyl decomposition

$${}^L\mathfrak{g}={}^L\mathfrak{h}\oplus{}^L\mathfrak{n}_+\oplus{}^L\mathfrak{n}_-=\mathfrak{h}^*\oplus{}^L\mathfrak{n}_+\oplus{}^L\mathfrak{n}_-.$$

We define the positive Borel subalgebra of ${}^L\mathfrak{g}$ as ${}^L\mathfrak{b}_+=\mathfrak{h}^*\oplus{}^L\mathfrak{n}_+$.

Heisenberg principal subalgebra and exponents. Let $\check{E}_i$'s and $\check{F}_i$'s $(i=0,\cdots,\ell)$ be the positive and negative Chevalley generators of ${}^L\mathfrak{g}$. We define the positive and negative principal elements of ${}^L\mathfrak{g}$ as

$$\check{p}_1=\sum_{i=0}^{\ell}a_i\check{E}_i \qquad \text{and} \qquad \check{p}_{-1}=\sum_{i=0}^{\ell}\check{F}_i.$$

We will need the following theorem, which can be found for example in [162].

Theorem 8.1.1. *There exist elements $\check{p}_d$'s in ${}^L\mathfrak{g}$, labelled by the exponents $d\in E$ of $\mathfrak{g}$, such that $\check{p}_1$ and $\check{p}_{-1}$ are given as above and*

$$\big[\check{p}_d,\check{p}_e\big] = d\,\delta_{d+e,0}\,\delta \tag{8.1.9}$$
$$\big[\rho,\check{p}_d\big] = d\,\check{p}_d. \tag{8.1.10}$$

These elements are the affine equivalent of the $\check{p}_d$'s introduced in Subsection 7.3.3 for finite algebras. Together with the central element δ, they form the so-called principal Heisenberg subalgebra of ${}^L\mathfrak{g}$. In the rest of this chapter, we will denote by E_+ the set of positive exponents.

8.2 Affine opers and hypergeometric integrals

In this section, we develop the theory of affine opers. As we have in mind the application to Gaudin models, we will consider here ${}^L\mathfrak{g}$-opers instead of $\mathfrak{g}$-opers. Affine opers were already studied by Drinfeld and Sokolov in [181] (see also [116, 187]). In the preprint [P5], we develop further this theory and in particular define hypergeometric functions on affine opers.

8.2.1 Opers and Miura opers associated with ${}^L\mathfrak{g}$

${}^L\mathfrak{g}$-opers. We will introduce ${}^L\mathfrak{g}$-opers in a way similar to the finite case, discussed in Subsection 7.3.4. We consider the space $\mathrm{Conn}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ of ${}^L\mathfrak{g}$-connections on the Riemann sphere. As for a finite algebra, it is composed of differential operators of the form

$$a\partial_z+A(z), \qquad \text{with } a\in\mathbb{C} \quad \text{and} \quad A\in{}^L\mathfrak{g}\otimes\mathcal{M},$$

where $\mathcal{M}$ denotes the algebra of meromorphic functions on $\mathbb{P}^1$ ($A(z)$ is thus a ${}^L\mathfrak{g}$-valued meromorphic function of z).

Recall the positive Borel subalgebra ${}^L\mathfrak{b}_+$ of ${}^L\mathfrak{g}$ introduced in Subsection 8.1.3. We shall be interested in the following particular subspace of $\mathrm{Conn}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$:

$$\mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1) = \{\partial_z + \check{p}_{-1} + A(z),\ A \in {}^L\mathfrak{b}_+ \otimes \mathcal{M}\}.$$

This is the equivalent of the space (7.3.17) defined for a finite Lie algebra.

Recall also the positive subalgebra ${}^L\mathfrak{n}_+$ of ${}^L\mathfrak{g}$. We define a Lie group ${}^LN_+$ whose Lie algebra is ${}^L\mathfrak{n}_+$ by exponentiation and we consider the group ${}^LN_+(\mathcal{M})$ of ${}^LN_+$-valued meromorphic functions[1]. The group ${}^LN_+(\mathcal{M})$ acts on the space $\mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ by gauge transformations. We then define the space of ${}^L\mathfrak{g}$-opers as the quotient

$$\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1) = \mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)/{}^LN_+(\mathcal{M}).$$

This definition is the equivalent for an affine algebra of the definition (7.3.19) of opers of a finite algebra.

${}^L\mathfrak{g}$-Miura opers. We define the space of ${}^L\mathfrak{g}$-Miura opers as

$$\mathrm{mOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1) = \{\partial_z + \check{p}_{-1} + A(z),\ A \in \mathfrak{h}^* \otimes \mathcal{M}\}.$$

As $\mathfrak{h}^*$ coincides with the Cartan subalgebra ${}^L\mathfrak{h}$ of ${}^L\mathfrak{g}$, this is the equivalent for the affine algebra ${}^L\mathfrak{g}$ of the definition (7.3.20) of Miura opers associated with a finite algebra.

The Miura opers considered originally by Drinfeld and Sokolov are connections of similar forms but with $A(z)$ valued in $\mathrm{Span}(\alpha_0, \cdots, \alpha_\ell)$ instead of the whole Cartan subalgebra ${}^L\mathfrak{h} = \mathfrak{h}^*$. In particular, this definition does not allow Miura opers for which the weight $A(z)$ contains a term proportional to ρ. As we will see, such a term will play a major role in our description of ${}^L\mathfrak{g}$-opers.

8.2.2 Quasi-canonical form and residual gauge transformation

Quasi-canonical form. Recall from Theorem 7.3.4 that finite opers possess a unique canonical form, expressed with the elements p_d labelled by the exponents d of the finite algebra. In Subsection 8.1.3, we defined the equivalent of these elements for the affine algebra ${}^L\mathfrak{g}$: the $\check{p}_d$'s, labelled by the exponents $d \in E$ of ${}^L\mathfrak{g}$. In particular, the objects $\check{p}_d$ for positive exponents $d \in E_+$ belong to the positive Borel subalgebra ${}^L\mathfrak{b}_+$. The affine equivalent of the existence of a canonical form is the following result:

Theorem 8.2.1. *Let $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ be a ${}^L\mathfrak{g}$-oper. Then $[\nabla]$ possesses a representative of the form*

$$\partial_z + \check{p}_{-1} + c^0(z)\rho + \sum_{d \in E_+} c^d(z)\check{p}_d, \tag{8.2.1}$$

where the $c^d(z)$'s ($d \in E_+ \cup \{0\}$) are meromorphic functions of z.

We refer to the preprint [P5] for the proof of this theorem, which is based on an algorithm similar to the one in the finite case (see the proof of Theorem 7.3.4). Let us comment on a few aspects related to this result. Note first that Theorem 8.2.1 does not state that the representative (8.2.1) is unique (as we shall see later, it is indeed not unique). We shall not call (8.2.1) a canonical form but a **quasi-canonical form** of $[\nabla]$, to stress this non-unicity.

Note also that contrarily to the finite case, the quasi-canonical form still possesses a component valued in the Cartan subalgebra ${}^L\mathfrak{h}$, namely the term $c^0(z)\rho$. This is due to the fact that ρ does not belong to the derived subalgebra of ${}^L\mathfrak{g}$: it can never be obtained as a linear combination of Lie brackets in ${}^L\mathfrak{g}$. Indeed, if ∇ is in $\mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$, because of that property, a gauge transformation of ∇ by any $g \in {}^LN_+(\mathcal{M})$ does not change the coefficient $c^0(z)$ of ρ in ∇. Thus, all representatives of the oper $[\nabla]$ share this same coefficient $c^0(z)$. In particular, one cannot find a quasi-canonical form of $[\nabla]$ where this coefficient vanishes.

[1] The rigorous definition of ${}^LN_+(\mathcal{M})$ requires a treatment of infinite sums by appropriate completions. More details about this can be found in the preprint [P5] joined with this thesis.

Residual gauge transformation. Let us prove that the quasi-canonical form ∇ given by (8.2.1) is not unique. For future convenience, we will write the coefficient c^0 of ρ in ∇ as

$$c^0(z) = -\frac{\varphi(z)}{h^\vee}.$$

For this subsection, one can consider the function $\varphi(z)$ to be any rational function ; for the applications to quantum AGM, it will be the twist function (8.1.8) of the model.

For each positive exponent $d \in E_+$, distinct from 1, let us consider a meromorphic function $f^d(z) \in \mathcal{M}$. The exponential

$$g(z) = \exp\left(-\sum_{d\in E_+\setminus\{1\}} f^d(z)\check{p}_d\right) \tag{8.2.2}$$

is an element of the group ${}^LN_+(\mathcal{M})$. Using the property $\mathrm{Ad}_{\exp(m)} = \exp(\mathrm{ad}_m)$ and the commutators of the $\check{p}_d$'s and ρ given by Theorem 8.1.1, one can compute the gauge transformation of ∇ by g. One then finds

$$\widetilde{\nabla} = g\nabla g^{-1} = \partial_z + \check{p}_{-1} - \frac{\varphi(z)}{h^\vee}\rho + \sum_{d\in E_+} \widetilde{c}^d(z)\check{p}_d,$$

where $\widetilde{c}^1(z) = c^1(z)$ and

$$\widetilde{c}^d(z) = c^d(z) + \partial_z f^d(z) - \frac{d\,\varphi(z)}{h^\vee} f^d(z).$$

We will call $\partial_z f^d(z) - \dfrac{d\,\varphi(z)}{h^\vee} f^d(z)$ the ***d*-twisted derivative** of $f^d(z)$.

The connection $\widetilde{\nabla}$ is of the same form than the initial connection ∇. In particular, it is also a quasi-canonical representative of the oper $[\nabla]$. We will call **residual gauge transformations** the gauge transformations which preserve the form of a quasi-canonical representative (they encode the remaining gauge freedom of the oper once we find its quasi-canonical form). The gauge transformations by elements of the form (8.2.2) are thus residual gauge transformations. In the preprint [P5], we also prove that these are the unique ones. Thus, the oper $[\nabla]$ is entirely characterised by the functions $\varphi(z)$ and $c^d(z)$'s, up to the transformation

$$c^d(z) \longmapsto c^d(z) + \partial_z f^d(z) - \frac{d\,\varphi(z)}{h^\vee} f^d(z), \qquad \text{for } \ d \in E_+ \setminus \{1\}. \tag{8.2.3}$$

Note that the exponent 1 has a singular behaviour compared to all other positive exponents: there is no residual gauge transformations modifying the coefficient $c^1(z)$ of $\check{p}_1$ in ∇. Indeed, if one considers a gauge transformation of ∇ by $g(z) = \exp\big(-f(z)\check{p}_1\big)$, the coefficient $c^1(z)$ is shifted by the 1-twisted derivative of f but the fact that

$$[\check{p}_1, \check{p}_{-1}] = \delta$$

creates a term $f(z)\delta$ in $g\nabla g^{-1}$. Thus, the representative $g\nabla g^{-1}$ of the oper is not in quasi-canonical form. We will come back on this fact later.

8.2.3 Hypergometric functions on ${}^L\mathfrak{g}$-opers

Recall that the FFR approach for finite $\mathfrak{g}$-Gaudin models involves functions on ${}^L\mathfrak{g}$-opers. To generalise this approach to quantum AGM, we will need to understand what are the functions on ${}^L\mathfrak{g}$-opers, for $\mathfrak{g}$ affine. As explained above, the coefficients $c^d(z)$ of a quasi-canonical form (8.2.1) are not gauge-invariant. Thus, they do not define a function on the space of opers. To find such functions, one needs to extract from $c^d(z)$ a quantity invariant under the residual gauge transformations (8.2.3).

As these residual gauge transformations involve the function $\varphi(z)$ appearing in the coefficient of ρ (which then depends on the considered oper), one expects gauge-invariant quantities to be defined in a way depending on $\varphi(z)$. Thus, such a gauge-invariant quantity would only define a function on the space $\mathrm{Op}^{\varphi}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ of ${}^L\mathfrak{g}$-opers sharing the same function $\varphi(z)$ (as this function appears in the coefficient of ρ, which is independent of the choice of representative, the definition of this space makes sense).

The case $\varphi = 0$. To gain some intuition, let us first consider the case where $\varphi(z) = 0$ (as explained in Subsection 8.2.1, this is the case considered initially by Drinfeld and Sokolov in [181]). The residual gauge transformation (8.2.3) simply reduces to

$$c^d(z) \longmapsto c^d(z) + \partial_z f^d(z).$$

It is then clear that one can construct gauge-invariant quantities by considering integrals

$$\oint_\gamma c^d(z)\,\mathrm{d}z$$

of $c^d(z)$ over any closed contour γ (on which $c^d(z)$ is regular).

The case $\varphi \neq 0$. Let us come back to the general case $\varphi \neq 0$. We introduce the function

$$\mathscr{P}(z) = \exp\left(\int \varphi(z)\,\mathrm{d}z\right),$$

where $\int \varphi(z)\,\mathrm{d}z$ denotes any primitive of φ. The function $\mathscr{P}$ satisfies

$$\partial_z \log\big(\mathscr{P}(z)\big) = \varphi(z).$$

The residual gauge transformation (8.2.3) can then be rewritten

$$\mathscr{P}(z)^{-d/h^\vee} c^d(z) \longmapsto \mathscr{P}(z)^{-d/h^\vee} c^d(z) + \partial_z\left(\mathscr{P}(z)^{-d/h^\vee} f^d(z)\right).$$

Thus, one can construct gauge-invariant quantities as integrals

$$\oint_\gamma \mathscr{P}(z)^{-d/h^\vee} c^d(z)\,\mathrm{d}z \tag{8.2.4}$$

over closed contours γ on which $\mathscr{P}(z)^{-d/h^\vee} c^d(z)$ is regular. The gauge-invariance of these quantities can be seen as the fact that for any function $f(z)$, one has

$$\oint_\gamma \mathscr{P}(z)^{-d/h^\vee}\left(\partial_z f(z) - \frac{d\,\varphi(z)}{h^\vee} f(z)\right)\mathrm{d}z = 0, \tag{8.2.5}$$

i.e. the integral of a d-twisted derivative pondered by the function $\mathscr{P}(z)^{-d/h^\vee}$ vanishes.

φ with simple poles and hypergeometric integrals. So far, we left an issue undiscussed. Indeed, as $\varphi(z)$ is a rational function of z, the function $\mathscr{P}(z)$ is in general multi-valued and should be more rigorously defined on a multi-sheeted covering of $\mathbb{P}^1$. We thus should restrict equation (8.2.4) to contours γ on which there exists a single-valued branch of the function $\mathscr{P}(z)$.

We will now restrict to the case where $\varphi(z)$ possesses simple poles at points $\boldsymbol{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$ (it is thus of the form (8.1.8), as the twist function of an AGM with simple poles at $\boldsymbol{z}$). We then consider the space

$$\mathrm{Op}^{\varphi}_{{}^L\mathfrak{g},\boldsymbol{z}}\big(\mathbb{P}^1\big) \subset \mathrm{Op}^{\varphi}_{{}^L\mathfrak{g}}\big(\mathbb{P}^1\big)$$

of ${}^L\mathfrak{g}$-opers which are regular everywhere except at the z_i's. For a function φ of the form (8.1.8), the function $\mathscr{P}$ is given formally by

$$\mathscr{P}(z) = \prod_{i=1}^{N} (z - z_i)^{k_i}.$$

In this case, examples of contours γ on which there exists a single-valued branch of this function are known. They are called the **Pochhammer contours**, which are closed contours winding in a particular way around the z_i's. We will not enter into more details about these contours here and refer to

the preprint [P5] at the end of this thesis.

Let us summarise and combine the observations made in this subsection. Let $[\nabla]$ be an oper in $\mathrm{Op}^{\varphi}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$. We consider a quasi-canonical form of $[\nabla]$:

$$\partial_z + \check{p}_{-1} - \frac{\varphi(z)}{h^\vee}\rho + \sum_{d\in E_+} c^d(z)\check{p}_d.$$

As $[\nabla]$ is regular on $\mathbb{C}\setminus\{z_1,\cdots,z_N\}$, the coefficients $c^d(z)$ can be chosen to be regular everywhere expect at the z_i's. Let γ be a Pochhammer contour and d a positive exponent greater than one. The quantity

$$I^d_\gamma([\nabla]) = \oint_\gamma \mathscr{P}(z)^{-d/h^\vee} c^d(z)\,\mathrm{d}z \tag{8.2.6}$$

is well-defined, as $\mathscr{P}$ has a single-valued branch along γ and c^d is regular on γ. It is called an **hypergeometric integral**. Moreover, it is gauge-invariant and thus depends only on the oper $[\nabla]$. We thus constructed a function

$$I^d_\gamma : \mathrm{Op}^{\varphi}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1) \longrightarrow \mathbb{C}$$

on the space of opers $\mathrm{Op}^{\varphi}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$ *via* an hypergeometric integral.

As the residual gauge transformation does not affect the coefficient $c^1(z)$, any quantity extracted from this coefficient (residue, integral, ...) also defines a function on $\mathrm{Op}^{\varphi}_{L_{\mathfrak{g},\boldsymbol{z}}}(\mathbb{P}^1)$.

8.3 Hierarchies of quantum AGM: some conjectures and a first result

Based on the description of functions on ${}^L\mathfrak{g}$-opers presented in the previous subsection, we present conjectures about hierarchies of quantum AGM and their spectrum. We support these conjectures by different observations and by presenting a first step towards their proofs, based on a second preprint [P6] of B. Vicedo, C.A.S. Young and myself. In this section, we then consider a quantum AGM with twist function (8.1.8).

8.3.1 Higher-degrees Hamiltonians

Conjecture 8.3.1. *There exist operators $S^d(z)$ of the Gaudin model, labelled by positive exponents $d\in E_+$ greater than one, which satisfy the following properties.*

(i) *$S^d(z)$ is of degree $d+1$;*

(ii) *For any $p,q\in E_+\setminus\{1\}$, we have*

$$\big[S^p(z),S^q(w)\big] = \big(h^\vee\partial_z - p\varphi(z)\big)A_{pq}(z,w) + \big(h^\vee\partial_w - q\varphi(w)\big)B_{pq}(z,w)$$

for some operators $A_{pq}(z,w)$ and $B_{pq}(z,w)$.

(iii) *For any $p\in E_+\setminus\{1\}$, we have*

$$\big[S^p(z),\mathscr{H}(w)\big] = \big(h^\vee\partial_z - p\varphi(z)\big)A_{p1}(z,w)$$

for some operators $A_{p1}(z,w)$.

If this conjecture holds, then for any Pochhammer contour γ, we define

$$\mathscr{H}^d_\gamma = \oint_\gamma \mathscr{P}(z)^{-d/h^\vee} S^d(z)\,\mathrm{d}z. \tag{8.3.1}$$

Using the property (8.2.5) that the integral of a d-twisted derivative pondered by $\mathscr{P}(z)^{-d/h^\vee}$ vanishes, we would then prove that for any $p,q\in E_+\setminus\{1\}$ and any Pochhammer contours γ and γ',

$$\big[\mathscr{H}^p_\gamma,\mathscr{H}^q_{\gamma'}\big] = 0.$$

Similarly, we would get

$$\big[\mathscr{H}^p_\gamma, \mathscr{H}(z)\big] = 0$$

for all $z \in \mathbb{C}$. This way, we would have constructed an infinite number of commuting higher-degrees Hamiltonians, labelled by the positive exponents of $\mathfrak{g}$ (which are the same as the ones of ${}^L\mathfrak{g}$) and by Pochhammer contours. This would define a quantum hierarchy of the model.

In a second preprint [P6] with B. Vicedo and C.A.S. Young, we obtained a first step in the proof of Conjecture 8.3.1. Indeed, we constructed for untwisted affine algebras of type A, which possess 2 as an exponent, the **cubic operator $S^2(z)$**. This operator satisfies the properties (ii) and (iii) of Conjecture 8.3.1, ensuring the existence of cubic Hamiltonians $\mathscr{H}^2_\gamma$, which commute between themselves and with the quadratic Hamiltonian $\mathscr{H}(z)$. The construction of the operator $S^2(z)$ and the proof of the properties (ii) and (iii) are based on vertex algebras techniques. We will not discuss this further here and refer to the preprint [P6], joined at the end of this thesis.

8.3.2 Spectrum of the affine Gaudin Hamiltonians

We now consider the Hilbert space $H_{\boldsymbol{\lambda}}$, as in Subsection 8.1.2. As the Hamiltonians $\mathscr{H}(z)$ and $\mathscr{H}^d_\gamma$ commute (assuming Conjecture 8.3.1), they can be diagonalised simultaneously on $H_{\boldsymbol{\lambda}}$. We already know how to diagonalise the quadratic Hamiltonian $\mathscr{H}(z)$, using the Bethe ansatz.

The eigenvalue of $\mathscr{H}(z)$ on an on-shell Bethe vector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ is given by equation (8.1.5) in terms of the weight (8.1.6) associated with $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$. From this weight, we define the ${}^L\mathfrak{g}$-Miura oper:

$$\nabla^{\boldsymbol{c}}_{\boldsymbol{w}} = \partial_z + \check{p}_{-1} - \lambda^{\boldsymbol{c}}(z, \boldsymbol{w}),$$

in a similar way to the finite case (see Section 7.3). According to equation (8.1.7), the coefficient of ρ in $\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}$ is

$$c^0(z) = -\frac{\varphi(z)}{h^\vee}.$$

Thus the oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ belongs to the space $\mathrm{Op}^{\varphi}_{L_{\mathfrak{g}}}\big(\mathbb{P}^1\big)$. Moreover, one shows (see [P5]) that the Bethe equations for $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ implies that $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ is regular at all Bethe roots w_j's (as in the finite case). The oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$ then belongs to the space $\mathrm{Op}^{\varphi}_{L_{\mathfrak{g},\boldsymbol{z}}}\big(\mathbb{P}^1\big)$. We consider a quasi-canonical form of this oper:

$$[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]_{\mathrm{qc}} = \partial_z + \check{p}_{-1} - \frac{\varphi(z)}{h^\vee} + \sum_{d \in E_+} \mathcal{C}^d_{\boldsymbol{c}}(z, \boldsymbol{w}). \tag{8.3.2}$$

One then shows (see also [P5]) that the eigenvalue of $\mathscr{H}(z)$ on $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ coincides with $\mathcal{C}^1_{\boldsymbol{c}}(z, \boldsymbol{w})$, up to a (model-independent) global factor. Thus, this eigenvalue can be read from the oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$.

We now formulate another conjecture about the Bethe ansatz for higher-degree Hamiltonians and the description of their eigenvalues in terms of the oper $[\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]$.

Conjecture 8.3.2. *Let d be a positive exponent greater than 1 and γ be a Pochhammer contour. The on-shell Bethe state $\Psi^{on}_{\boldsymbol{c}}(\boldsymbol{w})$ in $H_{\boldsymbol{\lambda}}$ is an eigenvector of $\mathscr{H}^d_\gamma$ and its eigenvalue is given by*

$$I^d_\gamma\big([\nabla^{\boldsymbol{c}}_{\boldsymbol{w}}]\big) = \oint_\gamma \mathscr{P}(z)^{-d/h^\vee} \mathcal{C}^d_{\boldsymbol{c}}(z, \boldsymbol{w})\, dz. \tag{8.3.3}$$

This conjecture is an affine version of the FRR approach developed in Section 7.3 for finite Gaudin models (or at least of the FFR reformulation of the Bethe ansatz). In our second preprint [P6], we have checked this conjecture for the cubic Hamiltonians $\mathscr{H}^2_\gamma$ (see discussion at the end of previous subsection) and for a Bethe vector $\Psi^{\mathrm{on}}_{\boldsymbol{c}}(\boldsymbol{w})$ with one excitation. We will not present this computation here.

As explained in Subsection 8.2, the exponent 1 has a different behaviour than other positive exponents in the quasi-canonical form. We recover this fact in Conjectures 8.3.1 and 8.3.2. Indeed, there

exists a quadratic Hamiltonian $\mathscr{H}(z)$ depending on the spectral parameter z whereas the construction of the higher-degree Hamiltonians $\mathscr{H}_\gamma^d$ necessitates an integral (8.3.1) on the spectral parameter. The analogue of this fact in terms of opers is that the coefficient $\mathcal{C}_{\boldsymbol{c}}^1(z, \boldsymbol{w})$ of the quasi-canonical form (8.3.2) is gauge-invariant whereas the coefficients $\mathcal{C}_{\boldsymbol{c}}^d(z, \boldsymbol{w})$ $(d > 1)$ are not and give gauge-invariant quantities only after taking integrals (8.3.3).

8.3.3 Classical limit

We close this section by a brief discussion about the classical limit of this conjectured quantum hierarchy. To take this classical limit, one has to reintroduce the Planck constant $\hbar$ in the computation. One of the effect of this is to rescale the levels k_i's of the model to $\frac{k_i}{\hbar}$. The expression (8.3.1) then becomes

$$\mathscr{H}_\gamma^d = \oint_\gamma \exp\left(-\frac{d\,\chi(z)}{\hbar}\right) S^d(z)\,\mathrm{d}z, \tag{8.3.4}$$

where $\chi(z)$ is a primitive of $\varphi(z)/h^\vee$. The classical limit $\hbar \to 0$ of this integral is given by a saddle-point approximation (we refer to [94] for the saddle-point approximation method for integrals over Pochhammer contours). In particular, the effect of this saddle point approximation is to localise the spectral parameter z to extrema of $\chi(z)$. By definition, these extrema are exactly the **zeros of the twist function $\boldsymbol{\varphi}$**. Moreover, for generic values of the sites z_i's and the levels k_i's there are as many zeros of φ as there are independent Pochhammer contours γ. One then expects that for any zero x of φ, there exists a Pochhammer contour γ such that the classical limit of the operator (8.3.4) is the evaluation $S^d(x)$.

The evaluation of the spectral parameter at zeros of the twist function is the method we used in [P3] (and Subsection 6.2.6 of this thesis) to construct an infinite hierarchy for classical AGM. We thus conjecture that the evaluation $S^d(x)$ obtained above as a classical limit of the quantum Hamiltonian $\mathscr{H}_\gamma^d$ corresponds to the charge of degree $d+1$ in the classical hierarchy constructed by evaluation at the zero x.

As explained in the introduction of this chapter, the construction of the classical hierarchy is faced with the non-existence of invariant polynomials on $\mathfrak{g}$ of degrees greater than 2: this difficulty is then overcome by the evaluation at zeros of the twist function. According to the discussion above, the quantum analogue of this procedure consists in taking integrals over Pochhammer contours pondered by powers of the function $\mathscr{P}(z)$. As there exists an invariant polynomial of degree two (the quadratic form $(\cdot,\cdot)$) and a corresponding quantum Casimir, one can construct a quadratic Hamiltonian $\mathscr{H}(z)$ depending on the spectral parameter, without needing to take such an integral. We recover here the difference between the exponent 1 and other positive exponents.

Chapter 9

Conclusion and perspectives

9.1 The key role of the twist function

The first part of this thesis concerned integrable field theories with **twist function**. It aimed to point out the fundamental role played by this function in the integrable structure of these theories. In particular, it exhibited how the general formalism based on this twist function allows the development of model-independent methods to study these field theories.

A key result in this direction is the content of Chapter 4, which shows the importance of the **zeros** of the twist function. Indeed, under some regularity condition, one can associate with any of these zeros an infinity of conserved local charges in involution. Moreover, these charges form an integrable hierarchy, in the sense that their Hamiltonian flows generate compatible integrable systems on the same phase space.

Chapter 5 focused on Yang-Baxter type deformations and revealed the role played by the **poles** of the twist function. Indeed, these deformations are characterised by the splitting of a double pole of the twist function into two simple poles. Studying the monodromy of the Lax matrix at these simple poles, we proved that, for all these models, one can extract conserved non-local charges forming a q-deformed Poisson-Hopf algebra. Moreover, we showed that these charges are associated with non-local Poisson-Lie deformed symmetries of these models.

Important examples of field theories with twist function are the integrable σ-models and their deformations. In particular, we were able to apply the results of Chapter 5 to all Yang-Baxter type deformations of σ-models and even more generally, the results of Chapter 4 to all integrable σ-models, deformed or not. Doing so, we recovered some already know results for particular models and exhibited that these are actually part of more general, model-independent, constructions, which rely on the twist function and apply to a larger class of models.

Note also that in Chapter 3, we showed that the Bi-Yang-Baxter model is a model with twist function and that it can be seen as the combination of two Yang-Baxter type deformations. Thus, this model enters in the framework of Chapters 4 and 5, showing that it admits an infinite local hierarchy and two deformed Poisson-Lie symmetries.

9.2 Affine Gaudin models

The second part of this thesis focused on Gaudin models and in particular **affine Gaudin models**. Indeed, the latter can classically be realised as integrable field theories with twist function. In this construction, the poles of the twist function are the sites of the Gaudin models. Moreover, as explained in Chapter 6, all zeros of the twist function naturally satisfy the regularity condition necessary to apply the construction of local charges in involution of Chapter 4. Thus, classical affine Gaudin models possess infinite integrable hierarchies.

One of the main subject addressed in this thesis is the quantisation of this hierarchy. More precisely, in Chapter 8, we conjectured that the quantum Hamiltonians in this hierarchy take the form of hypergeometric integrals (over contours determined by the poles of the twist function) and that their

spectrum is described by hypergeometric functions on affine opers. Moreover, we constructed the first higher-degree Hamiltonian in this hierarchy (the cubic one for algebras of type A) and verified that it agrees with the conjectures mentioned above.

These conjectures on quantum affine Gaudin models and their spectrum are motivated by an analogy with the situation for finite Gaudin models. Indeed, for these models, the higher-degrees Hamiltonians and their spectrum can be described in terms of finite opers by the Feigin-Frenkel-Reshetikhin approach. The above conjecture for affine Gaudin model is thus a conjecture about an affine version of this approach. In Chapter 7, we also lay the foundations for another generalisation of this approach, related to cyclotomic finite Gaudin models.

9.3 Perspectives

The work presented in this thesis can be pursued in various directions.

Integrable σ-models and their deformations. Although the Hamiltonian analysis of most deformed integrable σ-models has been carried out, it is still lacking for the multi-parameter deformation of the PCM introduced recently in [47]. It would be interesting to prove the Hamiltonian integrability of this model and determine its twist function. Another interesting question is the construction of deformations of $\mathbb{Z}_T$-coset σ-models for $T > 2$, which are believed to exist but which have never been explicitly constructed.

FFR approach beyond the Bethe ansatz. As explained in Section 7.3, the FFR approach gives a theoretical description of the entire spectrum of finite Gaudin models in terms of opers. In particular, it ensures the existence of such an oper for any eigenvector, including the ones which are not described by the Bethe ansatz (either because the Hilbert space is not a tensor product of highest-weight vectors or because the Bethe ansatz is not complete). It would be interesting to find explicit constructions of these opers, at least in particular examples.

Quantum hierarchy of affine Gaudin models. A natural project in the continuity of this PhD is to prove the conjectures made in the preprint [P5] (see Chapter 8) about the quantisation of the hierarchy of affine Gaudin models. More precisely, it would be interesting to construct all higher-degrees Hamiltonians of affine Gaudin models, for example using vertex algebra techniques, as done for the cubic charge in [P6] (see also Chapter 8). The next natural step would then be to describe the spectrum of this hierarchy, either through the Bethe ansatz or by proving the conjectured generalisation of the FFR approach for affine models.

The quantum hierarchy conjectured in Chapter 8 concerns the simplest affine Gaudin models: complex, without cyclotomy and with only simple poles. If this hierarchy exists, it is also natural to search for its generalisation when removing these constraints, aiming for the construction of a quantum hierarchy for the most general dihedral affine Gaudin model with arbitrary multiplicities.

ODE/IM correspondence. An affine FFR approach, as mentioned in the previous paragraph, would describe the spectrum of affine Gaudin models in terms of affine opers. There exists another result in the literature which relates the spectrum of some integrable field theories with affine opers, the so-called ODE/IM correspondence (initially proposed in [119] and [117], see also the review [188]). More precisely, this correspondence relates the value of quantum Integrals of Motion (IM) with some Ordinary Differential Equations (ODE), encoded in the form of affine opers [116]. Examples of models for which an ODE/IM correspondence has been proposed are the quantum KdV theory [117, 118], the quantum Boussinesq theory [189], affine Toda field theories [190–195] and the Fateev model [196, 187] (which can be seen as a deformed integrable σ-model [25], more precisely the Bi-Yang-Baxter model on $SU(2)$ [30]).

These models were realised as affine Gaudin models in [116] and [78]. It was proposed in these same articles that the ODE/IM correspondence for these models originates from an affine generalisation of the FFR approach, as the latter would also involve affine opers. However, the usual approach of ODE/IM uses affine opers in a different way than the FFR approach. More precisely, it relates spectral determinants of the ODE (encoded in affine opers) with the eigenvalues of particular operators associated with integrable systems (as for example Q-operators). This relation is based on the observation that these spectral determinants and these eigenvalues satisfy the same functional relations, which encode the Bethe equations of the underlying quantum integrable model (see the above references and [197–201]).

It would be interesting to understand how these results, which use spectral determinants of affine opers, can be related with the FFR approach, which involves functions on affine opers. In the long-term, this could provide a proof of an ODE/IM correspondence for affine Gaudin models through the FFR theory. A particularly interesting observation concerning this is the fact [187] that the ODE/IM correspondence for the Fateev model involves hypergometric integrals over Pochhammer contours. This is in striking resemblance with the description of functions on affine opers that we presented in [P5] (see Chapter 8).

It can be also interesting to note that functional relations like the ones appearing in the ODE/IM correspondence can be reinterpreted in a more abstract way as relations between representations of some algebraic objects. For example, the functional relations proved in [200, 201] have been recently related to representations of quantum affine Borel algebras in [202].

Part III

Appendices

Appendix A

Lie algebras

A.1 Generalities

In this section, we introduce a few basic facts and notations about Lie algebras. We will consider Lie algebras over a field $\mathbb{K}$, which we restrict to be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. A Lie algebra is then a $\mathbb{K}$-vector space $\mathfrak{g}$, equipped with a skew-symmetric bilinear map:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

which satisfies the Jacobi identity

$$\forall\, X, Y, Z \in \mathfrak{g}, \qquad \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0.$$

We will assume that $\mathfrak{g}$ is of finite dimension. Let us fix a basis $\{I^a\}$ $(a = 1, \cdots, n)$ of $\mathfrak{g}$. The **structure constants** of $\mathfrak{g}$ are defined by

$$[I^a, I^b] = f^{ab}{}_c\, I^c,$$

where a summation is implied on the repeated index c. The Jacobi identity is then equivalent to

$$f^{ae}{}_d\, f^{bc}{}_e + f^{be}{}_d\, f^{ca}{}_e + f^{ce}{}_d\, f^{ab}{}_e = 0.$$

For X in $\mathfrak{g}$, we define its adjoint action as the following endomorphism of $\mathfrak{g}$

$$\mathrm{ad}_X : \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ Y & \longmapsto & [X,Y] \end{array}.$$

By the Jacobi identity, the application $\mathrm{ad} : X \mapsto \mathrm{ad}_X$ is then a Lie algebra morphism from $\mathfrak{g}$ to $\mathrm{End}(\mathfrak{g})$ (the algebra of endomorphisms of $\mathfrak{g}$):

$$\mathrm{ad}_{[X,Y]} = \big[\mathrm{ad}_X, \mathrm{ad}_Y\big] = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X.$$

We define the Killing form on $\mathfrak{g}$ as the bilinear form

$$\kappa : \begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathbb{K} \\ (X,Y) & \longmapsto & \mathrm{Tr}\big(\mathrm{ad}_X \circ \mathrm{ad}_Y\big) \end{array}.$$

By cyclicity of the trace, it is clear that κ is symmetric. The main property of κ is its ad-invariance:

$$\kappa\big([X,Y],Z\big) = \kappa\big(X,[Y,Z]\big).$$

Writing the Killing form in the basis $\{I^a\}$ as

$$\kappa^{ab} = \kappa\big(I^a, I^b\big),$$

we then have

$$\kappa^{ab} = f^{ac}{}_d\, f^{bd}{}_c.$$

The ad-invariance equation then becomes

$$\kappa^{ad}\, f^{bc}{}_d + \kappa^{db}\, f^{ac}{}_d = 0. \tag{A.1.1}$$

A.2 Semi-simple finite-dimensional complex Lie algebras

A Lie algebra $\mathfrak{g}$ is said to be simple if it does not admit non trivial ideals and semi-simple if it is a direct sum of simple algebras. Semi-simple algebras are characterised by the Cartan criteria:

Theorem A.2.1. *$\mathfrak{g}$ is semi-simple if and only if its Killing form κ is non-degenerate.*

The semi-simple complex Lie algebras of finite dimension have been classified by Cartan using the notions of Cartan subalgebra and root system. Although it will be an important formalism for this thesis, we will not derive this classification here as it is a classical result (see for example [175]). We will simply summarise its main results and introduce the notations and conventions we use in this thesis.

A.2.1 Cartan-Weyl basis

Cartan-Weyl basis. The fundamental objects describing a semi-simple complex Lie algebra $\mathfrak{g}$ are the following:

- $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, of dimension ℓ the rank of $\mathfrak{g}$,
- $\Delta \subset \mathfrak{h}^*$ and $\Delta^\vee \subset \mathfrak{h}$ the associated sets of roots and coroots,
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ the Cartan-Weyl decomposition,
- $\{E_\alpha, \alpha > 0\}$ and $\{E_{-\alpha}, \alpha > 0\}$ the associated bases of the nilpotent subalgebras $\mathfrak{n}_\pm$,
- $\alpha_1, \cdots, \alpha_\ell \in \Delta$ the simple roots and $\check{\alpha}_1, \cdots, \check{\alpha}_\ell \in \Delta^\vee$ the associated coroots.

We consider the normalisation of the E_α's such that

$$\kappa\left(E_\alpha, E_\beta\right) = \delta_{\alpha,-\beta}, \tag{A.2.1}$$

For $\alpha \in \Delta$, we define $H_\alpha \in \mathfrak{h}$ *via* the Killing form isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$:

$$\forall X \in \mathfrak{h},\ \kappa(H_\alpha, X) = \alpha(X).$$

In particular, for simple roots, let us denote $H_i = H_{\alpha_i}$. Then, $\{H_i,\ i = 1, \cdots, \ell\}$ is a basis of $\mathfrak{h}$. The element H_α is related to the coroot $\check{\alpha}$ associated with α by the relation:

$$\check{\alpha} = \frac{2H_\alpha}{\kappa(H_\alpha, H_\alpha)}.$$

We will denote by $\Delta_\pm$ the set of positive and negative roots. For $\alpha \in \Delta_+$, we will sometimes use the notation

$$F_\alpha = E_{-\alpha}.$$

Commutation relations. The element E_α is characterised by the commutation relation

$$\forall X \in \mathfrak{h},\ [X, E_\alpha] = \alpha(X) E_\alpha.$$

Moreover, we have

$$[E_\alpha, E_{-\alpha}] = H_\alpha \quad \text{and} \quad \left[E_{\alpha_i}, E_{-\alpha_j}\right] = \delta_{ij} H_i. \tag{A.2.2}$$

Finally, the structure of $\mathfrak{n}_\pm$ is given by:

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \tag{A.2.3}$$

with $N_{\alpha,\beta}$ a real skew-symmetric normalisation constant. Moreover, one has $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$. For $i \in \{1, \cdots, \ell\}$, we define the so-called Chevalley generators

$$E_i = E_{\alpha_i} \quad \text{and} \quad F_i = F_{\alpha_i} = E_{-\alpha_i}.$$

Then, the E_i's and F_i's generate the whole algebra $\mathfrak{g}$.

A.2.2 Simple roots, coroots, weights and coweights.

In this appendix, we recall the main properties of some bases of the Cartan subalgebra $\mathfrak{h}$. As it is equipped with the non-degenerate Killing form $\kappa(\cdot,\cdot)$, there exists a natural isomorphism

$$\zeta : \mathfrak{h}^* \to \mathfrak{h},$$

between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. It is characterised by the relation

$$\forall\, \lambda \in \mathfrak{h}^*,\ \ \forall\, X \in \mathfrak{h},\ \ \kappa\big(\zeta(\lambda), X\big) = \lambda(X).$$

By this isomorphism, one induces a bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$:

$$\forall\, \lambda, \mu \in \mathfrak{h}^*,\ \ (\lambda,\mu) = \kappa\big(\zeta(\lambda), \zeta(\mu)\big).$$

Roots and co-roots. A basis of $\mathfrak{h}^*$ is given by the simple roots $\alpha_1, \cdots, \alpha_\ell$. Using the notations of Subsection A.2.1, the corresponding basis $\{\zeta(\alpha_i)\}$ of $\mathfrak{h}$ is $\{H_1, \cdots, H_\ell\}$. We will often use

$$d_i = \frac{(\alpha_i,\alpha_i)}{2} = \frac{\kappa(H_i,H_i)}{2}.$$

The simple co-roots are given by

$$\check{\alpha}_i = \frac{2H_i}{(\alpha_i,\alpha_i)} = d_i^{-1} H_i,$$

which form another basis of $\mathfrak{h}$. The Cartan matrix $A = \big(a_{ij}\big)_{i,j=1,\cdots,\ell}$ is then given by

$$a_{ij} = \alpha_j(\check{\alpha}_i) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}. \tag{A.2.4}$$

Let us introduce the diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_\ell)$. Then one has $A = DB$ with $B = \big(b_{ij}\big)_{i,j=1,\cdots,\ell}$ the symmetric matrix defined by

$$b_{ij} = (\alpha_i,\alpha_j) = \kappa(H_i,H_j).$$

Fundamental weights and co-weights. We define the fundamental weights $\omega_i \in \mathfrak{h}^*$ as the dual basis of the co-roots $\check{\alpha}_i$:

$$\omega_i(\check{\alpha}_i) = \delta_{ij}.$$

By the Killing form duality, the

$$P_i = \zeta(\omega_i)$$

form a basis of $\mathfrak{h}$. Moreover, we have the relation

$$\alpha_i(P_j) = d_i \delta_{ij}. \tag{A.2.5}$$

In the same way, one defines the fundamental co-weights $\check{\omega}_i \in \mathfrak{h}$ as the dual basis of the simple roots:

$$\alpha_j(\check{\omega}_i) = \delta_{ij},$$

which simply relates them to the weights by

$$\check{\omega}_i = d_i^{-1} P_i.$$

The scalar products between coweights are given by

$$\kappa(\check{\omega}_i, \check{\omega}_j) = m_{ij}, \qquad \text{where} \qquad M = \big(m_{ij}\big)_{i,j=1,\cdots,\ell} = B^{-1},$$

with the matrix B as defined in the previous paragraph.

A.2.3 Classification

A complex semi-simple Lie algebra is uniquely determined, up to isomorphisms, by its Cartan matrix (A.2.4). One can show that this matrix satisfies the following conditions

(a) $a_{ii} = 2$ for all $i \in \{1, \cdots, \ell\}$;

(b) $a_{ij} \in -\mathbb{N}$ for $i \neq j$;

(c) if $a_{ij} = 0$, then $a_{ji} = 0$;

(d) $A = DB$ with D diagonal and B symmetric definite positive.

Conversely, given any matrix A satisfying these conditions, there exists a semi-simple complex Lie algebra with Cartan matrix A. The classification of semi-simple complex Lie algebras is then reduced to the classification of such matrices.

This classification is a classic result (see for example [175]), that we shall not describe in detail here. Semi-simple complex Lie algebras are divided in seven different types, from A to G. Elements of type X are denoted X_ℓ, where ℓ is the size of the corresponding Cartan matrix (or equivalently the rank of the corresponding Lie algebra). The types A, B, C and D are infinite families corresponding to classical Lie algebras:

- A_ℓ is the special linear algebra $\mathfrak{sl}(\ell+1, \mathbb{C}) = \big\{M \in M_{\ell+1}(\mathbb{C}) \mid \mathrm{Tr}(M) = 0\big\}$,
- B_ℓ is the orthogonal algebra $\mathfrak{so}(2\ell+1, \mathbb{C}) = \big\{M \in M_{2\ell+1}(\mathbb{C}) \mid {}^tM + M = 0\big\}$,
- C_ℓ is the symplectic algebra $\mathfrak{sp}(2\ell, \mathbb{C}) = \big\{M \in M_{2\ell}(\mathbb{C}) \mid {}^tMJ + JM = 0\big\}$ with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$,
- D_ℓ is the orthogonal algebra $\mathfrak{so}(2\ell, \mathbb{C}) = \big\{M \in M_{2\ell}(\mathbb{C}) \mid {}^tM + M = 0\big\}$.

At the contrary, the so-called exceptional types E, F and G are finite families. More precisely, there are five exceptional Lie algebras: E_6, E_7, E_8, F_4 and G_2.

A.2.4 Split quadratic Casimir

Let us fix a basis $\{I^a\}$ of $\mathfrak{g}$. Recall the Killing form κ^{ab} expressed in this basis. As $\mathfrak{g}$ is semi-simple, it is non-degenerate (see Theorem A.2.1). We can then define the inverse κ_{ab} of the Killing form, satisfying $\kappa^{ac}\kappa_{cb} = \delta^a{}_b$. In this paragraph, we use the tensorial notations $\underline{\boldsymbol{i}}$ defined in Section 2.2. We define the split quadratic Casimir of $\mathfrak{g}$ as

$$C_{\underline{\mathbf{12}}} = \kappa_{ab}\, I^a \otimes I^b.$$

One checks that it does not depend on the choice of basis. The ad-invariance equation (A.1.1) translates to

$$\big[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{1}}} + X_{\underline{\mathbf{2}}}\big] = 0, \quad \forall\, X \in \mathfrak{g}. \tag{A.2.6}$$

Moreover, using the fact that κ_{ab} is the inverse of κ^{ab}, we get the following completeness relation:

$$\kappa_{\underline{\mathbf{2}}}\big(C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}\big) = X, \quad \forall\, X \in \mathfrak{g}. \tag{A.2.7}$$

In the Cartan-Weyl basis described in Subsection A.2.1, we get

$$C_{\underline{\mathbf{12}}} = H_{\underline{\mathbf{12}}} + \sum_{\alpha\in\Delta} E_\alpha \otimes E_{-\alpha} = H_{\underline{\mathbf{12}}} + \sum_{\alpha\in\Delta_+} \big(E_\alpha \otimes F_\alpha + F_\alpha \otimes E_\alpha\big),$$

where $H_{\underline{\mathbf{12}}}$ belongs to $\mathfrak{h}\otimes\mathfrak{h}$. More precisely, $H_{\underline{\mathbf{12}}}$ can be expressed in various way in terms of the bases of $\mathfrak{h}$ described in Subsection A.2.2. In particular, we have

$$H_{\underline{\mathbf{12}}} = \sum_{i,j=1}^{\ell} m_{ij} H_i \otimes H_j = \sum_{i=1}^{\ell} P_i \otimes \check{\alpha}_i.$$

A.3 Real forms

A.3.1 Generalities

Let us consider a complex Lie algebra $\mathfrak{g}$ of dimension n, with basis $\{I^a,\, a=1,\cdots,n\}$. It can be seen as a real Lie algebra of dimension $2n$, with basis $\{I^a,\, a=1,\cdots,n\} \sqcup \{i\, I^a,\, a=1,\cdots,n\}$. We will call this real algebra the realification $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$. A **real form** of $\mathfrak{g}$ is then a (real) subalgebra of $\mathfrak{g}_{\mathbb{R}}$ of dimension n.

In other words, a real form amounts to the choice of a basis $\{I^a,\, a=1,\cdots,n\}$ of $\mathfrak{g}$ whose structure constants are real. The real form is then $\mathfrak{g}_0 = \mathrm{Span}_{\mathbb{R}}\big(I^a,\, a=1,\cdots,\ell\big)$. In this case, we recover $\mathfrak{g}$ as the complexification of $\mathfrak{g}_0$:

$$\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i\, \mathfrak{g}_0.$$

Let us then define

$$\begin{array}{rccc} \tau: & \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \\ & X \otimes z & \longmapsto & X \otimes \bar{z} \end{array}.$$

One easily checks the following properties of τ:

(a) τ is antilinear, *i.e.* $\tau(\lambda X + \mu Y) = \bar{\lambda}\,\tau(X) + \bar{\mu}\,\tau(Y)$ for all $X, Y \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$;

(b) τ is an involution, *i.e.* $\tau^2 = \mathrm{Id}$;

(c) τ is a Lie algebra automorphism, *i.e.* $\tau\big([X,Y]\big) = \big[\tau(X), \tau(Y)\big]$;

(d) the subalgebra of fixed points of τ is the real form $\mathfrak{g}_0 = \mathfrak{g}^{\tau}$.

Conversely, such a antilinear involutive automorphism of $\mathfrak{g}$ defines a real form of $\mathfrak{g}$:

Theorem A.3.1. *The real forms of $\mathfrak{g}$ are in one-to-one correspondence with the antilinear involutive automorphisms of $\mathfrak{g}$ (as their subalgebra of fixed points).*

A natural question at this point is whether two real forms of $\mathfrak{g}$ can be isomorphic and whether we can classify the isomorphism equivalence classes of real forms of $\mathfrak{g}$. The answer is yes:

Proposition A.3.2. *Let τ and θ be two antilinear involutive automorphisms of $\mathfrak{g}$. Then the two real forms $\mathfrak{g}^{\tau}$ and $\mathfrak{g}^{\theta}$ are isomorphic if and only if there exists an automorphism $\sigma \in Aut(\mathfrak{g})$ of $\mathfrak{g}$ such that $\tau = \sigma \circ \theta \circ \sigma^{-1}$.*

A.3.2 Real semi-simple algebras

Proposition A.3.3. *A real Lie algebra is semi-simple if and only if it is the real form of a semi-simple complex Lie algebra.*

Combining this proposition with Theorem A.3.1 and Proposition A.3.2, we see that the classification of real semi-simple Lie algebras reduces to the classification of complex ones $\mathfrak{g}$ and of their antilinear involutive automorphisms, up to conjugacy in $\mathrm{Aut}(\mathfrak{g})$. The classification of complex semi-simple Lie algebras was described in Appendix A.2, using the notions of root system and Cartan-Weyl basis. Their antilinear involutive automorphisms, up to conjugacy in $\mathrm{Aut}(\mathfrak{g})$, have also been entirely classified [203].

We will not describe here this whole classification. More precisely, we will restrict ourselves to the so-called split and non-split real forms, which exist for all complex Lie algebra $\mathfrak{g}$. We consider the Cartan-Weyl basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ (cf. appendix A.2). As the $E_{\pm\alpha_i}$'s form generators of $\mathfrak{g}$, an automorphism τ of $\mathfrak{g}$ is completely described by its action on it.

Split real form. Let us consider the semi-linear involutive automorphism τ given by:

$$\tau\left(E_{\pm\alpha_i}\right) = E_{\pm\alpha_i}.$$

As the normalisation constants $N_{\alpha,\beta}$ in equation (A.2.3) are real, one gets for any root α:

$$\tau\left(E_\alpha\right) = E_\alpha. \tag{A.3.1}$$

Hence, one also has:

$$\tau\left(H_\alpha\right) = H_\alpha. \tag{A.3.2}$$

As a consequence, a basis of the real subalgebra fixed by τ is given by the Cartan-Weyl basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ itself. This way, we obtain the so-called split real form:

$$\mathfrak{g}_0 = \left(\bigoplus_{i=1}^{\ell} \mathbb{R}\, H_i\right) \oplus \left(\bigoplus_{\alpha\in\Delta} \mathbb{R}\, E_\alpha\right). \tag{A.3.3}$$

For example, for the simple algebra of type A, $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the split real form is simply $\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{R})$.

Non-split real forms. Another possibility for defining τ is:

$$\tau\left(E_{\pm\alpha_i}\right) = -\lambda_i E_{\mp\alpha_i},$$

where $\lambda_i = \pm 1$. Using equation (A.2.3), one obtains for any positive root $\alpha = p_1\alpha_1 + \cdots + p_\ell\alpha_\ell$:

$$\tau\left(E_\alpha\right) = -\lambda_\alpha E_{-\alpha}, \tag{A.3.4}$$

with $\lambda_\alpha = \lambda_1^{p_1}\cdots\lambda_\ell^{p_\ell} \in \{+1,-1\}$. In the same way, using (A.2.2), one has:

$$\tau(H_\alpha) = -H_\alpha. \tag{A.3.5}$$

The real subalgebra $\mathfrak{g}_0$ of elements fixed by τ is called a non-split real form of $\mathfrak{g}$. A basis of $\mathfrak{g}_0$ is given by:

$$T_i = iH_i, \quad B_\alpha = \frac{i}{\sqrt{2}}\left(E_\alpha + \lambda_\alpha E_{-\alpha}\right), \quad C_\alpha = \frac{1}{\sqrt{2}}\left(E_\alpha - \lambda_\alpha E_{-\alpha}\right). \tag{A.3.6}$$

If one chooses $\lambda_1 = \cdots = \lambda_\ell = 1$, then $\lambda_\alpha = 1$ for any root α and we get the so-called compact real form of $\mathfrak{g}$.

Let us discuss a few examples of non-split real forms. For a simple algebra of type A, $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the non-split real forms are the $\mathfrak{su}(p,q,\mathbb{R})$ (unitary algebra for a metric of signature (p,q), with $n = p+q$). In particular, the compact real form is the unitary algebra $\mathfrak{su}(n,\mathbb{R})$. In the same way, for simple algebras of type B and D, $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$, the non-split real forms are the $\mathfrak{so}(p,q,\mathbb{R})$ (orthogonal algebra for a metric of signature (p,q), with $n = p+q$). The compact real form is then $\mathfrak{so}(n,\mathbb{R})$.

A.4 Finite order automorphisms of Lie algebras

Generalities. Let us consider a complex Lie algebra $\mathfrak{g}$ and σ an automorphism of $\mathfrak{g}$ of finite order $T \in \mathbb{Z}_{\geq 1}$. We define

$$\omega = \exp\left(\frac{2i\pi}{T}\right).$$

We define the eigenspaces of σ:

$$\mathfrak{g}^{(p)} = \left\{x \in \mathfrak{g} \mid \sigma(x) = \omega^p x\right\}, \quad p = 0, \cdots, T-1.$$

These eigenspaces form a $\mathbb{Z}_T$-gradation of $\mathfrak{g}$ (where $\mathbb{Z}_T = \mathbb{Z}/T\mathbb{Z}$ is the cyclic group of order T):

$$\mathfrak{g} = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)}, \qquad \text{with} \qquad [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] = \mathfrak{g}^{(p+q \bmod T)}. \tag{A.4.1}$$

Conversely, a $\mathbb{Z}_T$-gradation (A.4.1) defines a unique automorphism σ of order T (we define σ as acting as the multiplication by ω^p on $\mathfrak{g}^{(p)}$). In particular, $\mathfrak{g}^{(0)}$ is a subalgebra of $\mathfrak{g}$ and, for any $p \in \{0, \cdots, T-1\}$, $\mathfrak{g}^{(p)}$ is a $\mathfrak{g}^{(0)}$-module.

We define $\pi^{(p)}$ to be the projection on $\mathfrak{g}^{(p)}$ in the decomposition (A.4.1). These projectors then satisfy $\pi^{(p)}\pi^{(q)} = \delta_{p,q}\pi^{(p)}$. They can be expressed in terms of σ as

$$\pi^{(p)} = \frac{1}{T}\sum_{k=0}^{T-1} \omega^{-kp}\sigma^k. \tag{A.4.2}$$

Inner automorphisms. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Any element $g \in G$ acts on $\mathfrak{g}$ by the adjoint (conjugacy) action:

$$\begin{array}{rccc} \mathrm{Ad}_g : & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & X & \longmapsto & gXg^{-1} \end{array}.$$

This is an automorphism of the Lie algebra $\mathfrak{g}$, called an inner automorphism. We will denote by $\mathrm{Inn}(\mathfrak{g})$ the set of such automorphism: it is a subgroup of $\mathrm{Aut}(\mathfrak{g})$ and moreover, the map

$$\begin{array}{rccc} \mathrm{Ad} : & G & \longrightarrow & \mathrm{Inn}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \end{array}$$

is a Lie group morphism. It induces an automorphism of Lie algebras

$$\begin{array}{rccc} \mathrm{ad} : & \mathfrak{g} & \longrightarrow & \mathrm{ad}(\mathfrak{g}) \\ & X & \longmapsto & \mathrm{ad}_X \end{array},$$

where

$$\begin{array}{rccc} \mathrm{ad}_X : & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & Y & \longmapsto & [X, Y] \end{array}.$$

We recall in particular that for all $X \in \mathfrak{g}$, we have

$$\mathrm{Ad}_{\exp(X)} = \exp\left(\mathrm{ad}_X\right).$$

Let us now suppose that $\mathfrak{g}$ is semi-simple. We will use the notations of Appendix A.2.1. If X is an element of $\mathfrak{h}$ and z is a complex number, we define the group element

$$z^X = \exp\left(\log(z)X\right),$$

where log is a determination of the logarithm on $\mathbb{C}$. We can then define $\mathrm{Ad}_{z^X} \in \mathrm{Inn}(\mathfrak{g})$, which acts on the basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ as

$$\mathrm{Ad}_{z^X}(H_i) = H_i \quad \text{and} \quad \mathrm{Ad}_{z^X}(E_\alpha) = z^{\alpha(X)}E_\alpha.$$

We will be particularly interested in the case where $z = \omega = \exp\left(\frac{2i\pi}{T}\right)$ and X is such that $\alpha(X) \in \mathbb{Z}$ for all $\alpha \in \Delta$, as the automorphism Ad_{ω^X} is then of order T. Recall the coweights $\{\tilde{\omega}_i, i = 1, \cdots, \ell\}$ defined in Subsection A.2.2. The space of $X \in \mathfrak{h}$ such as above is then the lattice of these coweights:

$$\Lambda(\mathfrak{h}) = \left\{X \in \mathfrak{h} \;\middle|\; \alpha(X) \in \mathbb{Z},\, \forall \alpha \in \Delta\right\} = \left\{\sum_{i=1}^{\ell} m_i\tilde{\omega}_i,\ m_i \in \mathbb{Z}\right\}. \tag{A.4.3}$$

Diagram automorphisms of semi-simple Lie algebras. We still suppose that $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra. Recall the Cartan matrix A of $\mathfrak{g}$, defined in (A.2.4). We say that a permutation μ of $\{1,\cdots,\ell\}$ is a diagram automorphism of A if

$$a_{\mu(i)\mu(j)} = a_{ij}, \quad \forall\, i,j \in \{1,\cdots,\ell\}.$$

Recall also the Chevalley generators $C = (E_i, F_i)_{i=1,\cdots,\ell}$ of $\mathfrak{g}$. As they generate the algebra $\mathfrak{g}$, an automorphism of $\mathfrak{g}$ is entirely defined by its action on the E_i's and F_i's. One can show that there exists a unique automorphism μ_C of $\mathfrak{g}$ such that

$$\mu_C(E_i) = E_{\mu(i)} \quad \text{and} \quad \mu_C(F_i) = F_{\mu(i)}.$$

μ_C is then called an outer automorphism of $\mathfrak{g}$. It has the same order than the permutation μ of $\{1,\cdots,\ell\}$: in particular, it is of finite order. Note here that the definition of μ_C from the permutation μ depends on the choice of the Chevalley generators E_i's and F_i's. Two choices $C = (E_i, F_i)_{i=1,\cdots,\ell}$ and $C' = (E'_i, F'_i)_{i=1,\cdots,\ell}$ of such generators are always related by an inner automorphism $\gamma \in \mathrm{Inn}(\mathfrak{g})$. The corresponding automorphisms μ_C and $\mu_{C'}$ of $\mathfrak{g}$ are then related by conjugacy by γ, *i.e.* $\mu' = \gamma \circ \mu \circ \gamma^{-1}$.

A choice of Chevalley generators $C = (E_i, F_i)_{i=1,\cdots,\ell}$ also determines a Cartan subalgebra $\mathfrak{h}$ by specifying the element H_i's (see Appendix A.2.1), as one have $[E_i, F_i] = H_i$. The action of μ_C on the H_i's is simply

$$\mu_C(H_i) = H_{\mu(i)}.$$

Classification of finite order automorphisms of semi-simple Lie algebras. The automorphisms of finite order of $\mathfrak{g}$ are classified by the following theorem [162].

Theorem A.4.1. *Let σ be an automorphism of order T of $\mathfrak{g}$. Then there exist a diagram automorphism μ, a choice of Chevalley generators $C = (E_i, F_i)_{i=1,\cdots,\ell}$ (hence also a choice of Cartan subalgebra $\mathfrak{h}$) and an element $X \in \Lambda(\mathfrak{h})$ (see previous paragraph), such that*

$$\sigma = \mu_C \circ Ad_{\omega^X} = Ad_{\omega^X} \circ \mu_C.$$

Moreover, one has $\mu_C(X) = X$. If we write $X = \sum_{i=1}^{\ell} m_i \check{\omega}_i$ (see equation (A.4.3)), this is equivalent to $m_i = m_{\mu(i)}$ for all $i \in \{1,\cdots,\ell\}$. The action of σ on the Chevalley generators is then given by

$$\sigma(E_i) = \omega^{m_i} E_{\mu(i)} \quad \text{and} \quad \sigma(F_i) = \omega^{-m_i} F_{\mu(i)}.$$

$\mathbb{Z}_T$-gradings of real Lie algebras. In the first paragraph of this section, we have seen that for a complex Lie algebra $\mathfrak{g}$, there is a one-to-one correspondence between $\mathbb{Z}_T$-gradings of $\mathfrak{g}$ and automorphisms of $\mathfrak{g}$ of finite order T. We will now discuss the $\mathbb{Z}_T$-gradings of real Lie algebras. The following lemma, whose demonstration is straightforward, relates them to the $\mathbb{Z}_T$-gradings of complex Lie algebras (recall the complexification of an algebra defined in Appendix A.3).

Lemma A.4.2. *Let $\mathfrak{g}_0$ be a real Lie algebra and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ be its complexification. If $\mathfrak{g}_0$ is equipped with a $\mathbb{Z}_T$-grading $\mathfrak{g}_0 = \bigoplus_{p=0}^{T-1} \mathfrak{g}_0^{(p)}$, then*

$$\mathfrak{g} = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)}, \qquad \mathfrak{g}^{(p)} = \mathfrak{g}_0^{(p)} \otimes \mathbb{C}$$

is a $\mathbb{Z}_T$-grading of $\mathfrak{g}$ and $\mathfrak{g}_0^{(p)} = \mathfrak{g}^{(p)} \cap \mathfrak{g}_0$. Conversely, if $\mathfrak{g}$ is equipped with a $\mathbb{Z}_T$-grading $\mathfrak{g} = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)}$ such that $\mathfrak{g}_0$ is a graded subalgebra of $\mathfrak{g}$, i.e.

$$\mathfrak{g}_0 = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)} \cap \mathfrak{g}_0,$$

then the $\mathfrak{g}_0^{(p)} = \mathfrak{g}^{(p)} \cap \mathfrak{g}_0$'s define a $\mathbb{Z}_T$-grading of $\mathfrak{g}$.

According to the lemma, classifying all $\mathbb{Z}_T$ gradings of $\mathfrak{g}_0$ is equivalent to classifying the $\mathbb{Z}_T$-gradings of the complexification $\mathfrak{g}$ such that $\mathfrak{g}_0$ is a graded subalgebra of $\mathfrak{g}$. Recall that $\mathfrak{g}_0$ can be seen as the subalgbera $\mathfrak{g}^\tau$ of fixed point of an antilinear involutive automorphism τ of $\mathfrak{g}$ and that the $\mathbb{Z}_T$-gradings of $\mathfrak{g}$ are in one-to-one correspondence with automorphisms of order T of $\mathfrak{g}$ (see first paragraph of this section).

Theorem A.4.3. *Let $\mathfrak{g}$ be a complex Lie algebra. We suppose that it possesses an antilinear involutive automorphism τ and a $\mathbb{Z}_T$-grading $\mathfrak{g} = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)}$, associated with an automorphism σ. Then the following points are equivalent:*

(i) $\mathfrak{g}^\tau$ is a graded subalgebra of $\mathfrak{g}$,

(ii) $\tau \circ \sigma = \sigma^{-1} \circ \tau$ (dihedrality condition),

(iii) τ stabilises the eigenspaces $\mathfrak{g}^{(p)}$ of σ,

(iv) τ commutes with the projections $\pi^{(p)}$ on $\mathfrak{g}^{(p)}$,

(v) there exists a common basis of eigenvectors of σ and τ.

Corollary A.4.4. *Let $\mathfrak{g}_0$ be a real Lie algebra. We denote by $\mathfrak{g}$ its complexificaton and by τ the antilinear involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{g}^\tau$. Then the $\mathbb{Z}_T$-gradings of $\mathfrak{g}_0$ are in one-to-one correspondence with the automorphisms σ of order T satisfying the dihedrality condition $\tau \circ \sigma = \sigma^{-1} \circ \tau$.*

Proof. The corollary is straightforward from the theorem and lemma A.4.2. Let us then prove the theorem. We will show the following cyclic sequence of implications: (i)⇒(v)⇒(ii)⇒(iv)⇒(iii)⇒(i).

(i)⇒(v): Let us suppose that $\mathfrak{g}^\tau$ is graded, *i.e.* that

$$\mathfrak{g}^\tau = \bigoplus_{p=0}^{T-1} \mathfrak{g}^\tau \cap \mathfrak{g}^{(p)}.$$

For $p \in \{0, \cdots, T-1\}$, let $X_1^{(p)}, \cdots, X_{m_p}^{(p)}$ be a basis of $\mathfrak{g}^\tau \cap \mathfrak{g}^{(p)}$, with $m_p = \dim\left(\mathfrak{g}^\tau \cap \mathfrak{g}^{(p)}\right)$. As the $X_k^{(p)}$'s are in $\mathfrak{g}^{(p)}$, they are eigenvectors of σ (of eigenvalue ω^p) and as they are in $\mathfrak{g}^\tau$, they are eigenvectors of τ (of eigenvalue 1). In the same way, $iX_k^{(p)}$ is also an eigenvector of σ (of eigenvalue ω^p) and of τ (of eigenvalue -1). the family $B = \bigsqcup_{p=0}^{T-1} \{X_k^{(p)},\, k = 1, \cdots, m_p\}$ is a basis of $\mathfrak{g}^\tau$ and $B' = \bigsqcup_{p=0}^{T-1} \{iX_k^{(p)},\, k = 1, \cdots, m_p\}$ is a basis of $i\mathfrak{g}^\tau$. Thus, $B \sqcup B'$ is a basis of $\mathfrak{g}$ composed of eigenvectors of σ and τ.

(v)⇒(ii): Let us suppose that there exists a common basis of eigenvectors of σ and τ. To prove the identity $\tau \circ \sigma = \sigma^{-1} \circ \tau$, it is enough to prove it on each element of this basis. Let then X be in this basis: there exist $\epsilon \in \{+1, -1\}$ and $p \in \{0, \cdots, T-1\}$ such that $\tau(X) = \epsilon X$ and $\sigma(X) = \omega^p X$ (and thus $\sigma^{-1}(X) = \omega^{-p} X$). As σ is linear and τ is antilinear, we have

$$\tau \circ \sigma(X) = \tau\big(\omega^p X\big) = \omega^{-p} \tau(X) = \omega^{-p} \epsilon X = \epsilon \sigma^{-1}(X) = \sigma^{-1}(\epsilon X) = \sigma^{-1} \circ \tau(X).$$

This then proves (ii).

(ii)⇒(iv): We suppose that $\tau \circ \sigma = \sigma^{-1} \circ \tau$. By recursion on $k \in \mathbb{Z}$, it implies $\tau \circ \sigma^k = \sigma^{-k} \circ \tau$. Recall the expression (A.4.2) of the projector $\pi^{(p)}$. For any $X \in \mathfrak{g}$, we then have

$$\begin{aligned}
\tau \circ \pi^{(p)}(X) &= \tau\left(\frac{1}{T} \sum_{k=0}^{T-1} \omega^{-kp}\, \sigma^k(X)\right) = \frac{1}{T} \sum_{k=0}^{T-1} \omega^{kp}\, \tau \circ \sigma^k(X) \\
&= \frac{1}{T} \sum_{k=0}^{T-1} \omega^{kp}\, \sigma^{-k} \circ \tau(X) = \frac{1}{T} \sum_{j=1}^{T} \omega^{Tp-jp}\, \sigma^{j-T} \circ \tau(X) \\
&= \frac{1}{T} \sum_{j=0}^{T-1} \omega^{-jp}\, \sigma^j \circ \tau(X) = \pi^{(p)} \circ \tau(X),
\end{aligned}$$

where we used the antilinearity of τ for the second equality, the identity $\tau \circ \sigma^k = \sigma^{-k} \circ \tau$ for the third, the map $k \mapsto j = T - k$ for the fourth and the facts that $\omega^T = 1$ and $\sigma^T = \mathrm{Id}$ for the fifth. This then proves (iv).

(iv)⇒(iii): Let us suppose that τ commutes with the projectors $\pi^{(p)}$'s. Let then X be in $\mathfrak{g}^{(p)}$. As $X = \pi^{(p)}(X)$, we have

$$\tau(X) = \tau \circ \pi^{(p)}(X) = \pi^{(p)} \circ \tau(X),$$

hence $\tau(X)$ belongs to the image of $\pi^{(p)}$, *i.e.* to $\mathfrak{g}^{(p)}$. This proves (iii).

(iii)⇒(i): As the $\mathfrak{g}^{(p)}$'s form a direct sum of $\mathfrak{g}$, it is clear that the subspaces $\mathfrak{g}^{(p)} \cap \mathfrak{g}^\tau$ of $\mathfrak{g}^\tau$ have a trivial intersection. Let us now suppose that τ stabilises the $\mathfrak{g}^{(p)}$'s. Let X be in $\mathfrak{g}^\tau$. As an element of $\mathfrak{g}$, it can be decomposed as

$$X = \sum_{p=0}^{T-1} X^{(p)}, \qquad \text{with} \qquad X^{(p)} = \pi^{(p)}(X) \in \mathfrak{g}^{(p)}.$$

Moreover, as $X \in \mathfrak{g}^\tau$, we also have

$$X = \tau(X) = \sum_{p=0}^{T-1} \tau\big(X^{(p)}\big).$$

Yet, by hypothesis, $\tau\big(X^{(p)}\big)$ belongs to $\mathfrak{g}^{(p)}$. By unicity of the decomposition along the direct sum of $\mathfrak{g}^{(p)}$'s, we then have $\tau\big(X^{(p)}\big) = X^{(p)}$, *i.e.* $X^{(p)} \in \mathfrak{g}^\tau$. Thus, we have

$$\mathfrak{g}^\tau = \bigoplus_{p=0}^{T-1} \mathfrak{g}^{(p)} \cap \mathfrak{g}^\tau.$$

This proves (i). □

A.5 Path-ordered exponentials

In this section, we recall some properties of path-ordered exponentials. Let $\mathfrak{g}$ be a Lie algebra and G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider a $\mathfrak{g}$-valued field $\mathcal{L}(x)$ and the path-ordered exponential

$$T(x,y) = \mathrm{P}\overleftarrow{\exp}\left(-\int_y^x \mathrm{d}z\, \mathcal{L}(z)\right).$$

$T(x,y)$ is a G-valued field verifying the differential equations

$$\big(\partial_x T(x,y)\big) T(x,y)^{-1} = -\mathcal{L}(x), \tag{A.5.1a}$$

$$T(x,y)^{-1}\big(\partial_y T(x,y)\big) = \mathcal{L}(y), \tag{A.5.1b}$$

and the initial condition $T(x,x) = \mathrm{Id}$. Under an infinitesimal transformation $\delta\mathcal{L}$ of $\mathcal{L}$, the path-ordered exponential is transformed by

$$\delta T(x,y) = -\int_y^x \mathrm{d}z\; T(x,z)\delta\mathcal{L}(z)T(z,y). \tag{A.5.2}$$

This formula allows one to compute the variations of $T(x,y)$ or Poisson brackets of $T(x,y)$ with other observables.

In particular, if $\mathcal{L}$ is the spatial component of a zero curvature equation

$$\partial_t \mathcal{L} - \partial_x \mathcal{M} + [\mathcal{M}, \mathcal{L}] = 0,$$

one finds from equation (A.5.2) that

$$\partial_t T(x,y) = T(x,y)\mathcal{M}(y) - \mathcal{M}(x)T(x,y).$$

Suppose now that we are given a Lie homomorphism $\sigma : G \to F$, from G to another Lie group F. It induces a Lie algebra homomorphism $\sigma_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{f}$. The image of the path-ordered exponential T by the homomorphism σ is simply

$$\sigma\big(T(x,y)\big) = \mathrm{P}\overleftarrow{\exp}_F \left(- \int_y^x \mathrm{d}z \, \sigma_{\mathfrak{g}}\big(\mathcal{L}(z)\big) \right), \tag{A.5.3}$$

where $\mathrm{P}\overleftarrow{\exp}_F$ designates the path-ordered exponential in the group F.

Appendix B

Poisson and symplectic geometries

In this appendix, we describe the basic notions of Poisson geometry and symplectic geometry. We will restrict by simplicity to the case of manifolds of finite dimension. The notation $\mathcal{F}[M]$ then stands for the smooth functions on M, valued in $\mathbb{R}$. The notions described here generalise to infinite dimensional manifolds, when taking appropriate care of the good definition(s) of infinite dimensional differential manifolds and of the subtleties which come with it. This is useful for field theories (where M is then the manifold of the fields configurations and $\mathcal{F}[M]$ the space of "smooth" functionals on it). However, we shall not enter into these technical considerations here to keep the appendix to a minimum and refer to [204] for a more complete presentation.

B.1 Poisson and symplectic manifolds

Definition B.1.1. *A Poisson manifold M is a differential manifold such that the space of functions $\mathcal{F}[M]$ on M is equipped with a Poisson bracket*

$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{F}[M]\times\mathcal{F}[M] & \longrightarrow & \mathcal{F}[M] \\ & (f,g) & \to & \{f,g\} \end{array},$$

which is a skew-symmetric bilinear derivation satisfying the Jacobi identity:

$$\forall f,g,h\in\mathcal{F}[M],\quad \{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0. \tag{B.1.1}$$

Let $\{x^i\}$ be (local) coordinates on M. Then, as the Poisson bracket is a derivation (on the left and on the right), one can rewrite it in these coordinates as

$$\{f,g\}=\sum_{i,j}P^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{B.1.2}$$

where $P^{ij}=-P^{ji}$ are functions in $\mathcal{F}[M]$. The Jacobi identity is then equivalent to

$$P^{il}\frac{\partial P^{jk}}{\partial x^l}+P^{jl}\frac{\partial P^{ki}}{\partial x^l}+P^{kl}\frac{\partial P^{ij}}{\partial x^l}=0, \tag{B.1.3}$$

where a summation on the repeated index l is implied. The Poisson bracket is said to be non-degenerate if the matrix P^{ij} is invertible (in any coordinate chart).

Definition B.1.2. *A symplectic manifold is a differential manifold M which possesses a closed non-degenerate 2-form ω.*

Proposition B.1.3. *The symplectic manifolds are exactly the Poisson manifolds with non-degenerate Poisson brackets.*

Proof. Suppose that we have a non-degenerate Poisson bracket $\{\cdot,\cdot\}$, given locally by some functional P^{ij}'s. The matrix P^{ij} admits an inverse P_{ij}, satisfying $P^{ik}P_{kj} = \delta^i_{\ j}$. Let us then define the 2-form:

$$\omega = P_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{B.1.4}$$

It is non-degenerate, as P_{ij} is invertible. Moreover, one checks that the Jacoby identity (B.1.3) translates in the closeness of the 2-form ω. Conversely, if one as a symplectic form ω on M, writing it in a local coordinates as (B.1.4) and inverting the matrix P_{ij}, one gets a non-degenerate Poisson bracket on M. □

B.2 The Kirillov-Kostant bracket

Let $\mathfrak{g}$ be a real Lie algebra (see Appendix A for definitions and basics results on Lie algebras). We consider the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, defined as the vector space of linear forms on $\mathfrak{g}$. As $\mathfrak{g}^*$ is a vector space, its tangent space at a point $\xi \in \mathfrak{g}^*$ is identified with $\mathfrak{g}^*$ itself. If f is a real-valued smooth function on $\mathfrak{g}^*$ (*i.e.* an element of $\mathcal{F}[\mathfrak{g}^*]$), the differential $\mathrm{d}_\xi f$ of f at a point $\xi \in \mathfrak{g}^*$ is then a linear map from $\mathfrak{g}^*$ to $\mathbb{R}$, hence an element of $(\mathfrak{g}^*)^* = \mathfrak{g}$. We then define the bracket of two functions f, g in $\mathcal{F}[\mathfrak{g}^*]$ as the function

$$\{f,g\}_*(\xi) = \xi\big([\mathrm{d}_\xi f, \mathrm{d}_\xi g]\big). \tag{B.2.1}$$

As the Lie bracket $[\cdot,\cdot]$ is linear and skew-symmetric, it is clear that the bracket $\{\cdot,\cdot\}_*$ is linear and skew-symmetric. Moreover, by the Leibniz rule $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for the differential, the bracket $\{\cdot,\cdot\}_*$ is a derivation. Finally, the Jacobi identity on the Lie bracket $[\cdot,\cdot]$ implies the one for the bracket $\{\cdot,\cdot\}_*$. The dual space $\mathfrak{g}^*$ is thus naturally equipped with a Poisson bracket $\{\cdot,\cdot\}_*$, called the **Kirillov-Kostant** bracket.

The definition (B.2.1) is an abstract definition of $\{\cdot,\cdot\}_*$ for all functions f and g in $\mathcal{F}[\mathfrak{g}^*]$. Let us now give a more explicit formula for the bracket $\{\cdot,\cdot\}_*$. As $\mathfrak{g}^*$ is a vector space, it is easy to find smooth functions on it: the linear forms in $(\mathfrak{g}^*)^*$. These can be canonically identified with elements of $\mathfrak{g}$: for any $X \in \mathfrak{g}$, the map

$$\begin{array}{rccc} X_\circ : & \mathfrak{g}^* & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \xi(X) \end{array}$$

is a (linear) function on $\mathfrak{g}^*$ and thus an element of $\mathcal{F}[\mathfrak{g}^*]$. On these linear functions, the Kirillov-Kostant bracket then coincides with the Lie bracket:

$$\forall\, X, Y \in \mathfrak{g},\quad \{X_\circ, Y_\circ\}_* = [X,Y]_\circ. \tag{B.2.2}$$

In particular, a choice of basis $\{I^a\}$ of $\mathfrak{g}$ defines a choice of coordinate functions $X^a = I^a_\circ$ on $\mathfrak{g}^*$. The "fundamental" brackets of these coordinate functions is then

$$\{X^a, X^b\}_* = f^{ab}_{\ \ c}\, X^c,$$

where the $f^{ab}_{\ \ c}$ are the structure constants of the Lie algebra $\mathfrak{g}$ (see Appendix A.1).

Suppose that $\mathfrak{g}$ is semi-simple. Then its Killing form κ^{ab} is non-degenerate and possesses an inverse κ_{ab}. One can encode all coordinates functions X^a in the $\mathfrak{g}$-valued function

$$X = \kappa_{ab} X^a I^b \in \mathcal{F}[\mathfrak{g}^*] \otimes \mathfrak{g}.$$

As X is a $\mathfrak{g}$-valued function, we can describe the Poisson bracket of its components using the tensorial notations $\underline{\boldsymbol{i}}$ introduced in Subsection 2.2. We then find that the Kirillov-Kostant bracket can be written in a compact way as

$$\{X_{\underline{\mathbf{1}}}, X_{\underline{\mathbf{2}}}\}_* = \big[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{1}}}\big] = -\big[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}\big]\,,$$

where $C_{\underline{\mathbf{12}}}$ is the split quadratic Casimir of $\mathfrak{g}$ (see Appendix A.2.4).

B.3 Poisson maps and canonical transformations

Definition B.3.1. *Let M and N be two Poisson manifolds, with brackets $\{\cdot,\cdot\}_M$ and $\{\cdot,\cdot\}_N$. A differentiable map $\varphi : M \to N$ is said to be a* ***Poisson map*** *if it preserves the Poisson bracket:*

$$\forall\, f, g \in \mathcal{F}[N], \quad \{f \circ \varphi, g \circ \varphi\}_M = \{f, g\}_N \circ \varphi.$$

Definition B.3.2. *Let M be a symplectic manifold with symplectic form ω. A diffeomorphism $\varphi : M \mapsto M$ is said to be a* ***canonical transformation*** *if it preserves the symplectic form:*

$$\varphi^*\omega = \omega,$$

where $\varphi^*\omega$ is the pullback of ω by the diffeomorphism φ.

Proposition B.3.3. *Let M be a symplectic manifold and thus a Poisson manifold (proposition B.1.3). Then the canonical transformations of M are exactly the Poisson diffeomorphisms from M to itself.*

Let us consider a one-parameter Lie group of diffeomorphisms $\{\varphi_\alpha, \alpha \in \mathbb{R}\}$ of M (*i.e.* satisfying $\varphi_\alpha \circ \varphi_\beta = \varphi_{\alpha+\beta}$ and $\varphi_0 = \mathrm{Id}_M$). Then the infinitesimal transformation around $\alpha = 0$ defines a vector field X on M:

$$\forall\, p \in M, \quad X(p) = \left.\frac{\partial \varphi_\alpha(p)}{\partial \alpha}\right|_{\alpha=0} \in T_pM.$$

Reciprocally, given a vector field X on M, its flow defines a one-parameter Lie group of diffeomorphisms of M. Recall that vector fields $\mathcal{X}[M]$ act on the space of smooth functions $\mathcal{F}[M]$ as derivations: we will denote this action by a dot.

Proposition B.3.4. *If the diffeomorphisms $\{\varphi_\alpha, \alpha \in \mathbb{R}\}$ are Poisson maps, the vector field X satisfies:*

$$\forall\, f, g \in \mathcal{F}[M], \quad \{X.f, g\} + \{f, X.g\} = X.\{f, g\}. \tag{B.3.1}$$

A vector field X satisfying (B.3.1) is said to be Poisson. We will denote by $\mathcal{P}[M]$ the Poisson vector fields of M. If M is a symplectic manifold, it is also a Poisson manifold by Proposition B.1.3. Then, the Poisson vector fields are the ones preserving the symplectic form:

Proposition B.3.5. *Let M be a symplectic manifold with symplectic form ω and $X \in \mathcal{X}[M]$ be a vector field on M. Then X is Poisson if and only if $\mathcal{L}_X\omega = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along X.*

Important examples of Poisson vector fields are given by the so-called Hamiltonian vector fields, which are associated with functions on the Poisson manifold.

Definition B.3.6. *Let M be a Poisson manifold and $f \in \mathcal{F}[M]$ be a function from M to $\mathbb{R}$. Then, the Hamiltonian vector field associated with f is the unique $V_f \in \mathcal{X}[M]$ such that $V_f.g = \{f, g\}$ for all $g \in \mathcal{F}[M]$ (this vector field exists as $\{f, \cdot\}$ acts as a derivation on $\mathcal{F}[M]$).*

Proposition B.3.7. *Let M be a Poisson manifold. The map*

$$\begin{array}{rccc} \mathcal{V}: & \mathcal{F}[M] & \longrightarrow & \mathcal{X}[M] \\ & f & \longmapsto & V_f \end{array}$$

is a derivation from the algebra $\mathcal{F}[M]$ to the $\mathcal{F}[M]$-module $\mathcal{X}[M]$. Moreover, it is a Lie homomorphism if $\mathcal{F}[M]$ is equipped with the Poisson bracket $\{\cdot,\cdot\}$ and $\mathcal{X}[M]$ is equipped with the Lie bracket $[\cdot,\cdot]$ of vector fields, i.e.

$$\forall\, f, g \in \mathcal{F}[M], \quad V_{\{f,g\}} = [V_f, V_g]\,. \tag{B.3.2}$$

Finally, for all $f \in \mathcal{F}[M]$, the Hamiltonian vector field V_f is Poisson. Thus $\mathcal{V}$ is $\mathcal{P}[M]$-valued.

Proof. It is clear that $\mathcal{V}$ is a derivation as $\{\cdot,\cdot\}$ is a derivation on the left. Moreover, for all $f, g, h \in \mathcal{F}[M]$, we have

$$\begin{aligned}
[V_f, V_g]\,.h &= V_f.(V_g.h) - V_g.(V_f.h) \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&= \{\{f,g\},h\} \qquad \text{by the Jacobi identity (B.1.1)} \\
&= V_{\{f,g\}}.h,
\end{aligned}$$

hence equation (B.3.2). In the same way, using the Jacoby identity, we have

$$\{V_f.g,h\} + \{g,V_f.h\} = \{\{f,g\},h\} + \{g,\{f,h\}\} = \{f,\{g,h\}\} = V_f.\{g,h\}.$$

This proves that V_f is a Poisson vector field. □

It is natural to ask whether every Poisson vector field can be written as a Hamiltonian vector field. The answer is yes when M is symplectic and simply connected.

Theorem B.3.8. *Let M be a symplectic manifold with symplectic form ω. Then, the Hamiltonian vector field associated with $f \in \mathcal{F}[M]$ is the unique vector field V_f such that*

$$df = \iota_{V_f}\omega,$$

where ι_{V_f} denotes the interior derivative with respect to V_f. Moreover, if M is simply connected, the map $\mathcal{V} : \mathcal{F}[M] \mapsto \mathcal{P}[M]$ is surjective.

Proof. Let us consider local coordinates x^i on M and the associated expressions (B.1.2) and (B.1.4) of the Poisson bracket and the symplectic form on M. For $f \in \mathcal{F}[M]$, we have $V_f.x^i = \{f,x^i\} = P^{ki}\frac{\partial f}{\partial x^k}$. Thus

$$\iota_{V_f}\omega = \omega(V_f,\cdot) = P_{ij}\,\mathrm{d}x^i(V_F)\,\mathrm{d}x^j = P_{ij}\big(V_f.x^i\big)\,\mathrm{d}x^j = \underbrace{P_{ij}P^{ki}}_{\delta^k_j}\frac{\partial f}{\partial x^k}\,\mathrm{d}x^j = \frac{\partial f}{\partial x^k}\mathrm{d}x^k = \mathrm{d}f,$$

which proves the first part of the theorem (the uniqueness of V_f defined this way comes from the non-degeneracy of ω).

Let us now consider a Poisson vector field X and let us define the 1-form $\alpha = \iota_X\omega$. By the Cartan identity, we have

$$\mathrm{d}\alpha = \mathrm{d}\big(\iota_X\omega\big) = -\iota_X\big(\mathrm{d}\omega\big) + \mathcal{L}_X\omega = 0,$$

as $\mathrm{d}\omega = 0$ (ω is closed) and $\mathcal{L}_X\omega = 0$ (X is a Poisson vector field, see Proposition B.3.5). Thus, the 1-form α is closed. If M is simply-connected, X is also exact by the Poincaré lemma, *i.e.* $\alpha = \mathrm{d}f$ for some f in $\mathcal{F}[M]$. We then conclude that $X = V_f$ by the first part of the theorem. □

B.4 Hamiltonian action of Lie groups

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. We suppose that it acts smoothly on a differentiable manifold M, through the action

$$\begin{array}{ccc} G\times M & \longrightarrow & M \\ (g,p) & \longmapsto & \rho(g,p) \end{array}.$$

This action can be seen as a group homomorphism $\rho : g \mapsto \rho(g,\cdot)$ from G to the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. Taking the differential of this homomorphism at the identity, we get a linear map

$$\begin{array}{rccc} \delta: & \mathfrak{g} & \longrightarrow & \mathcal{X}[M] \\ & \epsilon & \longmapsto & \delta_\epsilon \end{array}$$

from the Lie algebra $\mathfrak{g} = T_eG$ to the space of vector fields $\mathcal{X}[M] = T_{\mathrm{Id}}\mathrm{Diff}(M)$. As ρ is a group homomorphism, the map δ is an homomorphism of Lie algebra from $\mathfrak{g}$ to $\mathcal{X}[M]$, equipped with the Lie bracket of vector fields. In other words, we have for any $\epsilon, \epsilon' \in \mathfrak{g}$:

$$\delta_{[\epsilon,\epsilon']} = [\delta_\epsilon, \delta_{\epsilon'}]\,. \tag{B.4.1}$$

Let us now suppose that the action of G conserves the Poisson bracket, *i.e.* that for any fixed $g \in G$, the map $\rho(g,\cdot) : M \to M$ is Poisson. Then, the map δ is valued in $\mathcal{P}[M]$. We shall further suppose that the infinitesimal actions δ_ϵ are Hamiltonian for any $\epsilon \in \mathfrak{g}$ (this is directly the case if M is symplectic and simply connected, by Theorem B.3.8). Then there exists a linear map

$$\mu : \mathfrak{g} \longrightarrow \mathcal{F}[M]$$

such that for any $\epsilon \in \mathfrak{g}$,

$$\delta_\epsilon = V_{\mu(\epsilon)} = \{\mu(\epsilon), \cdot\}.$$

We call μ the **moment map**. It is easy to see that

$$\delta = \mathcal{V} \circ \mu, \tag{B.4.2}$$

as for any $\epsilon \in \mathfrak{g}$, $\mathcal{V} \circ \mu(\epsilon) = \mathcal{V}\big(\mu(\epsilon)\big) = V_{\mu(\epsilon)} = \delta_\epsilon$. As a linear map from $\mathfrak{g}$ to $\mathcal{F}[M]$, it can be seen as an element of $\mathfrak{g}^* \otimes \mathcal{F}[M]$, *i.e.* as a $\mathfrak{g}^*$-valued function on M:

$$Q : M \longrightarrow \mathfrak{g}^*.$$

In terms of this function, we have, for all $\epsilon \in \mathfrak{g}$ and $f \in \mathcal{F}[M]$,

$$\delta_\epsilon f = \langle \epsilon, \{Q, f\} \rangle, \tag{B.4.3}$$

where $\langle \cdot, \cdot \rangle$ represents the pairing on $\mathfrak{g} \times \mathfrak{g}^*$.

Let us now suppose that $\mathfrak{g}$ is semi-simple. Then there is a natural isomorphism between $\mathfrak{g}^*$ and $\mathfrak{g}$ *via* the Killing form κ on $\mathfrak{g}$, which is non-degenerate (see Appendix A.2). We can then consider the image m of Q under this isomorphism, which is a $\mathfrak{g}$-valued function on M. The infinitesimal action δ_ϵ of $\epsilon \in \mathfrak{g}$ can then be written

$$\delta_\epsilon = V_{\kappa(m,X)} = \kappa\big(\epsilon, \{m, \cdot\}\big). \tag{B.4.4}$$

If f is a $\mathfrak{g}$-valued function on M whose Poisson bracket vanishes with all other functions in $\mathcal{F}[M]$ (such a function is called a Casimir of $\big(\mathcal{F}[M], \{\cdot,\cdot\}\big)$), then note that one can replace m by $m + f$ in the equation (B.4.4). It is a classical result [204] that using this freedom, one can always choose m to satisfy the Kirillov-Kostant bracket defined in subsection B.2:

$$\big\{m_{\underline{\mathbf{1}}}, m_{\underline{\mathbf{2}}}\big\} = \big[C_{\underline{\mathbf{12}}}, m_{\underline{\mathbf{1}}}\big]\,. \tag{B.4.5}$$

This is due to the fact that δ is a Lie algebra homomorphism (and that a semi-simple Lie algebra has no non trivial central extension). Conversely, if one has a $\mathfrak{g}$-valued map m satisfying the Kostant-Kirillov bracket (B.4.5), one can define a Lie algebra homomorphism δ from $\mathfrak{g}$ to $\mathcal{X}[M]$ by equation (B.4.4), *i.e.* an infinitesimal action of $\mathfrak{g}$ on M.

The present subsection concerned left actions of the group G and thus of the Lie algebra $\mathfrak{g}$. One can also develop a similar formalism for right actions. In this case, there is still a moment map m such that equation (B.4.4) holds. However, the Poisson bracket (B.4.5) changes into minus itself.

Appendix C

$\mathcal{R}$-matrices and classical Yang-Baxter equations

C.1 Classical Yang-Baxter Equations

Let $\mathfrak{g}$ be a real Lie algebra, with Lie bracket $[\cdot,\cdot]$ (see Appendix A). In this appendix, we describe the (modified) Classical Yang-Baxter Equation, denoted (m)CYBE, on $\mathfrak{g}$. We will present both the "operator" form and the "matricial" form of this equation and explain the link between these two forms. We refer to [120, 15, 123, 148] for more details.

mCYBE in "operator" form. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear operator on $\mathfrak{g}$. We say that R is a solution of the **operator mCYBE on $\mathfrak{g}$** if it satisfies

$$\forall\, X, Y \in \mathfrak{g},\ \ [RX, RY] - R\big([RX, Y] + [X, RY]\big) = -\alpha[X, Y],$$

where α is a real number. We note that rescaling the operator R into λR sends solutions of the mCYBE for α to solutions for $\lambda^2\alpha$. Thus, by an appropriate rescaling, we reduce the study of the mCYBE to three cases: $\alpha = 0$, $\alpha = 1$ and $\alpha = -1$. We will then consider the equation

$$\forall\, X, Y \in \mathfrak{g},\ \ [RX, RY] - R\big([RX, Y] + [X, RY]\big) = -c^2[X, Y], \tag{C.1.1}$$

with $c = 0$ (homogeneous case), $c = 1$ (split case) and $c = i$ (non-split case). When in the homogeneous case $c = 0$, we then talk of the CYBE instead of the modified CYBE. Note that $R = \pm c\,\mathrm{Id}$ is a solution of the mCYBE. Solutions of the (resp. homogeneous, split, non-split) mCYBE (C.1.1) are called (resp. homogeneous, split, non-split) R-matrices. For R such a matrix, we define the R-bracket on $\mathfrak{g}$ as

$$[X, Y]_R = [RX, Y] + [X, RY], \quad \forall\, X, Y \in \mathfrak{g}.$$

This bracket possesses many interesting properties, including being a Lie bracket. As it is one of the main tool for the Chapter 5 of this thesis, it is described in detail in this chapter and we will not study it further in this appendix. We will just rewrite the mCYBE in a more compact way using the R-bracket:

$$\forall\, X, Y \in \mathfrak{g},\ \ [RX, RY] + c^2[X, Y] = R\big([X, Y]_R\big). \tag{C.1.2}$$

mCYBE in "matricial" form. In this section, we will use the tensorial notation $\underline{i}$ introduced in Section 2.2. Let us suppose that $\mathfrak{g}$ admits a non-degenerate invariant form κ, as for example the Killing form for semi-simple Lie algebras. Then, one can define the split quadratic Casimir $C_{\underline{12}}$ in $\mathfrak{g} \otimes \mathfrak{g}$ (see Appendix A.2.4).

Let $R_{\underline{12}}$ be a matrix in $\mathfrak{g} \otimes \mathfrak{g}$. We say that R is a solution of the **matricial mCYBE on $\mathfrak{g}$** if it satisfies

$$\big[R_{\underline{12}}, R_{\underline{13}}\big] + \big[R_{\underline{12}}, R_{\underline{23}}\big] + \big[R_{\underline{32}}, R_{\underline{13}}\big] = -c^2\left[C_{\underline{12}}, C_{\underline{13}}\right]. \tag{C.1.3}$$

As for the operator case, a rescaling of R allows to consider $c \in \{0, 1, i\}$. Using equation (2.3.4), one checks that $\pm c\, C_{\underline{12}}$ is a solution of the mCYBE.

It is often useful to restrict to skew-symmetric matrices $R_{\underline{12}} = -R_{\underline{21}}$. In this case, one can rewrite the mCYBE on R in (the more often used) form

$$R_{\underline{12}} = -R_{\underline{21}} \quad \text{and} \quad \big[R_{\underline{12}}, R_{\underline{13}}\big] + \big[R_{\underline{12}}, R_{\underline{23}}\big] + \big[R_{\underline{13}}, R_{\underline{23}}\big] = -c^2 \big[C_{\underline{12}}, C_{\underline{13}}\big] .$$

We end this subsection by a remark on the right-hand side of the mCYBE (C.1.3). Using the Jacoby identity and the fundamental property (A.2.6) of the split Casimir, one finds that

$$\Big[\big[C_{\underline{12}}, C_{\underline{13}}\big] , X_{\underline{1}} + X_{\underline{2}} + X_{\underline{3}}\Big] = 0, \quad \forall\, X \in \mathfrak{g}.$$

Kernels of operators. Let us consider a matrix $R_{\underline{12}} \in \mathfrak{g} \otimes \mathfrak{g}$. We associate with this matrix a linear operator R on $\mathfrak{g}$ by

$$R(X) = \kappa_{\underline{2}} \left(R_{\underline{12}}, X_{\underline{2}}\right) , \quad \forall\, X \in \mathfrak{g}.$$

We then say that $R_{\underline{12}}$ is the kernel of R. The following lemma collects some basic facts about kernels.

Lemma C.1.1. *We denote by ^{t} the transpose with respect to the bilinear form κ.*

(i) The kernel of a linear operator R on $\mathfrak{g}$ always exists and is unique.

(ii) The kernel of the identity is $C_{\underline{12}}$. More generally the kernel of R is $R_{\underline{1}} C_{\underline{12}}$.

(iii) The kernel of ${}^tR$ is $R_{\underline{21}}$. In particular R is skew-symmetric with respect to κ, i.e. $R = -{}^tR$, if and only if $R_{\underline{12}} = -R_{\underline{21}}$.

Proof. The point (i) is a consequence of the non-degeneracy of κ. The point (ii) is a rewriting of the completeness equation (A.2.7). Finally, let X, Y be any elements of $\mathfrak{g}$. We then have

$$\kappa\big({}^tR(X), Y\big) = \kappa\big(X, R(Y)\big) = \kappa_{\underline{1}} \left(X, \kappa_{\underline{2}}(R_{\underline{12}}, Y_{\underline{2}})\right) = \kappa_{\underline{2}} \left(\kappa_{\underline{1}}(X_{\underline{1}}, R_{\underline{12}}), Y_{\underline{2}}\right) = \kappa_{\underline{1}} \left(\kappa_{\underline{2}}(R_{\underline{21}}, X_{\underline{2}}), Y_{\underline{1}}\right) .$$

As this is true for all $Y \in \mathfrak{g}$, we get, by non-degeneracy of κ,

$${}^tR(X) = \kappa_{\underline{2}}(R_{\underline{21}}, X_{\underline{2}}),$$

which proves point (iii). □

The following proposition relates the operator and matricial forms of the mCYBE:

Proposition C.1.2. *Let R be a linear operator on $\mathfrak{g}$, with kernel $R_{\underline{12}}$. Then R is solution of the operator mCYBE (C.1.1) if and only if $R_{\underline{12}}$ is a solution of (C.1.3).*

Proof. We denote

$$\mathcal{Y}_{\underline{123}} = \big[R_{\underline{12}}, R_{\underline{13}}\big] + \big[R_{\underline{12}}, R_{\underline{23}}\big] + \big[R_{\underline{32}}, R_{\underline{13}}\big]$$

the left-hand side of (C.1.3). For X, Y in $\mathfrak{g}$, we have

$$\begin{aligned}
\kappa_{\underline{23}}\big(\mathcal{Y}_{\underline{123}}, X_{\underline{2}} Y_{\underline{3}}\big) &= \big[\kappa_{\underline{2}}\big(R_{\underline{12}}, X_{\underline{2}}\big), \kappa_{\underline{3}}\big(R_{\underline{13}}, Y_{\underline{3}}\big)\big] + \kappa_{\underline{2}} \left(\big[R_{\underline{12}}, \kappa_{\underline{3}}\big(R_{\underline{23}}, Y_{\underline{3}}\big)\big] , X_{\underline{2}}\right) \\
&\qquad + \kappa_{\underline{3}} \left(\big[\kappa_{\underline{2}}\big(R_{\underline{32}}, X_{\underline{2}}\big), R_{\underline{13}}\big] , Y_{\underline{3}}\right) \\
&= [RX, RY] + \kappa_{\underline{2}} \left(\big[R_{\underline{12}}, RY_{\underline{2}}\big], X_{\underline{2}}\right) + \kappa_{\underline{3}} \left(\big[RX_{\underline{3}}, R_{\underline{13}}\big], Y_{\underline{3}}\right) \\
&= [RX, RY] - \kappa_{\underline{2}}\big(R_{\underline{12}}, [R, RY]_{\underline{2}}\big) - \kappa_{\underline{3}}\big(R_{\underline{13}}, [RX, Y]_{\underline{3}}\big) \\
&= [RX, RY] - R\big([RX, Y] + [X, RY]\big).
\end{aligned}$$

In the same way, we have

$$\kappa_{\underline{23}}\big(\big[C_{\underline{12}}, C_{\underline{13}}\big] , X_{\underline{2}} Y_{\underline{3}}\big) = \big[\kappa_{\underline{2}}\big(C_{\underline{12}}, X_{\underline{2}}\big), \kappa_{\underline{3}}\big(C_{\underline{13}}, Y_{\underline{3}}\big)\big] = [X, Y].$$

Thus, one has

$$\kappa_{\underline{23}}\big(\mathcal{Y}_{\underline{123}} + c^2 \big[C_{\underline{12}}, C_{\underline{13}}\big] , X_{\underline{2}} Y_{\underline{3}}\big) = [RX, RY] - R\big([RX, Y] + [X, RY]\big) + c^2 [X, Y],$$

hence the proposition. □

C.2 Adler-Kostant-Symes scheme

AKS construction. In this appendix, we present a general scheme to construct R-matrices on a Lie algebra $\mathfrak{g}$, called the Adler-Kostant-Symes (AKS) scheme [205–207]. At least for the beginning of this section, we shall consider $\mathfrak{g}$ to be a complex Lie algebra (we will explain later how to apply this construction to real Lie algberas). We suppose that $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = A \oplus B \oplus C, \tag{C.2.1}$$

where A, B and C are three subalgebras of $\mathfrak{g}$ and $\oplus$ denotes a direct sum as a vector space. Note that we do not require A, B and C to be in direct sum as Lie algebras, *i.e.* to satisfy $[A,B] = [A,C] = [B,C] = 0$. However, we will impose the following conditions on C:

$$[C,C] = 0, \quad [C,A] \subset A \quad \text{and} \quad [C,B] \subset B. \tag{C.2.2}$$

In particular, C must be an abelian subalgebra of $\mathfrak{g}$. Note that if $\mathfrak{g} = A \oplus B$ with A and B subalgebras, we are in the situation described above by defining $C = \{0\}$ (without any assumptions on A and B).

