Doctoral School: ED 352
Physique et Science de la Matière

A Dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor in Philosophy at Aix–Marseille University

In subject of
Theoretical and Mathematical Physics

Conformal Gauge Theories, Cartan Geometry and Transitive Lie Algebroids

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Résumé

Le présent mémoire présente le travail effectué durant mes trois années de doctorat au Centre de Physique Théorique de l'Université d'Aix–Marseille.

Notre connaissance de l'Univers est basée sur l'existence de quatre *interactions fondamentales*, qui sont la gravitation d'un côté, et les interactions électromagnétique, faible et forte de l'autre. La description mathématique de ces interactions (au niveau classique, c'est-à-dire sans englober leur comportement quantique) se fait dans le cadre de ce qu'on appelle les *théories de jauge*, toutes basées, malgré leurs différences au niveau phénoménologique, sur un même principe de *symétrie* et formulées dans des structures mathématiques similaires. L'idée de la symétrie de jauge est que les objets fondamentaux de ces théories, les *champs*, admettent certaines transformations qui ne changent pas la description physique fournie par les-dites théories. Ainsi, il existe une redondance dans notre description, qui n'est pas complètement arbitraire mais au contraire est structurée par l'action d'un *groupe de jauge*. Les cadres mathématiques sous–jacents à la formulation des théories de jauge sont donc très similaires, même si, encore une fois, les différentes interactions ne se manifestent pas du tout de la même manière à notre échelle. Ce travail est une exploration et une construction explicite de nouveaux cadres possibles pour (re)formuler ces théories physiques, dans une démarche de *physique mathématique*. Ici, nous voyons en effet la physique mathématique comme l'analyse des structures mathématiques utilisées pour formuler les théories physiques décrivant les interactions fondamentales, avec un double objectif. D'une part, mieux comprendre ces cadres mathématiques afin de bien distinguer ce qui semble être *nécessaire* à la formulation de la–dite théorie, et ce qui ne semble relever que de la *contingence* (souvent historique), en la reformulant possiblement dans un autre language; d'autre part, une fois reformulées, généraliser ces constructions, dans une démarche plus mathématicienne, mais toujours dans l'optique d'obtenir un cadre plus à même de décrire de la physique et surtout de mieux la comprendre.

Après une introduction qui donne une vision de la physique moderne des inte-

ractions fondamentales, autant au niveau des cadres mathématiques utilisés que du statut philosophique des entités sous–jacentes, le corps de la thèse consiste en une présentation principalement mathématique de divers cadres possibles pour formuler les théories physiques décrivant les interactions fondamentales. Le premier chapitre consiste en un rappel de notions de base de géométrie différentielle ainsi qu'une présentation des algébroïdes de Lie transitifs, de leur intégration possible en des groupoïdes de Lie, et se termine sur la présentation de la méthode de l'habillage (*dressing*). Cette méthode a été développée ces dernières années et consiste en un changement de variables de champs tel que dans ces nouvelles variables, une partie de la symétrie de jauge a été effacée. C'est une méthode géométrique de réduction de symétrie, distincte du fixage de jauge ou d'une brisure spontanée de symétrie. Les quatre autres chapitres sont relativement indépendants. Le chapitre deux consiste en une présentation de la géométrie de Cartan, généralisation de la géométrie de Klein qui étudie les espaces homogènes via leurs groupes de symétrie. Ce cadre est ensuite utilisé pour formuler des théories de la gravitation (c'est–à–dire ce qu'on a appelé des théories de jauge à symétrie *externe*) dans le langage des fibrés, i.e. dans la même forme que les théories de jauge à symétrie *interne* utilisées pour formuler les trois autres interactions. Le chapitre trois est consacré à la géométrie conforme. Une vue d'ensemble de la géométrie conforme est donnée. Après avoir présenté les objets usuellement définis sur une variété conforme, on va les traduire dans le langage de la géométrie de Cartan associée. On donne aussi explicitement que possible toutes les manières de construire le fibré des 2–repères conformes, pour pouvoir ensuite définir une connexion de Cartan sur ce fibré qui soit équivalente à la donnée d'une classe conforme de métriques. Une fois posé ce cadre, on y applique la méthode de l'habillage (très bien adaptée à la formulation en termes de connexions sur des fibrés), et on retrouve d'une manière très naturelle des objets tels que les *Tractors* et les *Twistors*, avec une compréhension plus profonde de leur nature géométrique, notamment en ce qui concerne les transformations de jauge auxquelles ils sont soumis. Le chapitre quatre présente quant à lui deux manières d'utiliser les algébroïdes de Lie transitifs pour formuler des théories de jauge "unifiées", dans le sens où l'écriture naturelle d'un lagrangien dans ce cadre va automatiquement englober des termes qui usuellement sont mis côte à côte *à la main*. Le premier exemple reprend un résultat de N. Boroojerdian, reformulé dans notre langage, qui parvient, à l'aide d'une notion de connexion de Levi–Civita sur un algébroïde de Lie transitif, à obtenir un lagrangien où sont présents à la fois un terme de Yang–Mills pour une connexion ordinaire (donc décrivant une interaction de type électrofaible ou forte), et un terme de type Einstein–Hilbert pour la métrique sur l'espace tangent (donc décrivant la gravitation), ainsi qu'un terme de type constante cosmologique, d'origine algébrique. Le second exemple repose sur une notion de connexion généralisée, qui possède des degrés de liberté purement algébriques en plus de ses degrés de liberté géométriques qui va pouvoir décrire, dans un lagrangien généralisé de type Yang–Mills, un terme de potentiel de Higgs, naturellement présent, en plus du terme usuel pure jauge de Yang–Mills. Le dernier chapitre, finalement, présente un travail effectué récemment qui consiste en la formulation de la géométrie de Cartan en termes d'algébroïdes de Lie transitifs. Cette construction est basée sur un diagramme reposant sur les suites d'Atiyah associées aux fibrés principaux relatifs à la géométrie de Cartan considérée. Une conclusion est ensuite donnée, avec des possibles lignes de recherches futures pouvant généraliser encore plus radicalement ces différents cadres.

Cadre Géométrique et Algébrique

Le cadre géométrique naturel des théories de jauge à symétrie interne (qu'on appelera aussi simplement "théories de jauge internes"), qui décrivent toutes les interactions fondamentales sauf la gravitation, est celui des connexions sur des fibrés principaux. Ce premier chapitre commence par donner la définition, fondamentale, de variété différentielle de dimension n. Une telle variété M est une variété topologique (c'est–à–dire qui ressemble localement à $\mathbb{R}^n$, la "ressemblance" se construisant à l'aide d'un atlas consistant en des cartes locales recollées entre elles de manière continue) qui est telle que les changements de cartes, en plus d'être continus, sont de classe $\mathcal{C}^\infty$. Une structure différentielle est alors définie comme une classe d'équivalence d'atlas sur cette variété. C'est la notion fondamentale à l'élaboration de tout le cadre mathématique nécessaire à la formulation des théories de jauge.

Nous présentons ensuite la notion de fibré tangent, qui consiste en la donnée, en chaque point x de la variété, de l'espace tangent T_xM (qui est isomorphe à $\mathbb{R}^n$ en termes d'espace vectoriel), et qui est l'objet de base de l'étude de la géométrie de la variété M. Les sections du fibré tangent sont appelées les champs de vecteurs et sont notés $X \in \Gamma(TM)$. Ils peuvent être vus également comme les dérivations de l'algèbre des fonctions $\mathcal{C}^\infty(M)$. La dérivée de $f \in \mathcal{C}^\infty(M)$ par le vecteur $X \in \Gamma(TM)$ est notée $X \cdot f$. Les champs de vecteurs $\Gamma(TM)$ possèdent une structure d'algèbre de Lie, et donc un crochet $[\ ,\ ]$ qui vérifie:

$$[X, fY] = (X \cdot f)Y + f[X, Y].$$

Une notion plus générale est celle de fibré vectoriel, où, en chaque point de la variété, est donné un espace vectoriel $\mathbb{V}$ de dimension $k > 0$ a priori quelconque. Un cas particulier de fibré vectoriel est le cas d'un algébroïde de Lie A, dont les sections sont notées $\mathcal{A} = \Gamma(A)$. Dans ce cas, on munit le fibré vectoriel d'un morphisme $\rho : \mathcal{A} \to \Gamma(TM)$ dans les champs de vecteurs afin de pouvoir définir l'action d'un élément $\mathfrak{X} \in \mathcal{A}$ sur les fonctions: $\rho(\mathfrak{X}) \cdot f$, et d'un crochet de Lie $[\ ,\ ]$ qui imite le comportement des vecteurs sur les fonctions en ce qu'il vérifie la propriété fondamentale:

$$[\mathfrak{X}, f\mathfrak{Y}] = (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y} + f[\mathfrak{X}, \mathfrak{Y}].$$

Nous travaillons dans le cadre des algébroïdes de Lie transitifs, c'est–à–dire pour lesquels ρ est surjectif. Le noyau de ce morphisme, noté L (et dont les sections sont notées $\mathcal{L} = \Gamma(L)$), est un fibré vectoriel dont chaque fibre est une certaine algèbre de Lie $\mathfrak{g}$ (la même en chaque point du fait de la transitivité). On peut définir un algébroïde de Lie transitif par la suite exacte courte d'algèbres de Lie et de $\mathcal{C}^\infty(M)$–modules suivante:

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$$

ι est un morphisme d'algèbre de Lie tel que $\iota(\mathcal{L}) = \ker(\rho)$. Un algébroïde de Lie transitif peut donc être vu comme la généralisation du fibré tangent sur lequel on a rajouté une partie purement algébrique, dont l'élément fondamental est l'algèbre de Lie $\mathfrak{g}$. Cela permet de décrire la géométrie de la variété de base via son espace tangent TM, et *en même temps* celle correspondant aux données (au niveau infinitésimal) d'un fibré principal "abstrait" au dessus de cette variété. Un splitting $\sigma : \Gamma(TM) \to \mathcal{A}$, tel que $\rho \circ \sigma = \mathrm{id}_{\Gamma(TM)}$, permet de définir une 1–forme de connexion $\omega : \mathcal{A} \to \mathcal{L}$ telle que l'on puisse écrire n'importe quel élément $\mathfrak{X} \in \mathcal{A}$ comme

$$\mathfrak{X} = \sigma_{\rho(\mathfrak{X})} - \iota \circ \omega(\mathfrak{X}).$$

On verra que cette notion est une généralisation des connexions sur les fibrés principaux. On peut également définir une métrique $\hat{g}$ sur un algébroïde de Lie, qui consiste en une application bilinéaire symétrique $\hat{g} : \mathcal{A} \times \mathcal{A} \to \mathcal{C}^\infty(M)$. Il se trouve qu'une telle métrique, si elle est telle que $\iota^*\hat{g}$ est une métrique non dégénérée sur $\mathcal{L}$, est équivalente à la donnée d'un triplet $(h, \mathring{\omega}, g)$ où $\mathring{\omega}$ est l'unique connexion ordinaire dont le splitting $\mathring{\sigma}$ est tel que $\hat{g}(\iota(\gamma), \mathring{\sigma}_X) = 0$ pour tout $\gamma \in \mathcal{L}$ et tout $X \in \Gamma(TM)$, et g est une métrique sur $\Gamma(TM)$.

Nous discutons également rapidement de la possibilité d'intégrer ces algébroïdes de Lie transitifs en des *groupoïdes de Lie*, de la même façon que l'on peut intégrer une algèbre de Lie en certains groupes de Lie. Lorsque c'est possible (car, contrairement au cas des algèbres de Lie, il peut exister une obstruction à ce processus d'intégration), pour chaque groupoïde de Lie transitif qui intègre l'algébroïde, on sait qu'il existe un fibré principal sous–jacent d'où l'on peut reconstruire le groupoïde de Lie. Ainsi, tout algébroïde de Lie transitif qui est intégrable provient d'un fibré principal, et est appelé l'algébroïde de Lie d'Atiyah correspondant au fibré. Il est construit en considérant les vecteurs invariants à droite sur ce fibré, et on a la suite d'Atiyah:

$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{\iota_P} \Gamma_H(TP) \xrightarrow{\rho_P} \Gamma(TM) \longrightarrow 0$$

P, ici, est un fibré principal de groupe de structure H, où $\Gamma_H(TP)$ dénote l'espace des champs de vecteurs invariants à droite, et $\Gamma_H(P, \mathfrak{h})$ les applications sur P à valeurs dans l'algèbre de Lie $\mathfrak{h}$ qui sont équivariantes par rapport à l'action du groupe de jauge $\mathfrak{H}$. Toute construction existante sur ce fibré (comme la notion de connexion, par exemple) peut donc être encodée dans le langage des algébroïdes de Lie. Ce langage, pour autant, permet de *généraliser* les notions connues sur les fibrés principaux, exactement dans la démarche de physique mathématique décrite en introduction.

Finalement, nous présentons la méthode de l'habillage dans l'état actuel de son développement. Pour cela, nous rappelons le contenu mathématique usuel des théories de jauge, qui est la donnée:

- d'une variété différentiable M de dimension n;
- d'un fibré principal P de groupe de structure H;
- d'une connexion sur ce fibré (d'Ehresman ou de Cartan, selon le type d'interaction décrite) qui va jouer le rôle de variable dynamique;
- d'une représentation α de H sur un espace vectoriel $\mathbb{V}$;
- d'une section du fibré associé $P \times_\alpha \mathbb{V}$, qui va jouer le rôle de champ de matière (champ spinoriel ou assimilé pour les théories de jauge externes, champ de doublet d'isospin ou assimilé pour les autres).

La méthode de l'habillage consiste alors à construire, à partir des champs donnés à la base, un certain champ défini localement sur le fibré qui se transforme d'une certaine façon sous les transformations de jauge. À l'aide de ce champ particulier, on va alors habiller les autres champs de la théorie, par une transformation qui ressemble formellement à leur transformation de jauge respective. Une fois habillés, ces champs ne seront plus sensibles à l'action d'une partie du groupe de jauge: on dira qu'on a effacé une partie de la symétrie de jauge. Le nombre total de degrés de liberté n'a pas changé entre la version originelle et la version habillée de la théorie: ceux–ci ont simplement été redistribués. Certains champs ont disparus, exactement assez pour compenser le gain dû à l'effacement d'une partie de la symétrie.

Selon comment se transforme le champ d'habillage sous l'action du groupe de jauge résiduel, on saura que les champs habillés (appelés champs *composites*) seront des champs de jauge authentiques ou bien des champs de jauge d'un type nouveau. Ces derniers, que l'on rencontre par exemple en ce qui concerne les Tractors et les Twistors, ne se transforment pas dans une représentation du groupe de structure, mais *via* des objets plus généraux. Dans tous les cas, cette méthode a l'avantage de nous assurer à chaque étape de la nature géométrique des objets que l'on manipule, ce qui échappe parfois aux constructions plus *ad hoc* que l'on peut trouver dans la littérature.

Géométrie de Cartan et Théories de la Gravitation

La géométrie de Cartan est une manière d'étudier la géométrie d'une variété de base *via* celle d'un fibré principal bien particulier construit au–dessus. Le groupe de structure, dans ce cas–là, est un groupe de Lie H, sous–groupe d'un autre groupe de Lie G agissant de manière transitive sur une variété dite "homogène" M_0. Cette variété homogène peut alors s'écrire $M_0 = G/H$. La géométrie riemannienne consiste en l'étude du plan tangent de la variété en chaque point x, et en la façon de *déplacer* les objets définis en un point le long d'un chemin sur la variété. La géométrie de Cartan, quant à elle, substitue l'étude du plan tangent à celle de la variété homogène modèle M_0, et une connexion de Cartan permet de dire comment déplacer cette variété homogène le long de la variété, et d'en déduire des propriétés géométriques. Du point de vue de la physique, on dira que la géométrie de Cartan permet d'implémenter localement une symétrie de type externe sur une variété de base. Une géométrie de Cartan est moins générale que la donnée d'une connexion d'Ehresman sur un fibré principal abstrait, car dans le premier cas la géométrie du fibré en question est en partie contrainte par celle de la variété, alors que dans le second cas le fibré principal (notamment en termes de dimension d'espace) est totalement général. Ce cadre permet de formuler des théories de la gravitation, en particulier la Relativité Générale, en termes de fibré principal, c'est–à–dire dans un langage plus algébrique que la formulation usuelle. Plus précisément, à la donnée d'une structure géométrique sur la variété de base (par exemple une métrique $g_{\mu\nu}$ sur le plan tangent dans le cas de la gravitation usuelle), on substitue celle d'une connexion de Cartan $\varpi : TP \to \mathfrak{g}$ à valeurs dans l'algèbre de Lie du "gros" groupe G qui lui est équivalente. Dans le cas de la Relativité Générale, par exemple, la connexion s'écrit

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & 0 \end{pmatrix},$$

où ω est l'unique solution à d$\theta + [\omega, \theta] = 0$. Ainsi, tous les degrés de liberté de la théorie sont réunis dans la tétrade θ, qui est le substitut à la métrique. Une autre manière de formuler la Relativité Générale, due à MacDowell et Mansouri et reformulée récemment par D. Wise, est présentée également. Elle repose sur le même groupe de structure $H = SO(1,3)$, mais sur le groupe principal de symétrie $G = SO(1,4)$, appelé groupe de de Sitter. Dans cette géométrie, dont la variété modèle M_0 consiste en l'espace de de Sitter, la constante cosmologique est présente dès le début, et la Relativité Générale peut être formulée à l'aide d'une action de type Yang–Mills.

Géométrie Conforme, Tractors et Twistors

Ce chapitre est consacré à la géométrie conforme. Une variété conforme est une variété différentiable équipée d'une classe conforme de métriques, c'est–à–dire non pas seulement d'une métrique mais de son orbite sous l'action d'un *rescaling* de Weyl:

$$g_{\mu\nu} \mapsto \phi^2 g_{\mu\nu}.$$

On commence par une présentation des objets usuellement définis sur une telle variété. On passe ensuite à la construction de la géométrie de Cartan équivalente à la donnée d'une structure conforme. Pour cela, les différentes manières de construire le fibré des 2–repères conformes sont présentées et résumées d'une façon qui se veut pédagogique, sachant que dans la littérature on peut trouver chacune de ces formulations sans nécessairement que le lien soit fait entre elles. La connexion de Cartan *normale* équivalente à une structure conforme s'écrit alors:

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & \omega & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}.$$

Ici encore, les conditions de normalité font que tous les degrés de liberté sont concentrés dans θ, et que les autres termes de la connexion sont des fonctions de cet élément. Le groupe principal de symétrie de cette géométrie est le groupe de Lie des difféomorphismes conformes du compactifié conforme de l'espace de Minkowski $S^1 \times S^3/\mathbb{Z}^2$, appelé simplement *groupe conforme*. Le groupe de structure H, sous-groupe du groupe conforme, consiste en des éléments h qui s'écrivent:

$$h = \begin{pmatrix} \lambda & \lambda c^T \eta \Lambda & \lambda \frac{c^2}{2} \\ 0 & \Lambda & c \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

avec:

- $\lambda \in \mathbb{R}^*$, qui décrit les dilatations;
- $\Lambda \in SO(n-1,1)$, les transformations de Lorentz;
- $c \in \mathbb{R}^{n-1,1}$, les *transformations conformes spéciales*.

L'application de la méthode de l'habillage à cette géométrie permet d'effacer, premièrement, l'action des transformations conformes spéciales. La connexion composite ϖ_1 s'écrit plus simplement et le groupe de jauge effectif est celui trouvé dans la littérature dans la construction usuelle des *Tractors*. La formulation complexe, basée sur l'homomorphisme entre $SO(1,3)$ et $SL_2(\mathbb{C})$ (qui donne celui entre le groupe conforme et $SU(2,2)$), peut subir également le même processus d'habillage, et le résultat correspond aux *Twistors* usuellement trouvés dans la littérature. À noter que dans le cas réel, on peut aussi appliquer une nouvelle fois l'habillage et effacer la symétrie de Lorentz. Dans tous les cas, il est très intéressant de remarquer que les transformations de jauge *résiduelles*, c'est–à–dire celles après habillage, ou encore celles trouvées directement dans la construction usuelle, ne correspondent pas à une représentation du groupe de structure, mais au contraire possède une loi de composition *twistée*. Ainsi, la méthode de l'habillage permet une nouvelle fois de comprendre plus en profondeur la nature géométrique de certains objets, abondamment documentés par ailleurs dans la littérature.

Formulation de lagrangiens unifiés sur des Algébroïdes de Lie transitifs

Ce chapitre est dédié à la présentation de la formulation de théories de jauge dans le langage des algébroïdes de Lie transitifs. On rappelle qu'un algébroïde de Lie transitif est un fibré vectoriel A qui "imite" le comportement des champs de vecteurs d'une part, et qui possède une partie purement algébrique d'autre part. On travaille avec les sections $\mathcal{A} = \Gamma(A)$ qui possèdent une structure d'algèbre de Lie. Ce chapitre commence par une première section dans laquelle on construit des structures additionnelles sur les algébroïdes de Lie transitifs. En particulier, on définit, dans une trivialisation donnée, une base locale adaptée à une connexion donnée sur l'algébroïde. Cette base nous permet ensuite de définir un calcul tensoriel généralisé sur l'algébroïde, qui est tel que les versions locales des "tenseurs" généralisés ont deux types d'indices (internes/algébriques et externes/géométriques) et se transforment de manière homogène. On rappelle ensuite comment définir une notion d'intégration sur les algébroïdes de Lie, afin de pouvoir formuler des théories de jauge (c'est–à–dire des actions sous la forme $S = \int_A L$).

Le premier exemple de Lagrangien unifié présenté est inspiré du travail de N. Boroojerdian. Nous l'avons reformulé entièrement dans notre langage, où nous nous efforçons de définir proprement les différents objets utilisés. Cette approche consiste à prendre une métrique $\hat{g} \leftrightarrow (h, \omega, g)$ comme variable de champ, et d'encoder dans un certain lagrangien à la fois un terme de type Yang–Mills pour ω et la relativité générale avec constante cosmologique, c'est–à–dire le lagrangien d'Einstein–Hilbert pour g. Pour cela, on considère la connexion de Levi–Civita $\hat{\nabla} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, c'est–à–dire l'unique connexion affine sur $\mathcal{A}$ qui vérifie:

- $\hat{\nabla}_{\mathfrak{X}}\mathfrak{Y} - \hat{\nabla}_{\mathfrak{Y}}\mathfrak{X} = [\mathfrak{X}, \mathfrak{Y}]$;
- $\hat{\nabla}\hat{g} = 0$.

On peut alors calculer la courbure de Riemann associée:

$$\hat{R}(\mathfrak{X}, \mathfrak{Y}) := \hat{\nabla}_{\mathfrak{X}} \hat{\nabla}_{\mathfrak{Y}} - \hat{\nabla}_{\mathfrak{Y}} \hat{\nabla}_{\mathfrak{X}} - \hat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]},$$

ainsi que la courbure de Ricci $\hat{Ric}$ et finalement la courbure scalaire $\hat{R}$ associée, ainsi que leur lien avec les objets correspondant sur le fibré tangent. Pour cela on se place dans une trivialisation donnée au–dessus d'un certain ouvert, et on prend la base mixte définie dans la première partie du chapitre, qui est parfaitement adaptée à l'écriture de ces entités. Il se trouve alors que l'action d'Einstein–Hilbert associée

$$\int_M \hat{R} \,\mathrm{vol}_g$$

contient à la fois un terme de Yang–Mills pour ω et le terme d'Einstein–Hilbert avec constante cosmologique pour g.

Dans une troisième et dernière partie, le travail de C. Fournel et al. est résumé. Il consiste en la définition d'une connexion *généralisée* $\varpi \in \Omega^1(\mathcal{A}, \mathcal{L})$, qui ne vérifie pas en général la condition de normalisation propre aux connexions ordinaires sur les algébroïdes de Lie. Pour ϖ une telle connexion, il existe au contraire un endormorphism $\tau : \mathcal{L} \to \mathcal{L}$ tel que:

$$\varpi \circ \iota = -\mathrm{id}_{\mathcal{L}} + \tau.$$

Le champ (scalaire) τ encode une partie purement algébrique qui va permettre de formuler un Lagrangien très intéressant. En effet, une fois donnée une métrique $\hat{g}$ de fond sur l'algébroïde (c'est–à–dire qui n'est pas considérée comme une variable dynamique), ce qui est équivalent à la donnée d'un triplet $(h, \mathring{\omega}, g)$, l'objet

$$\omega = \varpi + \tau \circ \mathring{\omega}$$

est une connexion ordinaire, induite par ϖ. La courbure de la connexion généralisée s'écrit alors:

$$\overline{\Omega} = \mathcal{R} - (\mathcal{D}\tau) \circ \mathring{\omega} + \mathring{\omega}^* R_\tau$$

avec:

$$\mathcal{R} = \Omega - \tau \circ \mathring{\Omega}$$

$$(\mathcal{D}\tau) \circ \mathring{\omega} = [\Theta, \tau \circ \mathring{\omega}] - \tau \circ [\mathring{\Theta}, \mathring{\omega}]$$

$$\mathring{\omega}^* R_\tau = \frac{1}{2}(\tau \circ [\mathring{\omega}, \mathring{\omega}] - [\tau \circ \mathring{\omega}, \tau \circ \mathring{\omega}])$$

où:

- Ω est la courbure associée à la connexion ordinaire induite ω;

- $\mathring{\Omega}$ est la courbure associée à $\mathring{\omega}$;
- Θ: la dérivée covariante associée à ω;
- $\mathring{\Theta}$: la dérivée covariante associée à $\mathring{\omega}$.

Une difficulté est de définir, pour une connexion généralisée, une transformation de jauge qui soit pratique pour écrire des lagrangiens invariants. En effet, la transformation de jauge *géométrique*, i.e. celle qui peut être déduite directement de l'application de la dérivée de Lie, n'est pas adaptée à l'écriture d'une théorie de jauge. On définit alors une transformation de jauge *algébrique*, de par des considérations de dérivée covariante. On peut alors écrire le lagrangien de Yang–Mills pour la connexion généralisée

$$S(\varpi) = < \overline{\Omega}, \overline{\Omega} >_h = \int_A h(\overline{\Omega}, *\overline{\Omega}).$$

Il s'avère que ce lagrangien encode à la fois un terme de type Yang–Mills pour la connexion ordinaire induite ω, un terme cinétique pour le champ scalaire encodé par τ, ainsi qu'un terme de potentiel de type Higgs (potentiel quartique) d'origine purement algébrique. On obtient ainsi un lagrangien de type Yang–Mills généralisé unifiant l'écriture de plusieurs secteurs du modèle standard de la physique des particules.

Il s'agit donc d'un deuxième exemple où l'utilisation d'un algébroïde de Lie transitif permet de formuler un lagrangien unifiant différents secteurs des modèles usuels de la physique théorique.

Géométrie de Cartan et Algébroïdes de Lie transitifs

Le dernier chapitre présente un travail récent, lequel consiste en la formulation de la géométrie de Cartan dans le langage des algébroïdes de Lie. Nous faisons aussi la comparaison entre cette approche et celle de Crampin et Saunders dans leur récent ouvrage.

Une géométrie de Cartan est basée sur un fibré principal P de groupe de structure H, sous–groupe d'un autre groupe de Lie G. On peut également considérer le fibré principal $Q := P \times_H G$, de groupe de structure G, et à toute connexion de Cartan ϖ correspond une connexion d'Ehresman ω sur Q. À l'inverse, une connexion d'Ehresman ω sur Q doit vérifier une certaine condition pour pouvoir donner une connexion de Cartan sur P.

À partir des deux fibrés P et Q, on construit les suites d'Atiyah correspondantes:

$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{\iota_P} \Gamma_H(TP) \xrightarrow{\rho_P} \Gamma(TM) \longrightarrow 0$$
$$0 \longrightarrow \Gamma_G(Q, \mathfrak{g}) \xrightarrow{\iota_Q} \Gamma_G(TQ) \xrightarrow{\rho_Q} \Gamma(TM) \longrightarrow 0$$

Partant de là, on peut construire des applications allant d'un espace à un autre, et on obtient finalement le diagramme commutatif suivant:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{h}) & \xrightarrow{\iota_P} & \Gamma_H(TP) & \xrightarrow{\rho_P} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle J} & & \| & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{g}) & \xrightarrow{\iota_Q} & \Gamma_G(TQ) & \xrightarrow{\rho_Q} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle R} & & \downarrow & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{g}/\mathfrak{h}) & = & \Gamma_H(P,\mathfrak{g}/\mathfrak{h}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Une connexion de Cartan, dans ce contexte, est alors définie comme étant un isomorphisme d'espaces vectoriels et de modules sur les fonctions $\varpi_{Lie} : \Gamma_H(TP) \to \Gamma_H(P,\mathfrak{g})$ tel que

$$\varpi_{Lie} \circ \iota_P = i.$$

On montre qu'un tel objet est équivalent à la donnée d'une connexion de Cartan, au sens ordinaire, sur le fibré sous–jacent P. On réécrit également la condition pour une connection d'Ehresman ω de donner une connexion de Cartan sur P dans le langage des algébroïdes de Lie. On montre, en effet, que si $\omega \in \Omega^1(\Gamma_G(TQ), \Gamma_H(P,\mathfrak{g}))$ est une connexion ordinaire sur la suite exacte courte correspondant à Q, alors $\varpi_{Lie} := \omega \circ J$ est une connexion de Cartan si et seulement si

$$\text{ker}(\omega) \cap J(\Gamma_H(TP)) = \{0\}.$$

Nous présentons ensuite l'approche de Crampin et Saunders, et montrons que leur définition d'une connexion de Cartan sur un algébroïde de Lie est équivalente, dans notre langage, à une connexion ordinaire ω vérifiant la condition que nous venons d'énoncer. Leur définition est donc simplement une réécriture, dans le langage des algébroïdes de Lie, de la condition pour une connexion d'Ehresman sur Q de donner une connexion de Cartan sur P.

* * *

Mots clés: géométrie différentielle, géométrie de Cartan, symétrie conforme, théories de jauge, algébroïdes de Lie transitifs.

Contents

Notations and Conventions

- "i.e." (*id est*) means "that is to say".
- "e.g." (*exempli gratia*) means "for example".
- "mutatis mutandis" means "with the necessary changes having been made".
- "w.r.t." means "with respect to".
- The base manifold M playing the role of spacetime in physics will have dimension n. For physical cases we will take $n = 4$.
- The metric g taken on the base manifold M will always have a Minkowskian signature $(-1, 1, ..., 1)$, even if the results given are often valid in any signature. In particular, η will denote (in its context!) the Minkowski metric.
- The base manifold will be taken to be connected $(\pi_0(M) = 0)$ and simply–connected $(\pi_1(M) = 0)$ for physical purposes.

Interdependence of the Chapters

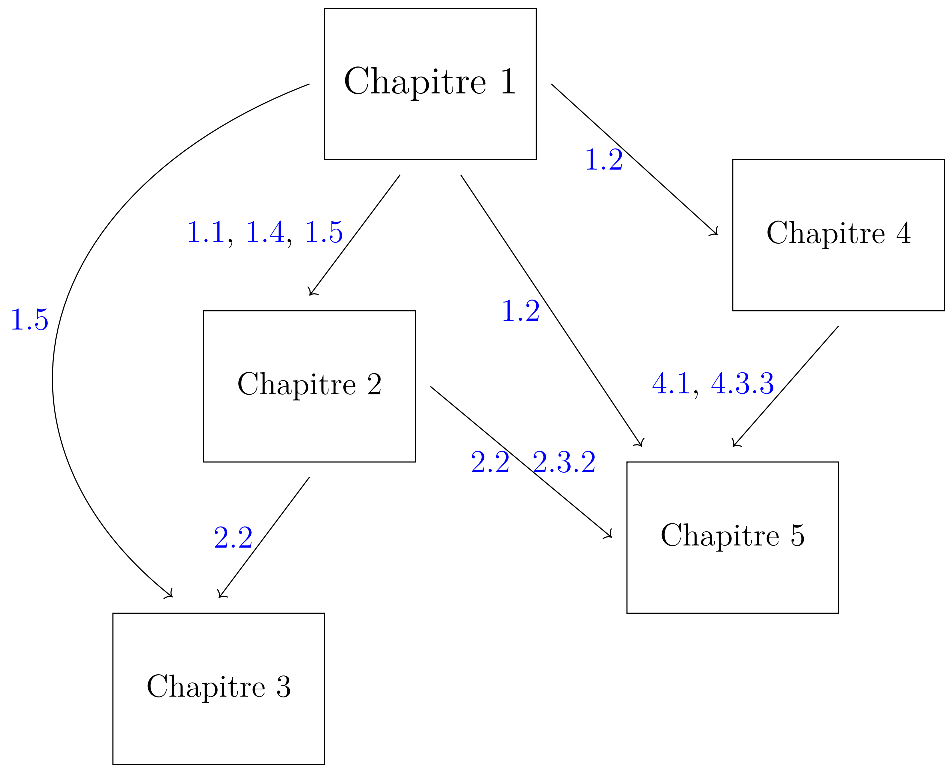


Figure 0.1: Interdependence of the chapters

Introduction[a]

Les Interactions Fondamentales et la Physique Moderne

Notre compréhension actuelle des phénomènes physiques, à toutes les échelles de distance et d'énergie auxquelles on a pu les explorer jusqu'à maintenant, repose sur l'existence de quatre *interactions fondamentales*. Cela signifie, en quelque sorte, que tout ce que l'on peut expliquer à ce jour dans le cadre de nos théories physiques se réduit à un certain agencement de ces quatre interactions. Il ne faut pas pour autant y voir un genre de dogme à la Aristote, où tout s'explique par les différents agencements possibles des *quatre éléments*. En effet, dans notre cas, rien n'interdit en principe que d'autres interactions encore inconnues existent, ou même qu'on puisse remettre en cause ces fondements à l'avenir au profit d'un cadre théorique plus à même de décrire de façon unifiée nos observations et nos expériences. Quoiqu'il en soit, dans l'état actuel de nos connaissances, notre description des phénomènes physiques observés et compris repose sur les quatre interactions suivantes:

- la gravitation, qui fait tomber les objets vers le centre de la Terre, qui fait tenir les galaxies entre elles ou encore qui courbe les rayons lumineux à leur approche d'un objet stellaire;
- l'interaction électromagnétique, qui fait s'attirer les charges électriques de signes différents et se repousser celles de même signe, et qui empêche les-dits objets soumis à la gravitation terrestre (et nous-mêmes, soit dit en passant) de passer à travers le sol et d'atteindre *effectivement* le centre de la Terre;

[a] English version below

- l'interaction faible, qui s'exerce au niveau des noyaux des atomes et qui est responsable de certaines désintégrations et de l'émission de neutrinos;
- l'interaction forte, qui s'exerce elle aussi à l'échelle nucléaire, entre les composants élémentaires de la matière appelés quarks, responsable de la stabilité des noyaux.

Nous faisons ici un abus de langage en disant "telle interaction, *responsable* de tel effet", qui consiste à attribuer à ces interactions un statut *ontologique*, c'est-à-dire prétendre qu'il existe un aspect de la réalité objective qui correspond effectivement à la description qu'on en fait en parlant d'interactions. Pour être plus prudents, nous devrions dire "telle interaction, qui permet de décrire tel effet d'une manière cohérente et unifiée".

La principale différence entre ces quatre interactions est l'échelle à laquelle elles s'appliquent. La gravitation est décrite comme ayant une portée infinie. En revanche, on ne possède pour l'instant aucune expérience permettant de décrire ce que la gravitation devient à une échelle inférieure à l'échelle microscopique. L'interaction électromagnétique, elle aussi, s'applique en principe à toutes les échelles de distance. Cependant, puisque la source de cette interaction est constituée des différentes charges électriques, positives et négatives, que celles-ci sont a priori en même nombre (l'Univers est sensé être globalement neutre) et qu'elles ont un effet inverse l'une de l'autre, très rapidement les deux effets se compensent et l'interaction électromagnétique ne s'applique plus à grande échelle. L'interaction faible, elle, s'applique à très petite échelle (le noyau), phénomène lié au fait que le champ qui véhicule l'interaction faible possède une masse non nulle. L'interaction forte, quant à elle, devrait pouvoir avoir une portée infinie également, car le champ d'interaction a une masse nulle, pour autant, dans les faits, elle ne s'applique qu'à très petite échelle pour des raisons d'ordre *quantique*, qui sont encore, en fait, mal comprises. Ce problème est connu sous le nom du problème du confinement des quarks.

Théories de Jauge et Théorie Quantique des Champs

Les théories qui forment le cadre descriptif de ces quatre interactions sont appelées les *théories de jauge*. Nous nous occupons dans cette thèse uniquement de leur aspect *classique*, par opposition à leur aspect *quantique*; pour autant, disons quelques mots à son sujet. Il est possible, en principe, d'appliquer aux théories de jauge les règles de la *théorie quantique des champs*, qui est le cadre théorique permettant de décrire le comportement quantique des systèmes au nombre infini de degrés de liberté, et en particulier d'unifier la description quantique au cadre relativiste, c'est-à-dire basé sur la symétrie de Poincaré et l'espace homogène associé: l'espace-temps de Minkowski. Les objets de base des théories de jauge ne sont pas des particules ni des points matériels comme en mécanique, mais des *champs*, c'est-à-dire l'attribution, en chaque point de l'espace-temps, d'un objet mathématique qui peut être un scalaire, un vecteur, un spineur, etc. La description de ces champs repose sur une certaine symétrie qui dicte l'interaction à laquelle ils vont être soumis, ainsi que leur structure en tant qu'objet mathématique. Appliquer la théorie quantique des champs à ces objets revient à considérer qu'un champ ne peut vibrer que selon certains modes, quantifiés, et il se trouve que ces vibrations sont localisées dans l'espace, à une échelle de l'ordre de la taille de Compton relatif au champ considéré, pour une masse donnée. On interprète ces différents modes de vibrations comme l'existence d'un certain nombre de *particules*, et le passage d'un mode à un autre comme la création ou l'annihilation d'un certain nombre de ces particules. Ici, la notion de particule est donc émergente de la notion plus fondamentale de champ. Les particules sont, en quelque sorte, les différentes façons suivant lesquelles l'information disponible sur un champ peut se structurer.

Les différentes interactions fondamentales citées plus haut ne sont pas égales face à l'application de ces règles quantiques. Rappelons que les règles de la physique quantique deviennent indispensables lorsque l'on veut décrire un système dont *l'action* caractéristique est de l'ordre de

$$\hbar = 1.054571800(13)10^{-34}\text{J.s.}$$ [b]

En effet, contrairement à une idée répandue, les effets quantiques ne se manifestent pas seulement à des échelles de distance microscopiques. Historiquement, il est vrai que les premiers effets quantiques qu'on a eu à considérer et à comprendre au sein d'une nouvelle théorie se situaient à cette échelle, comme le spectre de l'atome d'hydrogène, par exemple. Pour autant, la vraie échelle à prendre en compte n'est pas la taille caractéristique, mais bien l'action. Il y a nombre de systèmes macroscopiques dont le comportement est incompréhensible du point de vue classique, comme les lasers, les superfluides ou les supraconducteurs, mais aussi les étoiles à neutron (de plusieurs kilomètres de diamètre) dont la stabilité ne peut s'expliquer que par des considérations quantiques.

Concernant les interactions fondamentales, la physique quantique (la théorie quantique des champs, ici) est indispensable pour rendre prédictives les théories de jauge décrivant les interactions faibles et fortes: pour des raisons d'échelle (et non pas des raisons de principe, comme on l'a remarqué au paragraphe précédent), il n'a jamais été observé à des échelles classiques (qui correspondent ici à des échelles macroscopiques) de manifestation des interactions faibles et fortes. L'interaction électromagnétique, quant à elle, connaît les deux régimes. Le régime classique, décrit par les équations de Maxwell, redonne les phénomènes bien connus de l'électrostatique, de l'électricité, du magnétisme et des ondes électromagnétiques, et donc en particulier la description des phénomènes lumineux en terme d'ondes (interférences, diffraction, ...) Pour autant,

[b] https://physics.nist.gov/cuu/Constants/index.html

on peut aussi appliquer les règles quantiques à cette théorie et l'on obtient une description quantique du champs électromagnétique et de ses champs associés (électrons, ...) appelée électrodynamique quantique, qui est le cadre permettant d'expliquer certaines expériences comme l'émission de photons uniques ou bien le fonctionnement des lasers. La gravitation, pour finir, n'est connue, autant d'un point de vue expérimentale que théorique, qu'au régime classique. La théorie de jauge correspondante est la Relativité Générale d'Albert Einstein, élaborée au début du vingtième siècle et qui pour l'instant, comme la théorie quantique, d'ailleurs, n'a pas été mise en défaut – même si l'on observe certains phénomènes qui ne rentrent pas dans ce cadre théorique. La théorie de la relativité générale décrit la gravitation comme la manifestation de la courbure de l'espace-temps, qui n'est plus pensé comme une boite rigide dans laquelle les phénomènes "se passent" mais comme ayant une *géométrie dynamique* sur laquelle les objets (plus exactement le contenu en énergie et impulsion) peuvent influer. Le champ de gravitation est décrit par une métrique, qui est un tenseur symétrique d'ordre deux donné en chaque point d'une variété différentielle de dimension quatre décrivant l'espace-temps. La relativité générale, qui est donc une théorie purement classique, décrit tous les phénomènes gravitationnels connus de l'échelle microscopique (la plus petite sondée en terme de cette interaction) jusqu'à l'échelle cosmologique. La détection d'ondes gravitationnelles en 2015 a même confirmée la validité des équations d'Einstein en régime de champ fort, quand le régime en champ faible est bien testé à l'échelle du système solaire. Le seul petit "hic" de cette théorie est son inaptitude à rendre compte de phénomènes tels que l'expansion *accélérée* de l'univers ou encore le "problème de la matière noire", que l'on tente pour autant d'expliquer dans son cadre. À ce jour, il n'existe aucun phénomène connu, ni observation, ni expérience, qui permettrait de mesurer un quelconque comportement quantique de la gravitation. Cependant, il y a des raisons de penser qu'à certaines échelles, notamment lorsque le champ de gravitation est fort et que beaucoup d'énergie est concentrée dans une petite région de l'espace-temps (c'est-à-dire à l'intérieur d'un trou noir, ou au moment du big-bang, par exemple), une théorie quantique de la gravitation serait indispensable pour expliquer ces phénomènes (non encore observés, cela dit). L'application usuelle de la théorie quantique des champs à la relativité générale ne fonctionne pas, et toutes les théories qui prétendent pouvoir donner une description quantique des phénomènes gravitationnels, comme la théorie des cordes ou la gravitation quantique à boucles n'ont, pour l'instant, pas abouti à une quelconque prédiction qui ait été validée ou réfutée.

Quoiqu'il en soit, nous nous occupons, dans cette thèse, des théories de jauge dans leur aspect classique, c'est-à-dire sans jamais appliquer les règles de la physique quantique. Bien que la théorie quantique des champs est très mal définie mathématiquement, l'aspect classique des théories de jauge auxquelles elle s'applique, lui, est très bien compris. C'est ce cadre géométrique et algébrique, sous-jacent aux théories de jauge, qui est l'objet de notre étude. Les théories de jauge décrivant les quatre interactions fondamentales ont des structures mathématiques similaires qui reposent sur un principe de *symétrie de jauge*.

Principe de Symétrie de Jauge

Lorsque l'on décrit mathématiquement les interactions fondamentales, il se trouve que cette description possède invariablement une sorte de redondance, un surplus de structure qui fait que les objets de base (par exemple, les champs) ne sont définis qu'à *une certaine transformation près*, transformation caractéristique de l'interaction en question. Les quantités observables sont alors celles qui sont invariantes sous ces transformations, c'est-à-dire qui ne dépendent pas de la manière dont on les décrit. Imaginons, pour illustrer ce propos par une métaphore, que l'on veuille décrire un *cube*. Pour décrire ce cube, il faut nécessairement le tenir dans une main, dans une *certaine position*. Passer d'une position à une autre se fait à l'aide d'une rotation, et l'ensemble des rotations forme un groupe, noté $SO(3)$. Selon la position, il peut y avoir 1, 2 ou 3 faces apparentes, et de la même façon le nombre d'arêtes ou de sommets apparent(e)s varie lui aussi en fonction de la façon dont on tient le cube. Si l'on veut dire quelque chose à propos du cube qui lui soit propre, c'est-à-dire qui ne dépende pas de la façon dont on le tient, il faut trouver des quantités qui sont invariantes sous le groupe des rotations $SO(3)$. Par exemple, le nombre total de face ne dépend pas de la façon dont on tient le cube, c'est une caractéristique propre de celui-ci. C'est exactement cela l'esprit des théories de jauge: il existe une redondance dans notre description, qui fait que certaines quantités dépendent de la façon dont on les décrits, et dire quelque chose de propre au phénomène en question signifie s'émanciper de cette redondance, et trouver des quantités qui soient *invariantes sous les transformations qui structurent cette redondance*. On dit que c'est un principe de symétrie car les quantités sont inchangées lorsqu'on leur applique une certaine transformation, comme un objet possédant une symétrie axiale aura une image égale à lui-même dans un miroir.