We will denote by π_S ($S = A, B, C$) the projectors along the decomposition (C.2.1). Moreover, for X in $\mathfrak{g}$, we will denote by simplicity $X_S = \pi_S X$. Let ϕ_C be a linear operator from C to itself: we extend it to an operator of the whole algebra $\mathfrak{g}$ by letting $\phi_C(A) = \phi_C(B) = 0$. Let us then define the AKS operator R on $\mathfrak{g}$ as:

$$R = c\big(\pi_A - \pi_B\big) + \phi_C. \tag{C.2.3}$$

Theorem C.2.1. *R is a solution of the operator mCYBE (C.1.1).*

Proof. Let us first compute the R-bracket associated with the operator R. As $\mathfrak{g} = A \oplus B \oplus C$, we have $X = X_A + X_B + X_C$ for all $X \in \mathfrak{g}$. One then finds

$$\begin{aligned}
[X,Y]_R &= [RX,Y] + [X,RY] \\
&= [c(X_A - X_B) + \phi_C X, Y_A + Y_B + Y_C] + [X_A + X_B + X_C, c(Y_A - Y_B) + \phi_C Y] \\
&= c\left([X_A,Y_A] + \cancel{[X_A,Y_B]} - \cancel{[X_B,Y_A]} - [X_B,Y_B]\right) \\
&\qquad + c[X_A,Y_C] - c[X_B,Y_C] + [\phi_C X, Y_A] + [\phi_C X, Y_B] \\
&\quad + c\left([X_A,Y_A] - \cancel{[X_A,Y_B]} + \cancel{[X_B,Y_A]} - [X_B,Y_B]\right) \\
&\qquad + c[X_C,Y_A] - c[X_C,Y_B] + [X_A,\phi_C Y] + [X_B,\phi_C Y],
\end{aligned}$$

where we used the fact that $[C,C] = 0$. Thus, one has

$$\begin{aligned}
[X,Y]_R &= \underbrace{2c[X_A,Y_A] + c[X_A,Y_C] + c[X_C,Y_A] + [\phi_C X, Y_A] + [X_A, \phi_C X]}_{\in A} \\
&\quad - \underbrace{\Big(2c[X_B,Y_B] + c[X_B,Y_C] + c[X_C,Y_B] - [\phi_C X, Y_B] - [X_B, \phi_C X]\Big)}_{\in B},
\end{aligned} \tag{C.2.4}$$

using the fact that $[A,C] \subset A$ and $[B,C] \subset C$. As $[X,Y]_R$ does not contain any element of C, one has

$$R\big([X,Y]_R\big) = c(\pi_A - \pi_B)[X,Y]_R.$$

We then find

$$\begin{aligned}
R\big([X,Y]_R\big) &= c^2\Big(2[X_A,Y_A] + 2[X_B,Y_B] + [X_A,Y_C] + [X_C,Y_A] + c[X_B,Y_C] + c[X_C,Y_B]\Big) \\
&\qquad + c\Big([\phi_C X, Y_A - Y_B] + [X_A - X_B, \phi_C X]\Big).
\end{aligned}$$

In the same way, we have

$$[RX,RY] + c^2[X,Y] = [c(X_A - X_B) + \phi_C X, c(Y_A - Y_B) + \phi_C Y] + c^2[X_A + X_B + X_C, Y_A + Y_B + Y_C].$$

Developping this expression and using $[C,C] = 0$, one finds that

$$R\big([X,Y]_R\big) = [RX,RY] + c^2[X,Y],$$

i.e. R satisfies the operator mCYBE (C.1.1). □

Standard AKS operator. Let us consider the R-matrix (C.2.3) constructed from the AKS construction. We will say that R is standard if we chose $\phi_C = 0$ and $c \neq 0$, so that

$$R = c(\pi_A - \pi_B) \tag{C.2.5}$$

is either a split or a non-split R-matrix (not a homogeneous one).

Proposition C.2.2. *Let R be a solution of the operator mCYBE (C.1.1). Then R is a standard AKS operator if and only if $R^3 = c^2 R$.*

Proof. It is clear that $R = c(\pi_A - \pi_B)$ satisfies $R^3 = c^2 R$. Conversely let us suppose that we have a solution R of (C.1.1) satisfying $R^3 = c^2 R$. Then R is diagonalisable and have eigenvalues in $\{0, c, -c\}$. We then define $A = \mathrm{Ker}(R - c)$, $B = \mathrm{Ker}(R + c)$ and $C = \mathrm{Ker}(R)$. As R is diagonalisable, one has the vector space decomposition $\mathfrak{g} = A \oplus B \oplus C$ and R can be re-expressed as $R = c(\pi_A - \pi_B)$.

To prove that R is an AKS operator, we now need to show that A, B and C are subalgebras of $\mathfrak{g}$ and that C satisfies the condition (C.2.2). Recall the expression (C.1.2) of the mCYBE. We will distinguish several cases.

For example let us suppose that X and Y are in A. We then have $RX = cX$ and $RY = cY$. Thus, one has

$$[X, Y]_R = [RX, Y] + [X, RY] = 2c[X, Y] \quad \text{and} \quad [RX, RY] + c^2[X, Y] = 2c^2[X, Y].$$

We then get by (C.1.2) that

$$(R - c)\big([X, Y]\big) = 0.$$

Thus, $[X, Y] \in A = \mathrm{Ker}(R - c)$. We then deduce that A is a subalgebra of $\mathfrak{g}$.

In the same way, choosing appropriately X and Y either in A, B or C, one shows from (C.1.2) that B is also a subalgebra of $\mathfrak{g}$, that C is abelian and that C satisfies (C.2.2). This ends the demonstration of the proposition. □

One can note that a standard AKS operator R satisfies $R^2 = c^2 \mathrm{Id}$ if and only if $C = 0$.

Let us now suppose that $\mathfrak{g}$ is equipped with an invariant non-degenerate bilinear form κ (for example the Killing form if $\mathfrak{g}$ is semi-simple). We can then associate a kernel $R_{\underline{\mathbf{12}}} \in \mathfrak{g} \otimes \mathfrak{g}$ with the standard AKS operator R, which satisfies the matricial mCYBE (C.1.3) (see Section C.1). Recall from Lemma C.1.1 that $R_{\underline{\mathbf{12}}}$ is skew-symmetric if and only if R is skew-symmetric with respect to κ. The following proposition gives a necessary and sufficient condition to happen.

Proposition C.2.3. *The standard AKS operator R is skew-symmetric if and only if A and B are κ-isotropic and C is κ-orthogonal to A and B.*
In this case, κ pairs non-degenerately A and B and restricts to a non-degenerate form on C. If $\{I^a\}$ is a basis of A, there exists a corresponding dual basis $\{I_a\}$ in B. Then, the kernel $R_{\underline{\mathbf{12}}}$ of R is

$$R_{\underline{\mathbf{12}}} = c\big(I^a \otimes I_a - I_a \otimes I^a\big). \tag{C.2.6}$$

Proof. For E a subspace of $\mathfrak{g}$, we denote by $E^\perp$ the orthogonal of E with respect to κ:

$$E^\perp = \{X \in \mathfrak{g} \mid \kappa(X, Y) = 0, \forall Y \in E\}.$$

It is a standard result on quadratic forms that

$${}^t\pi_A = \pi_{A'}, \quad {}^t\pi_B = \pi_{B'} \quad \text{and} \quad {}^t\pi_C = \pi_{C'},$$

with $\pi_{A'}$, $\pi_{B'}$ and $\pi_{C'}$ the projections along the decomposition

$$\mathfrak{g} = A' \oplus B' \oplus C', \qquad \text{with} \qquad A' = (B \oplus C)^\perp,\ B' = (A \oplus C)^\perp,\ C' = (A \oplus B)^\perp.$$

Thus, one has

$$ {}^tR = c(\pi_{A'} - \pi_{B'}), $$

hence $R = -{}^tR$ if and only if $A = B' = (A \oplus C)^\perp$ and $B = A' = (B \oplus C)^\perp$.

If this the case, then one has $\kappa(A,A) = \kappa(B,B) = \kappa(A,C) = \kappa(B,C) = 0$, *i.e.* A and B are κ-isotropic and C is κ-orthogonal to A and B. Conversely, if we suppose the latter, we have $A \subseteq (A \oplus C)^\perp = B'$ and $B \subseteq (B \oplus C)^\perp = A'$: we then conclude that these inclusions are equalities from dimension considerations. This proves the first part of the proposition.

Let us now suppose that we are in the case described above. As κ is non-degenerate and A does not pair with $A \oplus C$, κ should pair non-degenerately A and B. In the same way, C does not pair with $A \oplus B$ so it should pair non-degenerately with itself. The expression (C.2.6) then follows from the fact that the kernel of π_A and π_B are respectively $I^a \otimes I_a$ and $I_a \otimes I^a$, which is straightforward to prove. □

Let us suppose that we are in the case described by Proposition C.2.3. Let us also consider a basis $\{J^b\}$ of C and the corresponding dual basis $\{J_b\}$, which is then also in C. The Casimir can then be written as

$$ C_{\underline{\mathbf{12}}} = I^a \otimes I_a + I_a \otimes I^a + J^b \otimes J_b. $$

We will use the two matrices

$$ R^+_{\underline{\mathbf{12}}} = R_{\underline{\mathbf{12}}} + c\, C_{\underline{\mathbf{12}}} = c\big(2 \underbrace{I^a \otimes I_a}_{\in A\otimes B} + \underbrace{J^b \otimes J_b}_{\in C\otimes C}\big), \tag{C.2.7a} $$

$$ R^-_{\underline{\mathbf{12}}} = R_{\underline{\mathbf{12}}} - c\, C_{\underline{\mathbf{12}}} = -c\big(2 \underbrace{I_a \otimes I^a}_{\in B\otimes A} + \underbrace{J^b \otimes J_b}_{\in C\otimes C}\big), \tag{C.2.7b} $$

which are the kernels of $R^\pm = R \pm c\,\mathrm{Id}$.

AKS matrices for real algebras. Let us consider a real Lie algebra $\mathfrak{g}_0$. We denote by $\mathfrak{g}$ its complexification and by τ the antilinear involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{g}_0$ is the real form $\mathfrak{g}_0 = \mathfrak{g}^\tau$ of $\mathfrak{g}$ (see Appendix A.3). Let us suppose that $\mathfrak{g}$ admits a R-matrix R.

Proposition C.2.4. *The operator R induces a R-matrix of $\mathfrak{g}_0$ if and only if $R \circ \tau = \tau \circ R$.*

Proof. If R stabilises $\mathfrak{g}_0$, the induced operator of $\mathfrak{g}_0$ also satisfies the mCYBE. Thus one needs to find a *sine qua non* condition for R to stabilise $\mathfrak{g}_0 = \mathfrak{g}^\tau = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. It is a classical fact from linear algebra that R and τ commute if and only if R stabilises all eigenspaces of τ (as τ is diagonalisable). Thus $R \circ \tau = \tau \circ R$ implies that R stabilises $\mathfrak{g}_0$. Conversely, if R stabilises $\mathfrak{g}_0$, it also stabilises the other eigenspace $i\mathfrak{g}_0$ of τ, as R is $\mathbb{C}$-linear: then R and τ commute. □

Conversely, if one has a R-matrix on $\mathfrak{g}_0$, it can be extended to a matrix R on $\mathfrak{g}$, also satisfying the mCYBE equation. In this case, this matrix R commutes with τ.

Let us now suppose that $\mathfrak{g}$ admits an AKS decomposition (C.2.1), as described in the first paragraph of this section. We can then apply the AKS scheme to find R matrices on $\mathfrak{g}$.

Proposition C.2.5. *Let R be the standard AKS operator (C.2.5) on $\mathfrak{g}$. Then R induces a R-matrix on $\mathfrak{g}_0$ if and only if:*

- *$\tau(A) = A$, $\tau(B) = B$ and $\tau(C) = C$, in the split case $c = 1$,*
- *$\tau(A) = B$ (hence also $\tau(B) = A$) and $\tau(C) = C$, in the non-split case $c = i$.*

Proof. For $S = A, B, C$, we define $\overline{S} = \tau(S)$ and $\overline{\pi}_S = \tau \circ \pi_S \circ \tau^{-1}$. We have

$$ \overline{\pi}_S \circ \overline{\pi}_{S'} = \tau \circ \big(\pi_S \circ \pi_{S'}\big) \circ \tau^{-1} = \delta_{SS'}\, \tau \circ \pi_S \circ \tau^{-1} = \delta_{SS'}\, \overline{\pi}_S $$

and

$$ \overline{\pi}_A + \overline{\pi}_B + \overline{\pi}_C = \tau \circ \big(\pi_A + \pi_B + \pi_C\big) \circ \tau^{-1} = \tau \circ \mathrm{Id} \circ \tau^{-1} = \mathrm{Id}. $$

Thus, the $\overline{\pi}_S$'s form a complete set of projectors. As $\mathrm{Im}(\overline{\pi}_S) = \tau\big(\mathrm{Im}(\pi_S)\big) = \overline{S}$, they are the projectors associated with the decomposition $\mathfrak{g} = \overline{A} \oplus \overline{B} \oplus \overline{C}$. We thus have

$$\tau \circ \pi_S \circ \tau^{-1} = \pi_{\overline{S}}.$$

In particular, we have

$$\tau \circ R \circ \tau^{-1} = \overline{c}\big(\pi_{\overline{A}} - \pi_{\overline{B}}\big).$$

We will then distinguish the split and non split cases.

If $c = 1$ (split case), we see that τ commutes with R if and only if $\pi_A - \pi_B = \pi_{\overline{A}} - \pi_{\overline{B}}$. As these two operators are diagonalisable with eigenvalues $\{0, 1, -1\}$, they coincide if and only if their eigenspaces are the same, thus if and only if $A = \overline{A}$, $B = \overline{B}$ and $C = \overline{C}$.

If $c = i$ (non-split case), we then have that τ commutes with R if and only if $\pi_A - \pi_B = \pi_{\overline{B}} - \pi_{\overline{A}}$. Using similar arguments as above, this is equivalent to $A = \overline{B}$ and $B = \overline{A}$ and $C = \overline{C}$. □

Note that in the split case, the subalgebras $S = A, B, C$ are stabilised under the action of τ. Thus one can define $S_0 = S^\tau$, which are subalgebras of $\mathfrak{g}_0$. We then get a decomposition $\mathfrak{g}_0 = A_0 \oplus B_0 \oplus C_0$, which satisfies the conditions of the AKS construction. The standard R-matrix on $\mathfrak{g}_0$ that one would obtain from this construction then coincides with the restriction of R on $\mathfrak{g}_0$.

At the contrary, in the non-split case, the subalgebras $S = A, B, C$ are not stabilised by τ. Thus, they do not possess real forms in $\mathfrak{g}_0$ and we cannot interpret the R-matrix on $\mathfrak{g}_0$ as an AKS operator constructed from real subalgebras of $\mathfrak{g}_0$.

C.3 Standard R-matrices on finite semi-simple Lie algebras.

Let us consider a finite dimensional complex semi-simple Lie algebra $\mathfrak{g}$. It is then described by the formalism developed in Section A.2. In particular, it possesses a Cartan-Weyl decomposition:

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{n}_- \oplus \mathfrak{h}.$$

The subalgebras $A = \mathfrak{n}_+$, $B = \mathfrak{n}_-$ and $C = \mathfrak{h}$ satisfy the conditions required for the AKS scheme described in Subsection C.2. Let us then consider the corresponding standard AKS operator

$$R = c(\pi_+ - \pi_-), \tag{C.3.1}$$

where we abbreviated the projections on $\mathfrak{n}_\pm$ as $\pi_\pm$.

The algebra $\mathfrak{g}$ is equipped with a non-degenerate invariant form, the Killing form κ. The nilpotent subalgebras $\mathfrak{n}_\pm$ are isotropic with respect to κ and are orthogonal to the Cartan subalgbera $\mathfrak{h}$ (see Subsection A.2.1). According to Proposition C.2.3, the matrix R is thus skew-symmetric.

A basis of $\mathfrak{n}_\pm$ is given by the E_α's with α in the positive root system Δ_+. According to equation (A.2.1), the corresponding dual basis in $\mathfrak{n}_-$ is $\{F_\alpha = E_{-\alpha}\}_{\alpha\in\Delta_+}$. The kernel of R is then given by Proposition C.2.3 as

$$R_{\underline{\mathbf{12}}} = c \sum_{\alpha\in\Delta_+} \big(E_\alpha \otimes F_\alpha - F_\alpha \otimes E_\alpha\big). \tag{C.3.2}$$

The matrices $R^\pm_{\underline{\mathbf{12}}}$ introduced in equation (C.2.7) are then given by

$$R^+_{\underline{\mathbf{12}}} = 2c \sum_{\alpha\in\Delta_+} E_\alpha \otimes F_\alpha + c\, H_{\underline{\mathbf{12}}} \quad \text{and} \quad R^-_{\underline{\mathbf{12}}} = -2c \sum_{\alpha\in\Delta_+} F_\alpha \otimes E_\alpha - c\, H_{\underline{\mathbf{12}}},$$

with $H_{\underline{\mathbf{12}}} \in \mathfrak{h} \otimes \mathfrak{h}$ as introduced in Subsection A.2.4.

So far, we constructed a standard AKS operator R on the complex Lie algebra $\mathfrak{g}$. In Subsection A.3.2, we discussed real forms of $\mathfrak{g}$ and in particularly the split and non-split real forms.

Let us consider the split real form. By equations (A.3.1) and (A.3.2), we see that in this case, the subalgebras $\mathfrak{n}_+$, $\mathfrak{n}_-$ and $\mathfrak{h}$ are stabilised by τ. According to Proposition C.2.5, the split AKS matrix (C.3.1) for $c = 1$ then induces a R-matrix on the split real form of $\mathfrak{g}$.

Let us consider now the non-split real form. By equations (A.3.4) and (A.3.5), we see that τ stabilises $\mathfrak{h}$ and exchange $\mathfrak{n}_+$ and $\mathfrak{n}_-$. By Proposition C.2.5, the non-split AKS matrix (C.3.1) for $c = i$ then restricts to the non-split real form of $\mathfrak{g}$.

C.4 Standard $\mathcal{R}$-matrices on loop algebras.

Loop algebras. Let us consider a complex Lie algebra $\mathfrak{g}$, equipped with an invariant non-degenerate bilinear form κ. We introduce the loop algebra

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g}(\!(\lambda)\!) = \mathfrak{g} \otimes \mathbb{C}(\!(\lambda)\!),$$

where $\mathbb{C}(\!(\lambda)\!)$ denotes the complex Laurent series in a formal variable λ (formal power series in λ with a finite number of negative powers). We define a Lie bracket on $\mathcal{L}(\mathfrak{g})$ by extending the one on $\mathfrak{g}$:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg, \quad \forall\, X, Y \in \mathfrak{g} \text{ and } \forall\, f, g \in \mathbb{C}(\!(\lambda)\!).$$

Moreover, we extend the bilinear form κ on $\mathcal{L}(\mathfrak{g})$ as

$$\langle X \otimes f, Y \otimes g \rangle = \kappa(X, Y) \operatorname*{res}_{\lambda=0} f(\lambda) g(\lambda) \mathrm{d}\lambda, \quad \forall\, X, Y \in \mathfrak{g} \text{ and } \forall\, f, g \in \mathbb{C}(\!(\lambda)\!).$$

One can see elements of $\mathcal{L}(\mathfrak{g})$ as $\mathfrak{g}$-valued Laurent series in the variable λ. In this case, the bracket defined above is the point-wise bracket

$$\forall\, M, N \in \mathcal{L}(\mathfrak{g}), \quad [M, N](\lambda) = \big[M(\lambda), N(\lambda)\big].$$

In the same way, the bilinear form above is

$$\forall\, M, N \in \mathcal{L}(\mathfrak{g}), \quad \langle M, N \rangle = \operatorname*{res}_{\lambda=0} \kappa\big(M(\lambda), N(\lambda)\big) \mathrm{d}\lambda = \oint \kappa\big(M(\lambda), N(\lambda)\big) \mathrm{d}\lambda, \tag{C.4.1}$$

where the integral is taken on a closed contour around 0. This bilinear form on $\mathcal{L}(\mathfrak{g})$ is invariant and non-degenerate, as κ is invariant and non-degenerate.

Standard AKS operator on $\mathcal{L}(\mathfrak{g})$. The loop algebra $\mathcal{L}(\mathfrak{g})$ admits a natural vector space decomposition

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g}(\!(\lambda)\!) = \mathfrak{g}[[\lambda]] \oplus \lambda^{-1}\mathfrak{g}[\lambda^{-1}] \tag{C.4.2}$$

into series of positive and strictly negative powers of the loop parameter λ, respectively. The subspaces $\mathfrak{g}[[\lambda]]$ and $\lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ are subalgebras of $\mathcal{L}(\mathfrak{g})$. Thus, the decomposition (C.4.2) satisfies the condition of the AKS scheme described in Section C.2, with $A = \mathfrak{g}[[\lambda]]$, $B = \lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ and $C = \{0\}$. We denote by π_+ and π_- the projectors along this decomposition. The standard AKS operator

$$\mathcal{R} = \pi_+ - \pi_- \tag{C.4.3}$$

is thus a solution of the (split) operator mCYBE (C.1.1) on $\mathcal{L}(\mathfrak{g})$.

The algebra $\mathcal{L}(\mathfrak{g})$ is equipped with the invariant non-degenerate bilinear form (C.4.1). If M, N belong to $\mathfrak{g}[[\lambda]]$, the power series $\kappa\big(M(\lambda), N(\lambda)\big)$ is composed of positive powers of λ and thus has no residue, hence $\langle M, N \rangle = 0$. Thus, the subalgebra $\mathfrak{g}[[\lambda]]$ is isotropic with respect to $\langle \cdot, \cdot \rangle$. In the same way, if M, N belong to $\lambda^{-1}\mathfrak{g}[\lambda^{-1}]$, the power series $\kappa\big(M(\lambda), N(\lambda)\big)$ is composed of powers of λ stritcly inferior to one and thus has no residue. Thus, the subalgebra $\lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ is also isotropic. According to Proposition C.2.3, the operator $\mathcal{R}$ is thus skew-symmetric.

Kernels and standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$. As $\mathcal{L}(\mathfrak{g})$ is equipped with a non-degenerate bilinear form, one should be able to consider kernels of operators on $\mathcal{L}(\mathfrak{g})$, as described in section C.1. However, due to the infinite dimensional nature of $\mathcal{L}(\mathfrak{g})$, the definition and manipulation of these kernels are quite subtle. Indeed, the kernel of an operator of $\mathcal{L}(\mathfrak{g})$ is not technically an element of the tensor product $\mathcal{L}(\mathfrak{g})\otimes\mathcal{L}(\mathfrak{g})$ but an element of a completion of this space. We will not enter into details here as this is a technical and subtle matter: we will only present the main ideas needed for this thesis and refer to [123] for details and rigorous proofs.

The completion of $\mathcal{L}(\mathfrak{g})\otimes\mathcal{L}(\mathfrak{g})$ mentioned above is composed of infinite series in two variables λ and μ, valued in $\mathfrak{g}\otimes\mathfrak{g}$ (where the variables λ and μ to be the loop variables of respectively the left and right tensor factor in $\mathcal{L}(\mathfrak{g})\otimes\mathcal{L}(\mathfrak{g})$). These can be seen as distributions in the two variables λ and μ. For example, the quadratic Casimir $\widetilde{C}_{\underline{\mathbf{12}}}$ of $\mathcal{L}(\mathfrak{g})$ is related to the Casimir $C_{\underline{\mathbf{12}}}$ of $\mathfrak{g}$ by

$$\widetilde{C}_{\underline{\mathbf{12}}}(\lambda,\mu) = C_{\underline{\mathbf{12}}}\,\delta(\lambda-\mu), \tag{C.4.4}$$

where δ is the Dirac δ-distribution. Without developping the rigorous mathematical formalism behind this, let us motivate this expression for $\widetilde{C}$. Let $M\in\mathcal{L}(\mathfrak{g})$, which can then be seen as a $\mathfrak{g}$-valued Laurent series in λ. By the second equality in (C.4.1), we then have

$$\langle\widetilde{C}_{\underline{\mathbf{12}}}, M_{\underline{\mathbf{2}}}\rangle_{\underline{\mathbf{2}}}(\lambda) = \oint \kappa_{\underline{\mathbf{2}}}\big(\widetilde{C}_{\underline{\mathbf{12}}}(\lambda,\mu), M_{\underline{\mathbf{2}}}(\mu)\big)\mathrm{d}\mu,$$

where the integral is taken on μ as it is the loop variable associated with the second tensor factor $\mathcal{L}(\mathfrak{g})$. Using the expression (C.4.4) of $\widetilde{C}$, one then gets

$$\langle\widetilde{C}_{\underline{\mathbf{12}}}, M_{\underline{\mathbf{2}}}\rangle_{\underline{\mathbf{2}}}(\lambda) = \oint \kappa_{\underline{\mathbf{2}}}\big(C_{\underline{\mathbf{12}}}, M_{\underline{\mathbf{2}}}(\mu)\big)\delta(\lambda-\mu)\mathrm{d}\mu = \kappa_{\underline{\mathbf{2}}}\big(C_{\underline{\mathbf{12}}}, M_{\underline{\mathbf{2}}}(\lambda)\big) = M(\lambda),$$

where we used the definition of the δ-distribution for the second equality and the completeness relation for the finite Casimir $C_{\underline{\mathbf{12}}}$ for the last one. Thus, we get

$$\langle\widetilde{C}_{\underline{\mathbf{12}}}, M_{\underline{\mathbf{2}}}\rangle_{\underline{\mathbf{2}}} = M,$$

as expected for the Casimir of $\mathcal{L}(\mathfrak{g})$, as it is the kernel of the identity operator.

Let $\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}$ denote the kernel of the AKS operator $\mathcal{R}$ defined in (C.4.3). It is valued in the completion of $\mathcal{L}(\mathfrak{g})\otimes\mathcal{L}(\mathfrak{g})$ mentioned above and thus is a $\mathfrak{g}\otimes\mathfrak{g}$-valued distribution in the two variables λ and μ. Moreover, it is a solution of the matricial mCYBE (C.1.3) on $\mathcal{L}(\mathfrak{g})$. According to the expression (C.4.4) of the Casimir of $\mathcal{L}(\mathfrak{g})$, this equation then reads:

$$\begin{aligned}\big[\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\widetilde{\mathcal{R}}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big] + \big[\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\widetilde{\mathcal{R}}_{\underline{\mathbf{23}}}(\lambda_2,\lambda_3)\big] + \big[\widetilde{\mathcal{R}}_{\underline{\mathbf{32}}}(\lambda_3,\lambda_2),\widetilde{\mathcal{R}}_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big] \\ = \big[C_{\underline{\mathbf{12}}}, C_{\underline{\mathbf{13}}}\big]\,\delta(\lambda_1-\lambda_2)\delta(\lambda_1-\lambda_3).\end{aligned} \tag{C.4.5}$$

The right-hand side of this equation is called a contact term. It is non-zero only if the loop variables λ_i's coincide. Computing the kernel $\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}$, one finds

$$\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda,\mu) = \text{p.v.}\,\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda},$$

where p.v. denotes the principal value (see [123] for the precise definition). $\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda,\mu)$ is thus a distribution on the two variables λ and μ, satisfying the mCYBE (C.4.5). Let us consider the same kernel but without the principal value: we then get the **standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$**,

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}$$

as introduced in (2.3.6). Contrarily to $\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}(\lambda,\mu)$, it is a meromorphic function of λ and μ, valued in $\mathfrak{g}\otimes\mathfrak{g}$, and not a distribution. For different values of the loop variables λ and μ, the fraction $(\mu-\lambda)^{-1}$

coincide with its principal value, hence the kernels $\mathcal{R}^0_{\underline{\mathbf{12}}}$ and $\widetilde{\mathcal{R}}_{\underline{\mathbf{12}}}$ coincide for $\lambda \neq \mu$. Computing the left-hand side of (C.4.5) for $\mathcal{R}^0$ instead of $\widetilde{\mathcal{R}}$, one can show that considering the meromorphic function instead of its principal value as for effect to discard the distribution-valued contact term, which is not zero only when the three loop variables are equal. Thus, one finds that $\mathcal{R}^0$ is a solution of the non-modified CYBE on $\mathcal{L}(\mathfrak{g})$

$$\big[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}^0_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]+\big[\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda_1,\lambda_2),\mathcal{R}^0_{\underline{\mathbf{23}}}(\lambda_2,\lambda_3)\big]+\big[\mathcal{R}^0_{\underline{\mathbf{32}}}(\lambda_3,\lambda_2),\mathcal{R}^0_{\underline{\mathbf{13}}}(\lambda_1,\lambda_3)\big]=0.$$

Twist function and kernel. In the previous paragraph, we constructed the standard $\mathcal{R}$-matrix on $\mathcal{L}(\mathfrak{g})$ by considering the kernel of the operator (C.4.3) with respect to the bilinear form (C.4.1). Let us slightly modify the bilinear form:

$$\forall\, M,N\in\mathcal{L}(\mathfrak{g}),\quad \langle M,N\rangle_\varphi=\operatorname*{res}_{\lambda=0}\kappa\big(M(\lambda),N(\lambda)\big)\varphi(\lambda)\mathrm{d}\lambda=\oint\kappa\big(M(\lambda),N(\lambda)\big)\varphi(\lambda)\mathrm{d}\lambda, \tag{C.4.6}$$

where φ is a rational functional of λ.

This bilinear form is also invariant and non-degenerate on $\mathcal{L}(\mathfrak{g})$. One can then define the corresponding kernel of the operator $\mathcal{R}$, which will then be a solution of the mCYBE (C.4.5). As in the previous paragraph, considering a meromorphic function instead of its principal value, one obtains a matrix $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)$ solution of the non-modified CYBE. This matrix reads

$$\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)=\frac{C_{\underline{\mathbf{12}}}}{\mu-\lambda}\varphi(\mu)^{-1}=\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\varphi(\mu)^{-1}. \tag{C.4.7}$$

We recognize here the $\mathcal{R}$-matrix (2.4.1) introduced in Chapter 2. The function φ is then what we called the **twist function** in this chapter.

Note that for φ different from a constant function, the subalgebras $\mathfrak{g}[[\lambda]]$ and $\lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ of $\mathcal{L}(\mathfrak{g})$ are not isotropic with respect to the bilinear form $\langle\cdot,\cdot\rangle_\varphi$. Thus, we get a non skew-symmetric kernel $\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu)$, as we can directly observe on equation (C.4.7).

Twisted $\mathcal{R}$-matrices. We end this section by discussing the loop interpretation of the twisted standard $\mathcal{R}$-matrices (2.3.8) on $\mathcal{L}(\mathfrak{g})$. Suppose that we are given an automorphism σ of $\mathfrak{g}$, of finite order T. Let ω be a Tth-root of unity. We define an endomorphism $\widehat{\sigma}$ on $\mathcal{L}(\mathfrak{g})$ by

$$\forall\, M\in\mathcal{L}(\mathfrak{g}),\quad \widehat{\sigma}(M)(\lambda)=\sigma\big(M(\omega^{-1}\lambda)\big).$$

This is an automorphism of $\mathcal{L}(\mathfrak{g})$, also of order T. We define the twisted loop algebra by σ as the subalgebra of fixed points of $\widehat{\sigma}$:

$$\mathcal{L}(\mathfrak{g},\sigma)=\mathcal{L}(\mathfrak{g})^{\widehat{\sigma}}.$$

The elements of $\mathcal{L}(\mathfrak{g},\sigma)$ are then equivariant $\mathfrak{g}$-valued Laurent series, in the sense that:

$$\forall\, M\in\mathcal{L}(\mathfrak{g}),\quad M\in\mathcal{L}(\mathfrak{g},\sigma)\Longleftrightarrow\sigma\big(M(\lambda)\big)=M(\omega\lambda).$$

The subalgebras $\mathfrak{g}[[\lambda]]$ and $\lambda^{-1}\mathfrak{g}[\lambda^{-1}]$ of $\mathcal{L}(\mathfrak{g})$ are stabilised by $\widehat{\sigma}$. Thus, the decomposition (C.4.2) of $\mathcal{L}(\mathfrak{g})$ induces the following decomposition of $\mathcal{L}(\mathfrak{g},\sigma)$;

$$\mathcal{L}(\mathfrak{g},\sigma)=\mathfrak{g}[[\lambda]]^{\widehat{\sigma}}\oplus\lambda^{-1}\mathfrak{g}[\lambda^{-1}]^{\widehat{\sigma}}.$$

This defines an AKS decomposition on $\mathcal{L}(\mathfrak{g},\sigma)$. We denote by π^σ_+ and π^σ_- the projectors along this decomposition. Applying the AKS scheme, we get an operator

$$\mathcal{R}^\sigma=\pi^\sigma_+-\pi^\sigma_-$$

of $\mathcal{L}(\mathfrak{g},\sigma)$, solution of the operator mCYBE.

Recall the invariant non-degenerate bilinear form (C.4.6) on $\mathcal{L}(\mathfrak{g})$, twisted by the function $\varphi(\lambda)$. This form reduces to an invariant non-degenerate form on $\mathcal{L}(\mathfrak{g},\sigma) = \mathcal{L}(\mathfrak{g})^{\widehat{\sigma}}$ if it is invariant under the action of $\widehat{\sigma}$, *i.e.* if

$$\langle \widehat{\sigma}(M), \widehat{\sigma}(N)\rangle_\varphi = \langle M, N\rangle_\varphi, \quad \forall\, M, N \in \mathcal{L}(\mathfrak{g}).$$

This is true if and only if the twist function φ satisfies the equivariance condition

$$\varphi(\omega\lambda) = \omega^{-1}\varphi(\lambda).$$

We recognize here the equivariance condition (2.4.5) discussed in Chapter 2. We will now suppose that this condition is verified, so that $\langle\cdot,\cdot\rangle_\varphi$ defines an invariant non-degenerate bilinear form on $\mathcal{L}(\mathfrak{g},\sigma)$. We can then construct the kernel of the operator $\mathcal{R}^\sigma$. As in the previous subsections, considering meromorphic functions instead of principal values, we get from this kernel a matrix solution of the non-modified CYBE:

$$\mathcal{R}_{\underline{\mathbf{12}}}(\lambda,\mu) = \mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu)\varphi(\mu)^{-1},$$

where

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\lambda,\mu) = \frac{1}{T}\sum_{k=0}^{T-1}\frac{\sigma^k_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}}{\mu - \omega^{-k}\lambda}.$$

We recognize here the standard $\mathcal{R}$-matrix twisted by σ, as introduced in (2.3.8).

Part IV

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Part V

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On the Hamiltonian integrability of the bi-Yang-Baxter σ-model

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Abstract: The bi-Yang-Baxter σ-model is a certain two-parameter deformation of the principal chiral model on a real Lie group G for which the left and right G-symmetries of the latter are both replaced by Poisson-Lie symmetries. It was introduced by C. Klimčík who also recently showed it admits a Lax pair, thereby proving it is integrable at the Lagrangian level. By working in the Hamiltonian formalism and starting from an equivalent description of the model as a two-parameter deformation of the coset σ-model on $G \times G/G_{\rm diag}$, we show that it also admits a Lax matrix whose Poisson bracket is of the standard r/s-form characterised by a twist function which we determine. A number of results immediately follow from this, including the identification of certain complex Poisson commuting Kac-Moody currents as well as an explicit description of the q-deformed symmetries of the model. Moreover, the model is also shown to fit naturally in the general scheme recently developed for constructing integrable deformations of σ-models. Finally, we show that although the Poisson bracket of the Lax matrix still takes the r/s-form after fixing the $G_{\rm diag}$ gauge symmetry, it is no longer characterised by a twist function.

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1 Introduction

The Yang-Baxter σ-model is a one-parameter deformation of the principal chiral model, first introduced by C. Klimčík more than twenty years ago [1]. Its name stems from the presence of a solution of the modified classical Yang-Baxter equation in its action. The classical integrability of this model at the Lagrangian level was later proved in [2] by exhibiting a Lax pair, the flatness of which reproduces the equations of motion. Recently, the Yang-Baxter σ-model was recovered as the simplest case of a general procedure developed to deform a broad class of integrable σ-models while preserving their integrability [3, 4]. The whole construction is deeply rooted in the Hamiltonian formalism. In particular, one of its salient features is that the integrability at the Hamiltonian level of the resulting deformed σ-models is ensured from the very outset.

Recall that proving Hamiltonian integrability requires more than determining a Lax pair. Indeed, the existence of a Lax pair only implies that there is an infinite number of conserved quantities. However, the Hamiltonian definition of integrability requires showing instead that there is an infinite number of quantities Poisson commuting with one another, not just with the Hamiltonian. Such a property is guaranteed if the Poisson bracket of the Lax matrix, defined as the spatial component of the Lax pair, can be put in the general r/s-form [5, 6]. Furthermore, it was shown in [7] for the principal chiral model, and in [8] for symmetric space σ-models and the $AdS_5 \times S^5$ superstring theory, that the algebraic structure behind the r/s-form of these σ-models is encoded in a so called twist function.

The twist function of a given integrable σ-model plays a key role in the study of its integrable deformations. Indeed, the one-parameter integrable deformations of the principal chiral model and (semi-)symmetric σ-models constructed in [3, 9] were obtained by deforming their twist functions. More precisely, the focus of [3, 9] was on the so called Yang-Baxter class of deformations, of which the Yang-Baxter σ-model is the prototype. There exists another way of deforming the σ-models in question, with a completely different Lagrangian description [10–19]. Nevertheless, in the Hamiltonian framework, the procedure for obtaining these alternative deformations may also be interpreted as deforming the corresponding twist functions [11, 20]. For completeness, let us also mention that within the Yang-Baxter class of integrable deformations there is also a way to deform a given σ-model by using a solution of the classical Yang-Baxter equation [21–31], but without changing its twist function [20].

The bi-Yang-Baxter σ-model was also proposed in [2] as a two-parameter deformation of the principal chiral model. Its Lagrangian integrability was only proved relatively recently in [32]. An interesting feature of this model is the following. Whereas the principal chiral model on a real Lie group G admits an invariance under $G \times G$ by left and right multiplications of the G-valued field, in the Yang-Baxter σ-model one of these two global symmetries gets deformed to $U_q^{\mathcal{P}}(\mathfrak{g})$, the Poisson algebra analogue of a quantum group. Here q is a function of the single deformation parameter. The bi-Yang-Baxter σ-model can be seen as a further deformation of the Yang-Baxter σ-model in which both left and right global G-symmetries get deformed [2].

In this article we will focus on the Hamiltonian analysis of the bi-Yang-Baxter σ-model. In section 2, we begin by recalling the action of the bi-Yang-Baxter σ-model. We start from

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its formulation as a two-parameter deformation of the coset σ-model on $G\times G/G_{\rm diag}$, where $G_{\rm diag}$ is the diagonal subgroup of $G\times G$. That is, when both deformation parameters are turned off we obtain the coset σ-model on $G\times G/G_{\rm diag}$. The principal chiral model on G is then recovered in a particular gauge. This point of view on the bi-Yang-Baxter σ-model was recently adopted in [33] where the corresponding Lax pair was introduced. Since the deformation preserves the gauge invariance under $G_{\rm diag}$, a first-class constraint appears in the canonical analysis. In the presence of such constraints, the Hamiltonian Lax matrix $\mathcal{L}(z)$, with z the spectral parameter, is not fully determined by its Lagrangian counterpart. Indeed, one has the freedom to add to the latter a term consisting of an arbitrary function $f(z)$ times the constraint. This freedom was first shown to play an important role in [34, 35] for the $AdS_5\times S^5$ superstring theory.

In section 3 we show that the Poisson bracket of $\mathcal{L}(z)$ and $\mathcal{L}(z')$ takes the desired r/s-form ensuring Hamiltonian integrability for a specific choice of the function $f(z)$. More precisely, since we are considering a deformation of the coset σ-model on $G\times G/G_{\rm diag}$, the Lax matrix naturally takes values in the twisted loop algebra of the real double $D\mathfrak{g}=\mathfrak{g}\oplus\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G. However, in this particular case it is possible to work instead with a Lax matrix taking values in the loop algebra of a single copy of $\mathfrak{g}$. The corresponding r- and s-matrices are the skew-symmetric and symmetric parts, respectively, of an $\mathcal{R}$-matrix of the standard form depending on a two-parameter twist function $\varphi_{\rm bYB}(z)$ which we determine.

To complete the analysis, in section 4 we indicate how the result obtained may be understood when working with a Lax matrix valued in the twisted loop algebra of $D\mathfrak{g}$. In this formalism, the Poisson bracket of the Lax matrix with itself is still of the r/s-form but where the $\mathcal{R}$-matrix takes on a novel form depending on both the twist function $\varphi_{\rm bYB}(z)$ and its "mirror" image $\varphi_{\rm bYB}(-z)$. This $\mathcal{R}$-matrix is shown to correspond to the kernel of the standard solution of the modified classical Yang-Baxter equation on the twisted loop algebra of $D\mathfrak{g}$ but with respect to an non-standard inner product on the latter. All these results show that the bi-Yang-Baxter σ-model belongs to the same class of deformations as those constructed in [3]. Indeed, it corresponds to a deformation of the twist function of the $G\times G/G_{\rm diag}$ coset σ-model.

In section 5, we recall the importance of studying the poles of the twist function. Specifically, we show that the Lax matrix $\mathcal{L}(z)$ evaluated at the poles of the twist function $\varphi_{\rm bYB}(z)$ yields a pair of Poisson commuting Kac-Moody currents valued in the complexification $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\otimes\mathbb{C}$ of the real Lie algebra $\mathfrak{g}$. We go on to show how the canonical fields of the bi-Yang-Baxter σ-model may be recovered from the Lax matrix at the poles of the twist function. The upshot of this analysis is that the bi-Yang-Baxter σ-model also fits the general scheme described in [20]. As another important output of studying the (gauge transformed) monodromy matrix at the poles of $\varphi_{\rm bYB}(z)$, it immediately follows that the global $G\times G$ symmetry of the principal chiral model gets deformed to $U_q^{\mathcal{P}}(\mathfrak{g})\times U_{\tilde{q}}^{\mathcal{P}}(\mathfrak{g})$. We indicate how we recover the values of q and $\tilde{q}$ first given in [33]. This generalises the situation in [3] recalled above, and which first appeared in the context of the Yang-Baxter σ-model on SU(2), also known as the squashed S^3 σ-model [36, 37].

Finally, in section 6 we study the fate of the r/s-form of the Lax matrix algebra when gauge fixing the local $G_{\rm diag}$-symmetry of the bi-Yang-Baxter σ-model. We do this

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by regarding the gauge fixing as a gauge transformation on the Lax matrix. This enables one to determine how the r/s-form behaves under this gauge fixing. We show that the r- and s-matrices are no longer fully determined by a twist function but depend also on the R-matrices characterising the Yang-Baxter type deformation.

2 The bi-Yang-Baxter σ-model

2.1 Lagrangian analysis

2.1.1 Action

Let G be a semi-simple real Lie group with Lie algebra $\mathfrak{g}$. Let R and $\tilde{R}$ be two skew-symmetric solutions of the modified classical Yang-Baxter equation (mCYBE) on $\mathfrak{g}$, i.e. endomorphisms of $\mathfrak{g}$ such that for every $x, y \in \mathfrak{g}$, we have

$$\kappa(x, Ry) = -\kappa(Rx, y), \tag{2.1a}$$

$$[Rx, Ry] = R\big([Rx, y] + [x, Ry]\big) + [x, y], \tag{2.1b}$$

and similarly for $\tilde{R}$. Here κ denotes the Killing form on $\mathfrak{g}$ defined as $\kappa(x, y) = -\operatorname{Tr}\left(\mathrm{ad}_x \mathrm{ad}_y\right)$ for any $x, y \in \mathfrak{g}$.

We then consider the bi-Yang-Baxter σ-model associated with R and $\tilde{R}$, defined by the following action for a field $(g, \tilde{g})$ valued in the double group $G \times G$ [33]

$$S[g, \tilde{g}] = K \int d\tau d\sigma \; \kappa \left(j_+ - \tilde{j}_+, \left(1 - \frac{\eta}{2} R_g - \frac{\tilde{\eta}}{2} \tilde{R}_{\tilde{g}}\right)^{-1} (j_- - \tilde{j}_-) \right). \tag{2.2}$$

K, η and $\tilde{\eta}$ are real parameters, $\partial_\pm = \partial_\tau \pm \partial_\sigma$, and we have introduced the following notations

$$\begin{aligned} j_\pm &= g^{-1}\partial_\pm g, & \tilde{j}_\pm &= \tilde{g}^{-1}\partial_\pm \tilde{g}, \\ R_g &= \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g, & \tilde{R}_{\tilde{g}} &= \mathrm{Ad}_{\tilde{g}}^{-1} \circ \tilde{R} \circ \mathrm{Ad}_{\tilde{g}}, \\ \mathrm{Ad}_g(M) &= gMg^{-1}. \end{aligned}$$

Let us notice here that R_g and $\tilde{R}_{\tilde{g}}$ are also skew-symmetric solutions of the mCYBE.

When $\eta = \tilde{\eta} = 0$ we recover the coset σ-model on the quotient $G \times G/G_{\mathrm{diag}}$ by the diagonal subgroup G_{diag} of $G \times G$. It is direct to check that, like the coset σ-model, the bi-Yang-Baxter σ-model is invariant under gauge transformations taking values in the subgroup G_{diag}, namely

$$g \mapsto gh^{-1} \quad \text{and} \quad \tilde{g} \mapsto \tilde{g}h^{-1}, \tag{2.3}$$

with h a field valued in the group G. We may impose the gauge fixing condition $\tilde{g} = \mathrm{Id}$, which is attained by performing the gauge transformation (2.3) with $h = \tilde{g}$. This leads to a model for the G-valued field $g' = g\tilde{g}^{-1}$, which coincides with the two-parameter deformation of the principal chiral model first introduced in [2].

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2.1.2 Equations of motion

The equation of motion for the field g derived from the action (2.2) can be written as

$$EOM = \partial_+ J_- + [a_+, J_-] + \partial_- J_+ + [a_-, J_+] = 0, \tag{2.4}$$

where we introduced

$$J_\pm = \left(1 \pm \frac{\eta}{2} R_g \pm \frac{\tilde{\eta}}{2} \tilde{R}_{\tilde{g}}\right)^{-1} (j_\pm - \tilde{j}_\pm) \tag{2.5}$$

and a "gauge field"

$$a_\pm = j_\pm \mp \frac{\eta}{2} R_g J_\pm = \left(1 \pm \frac{\tilde{\eta}}{2} \tilde{R}_{\tilde{g}}\right) J_\pm + \tilde{j}_\pm. \tag{2.6}$$

Notice that a transformation

$$a_\pm \mapsto a_\pm + \alpha J_\pm \tag{2.7}$$

of the gauge field does not change the equation of motion (2.4).

The action (2.2) is not changed when one exchanges η, R and g with $\tilde{\eta}$, $\tilde{R}$ and $\tilde{g}$. Thus the equation of motion for $\tilde{g}$ takes the same form:

$$\widetilde{EOM} = \partial_+ \tilde{J}_- + [\tilde{a}_+, \tilde{J}_-] + \partial_- \tilde{J}_+ + [\tilde{a}_-, \tilde{J}_+] = 0,$$

with

$$\tilde{J}_\pm = \left(1 \pm \frac{\eta}{2} R_g \pm \frac{\tilde{\eta}}{2} \tilde{R}_{\tilde{g}}\right)^{-1} (\tilde{j}_\pm - j_\pm),$$
$$\tilde{a}_\pm = \tilde{j}_\pm \mp \frac{\tilde{\eta}}{2} \tilde{R}_{\tilde{g}} \tilde{J}_\pm = \left(1 \pm \frac{\eta}{2} R_g\right) \tilde{J}_\pm + j_\pm.$$

It is then easy to check that

$$\tilde{J}_\pm = -J_\pm \qquad \text{and} \qquad \tilde{a}_\pm = a_\pm - J_\pm. \tag{2.8}$$

Thus, using the freedom (2.7) on $a_\pm$, we see that $\widetilde{EOM} = -EOM$. Therefore, the equation of motion for $\tilde{g}$ is equivalent to the one for g.

2.1.3 Lax pair

In this subsection, we recall that the equation of motion (2.4) can be cast in the form of a zero curvature equation

$$\partial_+ \mathcal{L}_-(z) - \partial_- \mathcal{L}_+(z) - [\mathcal{L}_+(z), \mathcal{L}_-(z)] = 0 \tag{2.9}$$

for a Lax pair $\mathcal{L}_\pm(z)$ depending on a spectral parameter z [33]. Starting from the Maurer-Cartan equation on $j_\pm$,

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0,$$

we re-express it in terms of $J_\pm$ and $a_\pm$ using (2.6), giving

$$\partial_+ a_- - \partial_- a_+ + [a_+, a_-] + \frac{\eta^2}{4} [J_+, J_-] - \frac{\eta}{2} R_g(EOM) = 0, \tag{2.10}$$

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where we used the mCYBE on R_g. In the same way, the Maurer-Cartan equation on $\tilde{j}_\pm$ reads

$$\partial_+\tilde{a}_- - \partial_-\tilde{a}_+ + [\tilde{a}_+, \tilde{a}_-] + \frac{\tilde{\eta}^2}{4}[J_+, J_-] + \frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}}(EOM) = 0, \tag{2.11}$$

where we have used $\tilde{J}_\pm = -J_\pm$ and $\widetilde{EOM} = -EOM$. Taking the difference between (2.10) and (2.11) and using (2.8), we obtain

$$\partial_+ J_- + [a_+, J_-] - \partial_- J_+ - [a_-, J_+] - \left(1 - \frac{\eta^2 - \tilde{\eta}^2}{4}\right)[J_+, J_-] - \frac{1}{2}(\eta R_g + \tilde{\eta}\tilde{R}_{\tilde{g}})(EOM) = 0. \tag{2.12}$$

We introduce new gauge fields

$$A_\pm = a_\pm - \frac{1}{2}\left(1 - \frac{\eta^2 - \tilde{\eta}^2}{4}\right) J_\pm. \tag{2.13}$$

In terms of these, the equation of motion (2.4) keeps the same form

$$EOM = \partial_+ J_- + [A_+, J_-] + \partial_- J_+ + [A_-, J_+] \tag{2.14}$$

and the equation (2.12) becomes

$$\partial_+ J_- + [A_+, J_-] - \partial_- J_+ - [A_-, J_+] - \frac{1}{2}(\eta R_g + \tilde{\eta}\tilde{R}_{\tilde{g}})(EOM) = 0. \tag{2.15}$$

Coming back to the expression (2.10) and using the definition (2.13) of $A_\pm$, we find

$$\begin{aligned} 0 = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] + \frac{\zeta^2}{4}[J_+, J_-] \\ + \frac{1}{4}\left(1 - \frac{\eta^2 - \tilde{\eta}^2}{4}\right)(\eta R_g + \tilde{\eta}\tilde{R}_{\tilde{g}})(EOM) - \eta R_g(EOM) \end{aligned} \tag{2.16}$$

where

$$\zeta = \sqrt{\left(1 + \frac{1}{4}(\eta + \tilde{\eta})^2\right)\left(1 + \frac{1}{4}(\eta - \tilde{\eta})^2\right)}. \tag{2.17}$$

Finally, taking the equation (2.16) on shell ($EOM = 0$) and the sum and the difference of equations (2.14) and (2.15) also on shell, we arrive at

$$\begin{gathered} \partial_+ A_- - \partial_- A_+ + [A_+, A_-] + \frac{\zeta^2}{4}[J_+, J_-] = 0, \\ \partial_+ J_- + [A_+, J_-] = 0 \qquad \text{and} \qquad \partial_- J_+ + [A_-, J_+] = 0. \end{gathered}$$

It is easy to see that these three equations are equivalent to the zero curvature equation (2.9) for the Lax pair:

$$\mathcal{L}_\pm(z) = -A_\pm - \frac{\zeta}{2} z^{\pm 1} J_\pm. \tag{2.18}$$

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2.2 Hamiltonian analysis

2.2.1 Conjugate momentum

Let us introduce a basis T_a of the Lie algebra $\mathfrak{g}$ and coordinates ϕ^i on the group G. We denote ∂_i the derivation with respect to the coordinate ϕ^i. We can then introduce L_i^a such that

$$g^{-1}\partial_i g = L_i^a T_a.$$

From the action (2.2), we compute the conjugate momenta π_i of the coordinates ϕ^i to be

$$\pi_i = KL_i^a \left[\kappa\left(T_a, \left(1 - \frac{\eta}{2}R_g - \frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}}\right)^{-1}(j_- - \tilde{j}_-)\right) + \kappa\left(j_+ - \tilde{j}_+, \left(1 - \frac{\eta}{2}R_g - \frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}}\right)^{-1} T_a\right)\right].$$

Using the skew-symmetry of R and (2.5), we have

$$\pi_i = KL_i^a \kappa(T_a, J_- + J_+). \tag{2.19}$$

with the metric $\kappa_{ab} = \kappa(T_a, T_b)$. It is more convenient to introduce the following $\mathfrak{g}$-valued field

$$X = L_a^i \pi_i \kappa^{ab} T_b, \tag{2.20}$$

where L_a^i is the inverse of L_i^a and κ^{ab} is the inverse of the metric κ_{ab}. In particular, one can check that these fields are independent of the choice of coordinates ϕ^i and of basis T_a. It is then easy to deduce the expression of X from (2.19) to be

$$X = K(J_+ + J_-). \tag{2.21}$$

In the same way, one would find $\tilde{X} = K(\tilde{J}_+ + \tilde{J}_-) = -K(J_+ + J_-)$. Thus, we have the constraint

$$X + \tilde{X} = 0. \tag{2.22}$$

This is a consequence of the gauge symmetry (2.3) of the model.

2.2.2 Poisson brackets and Hamiltonian density

We start with the canonical Poisson brackets

$$\{\pi_i(\sigma), \phi^j(\sigma')\} = \delta_i^j \delta_{\sigma\sigma'}. \tag{2.23}$$

where $\delta_{\sigma\sigma'}$ is the Dirac δ-distribution. From those canonical Poisson brackets and the definition (2.20) of X, we deduce the classical brackets on the fields g and X parametrising the cotangent bundle T^*LG, with LG the loop group associated with G, to be

$$\{g_{\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} = 0, \tag{2.24a}$$
$$\{g_{\underline{1}}(\sigma), X_{\underline{2}}(\sigma')\} = -g_{\underline{1}}(\sigma) C_{\underline{12}} \delta_{\sigma\sigma'}, \tag{2.24b}$$
$$\{X_{\underline{1}}(\sigma), X_{\underline{2}}(\sigma')\} = -\left[C_{\underline{12}}, X_{\underline{2}}(\sigma)\right] \delta_{\sigma\sigma'}. \tag{2.24c}$$

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We used standard tensorial notations with subscripts $\underline{1}$ and $\underline{2}$ and $C_{\underline{\mathbf{12}}} = \kappa^{ab} T_a \otimes T_b$ is the split Casimir. The fields $\tilde{g}$ and $\tilde{X}$ parametrising another copy of T^*LG verify the same Poisson brackets. All other brackets vanish. Moreover, as long as we are calculating Poisson brackets, we must consider X and $\tilde{X}$ as independent variables in the phase space, without imposing the constraint (2.22).

The Legendre transform of the Lagrangian in (2.2) is the "naive" Hamiltonian density

$$\mathcal{H}_0 = \frac{K}{2}\big(\kappa\left(J_+, J_+\right) + \kappa\left(J_-, J_-\right)\big). \tag{2.25}$$

As we are considering a constrained system, we have to follow the Dirac procedure and add a term proportional to the constraint to define the Hamiltonian density of the system

$$\mathcal{H} = \mathcal{H}_0 + \kappa\big(\Lambda, X + \tilde{X}\big), \tag{2.26}$$

where Λ is a $\mathfrak{g}$-valued field playing the role of a Lagrange multiplier. There is no secondary constraint.

2.2.3 Hamiltonian Lax matrix

Let us now determine the form of the Hamiltonian Lax matrix of the model. At the Lagrangian level, the Lax matrix is given by the spatial component of the Lax pair, i.e. by $\frac{1}{2}(\mathcal{L}_+ - \mathcal{L}_-)$. As we are considering a constrained Hamiltonian system, we have the freedom of adding a term proportional to the constraint, thus getting

$$\mathcal{L}(z) = \frac{1}{2}\big(\mathcal{L}_+(z) - \mathcal{L}_-(z)\big) + f(z)(X + \tilde{X}),$$

where f is some function of z, which will be fixed later to ensure the Hamiltonian integrability of the model. One could potentially add other extra terms, for instance proportional to $R_g(X + \tilde{X})$ and $\tilde{R}_{\tilde{g}}(X + \tilde{X})$, but as we will see in the next section, they turn out not to be necessary.

Using equations (2.18) and (2.13), we get

$$\mathcal{L}(z) = -\frac{1}{2}(a_+ - a_-) + \frac{1}{4}\left(1 - \frac{\eta^2 - \tilde{\eta}^2}{4}\right)(J_+ - J_-) - \frac{\zeta}{4}\left(zJ_+ - \frac{1}{z}J_-\right) + f(z)(X + \tilde{X}).$$

The definition (2.6) of $a_\pm$ can be re-written in a more symmetric way as

$$a_\pm = \frac{1}{2}\left(j_\pm + \tilde{j}_\pm + J_\pm \mp \frac{\eta}{2} R_g J_\pm \pm \frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}} J_\pm\right),$$

thus giving

$$a_+ - a_- = \frac{1}{2}\left(j_+ - j_- + \tilde{j}_+ - \tilde{j}_- + J_+ - J_- - \left(\frac{\eta}{2}R_g - \frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}}\right)(J_+ + J_-)\right).$$

Denoting $j = \frac{1}{2}(j_+ - j_-)$ and $\tilde{j} = \frac{1}{2}(\tilde{j}_+ - \tilde{j}_-)$, we obtain

$$\mathcal{L}(z) = -\frac{1}{2}(j + \tilde{j}) - \left(\frac{\eta^2 - \tilde{\eta}^2}{16} + \frac{\zeta}{8}\left(z + \frac{1}{z}\right)\right)(J_+ - J_-) - \frac{\zeta}{8}\left(z - \frac{1}{z}\right)(J_+ + J_-)$$

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$$+\left(\frac{\eta}{8}R_g-\frac{\tilde{\eta}}{8}\tilde{R}_{\tilde{g}}\right)(J_++J_-)+f(z)(X+\tilde{X}).$$

Using (2.5), we have

$$J_+-J_-=2j-2\tilde{j}-\left(\frac{\eta}{2}R_g+\frac{\tilde{\eta}}{2}\tilde{R}_{\tilde{g}}\right)(J_++J_-),$$

which gives

$$\begin{aligned}\mathcal{L}(z)=&-\frac{1}{2}\left(1+\frac{\eta^2-\tilde{\eta}^2}{4}+\frac{\zeta}{2}\left(z+\frac{1}{z}\right)\right)j-\frac{1}{2}\left(1+\frac{\tilde{\eta}^2-\eta^2}{4}-\frac{\zeta}{2}\left(z+\frac{1}{z}\right)\right)\tilde{j}\\&+\frac{\eta}{8}\left(1+\frac{\eta^2-\tilde{\eta}^2}{4}+\frac{\zeta}{2}\left(z+\frac{1}{z}\right)\right)R_g(J_++J_-)\\&-\frac{\tilde{\eta}}{8}\left(1+\frac{\tilde{\eta}^2-\eta^2}{4}-\frac{\zeta}{2}\left(z+\frac{1}{z}\right)\right)\tilde{R}_{\tilde{g}}(J_++J_-)\\&-\frac{\zeta}{8}\left(z-\frac{1}{z}\right)(J_++J_-)+f(z)(X+\tilde{X}).\end{aligned}\tag{2.27}$$

In order to finish re-expressing (2.27) in terms of the Hamiltonian fields alone, we make use of equations (2.21) and (2.22) namely $J_++J_-=X/K=-\tilde{X}/K$. For reasons of symmetry and simplicity, we will use X (respectively $\tilde{X}$) when R_g (respectively $\tilde{R}_{\tilde{g}}$) is applied to J_++J_-, and we will use the linear combination $\frac{1}{2}(X-\tilde{X})$ when J_++J_- stands alone. This last "prescription" does not change the expression of the Lax matrix, as any other choice can be re-absorbed in the function $f(z)$ which is so far arbitrary. Beyond the arguments of symmetry, the resulting form of the Hamiltonian Lax matrix will be justified in the following section to prove the Hamiltonian integrability of the model.

The final result can be written in terms of the set of fields $\mathcal{O}=\{j,X,R_gX,\tilde{j},\tilde{X},\tilde{R}_{\tilde{g}}\tilde{X}\}$ as

$$\mathcal{L}(z)=\sum_{Q\in\mathcal{O}}A_Q(z)Q,\tag{2.28}$$

with coefficients A_Q whose expressions are given in appendix A.

2.3 One-parameter deformation limit

By fixing $\eta=\tilde{\eta}$ we obtain a one-parameter deformation of the coset model on $G\times G/G_{\text{diag}}$. It is given by the action

$$S[g,\tilde{g}]=K\int d\sigma d\tau\,\kappa\left(j_+-\tilde{j}_+,\left(1-\frac{\eta}{2}R_g-\frac{\eta}{2}\tilde{R}_{\tilde{g}}\right)^{-1}(j_--\tilde{j}_-)\right).\tag{2.29}$$

Let us consider the double Lie group $DG=G\times G$ and the corresponding double Lie algebra $D\mathfrak{g}=\mathfrak{g}\oplus\mathfrak{g}$. The latter comes naturally equipped with the exchange automorphism

$$\begin{array}{rccc}\delta:&D\mathfrak{g}&\longrightarrow&D\mathfrak{g}\\&(x,y)&\longmapsto&(y,x)\end{array}.\tag{2.30}$$

We may decompose $D\mathfrak{g}$ into eigenspaces of this involution as $D\mathfrak{g}=D\mathfrak{g}^{(0)}\oplus D\mathfrak{g}^{(1)}$, with $D\mathfrak{g}^{(0)}=\ker(\delta-\mathrm{Id})$ and $D\mathfrak{g}^{(1)}=\ker(\delta+\mathrm{Id})$. We can notice here that $D\mathfrak{g}^{(0)}=\mathfrak{g}_{\text{diag}}$, the

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Lie algebra of the diagonal subgroup G_{diag}, so that the quotient $G \times G/G_{\text{diag}}$ is indeed the coset $DG/DG^{(0)}$.

We will denote P_0 and P_1 the projectors associated with this decomposition, defined by

$$\begin{array}{rccc} P_0: & D\mathfrak{g} & \longrightarrow & D\mathfrak{g} \\ & (x,y) & \longmapsto & \frac{1}{2}(x+y,x+y) \end{array} \quad \text{and} \quad \begin{array}{rccc} P_1: & D\mathfrak{g} & \longrightarrow & D\mathfrak{g} \\ & (x,y) & \longmapsto & \frac{1}{2}(x-y,y-x) \end{array} \tag{2.31}$$

In this formulation on the double Lie group and Lie algebra, it is natural to introduce the field $h = (g, \tilde{g}) \in DG$ and the solution $\mathfrak{R} = (R, \tilde{R}) \in \text{End}(D\mathfrak{g})$ of the mCYBE on $D\mathfrak{g}$. The action (2.29) can then be re-expressed as

$$S[g,\tilde{g}] = 2K \int d\sigma d\tau \; \kappa \left((h^{-1}\partial_+ h)^{(1)}, (1 - \eta\, \mathfrak{R}_h \circ P_1)^{-1} \, (h^{-1}\partial_- h)^{(1)} \right). \tag{2.32}$$

This is nothing but the one-parameter deformation of the coset σ-model introduced in [3] when the quotient considered is $G \times G/G_{\text{diag}}$ and with $K = \frac{1}{4}(1+\eta^2)$.

3 Hamiltonian integrability

In this section, we will compute the Poisson bracket of the Lax matrix (2.28) with itself and show that it can be cast in the r/s-form (more precisely an r/s-system involving twist function), thus proving the Hamiltonian integrability of the bi-Yang-Baxter σ-model.

3.1 r/s-form and twist functions

Let $\mathcal{R}_{\underline{\mathbf{12}}}(z,z')$ be a rational function of z and z' valued in $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$, where $\mathfrak{g}^{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, and satisfying the classical Yang-Baxter equation with spectral parameters. We do not assume that $\mathcal{R}_{\underline{\mathbf{12}}}(z,z')$ is skew-symmetric, i.e. that it has the property $\mathcal{R}_{\underline{\mathbf{12}}}(z,z') = -\mathcal{R}_{\underline{\mathbf{21}}}(z',z)$. We introduce its skew-symmetric and symmetric parts as

$$r_{\underline{\mathbf{12}}}(z,z') = \frac{1}{2}\big(\mathcal{R}_{\underline{\mathbf{12}}}(z,z') - \mathcal{R}_{\underline{\mathbf{21}}}(z',z)\big) \quad \text{and} \quad s_{\underline{\mathbf{12}}}(z,z') = \frac{1}{2}\big(\mathcal{R}_{\underline{\mathbf{12}}}(z,z') + \mathcal{R}_{\underline{\mathbf{21}}}(z',z)\big). \tag{3.1a}$$

The Poisson bracket of the Lax matrix with itself is said to be of the r/s-form, associated with this matrix $\mathcal{R}$, if it can be written as [5, 6]

$$\begin{aligned} \big\{\mathcal{L}_{\underline{\mathbf{1}}}(z,\sigma), \mathcal{L}_{\underline{\mathbf{2}}}(z',\sigma')\big\} = &\big[r_{\underline{\mathbf{12}}}(z,z'), \mathcal{L}_{\underline{\mathbf{1}}}(z,\sigma) + \mathcal{L}_{\underline{\mathbf{2}}}(z',\sigma')\big]\, \delta_{\sigma\sigma'} \\ &+ \big[s_{\underline{\mathbf{12}}}(z,z'), \mathcal{L}_{\underline{\mathbf{1}}}(z,\sigma) - \mathcal{L}_{\underline{\mathbf{2}}}(z',\sigma')\big]\, \delta_{\sigma\sigma'} + \; 2s_{\underline{\mathbf{12}}}(z,z')\delta'_{\sigma\sigma'}, \end{aligned} \tag{3.1b}$$

where $\delta'_{\sigma\sigma'} = \partial_\sigma \delta_{\sigma\sigma'}$.

The non-ultralocality of this Poisson bracket, namely the presence of δ'-terms, is completely characterised by the symmetric part of the $\mathcal{R}$-matrix being non-zero. For a very broad class of integrable σ-models, the $\mathcal{R}$-matrix $\mathcal{R}_{\underline{\mathbf{12}}}(z,z')$ is given by the kernel of an abstract solution of the mCYBE on the loop algebra $\mathfrak{g}((z))$, with respect to the standard inner product on $\mathfrak{g}((z))$ modified by a rational function $\varphi(z)$, called the twist function (see for instance [8] or section 4.2.1 for the case when $\mathfrak{g}$ is replaced by the double $D\mathfrak{g}$). In this

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situation the failure of $\mathcal{R}$ to be skew-symmetric is encoded in the twist function. In the simplest of cases, the kernel $\mathcal{R}_{\underline{12}}(z,z')$ takes the form

$$\mathcal{R}_{\underline{12}}(z,z') = \frac{C_{\underline{12}}}{z-z'}\varphi(z')^{-1}, \tag{3.1c}$$

and is therefore skew-symmetric if and only if φ is constant.

The simplest example of a model with such an $\mathcal{R}$-matrix is the principal chiral model [7]. Moreover, one can show from the results of [3] that the coset σ-model on $G \times G/G_{\text{diag}}$ and its one-parameter deformation also admit $\mathcal{R}$-matrices of this form.[1] The twist function of the coset σ-model on $G \times G/G_{\text{diag}}$ (which is, in the setting considered here, the limit $\eta = \tilde{\eta} = 0$ of the bi-Yang-Baxter σ-model) is

$$\varphi_{\text{coset}}(z) = \frac{16Kz}{(1-z^2)^2} \tag{3.2}$$

and the one of the Yang-Baxter deformation of this coset σ-model (which corresponds to $\eta = \tilde{\eta}$) is

$$\varphi_{\text{YB}}(z) = \frac{16Kz}{(1-z^2)^2 + \eta^2(1+z^2)^2}. \tag{3.3}$$

We will now show that the bi-Yang-Baxter σ-model also admits an $\mathcal{R}$-matrix of the form (3.1c) and will give the associated twist function.

3.2 Expected form of the Poisson bracket

We are seeking to put the Poisson bracket of the Lax matrix (2.28) in the r/s-form (3.1), with a twist function φ as in (3.1c). We will distinguish between two terms in this Poisson bracket: the ultralocal one, proportional to $\delta_{\sigma\sigma'}$, and the non-ultralocal one, proportional to $\delta'_{\sigma\sigma'}$. Let us write these as

$$\big\{\mathcal{L}_{\underline{1}}(z,\sigma), \mathcal{L}_{\underline{2}}(z',\sigma')\big\} = P^{\text{UL}}_{\underline{12}}(z,z',\sigma)\delta_{\sigma\sigma'} + P^{\text{NUL}}_{\underline{12}}(z,z',\sigma)\delta'_{\sigma\sigma'}.$$

According to (3.1b), the non-ultralocal term is directly proportional to the s-matrix. For a system with a twist function entering as in (3.1c), this term is thus given by

$$P^{\text{NUL}}_{\underline{12}}(z,z',\sigma) = -\frac{\varphi(z)^{-1} - \varphi(z')^{-1}}{z-z'} C_{\underline{12}}. \tag{3.4}$$

The ultralocal term is slightly more complicated. Considering the expressions (3.1c) of $\mathcal{R}$ and (2.28) of $\mathcal{L}$ and using the invariance property of the split Casimir, namely that for every $x \in \mathfrak{g}$ we have $[C_{\underline{12}}, x_{\underline{1}} + x_{\underline{2}}] = 0$, one can reduce the ultralocal term to the form

$$P^{\text{UL}}_{\underline{12}}(z,z',\sigma) = \sum_{Q\in\mathcal{O}} J_Q(z,z')[C_{\underline{12}}, Q_{\underline{2}}(\sigma)], \tag{3.5}$$

with the coefficients J_Q given by

$$J_Q(z,z') = \frac{\varphi(z)^{-1}A_Q(z') - \varphi(z')^{-1}A_Q(z)}{z-z'}. \tag{3.6}$$

[1] More precisely, [3] deals with a general coset σ-model F/G. In the case of the coset $G \times G/G_{\text{diag}}$, we get a Lax matrix in the double algebra $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$. We recover an r/s-system with an $\mathcal{R}$-matrix of the form (3.1c) by taking the projector of this Lax matrix on the left part of $D\mathfrak{g}$. This will be discussed in more details in section 4 of the present article.

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3.3 Poisson bracket of the Lax matrix

We will now compute the Poisson bracket of the Lax matrix explicitly and compare the result to the expected form discussed in the previous subsection. Using equation (2.28), this bracket is simply

$$\{\mathcal{L}_{\underline{\mathbf{1}}}(z,\sigma),\mathcal{L}_{\underline{\mathbf{2}}}(z',\sigma')\} = \sum_{Q,Q'\in\mathcal{O}} A_Q(z)A_{Q'}(z')\{Q_{\underline{\mathbf{1}}}(\sigma),Q'_{\underline{\mathbf{2}}}(\sigma')\}.$$

The Poisson brackets between the different fields $Q \in \mathcal{O} = \{j, X, R_g X, \tilde{j}, \tilde{X}, \tilde{R}_{\tilde{g}}\tilde{X}\}$ can be derived from the basic Poisson brackets (2.24). In particular, let us mention that we have

$$\{(R_gX)_{\underline{\mathbf{1}}}(\sigma),(R_gX)_{\underline{\mathbf{2}}}(\sigma')\} = \big[C_{\underline{\mathbf{12}}},X_{\underline{\mathbf{2}}}(\sigma)\big]\,\delta_{\sigma\sigma'}.$$

This follows from the fact that R_g is solution of the mCYBE. Any two fields from different copies of $\mathfrak{g}$ Poisson commute.

Non-ultralocal term. The non-ultralocal term is generated by the brackets of j and $\tilde{j}$ with the other fields. It reads

$$\begin{aligned}P^{\mathrm{NUL}}_{\underline{\mathbf{12}}}(z,z',\sigma,\sigma') = &-\big(A_j(z)A_X(z') + A_j(z')A_X(z) + A_{\tilde{j}}(z)A_{\tilde{X}}(z') + A_{\tilde{j}}(z')A_{\tilde{X}}(z)\big)C_{\underline{\mathbf{12}}}\\ &+\big(A_j(z)A_{R_gX}(z') - A_j(z')A_{R_gX}(z)\big)R_g(\sigma)_{\underline{\mathbf{12}}}\\ &+\big(A_{\tilde{j}}(z)A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z') - A_{\tilde{j}}(z')A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z)\big)\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}},\end{aligned}$$

where we defined $R_g(\sigma)_{\underline{\mathbf{12}}} = R_{g(\sigma)\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}}$ and $\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}} = \tilde{R}_{\tilde{g}(\sigma)\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}}$. One easily checks from (A.1) that the coefficients of $R_g(\sigma)_{\underline{\mathbf{12}}}$ and $\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}}$ in this expression vanish. As expected in (3.4), we find a non-ultralocal term proportional to the split Casimir $C_{\underline{\mathbf{12}}}$, namely

$$P^{\mathrm{NUL}}_{\underline{\mathbf{12}}}(z,z',\sigma,\sigma') = -\Big(A_j(z)A_X(z') + A_j(z')A_X(z) + A_{\tilde{j}}(z)A_{\tilde{X}}(z') + A_{\tilde{j}}(z')A_{\tilde{X}}(z)\Big)\,C_{\underline{\mathbf{12}}}. \tag{3.7}$$

Ultralocal term. We have in the ultralocal part three kinds of terms:

- Terms proportional to $[C_{\underline{\mathbf{12}}}, Q_{\underline{\mathbf{2}}}(\sigma)]$ with $Q \in \mathcal{O}$, as expected in (3.5).
- A term proportional to $[R_g(\sigma)_{\underline{\mathbf{12}}}, j_{\underline{\mathbf{2}}}(\sigma)]$.
- A term proportional to $[\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}}, \tilde{j}_{\underline{\mathbf{2}}}(\sigma)]$.

The coefficients of the last two terms are the same as the coefficients of $R_g(\sigma)_{\underline{\mathbf{12}}}$ and $\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}}$ in the non-ultralocal term. Thus, they also vanish. We are then left with an ultralocal term of the form (3.5). The expressions for the coefficients $J_Q(z,z')$ are given in appendix A.

3.4 Twist function of the model

To prove that the system admits a twist function, it remains to compare (3.4) with (3.7) and (3.6) with (A.2) and show that the different expressions match. We have shown that this is the case if we choose the function f to be

$$f(z) = -\frac{\zeta^2}{16K}(1+z^2) + \frac{1}{8K}\left(1 - \frac{(\eta^2-\tilde{\eta}^2)^2}{16}\right) - \frac{\zeta(\eta^2-\tilde{\eta}^2)}{64K}\left(3z+\frac{1}{z}\right),$$

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where ζ is defined by equation (2.17). The twist function is then

$$\varphi_{\mathrm{bYB}}(z) = \frac{1}{\zeta^2} \frac{16Kz}{z^4 + \dfrac{\eta^2 - \tilde{\eta}^2}{\zeta} z^3 + \left(2 + \dfrac{(\eta^2 - \tilde{\eta}^2)^2 - 16}{4\zeta^2}\right) z^2 + \dfrac{\eta^2 - \tilde{\eta}^2}{\zeta} z + 1}. \tag{3.8}$$

We will analyse the structure of this twist function in section 5.

4 Formulation in the double Lie algebra

Since we are considering a deformation of the coset σ-model on $G \times G/G_{\mathrm{diag}}$, we would expect the Lax matrix to be valued in the twisted loop algebra of the real double $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$, just as in the undeformed model [38]. However, the Lax matrix discussed so far only takes values in the loop algebra of $\mathfrak{g}$. We shall show in this section how the Hamiltonian integrability of the bi-Yang-Baxter σ-model can also be expressed using a formulation based on the double $D\mathfrak{g}$.

4.1 Lax pair in the double Lie algebra

We will use the formalism of the double Lie algebra $D\mathfrak{g}$ introduced in the subsection 2.3. Let us consider the loop algebra associated with $D\mathfrak{g}$, i.e. the space $D\mathfrak{g}(\!(z)\!) = D\mathfrak{g} \otimes \mathbb{C}(\!(z)\!)$ of Laurent series in a complex parameter z valued in the complexification of $D\mathfrak{g}$ and equipped with the natural Lie bracket. The exchange automorphism (2.30) on $D\mathfrak{g}$ induces an automorphism $\hat{\delta}$ on $D\mathfrak{g}(\!(z)\!)$ defined for all $X \in D\mathfrak{g}(\!(z)\!)$ by

$$\hat{\delta}(X)(z) = \delta\big(X(-z)\big).$$

Denote by $D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$ the twisted loop algebra, i.e. the subalgebra of $D\mathfrak{g}(\!(z)\!)$ formed by the fixed points of $\hat{\delta}$.

Recall that the Lax matrices of the coset σ-model (corresponding here to $\eta = \tilde{\eta} = 0$) and of its one-parameter deformation (corresponding here to $\eta = \tilde{\eta}$) belong to the twisted algebra $D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$. It is natural to expect such a Lax matrix to exist also for the bi-Yang-Baxter σ-model. The corresponding Lax pair can be constructed from the Lax pair $\mathcal{L}_\pm(z)$ valued in the loop algebra $\mathfrak{g}(\!(z)\!)$ of a single copy of $\mathfrak{g}$ in equation (2.18). Indeed, defining

$$L_\pm(z) = \big(\mathcal{L}_\pm(z), \mathcal{L}_\pm(-z)\big) \in D\mathfrak{g}(\!(z)\!),$$

we have automatically $L_\pm(z) \in D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$ and the Lax equation

$$\partial_+ L_-(z) - \partial_- L_-(z) - [L_+(z), L_-(z)] = 0$$

follows immediately from the one for $\mathcal{L}_\pm(z)$ in (2.9). The associated Hamiltonian Lax matrix is

$$L(z) = \big(\mathcal{L}(z), \mathcal{L}(-z)\big) \tag{4.1}$$

where $\mathcal{L}(z)$ is given by (2.28).

In the remainder of this section we will study the Hamiltonian properties of this Lax matrix, showing that its Poisson bracket is also of the r/s-form.

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4.2 Poisson bracket of the Lax matrix with itself

The Lax matrices of the coset σ-model on $G\times G/G_{\text{diag}}$ and of its one-parameter deformation have a Poisson bracket of the r/s-form in the double algebra $D\mathfrak{g}$. We will show that this is also the case for the bi-Yang-Baxter σ-model. As it turns out, however, the $\mathcal{R}$-matrix of the latter (which is a rational function of two spectral parameters z and z' valued in the complexification of $D\mathfrak{g}\otimes D\mathfrak{g}$) takes on a slightly non-standard form depending on both the twist function $\varphi_{\text{bYB}}(z)$ and on its mirror image $\varphi_{\text{bYB}}(-z)$. We will discuss the algebraic origin of this structure coming from the twisted loop algebra $D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$ by generalising the construction of [8].

4.2.1 $\mathcal{R}$-matrix and inner product

We begin by recalling the construction of [8] adapted to the present setting. The twisted loop algebra $D\mathfrak{g}(\!(z)\!)$ admits a natural decomposition

$$D\mathfrak{g}(\!(z)\!)^{\hat{\delta}} = D\mathfrak{g}[[z]]^{\hat{\delta}} \oplus \big(z^{-1}D\mathfrak{g}[z^{-1}]\big)^{\hat{\delta}} \tag{4.2}$$

into subalgebras of positive and strictly negative powers of the loop parameter z, respectively. Let π_+ and π_- denote the projection operators relative to this decomposition. The operator

$$\mathcal{R}^D = \pi_+ - \pi_- \tag{4.3}$$

defines a solution of the mCYBE on $D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$.

Suppose now that we are given an invariant inner product $\langle\cdot,\cdot\rangle$ on the twisted loop algebra $D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$. We define the kernel $\mathcal{R}^D_{\underline{\mathbf{12}}}(z,z')$ of the operator $\mathcal{R}^D$ in (4.3), with respect to $\langle\cdot,\cdot\rangle$, as the rational function $\mathcal{R}^D_{\underline{\mathbf{12}}}(z,z')$ of two complex variables and valued in the complexification of $D\mathfrak{g}\otimes D\mathfrak{g}$, such that for all $M\in D\mathfrak{g}(\!(z)\!)^{\hat{\delta}}$ we have

$$\langle \mathcal{R}^D_{\underline{\mathbf{12}}}(z,z'), M_{\underline{\mathbf{2}}}(z')\rangle_{\underline{\mathbf{2}}} = (\mathcal{R}^D M)(z). \tag{4.4}$$

This matrix is then a solution of the classical Yang-Baxter equation[2]

$$\big[\mathcal{R}^D_{\underline{\mathbf{12}}}(z_1,z_2),\mathcal{R}^D_{\underline{\mathbf{13}}}(z_1,z_3)\big] + \big[\mathcal{R}^D_{\underline{\mathbf{12}}}(z_1,z_2),\mathcal{R}^D_{\underline{\mathbf{23}}}(z_2,z_3)\big] + \big[\mathcal{R}^D_{\underline{\mathbf{32}}}(z_3,z_2),\mathcal{R}^D_{\underline{\mathbf{13}}}(z_1,z_3)\big] = 0. \tag{4.5}$$

The standard inner product on $D\mathfrak{g}(\!(z)\!)$ is defined for all $M,N\in D\mathfrak{g}(\!(z)\!)$ by

$$\langle M,N\rangle = \text{res}_{z=0}\,\kappa^D\big(M(z),N(z)\big)dz, \tag{4.6}$$

where κ^D is the Killing form on the double $D\mathfrak{g}$. Given any function $\varphi(z)$, one can also define a more general invariant inner product on $D\mathfrak{g}(\!(z)\!)$ as a "twist" of the standard one by φ, namely

$$\langle M,N\rangle_\varphi = \text{res}_{z=0}\,\kappa^D\big(M(z),N(z)\big)\varphi(z)dz, \tag{4.7}$$

for any $M,N\in D\mathfrak{g}(\!(z)\!)$. It is easy to check that this inner product is invariant under $\hat{\delta}$, i.e. $\langle\hat{\delta}M,\hat{\delta}N\rangle_\varphi = \langle M,N\rangle_\varphi$, and thus induces an inner product on the twisted loop algebra

[2] More precisely, it is a solution of the classical Yang-Baxter equation if we ignore contact terms by treating $\mathcal{R}^D_{\underline{\mathbf{12}}}(z,z')$ as a rational function. See, for instance, [8] for more details.

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$D\mathfrak{g}((z))^{\hat{\delta}}$, if and only if φ is an odd function. The kernel of the operator $\mathcal{R}^D$ defined in equation (4.3), with respect to this inner-product, is

$$\mathcal{R}^D_{\underline{\mathbf{12}}}(z,z') = 2\frac{z'C^{(00)}_{\underline{\mathbf{12}}} + zC^{(11)}_{\underline{\mathbf{12}}}}{z'^2 - z^2}\varphi(z')^{-1}, \tag{4.8}$$

with the graded components of the split Casimir

$$C^{(00)}_{\underline{\mathbf{12}}} = \frac{1}{2}\kappa^{ab}\big((T_a,0)+(0,T_a)\big)\otimes\big((T_b,0)+(0,T_b)\big), \tag{4.9a}$$

$$C^{(11)}_{\underline{\mathbf{12}}} = \frac{1}{2}\kappa^{ab}\big((T_a,0)-(0,T_a)\big)\otimes\big((T_b,0)-(0,T_b)\big). \tag{4.9b}$$

Expression (4.8) is the $\mathcal{R}$-matrix entering the r/s-form of the Poisson bracket of Lax matrices for the coset σ-model on $G\times G/G_{\text{diag}}$ as well as its one-parameter deformation, with the twist function φ given respectively by (3.2) and (3.3).

4.2.2 Inner product for the bi-Yang-Baxter σ-model

Let us now generalise the ideas presented in the previous subsections, to have a formalism that also describes the bi-Yang-Baxter σ-model. As we are considering the double Lie algebra $D\mathfrak{g}$, one can define an even more general inner product invariant under $\hat{\delta}$, by separating explicitly the left and right part of $D\mathfrak{g}$. That is, for any $M=(m,\tilde{m})$ and $N=(n,\tilde{n})$ in $D\mathfrak{g}$ we define

$$\langle M,N\rangle_\varphi = \text{res}_{z=0}\,\kappa\big(m(z),n(z)\big)\varphi(z)dz - \text{res}_{z=0}\,\kappa\big(\tilde{m}(z),\tilde{n}(z)\big)\varphi(-z)dz, \tag{4.10}$$

where κ is the Killing form on $\mathfrak{g}$. When φ is odd, we recover the twisted inner product (4.7). This construction allows to consider twist functions of any parity.

The kernel of $\mathcal{R}^D$ with respect to the inner product (4.10) is given by

$$\mathcal{R}^D_{\underline{\mathbf{12}}}(z,z') = \left(\frac{C^{LL}_{\underline{\mathbf{12}}}}{z'-z} + \frac{C^{RL}_{\underline{\mathbf{12}}}}{z+z'}\right)\varphi(z')^{-1} - \left(\frac{C^{RR}_{\underline{\mathbf{12}}}}{z'-z} + \frac{C^{LR}_{\underline{\mathbf{12}}}}{z+z'}\right)\varphi(-z')^{-1}, \tag{4.11}$$

where we defined the partial split Casimirs

$$C^{LL}_{\underline{\mathbf{12}}} = \kappa^{ab}(T_a,0)\otimes(T_b,0), \qquad C^{RR}_{\underline{\mathbf{12}}} = \kappa^{ab}(0,T_a)\otimes(0,T_b),$$
$$C^{LR}_{\underline{\mathbf{12}}} = \kappa^{ab}(T_a,0)\otimes(0,T_b), \qquad C^{RL}_{\underline{\mathbf{12}}} = \kappa^{ab}(0,T_a)\otimes(T_b,0).$$

The r/s form of the Poisson bracket of $\mathcal{L}(z)$ implies that the Lax matrix (4.1) in the double Lie algebra also has a Poisson bracket of the r/s-form. Furthermore, it is associated with the $\mathcal{R}$-matrix (4.11) for the twist function φ given by (3.8). The projection of this Poisson bracket onto the left part of the double Lie algebra gives back the Poisson bracket for $\mathcal{L}(z)$ of the r/s-form discussed in section 3.

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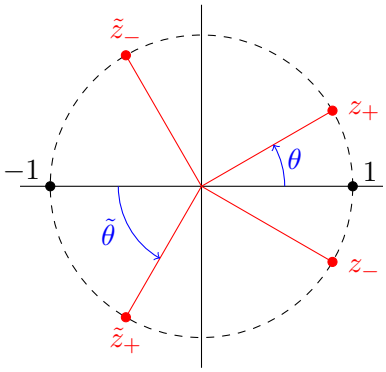

Figure 1. Poles of the twist function φ_{bYB} given by (3.8).

5 Analysis of the twist function and symmetries

As we will see later, the poles of the twist function characterises the model [20]. In the case of the bi-Yang-Baxter σ-model, the twist function (3.8) has four simple poles, disposed on the unit circle of the complex plane (cf figure 1):

$$z_\pm = \frac{1-\frac{1}{4}(\eta^2-\tilde{\eta}^2)\pm i\eta}{\zeta} = z_\mp^* \qquad \text{and} \qquad \tilde{z}_\pm = -\frac{1+\frac{1}{4}(\eta^2-\tilde{\eta}^2)\pm i\tilde{\eta}}{\zeta} = \tilde{z}_\mp^*.$$

Let us recall that

$$\zeta = \sqrt{\left(1+\frac{1}{4}(\eta+\tilde{\eta})^2\right)\left(1+\frac{1}{4}(\eta-\tilde{\eta})^2\right)}.$$

These poles can be re-expressed in a trigonometric form as $z_\pm = e^{\pm i\theta}$ and $\tilde{z}_\pm = -e^{\pm i\tilde{\theta}}$, with $\sin\theta = \eta/\zeta$ and $\sin\tilde{\theta} = \tilde{\eta}/\zeta$.

5.1 Lax matrix at the poles of the twist function

Evaluating the Lax matrix (2.28) at the poles of the twist function, one obtains:

$$\mathscr{J}_\pm \equiv \pm\frac{2iK}{\eta}\mathcal{L}(z_\pm) = \pm\frac{2iK}{\eta}\left(-j+\frac{\eta}{4K}(R_g\mp i)X\right), \tag{5.1a}$$

$$\widetilde{\mathscr{J}}_\pm \equiv \pm\frac{2iK}{\tilde{\eta}}\mathcal{L}(\tilde{z}_\pm) = \pm\frac{2iK}{\tilde{\eta}}\left(-\tilde{j}+\frac{\tilde{\eta}}{4K}(\tilde{R}_{\tilde{g}}\mp i)X\right). \tag{5.1b}$$

One can verify that $\mathscr{J}_\pm$ and $\widetilde{\mathscr{J}}_\pm$ are Poisson commuting Kac-Moody currents valued in $\mathfrak{g}^{\mathbb{C}}$ and with imaginary central charges

$$\{\mathscr{J}_\pm(\sigma)_{\underline{\mathbf{1}}}, \mathscr{J}_\pm(\sigma')_{\underline{\mathbf{2}}}\} = -\left[C_{\underline{\mathbf{12}}}, \mathscr{J}_\pm(\sigma)_{\underline{\mathbf{2}}}\right]\delta_{\sigma\sigma'} \pm \frac{2iK}{\eta}C_{\underline{\mathbf{12}}}\delta'_{\sigma\sigma'},$$

$$\{\widetilde{\mathscr{J}}_\pm(\sigma)_{\underline{\mathbf{1}}}, \widetilde{\mathscr{J}}_\pm(\sigma')_{\underline{\mathbf{2}}}\} = -[C_{\underline{\mathbf{12}}}, \widetilde{\mathscr{J}}_\pm(\sigma)_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'} \pm \frac{2iK}{\tilde{\eta}}C_{\underline{\mathbf{12}}}\delta'_{\sigma\sigma'}.$$

All the other Poisson brackets vanish. These brackets can also be seen more simply as a direct consequence of the r/s-system (3.1). Indeed, the form (3.1c) of the $\mathcal{R}$-matrix imposes

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that the values of the Lax matrix at each pole of the twist function define mutually Poisson commuting Kac-Moody currents, as already shown in [20].

Denote the gauge transformation of the Lax matrix by a G-valued field h as

$$\mathcal{L}^h(z) = h\mathcal{L}(z)h^{-1} - h\partial_\sigma h^{-1}.$$

One can eliminate the currents j and $\tilde{j}$ in (5.1) by performing a gauge transformation by the fields g and $\tilde{g}$, respectively,

$$\mathcal{L}^g(z_\pm) = \frac{\eta}{4K}(R \mp i)(gXg^{-1}), \tag{5.2a}$$

$$\mathcal{L}^{\tilde{g}}(\tilde{z}_\pm) = \frac{\tilde{\eta}}{4K}(\tilde{R} \mp i)(\tilde{g}\tilde{X}\tilde{g}^{-1}). \tag{5.2b}$$

5.2 Lift to the cotangent bundle $T^*L(G \times G)$

According to (5.2), $\mathcal{L}^g(z_\pm)$ belongs to the subalgebra $\mathfrak{g}_\mp = (R \mp i)\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Denote by $G_\mp$ the corresponding subgroup of $G^{\mathbb{C}}$. Let $\Psi^g_\pm(\sigma)$ be a solution belonging to $G_\mp$ of

$$\partial_\sigma \Psi^g_\pm(\sigma)\, \Psi^g_\pm(\sigma)^{-1} = \mathcal{L}^g(\sigma, z^\pm).$$

Then $\Psi_\pm(\sigma) = g(\sigma)^{-1}\Psi^g_\pm(\sigma)$ is a solution of

$$\partial_\sigma \Psi_\pm(\sigma)\, \Psi_\pm(\sigma)^{-1} = \mathcal{L}(\sigma, z^\pm).$$

We recover the result that $g(\sigma)^{-1}$ corresponds to the first factor in the Iwasawa decomposition $G^{\mathbb{C}} = GG_\mp$ of the extended solution $\Psi_\pm(\sigma)$ [2, 3, 20]. The same analysis can be carried out for $\tilde{z}_\pm$ and $\tilde{g}$.

Suppose we had started the construction of the 2-parameter deformation as in [3, 9, 20]. This means that we would have a twist function and an abstract Lax matrix, without having the expression of this matrix in terms of canonical fields. The analysis above proves that one could have derived the canonical fields g, $\tilde{g}$, X and $\tilde{X}$ from the values of the Lax matrix at the poles of the twist function. We shall address the problem of constructing the corresponding Hamiltonian defining the dynamics on phase space later, in subsection 5.4.

5.3 q-deformed symmetry algebra

We shall now discuss the symmetries of the bi-Yang-Baxter σ-model. For this we consider the case where the fields are defined on the real line i.e. $\sigma \in \mathbb{R}$. Let us consider the monodromy matrices of the Lax matrix and its gauge transformation, at the poles $z_\pm$ of the twist function

$$T_\pm = \overleftarrow{\mathrm{Pexp}}\left(\int_{-\infty}^{+\infty} d\sigma \mathcal{L}(z_\pm, \sigma)\right), \qquad T^g_\pm = \overleftarrow{\mathrm{Pexp}}\left(\int_{-\infty}^{+\infty} d\sigma \mathcal{L}^g(z_\pm, \sigma)\right),$$

and define similarly $\widetilde{T}_\pm$ and $\widetilde{T}^{\tilde{g}}_\pm$, at the poles $\tilde{z}_\pm$. As usual, the zero curvature equation (2.9) for the Lax pair implies the conservation of $T_\pm$ and $\widetilde{T}_\pm$. Moreover, we have

$$T_\pm = g(+\infty)^{-1}T^g_\pm g(-\infty) \qquad \text{and} \qquad \widetilde{T}_\pm = \tilde{g}(+\infty)^{-1}\widetilde{T}^{\tilde{g}}_\pm \tilde{g}(-\infty). \tag{5.3}$$

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Thus, if we suppose that the boundary conditions $g(\pm\infty)$ and $\tilde{g}(\pm\infty)$ are independent of τ, then $T^g_\pm$ and $\widetilde{T}^{\tilde{g}}_\pm$ are also conserved charges.

These charges are constructed as the path-ordered exponential of the currents $\mathcal{L}^g(z_\pm)$ and $\mathcal{L}^{\tilde{g}}(\tilde{z}_\pm)$, given by (5.2). This particular structure of the currents and the Poisson brackets (2.24) enable one to show [3] that the corresponding algebra of conserved charges forms the classical analogue of a quantum group. More precisely, applying the results of [3], one can extract from $T^g_\pm$ and $\widetilde{T}^{\tilde{g}}_\pm$ a set of non-local charges which generate the Poisson algebra $U^{\mathcal{P}}_q(\mathfrak{g}) \times U^{\mathcal{P}}_{\tilde{q}}(\mathfrak{g})$, analogue of a quantum group and where

$$q = \exp\left(-\frac{\eta}{4K}\right) \qquad \text{and} \qquad \tilde{q} = \exp\left(-\frac{\tilde{\eta}}{4K}\right).$$

One recovers the values already indicated in [33] and that in the one-paraneter deformation limit $\eta = \tilde{\eta}$ [3].

5.4 Reconstruction of the Hamiltonian

We will now show how to recover the Hamiltonian of the model from the Lax matrix and the twist function. Following [20], which treats the case of the one-parameter deformation $\eta = \tilde{\eta}$, we introduce the following Hamiltonian density[3]

$$\mathcal{H}_\varphi(\sigma) = \frac{1}{2}\left(\mathrm{Res}_{z=0} - \mathrm{Res}_{z=\infty}\right) \kappa\left(\mathcal{L}(z,\sigma), \mathcal{L}(z,\sigma)\right) \varphi(z)\mathrm{d}z.$$

One can show that this Hamiltonian density can be expressed in terms of the naive Hamiltonian density (2.25) and the constraint $X + \tilde{X}$ as

$$\mathcal{H}_\varphi = \mathcal{H}_0 + \kappa\big(\Lambda_\varphi, X + \tilde{X}\big),$$

where Λ_φ is a $\mathfrak{g}$-valued field, depending linearly on the fields j, $\tilde{j}$, X, $\tilde{X}$, $R_g X$ and $\tilde{R}_{\tilde{g}}\tilde{X}$. This Hamiltonian is indeed of the form (2.26), with a fixed Lagrange multiplier Λ_φ. Thus, it gives back the correct dynamics for all the fields.

6 Gauge fixing and Lax matrix

To analyse what happens when the bi-Yang-Baxter σ-model is formulated as in [2], one needs to gauge fix the G_{diag} gauge invariance. We do this by taking $\tilde{g} = \mathrm{Id}$. As already discussed in section 2, this gauge may be reached by the field-dependent gauge transformation (2.3) with $h = \tilde{g}$. As we shall see, this induces a gauge transformation on the Lax matrix. Let us first recall a general result [39] about the change in the Poisson bracket of the Lax matrix under a gauge transformation.

[3] In [20], the expression (3.23) for $\mathcal{H}_\varphi$ contains a factor $\frac{1}{4}$. Yet this expression is for the Lax matrix in the double Lie algebra. Here, for the Lax matrix in a simple copy of $\mathfrak{g}$, it translates to a factor $\frac{1}{2}$.

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A general result. Consider a Lax matrix taking values in $\mathfrak{g}^{\mathbb{C}}$ and whose Poisson bracket takes the r/s form (3.1b). Let us apply a gauge transformation

$$\mathcal{L} \to \mathcal{L}^h = h\mathcal{L}h^{-1} - h\partial_\sigma h^{-1}$$

by some G-valued field h constructed from the phase space fields. We suppose that the Poisson brackets of h with itself and with the Lax matrix take the form

$$\{h_{\underline{1}}(\sigma), h_{\underline{2}}(\sigma')\} = 0, \qquad \{\mathcal{L}_{\underline{1}}(z,\sigma), h_{\underline{2}}(\sigma')\}h_{\underline{2}}(\sigma')^{-1} = \omega_{\underline{12}}(z,\sigma)\delta_{\sigma\sigma'},$$

for some $\mathfrak{g}^{\mathbb{C}}\otimes\mathfrak{g}^{\mathbb{C}}$-valued (potentially field dependent) tensor $\omega_{\underline{12}}(z,\sigma)$. A direct computation shows that the Poisson bracket of the gauge transformed Lax matrix $\mathcal{L}^h(z)$ with itself is also of the r/s-form. More precisely, one has

$$\{\mathcal{L}^h_{\underline{1}}(z,\sigma), \mathcal{L}^h_{\underline{2}}(z',\sigma')\} = [\mathcal{R}^h_{\underline{12}}(z,z',\sigma), \mathcal{L}^h_{\underline{1}}(z,\sigma)]\delta_{\sigma\sigma'} - [\mathcal{R}^h_{\underline{21}}(z',z,\sigma), \mathcal{L}^h_{\underline{2}}(z',\sigma)]\delta_{\sigma\sigma'} \quad (6.1a)$$

$$+ \big(\mathcal{R}^h_{\underline{12}}(z,z',\sigma) + \mathcal{R}^h_{\underline{21}}(z',z,\sigma)\big)\delta'_{\sigma\sigma'}, \quad (6.1b)$$

where the $\mathcal{R}$-matrix $\mathcal{R}^h$ is given by:

$$\mathcal{R}^h_{\underline{12}}(z,z',\sigma) = h_{\underline{1}}(\sigma)h_{\underline{2}}(\sigma)\mathcal{R}_{\underline{12}}(z,z')h_{\underline{1}}(\sigma)^{-1}h_{\underline{2}}(\sigma)^{-1} - h_{\underline{2}}(\sigma)\omega_{\underline{21}}(z',\sigma)h_{\underline{2}}(\sigma)^{-1}. \quad (6.1c)$$

This $\mathcal{R}$-matrix may be dynamical i.e. field dependent.