Prenons l'exemple de l'électromagnétisme. Le champ électromagnétique $F_{\mu\nu}$ est un tenseur qui dérive du potentiel de jauge A_μ, ce que l'on écrit:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{0.1}$$

L'objet de base est le potentiel de jauge A_μ, tandis que la quantité mesurée en pratique est $F_{\mu\nu}$, le champ électromagnétique. Pour autant, on voit que si l'on change A_μ en $A_\mu + \partial_\mu \phi$, avec ϕ *n'importe quelle* fonction scalaire, $F_{\mu\nu}$, lui, ne varie pas. On a donc une liberté, dite "liberté de jauge", dans la description du potentiel: la quantité "A_μ" n'a pas vraiment de pertinence physique en elle-même, puisque l'ajout d'un gradient à cette quantité ne change pas ce qui effectivement mesuré (i.e. ce qui est donc *physique*). C'est cela l'idée de la symétrie de jauge: il y a des objets de base, comme A_μ, que l'on est obligé de prendre en compte dans notre description, mais qui pour autant n'ont pas de signification physique en eux-mêmes; il y a une redondance dans notre description. L'objectif, lorsque l'on étudie une théorie de jauge, est de mettre en lumière des quantités invariantes sous un certain groupe de transformations, que l'on va, elles, qualifier de *physiques*. Dans le cas de l'électromagnétisme, il se trouve que la transformation $A \to A + d\phi$ est une manifestation de l'action locale (c'est-à-dire pouvant être différente en chaque point de l'espace-temps) du groupe abélien $U(1)$, isomorphe au groupe des rotations en deux dimensions $SO(2)$. Pour autant, ce groupe n'agit pas sur l'espace-temps lui-même: c'est une "rotation" dans un espace abstrait *au-dessus* de chaque point de l'espace-temps. Cet espace abstrait est appelé une fibre, et la collection de tous ces espaces forme un *fibré*. C'est cet objet mathématique qui décrit la façon dont la redondance de notre description se structure. Il faut s'imaginer qu'au dessus de chaque point de l'espace-temps, on place une *fibre* qui ressemble (on dit qu'elle lui est *difféomorphe*) à la variété formant le groupe de structure (ici $U(1)$). L'interaction électromagnétique va être interprétée comme la manifestation d'une géométrie non triviale (et dynamique) de cet espace fibré. Les interactions faibles et fortes sont elles aussi basées sur exactement le même type de structure mathématique, seul le groupe de structure (c'est-à-dire la symétrie) change: l'interaction faible est gouvernée par le groupe $SU(2)$, tandis que l'interaction forte est gouvernée par $SU(3)$. De la même façon, ces "rotations" se réalisent dans un espace abstrait, absolument pas en tant que "vraies" rotations de l'espace-temps. On dira, par la suite, que les théories de jauge de ce type sont à *symétrie interne*, ou que ce sont des *théories de jauge internes*, car l'espace fibré est vu comme un espace interne, par opposition à l'espace-temps qui est vu comme "externe". Ces théories-là sont décrites dans le cadre de la géométrie différentielle des fibrés principaux et des connexions de Ehresman sur ces fibrés. On présentera ce cadre au chapitre 1.

Enfin, prenons l'exemple de la gravitation. La théorie de la relativité générale repose elle aussi sur un principe de symétrie de jauge. En effet, si M est la variété différentielle de dimension 4 sensée décrire l'espace-temps, alors les équations de la relativité générale[c] (équations d'Einstein et équation des géodésiques), sont invariantes sous le groupe des difféomorphismes Diff(M). Ce groupe est constitué de l'ensemble des applications $\phi : M \to M$ qui sont de classe $\mathcal{C}^\infty$, inversibles et dont l'inverse ϕ^{-1} est aussi $\mathcal{C}^\infty$. Ce groupe, contrairement aux autres interactions fondamentales, agit sur les points de l'espace-temps eux-mêmes, et pas sur un espace abstrait "interne". On dira que la relativité générale est une théorie de jauge *à symétrie externe*. La signification physique d'une telle invariance est assez troublante à première vue: elle mène à

[c]Ainsi, comme on le voit plus bas, que la *fonctionnelle d'action* redonnant ces équations d'après un principe variationnel.

conclure qu'un point de l'espace-temps *n'a pas de signification physique en lui-même*. Cela veut dire aussi que donner un statut ontologique à l'espace-temps, même si c'est très tentant cognitivement, n'a pas de sens du point de vue du cadre théorique de la relativité générale. La géométrie différentielle sous-jacente à cette théorie est la géométrie pseudo-riemannienne, où l'objet de base est une métrique $g_{\mu\nu}$ qui décrit la géométrie et qui va jouer le rôle de variable dynamique. Les équations d'Einstein relient, via une équation différentielle du second ordre, cette métrique au contenu en énergie et impulsion de la zone de l'espace-temps que l'on étudie, décrit par un tenseur $T_{\mu\nu}$ appelé tenseur énergie-impulsion. La géométrie (pseudo-)riemannienne est différente de la géométrie des connexions sur les fibrés, au sens où elle se consacre à la géométrie de la variété de base elle-même. Pour autant, on peut associer canoniquement à n'importe quelle variété étudiée un fibré principal d'un type particulier, appelé le fibré des repères, et retranscrire une certaine structure géométrique (comme une métrique pseudo-riemannienne) en termes de connexion sur ce fibré. Ce cadre d'étude s'appelle la géométrie de Cartan, et permet de formuler la relativité générale et toute autre théorie gravitationnelle sous la forme d'une théorie de jauge interne, c'est-à-dire avec un groupe de structure agissant localement sur une fibre. Dans le cas de la relativité générale, ce groupe de structure est le groupe de Lorentz, et la connexion de Cartan associée prend ses valeurs dans l'algèbre de Lie du groupe de Poincaré. Les difféomorphismes, même lorsqu'on rajoute cet espace "abstrait" (qui est tout de même plus concret que dans le cas des autres interactions, car on peut lui donner une signification physique directe) sont par ailleurs toujours à prendre en compte. Cette formulation a pour avantage de pouvoir facilement intégrer à la relativité générale des champs spinoriels, *via* les représentations du groupe de structure $SO(1,3)$.

En théorie quantique des champs, appliquée aux interactions fondamentales mise à part la gravitation, on travaille d'ordinaire en considérant la variété de base plate (au sens de la courbure de Riemann), c'est-à-dire en prenant M l'espace de Minkowski. La signature minkowskienne est d'ailleurs la source de beaucoup de problèmes du point de vue mathématique, pour définir proprement les quantités avec lesquelles on travaille autant que pour faire des calculs. Dans notre cas, nous allons considérer que la variété de base est quelconque, et que, dans l'esprit de la relativité générale, on prend aussi en considération les difféomorphismes même lorsqu'on décrira une théorie de jauge à symétrie interne. Le groupe de symétrie d'une telle théorie regroupera alors à la fois les difféomorphismes et le groupe de jauge associé au groupe de structure correspondant à la symétrie interne. On verra au chapitre 1 que ce groupe de symétrie peut être décrit de différentes manières, et en particulier en termes de bisections sur des groupoïdes de Lie ou encore, infinitésimalement, comme les sections des algébroïdes de Lie correspondants.

La Fonctionnelle d'Action

Une fois donné le cadre géométrique dans lequel on va décrire des champs en interactions (c'est-à-dire, *grosso modo*, une fois donné une symétrie de jauge), la dynamique de ces champs est donnée par une *fonctionnelle d'action*. Si on note ψ l'ensemble des champs de la théorie en question, une fonctionnelle d'action est une quantité scalaire qui dépend des champs: $S[\psi]$, et qui est *invariante* sous transformation de jauge. Si γ dénote une transformation de jauge dans le groupe de jauge choisi, alors on doit avoir $S[\psi^\gamma] = S[\psi]$. Il existe, dans les cas qui nous intéresse, un *lagrangien* L, c'est-à-dire une 4-forme sur la variété de base M, tel que $S[\psi]$ peut toujours s'écrire: $S[\psi] = \int_M L[\psi]$. Chercher à trouver les points extrêmes de S, c'est-à-dire les champs solutions du problème d'extremalisation de la fonctionnelle d'action, mène à des équations différentielles vérifiés par ces champs, en générale impossibles à résoudre en l'état. Dans le cas de la relativité générale, par exemple, les équations d'Einstein peuvent se dériver à partir d'une fonctionnelle d'action, appelée action d'Einstein–Hilbert. Des solutions sont ensuite données dans des cas particuliers pour lesquelles on a demandé des symétries plus restrictives que dans le cas général. Dans le cadre des théories de jauge internes, au niveau classique, les équations de champs[d] ne sont prédictives (c'est-à-dire, ne servent à quelque chose) que dans le cas de l'électromagnétisme, où elles sont bien évidement les équations de Maxwell. En effet, la théorie quantique des champs n'a besoin que du lagrangien pour faire des prédictions, les équations de Yang–Mills pour l'interaction faible et forte n'ont donc pas de portée prédictive[e]. Pour autant, il faut noter que parfois, les solutions de ces équations sont importantes, car il existe des techniques en théorie quantique des champs qui développent le champ étudié autour d'une solution *non triviale* de ces équations.[f]

Statut Philosophique des Symétries de Jauge, Brisure de Symétrie

Avant de présenter le travail effectué dans cette thèse en termes d'exploration de nouveaux cadres mathématiques pour une formulation de la physique des interactions fondamentales, arrêtons-nous d'abord sur les symétries de jauge, d'un point de vue philosophique, cette fois-ci. Quel statut donner à ces transformations? À première vue, puisqu'une symétrie de jauge traduit une redondance d'information dans notre description, elle ne semble avoir qu'un statut *épistémo-*

[d] Appelées dans ce cas *équations de Yang–Mills*

[e] Rappelons que pour ces deux interactions, il n'y a pas d'équivalent classique comme il en existe pour l'interaction électromagnétique

[f] Je remercie Laurent Lellouch pour avoir clarifier ce point lorsque je lui ai posé la question.

logique, c'est-à-dire être une caractéristique de *la façon dont on décrit* les interactions fondamentales, mais pas une caractéristique propre à la réalité objective de laquelle nos théories scientifiques sont sensées obtenir de l'information. De plus, comme on le verra, il existe une méthode de réduction de symétrie basée sur un simple changement de variables de champs, appelée méthode de l'habillage (ou *dressing*), par laquelle nous pouvons réécrire une même théorie de jauge dans de nouvelles variables sur lesquelles le groupe de jauge n'agit plus (agit trivialement). Dans ce cas, on dira qu'on a réduit la symétrie: on peut donc finir avec une théorie où la symétrie est encore moindre qu'au départ. Un habillage, comme on le verra, n'est pas une transformation de jauge, même si formellement il peut avoir la même forme. Par exemple, on montre que la symétrie de Lorentz locale avec laquelle on peut décrire la relativité générale peut être effacée par habillage: cette symétrie de jauge sera donc considérée comme artificielle. Dans tous les cas, que l'on puisse l'effacer ou non, une symétrie de jauge ne semble pas avoir de statut ontologique. Pour autant... le groupe de structure (qui donne le groupe de jauge par localisation de son action) est en réalité important pour décrire certaines de nos observations. En effet, les champs de matière sont décrits comme les sections de fibrés associés au fibré principal, et l'association se fait au moyen d'une *représentation* du groupe de structure. Et c'est effectivement ce que l'on observe: la structure en doublet d'isospin, par exemple, dans le cas de l'interaction électrofaible, est impossible à comprendre (en tout cas dans l'état actuel de nos connaissances) sans le groupe $SU(2)$. Une description de cette interaction nécessite d'effacer l'action du groupe $SU(2)$, ce qui peut être fait par habillage, pour autant, l'on ne pourrait pas non plus partir d'une théorie seulement basée sur $U(1)$. Même si on efface l'action du groupe $SU(2)$, celui-ci continue à se manifester via les champs de matière et leur organisation. Dans le cas de la symétrie locale de Lorentz, c'est pareil: on peut effacer cette symétrie par habillage et revenir à une description purement "riemannienne", pour autant, l'existence du *spin*, qui découle directement des représentations du groupe de Lorentz, montre que même si on l'efface, le groupe de Lorentz doit être pris en compte. Le statut ontologique des symétries de jauge n'est donc pas si trivial: celles-ci ne peuvent pas être considérées comme de simples caractéristiques de notre description, elles correspondent à quelque chose de plus profond, et ce, même si les quantités observables se doivent d'être aveugles à cette symétrie. Autrement dit, même si la symétrie de jauge traduit une redondance dans notre description et ne semble donc à première vue ne posséder qu'un statut proprement épistémologique, la structure de cette redondance (le groupe de symétrie) n'est pas arbitraire et se manifeste en particulier dans les champs de matière et les particules associées.

Quoiqu'il en soit, le principe de symétrie de jauge possède indéniablement un statut *heuristique* qu'il convient de mettre en lumière. On appelle principe heuristique, ici, un principe qui, une fois posé, va dérouler un cadre théorique qui va nous permettre de *trouver* ($\epsilon\upsilon\rho\iota\sigma\kappa\omega$ "je trouve"), c'est-à-dire de faire des prédictions possiblement réfutables par l'expérience. Prenons l'exemple de la conservation de l'énergie en mécanique classique. Cela a-t-il un sens de se demander s'il est *vrai* que "l'énergie d'un système isolé se conserve au cours du temps" ? Si cela avait un sens, cela signifierait que cette proposition est une *prédiction* de notre théorie, et donc qu'elle pourrait être en principe réfutée. Pour autant, pour réfuter une telle proposition, est-ce suffisant de trouver une expérience pour laquelle l'énergie n'est pas conservée? En réalité, on peut toujours imaginer qu'il existe un phénomène encore inconnu qui viendrait rétablir la conservation de l'énergie. Et c'est exactement ce qui s'observe historiquement: exiger la conservation de l'énergie a permit de trouver de nouveaux phénomènes (dissipation sous forme de chaleur, ...) ou de nouvelles particules (le neutrino, ...), alors que la conservation *semblait* violée. C'est pour cela qu'on parle d'un principe *heuristique*. Pour réfuter la conservation de l'énergie, il faudrait prouver qu'il n'existe aucun phénomène encore inconnu qui viendrait rétablir cette conservation: c'est tout aussi impossible que de montrer *qu'il n'existe pas* de licorne vivant quelque part dans l'univers. La proposition est donc irréfutable. Mais ça ne pose aucun problème, car en réalité, c'est un *principe*, et pas une prédiction de notre théorie. Insistons: le concept d'énergie n'aurait aucun intérêt si elle ne se conservait pas. Le principe de conservation de l'énergie est ce qui pose la base d'un cadre *à l'intérieur duquel* on va faire des prédictions qui, elles, devront être réfutables en principe. Depuis les travaux d'Emmy Noether au début du vingtième siècle, on sait que les quantités conservées en mécanique comme en théorie (classique) des champs proviennent d'une *symétrie* sous-jacente. Tout le raisonnement que l'on vient d'avoir dans le cas de la conservation de l'énergie s'applique donc aux symétries elles-mêmes, et en particulier aux symétries de jauge: celles-ci sont postulées, données dès le départ, et c'est à l'intérieur du cadre mathématique qu'elles permettent que l'on va faire des prédictions. Ainsi, le fait de briser *à la main* une symétrie, ou de voir une brisure de symétrie comme la manifestation d'un phénomène physique, est très critiquable de ce point de vue. Prenons le cas du fameux mécanisme de Higgs vue comme émanent de la brisure spontanée de symétrie $U(1) \times SU(2) \to U(1)$. En réalité, par la méthode de l'habillage, on peut montrer que $SU(2)$ est toujours effaçable, c'est-à-dire que le fait que la symétrie $SU(2)$ s'observe "avant brisure" n'est en fait due qu'aux variables de champ utilisées (c'est une sorte "d'illusion" due au fait qu'on n'utilise pas les variables adaptées) et la génération de la masse des bosons d'interaction se trouve être en réalité décorrélée de la brisure de $SU(2)$. La puissance prédictive du modèle standard, et notamment du secteur éléctrofaible, provient de l'application de la théorie quantique des champs (en particulier du calcul des diagrammes de Feynman) au Lagrangien ré-écrit "après brisure", qui est le même que le Lagrangien "habillé", dans notre approche. En d'autres termes, la puissance prédictive du modèle standard n'a pas besoin de la brisure spontanée de la symétrie $SU(2)$. Si l'on applique le raisonnement précédent à ce cas précis, on peut considérer que c'est un peu "tricher" que de briser une symétrie spontanément, dans le cadre des théories de jauge où le principe de symétrie est placé comme un fondement. En effet, le problème de la génération de masse dans le secteur électrofaible du modèle standard n'en est un que si la symétrie $SU(2)$ est supposée valide. Si on brise la symétrie $SU(2)$, on n'a pas résolu le problème: on est sorti du cadre dans lequel c'était un problème, ce qui est tout à fait différent.

Il se pourrait bien entendu qu'un cadre plus large existe dans lequel ces symétries (et donc les quantités conservées correspondantes) soient naturellement absentes d'une manière ou d'une autre. Ce qui est critiquable, par contre, c'est de briser une symétrie tout en restant à l'intérieur d'un cadre dont la pertinence, en réalité, *repose fondamentalement sur elle*.

La Physique Mathématique: Exploration de Nouveaux Cadres pour Formuler les Théories Physiques

La physique mathématique n'est pas autant une discipline en soit qu'une certaine démarche à l'intérieur de la physique théorique, ou, en tout cas, qui doit venir la nourrir directement. Cette démarche a un double objectif. D'une part, mettre en lumière les structures mathématiques sous-jacentes aux théories fondamentales (ici, les théories de jauge), pour séparer les caractéristiques qui leur sont propres et paraissent *nécessaires* à leur formulation, de celles qui ne sont que *contingentes*, par exemple en ayant été simplement héritée de leur formulation historique. Cela permet, en se réduisant à une écriture la plus minimale possible, de ne pas tomber dans des écueils de pensée qui pourraient nous faire mal interpréter certains phénomènes. Un exemple, comme on l'a déjà cité, est la réduction de symétrie dans le modèle standard de la physique des particules: attaché à la manière historique dont a été résolu le problème de la génération des masses dans ce modèle, on peut passer à coté du fait qu'en réalité, et ce, à la lumière de la méthode de l'habillage, il n'est nul besoin de supposer l'existence d'une brisure spontanée de symétrie pour décrire ce phénomène. D'autre part, ce premier processus de clarification du cadre mathématique et de mise en lumière des caractéristiques nécessaires aux théories étudiées permet ensuite d'imaginer de nouvelles formulations équivalentes, qui offriront peut-être une possibilité naturelle de généralisation, une nouvelle voie qui était invisible dans la première formulation.

C'est exactement cette approche que l'on a suivie dans cette thèse. Nous présentons notre exploration, principalement mathématique, de nouveaux cadres possibles pour formuler des théories de jauge, à symétries interne comme externe. Dans le chapitre 1, nous présentons les outils mathématiques de base de la géométrie différentielle. Comme il s'agit de choses relativement connues, nous avons opté pour une présentation plus originale: nous présentons tout d'abord les variétés différentielles, puis les fibrés vectoriels au dessus d'une variété de base M, pour ensuite introduire les algébroïdes de Lie transitifs comme des fibrés vectoriels particuliers. Après avoir présenté la notion de connexion sur de tels objets, nous présentons également, dans les grandes lignes, le problème d'intégration des algébroïdes de Lie transitifs en groupoïde(s) de Lie transitif(s). Quand l'intégration est possible, on sait alors qu'il existe, pour chaque groupoïde de Lie transitif intégrant l'algébroïde, un fibré principal sous-jacent tel que l'algébroïde de Lie est l'algébroïde d'Atiyah de ce fibré. Ce choix de présenter les fibrés principaux comme découlant de cas particuliers d'algébroïdes de Lie permet d'anticiper sur les possibles généralisations que peut offrir ce cadre géométrico-algébrique. Enfin, nous exposons la méthode de l'habillage dans l'état actuelle de son développement, dont un tour d'horizon peut être trouvé dans [2].

Les autres chapitres sont relativement indépendants; ils développent chacun un aspect de ce qui a été vu au premier chapitre.

Le chapitre 2 présente la géométrie de Klein, puis la géométrie de Cartan qui en est une généralisation, avant de donner deux exemples d'application de ce cadre à la formulation de théories de la gravitation. On réécrit la relativité générale, premièrement, dans le cadre de la géométrie de Poincaré, aussi connue sous le nom de "formulation en terme de tétrades", puis dans le cadre de la géométrie de de Sitter, redonnant la formulation "à la MacDowell Mansouri" présentée par D. Wise en 2009, où le lagrangien peut s'écrire sous la forme d'un lagrangien dit de "Yang–Mills".

Le chapitre 3 est consacré à la géométrie conforme, et aux outils mathématiques nécessaires à la formulation de théories de jauge (à symétrie externe, donc) dans ce cadre. C'est un cas où l'on essaie de généraliser ce qui a déjà été fait (en relativité générale, par exemple) en augmentant le groupe de symétrie, passant du groupe de Lorentz au groupe conforme. Nous présentons une vue d'ensemble de la géométrie conforme. En particulier, les différentes manières de construire le fibré des repères d'ordre deux conformes, ainsi que la géométrie homogène (de Klein) associée, c'est-à-dire le compactifié conforme de l'espace de Minkowski et ses groupes de symétrie, qui joue le même rôle dans la géométrie conforme que l'espace de Minkowski joue dans la géométrie riemannienne. Ces constructions aboutissent ensuite à la présentation de la géométrie conforme en termes de connexion de Cartan sur un fibré principal. L'application de la méthode de l'habillage à ce cadre nous permet alors de retrouver d'une manière claire et propre (d'un point de vue géométrique) les objets usuels de la géométrie conforme que sont les tenseurs de Weyl et de Schouten, les tractors, les twisteurs et les connexions qui leur sont associées. Ces résultats sont décrits en détails dans les articles: [18] et [17]. Nous finissons par présenter rapidement comment la gravité de Weyl, dans ce formalisme, s'écrit naturellement à l'aide d'un lagrangien de Yang–Mills, imitant en quelque sorte la démarche de

Wise pour la gravité de MacDowell–Mansouri. Ce dernier résultat est décrit plus en détail dans [4].

Le chapitre 4 est une exposition de deux exemples pour lesquels la formulation de théories de jauge en termes d'algébroïdes de Lie transitifs permet d'obtenir naturellement des Lagrangiens unifiés pour la physique. Ce chapitre commence par la présentation de structures additionnelles pouvant être construit sur les algébroïdes, comme la notion de base mixte locale, le calcul tensoriel, ou encore l'opérateur de Hodge et le processus d'intégration des formes différentielles. On expose ensuite un travail inspiré de celui de N. Boroojerdian, réécrit dans notre formalisme. On considère une métrique sur l'algébroïde de Lie comme une variable de champ. Une telle métrique est équivalente à un triplet constitué d'une connexion ordinaire (l'équivalent des connexions d'Ehresman sur les fibrés), d'une métrique sur l'algèbre de Lie qui dicte la symétrie interne, ainsi que d'une métrique sur le plan tangent de la variété de base, qui encode la gravitation. Une notion de connexion de Levi-Civita adaptée à ce type de métrique permet alors d'encoder à la fois un lagrangien de type Yang-Mills pour la connexion ordinaire du triplet équivalent à la métrique, et un lagrangien de type Einstein-Hilbert pour la métrique sur le plan tangent de la variété, avec en prime un terme de constante cosmologique, d'origine algébrique. On présente finalement le travail réalisé par Cédric Fournel et al. Cette approche repose aussi sur le choix d'une métrique du même type, cependant celle–ci (ou le triplet correspondant) est ici considérée comme fixe. Sur cette structure, on construit alors une connexion *généralisée*, qui possède un degré de liberté purement algébrique et qui va permettre *in fine* d'encoder dans un lagrangien de type Yang-Mills un terme de type potentiel de Higgs, sans aucune supposition *ad hoc*.