Gauge fixing as a suitable gauge transformation. Consider now the following gauge transformation of the Lax matrix (2.28) of the bi-Yang-Baxter σ-model,

$$\mathcal{L}^{\tilde{g}}(z) = \tilde{g}\mathcal{L}(z)\tilde{g}^{-1} - \tilde{g}\partial_\sigma\tilde{g}^{-1}.$$

Define then the gauge-invariant fields

$$g' = g\tilde{g}^{-1}, \qquad j' = g'^{-1}\partial_\sigma g' = \tilde{g}(j-\tilde{j})\tilde{g}^{-1}, \qquad X' = \tilde{g}X\tilde{g}^{-1}, \qquad \tilde{X}' = \tilde{g}\tilde{X}\tilde{g}^{-1}.$$

Using the relation $A_{\tilde{j}}(z) = -A_j(z) - 1$, one finds

$$\mathcal{L}^{\tilde{g}}(z) = A_j(z)j' + A_X(z)X' + A_{R_gX}(z)R_{g'}X' + A_{\tilde{X}}(z)\tilde{X}' + A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z)\tilde{R}\tilde{X}',$$

with the A_Q listed in appendix A. The Poisson brackets of g' and X' are the same as those of g and X, but the gauge transformed constraint $X' + \tilde{X}'$ Poisson commute with g' and X'. We may therefore impose the constraint $X' + \tilde{X}' = 0$ strongly in the Lax matrix, which becomes

$$\mathcal{L}^{\tilde{g}}(z) = A_j(z)j' + \big(A_X(z) - A_{\tilde{X}}(z)\big)X' + A_{R_gX}(z)R_{g'}X' - A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z)\tilde{R}X'.$$

The key property is that performing such a gauge transformation is equivalent to fixing the gauge by taking $\tilde{g} = \mathrm{Id}$ and replacing the canonical bracket by the Dirac bracket. Indeed, the Dirac bracket of g and X is the same as the canonical one, but the constraint $X + \tilde{X}$ has vanishing Dirac bracket with g and X, and may thus be set strongly to zero. The gauge fixed Lax matrix is just $\mathcal{L}^{\tilde{g}}$.

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Consequence. Viewing the gauge fixed Lax matrix as a suitable gauge transformation of the original Lax matrix allows us to use the result (6.1). It leads to an easy determination of its Poisson bracket. Applying (6.1) to the case at hand where $h = \tilde{g}$, we find that

$$\omega_{\underline{\mathbf{12}}}(z,\sigma) = \tilde{g}_{\underline{\mathbf{2}}}(\sigma)\big(A_{\tilde{X}}(z)C_{\underline{\mathbf{12}}} + A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z)\tilde{R}_{\tilde{g}}(\sigma)_{\underline{\mathbf{12}}}\big)\tilde{g}_{\underline{\mathbf{2}}}(\sigma)^{-1}.$$

As a consequence, the new $\mathcal{R}$-matrix is still non-dynamical and reads

$$\mathcal{R}^{\tilde{g}}_{\underline{\mathbf{12}}}(z,z') = \frac{C_{\underline{\mathbf{12}}}}{z-z'}\varphi_{\mathrm{bYB}}(z')^{-1} - A_{\tilde{X}}(z')C_{\underline{\mathbf{12}}} + A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z')\tilde{R}_{\underline{\mathbf{12}}}.$$

The new $\mathcal{R}$-matrix is not determined solely by the twist function and depends on the matrix $\tilde{R}$ appearing in the Lagrangian.

7 Conclusion

Let us end with a few comments on possible generalisations of this work.

It was shown in [17] that it is possible to apply a λ-deformation[4] to the Yang-Baxter σ-model. Just as the λ-deformation itself is known to coincide with the σ-model obtained by combining the effects of a Poisson-Lie T-duality and an analytic continuation on the Yang-Baxter σ-model [20, 40, 41], the λ-deformation of the Yang-Baxter σ-model itself should also be related in a similar fashion to the bi-Yang-Baxter σ-model. This relation has been shown for a specific example in [17]. It would be interesting to prove this in general.

We defined in [42] a two-parameter family of integrable deformations of the principal chiral model on an arbitrary compact Lie group, of a different nature to the bi-Yang-Baxter σ-model discussed here. The two limits of the model defined in [42], where one of the two parameters is taken to zero, correspond to the Yang-Baxter σ-model and the principal chiral model with a Wess-Zumino term. As already mentioned in [33], one expects to be able to combine this type of deformation with a bi-Yang-Baxter type deformation to obtain a three-parameter deformation of the principal chiral model on an arbitrary Lie group. In fact, a four-parameter deformation of the SU(2) principal chiral model has already been constructed in [43]. Yet from the point of view of the twist function we only expect to be able to construct a three-parameter deformation in the case of an arbitrary Lie group G. However, recall that it has also been suggested in [44] that the fourth parameter of the deformation in [43] is related to a TsT-transformation, and therefore shall correspond to a deformation where the twist function is not modified [20, 23, 25, 30].

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[4] It is also called k-deformation in the literature.

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A Coefficients A_Q and J_Q

The coefficients $A_Q(z)$ in the Lax matrix (2.28) read

$$A_j(z) = -\frac{1}{2}\left(1 + \frac{\eta^2 - \tilde{\eta}^2}{4} + \frac{\zeta}{2}\left(z + \frac{1}{z}\right)\right) \tag{A.1a}$$

$$A_X(z) = -\frac{\zeta}{16K}\left(z - \frac{1}{z}\right) + f(z) \tag{A.1b}$$

$$A_{R_g X}(z) = \frac{\eta}{8K}\left(1 + \frac{\eta^2 - \tilde{\eta}^2}{4} + \frac{\zeta}{2}\left(z + \frac{1}{z}\right)\right) \tag{A.1c}$$

$$A_{\tilde{j}}(z) = -\frac{1}{2}\left(1 + \frac{\tilde{\eta}^2 - \eta^2}{4} - \frac{\zeta}{2}\left(z + \frac{1}{z}\right)\right) \tag{A.1d}$$

$$A_{\tilde{X}}(z) = \frac{\zeta}{16K}\left(z - \frac{1}{z}\right) + f(z) \tag{A.1e}$$

$$A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z) = \frac{\tilde{\eta}}{8K}\left(1 + \frac{\tilde{\eta}^2 - \eta^2}{4} - \frac{\zeta}{2}\left(z + \frac{1}{z}\right)\right) \tag{A.1f}$$

The coefficients $J_Q(z, z')$ in the ultralocal term (3.5) are given by

$$J_j(z, z') = -A_j(z)A_X(z') - A_j(z')A_X(z) \tag{A.2a}$$

$$J_X(z, z') = -A_X(z)A_X(z') + A_{R_g X}(z)A_{R_g X}(z') \tag{A.2b}$$

$$J_{R_g X}(z, z') = -A_X(z)A_{R_g X}(z') - A_X(z')A_{R_g X}(z) \tag{A.2c}$$

$$J_{\tilde{j}}(z, z') = -A_{\tilde{j}}(z)A_{\tilde{X}}(z') - A_{\tilde{j}}(z')A_{\tilde{X}}(z) \tag{A.2d}$$

$$J_{\tilde{X}}(z, z') = -A_{\tilde{X}}(z)A_{\tilde{X}}(z') + A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z)A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z') \tag{A.2e}$$

$$J_{\tilde{R}_{\tilde{g}}\tilde{X}}(z, z') = -A_{\tilde{X}}(z)A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z') - A_{\tilde{X}}(z')A_{\tilde{R}_{\tilde{g}}\tilde{X}}(z) \tag{A.2f}$$

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Cyclotomic Gaudin Models, Miura Opers and Flag Varieties

Sylvain Lacroix and Benoît Vicedo

Abstract. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\nu \in \operatorname{Aut}\mathfrak{g}$ be a diagram automorphism whose order divides $T \in \mathbb{Z}_{\geq 1}$. We define *cyclotomic* $\mathfrak{g}$*-opers* over the Riemann sphere $\mathbb{P}^1$ as gauge equivalence classes of $\mathfrak{g}$-valued connections of a certain form, equivariant under actions of the cyclic group $\mathbb{Z}/T\mathbb{Z}$ on $\mathfrak{g}$ and $\mathbb{P}^1$. It reduces to the usual notion of $\mathfrak{g}$-opers when $T = 1$. We also extend the notion of Miura $\mathfrak{g}$-opers to the cyclotomic setting. To any cyclotomic Miura $\mathfrak{g}$-oper ∇, we associate a corresponding cyclotomic $\mathfrak{g}$-oper. Let ∇ have residue at the origin given by a ν-invariant rational dominant coweight $\check{\lambda}_0$ and be monodromy-free on a cover of $\mathbb{P}^1$. We prove that the subset of all cyclotomic Miura $\mathfrak{g}$-opers associated with the same cyclotomic $\mathfrak{g}$-oper as ∇ is isomorphic to the ϑ-invariant subset of the full flag variety of the adjoint group G of $\mathfrak{g}$, where the automorphism ϑ depends on ν, T and $\check{\lambda}_0$. The big cell of the latter is isomorphic to N^ϑ, the ϑ-invariant subgroup of the unipotent subgroup $N \subset G$, which we identify with those cyclotomic Miura $\mathfrak{g}$-opers whose residue at the origin is the same as that of ∇. In particular, the cyclotomic generation procedure recently introduced in Varchenko and Young (Sigma 11(091), 2015) is interpreted as taking ∇ to other cyclotomic Miura $\mathfrak{g}$-opers corresponding to elements of N^ϑ associated with simple root generators. We motivate the introduction of cyclotomic $\mathfrak{g}$-opers by formulating two conjectures which relate them to the cyclotomic Gaudin model of Vicedo and Young (Commun Math Phys 343(3):971–1024, 2016).

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1. Introduction and Motivation

The Gaudin model [12] is a quantum integrable long-range spin chain of any length $N \in \mathbb{Z}_{\geq 1}$ which can be associated with any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Among its many different possible generalisations, we shall be interested in the so-called *cyclotomic Gaudin model*. It can be regarded as a particular example of a general family of Gaudin models associated with *non*-skew-symmetric solutions of the classical Yang–Baxter equation, introduced in [21].

The algebra of observables of the cyclotomic Gaudin model, as introduced by C. Young and one of the present authors in [26], is the N-fold tensor product $U(\mathfrak{g})^{\otimes N}$ of the universal enveloping algebra $U(\mathfrak{g})$. Given any $\mathfrak{g}$-modules M_i for $i = 1, \dots, N$, the Hilbert space, or spin chain, is taken to be the N-fold tensor product $\bigotimes_{i=1}^{N} M_i$. To define the Hamiltonians, let $\{I^a\}_{a=1}^{\dim \mathfrak{g}}$ and $\{I_a\}_{a=1}^{\dim \mathfrak{g}}$ be dual bases of $\mathfrak{g}$ with respect to a chosen non-degenerate bilinear form on $\mathfrak{g}$. Let $T \in \mathbb{Z}_{\geq 1}$, pick a primitive Tth-root of unity ω and consider the cyclic group $\Gamma := \langle \omega \rangle \cong \mathbb{Z}/T\mathbb{Z}$. Let $\sigma \in \operatorname{Aut} \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ such that $\sigma^T = \mathrm{Id}$. Fix a collection of N distinct complex numbers $z_i \in \mathbb{C}^\times$, $i = 1, \dots, N$ with disjoint Γ-orbits, i.e. such that $z_i \neq \omega^k z_j$ for all distinct $i, j = 1, \dots, N$ and $k = 0, \dots, T-1$. The quadratic cyclotomic Gaudin Hamiltonians are defined as (see [22])

$$\mathcal{H}_i := \sum_{k=0}^{T-1} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{I^{a(i)} \sigma^k I_a^{(j)}}{z_i - \omega^{-k} z_j} + \sum_{k=1}^{T-1} \frac{I^{a(i)} \sigma^k I_a^{(i)}}{(1-\omega^k) z_i} \in U(\mathfrak{g})^{\otimes N}, \qquad i = 1, \dots, N \tag{1.1}$$

where for any $A \in U(\mathfrak{g})$ we let $A^{(i)}$ denote the element of $U(\mathfrak{g})^{\otimes N}$ with A in the ith tensor factor and the identity in every other factor. One checks directly that these Hamiltonians mutually commute, i.e. $[\mathcal{H}_i, \mathcal{H}_j] = 0$ for all $i, j = 1, \dots, N$, and commute with the diagonal action of the σ-invariant subalgebra $\mathfrak{g}^\sigma$. The Hamiltonians (1.1) belong to a large commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$, the so-called *cyclotomic Gaudin algebra*, whose definition we briefly recall below. For $\operatorname{rk} \mathfrak{g} \geq 2$, the latter contains also "higher" Gaudin Hamiltonians of degrees equal to the exponents of $\mathfrak{g}$ plus one. Taking $T = 1$, i.e. $\Gamma = \{1\}$, so that $\sigma = \mathrm{Id}$, we recover the usual quadratic Gaudin Hamiltonians

$$\mathcal{H}_i^{\mathrm{G}} := \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{I^{a(i)} I_a^{(j)}}{z_i - z_j} \in U(\mathfrak{g})^{\otimes N}, \qquad i = 1, \dots, N. \tag{1.2}$$

The cyclotomic Gaudin algebra reduces in this case to the *Gaudin algebra* [6] (see also [8]).

The construction of the Gaudin algebra by Feigin, Frenkel and Reshetikhin [6] exploits the commutative algebra structure of the subspace of singular vectors in the vacuum Verma module at the critical level over the untwisted affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$. This approach was recently generalised to construct the cyclotomic Gaudin algebra by C. Young and one of the present authors in [26]. The central ingredient in the construction of [6] is the notion of coinvariant of an N-fold tensor product of $\widehat{\mathfrak{g}}$-modules with respect to the algebra of rational functions $\mathbb{P}^1 \to \mathfrak{g}$ vanishing at infinity and with poles at most at the marked points z_i, $i = 1, \dots, N$. In the cyclotomic setting, this gets replaced by the notion of *cyclotomic* coinvariants [10,27], i.e. coinvariants with respect to the algebra of Γ-equivariant rational functions $\mathbb{P}^1 \to \mathfrak{g}$ vanishing at infinity and regular away from $\omega^k z_i$, $i = 1, \dots, N$, $k = 0, \dots, T-1$, where $\omega \in \Gamma$ acts on $\mathbb{P}^1$ by multiplication and on $\mathfrak{g}$ as σ. Let $u \in \mathbb{C}^\times \backslash \{z_1, \dots, z_N\}$ be such that its Γ-orbit is disjoint from those of the z_i, and denote by $\mathscr{O}_u$ and $\mathscr{K}_u$ the local ring and local field at u, respectively. Let $\widehat{\mathfrak{g}}_u := \mathfrak{g}(\mathscr{K}_u) \oplus \mathbb{C}K$ be the affine Kac–Moody algebra, where for any $\mathbb{C}$-algebra R we let $\mathfrak{g}(R) := \mathfrak{g} \otimes R$, and consider its subalgebra $\widehat{\mathfrak{g}}_u^+ := \mathfrak{g}(\mathscr{O}_u) \oplus \mathbb{C}K$. The vacuum Verma module $\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g}) := U(\widehat{\mathfrak{g}}_u) \otimes_{U(\widehat{\mathfrak{g}}_u^+)} \mathbb{C}v_0$ is the $\widehat{\mathfrak{g}}_u$-module induced from the one-dimensional $\widehat{\mathfrak{g}}_u^+$-module $\mathbb{C}v_0$ on which $\mathfrak{g}(\mathscr{O}_u)$ acts trivially and K acts by the critical level (whose specific value depends on the choice of normalisation of the bilinear form on $\mathfrak{g}$). The subspace $\mathfrak{z}\big(\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g})\big) := \{X \in \mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g}) \,|\, \mathfrak{g}(\mathscr{O}_u)X = 0\}$ of singular vectors in $\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g})$ comes naturally equipped with the structure of a commutative algebra and the upshot of the construction of [26], generalising that of [6] to the cyclotomic setting, is an algebra homomorphism

$$\Psi_{(z_i),u}^{\Gamma} : \mathfrak{z}\big(\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g})\big) \longrightarrow U(\mathfrak{g})^{\otimes N}.$$

Working at the critical level ensures that this homomorphism is non-trivial since it is only then that the vacuum Verma module admits singular vectors not proportional to the vacuum v_0. In particular, $\mathfrak{z}\big(\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g})\big)$ always contains the quadratic vector $S = \frac{1}{2} I^a(-1) I_a(-1) v_0$. Here, we fix once and for all a global coordinate t on $\mathbb{C} \subset \mathbb{P}^1$ and let $X(n) := X \otimes (t-u)^n \in \mathfrak{g}(\mathscr{K}_u)$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The image of S under $\Psi_{(z_i),u}^{\Gamma}$ gives rise to the quadratic cyclotomic Gaudin Hamiltonians (1.1) as $\mathcal{H}_j = \mathrm{res}_{z_j} \Psi_{(z_i),u}^{\Gamma}(S) du$ for each $j = 1, \dots, N$. The *cyclotomic Gaudin algebra* $\mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g})$ is defined as the image of $\Psi_{(z_i),u}^{\Gamma}$. We thus obtain a surjective homomorphism of commutative $\mathbb{C}$-algebras

$$\Psi_{(z_i),u}^{\Gamma} : \mathfrak{z}\big(\mathbb{V}_{0,u}^{\mathrm{crit}}(\mathfrak{g})\big) \twoheadrightarrow \mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g}). \tag{1.3}$$

Given any $\mathfrak{g}$-modules M_i, $i = 1, \dots, N$, one of the main problems in the study of the (cyclotomic) Gaudin model is to simultaneously diagonalise the quadratic (cyclotomic) Gaudin Hamiltonians (1.1), or indeed the entire (cyclotomic) Gaudin algebra $\mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g})$, on the tensor product $\bigotimes_{i=1}^N M_i$. If $\psi \in \bigotimes_{i=1}^N M_i$ is a joint eigenvector of the cyclotomic Gaudin algebra $\mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g})$, then the common eigenvalues of $\mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g})$ on ψ are encoded in a $\mathbb{C}$-algebra homomorphism $\eta_\psi : \mathscr{Z}_{(z_i)}^{\Gamma}(\mathfrak{g}) \to \mathbb{C}$, defined by $z\,\psi = \eta_\psi(z)\psi$ for every $z \in$

$\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$. In other words, the joint spectrum of $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ on any spin chain $\bigotimes_{i=1}^{N} M_i$ forms a subset of the maximal spectrum of $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$. It is therefore of interest to first obtain a description of the spectrum $\operatorname{Spec}\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ before attempting to diagonalise the cyclotomic Gaudin Hamiltonians on any tensor product of $\mathfrak{g}$-modules. In view of the homomorphism (1.3), let us begin by recalling the description of the spectrum of $\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big)$.

The commutative algebra $\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big)$ is naturally isomorphic, by a theorem of Feigin and Frenkel [5] (see also the book [9]), to the *classical W-algebra* $\mathcal{W}_u({}^L\mathfrak{g})$ for the Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$, whose Cartan matrix is the transpose of that of $\mathfrak{g}$. The classical W-algebra $\mathcal{W}_u(\mathfrak{g})$ is obtained by Drinfel'd–Sokolov reduction of the algebra of functions on the dual of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_u$ [4]. Its spectrum is isomorphic to the space of so-called $\mathfrak{g}$-*opers* on the formal disc $D_u = \operatorname{Spec}\mathscr{O}_u$ around the point u. In order to introduce the notion of $\mathfrak{g}$-oper, it is convenient to first recall the description, due to Kostant [14,15], of the classical finite W-algebra $\mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, p_{-1})$ associated with a principal nilpotent element p_{-1}. Embedding the latter into an $\mathfrak{sl}_2$-triple $\{p_{-1}, 2\check{\rho}, p_1\}$, we set $\mathfrak{n} := \bigoplus_{i>0}\mathfrak{g}_i$ and $\mathfrak{b} := \bigoplus_{i\geq 0}\mathfrak{g}_i$ using the $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ defined by $\mathrm{ad}_{\check{\rho}}$ and denote by N the unipotent subgroup of the adjoint group of $\mathfrak{g}$ with Lie algebra $\mathfrak{n}$. The algebra $\mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, p_{-1})$ is isomorphic to the algebra of N-invariant polynomial functions on $p_{-1}+\mathfrak{b}$, or equivalently to the algebra of polynomial functions on the Slodowy slice $p_{-1}+\mathfrak{a}$, where $\mathfrak{a}$ is the centraliser of p_1. In other words, we have isomorphisms $\operatorname{Spec}\mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, p_{-1}) \simeq (p_{-1}+\mathfrak{b})/N \simeq p_{-1}+\mathfrak{a}$. To emphasise the parallel with $\mathfrak{g}$-opers, we will call $\mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}} := (p_{-1}+\mathfrak{b})/N$ the space of *finite* $\mathfrak{g}$-*opers*. We denote by $[X]_{\mathfrak{g}}$ the class of $X \in p_{-1}+\mathfrak{b}$ in $\mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}}$.

A $\mathfrak{g}$-oper on D_u is then defined as an equivalence class of connections of the form $\mathrm{d} + p_{-1}\mathrm{d}t + v\,\mathrm{d}t$, with $v \in \mathfrak{b}(\mathscr{O}_u)$, modulo the gauge action of the group $N(\mathscr{O}_u)$ whose definition we recall in Sect. 4.2. Each class admits a canonical representative of the form $\mathrm{d} + p_{-1}\mathrm{d}t + c\,\mathrm{d}t$, with $c \in \mathfrak{a}(\mathscr{O}_u)$, which can be regarded as an affine analog of the Slodowy slice. Denoting the space of $\mathfrak{g}$-opers on D_u by $\mathrm{Op}_{\mathfrak{g}}(D_u)$, we have an isomorphism of varieties [5]

$$\operatorname{Spec}\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big) \simeq \mathrm{Op}_{{}^L\mathfrak{g}}(D_u). \tag{1.4}$$

Now the surjective homomorphism (1.3) induces an injective map

$$\operatorname{Spec}\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g}) \longhookrightarrow \operatorname{Spec}\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big) \tag{1.5}$$

between the corresponding spectra. We may thus regard $\operatorname{Spec}\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ as a subvariety of $\mathrm{Op}_{{}^L\mathfrak{g}}(D_u)$. In the non-cyclotomic case, $\Gamma = \{1\}$, it was shown by Frenkel [7] that the usual Gaudin algebra $\mathscr{Z}_{(z_i)}(\mathfrak{g}) := \mathscr{Z}^{\{1\}}_{(z_i)}(\mathfrak{g})$ is isomorphic to the algebra of functions on the space of *global* ${}^L\mathfrak{g}$-opers on the complex projective line $\mathbb{P}^1$ with regular singularities at the points z_i, $i = 1, \ldots, N$ and at infinity. In other words, denoting the space of such ${}^L\mathfrak{g}$-opers by $\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{(z_i),\infty}$ which we recall the definition of in Sect. 4, we have the isomorphism

$$\operatorname{Spec} \mathscr{Z}_{(z_i)}(\mathfrak{g}) \simeq \operatorname{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{(z_i),\infty}. \tag{1.6}$$

The injective map (1.5) therefore corresponds, in the case $\Gamma = \{1\}$, to the restriction of a global ${}^L\mathfrak{g}$-oper in $\operatorname{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{(z_i),\infty}$ to the disc around a regular point $u \in \mathbb{C}\backslash\{z_1, \ldots, z_N\}$.

The first purpose of this article is to conjecture an analog of the isomorphism (1.6) for describing the spectrum of the cyclotomic Gaudin algebra $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$. To this end, we will introduce a notion of *cyclotomic* $\mathfrak{g}$*-oper* on $\mathbb{P}^1$, which reduces to the usual notion of $\mathfrak{g}$-oper on $\mathbb{P}^1$, as given in [18], when $\Gamma = \{1\}$ (see also [1,7,8]). Fix a diagram automorphism $\nu \in \operatorname{Aut}\mathfrak{g}$ whose order divides T. Given any automorphism $\upsilon \in \operatorname{Aut}\mathfrak{g}$ in the same class as ν in $\operatorname{Aut}\mathfrak{g}/\operatorname{Inn}\mathfrak{g}$ and with the property that $\upsilon^T = \mathrm{Id}$, such as the automorphism σ entering the definition of the cyclotomic Gaudin model, we may consider the corresponding space $\operatorname{Conn}^{\hat{\upsilon}}_{\mathfrak{g}}(\mathbb{P}^1)$ of Γ-equivariant meromorphic $\mathfrak{g}$-valued connections on $\mathbb{P}^1$ where $\omega \in \Gamma$ acts on $\mathbb{P}^1$ by multiplication and on $\mathfrak{g}$ by υ. By constrast, consider meromorphic $\mathfrak{g}$-valued connections, or $\mathfrak{g}$-connections for short, of the form $\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v}$ with $\mathsf{v} \in \Omega(\mathfrak{b}) := \mathfrak{b}(\mathcal{M})\mathrm{d}t$, where $\mathcal{M}$ is the algebra of meromorphic functions on $\mathbb{P}^1$. Requiring the existence of Γ-equivariant $\mathfrak{g}$-connections of this form forces us to work with the representative $\varsigma = \operatorname{Ad}_{\omega^{-\check{\rho}}} \circ \nu \in \operatorname{Aut}\mathfrak{g}$ of the class of ν in $\operatorname{Aut}\mathfrak{g}/\operatorname{Inn}\mathfrak{g}$. We therefore define the space of cyclotomic $\mathfrak{g}$-opers as equivalence classes of Γ-equivariant $\mathfrak{g}$-connections of the above form with $\omega \in \Gamma$ acting on $\mathfrak{g}$ as ς, modulo the gauge action of a Γ-invariant subgroup $N^{\hat{\varsigma}}(\mathcal{M}) \subset N(\mathcal{M})$ whose definition is given in Sect. 4.2. Each such class also admits a unique canonical representative of the form $\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{c}$, with $\mathsf{c} \in \Omega^{\hat{\varsigma}}(\mathfrak{a})$, where $\Omega^{\hat{\varsigma}}(\mathfrak{a})$ denotes the space of Γ-equivariant $\mathfrak{a}$-valued differentials on $\mathbb{P}^1$. In Sect. 7, we will use the notion of cyclotomic $\mathfrak{g}$-opers to conjecture the analog of the isomorphism (1.6) for the cyclotomic Gaudin algebra $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ in Conjecture 7.2.

To motivate the central question concerning cyclotomic $\mathfrak{g}$-opers addressed in this article, we will use the analogy with classical finite W-algebras. Recall the Slodowy slice $p_{-1}+\mathfrak{a}$ which is transverse to the N-orbits in $p_{-1}+\mathfrak{b}$. Another transverse slice is given by $p_{-1} + \mathfrak{h}$ where $\mathfrak{h} = \mathfrak{g}_0$ is a Cartan subalgebra. Unlike the Slodowy slice, however, it does not intersect each N-orbit uniquely. In other words, the canonical map $p_{-1} + \mathfrak{h} \to \operatorname{Op}^{\mathrm{fin}}_{\mathfrak{g}}$ which sends an element $X \in p_{-1} + \mathfrak{h} \subset p_{-1} + \mathfrak{b}$ to its class $[X]_{\mathfrak{g}} \in \operatorname{Op}^{\mathrm{fin}}_{\mathfrak{g}}$ is surjective, but not injective. By comparison with the affine case discussed below, we shall refer to $p_{-1} + \mathfrak{h}$ as the space of *finite Miura* $\mathfrak{g}$*-opers*. If $\check{\lambda} \in \mathfrak{h}$ is dominant, i.e. $\langle \alpha_i, \check{\lambda}\rangle \geq 0$ for every simple root α_i, $i \in I := \{1, \ldots, \operatorname{rk}\mathfrak{g}\}$ of $\mathfrak{g}$, then the subset of all finite Miura $\mathfrak{g}$-opers whose class in $\operatorname{Op}^{\mathrm{fin}}_{\mathfrak{g}}$ coincides with $[p_{-1} - \check{\lambda} - \check{\rho}]_{\mathfrak{g}}$ can be shown using results of Kostant [14,15] to be in bijection with the Weyl group W of $\mathfrak{g}$.

Following the standard terminology in the non-cyclotomic setting [7,18], we define a *cyclotomic Miura* $\mathfrak{g}$*-oper* on $\mathbb{P}^1$ as a cyclotomic $\mathfrak{g}$-connection of the form $\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{u}$ with $\mathsf{u} \in \Omega^{\hat{\nu}}(\mathfrak{h})$. Note that a cyclotomic Miura $\mathfrak{g}$-oper ∇ on $\mathbb{P}^1$ is *not* a cyclotomic $\mathfrak{g}$-oper, but we can associate with it a cyclotomic

$\mathfrak{g}$-oper by taking its gauge equivalence class $[\nabla]_\Gamma$ under $N^{\hat{s}}(\mathcal{M})$. Let us fix a cyclotomic Miura $\mathfrak{g}$-oper ∇ with trivial monodromy representation and whose underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ has regular singularities at most at the points $z_i \in \mathbb{C}^\times$, $i = 1, \dots, N$ (and their Γ-orbits), the origin and infinity. We will further assume that ∇ is of the form

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{\check{\lambda}_0}{t}\mathrm{d}t + \mathsf{r} \tag{1.7}$$

for any ν-invariant integral dominant coweight $\check{\lambda}_0 \in \mathfrak{h}^\nu$ and with $\mathsf{r} \in \Omega^{\hat{\nu}}(\mathfrak{h})$ regular at the origin. We will show in Sect. 6.2 how the integrality assumption on $\check{\lambda}_0$ can be weakened by going to a cover of $\mathbb{P}^1$, which allows us to treat also the case of a ν-invariant dominant coweight $\check{\lambda}_0$ such that $\langle \alpha_i, \check{\lambda}_0 \rangle \in \mathbb{Q}$ for all $i \in I$. Note that in general r, and hence the cyclotomic $\mathfrak{g}$-connection ∇, may have simple poles at points $x_j \in \mathbb{C}^\times$, $j = 1, \dots, m$ (and their Γ-orbits) other than z_i, $i = 1, \dots, N$. Cyclotomic Miura $\mathfrak{g}$-opers of the above form whose residue at each x_j, $j = 1, \dots, m$ is a simple coroot and whose residue at each z_i, $i = 1, \dots, N$ is minus an integral dominant coweight, were shown in [26] to be described by solutions of the *cyclotomic Bethe ansatz equations* first introduced in [22]. Now as in the finite setting described above, a natural question in the present context is then to describe the space of all cyclotomic Miura $\mathfrak{g}$-opers whose underlying cyclotomic $\mathfrak{g}$-oper coincides with $[\nabla]_\Gamma$. In the non-cyclotomic case, $\Gamma = \{1\}$, it was proved independently by Mukhin and Varchenko [17,18] and Frenkel [7] that this space is isomorphic to the flag variety G/B_- where G is the adjoint group of $\mathfrak{g}$ and B_- is the Borel subgroup with Lie algebra $\mathfrak{b}_- := \bigoplus_{i \le 0} \mathfrak{g}_i$.

In applications to the cyclotomic Gaudin model, it follows from the construction of [26] that each cyclotomic Miura ${}^L\mathfrak{g}$-oper ∇ (not necessarily monodromy-free) with residues $-\lambda_i \in \mathfrak{h}^*$ at z_i for $i = 1, \dots, N$, $-\lambda_0 \in \mathfrak{h}^{*,\nu}$ at the origin and $\lambda_\infty \in \mathfrak{h}^{*,\nu}$ at infinity, built from a solution of the cyclotomic Bethe ansatz equations, corresponds to a joint eigenvector of $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ of weight $\lambda_\infty \in \mathfrak{h}^*$ in the tensor product $\bigotimes_{i=1}^N M_{\lambda_i}$ of Verma modules M_{λ_i} with highest weights λ_i. We will conjecture in Sect. 7 that the corresponding common eigenvalues of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ are determined by the underlying cyclotomic ${}^L\mathfrak{g}$-oper $[\nabla]_\Gamma$, see Conjecture 7.1. In the non-cyclotomic setting, it was conjectured in [8] that for integral dominant highest weights $\lambda_i \in \mathfrak{h}^*$, $i = 1, \dots, N$ and $\lambda_\infty \in \mathfrak{h}^*$, there is a bijection between monodromy-free ${}^L\mathfrak{g}$-opers in $\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{(z_i),\infty}$ with residues determined by the weights λ_i at z_i and λ_∞ at infinity, and eigenvalues of the Gaudin algebra $\mathscr{Z}_{(z_i)}(\mathfrak{g})$ on the tensor product $\bigotimes_{i=1}^N V_{\lambda_i}$ of finite-dimensional irreducible modules V_{λ_i} with highest weights λ_i. In particular, it is believed that when the Bethe ansatz is incomplete some of these eigenvalues correspond to Miura ${}^L\mathfrak{g}$-opers which do not arise from solutions of the Bethe ansatz equations.

Before proving the cyclotomic counterpart of the isomorphism between the space of Miura $\mathfrak{g}$-opers with a given underlying $\mathfrak{g}$-oper and the flag variety G/B_-, we begin in Sect. 5 by considering the effect of gauge transformations

on ∇ by elements of the form $e^{fE_k} \in N(\mathcal{M})$ for some $f \in \mathcal{M}$ and $k \in I$. Here E_i, H_i and F_i for $i \in I$ denote the Chevelley–Serre generators of $\mathfrak{g}$. It is well known that after such a gauge transformation, the new $\mathfrak{g}$-connection $e^{fE_k}\nabla e^{-fE_k}$ is still a Miura $\mathfrak{g}$-oper if and only if f is a solution of some Riccati equation. The result of going from the old Miura $\mathfrak{g}$-oper ∇ to the new one $e^{fE_k}\nabla e^{-fE_k}$ is known as a *reproduction* in the direction of the simple root α_k [17]. In general, however, the new Miura $\mathfrak{g}$-oper $e^{fE_k}\nabla e^{-fE_k}$ will no longer be cyclotomic. The problem of defining a cyclotomic version of the reproduction procedure, taking one cyclotomic Miura $\mathfrak{g}$-oper to another, was first studied by Varchenko and Young in [24] who considered so-called populations of solutions to the cyclotomic Bethe ansatz equations associated with an arbitrary Kac–Moody algebra $\mathfrak{g}$. It was shown there that under certain conditions on the coweight $\check{\lambda}_0$ (or rather the *weight* $\lambda_0 \in \mathfrak{h}^{*,\nu}$ since solutions of the cyclotomic Bethe ansatz equations used in [24] correspond to cyclotomic ${}^L\mathfrak{g}$-opers), it is possible to take one solution of the cyclotomic Bethe ansatz equations to another by performing a sequence of reproductions in the direction of other simple roots $\{\alpha_{i_j}\}_{j=2}^n$ for some $n \in \mathbb{Z}_{\geq 2}$ with $i_j \in I$ in the orbit $\mathcal{I} \in I/\nu$ of $i_1 := k \in I$ under the diagram automorphism $\nu : I \to I$.

When the Lie algebra $\mathfrak{g}$ is semisimple, as we are considering, there are only two possible types of orbits $\mathcal{I}$. We address the issue of existence of cyclotomic reproductions in the present language by studying gauge transformations by elements in the subgroup generated by simple roots along the orbit $\mathcal{I}$. Specifically, we consider Riccati equations built from the cyclotomic Miura $\mathfrak{g}$-oper ∇ given in (1.7). These are satisfied by the collection of meromorphic functions $f_j \in \mathcal{M}$ which appear in the individual gauge transformation parameters $e^{f_j E_{i_j}} \in N(\mathcal{M})$. Requiring the overall gauge transformation parameter $g = e^{f_n E_{i_n}} \dots e^{f_1 E_{i_1}}$ to live in the Γ-invariant subgroup $N^{\hat{\varsigma}}(\mathcal{M})$ imposes certain functional relations among the various functions f_j. If the latter are regular at the origin, then we find that these functional relations can be satisfied if and only if $\check{\lambda}_0$ satisfies the following relation

$$\ell_{\mathcal{I}}\langle \alpha_k, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \mod \frac{T}{|\mathcal{I}|}, \tag{1.8}$$

where $\ell_{\mathcal{I}} = 1$ or 2 depending on the type of the orbit $\mathcal{I}$. A key step in our analysis is considering a regularisation of the given Riccati equations at the origin. In this case, the new cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ takes the same form as ∇ in (1.7) but with some new differential $\mathsf{r} \in \Omega^{\hat{\nu}}(\mathfrak{h})$. On the other hand, without imposing any conditions on $\check{\lambda}_0$, we can always choose the overall gauge transformation parameter to be cyclotomic by letting one of the functions f_j be singular. Moreover, in this case we find that the new cyclotomic Miura $\mathfrak{g}$-oper takes the form

$$g\nabla g^{-1} = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{s^{\nu}_{\mathcal{I}} \cdot \check{\lambda}_0}{t}\mathrm{d}t + \widetilde{\mathsf{r}}$$

for some $\widetilde{\mathsf{r}} \in \Omega^{\hat{\rho}}(\mathfrak{h})$, with $s^{\nu}_{\mathcal{I}}$ the simple reflection of the ν-invariant Weyl group W^{ν} associated with the node $\mathcal{I} \in I/\nu$ of the folded diagram and where the dot denotes the shifted Weyl action.

Let us now consider the effect of a gauge transformation with a more general parameter $g \in N^{\hat{\varsigma}}(\mathcal{M})$ on the cyclotomic Miura $\mathfrak{g}$-oper ∇ in (1.7), not one corresponding merely to the orbit of a simple root. We begin in Sect. 6.1 by describing the space of all cyclotomic Miura $\mathfrak{g}$-opers $g\nabla g^{-1}$ of the same form as the original cyclotomic Miura $\mathfrak{g}$-oper ∇, namely (1.7) but for some different $\mathsf{r} \in \Omega^{\hat{\rho}}(\mathfrak{h})$. We refer to these cyclotomic Miura $\mathfrak{g}$-opers as being *generic* at the origin. Guided by the analysis of Sect. 5, we introduce the following *regularised* $\mathfrak{g}$-connection

$$\nabla_{\mathrm{r}} := t^{-\check{\lambda}_0} \nabla t^{\check{\lambda}_0} = \mathrm{d} + \sum_{k \in I} t^{\langle \alpha_k, \check{\lambda}_0 \rangle} F_k \, \mathrm{d}t + \mathsf{r}.$$

The regularisation has the effect of modifying the automorphism of $\mathfrak{g}$ in the Γ-equivariance property from ς to $\vartheta := \mathrm{Ad}_{\omega^{-\check{\lambda}_0 - \check{\rho}}} \circ \nu \in \mathrm{Aut}\,\mathfrak{g}$. In particular, regularising the new cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ yields the $\mathfrak{g}$-connection $(g\nabla g^{-1})_{\mathrm{r}} = g_{\mathrm{r}} \nabla_{\mathrm{r}} g_{\mathrm{r}}^{-1}$ where $g_{\mathrm{r}} = t^{-\check{\lambda}_0} \, g \, t^{\check{\lambda}_0} \in N^{\hat{\vartheta}}(\mathcal{M})$. Since we are assuming $g\nabla g^{-1}$ to be of the same form as in (1.7) it follows that $(g\nabla g^{-1})_{\mathrm{r}}$ is regular at the origin and hence so is g_{r}. Its initial condition $g_{\mathrm{r}}(0)$ then takes value in N^{ϑ}. Conversely, given any $g_0 \in N^{\vartheta}$ we construct in Theorem 6.2 an element $g \in N^{\hat{\varsigma}}(\mathcal{M})$ such that g_{r} is regular at the origin with $g_{\mathrm{r}}(0) = g_0$. The cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ is then of the same form as in (1.7). In this language, relation (1.8) on $\check{\lambda}_0$ for a given orbit $\mathcal{I} \in I/\nu$ can now be seen as the condition for the existence of a ϑ-invariant element in the subalgebra of $\mathfrak{n}$ generated by the E_i with $i \in \mathcal{I}$.

The residue at the origin of the cyclotomic $\mathfrak{g}$-oper $[\nabla]_{\Gamma}$, as defined in Sect. 4.5, corresponding to the cyclotomic Miura $\mathfrak{g}$-oper ∇ given in (1.7) is the finite $\mathfrak{g}^{\nu}$-oper $[p_{-1} - \check{\lambda}_0 - \check{\rho}]_{\mathfrak{g}^{\nu}} \in \mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}^{\nu}}$. The general form of a cyclotomic Miura $\mathfrak{g}$-oper $\widetilde{\nabla}$ whose corresponding cyclotomic $\mathfrak{g}$-oper $[\widetilde{\nabla}]_{\Gamma}$ has the same residue at the origin is

$$\widetilde{\nabla} = \mathrm{d} + p_{-1} \mathrm{d}t - \frac{w \cdot \check{\lambda}_0}{t} \mathrm{d}t + \widetilde{\mathsf{r}} \tag{1.9}$$

for some $w \in W^{\nu}$ and $\widetilde{\mathsf{r}} \in \Omega^{\hat{\rho}}(\mathfrak{h})$ regular at the origin. It follows from the above discussion that to reach such cyclotomic Miura $\mathfrak{g}$-opers from ∇ given in (1.7) one should apply a gauge transformation with parameter $g \in N^{\hat{\varsigma}}(\mathcal{M})$ for which $g_{\mathrm{r}} = t^{-\check{\lambda}_0} g t^{\check{\lambda}_0}$ is singular at the origin. We prove in Sect. 6.3 that the space of *all* cyclotomic Miura $\mathfrak{g}$-opers of the form $g\nabla g^{-1}$ with $g \in N^{\hat{\varsigma}}(\mathcal{M})$ is isomorphic to the ϑ-invariant subspace $(G/B_-)^{\vartheta}$ of the flag variety G/B_-. We show in Sect. 6.4 that this ϑ-invariant subspace has a cell decomposition

$$(G/B_-)^{\vartheta} = \bigsqcup_{w \in W^{\nu}} N^{\vartheta} \dot{w} B_-/B_-.$$

In particular, the big cell $N^{\vartheta}B_-/B_-$ is isomorphic to the space of generic cyclotomic Miura $\mathfrak{g}$-opers $g\nabla g^{-1}$, i.e. for which g_{r} is regular at the origin. We

expect the cyclotomic Miura $\mathfrak{g}$-opers $g\nabla g^{-1}$ of the form (1.9) to be isomorphic to the cell $N^{\vartheta}\dot{w}B_-/B_-$.

2. Notations and Conventions

2.1. Semisimple Lie Algebras

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by $\Phi \subset \mathfrak{h}^*$ the root system of $(\mathfrak{g}, \mathfrak{h})$. Fix a basis of simple roots α_i, $i \in I := \{1, \dots, \operatorname{rk} \mathfrak{g}\}$ and let $\Phi^{\pm}$ denote the corresponding set of positive and negative roots. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}, \qquad \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathbb{C}E_\alpha, \quad \mathfrak{n}_- := \bigoplus_{\alpha \in \Phi^+} \mathbb{C}F_\alpha. \tag{2.1}$$

Introduce also the positive and negative Borel subalgebras $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b}_- := \mathfrak{h} \oplus \mathfrak{n}_-$. For any positive root $\alpha \in \Phi^+$, we define the root spaces $\mathfrak{g}_\alpha := \mathbb{C}E_\alpha$ and $\mathfrak{g}_{-\alpha} := \mathbb{C}F_\alpha$. The Chevalley–Serre generators $\check{\alpha}_i \in \mathfrak{h}$, $E_i := E_{\alpha_i}$ and $F_i := F_{\alpha_i}$ for $i \in I$ satisfy the relations

$$[\check{\alpha}_i, \check{\alpha}_j] = 0, \qquad [\check{\alpha}_i, E_j] = a_{ij}E_j, \qquad [\check{\alpha}_i, F_j] = -a_{ij}F_j,$$
$$[E_i, F_j] = \delta_{ij}\check{\alpha}_i, \qquad \operatorname{ad}_{E_i}^{1-a_{ij}} E_j = 0, \qquad \operatorname{ad}_{F_i}^{1-a_{ij}} F_j = 0,$$

for all $i, j \in I$, where $A := (a_{ij})_{i,j\in I} := \big(\langle \alpha_j, \check{\alpha}_i \rangle\big)_{i,j\in I}$ is the Cartan matrix.

We fix a non-degenerate invariant symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$. Its restriction to the Cartan subalgebra $\mathfrak{h}$ is non-degenerate and hence induces an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. We use this to define a non-degenerate bilinear pairing on $\mathfrak{h}^*$ which we also denote $(\cdot|\cdot)$. For every $i \in I$ and $\lambda \in \mathfrak{h}^*$, we then have $\langle \lambda, \check{\alpha}_i \rangle = 2(\lambda|\alpha_i)/(\alpha_i|\alpha_i)$. A coweight $\check{\lambda} \in \mathfrak{h}$ is said to be *integral* if $\langle \alpha_i, \check{\lambda} \rangle \in \mathbb{Z}$ for every $i \in I$ and *dominant* if $\langle \alpha_i, \check{\lambda} \rangle \geq 0$ for every $i \in I$.

Let $\upsilon \in \operatorname{Aut} \mathfrak{g}$ be an automorphism of $\mathfrak{g}$. One can always find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that υ stabilises the associated Cartan decomposition (2.1). We will say that such a Cartan subalgebra is *adapted* to the automorphism υ. The restriction of υ to $\mathfrak{h}$ then induces a linear map $\upsilon \in \mathsf{GL}(\mathfrak{h})$. We define a linear map $\upsilon \in \mathsf{GL}(\mathfrak{h}^*)$ by requiring the υ-invariance of the bilinear pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, i.e. $\langle \upsilon\lambda, \upsilon\check{\mu} \rangle = \langle \lambda, \check{\mu} \rangle$ for every $\lambda \in \mathfrak{h}^*$ and $\check{\mu} \in \mathfrak{h}$. This linear map leaves invariant the set of all roots $\Phi \subset \mathfrak{h}^*$, the subset of positive and negative roots $\Phi^{\pm}$ and the subset of simple roots $\{\alpha_i\}_{i\in I}$. It therefore defines an automorphism $\upsilon : \Phi \to \Phi$ of the set of roots and a diagram automorphism $\upsilon : I \to I$. The action of the automorphism $\upsilon \in \operatorname{Aut} \mathfrak{g}$ on the Cartan–Weyl basis associated with the Cartan decomposition (2.1) is then given by

$$\upsilon(E_\alpha) = \tau_\alpha E_{\upsilon(\alpha)}, \qquad \upsilon(\check{\alpha}_i) = \check{\alpha}_{\upsilon(i)}, \qquad \upsilon(F_\alpha) = \tau_\alpha^{-1} F_{\upsilon(\alpha)}, \tag{2.2}$$

for each $i \in I$ and $\alpha \in \Phi$, where τ_α are complex numbers. The automorphism is entirely specified by its diagram part $\upsilon : I \to I$ and the collection of numbers τ_{α_i} corresponding to the simple roots.

The bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ is υ-invariant, i.e. $(\upsilon X|\upsilon Y) = (X|Y)$ for all $X, Y \in \mathfrak{g}$. Indeed, it can be expressed as a linear combination of the Killing forms $(X, Y) \mapsto \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ on each simple factor of $\mathfrak{g}$ and for every $X \in \mathfrak{g}$ we have $\mathrm{ad}_{\upsilon X} = \upsilon \circ \mathrm{ad}_X \circ \upsilon^{-1}$. If the Cartan subalgebra $\mathfrak{h}$ is adapted to υ, then the restriction of $(\cdot|\cdot)$ to $\mathfrak{h}$ and the induced bilinear form on $\mathfrak{h}^*$ are both υ-invariant.

2.2. Adjoint Group

Let $G = (\mathrm{Aut}\,\mathfrak{g})^\circ$ be the adjoint group associated with $\mathfrak{g}$, i.e. the connected component of the identity in the automorphism group $\mathrm{Aut}\,\mathfrak{g}$. It is a semisimple algebraic group, with Lie algebra $\mathrm{Lie}(G) = \mathrm{Lie}(\mathrm{Aut}\,\mathfrak{g}) = \mathrm{Der}\,\mathfrak{g} = \mathrm{ad}_{\mathfrak{g}}$ isomorphic to $\mathfrak{g}$. It is generated by the inner automorphisms of the form $\exp(\mathrm{ad}_X)$, for all ad-nilpotent $X \in \mathfrak{g}$.

By construction, G acts on the Lie algebra $\mathfrak{g}$ by the adjoint action. We define $H := Z_G(\mathfrak{h})$, the centraliser of the Cartan subalgebra $\mathfrak{h}$, i.e. the subgroup of G that fixes $\mathfrak{h}$ pointwise. It is the unique maximal torus of G with Lie algebra $\mathfrak{h}$, and is a closed connected abelian subgroup of G. In the same way, let $B := N_G(\mathfrak{b})$ and $B_- := N_G(\mathfrak{b}_-)$ be the normalisers of the Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}_-$, i.e. the subgroups of G whose adjoint action stabilises $\mathfrak{b}$ and $\mathfrak{b}_-$. They form Borel subgroups of G (maximal closed connected solvable subgroups) and have Lie algebras $\mathfrak{b}$ and $\mathfrak{b}_-$.

We define the respective derived subgroups $N := (B, B)$ and $N_- := (B_-, B_-)$. They are closed connected unipotent subgroups of G with Lie algebras $\mathfrak{n}$ and $\mathfrak{n}_-$. The Borel subgroups B and B_- then uniquely factorise as $B = HN$ and $B_- = HN_-$. We have the root decomposition

$$N = \prod_{\alpha \in \Phi^+} G_\alpha \quad \text{and} \quad N_- = \prod_{\alpha \in \Phi^-} G_\alpha,$$

where, for any $\alpha \in \Phi^+$, $G_\alpha := \{\exp(x\,\mathrm{ad}_{E_\alpha})\}_{x \in \mathbb{C}}$ and $G_{-\alpha} := \{\exp(x\,\mathrm{ad}_{F_\alpha})\}_{x \in \mathbb{C}}$ are one-dimensional closed connected subgroups of G with Lie algebras $\mathrm{ad}_{\mathfrak{g}_\alpha}$ and $\mathrm{ad}_{\mathfrak{g}_{-\alpha}}$. The above decompositions hold for any ordering of the roots $\alpha \in \Phi^\pm$.

Let $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ be any given automorphism of $\mathfrak{g}$. Since $G = (\mathrm{Aut}\,\mathfrak{g})^\circ$ is normal in $\mathrm{Aut}\,\mathfrak{g}$, we can lift υ to an automorphism of G, acting by conjugation $\eta \mapsto \upsilon \circ \eta \circ \upsilon^{-1}$, which by abuse of notation we shall also denote $\upsilon \in \mathrm{Aut}\,G$. The induced Lie algebra automorphism of $\mathrm{Lie}(G) = \mathrm{ad}_{\mathfrak{g}}$ coincides with $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ via the isomorphism between $\mathrm{ad}_{\mathfrak{g}}$ and $\mathfrak{g}$. If the Cartan subalgebra $\mathfrak{h}$ is chosen to be adapted to $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ (cf. Sect. 2.1), then the lift $\upsilon \in \mathrm{Aut}\,G$ stabilises the corresponding subgroups H, N and N_- of G. As a consequence, it also stabilises the Borel subgroups $B = HN$ and $B_- = HN_-$.

2.3. Weyl Group and Flag Varieties

The Weyl group $W \subset \mathsf{GL}(\mathfrak{h}^*)$ is generated by reflections $\lambda \mapsto s_i\lambda := \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i$ for all $i \in I$. We define an action of W on $\mathfrak{h}$ by requiring the invariance of the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, namely $\langle w(\lambda), w(\check{\mu}) \rangle = \langle \lambda, \check{\mu} \rangle$ for any $\lambda \in \mathfrak{h}^*$,

$\check{\mu} \in \mathfrak{h}$ and $w \in W$. Explicitly, for every $i \in I$ and $\check{\mu} \in \mathfrak{h}$ we have $s_i\check{\mu} = \check{\mu} - \langle \alpha_i, \check{\mu} \rangle \check{\alpha}_i$.

Let $N_G(\mathfrak{h})$ be the normaliser of $\mathfrak{h}$ for the adjoint action of G on the Lie algebra $\mathfrak{g}$. The restriction of the coadjoint action $G \to \mathsf{GL}(\mathfrak{g}^*)$ to $N_G(\mathfrak{h})$ gives rise to a map $N_G(\mathfrak{h}) \to \mathsf{GL}(\mathfrak{h}^*)$ which induces an isomorphism $\pi^{-1} : N_G(\mathfrak{h})/H \xrightarrow{\sim} W$. Given any $w \in W$ we fix a representative $\dot{w} \in N_G(\mathfrak{h})$ of the class $\pi(w) \in N_G(\mathfrak{h})/H$ (the other representatives are then of the form $\dot{w}h$ with $h \in H$). For any $\alpha \in \Phi$, the action of $\dot{w}$ by conjugation on the root subgroup G_α is simply $\dot{w}G_\alpha\dot{w}^{-1} = G_{w\alpha}$.

Consider the flag variety G/B_- associated with the group G. The Gauss decomposition of G into a disjoint union of cells $N\dot{w}B_-$ over $w \in W$ gives rise to the following cell decomposition of the flag variety

$$G/B_- = \bigsqcup_{w \in W} N\dot{w}B_-/B_- =: \bigsqcup_{w \in W} C_w. \tag{2.3}$$

Since the representative $\dot{w}$ of $w \in W$ in $N_G(\mathfrak{h})$ only differs from other choices of representatives through right multiplication by an element of the Cartan subgroup $H \subset B_-$, it is clear that the cell $C_w = N\dot{w}B_-/B_-$ does not depend on the choice of this representative. The big cell $C_{\mathrm{Id}} = NB_-/B_-$, which is isomorphic to N, is dense in G/B_-.

Let $\upsilon \in \operatorname{Aut}\mathfrak{g}$ be an automorphism of the Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ adapted to υ. We use the corresponding action of υ on $\mathfrak{h}^*$ to define an action of υ on W by conjugation $w \mapsto \upsilon(w) := \upsilon \circ w \circ \upsilon^{-1}$. By definition of the reflection $s_i \in W$, we have in particular $\upsilon(s_i) = s_{\upsilon(i)}$ for every $i \in I$. The map $\upsilon : W \to W$ constructed in this way is an automorphism of the group W. Note that it only depends on the restriction of υ to $\mathfrak{h}$, i.e. on the associated diagram automorphism $\upsilon : I \to I$. We introduce the subgroup

$$W^\upsilon := \{w \in W \,|\, \upsilon(w) = w\}$$

of υ-invariant elements in W.

The lift $\upsilon \in \operatorname{Aut} G$ of the automorphism $\upsilon \in \operatorname{Aut}\mathfrak{g}$ to the adjoint group G stabilises both the torus H and the normaliser $N_G(\mathfrak{h})$. It therefore descends to an automorphism υ of the subquotient $N_G(\mathfrak{h})/H$. The isomorphism $\pi : W \xrightarrow{\sim} N_G(\mathfrak{h})/H$ is equivariant with respect to the action of υ on W and $N_G(\mathfrak{h})/H$, i.e. $\pi\big(\upsilon(w)\big) = \upsilon\big(\pi(w)\big)$ for any $w \in W$. Since υ stabilises the Borel subgroup B_-, one can also define an automorphism υ of the flag variety G/B_-. This will play a central role in Sect. 6, so we postpone its definition and the study of its properties until then.

3. Finite $\mathfrak{g}$-Opers and Finite Miura $\mathfrak{g}$-Opers

We follow the conventions and notations of Sect. 2.

3.1. Principal $\mathfrak{sl}_2$ and $\mathbb{Z}$-Grading

Consider the regular nilpotent element

$$p_{-1} := \sum_{i\in I} F_i. \tag{3.1}$$

By the Jacobson–Morozov theorem, it can be embedded into an $\mathfrak{sl}_2$-triple. Let $\check{\omega}_i \in \mathfrak{h}$, $i \in I$, be the fundamental coweights of $\mathfrak{g}$ defined by $\alpha_i(\check{\omega}_j) = \delta_{ij}$ for all $i, j \in I$. The Weyl covector

$$\check{\rho} := \sum_{i\in I} \check{\omega}_i \tag{3.2}$$

then satisfies $\alpha_i(\check{\rho}) = 1$ for all $i \in I$, so that $[2\check{\rho}, p_{-1}] = -2p_{-1}$. The pair $\{p_{-1}, 2\check{\rho}\}$ extends uniquely to an $\mathfrak{sl}_2$-triple $\{p_{-1}, 2\check{\rho}, p_1\}$, i.e. with the relations

$$[p_1, p_{-1}] = 2\check{\rho}, \qquad [\check{\rho}, p_{\pm 1}] = \pm p_{\pm 1}.$$

Specifically, the regular nilpotent element p_1 is obtained by writing $2\check{\rho} = \sum_{i\in I} c_i \check{\alpha}_i$ in the basis of simple coroots $\check{\alpha}_i$ of $\mathfrak{g}$ and defining $p_1 := \sum_{i\in I} c_i E_i$. We will refer to $\{p_{-1}, 2\check{\rho}, p_1\}$ as the *principal* $\mathfrak{sl}_2$*-triple* of $\mathfrak{g}$ and to its span $\langle p_{-1}, \check{\rho}, p_1\rangle \subset \mathfrak{g}$ as the *principal* $\mathfrak{sl}_2$ *subalgebra* of $\mathfrak{g}$.

The *height* $\mathrm{ht}(\alpha) \in \mathbb{Z}$ of a root $\alpha \in \Phi$ is the eigenvalue of the corresponding root vector under the adjoint action of the Cartan element $\check{\rho}$, i.e. $[\check{\rho}, E_\alpha] = \mathrm{ht}(\alpha)E_\alpha$ and $[\check{\rho}, F_\alpha] = \mathrm{ht}(-\alpha)F_\alpha$ for any $\alpha \in \Phi^+$. The eigenspace decomposition of $\mathrm{ad}_{\check{\rho}}$ defines a $\mathbb{Z}$-grading of the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i = \bigoplus_{i=-h+1}^{h-1} \mathfrak{g}_i, \tag{3.3}$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$. Here, h denotes the Coxeter number of $\mathfrak{g}$, defined as the maximum of the Coxeter numbers of the simple factors in the semisimple decomposition of $\mathfrak{g}$ (for simple $\mathfrak{g}$, we have $h := \mathrm{ht}(\theta) + 1$ with θ the maximal root in Φ). The eigenspace $\mathfrak{g}_j := \{X \in \mathfrak{g} \,|\, [\check{\rho}, X] = jX\}$ of $\mathrm{ad}_{\check{\rho}}$ for each $j \in \mathbb{Z}$ is spanned by all root vectors of height j, explicitly

$$\mathfrak{g}_0 = \mathfrak{h}, \qquad \mathfrak{g}_i = \bigoplus_{\alpha\in\Phi^+ \,|\, \mathrm{ht}(\alpha)=i} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_{-i} = \bigoplus_{\alpha\in\Phi^+ \,|\, \mathrm{ht}(\alpha)=i} \mathfrak{g}_{-\alpha}$$

for all $i \in \mathbb{Z}_{\geq 1}$. In particular, we have $\mathfrak{g}_i = 0$ if $|i| \geq h$. Note also that $\mathfrak{b} = \oplus_{i\geq 0}\mathfrak{g}_i$ and $\mathfrak{n} = \oplus_{i>0}\mathfrak{g}_i$.

The centraliser of p_1 in $\mathfrak{g}$ is a $\mathbb{Z}_{\geq 1}$-graded abelian subalgebra $\mathfrak{a} := \ker(\mathrm{ad}_{p_1})$ of dimension $\mathrm{rk}\,\mathfrak{g}$. Let $\mathfrak{a}_j := \mathfrak{a} \cap \mathfrak{g}_j$ denote its grade-j subspace for each $j \in \{1, \ldots, h-1\}$. We let E denote the multiset containing each $j \in \{1, \ldots, h-1\}$ with multiplicity $\dim \mathfrak{a}_j$. Its elements are called the exponents of $\mathfrak{g}$. From the representation theory of $\mathfrak{sl}_2$, we then have, for every $i \in \{1, \ldots, h-1\}$,

$$\mathfrak{g}_i = \begin{cases} [p_{-1}, \mathfrak{g}_{i+1}] & \text{if} \quad i \notin E, \\ [p_{-1}, \mathfrak{g}_{i+1}] \oplus \mathfrak{a}_i & \text{if} \quad i \in E. \end{cases} \tag{3.4}$$

3.2. Finite $\mathfrak{g}$-Opers

Let G be the adjoint group of $\mathfrak{g}$, cf. Sect. 2.2. If $g \in G$ and $X \in \mathfrak{g}$, then we denote by gXg^{-1} the adjoint action of g on X. The affine subspace $p_{-1}+\mathfrak{b} \subset \mathfrak{g}$ is stabilised by the adjoint action of the unipotent subgroup N of G. Consider the quotient

$$\mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}} := (p_{-1} + \mathfrak{b})/N$$

and denote the class of any $X \in p_{-1} + \mathfrak{b}$ as $[X]_{\mathfrak{g}}$. We refer to the elements of $\mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}}$ as *finite* $\mathfrak{g}$*-opers*. The so-called *Slodowy slice* through p_{-1} is defined as the affine subspace $p_{-1} + \mathfrak{a} \subset p_{-1} + \mathfrak{b}$. It is transversal at every point to the adjoint orbit of N in $p_{-1} + \mathfrak{b}$. In fact, it is an important result of Kostant that the Slodowy slice intersects each N-orbit in $p_{-1} + \mathfrak{b}$ exactly once. For our purpose, it is convenient to formulate this statement as follows.

Theorem 3.1. *Every finite* $\mathfrak{g}$*-oper has a unique representative in* $p_{-1} + \mathfrak{a}$*. We shall refer to it as the* canonical representative *of the finite* $\mathfrak{g}$*-oper.*

Proof. This follows from [16, Theorem 1.2] which states that the map $N \times (p_{-1} + \mathfrak{a}) \to p_{-1} + \mathfrak{b}$, given by the adjoint action $(n, X) \mapsto nXn^{-1}$ is an isomorphism of affine varieties. Alternatively, it could also be proved by a simpler version of the argument used in the proof of Theorem 4.5. However, since the argument is so similar to the one given there, we do not repeat it here for the sake of brevity. □

We define a *finite Miura* $\mathfrak{g}$*-oper* as an element of the affine subspace $p_{-1} + \mathfrak{h} \subset \mathfrak{g}$. To any finite Miura $\mathfrak{g}$-oper is associated a finite $\mathfrak{g}$-oper, namely its class in $\mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}}$. It follows from the second half of the proof of [16, Theorem 1.2] that the corresponding map

$$\begin{aligned} p_{-1} + \mathfrak{h} &\longrightarrow \mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}}, \\ X &\longmapsto [X]_{\mathfrak{g}}, \end{aligned} \tag{3.5}$$

is surjective. The next theorem gives a necessary and sufficient condition for two finite Miura $\mathfrak{g}$-opers to correspond to the same finite $\mathfrak{g}$-oper.

Proposition 3.2. *For any* $\check{\lambda}, \check{\mu} \in \mathfrak{h}$*, we have* $[p_{-1} - \check{\lambda}]_{\mathfrak{g}} = [p_{-1} - \check{\mu}]_{\mathfrak{g}}$ *if and only if there exists* $w \in W$ *such that* $\check{\lambda} = w(\check{\mu})$*.*

Proof. Let $\check{\lambda}, \check{\mu} \in \mathfrak{h}$. Denote by $\boldsymbol{u} = (u_1, \ldots, u_{\mathrm{rk}\,\mathfrak{g}}) : \mathfrak{g} \to \mathbb{C}^{\mathrm{rk}\,\mathfrak{g}}$ the collection of $\mathrm{rk}\,\mathfrak{g}$ fundamental homogenous G-invariant polynomials on $\mathfrak{g}$. By [14, Lemma 9.2], there exists $w \in W$ such that $\check{\lambda} = w(\check{\mu})$ if and only if $\boldsymbol{u}(\check{\lambda}) = \boldsymbol{u}(\check{\mu})$. The latter is equivalent to $\boldsymbol{u}(p_{-1} - \check{\lambda}) = \boldsymbol{u}(p_{-1} - \check{\mu})$ by [15, Proposition 17]. Now since $p_{-1} - \check{\lambda}$ and $p_{-1} - \check{\mu}$ are regular by [15, Lemma 10], this in turn is equivalent to $p_{-1} - \check{\lambda}$ and $p_{-1} - \check{\mu}$ being conjugate under the adjoint action of G by [15, Theorem 3]. It is clear that if $p_{-1} - \check{\lambda}$ and $p_{-1} - \check{\mu}$ are conjugate under the adjoint action of N then they are also conjugate under the adjoint action of G. The converse of this statement follows from the second half of the proof of [16, Theorem 1.2]. Therefore, $p_{-1} - \check{\lambda}$ and $p_{-1} - \check{\mu}$ are conjugate under the adjoint action of G if and only if $[p_{-1} - \check{\lambda}]_{\mathfrak{g}} = [p_{-1} - \check{\mu}]_{\mathfrak{g}}$, and the theorem follows. □

We define the *shifted action* of W on $\mathfrak{h}$ by letting $w \in W$ send $\check{\lambda} \in \mathfrak{h}$ to $w \cdot \check{\lambda} := w(\check{\lambda} + \check{\rho}) - \check{\rho}$. Two coweights $\check{\lambda}, \check{\mu} \in \mathfrak{h}$ are then said to be *W-linked* if $\check{\mu} = w \cdot \check{\lambda}$ for some $w \in W$. The *W-linkage class* of a coweight $\check{\lambda} \in \mathfrak{h}$ is its orbit under this shifted action of W, which we denote by $[\check{\lambda}]_W$.

Corollary 3.3. *There is a bijection between the set $\mathfrak{h}/(W,\cdot)$ of W-linkage classes in $\mathfrak{h}$ and the set $\mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}}$ of finite $\mathfrak{g}$-opers, given explicitly by $[\check{\lambda}]_W \mapsto [p_{-1} - \check{\lambda} - \check{\rho}]_{\mathfrak{g}}$.*

Proof. The given map is well defined and injective by Proposition 3.2. It is surjective since (3.5) is. □

By abuse of notation, in what follows we always identify the W-linkage class $[\check{\lambda}]_W$ of a coweight $\check{\lambda} \in \mathfrak{h}$ with the corresponding finite $\mathfrak{g}$-oper $[p_{-1} - \check{\lambda} - \check{\rho}]_{\mathfrak{g}} \in \mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}}$. Let $\check{\lambda} \in \mathfrak{h}$ and consider the subset $\mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g},[\check{\lambda}]_W} := \big\{X \in p_{-1} + \mathfrak{h} \,\big|\, [X]_{\mathfrak{g}} = [\check{\lambda}]_W\big\}$ of the set $p_{-1} + \mathfrak{h}$ of all finite Miura $\mathfrak{g}$-opers.

Theorem 3.4. *If $\check{\lambda} \in \mathfrak{h}$ is dominant then $\mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g},[\check{\lambda}]_W} \simeq W$.*

Proof. By Proposition 3.2, every $X \in \mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g},[\check{\lambda}]_W}$ is of the form $X = p_{-1} - w(\check{\lambda} + \check{\rho})$ for some $w \in W$. We obtain a surjective map $W \twoheadrightarrow \mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g},[\check{\lambda}]_W}$ given by $w \mapsto p_{-1} - w(\check{\lambda} + \check{\rho})$. To prove injectivity, it is enough to show that if $\check{\lambda} + \check{\rho}$ is fixed by $w \in W$ then $w = \mathrm{Id}$. But since $\check{\lambda}$ is assumed dominant, we have $\langle \alpha_i, \check{\lambda} + \check{\rho} \rangle \geq 1$ for all $i \in I$. In other words, the coweight $\check{\lambda} + \check{\rho}$ lies inside the open fundamental Weyl chamber of $\mathfrak{h}$. The result now follows from the fact that W acts simply transitively on the set of all Weyl chambers. □

3.3. Folding and Finite $\mathfrak{g}^\nu$-Opers

Let $\nu : I \to I$ be a permutation of the simple roots of $\mathfrak{g}$ which preserves the Cartan matrix, i.e. such that

$$a_{\nu(i)\nu(j)} = a_{ij} \tag{3.6}$$

for all $i, j \in I$. Consider the associated diagram automorphism of $\mathfrak{g}$, which we also denote $\nu \in \mathrm{Aut}\,\mathfrak{g}$, defined by its action on the Chevalley–Serre generators as

$$\nu(E_i) = E_{\nu(i)}, \qquad \nu(\check{\alpha}_i) = \check{\alpha}_{\nu(i)}, \qquad \nu(F_i) = F_{\nu(i)}. \tag{3.7}$$

We are interested in describing the ν-invariant subalgebra $\mathfrak{g}^\nu := \{X \in \mathfrak{g} \,|\, \nu(X) = X\}$.

Let I/ν denote the set of orbits in I under ν. For each orbit $\mathcal{I} \in I/\nu$, we define (cf. [11])

$$\ell_{\mathcal{I}} := 3 - \sum_{i \in \mathcal{I}} a_{ij},$$

where j is any point in the orbit $\mathcal{I}$. The right-hand side depends only on the orbit $\mathcal{I}$ itself, and not on j, by virtue of the assumption (3.6) on ν. Moreover, by the properties $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i \neq j$ of the Cartan matrix it follows that $\ell_{\mathcal{I}} \in \mathbb{Z}_{\geq 1}$. Since $\mathfrak{g}$ is semisimple, we have $\ell_{\mathcal{I}} = 1$ or 2 for every $\mathcal{I} \in I/\nu$ (see e.g. [20, Lemma 4.3]). If $\ell_{\mathcal{I}} = 1$, then the restriction of the Dynkin diagram to the orbit $\mathcal{I}$ is isomorphic to the Dynkin diagram of $A_1^{\times|\mathcal{I}|}$. On the

other hand, if $\ell_{\mathcal{I}} = 2$, then $|\mathcal{I}|$ is even and the Dynkin diagram restricted to the orbit $\mathcal{I}$ is isomorphic to the Dynkin diagram of $A_2^{\times |\mathcal{I}|/2}$. Finally, note that in either case the relation $\ell_{\mathcal{I}} \sum_{i\in\mathcal{I}} a_{ij} = 2$ holds for any $j \in \mathcal{I}$.

The following result is well known (see, for instance, [2, § 9.5]).

Proposition 3.5. *The ν-invariant subalgebra $\mathfrak{g}^\nu$ is semisimple. Its Chevalley–Serre generators are*

$$\check{\alpha}^\nu_{\mathcal{I}} := \ell_{\mathcal{I}} \sum_{i\in\mathcal{I}} \check{\alpha}_i, \qquad E^\nu_{\mathcal{I}} := \ell_{\mathcal{I}} \sum_{i\in\mathcal{I}} E_i, \qquad F^\nu_{\mathcal{I}} := \sum_{i\in\mathcal{I}} F_i,$$

for $\mathcal{I} \in I/\nu$ with corresponding system of simple roots $\alpha^\nu_{\mathcal{I}} := \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} \alpha_i \subset \mathfrak{h}^{*,\nu} := (\mathfrak{h}^*)^\nu$. The Cartan matrix $A^\nu := (a^\nu_{\mathcal{I}\mathcal{J}})_{\mathcal{I},\mathcal{J}\in I/\nu}$ is given by $a^\nu_{\mathcal{I}\mathcal{J}} := \langle \alpha^\nu_{\mathcal{J}}, \check{\alpha}^\nu_{\mathcal{I}}\rangle = \ell_{\mathcal{I}} \sum_{i\in\mathcal{I}} a_{ij}$ for any $j \in \mathcal{J}$, and the Weyl group is the ν-invariant subgroup W^ν, cf. Sect. 2.3.

We shall need the following explicit description of the Weyl group W^ν of $\mathfrak{g}^\nu$ (see e.g. [24]).

Lemma 3.6. *The simple reflections $\{s^\nu_{\mathcal{I}}\}_{\mathcal{I}\in I/\nu}$ generating the ν-invariant subgroup $W^\nu \subset W$ read*

$$s^\nu_{\mathcal{I}} = \begin{cases} \prod_{i\in\mathcal{I}} s_i & \text{for} \quad \ell_{\mathcal{I}} = 1, \\ \prod_{i\in\mathcal{I}/2} s_i s_{\bar{\imath}} s_i & \text{for} \quad \ell_{\mathcal{I}} = 2, \end{cases}$$

where the product in the case $\ell_{\mathcal{I}} = 2$ is over half of the orbit, namely $\mathcal{I}/2 := \{k, \nu(k), \ldots, \nu^{|\mathcal{I}|/2-1}(k)\}$ for any $k \in \mathcal{I}$, and we define $\bar{\imath} := \nu^{|\mathcal{I}|/2}(i)$ for every $i \in \mathcal{I}$.

Proof. Let $\mathcal{I} \in I/\nu$. We determine $s^\nu_{\mathcal{I}}$ by computing its action on an arbitrary ν-invariant coweight $\check{\mu} \in \mathfrak{h}^\nu$. By definition, we have

$$s^\nu_{\mathcal{I}}\check{\mu} = \check{\mu} - \langle \alpha^\nu_{\mathcal{I}}, \check{\mu}\rangle \check{\alpha}^\nu_{\mathcal{I}} = \check{\mu} - \frac{\ell_{\mathcal{I}}}{|\mathcal{I}|} \sum_{i,j\in\mathcal{I}} \langle \alpha_j, \check{\mu}\rangle \check{\alpha}_i = \check{\mu} - \ell_{\mathcal{I}} \sum_{i\in\mathcal{I}} \langle \alpha_i, \check{\mu}\rangle \check{\alpha}_i, \tag{3.8}$$

where in the last step we used the relation $\langle \alpha_j, \check{\mu}\rangle = \langle \alpha_i, \check{\mu}\rangle$ which follows from the ν-invariance of $\check{\mu}$ and the fact that i, j lie on the same orbit $\mathcal{I}$.

If $\ell_{\mathcal{I}} = 1$, then the right-hand side of (3.8) can be rewritten as $\big(\prod_{i\in\mathcal{I}} s_i\big)(\check{\mu})$ since $\langle \alpha_j, \check{\alpha}_i\rangle = 0$ for all distinct $i, j \in \mathcal{I}$. On the other hand, if $\ell_{\mathcal{I}} = 2$, then a direct computation using the fact that $\langle \alpha_i, \check{\alpha}_{\bar{\imath}}\rangle = \langle \alpha_{\bar{\imath}}, \check{\alpha}_i\rangle = -1$ yields $s_i s_{\bar{\imath}} s_i(\check{\mu}) = \check{\mu} - 2\langle \alpha_i, \check{\mu}\rangle \check{\alpha}_i - 2\langle \alpha_{\bar{\imath}}, \check{\mu}\rangle \check{\alpha}_{\bar{\imath}}$. The result now follows. □

The fundamental coweights of $\mathfrak{g}^\nu$, defined by $\alpha^\nu_{\mathcal{I}}(\check{\omega}^\nu_{\mathcal{J}}) = \delta_{\mathcal{I}\mathcal{J}}$ for all $\mathcal{I}, \mathcal{J} \in I/\nu$, are

$$\check{\omega}^\nu_{\mathcal{I}} := \sum_{i\in\mathcal{I}} \check{\omega}_i.$$

Proposition 3.7. *The automorphism ν preserves the $\mathbb{Z}$-grading (3.3) and fixes the principal $\mathfrak{sl}_2$. In particular, $\{p_{-1}, 2\check{\rho}, p_1\}$ is the principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}^\nu$. Moreover, each $\mathfrak{a}_i$ for $i \in E$ is ν-stable.*

Proof. Since $\mathrm{ht}(\nu(\alpha)) = \mathrm{ht}(\alpha)$ for any $\alpha \in \Phi$, it follows that ν stabilises the eigenspaces $\mathfrak{g}_i$, $i \in \mathbb{Z}$. Moreover, $\nu(\check{\omega}_i) = \check{\omega}_{\nu(i)}$ for all $i \in I$ since $\langle \alpha_j, \nu(\check{\omega}_i)\rangle = \langle \alpha_{\nu^{-1}(j)}, \check{\omega}_i\rangle = \delta_{\nu^{-1}(j)i} = \delta_{j\nu(i)} = \langle \alpha_j, \check{\omega}_{\nu(i)}\rangle$ for any $j \in I$. Therefore, $\check{\rho}$ is ν-invariant. By definition of p_{-1} and ν, we clearly have $\nu(p_{-1}) = p_{-1}$. It follows that

$$[p_1, p_{-1}] = 2\check{\rho} = 2\nu(\check{\rho}) = \nu\big([p_1, p_{-1}]\big) = [\nu(p_1), \nu(p_{-1})] = [\nu(p_1), p_{-1}],$$

so $\mathrm{ad}_{p_{-1}}(p_1 - \nu(p_1)) = 0$. As ν stabilises $\mathfrak{g}_1$, p_1 and $\nu(p_1)$ are both in $\mathfrak{g}_1$. Yet $\mathrm{ad}_{p_{-1}} : \mathfrak{g}_{i+1} \to \mathfrak{g}_i$ is injective for $i \geq 0$ and hence $p_1 = \nu(p_1)$.

Recall the expressions (3.1) and (3.2) for p_{-1} and $\check{\rho}$, respectively. Noting that the sum over $i \in I$ in these expressions can be rewritten as a double sum over orbits $\mathcal{I} \in I/\nu$ and elements $i \in \mathcal{I}$ in each orbit, we can write these as

$$p_{-1} = \sum_{\mathcal{I} \in I/\nu} F^{\nu}_{\mathcal{I}}, \qquad \check{\rho} = \sum_{\mathcal{I} \in I/\nu} \check{\omega}^{\nu}_{\mathcal{I}}.$$

It follows that $\{p_{-1}, 2\check{\rho}, p_1\}$ is the principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}^\nu$.

Let $X \in \mathfrak{a}$. Then $[\nu(X), p_1] = [\nu(X), \nu(p_1)] = \nu\big([X, p_1]\big) = 0$ so that $\nu(X) \in \mathfrak{a}$. Thus ν stabilises both $\mathfrak{a}$ and $\mathfrak{g}_i$ and hence also $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{g}_i$. □

As a result of Propositions 3.5 and 3.7, the constructions of Sects. 3.1 and 3.2 apply directly to the ν-invariant subalgebra $\mathfrak{g}^\nu$. In particular, $\mathfrak{a}^\nu := \mathfrak{a} \cap \mathfrak{g}^\nu = \ker(\mathrm{ad}_{p_1} : \mathfrak{g}^\nu \to \mathfrak{g}^\nu)$ is a $\mathbb{Z}_{\geq 1}$-graded abelian subalgebra in $\mathfrak{g}^\nu$ of dimension $\mathrm{rk}\,\mathfrak{g}^\nu$, and the corresponding multiset E^ν of exponents of $\mathfrak{g}^\nu$ forms a sub-multiset of E.

Since ν fixes p_{-1} and stabilises $\mathfrak{b}$, we can consider the affine subspace $p_{-1} + \mathfrak{b}^\nu \subset \mathfrak{g}^\nu$ of ν-invariant elements in $p_{-1} + \mathfrak{b}$. It is stabilised by the adjoint action of N^ν, the ν-invariant subgroup of N. We may therefore define the corresponding space of finite $\mathfrak{g}^\nu$-opers

$$\mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}^\nu} = (p_{-1} + \mathfrak{b}^\nu)/N^\nu.$$

If $X \in p_{-1} + \mathfrak{b}^\nu$, then we denote by $[X]_{\mathfrak{g}^\nu}$ the associated finite $\mathfrak{g}^\nu$-oper. In particular, we recover the notion of finite $\mathfrak{g}$-opers from Sect. 3.2 when $\nu = \mathrm{Id}$. For completeness, let us end by stating the analogs of Theorem 3.1, Proposition 3.2 and Theorem 3.4 for finite $\mathfrak{g}^\nu$-opers.

Theorem 3.8. *Every finite $\mathfrak{g}^\nu$-oper has a unique representative in $p_{-1} + \mathfrak{a}^\nu$.*

For any ν-invariant coweights $\check{\lambda}, \check{\mu} \in \mathfrak{h}^\nu$, we have $[p_{-1} - \check{\lambda}]_{\mathfrak{g}^\nu} = [p_{-1} - \check{\mu}]_{\mathfrak{g}^\nu}$ if and only if there exists $w \in W^\nu$ such that $\check{\lambda} = w(\check{\mu})$.

If $\check{\lambda} \in \mathfrak{h}^\nu$ is dominant, then $\mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g}^\nu, [\check{\lambda}]_{W^\nu}} \simeq W^\nu$.

4. Cyclotomic $\mathfrak{g}$-Opers and Canonical Representatives

We pick and fix, once and for all, a diagram automorphism $\nu : I \to I$ as in Sect. 3.3. Let $T \in \mathbb{Z}_{\geq 1}$ be a multiple of the order of ν and choose a primitive Tth-root of unity ω. It generates a copy of the cyclic group of order T under multiplication, which we denote

$$\Gamma := \langle \omega \rangle \subset \mathbb{C}^\times.$$

In this section, we will define an action of Γ on various spaces defined over the Riemann sphere, including the spaces of $\mathfrak{g}$-valued meromorphic functions, differentials and connections. Elements of these spaces which are invariant under this action will be called *cyclotomic*.

4.1. Cyclotomic $\mathfrak{g}$-Valued Functions and Differentials

Let $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and fix a global coordinate t on $\mathbb{C} \subset \mathbb{P}^1$. There is a natural action $\mu : \Gamma \to \operatorname{Aut} \mathbb{P}^1$, $\alpha \mapsto \mu_\alpha$ of the cyclic group Γ on $\mathbb{P}^1$ which fixes ∞ and with $\alpha \in \Gamma$ acting by multiplication $t \mapsto \alpha t$ on $\mathbb{C}$.

In this subsection and the next, we let $\upsilon \in \operatorname{Aut} \mathfrak{g}$ be any automorphism whose order divides T, i.e. such that $\upsilon^T = \mathrm{Id}$, and with diagram part ν. In other words, let υ be defined on the Chevalley–Serre generators as

$$\upsilon(E_i) = \tau_{\alpha_i} E_{\nu(i)}, \qquad \upsilon(\check{\alpha}_i) = \check{\alpha}_{\nu(i)}, \qquad \upsilon(F_i) = \tau_{\alpha_i}^{-1} F_{\nu(i)}, \tag{4.1}$$

where τ_{α_i}, $i \in I$, are arbitrary Tth-roots of unity. We obtain an action of the cyclic group Γ on $\mathfrak{g}$ by letting ω act as υ. That is, we have a homomorphism $\Gamma \to \operatorname{Aut} \mathfrak{g}$, $\omega \mapsto \upsilon$.

Let $\mathcal{M}$ denote the algebra of meromorphic functions on $\mathbb{P}^1$. We introduce an action of Γ on $\mathcal{M}$ by letting $\alpha \in \Gamma$ act through the pullback $\mu^*_{\alpha^{-1}} : \mathcal{M} \to \mathcal{M}$, namely sending $f \in \mathcal{M}$ to the function $\alpha.f := f \circ \mu_{\alpha^{-1}} \in \mathcal{M}$. Let $\Omega := \mathcal{M}\mathrm{d}t$ be the space of meromorphic differentials on $\mathbb{P}^1$. It too comes naturally equipped with an action of Γ, letting $\alpha \in \Gamma$ act also by the pullback $\mu^*_{\alpha^{-1}} : \Omega \to \Omega$.

Consider the Lie algebra $\mathfrak{g}(\mathcal{M}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{M}$, i.e. the set of $\mathfrak{g}$-valued meromorphic functions equipped with the pointwise Lie bracket. Define an action of Γ on $\mathfrak{g}(\mathcal{M})$ by combining the action of Γ on $\mathfrak{g}$ with the above action on $\mathcal{M}$. That is, for any $X \otimes f \in \mathfrak{g}(\mathcal{M})$ we set

$$\hat{\upsilon}(X \otimes f) := \upsilon(X) \otimes \mu^*_{\omega^{-1}} f. \tag{4.2}$$

We denote the subalgebra of Γ-invariants as

$$\mathfrak{g}^{\hat{\upsilon}}(\mathcal{M}) := \{h \in \mathfrak{g}(\mathcal{M}) \,|\, \hat{\upsilon}(h) = h\},$$

which we shall call the space of cyclotomic $\mathfrak{g}$-valued meromorphic functions. Given any υ-stable subspace $\mathfrak{p} \subset \mathfrak{g}$, we define the $\hat{\upsilon}$-stable subspace $\mathfrak{p}(\mathcal{M}) := \mathfrak{p} \otimes_{\mathbb{C}} \mathcal{M}$ of $\mathfrak{g}(\mathcal{M})$. We denote the corresponding subspace of Γ-invariants by

$$\mathfrak{p}^{\hat{\upsilon}}(\mathcal{M}) := \{h \in \mathfrak{p}(\mathcal{M}) \,|\, \hat{\upsilon}(h) = h\}.$$

Given any subspace $\mathfrak{p} \subset \mathfrak{g}$, we denote by $\Omega(\mathfrak{p}) := \mathfrak{p} \otimes_{\mathbb{C}} \Omega$ the space of $\mathfrak{p}$-valued meromorphic differentials. If $\mathfrak{p}$ is υ-stable, we equip $\Omega(\mathfrak{p})$ with an action of Γ defined by letting ω act through the pullback by $\mu_{\omega^{-1}}$ on the second tensor factor, namely

$$\hat{\upsilon}(X \otimes \varpi) := \upsilon(X) \otimes \mu^*_{\omega^{-1}} \varpi,$$

for any $X \in \mathfrak{p}$ and $\varpi \in \Omega$. We define the subspace

$$\Omega^{\hat{\upsilon}}(\mathfrak{p}) := \{\mathsf{A} \in \Omega(\mathfrak{p}) \,|\, \hat{\upsilon}(\mathsf{A}) = \mathsf{A}\}$$

of cyclotomic (i.e. Γ-invariant) $\mathfrak{p}$-valued meromorphic differentials. We shall often abbreviate the notation $X \otimes f$ for an element in $\mathfrak{p}(\mathcal{M})$ to Xf for simplicity, and similarly, an element $Y \otimes \varpi$ of $\Omega(\mathfrak{p})$ will be denoted simply as $Y\varpi$.

4.2. Cyclotomic $\mathfrak{g}$-Connections

Let $\mathrm{d} : \mathcal{M} \to \Omega$ denote the de Rham differential and consider the affine space

$$\mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1) := \{\mathrm{d} + \mathsf{A} \,|\, \mathsf{A} \in \Omega(\mathfrak{g})\}$$

of $\mathfrak{g}$-valued meromorphic connections on $\mathbb{P}^1$, or $\mathfrak{g}$-*connections* for short. More generally, given any subspace $\mathfrak{p} \subset \mathfrak{g}$ we define the affine subspace $\mathrm{Conn}_{\mathfrak{p}}(\mathbb{P}^1)$ of $\mathfrak{p}$-valued mermorphic connections of the form $\mathrm{d} + \mathsf{A}$ with $\mathsf{A} \in \Omega(\mathfrak{p})$. Similarly, we define the affine subspace

$$\mathrm{Conn}^{\hat{\upsilon}}_{\mathfrak{g}}(\mathbb{P}^1) := \{\mathrm{d} + \mathsf{A} \,|\, \mathsf{A} \in \Omega^{\hat{\upsilon}}(\mathfrak{g})\} \subset \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$$

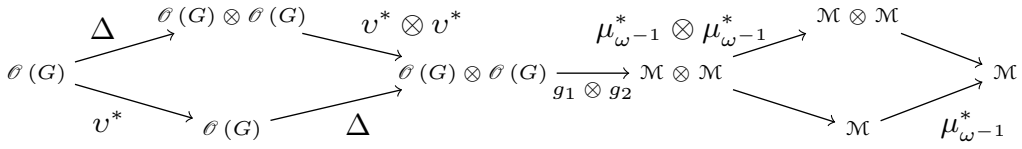
of cyclotomic $\mathfrak{g}$-valued meromorphic connections on $\mathbb{P}^1$, and if $\mathfrak{p} \subset \mathfrak{g}$ is a υ-stable subspace, we define the affine subspace $\mathrm{Conn}^{\hat{\upsilon}}_{\mathfrak{p}}(\mathbb{P}^1) \subset \mathrm{Conn}^{\hat{\upsilon}}_{\mathfrak{g}}(\mathbb{P}^1)$ of connections of the form $\mathrm{d} + \mathsf{A}$ with $\mathsf{A} \in \Omega^{\hat{\upsilon}}(\mathfrak{p})$. We shall refer to elements of $\mathrm{Conn}^{\hat{\upsilon}}_{\mathfrak{p}}(\mathbb{P}^1)$ as *cyclotomic* $\mathfrak{p}$*-connections*.

Consider the group $G(\mathcal{M})$ of $\mathcal{M}$-points of G, which can be thought of as the group of G-valued meromorphic functions on $\mathbb{P}^1$. It can be formally defined as the set of all $\mathbb{C}$-algebra homomorphisms $\mathscr{O}(G) \to \mathcal{M}$, where $\mathscr{O}(G)$ is the algebra of regular functions on G. The group structure on G makes $\mathscr{O}(G)$ into a commutative Hopf algebra with coproduct $\Delta : \mathscr{O}(G) \to \mathscr{O}(G \times G) \simeq \mathscr{O}(G) \otimes \mathscr{O}(G)$ given by $(\Delta\varphi)(x, y) = \varphi(xy)$. The counit and antipode maps are given by $\epsilon : \mathscr{O}(G) \to \mathbb{C}$, $\epsilon(\varphi) = \varphi(e)$ and $s : \mathscr{O}(G) \to \mathscr{O}(G)$, $s(\varphi)(x) = \varphi(x^{-1})$, respectively.

The Lie algebra of $G(\mathcal{M})$ is naturally isomorphic to $\mathfrak{g}(\mathcal{M})$. Lifting $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ to an automorphism $\upsilon \in \mathrm{Aut}\,G$ of G, cf. Sect. 2.2, this in turn induces an automorphism of $\mathscr{O}(G)$ via the pullback υ^*. In particular, when the order of υ divides T, we can endow $G(\mathcal{M})$ with an action of Γ. Specifically, we let ω act on $G(\mathcal{M})$ by sending any $g : \mathscr{O}(G) \to \mathcal{M}$ to the composition

$$\mathscr{O}(G) \xrightarrow{\upsilon^*} \mathscr{O}(G) \xrightarrow{g} \mathcal{M} \xrightarrow{\mu^*_{\omega^{-1}}} \mathcal{M},$$

which we write $\hat{\upsilon}(g) \in G(\mathcal{M})$. The map $\hat{\upsilon} : G(\mathcal{M}) \to G(\mathcal{M})$ so defined is an automorphism of $G(\mathcal{M})$ since the coproduct of $\mathscr{O}(G)$ and the multiplication in $\mathcal{M}$ are both Γ-equivariant. In other words, the following diagram



commutes for any $g_1, g_2 \in G(\mathcal{M})$, with the sequence of maps from left to right along the top of the diagram representing the element $\hat{\upsilon}(g_1)\hat{\upsilon}(g_2) \in G(\mathcal{M})$ and those along the bottom corresponding to $\hat{\upsilon}(g_1 g_2) \in G(\mathcal{M})$. The subgroup $G^{\hat{\upsilon}}(\mathcal{M})$ of Γ-invariant elements in $G(\mathcal{M})$ then consisting of all Γ-equivariant $\mathbb{C}$-algebra homomorphisms $g : \mathscr{O}(G) \to \mathcal{M}$, i.e. for which the diagram

$$\begin{array}{ccc} \mathscr{O}(G) & \xrightarrow{\quad g \quad} & \mathcal{M} \\ {\scriptstyle \upsilon^*}\downarrow & & \downarrow{\scriptstyle \mu_\omega^*} \\ \mathscr{O}(G) & \xrightarrow[\quad g \quad]{} & \mathcal{M} \end{array}$$

commutes. Given any $t \in \mathbb{P}^1$, by abuse of notation we write $g(t)$ for the composition $\mathrm{ev}_t \circ g$ with the evaluation at t map $\mathrm{ev}_t : \mathcal{M} \to \mathbb{P}^1$. We will say that an element $g \in G(\mathcal{M})$ is *regular* at $t \in \mathbb{P}^1$ if the composition $\mathrm{ev}_t \circ g \circ \varphi$ takes value in $\mathbb{C}$ for any $\varphi \in \mathscr{O}(G)$, so that $g(t) \in G(\mathbb{C}) \simeq G$. The property of Γ-equivariance of $g \in G^{\hat{\upsilon}}(\mathcal{M})$ may be expressed as $\upsilon\big(g(t)\big) = g(\omega^{-1}t)$. The Lie algebra of $G^{\hat{\upsilon}}(\mathcal{M})$ is naturally isomorphic to $\mathfrak{g}^{\hat{\upsilon}}(\mathcal{M})$.

Now for any $g \in G(\mathcal{M})$, we define the $\mathbb{C}$-linear map $dg : \mathscr{O}(G) \to \Omega$ by $\varphi \mapsto \mathrm{d}\big(g(\varphi)\big)$. In particular, we then have $dgg^{-1} \in \Omega(\mathfrak{g})$, where the product of dg with g^{-1} is defined as for $G(\mathcal{M})$ in terms of the coproduct on $\mathscr{O}(G)$. The affine space $\mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$ is equipped with an action of $G(\mathcal{M})$ by gauge transformations

$$\mathrm{d} + \mathsf{A} \longmapsto g(\mathrm{d} + \mathsf{A})g^{-1} := \mathrm{d} - dgg^{-1} + \mathrm{Ad}_g\, \mathsf{A}, \tag{4.3}$$

where $\mathrm{Ad} : G(\mathcal{M}) \to \mathsf{GL}\left(\Omega(\mathfrak{g})\right)$, $g \mapsto \mathrm{Ad}_g$ denotes the action of the adjoint group $G(\mathcal{M})$ on $\Omega(\mathfrak{g})$. If $g \in G^{\hat{\upsilon}}(\mathcal{M})$, then dg is seen to be Γ-equivariant so that $dgg^{-1} \in \Omega^{\hat{\upsilon}}(\mathfrak{g})$. It follows that (4.3) restricts to an action of $G^{\hat{\upsilon}}(\mathcal{M})$ on the affine subspace $\mathrm{Conn}_{\mathfrak{g}}^{\hat{\upsilon}}(\mathbb{P}^1)$ of cyclotomic $\mathfrak{g}$-connections.

We say that $\mathrm{d}+\mathsf{A} \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$ has a *pole* (resp. is *regular*) at some point $x \in \mathbb{P}^1$ if the $\mathfrak{g}$-valued differential $\mathsf{A} \in \Omega(\mathfrak{g})$ has a pole (resp. is regular) there. We define the residue of $\mathrm{d} + \mathsf{A}$ at $x \in \mathbb{P}^1$ as

$$\mathrm{res}_x(\mathrm{d} + \mathsf{A}) := \mathrm{res}_x\, \mathsf{A}.$$

Let $\{x_i\}_{i=1}^n \subset \mathbb{P}^1$ be the set of all poles of the $\mathfrak{g}$-connection $\mathrm{d} + \mathsf{A}$. Since a meromorphic connection on $\mathbb{P}^1$ is always flat, $\mathrm{d} + \mathsf{A}$ gives rise to a group homomorphism

$$M : \pi_1\big(\mathbb{P}^1 \backslash \{x_i\}_{i=1}^n\big) \longrightarrow G,$$

called the *monodromy representation*. To define this explicitly, we write $\mathsf{A} \in \Omega(\mathfrak{g})$ in coordinates as $\mathsf{A} = A(t)\mathrm{d}t$ with $A(t) \in \mathfrak{g}$. Given any path $\gamma : [0,1] \to \mathbb{P}^1 \backslash \{x_i\}_{i=1}^n$, the parallel transport of $\mathrm{d} + \mathsf{A}$ along γ is defined as the formal infinite sum, see e.g. [3],

$$\overleftarrow{P\exp} \int_0^1 \gamma^*\mathsf{A} := 1 + \sum_{n=1}^{\infty} \int \ldots \int_{1 \geq s_n \geq \ldots \geq s_1 \geq 0} A(\gamma(s_n)) \ldots A(\gamma(s_1)) ds_n \ldots ds_1, \tag{4.4}$$

where the product in each integrand is taken in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, and 1 is the identity in $U(\mathfrak{g})$. There is a natural pairing between $U(\mathfrak{g})$ and $\mathscr{O}(G)$ defined by regarding $\mathfrak{g}$ as the Lie algebra of left-invariant derivations of $\mathscr{O}(G)$ and setting $U(\mathfrak{g}) \times \mathscr{O}(G) \to \mathbb{C}$, $(u, \varphi) \mapsto (u.\varphi)(e)$. It can be shown, see, for instance, [3, Proposition 1.51 (1)&(2)], that the pairing of (4.4) with any element of $\mathscr{O}(G)$, defined using linearity, yields a finite sum and

gives rise to a well-defined homomorphism $\mathscr{O}(G) \to \mathbb{C}$, or in other words defines an element of $G(\mathbb{C}) \simeq G$. One can also show, cf. [3, Proposition 1.52], that (4.4) only depends on the homotopy class $[\gamma]$ of γ in $\mathbb{P}^1\backslash\{x_i\}_{i=1}^n$ with fixed start point at $\gamma(0)$ and end point at $\gamma(1)$. Moreover, for any $g \in G(\mathcal{M})$ the parallel transport of $g(\mathrm{d}+\mathsf{A})g^{-1}$ along γ is equal to $g(\gamma(1))\big(P\overleftarrow{\exp}\int_0^1 \gamma^*\mathsf{A}\big)g(\gamma(0))^{-1}$. Finally, by taking γ to be a closed path, i.e. $\gamma(0) = \gamma(1)$, the parallel transport (4.4) defines the monodromy $M([\gamma]) \in G$ of $\mathrm{d}+\mathsf{A}$ around γ.

The monodromy of $\mathrm{d}+\mathsf{A}$ at one of its poles x_i is then defined as $M([\gamma_i]) \in G$ where $[\gamma_i]$ is the homotopy class in $\mathbb{P}^1\backslash\{x_i\}_{i=1}^n$ of a small loop γ_i encircling the point x_i. The $\mathfrak{g}$-connection $\mathrm{d}+\mathsf{A}$ is said to have *trivial monodromy at* x_i if $M([\gamma_i]) = \mathrm{Id}$. Moreover, $\mathrm{d}+\mathsf{A}$ has *trivial monodromy* or is *monodromy-free* if the homomorphism M is trivial.[1]

Proposition 4.1. *Let* $\mathrm{d}+\mathsf{A} \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$ *be regular at* $x \in \mathbb{P}^1$. *If* $\mathrm{d}+\mathsf{A}$ *is monodromy-free, then there exists a unique solution* $Y \in G(\mathcal{M})$ *to* $dYY^{-1} = -\mathsf{A}$ *with* $Y(x) = \mathrm{Id}$. *Moreover, if there exists* $g \in G(\mathcal{M})$ *such that the gauge-transformed* $\mathfrak{g}$*-connection* $g(\mathrm{d}+\mathsf{A})g^{-1}$ *is regular at some point* $y \in \mathbb{P}^1$, *then* $\mathrm{d}+\mathsf{A}$ *has trivial monodromy at* y.

Proof. As noted above, the parallel transport (4.4) along any path $\gamma : [0,1] \to \mathbb{P}^1\backslash\{x_i\}_{i=1}^n$ depends only on the homotopy class of γ in $\mathbb{P}^1\backslash\{x_i\}_{i=1}^n$. If we suppose, moreover, that $\mathrm{d}+\mathsf{A}$ is monodromy-free, then (4.4) is actually independent of the path $\gamma : [0,1] \to \mathbb{P}^1\backslash\{x_i\}_{i=1}^n$ and depends only on its start and end points $\gamma(0)$ and $\gamma(1)$.

Let us fix the start point $\gamma(0)$ to be x and consider the end point $z := \gamma(1)$ to vary in $\mathbb{P}^1\backslash\{x_i\}_{i=1}^n$. It follows from the proof of [3, Proposition 1.56] that for any $\varphi \in \mathscr{O}(G)$, the pairing of (4.4) with φ depends holomorphically on z and has a finite order pole at each x_i, $i = 1, \dots, n$. Hence, this defines a homomorphism from $\mathscr{O}(G)$ to $\mathcal{M}$, i.e. an element $Y \in G(\mathcal{M})$. When $z = x$ we have $Y(x) = \mathrm{Id}$ and the fact that Y satisfies $\mathrm{d}YY^{-1} = -\mathsf{A}$ follows from [3, Proposition 1.51 (3)].

To show the last statement, suppose there exists $g \in G(\mathcal{M})$ such that $g(\mathrm{d}+\mathsf{A})g^{-1}$ is regular at $y \in \mathbb{P}^1$. The monodromy $M' \in G$ of $g(\mathrm{d}+\mathsf{A})g^{-1}$ at y is, by definition, the parallel transport of $g(\mathrm{d}+\mathsf{A})g^{-1}$ around a small loop γ encircling the point y. As such, M' only depends on the homotopy class $[\gamma]$ of γ in the Riemann sphere $\mathbb{P}^1$ with the singularities of $g(\mathrm{d}+\mathsf{A})g^{-1}$ removed. Since $g(\mathrm{d}+\mathsf{A})g^{-1}$ is regular at y, the loop γ is contractible on this punctured sphere. It then follows that $M' = \mathrm{Id}$. The monodromy of $\mathrm{d}+\mathsf{A}$ at y is then $M([\gamma]) = g(\gamma(1))^{-1}M'g(\gamma(0)) = \mathrm{Id}$, using the fact that $g(\gamma(0)) = g(\gamma(1))$. □

Given a $\mathfrak{g}$-connection $\nabla = \mathrm{d}+\mathsf{A} \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$ without monodromy, we shall say that $Y \in G(\mathcal{M})$ is a solution of the equation $\nabla Y = 0$ if $dYY^{-1} = -\mathsf{A}$. We will need the following straightforward corollary of Proposition 4.1 later.