Le chapitre 5, quant à lui, sera une présentation d'un travail réalisé récemment et qui cherche à formuler la géométrie de Cartan, dans son formalisme comme dans son idée, dans le cadre des algébroïdes de Lie transitifs. Pour cela, on commencera par présenter du matériel additionnel autour de la géométrie de Cartan. Puis, on encodera les deux fibrés principaux associés à une géométrie de Cartan en termes de leur algébroïdes de Lie d'Atiyah correspondant, puis on traduira dans ce cadre ce que signifie une connexion de Cartan. Un récent ouvrage (2016) de Crampin et Saunders ([9]) présente lui aussi la géométrie de Cartan en termes d'algébroïdes de Lie (et également des groupoïdes de Lie correspondants), dans une formulation légèrement différente. En particulier, ils choisissent une formulation que nous qualifions de géométrique, en voyant les algébroïdes de Lie en termes de fibrés vectoriels, alors que nous partons d'une formulation que nous pourrions qualifier de plus algébrique, en considérant les sections des algébroïdes de Lie, et donc en travaillant avec des modules et des algèbres de Lie au lieu des fibrés vectoriels. Nous réécrivons leur démarche pour la présenter, puis nous la reformulons dans nos notations pour la comparer ensuite à la nôtre. Tout ce chapitre est le résumé d'un article à paraître bientôt, [3].

La figure 0.1 montre les interdépendances entre des différents chapitres entre eux, avec le numéro de la section utile à la compréhension de tel chapitre.

En conclusion, nous présentons différentes généralisations possibles de ces cadres divers, nous dirigeant vers une *algébrisation* des théories physiques.

Introduction

Fundamental Interactions and Modern Physics

Our current understanding of physical phenomena, at all scales which have been probed so far, is based on the existence of four *fundamental interactions*. This means, in some sense, that everything which can be explained nowadays in the framework of our physical theories reduce to a certain combination of these four interactions. This is, however, something different from a kind of dogma *à la* Aristotle, where everything is explained by different possible combinations of *four elements*. Indeed, in our case, it is possible, at least in principle, that some unknown interactions exist, or even that these fundations be questionned in favour of a theoretical framework more efficient in describing our experiments and observations in a unified way. However, in our current state of knowledge, our description of physical phenomena rests on the following four interactions:

- gravitation, which makes objects fall down in the direction of the center of the Earth, which makes galaxies holding themselves, or which deviates the light rays when they get closer to a stellar object;
- electromagnetism, which makes electric charges of the opposite sign attract themselves, and those of the same sign repel each other, and which prevents objects attracted by the Earth from really reaching its center;
- weak interaction, which exists at the nuclear level and which is responsible for some desintegrations and for the emission of neutrinos;
- strong interaction, which is also applying at the nuclear scale, between elementary components of matter called quarks, responsible for the stability of nucleus.

Here, we make a language abuse saying "this interaction, *responsible* for this effect", which consists of attributing to these interactions an *ontological* status, i.e. saying that there exists an aspect of objective reality which really corresponds to the description we do by talking about interactions. To be more careful, we should say: "this interaction, which allows to describe this effect in a coherent and unified way".

The main difference between these four interactions is the scale at which they apply. The gravitation is described as having an infinite range of application. On the contrary, we do not have, so far, any experiment allowing a description of gravitational phenomena at the microscopic scale. Electromagnetic interaction also applies, in principle, at any scale of distance. Yet, since the source of this interaction consists of electric charges, positive and negative, that there is the same amount of both types in the Universe, and that they have a contrary effect, both effects compensate and electromagnetic interaction do not act anymore at large scale. Weak interaction applies at very low scale (nucleo), for the field which carries the interaction has a non-vanishing mass. Strong interaction, finally, should have an infinite range as well, for the interacting field is massless. However, in facts, it applies only at very low scale too, for quantum reasons, which are still, actually, not well understood. This question is known as the "quarks confinement issue".

Gauge Theories and Quantum Field Theory

The theories which form the descriptive framework of these four interactions are called *gauge theories*. We focus in this thesis only on their *classical* aspects, but let us say some words about their quantum aspects too. It is possible, in principle, to apply quantum rules to gauge theories, known as *quantum field theory*. It is the theoretical framework allowing to describe the quantum behaviour of systems with infinite number of degrees of freedom. In particular, it is possible to develop the quantum description of such system in relativistic framework, that is to say based on the Poincaré symmetry and the corresponding homogeneous space: Minkowski spacetime. The basic objects of gauge theories are not particles or material points as in mechanics, but *fields*. A field is the attribution of a mathematical object (like a scalar, a vector, a spinor, etc.) at each point of spacetime. The description of these fields rests on a certain symmetry which dictates the interaction to which they are exposed, together with their structure as mathematical entity. Applying quantum field theory to these objects leads to consider that a field can only vibrate according to certain modes, which are *quantized*. It turns out that these vibrations are localised in space, at a scale of the order of Compton length. We interpret these different modes of vibrations as the existence of a certain number of particles, and passing from a certain mode to the other is seen as the creation or the annihilation of a certain number of these particles. Here, the notion of particle is thus emerging from the more fundamental notion of field. Particles are, in some sense, the different possible ways according to which the available information about a field can structure itself.

The application of these quantum rules do not give the same result on all interactions. Let us recall that the quantum physics rules become unavoidable when one wants to describe a system with a characteristic *action* of the order of:

$$\hbar = 1.054571800(13)10^{-34}\text{J.s.}$$ [g]

Indeed, unlike a widespread idea, quantum effects do not manifest only at microscopic scales. Historically, this is true that the first quantum effects for which a new theory was needed were at this scale, like the hydrogen atom spectrum, for example. Still, the real scale one has to take in account is not the characteristic length, but the action. There are a lot of macroscopic systems the behaviour of which is not understandable for a classical viewpoint, like lasers, superfluids of supraconductors, but also neutron stars (the diameter of which can reach many kilometers!) the stability of which can only been explained, so far, by quantum considerations.

As for fundamental interactions, quantum physics (quantum field theory, here) is essential to render predictive gauge theories describing weak and strong interactions. Indeed, for scale reasons (and not principles reasons, as we remarked above), manifestations of weak and strong interactions have never been observed at classical scales (which corresponds, here, to macroscopic scales). For electromagnetic interaction, we need both regimes. The classical one, described by Maxwell equations, gives the well known phenomena of electrostatic, electricity, magnetism, electromagnetic waves, etc. However, quantum rules can also be applied to this theory and the result is a quantum description of electromagnetic field and its related fields (electrons, ...), called *quantum electrodynamics*. This is the framework allowing the explication of certain experiments as the emission of single photons, but also functioning of lasers. The gravitation, finally, at both experimental and theoretical level, is known only under a classical regime. The corresponding gauge theory is Einstein's General Relativity, constructed at the begining of the twentieth century and it has not been falsified so far. General Relativity describes gravitation as the manifesation of the curvature of space-time, which is not thought anymore as a rigid box in which phenomena "occur", but as having a *dynamical geometry* upon which objects can act. Gravitational field is described by a metric, which is a symmetric tensor of order two given at each point of a four dimensional smooth manifold describing spacetime. General Relativity, which is thus a purely classical theory, describes all gravitational phenomena that we know from microscopic scales to cosmological scales. Since the detection of gravitation waves in 2015, the strong field regime of Einstein equations has also been confirmed to be valid, not only the weak field regime desribing phenomena at the scale of solar system. The only "problem" is that phenomena like the accelerating expansion of the Universe, or the dark matter

[g] https://physics.nist.gov/cuu/Constants/index.html

issue do not enter into its framework. So far, there does not exist any known phenomenon, neither obervation nor experiment, which would allow to measure a quantum behaviour of gravitation. However, there are reasons to believe that at certain scales, a quantum theory of gravity is essential to explain some phenomena – neither measured nor observed yet. In particular, it could be the case when the gravitational field is strong and that a lot of energy is concentrated in a small region of spacetime, like in a black hole or at the Big Bang time. The usual application of quantum field theory to General Relativity do not work. Amongst the theories which claim building a quantum description of gravitational phenomena, like string theory of loop quantum theory, none of them, for the time being, has succeeded to predict something which may has been validated or falsified.

Anyways, we focus, in this thesis, only on the classical aspect of gauge theories, that is without applying quantum rules. Even though quantum field theory is very badly defined on a mathematical point of view, the classical aspect of gauge theory to which it applies is very well understood. This geometric and algebraic framework, underlying gauge theories, is the objects of our study. Gauge theories describing the fundamental interactions have very similar structures which rest on a *gauge symmetry principle*.

Gauge Symmetry Principle

When one mathematically describes fundamental interactions, it turns out that this description possesses a kind of redundance, a surplus of structure, such that the basic objects are defined only *up to a certain transformation*. This transformation turns out to be characteristic of the interaction in question. The physical (observable) quantities are thus those which are invariant under these transformations, that is which do not depend on the way they are described. Let us imagine, in order to illustrate this by a metaphor, that one wants to describe a *cube*. To describe this cube, one has to hold it in a hand, in a *certain position*. Passing from a position to another one is made by the help of a rotation, and the set of rotations forms a group, denoted $SO(3)$. For a given position, there can be 1, 2 or 3 apparent faces, and in the same way the number of apparent edges or corners can vary from a position to another one. If one wants to say something intrinsic about the cube, i.e. which do not depend on the way one is holding it, one has to find quantities invariant under the action of the group $SO(3)$. For example, the *total* number of faces do not depend on the way one holds the cube, it is an intrinsic caracterictic of it. This is exactly the spirit of gauge theories: there exists an unavoidable redundancy in our description, which makes some quantities depend on the way we describe it, and saying something proper to the phenomenon in question means emancipating ourselves from this redundance, and finally finding quantities which are invariant under the transformations which structure this redundance. It is said to be a symmetry principle for these quantities remain unchanged when a certain transformation is applied to them.

Let us take now the concrete example of electromagnetism. The electomagnetic field $F_{\mu\nu}$ is a tensor which derives from the gauge potential A_μ, which is written as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{0.2}$$

The basic object is the gauge potential A_μ, while the measured quantity is the electromagnetic field $F_{\mu\nu}$. However, one notices that if one changes A_μ to $A_\mu + \partial_\mu \phi$, with ϕ any scalar function, $F_{\mu\nu}$ do not vary. Thus, there is a freedom, called "gauge freedom", in the description of the potential: the quantity A_μ has no really physical relevance in itself, since adding a gradient to this quantity does not change what is measured in practice. This is the idea of gauge theory: there are basic objects, like A_μ, that one must take into account, but which have no physical meaning in themselves. The goal, when we study a gauge theory, is to highlight quantities which are invariant under a certain group of transformations. These quantities will be called *physical*. In the case of electromagnetism, it turns out that the transformation $A \to A + d\phi$ is a manifestation of the local action (i.e. which can be different at each point of spacetime) of the abelian group $U(1)$, isomorphic to the rotation group $SO(2)$. However, this group does not act on spacetime itself: it is a "rotation" in an abstract space *over* each point of spacetime. This abstract space is called a fibre, and the collection of all fibres is called a *fibre bundle*. This object describes the way the redundance of our description structures itself. One can imagine that over each point of spacetime, one puts a fibre which looks like (one says that it is *diffeomorphic* to) the manifold forming the structure group (here $U(1)$). The electromagnetic interaction is then interpreted as the manifestation of a non trivial (and dynamical) geometry of this fibred space. Weak and strong interactions are also based on this kind of mathematical structure. The only thing which changes is the structure group: weak interaction is governed by the group $SU(2)$, while strong interaction is ruled by $SU(3)$. These "rotations" realize themselves in an abstract space too, not like "true" rotations of spacetime. Gauge theories of this kind will be said to possess an "internal symmetry", or to be "internal gauge theories", for the fibred space is seen as an internal space, in contrast to spacetime which is seen as "external". These theories are described in the framework of differential geometry of principal fibre bundles and Ehresman connections on them. This framework is presented in chapter 1.

Finally, let us take the example of gravitation. General Relativity rests also on a gauge symmetry principle. Let M be the 4–dimensional smooth manifold describing spacetime. Then the equations of general relativity[h] (Einstein equations and geodesics equations) are invariant under the group of diffeomorphisms Diff(M). This group consists in the set of maps $\phi : M \to M$ which are of class $\mathcal{C}^\infty$, invertible and the inverse of which ϕ^{-1} is also $\mathcal{C}^\infty$. This group, unlike the

[h] So does the *action functional* itself, as it is explained later on.

other fundamental interactions, acts on points of spacetime themselves, and not on an abstract and internal space. General Relativity will be said to be a gauge theory with *external symmetry*. The physical meaning of such an invariance is a bit disturbing, at first: it leads to conclude that a point of spacetime has no physical meaning in itself. That means also that giving an ontological status to spacetime, even it is cognitively tempting, has no sense from the theoretical viewpoint based on diffeomorphisms invariance. The underlying differential geometry is pseudo–riemannian geometry, where the basic object is a metric $g_{\mu\nu}$ which describes the geometry and which plays the role of dynamical variable. Einstein's equations relate, via a second order differential equation, this metric to the energy–momentum content of the zone of spacetime under study, described by a symmetric tensor $T_{\mu\nu}$ called energy–momentum tensor. Riemanian geometry is different from the geometry of connections on fibre bundles, in the sense that it focuses on the geometry of the base manifold itself. Still, one can associate canonically a principal fibre bundle of a certain kind, called frame bundle, to any smooth manifold. Then, one can rewrite a certain geometric structure (like a pseudo–riemannian metric) in terms of a connection on this fibre bundle. This framework is called Cartan geometry, and allows the formulation of General Relativity and any other theory of gravity under the form of an internal gauge theory – i.e. with a structure group acting locally on a fibre. In the case of General Relativity, this structure group is the Lorentz group, and the associated Cartan connection takes values in the Lie algebra of the Poincaré group. Diffeomorphisms, even when this abstract space is added, are still to be taken into account. This formulation has the advantage to easily incorporate spinor fields into General Relativity, via the representations of the structure group $SO(1,3)$.

In quantum field theory, applied to fundamental interactions except gravitation, one works usually considering that the base manifold is flat (in the sense of Riemann curvature), that is to say taking M to be the Minkowski space. The minkowskian signature, in passing, is the source of a lot of problems from the mathematical viewpoint. In our case, one considers a general base manifold and, in the spirit of General Relativity, one takes also into account diffeomorphisms even if an internal gauge theory is described. The symmetry group of such a theory will then consist in the direct product of diffeomorphisms and of the gauge group associated to the structure group corresponding to the internal symmetry. In chapter 1, different ways of seeing this group will be presented. In particular, one can describe it in terms of bisections on Lie groupoids, or, infinitesimally, as sections of the corresponding Lie algebroids.

The Action Functional

Once given the geometric framework in which interacting fields are described, that is to say, roughly, once chosen a gauge symmetry, the field dynamics is given by an action functional. This entity is central since it will determine all what we are able to describe and predict with our theory. If one denotes ψ the set of fields of the theory in question, an action functionnal is a scalar quantity which depends on fields, $S[\psi]$, and which is *invariant* under a gauge transformation. If γ denotes a gauge transformation in the chosen gauge group, then we must have $S[\psi^\gamma] = S[\psi]$. There always exists, in our cases of interest, a lagrangian L, that is to say a 4–form on the base manifold M, such that $S[\psi]$ can always been written $S[\psi] = \int_M L[\psi]$. Searching for extremal points of S, i.e. the solutions of the extremalization problem of the action functional, leads to differential equations satisfied by these fields. In the case of General Relativity, for example, Einstein equations can be derived from an action functional, called Einstein–Hilbert action. Solutions are then given in particular cases for which more restrictive symmetries are demanded. In the framework of internal gauge theories, at the classical level, the field equations[i] are predictive only in the case of electromagnetism, where they reduce to Maxwell equations. Indeed, quantum field theory only needs the lagrangian to make predictions, Yang–Mills equations for weak and strong interactions have thus no predictive interest.[j] Let us remark, however, that solutions to these equations can also be important, sometimes. Indeed, there exists technics in quantum field theory which develop the field under study around a non trivial (classical) solution of these equations.[k]

Philosophical Status of Gauge Symmetry; Symmetry Breaking

Before presenting our work on new mathematical frameworks for formulations of fundamental interactions physics, let us first give a look to gauge symmetries from a philosophical viewpoint. Which status can be given to these transformations? At first sight, since a gauge symmetry is the manifestation of a redundancy of information in our description, it seems only to have an *epistemological* status. That is to say, it seems to be only a feature of *the way we describe* fundamental interactions, but not an intrinsic feature of the objective reality from which our theories are supposed to obtain information. Moreover, as we will see it, there exists a method of symmetry reduction based on a mere

[i] Called Yang–Mills equations in this case.

[j] Let us recall that for these interactions, there is not classical equivalent as it exists for electromagnetic interaction.

[k] I thank Laurent Lellouch for having pointing this out when I asked him the question.

change of field variables, called the dressing field method. By this method, it is sometimes possible to rewrite a given gauge theory in new variables on which a part of the gauge group does not act anymore (i.e. acts trivially). In this case, the symmetry will be said to be reduced: we finally end up with a theory with less symmetry. A dressing, as we will see it, is not a gauge transformation, even if formally it can have the same form. For example, we show that the local Lorentz symmetry, with which one can describe General Relativity, can be erased by dressing: this gauge symmetry is then considered as artificial. In any case, whether we can erase it or not, a gauge symmetry does not seem to possess any ontological status. However, the structure group (which gives the gauge group by localization of its action) is actually important to describe some of our observations. Indeed, matter fields are described as sections of associated fibre bundles, and the association is made with the help of a *representation* of the structure group. And this is what is observed, indeed: the structure of particles in isospin doublets, e.g., in the case of weak interaction, is impossible to understand[l] without the group $SU(2)$. For the description of this interaction, one needs to erase the action of $SU(2)$, which can be made by dressing. Yet, one could not start from a theory based only on $U(1)$ neither. Indeed, even if the action of $SU(2)$ is erased, it keeps manifest itself throught matter fields and the way they organise themselves. In the case of Lorentz symmetry, the same thing occurs: one can erase this symmetry by dressing and go back to a purely Riemannian description. Yet, the existence of *spin*, which comes directly from representations of Lorentz group, shows that even if one erases it, the Lorentz group must be taken into account. The ontological status of gauge symmetries is thus not so trivial. They cannot be considered as mere features of our description. At the contrary, they seem to correspond to something deeper about reality – even if the observable quantities have to be insensible to these symmetries. In summary, a gauge symmetry depicts a redundance in our description. Thus, it does not seem, at first sight, to possess any ontological status but only an epistemological one. However, the structure of this redundance is not arbitrary as it manifests itself in the *observed* matter fields and particles.

Anyway, the gauge symmetry principle undeniably has an *heuristic* status which sould be enlightened. Here, one calls heuristic principle a principle on which one can base a theoretical framework which will allow to *find*[m] new things, i.e. to make *falsifiable predictions*. Let us take the example of the energy conservation in classical mechanics. Does wondering whether it is *true* or not that "the energy of an isolated system is conserved in time" has any meaning? If this had a meaning, that would mean that this proposition is a prediction of our theoretical framework which can be in principle falsifiable by an experiment. However, in order to falsify such a claim, is it sufficient to find an experiment for which the energy is not conserved? Actually, one can always imagine that there

[l] At least in our current state of knowledge.
[m] $\epsilon \upsilon \rho \iota \sigma \kappa \omega$: "I find"

exists a phenomenon, still unknown, which would restablish the conservation of energy once incorporated in our model. This is exactly what is historically observed: to demand the conservation of energy led to find new phenomena (heat dissipation, ...) or new particles (neutrino, ...), while the conservation *seemed* to be violated at first sight. This is why we call it a heuristic principle. In order to falsify the conservation of energy, one should prove that there exists no phenomenon still unknown which would restablish the conservation. This is as impossible as proving that unicorns do not exist somewhere in the Universe. The claim is thus not falsifiable. But this is not a problem, because actually it is a principle, not a prediction of our theory. Let us insist on this point: the concept of energy itself would have no interest if energy was not conserved. The principle of conservation of energy is what sets the base of a framework *inside which* one does predictions. Then, these predictions will have to be falsifiable by an observation. From Emmy Noether's work at the beginning of the twentieth century, one knows that the conserved quantities in mechanics as in (classical) field theory come from an underlying *symmetry*. All the previous reasonning in the case of energy conservation holds also for symmetries themselves. In particular, it applies to gauge symmetries themselves: they are postulated, given at the beginning, and one makes experimental predictions inside the framework that these symmetries allow. Thus, the fact to break a symmetry *by hand*, or to see a symmetry breaking as the manifestation of physical phenomenon, is questionable from this viewpoint. Let us take the case of the famous Higgs mechanism, seen as emanent from the spontaneous symmetry breaking $U(1) \times SU(2) \rightarrow U(1)$. Actually, by the dressing field method, one can show that $SU(2)$ is always erasable. That is to say, the fact that the symmetry $SU(2)$ is observed "before symmetry breaking" is only due to the field variables in which one writes the theory. This is a kind of "illusion" due to the fact that one does not use the "good" variables. The generation of masses of interacting bosons turns out to be decorrelated from the $SU(2)$ breaking. The predictiveness of the Standard Model, and in particular of the Electroweak sector, rests on the application of quantum field theory (Feynman rules) to the lagrangian "after symmetry breaking". The latter is the same as the one obtained by dressing. Thus, the predictiveness of the Standard Model does not need $SU(2)$ to be *spontaneously* broken. Moreover, if we apply the previous reasonning to this precise case, one can consider that it is kind of "cheating" to spontaneously break a symmetry, in the case of gauge theories where the symmetry principle is placed as a fundation. Indeed, the masses generation in the electroweak sector of Standard Model is a *problem* only if the $SU(2)$ symmetry is supposed to be valid. If one breaks $SU(2)$, one did not solve the problem: one went out of the framework in which it was a problem, which is totally different.

Of course, a larger framework could exist, in which these symmetries (and thus all corresponding conserved quantities) would not be needed anymore. What is questionable, however, is to break a symmetry while remaining inside a framework the relevance of which, actually, *fundamentally rests on it*.

Mathematical Physics: Exploration of new Frameworks for Formulations of Physical Theories

Mathematical physics is not so much a field of study in itself as a certain approach inside theoretical physics, or, at least, which should feed it directly. This approach has a double aim. On the one hand, enlightening mathematical structures underlying these theories helps understanding them better by avoiding possible thought pitfalls. This leads to clean up the formalism on which a theory is based in order to separate the features that are only *contingent* from those which are *necessary* or at least more fundamental. This approach gives thus a clearer framework to think about physical phenomena as best as possible. On the other hand, such deeper understanding naturally leads to generalizations of these structures, that might eventually offer a new and wider physical framework.

We followed exactly this approach in this thesis. We present our exploration, mainly mathematics, of new possible frameworks to formulate gauge theories, with internal as external symmetry. In chapter 1, we present basic mathematical tools of differential geometry. Since this is quite well–known material, we have chosen a more original presentation. We present first smooth manifolds, then vector bundles over a base manifold M, and then we introduce transitive Lie algebroids as particular vector bundles. After having presented the notion of connection on such objects, we also present, roughly, the problem of integrating transitive Lie algebroids to transitive Lie groupoid(s). When the integration is possible, we then know that there exists, for each groupoid integrating the Lie algebroid, an underlying fibre bundle such that the Lie algebroid is the Atiyah Lie algebroid of this fibre bundle. This presentation of principal fibre bundles as coming from particular cases of Lie algebroids allows to anticipate on possible further generalizations that this geometrico–algebraic framework could offer. Finally, we present the dressing field method in its current state of development. An overview of it can be found in [2].

The other chapters are quite independant. Each of them develop an aspect of what has been seen in chapter 1.

Chapter 2 presents Klein geometry, and Cartan geometry which is a generalization of the latter. Then, we give two examples of formulation of gravitation theories in this framework. We rewrite General Relativity, first, in the framework of Poincaré geometry – also known as the "tetrad formulation". Second, in the framework of de Sitter geometry, we give the MacDowell–Mansouri–like formulation, as presented by D. Wise in 2009, where the lagrangian of the theory is written under a Yang–Mills form and is equivalent to General Relativity.

Chapter 3 is about conformal geometry, and about the mathematical tools necessary to the formulation of gauge theories (with external "conformal" symmetry) in this framework. This is a case of generalization of what has already been done (in General Relativity, e.g.) by increasing the symmetry group, passing from the Lorentz group to the conformal group. We present an overview of conformal geometry. In particular, the different ways of constructing 2–frame bundles are exposed, together with the associated homogeneous (Klein) geometry. The latter consists in the conformally compactified Minkowski space and its symmetry groups, which plays the same role in conformal geometry as Minkowski space plays in Riemannian geometry. These constructions then lead to the presentation of conformal geometry in terms of a Cartan connection on a principal fibre bundle. Applying the dressing method on this framework then allows to find on a clear and clean way (from a geometric viewpoint) the usual objects of conformal geometry as Weyl and Schouten tensors, Tractors, Twistors and their associated connections. These results are described in details in the papers: [18] and [17]. Eventually, we present promptly how Weyl gravity, in this formalism, is naturally written as a Yang–Mills–type lagrangian, mimicking in some sense the approach of Wise for MacDowell–Mansouri gravity. This last result is described more in details in [4].

Chapter 4 is a presention of two examples for which the formulation of gauge theories in terms of transitive Lie algebroids allows to naturally get unified lagrangian for physics. This chapter starts with the presentation of additional material on algebroids, as the notion of mixed local basis, tensorial calculus, Hodge operator or the process of integration of differential forms. We then expose a work inspired from N. Boroojerdian's one, rewritten in our formalism. In this work, a metric is choosen on a transitive Lie algebroid as a field variable. This kind of metric is shown to be equivalent to a triple consisting in an ordinary connection, a metric on the Lie algebra which dictates the internal symmetry, together with a metric on the tangent space of the base manifold, which encodes gravity. A notion of Levi–Civita connection adapted to this kind of metric allows then to encode in the same time a Lagrangian of Yang–Mills type for the ordinary connection of the triple, and the Einstein–Hilbert Lagrangian for the metric on the tangent space, with in addition a cosmological constant term of algebraic origin. Finally, one presents the work of C. Fournel et al. about generalized connections on transitive Lie algebroids. In this approach, a metric is also choosen, but as being fixed (not dynamical). On this background structure, is then built a generalized connection, which possesses a purely algebraic degree of freedom. This purely algebraic part allows to encode in a Yang–Mills type Lagrangian a Higgs potential term, with no *ad hoc* assumption.

Chapter 5 will be a presentation of a recent work which is a formulation of Cartan geometry, in its formalism as in its idea, in the framework of transitive Lie algebroids. For this aim, we start presenting additional material about Cartan geometry. Then, we encode both the principal fibre bundles associated to a given Cartan geometry in terms of their Atiyah algebroids. We give a definition, in this framework, of a Cartan connection. A recent book (2016) of Crampin and Saunders ([9]) presents also Cartan geometry in terms of Lie algebroids (and also in terms of the corresponding Lie groupoids), in a slightly different formulation. In particular, they choose a formulation that we call "geometric", seeing Lie algebroids in terms of vector bundles, while we start from a formulation more algebraic, considering the sections of Lie algebroids. Thus, we work with modules and Lie algebras instead of vector bundles. We rewrite their approach in order to present it, then we formulate it in our formulation in order to compare it to ours. All this chapter is the summary of a forthcoming paper [3].

Figure 0.1 shows the interdependences between chapters.

In conclusion, we present different possible generalizations of these frameworks, leading us to an *algebraization* of physical theories.

Chapter 1

Geometric and Algebraic Framework

1.1 Differential Geometry of Smooth Manifolds

Smooth manifolds are the basic tool in differential geometry. We present a definition here in order to set the notations used throughout this thesis. Extensive work on smooth manifolds and differential geometry (fibre bundles, manifolds, etc.) can be found in [30], [31], [38] and [36] (the latter is in French). A manifold M is a generalization of the notion of space on which differential calculus is provided. The way to do that is first defining the notion of "looking locally like $\mathbb{R}^n$", and then transposing the notion of differential calculus well known on $\mathbb{R}^n$ to M. Smooth manifolds became of primary importance in the description of fundamental interactions firstly with Einstein's General Relativity, in which gravitational phenomena are seen as a manifestation of the dynamical geometry of a base manifold of dimension 4 playing the role of spacetime. The geometrization of other interactions, in the framework of what we called *internal* gauge theories, is also based on smooth manifolds. One considers, over a base manifold still seen as spacetime, a bigger manifold structured in *fibres*, called a fibre bundle. These interactions, at the classical level, are also seen as the manifestation of dynamical geometry of this "internal" structure.

1.1.1 Definition

A $n-$ dimensional differentiable (or smooth) manifold M is a topological space (i.e. a space for which is given a notion of neighborhood at each of its point) provided with a collection of pairs $\{(U_i, \phi_i)\}$ such that

- $\phi_i : U_i \to U'_i$ is a homeomorphism, with U_i and U'_i open sets of M and $\mathbb{R}^n$ respectively;
- $\cup_i U_i = M$, a covering;

- For all i,j such that $U_i \cap U_j \neq 0$, the corresponding maps glue smoothly together, i.e. the map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a diffeomorphism.

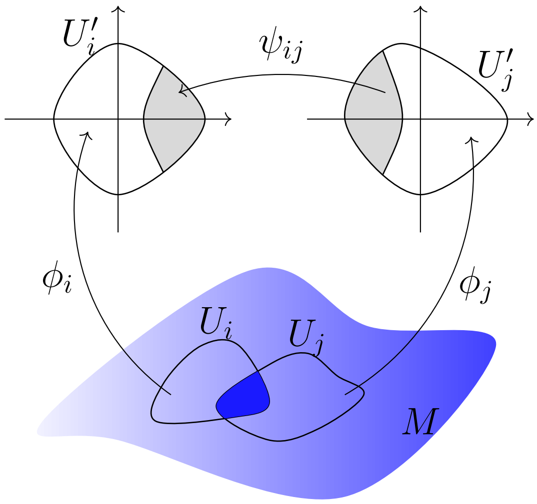


Figure 1.1: Charts on a Manifold

Recall that an homeomorphism is an invertible map of class $\mathcal{C}^0$ (continuous) such that its inverse is also $\mathcal{C}^0$. A diffeomorphism has the same definition replacing $\mathcal{C}^0$ by $\mathcal{C}^\infty$ (infinitely differentiable, or smooth).

1.1.1.1 Coordinate Systems

A pair (U_i, ϕ_i) is called a local chart, and can be thought as a local coordinate system. Indeed, if (U, ϕ) is a local chart, then for any $x \in U$, $\phi(x) \in U' \subset \mathbb{R}^n$, and thus $\phi(x)$ can be represented by n-coordinates $\phi^\mu(x)$, simply denoted x^μ. A change of coordinates, on the overlap of two open subsets to which x belongs, is called a passive diffeomorphism. From now on, a local chart will often be denoted (U, x^μ).

1.1.1.2 Differential Structure

A collection $\{(U_i, \phi_i)\}$ is called an atlas. Two atlases are said to be compatible if their union is again an atlas. Compatibility is an equivalence relation, and one defines a differentiable structure as an equivalence class of atlases. Let us notice that a given topological manifold can admit many distinct differentiable structures, see e.g. section 5.2.1. of [38]. From now on, when one talks about a "smooth manifold", one means "a given differential structure over a topological manifold".

1.1.2 Active Diffeomorphisms

A diffeomorphism $\phi : M \to N$ between two smooth manifolds M and N is an invertible map of class $\mathcal{C}^\infty$ such that its inverse is also $\mathcal{C}^\infty$. Two diffeomorphic smooth manifolds are "the same" in terms of differential structure, and will be denoted by the same letter. It is not easy to give concrete examples of two homeomorphic manifolds which would not be diffeomorphic, because this phenomenon appears only in dimension $n \geq 4$. (idem: section 5.2.1. of [38]).

In the context of General Relativity (GR), such a diffeomorphism is often called an active diffeomorphism, in constrast with passive ones, the latter describing a mere change of coordinates. There is however a 1–to–1 correspondance between passive and active diffeomorphisms. Indeed, let $x \in M$, (U, x^μ) a local chart and $\phi : M \to M$ a diffeomorphism such that $\phi(x) \in U$. Then, there exists a unique passive diffeomorphism, i.e. a change of local charts $(U, x^\mu) \to (V, y^\mu)$ such that $y^\mu(x) = x^\mu(\phi(x))$. A passive transformation is the description of the same point in another coordinate system, whereas an active one is the description of another point in (possibly) the same coordinate system. Let us remark that in general, an active diffeomorphism is defined without any reference to a local chart.

In GR, laws of physics are invariant under active diffeomorphisms of the base manifold M, representing spacetime. The physical meaning of this invariance is that a particular point of spacetime, thought as an "event" in special relativity, has no more any physical relevance. All what is relevant is the coincidence points of fields and trajectories over M, but not M itself. In the framework of gauge theories, physical quantities are those which are invariant under a change of gauge configuration – in GR, that means invariant under diffeomorphisms. The space Diff(M) of diffeomorphisms of a smooth manifold M has the structure of a group. For a pedagogical explanation of diffeomorphism invariance, and of the typical Einstein's Hole Argument, see e.g. section 2.2. of [48]. For a philosophy book (surely among others) about spacetime, see [11].

1.1.3 Tangent Space

1.1.3.1 Definitions

Several equivalent definitions can be given to the tangent space of a smooth manifold M at a point x, see e.g. [36] or [38]. Here, one presents promptly two of them: the first one, rather geometric, is based on tangent to curves, and the second one, a bit more algebraic, is based on derivations of functions.

Let $\mathcal{C}_x := \{\gamma : [-1, 1] \to M, \gamma(0) = x\}$, i.e. a set of curves passing throught a given $x \in M$. In a local coordinate system (U, x^μ), one sets $\gamma^\mu(t) := x^\mu(\gamma(t))$. Then, two curves γ and γ' are said to be equivalent if they define the same tangent vector at x, i.e. if

$$\frac{d\gamma^\mu(t)}{dt}|_{t=0} = \frac{d\gamma'^\mu(t)}{dt}|_{t=0}.$$

Let us remark that an object as $\frac{d\gamma^\mu(t)}{dt}|_{t=0}$ lives in $\mathbb{R}^n$ where it is well defined. The tangent space T_xM to M at x is then defined as the equivalence classes of this equivalence relation. In order to capture the vector space aspect of T_xM, let us present a second possible definition, in terms of derivations.

Let us consider $\mathcal{C}^\infty(M)$, the algebra of smooth functions over M. Given a $x \in M$, two functions f ang g are said to be equivalent if they coincide upon an open subset containing x. Then, one denotes $\mathcal{C}^\infty_x(M)$ the space of equivalence classes of this relation (the space of *germs*[a] at x), and $T_xM = \text{Der}(\mathcal{C}^\infty_x(M))$, i.e. consists in operators $\mathcal{D} : \mathcal{C}^\infty_x(M) \to \mathbb{R}$ such that $\mathcal{D}(\tilde{f}\tilde{g}) = \mathcal{D}(\tilde{f})\tilde{g}(x) + \tilde{f}(x)\mathcal{D}(\tilde{g})$ with $\tilde{f}, \tilde{g} \in \mathcal{C}^\infty_x(M)$. This space, equipped with coherent definitions of the sum of two derivations and the multiplication by a scalar, is a vector space. It can of course be shown that both definitions are equivalent, see e.g. [36], or [31]. Given a local coordinates system U, x^μ, one can define at $x \in U$ a natural (called holonomic) basis of the tangent space at x by $\partial_\mu := \frac{\partial}{\partial x^\mu}$.

1.1.3.2 Tangent Bundle

The collection of all T_xM for $x \in M$ is called the tangent bundle and is denoted $TM = \cup_{x\in M} T_xM$. It is a smooth manifold, a special case of more general objects called vector bundles, as one will see later. There is a natural projection $\pi : TM \to M$. A section of this bundle is a map $X : M \to TM$ such that $\pi \circ X(x) = x$ for all $x \in M$, and is called a vector field. The action of a vector field X on a smooth function f is another function denoted "$X \cdot f$", which in a holonomic basis reads $X \cdot f = X^\mu \partial_\mu f$.

[a] With germs, it can be easily shown that the definition of the tangent space at x indeed depends only on the point x.

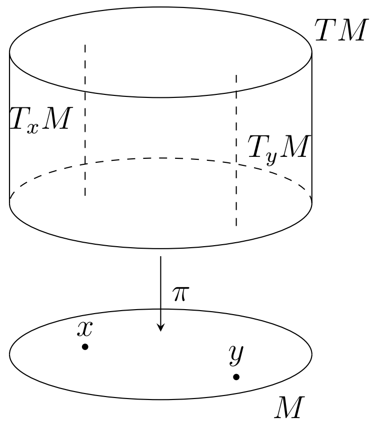


Figure 1.2: Tangent Bundle of a Manifold

The space of vector fields, denoted $\Gamma(TM)$, is equipped with a Lie bracket defined by: $[X,Y]\cdot f = X\cdot(Y\cdot f) - Y\cdot(X\cdot f)$ which turns it into an infinite dimensional Lie algebra. It is a straightforward computation to check that this bracket has the following property:

$$[X, fY] = (X\cdot f)Y + f[X,Y], \tag{1.1}$$

which one wants to mimick in the more general case of Lie algebroids.

1.1.3.3 Flow of a Vector Field and Diffeomorphisms

The vector fields over a manifold M and (the connected component to the identity of) the group of diffeomorphisms Diff(M) are closely related. Indeed, one can show that the latter is in some sense generated by the former: vector fields can be seen as infinitesimal diffeomorphisms. Let $X \in \Gamma(TM)$ be a vector field on M, the flow of X, denoted $t \longmapsto \phi_X(t,x)$, is the unique solution of the equation:

$$\frac{d\gamma(t,x)}{dt} = X_{|\gamma(t,x)} \tag{1.2}$$

such that (initial condition) $\gamma(0,x) = x$, for all $x \in M$.

For $t \in \mathbb{R}$, the application $\phi_X(t,.) : M \to M, x \mapsto \phi_X(t,x)$ is a diffeomorphism of M, and thus $t \to \phi(t,.)$ can be seen as a 1–parameter subgroup of Diff(M). One has $\phi_0(t,.) = \text{id}_M$. Thus, $\Gamma(TM)$ generate the connected component to the identity of the group of diffeomorphisms. As already mentionned, $\Gamma(TM)$ can actually be seen as the Lie algebra of the group of diffeomorphisms.

1.1.4 Vector Bundle over a Manifold

A vector bundle E over a smooth manifold M has the same structure as the tangent bundle but with an arbitrary vector space of dimension $k > 0$ over each point $x \in M$. More precisely, a vector bundle consists of a manifold E equipped with a smooth surjection $\pi : E \to M$ such that for any $x \in M$, the fibre $\pi^{-1}(x)$ has a vector space structure. This means that there exists a vector space $\mathbb{R}^k$ with $k > 0$, such that for any $x \in M$, there exists an open neighborhood U of x and a homeomorphism $\phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$. The local trivialization (U, ϕ) is such that $(\pi \circ \phi)(x, v) = x \; \forall v \in \mathbb{R}^k$ and $\forall x \in M, v \to \phi(x, v)$ is an isomorphism realizing

$$\pi^{-1}(x) \simeq \mathbb{R}^k.$$

The dimension of E is then $\dim(E) = n + k$.

1.1.5 Connections on a Vector Bundle

A (smooth) section of a vector bundle is a map $\mathfrak{X} : M \to E$, which smoothly assigns to each point $x \in M$ an element of E_x. The space of such sections is denoted $\Gamma(E)$. A real smooth function on M is an extreme case, since it can be seen as the smooth assignment of a real number to each point $x \in M$, thus as a section of the so–called line bundle $M \times \mathbb{R}$. Since vector fields $\Gamma(TM)$ acts on functions as derivations, which can be interpreted as derivations along certain directions, one could wonder in what extent this can be defined on a general vector bundle. One possible answer is the notion of *connection*.

Given a vector bundle $\pi : E \to M$, a connection on E is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

such that for any $X, X' \in \Gamma(TM)$, $\mathfrak{X}, \mathfrak{X}' \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, $\nabla_X : \Gamma(E) \to \Gamma(E)$ satisfies:

- $\nabla_{X+X'}\mathfrak{X} = \nabla_X \mathfrak{X} + \nabla_{X'}\mathfrak{X}$
- $\nabla_X(\mathfrak{X} + \mathfrak{X}') = \nabla_X \mathfrak{X} + \nabla_X \mathfrak{X}'$
- $\nabla_{fX}\mathfrak{X} = f\nabla_X \mathfrak{X}$
- $\nabla_X(f\mathfrak{X}) = f\nabla_X \mathfrak{X} + (X \cdot f)\mathfrak{X}$

It turns out that for any vector bundle, such an object always exists (but is far from being unique). $\nabla_X \mathfrak{X}$ corresponds to the idea of deriving a section $\mathfrak{X}$ in the direction of X.

1.1.6 Forms and Differential Complexes

We define here the notion of differential calculus on smooth manifolds. One takes throughout this section a n–dimensional smooth manifold M.

1.1.6.1 Definition

For $q \in \mathbb{N}$, a q–form ω with values in functions is an antisymmetric linear map which takes q–vector fields and gives a function. That is to say, $\omega : \Gamma(\wedge^q TM) \to \mathcal{C}^\infty(M)$. The space of such q–forms is denoted $\Omega^q(M)$. This space takes place in a differential complex:

$$\Omega^\bullet(M) = \bigoplus_{q\geq 0} \Omega^q(M) \tag{1.3}$$

with by definition $\Omega^0(M) = \mathcal{C}^\infty(M)$. There exists a differential operator d which increases the degree of form by +1, and which is defined on a q–form ω, given by the Koszul formula:

$$\begin{aligned}(\mathrm{d}\omega)(X_1, ..., X_{q+1}) = &\sum_{i=1}^{q+1}(-1)^{i+1} X_i \cdot \omega(X_1, ..., \overset{i}{\vee}, ..., X_{q+1}) \\ &+ \sum_{1\leq i<j\leq q+1} (-1)^{i+j}\omega([X_i, X_j], X_1, ..., \overset{i}{\vee}, ..., \overset{j}{\vee}, ..., X_{q+1})\end{aligned} \tag{1.4}$$

where $\overset{i}{\vee}$ denotes the omition of the $i-$th element. If (U, x^μ) is a local chart, then $\{\mathrm{d}x^\mu\}_{\mu=1..n}$ is a (holonomic) basis of the dual of TU, and any form ω of degree $q \leq n$ can be decomposed in this basis:

$$\omega = \omega_{\mu_1\mu_2...\mu_q} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} ... \wedge \mathrm{d}x^{\mu_q} \tag{1.5}$$

where the $\binom{q}{n}$ symbols $\omega_{\mu_1\mu_2...\mu_q}$ denote smooth functions and are the components of the q–form in the given basis.

1.1.6.2 Wedge product and commutator of two forms

Let ω and η be respectively a p–form and a q–form on M.

The *wedge product*, or exterior product, $\omega \wedge \eta$ defines a $(p+q)$–form on M as:

$$\begin{aligned}(\omega \wedge \eta)(X_1, ..., X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma\in S_{p+q}} &(-1)^{sign(\sigma)}\omega(X_{\sigma(1)}, ..., X_{\sigma(p)}) \\ &\cdot \eta(X_{\sigma(p+1)}, ..., X_{\sigma(p+q)})\end{aligned} \tag{1.6}$$

This product gives an algebra structure to the space $\Omega^\bullet(M)$. It has the following commutativity property:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \tag{1.7}$$

Moreover, it is also associative, what gives meaning to the basis used in 1.5.

The differential operator d is an anti–derivation of the algebra $\Omega^\bullet(M)$ in the sense that one has:

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta \tag{1.8}$$

For instance, for two 1–forms α and β, one gets:

$$\alpha \wedge \beta = -\beta \wedge \alpha \tag{1.9}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta - \alpha \wedge (d\beta) \tag{1.10}$$

1.1.6.3 Metric and Hodge Operator

$\Omega^q(M) = \Gamma(\wedge^q T^* M)$, and the fibre of the fibre bundle $\wedge^q T^* M$ is of dimension $\binom{n}{q}$, as well as that of the fibre bundle $\wedge^{n-q} T^* M$. If M is equipped with a non degenerate metric, i.e. a bilinear symmetric map $g : \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^\infty(M)$, such that $g(X, Y) = 0$ for all $X \in \Gamma(TM)$ implies $Y = 0$, then there exists an operator $*$ which realizes the isomorphism. Let us first define the totally antisymmetric tensor ϵ by:

$$\epsilon_{\mu_1\mu_2,\dots\mu_n} := \begin{cases} +1, & \text{if } (\mu_1, \mu_2, ..., \mu_n) \text{ is an even permutation of } (12...n) \\ -1, & \text{if } (\mu_1, \mu_2, ..., \mu_n) \text{ is an odd permutation of } (12...n) \\ 0, & \text{otherwise.} \end{cases} \tag{1.11}$$

With the help of the metric g, one has :

$$\epsilon^{\mu_1\dots\mu_p}{}_{\mu_{p+1}\dots\mu_n} = g^{\mu_1\nu_1} g^{\mu_2\nu_2} ... g^{\mu_p\nu_p} \epsilon_{\nu_1\dots\nu_p\mu_{p+1}\dots\mu_n} \tag{1.12}$$

for any $p \le n$. The Einstein convention of summation over repeated indices is used.