[1] Note that more generally, if G is any affine algebraic group with Lie algebra $\mathfrak{g}$, the monodromy representation is said to be trivial if its image lies in the centre of G. Since we always take G to be the adjoint group associated with $\mathfrak{g}$, cf. Sect. 2.2, its centre is trivial and so monodromy-free will always mean trivial monodromy.

Lemma 4.2. *Let $\nabla \in \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1)$ be monodromy-free and regular at $x \in \mathbb{C}$. Let $Y \in G(\mathcal{M})$ be a solution of $\nabla Y = 0$ such that $Y(x) \in B_-$. Then, $\nabla \in \mathrm{Conn}_{\mathfrak{b}_-}(\mathbb{P}^1)$ if and only if $Y \in B_-(\mathcal{M})$.*

Proof. Suppose first that $\nabla \in \mathrm{Conn}_{\mathfrak{b}_-}(\mathbb{P}^1)$. Since ∇ is monodromy-free, Proposition 4.1 yields a solution $\widetilde{Y} \in B_-(\mathcal{M})$ of $\nabla \widetilde{Y} = 0$ with initial condition $\widetilde{Y}(x) = \mathrm{Id}$. The given solution $Y \in G(\mathcal{M})$ with initial condition $Y(x) \in B_-$ is related to it by $Y = \widetilde{Y}\, Y(x)$ and therefore also lives in $B_-(\mathcal{M})$.

Conversely, suppose we have $Y \in B_-(\mathcal{M})$. Let us write $\nabla = \mathrm{d} + \mathsf{A}$ with $\mathsf{A} \in \Omega(\mathfrak{g})$. Then, as noted above (4.3) we have $\mathsf{A} = -dY Y^{-1} \in \Omega(\mathfrak{b}_-)$. In other words, $\nabla \in \mathrm{Conn}_{\mathfrak{b}_-}(\mathbb{P}^1)$. □

4.3. Cyclotomic $\mathfrak{g}$-Opers

Recall the principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ defined in Sect. 3.1 and the associated notion of finite $\mathfrak{g}$-oper introduced in Sect. 3.2.

Consider the following subset of $\mathfrak{g}$-valued connections

$$\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1) := \left\{ \mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} \,\middle|\, \mathsf{v} \in \Omega(\mathfrak{b}) \right\} \subset \mathrm{Conn}_{\mathfrak{g}}(\mathbb{P}^1),$$

where we recall that $p_{-1}\mathrm{d}t$ is used as a shorthand notation for $p_{-1} \otimes \mathrm{d}t$. This set is stable under the gauge action of the unipotent subgroup $N(\mathcal{M}) \subset G(\mathcal{M})$. The space of $\mathfrak{g}$-*opers* is defined as the quotient space

$$\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1) := \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)/N(\mathcal{M}).$$

We shall denote by $[\nabla]$ the class of $\nabla \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$.

In order to define a cyclotomic version of the space $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$, it is natural to first seek an analog of the subspace $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ within the space of cyclotomic $\mathfrak{g}$-connections $\mathrm{Conn}^{\hat{\upsilon}}_{\mathfrak{g}}(\mathbb{P}^1)$. However, it is clear that for a generic automorphism $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ of the form (4.1), the $\mathfrak{g}$-valued differential $p_{-1}\mathrm{d}t$ does not live in the $\hat{\upsilon}$-invariant subspace $\Omega^{\hat{\upsilon}}(\mathfrak{n}_-)$. In fact, the requirement that $p_{-1}\mathrm{d}t \in \Omega^{\hat{\upsilon}}(\mathfrak{n}_-)$ fixes the automorphism υ uniquely to be equal to

$$\varsigma := \mathrm{Ad}_{\omega^{-\check{\rho}}} \circ \nu \in \mathrm{Aut}\,\mathfrak{g}. \tag{4.5}$$

Explicitly, the action of this automorphism on the Chevalley–Serre generators is given by

$$\varsigma(E_i) = \omega^{-1} E_{\nu(i)}, \qquad \varsigma(\check{\alpha}_i) = \check{\alpha}_{\nu(i)}, \qquad \varsigma(F_i) = \omega F_{\nu(i)}.$$

In the remainder of this section, we shall specialise the constructions of Sects. 4.1 and 4.2 for an arbitrary automorphism υ of the form (4.1) to the specific automorphism ς defined by (4.5). We will need the following properties of the latter.

Proposition 4.3. *The automorphism ς preserves the $\mathbb{Z}$-grading (3.3) and its action on the principal $\mathfrak{sl}_2$-triple is given by $\varsigma(p_{\pm 1}) = \omega^{\mp 1} p_{\pm 1}$ and $\varsigma(\check{\rho}) = \check{\rho}$. Moreover, each $\mathfrak{a}_i$, $i \in E$, is ς-stable.*

Proof. This is a direct consequence of Proposition 3.7 and the definition (4.5) of ς. □

It follows from Proposition 4.3 that $\hat{\varsigma}(p_{-1}\mathrm{d}t) = p_{-1}\mathrm{d}t$ and therefore $p_{-1}\mathrm{d}t \in \Omega^{\hat{\varsigma}}(\mathfrak{n}_-)$, as required. We emphasise again that the automorphism $\varsigma \in \operatorname{Aut}\mathfrak{g}$ defined in (4.5) is the unique automorphism of the form (4.1) with this property.

We may now consider the following subset of Γ-invariant $\mathfrak{g}$-valued connections

$$\mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1) := \left\{\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} \,\middle|\, \mathsf{v} \in \Omega^{\hat{\varsigma}}(\mathfrak{b})\right\} \subset \mathrm{Conn}^{\hat{\varsigma}}_{\mathfrak{g}}(\mathbb{P}^1). \tag{4.6}$$

It is stable under the gauge action of the unipotent subgroup $N^{\hat{\varsigma}}(\mathcal{M}) \subset G^{\hat{\varsigma}}(\mathcal{M})$ on $\mathrm{Conn}^{\hat{\varsigma}}_{\mathfrak{g}}(\mathbb{P}^1)$. We define the space of *cyclotomic* $\mathfrak{g}$*-opers* as the quotient space

$$\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1) := \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)/N^{\hat{\varsigma}}(\mathcal{M}).$$

The class of ∇ in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ is denoted by $[\nabla]_\Gamma$. We note that the space $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ only depends on the diagram automorphism $\nu : I \to I$ and the choice of Tth-root of unity ω through the automorphism ς in (4.5). Any cyclotomic $\mathfrak{g}$-connection $\nabla \in \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ is also an element of $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$. As such, we can also consider its orbit in $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ under the action of $N(\mathcal{M})$, namely its class $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$. Moreover, it is clear that if $\nabla, \nabla' \in \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ are such that $[\nabla]_\Gamma = [\nabla']_\Gamma$ then $[\nabla] = [\nabla']$. This gives rise to a map

$$\begin{aligned} \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1) &\longrightarrow \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1), \\ [\nabla]_\Gamma &\longmapsto [\nabla]. \end{aligned} \tag{4.7}$$

It follows from Theorem 4.5 that this map is injective.

4.4. Canonical Representatives

Recall from Theorem 3.1 that each finite $\mathfrak{g}$-oper admits a unique representative in the Slodowy slice $p_{-1} + \mathfrak{a}$. Similarly, it is well known that a transverse slice in the space $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$ is given by the so-called Drinfel'd–Sokolov gauge [4]. We will prove a cyclotomic version of this result in Theorem 4.5.

The following lemma can be regarded as a (cyclotomic) affine analog of equation (3.4).

Lemma 4.4. *Let* $\mathsf{w} \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_i)$.

If $i \notin E$, *then there exists a unique* $m \in \mathfrak{g}^{\hat{\varsigma}}_{i+1}(\mathcal{M})$ *such that* $\mathsf{w} = [m, p_{-1}\mathrm{d}t]$.

If $i \in E$, *then there exists unique* $m \in \mathfrak{g}^{\hat{\varsigma}}_{i+1}(\mathcal{M})$ *and* $\mathsf{c} \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_i)$ *such that* $\mathsf{w} = [m, p_{-1}\mathrm{d}t] + \mathsf{c}$.

Proof. We will show this for $i \in E$, the case $i \notin E$ being similar. Suppose $\mathsf{w} \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_i)$. By definition, we can write w as a finite sum of terms of the form $X_j \otimes \varpi_j$ with $X_j \in \mathfrak{g}_i$ and $\varpi_j \in \Omega$. Applying the decomposition (3.4) to each X_j, we may write

$$\mathsf{w} = [m, p_{-1}\mathrm{d}t] + \mathsf{c} \tag{4.8}$$

for some unique $\mathsf{c} \in \Omega(\mathfrak{a}_i)$ and $[m, p_{-1}\mathrm{d}t] \in [\mathfrak{g}_{i+1}(\mathcal{M}), p_{-1}\mathrm{d}t]$. In turn, $m \in \mathfrak{g}_{i+1}(\mathcal{M})$ is then also unique by the injectivity of $\mathrm{ad}_{p_{-1}} : \mathfrak{g}_{i+1} \to \mathfrak{g}_i$ for $i \geq 0$. Applying $\hat{\varsigma}$ to both sides of (4.8), we obtain $\mathsf{w} = \left[\hat{\varsigma}(m), p_{-1}\mathrm{d}t\right] + \hat{\varsigma}(\mathsf{c})$ using the fact that w and $p_{-1}\mathrm{d}t$ are both $\hat{\varsigma}$-stable. It now follows by the uniqueness

of c and m in (4.8) together with the fact that $\mathfrak{a}_i$ and $\mathfrak{g}_{i+1}$ are ς-stable by Proposition 4.3, that $\mathsf{c} \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_i)$ and $m \in \mathfrak{g}^{\hat{\varsigma}}_{i+1}(\mathcal{M})$, as required. □

Theorem 4.5. *The action of $N^{\hat{\varsigma}}(\mathcal{M})$ on $\mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ (resp. of $N(\mathcal{M})$ on $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$) is free. Moreover, every cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ (resp. $\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$) has a unique representative of the form $\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{c}$ with $\mathsf{c} \in \Omega^{\hat{\varsigma}}(\mathfrak{a})$ (resp. $\mathsf{c} \in \Omega(\mathfrak{a})$). We shall call it the* canonical representative *of $[\nabla]_\Gamma$ (resp. $[\nabla]$).*

Proof. We consider only the case of a cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ since the proof in the case of a $\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$ is standard [4] and completely analogous.

Let $\nabla = \mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} \in \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$. We want to find $m \in \mathfrak{n}^{\hat{\varsigma}}(\mathcal{M})$ such that

$$\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} = e^{m}(\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{c})e^{-m} \tag{4.9}$$

for some $\mathsf{c} \in \Omega^{\hat{\varsigma}}(\mathfrak{a})$. Let us decompose m, v and c as follows

$$m = \sum_{i>0} m_i, \qquad \mathsf{v} = \sum_{i\geq 0} \mathsf{v}_i, \qquad \mathsf{c} = \sum_{i\in E} \mathsf{c}_i \tag{4.10}$$

with $m_i \in \mathfrak{g}^{\hat{\varsigma}}_i(\mathcal{M})$, $\mathsf{v}_i \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_i)$ and $\mathsf{c}_i \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_i)$. We can then write the three different terms on the right-hand side of (4.9) as

$$\mathrm{d} - \mathrm{d}(e^{m})e^{-m} = \mathrm{d} - \sum_{i>0} \mathrm{d}m_i - \frac{1}{2}\sum_{i,j>0}[m_j, \mathrm{d}m_i] - \frac{1}{6}\sum_{i,j,k}[m_k,[m_j,\mathrm{d}m_i]] + \dots$$

$$e^{\mathrm{ad}_m} p_{-1}\mathrm{d}t = p_{-1}\mathrm{d}t + \sum_{i>0}[m_i, p_{-1}\mathrm{d}t] + \frac{1}{2}\sum_{i,j>0}[m_j,[m_i,p_{-1}\mathrm{d}t]] + \dots$$

$$e^{\mathrm{ad}_m}\mathsf{c} = \sum_{i\in E}\mathsf{c}_i + \sum_{i\in E}\sum_{j>0}[m_j,\mathsf{c}_i] + \frac{1}{2}\sum_{i\in E}\sum_{j,k>0}[m_k,[m_j,\mathsf{c}_i]] + \dots$$

where the dots represent only finitely many terms since $m \in \mathfrak{n}^{\hat{\varsigma}}(\mathcal{M})$ is nilpotent. Here, we note that $\mathrm{Ad}_{e^m} = e^{\mathrm{ad}_m}$.

We determine m_i and c_i inductively. For this, we note that if $\mathsf{w} \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_j)$ for some $j \in \mathbb{Z}$ then

$$\big[m_{j_1},[m_{j_2},\dots[m_{j_n},\mathsf{w}]\dots]\big] \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_{j+j_1+\dots+j_n}), \tag{4.11}$$

for any $j_1,\dots,j_n \in \mathbb{Z}$. The terms of degree $i \in \mathbb{Z}_{\geq 0}$ in equation (4.9) then read

$$\mathsf{c}_i + [m_{i+1}, p_{-1}\mathrm{d}t] = \mathsf{v}_i + F_i\big(\{m_{j+1}, \mathrm{d}m_{j+1}, \mathsf{c}_j\}_{j<i}\big) \tag{4.12}$$

where F_i is a sum of commutators of the form (4.11) with $\mathsf{w} = \mathsf{c}_k$, $\mathrm{d}m_k$ or $p_{-1}\mathrm{d}t$ and hence lives in $\Omega^{\hat{\varsigma}}(\mathfrak{g}_i)$. When $i = 0$, the relation (4.12) simply says that $[m_1, p_{-1}\mathrm{d}t] = \mathsf{v}_0$. And since $\mathsf{v}_0 \in \Omega^{\hat{\varsigma}}(\mathfrak{g}_0)$, Lemma 4.4 yields $m_1 \in \mathfrak{g}^{\hat{\varsigma}}_1(\mathcal{M})$ uniquely. Now suppose that we have constructed the $\mathsf{c}_j \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_j)$ and $m_{j+1} \in \mathfrak{g}^{\hat{\varsigma}}_{j+1}(\mathcal{M})$ for $j < i$. By applying Lemma 4.4, we can then construct, in a unique way, $m_{i+1} \in \mathfrak{g}^{\hat{\varsigma}}_{i+1}(\mathcal{M})$ and $\mathsf{c}_i \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_i)$ for all $i \geq 1$ (with $\mathsf{c}_i = 0$ if $i \notin E$). In particular, the freeness of the action of $N^{\hat{\varsigma}}(\mathcal{M})$ on $\mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ follows from the uniqueness of $m_{i+1} \in \mathfrak{g}^{\hat{\varsigma}}_{i+1}(\mathcal{M})$. □

The proof of Theorem 4.5 provides an algorithm for constructing all the c_i and m_i from the v_i. In particular, a formula for c_1 can be obtained by considering the first few steps of the algorithm. For this, we need the explicit form of (4.12) for $i = 0, 1$, which read

$$[m_1, p_{-1}\mathrm{d}t] = \mathsf{v}_0, \tag{4.13a}$$

$$\mathsf{c}_1 + [m_2, p_{-1}\mathrm{d}t] = \mathsf{v}_1 - \tfrac{1}{2}\big[m_1, [m_1, p_{-1}\mathrm{d}t]\big] + \mathrm{d}m_1. \tag{4.13b}$$

Proposition 4.6. *Write* $\mathsf{v}_i = v_i\mathrm{d}t$ *with* $v_i \in \mathfrak{g}_i(\mathcal{M})$ *for* $i \in \mathbb{Z}_{\geq 0}$. *Then,* $\mathsf{c}_1 = u_1 p_1 \mathrm{d}t$ *with*

$$u_1 = \frac{1}{2(\check{\rho}|\check{\rho})}\big(\tfrac{1}{2}(v_0|v_0) + (\check{\rho}|v_0') + (p_{-1}|v_1)\big).$$

Proof. Let $\mathsf{w}_1 = w_1\mathrm{d}t$ with $w_1 \in \mathfrak{g}_1(\mathcal{M})$ denote the left-hand side of equation (4.13b). Then,

$$\begin{aligned}(p_{-1}|w_1) &= u_1(p_{-1}|p_1) + \big(p_{-1}\big|[m_2, p_{-1}]\big)\\ &= u_1(p_{-1}|p_1) - \big([p_{-1}, p_{-1}]\big|m_2\big) = 2u_1(\check{\rho}|\check{\rho}),\end{aligned}$$

where in the last step we used $(p_{-1}|p_1) = (p_{-1}|[\check{\rho}, p_1]) = ([p_1, p_{-1}]|\check{\rho}) = 2(\check{\rho}|\check{\rho})$. It now follows that

$$u_1 = \frac{(p_{-1}|w_1)}{2(\check{\rho}|\check{\rho})}.$$

Similarly, equation (4.13a) implies $(p_{-1}|m_1) = ([p_{-1}, \check{\rho}]|m_1) = (\check{\rho}|[m_1, p_{-1}]) = (\check{\rho}|v_0)$. So taking the derivative with respect to t yields $(p_{-1}|m_1') = (\check{\rho}|v_0')$. Next, we also have

$$\big(p_{-1}\big|\big[m_1, [m_1, p_{-1}]\big]\big) = -\big([m_1, p_{-1}]|[m_1, p_{-1}]\big) = -(v_0|v_0).$$

Equation (4.13b) then implies $(p_{-1}|w_1) = \frac{1}{2}(v_0|v_0) + (\check{\rho}|v_0') + (p_{-1}|v_1)$, as required. □

4.5. Regular Points and Regular Singularities

We shall say that a (cyclotomic) $\mathfrak{g}$-connection $\nabla = \mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} \in \mathrm{op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ is *regular* at $x \in \mathbb{C}$ if $\mathsf{v} \in \Omega^{\hat{\varsigma}}(\mathfrak{b})$ has no pole at x.

Proposition 4.7. *Let* $\nabla \in \mathrm{op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ *and* $x \in \mathbb{C}$. *The following are equivalent:*

(*i*) *the canonical representative of* $[\nabla]_\Gamma$ *is regular at* x
(*ii*) *there exists a representative of* $[\nabla]_\Gamma$ *which is regular at* x
(*iii*) *there exists a representative of* $[\nabla]$ *which is regular at* x

In this case, we will say that the cyclotomic $\mathfrak{g}$*-oper* $[\nabla]_\Gamma$ *itself is* regular *at the point* x.

Proof. It is obvious that (*i*) implies (*ii*) and (*iii*). To see that (*ii*) implies (*i*), we have to look back at the construction of the canonical representative (cf. Theorem 4.5).

Let $\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v} \in \mathrm{op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ be regular at x. In the inductive step of the proof of Theorem 4.5, if the right-hand side of (4.12) is regular at x, then so are c_i and m_{i+1} appearing on the left-hand side. We therefore conclude that the canonical representative of $[\mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{v}]_\Gamma$ is regular at x. Now if ∇ and

$\mathrm{d}+p_{-1}\mathrm{d}t+\mathsf{v}$ lie in the same $N^{\hat{\varsigma}}(\mathcal{M})$-orbit, then they share the same canonical representative, which proves that (ii) implies (i).

Finally, we prove that (iii) implies (i). Let $\mathrm{d}+p_{-1}\mathrm{d}t+\mathsf{c}$ and $\mathrm{d}+p_{-1}\mathrm{d}t+\tilde{\mathsf{c}}$ denote the canonical representatives of $[\nabla]_\Gamma$ and $[\nabla]$, respectively, which exist and are unique by Theorem 4.5. As $N^{\hat{\varsigma}}(\mathcal{M})$ is a subgroup of $N(\mathcal{M})$, $\mathrm{d}+p_{-1}\mathrm{d}t+\mathsf{c}$ is also a representative of $[\nabla]$ with $\mathsf{c}\in\Omega^{\hat{\varsigma}}(\mathfrak{a})\subset\Omega(\mathfrak{a})$. Yet, $\mathrm{d}+p_{-1}\mathrm{d}t+\tilde{\mathsf{c}}$ is the unique such representative of $[\nabla]$, hence $\tilde{\mathsf{c}}=\mathsf{c}$. Now, if there is a $\mathfrak{g}$-connection in the $N(\mathcal{M})$-orbit of ∇ which is regular at $x\in\mathbb{C}$, then the same argument as in the above proof that (ii) implies (i) applies so that $\tilde{\mathsf{c}}$ is regular at x, which concludes the proof since $\tilde{\mathsf{c}}=\mathsf{c}$. □

At any point on $\mathbb{P}^1$ where a cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma\in\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ is not regular, the mildest possible singularity it can have is a *regular singularity*. In the remainder of this section we define the notion of regular singularity at a nonzero finite point $x\in\mathbb{C}^\times$ and then at the origin and infinity. We shall say that $[\nabla]_\Gamma$ has *at most a regular singularity* at $x\in\mathbb{P}^1$ if it is either regular at x or has a regular singularity there.

Suppose for the moment that $x\in\mathbb{C}^\times$ and consider the subset of $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ given by

$$\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x := \left\{\mathrm{d}+p_{-1}\mathrm{d}t+\sum_{i\geq 0}(t-x)^{-i-1}\,\mathsf{v}_i\in\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)\,\middle|\,\mathsf{v}_i\in\Omega(\mathfrak{g}_i)\text{ regular at }x\right\}.$$

Let $\mathcal{O}_x\subset\mathcal{M}$ be the algebra of meromorphic functions on $\mathbb{P}^1$ which are regular at x. One checks using the definition of the $\mathbb{Z}$-grading in Sect. 3.1 that elements of $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ can equivalently be characterised as $\mathfrak{g}$-connections $\nabla\in\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ whose gauge transformation by $(t-x)^{\check{\rho}}$ takes the form

$$(t-x)^{\check{\rho}}\,\nabla\,(t-x)^{-\check{\rho}}=\mathrm{d}+\frac{1}{t-x}(p_{-1}-\check{\rho}+v)\mathrm{d}t \tag{4.14}$$

with $v=\sum_{i\geq 0}v_i\in\mathfrak{b}(\mathcal{O}_x)$. It is clear that the space $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ is stabilised by the gauge action of the group $(t-x)^{-\check{\rho}}N(\mathcal{O}_x)(t-x)^{\check{\rho}}$. We define the space $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ of *$\mathfrak{g}$-opers with regular singularity at x* as the corresponding quotient

$$\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x:=\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x\Big/(t-x)^{-\check{\rho}}N(\mathcal{O}_x)(t-x)^{\check{\rho}}.$$

Since $(t-x)^{-\check{\rho}}N(\mathcal{O}_x)(t-x)^{\check{\rho}}\subset N(\mathcal{M})$, we obtain a well-defined canonical map

$$\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x\longrightarrow\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1) \tag{4.15}$$

for each $x\in\mathbb{C}^\times$, which sends the class in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ of a $\mathfrak{g}$-connection $\nabla\in\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x\subset\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ to its class $[\nabla]$ in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$. By virtue of the next lemma, we will usually identify a class in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ with its image under (4.15).

Lemma 4.8. *The map (4.15) is injective. Moreover, its image consists of $\mathfrak{g}$-opers whose canonical representative $\mathrm{d}+p_{-1}\mathrm{d}t+\sum_{i\in E}\mathsf{c}_i$ is such that for each $i\in E$, $\mathsf{c}_i\in\Omega(\mathfrak{a}_i)$ has a pole of order at most $i+1$ at x.*

We will say that $x \in \mathbb{C}^\times$ *is a* regular singularity *of a cyclotomic* $\mathfrak{g}$*-oper* $[\nabla]_\Gamma \in \mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$ *if it is not regular there, cf. Proposition* 4.7*, and if the corresponding* $\mathfrak{g}$*-oper* $[\nabla]$ *lies in the image of* (4.15)*.*

Proof. In order to prove the lemma, it suffices to show that for any $\nabla \in \mathrm{op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$, the canonical representative of $[\nabla] \in \mathrm{Op}_\mathfrak{g}(\mathbb{P}^1)$ also lives in $\mathrm{op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$ and that the gauge transformation that brings ∇ to its canonical representative belongs to $(t-x)^{-\check\rho} N(\mathcal{O}_x)(t-x)^{\check\rho}$.

Let $\nabla = \mathrm{d} + p_{-1}\mathrm{d}t + \sum_{i\geq 0}(t-x)^{-i-1}\,\mathsf{v}_i \in \mathrm{op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$. The canonical representative of the $\mathfrak{g}$-oper $[\nabla]$ is obtained by the algorithm described in the proof of Theorem 4.5. Recall that one proceeds by induction on the degree by solving equation (4.12), where v_i there is replaced by $(t-x)^{-i-1}\,\mathsf{v}_i$ in the present case. As the latter has a pole of order at most $i+1$ at x, one can check by induction that, in this construction, m_i as a pole at most of order i and c_i as a pole at most of order $i+1$. By definition, this means that $e^{\sum_{i>0} m_i} \in (t-x)^{-\check\rho} N(\mathcal{O}_x)(t-x)^{\check\rho}$ and $\mathrm{d} + p_{-1}\mathrm{d}t + \sum_{i\geq 0}\mathsf{c}_i \in \mathrm{op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$, as required. □

Note that the gauge action by $(t-x)^{\check\rho}$ on classes in $\mathrm{Op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$ induces an obvious bijection

$$\mathrm{Op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x \xrightarrow{\sim} \left\{ \mathrm{d} + \frac{1}{t-x}\big(p_{-1} - \check\rho + v\big)\mathrm{d}t \,\middle|\, v \in \mathfrak{b}(\mathcal{O}_x) \right\} \Big/ N(\mathcal{O}_x).$$

The representative $\mathrm{d} + \frac{1}{t-x}\big(p_{-1} - \check\rho + v\big)\mathrm{d}t$ of a class in the latter quotient has residue $p_{-1} - \check\rho + v(x)$. Under the gauge action by an element $g \in N(\mathcal{O}_x)$, this residue transforms under the adjoint action of $g(x) \in N$. This yields a notion of *residue* at $x \in \mathbb{C}^\times$ for elements of $\mathrm{Op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x$ defined by

$$\begin{aligned} \mathrm{res}_x : \mathrm{Op}_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_x &\longrightarrow \mathrm{Op}^{\mathrm{fin}}_\mathfrak{g}, \\ [\nabla] &\longmapsto [p_{-1} - \check\rho + v(x)]_\mathfrak{g}, \end{aligned} \tag{4.16}$$

where $v \in \mathfrak{b}(\mathcal{O}_x)$ is defined in terms of ∇ through (4.14).

We now turn to the subset $\{0,\infty\} \subset \mathbb{P}^1$ of fixed points under the action $\mu : \Gamma \to \mathrm{Aut}\,\mathbb{P}^1$ introduced in Sect. 4.1. For each $z \in \{0,\infty\}$, we consider the subset of cyclotomic $\mathfrak{g}$-connections defined by

$$\mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_z := \left\{ \mathrm{d} + p_{-1}\mathrm{d}t + \sum_{i\geq 0} t^{-i-1}\,\mathsf{v}_i \in \mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1) \,\middle|\, \mathsf{v}_i \in \Omega(\mathfrak{g}_i) \text{ regular at } z \right\}.$$

By definition of the $\mathbb{Z}$-grading, an element of $\mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_z$ can alternatively be seen as a cyclotomic $\mathfrak{g}$-connection $\nabla \in \mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$ whose gauge transformation by $t^{\check\rho}$ reads

$$t^{\check\rho}\,\nabla\,t^{-\check\rho} = \mathrm{d} + \frac{1}{t}(p_{-1} - \check\rho + v)\mathrm{d}t \tag{4.17}$$

with $v = \sum_{i\geq 0} v_i \in \mathfrak{b}^{\hat\nu}(\mathcal{O}_z)$. The space $\mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_z$ is stabilised by the gauge action of the subgroup $t^{-\check\rho} N^{\hat\nu}(\mathcal{O}_z) t^{\check\rho} \subset N^{\hat\varsigma}(\mathcal{M})$. The corresponding quotient

$$\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_z := \mathrm{op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_z \Big/ t^{-\check\rho} N^{\hat\nu}(\mathcal{O}_z) t^{\check\rho},$$

defines the space of *cyclotomic* $\mathfrak{g}$*-opers with regular singularity at* z.

For each $z \in \{0, \infty\}$, we have a canonical map

$$\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z \longrightarrow \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1) \tag{4.18}$$

which sends the class in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z$ of a cyclotomic $\mathfrak{g}$-connection $\nabla \in \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z \subset \mathrm{op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ to its class $[\nabla]_\Gamma$ in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$. This is well defined since $t^{-\check{\rho}} N^{\hat{\nu}}(\mathcal{O}_z) t^{\check{\rho}} \subset N^{\hat{\varsigma}}(\mathcal{M})$. The following lemma is proved in exactly the same way as Lemma 4.8. Using this lemma, we will usually identify a class in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z$ with its image under (4.18).

Lemma 4.9. *The map* (4.18) *is injective. Moreover, its image consists of cyclotomic* $\mathfrak{g}$*-opers whose canonical representative* $\mathrm{d} + p_{-1}\mathrm{d}t + \sum_{i \in E} \mathsf{c}_i$ *is such that for each* $i \in E$, $\mathsf{c}_i \in \Omega^{\hat{\varsigma}}(\mathfrak{a}_i)$ *has a pole of order at most* $i+1$ *at the origin (if* $z = 0$*) or a zero of order at least* $i - 1$ *at infinity (if* $z = \infty$*).*

We will say that $z \in \{0, \infty\}$ *is a* regular singularity *of a cyclotomic* $\mathfrak{g}$*-oper* $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ *if it is not regular there, cf. Proposition 4.7, and it lies in the image of* (4.18).

The gauge action by $t^{\check{\rho}}$ on classes in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z$ induces a bijection

$$\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z \xrightarrow{\sim} \left\{ \mathrm{d} + \frac{1}{t}(p_{-1} - \check{\rho} + v)\mathrm{d}t \,\middle|\, v \in \mathfrak{b}^{\hat{\nu}}(\mathcal{O}_z) \right\} \Big/ N^{\hat{\nu}}(\mathcal{O}_z).$$

When $z = 0$ (resp. $z = \infty$), if we gauge transform the $\hat{\nu}$-invariant $\mathfrak{g}$-connection $\mathrm{d} + \frac{1}{t}(p_{-1} - \check{\rho} + v)\mathrm{d}t$ by an element $g \in N^{\hat{\nu}}(\mathcal{O}_0)$ (resp. $g \in N^{\hat{\nu}}(\mathcal{O}_\infty)$), then its residue $p_{-1} - \check{\rho} + v(0) \in p_{-1} + \mathfrak{b}^{\nu}$ (resp. $-p_{-1} + \check{\rho} - v(\infty) \in -p_{-1} + \mathfrak{b}^{\nu}$) transforms by the adjoint action of $g(0) \in N^{\nu}$ (resp. $g(\infty) \in N^{\nu}$). It is therefore natural to define the notion of *residue* at z on an element in $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_z$ to be (minus) a finite $\mathfrak{g}^{\nu}$-oper. Specifically, we define

$$\begin{aligned} \mathrm{res}_0 : \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_0 &\longrightarrow \mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}^{\nu}}, \\ [\nabla]_\Gamma &\longmapsto [p_{-1} - \check{\rho} + v(0)]_{\mathfrak{g}^{\nu}}, \end{aligned} \tag{4.19a}$$

for the origin, where $v \in \mathfrak{b}^{\hat{\nu}}(\mathcal{O}_0)$ is defined by (4.17). At infinity, we define instead

$$\begin{aligned} \mathrm{res}_\infty : \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_\infty &\longrightarrow -\mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}^{\nu}}, \\ [\nabla]_\Gamma &\longmapsto -[p_{-1} - \check{\rho} + v(\infty)]_{\mathfrak{g}^{\nu}}, \end{aligned} \tag{4.19b}$$

where $-\mathrm{Op}^{\mathrm{fin}}_{\mathfrak{g}^{\nu}}$ denotes the set $(-p_{-1} + \mathfrak{b}^{\nu})/N^{\nu}$ and $v \in \mathfrak{b}^{\hat{\nu}}(\mathcal{O}_\infty)$ is defined also by (4.17).

Note that if a cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ is regular at $x \in \mathbb{C}^\times$ then the corresponding $\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$, defined using the map (4.7), belongs to $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ [recall that we identify this space with its image under the injection (4.15)]. Likewise, if $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ is regular at the origin, then it belongs to $\mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_0$.

Lemma 4.10. *If* $[\nabla]_\Gamma \in \mathrm{Op}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ *is regular at* $x \in \mathbb{C}^\times$ *(resp. at the origin), then* $\mathrm{res}_x[\nabla]_\Gamma = [0]_W$ *(resp.* $\mathrm{res}_0[\nabla]_\Gamma = [0]_{W^{\nu}}$*).*

Proof. Let $\mathrm{d}+p_{-1}\mathrm{d}t+\sum_{i\in E}\mathsf{c}_i$ denote the canonical representative of $[\nabla]_\Gamma$, so that $\mathsf{c}_i\in\Omega(\mathfrak{a}_i)$ are all regular at x, cf. Proposition 4.7(i). Then,

$$(t-x)^{\check\rho}\bigg(\mathrm{d}+p_{-1}\mathrm{d}t+\sum_{i\in E}\mathsf{c}_i\bigg)(t-x)^{-\check\rho}=\mathrm{d}+\frac{1}{t-x}\bigg((p_{-1}-\check\rho)\mathrm{d}t+\sum_{i\in E}(t-x)^{i+1}\mathsf{c}_i\bigg),$$

from which the result follows. □

4.6. Cyclotomic Miura $\mathfrak{g}$-Opers

Recall the notion of a finite Miura $\mathfrak{g}$-oper from Sect. 3.2.

We will call *Miura* $\mathfrak{g}$*-oper* any $\mathfrak{g}$-connection of the form $\nabla=\mathrm{d}+p_{-1}\mathrm{d}t+\mathsf{u}$ where $\mathsf{u}\in\Omega(\mathfrak{h})$. Let $\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ denote the set of all Miura $\mathfrak{g}$-opers. Given a Miura $\mathfrak{g}$-oper $\nabla\in\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$, we will refer to $[\nabla]$ as the underlying $\mathfrak{g}$-oper.

Similarly, we define a *cyclotomic Miura* $\mathfrak{g}$*-oper* as a $\mathfrak{g}$-connection of the form $\nabla=\mathrm{d}+p_{-1}\mathrm{d}t+\mathsf{u}$ with $\mathsf{u}\in\Omega^{\hat\varsigma}(\mathfrak{h})=\Omega^{\hat\nu}(\mathfrak{h})$, where the last equality follows from the definition (4.5) of $\varsigma\in\mathrm{Aut}\,\mathfrak{g}$ and the fact that $\mathfrak{h}=\mathfrak{g}_0$ in the $\mathbb{Z}$-grading of Sect. 3.1 defined by $\mathrm{ad}_{\check\rho}$. Denote by $\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$, the set of all cyclotomic Miura $\mathfrak{g}$-opers. If $\nabla\in\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$, then we call $[\nabla]_\Gamma$ its underlying cyclotomic $\mathfrak{g}$-oper. In Sect. 6, we shall be interested in describing the preimage of a given cyclotomic $\mathfrak{g}$-oper under

$$\begin{aligned}\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)&\longrightarrow\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1),\\ \nabla&\longmapsto[\nabla]_\Gamma,\end{aligned}\tag{4.20}$$

which is a direct analog of (3.5) in the finite case.

There is an obvious bijection between cyclotomic $\mathfrak{h}$-connections and cyclotomic Miura $\mathfrak{g}$-opers

$$\begin{aligned}\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)&\xrightarrow{\sim}\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1),\\ \nabla&\longmapsto\nabla+p_{-1}\mathrm{d}t.\end{aligned}\tag{4.21}$$

Given a cyclotomic Miura $\mathfrak{g}$-oper $\nabla\in\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$, we denote the associated cyclotomic $\mathfrak{h}$-connection by $\overline{\nabla}:=\nabla-p_{-1}\mathrm{d}t\in\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)$.

In the remainder of this section, we turn to the study of singularities of cyclotomic Miura $\mathfrak{g}$-opers. For any $x\in\mathbb{P}^1$, we let $\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ denote the subspace of cyclotomic $\mathfrak{h}$-connections with a simple pole at x. Consider the composition of the above maps (4.20) and (4.21), namely

$$\begin{aligned}\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)&\xrightarrow{\sim}\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)\longrightarrow\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1),\\ \nabla&\longmapsto\nabla+p_{-1}\mathrm{d}t\longmapsto[\nabla+p_{-1}\mathrm{d}t]_\Gamma.\end{aligned}$$

We will call this the *cyclotomic Miura transform* by analogy with the non-cyclotomic case [7]. Its restriction to $\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ induces a map $\mathrm{Conn}^{\hat\nu}_{\mathfrak{h}}(\mathbb{P}^1)^{\mathrm{RS}}_x\to\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_x$ for each $x\in\mathbb{P}^1$. Recalling the notation $[\check\lambda]_W$ for the W-linkage class of a coweight $\check\lambda\in\mathfrak{h}$ introduced in Sect. 3.2, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Conn}_{\mathfrak{h}}^{\hat{\nu}}(\mathbb{P}^1)_x^{\mathrm{RS}} & \longrightarrow & \mathrm{Op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)_x^{\mathrm{RS}} \\ \downarrow{\scriptstyle -\operatorname{res}_x} & & \downarrow{\scriptstyle \operatorname{res}_x} \\ \mathfrak{h} & \xrightarrow[[\cdot]_W]{} & \mathrm{Op}_{\mathfrak{g}}^{\mathrm{fin}} \end{array}$$

for each $x \in \mathbb{C}^\times$, and for the origin and infinity, we have

$$\begin{array}{ccc} \mathrm{Conn}_{\mathfrak{h}}^{\hat{\nu}}(\mathbb{P}^1)_0^{\mathrm{RS}} & \longrightarrow & \mathrm{Op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)_0^{\mathrm{RS}} \\ \downarrow{\scriptstyle -\operatorname{res}_0} & & \downarrow{\scriptstyle \operatorname{res}_0} \\ \mathfrak{h}^\nu & \xrightarrow[[\cdot]_{W^\nu}]{} & \mathrm{Op}_{\mathfrak{g}^\nu}^{\mathrm{fin}} \end{array} \qquad \begin{array}{ccc} \mathrm{Conn}_{\mathfrak{h}}^{\hat{\nu}}(\mathbb{P}^1)_\infty^{\mathrm{RS}} & \longrightarrow & \mathrm{Op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)_\infty^{\mathrm{RS}} \\ \downarrow{\scriptstyle \operatorname{res}_\infty} & & \downarrow{\scriptstyle \operatorname{res}_\infty} \\ \mathfrak{h}^\nu & \xrightarrow[-[\cdot]_{W^\nu}]{} & -\mathrm{Op}_{\mathfrak{g}^\nu}^{\mathrm{fin}} \end{array}$$

Proposition 4.11. *Let* $\nabla \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$.

(*i*) *The underlying cyclotomic* $\mathfrak{g}$*-oper* $[\nabla]_\Gamma$ *has at most a regular singularity at* $x \in \mathbb{C}^\times$ *if and only if*

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{w \cdot \check{\lambda}}{t - x}\,\mathrm{d}t + \mathsf{r}$$

for some $w \in W$, $\mathsf{r} \in \Omega(\mathfrak{h})$ *regular at* x *and* $\check{\lambda} \in \mathfrak{h}$ *such that* $\operatorname{res}_x[\nabla]_\Gamma = [\check{\lambda}]_W$ *and* $\check{\lambda} + \check{\rho}$ *is dominant.*

(*ii*) *The underlying cyclotomic* $\mathfrak{g}$*-oper* $[\nabla]_\Gamma$ *has at most a regular singularity at* 0 *(resp.* ∞*) if and only if*

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{w \cdot \check{\lambda}}{t}\,\mathrm{d}t + \mathsf{r}$$

for some $w \in W^\nu$, $\mathsf{r} \in \Omega^{\hat{\nu}}(\mathfrak{h})$ *regular at* 0 *(resp.* ∞*) and* $\check{\lambda} \in \mathfrak{h}^\nu$ *such that* $\operatorname{res}_0[\nabla]_\Gamma = [\check{\lambda}]_{W^\nu}$ *(resp.* $\operatorname{res}_\infty[\nabla]_\Gamma = -[\check{\lambda}]_{W^\nu}$*) and* $\check{\lambda} + \check{\rho}$ *is dominant.*

Proof. Let $x \in \mathbb{C}^\times$. Note that $\nabla \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ lies in $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)_x^{\mathrm{RS}}$ if and only if it has the form

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{\check{\mu}}{t - x}\,\mathrm{d}t + \mathsf{r} \tag{4.22}$$

for some $\check{\mu} \in \mathfrak{h}$ and $\mathsf{r} \in \Omega(\mathfrak{h})$ regular at x. It follows from Lemma 4.8 that $[\nabla]_\Gamma$ has at most a regular singularity at x if and only if ∇ is of the form (4.22).

By definition of the residue at $x \in \mathbb{C}^\times$ in (4.16), we have $\operatorname{res}_x[\nabla]_\Gamma = [p_{-1} - \check{\rho} - \check{\mu}]_{\mathfrak{g}} = [\check{\mu}]_W$, from which it follows by Proposition 3.2 that $\operatorname{res}_x[\nabla]_\Gamma = [\check{\lambda}]_W$ if and only if $\check{\mu} = w \cdot \check{\lambda}$ for some $w \in W$, where $\check{\lambda} \in \mathfrak{h}$ can be chosen such that $\check{\lambda} + \check{\rho}$ is dominant.

Similarly, for $z \in \{0, \infty\}$ we have that $\nabla \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ lies in $\mathrm{op}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)_z^{\mathrm{RS}}$ if and only if it takes the form

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{\check{\mu}}{t}\,\mathrm{d}t + \mathsf{r} \tag{4.23}$$

for some $\check{\mu} \in \mathfrak{h}^\nu$ and $\mathsf{r} \in \Omega^{\hat{\nu}}(\mathfrak{h})$ regular at z. It follows from Lemma 4.9 that $[\nabla]_\Gamma$ has at most a regular singularity at z if and only if ∇ is of the form (4.23).

In this case, from the definition of the residue at 0 (resp. ∞) in (4.19) we have $\mathrm{res}_0[\nabla]_\Gamma = [\check{\mu}]_{W^\nu}$ (resp. $\mathrm{res}_\infty[\nabla]_\Gamma = -[\check{\mu}]_{W^\nu}$). It then follows from Theorem 3.8 that $\mathrm{res}_0[\nabla]_\Gamma = [\check{\lambda}]_{W^\nu}$ (resp. $\mathrm{res}_\infty[\nabla]_\Gamma = -[\check{\lambda}]_{W^\nu}$) if and only if $\check{\mu} = w \cdot \check{\lambda}$ for some $w \in W^\nu$, where $\check{\lambda} \in \mathfrak{h}^\nu$ can be chosen such that $\check{\lambda} + \check{\rho}$ is dominant. □

As a special case of Proposition 4.11, suppose $\nabla \in \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ is a cyclotomic Miura $\mathfrak{g}$-oper whose underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ is regular at $x \in \mathbb{C}^\times$. By Lemma 4.10, we have $\mathrm{res}_x[\nabla]_\Gamma = [0]_W$, and so applying Proposition 4.11, we deduce that ∇ is of the form

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{w \cdot 0}{t - x}\,\mathrm{d}t + \mathsf{r}, \tag{4.24}$$

for some $w \in W$ and $\mathsf{r} \in \Omega(\mathfrak{h})$ regular at x. In particular, it is possible for a cyclotomic Miura $\mathfrak{g}$-oper ∇ to have a simple pole at a point x even if its underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ is regular there. In the next proposition, we give a necessary condition for this to happen.

Proposition 4.12. *Let $\nabla \in \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ be of the form (4.24). If $[\nabla]_\Gamma$ is regular at x then*

$$(w \cdot 0 | \mathsf{r}(x)) = 0. \tag{4.25}$$

Proof. If $[\nabla]_\Gamma$ is regular at x, then, in particular, the component c_1 of its canonical representative is regular at x. Writing $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t$ and $\mathsf{r} = r\,\mathrm{d}t$, by Proposition 4.6 the latter is proportional to

$$\tfrac{1}{2}(u|u) + (\check{\rho}|u') = \frac{(w \cdot 0 | w \cdot 0 + 2\check{\rho})}{2(t-x)^2} - \frac{(w \cdot 0 | r(t))}{t - x} + \tfrac{1}{2}\,(r(t)|r(t)) + \big(\check{\rho}\big|r'(t)\big)\,.$$

The double-pole term vanishes identically using the identity $(w \cdot \check{\mu} | w \cdot \check{\mu} + 2\check{\rho}) = (\check{\mu} | \check{\mu} + 2\check{\rho})$ which is valid for any $\check{\mu} \in \mathfrak{h}$, and the last two terms are both regular at x. It follows that c_1 is regular at x only if (4.25) holds. □

Condition (4.25) on the cyclotomic Miura $\mathfrak{g}$-oper ∇ of the form (4.24) is, in general, not sufficient for $[\nabla]_\Gamma$ to be regular at a point $x \in \mathbb{C}^\times$. However, if the Weyl group element w is of length one, i.e. if w is the reflection s_i with respect to a simple root α_i (in which case $w \cdot 0 = -\check{\alpha}_i$), then (4.25) becomes a necessary and sufficient condition for the underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ to be regular at x.

Proposition 4.13. *Let $x \in \mathbb{C}^\times$. Suppose $\nabla \in \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ has the form*

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t + \frac{\check{\alpha}_i}{t - x}\,\mathrm{d}t + \mathsf{r}$$

for some simple root α_i and $\mathsf{r} \in \Omega(\mathfrak{h})$ regular at x. Then, $[\nabla]_\Gamma$ is regular at x if and only if

$$\langle \alpha_i, \mathsf{r}(x) \rangle = 0. \tag{4.26}$$

Proof. We have $\check{\alpha}_i = -s_i \cdot 0$ so that ∇ is of the form (4.24). Therefore, if $[\nabla]_\Gamma$ is regular at x, then applying Proposition 4.12 we deduce $(\check{\alpha}_i | \mathsf{r}(x)) = 0$, which is equivalent to (4.26).

Conversely, suppose (4.26) holds and let us show $[\nabla]_\Gamma$ is regular at x. According to Proposition 4.7, it is enough to find a (possibly non-cyclotomic) representative of the $\mathfrak{g}$-oper $[\nabla]$ which is regular at x. Applying a gauge transformation by $g = \exp\big(-\frac{1}{t-x}E_{\alpha_i}\big)$ to the given Miura $\mathfrak{g}$-oper ∇, cf. the proof of Lemma 5.1, we obtain

$$g\nabla g^{-1} = \mathrm{d} + p_{-1}\mathrm{d}t + \mathsf{r}(t) + \frac{\langle \alpha_i, r(t)\rangle}{t-x}E_{\alpha_i}\mathrm{d}t.$$

This is regular at x by virtue of Eq. (4.26), hence the proposition. □

Let $N \in \mathbb{Z}_{\geq 1}$ and fix a finite subset of $N+2$ points $\boldsymbol{z} := \{0, z_1, \ldots, z_N, \infty\} \subset \mathbb{P}^1$. We assume these points have disjoint Γ-orbits, i.e. $\omega^r z_i \neq z_j$ for all $i \neq j$ and $r = 0, \ldots, T-1$. Let $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z}}$ denote the set of all cyclotomic $\mathfrak{g}$-opers with at most regular singularities at the points in $\Gamma\boldsymbol{z}$ and which are regular elsewhere.

Let $\check{\boldsymbol{\lambda}} := \{\check{\lambda}_0, \check{\lambda}_1, \ldots, \check{\lambda}_N, \check{\lambda}_\infty\} \subset \mathfrak{h}$ be a collection of $N+2$ integral dominant coweights, which we think of as being attached to the corresponding points in $\boldsymbol{z}$. We assume that the coweights at the origin and infinity are both ν-invariant, namely $\check{\lambda}_0, \check{\lambda}_\infty \in \mathfrak{h}^\nu$. We denote by $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$ the subset of cyclotomic $\mathfrak{g}$-opers $[\nabla]_\Gamma \in \mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z}}$ whose residues at the points of $\boldsymbol{z}$ are given by

$$\mathrm{res}_0[\nabla]_\Gamma = [\check{\lambda}_0]_{W^\nu}, \qquad \mathrm{res}_{z_i}[\nabla]_\Gamma = [\check{\lambda}_i]_W, \qquad \mathrm{res}_\infty[\nabla]_\Gamma = -[\check{\lambda}_\infty]_{W^\nu}$$

for $i = 1, \ldots, N$. Let $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$ denote the further subset consisting of those cyclotomic $\mathfrak{g}$-opers in $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$ which are also monodromy-free.

Theorem 4.14. *Let $\boldsymbol{z} \subset \mathbb{P}^1$ and $\check{\boldsymbol{\lambda}} \subset \mathfrak{h}$ be as above and let $\nabla \in \mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$. The underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ lives in $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$ if and only if the following conditions hold:*

(i) ∇ *is of the form*

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{w_0\cdot\check{\lambda}_0}{t}\mathrm{d}t - \sum_{r=0}^{T-1}\left(\sum_{i=1}^{N}\frac{\nu^r(w_i\cdot\check{\lambda}_i)}{t-\omega^r z_i} + \sum_{j=1}^{m}\frac{\nu^r y_j\cdot 0}{t-\omega^r x_j}\right)\mathrm{d}t \quad (4.27)$$

for some $m \in \mathbb{Z}_{\geq 0}$, $w_0 \in W^\nu$, $w_i \in W$ for each $i = 1, \ldots, N$ and $x_j \in \mathbb{P}^1\backslash\Gamma\boldsymbol{z}$, $y_j \in W$ for each $j = 1, \ldots, m$,

(ii) there exists $w_\infty \in W^\nu$ such that

$$w_0\cdot\check{\lambda}_0 + \sum_{r=0}^{T-1}\left(\sum_{i=1}^{N}\nu^r(w_i\cdot\check{\lambda}_i) + \sum_{j=1}^{m}\nu^r y_j\cdot 0\right) = w_\infty\cdot\check{\lambda}_\infty, \quad (4.28)$$

(iii) $[\nabla]_\Gamma$ is regular at x_j for each $j = 1, \ldots, m$.

Proof. Suppose that the cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ belongs to $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$. Then, $[\nabla]_\Gamma$ has regular singularities at the points of $\Gamma\boldsymbol{z}$ and is regular everywhere else. However, as we have seen above, the cyclotomic Miura $\mathfrak{g}$-oper ∇ itself can have other simple poles at a set of points $\widetilde{\boldsymbol{x}} \subset \mathbb{P}^1\backslash\Gamma\boldsymbol{z}$. By virtue of cyclotomy,

if ∇ has a pole at a point in $\mathbb{P}^1$, it also has a pole at each element in the orbit of this point under the action of Γ. Thus, this set $\widetilde{\boldsymbol{x}}$ can be seen as the image $\Gamma\boldsymbol{x}$ of some minimal set $\boldsymbol{x} \subset \mathbb{P}^1 \backslash \Gamma\boldsymbol{z}$. Let $m \in \mathbb{Z}_{\geq 0}$ be the size of $\boldsymbol{x}$ and write $\boldsymbol{x} = \{x_1, \dots, x_m\}$.

According to Proposition 4.11, there exist $w_0 \in W^\nu$ and $w_i \in W$ for each $i = 1, \dots, N$ such that $\operatorname{res}_0 \overline{\nabla} = -w_0 \cdot \check{\lambda}_0$ and $\operatorname{res}_{z_i} \overline{\nabla} = -w_i \cdot \check{\lambda}_i$. Furthermore, from the discussion around (4.24) there exist $y_j \in W$ for each $j = 1, \dots, m$ such that $\operatorname{res}_{x_j} \overline{\nabla} = -y_j \cdot 0$. Moreover, since ∇ is cyclotomic, one has $\operatorname{res}_{\omega^r z_i} \overline{\nabla} = -\nu^r \left(w_i \cdot \check{\lambda}_i\right)$ and $\operatorname{res}_{\omega^r x_j} \overline{\nabla} = -\nu^r \left(y_j \cdot 0\right) = -\nu^r y_j \cdot 0$ for any $r = 0, \dots, T-1$, $i = 1, \dots, N$ and $j = 1, \dots, m$. Thus, ∇ is of the form (4.27).

The residue at infinity of the corresponding $\mathfrak{h}$-connection is then

$$\operatorname{res}_\infty \overline{\nabla} = w_0 \cdot \check{\lambda}_0 + \sum_{r=0}^{T-1} \left(\sum_{i=1}^{N} \nu^r (w_i \cdot \check{\lambda}_i) + \sum_{j=1}^{m} \nu^r y_j \cdot 0 \right).$$

This is related to the residue of the cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ by $\operatorname{res}_\infty [\nabla]_\Gamma = -\left[\operatorname{res}_\infty \overline{\nabla}\right]_{W^\nu}$. Yet, as $[\nabla]_\Gamma$ belongs to $\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$, we have $\operatorname{res}_\infty [\nabla]_\Gamma = -[\check{\lambda}_\infty]_{W^\nu}$ and hence the existence of $w_\infty \in W^\nu$ such that equation (4.28) holds.

Conversely, suppose that ∇ is of the form (4.27) and that we have the condition (4.28). It is then clear that, for $i = 1, \dots, N$,

$$\operatorname{res}_0 [\nabla]_\Gamma = [\check{\lambda}_0]_{W^\nu}, \qquad \operatorname{res}_{z_i} [\nabla]_\Gamma = [\check{\lambda}_i]_W, \qquad \operatorname{res}_\infty [\nabla]_\Gamma = -[\check{\lambda}_\infty]_{W^\nu}.$$

Moreover, suppose that $[\nabla]_\Gamma$ is regular at the points x_j, for $j = 1, \dots, m$. Then, by virtue of the cyclotomy, $[\nabla]_\Gamma$ is regular at all the points $\omega^r x_j$, for $r = 0, \dots, T-1$ and $j = 1, \dots, m$. Hence, $[\nabla]_\Gamma$ belongs to $\mathrm{Op}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$. □

Remark 1. Let $\nabla \in \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ be as in Theorem 4.14. According to Proposition 4.12, a necessary condition for $[\nabla]_\Gamma$ to be regular at x_k, $k \in \{1, \dots, m\}$ is the *generalised cyclotomic Bethe ansatz equation*

$$\frac{\left(y_k \cdot 0 | w_0 \cdot \check{\lambda}_0\right)}{x_k} + \sum_{r=0}^{T-1} \sum_{i=1}^{N} \frac{\left(y_k \cdot 0 | \nu^r (w_i \cdot \check{\lambda}_i)\right)}{x_k - \omega^r z_i} + \sum_{\substack{r=0 \\ (r,j) \neq (0,k)}}^{T-1} \sum_{j=1}^{m} \frac{\left(y_k \cdot 0 | \nu^r y_j \cdot 0\right)}{x_k - \omega^r x_j} = 0.$$

Moreover, Proposition 4.13 states that this condition is sufficient if $y_k \in W$ is a simple reflection. ◁

Remark 2. Let $\nabla \in \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)$ be as in Theorem 4.14. If $w_i = \mathrm{Id}$ for some $i = 1, \dots, N$ (resp. $w_0 = \mathrm{Id}$), then $[\nabla]_\Gamma$ has trivial monodromy at z_i (resp. at the origin). Indeed, in this case a gauge transformation of ∇ by $(t - z_i)^{\check{\lambda}_i}$ (resp. $t^{\check{\lambda}_0}$) yields a $\mathfrak{g}$-connection regular at z_i (resp. at the origin) and hence with trivial monodromy there by Proposition 4.1. ◁

To end this section, we present two explicit examples of cyclotomic Miura $\mathfrak{g}$-opers, for the simple Lie algebras $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sl}_4$, that we shall use to illustrate the various results from Sects. 5 and 6.

Example 1. Consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ in the fundamental representation. The two fundamental coweights of $\mathfrak{sl}_3$ are represented by the following diagonal matrices

$$\check{\omega}_1 = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \check{\omega}_2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Moreover, the principal nilpotent element p_{-1} is given by

$$p_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The unique non-trivial diagram automorphism $\nu : I \to I$ of $\mathfrak{sl}_3$ exchanges the labels 1 and 2. Let us consider the simplest possible cyclotomic Miura $\mathfrak{sl}_3$-oper ∇ with a pole only at the origin and whose residue there is given by a ν-invariant integral dominant coweight. The general form of such a coweight is $\check{\lambda}_0 = \eta(\check{\omega}_1 + \check{\omega}_2)$ with $\eta \in \mathbb{Z}_{\geq 0}$. Therefore, the cyclotomic Miura $\mathfrak{sl}_3$-oper in question takes the form

$$\nabla = \mathrm{d} + \left(p_{-1} - \frac{\check{\lambda}_0}{t}\right)\mathrm{d}t = \mathrm{d} + \begin{pmatrix} -\frac{\eta}{t} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \frac{\eta}{t} \end{pmatrix}\mathrm{d}t. \tag{4.29}$$

Note that the residue at infinity of the associated $\mathfrak{h}$-connection $\overline{\nabla}$ is simply $\check{\lambda}_0 = \eta(\check{\omega}_1 + \check{\omega}_2)$. The canonical representative of the associated cyclotomic $\mathfrak{sl}_3$-oper $[\nabla]_\Gamma$ reads

$$g\nabla g^{-1} = \mathrm{d} + \begin{pmatrix} 0 & \frac{\eta(\eta+2)}{2t} & 0 \\ 1 & 0 & \frac{\eta(\eta+2)}{2t} \\ 0 & 1 & 0 \end{pmatrix}\mathrm{d}t, \quad \text{with} \quad g(t) = \begin{pmatrix} 1 & \frac{\eta}{t} & \frac{\eta^2}{2t^2} \\ 0 & 1 & \frac{\eta}{t} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since we are working in the fundamental representation, the above group element g belongs to the three-dimensional representation of the special linear group SL_3. However, in this article we consider only the adjoint group $G = (\operatorname{Aut}\mathfrak{g})^\circ$ of the Lie algebra $\mathfrak{g}$, cf. Sect. 2.2. In the present example, we have $\mathfrak{g} = \mathfrak{sl}_3$, whose adjoint group is the projective linear group PGL_3 (the quotient of the linear group GL_3 by its centre $\{\lambda\,\mathrm{Id},\ \lambda \in \mathbb{C}^\times\}$). The adjoint group PGL_3 can also be seen as the quotient of SL_3 by its centre $Z = \{\lambda\,\mathrm{Id},\ \lambda \in \mathbb{C}^\times \mid \lambda^3 = 1\}$. Here and in Examples 4 and 5, all group elements written as 3×3 matrices are to be understood as the class of these matrices in the three-dimensional representation of SL_3 modulo multiplication by elements of Z. ◁

Example 2. Consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_4$. Denote by α_i and $\check{\omega}_i$ for $i = 1, 2, 3$, the simple roots and the associated coweights.

The unique non-trivial diagram automorphism of $\mathfrak{sl}_4$ is $\nu :\ 1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1$, of order 2. We choose the primitive Tth-root of unity $\omega = e^{\frac{2\pi i}{T}}$, with $T = 2S$ and $S \in \mathbb{Z}_{\geq 1}$. The general ν-invariant integral dominant coweight is of the form $\check{\lambda}_0 = \eta(\check{\omega}_1 + \check{\omega}_3) + \kappa\check{\omega}_2$ with $\eta, \kappa \in \mathbb{Z}_{\geq 0}$. We shall consider the

cyclotomic Miura $\mathfrak{sl}_4$-oper specified by the coweight $\check{\lambda}_0$ at the origin and the coweight $\check{\omega}_1$ at a point $z \in \mathbb{C}^\times$, namely

$$\nabla = \mathrm{d} + \left(p_{-1} - \frac{\check{\lambda}_0}{t} - \sum_{r=0}^{T-1} \frac{\nu^r \check{\omega}_1}{t - \omega^r z} \right) \mathrm{d}t.$$

This can be re-expressed as

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \left(\frac{\eta}{t} + \frac{St^{S-1}}{t^S - z^S} \right) \check{\omega}_1 \mathrm{d}t - \frac{\kappa}{t} \check{\omega}_2 \mathrm{d}t - \left(\frac{\eta}{t} + \frac{St^{S-1}}{t^S + z^S} \right) \check{\omega}_3 \mathrm{d}t. \quad (4.30)$$

The residues at the origin and infinity of the associated $\mathfrak{h}$-connection are

$$\operatorname{res}_0 \overline{\nabla} = -\eta(\check{\omega}_1 + \check{\omega}_3) - \kappa\check{\omega}_2, \qquad \operatorname{res}_\infty \overline{\nabla} = (\eta + S)\,(\check{\omega}_1 + \check{\omega}_3) + \kappa\check{\omega}_2.$$

◁

5. Cyclotomic Miura $\mathfrak{g}$-Opers and Reproductions

Given a Miura $\mathfrak{g}$-oper $\nabla \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ and any simple root α_k, $k \in I$, the following lemma provides a way of constructing a new Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$, where the gauge transformation parameter is of the form $g = e^{fE_k}$ with $f \in \mathcal{M}$. This is known as the *reproduction* or *generation procedure* in the direction α_k, see e.g. [18].

Lemma 5.1. *Let $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ and $g = e^{fE_k} \in N(\mathcal{M})$ for some $f \in \mathcal{M}$. We have $g\nabla g^{-1} \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ if and only if the function f satisfies the Riccati equation*

$$f'(t) + f(t)^2 + f(t)\langle \alpha_k, u(t) \rangle = 0.$$

In this case, $g\nabla g^{-1} = \nabla + f\check{\alpha}_k\,\mathrm{d}t$. Moreover, for any $x \in \mathbb{C}$ we have

$$\operatorname{res}_x \overline{g\nabla g^{-1}} = \begin{cases} -s_k \cdot (-\operatorname{res}_x \overline{\nabla}) & \text{if } f\mathrm{d}t \in \Omega \text{ has a pole at } x, \\ \operatorname{res}_x \overline{\nabla} & \text{otherwise.} \end{cases}$$

Similarly, at infinity, we have

$$\operatorname{res}_\infty \overline{g\nabla g^{-1}} = \begin{cases} s_k \cdot (\operatorname{res}_\infty \overline{\nabla}) & \text{if } f\mathrm{d}t \in \Omega \text{ has a pole at } \infty, \\ \operatorname{res}_\infty \overline{\nabla} & \text{otherwise.} \end{cases}$$

Moreover, if $\langle \alpha_k, \operatorname{res}_\infty \overline{\nabla} + \check{\rho} \rangle \geq 0$ and f is not identically zero, then $f\mathrm{d}t$ has a pole at ∞.

Proof. By a direct calculation, we find that

$$-dgg^{-1} = -f'(t)E_k\mathrm{d}t, \qquad gp_{-1}g^{-1} = p_{-1} + f(t)\check{\alpha}_k - f(t)^2E_k,$$
$$gu(t)g^{-1} = u(t) - f(t)\langle \alpha_k, u(t) \rangle E_k,$$

from which we obtain $g\nabla g^{-1} = \mathrm{d} + \big[p_{-1} + u(t) + f(t)\check{\alpha}_k - \big(f'(t) + f(t)^2 + f(t)\langle \alpha_k, u(t) \rangle\big)E_k\big]\mathrm{d}t$. The latter is a Miura $\mathfrak{g}$-oper if and only if the coefficient of E_k vanishes, which is equivalent to the Riccati equation. We then have $g\nabla g^{-1} = \nabla + f\check{\alpha}_k\,\mathrm{d}t$ and hence also $\overline{g\nabla g^{-1}} = \overline{\nabla} + f\check{\alpha}_k\,\mathrm{d}t$.

To prove the statement about the residue, suppose first that $x \in \mathbb{C}$ and define $\check{\mu} := -\operatorname{res}_x \overline{\nabla}$ so that $u(t) = -\frac{\check{\mu}}{t-x} + r(t)$ for some $r \in \mathfrak{h}(\mathcal{M})$ regular at

x. By considering the Laurent series expansion at x of the solution $f \in \mathcal{M}$, we find that it is consistent with the Riccati equation provided f has at most a simple pole there, namely

$$f(t) = \frac{a}{t-x} + \mathcal{O}\big((t-x)^0\big)$$

with either $a = 0$ or $a = \langle \alpha_k, \check{\mu} + \check{\rho} \rangle$. In the first case, $f\mathrm{d}t$ is regular at x so that $\operatorname{res}_x f\mathrm{d}t = 0$ and hence $\operatorname{res}_x \overline{g\nabla g^{-1}} = \operatorname{res}_x \overline{\nabla}$. In the second case, $f\mathrm{d}t$ has a simple pole at x with $\operatorname{res}_x f\mathrm{d}t = \langle \alpha_k, \check{\mu} + \check{\rho} \rangle$ so that $\operatorname{res}_x \overline{g\nabla g^{-1}} = -\check{\mu} + \langle \alpha_k, \check{\mu} + \check{\rho} \rangle \check{\alpha}_k = -s_k \cdot \check{\mu} = -s_k \cdot (-\operatorname{res}_x \overline{\nabla})$.

Consider now the point at infinity and define $\check{\mu} := \operatorname{res}_\infty \overline{\nabla}$ so that $u(t) = -\frac{\check{\mu}}{t} + \mathcal{O}\big(\frac{1}{t^2}\big)$. It then follows that the asymptotic behaviour of the solution f to the Riccati equation for large t reads

$$f(t) = \frac{a}{t} + \mathcal{O}\left(\frac{1}{t^2}\right),$$

with either $a = 0$ or $a = \langle \alpha_k, \check{\mu} + \check{\rho} \rangle$. As above, in the first case, $f\mathrm{d}t$ is regular at infinity and we find $\operatorname{res}_\infty \overline{g\nabla g^{-1}} = \operatorname{res}_\infty \overline{\nabla}$. In the second case, $f\mathrm{d}t$ has a simple pole at infinity and we deduce that $\operatorname{res}_\infty \overline{g\nabla g^{-1}} = s_k \cdot (\operatorname{res}_\infty \overline{\nabla})$, as required.

Finally, let us suppose that $\langle \alpha_k, \operatorname{res}_\infty \overline{\nabla} + \check{\rho} \rangle$ is non-negative, *i.e.* that $\langle \alpha_k, \check{\mu} \rangle \geq -1$. In order to prove the last statement of the lemma, we shall show that if f does not have a pole at infinity (*i.e.* if $a = 0$), then f is identically zero. Indeed, if f were different from zero, we could write

$$f(t) = \frac{b}{t^p} + \mathcal{O}\left(\frac{1}{t^{p+1}}\right),$$

for some nonzero $b \in \mathbb{C}^\times$ and $p \in \mathbb{Z}_{\geq 2}$. The cancellation of the terms of order t^{-p-1} in the Riccati equation yields the condition $\big(p + \langle \alpha_k, \check{\mu} \rangle\big) b = 0$. However, $p + \langle \alpha_k, \check{\mu} \rangle \geq 2 - 1 = 1$ so that $b = 0$, hence a contradiction. □

Suppose now that the Miura $\mathfrak{g}$-oper ∇ we start with is cyclotomic. More precisely, let us consider $\nabla \in \mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$ such that the underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$ lives in $\mathrm{Op}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\check{\boldsymbol{\lambda}}}$, as defined in Sect. 4.6. Then, ∇ is described by Theorem 4.14 and in particular is of the form (4.27). Moreover, using the notations of Theorem 4.14, we suppose $w_0 = \mathrm{Id}$ so that $-\operatorname{res}_0 \overline{\nabla} = \check{\lambda}_0$ is a ν-invariant integral dominant coweight. We can thus write $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t$ with

$$u(t) := -\frac{\check{\lambda}_0}{t} - \sum_{r=0}^{T-1} \left(\sum_{i=1}^{N} \frac{\nu^r(w_i \cdot \check{\lambda}_i)}{t - \omega^r z_i} + \sum_{j=1}^{m} \frac{\nu^r y_j \cdot 0}{t - \omega^r x_j} \right). \tag{5.1}$$

The difficulty with applying the generation procedure in the cyclotomic setting is that the new Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ will in general no longer be cyclotomic, since the parameter $g \in N(\mathcal{M})$ of the gauge transformation we applied need not be $\hat{\varsigma}$-invariant. This problem was first studied in [24], when $w_i = \mathrm{Id}$ for each $i = 1, \ldots, N$ and y_j are simple reflections for each $j = 1, \ldots, m$, using the language of populations of solutions to the Bethe ansatz equations (the

collection of algebraic equations ensuring the regularity of (5.1) at the set of points $\{x_j\}_{j=1}^m$, cf. Proposition 4.13). As observed there, it is possible to move from one solution of the *cyclotomic* Bethe equations [26] to another by applying instead a sequence of reproductions in the directions of the various roots in the ν-orbit of α_k. Moreover, the existence of such a *cyclotomic generation procedure* was shown to require imposing certain conditions on the coweight $\check{\lambda}_0$, cf. Theorems 5.3 and 5.5. Note that for the case discussed in Sect. 5.2 $\check{\lambda}_0$ was only assumed *half*-integral dominant in [24]. Although we assume throughout this section that $\check{\lambda}_0$ is integral dominant, we shall show later in Sect. 6.2 that the integrality condition can in fact be relaxed.

In this section, we will describe cyclotomic reproduction within the framework of cyclotomic Miura $\mathfrak{g}$-opers by studying the properties of Riccati-type equations. Specifically, we will prove cyclotomic generalisations of Lemma 5.1, corresponding to reproduction in the direction of a simple root $\alpha_{\mathcal{I}}^{\nu}$ of the folded Lie algebra $\mathfrak{g}^{\nu}$, in the case when the orbit $\mathcal{I} \in I/\nu$ is of type $A_1^{\times|\mathcal{I}|}$ or $A_2^{\times|\mathcal{I}|/2}$.

Recall the setting of Sect. 3.3. Let $\mathcal{I} \in I/\nu$ be an orbit in I under the diagram automorphism $\nu : I \to I$. Consider the corresponding orbit of simple roots

$$\{\alpha_i\}_{i\in\mathcal{I}} \subset \Phi^+.$$

Unless the orbit is of size one, i.e. $|\mathcal{I}| = 1$, applying a single gauge transformation by an element of the form $e^{f_i E_i}$ with $i \in \mathcal{I}$ and $f_i \in \mathcal{M}$ will certainly break the Γ-invariance of the cyclotomic Miura $\mathfrak{g}$-oper ∇. In order to restore it, we will have to apply a series of successive gauge transformations by $e^{f_i E_i}$, with some $f_i \in \mathcal{M}$, corresponding to the other points i in the orbit $\mathcal{I}$. The specific combination of gauge transformations required will depend on whether the orbit $\mathcal{I}$ is of type $A_1^{\times|\mathcal{I}|}$ or $A_2^{\times|\mathcal{I}|/2}$, cf. Sect. 3.3. We will treat these two cases separately.

5.1. Orbit of Type $A_1^{\times|\mathcal{I}|}$

Let $\mathcal{I}$ be an orbit of type $A_1^{\times|\mathcal{I}|}$, i.e. such that $\ell_{\mathcal{I}} = 1$ in the notation of Sect. 3.3. As mentioned above, the case $|\mathcal{I}| = 1$ is easy to treat, so we shall assume $|\mathcal{I}| \geq 2$.

Let $i, j \in \mathcal{I}$ be distinct. If we perform a gauge transformation of $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ by $e^{f_i E_i}$ with $f_i \in \mathcal{M}$ a solution to the Riccati equation

$$f'(t) + f(t)^2 + f(t)\langle \alpha_i, u(t)\rangle = 0,$$

then by Lemma 5.1 we obtain the new Miura $\mathfrak{g}$-oper $e^{f_i E_i}\nabla e^{-f_i E_i} = \nabla + f_i\check{\alpha}_i\,\mathrm{d}t$. If we perform a second gauge transformation of the latter by $e^{f_j E_j}$, then by Lemma 5.1 the resulting $\mathfrak{g}$-connection will again be a Miura $\mathfrak{g}$-oper provided $f_j \in \mathcal{M}$ is chosen to satisfy the Riccati equation

$$f'(t) + f(t)^2 + f(t)\langle \alpha_j, u(t) + f_i\check{\alpha}_i\rangle = 0.$$

However, since we are assuming $\mathcal{I}$ to be of type $A_1^{\times|\mathcal{I}|}$, we have $\langle \alpha_j, \check{\alpha}_i\rangle = 0$ and hence the Riccati equation satisfied by f_j only depends on the original Miura $\mathfrak{g}$-oper ∇ and not on f_i.

We are therefore seeking a gauge transformation by $g := \prod_{i\in\mathcal{I}} e^{f_i E_i}$ where the functions $f_i \in \mathcal{M}$ satisfy the Riccati equations

$$(R^i): \qquad f_i'(t) + f_i(t)^2 + f_i(t)\langle \alpha_i, u(t)\rangle = 0, \tag{5.2}$$

for $i \in \mathcal{I}$. Note that the order of the factors in the expression for g does not matter by the assumption on the nature of the orbit $\mathcal{I}$. By Lemma 5.1, the resulting Miura $\mathfrak{g}$-oper is given by

$$g\nabla g^{-1} = \nabla + \sum_{i\in\mathcal{I}} f_i(t)\,\check{\alpha}_i\,\mathrm{d}t. \tag{5.3}$$

The sum over the orbit $\mathcal{I}$ is $\hat{\nu}$-invariant, and hence $g\nabla g^{-1}$ is cyclotomic, if and only if

$$\omega^{-1} f_i(\omega^{-1}t) = f_{\nu(i)}(t), \qquad \text{for all } \ i \in \mathcal{I}. \tag{5.4}$$

Let us fix a reference point $k \in \mathcal{I}$ on the orbit so that

$$\mathcal{I} = \big\{k, \nu(k), \ldots, \nu^{|\mathcal{I}|-1}(k)\big\}.$$

Noting that $\omega^{-1}\nu\big(u(\omega^{-1}t)\big) = u(t)$, we have $\omega^{-1}\langle\alpha_i, u(\omega^{-1}t)\rangle = \langle\alpha_{\nu(i)}, u(t)\rangle$, from which it follows that $\omega^{-1}f_i(\omega^{-1}t)$ is a solution of the Riccati equation $(R^{\nu(i)})$. So starting from *any* solution $f_k \in \mathcal{M}$ of (R^k), we may define solutions $f_i \in \mathcal{M}$ of (R^i) recursively by

$$f_i(t) := \omega^{-1} f_{\nu^{-1}(i)}(\omega^{-1}t), \tag{5.5}$$

for $i \in \mathcal{I}\backslash\{k\}$. By construction, these satisfy the relations in (5.4) for all $i \in \mathcal{I}\backslash\{\nu^{|\mathcal{I}|-1}(k)\}$, so we just need to ensure that it also holds for $i = \nu^{|\mathcal{I}|-1}(k)$, namely

$$\omega^{-1} f_{\nu^{|\mathcal{I}|-1}(k)}(\omega^{-1}t) = f_k(t).$$

Using the recurrence relation (5.5) to rewrite the left-hand side of this relation, we can express the condition for (5.3) to be cyclotomic as the following functional relation on f_k alone

$$\omega^{-|\mathcal{I}|} f_k(\omega^{-|\mathcal{I}|}t) = f_k(t). \tag{5.6}$$

Now since both sides are solutions of the same differential equation (R^k), to check they are equal it suffices to compare their values at a single point. However, to avoid having to explicitly solve (R^k) it would be preferable to evaluate them at the origin since $t = 0$ is a fixed point of $t \mapsto \omega^{-|\mathcal{I}|}t$.

If the original cyclotomic Miura $\mathfrak{g}$-oper ∇ happens to be regular at the origin, i.e. $\check{\lambda}_0 = 0$, so that $u(t)$ is also, then the Riccati equation (R^k) is regular at 0 and therefore admits solutions regular at the origin for any initial value $f_k(0) \in \mathbb{C}$. In particular, two solutions of (5.6) are equal if and only if they agree at the origin. It therefore follows that the gauge-transformed Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ is cyclotomic if and only if $\omega^{|\mathcal{I}|} = 1$, that is to say $|\mathcal{I}| = T$ since ω is a primitive Tth-root of unity.

In order to treat the general case of an arbitrary ν-invariant integral dominant coweight $\check{\lambda}_0 \in \mathfrak{h}^\nu$, we will make use of the following.

Lemma 5.2. *Let f be a nonzero meromorphic solution of the Riccati equation*

$$f'(t) + f(t)^2 + f(t)\left(-\frac{\eta}{t} + r(t)\right) = 0, \tag{5.7}$$

where $r \in \mathcal{M}$ is regular at the origin. If $\eta \in \mathbb{Z}_{\geq 0}$, then either

(i) f is regular at the origin, in which case it reads $f(t) = t^\eta h(t)$ for some $h \in \mathcal{M}$ regular at 0, with $h(0) \in \mathbb{C}^\times$ arbitrary, and satisfying

$$h'(t) + t^\eta h(t)^2 + h(t)r(t) = 0. \tag{5.8}$$

We call (5.8) the regularised Riccati equation *associated with (5.7).*

(ii) f is singular at the origin, in which case it takes the form $f(t) = \frac{\eta+1}{t} + k(t)$ for some unique $k \in \mathcal{M}$ regular at 0. □

Proof. Consider the Laurent expansion of $f \in \mathcal{M}$ at the origin. As in the proof of Lemma 5.1, for this to be consistent with equation (5.7) we find that f can have at most a first-order pole at the origin, namely

$$f(t) = \frac{a}{t} + \sum_{n\geq 0} a_n t^n$$

with either $a = 0$ or $a = \eta + 1$.

Let us now discuss how the Riccati equation (5.7) constrains the coefficients a_n. Taylor expanding r as $r(t) = \sum_{n\geq 0} r_n t^n$, we can write the Laurent expansion of the different terms appearing in this equation as

$$\begin{aligned}
f'(t) &= -\frac{a}{t^2} + \sum_{n\geq 0}(n+1)a_{n+1}t^n \\
f(t)^2 &= \frac{a^2}{t^2} + \frac{2aa_0}{t} + \sum_{n\geq 0}\left(2aa_{n+1} + \sum_{k=0}^{n} a_k a_{n-k}\right)t^n \\
f(t)\left(-\frac{\eta}{t} + r(t)\right) &= -\frac{\eta a}{t^2} + \frac{ar_0 - \eta a_0}{t} \\
&\quad + \sum_{n\geq 0}\left(ar_{n+1} - \eta a_{n+1} + \sum_{k=0}^{n} a_k r_{n-k}\right)t^n.
\end{aligned}$$

The sum of these three terms must vanish by virtue of the Riccati equation (5.7). The cancellation of the double pole gives $a = 0$ or $a = \eta + 1$, as stated above. Those of the simple pole and of the Taylor expansion are equivalent to

$$\begin{cases}
(2a - \eta)a_0 = -ar_0, \\
(2a + n - \eta)a_n = -ar_n - \sum_{k=0}^{n-1} a_k\left(a_{n-1-k} + r_{n-1-k}\right) \qquad \text{for } n \geq 1.
\end{cases}$$

If $a = 0$, then f is regular at 0, corresponding to part (i). The first equation then gives $a_0 = 0$. For $0 < n < \eta$, the second equation also yields $a_n = 0$ by induction, as $n - \eta \neq 0$. For $n = \eta$, the equation is verified for arbitrary $a_\eta \in \mathbb{C}$. Finally, all a_n for $n > \eta$ are fixed in terms of a_η and the coefficients in the Taylor expansion of $r(t)$ at the origin. In particular, f is of the form $f(t) = t^\eta h(t)$ for some $h \in \mathcal{M}$ with $h(0) = a_\eta \in \mathbb{C}$ arbitrary. Moreover, by direct substitution into (5.7) we find that h satisfies (5.8).

On the other hand, if $a = \eta+1$ then f is singular at 0, which corresponds to part (ii). In this case, the first equation reads $a_0 = -\frac{\eta+1}{\eta+2} r_0$. By induction on $n > 0$, the second one fixes uniquely the coefficients a_n in terms of the coefficients r_k, as for any $n > 0$ the coefficient $2a + n - \eta = n + \eta + 2$ is nonzero. □

5.1.1. Singular Reproduction Procedure. Since $\check{\lambda}_0$ is assumed integral dominant, we may apply Lemma 5.2 to the solution f_k of (R^k), with $\eta = \langle \alpha_k, \check{\lambda}_0 \rangle$. If f_k has a pole at the origin, then so does the function $t \mapsto \omega^{-|\mathcal{I}|} f_k(\omega^{-|\mathcal{I}|} t)$. However, since both are solutions to the same Riccati equation of the form (5.7), by the uniqueness of such a solution in Lemma 5.2(ii) it follows that they are equal, i.e. the condition (5.6) holds.

Moreover, in this case, all the functions f_i for $i \in \mathcal{I}$ have a pole at the origin. Thus, using the explicit form $g = \prod_{i \in \mathcal{I}} e^{f_i E_i}$ of the gauge transformation parameter and applying Lemma 5.1 for each $i \in \mathcal{I}$, followed by Lemma 3.6, we find that

$$-\operatorname{res}_0 \overline{g \nabla g^{-1}} = \left(\prod_{i \in \mathcal{I}} s_i \right) \cdot (-\operatorname{res}_0 \overline{\nabla}) = s_{\mathcal{I}}^{\nu} \cdot (-\operatorname{res}_0 \overline{\nabla}).$$

5.1.2. Regular Reproduction Procedure. Suppose now that f_k is regular at the origin, corresponding to case (i) of Lemma 5.2. It remains to check when the condition (5.6) holds. Introducing the function

$$h_k(t) := t^{-\eta} f_k(t),$$

regular at the origin, the condition (5.6) for $g \nabla g^{-1}$ to be cyclotomic may then be rewritten as

$$\omega^{-|\mathcal{I}|(\eta+1)} h_k(\omega^{-|\mathcal{I}|} t) = h_k(t). \tag{5.9}$$

Now since both sides of (5.9) satisfy the same regularised Riccati equation of the form (5.8), they are equal if and only if they agree at the origin. Note that we may as well assume $h_k(0) \neq 0$ since if $h_k(0) = 0$ then by the homogeneity of the regularised Riccati equation h_k would be identically zero and hence g the identity. It follows that $g \nabla g^{-1}$ is cyclotomic if and only if $\omega^{|\mathcal{I}|(\eta+1)} = 1$. Recalling that ω is a primitive Tth-root of unity, this condition is equivalent to $|\mathcal{I}|(\eta + 1) \equiv 0 \bmod T$. Or since the size $|\mathcal{I}|$ of the orbit $\mathcal{I}$ necessarily divides the order of ν which by definition divides T, we can also rewrite this condition as

$$\langle \alpha_k, \check{\lambda}_0 + \check{\rho} \rangle \equiv 0 \mod \frac{T}{|\mathcal{I}|}.$$

Note that, in this case, all the functions f_i are regular at the origin, so that $\operatorname{res}_0 \overline{g \nabla g^{-1}} = \operatorname{res}_0 \overline{\nabla}$.

5.1.3. Residue at Infinity. Finally, let us discuss the residue at infinity of the new connection $g \nabla g^{-1}$. We suppose that $\langle \alpha_k, \operatorname{res}_\infty \overline{\nabla} + \check{\rho} \rangle$ (and thus all the $\langle \alpha_i, \operatorname{res}_\infty \overline{\nabla} + \check{\rho} \rangle$ for $i \in \mathcal{I}$) is non-negative. One can see the gauge transformation by g as successive gauge transformations by $e^{f_i E_i}$, for each $i \in \mathcal{I}$. As $\langle \alpha_i, \check{\alpha}_j \rangle = 0$ for any distinct $i, j \in \mathcal{I}$, the condition on the residue at infinity of the

connection still holds after each step. Thus, applying the last part of Lemma 5.1 for each $i \in \mathcal{I}$, followed by Lemma 3.6, we find that

$$\operatorname{res}_\infty \overline{g\nabla g^{-1}} = \left(\prod_{i\in\mathcal{I}} s_i\right) \cdot (\operatorname{res}_\infty \overline{\nabla}) = s_{\mathcal{I}}^{\nu} \cdot (\operatorname{res}_\infty \overline{\nabla}).$$

Note that this discussion holds whether f_k is regular or singular at 0. We have proved the following.

Theorem 5.3. *Let $\mathcal{I} \in I/\nu$ be an orbit of type $A_1^{\times|\mathcal{I}|}$. Fix a $k \in \mathcal{I}$ and let $f_k \in \mathcal{M}$ be any nonzero solution of the Riccati equation*

$$f_k'(t) + f_k(t)^2 + f_k(t)\langle \alpha_k, u(t)\rangle = 0,$$

with $u(t)$ as in (5.1). Define $f_i \in \mathcal{M}$ for all $i \in \mathcal{I}\backslash\{k\}$ recursively by $f_i(t) := \omega^{-1} f_{\nu^{-1}(i)}(\omega^{-1}t)$ and let $g = \prod_{i\in\mathcal{I}} e^{f_i E_i} \in N(\mathcal{M})$.

If f_k is regular at the origin, then $g \in N^{\hat{\varsigma}}(\mathcal{M})$, and hence $g\nabla g^{-1} \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$, if and only if

$$\langle \alpha_k, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \mod \frac{T}{|\mathcal{I}|}.$$

Moreover, in this case we have $\operatorname{res}_0 \overline{g\nabla g^{-1}} = \operatorname{res}_0 \overline{\nabla}$.

If f_k has a pole at the origin, then we have $g\nabla g^{-1} \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ without any condition on $\check{\lambda}_0$ and $-\operatorname{res}_0 \overline{g\nabla g^{-1}} = s_{\mathcal{I}}^{\nu} \cdot (-\operatorname{res}_0 \overline{\nabla})$.

Finally, if $\langle \alpha_k, \operatorname{res}_\infty \overline{\nabla} + \check{\rho}\rangle$ is non-negative, then $\operatorname{res}_\infty \overline{g\nabla g^{-1}} = s_{\mathcal{I}}^{\nu} \cdot (\operatorname{res}_\infty \overline{\nabla})$ (whether the function f_k is regular or singular at the origin). □

Remark 3. It is interesting to note that, just as in the case of an ordinary reproduction, we have here a one-parameter family of transformations (the parameter being the initial value of h_k at the origin). In other words, even though we performed $|\mathcal{I}|$ different successive gauge transformations, we do not have $|\mathcal{I}|$ free parameters since the Γ-equivariance imposes relations between them. ◁

Example 3. Let us illustrate Theorem 5.3 with an example. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_4$ and the cyclotomic Miura $\mathfrak{sl}_4$-oper ∇ described in (4.30). We use the notations and conventions introduced in Example 2. To perform a reproduction in the direction of the orbit $\{\alpha_1, \alpha_3\}$, we use a gauge transformation by the unipotent element $g = \exp(f_3 E_3)\exp(f_1 E_1)$. The functions f_1 and f_3 must satisfy the Riccati equations

$$(R^i): \quad f_i'(t) + f_i(t)^2 + q_i(t) f_i(t) = 0,$$

with

$$q_1(t) := -\frac{\eta}{t} - \frac{St^{S-1}}{t^S - z^S}, \qquad q_3(t) := -\frac{\eta}{t} - \frac{St^{S-1}}{t^S + z^S}.$$

Introducing the functions

$$Q_1(t) := e^{-\int q_1(t)\mathrm{d}t} = t^\eta (t^S - z^S),$$

$$R_1(t) := \int Q_1(t)\mathrm{d}t = \frac{1}{\eta + S + 1} t^{\eta+S+1} - \frac{z^S}{\eta+1} t^{\eta+1},$$

one can check that the most general solution of (R^1) is

$$f_1(t) = \frac{Q_1(t)}{R_1(t) + \widetilde{A}} = \frac{(\eta+1)(S+\eta+1)t^\eta(t^S - z^S)}{(\eta+1)t^{S+\eta+1} - (S+\eta+1)z^S t^{\eta+1} + A},$$

where $A := (S+\eta+1)(\eta+1)\widetilde{A}$ is an arbitrary integration constant. Following Theorem 5.3, we introduce

$$f_3(t) := \omega^{-1} f_1(\omega^{-1}t) = \frac{(\eta+1)(S+\eta+1)t^\eta(t^S + z^S)}{(\eta+1)t^{S+\eta+1} + (S+\eta+1)z^S t^{\eta+1} - \omega^{\eta+1}A},$$

where in the second equality we used the fact that $\omega^{-S} = -1$. One can check that f_3 is a solution of the Riccati equation (R^3). The resulting unipotent element g is cyclotomic if and only if

$$f_1(t) = \omega^{-1} f_3(\omega^{-1}t) = \frac{(\eta+1)(S+\eta+1)t^\eta(t^S - z^S)}{(\eta+1)t^{S+\eta+1} - (S+\eta+1)z^S t^{\eta+1} + \omega^{2(\eta+1)}A}.$$

If $A = 0$, then the above condition always holds. On the other hand, if $A \neq 0$, then this condition is equivalent to $\omega^{2(\eta+1)} = 1$, *i.e.* to

$$\langle \alpha_1, \check{\lambda}_0 + \check{\rho} \rangle \equiv \eta + 1 \equiv 0 \bmod \frac{T}{2},$$

in agreement with Theorem 5.3.

Let us also determine the residues at the origin and infinity of the connection $\overline{g\nabla g^{-1}}$ obtained after reproduction. We have seen in example 2 that $\operatorname{res}_\infty \overline{\nabla} = (\eta+S)(\check{\omega}_1 + \check{\omega}_3) + \kappa\check{\omega}_2$. After the reproduction procedure, we get

$$\operatorname{res}_\infty \overline{g\nabla g^{-1}} = (\eta+S)(\check{\omega}_1 + \check{\omega}_3) + \kappa\check{\omega}_2 - (\eta+S+1)(\check{\alpha}_1 + \check{\alpha}_3) = s_1 s_3 \cdot \left(\operatorname{res}_\infty \overline{\nabla}\right).$$

When f_1 and f_3 are both regular at the origin (*i.e.* when $A \neq 0$), the residue at the origin does not change after the reproduction procedure: $\operatorname{res}_0 \overline{g\nabla g^{-1}} = \operatorname{res}_0 \overline{\nabla} = -\eta(\check{\omega}_1 + \check{\omega}_3) - \kappa\check{\omega}_2$. However, if f_1 and f_3 are singular at the origin (*i.e.* $A = 0$), we find

$$-\operatorname{res}_0 \overline{g\nabla g^{-1}} = \eta(\check{\omega}_1 + \check{\omega}_3) + \kappa\check{\omega}_2 - (\eta+1)(\check{\alpha}_1 + \check{\alpha}_3) = s_1 s_3 \cdot \left(-\operatorname{res}_0 \overline{\nabla}\right).$$

◁

5.2. Orbit of Type $A_2^{\times|\mathcal{I}|/2}$

Suppose now that $\mathcal{I} \in I/\nu$ is an orbit of type $A_2^{\times|\mathcal{I}|/2}$ with $|\mathcal{I}|$ even, corresponding to the case $\ell_{\mathcal{I}} = 2$, cf. Sect. 3.3. In the setting of simple Lie algebras of finite type, such orbits occur only in type A_{2n}, in which case the orbit is of size $|\mathcal{I}| = 2$.

Let us fix a reference point $k \in \mathcal{I}$ and denote one half of the orbit by

$$\mathcal{I}/2 = \{k, \nu(k), \ldots, \nu^{|\mathcal{I}|/2-1}(k)\},$$

cf. Lemma 3.6. Recall the notation $\bar{\imath} = \nu^{|\mathcal{I}|/2}(i)$. The full orbit $\mathcal{I}$ then consists of $|\mathcal{I}|/2$ distinct pairs $\{i, \bar{\imath}\}$ for each $i \in \mathcal{I}/2$ such that $\langle \alpha_i, \check{\alpha}_{\bar{\imath}} \rangle = \langle \alpha_{\bar{\imath}}, \check{\alpha}_i \rangle = -1$ and $\langle \alpha_i, \check{\alpha}_j \rangle = 0$ for any $j \in \mathcal{I}$ with $j \neq i, \bar{\imath}$. For each $i \in \mathcal{I}/2$, the subalgebra generated by E_i and $E_{\bar{\imath}}$ has dimension 3 and is spanned by E_i, $E_{\bar{\imath}}$ and $[E_i, E_{\bar{\imath}}]$.

In this case, performing two successive reproductions in the directions of α_i and $\alpha_{\bar{\imath}}$ is not enough to restore the Γ-equivariance of the Miura $\mathfrak{g}$-oper. In

order to obtain a cyclotomic Miura $\mathfrak{g}$-oper, we will need to perform a third reproduction in the direction of α_i, so that the total gauge transformation parameter is of the form $g = \prod_{i\in\mathcal{I}/2} e^{k_{i,3}E_i} e^{k_{i,2}E_{\bar{\imath}}} e^{k_{i,1}E_i}$ for some $k_{i,1}, k_{i,2}, k_{i,3} \in \mathcal{M}$. Equivalently, we can write the latter as

$$g := \prod_{i\in\mathcal{I}/2} e^{f_{i,1}(E_i+E_{\bar{\imath}})+f_{i,2}(E_i-E_{\bar{\imath}})+f_{i,3}[E_i,E_{\bar{\imath}}]} \tag{5.10}$$

for some $f_{i,1}, f_{i,2}, f_{i,3} \in \mathcal{M}$, where we have used the combinations $E_i \pm E_{\bar{\imath}}$ rather than E_i and $E_{\bar{\imath}}$ for later convenience. Note that the order in the above product over $\mathcal{I}/2$ does not matter since the generators $E_i, E_{\bar{\imath}}$ commute with $E_j, E_{\bar{\jmath}}$ for any distinct $i, j \in \mathcal{I}/2$. The calculation can be done using either of the above expressions for g. If we consider the product of exponentials of simple root generators, then we have to study three Riccati equations but, since now $(\alpha_i|\alpha_{\bar{\imath}}) \neq 0$, the argument used in Sect. 5.1 no longer applies. We will use the second form (5.10) for g.

Let $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$. A lengthy but straightforward computation gives the expression of the gauge-transformed $\mathfrak{g}$-connection

$$\begin{aligned} g\nabla g^{-1} = \nabla + \sum_{i\in\mathcal{I}/2} \Big(& (f_{i,1}+f_{i,2})\check{\alpha}_i + (f_{i,1}-f_{i,2})\check{\alpha}_{\bar{\imath}} \\ & - \tfrac{1}{2}\big(f_{i,1}^2 + 3f_{i,2}^2 + (q_i+q_{\bar{\imath}})f_{i,1} + (q_i-q_{\bar{\imath}})f_{i,2} + 2f_{i,1}'\big)(E_i+E_{\bar{\imath}}) \\ & - \tfrac{1}{2}\big(4f_{i,1}f_{i,2} + (q_i+q_{\bar{\imath}})f_{i,2} + (q_i-q_{\bar{\imath}})f_{i,1} - 2f_{i,3} + 2f_{i,2}'\big)(E_i-E_{\bar{\imath}}) \\ & - \big(2f_{i,1}f_{i,3} - f_{i,1}f_{i,2}' + f_{i,2}f_{i,1}' + f_{i,2}(f_{i,2}^2 - f_{i,1}^2) + f_{i,3}' \\ & + (q_i+q_{\bar{\imath}})f_{i,3} - \tfrac{1}{2}(q_i-q_{\bar{\imath}})(f_{i,1}^2 - f_{i,2}^2)\big)[E_i, E_{\bar{\imath}}]\Big)\mathrm{d}t, \end{aligned}$$

where $q_i(t) := \langle \alpha_i, u(t)\rangle$ and $q_{\bar{\imath}}(t) := \langle \alpha_{\bar{\imath}}, u(t)\rangle$.

Since we want $g\nabla g^{-1}$ to be a Miura $\mathfrak{g}$-oper, we should impose that the coefficients of $E_i \pm E_{\bar{\imath}}$ and $[E_i, E_{\bar{\imath}}]$ all vanish. This gives rise to a set of three coupled Riccati-type differential equations, which after some rearranging take the form

$$(R^i): \quad \begin{cases} 2f_{i,1}' + f_{i,1}^2 + 3f_{i,2}^2 + (q_i+q_{\bar{\imath}})f_{i,1} + (q_i-q_{\bar{\imath}})f_{i,2} = 0, \\ 2f_{i,2}' + 4f_{i,1}f_{i,2} - 2f_{i,3} + (q_i+q_{\bar{\imath}})f_{i,2} + (q_i-q_{\bar{\imath}})f_{i,1} = 0, \\ 2f_{i,3}' + 2f_{i,1}f_{i,3} + f_{i,2}(f_{i,1}^2 - f_{i,2}^2) + 2(q_i+q_{\bar{\imath}})f_{i,3} = 0. \end{cases} \tag{5.11}$$

Noting that $\omega^{-1}\nu\big(u(\omega^{-1}t)\big) = u(t)$ since $u(t)\mathrm{d}t \in \Omega^{\hat{\nu}}(\mathfrak{h})$, which also implies that for every $i \in \mathcal{I}/2$ we have $\omega^{-1}q_{\nu^{-1}(i)}(\omega^{-1}t) = q_i(t)$ and $\omega^{-1}q_{\nu^{-1}(\bar{\imath})}(\omega^{-1}t) = q_{\bar{\imath}}(t)$, we deduce that the three functions $\omega^{-1}f_{i,1}(\omega^{-1}t)$, $\omega^{-1}f_{i,2}(\omega^{-1}t)$ and $\omega^{-2}f_{i,3}(\omega^{-1}t)$ satisfy the coupled system of equations $(R^{\nu(i)})$. Therefore, starting with *any* solution $f_{k,1}, f_{k,2}, f_{k,3} \in \mathcal{M}$ for the reference point $k \in \mathcal{I}$ we can define a solution $f_{i,1}, f_{i,2}, f_{i,3} \in \mathcal{M}$ of (R^i) recursively by

$$\begin{aligned} f_{i,1}(t) &:= \omega^{-1}f_{\nu^{-1}(i),1}(\omega^{-1}t), \\ f_{i,2}(t) &:= \omega^{-1}f_{\nu^{-1}(i),2}(\omega^{-1}t), \\ f_{i,3}(t) &:= \omega^{-2}f_{\nu^{-1}(i),3}(\omega^{-1}t). \end{aligned} \tag{5.12}$$

for every $i \in \mathfrak{I}/2\backslash\{k\}$. Now since $\nu^{|\mathfrak{I}|/2}(k) = \bar{k}$ we have

$$\varsigma\big(E_{\nu^{|\mathfrak{I}|/2-1}(k)} \pm E_{\nu^{|\mathfrak{I}|/2-1}(\bar{k})}\big) = \pm\omega^{-1}(E_k \pm E_{\bar{k}}),$$
$$\varsigma\big(\big[E_{\nu^{|\mathfrak{I}|/2-1}(k)}, E_{\nu^{|\mathfrak{I}|/2-1}(\bar{k})}\big]\big) = -\omega^{-2}[E_k, E_{\bar{k}}].$$

So by construction, we find that $g \in N(\mathcal{M})$ defined in (5.10) using the functions $f_{i,1}, f_{i,2}, f_{i,3}$ for all $i \in \mathfrak{I}/2$ is invariant under $\hat{\varsigma}$ if and only if

$$f_{k,1}(t) = \omega^{-1} f_{\nu^{|\mathfrak{I}|/2-1}(k),1}(\omega^{-1}t), \qquad f_{k,2}(t) = -\omega^{-1} f_{\nu^{|\mathfrak{I}|/2-1}(k),2}(\omega^{-1}t),$$
$$f_{k,3}(t) = -\omega^{-2} f_{\nu^{|\mathfrak{I}|/2-1}(k),3}(\omega^{-1}t).$$

Equivalently, using the above recurrence relations to rewrite the right-hand sides of these equations we obtain conditions on the functions $f_{k,1}$, $f_{k,2}$ and $f_{k,3}$ alone, namely

$$f_{k,1}(t) = \omega^{-|\mathfrak{I}|/2} f_{k,1}(\omega^{-|\mathfrak{I}|/2}t), \quad f_{k,2}(t) = -\omega^{-|\mathfrak{I}|/2} f_{k,2}(\omega^{-|\mathfrak{I}|/2}t),$$
$$f_{k,3}(t) = -\omega^{-|\mathfrak{I}|} f_{k,3}(\omega^{-|\mathfrak{I}|/2}t). \tag{5.13}$$

We now consider the issue of the regularity of Eqs. (5.11) at the origin. The functions q_i and $q_{\bar{\imath}}$ have the form $q_i(t) = -\frac{\eta}{t} + r_i(t)$ and $q_{\bar{\imath}}(t) = -\frac{\eta}{t} + r_{\bar{\imath}}(t)$, with $\eta = \langle \alpha_i, \check{\lambda}_0 \rangle = \langle \alpha_{\bar{\imath}}, \check{\lambda}_0 \rangle$ and $r_i, r_{\bar{\imath}} \in \mathfrak{h}(\mathcal{M})$ both regular at the origin. Note that η is independent of i and $\bar{\imath}$ by the ν-invariance of the coweight $\check{\lambda}_0$ and is a non-negative integer, as $\check{\lambda}_0$ is a dominant integral coweight. We will need the following result, which is the analog, for an orbit of type $A_2^{\times|\mathfrak{I}|/2}$, of Lemma 5.2.