The hodge star operator has the following definition acting on a holonomic basis $\{\mathrm{d}x^\mu\}_{\mu=1..n}$,

$$*(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} ... \wedge \mathrm{d}x^{\mu_q}) = \frac{\sqrt{|g|}}{(n-q)!} \epsilon^{\mu_1\dots\mu_q}{}_{\nu_{q+1}\dots\nu_n} \mathrm{d}x^{\nu_{q+1}} \wedge \mathrm{d}x^{\nu_{q+2}} ... \wedge \mathrm{d}x^{\nu_n} \tag{1.13}$$

where $|g| := |\mathrm{det}(g)|$.

1.1.6.4 Generalized Kronecker delta

It is worth defining the generalized Kronecker delta, and giving some useful computational properties. The Kronecker delta of order $2p$, $p > 0$, is defined as:

$$\delta^{\mu_1\mu_2...\mu_p}_{\nu_1\nu_2,...\nu_p} := \begin{cases} +1, \text{ if } \nu_1, \nu_2, ..., \nu_p \text{ are distinct integers} \\ \quad \text{and is an even permutation of } (\mu_1, \mu_2, ..., \mu_p) \\ -1, \text{ if } \nu_1, \nu_2, ..., \nu_p \text{ are distinct integers} \\ \quad \text{and is an odd permutation of } (\mu_1, \mu_2, ..., \mu_p) \\ 0, \text{ otherwise.} \end{cases} \tag{1.14}$$

In particular, one has:

$$\delta^{\mu_1\mu_2...\mu_n}_{\nu_1\nu_2...\nu_n} = \epsilon^{\mu_1\mu_2,...\mu_n}\epsilon_{\nu_1\nu_2...\nu_n}. \tag{1.15}$$

in the case of $p = n$.

In dimension $n = p = 4$, it is useful to have in mind the following relations:

$$\epsilon^{aijk}\epsilon_{bijk} = -6\delta^a_b \tag{1.16}$$

$$\epsilon^{abij}\epsilon_{cdij} = -2(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d) \tag{1.17}$$

$$\epsilon^{abci}\epsilon_{rsti} = -(\delta^a_r\delta^b_s\delta^c_t + \delta^b_r\delta^c_s\delta^a_t + \delta^c_r\delta^a_s\delta^b_t - \delta^a_r\delta^c_s\delta^b_t - \delta^b_r\delta^a_s\delta^c_t - \delta^c_r\delta^b_s\delta^a_t) \tag{1.18}$$

1.1.6.5 Forms with values in a Lie algebra

An important kind of differential forms is the generalized notion of differential forms with values in a Lie algebra $\mathfrak{g}$. One restricts here to finite dimensional Lie algebras, so that they are always isomorphic to some matrix algebras, and thus an associative product together with a natural commutator is always defined on them. Everything which has been defined for ordinary forms (i.e. with values in function) can also be defined in this case, with slight modifications of their properties.

If $\{E_a\}_a$ is a basis of the (finite dimensional) Lie algebra $\mathfrak{g}$, then any $\mathfrak{g}$–valued p–form ω can be written:

$$\omega = \omega^a \otimes E_a \tag{1.19}$$

with $\omega^a \in \Omega^p(M)$. The space of such $\mathfrak{g}$–valued q–forms thus reads:

$$\Omega^q(M, \mathfrak{g}) = \Omega^q(M) \otimes \mathfrak{g} \tag{1.20}$$

The *graded commutator* of two $\mathfrak{g}$–valued forms ω and η (with respective degree p and q) is defined as:

$$[\omega, \eta] := \omega \wedge \eta - (-1)^{pq} \eta \wedge \omega. \tag{1.21}$$

It fulfills the following graded Jacobi identity, with ξ a r–form:

$$(-1)^{pr}[\omega, [\eta, \xi]] + (-1)^{pq}[\eta, [\xi, \omega]] + (-1)^{qr}[\xi, [\omega, \eta]] = 0 \tag{1.22}$$

$\Omega^\bullet(M, \mathfrak{g})$ is said to be a graded Lie algebra.

Decomposing the two forms in the basis E_a leads to the following decomposition of the commutator:

$$[\omega, \eta] = \omega^a \wedge \omega^b \otimes [E_a, E_b]. \tag{1.23}$$

The Lie algebra in which a form ω takes values being supposed to be an algebra of matrices, one has the following useful property of the wedge product of ω with itself:

$$[\omega, \omega] = 2\omega \wedge \omega. \tag{1.24}$$

All these notions will be defined in the more general framework of transitive Lie algebroids, as one is going to see right now.

1.2 Transitive Lie Algebroids

1.2.1 Introduction

Transitive Lie algebroids over a (connected) smooth manifold M are a particular kind of vector bundles and can be seen as generalizations of either:

- the tangent bundle TM;
- Lie algebras;
- (infinitesimal data related to) principal fibre bundles over M.

Indeed, a transitive Lie algebroid A is given with a surjective homomorphism $\rho : A \to TM$ which satisfies some properties. By this way, it both encodes the data related to the geometry of the base manifold M and contains a purely algebraic part. A Lie algebroid over a point ($M = \{\star\}$) is a mere Lie algebra. By the third Lie theorem, any finite dimensional Lie algebra is the Lie algebra of a certain Lie group. In the case of Lie algebroids, it is not always true. Some transitive Lie algebroids are the Lie algebroids associated to an underlying structure called a Lie groupoid, but not all of them. Lie algebroids which are the infinitesimal version of a Lie groupoid are said to be integrable. Thus, in other words, not all transitive Lie algebroids are integrable, unlike (finite-dimensional) Lie algebras. However, the integrable and transitive Lie algebroids are in 1:1 correspondance with principal fibre bundles. Indeed, there is a particular class of transitive Lie algebroids, called Atiyah Lie algrebroids, which are transitive Lie algebroids built from the data of (possibly many) principal bundles. It turns out that any transitive and integrable Lie algebroid is the Atiyah Lie algebroid related to some principal fibre bundles. This class of transitive Lie algebroids is of primary importance since principal fibre bundles provide a natural geometric framework of gauge theories describing fundamental interactions in physics.

In this section, one presents transitive Lie algebroids in a general way. In the subsequent sections, one studies the integrability conditions and present principal fibre bundles as particular cases, rederived from integrable transitive Lie algebroids. A general introduction to Lie algebroids and Lie groupoids can be found in [25], [8] and [9]. Both the geometric and differentiable stuctures constructed on transitive Lie algebroids and presented further can be found with more precision and details in [16] (C. Fournel's PhD thesis), and [15].

1.2.2 Definition

1.2.2.1 A particular vector bundle

A Lie algebroid A over a connected smooth manifold M is a vector bundle $\pi_A : A \to M$ equipped with a map $\rho : A \to TM$ called an **anchor**. The space of sections of A, $\Gamma(A)$, has a Lie algebra structure, and ρ is naturally extended to this space. By definition, its Lie bracket $[\ ,\ ]_A$ verifies:

$$[\mathfrak{X}, f\mathfrak{Y}]_A = f[\mathfrak{X}, \mathfrak{Y}]_A + (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y} \tag{1.25}$$

This is the way one can mimick the behaviour of vector fields on this more general vector bundle. The multiplication of an element of the Lie algebroid by a function is denoted without any symbol, and $X \cdot f$ denotes as usual the action of a vector field $X \in \Gamma(TM)$ on a function $f \in \mathcal{C}^\infty(M)$.

Defined on $\Gamma(A)$, ρ is a morphism of Lie algebra, i.e.:

$$\rho([\mathfrak{X}, \mathfrak{Y}]_A) = [\rho(\mathfrak{X}), \rho(\mathfrak{Y})]_{TM}. \tag{1.26}$$

From now on, neither the label A nor TM will be specified anymore on the Lie bracket, the context making it unambigous.

1.2.2.2 A–paths and transitivity

An A–path a, is a path $a : [0,1] \to A$ such that the vector field defined by the projection of a on M corresponds to the image of a by the anchor ρ, i.e. $\frac{d}{dt}\pi_A \circ a(t) = \rho \circ a(t)$.

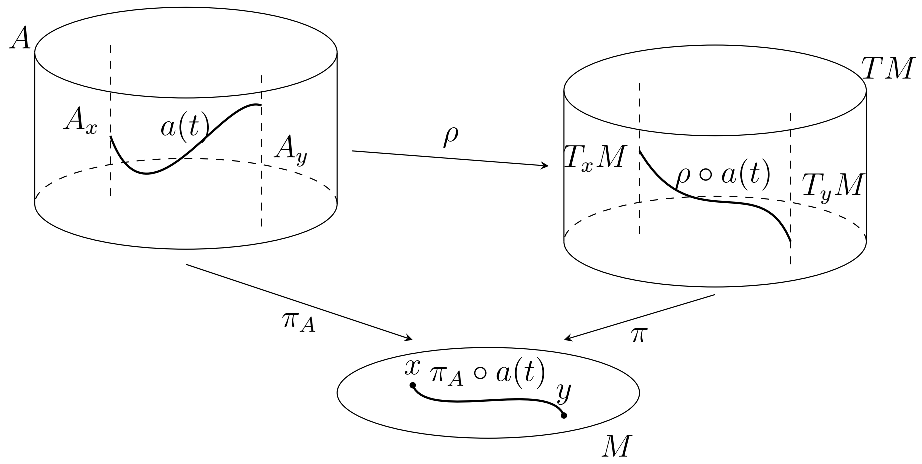


Figure 1.3: A–paths on a Lie algebroid.

The space of A–paths is denoted $\mathcal{P}(A)$. A can act on M via $\mathcal{P}(A)$: let $x \in M$, and $a \in \mathcal{P}(A)$ such that $\pi_A \circ a(0) = x$. Then the image of x by the action of a is $y = \pi_A \circ a(1)$. One can define orbits of the action of A in M, as the different sets of points related by the action of an A–path. The Lie algebroid is said to be **transitive** if there is only one orbit, i.e. if for any $x, y \in M$, there exists $a \in \mathcal{P}(A)$ such that $\pi_A \circ a(0) = x$ and $\pi_A \circ a(1) = y$. The transitivity of the Lie algebroid is equivalent to the **surjectivity** of the anchor ρ. Indeed, for $x, y \in M$, one can always find a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$ because M is supposed to be connected, and one can always get the corresponding tangent field $\frac{d}{dt}\gamma(t) \in TM$. x and y are related by an A–action if and only if $\frac{d}{dt}\gamma(t) \in TM$ is in Im(ρ). It is always the case if ρ is surjective, and conversely.

The Lie algebroids we are studying are always supposed to be transitive. This can be justified by physical considerations. In the physical theoretical framework we are considering, the base manifold M plays the role of spacetime, a particular point of which has no physical meaning.[b] Considering a non transitive Lie algebroid would mean having M foliated in different zones (corresponding to the different orbits of the action), which seems to be hardly interpretable from this "no physical relevance of spacetime point" viewpoint.

Ker(ρ) is an ideal of A. One has the following short exact sequence caracterizing a transitive Lie algebroid:

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} TM \longrightarrow 0 \tag{1.27}$$

where L and ι are such that $\iota(L) = \ker(\rho)$. This sequence extends to a short

[b]As it has already been mentionned, this fact is in particular expressed by the active diffeomorphism invariance in General Relativity.

exact sequence of $\mathcal{C}^\infty(M)$–modules and Lie algebras:

$$0 \longrightarrow \mathcal{L} \xrightarrow{\ \iota\ } \mathcal{A} \xrightarrow{\ \rho\ } \Gamma(TM) \longrightarrow 0 \tag{1.28}$$

with $\mathcal{A} = \Gamma(A)$ and $\mathcal{L} = \Gamma(L)$. From now on, we will work without distinction with one or the other formulation.

1.2.3 Trivialization

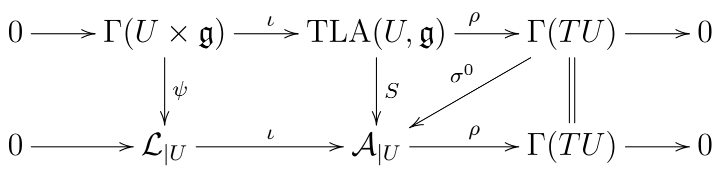
Transitive Lie algebroids look locally (over an open set $U \subset M$) like a trivial Lie algebroid, denoted TLA$(U, \mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. TLA$(U, \mathfrak{g})$ is defined by the following short exact sequence:

$$0 \longrightarrow \Gamma(U \times \mathfrak{g}) \xrightarrow{\ \iota\ } \text{TLA}(U, \mathfrak{g}) \xrightarrow{\ \rho\ } \Gamma(TU) \longrightarrow 0 \tag{1.29}$$

where TLA$(U, \mathfrak{g}) := \Gamma(TU) \oplus \Gamma(U \times \mathfrak{g})$, and with $\iota(\gamma) = 0 \oplus \gamma$ and $\rho(X \oplus \gamma) = X$ for $\gamma \in \Gamma(U \times \mathfrak{g})$ and $X \in \Gamma(TU)$. The Lie bracket between two elements of this trivial Lie algebroid reads:

$$[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta]) \tag{1.30}$$

Then, a trivialization of a transitive Lie algebroid $\mathcal{A}$ over U is an isomorphism S : TLA$(U, \mathfrak{g}) \to \mathcal{A}_{|U}$. The map S splits into in two maps $\sigma^0 : TU \to \mathcal{A}_{|U}$, and $\psi : \Gamma(U \times \mathfrak{g}) \to \mathcal{L}_{|U}$.



The map σ^0 is defined such that $\sigma^0_X = S(X \oplus 0)$, for any $X \in \Gamma(TU)$, and has the following properties, for any $X, Y \in \Gamma(TU)$ and $f \in \mathcal{C}^\infty(M)$:

- $[\sigma^0_X, \sigma^0_Y] = \sigma_{[X,Y]}$,
- $\sigma^0_{fX} = f\sigma^0_X$,
- $\rho \circ \sigma^0_X = X$,

The map ψ is defined by $\iota \circ \psi(\gamma) = S(0 \oplus \gamma)$, for any $\gamma \in \Gamma(U \times \mathfrak{g})$ and has the following properties:

- $\psi([\gamma, \eta]) = [\psi(\gamma), \psi(\eta)]$,

- $\psi(f\gamma) = f\psi(\gamma)$,

for any $\gamma, \eta \in \Gamma(U \times \mathfrak{g})$ and any $f \in \mathcal{C}^\infty(M)$. In other words, ψ is a morphism of Lie algebras and of $\mathcal{C}^\infty(M)$–modules.

These two formulations are related by the formula:

$$S(X \oplus \gamma) = \sigma^0_X + \iota \circ \psi(\gamma) \tag{1.31}$$

and moreover, there is a compatibility condition fulfilled by the two maps with respect to the Lie bracket of the trivial Lie algebroid:

$$[\sigma^0_X, \iota \circ \psi(\gamma)] = \iota \circ \psi(X \cdot \gamma). \tag{1.32}$$

Like in the definition of an atlas of a manifold, one can define an atlas of Lie algebroids as the pairs $\{(U_i, S_i)\}_i = \{(U_i, \sigma_i, \psi_i)\}_i$, assuming that the charts $\{U_i\}_i$ coincide with the charts of the manifold itself. Changes of trivialization of a Lie algebroid, however, have to be computed separately that changes of local charts. Let us give the gluing relations that a family of pairs as $\{X_i \oplus \omega_i\}_i$ has to fulfill to be the local trivialization of a global element $\mathfrak{X} \in \mathcal{A}$. Let (U, S) and (U, S') be two local trivializations over the same open subset $U \subset M$.Then, there exist two maps $\alpha \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$ and $\chi \in \Omega^1(U) \otimes \mathfrak{g}$ such that:

$$(S^{-1} \circ S')(X \oplus \gamma) = X \oplus (\alpha(\gamma) + \chi(X)) \tag{1.33}$$

for $X \oplus \gamma \in TLA(U, \mathfrak{g})$. The tangent vector X is invariant under a change of trivialization. The algebraic part, however, is moved by an endomorphism α and then lifted by the action of X.

1.2.4 Representations

One first presents the particular transitive Lie algebroid of derivations of a given vector bundle. Then, one defines a representation of a general transitive Lie algebroid as a morphism from this algebroid to the algebroid of derivations of a certain vector bundle.

1.2.4.1 The transitive Lie algebroid of derivations of a vector bundle

Let $E \to M$ be a vector bundle over a smooth manifold M. An *endomorphism* of E, denoted $\phi \in \text{End}(E)$, also called 0^{th}–order operator, is an homomorphism of E into itself which is a linear map on each fibre E_x, $x \in M$. In other words: $\phi(f\mu) = f\phi(\mu)$ for any $f \in \mathcal{C}^\infty(M)$ and $\mu \in \Gamma(E)$.

A *first order differential operator* on E is a map $D : \Gamma(E) \to \Gamma(E)$ such that for $f \in \mathcal{C}^\infty(M)$, the map $\mu \to D(f\mu) - fD(\mu)$ is a 0^{th}–order operator. The space of all such operators is denoted Diff$^1(E)$. A *derivation* D is a first order differential operator for which there exists a map $\alpha : D \mapsto \alpha(D) \in \Gamma(TM)$ such that

$$D(f\mu) = fD(\mu) + (\alpha(D).f)\mu \tag{1.34}$$

In other words, its action on the algebra of functions is specified. The space of all derivations of E is denoted $\mathcal{D}(E)$. The commutator of two elements $D, D' \in \mathcal{D}(E)$ is defined as: $[D, D'] = D \circ D' - D' \circ D$. The bracket $[D, D']$ is a derivation and $\alpha([D, D']) = [\alpha(D), \alpha(D')]$. Indeed, a direct computation shows that

$$[D, D'](f\mu) = f[D, D'](\mu) + ([\alpha(D), \alpha(D')].f)\mu,$$

which proves both assertions. $\mathcal{D}(E)$ has thus a Lie algebra structure, and α is a Lie algebra homomorphism which makes the correspondence with the Lie algebra structure of vector fields. Moreover, α is surjective. Indeed, as it has been claimed (without proof), on any vector bundle it is always possible to define a connection $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$. Take any of them, then for any $X \in \Gamma(TM)$, $\nabla_X \in \mathcal{D}(E)$, i.e. any $X \in \Gamma(TM)$ has an antecedent by α.

A straightforward computation shows the important property of the homomorphism with respect to the bracket:

$$[D, fD'](\mu) = f[D, D'](\mu) + (\alpha(D).f)D'(\mu) \tag{1.35}$$

Due to the very definition of α, it is straightforward to see that ker(α) is identified with End(E). Finally, $\mathcal{D}(E)$ takes place in the following short exact sequence:

$$0 \longrightarrow \text{End}(E) \xrightarrow{\iota_E} \mathcal{D}(E) \xrightarrow{\alpha} \Gamma(TM) \longrightarrow 0 \tag{1.36}$$

where ι_E denotes the identification End$(E) =$ ker$(\alpha) \subset \mathcal{D}(E)$. This sequence defines the *transitive Lie algebroid of derivations* associated to a vector bundle E.

1.2.4.2 Representations on vector bundles

Transitive Lie algebroids are represented as operators which act as derivations on vector bundles. Let $(\mathcal{A}, \rho)$ be a transitive Lie algebroid and $E \to M$ a vector bundle, a representation of A on E is a morphism of Lie algebroids

$$\phi : \mathcal{A} \to \mathcal{D}(E).$$

ϕ reduces to another morphism $\phi_{\mathcal{L}} : \mathcal{L} \to$ End(E) on the kernel, such that $\iota_E \circ \phi_{\mathcal{L}} = \phi \circ \iota$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \xrightarrow{\iota} & \mathcal{A} & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi_{\mathcal{L}}} & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & \mathsf{End}(E) & \xrightarrow{\iota_E} & \mathcal{D}(E) & \xrightarrow{\alpha} & \Gamma(TM) & \longrightarrow & 0 \end{array}$$

1.2.4.3 Representations on the kernel

Since $L \simeq \mathsf{ker}(\rho)$ is a vector bundle, one can consider a representation of $\mathcal{A}$ on its own kernel $\mathcal{L}$ realized by the commutator. That is to say, one associates to any $\mathfrak{X} \in \mathcal{A}$ the derivation of $\mathcal{L}$ defined by $\eta \longmapsto \mathfrak{X} \cdot \eta$, where $\mathfrak{X} \cdot \eta$ is defined as the only element of $\mathcal{L}$ such that

$$\iota(\mathfrak{X} \cdot \eta) = [\mathfrak{X}, \iota(\eta)]. \tag{1.37}$$

It is always convenient to have an idea of how things work in a trivialization. In the case of formula 1.37, one gets

$$(X \oplus \gamma) \cdot \eta = X \cdot \eta + [\gamma, \eta],$$

that is to say the geometric part derives the element η and the algebraic part acts on it via the commutator, which is also a form of derivation.