Lemma 5.4. *Let f_1, f_2 and f_3 be meromorphic solutions of the system of differential equations*

$$2f_1' + f_1^2 + 3f_2^2 + (q+\overline{q})f_1 + (q-\overline{q})f_2 = 0, \tag{5.14a}$$
$$2f_2' + 4f_1f_2 - 2f_3 + (q+\overline{q})f_2 + (q-\overline{q})f_1 = 0, \tag{5.14b}$$
$$2f_3' + 2f_1f_3 + f_2(f_1^2 - f_2^2) + 2(q+\overline{q})f_3 = 0. \tag{5.14c}$$

Suppose $q(t) = -\frac{\eta}{t} + r(t)$ and $\overline{q}(t) = -\frac{\eta}{t} + \overline{r}(t)$, with $r, \overline{r} \in \mathcal{M}$ regular at 0. If $\eta \in \mathbb{Z}_{\geq 0}$ then either

(*i*) *the functions f_i are regular at 0, in which case they read $f_1(t) = t^\eta h_1(t)$, $f_2(t) = t^\eta h_2(t)$ and $f_3(t) = t^{2\eta} h_3(t)$ for some $h_i \in \mathcal{M}$ regular at 0, with $h_i(0) \in \mathbb{C}$ arbitrary.*

(*ii*) *at least one of the functions f_i is singular at 0, in which case they are of the form*

$$f_1(t) = \frac{a}{t} + k_1(t), \quad f_2(t) = \frac{b}{t} + k_2(t), \quad f_3(t) = \frac{c}{t^2} + \frac{\mathrm{d}}{t} + k_3(t), \tag{5.15}$$

where $k_i \in \mathcal{M}$ are regular at 0 and $a, b, c, \mathrm{d} \in \mathbb{C}$. Moreover, the coefficients a, b, c fall into one of two possible classes

(*a*) *$(a,b,c) = \big(2(\eta+1), 0, 0\big)$, in which case $\mathrm{d} \in \mathbb{C}$ and $k_i \in \mathcal{M}$ are unique,*

(*b*) *$(a,b,c) = \big(\frac{\eta+1}{2}, \pm\frac{\eta+1}{2}, 0\big)$ or $(a,b,c) = \Big(\frac{3(\eta+1)}{2}, \pm\frac{\eta+1}{2}, \pm(\eta+1)^2\Big)$.*

Proof. Consider the Laurent expansions of the functions f_i around the origin. The consistency of these expansions with Eqs. (5.14) requires f_1 and f_2 to have at most a simple pole at 0 and f_3 at most a double pole at 0. Thus the f_i's can be written in the form of Eq. (5.15). Moreover, one finds that the cancellation of the double poles in the first two equations and of the triple pole in the third one leads to the set of conditions

$$a^2-2(\eta+1)a+3b^2=0, \quad 2ab-(\eta+1)b-c=0, \quad 2ac+b(a^2-b^2)-4(\eta+1)c=0. \tag{5.16}$$

There are six solutions to this system of equations: $(a,b,c)=(0,0,0)$, $(a,b,c)=\big(2(\eta+1),0,0\big)$, $(a,b,c)=\big(\frac{\eta+1}{2},\pm\frac{\eta+1}{2},0\big)$ and $(a,b,c)=\big(\frac{3(\eta+1)}{2},\pm\frac{\eta+1}{2},\pm(\eta+1)^2\big)$.

In the case where $(a,b,c)=(0,0,0)$, one finds that d is also zero. This corresponds to case (i) of the lemma, where the functions f_i are all regular at the origin. Consider then the Taylor expansions $f_1(t)=\sum_{n\geq 0}a_nt^n$, $f_2(t)=\sum_{n\geq 0}b_nt^n$ and $f_3(t)=\sum_{n\geq 0}c_nt^n$. Substituting these into (5.14) and working order by order in powers of t, one finds that $a_0=b_0=c_0=0$ and that the coefficients a_n, b_n and c_n for $n\in\mathbb{Z}_{\geq 1}$ must satisfy the recurrence relations

$$\begin{cases} 2(n+1-\eta)a_{n+1} = -\sum_{k=0}^{n}\Big(a_k(a_{n-k}+r_{n-k}+\overline{r}_{n-k}) \\ \qquad\qquad + b_k(3b_{n-k}+r_{n-k}-\overline{r}_{n-k})\Big), \\ 2(n+1-\eta)b_{n+1} = 2c_n-\sum_{k=0}^{n}\Big(a_k(4b_{n-k}+r_{n-k}-\overline{r}_{n-k}) \\ \qquad\qquad + b_k(r_{n-k}+\overline{r}_{n-k})\Big), \\ 2(n+1-2\eta)c_{n+1} = -\sum_{k=0}^{n}c_k(a_{n-k}+2r_{n-k}+2\overline{r}_{n-k}) \\ \qquad\qquad + \sum_{k_1+k_2+k_3=n}\big(b_{k_1}b_{k_2}b_{k_3}-b_{k_1}a_{k_2}a_{k_3}\big), \end{cases}$$

where r_n and $\overline{r}_n$ are coefficients in the Taylor expansion of r and $\overline{r}$.

We have $a_0=b_0=c_0=0$. For $n+1<\eta$, the coefficients $n+1-\eta$ and $n+1-2\eta$ in the left-hand sides of these equations are nonzero, so we conclude by induction that $a_n=b_n=c_n=0$ for $n<\eta-1$. When $n=\eta-1$, the first two equations are verified for arbitrary values of $a_{n+1}=a_\eta\in\mathbb{C}$ and $b_{n+1}=b_\eta\in\mathbb{C}$. Similarly, the third equation yields $c_n=0$ for all $n<2\eta-1$ and $c_{2\eta}\in\mathbb{C}$ is arbitrary. The coefficients a_n and b_n for $n>\eta$ and c_n for $n>2\eta$ are then determined uniquely in terms of the r_n's, the $\overline{r}_n$'s and the arbitrary coefficients a_η, b_η and $c_{2\eta}$. This concludes the discussion of case (i) of the lemma.

Next, we turn to the five other solutions of Eq. (5.16), corresponding to the case where at least one of the f_i's is singular, i.e. the case (ii) of the lemma. We shall focus on case (a), with $(a,b,c)=\big(2(\eta+1),0,0\big)$. Substituting

the Taylor expansions of the regular functions k_i into (5.14), one finds unique expressions for the coefficients d, a_0, b_0 and c_0. After a tedious but straightforward computation, one finds recurrence relations similar to the ones above. In the case at hand, however, the coefficients in front of a_{n+1}, b_{n+1} and c_{n+1} are nonzero for any $n \in \mathbb{Z}_{\geq 0}$. Thus, all the coefficients a_n, b_n and c_n are determined uniquely by these recursion relations and the initial values a_0, b_0 and c_0 previously obtained. Therefore, in this case, the functions k_i are unique. □

Let us now come back to the discussion of cyclotomic reproductions along the orbit $\mathcal{I}$. We began with solutions $f_{k,1}$, $f_{k,2}$ and $f_{k,3}$ of equations (R^k). As explained above, we constructed solutions $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$ of equations (R^i) for any $i \in \mathcal{I}/2$ recursively using formula (5.12). The condition for the corresponding reproduction to be cyclotomic is then given by equation (5.13).

5.2.1. Regular Reproduction Procedure. Suppose that the functions $f_{k,1}$, $f_{k,2}$ and $f_{k,3}$ (and thus also the functions $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$ for any $i \in \mathcal{I}/2$) are regular at the origin, so that they are described by case (i) of Lemma 5.4. Then, $h_{i,1}(t) := t^{-\eta} f_{i,1}(t)$, $h_{i,2}(t) := t^{-\eta} f_{i,2}(t)$ and $h_{i,3}(t) := t^{-2\eta} f_{i,3}(t)$ are regular at the origin. They obey the following regularised equations

$$2h'_{i,1} + t^\eta(h_{i,1}^2 + 3h_{i,2}^2) + (r_i + r_{\bar{\imath}})h_{i,1} + (r_i - r_{\bar{\imath}})h_{i,2} = 0, \tag{5.17a}$$

$$2h'_{i,2} + t^\eta(4h_{i,1}h_{i,2} - 2h_{i,3}) + (r_i + r_{\bar{\imath}})h_{i,2} + (r_i - r_{\bar{\imath}})h_{i,1} = 0, \tag{5.17b}$$

$$2h'_{i,3} + t^\eta\big(2h_{i,1}h_{i,3} + h_{i,2}(h_{i,1}^2 - h_{i,2}^2)\big) + 2(r_i + r_{\bar{\imath}})h_{i,3} = 0, \tag{5.17c}$$

whose coefficients are regular at 0 since $\eta \in \mathbb{Z}_{\geq 0}$. The coupled system of equations (5.17) admits a solution regular at 0 for every set of initial conditions $h_{i,1}(0), h_{i,2}(0), h_{i,3}(0) \in \mathbb{C}$. It remains to check whether there are solutions for which the corresponding gauge transformation parameter g defined in (5.10) is $\hat{\varsigma}$-invariant so that $g\nabla g^{-1} \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$.

When phrased in terms of the regularised functions $h_{k,1}$, $h_{k,2}$ and $h_{k,3}$, conditions (5.13) for $g \in N(\mathcal{M})$ to be $\hat{\varsigma}$-invariant read

$$h_{k,1}(t) = \omega^{-(1+\eta)|\mathcal{I}|/2} h_{k,1}(\omega^{-|\mathcal{I}|/2}t), \quad h_{k,2}(t) = -\omega^{-(1+\eta)|\mathcal{I}|/2} h_{k,2}(\omega^{-|\mathcal{I}|/2}t),$$
$$h_{k,3}(t) = -\omega^{-(1+\eta)|\mathcal{I}|} h_{k,3}(\omega^{-|\mathcal{I}|/2}t). \tag{5.18}$$

Now by construction, both sides of the relations (5.18) satisfy the same system of coupled Eqs. (5.17). Thus, (5.18) hold for all $t \in \mathbb{C}$ if and only if they hold at the origin. It follows that $g \in N^{\hat{\varsigma}}(\mathcal{M})$ if and only if

$$h_{k,1}(0) = \omega^{-(1+\eta)|\mathcal{I}|/2} h_{k,1}(0), \qquad h_{k,2}(0) = -\omega^{-(1+\eta)|\mathcal{I}|/2} h_{k,2}(0),$$
$$h_{k,3}(0) = -\omega^{-(1+\eta)|\mathcal{I}|} h_{k,3}(0). \tag{5.19}$$

Since the coupled Eqs. (5.17) are homogeneous, for the set of functions $h_{k,1}$, $h_{k,2}$ and $h_{k,3}$ to be non-trivial, and hence $g \neq \mathrm{Id}$, we should certainly have at least one of the initial conditions $h_{k,1}(0)$, $h_{k,2}(0)$ and $h_{k,3}(0)$ be nonzero. However, conditions (5.19) are consistent only if at most one of these initial conditions is nonzero. We therefore have three distinct possibilities:

- $\omega^{(1+\eta)|\mathcal{I}|/2} = 1 \quad$ if $\quad h_{k,2}(0) = h_{k,3}(0) = 0,$
- $\omega^{(1+\eta)|\mathcal{I}|/2} = -1 \quad$ if $\quad h_{k,1}(0) = h_{k,3}(0) = 0,$
- $\omega^{(1+\eta)|\mathcal{I}|} = -1 \quad$ if $\quad h_{k,1}(0) = h_{k,2}(0) = 0.$

In each case, the $\hat{\varsigma}$-invariance of g reduces the number of free initial conditions to one. Therefore, we have once again a one-parameter family of cyclotomic gauge transformations. We note that the above three cases can be combined into the single condition $\omega^{2|\mathcal{I}|(1+\eta)} = 1$. Or since ω is a primitive Tth-root of unity, this is equivalent to $2|\mathcal{I}|(1+\eta) \equiv 0 \bmod T$. Using the definition of η, we can also rewrite this as $|\mathcal{I}|\langle \alpha_k + \alpha_{\bar{k}}, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \bmod T$ and in turn, since $|\mathcal{I}|$ divides T, as

$$\langle \alpha_k + \alpha_{\bar{k}}, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \bmod \frac{T}{|\mathcal{I}|}.$$

Since all the functions $f_{i,1}$ and $f_{i,2}$ are regular at 0, it is clear that this cyclotomic reproduction does not change the residue at the origin, namely

$$\operatorname{res}_0 \overline{g\nabla g^{-1}} = \operatorname{res}_0 \overline{\nabla} = -\check{\lambda}_0.$$

5.2.2. Singular Reproduction Procedure. Suppose now that at least one of the functions $f_{k,1}$, $f_{k,2}$ and $f_{k,3}$ is singular at 0. They are then described by case (ii) of Lemma 5.4 and are of the form (5.15). Let us see when these solutions correspond to a cyclotomic reproduction, i.e. when they verify the functional relation (5.13). It is clear that this relation is compatible with the form (5.15) only if the coefficients b and c are zero. According to Lemma 5.4, this only leaves case (a) as a possibility, where $(a, b, c) = (2(\eta+1), 0, 0)$.

By construction, the sets of functions on the left- and right-hand sides of (5.13) are solutions of the same equation (R^k). Moreover, they are both of the form given in case $(ii)(a)$ of Lemma 5.4. By uniqueness of the solutions of this form, these two sets of functions must be equal and thus the corresponding reproduction procedure is cyclotomic.

Finally, recall that the new $\mathfrak{g}$-connection is given by

$$g\nabla g^{-1} = \nabla + \sum_{i\in\mathcal{I}/2} \Big(f_{i,1}\,(\check{\alpha}_i + \check{\alpha}_{\bar{\imath}}) + f_{i,2}\,(\check{\alpha}_i - \check{\alpha}_{\bar{\imath}}) \Big) \mathrm{d}t.$$

Using the fact that

$$\operatorname{res}_0 f_{i,1}\mathrm{d}t = 2(\eta+1) = \langle \alpha_i + \alpha_{\bar{\imath}}, \check{\lambda}_0 + \check{\rho}\rangle, \quad \operatorname{res}_0 f_{i,2}\mathrm{d}t = 0 \ \text{ and } \ \operatorname{res}_0 f_{i,3}\mathrm{d}t = 0,$$

and Lemma 3.6, one finds

$$-\operatorname{res}_0 \overline{g\nabla g^{-1}} = \left(\prod_{i\in\mathcal{I}/2} s_i s_{\bar{\imath}} s_i \right) \cdot \check{\lambda}_0 = s_{\mathcal{I}}^{\nu} \cdot (-\operatorname{res}_0 \overline{\nabla}).$$

5.2.3. Residue at Infinity. The following discussion applies whether the reproduction procedure is regular or singular at the origin. In order to determine the residue of the new Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ at infinity, recall that the gauge transformation parameter can also be written in the form of a product of exponentials of simple root generators as $g = \prod_{i\in\mathcal{I}/2} e^{k_{i,3}E_i} e^{k_{i,2}E_{\bar{\imath}}} e^{k_{i,1}E_i}$, where

$k_{i,1}, k_{i,2}, k_{i,3} \in \mathcal{M}$ can be expressed in terms of the functions $f_{i,1}, f_{i,2}, f_{i,3} \in \mathcal{M}$ constructed above. In particular, if we started with functions $f_{k,1}$, $f_{k,2}$ and $f_{k,3}$ not all zero and verifying the cyclotomy condition (5.13), then one checks that the functions $k_{i,1}$, $k_{i,2}$ and $k_{i,3}$ for $i \in \mathcal{I}/2$ are all nonzero.

Let $\check{\mu} = \mathrm{res}_\infty \overline{\nabla}$ and fix $i \in \mathcal{I}/2$. Suppose that $\kappa = \langle \alpha_i, \check{\mu} + \check{\rho} \rangle = \langle \alpha_{\bar{\imath}}, \check{\mu} + \check{\rho} \rangle$ is non-negative and consider the three successive reproductions with overall parameter $g_i = e^{k_{i,3}E_i} e^{k_{i,2}E_{\bar{\imath}}} e^{k_{i,1}E_i}$. According to the last part of Lemma 5.1, since $k_{i,1}$ is nonzero, it has a simple pole at infinity and the first reproduction then yields a new $\mathfrak{g}$-connection with residue $s_i \cdot \check{\mu}$ at infinity. Recalling that $\langle \alpha_i, \check{\alpha}_{\bar{\imath}} \rangle = \langle \alpha_{\bar{\imath}}, \check{\alpha}_i \rangle = -1$, one then finds $\langle \alpha_{\bar{\imath}}, s_i \cdot \check{\mu} + \check{\rho} \rangle = 2\kappa$. Thus, the last part of Lemma 5.1 still applies to the second reproduction and the new $\mathfrak{g}$-connection has residue $s_{\bar{\imath}} s_i \cdot \check{\mu}$ at infinity. Finally, one has $\langle \alpha_i, s_{\bar{\imath}} s_i \cdot \check{\mu} + \check{\rho} \rangle = \kappa$. Applying Lemma 5.1 one last time, one finds that after the gauge transformation by g_i, the residue at infinity of the new $\mathfrak{g}$-connection $\overline{g_i \nabla g_i^{-1}}$ is $s_i s_{\bar{\imath}} s_i \cdot \big(\mathrm{res}_\infty \overline{\nabla}\big)$.

If $\langle \alpha_k, \check{\mu} + \check{\rho} \rangle$ is non-negative for some $k \in \mathcal{I}/2$, then $\langle \alpha_i, \check{\mu} + \check{\rho} \rangle$ is non-negative for all $i \in \mathcal{I}/2$. As $\langle \alpha_i, \check{\alpha}_j \rangle = 0$ for any $j \neq i, \bar{\imath}$, the group elements g_i, $i \in \mathcal{I}/2$, commute with one another and

$$\langle \alpha_i, \mathrm{res}_\infty \overline{g_j \nabla g_j^{-1}} + \check{\rho} \rangle = \langle \alpha_i, \mathrm{res}_\infty \overline{\nabla} + \check{\rho} \rangle. \tag{5.20}$$

Thus, we can apply successively (and in any order) the reasoning of the previous paragraph to the gauge transformations with parameters g_i, for all $i \in \mathcal{I}/2$. We then deduce that

$$\mathrm{res}_\infty \overline{g \nabla g^{-1}} = \left(\prod_{i \in \mathcal{I}/2} s_i s_{\bar{\imath}} s_i \right) \cdot (\mathrm{res}_\infty \overline{\nabla}) = s_{\mathcal{I}}^\nu \cdot (\mathrm{res}_\infty \overline{\nabla}),$$

where we used Lemma 3.6. We have thus proved the following.

Theorem 5.5. *Let $\mathcal{I} \in I/\nu$ be an orbit of type $A_2^{\times |\mathcal{I}|/2}$. Fix a $k \in \mathcal{I}$ and let $f_{k,1}, f_{k,2}, f_{k,3} \in \mathcal{M}$ be solutions of the coupled differential equations (5.11), not all zero. Define $f_{i,1}, f_{i,2}, f_{i,3} \in \mathcal{M}$ for all $i \in (\mathcal{I}/2) \backslash \{k\}$ recursively by*

$$\begin{aligned} f_{i,1}(t) &:= \omega^{-1} f_{\nu^{-1}(i),1}(\omega^{-1} t), \\ f_{i,2}(t) &:= \omega^{-1} f_{\nu^{-1}(i),2}(\omega^{-1} t), \\ f_{i,3}(t) &:= \omega^{-2} f_{\nu^{-1}(i),3}(\omega^{-1} t). \end{aligned}$$

and let $g := \prod_{i \in \mathcal{I}/2} e^{f_{i,1}(E_i + E_{\bar{\imath}}) + f_{i,2}(E_i - E_{\bar{\imath}}) + f_{i,3}[E_i, E_{\bar{\imath}}]}$.

If we consider functions $f_{k,1}, f_{k,2}, f_{k,3}$ regular at the origin, then they can be chosen so that g belongs to $N^{\hat{\varsigma}}(\mathcal{M})$ if and only if

$$\langle \alpha_k + \alpha_{\bar{k}}, \check{\lambda}_0 + \check{\rho} \rangle \equiv 0 \mod \frac{T}{|\mathcal{I}|}.$$

Moreover, in this case we have $g \nabla g^{-1} \in \mathrm{MOp}_{\mathfrak{g}}^{\Gamma}(\mathbb{P}^1)$ with $\mathrm{res}_0 \overline{g \nabla g^{-1}} = \mathrm{res}_0 \overline{\nabla}$.

Suppose now that at least one of the functions $f_{k,1}, f_{k,2}, f_{k,3}$ is singular at the origin, so that they are described by case (ii) of Lemma 5.4. Then, g is in $N^{\hat{\varsigma}}(\mathcal{M})$ if and only if the singular behaviour of $f_{k,1}, f_{k,2}, f_{k,3}$ is as in the

subcase (a)*, without any conditions on the coweight* $\check{\lambda}_0$*. Moreover, in this case we have* $g\nabla g^{-1} \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ *with* $-\operatorname{res}_0 \overline{g\nabla g^{-1}} = s^{\nu}_{\mathcal{J}} \cdot (-\operatorname{res}_0 \overline{\nabla})$.

Finally, if $\langle \alpha_k, \operatorname{res}_\infty \overline{\nabla} + \check{\rho} \rangle$ *is non-negative, then we have* $\operatorname{res}_\infty \overline{g\nabla g^{-1}} = s^{\nu}_{\mathcal{J}} \cdot (\operatorname{res}_\infty \overline{\nabla})$ *(whether we started with functions regular or singular at the origin).* □

Remark 4. Note that the approach of this section may equally be applied to the case of an orbit of type $A_1^{\times|\mathcal{J}|}$ for even $|\mathcal{J}|$: in this case, there are no functions $f_{i,3}$ and we only get the first two conditions $\omega^{(1+\eta)|\mathcal{J}|/2} = \pm 1$ above, which are indeed equivalent to the condition $\langle \alpha_k, \check{\lambda}_0 + \check{\rho} \rangle \equiv 0 \bmod \frac{T}{|\mathcal{J}|}$ as given in Theorem 5.3. ◁

Example 4. Let us illustrate Theorem 5.5 with an example. We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ and the cyclotomic Miura $\mathfrak{sl}_3$-oper ∇ defined in equation (4.29) of example 1.

We consider the orbit of simple roots $\{\alpha_1, \alpha_2\}$, which is of type A_2, and the unipotent element

$$g = e^{\tilde{f}_1 E_1 + \tilde{f}_2 E_2 + \tilde{f}_3 E_{12}}, \tag{5.21}$$

expressed in the basis $E_1, E_2, E_{12} := [E_1, E_2]$. In Theorem 5.5 and the discussion above, g is given by $g = e^{f_1(E_1+E_2)+f_2(E_1-E_2)+f_3 E_{12}}$, in a basis of eigenvectors of ς. The two expressions are simply related by $\tilde{f}_1 = f_1 + f_2$, $\tilde{f}_2 = f_1 - f_2$ and $\tilde{f}_3 = f_3$. In the fundamental representation, we have

$$g = \begin{pmatrix} 1 & \tilde{f}_1 & \tilde{f}_3 + \frac{1}{2}\tilde{f}_1\tilde{f}_2 \\ 0 & 1 & \tilde{f}_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The gauge transformation $g\nabla g^{-1}$ is a Miura $\mathfrak{sl}_3$-oper if and only if the functions $\tilde{f}_i$ are solutions of the following system of coupled differential equations

$$\begin{aligned}
\tilde{f}_1'(t) &= \frac{\eta}{t}\tilde{f}_1(t) - \tilde{f}_1(t)^2 - \frac{1}{2}\tilde{f}_1(t)\tilde{f}_2(t) - \tilde{f}_3(t), \\
\tilde{f}_2'(t) &= \frac{\eta}{t}\tilde{f}_2(t) - \tilde{f}_2(t)^2 - \frac{1}{2}\tilde{f}_1(t)\tilde{f}_2(t) + \tilde{f}_3(t), \\
\tilde{f}_3'(t) &= 2\frac{\eta}{t}\tilde{f}_3(t) - \Big(\tilde{f}_1(t) + \tilde{f}_2(t)\Big)\tilde{f}_3(t) + \frac{1}{2}\Big(\tilde{f}_1(t)^2 + \tilde{f}_1'(t)\Big)\tilde{f}_2(t) \\
&\quad - \frac{1}{2}\Big(\tilde{f}_2(t)^2 + \tilde{f}_2'(t)\Big)\tilde{f}_1(t).
\end{aligned}$$

The regularised functions $\tilde{h}_1(t) = t^{-\eta}\tilde{f}_1(t)$, $\tilde{h}_2(t) = t^{-\eta}\tilde{f}_2(t)$ and $\tilde{h}_3(t) = t^{-2\eta}\tilde{f}_3(t)$ are solutions of differential equations regular at the origin. The solution of the above differential system with initial conditions $\tilde{h}_1(0) = a$, $\tilde{h}_2(0) = b$ and $\tilde{h}_3(0) = c$ is

$$\tilde{f}_1(t) = 2\mu\, t^{\mu-1} \frac{(ab+2c)\, t^\mu + 2\mu a}{(ab+2c)\, t^{2\mu} + 4\mu a\, t^\mu + 4\mu^2}, \tag{5.22a}$$

$$\tilde{f}_2(t) = 2\mu\, t^{\mu-1} \frac{(ab-2c)\, t^\mu + 2\mu b}{(ab-2c)\, t^{2\mu} + 4\mu b\, t^\mu + 4\mu^2}, \tag{5.22b}$$

$$\tilde{f}_3(t) = 4\mu^3\, t^{2\mu-2} \frac{\big((ab+2c)b - (ab-2c)a\big)\, t^\mu + 4\mu c}{\big((ab+2c)\, t^{2\mu} + 4\mu a\, t^\mu + 4\mu^2\big)\big((ab-2c)\, t^{2\mu} + 4\mu b\, t^\mu + 4\mu^2\big)}. \tag{5.22c}$$

where we defined $\mu = \eta + 1$.

To simplify the analysis of this solution, we define functions A and B such that $\tilde{f}_1(t) = A(t; a, b, c)$, $\tilde{f}_2(t) = A(t; b, a, -c)$ and $\tilde{f}_3(t) = B(t; a, b, c) = -B(t; b, a, -c)$. One checks easily that

$$\omega^{-1} A(\omega^{-1} t; a, b, c) = A(t; \omega^{-\mu} a, \omega^{-\mu} b, \omega^{-2\mu} c),$$
$$\omega^{-2} B(\omega^{-1} t; a, b, c) = B(t; \omega^{-\mu} a, \omega^{-\mu} b, \omega^{-2\mu} c).$$

Noting that g is $\hat{\varsigma}$-invariant if and only if we have $\omega^{-1}\tilde{f}_1(\omega^{-1}t) = \tilde{f}_2(t)$, $\omega^{-1}\tilde{f}_2(\omega^{-1}t) = \tilde{f}_1(t)$ and $\omega^{-2}\tilde{f}_3(\omega^{-1}t) = -\tilde{f}_3(t)$, the condition for the reproduction to be cyclotomic can be re-expressed as

$$A(t; \omega^{-\mu} a, \omega^{-\mu} b, \omega^{-2\mu} c) = A(t; b, a, -c),$$
$$A(t; \omega^{-\mu} b, \omega^{-\mu} a, -\omega^{-2\mu} c) = A(t; a, b, c),$$
$$B(t; \omega^{-\mu} a, \omega^{-\mu} b, \omega^{-2\mu} c) = B(t; b, a, -c).$$

These conditions are equivalent to $a = \omega^\mu b$, $b = \omega^\mu a$ and $c = -\omega^{2\mu} c$. As in the general discussion above, we distinguish three possible cases where these conditions have non-trivial solutions:

- $\omega^\mu = 1$, $a = b \in \mathbb{C}^\times$ and $c = 0$, i.e. $h_2(0) = h_3(0) = 0$, in which case

$$f_1(t) = \frac{2\mu a\, t^{\mu-1}}{a\, t^\mu + 2\mu}, \quad f_2(t) = 0, \quad f_3(t) = 0,$$

- $\omega^\mu = -1$, $a = -b \in \mathbb{C}^\times$ and $c = 0$, i.e. $h_1(0) = h_3(0) = 0$, in which case

$$f_1(t) = \frac{2\mu a^2\, t^{2\mu-1}(a^2\, t^{2\mu} - 12\mu^2)}{R(t)},$$
$$f_2(t) = \frac{4\mu^2 a\, t^{\mu-1}(a^2\, t^{2\mu} + 4\mu^2)}{R(t)},$$
$$f_3(t) = \frac{8\mu^3 a^3\, t^{3\mu-2}}{R(t)},$$

  where $R(t) = a^4\, t^{4\mu} - 24\mu^2 a^2\, t^{2\mu} + 16\mu^4$,

- $\omega^{2\mu} = -1$, $a = b = 0$ and $c \in \mathbb{C}^\times$, i.e. $h_1(0) = h_2(0) = 0$, in which case

$$f_1(t) = \frac{2\mu c^2\, t^{4\mu-1}}{c^2\, t^{4\mu} - 4\mu^4}, \quad f_2(t) = -\frac{4\mu^3 c\, t^{2\mu-1}}{c^2\, t^{4\mu} - 4\mu^4}, \quad f_3(t) = -\frac{4\mu^4 c\, t^{2\mu-2}}{c^2\, t^{4\mu} - 4\mu^4}, \tag{5.23}$$

where the regularised functions $h_1 = \frac{1}{2}(\tilde{h}_1 + \tilde{h}_2)$, $h_2 = \frac{1}{2}(\tilde{h}_1 - \tilde{h}_2)$ and $h_3 = \tilde{h}_3$ correspond to the basis of eigenvectors $\{E_1 + E_2, E_1 - E_2, [E_1, E_2]\}$. In each of these cases, the solution depends on an arbitrary parameter in $\mathbb{C}^\times$.

To end this example, we discuss also the solutions of the differential system singular at the origin. Starting from any of the above three types of regular solutions, a singular solution can be obtained by considering the limit where the free parameter goes to infinity. One can check that each of the three cases yields the same singular solution

$$g(t) = \exp\left(\frac{2(\eta+1)}{t}\,(E_1 + E_2)\right). \tag{5.24}$$

Note that this solution is always cyclotomic, without imposing any restriction on ω or η. ◁

6. Cyclotomic Miura $\mathfrak{g}$-Opers and Flag Varieties

Recall the setup of Sect. 5. Throughout this section, we will also consider a cyclotomic Miura $\mathfrak{g}$-oper of the form $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t$ with $u\,\mathrm{d}t \in \Omega^{\hat{\nu}}(\mathfrak{h})$ as in (5.1). In particular, we still assume for the time being that $-\operatorname{res}_0 \overline{\nabla} = \check{\lambda}_0$ is a ν-invariant integral dominant coweight. We will further assume here that ∇ is monodromy-free. In Sect. 6.2, however, we will show how to weaken the integrality assumption on the coweight $\check{\lambda}_0$ together with the related assumption on the trivial monodromy at the origin.

Consider the set

$$\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma} := \left\{\widetilde{\nabla} \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1) \,\middle|\, [\widetilde{\nabla}]_\Gamma = [\nabla]_\Gamma\right\} \tag{6.1}$$

of cyclotomic Miura $\mathfrak{g}$-opers with underlying cyclotomic $\mathfrak{g}$-oper $[\nabla]_\Gamma$. By definition, any cyclotomic Miura $\mathfrak{g}$-oper in $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ is a representative of $[\nabla]_\Gamma$ and hence is of the form $\nabla^g := g\nabla g^{-1}$ for some $g \in N^{\hat{\varsigma}}(\mathcal{M})$. The goal of this section is to describe the space $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$.

A finite analog of (6.1) is the set $\mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g}^\nu,[\check{\lambda}_0]_{W^\nu}}$, introduced in Sect. 3.2, consisting of all finite Miura $\mathfrak{g}^\nu$-opers which are connected to the given finite Miura $\mathfrak{g}^\nu$-oper $p_{-1} - \check{\lambda}_0 - \check{\rho}$ by the adjoint action of N^ν. We saw in Theorem 3.4 that $\mathrm{MOp}^{\mathrm{fin}}_{\mathfrak{g}^\nu,[\check{\lambda}_0]_{W^\nu}}$ is in bijection with the ν-invariant subgroup W^ν of the Weyl group W. The affine counterpart of this statement is Theorem 6.5 which provides an isomorphism between $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ and the subset $(G/B_-)^\vartheta$ of the flag variety G/B_- consisting of points fixed by some automorphism ϑ which we introduce in Sect. 6.1. In Sect. 6.4, we show that $(G/B_-)^\vartheta$ admits a cell decomposition where the cells are labelled by elements of W^ν.

6.1. Generic Cyclotomic Miura $\mathfrak{g}$-Opers

If $\check{\lambda}_0 = 0$, then the origin is a regular point of the given cyclotomic Miura $\mathfrak{g}$-oper ∇. In this case, we let $Y \in B_-(\mathcal{M})$ be the solution of $\nabla Y = 0$ with $Y(0) = \mathrm{Id}$, which exists by Lemma 4.2. Then, for any $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ such that $g \in N^{\hat{\varsigma}}(\mathcal{M})$ is also regular at 0, we will denote by $Y_g \in B_-(\mathcal{M})$ the solution of $\nabla^g Y_g = 0$ with $Y_g(0) = \mathrm{Id}$. If instead $\check{\lambda}_0 \neq 0$, then ∇ is singular

at 0 so we cannot consider such a solution. In this case, we will work with a suitable regularisation of ∇ defined as follows. Let

$$\begin{aligned}\nabla_{\mathrm{r}} &:= t^{-\check{\lambda}_0}\nabla t^{\check{\lambda}_0}\\ &= \mathrm{d} + \sum_{k\in I} t^{\langle \alpha_k, \check{\lambda}_0\rangle} F_k \,\mathrm{d}t - \sum_{r=0}^{T-1}\left(\sum_{i=1}^{N}\frac{\nu^r(w_i\cdot\check{\lambda}_i)}{t-\omega^r z_i} + \sum_{j=1}^{m}\frac{\nu^r y_j\cdot 0}{t-\omega^r x_j}\right)\mathrm{d}t.\end{aligned} \tag{6.2}$$

The assumption that the coweight $\check{\lambda}_0$ is integral dominant ensures this $\mathfrak{g}$-connection is regular at the origin. We shall refer to ∇_{r} as the *regularisation* of ∇ at the origin. By Lemma 4.2, we can then consider the solution $Y \in B_-(\mathcal{M})$ of $\nabla_{\mathrm{r}} Y = 0$ such that $Y(0) = \mathrm{Id}$. For any $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$, we consider its regularisation $(\nabla^g)_{\mathrm{r}} = (g\nabla g^{-1})_{\mathrm{r}} = g_{\mathrm{r}}\nabla_{\mathrm{r}} g_{\mathrm{r}}^{-1}$ where $g_{\mathrm{r}} := t^{-\check{\lambda}_0} g t^{\check{\lambda}_0} \in N(\mathcal{M})$. Provided the latter is regular at the origin, we can define the solution Y_g of $(\nabla^g)_{\mathrm{r}} Y_g = 0$ with $Y_g(0) = \mathrm{Id}$.

An important feature of the regularisation procedure is that all the regularised objects such as the $\mathfrak{g}$-connection (6.2) are Γ-equivariant not with respect to the automorphism $\varsigma \in \mathrm{Aut}\, G$ but rather with respect to the automorphism

$$\vartheta := \mathrm{Ad}_{\omega^{-\check{\lambda}_0}} \circ \varsigma \in \mathrm{Aut}\, G. \tag{6.3}$$

The following proposition describes the Γ-equivariance properties of the regularised $\mathfrak{g}$-connection ∇_{r}, of the regularised gauge transformation parameter $g_{\mathrm{r}} \in N(\mathcal{M})$ and of a solution $Y \in G(\mathcal{M})$ of the equation $\nabla_{\mathrm{r}} Y = 0$.

Proposition 6.1. *Define* $D_{\mathrm{r}} : G(\mathcal{M}) \to \Omega(\mathfrak{g})$, $Z \mapsto (\nabla_{\mathrm{r}} Z)Z^{-1}$ *and* $\mathrm{reg} : N(\mathcal{M}) \to N(\mathcal{M})$, $g \mapsto g_{\mathrm{r}}$. *Both are* Γ*-equivariant in the sense that the following diagrams are commutative*

$$\begin{array}{ccc} G(\mathcal{M}) & \xrightarrow{D_{\mathrm{r}}} & \Omega(\mathfrak{g}) \\ {\scriptstyle\hat{\vartheta}}\downarrow & & \downarrow{\scriptstyle\hat{\vartheta}} \\ G(\mathcal{M}) & \xrightarrow[D_{\mathrm{r}}]{} & \Omega(\mathfrak{g}) \end{array} \qquad \begin{array}{ccc} N(\mathcal{M}) & \xrightarrow{\mathrm{reg}} & N(\mathcal{M}) \\ {\scriptstyle\hat{\varsigma}}\downarrow & & \downarrow{\scriptstyle\hat{\vartheta}} \\ N(\mathcal{M}) & \xrightarrow[\mathrm{reg}]{} & N(\mathcal{M}) \end{array}$$

Let $Y \in G(\mathcal{M})$ *be any solution of* $\nabla_{\mathrm{r}} Y = 0$ *regular at* 0. *Then,* $Y \in G^{\hat{\vartheta}}(\mathcal{M})$ *if and only if* $Y(0) \in G^{\vartheta}$. *In particular,* $Y \in G^{\hat{\vartheta}}(\mathcal{M})$ *if* $Y(0) = \mathrm{Id}$.

Proof. The Γ-equivariance of the map D_{r} is the statement that $\hat{\vartheta}\big((\nabla_{\mathrm{r}} Z)Z^{-1}\big) = (\nabla_{\mathrm{r}}\hat{\vartheta}Z)(\hat{\vartheta}Z)^{-1}$ for every $Z \in G(\mathcal{M})$. This follows from expression (6.2) for ∇_{r} noting that $\hat{\vartheta}\big(dZZ^{-1}\big) = (\mathrm{d}\hat{\vartheta}Z)(\hat{\vartheta}Z)^{-1}$, the action of $\hat{\vartheta}$ on $\Omega^{\hat{\varsigma}}(\mathfrak{h})$ coincides with that of $\hat{\varsigma}$ and

$$\begin{aligned}&\hat{\vartheta}\Big(t^{-\check{\lambda}_0} p_{-1} t^{\check{\lambda}_0}\mathrm{d}t\Big)\\ &\quad = \mathrm{Ad}_{\omega^{-\check{\lambda}_0}}\left((\omega^{-1}t)^{-\check{\lambda}_0}\varsigma p_{-1}(\omega^{-1}t)^{\check{\lambda}_0}\mathrm{d}(\omega^{-1}t)\right) = t^{-\check{\lambda}_0} p_{-1} t^{\check{\lambda}_0}\mathrm{d}t\end{aligned}$$

using the ς-invariance of $\check{\lambda}_0$ and Proposition 4.3. Similarly, the Γ-equivariance of the map $g \mapsto g_{\mathrm{r}}$ is seen as follows

$$\hat{\vartheta} g_{\mathrm{r}} = \mathrm{Ad}_{\omega^{-\check{\lambda}_0}} \circ \hat{\varsigma}\big(t^{-\check{\lambda}_0} g t^{\check{\lambda}_0}\big) = \mathrm{Ad}_{\omega^{-\check{\lambda}_0}} \big((\omega^{-1}t)^{-\check{\lambda}_0} \hat{\varsigma} g (\omega^{-1}t)^{\check{\lambda}_0}\big) = (\hat{\varsigma} g)_{\mathrm{r}}.$$

For the last statement, suppose first that $Y \in G^{\hat{\vartheta}}(\mathcal{M})$ is a solution to $\nabla_{\mathrm{r}} Y = 0$ which is regular at 0. This implies $\hat{\vartheta} Y = Y$ or in other words $\vartheta \circ Y \circ \mu^*_{\omega^{-1}} = Y$. Evaluating this at the origin we obtain $\vartheta Y(0) = Y(0)$. Conversely, let $Y \in G(\mathcal{M})$ be a solution of $\nabla_{\mathrm{r}} Y = 0$ such that $\vartheta Y(0) = Y(0)$. Using the first part of the proposition, or more precisely the Γ-equivariance of D_{r}, we deduce that $\hat{\vartheta} Y$ also satisfies the same equation since $(\nabla_{\mathrm{r}} \hat{\vartheta} Y)(\hat{\vartheta} Y)^{-1} = \hat{\vartheta}\big((\nabla_{\mathrm{r}} Y) Y^{-1}\big) = 0$. However, by assumption the two solutions Y and $\hat{\vartheta} Y$ coincide at the origin and therefore are equal. □

Introduce the following subset of cyclotomic Miura $\mathfrak{g}$-opers, which we call *generic* at the origin,

$$\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0} := \{\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma} \,|\, g_{\mathrm{r}} \text{ is regular at } 0\}. \tag{6.4}$$

The following theorem describes $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{[\nabla]_\Gamma,0}$ as an N^{ϑ}-orbit in $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ through ∇.

Theorem 6.2. *Let $\nabla \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$. If $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}$ then $Y g_{\mathrm{r}}(0)^{-1} = g_{\mathrm{r}}^{-1} Y_g$ and $g_{\mathrm{r}}(0) \in N^{\vartheta}$. Conversely, for every $g_0 \in N^{\vartheta}$ there exists unique $n \in N^{\hat{\vartheta}}(\mathcal{M})$ and $\widetilde{Y} \in B_-^{\hat{\vartheta}}(\mathcal{M})$ such that*

$$Y g_0^{-1} = n^{-1} \widetilde{Y}.$$

Defining $g := t^{\check{\lambda}_0} n t^{-\check{\lambda}_0}$ so that $n = g_{\mathrm{r}}$, we have $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}$, $g_{\mathrm{r}}(0) = g_0$ and $\widetilde{Y} = Y_g$.

Proof. Suppose that $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}$. By definition g_{r} is regular at the origin so $g_{\mathrm{r}}(0) \in N$ and clearly $\nabla_{\mathrm{r}} Y g_{\mathrm{r}}(0)^{-1} = 0$. On the other hand, we have $\nabla_{\mathrm{r}} g_{\mathrm{r}}^{-1} Y_g = g_{\mathrm{r}}^{-1} (\nabla^g)_{\mathrm{r}} Y_g = 0$. It therefore follows by the uniqueness of the solution to $\nabla_{\mathrm{r}} Z = 0$ with $Z(0) = g_{\mathrm{r}}(0)^{-1}$ that $Y g_{\mathrm{r}}(0)^{-1} = g_{\mathrm{r}}^{-1} Y_g$. It remains to show that $g_{\mathrm{r}}(0) \in N^{\vartheta}$. However, since $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$, we have $g \in N^{\hat{\varsigma}}(\mathcal{M})$, i.e. $\hat{\varsigma} g = g$, and therefore, by Proposition 6.1 we deduce that $\hat{\vartheta} g_{\mathrm{r}} = g_{\mathrm{r}}$. In particular, evaluating the latter at the origin gives $g_{\mathrm{r}}(0) \in N^{\vartheta}$.

Conversely, let $g_0 \in N$ and consider the element $Y g_0^{-1} \in G(\mathcal{M})$. Let $\varphi \in \mathscr{O}(G)$ be the coordinate function on G whose nonzero set is the open cell $NB_- = \{x \in G | \varphi(x) \neq 0\}$ in G. Let $f \in \mathcal{M}$ denote the image of $\varphi \in \mathscr{O}(G)$ under the homomorphism $Y g_0^{-1} : \mathscr{O}(G) \to \mathcal{M}$. Note that f is not identically zero since $f(0) = \varphi(Y(0) g_0^{-1}) \neq 0$, which follows from $Y(0) g_0^{-1} = g_0^{-1} \in NB_-$. Since the number of zeroes of a nonzero meromorphic function on $\mathbb{P}^1$ is finite, we have $\varphi(Y(t) g_0^{-1}) = f(t) \neq 0$ for all but finitely many $t \in \mathbb{P}^1$. In other words, $Y(t) g_0^{-1} \in NB_-$ for all but finitely many $t \in \mathbb{P}^1$. We may therefore write $Y g_0^{-1} = n^{-1} \widetilde{Y}$, for some $n \in N(\mathcal{M})$ and $\widetilde{Y} \in B_-(\mathcal{M})$.

By definition of $g \in N(\mathcal{M})$, we have $n = t^{-\check{\lambda}_0} g t^{\check{\lambda}_0} = g_{\mathrm{r}}$ so that $(\nabla^g)_{\mathrm{r}}\widetilde{Y} = g_{\mathrm{r}}\nabla_{\mathrm{r}}g_{\mathrm{r}}^{-1}\widetilde{Y} = n\nabla_{\mathrm{r}}n^{-1}\widetilde{Y} = n\nabla_{\mathrm{r}}Yg_0^{-1} = 0$. Since $\widetilde{Y} \in B_-(\mathcal{M})$, it follows by the converse in Lemma 4.2 that $(\nabla^g)_{\mathrm{r}} \in \mathrm{Conn}_{\mathfrak{b}_-}(\mathbb{P}^1)$ and hence $\nabla^g = t^{\check{\lambda}_0}(\nabla^g)_{\mathrm{r}}t^{-\check{\lambda}_0} \in \mathrm{Conn}_{\mathfrak{b}_-}(\mathbb{P}^1)$. But since $\nabla^g \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$, we conclude that $\nabla^g \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ and hence $\nabla^g \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]}$. Moreover, letting $t = 0$ we obtain $n(0)^{-1}\widetilde{Y}(0) = g_0^{-1}$, or equivalently $\widetilde{Y}(0) = g_{\mathrm{r}}(0)g_0^{-1}$. But $N \cap B_- = \{\mathrm{Id}\}$ from which we deduce $g_{\mathrm{r}}(0) = g_0$ and $\widetilde{Y}(0) = \mathrm{Id}$. The first equality shows g_{r} is regular at 0. The second equality implies that $\widetilde{Y} = Y_g$ by the uniqueness of the solution to $(\nabla^g)_{\mathrm{r}}Z = 0$ with $Z(0) = \mathrm{Id}$. It just remains to show that $\hat{\vartheta}n = n$ and $\hat{\vartheta}\widetilde{Y} = \widetilde{Y}$. But from Proposition 6.1, we know that $\hat{\vartheta}Y = Y$. So since $g_0 \in N^{\vartheta}$ it follows that $\hat{\vartheta}(Yg_0^{-1}) = Yg_0^{-1}$. Hence, $n^{-1}\widetilde{Y} = \hat{\vartheta}(n^{-1}\widetilde{Y}) = (\hat{\vartheta}n)^{-1}\hat{\vartheta}\widetilde{Y}$, or equivalently $(\hat{\vartheta}n)n^{-1} = (\hat{\vartheta}\widetilde{Y})\widetilde{Y}^{-1}$. Now ϑ stabilises N and B_- so that $\hat{\vartheta}n \in N(\mathcal{M})$ and $\hat{\vartheta}\widetilde{Y} \in B_-(\mathcal{M})$. Finally, since $N \cap B_- = \{\mathrm{Id}\}$ we deduce that $\hat{\vartheta}n = n$ and $\hat{\vartheta}\widetilde{Y} = \widetilde{Y}$, as required. □

Remark 5. In Theorems 5.3 and 5.5, we found necessary and sufficient conditions on $\check{\lambda}_0$ for the existence of a cyclotomic gauge transformation along the Γ-orbit $\{\alpha_i\}_{i\in\mathcal{I}}$ of some simple root α_k, $k \in I$. These results can now be seen as a particular case of Theorem 6.2: they correspond to the conditions on $\check{\lambda}_0$, and thus on ϑ, for the existence of a ϑ-invariant element in the subgroup of N generated by the G_{α_i}, $i \in \mathcal{I}$, cf. Sect. 2.2.

Consider the concrete example of a simple root α with an orbit $\{\alpha, \beta\}$ of type $A_1 \times A_1$. Theorem 5.3 asserts that there exists a cyclotomic reproduction in the direction of this orbit if and only if $\langle \alpha, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \mod \frac{T}{2}$. In the above language, there exists such a cyclotomic reproduction if and only if a linear combination of E_α and E_β lives in $\mathfrak{n}^{\vartheta}$. Yet, we have $\vartheta(E_\alpha \pm E_\beta) = \pm\omega^{-\langle\alpha,\check{\lambda}_0+\check{\rho}\rangle}(E_\alpha \pm E_\beta)$, so this condition is indeed equivalent to $\omega^{-\langle\alpha,\check{\lambda}_0+\check{\rho}\rangle} = \pm 1$, and thus to $\langle \alpha, \check{\lambda}_0 + \check{\rho}\rangle \equiv 0 \mod \frac{T}{2}$. See also example 5. ◁

Example 5. Recall the cyclotomic Miura $\mathfrak{sl}_3$-oper ∇ of example 1 given by (4.29), whose cyclotomic reproduction we studied in example 4. We now return to this example in light of Theorem 6.2.

The regularised connection associated with ∇ is simply

$$\nabla_{\mathrm{r}} = t^{-\check{\lambda}_0}\nabla t^{\check{\lambda}_0} = \mathrm{d} + \begin{pmatrix} 0 & 0 & 0 \\ t^{\eta} & 0 & 0 \\ 0 & t^{\eta} & 0 \end{pmatrix} \mathrm{d}t, \quad \text{with} \quad t^{-\check{\lambda}_0} = \begin{pmatrix} t^{-\eta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{\eta} \end{pmatrix}. \tag{6.5}$$

The solution of equation $\nabla_{\mathrm{r}}Y = 0$ with initial condition $Y(0) = \mathrm{Id}$ is (recall that $\mu = \eta + 1$)

$$Y(t) = \begin{pmatrix} 1 & 0 & 0 \\ -\dfrac{t^{\mu}}{\mu} & 1 & 0 \\ \dfrac{t^{2\mu}}{2\mu^2} & -\dfrac{t^{\mu}}{\mu} & 1 \end{pmatrix}. \tag{6.6}$$

Let us now consider an arbitrary element g_0 of N which we parametrise as

$$g_0 = e^{aE_1+bE_2+cE_{12}} = \begin{pmatrix} 1 & a & c+\frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Following Theorem 6.2, we perform the NB_- factorisation of $Y(t)g_0^{-1}$ and get $Y(t)g_0^{-1} = n(t)^{-1}\widetilde{Y}(t)$ with $n \in N(\mathcal{M})$ and $\widetilde{Y} \in B_-(\mathcal{M})$. In particular, we have

$$n(t) = e^{\tilde{h}_1(t)E_1+\tilde{h}_2(t)E_2+\tilde{h}_3(t)E_{12}} = \begin{pmatrix} 1 & \tilde{h}_1(t) & \tilde{h}_3(t)+\frac{1}{2}\tilde{h}_1(t)\tilde{h}_2(t) \\ 0 & 1 & \tilde{h}_2(t) \\ 0 & 0 & 1 \end{pmatrix},$$

with

$$\tilde{h}_1(t) = 2\mu\frac{(ab+2c)\,t^\mu + 2\mu a}{(ab+2c)\,t^{2\mu} + 4\mu a\,t^\mu + 4\mu^2},$$

$$\tilde{h}_2(t) = 2\mu\frac{(ab-2c)\,t^\mu + 2\mu b}{(ab-2c)\,t^{2\mu} + 4\mu b\,t^\mu + 4\mu^2},$$

$$\tilde{h}_3(t) = 4\mu^3\frac{\big((ab+2c)b-(ab-2c)a\big)\,t^\mu + 4\mu c}{\big((ab+2c)\,t^{2\mu} + 4\mu a\,t^\mu + 4\mu^2\big)\big((ab-2c)\,t^{2\mu} + 4\mu b\,t^\mu + 4\mu^2\big)}.$$

Defining $g = t^{\check{\lambda}_0}nt^{-\check{\lambda}_0}$, we recover the element (5.21) with the functions $\tilde{f}_i$ given by (5.22). We have thus checked that the NB_--factorisation of Yg_0^{-1} allows one to find the most general reproduction (regular at the origin) of the Miura $\mathfrak{sl}_3$-oper ∇. Moreover, one also checks that $g_0 = n(0)$ is indeed the initial condition of the regularised transformation $n = t^{-\check{\lambda}_0}gt^{\check{\lambda}_0}$.

We now discuss the condition under which this reproduction is cyclotomic, following Theorem 6.2, and compare it to the results found in Example 4. A straightforward calculation gives

$$\vartheta g_0 = \begin{pmatrix} 1 & \omega^{-\mu}b & \omega^{-2\mu}\left(-c+\frac{1}{2}ab\right) \\ 0 & 1 & \omega^{-\mu}a \\ 0 & 0 & 1 \end{pmatrix}.$$

According to Theorem 6.2, the reproduction is cyclotomic if the initial condition g_0 is ϑ-invariant. Thus, we recover the conditions

$$a = \omega^{-\mu}b, \qquad b = \omega^{-\mu}a \qquad \text{and} \qquad c = -\omega^{-2\mu}c,$$

which we found in Example 4. We verified on an explicit example the general statement of remark 5, namely that the condition $\langle\alpha_1+\alpha_2, \check{\lambda}_0+\check{\rho}\rangle \equiv 0 \bmod \frac{T}{2}$ given in Theorem 5.5 is simply the condition for the ϑ-invariant subgroup N^ϑ to be non-trivial. ◁

6.2. Non-Integral Coweight $\check{\lambda}_0$

In Sects. 5 and 6.1, we studied the reproduction of a cyclotomic Miura $\mathfrak{g}$-oper with a dominant integral coweight $\check{\lambda}_0$ at the origin. By virtue of the Γ-equivariance, this procedure can be extended to any dominant coweight $\check{\lambda}_0$ such that for all $i \in I$,

$$\langle\alpha_i, \check{\lambda}_0\rangle \in \mathbb{Q}. \tag{6.7}$$

We start with a cyclotomic Miura $\mathfrak{g}$-oper of the form

$$\nabla = \mathrm{d} + p_{-1}\mathrm{d}t - \frac{\check{\lambda}_0}{t}\mathrm{d}t + \mathsf{r}, \tag{6.8}$$

with $\check{\lambda}_0$ as above and $\mathsf{r} \in \mathfrak{h}(\mathcal{M})\mathrm{d}t$ regular at the origin. One can always find a $q \in \mathbb{Z}_{\geq 1}$ such that $q\langle \alpha_i, \check{\lambda}_0\rangle \in \mathbb{Z}$ for any $i \in I$, i.e. such that $q\check{\lambda}_0$ is integral dominant. Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be a q-sheeted cover branched over the origin and infinity, given by $u \mapsto t = u^q$ in terms of a global coordinate u on $\mathbb{C} \subset \mathbb{P}^1$. Let $\widetilde{\mathcal{M}}$ denote the algebra of rational functions in u. We can regard $\mathcal{M}$ as a subalgebra of $\widetilde{\mathcal{M}}$ via the pullback $\pi^* : \mathcal{M} \hookrightarrow \widetilde{\mathcal{M}}$.

Let $\nabla = \mathrm{d} + \mathsf{A}$ be a $\mathfrak{g}$-connection meromorphic in t, i.e. with $\mathsf{A} \in \mathfrak{g}(\mathcal{M})\mathrm{d}t$, and consider the associated $\mathfrak{g}$-connection $\widetilde{\nabla} := \mathrm{d} + \pi^*\mathsf{A}$ on the cover, noting that $\pi^*\mathsf{A} \in \mathfrak{g}(\widetilde{\mathcal{M}})du$. Define the primitive (qT)th-root of unity $\widetilde{\omega} := \omega^{\frac{1}{q}}$ and let

$$\widetilde{\Gamma} := \langle \widetilde{\omega} \rangle \subset \mathbb{C}^\times$$

denote the corresponding copy of the cyclic group of order qT. If the $\mathfrak{g}$-connection ∇ is cyclotomic, then its Γ-equivariance property $\hat{\varsigma}(\mathsf{A}) = \mathsf{A}$ translates to the $\widetilde{\Gamma}$-equivariance property of $\widetilde{\nabla}$, namely $\widetilde{\varsigma}(\pi^*\mathsf{A}) = \pi^*\mathsf{A}$, where for any automorphism $\upsilon \in \operatorname{Aut}\mathfrak{g}$ and any $X \otimes \varpi \in \mathfrak{g}(\widetilde{\mathcal{M}})du$ we define

$$\widetilde{\upsilon}(X \otimes \varpi) := \upsilon(X) \otimes \mu^*_{\widetilde{\omega}^{-1}}\varpi.$$

Here, we have used the fact that $\pi \circ \mu_{\widetilde{\omega}} = \mu_\omega \circ \pi$ and we note that $\varsigma^{qT} = \mathrm{Id}$. The cyclotomic Miura $\mathfrak{g}$-oper (6.8) pulls back to the following $\mathfrak{g}$-connection on the cover

$$\widetilde{\nabla} = \mathrm{d} + qu^{q-1}p_{-1}du - \frac{q\check{\lambda}_0}{u}du + \pi^*\mathsf{r},$$

where we note that $\pi^*\mathsf{r} \in \mathfrak{h}(\widetilde{\mathcal{M}})du$ is regular at the origin on the cover.

Although we could not define $t^{\check{\lambda}_0}$ in $G(\mathcal{M})$, since $q\check{\lambda}_0$ is an integral coweight we can define $u^{q\check{\lambda}_0}$ in $G(\widetilde{\mathcal{M}})$ and compute the corresponding regularisation of the $\mathfrak{g}$-connection $\widetilde{\nabla}$ over the cover, namely

$$\widetilde{\nabla}_{\mathrm{r}} = u^{-q\check{\lambda}_0}\widetilde{\nabla}u^{q\check{\lambda}_0} = \mathrm{d} + q\sum_{i\in I} u^{q-1+q\langle \alpha_i, \check{\lambda}_0\rangle}F_i\, du + \pi^*\mathsf{r}.$$

This $\mathfrak{g}$-connection is clearly regular at the origin, so we may apply the methods developed in Sect. 6.1. We consider the solution $Y \in B_-(\widetilde{\mathcal{M}})$ of $\widetilde{\nabla}_{\mathrm{r}}Y = 0$ such that $Y(0) = \mathrm{Id}$, which will be $\widetilde{\Gamma}$-equivariant but with respect to the automorphism ϑ. Now given any $g_0 \in N^\vartheta$, we perform the NB_- factorisation $Y(u)g_0^{-1} = n(u)^{-1}\widetilde{Y}(u)$ as in Theorem 6.2. Defining $\widetilde{g} := u^{q\check{\lambda}_0}nu^{-q\check{\lambda}_0}$, we have that $\widetilde{g} \in N^{\widetilde{\varsigma}}(\widetilde{\mathcal{M}})$ and $\widetilde{g}\widetilde{\nabla}\widetilde{g}^{-1}$ is a cyclotomic Miura $\mathfrak{g}$-oper. The $\widetilde{\Gamma}$-equivariance of $\widetilde{g}$ means that $\varsigma\big(\widetilde{g}(u)\big) = \widetilde{g}(\widetilde{\omega}u)$, which implies that $\widetilde{g}(u) = \widetilde{g}(\widetilde{\omega}^T u)$. And since $\widetilde{\omega}^T$ is a primitive qth-root of unity, it therefore follows that $\widetilde{g}(u) = g\big(\pi(u)\big) = g(t)$ for some $g \in N(\mathcal{M})$. Moreover, $\varsigma\big(g(t)\big) = \varsigma\big(\widetilde{g}(u)\big) = \widetilde{g}(\widetilde{\omega}u) = g(\widetilde{\omega}^q t) = g(\omega t)$, so that $g \in N^{\hat{\varsigma}}(\mathcal{M})$. The cyclotomic Miura $\mathfrak{g}$-oper $\widetilde{g}\widetilde{\nabla}\widetilde{g}^{-1}$ on the cover is therefore the pullback by π of the cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$.

In summary, even when the dominant coweight $\check{\lambda}_0 \in \mathfrak{h}^\nu$ is not integral but satisfies the weaker condition (6.7), we are still able to construct a new cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$ from any given $g_0 \in N^\vartheta$. In this case, g_0 can still be interpreted as the initial condition of the regularised gauge transformation parameter g_r, but where g_r has to be considered as an element in $N(\widetilde{\mathcal{M}})$.

Example 6. Let us illustrate this construction with the help of Examples 4 and 5. We consider the cyclotomic Miura $\mathfrak{sl}_3$-oper ∇ defined in Eq. (4.29), but we relax here the condition that $\check{\lambda}_0$ is integral dominant, *i.e.* that $\eta \in \mathbb{Z}_{\geq 0}$. Instead, we will consider η positive and rational and thus write $\eta = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{\geq 1}$ coprime.

We can consider the regularisation ∇_r of ∇ and its extended solution Y as in Eqs. (6.5) and (6.6), but over a q-sheeted cover of $\mathbb{P}^1$, with coordinates $u = t^{\frac{1}{q}}$, so that the expression t^η in these equations can be understood as u^p. Let us recall that the ϑ-invariant subgroup N^ϑ is non-trivial if $\mu = \eta + 1$ satisfies one of the following three conditions:

- $\omega^\mu = 1$: In this case, μ must be integer (a multiple of T), hence η is integer and we recover the usual setting.
- $\omega^\mu = -1$: In this case, μ is also integer (a multiple of $\frac{T}{2}$).
- $\omega^{2\mu} = -1$: This is the interesting case since μ can now be half-integer. Working with ∇_r thus requires working in the variable $u = \sqrt{t}$. However, in this case the transformation parameter g is described by the functions f_i as given in Eq. (5.23). We see that they only depend on t^μ through the expression $t^{2\mu}$ and so are indeed meromorphic functions of t.

Finally, let us note that, whatever the value of η we consider (possibly non-integer), the singular reproduction (5.24) is always meromorphic in t. ◁

We end this subsection with a discussion of the monodromy around 0 of the connection ∇. As explained above, working with the cover $\pi : u \mapsto t = u^q$, one can consider a solution $Y \in B_-(\widetilde{\mathcal{M}})$ of the regularised equation $\widetilde{\nabla}_\mathrm{r} Y = 0$ with $Y(0) = \mathrm{Id}$. And since $\widetilde{\nabla} = u^{q\check{\lambda}_0} \widetilde{\nabla}_\mathrm{r} u^{-q\check{\lambda}_0}$, it follows that $X(u) := u^{q\check{\lambda}_0} Y(u)$ is a solution of the equation $\widetilde{\nabla} X = 0$.

Introducing the qth-root of unity $\zeta := e^{2\pi i/q}$, we find that $\widetilde{X}(u) := X(\zeta u)$ is also a solution of $\widetilde{\nabla}\widetilde{X} = 0$. The monodromy of ∇ around 0 is then the unique $M_0 \in G$ such that $X(\zeta u) = X(u) M_0$. To work out M_0 explicitly, we consider the corresponding regularised solution $\widetilde{Y}(u) := u^{-q\check{\lambda}_0} \widetilde{X}(u)$ of $\widetilde{\nabla}_\mathrm{r} \widetilde{Y} = 0$ which is related to Y as $\widetilde{Y}(u) = u^{-q\check{\lambda}_0} X(\zeta u) = \zeta^{q\check{\lambda}_0} (\zeta u)^{-q\check{\lambda}_0} X(\zeta u) = \zeta^{q\check{\lambda}_0} Y(\zeta u)$. Thus $\widetilde{Y}(0) = \zeta^{q\check{\lambda}_0}$ and hence $\widetilde{Y} = Y \zeta^{q\check{\lambda}_0}$, or equivalently $\widetilde{X} = X \zeta^{q\check{\lambda}_0}$. Therefore, the monodromy of ∇ around the origin is

$$M_0 = \zeta^{q\check{\lambda}_0} = e^{2\pi i \check{\lambda}_0}. \tag{6.9}$$

In this equation, the expression $e^{2\pi i \check{\lambda}_0}$ has to be considered as a formal notation for $\zeta^{q\check{\lambda}_0}$, which is a well-defined element of H as $q\check{\lambda}_0$ is an integral coweight. When $\check{\lambda}_0$ is an integral coweight, we find that $M_0 = \mathrm{Id}$ as expected from Remark 2.

6.3. Isomorphism with $(G/B_-)^\vartheta$

Given any $\nabla \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$, consider the set

$$\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]} := \big\{ \widetilde{\nabla} \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1) \,\big|\, [\widetilde{\nabla}] = [\nabla] \big\}$$

of all Miura $\mathfrak{g}$-opers with underlying $\mathfrak{g}$-oper $[\nabla]$.

Recall the definition (6.2) of the regularisation of a cyclotomic Miura $\mathfrak{g}$-oper at the origin. If ∇ is any Miura $\mathfrak{g}$-oper with residue given by minus an integral dominant coweight $-\check{\lambda} \in \mathfrak{h}$ at some point $x \in \mathbb{C}$, then we can similarly introduce its regularisation $\nabla^x_{\mathrm{r}} := (t-x)^{-\check{\lambda}} \nabla (t-x)^{\check{\lambda}}$ at x and for any $g \in N(\mathcal{M})$ we let $g^x_{\mathrm{r}} := (t-x)^{-\check{\lambda}} g (t-x)^{\check{\lambda}}$. For any $x \in \mathbb{C}$, we introduce the subset of Miura $\mathfrak{g}$-opers which are generic at x as, cf. (6.4) in the cyclotomic setting,

$$\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,x} := \{ \nabla^g \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]} \,|\, g^x_{\mathrm{r}} \text{ is regular at } x \}.$$

The following theorem is proved in [18] in the case when $\check{\lambda} = 0$, i.e. when the Miura $\mathfrak{g}$-oper ∇ is regular at $x \in \mathbb{C}$, but similar arguments apply also when ∇ has a regular singularity at the given point x by working with its regularisation ∇^x_{r}. Recall the notation of Sect. 2.3.

Theorem 6.3. *Let $\nabla \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)$ be monodromy-free. For any $x \in \mathbb{C}$, the following map*

$$\begin{aligned} \Phi : \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]} &\longrightarrow G/B_- \\ \nabla^g &\longmapsto \lim_{t \to x} g^x_{\mathrm{r}}(t)^{-1} B_- \end{aligned}$$

is an isomorphism. Moreover, $\Phi\big(\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{[\nabla],x}\big) = C_{\mathrm{Id}}$ and $\overline{\mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{[\nabla],x}} = \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]}$. □

In this section, we derive a cyclotomic analog of Theorem 6.3. Specifically, from now on we restrict attention to the point $x = 0$, the fixed point of $\mathbb{C}$ under the action of Γ. Given a *cyclotomic* Miura $\mathfrak{g}$-oper ∇ of the form specified at the start of this section, our aim is to describe the image of $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma} \subset \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]}$ under the isomorphism Φ.

Since the automorphism $\varsigma \in \mathrm{Aut}\, G$ stabilises the Borel subgroup B_-, so does the automorphism $\vartheta \in \mathrm{Aut}\, G$ defined in (6.3). Hence, we obtain a well-defined induced action on the quotient G/B_-, which by abuse of notation we also denote ϑ, given by

$$\begin{aligned} \vartheta : G/B_- &\longrightarrow G/B_- \\ gB_- &\longmapsto \vartheta(g) B_-. \end{aligned}$$

Note that the canonical projection $G \to G/B_-$ is Γ-equivariant with respect to this action, i.e. the following diagram is commutative

$$\begin{array}{ccc} G & \longrightarrow & G/B_- \\ {\scriptstyle\vartheta}\downarrow & & \downarrow{\scriptstyle\vartheta} \\ G & \longrightarrow & G/B_- \end{array}$$

We denote by $(G/B_-)^\vartheta$ the subset of fixed points of G/B_- under ϑ.

Lemma 6.4. *Let $\nabla \in \mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$ be monodromy-free. Define $\gamma : N(\mathcal{M}) \to \mathrm{MOp}_\mathfrak{g}(\mathbb{P}^1)_{[\nabla]}$, $g \mapsto \nabla^g$. The composition $\Phi \circ \gamma : N(\mathcal{M}) \to G/B_-$ is Γ-equivariant in the sense that the following diagram is commtuative*

$$\begin{array}{ccc} N(\mathcal{M}) & \xrightarrow{\Phi\circ\gamma} & G/B_- \\ {\scriptstyle\hat\varsigma}\downarrow & & \downarrow{\scriptstyle\vartheta} \\ N(\mathcal{M}) & \xrightarrow[\Phi\circ\gamma]{} & G/B_- \end{array}$$

In other words, for all $g \in N(\mathcal{M})$ we have $\Phi(\nabla^{\hat\varsigma g}) = \vartheta\big(\Phi(\nabla^g)\big)$. In particular,

$$\Phi\big(\mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)_{[\nabla]_\Gamma}\big) \subset (G/B_-)^\vartheta$$

since $\mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)_{[\nabla]_\Gamma} = \gamma\big(N^{\hat\varsigma}(\mathcal{M})\big)$.

Proof. Using Proposition 6.1 and the definition and continuity of ϑ, we have

$$\begin{aligned}\Phi(\nabla^{\hat\varsigma g}) &= \lim_{t\to 0}(\hat\varsigma g)_\mathrm{r}(t)^{-1}B_- = \lim_{t\to 0}\big(\hat\vartheta g_\mathrm{r}\big)(t)^{-1}B_- = \lim_{t\to 0}\vartheta\left[g_\mathrm{r}(\omega^{-1}t)^{-1}\right]B_- \\ &= \lim_{t\to 0}\vartheta\left[g_\mathrm{r}(\omega^{-1}t)^{-1}B_-\right] = \vartheta\left[\lim_{t\to 0} g_\mathrm{r}(t)^{-1}B_-\right] = \vartheta\big(\Phi(\nabla^g)\big). \qquad \square\end{aligned}$$

In general, $(G/B_-)^\vartheta$ is *not* isomorphic to the flag variety $G^\vartheta/B_-^\vartheta$. However, we can always realise $G^\vartheta/B_-^\vartheta$ as a subset of $(G/B_-)^\vartheta$. Indeed, we have a well-defined injection

$$\begin{aligned} G^\vartheta/B_-^\vartheta &\hookrightarrow (G/B_-)^\vartheta \\ gB_-^\vartheta &\longmapsto gB_-. \end{aligned}$$

The main result of this section is the following cyclotomic version of Theorem 6.3.

Theorem 6.5. *Let $\nabla \in \mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)$ be monodromy-free and of the form $\nabla = \mathrm{d} + (p_{-1} + u)\mathrm{d}t$ with $u\,\mathrm{d}t \in \Omega^{\hat\nu}(\mathfrak{h})$ as in (5.1). The following map*

$$\begin{aligned}\Phi : \mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)_{[\nabla]_\Gamma} &\longrightarrow (G/B_-)^\vartheta \\ \nabla^g &\longmapsto \lim_{t\to 0} g_\mathrm{r}(t)^{-1}B_-\end{aligned}$$

is an isomorphism. Moreover, we have $\Phi\big(\mathrm{MOp}^\Gamma_\mathfrak{g}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}\big) = N^\vartheta B_-/B_-$.

Proof. This map is the restriction to $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ of the one of Theorem 6.3 with $x = 0$ and so is injective. Moreover, we know that it is valued in $(G/B_-)^{\vartheta}$ by Lemma 6.4. It therefore remains to prove that it is surjective.

Let $p \in (G/B_-)^{\vartheta}$. By Theorem 6.3, there exists a $\nabla^g \in \mathrm{MOp}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]}$ such that $\Phi(\nabla^g) = p$. And according to Lemma 6.4, we have $\Phi(\nabla^{\hat{\varsigma}g}) = \vartheta\big(\Phi(\nabla^g)\big) = \vartheta(p) = p = \Phi(\nabla^g)$. The injectivity of Φ implies that $\nabla^{\hat{\varsigma}g} = \nabla^g$. And since the action of $N(\mathcal{M})$ on $\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ is free by Theorem 4.5, we deduce that $\hat{\varsigma}g = g$. In other words, $g \in N^{\hat{\varsigma}}(\mathcal{M})$ so that $\nabla^g \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$, which proves the surjectivity.

The last claim follows at once from Theorem 6.2. □

Remark 6. In general, by contrast with the non-cyclotomic case, $\overline{\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}} \neq \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$. This feature of the cyclotomic case will be illustrated in example 7. ◁

6.4. Cell Decomposition of $(G/B_-)^{\upsilon}$

Given any cyclotomic Miura $\mathfrak{g}$-oper $\nabla \in \mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)$ of the form considered in this section, Theorem 6.5 establishes an isomorphism between $\mathrm{MOp}^{\Gamma}_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$ and the ϑ-invariant subvariety $(G/B_-)^{\vartheta}$ of the flag variety G/B_-. Recall the cell decomposition (2.3) of the latter. Let $\upsilon \in \mathrm{Aut}\,\mathfrak{g}$ be any automorphism of $\mathfrak{g}$, such that $\upsilon^T = \mathrm{Id}$, whose diagram part is given by ν, and which stabilises the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$. An example of such an automorphism is provided by ϑ. In this subsection, we establish a similar cell decomposition for the υ-invariant subvariety $(G/B_-)^{\upsilon}$. We first need the following lemma.

Lemma 6.6. *The map $\mathcal{K} : W \to G/B_-$ defined by $\mathcal{K}(w) := \dot{w}B_-$ is Γ-equivariant, i.e. the diagram*

$$\begin{array}{ccc} W & \xrightarrow{\mathcal{K}} & G/B_- \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \upsilon} \\ W & \xrightarrow[\mathcal{K}]{} & G/B_- \end{array}$$

is commutative.

Proof. Recall from Sect. 2.3 that there is an isomorphism $\pi : W \xrightarrow{\sim} N_G(\mathfrak{h})/H$, given by $w \mapsto \dot{w}H$ and that this isomorphism is υ-equivariant with respect to the actions of υ on W and $N_G(\mathfrak{h})/H$. Moreover, the map

$$\begin{aligned} \iota : N_G(\mathfrak{h})/H &\longmapsto G/B_- \\ gH &\longmapsto gB_- \end{aligned}$$

is well defined and υ-equivariant. Therefore, $\mathcal{K} = \iota \circ \pi$ is also υ-equivariant. Finally, note that the action of υ on the Weyl group W only depends on the diagram part $\nu : I \to I$ of the automorphism υ, which coincide with that of $\nu \in \mathrm{Aut}\,\mathfrak{g}$, hence the lemma. □

To obtain the cell decomposition of $(G/B_-)^\upsilon$, we begin by recalling some facts about the Schubert cells of G/B_-. Let w be an element of the Weyl group W. We define the subset of positive roots $R(w) := \{\alpha \in \Phi^+ \,|\, w^{-1}\alpha \in \Phi^-\}$. Its cardinality $|R(w)|$ is equal to the length $l(w)$ of the Weyl group element w. Consider the subgroup $U_w := \prod_{\alpha\in R(w)} G_\alpha$ of N which does not depend on the chosen ordering on the roots $\alpha \in R(w)$.

Let $w_\circ$ be the longest element of W (the unique element of W of maximal length). Then, the cell C_w is isomorphic to $U_{w_\circ w}$. More precisely, every point of C_w is of the form $n\dot{w}B_- = n\mathcal{K}(w)$, with $n \in U_{w_\circ w}$, and this form is unique. The following lemma describes the action of the automorphism υ on the subgroups U_w.

Lemma 6.7. *Let $w \in W$. We have $R(\nu w) = \nu\big(R(w)\big)$ and $\upsilon(U_w) = U_{\nu w}$. In particular, if $w \in W^\nu$ then υ stabilises U_w. Moreover, $\nu w_\circ = w_\circ$.*

Proof. Let $\alpha \in R(w)$. By definition of νw, we have $(\nu w)^{-1}(\nu\alpha) = \nu \circ w^{-1} \circ \nu^{-1}(\nu\alpha) = \nu(w^{-1}\alpha)$. Yet, $\alpha \in R(w)$, hence $w^{-1}\alpha \in \Phi^-$. Thus $(\nu w)^{-1}(\nu\alpha) = \nu(w^{-1}\alpha)$ belongs to Φ^-, i.e. $\nu\alpha \in R(\nu w)$. Therefore, $\nu\big(R(w)\big) \subset R(\nu w)$. Applying this result to $\nu w \in W$ and the automorphism ν^{-1}, we get $\nu^{-1}\big(R(\nu w)\big) \subset R(w)$, hence (applying ν to this inclusion) $R(\nu w) \subset \nu\big(R(w)\big)$. By double inclusion, we conclude that $R(\nu w) = \nu\big(R(w)\big)$. The claim that $\upsilon(U_w) = U_{\nu w}$ follows directly, noting that, for any root $\alpha \in \Phi$, $G_{\nu\alpha} = \nu(G_\alpha) = \upsilon(G_\alpha)$ [see Eq. (2.2)].

As $R(\nu w) = \nu\big(R(w)\big)$, we have $l(\nu w) = |R(\nu w)| = |R(w)| = l(w)$, i.e. the action of ν on the Weyl group W preserves the length. In particular, $l(\nu w_\circ) = l(w_\circ)$. As $w_\circ$ is the unique element of W of maximal length, we deduce that $\nu w_\circ = w_\circ$. □

We are now in a position to state and prove the sought after cell decomposition of $(G/B_-)^\upsilon$.

Theorem 6.8. *The space $(G/B_-)^\upsilon$ admits the following cell decomposition*

$$(G/B_-)^\upsilon = \bigsqcup_{w\in W^\nu} N^\upsilon \dot{w} B_-/B_- =: \bigsqcup_{w\in W^\nu} C_w^\upsilon. \tag{6.10}$$

Moreover, each element of the cell C_w^υ can be written uniquely in the form $n\dot{w}B_-$ with $n \in U_{w_\circ w}^\upsilon$.

Proof. Let $w \in W^\nu$ and $n \in N^\upsilon$. Using Lemma 6.6, we have $\upsilon\big(\mathcal{K}(w)\big) = \mathcal{K}(\nu w) = \mathcal{K}(w)$ and hence $\upsilon(n\dot{w}B_-) = \upsilon(n)\upsilon\big(\mathcal{K}(w)\big) = n\mathcal{K}(w) = n\dot{w}B_-$. In other words, $n\dot{w}B_-$ belongs to $(G/B_-)^\upsilon$. This proves the inclusion $\bigsqcup_{w\in W^\nu} N^\upsilon \dot{w} B_-/B_- \subset (G/B_-)^\upsilon$.

Conversely, let us consider an element $p \in (G/B_-)^\upsilon$. As an element of G/B_-, it belongs to a cell C_w, for some $w \in W$. Thus, we can write $p = n\dot{w}B_- = n\mathcal{K}(w)$ for some unique $n \in U_{w_\circ w}$. Using Lemma 6.6, we get $p = \upsilon(p) = \upsilon(n)\upsilon\big(\mathcal{K}(w)\big) = \upsilon(n)\mathcal{K}(\nu w)$. As υ stabilises N, $\upsilon(n) \in N$ and so $p \in C_{\nu w}$. Yet, the cells $C_{w'}$ are disjoint in G/B_-, hence $w = \nu w$, i.e. $w \in W^\nu$. Lemma 6.7 implies that $w_\circ w \in W^\nu$ and that $U_{w_\circ w}$ is stabilised by υ. Thus,

we have $p = n\mathcal{K}(w) = \upsilon(n)\mathcal{K}(w)$, with n and $\upsilon(n)$ in $U_{w_\circ w}$. By unicity of this form, we deduce that $n \in U^{\upsilon}_{w_\circ w}$. This provides the opposite inclusion $(G/B_-)^{\upsilon} \subset \bigsqcup_{w\in W^\nu} N^{\upsilon}\dot{w}B_-/B_-$ and proves also the last statement. □

Example 7. Consider the cyclotomic Miura $\mathfrak{sl}_3$-oper ∇ studied in Examples 1, 4 and 5. We will illustrate Theorem 6.5 for ∇, with the help of Theorem 6.8. The ν-invariant Weyl group W^ν is composed of two elements: the identity and the simple reflection $s^\nu = s_1 s_2 s_1$ associated with the simple folded root $\alpha^\nu = \frac{1}{2}(\alpha_1 + \alpha_2)$. By Theorem 6.8, the variety $(G/B_-)^\vartheta$ must be composed of two cells $C^\vartheta_{\mathrm{Id}}$ and $C^\vartheta_{s^\nu}$.

In Examples 4 and 5, we described all possible cyclotomic reproductions of ∇. They are of the following two types:

- the regular ones, characterised by a choice of initial condition $g_0 \in N^\vartheta$,
- the (unique) singular one, given by equation (5.24).

As explained in Sect. 6.1, the regular reproductions generate the subset of generic cyclotomic Miura $\mathfrak{sl}_3$-opers $\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}$. Moreover, it follows from Theorem 6.5 that this subset corresponds to the big cell $C^\vartheta_{\mathrm{Id}} = N^\vartheta B_-/B_-$ of $(G/B_-)^\vartheta$. More precisely, if $g \in N^{\hat{\varsigma}}(\mathcal{M})$ is such that g_{r} is regular with initial condition $g_{\mathrm{r}}(0) = g_0 \in N^\vartheta$, then $\Phi(g\nabla g^{-1}) = g_0 B_-$.

The singular reproduction, with parameter g given by (5.24), gives rise to a non-generic cyclotomic Miura $\mathfrak{g}$-oper $g\nabla g^{-1}$, corresponding to the point $\Phi(g\nabla g^{-1}) = \dot{s}^\nu B_-$ in the variety $(G/B_-)^\vartheta$ forming the one-point cell $C^\vartheta_{s^\nu} = \{\dot{s}^\nu B_-\}$.

As explained in example 5, the ϑ-invariant subgroup N^ϑ is non-trivial if and only if $\omega^{4(\eta+1)} = 1$. In the case where it is non-trivial, the singular reproduction (5.24) can be obtained as a limit of the regular ones (cf. example 4): in the flag variety, this corresponds to the density of the big cell $N^\vartheta B_-/B_-$ in the variety $(G/B_-)^\vartheta$. However, when N^ϑ is trivial, the variety $(G/B_-)^\vartheta$ is composed of only two points: B_- and $\dot{s}B_-$. In this case, the big cell $\{B_-\}$ is not dense in $(G/B_-)^\vartheta$, which illustrates the point made in remark 6 that in general $\overline{\mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{gen}}_{\nabla,0}} \neq \mathrm{MOp}^\Gamma_{\mathfrak{g}}(\mathbb{P}^1)_{[\nabla]_\Gamma}$. ◁

7. Cyclotomic Gaudin Models

In this section, we use the notion of cyclotomic $\mathfrak{g}$-oper from Sect. 4 to formulate the conjecture relating the (maximal) spectrum of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$, introduced in [26] and whose definition was briefly recalled in the introduction, to a certain space of cyclotomic ${}^L\mathfrak{g}$-opers for the Langlands dual Lie algebra ${}^L\mathfrak{g}$ of $\mathfrak{g}$.

We begin recalling the Bethe ansatz solution of the cyclotomic Gaudin model obtained in [26] when the spin chain is a tensor product of Verma modules. The joint eigenvectors of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ are characterised in this case by certain Γ-equivariant rational functions $\mathbb{P}^1 \to \mathfrak{h}^*$ valued in weight space $\mathfrak{h}^*$ with poles at the points z_i, $i = 1, \ldots, N$ of the

cyclotomic Gaudin model (and their Γ-orbits), at a further collection of *Bethe roots* x_j, $j = 1, \dots, m$ (and their Γ-orbits) satisfying the cyclotomic Bethe ansatz equations, as well as an additional pole at the origin with residue given by a special weight depending on the cyclotomic Gaudin model datum.

Noting that $\mathfrak{h}^*$ is canonically identified with the Cartan subalgebra of the Langlands dual Lie algebra ${}^L\mathfrak{g}$, such a rational function $\mathbb{P}^1 \to \mathfrak{h}^*$ can be used to define a cyclotomic Miura ${}^L\mathfrak{g}$-oper. We conjecture in this case that the eigenvalues of the cyclotomic Gaudin Hamiltonians can be read off from the canonical form of the underlying cyclotomic ${}^L\mathfrak{g}$-oper.

7.1. Bethe Ansatz

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. We follow the notation and conventions of Sect. 2.1. As in Sect. 4 we let $T \in \mathbb{Z}_{\geq 1}$, we pick a primitive Tth-root of unity ω and let $\Gamma := \langle \omega \rangle \cong \mathbb{Z}/T\mathbb{Z}$ be a copy of the cyclic group of order T acting on $\mathbb{P}^1$ by multiplication. Let $\sigma \in \operatorname{Aut}\mathfrak{g}$ be such that $\sigma^T = \mathrm{Id}$ with diagram part $\nu : I \to I$. We can always choose a Cartan subalgebra $\mathfrak{h}$ adapted to σ with corresponding Chevalley–Serre generators E_i, $\check{\alpha}_i$, F_i for $i \in I$ such that, see e.g. [13],

$$\sigma(E_i) = \tau_{\alpha_i} E_{\nu(i)}, \qquad \sigma(\check{\alpha}_i) = \check{\alpha}_{\nu(i)}, \qquad \sigma(F_i) = \tau_{\alpha_i}^{-1} F_{\nu(i)},$$

for some Tth-roots of unity $\tau_{\alpha_i} \in \Gamma$.

Let $N \in \mathbb{Z}_{\geq 1}$ and fix a set of N distinct finite nonzero points $z_i \in \mathbb{C}^\times$, $i = 1, \dots, N$ with disjoint Γ-orbits. Let $u \in \mathbb{C}^\times \backslash \{z_1, \dots, z_N\}$ be any other point whose Γ-orbit is also disjoint from those of the z_i. The *cyclotomic Gaudin algebra* $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ is defined in [26] as the image of a homomorphism of commutative $\mathbb{C}$-algebras

$$\Psi^\Gamma_{(z_i),u} : \mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big) \longrightarrow U(\mathfrak{g})^{\otimes N}, \tag{7.1}$$

where $\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big)$ denotes the space of singular vectors in the vacuum Verma module $\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the *critical level*. The construction of (7.1) in [26], which generalises arguments of [6] to the cyclotomic setting, makes essential use of the notion of *cyclotomic coinvariants* of a tensor product of $\widehat{\mathfrak{g}}$-modules introduced in [27] (see also [10]).

Pick a collection of weights $\lambda_i \in \mathfrak{h}^*$ for $i = 1, \dots, N$. Let $M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\, v_\lambda$ be the Verma module of highest weight $\lambda \in \mathfrak{h}^*$. The problem of diagonalising the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ on the tensor product $\bigotimes_{i=1}^N M_{\lambda_i}$ was addressed in [26] by using a generalisation of the Bethe ansatz construction of [6] to the case $\Gamma \neq \{1\}$. The central idea of this approach is to utilise the rich structure of modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level. In particular, one has access to the so-called *Wakimoto modules* which are parametrised by $\mathfrak{h}$-valued formal Laurent series. The eigenvectors of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ in $\bigotimes_{i=1}^N M_{\lambda_i}$ are then obtained from cyclotomic coinvariants of tensor products of such modules. The main novelty in the cyclotomic setting compared to the usual case is the need to also introduce a Wakimoto module over the twisted affine Kac–Moody algebra $\widehat{\mathfrak{g}}^\sigma$ at the origin, i.e. the fixed point of the action of Γ in $\mathbb{C}$. Below we will only recall the

end result of the construction, referring the interested reader to [26,27] and [6] for further details.

Let $m \in \mathbb{Z}_{\geq 0}$ and suppose we are given any map $c : \{1, \ldots, m\} \to I$, the so-called *colour function*. The corresponding *cyclotomic Bethe ansatz equations* are a collection of m algebraic equations on a subset of m points $\{x_j\}_{j=1}^m \subset \mathbb{C}^\times$ given by

$$0 = \sum_{r=0}^{T-1} \sum_{i=1}^{N} \frac{(\alpha_{c(j)}|\nu^r \lambda_i)}{x_j - \omega^r z_i} - \sum_{\substack{r=0 \\ (r,k)\neq(0,j)}}^{T-1} \sum_{k=1}^{m} \frac{(\alpha_{c(j)}|\nu^r \alpha_{c(k)})}{x_j - \omega^r x_k} + \frac{1}{x_j}(\alpha_{c(j)}|\lambda_0),$$
$$j = 1, \ldots, m. \tag{7.2}$$

Here, the *weight at the origin* $\lambda_0 \in \mathfrak{h}^*$ is defined in terms of the automorphism $\sigma \in \operatorname{Aut} \mathfrak{g}$ and the root of unity ω as

$$\lambda_0(h) := \sum_{r=1}^{T-1} \frac{\operatorname{tr}_{\mathfrak{n}}(\sigma^{-r} \circ \operatorname{ad}_h)}{1 - \omega^r}. \tag{7.3}$$

One checks that this weight is ν-invariant using the equality $\operatorname{ad}_{\nu h} = \sigma \circ \operatorname{ad}_h \circ \sigma^{-1}$ as endomorphisms of the nilpotent Lie algebra $\mathfrak{n}$, together with the cyclicity of the trace $\operatorname{tr}_{\mathfrak{n}}$ over $\mathfrak{n}$.

To any solution of (7.2), we can associate an eigenvector $\psi_{(x_j)} \in \bigotimes_{i=1}^N M_{\lambda_i}$ of the cyclotomic Gaudin algebra $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$. Such a *Bethe vector* can be expressed as a linear combination of vectors of the form $\bigotimes_{i=1}^N F_{c(p_1^i)} \ldots F_{c(p_{n_i}^i)} v_{\lambda_i}$, where $n_1 + \ldots + n_N = m$ is a composition of the integer m and $(p_1^1, \ldots, p_{n_1}^1; \ldots; p_1^N, \ldots, p_{n_N}^N)$ is a permutation of $(1, 2, \ldots, m)$, with the coefficients depending rationally on all the Bethe roots x_j, $j = 1, \ldots, m$ and on the points z_i, $i = 1, \ldots, N$. See [26, Proposition 4.6] for the explicit expression of $\psi_{(x_j)}$. The eigenvalues of the family of N quadratic cyclotomic Gaudin Hamiltonians, cf. (1.1), on this eigenvector read

$$\mathcal{E}_i = \sum_{\substack{r=0 \\ (r,k)\neq(0,j)}}^{T-1} \sum_{j=1}^{N} \frac{(\lambda_i|\nu^r \lambda_j)}{z_i - \omega^r z_j} - \sum_{r=0}^{T-1} \sum_{j=1}^{m} \frac{(\lambda_i|\nu^r \alpha_{c(j)})}{z_i - \omega^r x_j} + \frac{1}{z_i}(\lambda_i|\lambda_0), \quad i = 1, \ldots, N. \tag{7.4}$$

In other words, we have $\mathcal{H}_i \psi_{(x_j)} = \mathcal{E}_i \psi_{(x_j)}$ for all $i = 1, \ldots, N$ provided the cyclotomic Bethe ansatz equations (7.2) hold. Note that the question of whether the Bethe vectors $\psi_{(x_j)}$ so constructed are nonzero was not addressed in [26]. This problem was considered recently in [25] where it was proved, at least in the case when $\sigma \in \operatorname{Aut} \mathfrak{g}$ is a diagram automorphism, that the Bethe vectors $\psi_{(x_j)}$ are indeed nonzero (see also [19,23] in the non-cyclotomic case).

7.2. Cyclotomic Miura ${}^L\mathfrak{g}$-Opers

In the non-cyclotomic case, the eigenvectors $\psi_{(x_j)}$ obtained by the Bethe ansatz, or Wakimoto construction, are naturally parameterised by an $\mathfrak{h}^*$-valued connection on $\mathbb{P}^1$ with poles at each z_i, $i = 1, \ldots, N$ with residue $-\lambda_i$ and at

each Bethe root x_j, $j = 1, \dots, m$ with residue $\alpha_{c(j)}$ [8]. Likewise, in the cyclotomic analog of the Wakimoto construction [26], each Bethe vector $\psi_{(x_j)}$ corresponds to a Γ-equivariant $\mathfrak{h}^*$-valued meromorphic differential on $\mathbb{P}^1$,

$$\lambda(t)\mathrm{d}t := \frac{\lambda_0}{t}\mathrm{d}t + \sum_{r=0}^{T-1}\left(\sum_{i=1}^{N}\frac{\nu^r\lambda_i}{t-\omega^r z_i} - \sum_{j=1}^{m}\frac{\nu^r\alpha_{c(j)}}{t-\omega^r x_j}\right)\mathrm{d}t \in \Omega^{\hat{\nu}}(\mathfrak{h}^*). \tag{7.5}$$

In particular, taking the Laurent expansion of $\lambda(t)$ at x_j we find

$$\lambda(t) = -\frac{\alpha_{c(j)}}{t-x_j} + r(t)$$

where $r \in \mathfrak{h}(\mathcal{M})$ is regular at x_j, and the cyclotomic Bethe ansatz equations (7.2) then simply read $(\alpha_{c(j)}|r(x_j)) = 0$. Similarly, the energies $\mathcal{E}_i$ of the quadratic cyclotomic Gaudin Hamiltonians $\mathcal{H}_i$ are given in terms of (7.5) simply by

$$\mathcal{E}_i = \mathrm{res}_{z_i}\left(\tfrac{1}{2}(\lambda(t)|\lambda(t)) - (\lambda'(t)|\rho)\right)\mathrm{d}t. \tag{7.6}$$

Both of these statements can be formulated as properties of a cyclotomic Miura ${}^L\mathfrak{g}$-oper built from (7.5), where ${}^L\mathfrak{g}$ denotes the *Langlands dual* of $\mathfrak{g}$.

The set $\Phi^\vee := \{\check{\alpha} \in \mathfrak{h} \,|\, \alpha \in \Phi\}$ of all coroots forms a root system in $\mathfrak{h}$. The Langlands dual of $\mathfrak{g}$ is defined as the finite-dimensional complex semisimple Lie algebra ${}^L\mathfrak{g}$ with this root system $\Phi^\vee$. Its Cartan matrix is then the transpose of that of $\mathfrak{g}$. The Cartan subalgebra of ${}^L\mathfrak{g}$ is canonically identified with the dual $\mathfrak{h}^*$ of $\mathfrak{h}$. Denote by $\check{E}_i$, α_i, $\check{F}_i$ for $i \in I$ the Chevalley–Serre generators of ${}^L\mathfrak{g}$. We let ω_i, $i \in I$ denote the basis of fundamental weights of $\mathfrak{g}$, i.e. fundamental coweights of ${}^L\mathfrak{g}$.

As in Sect. 3.1, the principal $\mathfrak{sl}_2$-triple is defined by its regular nilpotent and semisimple elements

$$\check{p}_{-1} := \sum_{i\in I}\check{F}_i, \qquad \rho := \sum_{i\in I}\omega_i.$$

We denote this $\mathfrak{sl}_2$-triple by $\{\check{p}_{-1}, \rho, \check{p}_1\}$. To define the notion of cyclotomic ${}^L\mathfrak{g}$-oper, we introduce a specific automorphism $\varsigma \in \mathrm{Aut}\,{}^L\mathfrak{g}$ of ${}^L\mathfrak{g}$, as in Sect. 4.3, by letting

$$\varsigma(\check{E}_i) = \omega^{-1}\check{E}_{\nu(i)}, \qquad \varsigma(\alpha_i) = \alpha_{\nu(i)}, \qquad \varsigma(\check{F}_i) = \omega\check{F}_{\nu(i)}, \tag{7.7}$$

for all $i \in I$. We may use (7.5) to define a cyclotomic Miura ${}^L\mathfrak{g}$-oper as

$$\nabla := \mathrm{d} + \check{p}_{-1}\mathrm{d}t - \lambda(t)\mathrm{d}t \in \mathrm{MOp}^{\Gamma}_{{}^L\mathfrak{g}}(\mathbb{P}^1). \tag{7.8}$$

It follows from Proposition 4.13 that the cyclotomic Bethe ansatz equations (7.2) are precisely the conditions for the underlying cyclotomic ${}^L\mathfrak{g}$-oper $[\nabla]_\Gamma$ to be regular at all the Bethe roots x_j, $j = 1, \dots, m$, see also remark 1. We define the *weight at infinity* as the unique ν-invariant weight $\lambda_\infty \in \mathfrak{h}^{*,\nu}$ in the shifted W^ν-orbit of $-\mathrm{res}_\infty \lambda(t)\mathrm{d}t$ such that $\lambda_\infty + \rho$ is dominant. We then have

$$\lambda_0 + \sum_{r=0}^{T-1}\left(\sum_{i=1}^{N}\nu^r\lambda_i - \sum_{j=1}^{m}\nu^r\alpha_{c(j)}\right) = w_\infty \cdot \lambda_\infty \tag{7.9}$$

for some $w_\infty \in W^\nu$. We deduce at once from Theorem 4.14 that the cyclotomic ${}^L\mathfrak{g}$-oper $[\nabla]_\Gamma$ lives in the space of ${}^L\mathfrak{g}$-opers $\mathrm{Op}^\Gamma_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\boldsymbol{\lambda}}$ introduced at the end of Sect. 4, where $\boldsymbol{z} := \{0, z_1, \ldots, z_N, \infty\} \subset \mathbb{P}^1$ and $\boldsymbol{\lambda} := \{\lambda_0, \lambda_1, \ldots, \lambda_N, \lambda_\infty\} \subset \mathfrak{h}^*$. If, moreover, the weights $\lambda_i \in \mathfrak{h}^*$, $i = 1, \ldots, N$ and ν-invariant weight $\lambda_0 \in \mathfrak{h}^{*,\nu}$ are all integral dominant, then so are all the weights $\nu^r\lambda_i \in \mathfrak{h}^*$ for $r = 0, \ldots, T-1$ and $i = 1, \ldots, N$ from which it follows that ∇ is monodromy-free, cf. remark 2. In this case, $[\nabla]_\Gamma$ defines an element of the subset $\mathrm{Op}^\Gamma_{{}^L\mathfrak{g}}(\mathbb{P}^1)_{\boldsymbol{z};\boldsymbol{\lambda}} \subset \mathrm{Op}^\Gamma_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\boldsymbol{\lambda}}$ introduced at the end of Sect. 4. In fact, it follows from Sect. 6.2 that the integrality assumption on λ_0 can be relaxed to $\langle \check{\alpha}_i, \lambda_0\rangle \in \mathbb{Q}$ for all $i \in I$, in which case ∇ has monodromy $e^{2\pi i\lambda_0} \in {}^LG$ at the origin, where ${}^LG$ denotes the adjoint group of the Langlands dual Lie algebra ${}^L\mathfrak{g}$. The meromorphic ${}^L\mathfrak{g}$-connection ∇ becomes a monodromy-free when pulled back to a cover of $\mathbb{P}^1$.

7.3. Spectrum of $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$

In this final section, we state two conjectures relating the spectrum of the cyclotomic Gaudin model to cyclotomic ${}^L\mathfrak{g}$-opers. The first gives a description of the common eigenvalues of elements of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ on a joint eigenvector $\psi_{(x_j)}$ in the tensor product of Verma modules $\bigotimes_{i=1}^N M_{\lambda_i}$, given the corresponding cyclotomic Miura ${}^L\mathfrak{g}$-oper. The second describes the maximal spectrum of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ in terms of cyclotomic ${}^L\mathfrak{g}$-opers with regular singularities at the z_i, $i = 1, \ldots, N$, the origin and infinity.

Recall that the centre $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ is isomorphic to the polynomial algebra $\mathbb{C}[P_k]_{k\in E}$ in $\operatorname{rk}\mathfrak{g}$ generators, where P_k has degree $k+1$ in the canonical filtration on $U(\mathfrak{g})$. Recall here that E denotes the *multiset* of exponents of the semisimple Lie algebra $\mathfrak{g}$. Similarly, in the affine setting it follows from the results of Feigin and Frenkel [5] that, as a commutative algebra, the subspace of singular vectors $\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big) \subset \mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ in the vacuum Verma module $\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ at critical level over $\widehat{\mathfrak{g}}_u$ is isomorphic to the polynomial algebra in infinitely many variables

$$\mathfrak{z}\big(\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})\big) \simeq \mathbb{C}[\mathcal{D}^r S_k]_{r\in\mathbb{Z}_{\geq 0};k\in E},$$

where $\mathcal{D} \in \operatorname{End}\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ is defined by the property $[\mathcal{D}, a(n)] = -n\,a(n-1)$ for all $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, where $a(n) := a \otimes (t-u)^n \in \widehat{\mathfrak{g}}_u$, and $\mathcal{D}v_0 = 0$. The element $S_k \in \mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ is of degree $k+1$ in the natural filtration on the $\widehat{\mathfrak{g}}_u$-module $\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$.

Conjecture 7.1. *Let $\nabla \in \mathrm{MOp}^\Gamma_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ correspond to an eigenvector $\psi_{(x_j)}$ of the cyclotomic Gaudin algebra $\mathscr{Z}^\Gamma_{(z_i)}(\mathfrak{g})$ on the tensor product of Verma modules $\bigotimes_{i=1}^N M_{\lambda_i}$, as considered in Sects. 7.1 and 7.2. Let $\mathrm{d} + \check{p}_{-1}\mathrm{d}t + \sum_{k\in E}\mathsf{c}_k$ be the canonical representative of the underlying cyclotomic ${}^L\mathfrak{g}$-oper $[\nabla]_\Gamma$. There exists a basis*[2] *$\{\check{p}_i\}_{i\in E}$ of the centraliser of $\check{p}_1$ in ${}^L\mathfrak{g}$, which is independent of*

[2] Note that bases of the centraliser of $\check{p}_1$ in ${}^L\mathfrak{g}$ are naturally labelled by elements of the multiset of exponents $\check{E}$ of ${}^L\mathfrak{g}$, cf. Sect. 3.1. Here, we implicitly make use of the fact that $\check{E}$ coincides with the multiset E of exponents of $\mathfrak{g}$.

∇, *such that for every* $r \in \mathbb{Z}_{\geq 0}$ *and* $k \in E$, *the eigenvalue of* $\Psi^{\Gamma}_{(z_i),t}(\mathcal{D}^r S_k)$ *on* $\psi_{(x_j)}$ *is given by* $\partial_t^r u_k(t)$, *where* $\mathsf{c}_k = u_k(t)\,\check{p}_k\,\mathrm{d}t$.

To motivate the conjecture, consider the quadratic singular vector $S := S_1 = \frac{1}{2} I_a(-1) I^a(-1) v_0$ in $\mathbb{V}^{\mathrm{crit}}_{0,u}(\mathfrak{g})$ corresponding to the exponent 1. The quadratic cyclotomic Gaudin Hamiltonians are obtained from it as $\mathcal{H}_j = \mathrm{res}_{z_j} \Psi^{\Gamma}_{(z_i),t}(S)\mathrm{d}t$ for each $j = 1, \dots, N$. According to Proposition 4.6, the component $\mathsf{c}_1 \in \Omega^{\hat{\varsigma}}(\mathfrak{a})$ of the canonical representative of $\nabla = \mathrm{d} + \check{p}_{-1}\mathrm{d}t - \lambda(t)\mathrm{d}t \in \mathrm{MOp}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)$ is given by $\mathsf{c}_1 = u_1\,\check{p}_1 \mathrm{d}t$ where

$$u_1 = \frac{1}{2(\check{\rho}|\check{\rho})}\big(\tfrac{1}{2}(\lambda(t)|\lambda(t)) - (\lambda'(t)|\rho)\big).$$

By suitably normalising the basis element $\check{p}_1$, we can remove the overall factor of $1/2(\check{\rho}|\check{\rho})$ from u_1. It then follows from (7.6) that the eigenvalue of $\mathcal{H}_i$ on $\psi_{(x_j)}$ is given by the residue $\mathrm{res}_{z_i} u_1(t)\mathrm{d}t$, in agreement with Conjecture 7.1 .

At the end of Sect. 4, we defined the set $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z}}$ of cyclotomic ${}^L\mathfrak{g}$-opers with regular singularities at the points in the set $\boldsymbol{z} = \{0, z_1, \dots, z_N, \infty\}$. Let $\lambda_0 \in \mathfrak{h}^{*,\nu}$ be the ν-invariant weight defined in (7.3). We shall assume that λ_0 is dominant and such that $\langle \check{\alpha}_i, \lambda_0 \rangle \in \mathbb{Q}$ for all $i \in I$, as in the setting of Sect. 6.2. Let $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\lambda_0}$ be the subset of cyclotomic ${}^L\mathfrak{g}$-opers in $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z}}$ whose residue at the origin is given by the finite ${}^L\mathfrak{g}^{\nu}$-oper $[\lambda_0]_{W^\nu}$ and with monodromy $e^{2\pi i \lambda_0} \in {}^L G$ at the origin.

Conjecture 7.2. *The cyclotomic Gaudin algebra* $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ *is isomorphic to the algebra of functions on the space* $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\lambda_0}$. *In other words,*

$$\mathrm{Spec}\, \mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g}) \simeq \mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\lambda_0}. \tag{7.10}$$

In the non-cyclotomic case where $\Gamma = \{1\}$, the cyclotomic Gaudin algebra $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ reduces to the ordinary Gaudin algebra $\mathscr{Z}_{(z_i)}(\mathfrak{g})$. On the other hand, since $T = 1$ the weight at the origin becomes trivial, i.e. $\lambda_0 = 0$. Hence, $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\lambda_0}$ consists of ${}^L\mathfrak{g}$-opers with regular singularity at the points of $\boldsymbol{z}$, with trivial monodromy at the origin and residue there equal to $[0]_{W^\nu}$. It follows from [7, Lemma 2.4] that such ${}^L\mathfrak{g}$-opers are regular at the origin. Thus, Conjecture 7.2 reduces in the case $\Gamma = \{1\}$ to [8, Theorem 2.7(1)] describing the spectrum of the usual Gaudin algebra $\mathscr{Z}_{(z_i)}(\mathfrak{g})$.

It is also interesting to note, in the cyclotomic setting, how the dependence on the automorphism $\sigma \in \mathrm{Aut}\,\mathfrak{g}$ with $\sigma^T = \mathrm{id}$ is encoded in the space of ${}^L\mathfrak{g}$-opers $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z};\lambda_0}$. Indeed, recall that the cyclotomic Gaudin algebra $\mathscr{Z}^{\Gamma}_{(z_i)}(\mathfrak{g})$ is defined for any choice of such automorphism. On the other hand, the set $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)$ of cyclotomic ${}^L\mathfrak{g}$-opers, and in particular, the subset $\mathrm{Op}^{\Gamma}_{L\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\boldsymbol{z}}$, is defined relative to a particular automorphism $\varsigma \in \mathrm{Aut}\,{}^L\mathfrak{g}$, cf. the definition (7.7), which depends only on $T \in \mathbb{Z}_{\geq 1}$ and the diagram part ν of σ. According to Conjecture 7.2, the additional information on the automorphism $\sigma \in \mathrm{Aut}\,\mathfrak{g}$ is encoded through the weight at the origin λ_0 as defined in (7.3).

The difference in the descriptions of the cyclotomy on both sides of the isomorphism (7.10) can also be understood from the fact that there is no

canonical way of defining an automorphism of ${}^L\mathfrak{g}$ from the automorphism $\sigma \in \operatorname{Aut}\mathfrak{g}$ characterising the cyclotomic Gaudin model. Indeed, when passing from $\mathfrak{g}$ to its Langlands dual ${}^L\mathfrak{g}$, the automorphism $\sigma \in \operatorname{Aut}\mathfrak{g}$ induces an automorphism of the Cartan subalgebra ${}^L\mathfrak{h}$ of ${}^L\mathfrak{g}$ since this is canonically identified with $\mathfrak{h}^*$. However, the automorphism of ${}^L\mathfrak{h}$ obtained in this way only depends on the diagram part ν of σ. The freedom in extending it to an automorphism $\varsigma \in \operatorname{Aut}{}^L\mathfrak{g}$, corresponding to the choice of inner part of the automorphism ς, is uniquely fixed by requiring that the differential $p_{-1}\mathrm{d}t$ be invariant under the action of $\hat{\varsigma}$, see Sect. 4.3.

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AFFINE GAUDIN MODELS AND HYPERGEOMETRIC FUNCTIONS ON AFFINE OPERS

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Abstract. We conjecture that quantum Gaudin models in affine types admit families of higher Hamiltonians, labelled by the (countably infinite set of) exponents, whose eigenvalues are given by functions on a space of meromorphic opers associated with the Langlands dual Lie algebra. This is in direct analogy with the situation in finite types. However, in stark contrast to finite types, we prove that in affine types such functions take the form of hypergeometric integrals, over cycles of a twisted homology defined by the levels of the modules at the marked points. That result prompts the further conjecture that the Hamiltonians themselves are naturally expressed as such integrals.

We go on to describe the space of meromorphic affine opers on an arbitrary Riemann surface. We prove that it fibres over the space of meromorphic connections on the canonical line bundle Ω. Each fibre is isomorphic to the direct product of the space of sections of the square of Ω with the direct product, over the exponents j not equal to 1, of the twisted cohomology of the jth tensor power of Ω.

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1. Introduction and overview

Let $\mathfrak{g}$ be any symmetrizable Kac-Moody Lie algebra. Pick a collection $z_1, \dots, z_N$ of distinct points in the complex plane. The quadratic Hamiltonians of the quantum *Gaudin model* for these data are the elements

$$\mathcal{H}_i := \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\Xi_{ij}}{z_i - z_j}, \qquad i = 1, \dots, N, \tag{1.1}$$

of the (suitably completed) tensor product $U(\mathfrak{g})^{\otimes N}$, where the notation Ξ_{ij} means Ξ acting in tensor factors i and j. Here $\Xi = \sum_\alpha \Xi_{(\alpha)}$ is the (possibly infinite) sum over all root spaces of $\mathfrak{g}$ of the canonical elements $\Xi_{(\alpha)} \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ defined by the standard bilinear form on $\mathfrak{g}$ [Kac90, Chapter 2]. The action of Ξ is well-defined on tensor products of highest-weight $\mathfrak{g}$-modules. Let L_λ denote the irreducible $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^* = \mathfrak{g}_0^*$, and pick a collection $\lambda_1, \dots, \lambda_N$ of weights. Then in particular $\mathcal{H}_i$ are well-defined as linear maps in $\mathrm{End}(\bigotimes_{i=1}^N L_{\lambda_i})$. These maps commute amongst themselves. The *Bethe ansatz* is a technique for finding their joint eigenvectors and eigenvalues. One constructs a vector ψ called the *weight function* or *Schechtman-Varchenko vector*, which depends on variables called *Bethe roots*. Provided these variables obey certain *Bethe ansatz equations*, then ψ is a joint eigenvector of the $\mathcal{H}_i$, with certain explicit eigenvalues. Let us stress that this statement is known to hold for arbitrary symmetrizable Kac-Moody algebras $\mathfrak{g}$. Indeed, it follows from results in [SV91, RV95], as we recall in an appendix.

In the special case where $\mathfrak{g}$ is of finite type, much more is known. Namely, in that case the quadratic Gaudin Hamiltonians $\mathcal{H}_i$ belong to a commutative subalgebra $\mathcal{B} \subset U(\mathfrak{g})^{\otimes N}$ called the *Gaudin* [Fre05] or *Bethe* [MTV06] subalgebra. The Schechtman-Varchenko vector is a joint eigenvector for this commutative algebra $\mathcal{B}$ [FFR94], and the eigenvalues are encoded as functions on a space of *opers* (see below for the definition). In fact there is even a stronger result that the image of $\mathcal{B}$ in $\mathrm{End}(\bigotimes_{i=1}^N L_{\lambda_i})$ can be identified with the algebra of functions on a certain space of monodromy-free opers whose singularities are at the marked points z_i and whose residues at these singularities are given by the highest weights λ_i – see [MTV09] in type A and [Ryb] in all finite types.

Now suppose $\mathfrak{g}$ is of untwisted affine type. Two natural questions arise [FF11]:

1. Are there higher Gaudin Hamiltonians? *i.e.* are the quadratic Hamiltonians above part of some larger commutative subalgebra of (a suitable completion of) $U(\mathfrak{g})^{\otimes N}$, such that ψ is still a common eigenvector?
2. If yes, then what parameterizes the eigenvalues of these higher Hamiltonians?

In this paper we shall provisionally assume that the answer to the first question is yes, and give a conjectural answer to the second. Namely, we introduce a notion of meromorphic affine opers on $\mathbb{P}^1$ (affine opers have been defined previously in [BZF01, Fre04]), and then the main result of the paper is that:

(i) There is a notion of quasi-canonical form for affine opers which is the direct generalisation of the canonical form in finite type, and yet
(ii) The functions on the space of affine opers turn out to be of a very different character than in the finite case. Namely, they are given by hypergeometric integrals, over cycles of a certain twisted homology defined by the levels of the modules at the marked points.

We conjecture that these hypergeometric integrals give the eigenvalues of ("local") higher affine Gaudin Hamiltonians. This observation in turn allows us to make a conjecture about the form of the higher affine Gaudin Hamiltonians themselves.

To explain these statements, let us begin by recalling the situation in finite types. Consider first $\mathfrak{g}$ of finite type of rank ℓ. The spectrum of the Gaudin algebra for $\mathfrak{g}$ is described by ${}^L\mathfrak{g}$-opers, where ${}^L\mathfrak{g}$ is the Langlands dual of $\mathfrak{g}$, *i.e.* the Kac-Moody algebra with transposed Cartan matrix. Let ${}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$ be a Cartan decomposition and $\bar{p}_{-1} \coloneqq \sum_{i=1}^{\ell} \check{f}_i \in {}^L\mathfrak{n}_-$ the corresponding principal nilpotent element ($\check{f}_i$ are Chevalley generators). A *Miura* ${}^L\mathfrak{g}$*-oper* is a connection of the form

$$d + (\bar{p}_{-1} + u(z))\, dz \tag{1.2a}$$

where $u(z)$ is a meromorphic function valued in ${}^L\mathfrak{h} = \mathfrak{h}^*$. Let $\alpha_i \in \mathfrak{h}^*$, $i = 1, \dots, \ell$ be the simple roots of $\mathfrak{g}$; they are also the simple coroots of ${}^L\mathfrak{g}$. For the Gaudin model with regular singularities as described above, $u(z)$ generically takes the form

$$u(z) = -\sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} + \sum_{j=1}^{m} \frac{\alpha_{c(j)}}{z - w_j}, \tag{1.2b}$$

where $w_1, \dots, w_m$ are the $m \in \mathbb{Z}_{\geq 0}$ Bethe roots, with "colours" $\{c(j)\}_{j=1}^m \subset \{1, \dots, \ell\}$. An ${}^L\mathfrak{g}$*-oper* is a gauge equivalence class of connections of the form

$$d + (\bar{p}_{-1} + b(z))dz,$$

where $b(z)$ is a meromorphic function valued in ${}^L\mathfrak{b}_+ = {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$, under the gauge action of the unipotent subgroup ${}^LN = \exp({}^L\mathfrak{n}_+)$. So in particular each Miura ${}^L\mathfrak{g}$-oper defines an underlying ${}^L\mathfrak{g}$-oper, namely the equivalence class to which it belongs. It is known that each ${}^L\mathfrak{g}$-oper has a *unique* representative of the form

$$d + \left(\bar{p}_{-1} + \sum_{k \in \bar{E}} \bar{v}_k(z) \bar{p}_k \right) dz. \tag{1.3}$$

Here the sum is over the (finite) set[1] $\bar{E}$ of exponents of ${}^L\mathfrak{g}$. For each exponent $k \in \bar{E}$, $\bar{p}_k \in {}^L\mathfrak{n}_+$ is a certain nonzero element of grade k in the principal gradation of ${}^L\mathfrak{g}$. Its coefficient $\bar{v}_k(z)$ is a meromorphic function valued in $\mathbb{C}$. Since this representative is unique, these functions $\{\bar{v}_k(z)\}_{k \in \bar{E}}$ are good coordinates on the space of ${}^L\mathfrak{g}$-opers. On the underlying ${}^L\mathfrak{g}$-oper of the Miura ${}^L\mathfrak{g}$-oper in (1.2), the functions $\{\bar{v}_k(z)\}_{k \in \bar{E}}$ will generically have poles at all the poles of $u(z)$. The Bethe equations are precisely the equations needed to ensure they in fact *only* have poles at the marked points $\{z_i\}_{i=1}^N$ and *not* at the Bethe roots $\{w_j\}_{j=1}^m$. Suppose the Bethe equations hold. Then the Schechtman-Varchenko vector ψ obeys $S_k(z)\psi = \bar{v}_k(z)\psi$ for all $k \in \bar{E}$, where $\{S_k(z)\}_{k \in \bar{E}}$ are certain generating functions of the Gaudin algebra.

Let us now turn to affine types and try to follow the steps above as closely as possible. Suppose $\mathfrak{g}$ is an untwisted affine Kac-Moody algebra with Cartan matrix of rank ℓ. Let ${}^L\mathfrak{g}$ be its Langlands dual, with Cartan decomposition ${}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$. Define a *Miura* ${}^L\mathfrak{g}$*-oper* to be a connection of the form

$$d + (p_{-1} + u(z))\, dz \tag{1.4}$$

[1]In exactly one case, that of type D_{2n}, $\bar{E}$ is a multiset.

where $u(z)$ is again a meromorphic function valued in ${}^L\mathfrak{h} = \mathfrak{h}^*$, and where now $p_{-1} \coloneqq \sum_{i=0}^{\ell} \check{f}_i \in {}^L\mathfrak{n}_-$. The Cartan subalgebra ${}^L\mathfrak{h} = \mathfrak{h}^*$ of ${}^L\mathfrak{g}$ is now of dimension $\ell + 2$. As a basis, we may choose the simple roots α_i, $i = 0, 1, \dots, \ell$, of $\mathfrak{g}$ (which are the simple coroots of ${}^L\mathfrak{g}$) together with a choice of derivation element. It is natural to choose the derivation corresponding to the principal gradation of ${}^L\mathfrak{g}$. So let us pick a derivation element $\rho \in {}^L\mathfrak{h}$ such that $[\rho, \check{e}_i] = \check{e}_i$, $[\rho, \check{f}_i] = -\check{f}_i$ for each $i = 0, 1, \dots, \ell$.

By analogy with (1.2) one can expect that for the Gaudin model with regular singularities at the marked points $\{z_i\}_{i=1}^N$ the relevant Miura ${}^L\mathfrak{g}$-opers are those with $u(z)$ just as in (1.2b) except that now the "colours" of the Bethe roots $\{c(j)\}_{j=1}^m \subset \{0, 1, \dots, \ell\}$ can include 0. We can write $u(z)$ in our basis as

$$u(z) = \sum_{i=0}^{\ell} u_i(z)\alpha_i - \frac{\varphi(z)}{h^\vee}\rho$$

where $\{u_i(z)\}_{i=0}^{\ell}$ and $\varphi(z)$ are meromorphic functions valued in $\mathbb{C}$. (It proves convenient to include the factor of one over the Coxeter number $h^\vee$ of ${}^L\mathfrak{g}$.) The function $\varphi(z)$ depends only on the levels k_i of the $\mathfrak{g}$-modules L_{λ_i}, *i.e.* the values of the central element of $\mathfrak{g}$ on these modules:

$$\varphi(z) = \sum_{i=1}^{N} \frac{k_i}{z - z_i}. \tag{1.5}$$

Now define an ${}^L\mathfrak{g}$-*oper* to be a gauge equivalence class of connections of the form

$$d + (p_{-1} + b(z))dz,$$

where $b(z)$ is a meromorphic function valued in ${}^L\mathfrak{b}_+ = {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$, under the gauge action of the subgroup ${}^LN_+ = \exp({}^L\mathfrak{n}_+)$. This subgroup is no longer unipotent, but it is still easy to make sense of gauge transformations grade-by-grade in the principal gradation. See §3.3 below. In this way we shall show that each ${}^L\mathfrak{g}$-oper has a representative of the form

$$d + \left(p_{-1} - \frac{\varphi(z)}{h^\vee}\rho + \sum_{r \in E} v_r(z)p_r\right)dz. \tag{1.6}$$

Here E denotes the set of positive exponents of ${}^L\mathfrak{g}$, which is now an infinite set[2]. For each $r \in \pm E$, p_r is a certain nonzero element of grade r in the principal gradation of ${}^L\mathfrak{g}$, and its coefficient $v_r(z)$ is a meromorphic function valued in $\mathbb{C}$. In particular, the underlying ${}^L\mathfrak{g}$-oper of the Miura ${}^L\mathfrak{g}$-oper in (1.4) has a representative of this form.

However, in stark contrast to the case of finite type algebras above, the representative (1.6) is not unique, because there is a residual gauge freedom. This freedom is generated by gauge transformations of the form $\exp(\sum_{r \in E_{\geq 2}} g_r(z)p_r)$, where $g_r(z)$ are meromorphic functions valued in $\mathbb{C}$ and $E_{\geq 2}$ is the set of positive exponents of ${}^L\mathfrak{g}$ excluding 1. Such a transformation preserves the form of the connection (1.6) and

[2]once again, multiset, in the case of type ${}^1D_{2n}$.

the function $\varphi(z)$ while sending, for each[3] $r \in E_{\geq 2}$,

$$v_r(z) \longmapsto v_r(z) - g_r'(z) + \frac{r\varphi(z)}{h^\vee} g_r(z). \tag{1.7}$$

Consequently, these $v_r(z)$ are not themselves well-defined functions on the space of ${}^L\mathfrak{g}$-opers, and one should not expect them to parameterize eigenvalues of Gaudin Hamiltonians. Rather, one should take appropriate integrals of them. Indeed, consider the multivalued (for generic k_i) function on $\mathbb{C} \setminus \{z_1, \dots, z_N\}$ defined as

$$\mathcal{P}(z) \coloneqq \prod_{i=1}^{N} (z - z_i)^{k_i}.$$

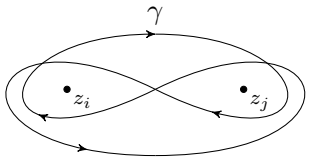
If we multiply $v_r(z)$ by $\mathcal{P}(z)^{-r/h^\vee}$ then its transformation property (1.7) can equivalently be written as

$$\mathcal{P}(z)^{-r/h^\vee} v_r(z) \longmapsto \mathcal{P}(z)^{-r/h^\vee} v_r(z) - \partial_z\big(\mathcal{P}(z)^{-r/h^\vee} g_r(z)\big).$$

We now see that in order to get gauge-invariant quantities we should consider integrals

$$I_r^\gamma \coloneqq \int_\gamma \mathcal{P}(z)^{-r/h^\vee} v_r(z) dz \tag{1.8}$$

over any cycle γ along which $\mathcal{P}^{-r/h^\vee}$ has a single-valued branch. The prototypical example of such a cycle is a Pochhammer contour, drawn below around two distinct points z_i and z_j, $i, j = 1, \dots, N$:



Another way of formulating the above, described in more detail in §7, is to note that the transformation property (1.7) says that the 1-form $v_r(z)dz$ is really an element of some suitably defined twisted cohomology, and (1.8) represents its integral over the class of a cycle γ in the dual twisted homology. (For an introduction to twisted homology and local systems see *e.g.* [EFK98]. Note, though, that the local system underlying the twisted homology described above is conceptually distinct from the "usual" local system associated to Gaudin models, namely the local system defined by the KZ connection.)

Next one should ask about the role of the Bethe equations. Consider the underlying ${}^L\mathfrak{g}$-oper of the Miura ${}^L\mathfrak{g}$-oper (1.4) with $u(z)$ as in (1.2b). We shall show that the Bethe equations are precisely the equations needed to ensure that there exists a choice of gauge in which the functions $v_r(z)$ only have poles at the marked points $\{z_i\}_{i=1}^n$ and not at the Bethe roots $\{w_j\}_{j=1}^m$. The Bethe equations thus ensure that the integrands $\mathcal{P}(z)^{-r/h^\vee} v_r(z)dz$ in (1.8) have no residues at the Bethe roots. Thus, in particular,

[3]There is a subtlety for $r = 1$; see Corollary 3.9 below.

if the Bethe equations hold then the integrals (1.8) do not depend on the position of the chosen contour γ relative to these Bethe roots.

The form of the functions (1.8) on the space of ${}^L\mathfrak{g}$-opers leads us to conjecture the existence of a collection $\{\mathcal{S}_r(z)\}_{r\in E}$ of meromorphic functions valued in (a suitable completion of) $U(\mathfrak{g})^{\otimes N}$, whose properties are listed in Conjecture 5.1. These ensure, in particular, that the corresponding integrals

$$\hat{Q}_r^\gamma := \int_\gamma \mathcal{P}(z)^{-r/h^\vee} \mathcal{S}_r(z) dz \tag{1.9}$$

mutually commute in the quotient of (the completion of) $U(\mathfrak{g})^{\otimes N}$ in which the central elements act by the levels k_i. Moreover, we conjecture that the Schechtman-Varchenko vector ψ is a simultaneous eigenvector of the $\hat{Q}_r^\gamma$ with eigenvalues given by (1.8). That is, $\hat{Q}_r^\gamma \psi = I_r^\gamma \psi$ for any choice of contour γ as above and any $r \in E$.

A first non-trivial check of these conjectures is to show that in the case $\mathfrak{g}' = \widehat{\mathfrak{sl}}_M$, for $M \geq 3$, there are commuting *cubic* Hamiltonians fitting this pattern. In a forthcoming paper [LVY], we explicitly construct such cubic Hamiltonians and prove that they commute; we also check that ψ is an eigenvector with the expected eigenvalues as in (1.8), at least for 0 and 1 Bethe roots.

Our conjecture on the general form of the "local" higher affine Gaudin Hamiltonians in (1.9) is motivated by the recent construction of local integrals of motion in *classical* affine Gaudin models. Specifically, it was shown in [Vic] that classical affine Gaudin models provide a unifying framework for describing a broad family of classical integrable field theories. One of the defining features of such theories is that the Poisson bracket of their Lax matrix is characterised by a certain rational function, called the *twist function* $\varphi(z)$. We restrict attention in this article to those with twist function of the form (1.5). It was subsequently shown in [LMV17], in the case when $\mathfrak{g}$ is the untwisted affine Kac-Moody algebra associated with a semisimple Lie algebra of classical type, how to associate an infinite set $\{Q_r^x\}_{r\in E}$ of local integrals of motion in such a theory to each zero x of the twist function. These local charges were obtained by generalising the original procedure of [EHMM99] for classical principal chiral models on compact Lie groups of classical type, which had later also been extended to various other classical integrable field theories in [EHMM00, EM00, EY05], see also [Eva01]. As we argue in §8.3, the integral over the contour γ in (1.9) localises in the classical limit to critical points of the function $\mathcal{P}(z)$, in other words to zeroes of the twist function $\varphi(z)$. In this sense, the operators (1.9) provide natural quantisations of the local integrals of motion Q_r^x in the classical affine Gaudin model.

Let us finally note that the appearance of hypergeometric integrals, as in (1.8), is very suggestive in relation to recent work on the massive ODE/IM correspondence for the Fateev model [Luk13, BL14].

The paper is organised as follows.

In §2, to set the notation we recall the definition of an affine Kac-Moody algebra $\mathfrak{g}$ and its Langlands dual ${}^L\mathfrak{g}$, focusing on the latter for the purpose of this paper. In particular, we recall the definition and main properties of its principal subalgebra.

In §3 we introduce the space of meromorphic ${}^L\mathfrak{g}$-opers on $\mathbb{P}^1$, working in a fixed global coordinate on $\mathbb{C} \subset \mathbb{P}^1$. The main result of this section is Theorem 3.7 which describes the quasi-canonical form of an ${}^L\mathfrak{g}$-oper $[\nabla]$. This allows us to describe gauge invariant functions on the space of ${}^L\mathfrak{g}$-opers as hypergeometric integrals of the form (1.8) in Corollary 3.13.

In §4 we introduce a class of Miura ${}^L\mathfrak{g}$-opers with simple poles at the marked points z_i, $i = 1, \dots, N$ with residues $\lambda_i \in \mathfrak{h}^*$, and additional simple poles at the Bethe roots w_j, $j = 1, \dots, m$. The ${}^L\mathfrak{g}$-oper $[\nabla]$ underlying such a Miura ${}^L\mathfrak{g}$-oper ∇ is shown to be regular at each of the Bethe roots w_j if and only if the Bethe equations hold. Moreover, we show that the eigenvalues of the quadratic Gaudin Hamiltonians (1.1) on the tensor product $\bigotimes_{i=1}^N L_{\lambda_i}$ appear as the residues at the z_i in the coefficient of p_1 in any quasi-canonical form of the ${}^L\mathfrak{g}$-oper $[\nabla]$.

Based on the description of functions on the space of ${}^L\mathfrak{g}$-opers from Corollary 3.13, in §5 we formulate our main conjecture about the form of the higher Gaudin Hamiltonians of an affine Gaudin model associated with the affine Kac-Moody algebra $\mathfrak{g}$. See Conjecture 5.1.

In §6 we give a coordinate-independent definition of meromorphic ${}^L\mathfrak{g}$-opers on an arbitrary Riemann surface Σ. In particular, we compare and contrast the description of the space of ${}^L\mathfrak{g}$-opers in the cases when ${}^L\mathfrak{g}$ is of finite and affine type.

Specialising the discussion of §6 to the case $\Sigma = \mathbb{P}^1$, §7 is devoted to a coordinate-independent description of the functions on the space of ${}^L\mathfrak{g}$-opers from Corollary 3.13.

In §8 we discuss various connections between the present work and the literature. In particular, we compare our main Theorem 3.7 with the procedure of Drinfel'd and Sokolov [DS85] for constructing classical integrals of motion of generalised (m)KdV. We also mention connections with the (massive) ODE/IM correspondence. We provide motivation for Conjecture 5.1 by relating the form of the classical limit of the higher Gaudin Hamiltonians with the existing hierarchy of classical integrals of motion in classical affine Gaudin models.

Finally, in appendix A we briefly review the work of Schechtmann and Varchenko [SV91] on the diagonalisation of the quadratic Gaudin Hamiltonians for an arbitrary Kac-Moody algebra $\mathfrak{g}$ by the Bethe ansatz.

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2. The affine algebra ${}^L\mathfrak{g}$

2.1. **Cartan data and defining relations.** Let $\mathfrak{g} \coloneqq \mathfrak{g}(A)$ be an untwisted affine Kac-Moody algebra with indecomposable Cartan matrix $A \coloneqq (A_{ij})_{i,j=0}^{\ell}$, and let ${}^L\mathfrak{g} \coloneqq \mathfrak{g}({}^tA)$ be its Langlands dual, namely the affine Kac-Moody algebra associated with the transposed Cartan matrix. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{h}$ is a complex vector space of dimension $\dim \mathfrak{h} = \ell + 2$. The sets of simple roots $\{\alpha_i\}_{i=0}^{\ell}$ and simple coroots $\{\check{\alpha}_i\}_{i=0}^{\ell}$ of $\mathfrak{g}$ are by definition linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, such that $A_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$ for $i, j \in I := \{0, \dots, \ell\}$. Here $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denotes the canonical pairing. In the Cartan decomposition of ${}^L\mathfrak{g}$,

$$ {}^L\mathfrak{g} = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+, $$

we may identify ${}^L\mathfrak{h} = \mathfrak{h}^*$. Then $\{\alpha_i\}_{i=0}^{\ell}$ is a set of simple *coroots* of ${}^L\mathfrak{g}$, and $\{\check{\alpha}_i\}_{i=0}^{\ell}$ a set of simple *roots* of ${}^L\mathfrak{g}$. In terms of the Chevalley generators $\check{e}_i$, $i \in I$, of ${}^L\mathfrak{n}_+$ and $\check{f}_i$, $i \in I$, of ${}^L\mathfrak{n}_-$, the defining relations of ${}^L\mathfrak{g}$ are given by

$$ [x, \check{e}_i] = \langle x, \check{\alpha}_i \rangle \check{e}_i, \qquad [x, \check{f}_i] = -\langle x, \check{\alpha}_i \rangle \check{f}_i, \tag{2.1a} $$

$$ [x, x'] = 0, \qquad [\check{e}_i, \check{f}_j] = \alpha_i \delta_{i,j}, \tag{2.1b} $$

for any $x, x' \in {}^L\mathfrak{h}$, together with the Serre relations

$$ (\operatorname{ad} \check{e}_i)^{1-A_{ji}} \check{e}_j = 0, \qquad (\operatorname{ad} \check{f}_i)^{1-A_{ji}} \check{f}_j = 0. \tag{2.1c} $$

Remark 2.1. We shall be mostly concerned with the Lie algebra ${}^L\mathfrak{g}$ rather than $\mathfrak{g}$. Nevertheless, since we have in mind applications to the Gaudin model for $\mathfrak{g}$, we prefer to keep the notation adapted to $\mathfrak{g}$, at the cost of the somewhat non-standard appearance of these relations (2.1) and others below. ◁

Let a_i (resp. $\check{a}_i$), $i \in I$, be the unique positive relatively prime integers such that $A\,{}^t(a_0, \dots, a_\ell) = 0$ (resp. ${}^tA\,{}^t(\check{a}_0, \dots, \check{a}_\ell) = 0$). Define

$$ h := \sum_{i=0}^{\ell} a_i \qquad h^\vee := \sum_{i=0}^{\ell} \check{a}_i. $$

Then h is the Coxeter number of $\mathfrak{g}$ (and the dual Coxeter number of ${}^L\mathfrak{g}$) while $h^\vee$ is the Coxeter number of ${}^L\mathfrak{g}$ (and the dual Coxeter number of $\mathfrak{g}$). Define also

$$ \delta := \sum_{i=0}^{\ell} a_i \alpha_i, \qquad \mathsf{k} := \sum_{i=0}^{\ell} \check{a}_i \check{\alpha}_i. $$

Then δ spans the centre of ${}^L\mathfrak{g}$ while k spans the centre of $\mathfrak{g}$. Denote by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and ${}^L\mathfrak{g}' := [{}^L\mathfrak{g}, {}^L\mathfrak{g}]$ the derived subalgebras of $\mathfrak{g}$ and ${}^L\mathfrak{g}$, respectively.

We shall suppose that $\check{a}_0 = 1$ and $a_0 = 1$.

Remark 2.2. One has $\check{a}_0 = 1$ and $a_0 = 1$ for all affine Kac-Moody algebras except for type ${}^2A_{2k}$. In type ${}^2A_{2k}$ one can choose to take either $\check{a}_0 = 1$ and $a_0 = 2$ or vice versa $\check{a}_0 = 2$ and $a_0 = 1$. The Cartan matrices in these two descriptions are transposes of one another so that in this case ${}^L\mathfrak{g}$ and $\mathfrak{g}$ are both twisted, of type ${}^2A_{2k}$. Since we have in mind applications to the Gaudin model for an untwisted affine Kac-Moody algebra $\mathfrak{g}$, we shall not consider this case. ◁

Recall that given any $d \in \mathfrak{h}$ such that $\langle \delta, d \rangle \neq 0$, $\{\check{\alpha}_i\}_{i=0}^{\ell} \cup \{d\}$ forms a basis of $\mathfrak{h}$; and similarly, given any $\Lambda \in {}^L\mathfrak{h}$ such that $\langle \Lambda, \mathsf{k} \rangle \neq 0$, $\{\alpha_i\}_{i=0}^{\ell} \cup \{\Lambda\}$ provides a basis for ${}^L\mathfrak{h}$. We call such elements d and Λ *derivation elements* of $\mathfrak{h}$ and ${}^L\mathfrak{h}$, respectively.

Let $\mathsf{d} \in \mathfrak{h}$ be a derivation element of $\mathfrak{g}$ such that

$$\langle \alpha_i, \mathsf{d} \rangle = \delta_{i,0}, \qquad i \in I.$$

Such a d is unique up to the addition of a multiple of k. Having made such a choice we define a non-degenerate symmetric bilinear form $(\cdot|\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ on $\mathfrak{h}$ by

$$(\check{\alpha}_i|x) = a_i \check{a}_i^{-1} \langle \alpha_i, x \rangle, \qquad (\mathsf{d}|\mathsf{d}) = 0 \tag{2.2}$$

for any $i \in I$ and $x \in \mathfrak{h}$. It extends uniquely to an invariant symmetric bilinear form on the whole of $\mathfrak{g}$ [Kac90, Proposition 2.2], which we also denote $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. It also induces a linear isomorphism $\nu : \mathfrak{h} \xrightarrow{\sim} {}^L\mathfrak{h}$, and hence we have a non-degenerate symmetric bilinear form $(\nu^{-1}(\cdot)|\nu^{-1}(\cdot)) : {}^L\mathfrak{h} \times {}^L\mathfrak{h} \to \mathbb{C}$ on ${}^L\mathfrak{h}$, which henceforth we shall also denote by $(\cdot|\cdot)$. The latter then extends uniquely to an invariant symmetric bilinear form $(\cdot|\cdot) : {}^L\mathfrak{g} \times {}^L\mathfrak{g} \to \mathbb{C}$ on the whole of ${}^L\mathfrak{g}$.

There exists a unique set $\{\Lambda_i\}_{i=0}^{\ell} \subset {}^L\mathfrak{h}$ of derivation elements of ${}^L\mathfrak{h}$, the fundamental coweights of ${}^L\mathfrak{g}$ (and the fundamental weights of $\mathfrak{g}$) relative to our choice of d, such that

$$\langle \Lambda_i, \mathsf{d} \rangle = 0 \qquad \text{and} \qquad \langle \Lambda_i, \check{\alpha}_j \rangle = \delta_{i,j}, \qquad i,j \in I. \tag{2.3}$$

Likewise, there exists a unique set $\{\check{\Lambda}_i\}_{i=0}^{\ell} \subset \mathfrak{h}$ of derivation elements of $\mathfrak{h}$, the fundamental coweights of $\mathfrak{g}$ (and the fundamental weights of ${}^L\mathfrak{g}$) such that

$$\langle \Lambda_0, \check{\Lambda}_i \rangle = 0 \qquad \text{and} \qquad \langle \alpha_i, \check{\Lambda}_j \rangle = \delta_{i,j}, \qquad i,j \in I. \tag{2.4}$$

In particular, we have $\check{\Lambda}_0 = \mathsf{d}$.

2.2. **Principal gradation.** Let $\check{Q} \coloneqq \bigoplus_{i=0}^{\ell} \mathbb{Z}\check{\alpha}_i$ be the root lattice of ${}^L\mathfrak{g}$. We have the root space decomposition

$${}^L\mathfrak{g} = \bigoplus_{\check{\alpha} \in \check{Q}} {}^L\mathfrak{g}_{\check{\alpha}},$$

where ${}^L\mathfrak{g}_{\check{\alpha}} \coloneqq \{x \in {}^L\mathfrak{g} \,|\, [h,x] = \langle h, \check{\alpha} \rangle x \text{ for all } h \in {}^L\mathfrak{h}\}$. In particular, for the origin of the root lattice $0 \in \check{Q}$ we have ${}^L\mathfrak{g}_0 = {}^L\mathfrak{h}$. The *height* of a root $\check{\alpha} = \sum_{i=0}^{\ell} r_i \check{\alpha}_i \in \check{Q}$ is $\mathrm{ht}(\check{\alpha}) \coloneqq \sum_{i=0}^{\ell} r_i$. The *principal gradation* of ${}^L\mathfrak{g}$ is the $\mathbb{Z}$-gradation defined by

$${}^L\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} {}^L\mathfrak{g}_n, \qquad {}^L\mathfrak{g}_n \coloneqq \bigoplus_{\substack{\check{\alpha} \in \check{Q} \\ \mathrm{ht}(\check{\alpha}) = n}} {}^L\mathfrak{g}_{\check{\alpha}}.$$

Equivalently, the principal gradation is the $\mathbb{Z}$-gradation defined by

$$\deg(\check{e}_i) = 1, \qquad \deg(\check{f}_i) = -1, \qquad i \in I,$$

and $\deg({}^L\mathfrak{h}) = 0$. In particular ${}^L\mathfrak{g}_0 = {}^L\mathfrak{h}$, so that the notation ${}^L\mathfrak{g}_0$, where the subscript 0 could stand for either $0 \in \check{Q}$ or $0 \in \mathbb{Z}$, is unambiguous.

Let $\rho \in {}^L\mathfrak{h}$ be the unique derivation element of ${}^L\mathfrak{h}$ such that

$$\langle \rho, \check{\alpha}_i \rangle = 1, \qquad (\rho|\rho) = 0,$$

for every $i \in I$. By the first property we have $\langle \rho, \mathsf{k} \rangle = h^\vee$. The ad-eigenspaces of ρ are the subspaces ${}^L\mathfrak{g}_n$, $n \in \mathbb{Z}$. Indeed, we have

$$[\rho, \check{e}_i] = \check{e}_i, \qquad [\rho, \check{f}_i] = -\check{f}_i, \qquad i \in I.$$

2.3. **Principal subalgebra and exponents.** Define p_{-1}, the *cyclic element* of ${}^L\mathfrak{g}$, as

$$p_{-1} \coloneqq \sum_{i=0}^{\ell} \check{f}_i. \tag{2.5}$$

It belongs to the $(-1)^{\text{st}}$-grade of the derived subalgebra ${}^L\mathfrak{g}' = [{}^L\mathfrak{g}, {}^L\mathfrak{g}]$. There is a realization of ${}^L\mathfrak{g}'$ as the central extension of a certain twisted loop algebra $\mathcal{L}$, in such a way that the power of the formal loop variable t measures the grade in the principal gradation. (Equivalently, the derivation element $\rho \in {}^L\mathfrak{g}$ is realized as $t\partial_t$.) By studying this realization, one establishes some important facts about the adjoint action of p_{-1}. Here we shall merely recall these facts; for more details see [Kac90, Chapter 14]. Let

$$\pi : {}^L\mathfrak{g}' \longrightarrow \mathcal{L} \cong {}^L\mathfrak{g}'/\mathbb{C}\delta$$

be the canonical projection. The twisted loop algebra $\mathcal{L}$ is the direct sum of the image and the kernel of the adjoint action of $\pi(p_{-1})$:

$$\mathcal{L} = \ker(\mathrm{ad}_{\pi(p_{-1})}) \oplus \mathrm{im}(\mathrm{ad}_{\pi(p_{-1})}), \tag{2.6a}$$

and this decomposition respects the principal gradation, *i.e.* for each $n \in \mathbb{Z}$,

$$\mathcal{L}_n = \ker(\mathrm{ad}_{\pi(p_{-1})})_n \oplus \mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})_n, \tag{2.6b}$$

where $\mathcal{L}_n \coloneqq \pi({}^L\mathfrak{g}'_n)$.

The graded subspaces $\mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})_n$ are all of dimension ℓ and moreover

$$\mathrm{ad}_{\pi(p_{-1})} : \mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})_n \xrightarrow{\sim} \mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})_{n-1}$$

is a linear isomorphism for each n. The graded subspaces $\ker(\mathrm{ad}_{\pi(p_{-1})})_n$ have dimensions encoded by the exponents. Indeed, the multiset of *exponents* of ${}^L\mathfrak{g}$ is by definition the multiset consisting of each integer n with multiplicity $\dim(\ker(\mathrm{ad}_{\pi(p_{-1})})_n)$. One has $\dim(\ker(\mathrm{ad}_{\pi(p_{-1})})_n) = \dim(\ker(\mathrm{ad}_{\pi(p_{-1})})_{-n})$ and $\dim(\ker(\mathrm{ad}_{\pi(p_{-1})})_0) = 0$. So the multiset of exponents is of the form $\pm E$, where we denote by E the multiset of strictly positive exponents. The kernel $\ker(\mathrm{ad}_{\pi(p_{-1})})$ forms an abelian Lie subalgebra of the twisted loop algebra $\mathcal{L}$, called the *principal subalgebra*.

We need the "lift" to ${}^L\mathfrak{g}$ of the decomposition (2.6). For each $n \in \mathbb{Z}_{\neq 0}$, we have ${}^L\mathfrak{g}_n = {}^L\mathfrak{g}'_n$, the map $\pi|_{{}^L\mathfrak{g}'_n} : {}^L\mathfrak{g}'_n \xrightarrow{\sim} \mathcal{L}_n$ is a linear isomorphism, and one defines

$$\mathfrak{a}_n \coloneqq (\pi|_{{}^L\mathfrak{g}'_n})^{-1}\left(\ker(\mathrm{ad}_{\pi(p_{-1})})_n\right), \qquad \mathfrak{c}_n \coloneqq (\pi|_{{}^L\mathfrak{g}'_n})^{-1}\left(\mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})_n\right).$$

Meanwhile the subspaces $\mathfrak{a}_0$ and $\mathfrak{c}_0$ of ${}^L\mathfrak{g}_0 = {}^L\mathfrak{h}$ are defined as

$$\mathfrak{a}_0 \coloneqq \mathbb{C}\delta \oplus \mathbb{C}\rho, \qquad \mathfrak{c}_0 \coloneqq \mathrm{ad}_{p_{-1}}(\mathfrak{c}_1).$$

Then for each $n \in \mathbb{Z}$ we have the direct sum decomposition

$${}^L\mathfrak{g}_n = \mathfrak{a}_n \oplus \mathfrak{c}_n. \tag{2.7}$$

Let $\mathfrak{a} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}_n$ and $\mathfrak{c} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{c}_n$, so ${}^L\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{c}$. One has $\dim(\mathfrak{c}_n) = \ell$ for each $n \in \mathbb{Z}$ and the linear map

$$\mathrm{ad}_{p_{-1}} : \mathfrak{c}_n \xrightarrow{\sim} \mathfrak{c}_{n-1}$$

is an isomorphism for every $n \in \mathbb{Z}$.

The subspace $\mathfrak{a}$ is a Lie subalgebra, the *principal subalgebra of* ${}^L\mathfrak{g}$. It is the central extension, by a one-dimensional centre $\mathbb{C}\delta$, of the principal subalgebra $\mathrm{im}(\mathrm{ad}_{\pi(p_{-1})})$ of $\mathcal{L}$, equipped with a derivation element ρ. Indeed, we may pick a basis $\{p_n\}_{n\in\pm E}\cup\{\delta,\rho\}$ of $\mathfrak{a}$ where for each exponent $n\in\pm E$, $p_n\in\mathfrak{a}_n$. This basis can be so chosen that the non-trivial Lie algebra relations of $\mathfrak{a}$ are given by

$$[p_m,p_n]=m\delta_{m+n,0}\,\delta,\qquad [\rho,p_n]=n\,p_n,\qquad m,n\in\pm E. \tag{2.8}$$

The restriction to $\mathfrak{a}$ of the bilinear form $(\cdot|\cdot)$ on ${}^L\mathfrak{g}$ is non-degenerate, with the non-trivial pairings given by

$$(\delta|\rho)=(\rho|\delta)=h^\vee,\qquad (p_m|p_n)=h^\vee\delta_{m+n,0},\qquad m,n\in\pm E.$$

Remark 2.3.

(a) ±1 are always exponents with multiplicity 1. We keep p_{-1} as in (2.5) and set

$$p_1=\sum_{i=0}^{\ell}a_i\check{e}_i.$$

(b) The pattern of exponents is periodic with period $rh^\vee$, where ${}^rX_{\mathsf{N}}$ is the type of ${}^L\mathfrak{g}$ in Kac's notation. For a table of the patterns of exponents in all types see e.g. [Kac90, Chapter 14] or [DS85, §5].

(c) The exponents of $\mathfrak{g}$ and ${}^L\mathfrak{g}$ are the same [Kac90, Corollary 14.3], which is important for Conjecture 5.1 below. (Consequently, if ${}^sY_{\mathsf{M}}$ is the type of $\mathfrak{g}$ then the pattern of exponents is also periodic with period sh, which need not equal $rh^\vee$. For us $s=1$ since $\mathfrak{g}$ is untwisted.)

(d) In all types except ${}^1D_{2k}$, the multiset E of positive exponents is actually a set, *i.e.* $\dim(\mathfrak{a}_n)\in\{0,1\}$ for all $n\in\mathbb{Z}_{\neq0}$. In such cases, for each $j\in E$ the basis element $p_j\in\mathfrak{a}_j$ is unique up to rescaling. Exceptionally, in type ${}^1D_{2k}$ one has $\dim(\mathfrak{a}_{2k-1+(4k-2)n})=2$ for every $n\in\mathbb{Z}$. For each $n\geq0$ one must therefore pick two basis vectors, each one labelled by one of the two distinct copies of $2k-1+(4k-2)n$ in E. (The basis vectors for $n\leq0$ are then fixed by the form of the bilinear form above.)

(e) The action of the $\mathbb{C}$-linear map $\mathrm{ad}_{p_{-1}}:{}^L\mathfrak{g}\to{}^L\mathfrak{g}$ on the subspaces ${}^L\mathfrak{g}_n$, $n\in\mathbb{Z}$, of the principal gradation of ${}^L\mathfrak{g}$ can be summarised in the following diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbb{C}\rho & \longrightarrow & \mathfrak{a}_{-1} & & \mathfrak{a}_{-2} & & \cdots \\
 & & & & & & \oplus & & \oplus & & \oplus & & \\
\cdots & \xrightarrow{\sim} & \mathfrak{c}_2 & \xrightarrow{\sim} & \mathfrak{c}_1 & \xrightarrow{\sim} & \mathfrak{c}_0 & \xrightarrow{\sim} & \mathfrak{c}_{-1} & \xrightarrow{\sim} & \mathfrak{c}_{-2} & \xrightarrow{\sim} & \cdots \\
 & & \oplus & & \oplus & & \oplus & & & & & & \\
\cdots & & \mathfrak{a}_2 & & \mathfrak{a}_1 & \longrightarrow & \mathbb{C}\delta & & & & & &
\end{array}$$

where each column corresponds to a subspace ${}^L\mathfrak{g}_n$ decomposed as in (2.7).

(f) Recall the decomposition (2.6) of the subquotient $\mathcal{L}$. In fact $\mathfrak{c}_n=\mathrm{im}(\mathrm{ad}_{p_{-1}})_n$ and $\mathfrak{a}_n=\ker(\mathrm{ad}_{p_{-1}})_n$ for every $n\in\mathbb{Z}$, with precisely the following exceptions: $\mathfrak{c}_0\neq(\mathrm{im}\,\mathrm{ad}_{p_{-1}})_0$ and $\mathfrak{c}_{-1}\neq(\mathrm{im}\,\mathrm{ad}_{p_{-1}})_{-1}$ since $\delta=[p_1,p_{-1}]$ and $p_{-1}=[p_{-1},\rho]$ both belong to the image of $\mathrm{ad}_{p_{-1}}:{}^L\mathfrak{g}\to{}^L\mathfrak{g}$; and $\mathfrak{a}_1\neq(\ker\mathrm{ad}_{p_{-1}})_1$ since $\mathrm{ad}_{p_{-1}}:{}^L\mathfrak{g}_1\hookrightarrow{}^L\mathfrak{g}_0$ is injective. ◁

3. ${}^L\mathfrak{g}$-opers and quasi-canonical form

3.1. **Inverse limits.** Recall the subalgebras ${}^L\mathfrak{h}$ and ${}^L\mathfrak{n}_+$ of ${}^L\mathfrak{g}$ from §2.1. We introduce also the Borel subalgebra ${}^L\mathfrak{b}_+ \coloneqq {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+ \subset {}^L\mathfrak{g}$. These can be described in terms of the principal gradation of ${}^L\mathfrak{g}$ as ${}^L\mathfrak{n}_+ = \bigoplus_{n>0} {}^L\mathfrak{g}_n$ and ${}^L\mathfrak{b}_+ = \bigoplus_{n\geq 0} {}^L\mathfrak{g}_n$. Moreover, there is a natural descending $\mathbb{Z}_{>0}$-filtration on ${}^L\mathfrak{n}_+$ (and ${}^L\mathfrak{b}_+$) by Lie ideals

$$ {}^L\mathfrak{n}_k = \bigoplus_{n\geq k} {}^L\mathfrak{g}_n, \qquad k \in \mathbb{Z}_{>0}. $$

Since ${}^L\mathfrak{n}_k \subset {}^L\mathfrak{g}'$ for each $k \in \mathbb{Z}_{>0}$, these ideals also define a descending $\mathbb{Z}_{>0}$-filtration on the derived subalgebra ${}^L\mathfrak{b}'_+ \coloneqq {}^L\mathfrak{b}_+ \cap {}^L\mathfrak{g}'$.

Let $\mathcal{M}$ be the field of meromorphic functions on $\mathbb{P}^1 \coloneqq \mathbb{C} \cup \{\infty\}$. For any Lie subalgebra $\mathfrak{p} \subset {}^L\mathfrak{g}$ we introduce the Lie algebra $\mathfrak{p}(\mathcal{M}) \coloneqq \mathfrak{p} \otimes \mathcal{M}$ of $\mathfrak{p}$-valued meromorphic functions on $\mathbb{P}^1$.

The Lie algebras ${}^L\mathfrak{n}_k(\mathcal{M})$, $k \in \mathbb{Z}_{>0}$ endow ${}^L\mathfrak{n}_+(\mathcal{M})$ with a descending $\mathbb{Z}_{>0}$-filtration by ideals such that the quotient Lie algebras ${}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$, $k \in \mathbb{Z}_{>0}$ are nilpotent. Consider the Lie algebra defined as the inverse limit

$$ {}^L\hat{\mathfrak{n}}_+(\mathcal{M}) \coloneqq \varprojlim {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M}). $$

By definition, its elements are infinite sums $\sum_{n>0} y_n$, with $y_n \in {}^L\mathfrak{g}_n(\mathcal{M})$, which truncate to finite sums when working in the quotient ${}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$ for any $k \in \mathbb{Z}_{>0}$.

Remark 3.1. It should be stressed that for a given element $\sum_{n>0} y_n$ of ${}^L\hat{\mathfrak{n}}_+(\mathcal{M})$, the orders of the poles of the ${}^L\mathfrak{g}_n$-valued meromorphic functions y_n are allowed to increase without bound as n increases. Thus ${}^L\hat{\mathfrak{n}}_+(\mathcal{M})$ is strictly larger than ${}^L\hat{\mathfrak{n}}_+ \otimes \mathcal{M}$, where ${}^L\hat{\mathfrak{n}}_+ \coloneqq \varprojlim {}^L\mathfrak{n}_+/{}^L\mathfrak{n}_k$ is the completion of ${}^L\mathfrak{n}_+$. ◁

We also have the inverse limits

$$ \begin{aligned} {}^L\hat{\mathfrak{b}}_+(\mathcal{M}) &\coloneqq {}^L\mathfrak{h}(\mathcal{M}) \oplus {}^L\hat{\mathfrak{n}}_+(\mathcal{M}) = \varprojlim {}^L\mathfrak{b}(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M}), \\ {}^L\hat{\mathfrak{g}}(\mathcal{M}) &\coloneqq {}^L\mathfrak{n}_-(\mathcal{M}) \oplus {}^L\mathfrak{h}(\mathcal{M}) \oplus {}^L\hat{\mathfrak{n}}_+(\mathcal{M}) = \varprojlim {}^L\mathfrak{g}(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M}). \end{aligned} $$

The latter is an inverse limit of vector spaces only, since the ${}^L\mathfrak{n}_k(\mathcal{M})$ are not Lie ideals in ${}^L\mathfrak{g}(\mathcal{M})$. Nonetheless, ${}^L\hat{\mathfrak{g}}(\mathcal{M})$ is a Lie algebra, with ${}^L\mathfrak{g}(\mathcal{M})$ as a subalgebra.[4]

3.2. **The group ${}^L\hat{N}_+(\mathcal{M})$.** For every $k \in \mathbb{Z}_{>0}$, the Baker-Campbell-Hausdorff formula then endows the vector space ${}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$ with the structure of a group. Specifically, we denote this group by

$$ \exp\big({}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})\big) \coloneqq \{\exp(m) \,|\, m \in {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})\}, $$

whose elements are denoted formally as exponentials of elements in ${}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$. The group operation is then defined as

$$ \exp(x)\exp(y) \coloneqq \exp(x \bullet y) = \exp(x + y + \tfrac{1}{2}[x,y] + \ldots) \tag{3.1} $$

[4]Given any two elements $x = \sum_{n>-N} x_n$, $y = \sum_{n>-M} y_n$ of the vector space ${}^L\hat{\mathfrak{g}}(\mathcal{M})$, with each $x_n, y_n \in {}^L\mathfrak{g}_n(\mathcal{M})$, their Lie bracket $[x,y] = \sum_{k>-N-M} \sum_{\substack{n>-N, m>-M \\ n+m=k}} [x_n, y_m]$ is a well-defined element of ${}^L\hat{\mathfrak{g}}(\mathcal{M})$ since the inner sum is finite for each k. This bracket obeys the Jacobi identity and agrees with the usual bracket on ${}^L\mathfrak{g}(\mathcal{M}) \subset {}^L\hat{\mathfrak{g}}(\mathcal{M})$.

for all $x, y \in {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$. Here $x \bullet y$ is given by the Baker-Campbell-Hausdorff formula, whose first few term are shown in the exponent on the right hand side of (3.1). The sum is finite because ${}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})$ is nilpotent.

Now the formal exponential map $\exp : {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M}) \xrightarrow{\sim} \exp\left({}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})\right)$ is a bijection by definition, and there are canonical group homomorphisms π_k^m making the following diagram commutative:

$$\begin{array}{ccc} \exp\left({}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_m(\mathcal{M})\right) & \xrightarrow{\pi_k^m} & \exp\left({}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})\right) \\ \exp \wr \uparrow & & \exp \wr \uparrow \\ {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_m(\mathcal{M}) & \longrightarrow & {}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M}) \end{array}$$

for all $m \geq k > 0$. We define a group ${}^L\hat{N}_+(\mathcal{M})$ as the corresponding inverse limit

$${}^L\hat{N}_+(\mathcal{M}) \coloneqq \varprojlim \exp\left({}^L\mathfrak{n}_+(\mathcal{M})/{}^L\mathfrak{n}_k(\mathcal{M})\right). \tag{3.2}$$

The above commutative diagram defines an exponential map $\exp : {}^L\hat{\mathfrak{n}}_+(\mathcal{M}) \to {}^L\hat{N}_+(\mathcal{M})$.

3.3. **Definition of an ${}^L\mathfrak{g}$-oper.** Now, and until §6 below, we shall pick and fix a global coordinate z on $\mathbb{C} \subset \mathbb{P}^1$. Thus, for any $f \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M})$ its holomorphic de Rham differential is $df = dz\partial_z f$.

Define $\mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ to be the affine space of connections of the form

$$\nabla = d + p_{-1}dz + bdz, \qquad b \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M}). \tag{3.3}$$

Remark 3.2. This is an affine space over ${}^L\hat{\mathfrak{b}}_+(\mathcal{M})$. For the moment, in calling it a space of connections we mean merely that it admits an action of the group ${}^L\hat{N}_+(\mathcal{M})$ by gauge transformations, as we shall now describe. In §6 we will discuss its behaviour under coordinate transformations. ◁

Define the adjoint action of the group ${}^L\hat{N}_+(\mathcal{M})$ on the vector space ${}^L\hat{\mathfrak{g}}(\mathcal{M})$ as follows. Let $g = \exp(m) \in {}^L\hat{N}_+(\mathcal{M})$ with $m = \sum_{n>0} m_n \in {}^L\hat{\mathfrak{n}}_+(\mathcal{M})$. For any $u \in {}^L\hat{\mathfrak{g}}(\mathcal{M})$, which we write as $u = \sum_{n \geq M} u_n$ for some $M \in \mathbb{Z}$, we define the adjoint action of g on u as

$$gug^{-1} \coloneqq \sum_{k\geq 0} \frac{1}{k!} \operatorname{ad}_m^k u = \sum_{n \geq M} u_n + \sum_{n \geq M} \sum_{r>0} [m_r, u_n] + \frac{1}{2} \sum_{n \geq M} \sum_{r,s>0} \big[m_s, [m_r, u_n]\big] + \dots \tag{3.4a}$$

where the dots represent terms involving an increasing number of m_n's with $n \geq 1$. Since $\deg m_n = n$ in the principal gradation of ${}^L\mathfrak{g}$, it follows that for each $k \in \mathbb{Z}_{>0}$ there are only finitely many terms of degree less than k in the expression on the right hand side. Therefore the sum on the right hand side of (3.4a) is a well-defined element of ${}^L\hat{\mathfrak{g}}(\mathcal{M})$.

Lemma 3.3. *The definition* (3.4a) *defines an action of the group* ${}^L\hat{N}_+(\mathcal{M})$ *on* ${}^L\hat{\mathfrak{g}}(\mathcal{M})$.

Proof. By the Baker-Campbell-Hausdorff formula we have

$$\left(\sum_{k\geq 0} \frac{1}{k!} \operatorname{ad}_m^k\right)\left(\sum_{\ell \geq 0} \frac{1}{\ell!} \operatorname{ad}_n^\ell\right) u = \sum_{k \geq 0} \frac{1}{k!} \operatorname{ad}_{m \bullet n}^k u,$$

for any $m, n \in {}^L\hat{\mathfrak{n}}_+(\mathcal{M})$ and $u \in {}^L\hat{\mathfrak{g}}(\mathcal{M})$. □

Now we define also

$$(dg)g^{-1} \coloneqq \sum_{k\geq 1} \frac{1}{k!} \operatorname{ad}_m^{k-1} dm = \sum_{n>0} dm_n + \frac{1}{2} \sum_{n,r>0} [m_r, dm_n] + \dots, \tag{3.4b}$$

which is a well-defined sum in ${}^L\hat{\mathfrak{n}}_+(\mathcal{M})dz$.

Lemma 3.4. *For any $g, h \in {}^L\hat{N}_+(\mathcal{M})$, we have*

$$d(gh)(gh)^{-1} = g\left((dh)h^{-1}\right)g^{-1} + (dg)g^{-1}.$$

Proof. By direct calculation from the definitions (3.4) one verifies that

$$d(gyg^{-1}) = \left[dgg^{-1}, gyg^{-1}\right] + g(dy)g^{-1},$$

for any $y \in {}^L\hat{\mathfrak{g}}(\mathcal{M})$ and any $g \in {}^L\hat{N}_+(\mathcal{M})$. By Lemma 3.3, we have $(gh)y(gh)^{-1} = g(hyh^{-1})g^{-1}$, and on applying d to both sides we obtain

$$\left[-d(gh)(gh)^{-1} + g\left((dh)h^{-1}\right)g^{-1} + (dg)g^{-1}, x\right] = 0$$

where $x = gyg^{-1}$ is arbitrary. Since the centre of ${}^L\hat{\mathfrak{n}}_+(\mathcal{M})$ is trivial, the result follows. □

Observe that $gp_{-1}g^{-1} - p_{-1} \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M})$, so that if $u \in {}^L\hat{\mathfrak{g}}(\mathcal{M})$ is of the form $u = p_{-1} + b$ with $b \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M})$ then so is gug^{-1}. Hence, from Lemmas 3.3 and 3.4, we have the following.

Proposition 3.5. *We have an action of ${}^L\hat{N}_+(\mathcal{M})$ on $\operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)$ defined by*

$$\begin{aligned} &{}^L\hat{N}_+(\mathcal{M}) \times \operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1) \longrightarrow \operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1), \\ &(g, d + p_{-1}dz + bdz) \longmapsto d + gp_{-1}g^{-1}dz - (dg)g^{-1} + gbg^{-1}dz, \end{aligned}$$

which we refer to as the action by gauge transformations. *If $\nabla \in \operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)$ then we denote by $\nabla^g \in \operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)$ its gauge transformation by an element $g \in {}^L\hat{N}_+(\mathcal{M})$.* □

Our main object of interest, the space of ${}^L\mathfrak{g}$-*opers*, can now be defined as the quotient of the affine space (3.3) by this gauge action

$$\operatorname{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1) \coloneqq \operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1) \big/ {}^L\hat{N}_+(\mathcal{M}).$$

In fact, we shall be interested in certain affine subspaces of $\operatorname{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)$ defined as follows.

3.4. **Twist function φ.** Fix a choice of meromorphic function φ on $\mathbb{P}^1$, called the *twist function*. We call a derivation element Λ of ${}^L\mathfrak{h}$ *normalised* if $\langle \Lambda, \mathsf{k} \rangle = 1$. Define $\operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^\varphi$ to be the affine subspace of $\operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)$ consisting of connections of the form

$$d + p_{-1}dz - \Lambda\varphi dz + b'dz, \qquad b' \in {}^L\hat{\mathfrak{b}}'_+(\mathcal{M}).$$

Lemma 3.6. *The affine subspace $\operatorname{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^\varphi$ is independent of the choice of normalised derivation element Λ, and it is stable under ${}^L\hat{N}_+(\mathcal{M})$-valued gauge transformations.*

Proof. Let Λ and Λ' be two choices of normalised derivation element of ${}^L\mathfrak{h}$. Then we have $\langle \Lambda, \mathsf{k}\rangle - \langle \Lambda', \mathsf{k}\rangle = 0$ so that $\Lambda - \Lambda'$ is in the span of the simple roots α_i, $i \in I$ and hence $\Lambda - \Lambda' \in {}^L\mathfrak{h} \cap {}^L\mathfrak{b}'_+$. It follows that $\mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ is independent of Λ.

Since we have the direct sum of vector spaces ${}^L\mathfrak{g} = {}^L\mathfrak{g}' \oplus \mathbb{C}\Lambda$, and so in particular ${}^L\mathfrak{b}_+ = {}^L\mathfrak{b}'_+ \oplus \mathbb{C}\Lambda$, it follows from the definition of the action of ${}^L\hat{N}_+(\mathcal{M})$ on $\mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ by gauge transformations that $\mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ is stable. □

Given a choice of twist function φ, we may now define the corresponding affine subspace of ${}^L\mathfrak{g}$-opers as

$$\mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi \coloneqq \mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi \big/ {}^L\hat{N}_+(\mathcal{M}). \tag{3.5}$$

If $\nabla \in \mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ then we shall denote its class in $\mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ by $[\nabla]$.

We introduce also the *twisted de Rham differential* corresponding to the twist function φ. For every $f \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M})$,

$$\begin{aligned} d^\varphi f &\coloneqq df - h^{\vee -1}\varphi(\mathrm{ad}_\rho f)dz \\ &= dz\left(\partial_z f - h^{\vee -1}\varphi(\mathrm{ad}_\rho f)\right). \end{aligned} \tag{3.6}$$

3.5. **Quasi-canonical form of an ${}^L\mathfrak{g}$-oper.** Recall, from §2.3, the definition of the principal subalgebra $\mathfrak{a}$ of ${}^L\mathfrak{g}$, and its basis $\{p_j\}_{j\in\pm E} \cup \{\delta, \rho\}$ where E is the multiset of positive exponents of ${}^L\mathfrak{g}$.

Let $\hat{\mathfrak{a}}(\mathcal{M})$ denote the completion of the algebra $\mathfrak{a}(\mathcal{M})$ of $\mathfrak{a}$-valued mermorphic functions on $\mathbb{P}^1$:

$$\hat{\mathfrak{a}}(\mathcal{M}) \coloneqq \varprojlim \mathfrak{a}(\mathcal{M})/(\mathfrak{a} \cap \mathfrak{n}_k)(\mathcal{M}).$$

For each $n \in \mathbb{Z}_{\geq 0}$, let $\hat{\mathfrak{a}}_{\geq n}(\mathcal{M}) \coloneqq \varprojlim \mathfrak{a}_{\geq n}(\mathcal{M})/(\mathfrak{a}_{\geq n} \cap \mathfrak{n}_k)(\mathcal{M})$, where $\mathfrak{a}_{\geq n} \coloneqq \bigoplus_{j=n}^\infty \mathfrak{a}_j$. These are Lie subalgebras of $\hat{\mathfrak{a}}(\mathcal{M})$.

Theorem 3.7. *Every class $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ has a representative $\nabla \in \mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ of the form*

$$\nabla = d + p_{-1}dz - h^{\vee -1}\rho\,\varphi dz + a dz, \qquad a \in \hat{\mathfrak{a}}_{\geq 1}(\mathcal{M}).$$

We say that such a representative is in quasi-canonical form. *For any $g \in {}^L\hat{N}_+(\mathcal{M})$, ∇^g is still in quasi-canonical form if and only if $g = \exp(f) \in \exp(\hat{\mathfrak{a}}_{\geq 2}(\mathcal{M}))$, in which case $\nabla^g = \nabla - d^\varphi f$.*

Equivalently but more explicitly, every class $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ has a quasi-canonical *representative of the form*

$$\nabla = d + \left(p_{-1} - \frac{\varphi}{h^\vee}\rho + \sum_{j\in E} v_j p_j\right)dz, \tag{3.7}$$

where v_j is a meromorphic function on $\mathbb{P}^1$ for each positive exponent $j \in E$. The gauge transformations in ${}^L\hat{N}_+(\mathcal{M})$ preserving quasi-canonical form are precisely those of the form $\exp\left(\sum_{j\in E_{\geq 2}} f_j p_j\right)$ with f_j meromorphic functions on $\mathbb{P}^1$. The effect of such gauge transformations on the functions v_j is to send

$$v_j \longmapsto v_j - f_j' + \frac{j\varphi}{h^\vee} f_j \tag{3.8}$$

for all $j \in E_{\geq 2}$, and to leave v_1 invariant.

Proof. Let $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$. Since $h^{\vee-1}\rho$ is a normalised derivation element of ${}^L\mathfrak{h}$, see §2.2, it follows using Lemma 3.6 that there is a representative of $[\nabla]$ of the form $\nabla = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + \sum_{n\geq 0} u_n dz \in \mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$ for some functions $u_n \in {}^L\mathfrak{g}'_n(\mathcal{M})$.

Let $g \in {}^L\hat{N}_+(\mathcal{M})$ be of the form $g = \exp(m)$ with $m = \sum_{n>0} m_n$ where $m_n \in \mathfrak{c}_n(\mathcal{M})$ for each $n > 0$. Using (3.4) we determine the gauge transformation of ∇ by g to be

$$\nabla^g = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + \sum_{n\geq 0} a_n dz$$

where $a_n \in {}^L\mathfrak{g}'_n(\mathcal{M})$ for each $n \geq 0$ are of the form

$$\begin{aligned} a_n dz = u_n dz + [m_{n+1}, p_{-1}]dz + F_n\big(\{u_k, dm_k, m_k\}_{k<n}\big) \\ - d^\varphi m_n + [m_n, u_0] + \tfrac{1}{2}\big[m_n, [m_1, p_{-1}]\big]dz + \tfrac{1}{2}(1-\delta_{n,1})\big[m_1, [m_n, p_{-1}]\big]dz. \end{aligned} \tag{3.9}$$

The last term on the first line of the right hand side contains all the terms involving only m_k and u_k with $k < n$, and the second line contains those terms involving m_n. Let $w_n dz$, $w_n \in {}^L\mathfrak{g}'_n(\mathcal{M})$, denote the sum of all these terms, *i.e.* we rewrite (3.9) as

$$a_n = u_n + [m_{n+1}, p_{-1}] + w_n. \tag{3.10}$$

We can now use (3.10) to determine $m_n \in \mathfrak{c}_n(\mathcal{M})$ recursively for all $n > 0$ by requiring that $a_n \in \mathfrak{a}_n(\mathcal{M})$ for each $n \geq 0$. Indeed, suppose m_k has been determined for each $k \leq n$. Then w_n is known (in fact $w_0 = 0$ for the base case) and so decomposing $u_n + w_n$ relative to the direct sum (2.7) (or rather ${}^L\mathfrak{g}'_0 = \mathbb{C}\delta \oplus \mathfrak{c}_0$ in the case $n = 0$) we can use the injectivity of $\mathrm{ad}_{p_{-1}} : \mathfrak{c}_{n+1} \to \mathfrak{c}_n$ to fix m_{n+1} uniquely so as to cancel the component of $u_n + w_n$ in $\mathfrak{c}_n$, thereby ensuring that $a_n \in \mathfrak{a}_n(\mathcal{M})$ for all $n > 0$ or $a_0 \in (\mathbb{C}\delta)(\mathcal{M})$. This proves $\nabla^g \in d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + (\mathbb{C}\delta \oplus \hat{\mathfrak{a}}_{\geq 1})(\mathcal{M})dz$.

Let us write $\nabla^g = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + \delta\chi dz + a'dz$ with $\chi \in \mathcal{M}$ and $a' \in \hat{\mathfrak{a}}_{\geq 1}(\mathcal{M})$. In order to remove the term in δ, we can apply a further gauge transformation by $h = \exp(-\chi p_1)$, which yields

$$\nabla^{hg} = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + a'dz + d^\varphi(\chi p_1).$$

The last two terms belong to $\hat{\mathfrak{a}}_{\geq 1}(\mathcal{M})dz$, which completes the proof of the first statement.

Finally, suppose $\nabla = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + udz$ where $u \in \hat{\mathfrak{a}}_{\geq 1}(\mathcal{M})$ and let $g = \exp(m)$ for some $m \in {}^L\hat{\mathfrak{n}}_+(\mathcal{M})$. Then $\nabla^g = d + p_{-1}dz - h^{\vee-1}\rho\,\varphi dz + vdz$ where $v = \sum_{n\geq 0} a_n \in {}^L\hat{\mathfrak{b}}'_+(\mathcal{M})$ is given by (3.9). We want to recursively determine the components $m_n \in {}^L\mathfrak{g}_n(\mathcal{M})$ of m so that $v \in \hat{\mathfrak{a}}_{\geq 1}(\mathcal{M})$. Considering first the case $n = 0$ we have $u_0 = a_0 = 0$ so that (3.9) reduces to $[m_1, p_{-1}] = 0$, and therefore $m_1 = 0$ since $\mathrm{ad}_{p_{-1}} : {}^L\mathfrak{g}_1 \to {}^L\mathfrak{g}_0$ is injective. In particular, for every $n \geq 0$ the last two terms on the right hand side of (3.9) are now absent. Suppose that having $a_k \in \mathfrak{a}_k(\mathcal{M})$ for all $k < n$ requires that $m_k \in \mathfrak{a}_k(\mathcal{M})$ for each $k \leq n$. It just remains to show that the condition $a_n \in \mathfrak{a}_n(\mathcal{M})$ also implies $m_{n+1} \in \mathfrak{a}_{n+1}(\mathcal{M})$. For this we note that all the terms contained in $F_n\big(\{u_k, dm_k, m_k\}_{k<n}\big)$ are commutators, which vanish using the

fact that $u \in \hat{\mathfrak{a}}_{\geq 1}(\mathcal{M})$, $m_1 = 0$, $m_k \in \mathfrak{a}_k(\mathcal{M})$ for $1 < k \leq n$ and $\mathfrak{a}_{\geq 1}$ is abelian. So (3.9) now simply reads

$$a_n dz - u_n dz + d^{\varphi} m_n = [m_{n+1}, p_{-1}]dz.$$

The left hand side clearly belongs to $\mathfrak{a}_n(\mathcal{M})dz$, using the fact that $\mathrm{ad}_\rho\, m_n = n m_n$. On the other hand, the right hand side belongs instead to $\mathfrak{c}_n(\mathcal{M})dz$ since $\mathfrak{c}_n = (\mathrm{im}\,\mathrm{ad}_{p_{-1}})_n$ for every $n > 0$, cf. Remark 2.3(f). Hence both sides vanish so that, in particular, $m_{n+1} \in \mathfrak{a}_{n+1}(\mathcal{M})$. The vanishing of the left hand side is the final statement about the form of $\nabla^g - \nabla$. □

Although the quasi-canonical form of an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\varphi}$ is not unique, the coefficient v_1 of p_1 in any quasi-canonical form is the same. To emphasise the origin of this distinction between v_1 and all the remaining coefficients v_j, $j \in E_{\geq 2}$, the following is helpful.

Proposition 3.8. *Every class $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\varphi}$ has a representative of the form*

$$\nabla = d + \left(p_{-1} - \frac{\varphi}{h^\vee}\rho + v_0\delta + \sum_{j \in E} v_j p_j \right) dz, \tag{3.11}$$

where v_j is a meromorphic function on $\mathbb{P}^1$ for each $j \in \{0\} \cup E$. The gauge transformations in ${}^L\hat{N}_+(\mathcal{M})$ preserving this form are precisely those of the form $\exp\big(\sum_{j\in E} f_j p_j\big)$ with f_j meromorphic functions on $\mathbb{P}^1$. The effect of such gauge transformations on the functions v_j is as in (3.8) *for all $j \in E_{\geq 2}$, and now also*

$$v_0 \longmapsto v_0 + f_1$$
$$v_1 \longmapsto v_1 - f_1' + \frac{\varphi}{h^\vee} f_1.$$

Proof. The proof is very similar to that of Theorem 3.7. □

Consequently, if one works not with ${}^L\mathfrak{g}$ but with the quotient by the centre ${}^L\mathfrak{g}/\mathbb{C}\delta$ then the distinction between v_1 and the rest disappears, as follows. (We return to this point in §5.3 below.)

Corollary 3.9. *For an $({}^L\mathfrak{g}/\mathbb{C}\delta)$–oper $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(\mathbb{P}^1)^{\varphi}$, there is always a quasi-canonical representative of the form* (3.7). *The gauge transformations in ${}^L\hat{N}_+(\mathcal{M})$ preserving this form are precisely those of the form $\exp\big(\sum_{j\in E} f_j p_j\big)$ with f_j meromorphic functions on $\mathbb{P}^1$. The effect of such gauge transformations on the functions v_j is as in* (3.8) *but now for all $j \in E$ (including 1).* □

Returning to ${}^L\mathfrak{g}$-opers, we have the following explicit expression for the coefficient v_1 in any quasi-canonical form.

Proposition 3.10. *The coefficient of $p_1 \in \mathfrak{a}_1$ of any quasi-canonical form of an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^{\varphi}$ is*

$$v_1 = h^{\vee -1}\big(\tfrac{1}{2}(u_0|u_0) + (\rho|u_0') - h^{\vee -1}\varphi(\rho|u_0) + (p_{-1}|u_1)\big),$$

where

$$\nabla = d + p_{-1}dz - h^{\vee -1}\rho\,\varphi dz + \sum_{n\geq 0} u_n dz \in \mathrm{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^{\varphi},$$

with $u_n \in {}^L\mathfrak{g}'_n$*, is any representative of* $[\nabla]$*.*

Proof. In the present case, the recursion relation (3.9) for $n = 0$ gives $u_0 = -[m_1, p_{-1}]$. Note that here we are including in m_1 the term $-\chi p_1$ coming from the subsequent gauge transformation performed in the second step of the proof of Theorem 3.7. Using this, the relation (3.9) for $n = 1$ then reads

$$\begin{aligned} a_1 dz &= u_1 dz + [m_2, p_{-1}]dz + \tfrac{1}{2}\big[m_1, [m_1, p_{-1}]\big]dz + [m_1, u_0]dz - d^{\varphi} m_1 \\ &= u_1 dz + [m_2, p_{-1}]dz - \tfrac{1}{2}\big[m_1, [m_1, p_{-1}]\big]dz - d^{\varphi} m_1. \end{aligned}$$

By applying the linear map $(p_{-1}|\cdot)$ to both sides we find

$$(p_{-1}|a_1)dz = (p_{-1}|u_1)dz + \tfrac{1}{2}(u_0|u_0)dz - (p_{-1}|d^{\varphi} m_1),$$

where to obtain the second term on the right hand side we have used again the fact that $u_0 = -[m_1, p_{-1}]$. To evaluate further the last term above, we note that

$$(p_{-1}|d^{\varphi} m_1) = ([p_{-1}, \rho]|d^{\varphi} m_1) = (\rho|[d^{\varphi} m_1, p_{-1}]) = -(\rho|du_0) + h^{\vee -1}\varphi(\rho|u_0)dz,$$

where in the last step we used the definition (3.6) of the twisted de Rham differential. Since $(p_{-1}|p_1) = h^{\vee}$, we arrive at the desired expression for $v_1 = h^{\vee -1}(p_{-1}|a_1)$. □

Remark 3.11. Let $\nabla \in \mathrm{op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^{\varphi}$ be as in the statement of Proposition 3.10 and introduce $\widetilde{u}_0 := -h^{\vee -1}\rho\,\varphi + u_0 \in {}^L\mathfrak{g}_0(\mathcal{M})$ and $\widetilde{u}_n := u_n \in {}^L\mathfrak{g}_n(\mathcal{M})$ for every $n > 0$. Then we have

$$\nabla = d + p_{-1}dz + \sum_{n\geq 0} \widetilde{u}_n dz,$$

and, using the fact that $(\rho|\rho) = 0$, cf. §2.2, the expression for the coefficient v_1 in any quasi-canonical form of $[\nabla]$ can be rewritten as

$$v_1 = h^{\vee -1}\big(\tfrac{1}{2}(\widetilde{u}_0|\widetilde{u}_0) + (\rho|\widetilde{u}'_0) + (p_{-1}|\widetilde{u}_1)\big)dz. \qquad \triangleleft$$

3.6. **Twisted homology and functions on the space of affine opers.** Our goal is to describe functions $\mathrm{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^{\varphi} \to \mathbb{C}$ on the space of meromorphic ${}^L\mathfrak{g}$-opers on $\mathbb{P}^1$.

Theorem 3.7 shows that one well-defined map $\mathrm{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)^{\varphi} \to \mathcal{M}$ is given by extracting the coefficient v_1 of p_1 in any quasi-canonical form (and Proposition 3.10 gives the explicit formula). Obviously we can then "pair" this function with any point $p \in \mathbb{P}^1$ where v_1 doesn't have a pole, by simply evaluating it there, $v_1 \mapsto v_1(p)$.

Yet Theorem 3.7 also shows that the remaining data in the oper comes in the form of functions v_i, $i \in E_{\geq 2}$, defined only up to certain "twisted" derivatives. So they are in some sense cohomology elements. In §6 we shall make that idea precise by showing that each of the functions v_i, $i \in E_{\geq 2}$, represents a cocycle in the cohomology of the de Rham complex with coefficients in a certain local system. A generalization of the usual de Rham theorem states that there is a pairing (given by integrating) between such cocycles and the cycles of the singular homology with coefficients in the dual local system. For the moment though, we are not quite in a position to invoke such

results: a local system is a vector bundle with a flat connection and we cannot yet identify the correct bundle, since we have no handle on its transition functions.

Nonetheless, it is already possible to define the integrals one should take to obtain functions $\mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi \to \mathbb{C}$, as follows.

First, let us now and for the remainder of this article restrict attention to the case when the twist function φ has only simple poles, *i.e.* we shall take it to be of the form

$$\varphi(z) \coloneqq \sum_{i=1}^N \frac{k_i}{z - z_i}, \tag{3.12}$$

for some $k_i \in \mathbb{C}^\times$, $i = 1, \dots, N$. It has simple poles in the subset $\{z_i\}_{i=1}^N \subset \mathbb{P}^1$.

Remark 3.12. Based on the situation in finite types, [FFTL10, VY17b], our expectation is that introducing a pole of order $p \geq 2$ at z_i in the twist function φ (and more generally in the Miura ${}^L\mathfrak{g}$-opers of §4.1 below) will correspond to a Gaudin model in which one assigns to the marked point z_i a representation of a *Takiff algebra* $\mathfrak{g}[t]/t^p\mathfrak{g}[t]$ over the affine Kac-Moody algebra $\mathfrak{g}$. ◁

We denote the complement of the set of marked points $\{z_i\}_{i=1}^N$ as

$$X \coloneqq \mathbb{C} \setminus \{z_i\}_{i=1}^N. \tag{3.13}$$

Consider the multivalued holomorphic function $\mathcal{P}$ on X defined by

$$\mathcal{P}(z) \coloneqq \prod_{i=1}^N (z - z_i)^{k_i}, \tag{3.14}$$

which is related to the twist function as $\varphi(z) = \partial_z \log \mathcal{P}(z)$. Observe that the ambiguity in the function v_j, namely (3.8), can be expressed as

$$\mathcal{P}(z)^{-j/h^\vee} v_j(z) \longmapsto \mathcal{P}(z)^{-j/h^\vee} v_j(z) - \partial_z\big(\mathcal{P}(z)^{-j/h^\vee} f_j(z)\big).$$

We therefore obtain the following corollary of Theorem 3.7.

Corollary 3.13. *Suppose*

$$\nabla = d + p_{-1}dz - h^{\vee -1}\rho\,\varphi dz + \sum_{j \in E} v_j p_j dz$$

is a quasi-canonical form of an oper $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$.

Let $r \in E_{\geq 2}$ be a positive exponent greater than or equal to 2. Let γ be any contour in $X = \mathbb{C} \setminus \{z_i\}_{i=1}^N$ such that

1. *γ is closed;*
2. *there exists a single-valued branch of the function $\mathcal{P}^{-r/h^\vee}$ along γ;*
3. *v_r has no poles (and is therefore holomorphic) along γ.*

Then the following integral is gauge-invariant, i.e. *it depends only on the oper* $[\nabla]$ *and is independent of the choice of quasi-canonical form:*

$$I_r^\gamma([\nabla]) \coloneqq \int_\gamma \mathcal{P}(z)^{-r/h^\vee} v_r(z) dz.$$

This function is invariant under smooth deformations of the contour γ which do not cross any pole of v_r or any of the marked points $\{z_i\}_{n=1}^\infty$. □

4. Miura ${}^L\mathfrak{g}$-opers and the Bethe equations

4.1. **A class of Miura ${}^L\mathfrak{g}$-opers.** Define a *Miura ${}^L\mathfrak{g}$-oper* as a connection of the form

$$\nabla \coloneqq d + p_{-1}dz + u\,dz \in \mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1) \tag{4.1}$$

where $u \in {}^L\mathfrak{h}(\mathcal{M}) = \mathfrak{h}^*(\mathcal{M})$, using the natural identification ${}^L\mathfrak{h} = \mathfrak{h}^*$. Let $\mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)$ denote the affine space of all Miura ${}^L\mathfrak{g}$-opers. Given a Miura ${}^L\mathfrak{g}$-oper $\nabla \in \mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)$ we refer to its class $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ as the underlying ${}^L\mathfrak{g}$-oper.

Recall the twist function $\varphi \in \mathcal{M}$ defined in (3.12). Given any choice of normalised derivation element Λ of ${}^L\mathfrak{h}$, cf. §3.3, we introduce the affine subspace

$$\mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi \coloneqq d + p_{-1}dz - \Lambda\varphi dz + {}^L\mathfrak{h}'(\mathcal{M})dz \tag{4.2}$$

of $\mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)$ where ${}^L\mathfrak{h}'$ is the span of the simple roots $\{\alpha_i\}_{i=0}^\ell$. It follows from the first part of the proof of Lemma 3.6 that $\mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ is independent of the choice of normalised derivation Λ.

In this paper we shall be interested in Miura ${}^L\mathfrak{g}$-opers (4.1) where the meromorphic $\mathfrak{h}^*$-valued function $u \in \mathfrak{h}^*(\mathcal{M})$ has at most simple poles. Fix a collection of weights $\lambda_1, \dots, \lambda_N \in \mathfrak{h}^*$. We shall, more specifically, be interested in the case when u has a simple pole at each marked point z_i, $i = 1, \dots, N$, with residue $-\lambda_i \in {}^L\mathfrak{h}$. We will furthermore allow the function u to have simple poles at some additional $m \in \mathbb{Z}_{\geq 0}$ marked points w_j, $j = 1, \dots, m$, with residues there given by simple roots $\alpha_{c(j)}$, for some function $c : \{1, \dots, m\} \to I = \{0, \dots, \ell\}$. In other words, we shall consider Miura ${}^L\mathfrak{g}$-opers of the form

$$\nabla = d + p_{-1}dz - \sum_{i=1}^N \frac{\lambda_i}{z - z_i}dz + \sum_{j=1}^m \frac{\alpha_{c(j)}}{z - w_j}dz. \tag{4.3}$$

The residue of ∇ at infinity is the weight $\lambda_\infty \coloneqq \sum_{i=1}^N \lambda_i - \sum_{j=1}^m \alpha_{c(j)} \in {}^L\mathfrak{h}$.

Decomposing each weight $\lambda_i \in \mathfrak{h}^*$ with respect to the basis $\{\alpha_i\}_{i=1}^\ell \cup \{\rho, \delta\}$, we may write it as

$$\lambda_i = \dot{\lambda}_i + \frac{k_i}{h^\vee}\rho - \Delta_i\delta \tag{4.4}$$

for some $\dot{\lambda}_i \in \dot{\mathfrak{h}}^* \coloneqq \mathrm{span}_\mathbb{C}\{\alpha_j\}_{j=1}^\ell$, $k_i \coloneqq \langle \lambda_i, \mathsf{k}\rangle \in \mathbb{C}$ and $\Delta_i \coloneqq -\langle \lambda_i, \mathsf{d}\rangle \in \mathbb{C}$. Since $\dot{\lambda}_i$, δ and the simple roots $\alpha_{c(j)}$ all lie in ${}^L\mathfrak{h}'$, it follows that ∇ belongs to the space $\mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ with the twist function φ defined as in (3.12) in terms of the k_i, $i = 1, \dots, N$.

4.2. **Regular points.** Let $\mathcal{M}_x^{\mathrm{reg}}$ be the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{P}^1$ which are holomorphic at x. We shall say that an ${}^L\mathfrak{g}$-connection $\nabla = d + p_{-1}dz + b\,dz$ in $\mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ is *regular* at a point $x \in \mathbb{C}$ if in fact $b \in {}^L\hat{\mathfrak{b}}_+(\mathcal{M}_x^{\mathrm{reg}})$, *i.e.* b has no pole at x. Let $\mathrm{op}_{L_\mathfrak{g}}^{\mathrm{reg}}(\mathbb{P}^1)_x$ denote the set of all such ${}^L\mathfrak{g}$-connections. It is stabilised by the action of the subgroup ${}^L\hat{N}_+(\mathcal{M}_x^{\mathrm{reg}}) \subset {}^L\hat{N}_+(\mathcal{M})$ on $\mathrm{op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ by gauge transformations. In particular, we can define the quotient space

$$\mathrm{Op}_{L_\mathfrak{g}}^{\mathrm{reg}}(\mathbb{P}^1)_x \coloneqq \mathrm{op}_{L_\mathfrak{g}}^{\mathrm{reg}}(\mathbb{P}^1)_x \big/ {}^L\hat{N}_+(\mathcal{M}_x^{\mathrm{reg}}).$$

If x is not a pole of the twist function φ we may similarly define the space $\mathrm{op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_x$ of ${}^L\mathfrak{g}$-connections of the form $\nabla = d + p_{-1}dz - \Lambda\varphi dz + b'dz$ where Λ is a normalised derivation element of ${}^L\mathfrak{h}$ and $b' \in {}^L\hat{\mathfrak{b}}'_+(\mathcal{M}^{\mathrm{reg}}_x)$. We then also define

$$\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_x \coloneqq \mathrm{op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_x / {}^L\hat{N}_+(\mathcal{M}^{\mathrm{reg}}_x).$$

Lemma 4.1. *For each $x \in \mathbb{C}$ there is a canonical injection*

$$\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x \longhookrightarrow \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1). \tag{4.5}$$

When x is not a pole of φ there is a canonical injection $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_x \hookrightarrow \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$.

Proof. Since ${}^L\hat{N}_+(\mathcal{M}^{\mathrm{reg}}_x) \subset {}^L\hat{N}_+(\mathcal{M})$ we certainly have a well-defined canonical map $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x \to \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)$. Suppose that two ${}^L\mathfrak{g}$-connections $\nabla, \nabla' \in \mathrm{op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$, regular at x, define the same class $[\nabla] = [\nabla']$ in $\mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)$. We must show that they also define the same class in $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$.

Applying the procedure in the first half of the proof of Theorem 3.7 to both of the ${}^L\mathfrak{g}$-connections $\nabla, \nabla' \in \mathrm{op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$, with $\mathcal{M}$ there replaced by $\mathcal{M}^{\mathrm{reg}}_x$, we find that they can each be brought to a quasi-canonical form which is regular at x using a gauge transformation in ${}^L\hat{N}_+(\mathcal{M}^{\mathrm{reg}}_x)$. On the other hand, by the argument in the second half of the proof of Theorem 3.7 with $\mathcal{M}$ there replaced by $\mathcal{M}^{\mathrm{reg}}_x$, we also deduce that these two quasi-canonical forms are related by a gauge transformation in $\exp(\hat{\mathfrak{a}}_{\geq 2}(\mathcal{M}^{\mathrm{reg}}_x))$. It now follows that ∇ and ∇' define the same class in $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$. □

We will identify $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$ with its image in $\mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ under the injection (4.5). We then say that an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)$ is *regular* at $x \in \mathbb{P}^1$ if it lies in $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$. More concretely, this means that there exists a representative of the class $[\nabla]$ in $\mathrm{op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x$, *i.e.* which has no pole at x.

Recall the set $X = \mathbb{C} \setminus \{z_i\}_{i=1}^N$ introduced in §3.6. We define the space of ${}^L\mathfrak{g}$-*opers regular on* X as

$$\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_X \coloneqq \bigcap_{x \in X} \mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_x \subset \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1).$$

Since the twist function has no poles in X, we may also define the space of ${}^L\mathfrak{g}$-*opers with twist function φ regular on* X as

$$\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_X \coloneqq \bigcap_{x \in X} \mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi_x \subset \mathrm{Op}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi.$$

4.3. **Bethe equations.**

Proposition 4.2. *Let $x \in X$ and suppose $\nabla \in \mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)^\varphi$ has the form*

$$\nabla = d + \Big(p_{-1} - h^{\vee -1}\varphi\,\rho + \frac{\alpha_i}{z-x} + r\Big)dz$$

for some simple root α_i, $i \in I$, where $r \in {}^L\mathfrak{h}'(\mathcal{M})$ is regular at x. Then $[\nabla]$ is regular at x, i.e. *there is a representative of $[\nabla]$ which is regular at x, if and only if*

$$h^\vee \langle r(x), \check{\alpha}_i \rangle = \varphi(x). \tag{4.6}$$

Proof. Suppose first that $\widetilde{\nabla}$ is a representative of $[\nabla]$ which is regular at x. Then the gauge transformation parameter $g \in {}^L\hat{N}_+(\mathcal{M})$ determined by following the recursive procedure of Theorem 3.7 is of the form $g = \exp(-\chi\delta)\exp(\sum_{n>0} m_n)$ where $\chi \in \mathcal{M}_x^{\mathrm{reg}}$ and $m_n \in \mathfrak{c}_n(\mathcal{M}_x^{\mathrm{reg}})$, $n > 0$ are all regular at x. Therefore, the quasi-canonical form $\widetilde{\nabla}^g$ of $[\nabla]$ is regular at x. Then, in particular, its component in $\mathfrak{a}_1$ must be regular. Yet by Proposition 3.10 the latter is proportional to (note that in the notation of Proposition 3.10 we have $u_0 = \frac{\alpha_i}{z-x} + r$ and $u_1 = 0$ in the present case)

$$\frac{(-\alpha_i| - \alpha_i + 2\rho)}{2(z-x)^2}dz + \frac{(\alpha_i|r(x)) - h^{\vee -1}\varphi(x)(\alpha_i|\rho)}{z-x}dz + \dots$$

where the dots represent terms regular at $z = x$. Recalling that $\langle \rho, \check{\alpha}_i\rangle = 1$ for all $i \in I$, and in view of (2.2), we see that the double pole term here vanishes and the simple pole term vanishes only if the equation (4.6) holds.

Conversely, suppose (4.6) holds. Let $g = \exp\big(-\frac{1}{z-x}\check{e}_i\big)$. For all $u \in {}^L\mathfrak{h}(\mathcal{M})$ we have

$$(dg)g^{-1} = \check{e}_i\frac{dz}{(z-x)^2}, \qquad gug^{-1} = u - \frac{1}{z-x}[\check{e}_i, u] = u + \frac{\langle u, \check{\alpha}_i\rangle}{z-x}\check{e}_i,$$
$$gp_{-1}g^{-1} = p_{-1} - \frac{1}{z-x}[\check{e}_i, p_{-1}] + \frac{1}{2(z-x)^2}\big[\check{e}_i, [\check{e}_i, p_{-1}]\big] = p_{-1} - \frac{\alpha_i}{z-x} - \frac{\check{e}_i}{(z-x)^2}.$$

Therefore, with $u = -h^{\vee -1}\varphi\,\rho + \frac{\alpha_i}{z-x} + r$ we find

$$\begin{aligned}\nabla^g &= d - (dg)g^{-1} + gp_{-1}g^{-1}dz + gug^{-1}dz \\ &= d + \Big(p_{-1} - h^{\vee -1}\varphi\,\rho + r(z) + \frac{\langle r(z), \check{\alpha}_i\rangle - h^{\vee -1}\varphi(z)}{z-x}\check{e}_i\Big)dz.\end{aligned}$$

(The coefficient of the $(z-x)^{-2}$ term is $-1 - 1 + \langle \alpha_i, \check{\alpha}_i\rangle = 0$.) This is regular at x by virtue of (4.6), so the ${}^L\mathfrak{g}$-oper $[\nabla] = [\nabla^g]$ is regular at x. □

Remark 4.3. In the statement of Proposition 4.2, if we write $\nabla \in \mathrm{op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$ as

$$\nabla = d + \Big(p_{-1} + \frac{\alpha_i}{z-x} + \widetilde{r}\Big)dz$$

where $\widetilde{r} \in {}^L\mathfrak{h}(\mathcal{M})$ is regular at x, noting that φ is regular at $x \in X$, then the Bethe equation (4.6) for the regularity of $[\nabla]$ at x simply reads

$$\langle \widetilde{r}(x), \check{\alpha}_i\rangle = 0. \qquad \triangleleft$$

Recall the subset $X = \mathbb{C} \setminus \{z_i\}_{i=1}^N$ of $\mathbb{P}^1$ introduced in §3.6.

Corollary 4.4. *Let $\nabla \in \mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ be of the form* (4.3). *We have $[\nabla] \in \mathrm{Op}^{\mathrm{reg}}_{{}^L\mathfrak{g}}(\mathbb{P}^1)_X$ if and only if*

$$-\sum_{i=1}^N \frac{(\lambda_i|\alpha_{c(j)})}{w_j - z_i} + \sum_{\substack{i=1\\ i\neq j}}^m \frac{(\alpha_{c(i)}|\alpha_{c(j)})}{w_j - w_i} = 0, \qquad j = 1, \dots, m. \tag{4.7}$$

We refer to these as the Bethe equations.

In particular, when the Bethe equations hold then there exists a quasi-canonical representative of $[\nabla]$ *in which the coefficient functions* $v_i(z)$, $i \in E$, *have no singularities at the Bethe roots* w_j, $j = 1, \dots, m$.

Proof. The ${}^L\mathfrak{g}$-oper $[\nabla]$ is certainly regular away from the points z_i, $i = 1, \dots, N$ and w_j, $j = 1, \dots, m$, since the defining representative ∇ in (4.3) is regular there.

And by Proposition 4.2, see also Remark 4.3, the ${}^L\mathfrak{g}$-oper $[\nabla]$ is also regular at each of the w_j if and only if the jth Bethe equation (4.7) holds. □

Define the *master function* to be

$$\Phi := \sum_{\substack{i,j=1\\ i<j}}^{N} (\lambda_i|\lambda_j) \log(z_i - z_j) - \sum_{i=1}^{N}\sum_{j=1}^{m} (\lambda_i|\alpha_{c(j)}) \log(z_i - w_j) + \sum_{\substack{i,j=1\\ i<j}}^{m} (\alpha_{c(i)}|\alpha_{c(j)}) \log(w_i - w_j). \tag{4.8}$$

It is a multivalued function on $\mathbb{C} \setminus \{z_1, \dots, z_N, w_1, \dots, w_m\}$. One sees that the Bethe equations (4.7) are given by

$$\frac{\partial \Phi}{\partial w_j} = 0, \qquad j = 1, \dots, m.$$

Moreover it is known – see Appendix A for a brief review – that the eigenvalues of the quadratic Hamiltonians (1.1) are given in terms of the partial derivates $\partial\Phi/\partial z_i$.

The following result shows that the partial derivatives of the master function can be read off from the ${}^L\mathfrak{g}$-oper underlying the Miura ${}^L\mathfrak{g}$-oper ∇ of (4.3).

Theorem 4.5. *Let* $\nabla \in \mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ *be a Miura oper of the form* (4.3). *The coefficient of* p_1 *in any quasi-canonical form of the underlying* ${}^L\mathfrak{g}$*-oper* $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ *is*

$$\frac{1}{h^\vee}\left(\sum_{i=1}^{N} \frac{\frac{1}{2}(\lambda_i|\lambda_i + 2\rho)}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{\partial\Phi/\partial z_i}{z - z_i} + \sum_{j=1}^{m} \frac{\partial\Phi/\partial w_j}{z - w_j} \right) dz.$$

Proof. Let us write the Miura ${}^L\mathfrak{g}$-oper in (4.3) as $\nabla = d + p_{-1}dz + u(z)dz$ with

$$u(z) = \frac{\alpha_{c(j)}}{z - w_j} + \widetilde{r}(z), \qquad \widetilde{r}(z) := -\sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} + \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{\alpha_{c(i)}}{z - w_i}.$$

The result follows from a direct computation, using the expression given in Remark 3.11 for the coefficient of p_1 in any quasi-canonical form of the ${}^L\mathfrak{g}$-oper $[\nabla]$, with $\widetilde{u}_0 = u$ given above and $\widetilde{u}_1 = 0$. Explicitly, we find on the one hand

$$\tfrac{1}{2}(u(z)|u(z)) = \frac{1}{2}\sum_{i=1}^{N} \frac{(\lambda_i|\lambda_i)}{(z - z_i)^2} + \frac{1}{2}\sum_{j=1}^{m} \frac{(\alpha_{c(j)}|\alpha_{c(j)})}{(z - w_j)^2} + \sum_{i=1}^{N} \frac{\partial\Phi/\partial z_i}{z - z_i} + \sum_{j=1}^{m} \frac{\partial\Phi/\partial w_j}{z - w_j},$$

where the derivatives of the master function (4.8) with respect to the variables z_i, $i = 1, \dots, N$ and w_j, $j = 1, \dots, m$ read

$$\frac{\partial \Phi}{\partial z_i} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{(\lambda_i|\lambda_j)}{z_i - z_j} - \sum_{j=1}^{m} \frac{(\lambda_i|\alpha_{c(j)})}{z_i - w_j}, \qquad \frac{\partial \Phi}{\partial w_j} = -\sum_{i=1}^{N} \frac{(\lambda_i|\alpha_{c(j)})}{w_j - z_i} + \sum_{\substack{i=1 \\ i \neq j}}^{m} \frac{(\alpha_{c(i)}|\alpha_{c(j)})}{w_j - w_i}.$$

On the other hand, we also have

$$(\rho|u'(z)) = \sum_{i=1}^{N} \frac{(\rho|\lambda_i)}{(z - z_i)^2} - \sum_{j=1}^{m} \frac{(\rho|\alpha_{c(j)})}{(z - w_j)^2}.$$

Adding the above and using the fact that $2(\alpha_i|\rho) = (\alpha_i|\alpha_i)$ for any simple root α_i (since $\langle \rho, \check{\alpha}_i \rangle = 1$) we obtain the result. □

5. Conjectures on affine Gaudin Hamiltonians

Before turning to the affine case, let us recall some features of the situation in finite types. When $\mathfrak{g}$ is a Kac-Moody algebra of finite type, the *quantum Gaudin algebra* is a commutative subalgebra of $U(\mathfrak{g}^{\oplus N})$ generated by the coefficients in the partial fraction decompositions of a finite collection of $U(\mathfrak{g}^{\oplus N})$-valued meromorphic functions $S_k(z)$, indexed by the exponents $k \in \bar{E}$. The $S_k(z)$ have poles at the marked points $z_1, \dots, z_N$. They commute amongst themselves and with the diagonal action of $\mathfrak{g}$. In particular, $1 \in \bar{E}$, and the explicit form of $S_1(z)$ is

$$S_1(z) = \sum_{i=1}^{N} \frac{\mathcal{C}^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{\mathcal{H}_i}{z - z_i}, \tag{5.1}$$

where the $\mathcal{H}_i$ are the quadratic Hamiltonians in (1.1) and where $\mathcal{C}^{(i)}$ is the copy of the Casimir element $\mathcal{C} \in U(\mathfrak{g})^{\mathfrak{g}}$ in the ith tensor factor of $U(\mathfrak{g}^{\oplus N})$. More generally, the pole terms of highest order in each $S_k(z)$ are $\sum_{i=1}^{N} \mathcal{C}^{(i)}_{k+1}/(z - z_i)^{k+1}$, where $\mathcal{C}_{k+1} \in U(\mathfrak{g})^{\mathfrak{g}}$ is a central element – as indeed it must be for $S_k(z)$ to commute with the diagonal action of $\mathfrak{g}$. Each $S_k(z)$ has degree $k + 1$ as an element of $U(\mathfrak{g}^{\oplus N})$.