1.2.5 Differential Structures on transitive Lie algebroids

1.2.5.1 Definitions

Forms and their underlying differential complex can be also defined in the case of a Lie algebroid. The latter needs not to be transitive, even if it will be always assumed in the following. Let E be a vector bundle and $\phi : \mathcal{A} \to \mathcal{D}(E)$ a representation of a transitive Lie algebroid $\mathcal{A}$. For $q \in \mathbb{N}$, a differential q–form is a $\mathcal{C}^\infty(M)$–linear antisymmetric map defined on $\wedge^q \mathcal{A}$ (antisymmetrized product of q elements of $\mathcal{A}$) with values in $\Gamma(E)$. One denotes $\Omega^q(\mathcal{A}, E)$ the space of such forms, and $\Omega^\bullet(\mathcal{A}, E)$ the differential complex of forms of any degree defined on $\mathcal{A}$ with values in $\Gamma(E)$, which can be written as (with $\Omega^0(\mathcal{A}, E) := \Gamma(E)$ by convention):

$$\Omega^\bullet(\mathcal{A}, E) := \bigoplus_{q \geq 0} \Omega^q(\mathcal{A}, E) \tag{1.38}$$

1.2.5.2 Differential Operator

One can then define on such a differential complex a differential operator $\hat{\mathsf{d}}_\phi$, according to the Koszul formula, which increases the degree of forms by +1:

$$\begin{aligned}(\hat{\mathsf{d}}_\phi\omega)(\mathfrak{X}_1,...,\mathfrak{X}_{q+1}) &= \sum_{i=1}^{q+1}(-1)^{i+1}\phi(\mathfrak{X}_i)\cdot\omega(\mathfrak{X}_1,...,\overset{i}{\vee},...,\mathfrak{X}_{q+1}) \\ &+ \sum_{1\leq i<j\leq q+1}(-1)^{i+j}\omega([\mathfrak{X}_i,\mathfrak{X}_j],\mathfrak{X}_1,...,\overset{i}{\vee},...,\overset{j}{\vee}),...,\mathfrak{X}_{q+1})\end{aligned} \tag{1.39}$$

1.2.5.3 Example for a $\mathcal{L}$–valued 1–form

As one shall see later, an important kind of form is a 1–form with values in the kernel, i.e. $\omega \in \Omega^1(\mathcal{A},\mathcal{L})$. (That is to say, one considers the representation of the Lie algebroid on its kernel (see above).) For such a form, $\hat{\mathsf{d}}\omega$ (without label) reads:

$$\hat{\mathsf{d}}\omega(\mathfrak{X},\mathfrak{Y}) = \mathfrak{X}\cdot\omega(\mathfrak{Y}) - \mathfrak{Y}\cdot\omega(\mathfrak{X}) - \omega([\mathfrak{X},\mathfrak{Y}]) \tag{1.40}$$

1.2.5.4 Cartan Operations on $(\Omega^\bullet(\mathcal{A},E),\hat{d}_\phi)$

Let $(\Omega^\bullet(\mathcal{A},E),\hat{d}_\phi)$ be the graded differential space associated to a representation ϕ of the Lie algebroid $\mathcal{A}$ on a vector bundle E. For any $\mathfrak{X}\in\mathcal{A}$, and $q>0$, one defines the following Cartan operations on a q–form ω:

- $(i_\mathfrak{X}\omega)(\mathfrak{X}_1,...,\mathfrak{X}_{q-1}) = \omega(\mathfrak{X},\mathfrak{X}_1,...,\mathfrak{X}_{q-1})$, and
- $L_\mathfrak{X} := \hat{d}\circ i_\mathfrak{X} + i_\mathfrak{X}\circ\hat{d}$.

$i_\mathfrak{X}:\Omega^q(\mathcal{A},E)\to\Omega^{q-1}(\mathcal{A},E)$ is called the *inner operation*, and $L_\mathfrak{X}:\Omega^q(\mathcal{A},E)\to\Omega^q(\mathcal{A},E)$ the *Lie derivative*. These Cartan operations satisfy the following properties:

$$i_{f\mathfrak{X}} = fi_\mathfrak{X},\ i_\mathfrak{X}i_\mathfrak{Y} + i_\mathfrak{Y}i_\mathfrak{X} = 0,\ [L_\mathfrak{X},L_\mathfrak{Y}] = L_{[\mathfrak{X},\mathfrak{Y}]},\ [L_\mathfrak{X},i_\mathfrak{Y}] = i_{[\mathfrak{X},\mathfrak{Y}]} \tag{1.41}$$

The Lie derivative is of primary importance to formulate gauge theories. Indeed, the restriction of L to elements of the kernel describe infinitesimal internal gauge transformations, while objects like L_X with $X\in\Gamma(TM)$ is the infinitesimal version of a diffeomorphism, i.e. describe external gauge transformations.[c]

[c]Let us notice that a slight abuse of language is made here, because for a certain $X\in\Gamma(TM)$, there is not a unique corresponding $\mathfrak{X}\in\mathcal{A}$: a (non-unique) correspondence is made with the notion of splitting, defined later.

1.2.5.5 On a trivial transitive Lie algebroid

Let us consider the differential complex of forms defined on a trivial transitive Lie algebroid TLA$(U, \mathfrak{g})$ with values in functions. Here, U is an open subset of M equipped with a coordinate system x^μ. The corresponding differential complex splits into two parts:

$$\Omega^\bullet_{\mathsf{TLA}}(U) = \Omega^\bullet(U) \otimes \Omega^\bullet(\mathfrak{g}) \tag{1.42}$$

Denote $(\mathsf{d}x^1, ..., \mathsf{d}x^n)$ a basis of T^*U and $(\theta^1, ..., \theta^m)$ a basis of $\mathfrak{g}^*$, with $m = \dim(\mathfrak{g})$. Then, any $\omega \in \Omega^q_{TLA}(U)$ can be decomposed as

$$\omega = \sum_{r+s=q} \omega_{\mu_1 ... \mu_r a_1 ... a_s} \mathsf{d}x^{\mu_1} \wedge \mathsf{d}x^{\mu_2} ... \wedge \mathsf{d}x^{\mu_r} \otimes \theta^{a_1} ... \wedge \theta^{a_2} ... \wedge \theta^{a_s} \tag{1.43}$$

Such a q–form can thus be thought as a sum of forms, each of which eating r vector fields and s Lie algebra elements such that $r + s = q$. r is the *geometric* degree of form, while s is the *algebraic* degree of form of each component.

The differential operator $\hat{\mathsf{d}}_\phi$ also splits into two parts, corresponding to the representation of vector fields on functions and to the trivial representation of the Lie algebra $\mathfrak{g}$. It is denoted δ in the present case of a trivial Lie algebroid, and for $\omega \in \Omega^q_{TLA}(U)$ it reads, according to the Koszul formula:

$$\begin{aligned} &(\delta\omega)(X_1 \oplus \gamma_1, ..., X_{q+1} \oplus \gamma_{q+1}) = \\ &\sum_{i=1}^{q+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1}) + \\ &\sum_{1 \le i < j \le q+1} (-1)^{i+j} \omega([X_i \oplus \gamma_i, X_j \oplus \gamma_j], X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., \overset{j}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1}) \end{aligned} \tag{1.44}$$

δ can also be written as:

$$\delta = \mathsf{d} + s \tag{1.45}$$

where d is the Koszul derivative associated to the representation of $\Gamma(TM)$ on

$\mathcal{C}^\infty(M)$. Due to the form of the commutator 1.30, one gets:

$$\begin{aligned}(\mathrm{d}\omega)(X_1 \oplus \gamma_1, ..., X_{q+1} \oplus \gamma_{q+1}) = \\ \sum_{i=1}^{q+1}(-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1}) + \\ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([X_i, X_j] \oplus (X_i \cdot \gamma_j - X_j \cdot \gamma_i), X_1 \oplus \gamma_1, \\ ..., \overset{i}{\vee}, ..., \overset{j}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1})\end{aligned} \tag{1.46}$$

and

$$\begin{aligned}(s\omega)(X_1 \oplus \gamma_1, ..., X_{q+1} \oplus \gamma_{q+1}) = \\ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega(0 \oplus [\gamma_i, \gamma_j], X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., \overset{j}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1})\end{aligned} \tag{1.47}$$

Thus, d increases the geometric degree of forms by +1, and s increases the algebraic degree of forms by +1. $\mathrm{d}^2 = 0$, $s^2 = 0$ and $\mathrm{d} \circ s + s \circ \mathrm{d} = 0$.

In the case of a differential form ω with values in the kernel, that is to say in $\Gamma(U \times \mathfrak{g})$, its decomposition in a basis changes sightly according to:

$$\omega = \sum_{r+s=q} \omega^a_{\mu_1 ... \mu_r a_1 ... a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} ... \wedge \mathrm{d}x^{\mu_r} \otimes \theta^{a_1} ... \wedge \theta^{a_2} ... \wedge \theta^{a_s} \otimes E_a \tag{1.48}$$

where $\{E_a\}_a$ is a basis of $\mathfrak{g}$. The corresponding differential operator $\hat{\mathrm{d}}_{TLA}$ splits also into two terms, as in the previous case: $\hat{\mathrm{d}}_{TLA} = \mathrm{d} + s'$, where d is still the Koszul derivative associated to the representation of vector fields on functions, and s' is the Chevalley–Eilenberg derivative associated to the adjoint representation of $\mathfrak{g}$ on $\Gamma(U \times \mathfrak{g})$, taking into account that ω takes values in $\mathfrak{g}$. In this case, s' reads:

$$\begin{aligned}(s'\omega)(X_1 \oplus \gamma_1, ..., X_{q+1} \oplus \gamma_{q+1}) = \\ \sum_{i=1}^{q+1}(-1)^{i+1} [\gamma_i, \omega(X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1})] + \\ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega(0 \oplus [\gamma_i, \gamma_j], X_1 \oplus \gamma_1, ..., \overset{i}{\vee}, ..., \overset{j}{\vee}, ..., X_{q+1} \oplus \gamma_{q+1})\end{aligned} \tag{1.49}$$

These two operators are also nilpotent and anticommute.

1.2.6 Geometric Structures on transitive Lie algebroids

In all this subsection, one considers a transitive Lie algebroid:

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$$

as a short exact sequence of $\mathcal{C}^\infty(M)$–modules and Lie algebras.

1.2.6.1 Splitting and connection 1–form

A splitting is a map $\sigma : \Gamma(TM) \to \mathcal{A}$ such that $\rho \circ \sigma = \mathrm{id}_{\Gamma(TM)}$. It allows to (arbitrarily) split any element $\mathfrak{X} \in \mathcal{A}$ into a geometric part and an algebraic part. Indeed, for any $\mathfrak{X} \in \mathcal{A}$, $\mathfrak{X}$ and $\sigma_{\rho(\mathfrak{X})}$ have the same image by ρ, i.e. $\rho(\mathfrak{X} - \sigma_{\rho(\mathfrak{X})}) = 0$ and thus there exists an element $\omega(\mathfrak{X}) \in \mathcal{L}$ such that:

$$\mathfrak{X} = \sigma_{\rho(\mathfrak{X})} - \iota \circ \omega(\mathfrak{X}) \tag{1.50}$$

(the minus sign is conventional). The map $\omega : \mathcal{A} \to \mathcal{L}$ is called the connection 1–form related to the splitting σ, $\omega \in \Omega^1(\mathcal{A}, \mathcal{L})$ and satisfies the normalization condition:

$$\omega \circ \iota(\gamma) = -\gamma \tag{1.51}$$

for any $\gamma \in \mathcal{L}$. (Proof: set $\mathfrak{X} = \iota(\gamma)$ in 1.50.)

Let us remark that once a splitting σ is given, one gets the following short exact sequence:

$$0 \longleftarrow \mathcal{L} \xleftarrow{\omega} \mathcal{A} \xleftarrow{\sigma} \Gamma(TM) \longleftarrow 0$$

i.e. $\omega \circ \sigma = 0$ (Proof: take $\omega(\mathfrak{X})$ with $\mathfrak{X}$ written as in 1.50.)

The curvature related to a splitting σ is defined as a map $R : \Gamma(TM) \times \Gamma(TM) \to \mathcal{A}$ by:

$$R(X,Y) := [\sigma_X, \sigma_Y] - \sigma_{[X,Y]} \tag{1.52}$$

R measures the obstruction of σ to be a morphism of Lie algebras. One can also define the corresponding curvature 2–form $\Omega : \mathcal{A} \times \mathcal{A} \to \mathcal{L}$ by the standard formula:

$$\Omega := \hat{d}\omega + \frac{1}{2}[\omega, \omega] \tag{1.53}$$

Both definitions are deeply related:

$$\iota \circ \Omega = \rho^* R. \tag{1.54}$$

(Proof: compute $R(\rho(\mathfrak{X}), \rho(\mathfrak{Y}))$ and $\iota \circ \Omega(\mathfrak{X}, \mathfrak{Y})$, and use $\sigma_{[\rho(\mathfrak{X}),\rho(\mathfrak{Y})]} = \sigma_{\rho([\mathfrak{X},\mathfrak{Y}])} = [\mathfrak{X}, \mathfrak{Y}] + \iota \circ \omega([\mathfrak{X}, \mathfrak{Y}])$ to see they are equal.)

1.2.6.2 Affine Connections

There is another kind of connection that one can build on a transitive Lie algebroid, which are closer to the spirit of Riemannian geometry. A more detailed study of Riemannian geometry on Lie algebroids can be found in [7]. An *affine connection* on a Lie algebroid is a map $\hat{\nabla} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that:

- $\hat{\nabla}_{f\mathfrak{X}}\mathfrak{Y} = f\hat{\nabla}_{\mathfrak{X}}\mathfrak{Y}$;
- $\hat{\nabla}_{\mathfrak{X}}(f\mathfrak{Y}) = f\hat{\nabla}_{\mathfrak{X}}\mathfrak{Y} + (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y}$

for $\mathfrak{X}, \mathfrak{Y} \in \mathcal{A}$, and $f \in \mathcal{C}^\infty(M)$.

By convention, one defines $\hat{\nabla}_{\mathfrak{X}} f := \rho(\mathfrak{X}) \cdot f$. It is a natural generalization of the notion of affine connection defined on the tangent bundle. It cannot be, however, defined on any vector bundle: to realize the Leibniz rule, one needs to have a specific action on functions, like $\rho(\mathfrak{X}) \cdot f$ in the latter case.

1.2.6.3 Metric

Equivalent Triple Let $\rho : \mathcal{A} \to \Gamma(TM)$ be a transitive Lie algebroid. A metric on $\mathcal{A}$ is a bilinear symmetric map $\hat{g} : \mathcal{A} \times \mathcal{A} \to \mathcal{C}^\infty(M)$. $\hat{g}$ is said to be non-degenerate if it fulfills the condition: $\hat{g}(\mathfrak{X}, \mathfrak{Y}) = 0 \; \forall \; \mathfrak{X} \in \mathcal{A} \Rightarrow \mathfrak{Y} = 0$. $\hat{g}$ reduces to a metric $h : \mathcal{L} \times \mathcal{L} \to \mathcal{C}^\infty(M)$ on the kernel, with $h = \iota^*\hat{g}$. Moreover, $\hat{g}$ is said to be *inner non degenerate* if h is non-degenerate.

An inner non degenerate metric $\hat{g}$ defines a unique triple (h, σ, g) where $h = \iota^*\hat{g}$, σ is a splitting such that $\hat{g}(\sigma_X, \iota(\gamma)) = 0, \; \forall \; X \in \Gamma(TM)$ and $\gamma \in \mathcal{L}$, and $g : \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^\infty(M)$ is a metric on the tangent bundle.
Proof:

- unicity of the splitting: if σ and σ' are two such splittings, then $\rho(\sigma_X - \sigma'_X) = 0$, so there exists a element $\alpha \in \mathcal{L}$ such that $\sigma_X - \sigma'_X = \iota(\alpha)$. By the orthogonality property of σ and σ', one gets that for all $\omega \in \mathcal{L}$, $\hat{g}(\iota(\alpha), \iota(\gamma)) = 0$, i.e. $h(\alpha, \gamma) = 0$ and thus $\alpha = 0$ for h is non degenerate, so $\sigma = \sigma'$.

- Existence of such a splitting: take any splitting $\tilde{\sigma} : X \longmapsto \tilde{\sigma}_X \in \mathcal{A}$. Let $\mathcal{L}^\perp := \{\mathfrak{X} \in \mathcal{A}, \hat{g}(\iota(\gamma), \mathfrak{X}) = 0\}$, in some sense the subspace of $\mathcal{A}$ orthogonal to the kernel, and $\mathsf{p}_\perp : \mathcal{A} \to \mathcal{L}^\perp$ the projection operator to this subspace. Then, the splitting $\sigma := \mathsf{p}_\perp \circ \tilde{\sigma}$ is well-defined (using another $\tilde{\sigma}'$ gives the same result for a given $X \in \Gamma(TM)$) and fulfills the orthogonality condition (by construction).

- Let us define $g := \sigma^*\hat{g}$. Then g is a well defined metric, uniquely determined by σ (which is unique, given $\hat{g}$) and $\hat{g}$ itself.

- Decomposition of $\hat{g}$: one has:

– $h(u,v) = \hat{g}(\iota(u), \iota(v))$

– $g(X,Y) = \hat{g}(\sigma_X, \sigma_Y)$

Recalling that $\mathfrak{X} = \sigma_X - \iota \circ \omega(\mathfrak{X})$, (where $X = \rho(\mathfrak{X})$ let us remark that this decomposition is an orthogonal decomposition due to the definition of σ, ω being the connection 1–form associated to σ. Thus, $\hat{g}(\mathfrak{X}, \mathfrak{Y}) = g(X,Y) + h(\iota(u), \iota(v))$.

- The converse is also true: given a triple (h, σ, g) satisfying the conditions, one can define a unique inner non degenerate metric by $\hat{g} := \rho^* g + \omega^* h$. One can thus use both descriptions equivalently.

In summary, once given such a metric $\hat{g}$, one can write:

$$\mathcal{A} = \sigma(TM) \overset{\perp}{\oplus} \iota(\mathcal{L}) \tag{1.55}$$

Covariant derivative of the metric Given an affine connection $\hat{\nabla}$, one can define its action on a metric $\hat{g}$ using the Leibniz rule. Indeed, for $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \mathcal{A}$, $\hat{g}(\mathfrak{Y}, \mathfrak{Z}) \in \mathcal{C}^\infty(M)$ and thus $\hat{\nabla}_{\mathfrak{X}}(\hat{g}(\mathfrak{Y}, \mathfrak{Z})) = \rho(\mathfrak{X}) \cdot \hat{g}(\mathfrak{Y}, \mathfrak{Z}) = (\hat{\nabla}_{\mathfrak{X}} \hat{g})(\mathfrak{Y}, \mathfrak{Z}) + \hat{g}(\hat{\nabla}_{\mathfrak{X}} \mathfrak{Y}, \mathfrak{Z}) + \hat{g}(\mathfrak{Y}, \hat{\nabla}_{\mathfrak{X}} \mathfrak{Z})$.

Then, one can define:

$$(\tilde{\nabla}_{\mathfrak{X}} \hat{g})(\mathfrak{Y}, \mathfrak{Z}) := \rho(\mathfrak{X}) \cdot \hat{g}(\mathfrak{Y}, \mathfrak{Z}) - \hat{g}(\tilde{\nabla}_{\mathfrak{X}} \mathfrak{Y}, \mathfrak{Z}) - \hat{g}(\mathfrak{Y}, \tilde{\nabla}_{\mathfrak{X}} \mathfrak{Z}). \tag{1.56}$$

1.2.6.4 Levi-Civita Connection

Given a non degenerate metric $\hat{g}$ on the Lie algebroid, there is a unique affine connection $\hat{\nabla}$ which is such that:

- $\hat{\nabla}_{\mathfrak{X}} \mathfrak{Y} - \hat{\nabla}_{\mathfrak{Y}} \mathfrak{X} = [\mathfrak{X}, \mathfrak{Y}]$ (Torsionfree),
- $\hat{\nabla}_{\mathfrak{X}} \hat{g} = 0$ (Metric Compatible),

for all $\mathfrak{X}, \mathfrak{Y} \in \mathcal{A}$. This particular connection is called the Levi-Civita connection related to the non-degenerate metric $\hat{g}$, and using these various relations, one can compute $\hat{\nabla}_{\mathfrak{X}} \mathfrak{Y}$ implicitly as the unique element such that:

$$\begin{aligned} 2\hat{g}(\hat{\nabla}_{\mathfrak{X}} \mathfrak{Y}, \mathfrak{Z}) =& \rho(\mathfrak{X}) \cdot \hat{g}(\mathfrak{Y}, \mathfrak{Z}) + \rho(\mathfrak{Y}) \cdot \hat{g}(\mathfrak{Z}, \mathfrak{X}) - \rho(\mathfrak{Z}) \cdot \hat{g}(\mathfrak{X}, \mathfrak{Y}) \\ &+ \hat{g}([\mathfrak{X}, \mathfrak{Y}], \mathfrak{Z}) - \hat{g}([\mathfrak{Y}, \mathfrak{Z}], \mathfrak{X}) - \hat{g}([\mathfrak{Z}, \mathfrak{X}], \mathfrak{Y}). \end{aligned} \tag{1.57}$$

Given the Levi-Civita connection $\hat{\nabla}$ on $\mathcal{A}$, one can define an affine connection on the vector fields $\Gamma(TM)$ by:

$$\nabla_X Y := \rho(\hat{\nabla}_{\sigma_X} \sigma_Y) \tag{1.58}$$

where σ is the splitting of the triple equivalent to $\hat{g}$. ∇ turns out to be the Levi-Civita connection related to the metric g, usually defined in Riemannian geometry. Indeed, $\nabla_X Y$ satisfies:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y) \\ &+ g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y). \end{aligned} \tag{1.59}$$

Indeed:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= 2g(\rho(\hat{\nabla}_{\sigma_X} \sigma_Y), \rho \circ \sigma_Z) \\ &= 2\hat{g}(\hat{\nabla}_{\sigma_X} \sigma_Y, \sigma_Z) - 2h(\omega \circ \hat{\nabla}_{\sigma_X} \sigma_Y), \omega \circ \sigma_Z) \end{aligned}$$

The second term vanishes due to the fact that the image of the splitting is the kernel of its corresponding connection 1–form. Thus,

$$2g(\nabla_X Y, Z) = 2\hat{g}(\hat{\nabla}_{\sigma_X} \sigma_Y, \sigma_Z)$$

and one can use the defining relation 1.57. There are two kinds of terms, terms like $\rho(\sigma_X) \cdot \hat{g}(\sigma_X, \sigma_Z)$ which gives directly $X \cdot g(Y, Z)$ (i.e. the good term), and other ones like $\hat{g}([\sigma_X, \sigma_Y], \sigma_Z)$. The latter can be written using

$$R(X, Y) = \iota \circ \Omega(\mathfrak{X}, \mathfrak{Y})$$

(where $\mathfrak{X}$ and $\mathfrak{Y}$ are any antecedents of X and Y by ρ). One has

$$\begin{aligned} \hat{g}([\sigma_X, \sigma_Y], \sigma_Z) &= \hat{g}(\sigma_{[X,Y]}, \sigma_Z) + \hat{g}(R(X, Y), \sigma_Z) \\ &= \hat{g}(\sigma_{[X,Y]}, \sigma_Z) + \hat{g}(\iota \circ \Omega(\mathfrak{X}, \mathfrak{Y}), \sigma_Z) \\ &= \hat{g}(\sigma_{[X,Y]}, \sigma_Z) = g([X, Y], Z) \end{aligned}$$

from the orthogonality condition. Thus, ∇ so defined satisfies 1.59, i.e. it is the Levi Civita connection related to g.

Thus, from a more general framework, one is thus able to recover some notions usually defined in Riemannian geometry, like a metric and its related Levi-Civita connection. In chapter 4, one will present also the corresponding generalized Riemannian curvature, Ricci curvature and scalar curvature, written in local coordinates.

1.3 Lie Groupoids and Integrability of Lie Algebroids

1.3.1 Lie Groupoids

1.3.1.1 Definition

A topological groupoid consists of two topological manifolds $\mathcal{G}$ and M, together with two surjective maps $\alpha : \mathcal{G} \to M$ and $\beta : \mathcal{G} \to M$ called the source and the target respectively, and the object inclusion map $1 : M \to \mathcal{G}$. A groupoid can be thought as a group in which not every pair (g, g') can be composed. Indeed, one defines a composition law on the space $\mathcal{G} \star \mathcal{G} := \{(g, g') \in \mathcal{G} \times \mathcal{G}, \alpha(g) = \beta(g')\}$ which satisfies some natural properties. (see [25] or [8] for all details). Let us just recall that any element $g \in \mathcal{G}$ has a two–sided inverse g^{-1} such that $\alpha(g^{-1}) = \beta(g), \beta(g^{-1}) = \alpha(g), g^{-1}g = 1_{\alpha(g)}$ and $gg^{-1} = 1_{\beta(g)}$.