As we sketched in the introduction, to each Miura ${}^L\mathfrak{g}$-oper of the form (1.2), with the Bethe roots w_i obeying the Bethe equations, there corresponds a joint eigenvector ψ of the functions $S_k(z)$, and the joint eigenvalues are given by the coefficients $\bar{v}_k(z)$ of the $\bar{p}_k$ in the canonical form of the underlying ${}^L\mathfrak{g}$-oper.

Now, not all these features can be precisely preserved in the affine case. Indeed, in finite types the centre $U(\mathfrak{g})^{\mathfrak{g}}$ is isomorphic (via the Harish-Chandra isomorphism) to a graded polynomial algebra $\mathbb{C}[\{\mathcal{C}_{k+1}\}_{k \in \bar{E}}]$ in rank $\mathfrak{g}$ generators of the correct degrees. But in affine types the centre is much smaller. Namely, the centre of the (completed, as in §5.1 below) envelope of an affine Kac-Moody algebra is isomorphic to the graded polynomial algebra in only two generators, k and $\mathcal{C}$ (of degrees 0 and 2; the definition of $\mathcal{C}$ in the affine case is in (5.7) below) [CI84]. Thus, one should not expect to find meromorphic functions $S_k(z)$, indexed by the positive exponents $k \in E$, such that

they commute with the diagonal action of $\mathfrak{g}$ for each $z \in X = \mathbb{C} \setminus \{z_1, \dots, z_N\}$ *and* have degrees $k+1$.[5]

This is consistent with the results in §3 – §4 above: we saw in Theorem 3.7 that the coefficients $v_k(z)$ of the quasi-canonical form of an ${}^L\mathfrak{g}$-oper are defined, for $k \in E_{\geq 2}$, only up to the addition of twisted derivatives. So they themselves are not good candidates for the eigenvalues of such would-be generating functions. But we also saw that there are well-defined functions on the space of opers given by integrals, as in Corollary 3.13. It is natural to think that these functions are the eigenvalues of higher Gaudin Hamiltonians. That in turn suggests that such Hamiltonians are *themselves* given by such integrals. This is the content of Conjecture 5.1 below. To state it, we must define an appropriate completion of $U(\mathfrak{g}^{\oplus N})$ when $\mathfrak{g}$ is of untwisted affine type.

5.1. **Completion of $U(\mathfrak{g}^{\oplus N})$.** Let $\mathfrak{g} = \mathfrak{g}(A)$ be an untwisted affine Kac-Moody algebra as in §2.1. Let $\dot{A} = (A_{ij})_{i,j=1}^{\ell}$ denote the Cartan matrix of finite type obtained from the Cartan matrix A of affine type by removing the 0th row and column, and $\dot{\mathfrak{g}} := \mathfrak{g}(\dot{A})$ the corresponding finite-dimensional simple Lie algebra. The Lie algebra $\mathfrak{g}$ can be realised as the semi-direct product $\hat{\mathcal{L}}\dot{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$ of the central extension $\hat{\mathcal{L}}\dot{\mathfrak{g}} \cong_{\mathbb{C}} \mathcal{L}\dot{\mathfrak{g}} \oplus \mathbb{C}\mathsf{k}$ of the loop algebra $\mathcal{L}\dot{\mathfrak{g}} := \dot{\mathfrak{g}}[t, t^{-1}]$ with derivation element d acting as the derivative $t\partial_t$ in the formal loop variable t. In what follows we shall identify $\mathfrak{g}$ with $\hat{\mathcal{L}}\dot{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$.

Let $\mathfrak{g}^{\oplus N}$ denote the N-fold direct sum of $\mathfrak{g}$. We denote by $X^{(i)}$ the copy of any $X \in \mathfrak{g}$ in the ith summand, for $i = 1, \dots, N$. Consider the left ideals $\mathcal{I}_n := U(\mathfrak{g}^{\oplus N})(t^n\dot{\mathfrak{g}}[t])^{\oplus N}$, for $n \in \mathbb{Z}_{\geq 0}$, of the universal enveloping algebra $U(\mathfrak{g}^{\oplus N})$. They define a descending $\mathbb{Z}_{\geq 0}$-filtration on $U(\mathfrak{g}^{\oplus N})$, that is to say we have $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$ with $\cap_{n \geq 0}\mathcal{I}_n = \{0\}$. Define the corresponding completion of $U(\mathfrak{g}^{\oplus N})$ as the inverse limit

$$\hat{U}(\mathfrak{g}^{\oplus N}) := \varprojlim_n U(\mathfrak{g}^{\oplus N})/\mathcal{I}_n.$$

By definition, an element of $\hat{U}(\mathfrak{g}^{\oplus N})$ is a possibly infinite sum

$$x = \sum_{m \geq 0} x_m \tag{5.2}$$

of elements in $U(\mathfrak{g}^{\oplus N})$, with $x_m \in \mathcal{I}_m$ for all $m > 0$ so that only finitely many terms contribute when one works modulo any $\mathcal{I}_n$. Since the $\mathcal{I}_n$, $n \geq 0$ are only left ideals, the quotients $U(\mathfrak{g}^{\oplus N})/\mathcal{I}_n$ are not associative algebras. However, the multiplication in $U(\mathfrak{g}^{\oplus N})$ is continuous with respect to the linear topology whose basis of open neighbourhoods for 0 is $\{\mathcal{I}_n\}_{n \geq 0}$. So the completion $\hat{U}(\mathfrak{g}^{\oplus N})$ is an associative algebra.

The tensor product $\bigotimes_{j=1}^N L_{\lambda_j}$ of irreducible $\mathfrak{g}$-modules is *smooth* as a module over $U(\mathfrak{g}^{\oplus N})$, meaning that for every $v \in \bigotimes_{j=1}^N L_{\lambda_j}$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{I}_n v = 0$. Therefore $\bigotimes_{j=1}^N L_{\lambda_j}$ is a module over the completion $\hat{U}(\mathfrak{g}^{\oplus N})$. Let $\hat{U}_{\boldsymbol{k}}(\mathfrak{g}^{\oplus N})$, with $\boldsymbol{k} := (k_i)_{i=1}^{\ell}$, denote the quotient of the algebra $\hat{U}(\mathfrak{g}^{\oplus N})$ by the ideal $J_{\boldsymbol{k}}$ generated by

[5]That would be impossible in any type with an even exponent, since any polynomial in k and $\mathcal{C}$ has degree $k+1$ with k odd; in other types these considerations merely make it seem unnatural.

the elements $\mathsf{k}^{(i)} - k_i$, namely

$$\hat{U}_{\boldsymbol{k}}(\mathfrak{g}^{\oplus N}) \coloneqq \hat{U}(\mathfrak{g}^{\oplus N})/J_{\boldsymbol{k}}.$$

The action of $\hat{U}(\mathfrak{g}^{\oplus N})$ on $\bigotimes_{j=1}^N L_{\lambda_j}$ factors through the quotient $\hat{U}_{\boldsymbol{k}}(\mathfrak{g}^{\oplus N})$. In particular, if we define

$$\mathsf{k}(z) \coloneqq \sum_{i=1}^N \frac{\mathsf{k}^{(i)}}{z - z_i} \tag{5.3}$$

then the image of $\mathsf{k}(z)$ in $\hat{U}_{\boldsymbol{k}}(\mathfrak{g}^{\oplus N})$ is the twist function $\varphi(z)$ as in (3.12), cf. (4.4).

We have the usual ascending filtration $\mathbb{C}1 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ of the universal enveloping algebra $U(\mathfrak{g}^{\oplus N})$ of the Lie algebra $\mathfrak{g}^{\oplus N}$. Every $x \in U(\mathfrak{g}^{\oplus N})$ belongs to some filtered subspace; the *degree* of x is by definition the smallest $k \in \mathbb{Z}_{\geq 0}$ such that $x \in \mathcal{F}_k$. Let us say that an element $x \in \hat{U}(\mathfrak{g}^{\oplus N})$ has *(finite) degree* $k \in \mathbb{Z}_{\geq 0}$ if, when x is written as a sum as in (5.2), the degrees of the x_m are bounded above and k is their maximum.

5.2. **Conjectures.**

Conjecture 5.1. *There exist nonzero $\hat{U}(\mathfrak{g}'^{\oplus N})$-valued meromorphic functions $\mathcal{S}_i(z)$, $i \in E$, on $\mathbb{P}^1$ with the following properties:*

(i) For each $i \in E$, $\mathcal{S}_i(z)$ has degree $i+1$.

(ii) For any $i, j \in E$ we have

$$[\mathcal{S}_i(z), \mathcal{S}_j(w)] = \big(h^\vee \partial_z - i\mathsf{k}(z)\big)\, \mathcal{A}_{ij}(z,w) + \big(h^\vee \partial_w - j\mathsf{k}(w)\big)\, \mathcal{B}_{ij}(z,w),$$

for some $\hat{U}(\mathfrak{g}'^{\oplus N})$-valued meromorphic functions $\mathcal{A}_{ij}(z,w), \mathcal{B}_{ij}(z,w)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

(iii) For each $i \in E$ and each $j = 1, \dots, N$ we have

$$[\mathcal{H}_j, \mathcal{S}_i(z)] = \big(h^\vee \partial_z - i\mathsf{k}(z)\big)\, \mathcal{D}_i^j(z),$$

for some $\hat{U}(\mathfrak{g}'^{\oplus N})$-valued meromorphic function $\mathcal{D}_i^j(z)$ on $\mathbb{P}^1$.

(iv) For each $i \in E$ and any $x \in \mathfrak{g}$ we have, writing $\Delta x \coloneqq \sum_{j=1}^N x^{(j)}$,

$$[\Delta x, \mathcal{S}_i(z)] = \big(h^\vee \partial_z - i\mathsf{k}(z)\big)\, \mathcal{C}_i^x(z),$$

for some $\hat{U}(\mathfrak{g}'^{\oplus N})$-valued meromorphic function $\mathcal{C}_i^x(z)$ on $\mathbb{P}^1$. □

Suppose such functions $\mathcal{S}_i(z)$ do exist. For any contour γ as in Corollary 3.13 and any $i \in E$, denote by $\hat{Q}_i^\gamma$ the image of

$$\int_\gamma \mathcal{P}(z)^{-i/h^\vee} \mathcal{S}_i(z) dz \tag{5.4}$$

in the quotient $\hat{U}_{\boldsymbol{k}}(\mathfrak{g}^{\oplus N})$. Then these $\hat{Q}_i^\gamma$ are commuting Hamiltonians, as follows.

Corollary 5.2. *Given Conjecture 5.1, one has*

$$[\hat{Q}_i^\gamma, \hat{Q}_j^\eta] = 0$$

for any $i, j \in E$ and any pair of contours γ, η. Moreover, each $\hat{Q}_i^\gamma$ commutes with the diagonal action of $\mathfrak{g}$ and with the quadratic Hamiltonians $\mathcal{H}_j$, $j = 1, \dots, N$. □

Conjecture 5.3. *Let $\psi \in \left(\bigotimes_{j=1}^N L_{\lambda_j}\right)_{\lambda_\infty}$ be the Schechtman-Varchenko vector associated with the Miura ${}^L\mathfrak{g}$-oper $\nabla \in \mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ in (4.3). For every $j \in E$, let $v_j(z)$ be the coefficient of p_j in any quasi-canonical form of the ${}^L\mathfrak{g}$-oper $[\nabla]$.*

If the Bethe roots w_j, $j = 1, \dots, m$, satisfy the Bethe equations (4.7) then

$$\hat{Q}_i^\gamma \psi = \int_\gamma \mathcal{P}(z)^{-i/h^\vee} v_i(z) dz \; \psi$$

for every $i \in E$ and any choice of contour γ as in Corollary 3.13. □

The remainder of this section is devoted to showing that these conjectures are consistent with what is known about the quadratic Hamiltonians for affine Gaudin models.

In a separate paper [LVY] we explicitly construct $\mathcal{S}_2(z)$ in the case $\mathfrak{g}' = \widehat{\mathfrak{sl}}_M$, $M \geq 3$ and show that the statements of Conjecture 5.1 hold for $i = 1, 2$. In these cases we also verify Conjecture 5.3 for $m = 0, 1$.

5.3. **Quadratic affine Gaudin Hamiltonians.** Recall $\mathfrak{g} \cong \hat{\mathcal{L}}\dot{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$. Fix a basis I^a, for $a = 1, \dots, \dim \dot{\mathfrak{g}}$, of $\dot{\mathfrak{g}}$. Recall the non-degenerate bilinear form $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$ from §2.1. It restricts to a non-degenerate bilinear form $\dot{\mathfrak{g}} \times \dot{\mathfrak{g}} \to \mathbb{C}$ on $\dot{\mathfrak{g}}$. Let I_a be the dual basis of $\dot{\mathfrak{g}}$ with respect to this restriction. A basis of $\mathfrak{g}$ is then given by $I_n^a := I^a \otimes t^n$, for $a = 1, \dots, \dim \dot{\mathfrak{g}}$ and $n \in \mathbb{Z}$, together with k and d. The corresponding dual basis of $\mathfrak{g}$, with respect to $(\cdot|\cdot)$, is given by $I_{a,-n} := I_a \otimes t^{-n}$, for $a = 1, \dots, \dim \dot{\mathfrak{g}}$ and $n \in \mathbb{Z}$, together with d and k. In terms of these bases, the quadratic Gaudin Hamiltonians (1.1) in this untwisted affine case then take the form

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\mathsf{k}^{(i)}\mathsf{d}^{(j)} + \mathsf{d}^{(i)}\mathsf{k}^{(j)} + \sum_{n \in \mathbb{Z}} I_{a,-n}^{(i)} I_n^{a(j)}}{z_i - z_j} \in \hat{U}(\mathfrak{g}^{\oplus N}), \qquad i = 1, \dots, N. \tag{5.5}$$

(Here and below we employ summation convention: $I_a I^a := \sum_{a=1}^{\dim \dot{\mathfrak{g}}} I_a I^a$.)

For every $i = 1, \dots, N$, the completed enveloping algebra $\hat{U}(\mathfrak{g}^{\oplus N})$ also contains the ith copy of the *quadratic Casimir* of $\mathfrak{g}$, namely the element of $\hat{U}(\mathfrak{g}^{\oplus N})$ defined as

$$\mathcal{C}^{(i)} := (\mathsf{k}^{(i)} + h^\vee)\mathsf{d}^{(i)} + \tfrac{1}{2} I_{a,0}^{(i)} I_0^{a(i)} + \sum_{n>0} I_{a,-n}^{(i)} I_n^{a(i)}. \tag{5.6}$$

Proposition 5.4. *Each $\mathcal{C}^{(i)}$, $i = 1, \dots, N$ is central in $\hat{U}(\mathfrak{g}^{\oplus N})$.*

Its action on the tensor product of irreducible highest weight $\mathfrak{g}$-module $\bigotimes_{j=1}^N L_{\lambda_j}$, for any $\lambda_1, \dots, \lambda_N \in \mathfrak{h}^*$, is given by multiplication by $\frac{1}{2}(\lambda_i | \lambda_i + 2\rho)$.

Proof. It suffices to consider the case $N = 1$, for which we can drop all superscripts labelling the copy of $\mathfrak{g}$ in the direct sum $\mathfrak{g}^{\oplus N}$. For the first statement we will simply show that $\mathcal{C}$ as defined in (5.6) coincides with the quadratic Casimir for a general Kac-Moody algebra [Kac90, §2.5] in the affine case. The second part of the statement will then follow from [Kac90, Corollary 2.6].

We have the Cartan subalgebra $\dot{\mathfrak{h}} = \mathfrak{h} \cap \dot{\mathfrak{g}}$ of $\dot{\mathfrak{g}}$ and the root space decomposition

$$\dot{\mathfrak{g}} = \dot{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \dot{\Delta}} \dot{\mathfrak{g}}_\alpha,$$

where $\dot{\Delta}$ denotes the root system of $\dot{\mathfrak{g}}$. Let $\dot{\Delta}_+ \subset \dot{\Delta}$ denote the subset of positive roots. The corresponding root space decomposition of the untwisted affine Kac-Moody algebra $\mathfrak{g}$ reads

$$\mathfrak{g} = \bigoplus_{\widetilde{\alpha} \in \Delta} \mathfrak{g}_{\widetilde{\alpha}},$$

where $\Delta := \{\alpha + n\delta \,|\, \alpha \in \dot{\Delta}, n \in \mathbb{Z}\}$ is the root system of $\mathfrak{g}$. We denote the subset of positive roots by $\Delta_+ := \{\alpha + n\delta \,|\, \alpha \in \dot{\Delta}, n > 0\} \cup \dot{\Delta}_+$. Explicitly, $\mathfrak{g}_{\pm\alpha+n\delta} = \dot{\mathfrak{g}}_{\pm\alpha} \otimes t^n$ for every $\alpha \in \dot{\Delta}_+$ and $n \in \mathbb{Z}$, while $\mathfrak{g}_{n\delta} = \dot{\mathfrak{h}} \otimes t^n$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $\mathfrak{g}_0 = \mathfrak{h}$. We fix a basis $e^s_{\widetilde{\alpha}}$, $s = 1, \dots, \dim \mathfrak{g}_{\widetilde{\alpha}}$ of the root space $\mathfrak{g}_{\widetilde{\alpha}}$ for each $\widetilde{\alpha} \in \Delta_+$ and denote by $e^s_{-\widetilde{\alpha}}$, $s = 1, \dots, \dim \mathfrak{g}_{\widetilde{\alpha}}$ its dual basis in $\mathfrak{g}_{-\widetilde{\alpha}}$. Also fix a basis $\{u_i\}_{i=1}^{\dim \mathfrak{h}} = \{h_i\}_{i=1}^{\ell} \cup \{\mathsf{k}, \mathsf{d}\}$ of $\mathfrak{h}$, where $\{h_i\}_{i=1}^{\ell}$ is a basis of $\dot{\mathfrak{h}}$, and let $\{u^i\}_{i=1}^{\dim \mathfrak{h}} = \{h^i\}_{i=1}^{\ell} \cup \{\mathsf{d}, \mathsf{k}\}$ be its dual basis, where $\{h^i\}_{i=1}^{\ell}$ is the basis of $\dot{\mathfrak{h}}$ dual to $\{h_i\}_{i=1}^{\ell}$.

We may now rewrite the expression (5.6) for the quadratic Casimir using the above dual bases of $\mathfrak{g}$. For the second term on the right hand side of (5.6) we have

$$I_{a,0} I_0^a = \sum_{i=1}^{\ell} h_i h^i + \sum_{\alpha \in \dot{\Delta}_+} \big(e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha\big) = 2\nu^{-1}(\dot{\rho}) + \sum_{i=1}^{\ell} h_i h^i + 2 \sum_{\alpha \in \dot{\Delta}_+} e_{-\alpha} e_\alpha,$$

where we dropped the superscript 's' on the basis elements $e^s_{\pm\alpha}$ for $\alpha \in \dot{\Delta}_+$ since in this case $\dim \mathfrak{g}_{\pm\alpha} = 1$. In the second equality we used the relation $[e_\alpha, e_{-\alpha}] = \nu^{-1}(\alpha)$ and set $\dot{\rho} := \frac{1}{2} \sum_{\alpha \in \dot{\Delta}_+} \alpha$. On the other hand, the infinite sum over $n > 0$ in (5.6) can be written as

$$\sum_{n>0} I_{a,-n} I_n^a = \sum_{\widetilde{\alpha} \in \Delta_+ \setminus \dot{\Delta}_+} \sum_{s=1}^{\dim \mathfrak{g}_{\widetilde{\alpha}}} e^s_{-\widetilde{\alpha}} e^s_{\widetilde{\alpha}}.$$

Recall the set of fundamental coweights $\{\check{\Lambda}_i\}_{i=0}^{\ell}$ of $\mathfrak{g}$ defined by (2.4). The set of fundamental coweights $\{\check{\omega}_i\}_{i=1}^{\ell}$ of $\dot{\mathfrak{g}}$ can be identified with $\check{\omega}_i = \check{\Lambda}_i - a_i \check{\Lambda}_0$ for each $i = 1, \dots, \ell$. If we set $\epsilon_i := a_i \check{a}_i^{-1}$ for $i = 0, \dots, \ell$, then

$$\nu^{-1}(\dot{\rho}) = \sum_{i=1}^{\ell} \epsilon_i^{-1} \check{\omega}_i = \sum_{i=0}^{\ell} \epsilon_i^{-1} (\check{\Lambda}_i - a_i \check{\Lambda}_0) = \nu^{-1}(\rho) - h^\vee \mathsf{d}$$

where in the second step we used the assumption that $a_0 = 1$, cf. Remark 2.2. Therefore, combining all the above we can rewrite the quadratic Casimir (5.6) as

$$\mathcal{C} = \nu^{-1}(\rho) + \tfrac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} u_i u^i + \sum_{\widetilde{\alpha} \in \Delta_+} \sum_{s=1}^{\dim \mathfrak{g}_{\widetilde{\alpha}}} e^s_{-\widetilde{\alpha}} e^s_{\widetilde{\alpha}}, \tag{5.7}$$

which coincides with its expression given in [Kac90, §2.5], as required. □

By direct analogy with the finite-dimensional case, cf. (5.1), it is natural to introduce the $\hat{U}(\mathfrak{g}^{\oplus N})$-valued meromorphic function

$$S_1(z) \coloneqq \sum_{i=1}^{N} \frac{\mathcal{C}^{(i)}}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{\mathcal{H}_i}{z-z_i}. \tag{5.8}$$

We then have the following direct generalisation of the finite-dimensional case.

Theorem 5.5. *Let $\nabla \in \mathrm{MOp}_{L_\mathfrak{g}}(\mathbb{P}^1)$ be of the form (4.3). If the set of Bethe roots w_j, $j = 1, \dots, m$ satisfy the Bethe equations (4.7), then the eigenvalue of $S_1(z)$ on the subspace $\big(\bigotimes_{j=1}^N L_{\lambda_j}\big)_{\lambda_\infty}$ of weight $\lambda_\infty = \sum_{i=1}^N \lambda_i - \sum_{j=1}^m \alpha_{c(j)} \in \mathfrak{h}^*$, is given by $h^\vee$ times the coefficient of p_1 in any quasi-canonical form of the underlying ${}^L\mathfrak{g}$-oper $[\nabla]$.*

Proof. This follows form Theorem 4.5 together with Proposition 5.4 and (A.6). □

The expression (5.8) can alternatively be described as follows. For any $x \in \mathfrak{g}$ we define the $\mathfrak{g}^{\oplus N}$-valued meromorphic function, cf. (5.3),

$$x(z) \coloneqq \sum_{i=1}^{N} \frac{x^{(i)}}{z-z_i}.$$

We then introduce the *formal Lax matrix* of the Gaudin model associated with $\mathfrak{g}$ as the element

$$\mathcal{L}(z) \coloneqq \mathsf{k} \otimes \mathsf{d}(z) + \mathsf{d} \otimes \mathsf{k}(z) + \sum_{n \in \mathbb{Z}} I_{a,-n} \otimes I^a_n(z) \tag{5.9}$$

of the completed tensor product $\mathfrak{g} \hat{\otimes} \mathfrak{g}^{\oplus N}$. Then the generating function (5.8) for the quadratic affine Gaudin Hamiltonians can be rewritten as

$$S_1(z) = \tfrac{1}{2} :\big(\mathcal{L}(z)\big|\mathcal{L}(z)\big): - h^\vee \mathsf{d}'(z), \tag{5.10}$$

where $:\cdot\cdot:$ denotes normal ordering by mode numbers, *i.e.* $:I^a_m I^b_n:$ is $I^b_n I^a_m$ if $m \geq 0$ and $I^a_m I^b_n$ otherwise.

Remark 5.6. This expression can be regarded as a quantisation of the generating function for the quadratic Hamiltonians of a *classical* affine Gaudin model [Vic]. Indeed, the latter is a meromorphic function valued in a completion of the symmetric algebra $S(\mathfrak{g}^{\oplus N})$, given explicitly by $\frac{1}{2}(\mathcal{L}(z)|\mathcal{L}(z))$. In fact, the above expression for $S_1(z)$ can be heuristically obtained from $\frac{1}{2}(\mathcal{L}(z)|\mathcal{L}(z))$ by using the commutation relations of $\mathfrak{g}$ to rewrite each term so that all raising operators, *i.e.* I^a_n with $n > 0$, appear on the right. This procedure results in a meaningless infinite sum, but the term $-h^\vee \mathsf{d}'(z)$ can be thought of as its regularisation; see [Kac90, §2.11] for a similar motivation of the linear term $\nu^{-1}(\rho)$ in the expression (5.7) for the quadratic Casimir of $\mathfrak{g}$ in the proof of Proposition 5.4. ◁

Now we explain how the generating function $S_1(z)$ of the quadratic Hamiltonians fits into our conjecture on the form of the higher affine Gaudin Hamiltonians. We first reinterpret it in light of Corollary 3.9. Define the *local Lax matrix* as the part of the formal Lax matrix (5.9) involving only the loop generators of $\mathfrak{g}$, namely

$$L(z) \coloneqq \sum_{n \in \mathbb{Z}} I_{a,-n} \otimes I^a_n(z). \tag{5.11}$$

The expression (5.10) for $S_1(z)$ can now be rewritten as follows

$$S_1(z) = \mathcal{S}_1(z) - \big(h^\vee\partial_z - \mathsf{k}(z)\big)\mathsf{d}(z), \tag{5.12}$$

where we defined

$$\mathcal{S}_1(z) := \tfrac{1}{2}:\big(L(z)\big|L(z)\big):.$$

Let $\psi \in \big(\bigotimes_{j=1}^N L_{\lambda_j}\big)_{\lambda_\infty}$ denote the Schechtman-Varchenko vector corresponding to the Miura ${}^L\mathfrak{g}$-oper $\nabla \in \mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)$ in (4.3). Recall the expression (4.4) for the weights $\lambda_i \in \mathfrak{h}^*$, $i = 1, \dots, N$. On $\bigotimes_{j=1}^N L_{\lambda_j}$, $\mathsf{k}(z)$ acts as $\varphi(z)$ as we noted above, and the action of $\mathsf{d}(z)$ on ψ is given by

$$\mathsf{d}(z)\psi = \sum_{i=1}^N \frac{\Delta_i - m_0}{z - z_i}\psi =: \Delta(z)\psi,$$

where m_0 is the number of Bethe roots associated to the affine simple root α_0. In other words, the action of (5.12) on the Schechtman-Varchenko vector reads

$$S_1(z)\psi = \mathcal{S}_1(z)\psi - h^\vee\bigg(\Delta'(z) - \frac{\varphi(z)}{h^\vee}\Delta(z)\bigg)\psi. \tag{5.13}$$

Observe that the final term is a twisted derivative of degree 1.

Now recall from Corollary 3.9 that if instead of working with ${}^L\mathfrak{g}$-opers we were to consider ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers, then the coefficients of all the p_i's, $i \in E$ in a quasi-canonical form would be on an equal footing since the coefficient $v_1(z)$ of p_1 would also only be defined up to a twisted derivative

$$v_1 \longmapsto v_1 - f_1' + \frac{\varphi}{h^\vee}f_1,$$

with $f_1 \in \mathcal{M}$. In particular, only its integral

$$\int_\gamma \mathcal{P}(z)^{-1/h^\vee} v_1(z)dz$$

over a cycle γ as in Corollary 3.13 would provide a well-defined function on the space of ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers.

In exactly the same way as we conjecture the spectrum of an affine Gaudin model associated with $\mathfrak{g}$ to be described by ${}^L\mathfrak{g}$-opers, the space of ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers should describe the spectrum of an affine Gaudin model associated with the derived algebra $\mathfrak{g}'$. Indeed, since the weights appearing as the residues in a Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper are classes in ${}^L\mathfrak{h}/\mathbb{C}\delta = \mathfrak{h}^*/\mathbb{C}\delta$, *i.e.* weights in $\mathfrak{h}^*$ defined up to an arbitrary multiple of δ, we should not include the generator d on the Gaudin model side since it pairs non-trivially with the weight δ. And one way to disregard the generator d from the expression (5.12) for $S_1(z)$ is to consider its integral

$$\int_\gamma \mathcal{P}(z)^{-1/h^\vee} S_1(z)dz$$

over a contour γ as in Corollary 3.13. Indeed, it follows from (5.13) that the action of this operator on the Schechtman-Varchenko vector ψ coincides with the action of

the operator

$$\int_\gamma \mathcal{P}(z)^{-1/h^\vee} \mathcal{S}_1(z) dz.$$

The following Lemma shows that Conjecture 5.1 holds at least for $i = 1$.

Lemma 5.7. *For any distinct $z, w \in X$ we have*

$$\big[\mathcal{S}_1(z), \mathcal{S}_1(w)\big] = \big(h^\vee \partial_z - \mathsf{k}(z)\big)\mathcal{A}(z,w) - \big(h^\vee \partial_w - \mathsf{k}(w)\big)\mathcal{A}(z,w),$$

where $\mathcal{A}(z,w)$ is the $\hat{U}(\mathfrak{g}'^{\oplus N})$-valued meromorphic function on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\mathcal{A}(z,w) \coloneqq \frac{1}{z-w} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\sum_{n\in\mathbb{Z}} n I^{(i)}_{a,-n} I^{a(j)}_n}{(z-z_i)(w-z_j)}.$$

Also, for each $j = 1, \dots, N$,

$$\big[\mathcal{H}_j, \mathcal{S}_1(w)\big] = \big(h^\vee \partial_w - \mathsf{k}(w)\big) \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{\sum_{n\in\mathbb{Z}} n I^{(j)}_{a,-n} I^{a(i)}_n}{(w-z_i)(w-z_j)}.$$

Moreover, for any $x \in \mathfrak{g}$ we have $[\Delta x, \mathcal{S}_1(z)] = \big(h^\vee \partial_z - \mathsf{k}(z)\big)[x, \mathsf{d}](z)$.

Proof. Since the quadratic Gaudin Hamiltonians $\mathcal{H}_i$, $i = 1, \dots, N$ mutually commute and the $\mathcal{C}^{(i)}$, $i = 1, \dots, N$ are central in $\hat{U}(\mathfrak{g}^{\oplus N})$ by Proposition 5.4, it follows that $[S_1(z), S_1(w)] = 0$. Noting that $[\mathsf{d}(z), \mathcal{S}_1(w)] = \mathcal{A}(z,w)$ one has

$$\big[h^\vee \mathsf{d}'(z) - \mathsf{k}(z)\mathsf{d}(z), \mathcal{S}_1(w)\big] = \big(h^\vee \partial_z - \mathsf{k}(z)\big)\mathcal{A}(z,w)$$

Using the relation (5.12) we get the first result. It also follows that $[S_1(z), \mathcal{S}_1(w)] = -(h^\vee \partial_w - \mathsf{k}(w))\mathcal{A}(z,w)$, from which, taking the residue at $z = z_j$, we obtain the commutators with $\mathcal{H}_j$. The last part follows similarly from the relation $[\Delta x, S_1(z)] = 0$ which is a consequence of the fact that both $\mathcal{C}^{(i)}$ and $\mathcal{H}_i$, for each $i = 1, \dots, N$, commute with the diagonal action of $\mathfrak{g}$. □

6. Coordinate invariance and meromorphic ${}^L\mathfrak{g}$-opers on curves

Throughout §3–4 we fixed a global coordinate z on $\mathbb{C} \subset \mathbb{P}^1$ and studied meromorphic ${}^L\mathfrak{g}$-opers in that coordinate. Let us now consider meromorphic ${}^L\mathfrak{g}$-opers in local charts, and discuss their behaviour under changes in coordinate. In this section only, we shall work over an arbitrary Riemann surface Σ.

When ${}^L\mathfrak{g}$ is of finite type, an ${}^L\mathfrak{g}$-oper on Σ is a triple $(\mathcal{F}, \mathcal{B}, \nabla)$ where $\mathcal{F}$ is a principal ${}^LG$-bundle, $\mathcal{B}$ is an ${}^LB$-reduction and ∇ is a connection on $\mathcal{F}$ with certain properties; see *e.g.* [BD, Fre07]. Concretely, such a triple can be constructed by gluing together trivial ${}^LG$-bundles over coordinate patches, each equipped with a connection given by an ${}^L\mathfrak{g}$-oper in canonical form, using the ${}^LB$-valued transition functions relating canonical forms in different coordinates (see equation (6.12) below) [Fre04, Fre07].

The abstract definition of an ${}^L\mathfrak{g}$-oper as a triple can be generalised to the case when ${}^L\mathfrak{g}$ is of affine type [Fre04]. However, since the quasi-canonical form is not unique in this case by Theorem 3.7, and there is no naturally preferred quasi-canonical form, it is less clear how to construct such a triple explicitly. We therefore proceed differently:

we first define the space of ${}^L\mathfrak{g}$-opers over any coordinate patch as in §3 and then glue these together to form a sheaf, the sheaf of ${}^L\mathfrak{g}$-opers over Σ.

6.1. **The sheaf of ${}^L\mathfrak{g}$-opers $\mathrm{Op}_{{}^L\mathfrak{g}}$.** For any open subset $U \subset \Sigma$ we let $\mathcal{K}(U)$ denote the field of meromorphic functions on U. We denote by $\mathcal{K}$ the sheaf $U \mapsto \mathcal{K}(U)$ of meromorphic functions on Σ. When $\Sigma = \mathbb{P}^1$, the field $\mathcal{M}$ of meromorphic functions on $\mathbb{P}^1$, introduced in §3.1, is the field of global sections of $\mathcal{K}$. For any open subset $U \subset \Sigma$, we define the Lie algebra ${}^L\hat{\mathfrak{n}}_+(\mathcal{K}(U))$ and the group ${}^L\hat{N}_+(\mathcal{K}(U))$ as in §3.3. We also set ${}^L\hat{\mathfrak{b}}_+(\mathcal{K}(U)) \coloneqq {}^L\mathfrak{h}(\mathcal{K}(U)) \oplus {}^L\hat{\mathfrak{n}}_+(\mathcal{K}(U))$.

To begin with, let us suppose that $U \subset \Sigma$ is an open subset equipped with a holomorphic coordinate $t : U \to \mathbb{C}$. Define $\mathrm{op}_{{}^L\mathfrak{g}}(U)$ to be the affine space of connections of the form

$$\nabla \coloneqq d + p_{-1}dt + bdt, \quad b \in {}^L\hat{\mathfrak{b}}_+(\mathcal{K}(U)). \tag{6.1}$$

As in §3.3, it admits an action of the group ${}^L\hat{N}_+(\mathcal{K}(U))$ by gauge transformations, and we define the space of meromorphic ${}^L\mathfrak{g}$-opers on U to be the quotient

$$\mathrm{Op}_{{}^L\mathfrak{g}}(U) \coloneqq \mathrm{op}_{{}^L\mathfrak{g}}(U)\big/{}^L\hat{N}_+(\mathcal{K}(U)). \tag{6.2}$$

The proof of the following is as for Theorem 3.7.

Theorem 6.1. *Let $U \subset \Sigma$ be open with a holomorphic coordinate $t : U \to \mathbb{C}$. Every class $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(U)$ has a representative of the form*

$$\nabla = d + \left(p_{-1} - \frac{\varphi}{h^\vee}\rho + \sum_{i\in E} v_i p_i\right)dt$$

where $\varphi \in \mathcal{K}(U)$ and $v_i \in \mathcal{K}(U)$ for each $i \in E$. We call such a form quasi-canonical. *It is unique up to residual gauge transformations as in Theorem 3.7.* □

We would like to understand the behaviour of such quasi-canonical representatives under changes in local coordinate. We will come back to this in §6.2 below. The first problem is to formulate the definition of $\mathrm{Op}_{{}^L\mathfrak{g}}(U)$ itself in a coordinate-independent fashion. Indeed, suppose $s : U \to \mathbb{C}$ is another holomorphic coordinate on the same open set $U \subset \Sigma$, with $t = \mu(s)$. The connection in (6.1) becomes

$$\nabla = d + p_{-1}\mu'(s)ds + b\mu'(s)ds. \tag{6.3}$$

This is no longer of the form (6.1) in the new coordinate s, and in this sense the definition of $\mathrm{op}_{{}^L\mathfrak{g}}(U)$ is coordinate dependent. However, it is possible to re-express $\mathrm{Op}_{{}^L\mathfrak{g}}(U)$ as the quotient of a suitably larger affine space of connections $\widetilde{\mathrm{op}}_{{}^L\mathfrak{g}}(U) \supset \mathrm{op}_{{}^L\mathfrak{g}}(U)$, which itself is coordinate *independent*, by some larger group of gauge transformations ${}^L\hat{B}_+(\mathcal{K}(U)) \supset {}^L\hat{N}_+(\mathcal{K}(U))$ to be defined below.

Indeed, let $\widetilde{\mathrm{op}}_{{}^L\mathfrak{g}}(U)$ be the affine space consisting of all connections of the form

$$\widetilde{\nabla} = d + \left(\sum_{i=0}^{\ell} \psi_i \check{f}_i + b\right)dt \tag{6.4}$$

with ψ_i a nonzero element of $\mathcal{K}(U)$ for each $i \in I$, and $b \in {}^L\hat{\mathfrak{b}}_+(\mathcal{K}(U))$. Observe that the definition of $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$ is independent of the choice of coordinate. (The derivative μ' in (6.3) belongs to $\mathcal{K}(U)$ since it is holomorphic and non-vanishing on U.)

Now we define the group ${}^L\hat{B}_+(\mathcal{K}(U))$ and its action on $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$ by gauge transformations. First, let $P := \bigoplus_{i=0}^{\ell} \mathbb{Z}\Lambda_i \subset {}^L\mathfrak{h}$ denote the lattice of integral coweights of ${}^L\mathfrak{g}$, where $\{\Lambda_i\}_{i=0}^{\ell}$ are the fundamental coweights of ${}^L\mathfrak{g}$ defined in (2.3). Let ${}^LH(\mathcal{K}(U))$ denote the abelian group generated by elements of the form ϕ^λ, $\phi \in \mathcal{K}(U) \setminus \{0\}$, $\lambda \in P$, subject to the relations $\phi^\lambda \psi^\lambda = (\phi\psi)^\lambda$, $\phi^{\lambda+\mu} = \phi^\lambda \phi^\mu$ for all $\phi, \psi \in \mathcal{K}(U) \setminus \{0\}$ and $\lambda, \mu \in P$. (Note that this definition makes sense for *any* open subset $U \subset \Sigma$, but to describe the action of the group ${}^LH(\mathcal{K}(U))$ on $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$ we shall only need the case when U is a coordinate chart.)

For each $\check{\alpha} \in \check{Q}$ in the root lattice of ${}^L\mathfrak{g}$, we have the (adjoint) action of the group ${}^LH(\mathcal{K}(U))$ on the space ${}^L\mathfrak{g}_{\check{\alpha}}(\mathcal{K}(U))$ of meromorphic functions on U valued in the root space ${}^L\mathfrak{g}_{\check{\alpha}}$, given by

$$\phi^\lambda n \phi^{-\lambda} := \phi^{\langle \lambda, \check{\alpha} \rangle} n, \tag{6.5}$$

for all $n \in {}^L\mathfrak{g}_{\check{\alpha}}(\mathcal{K}(U))$, $\phi \in \mathcal{K}(U) \setminus \{0\}$ and $\lambda \in P$. Here $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$, by definition of P, so that $\phi^{\langle \lambda, \check{\alpha} \rangle} \in \mathcal{K}(U)$. Hence we get an action on the Lie algebra ${}^L\hat{\mathfrak{n}}_+(\mathcal{K}(U))$. Then ${}^LH(\mathcal{K}(U))$ acts also on the group ${}^L\hat{N}_+(\mathcal{K}(U))$, with $\phi^\lambda \exp(n) \phi^{-\lambda} := \exp(\phi^\lambda n \phi^{-\lambda})$. We may now define the desired group to be the semi-direct product

$${}^L\hat{B}_+(\mathcal{K}(U)) := {}^L\hat{N}_+(\mathcal{K}(U)) \rtimes {}^LH(\mathcal{K}(U)).$$

That is, ${}^L\hat{B}_+(\mathcal{K}(U))$ is the group generated by elements of the form $\exp(n)\phi^\lambda$ with $n \in {}^L\hat{\mathfrak{n}}_+(\mathcal{K}(U))$, $\phi \in \mathcal{K}(U) \setminus \{0\}$ and $\lambda \in P$, with the group product given by

$$(\exp(n)\phi^\lambda)(\exp(m)\psi^\mu) := \left(\exp(n) \exp\left(\phi^\lambda m \phi^{-\lambda}\right)\right)\left(\phi^\lambda \psi^\mu\right),$$

for any $m, n \in {}^L\hat{\mathfrak{n}}_+(\mathcal{K}(U))$, $\phi, \psi \in \mathcal{K}(U) \setminus \{0\}$ and $\lambda, \mu \in P$.

Finally, we define the gauge action of ${}^LH(\mathcal{K}(U))$ on connections in $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$, of the form (6.4), by

$$\phi^\lambda \left(d + \sum_{i=0}^{\ell} \psi_i \check{f}_i dt + b dt \right) \phi^{-\lambda} := d + \sum_{i=0}^{\ell} \phi^{-\langle \lambda, \check{\alpha}_i \rangle} \psi_i \check{f}_i dt - \lambda \phi^{-1} d\phi + \phi^\lambda b \phi^{-\lambda} dt, \tag{6.6}$$

where, again, $\phi^\lambda b \phi^{-\lambda}$ is defined by extending (6.5) to ${}^L\hat{\mathfrak{b}}_+(\mathcal{K}(U))$ by linearity.

Lemma 6.2. *Equation* (6.6) *defines an action of the group* ${}^LH(\mathcal{K}(U))$ *on the space of connections* $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$. *Combining it with the action of* ${}^L\hat{N}_+(\mathcal{K}(U))$ *defined as in §3.3, we obtain a well-defined action of* ${}^L\hat{B}_+(\mathcal{K}(U))$ *on* $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$. □

Lemma 6.3. *The space of meromorphic* ${}^L\mathfrak{g}$*-opers on a coordinate chart* U *is equal to the quotient of* $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$ *by this gauge action of* ${}^L\hat{B}_+(\mathcal{K}(U))$*:*

$$\mathrm{Op}_{L_\mathfrak{g}}(U) = \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U) \big/ {}^L\hat{B}_+(\mathcal{K}(U)).$$

Proof. Let $\widetilde{\nabla} \in \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)$ be as in (6.4). On inspecting (6.6), we see that its ${}^LH(\mathcal{K}(U))$-orbit has a unique representative in $\mathrm{op}_{L_\mathfrak{g}}(U)$, namely $(\prod_{i=0}^{\ell} \psi_i^{\Lambda_i}) \widetilde{\nabla} (\prod_{i=0}^{\ell} \psi_i^{-\Lambda_i})$. □

Remark 6.4. If we were to replace P by $P \oplus \mathbb{C}\delta$ in the definition of ${}^L\tilde{H}(\mathcal{K}(U))$ then the quotient $\widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U)/{}^L\hat{B}_+(\mathcal{K}(U))$ would be smaller than $\mathrm{Op}_{L_\mathfrak{g}}(U)$; in fact it would be isomorphic to $\mathrm{Op}_{L_{\mathfrak{g}/\mathbb{C}\delta}}(U)$. ◁

Now suppose $U \subset \Sigma$ is *any* open subset, not necessarily a coordinate chart. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of U by coordinate charts, *i.e.* open subsets $U_\alpha \subset \Sigma$ for each α in some indexing set A with holomorphic coordinates $t_\alpha : U_\alpha \to \mathbb{C}$ such that $U = \cup_{\alpha\in A} U_\alpha$. We define $\mathrm{Op}_{L_\mathfrak{g}}(U)$ to be the set of collections $\{[\nabla_\alpha] \in \mathrm{Op}_{L_\mathfrak{g}}(U_\alpha)\}_{\alpha\in A}$ with the following property: for any pair of overlapping charts $U_\alpha \cap U_\beta \neq \emptyset$ and any choice of representatives $\nabla_\alpha \in \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U_\alpha)$ and $\nabla_\beta \in \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U_\beta)$, their restrictions $\nabla_\alpha|_{U_\alpha\cap U_\beta}$ and $\nabla_\beta|_{U_\alpha\cap U_\beta}$ define the same class in $\mathrm{Op}_{L_\mathfrak{g}}(U_\alpha \cap U_\beta)$. That is, the pair of representatives $\nabla_\alpha \in \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U_\alpha)$ and $\nabla_\beta \in \widetilde{\mathrm{op}}_{L_\mathfrak{g}}(U_\beta)$ considered on the overlap $U_\alpha \cap U_\beta$ are related by a gauge transformation in ${}^L\hat{B}_+(\mathcal{K}(U_\alpha \cap U_\beta))$. Since ${}^L\hat{B}_+(\mathcal{K}(U_\alpha))$ and ${}^L\hat{B}_+(\mathcal{K}(U_\beta))$ are naturally subgroups of ${}^L\hat{B}_+(\mathcal{K}(U_\alpha \cap U_\beta))$, the above property does not depend on the choice of representatives of the ${}^L\mathfrak{g}$-opers $[\nabla_\alpha] \in \mathrm{Op}_{L_\mathfrak{g}}(U_\alpha)$ for each $\alpha \in A$. This defines the *sheaf of* ${}^L\mathfrak{g}$*-opers* $\mathrm{Op}_{L_\mathfrak{g}}$.

6.2. **Quasi-canonical form.** Let $U \subset \Sigma$ be open and $[\nabla] \coloneqq \{[\nabla_\alpha] \in \mathrm{Op}_{L_\mathfrak{g}}(U_\alpha)\}_{\alpha\in A}$ be an element of $\mathrm{Op}_{L_\mathfrak{g}}(U)$. Call $\widetilde{\nabla} \coloneqq \{\widetilde{\nabla}_\alpha \in \mathrm{op}_{L_\mathfrak{g}}(U_\alpha)\}_{\alpha\in A}$ a representative of $[\nabla]$ if $[\widetilde{\nabla}_\alpha] = [\nabla_\alpha]$ for each $\alpha \in A$. We shall say that this representative is in *quasi-canonical form* if for each $\alpha \in A$, $\widetilde{\nabla}_\alpha$ is a quasi-canonical form as in Theorem 6.1 with respect to the local coordinate $t_\alpha : U_\alpha \to \mathbb{C}$, *i.e.* for each $\alpha \in A$ we have

$$\widetilde{\nabla}_\alpha = d + \left(p_{-1} - \frac{\varphi_\alpha(t_\alpha)}{h^\vee}\rho + \sum_{i\in E} v_{\alpha,i}(t_\alpha) p_i \right) dt_\alpha \tag{6.7}$$

for some $\varphi_\alpha \in \mathcal{K}(U_\alpha)$ and $v_{\alpha,i} \in \mathcal{K}(U_\alpha)$, $i \in E$. In this section we identify which sheaves the collections of functions $\{\varphi_\alpha\}_{\alpha\in A}$ and $\{v_{\alpha,i}\}_{\alpha\in A}$, $i \in E$ define sections of.

It suffices to consider an open subset $U \subset \Sigma$ equipped with a pair of holomorphic coordinates $t : U \to \mathbb{C}$ and $s : U \to \mathbb{C}$, and to determine the gauge transformation parameter in ${}^L\hat{B}_+(\mathcal{K}(U))$ relating quasi-canonical forms in each coordinate. In the above notation, U corresponds to the overlap $U_\alpha \cap U_\beta$ of the open sets U_α and U_β with coordinates $t = t_\alpha : U_\alpha \to \mathbb{C}$ and $s = t_\beta : U_\beta \to \mathbb{C}$, respectively. So suppose that we start with a representative of an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{L_\mathfrak{g}}(U)$ which is in quasi-canonical form in the t coordinate, as in Theorem 6.1:

$$\nabla = d + p_{-1}dt - \frac{\varphi(t)}{h^\vee}\rho dt + \sum_{i\in E} v_i(t) p_i dt.$$

In terms of the other coordinate s with $t = \mu(s)$ we have

$$\nabla = d + p_{-1}\mu'(s)ds - \frac{\varphi(\mu(s))}{h^\vee}\rho\mu'(s)ds + \sum_{i\in E} v_i(\mu(s)) p_i \mu'(s) ds.$$

This can be brought into quasi-canonical form in the s coordinate by performing a gauge transformation by $\mu'(s)^\rho \in {}^L H(\mathcal{K}(U))$. Indeed, one finds that

$$\mu'(s)^\rho \nabla \mu'(s)^{-\rho} = d + p_{-1}ds - \frac{\tilde{\varphi}(s)}{h^\vee}\rho ds + \sum_{i\in E} \tilde{v}_i(s) p_i ds$$

making use of the second relation in (2.8), and where we defined

$$-\frac{1}{h^\vee}\tilde{\varphi}(s) \coloneqq -\frac{1}{h^\vee}\varphi(\mu(s))\mu'(s) - \frac{\mu''(s)}{\mu'(s)}, \tag{6.8a}$$

$$\tilde{v}_i(s) \coloneqq v_i(\mu(s))\mu'(s)^{i+1}, \qquad i \in E. \tag{6.8b}$$

Here we will interpret the transformation property (6.8a) as well as (6.8b) in the case $i = 1$. We will come back to (6.8b) for $i \in E_{\geq 2}$ in §6.4 below.

We shall need the following notation. For any $k \in \mathbb{Z}$, let us denote by $\Omega^k \coloneqq \Omega^{\otimes k}$ the kth tensor power[6] of the canonical line bundle (*i.e.* the cotangent bundle) Ω over Σ. We denote by $U \mapsto \Gamma(U, \Omega^k)$ the sheaf of meromorphic sections of Ω^k. Also let $U \mapsto \mathrm{Conn}(U, \Omega)$ denote the sheaf of meromorphic connections on Ω.

Theorem 6.5. *Let $U \subset \Sigma$ be open and ∇ be any quasi-canonical form of an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(U)$. The coefficient of ρ is the component of a connection in* $\mathrm{Conn}(U, \Omega)$.

Proof. The coefficient of ρ in ∇, or to be more precise the collection of coefficients of ρ for every $\nabla_\alpha \in \mathrm{op}_{{}^L\mathfrak{g}}(U_\alpha)$ where $\nabla = \{\nabla_\alpha\}_{\alpha\in A}$ relative to a cover $\{U_\alpha\}_{\alpha\in A}$ of U, is independent of the choice of quasi-canonical form ∇ by Lemma 3.6.

In the local trivialization defined by the coordinate t, a meromorphic section of Ω^k is given by a meromorphic function $f(t)$, and a connection $\Gamma(U, \Omega^k) \to \Gamma(U, \Omega \otimes \Omega^k)$ is a differential operator $f(t) \mapsto f'(t) + A(t)f(t)$, specified by a meromorphic function $A(t)$, the component of the connection in this local trivialization. Here $f'(t) + A(t)f(t)$ must transform as a section of $\Omega \otimes \Omega^k$, which is to say that

$$\tilde{f}'(s) + \tilde{A}(s)\tilde{f}(s) = \big(f'(t) + A(t)f(t)\big)\,\mu'(s)^{k+1}.$$

Now in fact

$$\begin{aligned}\tilde{f}'(s) + \tilde{A}(s)\tilde{f}(s) &= \partial_s\big(f(t)\mu'(s)^k\big) + \tilde{A}(s)f(t)\mu'(s)^k \\ &= f'(t)\mu'(s)^{k+1} + kf(t)\mu'(s)^{k-1}\mu''(s) + \tilde{A}(s)f(t)\mu'(s)^k\end{aligned}$$

and we see that indeed A must transform as $\tilde{A}(s) = A(t)\mu'(s) - k\mu''(s)/\mu'(s)$. On comparing with (6.8a) the first result follows. □

6.3. **Interlude: Comparison with finite type opers.** Before proceeding, it is interesting to compare the transformation properties (6.8) with those of opers of finite type. To that end, we now briefly recall the situation for finite type opers. Suppose, for this subsection only, that ${}^L\mathfrak{g}$ is of finite type. Let $U \subset \Sigma$ be an open subset with

[6]Recall that over $\mathbb{P}^1$, Ω^k is defined for all $k \in \mathbb{Z}/2$. In particular $\Omega^{1/2}$ is the tautological line bundle. For our purposes with ${}^L\mathfrak{g}$ affine we shall need only integer powers.

two coordinates $t : U \to \mathbb{C}$ and $s : U \to \mathbb{C}$. Define $\mathrm{op}_{L\mathfrak{g}}(U)$ to be the affine space of connections of the form

$$\nabla \coloneqq d + \bar{p}_{-1}dt + bdt, \quad b \in {}^L\mathfrak{b}_+(\mathcal{K}(U)).$$

with $\bar{p}_{-1} \coloneqq \sum_{i=1}^{\ell} \check{f}_i$, and define $\widetilde{\mathrm{op}}_{L\mathfrak{g}}(U)$ to be the affine space of connections of the form

$$\widetilde{\nabla} \coloneqq d + \left(\sum_{i=1}^{\ell} \psi_i \check{f}_i + b \right) dt$$

with ψ_i a nonzero element of $\mathcal{K}(U)$ for each $i \in I \setminus \{0\}$, and with $b \in {}^L\mathfrak{b}_+(\mathcal{K}(U))$. (In finite type ${}^L\hat{\mathfrak{b}}_+ = {}^L\mathfrak{b}_+$ since ${}^L\mathfrak{n}_m = \{0\}$ for all $m \geq h^\vee$.) Then the definition of the space $\mathrm{Op}_{L\mathfrak{g}}(U)$ of ${}^L\mathfrak{g}$-opers on U in (6.2) and Lemma 6.3 remains correct as written. Let $\bar{\rho} \coloneqq \sum_{i=1}^{\ell} \bar{\Lambda}_i$ be the sum of the fundamental coweights $\bar{\Lambda}_i$ of ${}^L\mathfrak{g}$. There is a unique element $\bar{p}_1 \in {}^L\mathfrak{n}_+$ such that $\{\bar{p}_{-1}, 2\bar{\rho}, \bar{p}_1\}$ form an $\mathfrak{sl}_2$-triple:

$$[\bar{p}_{-1}, \bar{p}_1] = -2\bar{\rho}, \qquad [2\bar{\rho}, \bar{p}_{\pm 1}] = \pm 2\bar{p}_{\pm 1}. \tag{6.9}$$

The analogue of Theorem 6.1 in finite type is the following statement: Each gauge equivalence class $[\nabla] \in \mathrm{Op}_{L\mathfrak{g}}(U)$ contains a unique representative of the form

$$\nabla = d + \bar{p}_{-1}dt + \sum_{i \in \bar{E}} \bar{v}_i(t)\bar{p}_i dt, \tag{6.10}$$

where the (multi)set $\bar{E}$ of exponents is now finite and where, for each $i \in \bar{E}$ we have $\bar{v}_i \in \mathcal{K}(U)$ and $\bar{p}_i \in {}^L\mathfrak{n}_+$ are elements such that

$$[\bar{p}_1, \bar{p}_i] = 0, \qquad [\bar{\rho}, \bar{p}_i] = i\bar{p}_i. \tag{6.11}$$

In terms of the new coordinate s with $t = \mu(s)$ we have

$$\nabla = d + \bar{p}_{-1}\mu'(s)ds + \sum_{i \in \bar{E}} \bar{v}_i(\mu(s))\bar{p}_i\mu'(s)ds.$$

Similarly to the affine case above, we may first perform a gauge transformation by $\mu'(s)^{\bar{\rho}} \in {}^LH(\mathcal{K}(U))$ to bring the $\bar{p}_{-1}$ term into the canonical form $\bar{p}_{-1}ds$, namely

$$\mu'(s)^{\bar{\rho}} \nabla \mu'(s)^{-\bar{\rho}} = d + \bar{p}_{-1}ds - \bar{\rho}\,\frac{\mu''(s)}{\mu'(s)}ds + \sum_{i \in \bar{E}} \bar{v}_i(\mu(s))\bar{p}_i\mu'(s)^{i+1}ds.$$

However, in contrast to the affine case, we are not yet done, because it is necessary – in order to reach the canonical form – to remove the $\bar{\rho}$ term by performing a further gauge transformation by $\exp\left(\bar{p}_1 \frac{\mu''(s)}{2\mu'(s)}\right)$. One finds that

$$\exp\left(\bar{p}_1\frac{\mu''(s)}{2\mu'(s)}\right) \mu'(s)^{\bar{\rho}} \nabla \mu'(s)^{-\bar{\rho}} \exp\left(-\bar{p}_1\frac{\mu''(s)}{2\mu'(s)}\right) = d + \bar{p}_{-1}ds + \sum_{i \in \bar{E}} \tilde{\bar{v}}_i(s)\bar{p}_i ds \tag{6.12}$$

using both the relations (6.11) and (6.9), and where we defined

$$\tilde{\bar{v}}_1(s) \coloneqq \bar{v}_1(\mu(s))\mu'(s)^2 - \tfrac{1}{2}(S\mu)(s), \tag{6.13a}$$

$$\tilde{\bar{v}}_i(s) \coloneqq \bar{v}_i(\mu(s))\mu'(s)^{i+1}, \qquad i \in \bar{E}, i > 1. \tag{6.13b}$$

Here $S\mu$ is the *Schwarzian derivative* of μ,

$$S\mu := \frac{\mu'''}{\mu'} - \frac{3}{2}\left(\frac{\mu''}{\mu'}\right)^2.$$

Now, what (6.13) shows is that each of the $\bar{v}_i$, $i > 1$ transforms as a section of the power Ω^{i+1} of the canonical bundle, but that $\bar{v}_1$ transforms as a *projective connection*; see [Fre04, Fre07]. Since (6.10) was the unique canonical form of the ${}^L\mathfrak{g}$-oper in this chart, that means there is an isomorphism, in finite type ${}^L\mathfrak{g}$,

$$\mathrm{Op}_{{}^L\mathfrak{g}}(U) \simeq \mathrm{Proj}(U) \times \prod_{\substack{i\in E\\ i>1}} \Gamma(U, \Omega^{i+1}), \tag{6.14}$$

where $\mathrm{Proj}(U)$ denotes the space of projective connections on U.

Remark 6.6. Let us emphasise why, in the affine case, there is no analogue of the second gauge transformation by $\exp\left(\bar{p}_1 \frac{\mu''(s)}{2\mu'(s)}\right)$ performed above. When ${}^L\mathfrak{g}$ is of affine type it is the central element δ (and not ρ) which appears in the bracket $[p_1, p_{-1}] = \delta$. The derivation element ρ is not in the derived subalgebra. Hence, as we saw in Lemma 3.6, there is no way to remove the term $-\varphi/h^\vee \rho\, dt$ from a quasi-canonical form via ${}^L\hat{N}_+(\mathcal{K}(U))$-valued gauge transformations. Rather, the twist function φ forms part of the data defining the underlying ${}^L\mathfrak{g}$-oper (and Theorem 6.5 gives its properties under coordinate transformations). ◁

6.4. **Twisted cohomologies.** Now we return to the case in which ${}^L\mathfrak{g}$ is of affine type. We would like to give a coordinate-independent description of the space of affine opers analogous to (6.14) in finite types.

Let $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}}(U)$ be a meromorphic ${}^L\mathfrak{g}$-oper on an open subset $U \subset \Sigma$ and let ∇ be a representative in quasi-canonical form. According to Theorem 6.5, the coefficient of ρ in ∇ defines a (trivially flat, since we are working on a curve) connection on Ω over U, which we denote by

$$\nabla|_\rho : \Gamma(U, \Omega) \longrightarrow \Gamma(U, \Omega \otimes \Omega),$$

If $t : U \to \mathbb{C}$ is a coordinate on U then it can be written as $f \mapsto \nabla|_\rho f = df - h^{\vee -1}\varphi f dt$ for some $\varphi \in \mathcal{K}(U)$. We therefore obtain a surjective map

$$\mathrm{Op}_{{}^L\mathfrak{g}}(U) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Conn}(U, \Omega), \qquad [\nabla] \longmapsto \nabla|_\rho \tag{6.15}$$

into the space of meromorphic connections on Ω over U. Given any $\overline{\nabla} \in \mathrm{Conn}(U, \Omega)$, we denote its preimage in $\mathrm{Op}_{{}^L\mathfrak{g}}(U)$ under the map (6.15) by $\mathrm{Op}_{{}^L\mathfrak{g}}(U)^{\overline{\nabla}}$. This can be seen as a coordinate-independent version, over an open subset of an arbitrary curve $U \subset \Sigma$, of the space $\mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$ introduced in §3.4.

For each $j \in \mathbb{Z}$, $\nabla|_\rho$ induces a connection on the line bundle Ω^j,

$$\nabla|_\rho : \Gamma(U, \Omega^j) \longrightarrow \Gamma(U, \Omega \otimes \Omega^j).$$

In a local coordinate $t : U \to \mathbb{C}$ it takes the form $f \mapsto \nabla|_\rho f = df - jh^{\vee -1}\varphi f dt$. The transformation property (6.8b) suggests that the coefficient of p_j in ∇, for each $j \in E$, defines a section of Ω^{j+1}. This is indeed the case for $j = 1$ since the coefficient of p_1

in ∇, *i.e.* the collection of coefficients of p_1 for every $\nabla_\alpha \in \mathrm{op}_{L\mathfrak{g}}(U_\alpha)$, is independent of the choice of quasi-canonical form ∇ by Proposition 3.10. However, it is not quite true for $j \in E_{\geq 2}$ since, unlike the coefficient φ_α of ρ and $v_{\alpha,1}$ of p_1 in (6.7), the coefficient $v_{\alpha,j}$ of p_j, $j \in E_{\geq 2}$ in each chart U_α depends on the choice of quasi-canonical representative ∇_α by Theorem 6.1. More precisely, as formulated in Theorem 3.7, the coefficient of p_j in ∇ is defined up to the addition of an element $\nabla|_\rho f$.

Recall that a *local system* is a vector bundle equipped with a flat connection. In our case, for each $j \in E$ we have the local system $(\Omega^j, \nabla|_\rho)$ consisting of the line bundle Ω^j equipped with the connection $\nabla|_\rho$. Given any local system one has the associated de Rham complex with coefficients in that local system. In our case it is

$$0 \longrightarrow \Gamma(U, \Omega^j) \xrightarrow{\nabla|_\rho} \Gamma(U, \Omega \otimes \Omega^j) \longrightarrow 0.$$

The *de Rham cohomology of U with coefficients in* $(\Omega^j, \nabla|_\rho)$, or simply the *twisted de Rham cohomology*, is then by definition the quotient

$$H^1(U, \Omega^j, \nabla|_\rho) \coloneqq \Gamma(U, \Omega \otimes \Omega^j) / \nabla|_\rho \Gamma(U, \Omega^j).$$

We now see that, for each $j \in E_{\geq 2}$, the coefficient of p_j in ∇ defines an element of this twisted de Rham cohomology which is independent of the choice of quasi-canonical form ∇. In other words, we have the following analogue of Theorem 6.5.

Proposition 6.7. *Let $U \subset \Sigma$ be open and ∇ be any quasi-canonical form of an ${}^L\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{L\mathfrak{g}}(U)$. For each $j \in E_{\geq 2}$, the coefficient of p_j belongs to $H^1(U, \Omega^j, \nabla|_\rho)$. The coefficient of p_1 belongs to $\Gamma(U, \Omega^2)$.* □

Combining this with Theorem 6.5 we arrive at the following.

Theorem 6.8. *For any open subset $U \subset \Sigma$, the space $\mathrm{Op}_{L\mathfrak{g}}(U)$ fibres over $\mathrm{Conn}(U, \Omega)$ and we have the isomorphism*

$$\mathrm{Op}_{L\mathfrak{g}}(U)^{\overline{\nabla}} \simeq \Gamma(U, \Omega^2) \times \prod_{j \in E_{\geq 2}} H^1(U, \Omega^j, \overline{\nabla})$$

for the fibre over any connection $\overline{\nabla} \in \mathrm{Conn}(U, \Omega)$. □

Remark 6.9. Recall Corollary 3.9. One can also define the sheaf $\mathrm{Op}_{L\mathfrak{g}/\mathbb{C}\delta}$ of $({}^L\mathfrak{g}/\mathbb{C}\delta)$-opers over Σ, and one has the analogue of the above theorem, with

$$\mathrm{Op}_{L\mathfrak{g}/\mathbb{C}\delta}(U)^{\overline{\nabla}} \simeq \prod_{j \in E} H^1(U, \Omega^j, \overline{\nabla})$$

for every $\overline{\nabla} \in \mathrm{Conn}(U, \Omega)$. ◁

In the present paper, our main interest lies in ${}^L\mathfrak{g}$-opers over $\mathbb{P}^1$. What we need is the analogous statement to Theorem 6.8 for the space $\mathrm{Op}^{\mathrm{reg}}_{L\mathfrak{g}}(\mathbb{P}^1)^\varphi_X$ of global meromorphic ${}^L\mathfrak{g}$-opers which are holomorphic on the complement $X = \mathbb{C} \setminus \{z_i\}_{i=1}^N$ of the set of marked points, as defined in §4.2.

For every $j \in E$, let us denote by $\Gamma_X(\mathbb{P}^1, \Omega^{j+1})$ the space of global meromorphic sections of Ω^{j+1} that are holomorphic on X. Let $H^1_X(\mathbb{P}^1, \Omega^j, \nabla|_\rho)$ be the corresponding twisted cohomologies. Also let $\mathrm{Conn}_X(\mathbb{P}^1, \Omega)$ denote the space of global meromorphic connections on Ω which are holomorphic on X.

Theorem 6.10. $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)_X$ *fibres over* $\mathrm{Conn}_X(\mathbb{P}^1,\Omega)$ *and we have the isomorphism*

$$\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^{\overline{\nabla}}_X \simeq \Gamma_X(\mathbb{P}^1,\Omega^2) \times \prod_{j\in E_{\geq 2}} H^1_X(\mathbb{P}^1,\Omega^j,\overline{\nabla})$$

for the fibre over any $\overline{\nabla} \in \mathrm{Conn}_X(\mathbb{P}^1,\Omega)$. □

7. Twisted homology and the integral pairing

Recall the integrals from Corollary 3.13. With Theorem 6.10 in hand we can give a coordinate-independent description of functions on $\mathrm{Op}^{\mathrm{reg}}_{L_\mathfrak{g}}(\mathbb{P}^1)^{\overline{\nabla}}_X$.

7.1. **Twisted homology.** Suppose $U \subset X$ is an open subset with a holomorphic coordinate $t : U \to \mathbb{C}$. In the trivialization defined by the coordinate t, a horizontal section of the local system $(\Omega^j, \nabla|_\rho)$, for any $j \in \mathbb{Z}$, is given by a holomorphic function f on U such that $df - jh^{\vee -1}\varphi f dt = 0$. Recall the multivalued holomorphic function $\mathcal{P}$ on the complement $X = \mathbb{C} \setminus \{z_i\}_{i=1}^N$ from (3.14): concretely, f is a constant multiple of a univalued branch of $\mathcal{P}^{j/h^\vee}$ over U.

Recall that a *singular p-simplex in X* is a continuous map $\sigma : \Delta_p \to X$ from the standard p-simplex Δ_p to X. (We shall need only $p \in \{0,1,2\}$.) Define a *twisted p-simplex in X of degree j* to be a pair $\sigma \otimes f$ consisting of a singular p-simplex σ in X together with a horizontal section f of $(\Omega^j, \nabla|_\rho)$ over an open neighbourhood of $\sigma(\Delta_p)$ in X. A *twisted p-chain of degree j* is then a finite formal sum

$$\gamma = \sum_k \sigma_k \otimes g_k$$

of twisted p-simplices in X of degree j. Let $C_p(X, \Omega^j, \nabla|_\rho)$ be the (infinite-dimensional) complex vector space of twisted p-chains in X of degree j, where scalar multiplication of chains is by scalar multiplication of the horizontal sections. Recall that the usual boundary operator sends a singular p-simplex σ to $\sum_{k=0}^p (-1)^k s_k$ where s_k is the restriction of σ to the kth face of Δ_p (which is canonically identified with Δ_{p-1}). In our twisted setting, the boundary operator ∂ is the linear map

$$\partial : C_p(X, \Omega^j, \nabla|_\rho) \longrightarrow C_{p-1}(X, \Omega^j, \nabla|_\rho), \tag{7.1}$$

defined by $\sigma \otimes f \mapsto \sum_{k=0}^p (-1)^k s_k \otimes f_k$ where f_k is the restriction of f to an open neighbourhood of $s_k(\Delta_{p-1})$ in X. The property $\partial^2 = 0$ follows from the same property in the usual setting. The kernel of the map (7.1), which we denote by $Z_p(X, \Omega^j, \nabla|_\rho)$, is the space of *closed* twisted p-chains of degree j. The *twisted homology of X* is then the quotient of vector spaces

$$H_1(X, \Omega^j, \nabla|_\rho) \coloneqq Z_1(X, \Omega^j, \nabla|_\rho)/\partial C_2(X, \Omega^j, \nabla|_\rho).$$

and its elements are *twisted cycles of degree j*.

7.2. **Twisted de Rham theorem.** Let $\omega \in \Gamma_X(\mathbb{P}^1, \Omega \otimes \Omega^j)$ and (σ, f) be a twisted 1-simplex in X of degree $-j$, for any $j \in E$. On an open neighbourhood U of $\sigma(\Delta_1)$ we have the holomorphic 1-form $f\omega \in \Gamma_X(U, \Omega)$. Define the integral of ω over $\sigma \otimes f$ to be the usual integral of $f\omega$ over the singular 1-simplex σ:

$$\int_{\sigma\otimes f} \omega := \int_\sigma f\omega.$$

Extending by linearity we have the integral $\int_\gamma \omega$ over any $\gamma \in C_1(X, \Omega^{-j}, \nabla|_\rho)$.

In the same way, one defines the integral of a 0-form $\omega \in \Gamma_X(\mathbb{P}^1, \Omega^j)$ over a twisted 0-chain $\gamma \in C_0(X, \Omega^{-j}, \nabla|_\rho)$; and also in principle of a 2-form over a twisted 2-chain, although of course since we are on a curve the only meromorphic 2-form we have to integrate is the zero 2-form.

Proposition 7.1 (Twisted Stokes's theorem)**.** *Let $p \in \{0, 1\}$. For any p-form $\omega \in \Gamma_X(\mathbb{P}^1, \Omega^{\wedge p} \otimes \Omega^j)$ and any twisted $(p+1)$-chain $\gamma \in C_{p+1}(X, \Omega^{-j}, \nabla|_\rho)$,*

$$\int_\gamma \nabla|_\rho \omega = \int_{\partial\gamma} \omega.$$

Proof. By linearity it suffices to consider $\gamma = \sigma \otimes f$ with σ a singular $(p+1)$-simplex on X and f a horizontal section of Ω^{-j} over an open neighbourhood of $\sigma(\Delta_{p+1})$. We then have

$$\int_\gamma \nabla|_\rho \omega = \int_\sigma f\nabla|_\rho \omega = \int_\sigma d(f\omega) = \int_{\partial\sigma} f\omega = \int_{\partial(\sigma\otimes f)} \omega = \int_{\partial\gamma} \omega,$$

where in the second equality we used $d(f\omega) = (\nabla|_\rho f)\omega + f\nabla|_\rho\omega = f\nabla|_\rho\omega$ (since f is horizontal) and in the third equality we used the usual Stokes's theorem. □

Proposition 7.2 (Complex de Rham theorem)**.** *There is a bilinear pairing between twisted homologies and cohomologies, given by integrating twisted forms over twisted chains:*

$$H_i(X, \Omega^{-j}, \nabla|_\rho) \times H^i_X(\mathbb{P}^1, \Omega^j, \nabla|_\rho) \longrightarrow \mathbb{C}, \qquad (\gamma, \omega) \longmapsto \int_\gamma \omega.$$

for $i \in \{0, 1\}$ and any $j \in E$.

Proof. The fact that the integral pairing between forms and chains descends to a well-defined pairing between homologies and cohomologies follows from the twisted version of Stokes's theorem above. □

8. Discussion

8.1. **Smooth opers, Drinfel'd-Sokolov and (m)KdV.** The present paper concerned the role of affine opers in describing the spectrum of a (conjectured) family of higher Hamiltonians for affine Gaudin models. Affine Miura opers also play a conceptually quite distinct role in mathematical physics: namely they serve as the phase space of classical generalized mKdV theories. In the latter context, the procedure of Theorem 3.7 for putting an affine oper into quasi-canonical form essentially appears

in the paper of Drinfel'd and Sokolov [DS85, §6]. Specifically, if one replaces meromorphic functions with smooth functions on the circle, and – crucially – if one sets the twist function φ to zero, then our procedure in §3.5 coincides with the procedure of [DS85] to construct the densities of Hamiltonians of the classical generalised ${}^L\mathfrak{g}$-(m)KdV hierarchy. In what follows we elaborate on this last statement, and contrast the two settings.

In our earlier definition of ${}^L\mathfrak{g}$-opers from §3 and of Miura ${}^L\mathfrak{g}$-opers from §4.1, one can replace the algebra $\mathcal{M}$ of meromorphic functions on $\mathbb{P}^1$ by the algebra $C^\infty(S^1,\mathbb{C})$ of smooth functions on the circle. On doing so, we obtain the spaces of smooth ${}^L\mathfrak{g}$-opers $\mathrm{Op}_{{}^L\mathfrak{g}}(S^1)$ and Miura ${}^L\mathfrak{g}$-opers $\mathrm{MOp}_{{}^L\mathfrak{g}}(S^1)$ on S^1. Furthermore, one may also consider the spaces of smooth ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers $\mathrm{Op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)$ and Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)$, cf. Corollary 3.9 and §5.3. The phase space of ${}^L\mathfrak{g}$-mKdV can then be identified with the set $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ of Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-opers with zero twist function $\varphi=0$. Indeed, let $\sigma: S^1\to(0,2\pi)$ denote the natural coordinate on the circle $S^1=\mathbb{R}/2\pi\mathbb{Z}$. Then a connection $\nabla\in\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ takes the form

$$\nabla = d + p_{-1}d\sigma + \sum_{i=1}^{\ell} u_i(\sigma)\alpha_i d\sigma \tag{8.1}$$

where $u_i(\sigma)\in C^\infty(S^1,\mathbb{C})$ are smooth functions on the circle, the *classical ${}^L\mathfrak{g}$-mKdV fields*. Recalling that ${}^L\mathfrak{h}'$ is the span of the simple roots $\{\alpha_i\}_{i=0}^{\ell}$, here we implicitly identify the quotient ${}^L\mathfrak{h}'/\mathbb{C}\delta$ with the span of the subset $\{\alpha_i\}_{i=1}^{\ell}$.

To go from mKdV to KdV we first need some definitions. Let $\overline{{}^L\mathfrak{n}_+}$ be the finite-dimensional nilpotent Lie subalgebra of ${}^L\mathfrak{n}_+$ generated by $\check{e}_i$, $i=1,\dots,\ell$. We may form the infinite-dimensional nilpotent subalgebra $\overline{{}^L\mathfrak{n}}_+(C^\infty(S^1,\mathbb{C}))$ of the Lie algebra ${}^L\hat{\mathfrak{n}}_+(C^\infty(S^1,\mathbb{C}))$ defined as a completion of ${}^L\mathfrak{n}_+\otimes C^\infty(S^1,\mathbb{C})$ as in §3.1. The Baker-Campbell-Hausdorff formula (3.1) then endows the vector space $\overline{{}^L\mathfrak{n}}_+(C^\infty(S^1,\mathbb{C}))$ with the structure of a group which we denote by $\overline{{}^LN}_+(C^\infty(S^1,\mathbb{C}))$. This is a subgroup of ${}^L\hat{N}_+(C^\infty(S^1,\mathbb{C}))$ defined just as in §3.2 but with $\mathcal{M}$ replaced by $C^\infty(S^1,\mathbb{C})$.

The canonical map $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0\to\mathrm{Op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0, \nabla\mapsto[\nabla]$ factors through

$$\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0 \hookrightarrow \mathrm{op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0 \twoheadrightarrow \mathscr{M} := \mathrm{op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0\big/\overline{{}^LN}_+(C^\infty(S^1,\mathbb{C})). \tag{8.2}$$

The phase space of ${}^L\mathfrak{g}$-KdV is by definition the quotient space $\mathscr{M}$. Consider now the connection $\nabla\in\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ given in (8.1) which we write as

$$\nabla = d + \bar{p}_{-1}d\sigma + \check{f}_0 d\sigma + \sum_{i=1}^{\ell} u_i(\sigma)\alpha_i d\sigma \in \mathrm{op}_{{}^L\mathfrak{g}}(S^1)^0$$

with $\bar{p}_{-1}=\sum_{i=1}^{\ell}\check{f}_i$. Since $[\check{e}_i,\check{f}_0]=0$ for $i=1,\dots,\ell$ it follows that $\check{f}_0$ is invariant under the adjoint action of $\overline{{}^LN}_+(C^\infty(S^1,\mathbb{C}))$ on ${}^L\hat{\mathfrak{g}}(C^\infty(S^1,\mathbb{C}))$. Therefore, exactly as in the finite-dimensional setting recalled in §6.3, the image of ∇ under the map (8.2) has a unique representative of the form

$$d + p_{-1}d\sigma + \sum_{i\in\bar{E}} v_i(\sigma)\bar{p}_i d\sigma. \tag{8.3}$$

Here $\bar{p}_1$ denotes the unique element in $\overline{{}^L\mathfrak{n}_+}$ such that $\{\bar{p}_{-1}, 2\rho - 2h^\vee\Lambda_0, \bar{p}_1\} \subset {}^L\mathfrak{g}/\mathbb{C}\delta$ forms an $\mathfrak{sl}_2$-triple, and the $\bar{p}_i$, $i \in \bar{E}$ span the kernel of $\operatorname{ad}\bar{p}_1 : \overline{{}^L\mathfrak{n}_+} \to \overline{{}^L\mathfrak{n}_+}$. The smooth functions $v_i(\sigma) \in C^\infty(S^1, \mathbb{C})$ are the *classical ${}^L\mathfrak{g}$-KdV fields*.

Now Theorem 3.7, and in particular also Corollary 3.9, generalises to the smooth setting. Therefore, starting from the smooth Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper ∇ in (8.1) we obtain a quasi-canonical form

$$d + p_{-1}d\sigma + \sum_{i\in E} h_i(\sigma)p_i d\sigma \tag{8.4}$$

of the underlying smooth ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper $[\nabla] \in \mathrm{Op}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$. Let $g \in {}^L\hat{N}_+(C^\infty(S^1,\mathbb{C}))$ denote the gauge transformation parameter sending the Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper (8.1) to the quasi-canonical form (8.4). For any $i \in E$, the ith *mKdV flow* is given by

$$\frac{\partial}{\partial t_i}\nabla_{\partial_\sigma} = \big[(g^{-1}p_{-i}g)_+, \nabla_{\partial_\sigma}\big] \tag{8.5}$$

where ${}^L\mathfrak{g} \mapsto {}^L\mathfrak{n}_+$, $X \mapsto X_+$ is the canonical projection onto ${}^L\mathfrak{n}_+$ relative to the Cartan decomposition of ${}^L\mathfrak{g}$. Note that since p_{-i} commutes with $\mathfrak{a}_{\geq 2}$ in the quotient ${}^L\mathfrak{g}/\mathbb{C}\delta$, the expression $g^{-1}p_{-i}g$ does not depend on the ambiguity in the quasi-canonical form described in Theorem 3.7. Now the right hand side of (8.5) takes values in ${}^L\mathfrak{h}'/\mathbb{C}\delta$ by [DS85, Lemma 6.7] so that equation (8.5) indeed defines a flow, for each $i \in E$, on the phase space $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ of ${}^L\mathfrak{g}$-mKdV. Furthermore, these flows are mutually commuting [DS85, Proposition 6.5] and define the *${}^L\mathfrak{g}$-mKdV hierarchy*. Similarly, one can also define commuting flows on the phase space $\mathscr{M}$ of ${}^L\mathfrak{g}$-KdV [DS85, §6.2], giving rise to the *${}^L\mathfrak{g}$-KdV hierarchy*.

The smooth functions $h_i(\sigma) \in C^\infty(S^1, \mathbb{C})$ appearing in the quasi-canonical form (8.4) are the *densities of the ${}^L\mathfrak{g}$-(m)KdV Hamiltonians*. Indeed, as in Theorem 3.7, the h_i are defined up to the addition of exact derivatives (now not twisted, since $\varphi = 0$). To get gauge-invariant functions we should integrate them over a cycle, and on the circle that leaves only one possibility. Thus, we obtain the following functions on the phase space $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ of ${}^L\mathfrak{g}$-mKdV, or alternatively on the phase space $\mathscr{M}$ of ${}^L\mathfrak{g}$-KdV:

$$\mathscr{H}_i := \int_0^{2\pi} h_i(\sigma)d\sigma, \qquad i \in E.$$

These are the *${}^L\mathfrak{g}$-(m)KdV Hamiltonians*. They are conserved quantities under the flows of both the ${}^L\mathfrak{g}$-mKdV hierarchy (8.5) and the ${}^L\mathfrak{g}$-KdV hierarchy [DS85, Proposition 6.6]. Moreover, they generate these flows with respect to certain Poisson brackets on $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^0$ and $\mathscr{M}$, respectively [DS85, Propositions 6.11 and 6.10].

It is interesting to note that the equation (8.5) also defines a flow on $\mathrm{MOp}_{{}^L\mathfrak{g}/\mathbb{C}\delta}(S^1)^\varphi$ for any choice of smooth function $\varphi \in C^\infty(S^1, \mathbb{C})$ since $\rho \notin {}^L\mathfrak{g}'$. In fact, one could define ${}^L\mathfrak{g}$-mKdV flows in the analytic setting of the present paper with non-zero twist function φ. In the case $\varphi = 0$, the ${}^L\mathfrak{g}$-(m)KdV flows have previously been discussed in the analytic setting in [VW14] when ${}^L\mathfrak{g} = \widehat{\mathfrak{sl}}_N$ and in [VWW14] for ${}^L\mathfrak{g}$ of type ${}^2A_2$.

8.2. **${}^L\mathfrak{g}$-(m)KdV on polygonal domains.** As recalled in §8.1, the affine space of Miura ${}^L\mathfrak{g}$-opers $\mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$ considered in the present article does not quite correspond to the phase space of ${}^L\mathfrak{g}$-mKdV. Indeed, for us the twist function φ plays a central role in characterising the affine Gaudin model, just as in the classical case [Vic].

In the present section we show that the simple class of rational Miura ${}^L\mathfrak{g}$-opers in $\mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$ introduced in §4.1 can, nevertheless, alternatively be described in terms of Miura ${}^L\mathfrak{g}$-opers of the mKdV-type as in (8.1), *i.e.* with zero twist function. The price to pay, however, is that one then needs to work with multivalued Miura ${}^L\mathfrak{g}$-opers defined over a polygonal region of the complex plane (whose shape now encodes the same information as the twist function did). For this reason we prefer to keep the twist function explicit and work with the space $\mathrm{MOp}_{{}^L\mathfrak{g}}(\mathbb{P}^1)^\varphi$.

Let us remark, in passing, that the situation described below is very reminiscent of the relation between the modified sinh-Gordon equation and the sinh-Gordon equation in the context of the massive ODE/IM correspondence [LZ10].

Recall the class of Miura ${}^L\mathfrak{g}$-opers of the form (4.3) introduced in §4.1. We consider the analogous class of Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper of the form

$$\nabla = d + p_{-1}dz + \widetilde{u}(z)dz - \frac{\varphi(z)}{h^\vee}\rho\, dz, \tag{8.6a}$$

where $\widetilde{u} \in ({}^L\mathfrak{h}'/\mathbb{C}\delta)(\mathcal{M})$ is the rational function valued in ${}^L\mathfrak{h}'/\mathbb{C}\delta$ defined as

$$\widetilde{u}(z) \coloneqq -\sum_{i=1}^{N} \frac{\dot{\lambda}_i}{z-z_i}dz + \sum_{i=1}^{\ell}\left(\sum_{j=1}^{m_i}\frac{1}{z-w_j^i} - \sum_{j=1}^{m_0}\frac{1}{z-w_j^0}\right)\alpha_i dz. \tag{8.6b}$$

Here, by comparison with the expression (4.3), we have split the sum over the simple poles at the Bethe roots into separate sums over the collection of Bethe roots w_j^i, $j = 1, \dots, m_i$ of the same colour $i \in I = \{0, \dots, \ell\}$. We also implicitly identify the subspace ${}^L\mathfrak{h}'/\mathbb{C}\delta$ of ${}^L\mathfrak{g}'/\mathbb{C}\delta$ with the span of the simple roots $\{\alpha_i\}_{i=1}^\ell$ and the subspaces ${}^L\mathfrak{g}_n/\mathbb{C}\delta$ for $n \neq 0$ with ${}^L\mathfrak{g}_n$, as we did in §8.1.

Recall from §3.6 that $\varphi(z) = \partial_z \log \mathcal{P}(z)$. Fix a collection of cuts $C \subset \mathbb{P}^1$ between the branch points z_i of the multivalued function $\mathcal{P}^{1/h^\vee}$ on the Riemann sphere $\mathbb{P}^1$ and let $\overline{\mathcal{M}}$ denote the field of meromorphic functions on $\mathbb{P}^1 \setminus C$. From now on we fix a branch of $\mathcal{P}^{1/h^\vee}$ which by abuse of notation we also denote by $\mathcal{P}^{1/h^\vee} \in \overline{\mathcal{M}}$.

By treating (8.6) as a Miura ${}^L\mathfrak{g}/\mathbb{C}\delta$-oper on $\mathbb{P}^1 \setminus C$ and working over the larger field $\overline{\mathcal{M}} \supset \mathcal{M}$ one can then remove the ρ term by performing a gauge transformation by $\mathcal{P}(z)^{-\rho/h^\vee} \in {}^L H(\overline{\mathcal{M}})$. Using the second relation in (2.8) with $n = -1$ we find

$$\widetilde{\nabla} \coloneqq \mathcal{P}(z)^{-\rho/h^\vee}\nabla\mathcal{P}(z)^{\rho/h^\vee} = d + p_{-1}\mathcal{P}(z)^{1/h^\vee}dz + \widetilde{u}(z)dz. \tag{8.7}$$

In order to bring $\widetilde{\nabla}$ back to the form of a Miura ${}^L\mathfrak{g}'/\mathbb{C}\delta$-oper, consider the new variable x defined as the indefinite integral

$$x \coloneqq \int \mathcal{P}(z)^{1/h^\vee}dz = \int \prod_{i=1}^{N}(z-z_i)^{k_i/h^\vee}dz, \tag{8.8}$$

where in the second equality we used the explicit form (3.14) of the function $\mathcal{P}$.

Suppose, for simplicity, that all the z_i, $i = 1, \dots, N$ are real. It is then convenient to relabel them so that they are ordered as $z_1 < z_2 < \dots < z_N$. For generic $k_i \in \mathbb{C}$, $i = 1, \dots, N$, a possible choice of cuts $C \subset \mathbb{P}^1$ is to take the following union of open intervals along the real axis in the z-plane

$$C = (-\infty, z_1) \cup \bigcup_{i=2}^{N-1} (z_i, z_{i+1}) \cup (z_N, \infty).$$

Let $x_i \in \mathbb{P}^1$, for each $i = 1, \dots, N$, be the image of z_i under the transformation (8.8), and let $x_{N+1} \in \mathbb{P}^1$ denote the image of $z = \infty$. Suppose the x_i, $i = 1, \dots, N+1$ are all distinct and the ordered set $(x_i)_{i=1}^N$ describes the adjacent vertices of a simple polygonal domain P in the x-plane, where one of the x_i's could be infinite. In this case, the transformation $z \mapsto x$ given in (8.8) defines a Schwartz-Christoffel mapping. It is a biholomorphic map $\mathbb{H} \to P$, *i.e.* a bijective holomorphic map whose inverse is also holomorphic, from the upper-half $\mathbb{H} \coloneqq \{z \in \mathbb{C} \,|\, \Im z \geq 0\}$ of the z-plane to the polygon P in the x-plane. Each interval (z_i, z_{i+1}) for $i = 1, \dots, N-1$ is sent to the straight edge from x_i to x_{i+1}, while the semi-infinite intervals $(-\infty, z_1)$ and (z_N, ∞) are sent to the edges connecting x_{N+1} to x_1 and x_N, respectively. The interior angles α_j of P at each of its vertices x_j, for $j = 1, \dots, N+1$, are given by

$$\alpha_i = \frac{k_i + h^\vee}{h^\vee}\pi, \quad \text{for } i = 1, \dots, N \qquad \text{and} \quad \alpha_{N+1} = -\frac{\sum_{i=1}^N k_i + h^\vee}{h^\vee}\pi.$$

Furthermore, the transformation $z \mapsto x$ maps the lower-half $\overline{\mathbb{H}} = \{z \in \mathbb{C} \,|\, \Im z \leq 0\}$ of the z-plane to the reflection P' in the x-plane of the polygonal domain P through its edge connecting the vertices x_1 and x_2. The map $z \mapsto x$ in (8.8) therefore sends the Riemann sphere $\mathbb{P}^1$, equipped with the global coordinate z on the dense open subset $\mathbb{C} \subset \mathbb{P}^1$, to another copy of the Riemann sphere $\mathbb{P}^1$, equipped with the global coordinate x on an open and dense subset identified with the interior of the domain $P \cup P'$ in the x-plane. The case $N = 2$ is depicted in Figure 1. A very similar change of coordinate to this particular example was considered in [Luk13] in the context of the massive ODE/IM correspondence for the Fateev model.

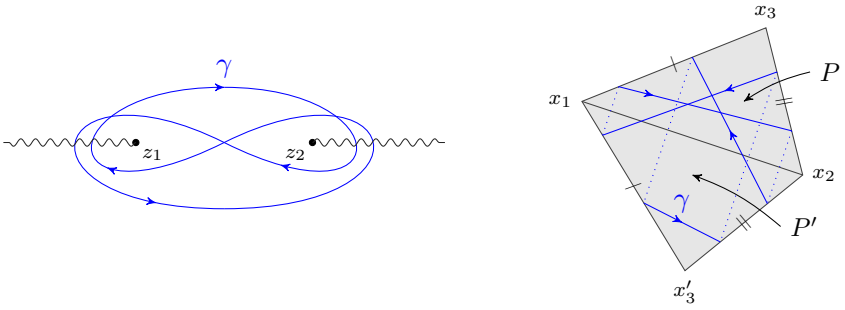

FIGURE 1. A pochhammer contour γ in the case of two marked points z_1, z_2 in the z-plane (left) and its image in the polygonal region $P \cup P'$ in the x-plane (right). The edge $x_i x_3$ is identified with $x_i x_3'$ for $i = 1, 2$.

Coming back to the connection (8.7), the pullback of the meromorphic differential $\widetilde{u}(z)dz$ by the inverse transformation $\mathbb{P}^1 \to \mathbb{P}^1$, $x \mapsto z$ gives a multivalued differential $\widehat{u}(x)dx$, where $\widehat{u}$ is the multivalued function on $\mathbb{P}^1$ given in the interior of the domain $P \cup P'$ of the x-plane by $\widehat{u}(x) \coloneqq \mathcal{P}(z(x))^{-1/h^\vee}\widetilde{u}(z(x))$. Here we have used the fact that $dx/dz = \mathcal{P}(z)^{1/h^\vee}$. Therefore (8.7) can now be re-expressed as a multivalued Miura ${}^L\mathfrak{g}'/\mathbb{C}\delta$-oper on $\mathbb{P}^1$ which is meromorphic on $\mathbb{P}^1 \setminus \{x_i\}_{i=1}^{N+1}$ and given in the interior of the polygonal domain $P \cup P'$ of the x-plane by

$$\widetilde{\nabla} = d + p_{-1}dx + \sum_{i=1}^{\ell} \widehat{u}_i(x)\alpha_i dx. \tag{8.9}$$

Here we wrote $\widehat{u}(x) = \sum_{i=1}^{\ell} \widehat{u}_i(x)\alpha_i$ in the basis of simple roots $\{\alpha_i\}_{i=1}^{\ell}$. Comparing the above expression (8.9) with the connection (8.1), it is tempting to regard the $\widehat{u}_i(x)$ for $i = 1, \dots, \ell$ as classical ${}^L\mathfrak{g}$-mKdV fields on the interior of the polygonal domain $P \cup P'$ in the x-plane.

Just as in §8.1, one could also consider bringing the Miura ${}^L\mathfrak{g}$-oper (8.9) to a form analogous to (8.3) in the smooth setting and define classical ${}^L\mathfrak{g}$-KdV fields $\widehat{v}_r(x)$, $r \in E$ on the interior of the polygonal domain $P \cup P'$ as the coefficients of the $\bar{p}_r$, $r \in E$, *i.e.*

$$d + p_{-1}dx + \sum_{r \in \bar{E}} \widehat{v}_r(x)\bar{p}_r dx. \tag{8.10}$$

Suppose that the collection of Bethe roots w_j^i, $j = 1, \dots, m_i$ for $i \in \{1, \dots, \ell\}$ satisfy the Bethe equations (4.7), or more explicitly

$$-\sum_{k=1}^{N} \frac{(\lambda_k|\alpha_{c(j)})}{w_j^i - z_k} + \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{(\alpha_{c(k)}|\alpha_{c(j)})}{w_j^i - w_k^i} = 0, \qquad j = 1, \dots, m_i,$$

for every $i \in \{1, \dots, \ell\}$. Then it follows from Proposition 4.2 and the explicit form (8.6) of the connection ∇ we started with in the z-plane, that the $\widehat{v}_r(x)$, $r \in \bar{E}$ are holomorphic at the images $x(w_j^i)$ of these Bethe roots under the Schwarz-Christoffel transformation (8.8). However, each $\widehat{v}_r(x)$, $r \in \bar{E}$ will generically still be singular at the images $x(w_j^0)$, $j = 1, \dots, m_0$, of the Bethe roots of "colour" 0. (By contrast, let us stress that there exists a quasi-canonical form of the affine ${}^L\mathfrak{g}$-oper in which *all* Bethe roots are erased, as in Corollary 4.4.)

Multivalued ${}^L\mathfrak{g}$-opers of the form (8.10) but with a certain irregular singularity were conjectured in [FF11] to describe the spectrum of quantum $\mathfrak{g}$-KdV. Specifically, it was shown that in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ such ${}^L\mathfrak{g}$-opers are equivalent to the Schrödinger operators with 'monster' potentials used to describe the spectrum of both local and non-local integrals of motion in quantum KdV theory via the ODE/IM correspondence [DT99, BLZ01, BLZ03, DDT07].

In the ODE/IM setting, it was recently shown in [FH16] and [MRV16, MRV17] that a certain $Q\widetilde{Q}$-system can be extracted from both sides of the 'KdV-oper' correspondence proposed in [FF11], providing strong evidence in support of the conjecture. By contrast, the proposal of the present work is a direct approach to establishing a correspondence between the spectra of quantum Gaudin models of affine type and

opers of Langlands-dual affine type. It relies on the idea that, in close parallel with the well-established correspondence in finite types, the spectrum can be obtained from a (quasi-)canonical form of the (affine) oper.

8.3. **Classical limit of higher Gaudin Hamiltonians.** One of the motivations for Conjectures 5.1 and 5.3 comes from the structure of higher local integrals of motion in *classical* affine Gaudin models [Vic], constructed in [LMV17] when the underlying finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$, cf. §5.1, is of classical type.

Specifically, to every exponent $i \in E$ and every zero x of the twist function, *i.e.* an $x \in \mathbb{C}$ such that $\varphi(x) = 0$, is assigned an element Q_i^x of a certain completion $\hat{S}_{\boldsymbol{k}}(\mathfrak{g}'^{\oplus N})$ of the quotient of the symmetric algebra of $\mathfrak{g}'^{\oplus N}$ by the ideal generated by the elements $\mathsf{k}^{(j)} - k_j$, $j = 1, \dots, N$. These were obtained by generalising the approach of [EHMM99], where certain $\dot{\mathfrak{g}}$-invariant homogeneous polynomials $P_i : \dot{\mathfrak{g}}^{\times(i+1)} \to \mathbb{C}$ of degree $i+1$ were constructed for each $i \in E$. Extending these to $\mathfrak{g}'/\mathbb{C}\mathsf{k} \cong_{\mathbb{C}} \mathcal{L}\dot{\mathfrak{g}}$ as $P_i(x_m, \dots, y_n) := P_i(x, \dots, y)\delta_{m+\dots+n,0}$ for any $x, \dots, y \in \dot{\mathfrak{g}}$ and $m, \dots, n \in \mathbb{Z}$, they can be applied to the first tensor factor of the local Lax matrix (5.11). The resulting $\hat{S}_{\boldsymbol{k}}(\mathfrak{g}'^{\oplus N})$-valued meromorphic functions on $\mathbb{P}^1$ are then evaluated at any zero x of the twist function to produce the charges Q_i^x, $i \in E$. The collection of these charges was shown in [LMV17] to form a Poisson commutative subalgebra of $\hat{S}_{\boldsymbol{k}}(\mathfrak{g}'^{\oplus N})$, *i.e.* $\{Q_i^x, Q_j^{x'}\} = 0$ for every $i, j \in E$ and any pair of zeroes x, x' of the twist function.

We also expect the operators $\mathcal{S}_j(z)$, $j \in E$ in Conjecture 5.1 to be built from the local Lax matrix $L(z)$ in a similar way. Furthermore, reintroducing Planck's constant $\hbar$ to take the classical limit, we expect the dependence of the integral (5.4) on $\hbar$ to come in the form

$$\int_\gamma \mathcal{P}(z)^{-i/\hbar h^\vee} \mathcal{S}_i(z) dz.$$

In the classical limit $\hbar \to 0$ such an integral localises, by the steepest descent method, at the critical points of the function $\mathcal{P}(z)$. Yet these are nothing but the zeroes of the twist function $\varphi(z) = \partial_z \log \mathcal{P}(z)$. Moreover, for generic z_i, $i = 1, \dots, N$ the number of zeroes of the twist function is $N-1$, which coincides with the number of linearly independent cycles from Corollary 3.13. We thus expect that the higher affine Gaudin Hamiltonians $\hat{Q}_i^\gamma$ of Conjecture 5.1 provide a quantisation of the local integrals of motion Q_i^x for classical affine Gaudin models in [LMV17].

Strictly speaking, the classical field theories correspond to classical affine Gaudin models with reality conditions and various other generalizations [Vic]. To understand the corresponding quantum field theories in the present framework, one would need to extend the constructions above to, in particular, the cyclotomic case and the case of irregular singularities. In finite types, such generalizations of quantum Gaudin models were studied in [VY16, VY17d] and [FFTL10, VY17c] respectively.

8.4. **Two-point case.** One natural arena in which to test the conjectures in §5.2 is the special case of $N = 2$ marked points and $\mathfrak{g}' = \widehat{\mathfrak{sl}}_2$. As was noted in [FF11, §6.4], in that case the GKO coset construction [GKO86] means that one already has candidates for the higher affine Gaudin Hamiltonians, namely the Integrals of Motion of quantum KdV acting in multiplicity spaces. With that in mind, it is interesting

to note that Bethe equations of a two-point Gaudin model of type $\widehat{\mathfrak{sl}}_2$ appeared in [FJM17, §7.3] as limits of Bethe equations for quantum toroidal algebras.

Appendix A. Hyperplane arrangements and quadratic Hamiltonians

A.1. **The Aomoto complex.** Fix coordinates $z_1, \dots, z_N, w_1, \dots, w_m$ on $\mathbb{C}^N \times \mathbb{C}^m$. Let $\mathscr{C}_{N+m}$ denote the hyperplane arrangement in $\mathbb{C}^N \times \mathbb{C}^m$ consisting of the following affine hyperplanes:

$$\begin{aligned} H_{ij} &: w_i - w_j = 0, \quad 1 \le i < j \le m, \\ H_i^j &: w_i - z_j = 0, \quad 1 \le i \le m, \quad 1 \le j \le N, \\ H^{ij} &: z_i - z_j = 0, \quad 1 \le i < j \le N. \end{aligned}$$

A *weighting* of the hyperplane arrangement $\mathscr{C}_{N+m}$ means a map $a : \mathscr{C}_{N+m} \to \mathbb{C}$, *i.e.* an assignment to every hyperplane H of a number $a(H) \in \mathbb{C}$ called its *weight*. Suppose we fix such a weighting. The corresponding *master function* is by definition

$$\Phi \coloneqq \sum_{H \in \mathscr{C}_{N+m}} a(H) \log l_H,$$

where $l_H = 0$ is an affine equation for the hyperplane H. It is a multivalued function on the complement of the arrangement,

$$U(\mathscr{C}_{N+m}) \coloneqq (\mathbb{C}^N \times \mathbb{C}^m) \setminus \mathscr{C}_{N+m}.$$

Let $\mathscr{L}$ denote the trivial line bundle, over this complement $U(\mathscr{C}_{N+m})$, equipped with the flat connection given by

$$d_\Phi f = df + (d\Phi) f,$$

where f is any holomorphic function on $U(\mathscr{C}_{N+m})$ and d is the de Rham differential. Let $\Omega^\bullet(\mathscr{L})$ denote the complex of $U(\mathscr{C}_{N+m})$-sections of the holomorphic de Rham complex of $\mathscr{L}$. That means, $\Omega^\bullet(\mathscr{L})$ is isomorphic as a vector space to the usual de Rham complex of holomorphic forms on $U(\mathscr{C}_{N+m})$, but with the differential

$$d_\Phi x = dx + d\Phi \wedge x.$$

The *Orlik-Solomon algebra* [OS80] $\mathcal{A}^\bullet = \bigoplus_{p=0}^m \mathcal{A}^p$ can be defined as the $\mathbb{C}$-algebra of differential forms generated by 1 and the one-forms

$$d \log l_H = dl_H / l_H, \qquad H \in \mathscr{C}_{N+m}.$$

Notice that $dx = 0$ for all $x \in \mathcal{A}^\bullet$. We have $d_\Phi(\mathcal{A}^k) \subset \mathcal{A}^{k+1}$. The complex $(\mathcal{A}^\bullet, d_\Phi)$ is called the *Aomoto complex* or the *complex of hypergeometric differential forms of weight* a. There is an inclusion of complexes

$$(\mathcal{A}^\bullet, d_\Phi) \hookrightarrow (\Omega^\bullet(\mathscr{L}), d_\Phi).$$

A.2. **Kac-Moody data.** Let $\mathfrak{g}$ be any Kac-Moody Lie algebra with symmetrizable Cartan matrix of size r. Let B be the (non-extended) symmetrized Cartan matrix of $\mathfrak{g}$. A realization of this Kac-Moody Lie algebra involves the choice of the following data:

(1) A finite-dimensional complex vector space $\mathfrak{h}$;
(2) A non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$;
(3) A collection $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ of linearly independent elements (the *simple roots*) such that $B = ((\alpha_i, \alpha_j))_{i,j=1}^r$.

Let $\tilde{\mathfrak{g}}$ denote "the Kac-Moody algebra $\mathfrak{g}$ but without Serre relations" – see [SV91, Section 6] for the precise definition. One has $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$, where $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ are free Lie algebras in generators F_i and E_i, $i = 1, \dots, r$, respectively. For any weight $\lambda \in \mathfrak{h}^*$ let M_λ denote the Verma module over $\tilde{\mathfrak{g}}$ of highest weight λ and let M_λ^* denote its contragredient dual. The *Shapovalov form* is a certain bilinear form defined on $\tilde{\mathfrak{g}}$ and on each weight subspace of M_λ. The quotient $\tilde{\mathfrak{g}}/\ker S$ is the Kac-Moody algebra $\mathfrak{g}$. The quotient $L_\lambda \cong M_\lambda/\ker S$ is the irreducible $\mathfrak{g}$ module of highest weight λ. We can regard the Shapovalov form S on M_λ as a map $M_\lambda \to M_\lambda^*$. Then we can identify L_λ with the image of M_λ in M_λ^*, *i.e.*

$$L_\lambda = S(M_\lambda) \subset M_\lambda^*.$$

Now fix weights $\lambda_1, \dots, \lambda_N \in \mathfrak{h}^*$ and a tuple $(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 0}^r$ and consider the subspace

$$\left(\bigotimes_{i=1}^N M_{\lambda_i}^*\right)_{\lambda_\infty}$$

of weight

$$\lambda_\infty := \sum_{i=1}^N \lambda_i - \sum_{j=1}^r k_j \alpha_j.$$

Set $m = k_1 + \cdots + k_r$ and consider the arrangement $\mathscr{C}_{N+m}$ as above. We assign "colours" $1, \dots, r$ to the coordinates $w_1, \dots, w_m$ in such a way that k_i of them have colour i, for each i. Pick any $\kappa \in \mathbb{C}^\times$. The inner products amongst the weights λ_i and roots α_i defines a weighting of the arrangement $\mathscr{C}_{N+m}$. Namely:

$$a(H^{ij}) = (\lambda_i, \lambda_j)/\kappa, \quad a(H_j^i) = -(\lambda_i, \alpha_{c(j)})/\kappa, \quad a(H_{ij}) = (\alpha_{c(i)}, \alpha_{c(j)})/\kappa, \tag{A.1}$$

where $c(i)$ is the colour of the coordinate w_i for each $i = 1, \dots, m$, cf. (4.8).

A.3. **Eigenvectors of the quadratic Gaudin Hamiltonians.** In [SV91, Section 7] Schechtman and Varchenko define a map

$$\tilde{\eta} : \left(\bigotimes_{i=1}^N M_{\lambda_i}^*\right)_{\lambda_\infty} \longrightarrow \Omega^m(\mathscr{L})$$

and prove that it obeys

$$d_\Phi \tilde{\eta}(x) = \frac{1}{\kappa} \sum_{i<j} \tilde{\eta}(\Omega_{ij} x) \wedge d\log(z_i - z_j). \tag{A.2}$$

Here Ω_{ij} are certain endomorphisms of $M^*_{\lambda_i} \otimes M^*_{\lambda_j}$ whose definition can be found in [SV91, Section 7]. Their important property here is that on the subspace $L_{\lambda_i} \otimes L_{\lambda_j}$, Ω_{ij} coincides with the action of the element Ξ of the Kac-Moody algebra $\mathfrak{g}$. (Recall, the element Ξ is the element of the formal sum $\sum_\alpha \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ over root spaces, defined by the standard bilinear form on $\mathfrak{g}$ of [Kac90, Chapter 2].)

The statement (A.2) is Theorem 7.2.5″ in [SV91]. Note that the differential $d\tilde{\eta}$ in the statement of Theorem 7.2.5″ is the differential in $\Omega^\bullet(\mathscr{L})$, *i.e.* $d_\Phi \eta$ in our notation. (This is stated explicitly in Theorem 7.2.5′.)

Let

$$\Theta \in \left(\bigotimes_{i=1}^N M_{\lambda_i}\right)_{\lambda_\infty} \otimes \left(\bigotimes_{i=1}^N M^*_{\lambda_i}\right)_{\lambda_\infty}$$

be the canonical element. Consider the image of Θ under the map $S \otimes \tilde{\eta}$. The result is an element

$$\Psi \coloneqq (S \otimes \tilde{\eta})\Theta \in \left(\bigotimes_{i=1}^N L_{\lambda_i}\right)_{\lambda_\infty} \otimes \Omega^m(\mathscr{L})$$

which obeys

$$d_\Phi \Psi = \left(\frac{1}{\kappa} \sum_{i<j} \Xi_{ij}\, d\log(z_i - z_j)\right) \wedge \Psi. \tag{A.3}$$

We have the trivial fibre bundle $\pi : \mathbb{C}^N \times \mathbb{C}^m \twoheadrightarrow \mathbb{C}^N$ given by projecting along $\mathbb{C}^m$. Now Ψ can be uniquely written in the form

$$\Psi = \psi dw_1 \wedge \cdots \wedge dw_m + \psi',$$

where ψ is a $\left(\bigotimes_{i=1}^N L_{\lambda_i}\right)_{\lambda_\infty}$-valued holomorphic function on the complement $U(\mathscr{C}_{N+m})$ of the arrangement and where ψ' is a $\left(\bigotimes_{i=1}^N L_{\lambda_i}\right)_{\lambda_\infty}$-valued holomorphic m-form on $U(\mathscr{C}_{N+m})$ whose pull-back to each fibre vanishes. (That is, informally, ψ' has factors of dz_i.)

The function ψ is the *weight function*, or *Schechtman-Varchenko vector*. Now the image of $\tilde{\eta}$ is in fact in the subspace $\mathcal{A}^m \hookrightarrow \Omega^m(\mathscr{L})$ of hypergeometric forms and these are closed as we noted above. So $d\Psi = 0$ and hence $d_\Phi \Psi = d\Phi \wedge \Psi$. Therefore the identity (A.3) becomes

$$\sum_{i=1}^m \frac{\partial \Phi}{\partial w_i} dw_i \wedge \psi' + \sum_{i=1}^N \frac{\partial \Phi}{\partial z_i} dz_i \wedge \Psi = \frac{1}{\kappa} \sum_{i=1}^N \sum_{j \neq i} \frac{\Xi_{ij}}{z_i - z_j} dz_i \wedge \Psi. \tag{A.4}$$

Assume that

$$\frac{\partial \Phi}{\partial w_i} = 0, \quad \text{for} \quad i = 1, \ldots, m, \tag{A.5}$$

i.e. assume we are at a critical point of the pull-back of the master function to the fibre. On inspecting the weighting of the arrangement in (A.1), one sees that equations (A.5) are the Bethe ansatz equations. Now take the inner derivative $\neg\partial/\partial z_i$ of (A.4) and

then pull back to the fibre. (That is, consider the coefficient of $dz_i \wedge dw_1 \wedge \cdots \wedge dw_m$.) The resulting equality of top-degree forms on the fibre gives

$$\kappa \frac{\partial \Phi}{\partial z_i} \psi = \sum_{j \neq i} \frac{\Xi_{ij}}{z_i - z_j} \psi. \tag{A.6}$$

In other words, ψ is an eigenvector of the quadratic Gaudin Hamiltonians, with the eigenvalue $\kappa \frac{\partial \Phi}{\partial z_i}$. It is known that if the critial point (A.5) is isolated then this eigenvector ψ is nonzero. Indeed, this follows from [Var11, Theorem 9.13]; see, for example, the proof of [VY17a, Theorem 9.17].

A.4. **KZ solutions.** A related but slightly less direct way to derive the Bethe ansatz for the quadratic Gaudin Hamilitonians starting with (A.3) is to go via asymptotics of solutions of the KZ equations in terms of hypergeometric integrals, as in [RV95]. For comparision, and also because hypergeometric integrals appear in a quite different role in the present paper, we now briefly recall this method. Let us re-write (A.4) but now include the term $d\Psi$ (which as we noted above is actually zero). We obtain

$$\sum_{i=1}^{N} \frac{\partial \psi}{\partial z_i} dz_i \wedge d\boldsymbol{w} + d\psi' + \sum_{i=1}^{m} \frac{\partial \Phi}{\partial w_i} dw_i \wedge \psi' + \sum_{i=1}^{N} \frac{\partial \Phi}{\partial z_i} dz_i \wedge \Psi = \frac{1}{\kappa} \sum_{i=1}^{N} \sum_{j \neq i} \frac{\Xi_{ij}}{z_i - z_j} dz_i \wedge \Psi,$$

where $d\boldsymbol{w} := dw_1 \wedge \ldots \wedge dw_m$. Now upon taking the inner derivative $\neg \partial / \partial z_i$ of this equation and pulling back to the fibres of the trivial fibre bundle $\pi : \mathbb{C}^N \times \mathbb{C}^m \twoheadrightarrow \mathbb{C}^N$ we obtain the equation

$$\frac{\partial \psi}{\partial z_i} d\boldsymbol{w} + \frac{\partial \Phi}{\partial z_i} \psi d\boldsymbol{w} - \sum_{j=1}^{m} dw_j \wedge \left(\frac{\partial}{\partial w_j} \chi_i + \frac{\partial \Phi}{\partial w_j} \chi_i \right) = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Xi_{ij}}{z_i - z_j} \psi d\boldsymbol{w},$$

where χ_i denotes the pullback to the fibre of the inner derivative $\neg \partial / \partial z_i$ applied to ψ'. Next, we multiply this last equation through by the multivalued function e^{Φ} and integrate over an m-dimensional twisted cycle Γ in the fibre, along which this function is single-valued. Since the bracketed sum on the left-hand side is the twisted derivative of the function χ_i, it vanishes once we take the integral over Γ. We therefore end up with

$$\kappa \frac{\partial}{\partial z_i} \int_\Gamma e^{\Phi} \psi d\boldsymbol{w} = \sum_{j \neq i} \frac{\Xi_{ij}}{z_i - z_j} \int_\Gamma e^{\Phi} \psi d\boldsymbol{w},$$

which expresses the fact that the integral $\int_\Gamma e^{\Phi} \psi d\boldsymbol{t}$ is a solution of the KZ equation. The eigenvalue equation (A.6) can be recovered in the limit $\kappa \to 0$ by applying the method of steepest descent to this integral solution, which localises in this limit to the isolated zeroes of the Bethe equations (A.5). This is to be contrasted with the discussion of the classical limit of the higher affine Gaudin Hamiltonians in §8.3.

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CUBIC HYPERGEOMETRIC INTEGRALS OF MOTION IN AFFINE GAUDIN MODELS

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Abstract. We construct cubic Hamiltonians for quantum Gaudin models of affine types $\widehat{\mathfrak{sl}}_M$. They are given by hypergeometric integrals of a form we recently conjectured in [LVY]. We prove that they commute amongst themselves and with the quadratic Hamiltonians. We prove that their vacuum eigenvalues, and their eigenvalues for one Bethe root, are given by certain hypergeometric functions on a space of affine opers.

1. Introduction

The *quantum Gaudin model* [Gau14] can be defined for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. One chooses a collection $z_1, \dots, z_N$ of distinct points in the complex plane and the model is defined by its quadratic Hamiltonians, which are the elements

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\Xi^{(ij)}}{z_i - z_j}, \qquad i = 1, \dots, N,$$

of the (suitably completed) tensor product $U(\mathfrak{g}^{\oplus N})$, where Ξ is the canonical element of $\mathfrak{g}\otimes\mathfrak{g}$ coming from the invariant bilinear form on $\mathfrak{g}$. In finite types $\mathfrak{g}$, these quadratic Hamiltonians are known to belong to a large commutative subalgebra $\mathcal{B} \subset U(\mathfrak{g}^{\oplus N})$ called the *Bethe* or *Gaudin* algebra, which is generated by the $\mathcal{H}_i$ together with the central elements of $U(\mathfrak{g}^{\oplus N})$ and also (when $\mathrm{rank}(\mathfrak{g}) > 1$) certain families of *higher Gaudin Hamiltonians* [FFR94, Fre05, Tal06, MTV06, Ryb08, Mol13, Ryb]. It is an interesting open question whether analogues of these higher Hamiltonians also exist when $\mathfrak{g}$ is of affine type. Of particular interest is the case when $\mathfrak{g}$ is of untwisted affine type since it is expected that quantum integrable field theories can be described as affine Gaudin models associated with such Kac-Moody Lie algebras [FF11, Vic]. In a recent paper, [LVY], we conjectured that higher Hamiltonians do exist for $\mathfrak{g}$ of untwisted affine type, and gave a broad conjecture for the form they should take as well as a precise conjecture for their eigenvalues on tensor products of irreducible highest-weight $\mathfrak{g}$-modules. According to [LVY] both the higher Hamiltonians and their eigenvalues should be given by integrals of hypergeometric type in the spectral plane (*i.e.* the copy of $\mathbb{C}$ containing the marked points $z_1, \dots, z_N$).

In the present paper, we check the simplest case of those conjectures. The prediction is that there is a family of higher Hamiltonians for each *exponent* of $\mathfrak{g}$. Recall that, when $\mathfrak{g}$ is of affine type, the exponents are a countably infinite (multi)set of integers whose pattern repeats modulo the Coxeter number (see *e.g.* [Kac90]). The quadratic

Hamiltonians are associated with the exponent 1. The next case to check is that of Hamiltonians associated to cubic symmetric invariant tensors on the underlying finite-type Lie algebra. Such tensors in fact exist only in types A_{M-1} (*i.e.* $\mathfrak{sl}_M$) with $M \geq 3$. Equivalently, ${}^1\mathsf{A}_{M-1}$ are the only untwisted affine types for which 2 is an exponent. In this paper we specialize to those types. We construct the cubic Hamiltonians and show that they commute amongst themselves and with the quadratic Hamiltonians. We also show that their eigenvalues are as predicted in [LVY] at least for Bethe vectors corresponding to 0 or 1 Bethe roots.

The paper is structured as follows.

In §2 we recall details of the affine algebra $\widehat{\mathfrak{sl}}_M$, its local completion and its vacuum Verma module. We also recall very briefly some concepts about vertex algebras.

In §3 we define the algebra of observables of the quantum Gaudin model, and define the states $\varsigma_1(z)$ and $\varsigma_2(z)$ used to construct the quadratic and cubic Hamiltonians respectively. The main technical results of the paper are then Theorems 3.3 and 3.4. The former shows that the ${}_{(0)}$th products (in the sense of vertex algebras) between these states vanish modulo certain *twisted derivatives* and *translates*. The latter shows that the same is true of the action of the diagonal copy of $\widehat{\mathfrak{sl}}_M$ on $\varsigma_i(z)$, $i = 1, 2$.

These statements allow us to prove our main result, Theorem 3.8, which establishes that the quadratic and cubic Hamiltonians commute amongst themselves and with the diagonal action of $\widehat{\mathfrak{sl}}_M$. The Hamiltonians $\hat{Q}_i^\gamma$, $i = 1, 2$, are defined in (3.14): the superscript γ denotes a Pochhammer contour in $\mathbb{C} \setminus \{z_1, \dots, z_N\}$.

(There is a slight subtlety because the $\hat{Q}_1^\gamma$ are not exactly the standard quadratic Hamiltonians $\mathcal{H}_j$. Thus, we also show in Theorem 3.8 that the $\hat{Q}_i^\gamma$, $i = 1, 2$, commute with the $\mathcal{H}_j$. We do so using another result, Theorem 3.5.)

In §4 we recall the conjectured eigenvalues for the Hamiltonians $\hat{Q}_i^\gamma$, $i = 1, 2$, from [LVY]. Namely, the eigenvalues are obtained by putting a certain *affine oper* (coming from an affine *Miura oper*) into *quasi-canonical form*, and then integrating the resulting coefficient functions $v_i(z)$, $i = 1, 2$, along the same contour γ.

In §5 we check that the predicted eigenvalues of the cubic Hamiltonians are correct (for the quadratic Hamiltonians see [LVY]) for Bethe vectors with 0 and 1 Bethe roots, *i.e.* for the vacuum state and for Bethe eigenstates at one step down in the principal gradation.

Finally, in §6 we consider the special case of only $N = 2$ marked points. In that case we show that our Hamiltonians $\hat{Q}_i^\gamma$, $i = 1, 2$ coincide up to rescaling with the zero modes of, respectively, the GKO coset conformal vector ω, [GKO86], and a state W constructed in [BBSS88]. It is known that, after certain further specializations, these generate a copy of the W_3 algebra. This provides an interesting arena for checking conjectures about higher Hamiltonians, since higher integrals of motion of the W_3 algebra are already known.

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2. Vacuum verma modules for $\widehat{\mathfrak{sl}}_M$

2.1. **Loop realization of $\widehat{\mathfrak{sl}}_M$.** We work over $\mathbb{C}$. Pick and fix an integer $M \geq 3$. Let $\mathfrak{sl}_M[t,t^{-1}] = \mathfrak{sl}_M \otimes \mathbb{C}[t,t^{-1}]$ denote the Lie algebra of Laurent polynomials, in a formal variable t, with coefficients in the finite-dimensional simple Lie algebra $\mathfrak{sl}_M$. The Lie bracket on $\mathfrak{sl}_M[t,t^{-1}]$ is given by $[a \otimes f(t), b \otimes g(t)] \coloneqq [a,b] \otimes f(t)g(t)$ for any $a, b \in \mathfrak{sl}_M$ and $f(t), g(t) \in \mathbb{C}[t,t^{-1}]$. Let $(\cdot|\cdot) : \mathfrak{sl}_M \times \mathfrak{sl}_M \to \mathbb{C}$ be the standard bilinear form on $\mathfrak{sl}_M$. It is given by

$$(X|Y) \coloneqq \operatorname{tr}(XY) \tag{2.1}$$

where tr denotes the trace in the defining $M \times M$ matrix representation. The affine algebra $\widehat{\mathfrak{sl}}_M$ is the central extension of $\mathfrak{sl}_M[t,t^{-1}]$ by a one dimensional centre $\mathbb{C}\mathsf{k}$,

$$0 \longrightarrow \mathbb{C}\mathsf{k} \longrightarrow \widehat{\mathfrak{sl}}_M \longrightarrow \mathfrak{sl}_M[t,t^{-1}] \longrightarrow 0,$$

whose commutation relations are given by $[\mathsf{k}, \cdot] = 0$ and

$$[a \otimes f(t), b \otimes g(t)] \coloneqq [a,b] \otimes f(t)g(t) - (\operatorname{res}_t f dg)(a|b)\mathsf{k}.$$

Let $a_n \coloneqq a \otimes t^n$ for $a \in \mathfrak{sl}_M$ and $n \in \mathbb{Z}$. The commutation relations can equivalently be written as

$$[a_m, b_n] = [a,b]_{n+m} - n\delta_{n+m,0}(a|b)\mathsf{k}.$$

Define the Lie algebra

$$\mathfrak{g} \coloneqq \widehat{\mathfrak{sl}}_M \oplus \mathbb{C}\mathsf{d},$$

by declaring that the *derivation element* d obeys $[\mathsf{d}, \mathsf{k}] = 0$ and $[\mathsf{d}, a \otimes f(t)] = a \otimes t\partial_t f(t)$ for all $a \in \mathfrak{sl}_M$ and $f(t) \in \mathbb{C}[t,t^{-1}]$.

2.2. **Kac-Moody data.** Recall, for example from [Kac90], that the Lie algebra $\mathfrak{g}$ is isomorphic to the Kac-Moody algebra over $\mathbb{C}$ of type ${}^1\mathsf{A}_{M-1}$. The Cartan matrix is $A \coloneqq (a_{ij})_{i,j=0}^{M-1} = (2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j=0}^{M-1}$, where addition of indices is modulo M. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and sets of Chevalley-Serre generators $\{e_i\}_{i=0}^{M-1} \subset \mathfrak{n}_+$, $\{f_i\}_{i=0}^{M-1} \subset \mathfrak{n}_-$. Let $\{\check{\alpha}_i\}_{i=0}^{M-1} \subset \mathfrak{h}$ be the simple coroots of $\mathfrak{g}$ and $\{\alpha_i\}_{i=0}^{M-1} \subset \mathfrak{h}^*$ the simple roots. They obey $a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$ for $i,j \in \{0,1,\ldots,M-1\}$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the canonical pairing of the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. The defining relations of $\mathfrak{g}$ are then

$$[x, e_i] = \langle \alpha_i, x \rangle e_i, \qquad [x, f_i] = -\langle \alpha_i, x \rangle f_i, \tag{2.2a}$$

$$[x, x'] = 0, \qquad [e_i, f_j] = \check{\alpha}_i \delta_{i,j}, \tag{2.2b}$$

for any $x, x' \in \mathfrak{h}$ and $i,j \in \{0,1,\ldots,M-1\}$, together with the Serre relations

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0. \tag{2.2c}$$

The centre of $\mathfrak{g}$ is one dimensional and spanned by the *central element* $\mathsf{k} \coloneqq \sum_{i=0}^{M-1} \check{\alpha}_i$. The Cartan subalgebra has a basis consisting of the simple coroots $\{\check{\alpha}_i\}_{i=0}^{M-1}$ together with the derivation element d, which obeys

$$\langle \alpha_i, \mathsf{d} \rangle = \delta_{i,0}.$$

(This condition fixes d uniquely up to the addition of a multiple of k.)

The matrix $(a_{ij})_{i,j=1}^{M-1}$ obtained by removing the zeroth row and column of A is the Cartan matrix of finite type A_{M-1}, with $\mathfrak{sl}_M$ the corresponding finite-dimensional simple Lie algebra. We can identify $\mathfrak{sl}_M$ with the subalgebra of $\mathfrak{g}$ generated by $\{e_i\}_{i=1}^{M-1} \subset \mathfrak{n}_+$ and $\{f_i\}_{i=1}^{M-1} \subset \mathfrak{n}_-$, and then the defining relations are as above. Let $\dot{\mathfrak{h}} \coloneqq \mathrm{span}_{\mathbb{C}}\{\check{\alpha}_i\}_{i=1}^{M-1} \subset \mathfrak{sl}_M$ denote its Cartan subalgebra and $\dot{\mathfrak{h}}^* = \mathrm{span}_{\mathbb{C}}\{\alpha_i\}_{i=1}^{M-1}$ its dual. Let $\{\omega_i\}_{i=1}^{M-1} \subset \dot{\mathfrak{h}}^*$ and $\{\check{\omega}_i\}_{i=1}^{M-1} \subset \dot{\mathfrak{h}}$ be respectively the fundamental weights and coweights of $\mathfrak{sl}_M$.

2.3. **Local completion and vacuum Verma module.** For any $k \in \mathbb{C}$, let $U_k(\widehat{\mathfrak{sl}}_M)$ denote quotient of the enveloping algebra of $\widehat{\mathfrak{sl}}_M$ by the two-sided ideal generated by $\mathsf{k}-k$. For each $n \in \mathbb{Z}_{\geq 0}$ define the left ideal $J_n \coloneqq U_k(\widehat{\mathfrak{sl}}_M)\cdot(\mathfrak{sl}_M \otimes t^n\mathbb{C}[t])$. That is, J_n is the linear span of monomials of the form $a_p \dots b_q c_r$ with $r \geq n$, $c \in \mathfrak{sl}_M$, and some non-negative number of elements $a, \dots, b \in \mathfrak{sl}_M$ and mode numbers $p, \dots, q \in \mathbb{Z}$. The inverse limit $\tilde{U}_k(\widehat{\mathfrak{sl}}_M) \coloneqq \varprojlim U_k(\widehat{\mathfrak{sl}}_M)/J_n$ is a complete topological algebra, called the *local completion of* $U(\widehat{\mathfrak{sl}}_M)$ *at level* k. By definition, elements of $\tilde{U}_k(\widehat{\mathfrak{sl}}_M)$ are (possibly infinite) sums $\sum_{m\geq 0} X_m$ of elements $X_m \in U_k(\widehat{\mathfrak{sl}}_M)$ which truncate to finite sums when one works modulo any J_n, *i.e.* for every n, $X_m \in J_n$ for all sufficiently large m.

A module $\mathcal{M}$ over $\widehat{\mathfrak{sl}}_M$ is called *smooth* if, for all $a \in \mathfrak{sl}_M$ and all $v \in \mathcal{M}$, $a_n v = 0$ for all sufficiently large n. A module $\mathcal{M}$ has *level* k if $\mathsf{k} - k$ acts as zero on $\mathcal{M}$. Any smooth module of level k over $\widehat{\mathfrak{sl}}_M$ is also a module over $\tilde{U}_k(\widehat{\mathfrak{sl}}_M)$.

We have the Lie subalgebra $\mathfrak{sl}_M[t]\oplus\mathbb{C}\mathsf{k} \subset \widehat{\mathfrak{sl}}_M$. Let $\mathbb{C}\,|0\rangle^k$ denote the one-dimensional representation of this Lie algebra given by $(\mathsf{k}-k)\,|0\rangle^k = 0$ and $a_n\,|0\rangle^k = 0$ for all $n \geq 0$ and all $a \in \mathfrak{sl}_M$. Let $\mathbb{V}_0^k$ denote the induced $\widehat{\mathfrak{sl}}_M$-module:

$$\mathbb{V}_0^k = U(\widehat{\mathfrak{sl}}_M) \otimes_{U(\mathfrak{sl}_M[t]\oplus\mathbb{C}\mathsf{k})} \mathbb{C}\,|0\rangle^k\,.$$

This module $\mathbb{V}_0^k$ is called the *vacuum Verma module of level* k. It is a smooth module. Concretely, $\mathbb{V}_0^k$ is the linear span of vectors of the form $a_p \dots b_q\,|0\rangle^k$ with $a, \dots, b \in \mathfrak{sl}_M$ and strictly negative mode numbers $p, \dots, q \in \mathbb{Z}_{\leq -1}$. Elements of $\mathbb{V}_0^k$ are called *states*.

Let $[T, \cdot]$ be the derivation on $U_k(\widehat{\mathfrak{sl}}_M)$ defined by $[T, a_n] \coloneqq -n a_{n-1}$ and $[T, 1] \coloneqq 0$. By setting $T(X\,|0\rangle^k) \coloneqq [T, X]\,|0\rangle^k$ for any $X \in U_k(\widehat{\mathfrak{sl}}_M)$, one can then regard T also as a linear map $\mathbb{V}_0^k \to \mathbb{V}_0^k$, called the *translation operator*.

2.4. **Vertex algebra structure.** For every state $A \in \mathbb{V}_0^k$ and every $n \in \mathbb{Z}$, there is an element $A_{(n)} \in \tilde{U}_k(\widehat{\mathfrak{sl}}_M)$, the nth *formal mode of* A. These modes can be arranged into a formal power series

$$Y[A, u] \coloneqq \sum_{n\in\mathbb{Z}} A_{(n)} u^{-n-1} \tag{2.3}$$

where $Y[\cdot, u]$ is called the *formal state-field map*. Their definition is as follows.

First, if $A = a_{-1}\,|0\rangle^k$ for some $a \in \mathfrak{sl}_M$ then $A_{(n)} \coloneqq a_n$ for all $n \in \mathbb{Z}$, *i.e.*

$$Y[a_{-1}\,|0\rangle^k, u] \coloneqq \sum_{n\in\mathbb{Z}} a_n u^{-n-1}.$$

Next, for all states $A, B \in \mathbb{V}_0^k$,

$$Y[TA,u] \coloneqq \partial_u Y[A,u] \qquad \text{and} \qquad Y[A_{(-1)}B,u] \coloneqq :Y[A,u]Y[B,u]:$$

where

$$:Y[A,u]Y[B,u]: \coloneqq \left(\sum_{m<0} A_{(m)}u^{-m-1}\right)Y[B,u] + Y[B,u]\left(\sum_{m\geq 0} A_{(m)}u^{-m-1}\right) \quad (2.4)$$

is called the *normal ordered product*. These assignments together recursively define $Y[C,u]$ for all $C \in \mathbb{V}_0^k$ (by writing $C = a_{-n}B$ for some $a \in \mathfrak{sl}_M$, $n \in \mathbb{Z}_{\geq 1}$ and $B \in \mathbb{V}_0^k$).

The *state-field map* $Y(\cdot,u) : \mathbb{V}_0^k \to \mathrm{Hom}(\mathbb{V}_0^k, \mathbb{V}_0^k((u)))$ is obtained from $Y[A,u]$ by sending each formal mode to its image in $\mathrm{End}(\mathbb{V}_0^k)$. This map $Y(\cdot,u)$ obeys a collection of axioms that make $\mathbb{V}_0^k$ into a *vertex algebra*; see *e.g.* [FBZ04].

3. Quadratic and cubic Hamiltonians

3.1. **The algebra of observables.** Let $\boldsymbol{k} \coloneqq (k_i)_{i=1}^N$ be a collection of $N \in \mathbb{Z}_{\geq 1}$ complex numbers $k_i \neq -M$ for $i = 1, \ldots, N$. Consider the tensor product

$$\mathbb{V}_0^{\boldsymbol{k}} \coloneqq \mathbb{V}_0^{k_1} \otimes \ldots \otimes \mathbb{V}_0^{k_N}$$

of vacuum Verma modules. We can regard it as a module over the direct sum $\widehat{\mathfrak{sl}}_M^{\oplus N}$ of N copies of $\widehat{\mathfrak{sl}}_M$. Let $A^{(i)} \in \widehat{\mathfrak{sl}}_M^{\oplus N}$ denote the copy of $A \in \widehat{\mathfrak{sl}}_M$ in the ith direct summand. Let $\mathbb{C}\,|0\rangle^{\boldsymbol{k}}$ denote the one-dimensional representation of the Lie subalgebra $(\mathfrak{sl}_M[t] \oplus \mathbb{C}\mathsf{k})^{\oplus N} \subset \widehat{\mathfrak{sl}}_M^{\oplus N}$ defined by $(\mathsf{k}^{(i)} - k_i)\,|0\rangle^{\boldsymbol{k}} = 0$ and $a_n^{(i)}\,|0\rangle^k = 0$, for all $n \geq 0$, all $a \in \mathfrak{sl}_M$, and all $i \in \{1, \ldots, N\}$. Then $\mathbb{V}_0^{\boldsymbol{k}}$ is the induced $\widehat{\mathfrak{sl}}_M^{\oplus N}$-module:

$$\mathbb{V}_0^{\boldsymbol{k}} = U(\widehat{\mathfrak{sl}}_M^{\oplus N}) \otimes_{U((\mathfrak{sl}_M[t]\oplus\mathbb{C}\mathsf{k})^{\oplus N})} \mathbb{C}\,|0\rangle^{\boldsymbol{k}}.$$

Let $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ denote the quotient of $U(\widehat{\mathfrak{sl}}_M^{\oplus N})$ by the two-sided ideal generated by $\mathsf{k}^{(i)} - k_i$ for all $i \in \{1, \ldots, N\}$. We have the isomorphism

$$U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}) \cong U_{k_1}(\widehat{\mathfrak{sl}}_M) \otimes \cdots \otimes U_{k_N}(\widehat{\mathfrak{sl}}_M),$$

which sends $A^{(i)} \in \widehat{\mathfrak{sl}}_M^{\oplus N} \subset U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ to $1^{\otimes i-1} \otimes A \otimes 1^{\otimes N-i}$. Let J_n^N denote the left ideal in $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ generated by $a_r^{(i)}$ for all $r \geq n$, $a \in \mathfrak{sl}_M$ and all $i \in \{1, \ldots, N\}$. Let $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}) \coloneqq \varprojlim U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})/J_n^N$ denote the inverse limit. It is a complete topological algebra, and

$$\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}) \cong \tilde{U}_{k_1}(\widehat{\mathfrak{sl}}_M) \hat{\otimes} \cdots \hat{\otimes} \tilde{U}_{k_N}(\widehat{\mathfrak{sl}}_M),$$

where $\hat{\otimes}$ denotes the completed tensor product. We call $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ the *algebra of observables* of the Gaudin model.

We have the formal state-field map $Y[\cdot,u] : \mathbb{V}_0^{\boldsymbol{k}} \to \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})[[u,u^{-1}]]$ and translation operator $T \in \mathrm{End}(\mathbb{V}_0^{\boldsymbol{k}})$ defined as above. $\mathbb{V}_0^{\boldsymbol{k}}$ is a module over $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, and we get the state-field map $Y(\cdot,u) : \mathbb{V}_0^{\boldsymbol{k}} \to \mathrm{Hom}(\mathbb{V}_0^{\boldsymbol{k}}, \mathbb{V}_0^{\boldsymbol{k}}((u)))$.

Let $\Delta : \widehat{\mathfrak{sl}}_M \hookrightarrow \widehat{\mathfrak{sl}}_M^{\oplus N}$ denote the diagonal embedding of $\widehat{\mathfrak{sl}}_M$ into $\widehat{\mathfrak{sl}}_M^{\oplus N}$,

$$\Delta x = \sum_{i=1}^{N} x^{(i)}$$

It extends uniquely to an embedding $\Delta : U(\widehat{\mathfrak{sl}}_M) \hookrightarrow U(\widehat{\mathfrak{sl}}_M^{\oplus N}) = U(\widehat{\mathfrak{sl}}_M)^{\otimes N}$ (the usual N-fold coproduct). For all $a, b \in \mathfrak{sl}_M$ and $m, n \in \mathbb{Z}$,

$$[\Delta a_m, \Delta b_n] = \Delta[a, b]_{n+m} - n\delta_{n+m,0}(a|b) \sum_{i=1}^{n} \mathsf{k}^{(i)}$$

in $U(\widehat{\mathfrak{sl}}_M^{\oplus N})$. Therefore Δ descends to an embedding of the quotients, $\Delta : U_{|\boldsymbol{k}|}(\widehat{\mathfrak{sl}}_M) \hookrightarrow U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, where $|\boldsymbol{k}| = \sum_{i=1}^{N} k_i$, and thence of their completions,

$$\Delta : \tilde{U}_{|\boldsymbol{k}|}(\widehat{\mathfrak{sl}}_M) \longhookrightarrow \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}).$$

3.2. **Invariant tensors on $\mathfrak{sl}_M$.** Let I^a, $a = 1, \dots, \dim(\mathfrak{sl}_M)$, be a basis of $\mathfrak{sl}_M$ and I_a, $a = 1, \dots, \dim(\mathfrak{sl}_M)$ the dual basis with respect to the non-degenerate bilinear form $(\cdot|\cdot)$ from (2.1). Let $f^{ab}{}_c$ denote the structure constants,

$$[I^a, I^b] = f^{ab}{}_c I^c \coloneqq \sum_c f^{ab}{}_c I^c, \tag{3.1}$$

where, from now on, we employ summation convention over repeated pairs of Lie algebra indices. Since the symmetric bilinear form $(\cdot|\cdot)$ is non-degenerate, we may and shall suppose that the basis is chosen such that

$$(I_a|I_b) = \delta_{ab}.$$

Then $I_a = I^a$ for each a and we no longer need to distinguish between upper and lower indices, and we shall write for example $f_{abc} = f^{ab}{}_c$. We have

$$\delta_{ab} = \operatorname{tr} I_a I_b, \qquad f_{abc} = \operatorname{tr}[I_a, I_b] I_c = \operatorname{tr}(I_a I_b I_c - I_b I_a I_c).$$

Recall that in general a tensor $t : \mathfrak{sl}_M \times \cdots \times \mathfrak{sl}_M \to \mathbb{C}$ is *invariant* if

$$t([a, x], y, \dots, z) + t(x, [a, y], \dots, z) + \cdots + t(x, y, \dots, [a, z]) = 0$$

for all $x, y, \dots, z$ and a in $\mathfrak{sl}_M$. In particular, δ_{ab} and f_{abc} are the components of respectively a symmetric second-rank invariant tensor and a totally skew-symmetric third-rank invariant tensor.

The reason for specialising to $\mathfrak{sl}_M$ in this paper is that in these types, and in no others, there exists a nonzero totally symmetric third-rank tensor. It is unique up to normalization, and is given by $t(x, y, z) \coloneqq \operatorname{tr}(xyz + yxz)$. Let t_{abc} denote its components,

$$t_{abc} \coloneqq t(I_a, I_b, I_c).$$

The Coxeter number and dual Coxeter number of $\mathfrak{sl}_M$ (and of $\widehat{\mathfrak{sl}}_M$) are equal to M.

Lemma 3.1. *We have the following tensor identities*

$$f_{ade}f_{bdg}t_{ceg} = -Mt_{abc} \qquad t_{abc}t_{dbc} = 2\frac{M^2-4}{M}\delta_{ad},$$
$$f_{abc}f_{dbc} = -2M\delta_{ad}, \qquad f_{ade}t_{bgd}t_{ceg} = \frac{M^2-4}{M}f_{abc},$$

and the tensors $f_{ade}f_{dfg}f_{ehi}t_{bfh}t_{cgi}$, $f_{acd}f_{def}f_{egh}f_{fij}t_{bgi}t_{chj}$ *and* $f_{acd}f_{def}f_{egh}f_{fij}t_{bhj}t_{cgi}$ *are identically zero. Moreover, we have the tensor identity*

$$t_{ea(b}t_{cd)e} = t_{e(ab}t_{cd)e}. \tag{3.2}$$

Here $X_{(a_1\ldots a_p)} \coloneqq \sum_{\sigma\in S_p}\frac{1}{p!}X_{\sigma(a_1)\ldots\sigma(a_p)}$.

Proof. By direct calculation in the defining matrix representation. □

3.3. **Quadratic and cubic states.** With $N \in \mathbb{Z}_{\geq 1}$ as above, let $z_1, \ldots, z_N \in \mathbb{C}$ be a collection of distinct points in the complex plane. For any $A \in \widehat{\mathfrak{sl}}_M$ let us define the $\widehat{\mathfrak{sl}}_M^{\oplus N}$-valued rational function

$$A(z) \coloneqq \sum_{i=1}^{N}\frac{A^{(i)}}{z-z_i}. \tag{3.3}$$

Lemma 3.2. *For any* $A, B \in \widehat{\mathfrak{sl}}_M$,

$$[A(z), B(w)] = -\frac{[A,B](z)-[A,B](w)}{z-w}, \qquad z \neq w,$$
$$[A(z), B(z)] = -[A,B]'(z)$$

where in the second line the $'$ *denotes derivative with respect to the argument,* z. □

Now we define states $\varsigma_1(z)$ and $\varsigma_2(z)$ in $\mathbb{V}_0^{\boldsymbol{k}}$, depending rationally on z, as follows

$$\varsigma_1(z) \coloneqq \tfrac{1}{2}I^a_{-1}(z)I^a_{-1}(z)\,|0\rangle^{\boldsymbol{k}},$$
$$\varsigma_2(z) \coloneqq \tfrac{1}{3}t_{abc}I^a_{-1}(z)I^b_{-1}(z)I^c_{-1}(z)\,|0\rangle^{\boldsymbol{k}}.$$

Define also the *twist function*

$$\varphi(z) \coloneqq \sum_{i=1}^{N}\frac{k_i}{z-z_i}. \tag{3.4}$$

and, for any integer j, the *twisted derivative operator of degree* j *with respect to* z,

$$D_z^{(j)} \coloneqq \partial_z - \frac{j}{M}\varphi(z). \tag{3.5}$$

Recall the translation operator $T \in \mathrm{End}(\mathbb{V}_0^{\boldsymbol{k}})$.

3.4. **Main result.** We can now state the main results of the paper.

Theorem 3.3. *For all $i,j \in \{1,2\}$, there exist $\mathbb{V}_0^{\boldsymbol{k}}$-valued rational functions $A_{ij}(z,w)$ and $B_{ij}(z,w)$ such that*

$$\varsigma_i(z)_{(0)}\varsigma_j(w) = \big(jD_z^{(i)} - iD_w^{(j)}\big)A_{ij}(z,w) + TB_{ij}(z,w). \tag{3.6}$$

Moreover, $A_{ij}(z,w)$ are regular at $z=w$ up to terms of the form TZ with $Z \in \mathbb{V}_0^{\boldsymbol{k}}$.

Proof. The first statement follows from a (lengthy) direct computation using the various identities from Lemma 3.1. Explicitly, we find that one choice of functions $A_{ij}(z,w)$ and $B_{ij}(z,w)$ is given by:

$$A_{11}(z,w) = \frac{M}{z-w} I^a_{-2}(z) I^a_{-1}(w)\,|0\rangle^{\boldsymbol{k}}\,, \qquad B_{11}(z,w) = \frac{M}{(z-w)^2} I^a_{-1}(z) I^a_{-1}(w)\,|0\rangle^{\boldsymbol{k}}$$

for $i=j=1$,

$$\begin{aligned} A_{12}(z,w) &= \frac{M}{2(z-w)} t_{abc} I^a_{-2}(z) I^b_{-1}(w) I^c_{-1}(w)\,|0\rangle^{\boldsymbol{k}}\,, \\ B_{12}(z,w) &= \frac{M}{2(z-w)^2} t_{abc} I^a_{-1}(z) I^b_{-1}(w) I^c_{-1}(w)\,|0\rangle^{\boldsymbol{k}}\,, \end{aligned}$$

for $i=1$ and $j=2$,

$$\begin{aligned} A_{21}(z,w) &= \frac{M}{z-w} t_{abc} I^a_{-2}(z) I^b_{-1}(z) I^c_{-1}(w)\,|0\rangle^{\boldsymbol{k}}\,, \\ B_{21}(z,w) &= \frac{M}{(z-w)^2} t_{abc} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(w)\,|0\rangle^{\boldsymbol{k}}\,, \end{aligned}$$

for $i=2$ and $j=1$, and finally

$$\begin{aligned} A_{22}(z,w) = {} & \frac{M}{2(z-w)} t_{abe} t_{cde} I^a_{-2}(z) I^b_{-1}(z) I^c_{-1}(w) I^d_{-1}(w)\,|0\rangle^{\boldsymbol{k}} \\ & - \frac{M(M^2-4)}{(z-w)^2}\bigg(\frac{\varphi(z)-\varphi(w)}{M} I^a_{-4}(z) I^a_{-1}(w) \\ & + \tfrac{1}{2}\big(I^a_{-4}{}'(z) I^a_{-1}(w) + I^a_{-4}(z) I^a_{-1}{}'(w)\big) - \frac{1}{M} f_{abc} I^a_{-3}(z) I^b_{-1}(z) I^c_{-1}(w)\bigg)\,|0\rangle^{\boldsymbol{k}} \\ & + \frac{M(M^2-4)}{(z-w)^3}\big(I^a_{-4}(z) I^a_{-1}(z) - 3 I^a_{-4}(z) I^a_{-1}(w) - I^a_{-3}(z) I^a_{-2}(z)\big)\,|0\rangle^{\boldsymbol{k}} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}B_{22}(z,w) = {}& \frac{M}{2(z-w)^2} t_{abe}t_{cde} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(w) I^d_{-1}(w)\,|0\rangle^{\boldsymbol{k}} \\ & - \frac{2M(M^2-4)}{(z-w)^3}\bigg(2\frac{\varphi(z)-\varphi(w)}{M} I^a_{-3}(z) I^a_{-1}(w) + I^a_{-3}(z) I^a_{-1}{}'(w) \\ & \qquad - \frac{1}{M} f_{abc} I^a_{-2}(z) I^b_{-1}(z) I^c_{-1}(w)\bigg)\,|0\rangle^{\boldsymbol{k}} \\ & + \frac{2M(M^2-4)}{(z-w)^4}\big(I^a_{-3}(z) I^a_{-1}(z) - 5 I^a_{-3}(z) I^a_{-1}(w) \\ & \qquad - I^a_{-2}(z) I^a_{-2}(z) + \tfrac{1}{2} I^a_{-2}(z) I^a_{-2}(w)\big)\,|0\rangle^{\boldsymbol{k}}\,.\end{aligned}$$

in the case $i = j = 2$.

To show the 'moreover' part, it suffices to expand the expressions for $A_{ij}(z,w)$ given above in w near z and express all singular terms in the desired form. When $i = j = 1$ we have

$$A_{11}(z,w) = \frac{M}{2(z-w)} T I^a_{-1}(z) I^a_{-1}(z)\,|0\rangle^{\boldsymbol{k}} + \dots$$

where the dots represent terms regular at $z = w$. Likewise, for $i = 1$, $j = 2$ and $i = 2$, $j = 1$ we find, respectively,

$$\begin{aligned}A_{12}(z,w) &= \frac{M}{3(z-w)} t_{abc} T I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z)\,|0\rangle^{\boldsymbol{k}} + \dots \\ A_{21}(z,w) &= \frac{M}{3(z-w)} t_{abc} T I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z)\,|0\rangle^{\boldsymbol{k}} + \dots\end{aligned}$$

as required.

Now consider the case $i = j = 2$. The terms of order $(z-w)^{-3}$ and $(z-w)^{-2}$ in $A_{22}(z,w)$ can be written as

$$\begin{aligned}& - \frac{M(M^2-4)}{3(z-w)^3} T\big(2 I^a_{-3}(z) I^a_{-1}(z) + \tfrac{1}{4} I^a_{-2}(z) I^a_{-2}(z)\big)\,|0\rangle^{\boldsymbol{k}} \\ & \quad + \frac{M(M^2-4)}{6(z-w)^2} T\big(5 I^a_{-3}(z) I^a_{-1}{}'(z) - I^a_{-3}{}'(z) I^a_{-1}(z) + \tfrac{1}{2} I^a_{-2}{}'(z) I^a_{-2}(z)\big)\,|0\rangle^{\boldsymbol{k}}\,.\end{aligned}$$

The terms of order $(z-w)^{-1}$ in $A_{22}(z,w)$ can also be written as TZ for some state $Z \in \mathbb{V}_0^{\boldsymbol{k}}$. In order to see this, one needs to note that the singular part of the first term on the right hand side of (3.7) can be written as

$$\begin{aligned}& \frac{M}{z-w} t_{abe}t_{cde} I^a_{-2}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z)\,|0\rangle^{\boldsymbol{k}} \\ & \quad = \frac{M}{z-w} t_{abe}t_{cde} I^a_{-2}(z) I^{(b}_{-1}(z) I^c_{-1}(z) I^{d)}_{-1}(z)\,|0\rangle^{\boldsymbol{k}} \\ & \qquad - \frac{M(M^2-4)}{z-w} f_{abc}\bigg(\frac{2}{3} I^a_{-2}(z) I^c_{-2}{}'(z) I^b_{-1}(z) + \frac{1}{3} I^a_{-2}(z) I^b_{-1}(z) I^c_{-2}{}'(z)\bigg)\,|0\rangle^{\boldsymbol{k}}\,.\end{aligned}$$

The first term on the right hand side is proportional to

$$t_{abe}t_{cde}I^a_{-2}(z)I^{(b}_{-1}(z)I^c_{-1}(z)I^{d)}_{-1}(z)\,|0\rangle^{\boldsymbol{k}} = t_{e(ab}t_{cd)e}I^a_{-2}(z)I^b_{-1}(z)I^c_{-1}(z)I^d_{-1}(z)\,|0\rangle^{\boldsymbol{k}}$$
$$= \frac{1}{4}t_{e(ab}t_{cd)e}TI^a_{-1}(z)I^b_{-1}(z)I^c_{-1}(z)I^d_{-1}(z)\,|0\rangle^{\boldsymbol{k}}$$

where in the first equality we used the identity (3.2). □

Theorem 3.4. *For any $x \in \mathfrak{sl}_M$ and $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\Delta x_n\varsigma_1(z) = D^{(1)}_z\big(-Mx_{-1}(z)\,|0\rangle\,\delta_{n,1}\big)$$
$$\Delta x_n\varsigma_2(z) = D^{(2)}_z\big(-\tfrac{1}{2}Mt_{abc}(x|I^a)I^b_{-1}(z)I^c_{-1}(z)\,|0\rangle\,\delta_{n,1}\big).$$

Proof. As for Theorem 3.3, the proof is by direct calculation using the identities in Lemma 3.1. □

3.5. **Quadratic Gaudin Hamiltonians.** Recall from §3.1 that we are assuming the levels to be non-critical, *i.e.* $k_i \neq -M$ for each $i \in \{1,\dots,N\}$. Let

$$\omega^{(i)} := \frac{1}{2(k_i+M)}I^{a(i)}_{-1}I^{a(i)}_{-1}\,|0\rangle^{\boldsymbol{k}} \in \mathbb{V}^{\boldsymbol{k}}_0 \tag{3.8}$$

be the corresponding Segal-Sugawara state at site i. It has the property that for any $A = \bigotimes_{i=1}^N A_i \in \mathbb{V}^{\boldsymbol{k}}_0$ we have $\big(\omega^{(i)}\big)_{(0)}A = A_1 \otimes \ldots \otimes A_{i-1} \otimes TA_i \otimes A_{i+1} \otimes \ldots \otimes A_N$. Let us also introduce the $\mathbb{V}^{\boldsymbol{k}}_0$-valued rational function

$$\omega(z) := \sum_{i=1}^N \frac{\omega^{(i)}}{z-z_i}.$$

Consider the state

$$s_1(z) := \varsigma_1(z) + MD^{(1)}_z\omega(z) \in \mathbb{V}^{\boldsymbol{k}}_0 \tag{3.9}$$

depending rationally on z.

Theorem 3.5. *For $i \in \{1,2\}$, we have*

$$s_1(z)_{(0)}\varsigma_i(w) = -D^{(i)}_w A_{1i}(z,w) + T\bigg(B_{1i}(z,w) + D^{(1)}_z\frac{M\varsigma_i(w)}{z-w}\bigg),$$

with the $\mathbb{V}^{\boldsymbol{k}}_0$-valued rational functions $A_{1i}(z,w)$ and $B_{1i}(z,w)$ as in Theorem (3.3).

Proof. We have

$$\omega(z)_{(0)}\varsigma_2(w) = -\frac{1}{z-w}t_{abc}\big(I^a_{-2}(z) - I^a_{-2}(w)\big)I^b_{-1}(w)I^c_{-1}(w)\,|0\rangle^{\boldsymbol{k}}$$
$$= -\frac{2}{M}A_{12}(z,w) + T\bigg(\frac{\varsigma_2(w)}{z-w}\bigg).$$

Similarly, one finds that

$$\omega(z)_{(0)}\varsigma_1(w) = -\frac{1}{M}A_{11}(z,w) + T\bigg(\frac{\varsigma_1(w)}{z-w}\bigg).$$

Acting with the twisted derivative $D_z^{(1)}$ on the above equations we obtain

$$\big(MD_z^{(1)}\omega(z)\big)_{(0)}\varsigma_j(w) = -jD_z^{(1)}A_{1j}(z,w) + TD_z^{(1)}\frac{M\varsigma_j(w)}{z-w},$$

for $j \in \{1,2\}$. The result now follows from adding this to (3.6) with $i = 1$ and using the definition (3.9) of the state $s_1(z)$. □

3.6. **Twisted cycles.** Let us define

$$\mathcal{P}(z) \coloneqq \prod_{j=1}^{N}(z-z_j)^{k_j}.$$

It is a multivalued function on $\mathbb{C} \setminus \{z_1, \dots, z_N\}$.

Pick $i \in \mathbb{Z}$. Let γ be a closed contour in $\mathbb{C} \setminus \{z_1, \dots, z_N\}$ along which there exists a univalued branch of the function $\mathcal{P}(z)^{i/M}$. For example one can let γ be a Pochhammer contour about any pair of the marked points $z_1, \dots, z_N$. The pair, consisting of such a contour γ and univalued branch of $\mathcal{P}(z)^{i/M}$ along it, defines a cycle (possibly the zero cycle) in the twisted homology corresponding to $D_z^{(i)}$ – see [LVY] for the precise definition.

Lemma 3.6. *One has*

$$\int_\gamma \mathcal{P}(z)^{-i/M}D_z^{(i)}f(z)dz = 0$$

for any meromorphic f which is nonsingular on γ.

Proof. This follows by noting that $D_z^{(i)}f(z) = \mathcal{P}(z)^{i/M}\partial_z\big(\mathcal{P}(z)^{-i/M}f(z)\big)$. □

3.7. **Quadratic and cubic Hamiltonians.** Suppose $X, Y \in \mathbb{V}_0^{\boldsymbol{k}}$ are any two states. Consider the formal modes $X_{(0)}, Y_{(0)} \in \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$. One can show that

$$[X_{(0)}, Y_{(0)}] = (X_{(0)}Y)_{(0)}.$$

This is actually a general statement which holds for any vertex algebra. See, for example, [FBZ04, Chapter 4].[1]

It follows that, given a collection of states in $\mathbb{V}_0^{\boldsymbol{k}}$ whose zeroth products vanish, the formal zero modes of these states define a commutative subalgebra of the algebra of observables $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$. In fact the zeroth products do not even need to vanish for this to be true: one can show that $(TZ)_{(0)} = 0$ for any state Z, so it is sufficient that the zeroth products are translates, *i.e.* lie in the image of T.

In our case we have the states $\varsigma_1(z)$ and $\varsigma_2(z)$ which depend rationally on the spectral parameter $z \in \mathbb{C}\setminus\{z_1, \dots, z_N\}$. Their zeroth products, according to Theorem 3.3, are zero modulo translates *and* twisted derivatives in the spectral parameters. Let γ_i and η_i be any cycles of the twisted homology corresponding to $D_z^{(i)}$, for $i \in \{1,2\}$. In view of Lemma 3.6, we have

$$\left[\int_{\gamma_i}\mathcal{P}(z)^{-i/M}\varsigma_i(z)_{(0)}dz, \int_{\eta_j}\mathcal{P}(w)^{-j/M}\varsigma_j(w)_{(0)}dw\right] = 0 \tag{3.10}$$

[1]Note that there the notation for formal modes is *e.g.* $X_{[0]}$, with $X_{(0)}$ reserved for the representative in $\operatorname{End}\mathbb{V}$.

for $i,j \in \{1,2\}$, and we get a commutative subalgebra of the algebra of observables $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, generated by such integrals over twisted cycles.

However, these are not quite the Hamiltonians we want, because their action on highest weight modules is not diagonalizable. Indeed, recall that the *homogeneous gradation* on $U(\widehat{\mathfrak{sl}}_M^{\oplus N})$ is the $\mathbb{Z}$-gradation in which $\deg(X_n^{(i)}) = n$. It induces a $\mathbb{Z}_{\leq 0}$-gradation on $\mathbb{V}_0^{\boldsymbol{k}}$, if we set $\deg(|0\rangle^{\boldsymbol{k}}) = 0$. One can show that if $X \in \mathbb{V}_0^{\boldsymbol{k}}$ has $\deg(X) = k$ then $\deg(X_{(n)}) = 1 + k + n$. Now in our case $\deg(\varsigma_i(z)) = -i - 1$, and hence

$$\deg(\varsigma_i(z)_{(0)}) = -i$$

for $i \in \{1,2\}$. Thus the operators $\int_{\gamma_i} \mathcal{P}(z)^{-i/M} \varsigma_i(z)_{(0)} dz \in \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ are of degree $-i \neq 0$ in the homogeneous gradation. Consequently if M is any graded module over $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ whose subspace of grade n is trivial for sufficiently large n then the operator $\int_{\gamma_i} \mathcal{P}(z)^{-i/M} \varsigma_i(z)_{(0)} dz$ has no non-zero generalised eigenvalue in M.

Instead, what we want are commuting operators in $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ that are of degree zero in the homogeneous gradation. There is a general procedure to construct such operators, which roughly speaking involves a change of coordinate[2] to go from the plane to the cylinder [Nah91]. For our present purposes we just recall the following facts: there is a notion of what we shall call the *Fourier modes*, $X_n \in \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, of any state $X \in \mathbb{V}_0^{\boldsymbol{k}}$. These modes have the property that X_n always has grade n. They obey the commutator formula

$$[X_m, Y_n] = \left(X_{(0)}Y + \sum_{k=1}^{\infty} \frac{m^k X_{(k)} Y}{k!} \right)_{m+n} \tag{3.11}$$

for any states $X, Y \in \mathbb{V}_0^{\boldsymbol{k}}$ and any $n, m \in \mathbb{Z}$. In particular

$$[X_0, Y_0] = (X_{(0)}Y)_0.$$

Also, $(TZ)_0 = 0$ for any state $Z \in \mathbb{V}_0^{\boldsymbol{k}}$.

One has $(x_{-1}^{(i)} |0\rangle^{\boldsymbol{k}})_n = x_n$ for $x \in \mathfrak{sl}_M$ and there is a formula, analogous to the normal ordered product formula (2.4), which allows one to compute by recursion the Fourier modes of a general state $X \in \mathbb{V}_0^{\boldsymbol{k}}$. (See for example equation (7) in [BLZ96].) In general the resulting expressions for modes of states are rather intricate but for $\varsigma_1(z)_0$ and $\varsigma_2(z)_0$ they are simple: one finds

$$\varsigma_1(z)_0 = \frac{\dim(\mathfrak{sl}_M)}{24} \varphi'(z) + \tfrac{1}{2} I_0^a(z) I_0^a(z) + \sum_{n>0} I_{-n}^a(z) I_n^a(z), \tag{3.12}$$

$$\varsigma_2(z)_0 = \tfrac{1}{3} t_{abc} \sum_{j,k \geq 0} \Big(I_{-1-k}^a(z) I_{-1-j}^b(z) I_{2+j+k}^c(z) + 2 I_{-1-k}^a(z) I_{1+k-j}^b(z) I_j^c(z) + I_{-j-k}^a(z) I_k^b(z) I_j^c(z) \Big). \tag{3.13}$$

[2](in the "loop variable", t, not the "spectral variable" z)

Let us now define

$$\hat{Q}_i^{\gamma} := \int_{\gamma} \mathcal{P}(z)^{-i/M} \varsigma_i(z)_0 dz \tag{3.14}$$

for $i \in \{1,2\}$ and for γ any cycle of the twisted homology corresponding to $D_z^{(i)}$.

Recall the Lie algebra $\mathfrak{g} := \widehat{\mathfrak{sl}}_M \oplus \mathbb{C}\mathsf{d}$ from §2.1. Let $\tilde{U}(\mathfrak{g}^{\oplus N}) := \varprojlim U(\mathfrak{g}^{\oplus N})/J_n^N$ be the completion of $U(\mathfrak{g}^{\oplus N})$ where for each $n \geq 0$, J_n^N is the ideal of $U(\mathfrak{g}^{\oplus N})$ generated by $a_r^{(i)}$ for all $r \geq n$, $a \in \mathfrak{sl}_M$ and $i \in \{1,\dots,N\}$. The quadratic Gaudin Hamiltonians are the elements of $\tilde{U}(\mathfrak{g}^{\oplus N})$ defined as

$$\mathcal{H}_i := \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\mathsf{k}^{(i)}\mathsf{d}^{(j)} + \mathsf{d}^{(i)}\mathsf{k}^{(j)} + \sum_{n\in\mathbb{Z}} I_{-n}^{a(i)} I_n^{a(j)}}{z_i - z_j}, \qquad i = 1,\dots,N.$$

For each $i \in \{1,\dots,N\}$ we also have the ith copy of the quadratic Casimir of $\mathfrak{g}$ in $\tilde{U}(\mathfrak{g}^{\oplus N})$, which is defined as

$$\mathcal{C}^{(i)} := (\mathsf{k}^{(i)} + M)\mathsf{d}^{(i)} + \tfrac{1}{2} I_0^{a(i)} I_0^{a(i)} + \sum_{n>0} I_{-n}^{a(i)} I_n^{a(i)}.$$

The algebra $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, introduced in §3.1, is isomorphic to the quotient of $\tilde{U}(\mathfrak{g}^{\oplus N})$ by the ideal generated by $\mathsf{k}^{(i)} - k_i$ and $\mathcal{C}^{(i)}$ for each $i \in \{1,\dots,N\}$.

Recall the state $s_1(z) \in \mathbb{V}_0^{\boldsymbol{k}}$ defined in (3.9).

Lemma 3.7. *The operator $s_1(z)_0$ is the image in $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ of the operator*

$$\sum_{i=1}^{N} \frac{\mathcal{C}^{(i)}}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{\mathcal{H}_i}{z - z_i} \in \tilde{U}(\mathfrak{g}^{\oplus N}).$$

Proof. Computing the Fourier zero mode of the state $\varsigma_1(z)$ we find

$$\varsigma_1(z)_0 = \sum_{i=1}^{N} \frac{1}{(z-z_i)^2}\left(-\frac{\dim(\mathfrak{sl}_M)}{24} k_i + \tfrac{1}{2} I_0^{a(i)} I_0^{a(i)} + \sum_{n>0} I_{-n}^{a(i)} I_n^{a(i)}\right) + \sum_{i=1}^{N} \frac{1}{z-z_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{z_i - z_j} \sum_{n\in\mathbb{Z}} I_{-n}^{a(i)} I_n^{a(j)}.$$

On the other hand, we also have

$$M D_z^{(1)} \omega(z)_0 = -\sum_{i=1}^{N} \frac{(k_i + M)\big(\omega^{(i)}\big)_0}{(z-z_i)^2} - \sum_{i=1}^{N} \frac{1}{z-z_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{k_i\big(\omega^{(j)}\big)_0 + k_j\big(\omega^{(i)}\big)_0}{z_i - z_j}.$$

The result now follows from combining the above and noting that the Fourier zero mode of the Segal-Sugawara state $\omega^{(i)}$, for each $i \in \{1,\dots,N\}$, as defined in (3.8), reads

$$\big(\omega^{(i)}\big)_0 = \frac{\frac{1}{2} I_0^{a(i)} I_0^{a(i)} + \sum_{n>0} I_{-n}^{a(i)} I_n^{a(i)}}{k_i + M} - \frac{k_i \dim(\mathfrak{sl}_M)}{24(k_i + M)}. \qquad \square$$

We now have the following corollary of Theorems 3.3, 3.4 and 3.5.

Theorem 3.8. *The operators $\hat{Q}_i^\gamma \in \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ defined in (3.14) are of grade 0 in the homogeneous gradation. They have the following properties:*

(*i*) *For $i,j \in \{1,2\}$ and any cycles γ and η of the twisted homology corresponding to $D_z^{(i)}$ and $D_z^{(j)}$, respectively, we have*

$$\big[\hat{Q}_i^\gamma, \hat{Q}_j^\eta\big] = 0.$$

(*ii*) *For all $i \in \{1,2\}$, any cycle γ of the twisted homology corresponding to $D_z^{(i)}$ and $j \in \{1,\dots,N\}$ we have*

$$[\mathcal{H}_j, \hat{Q}_i^\gamma] = 0.$$

(*iii*) *For all $i \in \{1,2\}$, any cycle γ of the twisted homology corresponding to $D_z^{(i)}$ and all $x \in \mathfrak{sl}_M$ and $n \in \mathbb{Z}$, we have*

$$[\Delta x_n, \hat{Q}_i^\gamma] = 0. \qquad \square$$

According to the conjecture in [LVY], the Hamiltonians $\hat{Q}_i^\gamma$ can be diagonalized by means of the Bethe ansatz and have certain specific eigenvalues encoded by affine opers. For $\hat{Q}_1^\gamma$ this is known; see [LVY, §5]. In the next two sections we perform a simple Bethe ansatz calculation to check that the conjecture is correct also for $\hat{Q}_2^\gamma$ at least for zero and one Bethe roots.

4. Opers and eigenvalues

The general expectation is that, following the situation in finite types – for which see *e.g.* [Fre05] – the spectrum of Hamiltonians for the Gaudin model associated with $\mathfrak{g}$ should be described in terms of *opers* for the Langlands dual Lie algebra ${}^L\mathfrak{g}$, *i.e.* the Lie algebra with transposed Cartan matrix. Since $\widehat{\mathfrak{sl}}_M$ has symmetric Cartan matrix it is self-dual: ${}^L\widehat{\mathfrak{sl}}_M \cong \widehat{\mathfrak{sl}}_M$. Nevertheless, to keep the general structure in sight and to follow the notation of [LVY] we prefer to distinguish the two copies of $\widehat{\mathfrak{sl}}_M$. Thus, let $\check{e}_i$, $\check{f}_i$ be the Chevalley-Serre generators of the dual copy ${}^L\widehat{\mathfrak{sl}}_M$. They obey the same relations (2.2) as the generators e_i, f_i above but with the roots and coroots α_i and $\check{\alpha}_i$ interchanged. The Cartan subalgebra of ${}^L\widehat{\mathfrak{sl}}_M$ is ${}^L\mathfrak{h} := \mathfrak{h}^*$. Let ${}^L\widehat{\mathfrak{sl}}_M = {}^L\mathfrak{n}_- \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$ be the Cartan decomposition.

The paper [LVY] contains an explicit conjecture of how ${}^L\widehat{\mathfrak{sl}}_M$-opers encode the eigenvalues of the Hamiltonians above, as we now recall.

4.1. **Opers and Miura opers of type ${}^L\widehat{\mathfrak{sl}}_M$.** Pick a derivation element $\rho \in {}^L\mathfrak{h}$ such that $[\rho, \check{e}_i] = \check{e}_i$, $[\rho, \check{f}_i] = -\check{f}_i$ for each $i = 0, 1, \dots, M-1$. Explicitly, one has

$$\rho := M\mathsf{d} + \dot{\rho} \tag{4.1}$$

where $\dot{\rho} := \sum_{i=1}^{M-1} \omega_i \in \dot{\mathfrak{h}}^*$ is the Weyl vector of $\mathfrak{sl}_M$.

Let $p_{-1} := \sum_{i=0}^{M-1} \check{f}_i \in {}^L\mathfrak{n}_-$. A *Miura oper* of type ${}^L\widehat{\mathfrak{sl}}_M$ is a connection of the form

$$d + (p_{-1} + u(z))\,dz \tag{4.2a}$$

where $u(z)$ is a meromorphic function valued in ${}^L\mathfrak{h} = \mathfrak{h}^*$. Since $\{\alpha_i\}_{i=0}^{M-1} \cup \{\rho\}$ is a basis of ${}^L\mathfrak{h}$ we have

$$u(z) = \sum_{i=0}^{M-1} u_i(z)\alpha_i - \frac{\varphi(z)}{M}\rho \tag{4.2b}$$

for some meromorphic functions $\{u_i(z)\}_{i=0}^{M-1}$ and $\varphi(z)$. An ${}^L\widehat{\mathfrak{sl}}_M$-*oper* is a gauge equivalence class of connections of the form

$$d + (p_{-1} + b(z))dz,$$

where $b(z)$ is a meromorphic function valued in ${}^L\mathfrak{b}_+ = {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_+$, under the gauge action of the group ${}^LN_+ = \exp({}^L\mathfrak{n}_+)$. Any ${}^L\widehat{\mathfrak{sl}}_M$-oper has a *quasi-canonical* representative [LVY]: namely, by suitable gauge transformation the connection can be brought to the form

$$\nabla = d + \left(p_{-1} - \frac{\varphi}{M}\rho + \sum_{j\in E} v_j p_j\right)dz, \tag{4.3}$$

where the sum is over the set

$$E = \mathbb{Z}_{\geq 1} \setminus M\mathbb{Z}_{\geq 1}$$

of positive *exponents* of ${}^L\widehat{\mathfrak{sl}}_M$, and where for each $j \in E$, v_j is a meromorphic function. For each $j \in \pm E$, p_j is a certain element of ${}^L\mathfrak{n}_+$ of grade j in the principal gradation, *i.e.*

$$[\rho, p_j] = j\, p_j,$$

such that the following commutation relations hold with p_{-1}:

$$[p_{-1}, p_j] = \begin{cases} 0 & j \in \pm E \setminus \{1\} \\ -\delta & j = 1, \end{cases}$$

where $\delta \coloneqq \sum_{i=0}^{M-1} \alpha_i$ is central in ${}^L\widehat{\mathfrak{sl}}_M$. It can be shown that, for each $j \in \pm E$, a non-zero such p_j exists and is unique up to normalization. Let $\mathfrak{a} \coloneqq \mathbb{C}\{p_j\}_{j\in\pm E} \oplus \mathbb{C}\delta \oplus \mathbb{C}\rho \subset {}^L\widehat{\mathfrak{sl}}_M$. Then $\mathfrak{a}$ is a Lie subalgebra of ${}^L\widehat{\mathfrak{sl}}_M$ called the *principal subalgebra* (the remaining non-trivial commutation relations are $[p_m, p_n] = m\delta_{m+n,0}\delta$). The normalization of the p_j can be so chosen that the restriction to $\mathfrak{a}$ of the bilinear form $(\cdot|\cdot)$ is given by

$$(\delta|\rho) = (\rho|\delta) = M, \qquad (p_m|p_n) = M\delta_{m+n,0}, \qquad m, n \in \pm E.$$

The expression for v_1 below can be found in [LVY, Proposition 3.10].

Proposition 4.1. *The coefficients of p_1 and p_2 of any quasi-canonical form of the ${}^L\widehat{\mathfrak{sl}}_M$-oper defined by the Miura ${}^L\widehat{\mathfrak{sl}}_M$-oper* (4.2) *are given by*

$$v_1 = \frac{1}{M}\big(\tfrac{1}{2}(u|u) + D_z^{(1)}(\rho|u)\big),$$
$$v_2 = \frac{1}{M}\left(-\sum_{i=0}^{M-1} u_i(u_{i+1}^2 - u_{i-1}^2) - \tfrac{1}{2}\sum_{i=0}^{M-1} u_i(u'_{i+1} - u'_{i-1}) + D_z^{(2)} f_2\right),$$

where f_2 is an arbitrary meromorphic function.

Proof. Let $g(z) = \exp(m(z)) \in {}^L N_+$ with m a meromorphic function valued in ${}^L\mathfrak{n}_+$. Let $m = \sum_{n=1}^{\infty} m_n$ be the decomposition of m in the principal gradation.

Let $\nabla_{\partial_z} = \partial_z + p_{-1} + u - \frac{\varphi}{M}\rho$, as in (4.2). We demand that

$$g\nabla_{\partial_z} g^{-1} = \partial_z + p_{-1} - \frac{\varphi}{M}\rho + v_1 p_1 + v_2 p_2 + \dots \tag{4.4}$$

where the dots denote terms of degree at least three in the principal gradation. In grade 0 we have $u + [m_1, p_{-1}] - \frac{\varphi}{M}\rho = -\frac{\varphi}{M}\rho$ and hence $u = -[m_1, p_{-1}]$. Yet since $u = \sum_{i=0}^{M-1} u_i \alpha_i$, we must set

$$m_1 = -\sum_{i=0}^{M-1} u_i \check{e}_i.$$

Now consider the component of equation (4.4) in grade 1. We need

$$[m_2, p_{-1}] + \tfrac{1}{2}[m_1, [m_1, p_{-1}]] + [m_1, u] + \frac{\varphi}{M} m_1 - m_1' = v_1 p_1. \tag{4.5}$$

Consider taking the overlap $(p_{-1}|\cdot)$ of both sides of this equation. Recall the definition of $D_z^{(j)}$ from (3.5). Since $(p_{-1}|p_1) = M$ we find, using the invariance of the bilinear form, that

$$\begin{aligned} Mv_1 &= -\tfrac{1}{2}\big([m_1, p_{-1}]\big|[m_1, p_{-1}]\big) - \big([m_1, p_{-1}]\big|u\big) - D_z^{(1)}(p_{-1}|m_1) \\ &= \tfrac{1}{2}(u|u) + D_z^{(1)}(\rho|u), \end{aligned}$$

where $(p_{-1}|m_1) = ([p_{-1}, \rho]|m_1) = (\rho|[m_1, p_{-1}]) = -(\rho|u)$. (The equation (4.5) also fixes the component of m_2 in the orthogonal complement $\mathfrak{c}_2$ of $\mathfrak{a}_2$, but we won't need that here.)

We turn to the component of equation (4.4) in grade 2. We have

$$\begin{aligned} (g\partial_z g^{-1})_2 &= -m_2' - \tfrac{1}{2}[m_1, m_1'], \\ (gp_{-1}g^{-1})_2 &= [m_3, p_{-1}] + \tfrac{1}{2}[m_1, [m_2, p_{-1}]] + \tfrac{1}{2}[m_2, [m_1, p_{-1}]] + \tfrac{1}{6}[m_1, [m_1, [m_1, p_{-1}]]] \\ &= [m_3, p_{-1}] + \tfrac{1}{2}[[m_1, m_2], p_{-1}] - [m_2, u] - \tfrac{1}{6}[m_1, [m_1, u]], \\ (gug^{-1})_2 &= [m_2, u] + \tfrac{1}{2}[m_1, [m_1, u]], \\ -(g\rho g^{-1})_2 &= -[m_2, \rho] - \tfrac{1}{2}[m_1, [m_1, \rho]] = 2m_2. \end{aligned}$$

Hence we need

$$\big[m_3 + \tfrac{1}{2}[m_1, m_2], p_{-1}\big] + \tfrac{1}{3}[m_1, [m_1, u]] - \tfrac{1}{2}[m_1, m_1'] - D_z^{(2)} m_2 = v_2 p_2.$$

Taking $(p_{-2}|\cdot)$ of both sides and then using $(p_{-2}|p_2) = M$, the invariance of $(\cdot|\cdot)$ and the relation $[p_{-1}, p_{-2}] = 0$, we find

$$Mv_2 = -\tfrac{1}{3}\left([m_1, p_{-2}]|[m_1, u]\right) + \tfrac{1}{2}\left([m_1, p_{-2}]\big|m_1'\right) - D_z^{(2)}(p_{-2}|m_2).$$

Now in fact, in type A, we have $p_{-2} = -\sum_{i=0}^{M-1}[\check{f}_i, \check{f}_{i+1}]$ (addition of indices modulo M). Hence

$$[m_1, p_{-2}] = \sum_{i,j=0}^{M-1} u_i [\check{e}_i, [\check{f}_j, \check{f}_{j+1}]] = -\sum_{i=0}^{M-1} u_i (\check{f}_{i-1} - \check{f}_{i+1})$$

and

$$[m_1, u] = -\sum_{i,j=0}^{M-1} u_i u_j [\check{e}_i, \alpha_j] = \sum_{i,j=0}^{M-1} a_{ij} u_i u_j \check{e}_i.$$

Since the coefficient of p_2 of an ${}^L\widehat{\mathfrak{sl}}_M$-oper in quasi-canonical form is defined only up to twisted derivative, we therefore have

$$\begin{aligned} Mv_2 &= \tfrac{1}{3} \sum_{i,j,k=0}^{M-1} a_{ij}(\delta_{i,k-1} - \delta_{i,k+1}) u_i u_j u_k - \tfrac{1}{2} \sum_{i,j=0}^{M-1} (\delta_{i,j-1} - \delta_{i,j+1}) u_i u'_j + D_z^{(2)} f_2 \\ &= -\sum_{i=0}^{M-1} u_i (u_{i+1}^2 - u_{i-1}^2) - \tfrac{1}{2} \sum_{i=0}^{M-1} u_i (u'_{i+1} - u'_{i-1}) + D_z^{(2)} f_2, \end{aligned}$$

for an arbitrary meromorphic function f_2. Here we used the fact that the Chevalley-Serre generators of ${}^L\widehat{\mathfrak{sl}}_M$ are normalised as $(\check{e}_i | \check{f}_j) = \delta_{ij}$. □

5. Bethe Ansatz for 0 and 1 roots

5.1. **Homogeneous vs. principal gradations.** Let us write $q(z) \in \tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ for the Fourier zero mode of the state $\varsigma_2(z)$ as in (3.13),

$$\begin{aligned} q(z) \coloneqq \varsigma_2(z)_0 = \tfrac{1}{3} t_{abc} \sum_{j,k \geq 0} \Big(& I^a_{-1-k}(z) I^b_{-1-j}(z) I^c_{2+j+k}(z) \\ & + 2 I^a_{-1-k}(z) I^b_{1+k-j}(z) I^c_j(z)) + I^a_{-j-k}(z) I^b_k(z) I^c_j(z) \Big). \end{aligned}$$

By definition of $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$, $q(z)$ is an infinite sum

$$q(z) = q(z)_0^{\mathrm{hom}} + q(z)_1^{\mathrm{hom}} + q(z)_2^{\mathrm{hom}} + \dots$$

such that, for each $n \geq 1$ we have $q(z) = q(z)_0^{\mathrm{hom}} + \dots + q(z)_{n-1}^{\mathrm{hom}}$ modulo J_n^N. Recall that J_n^N is the left ideal in $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ generated by elements of grade at least n in the homogeneous gradation. Let $J_n^{N,\mathrm{prin}}$ denote the left ideal generated by elements of grade at least n in the principal gradation. For each $n \geq 1$, there is a nesting of ideals $J_{Mn}^{N,\mathrm{prin}} \subset J_n^N \subset J_{Mn-M+1}^{N,\mathrm{prin}}$. So we could equivalently have defined $\tilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N})$ as the completion with respect to the inverse system $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}) \big/ J_n^{N,\mathrm{prin}}$.

Let $\mathfrak{b}_- \coloneqq \mathfrak{n}_- \oplus \mathfrak{h}$. Denote by $U_{\boldsymbol{k}}(\mathfrak{b}_-^{\oplus N})$ the quotient of $U(\mathfrak{b}_-^{\oplus N})$ by the two-sided ideal generated by $\mathsf{k}^{(i)} - k_i$ for all $i \in \{1, \dots, N\}$, cf. §3.1. We have the vector space isomorphism $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_M^{\oplus N}) \cong_{\mathbb{C}} U_{\boldsymbol{k}}(\mathfrak{b}_-^{\oplus N}) \otimes U(\mathfrak{n}_+^{\oplus N})$. Let $U(\mathfrak{n}_+^{\oplus N})_n$ denote the subspace of $U(\mathfrak{n}_+^{\oplus N})$ spanned by elements of total degree n in the principal gradation. We have the unique decomposition

$$q(z) = q(z)_0 + q(z)_1 + q(z)_2 + \dots \tag{5.1}$$

where $q(z)_n \in U_{\boldsymbol{k}}(\mathfrak{b}_-^{\oplus N}) \otimes U(\mathfrak{n}_+^{\oplus N})_n$.

Recall the definition (4.1) of the derivation element ρ which measures the grade in the principal gradation. Let $\check{\eta}_i \in \mathfrak{h}$ be the set of fundamental coweights adapted

to the principal gradation; namely, such that $\{\check\eta_i\}_{i=0}^{M-1} \cup \{\mathsf{k}\} \subset \mathfrak{h}$ is the dual basis to $\{\alpha_i\}_{i=0}^{M-1} \cup \{\rho/M\} \subset \mathfrak{h}^*$:

$$\langle \alpha_i, \check\eta_j \rangle = \delta_{ij}, \quad \langle \rho, \check\eta_j \rangle = 0, \quad \langle \alpha_i, \mathsf{k} \rangle = 0, \quad \langle \rho, \mathsf{k} \rangle = M$$

for all $i, j \in \{0, 1, \dots, M-1\}$.

Proposition 5.1. *The first two terms in the sum* (5.1) *are*

$$q(z)_0 = -\sum_{i,j,k=0}^{M-1} \check\eta_i(z)\big(\check\eta_{i+1}(z)^2 - \check\eta_{i-1}(z)^2\big) + \tfrac{1}{2}\sum_{i,j=0}^{M-1} \check\eta_i(z)\left(\check\eta'_{i+1}(z) - \check\eta'_{i-1}(z)\right),$$
$$q(z)_1 = \sum_{i,j=0}^{M-1} f_i(z)\left(\check\eta_{i+1}(z) - \check\eta_{i-1}(z)\right) e_i(z), \tag{5.2}$$

modulo twisted derivatives of degree 2.

Proof. By direct calculation. □

5.2. **Spin chain $\mathbb{M}_{\boldsymbol\lambda}$.** For each $i \in \{1, \dots, N\}$ let $\lambda_i \in \mathfrak{h}^*$ be a weight of $\widehat{\mathfrak{sl}}_M$ such that

$$\langle \lambda_i, \mathsf{k} \rangle = k_i. \tag{5.3}$$

Let $\mathbb{M}_{\lambda_i}$ denote the Verma module over $\mathfrak{g}$ of highest weight λ_i and define

$$\mathbb{M}_{\boldsymbol\lambda} \coloneqq \mathbb{M}_{\lambda_1} \otimes \ldots \otimes \mathbb{M}_{\lambda_N}.$$

It is a smooth module over $\widehat{\mathfrak{sl}}_M^{\oplus N}$ of level $\boldsymbol{k}$, and therefore a module over the algebra of observables $\tilde U_{\boldsymbol k}(\widehat{\mathfrak{sl}}_M^{\oplus N})$. Let $|\boldsymbol\lambda\rangle$ denote the highest weight vector in $\mathbb{M}_{\boldsymbol\lambda}$.

5.3. **Vacuum eigenvalues.** For this subsection, consider the Miura ${}^L\widehat{\mathfrak{sl}}_M$-oper (4.2), with

$$u(z) = u^{(0)}(z) \coloneqq -\sum_{j=1}^{N} \frac{\lambda_j}{z - z_j}.$$

That is, $u_i(z) = -\sum_{j=1}^N \frac{\langle \lambda_j, \check\eta_i \rangle}{z - z_j}$ for each $i \in \{0, 1, \dots, M-1\}$, and $\varphi(z)$ is the twist function as in (3.4), by virtue of (5.3). Let $v_1(z)$ and $v_2(z)$ be the coefficients of a quasi-canonical form of the underlying ${}^L\widehat{\mathfrak{sl}}_M$-oper, as in Proposition 4.1.

Proposition 5.2. *Up to twisted derivatives of degree 2, the vacuum eigenvalue of $q(z)$ is $-Mv_2(z)$,* i.e.

$$q(z)\,|\boldsymbol\lambda\rangle = -Mv_2(z)\,|\boldsymbol\lambda\rangle + D_z^{(2)}|\varepsilon(z)\rangle$$

for some vector $|\varepsilon(z)\rangle \in \mathbb{M}_{\boldsymbol\lambda}$ depending rationally on z. In particular, for any cycle γ in the twisted homology corresponding to $D_z^{(2)}$ we have

$$\hat Q_2^\gamma\,|\boldsymbol\lambda\rangle = -M\int_\gamma \mathcal{P}(z)^{-2/M} v_2(z)dz\ \ |\boldsymbol\lambda\rangle\,.$$

Proof. We have $q(z)_n \,|\boldsymbol{\lambda}\rangle = 0$ for all $n \geq 1$, *i.e.* $q(z)\,|\boldsymbol{\lambda}\rangle = q(z)_0\,|\boldsymbol{\lambda}\rangle$. By definition of the $\check{\eta}_i$,

$$\check{\eta}_i(z)\,|\boldsymbol{\lambda}\rangle = -\langle u(z), \check{\eta}_i\rangle\,|\boldsymbol{\lambda}\rangle = -u_i(z)\,|\boldsymbol{\lambda}\rangle$$

and then the statement follows from Propositions 4.1 and 5.1. □

5.4. **Eigenvalues at depth 1.** Let $w \in \mathbb{C}\setminus\{z_1,\dots,z_N\}$ be an additional point in the complex plane, distinct from the marked points $z_1,\dots,z_N$. Pick $n \in \{0,1,\dots,M-1\}$. Now, for this subsection, we consider the Miura ${}^L\widehat{\mathfrak{sl}}_M$-oper, (4.2), with

$$u(z) = u^{(0)}(z) + \frac{\alpha_n}{z-w} = -\sum_{j=1}^{N}\frac{\lambda_i}{z-z_i} + \frac{\alpha_n}{z-w} \tag{5.4}$$

and let $v_1(z)$ and $v_2(z)$ be the coefficients of a quasi-canonical form of its underlying ${}^L\widehat{\mathfrak{sl}}_M$-oper, as in Proposition 4.1.

The *weight function* or *Schechtman-Varchenko vector* for one Bethe root of colour $n \in \{0,1,\dots,M-1\}$ is by definition $f_n(w)\,|\boldsymbol{\lambda}\rangle$, where

$$f_n(w) = \sum_{j=1}^{N}\frac{f_n^{(i)}}{w-z_i}$$

using the notation (3.3).

Proposition 5.3. *Suppose w obeys the Bethe equation*

$$0 = \sum_{j=1}^{N}\frac{\langle\lambda_j,\check{\alpha}_n\rangle}{w-z_j}.$$

Then, up to twisted derivatives of degree 2, the weight function $f_n(w)\,|\boldsymbol{\lambda}\rangle$ is an eigenstate of $q(z)$ with eigenvalue $-Mv_2(z)$, i.e.

$$q(z)f_n(w)\,|\boldsymbol{\lambda}\rangle = -Mv_2(z)f_n(w)\,|\boldsymbol{\lambda}\rangle + D_z^{(2)}|\tilde{\varepsilon}(z)\rangle,$$

for some vector $|\tilde{\varepsilon}(z)\rangle \in \mathbb{M}_{\boldsymbol{\lambda}}$ depending rationally on z. In particular, for any cycle γ in the twisted homology corresponding to $D_z^{(2)}$ we have

$$\hat{Q}_2^{\gamma} f_n(w)\,|\boldsymbol{\lambda}\rangle = -M\int_{\gamma}\mathcal{P}(z)^{-2/M}v_2(z)dz\; f_n(w)\,|\boldsymbol{\lambda}\rangle\,.$$

Proof. By definition $[\check{\eta}_i, f_n] = -f_n\langle\alpha_n,\check{\eta}_i\rangle = -\delta_{in}f_n$. Using this and Lemma 3.2, one computes, starting from the expressions for $q(z)_0$ in Proposition 5.1,

$$\begin{aligned}[q(z)_0, f_n(w)]\,|\boldsymbol{\lambda}\rangle = \Bigg(&-\sum_{i=0}^{M-1}a_{ni}u_i^{(0)}(z)\frac{u_{n+1}^{(0)}(z)-u_{n-1}^{(0)}(z)}{z-w}\\ &-\frac{u_{n+1}^{(0)}(z)-u_{n-1}^{(0)}(z)}{2(z-w)^2}+\frac{u_{n+1}^{(0)\prime}(z)-u_{n-1}^{(0)\prime}(z)}{2(z-w)}\Bigg)f_n(w)\,|\boldsymbol{\lambda}\rangle\\ +\Bigg(&\sum_{i=0}^{M-1}a_{ni}u_i^{(0)}(z)\frac{u_{n+1}^{(0)}(z)-u_{n-1}^{(0)}(z)}{z-w}-\partial_z\frac{u_{n+1}^{(0)}(z)-u_{n-1}^{(0)}(z)}{z-w}\Bigg)f_n(z)\,|\boldsymbol{\lambda}\rangle\,.\end{aligned}$$

Here one recognizes the first term on the right hand side as the required correction to the vacuum value of $-Mv_2(z)$, cf. (5.4) and Proposition 4.1. Next, one finds

$$\begin{aligned}[q(z)_1, f_n(w)]\,|\boldsymbol{\lambda}\rangle &= \sum_{i=0}^{M-1} [f_i(z)\,(\check\eta_{i+1}(z) - \check\eta_{i-1}(z))\,e_i(z), f_n(w)]\,|\boldsymbol{\lambda}\rangle \\ &= -f_n(z)\,(\check\eta_{n+1}(z) - \check\eta_{n-1}(z))\,\frac{\check\alpha_n(z) - \check\alpha_n(w)}{z-w}\,|\boldsymbol{\lambda}\rangle \\ &= -f_n(z)\,|\boldsymbol{\lambda}\rangle\,\big(u^{(0)}_{n+1}(z) - u^{(0)}_{n-1}(z)\big)\frac{\langle u^{(0)}(z), \check\alpha_n\rangle - \langle u^{(0)}(w), \check\alpha_n\rangle}{z-w}.\end{aligned}$$

Now we have

$$\langle u(z)^{(0)}, \check\alpha_n\rangle = -\frac{\varphi(z)}{M}\langle\rho, \check\alpha_n\rangle + \sum_{i=0}^{M-1}\langle\alpha_i, \check\alpha_n\rangle u_i^{(0)}(z) = -\frac{\varphi(z)}{M} + \sum_{i=0}^{M-1} a_{ni}u_i^{(0)}(z).$$

Hence, one arrives at

$$\begin{aligned}q(z)f_n(w)\,|\boldsymbol{\lambda}\rangle = -Mv_2(z)f_n(w)\,|\boldsymbol{\lambda}\rangle - D_z^{(2)}\Big(\tfrac{1}{2}\big(u^{(0)}_{n+1}(z) - u^{(0)}_{n-1}(z)\big)\frac{f_n(z)}{z-w}\,|\boldsymbol{\lambda}\rangle\Big) \\ + \langle u^{(0)}(w), \check\alpha_n\rangle\big(u^{(0)}_{n+1}(z) - u^{(0)}_{n-1}(z)\big)\frac{f_n(z)}{z-w}\,|\boldsymbol{\lambda}\rangle\,.\end{aligned}$$

The second line on the right hand side vanishes by the Bethe equation. □

6. Two-point case, coset construction and the W3 algebra

In this section we specialize to the case of $N = 2$ marked points. For convenience, let us choose them to be $z_1 = 0$ and $z_2 = 1$. Let γ denote a Pochhammer contour around these two points. For $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$ let $(a)_n$ denote the falling factorial

$$(a)_n := \prod_{k=0}^{n-1}(a-k) = a(a-1)\dots(a-n+1)$$

with $(a)_0 := 1$. For $a, b \in \mathbb{C}$, define

$$B(a,b) := \int_\gamma z^a(z-1)^b dz.$$

Lemma 6.1.

$$B(a-1,b) = \frac{a+b+1}{a}B(a,b), \qquad B(a,b-1) = -\frac{a+b+1}{b}B(a,b).$$

Proof. Since the contour γ has no boundary, $aB(a-1,b)+bB(a,b-1) = 0$. From the identity $\frac{1}{z(z-1)} - \frac{1}{z-1} + \frac{1}{z} = 0$, one sees that $B(a-1,b-1) - B(a,b-1) + B(a-1,b) = 0$. The result follows. □

Define

$$\begin{aligned}\omega &\coloneqq \omega^{(1)} + \omega^{(2)} - \omega^{(diag)} \\ &= \frac{1}{2(k_1+M)} I_{-1}^{a(1)} I_{-1}^{a(1)} \left|0\right\rangle^{\boldsymbol{k}} + \frac{1}{2(k_1+M)} I_{-1}^{a(2)} I_{-1}^{a(2)} \left|0\right\rangle^{\boldsymbol{k}} \\ &\qquad - \frac{1}{2(k_1+k_2+M)} (I_{-1}^{a(1)} + I_{-1}^{a(2)})(I_{-1}^{a(1)} + I_{-1}^{a(2)}) \left|0\right\rangle^{\boldsymbol{k}}.\end{aligned}$$

Define also the state $\mathsf{W} \in \mathbb{V}_0^{\boldsymbol{k}}$ by

$$\begin{aligned}\mathsf{W} \coloneqq C(k_1,k_2)\Big(&\tfrac{1}{3} t_{abc} I_{-1}^{a(1)} I_{-1}^{b(1)} I_{-1}^{c(1)} \left|0\right\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_2\right)\left(-\tfrac{2}{M}k_2 - 1\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &- t_{abc} I_{-1}^{a(1)} I_{-1}^{b(1)} I_{-1}^{c(2)} \left|0\right\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1 - 2\right)\left(-\tfrac{2}{M}k_2 - 1\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &+ t_{abc} I_{-1}^{a(1)} I_{-1}^{b(2)} I_{-1}^{c(2)} \left|0\right\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1 - 1\right)\left(-\tfrac{2}{M}k_1 - 2\right)\left(-\tfrac{2}{M}k_2 - 2\right) \\ &- \tfrac{1}{3} t_{abc} I_{-1}^{a(2)} I_{-1}^{b(2)} I_{-1}^{c(2)} \left|0\right\rangle^{\boldsymbol{k}} \left(-\tfrac{2}{M}k_1\right)\left(-\tfrac{2}{M}k_1 - 1\right)\left(-\tfrac{2}{M}k_1 - 2\right)\Big),\end{aligned}$$

where the normalization factor $C(k_1,k_2)$ is given by

$$\begin{aligned}C(k_1,k_2) &\coloneqq -\frac{M^3}{4} \frac{1}{(k_1+M)(k_2+M)(k_1+k_2+M)} D(k_1,k_2), \\ D(k_1,k_2)^2 &\coloneqq \frac{-M}{2(M+2k_1)(M+2k_2)(3M+2k_1+2k_2)(M^2-4)}.\end{aligned}$$

Proposition 6.2. *Up to normalization factors depending only on the levels k_1, k_2, the states ω and W are equal to $\int_\gamma \mathcal{P}(z)^{-1/M}\varsigma_1(z)dz$ and $\int_\gamma \mathcal{P}(z)^{-2/M}\varsigma_2(z)dz$ respectively. Namely,*

$$\begin{aligned}\omega &= \frac{k_1 k_2}{(k_1+k_2+M)(k_1+k_2)(k_1+k_2-M)} \frac{\int_\gamma \mathcal{P}(z)^{-1/M}\varsigma_1(z)dz}{\int_\gamma \mathcal{P}(z)^{-1/M}dz}, \\ \mathsf{W} &= \qquad \frac{C(k_1,k_2)(-\frac{2}{M}k_1)_3(-\frac{2}{M}k_2)_3}{(-\frac{2}{M}k_1 - \frac{2}{M}k_2 + 1)_3} \qquad \frac{\int_\gamma \mathcal{P}(z)^{-2/M}\varsigma_2(z)dz}{\int_\gamma \mathcal{P}(z)^{-2/M}dz}.\end{aligned}$$

Proof. From the definition, §3.3, of $\varsigma_1(z)$ we have, using Lemma 6.1,

$$\int_\gamma \mathcal{P}(z)^{-1/M}\varsigma_1(z)dz = -\int_\gamma \mathcal{P}(z)^{-1/M}\left(\frac{1}{z} - \frac{1}{z-1}\right)dz \times \Xi \tag{6.1}$$

where one finds

$$\Xi = -k_1\omega^{(2)} - k_2\omega^{(1)} + I_{-1}^{a(1)} I_{-1}^{a(2)} \left|0\right\rangle^{\boldsymbol{k}} = -(k_1+k_2+M)\omega.$$

The second equality is by a short calculation starting from the definition of ω above. By further use of Lemma 6.1, one has the first part. Similarly, from the definition of

$\varsigma_2(z)$ in §3.3 we have

$$\begin{aligned}\int_\gamma \mathcal{P}(z)^{-2/M}\varsigma_2(z)dz = \tfrac{1}{3}t_{abc}I_{-1}^{a(1)}I_{-1}^{b(1)}I_{-1}^{c(1)}\,|0\rangle^{\boldsymbol{k}}\,B(-\tfrac{2}{M}k_1-3,-\tfrac{2}{M}k_2)\\ + t_{abc}I_{-1}^{a(1)}I_{-1}^{b(1)}I_{-1}^{c(2)}\,|0\rangle^{\boldsymbol{k}}\,B(-\tfrac{2}{M}k_1-2,-\tfrac{2}{M}k_2-1)\\ + t_{abc}I_{-1}^{a(1)}I_{-1}^{b(2)}I_{-1}^{c(2)}\,|0\rangle^{\boldsymbol{k}}\,B(-\tfrac{2}{M}k_1-1,-\tfrac{2}{M}k_2-2)\\ + \tfrac{1}{3}t_{abc}I_{-1}^{a(2)}I_{-1}^{b(2)}I_{-1}^{c(2)}\,|0\rangle^{\boldsymbol{k}}\,B(-\tfrac{2}{M}k_1,-\tfrac{2}{M}k_2-3)\end{aligned}$$

and the second result follows by repeated use of Lemma 6.1. □

By the coset construction [GKO86], the state $\omega \in \mathbb{V}_0^{\boldsymbol{k}}$ is a conformal vector: it generates a copy of the Virasoro vertex algebra with central charge

$$c = \dim(\mathfrak{sl}_M)\left(\frac{k_1}{k_1+M}+\frac{k_2}{k_2+M}-\frac{k_1+k_2}{k_1+k_2+M}\right).$$

That is,

$$\omega_{(n)}\omega = \delta_{n,0}T(\omega)+\delta_{n,1}2\omega+\delta_{n,3}\tfrac{1}{2}c\,|0\rangle^{\boldsymbol{k}}$$

for $n \in \mathbb{Z}_{\geq 0}$.

The state W was constructed in [BBSS88] (see equation 2.8 of that paper, where the field $Y[W,u]$ is given). Suppose we specialize further to the case of $M=3$, *i.e.* $\widehat{\mathfrak{sl}}_3$, and choosing the irreducible module at the marked point $z_2=1$ to have highest weight

$$\lambda_2 = \Lambda_0.$$

so that, in particular, $k_2=1$. It was shown in [BBSS88] that in that case the states W and ω generate a copy of the W_3 algebra. That is, one has – see *e.g.* [BMP96] –

$$\omega_{(n)}\mathsf{W} = \delta_{n,0}T(\mathsf{W})+\delta_{n,1}3\mathsf{W},$$

$$\begin{aligned}\mathsf{W}_{(n)}\mathsf{W} = \delta_{n,0}\left(\beta T(\Lambda)+\frac{1}{15}T^3(\omega)\right)+\delta_{n,1}\left(2\beta\Lambda+\frac{3}{10}T^2(\omega)\right)\\ +\delta_{n,2}T(\omega)+\delta_{n,3}2\omega+\delta_{n,5}\tfrac{1}{3}c\,|0\rangle^{\boldsymbol{k}},\end{aligned}\quad (6.2)$$

for $n \in \mathbb{Z}_{\geq 0}$. Here

$$\Lambda \coloneqq \omega_{(-1)}\omega-\frac{3}{10}T^2(\omega),\quad \text{and}\quad \beta=\frac{16}{22+5c}.$$

Remark. The W_3 algebra is known to have a commutative algebra of Quantum Integral of Motions (of $\widehat{\mathfrak{sl}}_3$-(m)-KdV, *i.e.* quantum Boussinesq theory) indexed by the exponents of $\widehat{\mathfrak{sl}}_3$. The first few integrals of motion I_1, I_2, I_4, I_5 can be found in [BHK02].[3] The first two are $I_1=\omega_0$ and $I_2=\mathsf{W}_0$. By Proposition 6.2, these are proportional to our first two Hamiltonians $\hat{Q}_1^\gamma$ and $\hat{Q}_2^\gamma$. It is natural to think that the higher integrals of motion are likewise the two-point specializations of the (conjectural) higher Gaudin Hamiltonians. (This was suggested in the $\widehat{\mathfrak{sl}}_2$ case in [FF11, §6.4]). We have checked that the vacuum eigenvalues of I_4 and I_5 indeed agree with the eigenvalues

[3]Note that $\mathsf{W}_{BBSS}=\sqrt{3\beta}\mathsf{W}_{BHK}$.

$\int_\gamma \mathcal{P}(z)^{-4/3} v_4(z)dz$ and $\int_\gamma \mathcal{P}^{-5/3}(z)v_5(z)dz$ (see §4) predicted in [LVY], up to overall factors depending on the remaining weight λ_1 only through the level k_1. The analogous check also works for I_3 and I_5 in the $\widehat{\mathfrak{sl}}_2$ case. ◁

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