The manifold M is called the base manifold, and $\mathcal{G}$ is just called the groupoid. It is often denoted with a double arrow $\mathcal{G} \rightrightarrows M$. For $x, y \in M$, $\mathcal{G}_x = \alpha^{-1}(x) \subset \mathcal{G}$ is called the α–fibre at x, and $\mathcal{G}^y = \beta^{-1}(y) \subset \mathcal{G}$ is called the β–fibre at y. One denotes $\mathcal{G}_x^y = \mathcal{G}_x \cap \mathcal{G}^y$. $\mathcal{G}_x^x$ is a topological group, called the vertex group, or isotropy group. For each $x \in M$, $1_x \in \mathcal{G}$ is the identity element of $\mathcal{G}_x^x$.

A Lie groupoid is a topological groupoid for which $\mathcal{G}$ and M are smooth manifolds, α, β and 1 are smooth, and the composition law $\mathcal{G} \star \mathcal{G} \to \mathcal{G}$ is also smooth. In this case, obviously, the vertex group is a Lie group.

1.3.1.2 Transitivity

One can define an action of $\mathcal{G}$ on the base M. For $x \in M$ the action of an element $g \in \mathcal{G}_x$ is defined by $g \cdot x := \beta(g)$. For x, y in the same orbit (i.e. there exists $g \in \mathcal{G}_x^y$, i.e. an element $g \in \mathcal{G}$ such that $\alpha(g) = x$ and $\beta(g) = y$), the α–fibres $\mathcal{G}_x$ and $\mathcal{G}_y$ are diffeomorphic closed submanifolds of $\mathcal{G}$.

The Lie groupoid is said to be transitive if this action has only one orbit. In this case, all α–fibres are diffeomorphic to each other, and all vertex groups as well. In his book [25] K. Mackenzie calls it also a *locally trivial* groupoid.

1.3.1.3 The pair groupoid and the anchor map

Let M be a smooth manifold. The pair groupoid is defined as $M \times M \rightrightarrows M$, with $\alpha((x, y)) = x, \beta((x, y)) = y$, $1_x = (x, x)$ for any $x, y \in M$. Let $\mathcal{G} \rightrightarrows M$ over M, the anchor map is defined as $r : \mathcal{G} \to M \times M$ from the Lie groupoid to the pair groupoid, with $r := \alpha \times \beta$. It is easy to see that the anchor map of a transitive Lie groupoid is surjective.

1.3.1.4 The short exact sequence of Lie groupoids

Let $\mathcal{G} \rightrightarrows M$ be a transitive Lie groupoid, and let $\mathcal{H} :=$ ker(r). By definition, $\mathcal{H} = \{g \in \mathcal{G},\ \exists\ x \in M,\ r(g) = 1_x = (x,x)\} = \cup_{x\in M}\mathcal{H}_x$ where $\mathcal{H}_x := \{g \in \mathcal{G},\ r(g) = 1_x\} = \{g \in \mathcal{G},\ \alpha(g) = \beta(g) = x\} = \mathcal{G}_x^x$. Thus, ker$(r)$ is a bundle of Lie groups of the form $\mathcal{G}_x^x$, i.e. a Lie groupoid with equal target and source maps. One has the following short exact sequence of Lie groupoids defining a transitive Lie groupoid:

$$1 \longrightarrow \cup_{x\in M}\mathcal{G}_x^x \xrightarrow{i} \mathcal{G} \xrightarrow{r} M \times M \longrightarrow 1$$

1.3.1.5 Bisections

A bisection $s : M \to \mathcal{G}$ of α is a map such that: $\alpha \circ s =$ id$_M$ and $\beta \circ s : M \to M$ is a diffeomorphism. The map $r' : s \longmapsto \beta \circ s$ is a morphism of groups, between the group of bisections of $\mathcal{G} \rightrightarrows M$, denoted Bisect$(\mathcal{G})$ and the group of diffeomorphisms of M. There is a short exact sequence at the level of sections as well, the map r' playing the role of the anchor (and one will call i' the injection of the corresponding kernel). Let us show that it is surjective and that its kernel is composed of sections of $\mathcal{H}$.

- Surjectivity: take $\phi \in$ Diff(M). For any $x \in M$, $(x, \phi(x)) \in M \times M$, so there exists $g_x \in \mathcal{G}$ such that $r(g_x) = (x, \phi(x))$, from the surjectivity of r. Let s be a map from M to $\mathcal{G}$ defined by $s(x) := g_x$. Then, by definition $r \circ s(x) = (x, \phi(x))$, i.e. (since $r = \alpha \times \beta$), $\alpha \circ s = id_M$ and $\beta \circ s = \phi \in$ Diff(M), i.e. s is a bisection corresponding to the diffeomorphism ϕ.
- Kernel: if $s \in$ Bisect$(\mathcal{G})$, that means $\beta \circ s(x) = x = \alpha \circ s(x)$, i.e. $s(x) \in \mathcal{G}_x^x$, i.e. s is a map from M to $\cup_x\mathcal{G}_x^x$. Thus the kernel of this map is $\mathcal{H}$.

Thus, one has the following short exact sequence in terms of bisections:

$$1 \longrightarrow \Gamma(\cup_{x\in M}\mathcal{G}_x^x) \xrightarrow{i'} \mathrm{Bisect}(\mathcal{G}) \xrightarrow{r'} \mathrm{Diff}(M) \longrightarrow 1$$

1.3.2 Lie Algebroid of a Lie Groupoid

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, one can define its corresponding Lie algebroid as encoding its infinitesimal data much in the same way one can define the Lie algebra of a Lie group. A Lie algebroid is a vector bundle, thus it will be defined fibre by fibre. The fibre A_x over $x \in M$ is defined as the tangent space to the α–fibre $\alpha^{-1}(x)$ (which is a smooth manifold) at 1_x. The Lie algebroid is the union of all these fibres. In other words:

$$A := \cup_{x\in M}T_{1_x}\alpha^{-1}(x) \tag{1.60}$$

The anchor ρ of this Lie algebroid is then defined thanks to the restriction of the tangent map $T\beta : T\mathcal{G} \to TM$ to the Lie algebroid. Let us remark that if the groupoid is transitive, then its corresponding Lie algebroid is transitive too.

1.3.3 Condition of Integrability

Now, given a transitive Lie algebroid A, one can wonder under which conditions there exists a transitive Lie groupoid $\mathcal{G}$ such that A is the Lie algebroid corresponding to $\mathcal{G}$ by the latter construction. In these cases, the Lie algebroid will be called integrable. Let us notice that any Lie groupoid which integrates a transitive Lie algebroid is necessarily transitive. Thus, the question of integrability is equivalent whether the Lie algebroid is transitive or not.

The integrability condition has been studied quite recently by Crainic and Fernandes in the general case in their 2011's lectures ([8]),[d] and by Mackenzie in the transitive case ([25]). It is quite technical and we just schetch the idea here. The fact is that, given a Lie algebroid A, it is always possible to construct a *topological* groupoid $\mathcal{W}(A)$, called the Weinstein groupoid. Recall that one has the short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & TM & \longrightarrow & 0 \\ & & \Big\downarrow ? & & \Big\downarrow ? & & \Big\downarrow \text{integration} & & \\ 1 & \longrightarrow & \cup_{x\in M}\mathcal{W}(A)_x^x & \xrightarrow{i} & \mathcal{W}(A) & \xrightarrow{r} & M \times M & \longrightarrow & 1 \end{array}$$

TM is the Lie algebroid of the pair groupoid. Indeed, if one calls $\mathsf{p}_1 : (x,y) \longmapsto x$ the source map of the pair groupoid, then for $x \in M, \mathsf{p}_1^{-1}(x) = \{(x,y) \in \{x\} \times M\}$, i.e. $\mathsf{p}_1^{-1}(x) = M$. Since $1_x = (x,x), T_{1_x}\mathsf{p}_1^{-1}(x) = T_xM$, and then the corresponding algebroid is TM. Thus, the right hand part of both sequences corresponds to an integration of Lie algebroid. It turns out that $\mathcal{W}(A)$ integrates A if and only if each (topological) isotropy group $\mathcal{W}(A)_x^x$ is actually a Lie group integrating $L_x \simeq \mathfrak{g}$ (recall that the kernel L is a bundle of Lie algebras modeled on $\mathfrak{g}$), i.e. if the left hand part of both sequences corresponds also to an integration.

Since $\mathfrak{g}$ is given with the Lie algebroid, and is finite dimensional, there already exists a Lie group G integrating $\mathfrak{g}$. Studying the integrability of the Lie algebroid amounts to studying the relation between the isotropy groups $\mathcal{W}(A)_x^x$ and the Lie group G. To this aim, Crainic and Fernandes end up constructing the monodromy map:

$$\partial : \pi_2(M) \to \mathcal{Z}(\mathfrak{g}) \tag{1.61}$$

where $\pi_2(M)$ is the second homotopy group of the base manifold M, and $\mathcal{Z}(\mathfrak{g})$ is the center of the Lie algebra associated to the Lie algebroid. The image of

[d] I thank Rui Loja Fernandes for his help by emails about this topic.

this monodromy map is the monodromy group $\mathcal{N}(A)$ and this group controls the integrability of the Lie algebroid. For a transitive Lie algebroid A,

$$A \text{ is integrable if and only if } \mathcal{N}(A) \text{ is discrete in } \mathcal{Z}(\mathfrak{g}).$$

Directly, one sees two cases where the transitive Lie algebroid will be automatically integrable:

- $\pi_2(M) = 0$,
- $\mathcal{Z}(g) = 0$, or even more restrictively: $\mathfrak{g}$ is semisimple.

If one wants to work explicitly with a non-integrable transitive Lie algebroid, M has to have a non trivial topology, and $\mathcal{Z}(g)$ cannot be trivial neither. Of course, these conditions are not sufficient but just necessary. Remark: there could be several Lie groupoids integrating a given Lie algebroid. However, there exists a unique source–simply connected (i.e. the submanifold $\alpha^{-1}(x)$ is connected and simply connected for all $x \in M$) Lie groupoid which integrates it, and this is the Weinstein Lie groupoid.

1.4 Integrable Transitive Lie Algebroid, Principal Fibre Bundle and Atiyah Sequence

1.4.1 Principal Fibre Bundle from an Integrable Transitive Lie Algebroid

1.4.1.1 Principal Fibre Bundle

A principal fibre bundle P over a smooth manifold M is a smooth manifold equipped with a surjection $\pi : P \to M$; there exists a Lie group G such that P looks locally like $U \times G$, where U is an open subset of M. More concretely, P consists of fibres $P_x = \pi^{-1}(x)$ over each $x \in M$, which are all diffeomorphic to G. G acts transitively on the right on the fibres. The action of $g \in G$ on $p \in P_x$ is denoted $p \cdot g$ and is such that $\pi(p \cdot g) = \pi(p)$, and for p and p' in the same fibre, there exists $g \in G$ such that $p' = p \cdot g$. A principal fibre bundle is also equipped with a local trivialization system $\{(U_i, \phi_i)\}_i$, where $\{U_i\}_i$ cover M and ϕ_i is a diffeomorphism trivializing $\pi^{-1}(U)$: $\phi : U \times G \to \pi^{-1}(U)$, such that:

- $\pi(\phi(x, g)) = x$
- ϕ is compatible with the action of G on P: if $\phi_x := g \longmapsto \phi(x, g)$, then $\phi_x(ga) = \phi_x(g) \cdot a$.

In order to glue smoothly together the different local trivializations over an intersection $U_i \cap U_j \neq \emptyset$, $P \to M$ is also provided with transition function $g_{ij} : M \to G$ which are such that for each $x \in U_i \cap U_j$, $g_{ij}(x) = \phi_{i,x}^{-1} \circ \phi_{j,x}(e)$, where e is the identity element of G. G is called the structure group of the principal fibre bundle. One shall see later that it must not be confused with the *gauge group*, the symmetry group of so-called gauge theories, even if they are of course deeply related (see below). A principal fibre bundle with structure group G is often denoted $G \to P \to M$, or $\pi : P \to M$, or simply $P(M, G)$.

1.4.1.2 Associated Bundles

Let $G \to P \to M$ be a principal bundle, and N a smooth manifold on which G acts via a left–action $\alpha : G \times N \to N$. Then, from the bundle P, on can define a new bundle with typical fibre N, and structure group still G. For this purpose, consider $P \times N$, and define on this space the equivalence relation:

$$(p, n) \sim (p \cdot g, \alpha(g)^{-1} n).$$

The result $(P \times N)/\sim$ is a bundle over M with typical fibre N. A case worth noting is when α is a representation of the structure group on a vector space $\mathbb{V}$. In this case, the resulting bundle $P \times_\alpha \mathbb{V}$ is called an *associated vector bundle*. In the framework of gauge theories, as one shall see, the space of sections of this fibre bundle is of primary importance. Let $\psi \in \Gamma(P \times_\rho \mathbb{V})$. It turns out that this space is isomorphic to the space of G–equivariant maps from P to $\mathbb{V}$, i.e. maps $\psi : P \to \mathbb{V}$ such that $\psi(p \cdot g) = \rho(g^{-1})\psi(p)$.

1.4.1.3 The Gauge Group

Let $G \to P \to M$ be a principal fibre bundle. There are three ways of defining the *gauge group* of this geometric structure, denoted with a gothic letter $\mathfrak{G}$. First, $\mathfrak{G}$ is the group of vertical automorphisms of P, which means automorphisms $f : P \to P$ such that the image of $p \in P_x$ is in P_x and f is compatible with the action of G: $f(p \cdot g) = f(p) \cdot g$. Since $f(p)$ is in the same fibre as p, there exists a group element $\gamma(p) \in G$ such that $f(p) = p \cdot \gamma(p)$. $f(p \cdot g) = p \cdot g\gamma(p \cdot g) = p \cdot \gamma(p) \cdot g$ and thus $\gamma(p \cdot g) = g^{-1}\gamma(p)g$. Thus, the map $\gamma : P \to G$ is equivariant with respect to the action of the structure group G. This is the second way of considering the gauge group $\mathfrak{G}$. The third way is to see these maps γ as the sections of an associated bundle with fibre type G associated with the *adjoint action* of G on itself: $\mathfrak{G} \simeq \Gamma(P \times_{Ad_G} G)$.

As one shall see, any object defined on a principal fibre bundle will have a certain equivariance property, which is given by the right action of an element of the structure group $g \in G$ by R_g^* or R_{*g}, and will transform also under a gauge transformation, i.e. via an element of the gauge group $\gamma \in \mathfrak{G}$.

1.4.1.4 The gauge groupoid and Transitive Lie groupoids

Given a principal fibre bundle $\pi : P \to M$, one can naturally associate to it a transitive Lie groupoid called the gauge groupoid and constructed as follows. Let define the equivalence relation on $P \times P$: $(p, q) \sim (p \cdot g, p \cdot g)$ for some $g \in G$. Then, the gauge groupoid is defined as $\mathcal{G} := (P \times P)/\sim$. An element (an equivalence class) is naturally denoted $[p, q]$. The source map is $\alpha([p, q]) := \pi(p)$, and the target: $\beta([p, q]) := \pi(q)$. An element can be seen as a class of arrows from a fibre to another, all "translated" from each other by the group G. One has the following short exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P \times_{Ad_G} G & \xrightarrow{i} & (P \times P)/G & \xrightarrow{r} & M \times M & \longrightarrow & 1 \\ & & & \searrow^{\pi} & \downarrow{\scriptstyle \alpha,\beta} & \swarrow_{p_1,p_2} & & & \\ & & & & M & & & & \end{array}$$

This claims that

$$ker(r) = P \times_{Ad_G} G. \tag{1.62}$$

Let us verify it: $[p, q] \in ker(r)$ means that $\exists\ x \in M$ such that $r([p, q]) = 1_x = (x, x)$, i.e. p and q are in the same fibre: $q = p.g$ for a certain $g \in G$. That means $ker(r)$ is made of pairs of the form $(p, g) \in P \times G$. How does the equivalence relation on $(P \times P)/G$ passes on $ker(r)$? On $(P \times P)/G, (p, q) \sim (p.g', q.g')$, but if $q = p.g$ then $(p, q) \sim (p.g', p.gg') \sim (p.g', (p.g')g'^{-1}gg')$. Then on $ker(r)$ there is the natural equivalence relation: $(p, g) \sim (p.g', Ad_{g'^{-1}}g)$, that is to say one can identify $ker(r)$ with the associated bundle $P \times_{Ad_G} G$. This is a Lie groupoid with source and target maps equal to π, i.e. a bundle of Lie groups as expected.

One notices that the space of sections of the kernel (i.e. maps $\gamma : M \to P \times_{Ad_G} G$ such that $\pi \circ s = \mathrm{id}_M$) is exactly the gauge group of the principal bundle P defined in 1.4.1.3.

An important result is the following:

Any transitive Lie groupoid $\mathcal{G} \rightrightarrows M$ is the gauge groupoid
of a certain principal fibre bundle $P \to M$.

The explicit construction can be found in section 1.3., *Local Triviality*, of Mackenzie's book [25]. This construction from the groupoid to the principal fibre bundle is based on a choosen point $x \in M$, but the result does not depend on the base point. The idea is to take the isotropy group $\mathcal{G}_x^x$ (which is a Lie group, and the transitivity implies that all isotropy groups are diffeomorphic) as the structure group, and define an action $\mathcal{G}_x^x \times \mathcal{G}_x \to \mathcal{G}_x$ on $\mathcal{G}_x = \{g \in \mathcal{G}, \alpha(g) = x\}$. The orbits of the action naturally play the role of fibres, and the map β restricted to $\mathcal{G}_x$ plays the role of the projection on the base manifold. All in all, one constructs a principal fibre bundle $G[x] \to P[x] \to M$, depending on the point $x \in M$. If another point $y \in M$ is choosen, both the principal bundles $P[x]$ and $P[y]$ are isomorphic, the isomorphism being constructed with the help of an element $g[x,y] \in \mathcal{G}_x^y$.

1.4.2 Atiyah Lie Algebroids

Atiyah Lie algebroids are transitive Lie algebroids associated to a certain principal fibre bundle, and correspond to the infinitesimal encoding of gauge groupoids, i.e. of transitive Lie groupoids. One first presents the construction from the principal fibre bundle, and then as the Lie algebroid of the gauge groupoid.

Let $\pi : P \to M$ a principal fibre bundle. Among the vector fields $\mathfrak{X} \in \Gamma(TP)$ on P, consider those which are right–invariant in terms of the action of the structure group G, i.e. such that $R_{*g}\mathfrak{X}_p = \mathfrak{X}_{p \cdot g}$. The space of right–invariant vector fields will be denoted $\Gamma_G(TP)$, and TP/G will denote the vector bundle such that $\Gamma_G(TP) = \Gamma(TP/G)$. For $\mathfrak{X} \in \Gamma_G(TP)$, the projection $\pi_* : TP/G \to TM$ is well defined (by construction) by $T_xM \ni \pi_*(\mathfrak{X})_x := \pi_*(\mathfrak{X}_p)$, with any $p \in \pi^{-1}(x)$. $\rho := \pi_*$ is the anchor. The kernel of ρ consists on vertical vector fields, i.e. right–invariant vector fields being at each point tangent to the corresponding fibre. It turns out that the space of vertical vector fields is isomorphic to the space of sections of the associated vector bundle $P \times_{Ad_G} \mathfrak{g}$, which will be denoted $\Gamma_G(P, \mathfrak{g})$. Let $\# : \gamma \longmapsto \gamma^{\#}$ denote this isomorphism, for $\gamma \in P \times_{Ad_G} \mathfrak{g}$ and $\gamma^{\#}$ its corresponding vertical vector field. Then, define the inclusion $\iota : P \times_{Ad_G} \mathfrak{g} \to TP/G$ by $\iota(\gamma) := -\gamma^{\#}$ (the minus sign is conventional). One then gets the Atiyah sequence of Lie algebroid:

$$0 \longrightarrow \Gamma_G(P, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(TP) \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$$

Let us now see quickly that given a transitive Lie groupoid $\mathcal{G} \rightrightarrows M$, i.e. a gauge groupoid of a certain principal fibre bundle P, its associated transitive Lie algebroid is an Atiyah Lie algebroid. For $x \in M, \alpha^{-1}(x) = \{[p,q], p \in P_x, q \in P\} = \{(p,q), q \in P\}$, with $p \in P_x$ fixed. Thus $\alpha^{-1}(x) \simeq P$. $1_x = \{(p.g, p.g), p \in P_x, g \in G\} = x$. Then the corresponding Lie algebroid is $A = TP/G$. β is lifted to $T\beta$ on $T\mathcal{G}$ and thus its restriction to $\mathcal{A} \subset T\mathcal{G}$ defines ρ. $T\alpha$ defines the bundle projection $\mathcal{A} \to M$.

This kind of transitive Lie algebroid is of primary importance. It follows from the latter considerations that:

any integrable and transitive Lie algebroid is the Atiyah Lie algebroid of some underlying principal fibre bundles.

As is noticed here, there could be several principal fibre bundles associated to an integrable and transitive Lie algebroid. For example, if one takes the Lie algebroid TM (with anchor equals to the identity), it can be integrated into the pair groupoid $M \times M$, the fibre bundle related to which has trivial structure group, but also into the fundamental groupoid $\Pi_1(M)$, which is the homotopy class of paths in M (homotopy related to fixed end point). In this second case, the related fibre bundle has the first homotopy group $\pi_1(M)$ as structure group.[e] If M is 1–connected, both are equivalent, but it is not a general result.

In classical gauge theory, we usually use the one the structure group of which is simply connected, i.e. the fibre bundle coming from the Weinstein groupoid. At a classical level, as we shall show later, one can recast gauge theories in the framework of Lie algebroids without loss of information, only the infinitesimal data seems to be relevant, and thus, it does not change anything at the physical level. Yet, maybe something changes at the quantization level, further investigations have to be pursued in this direction in order to give a clear answer to this question.

1.4.3 Local Basis of Right-Invariant Vector Fields

Let us set an important result which will be used several time in this thesis: the proposition 4.4. of [33]. Let $U \subset M$ an open subset of M, trivializing the bundle P; then, there exists a family of right–invariant vector fields $\{\mathfrak{X}^i\}_{i=1..\text{dim}(P)}$, $X^i \in \Gamma_H(TP_{|U})$, such that:

- For any $p \in P$, $\{\mathfrak{X}^i_p\}_i$ is a basis of T_pP,
- $\{\mathfrak{X}^i\}_i$ generates $\Gamma_H(TP_{|U})$ as a $\mathcal{C}^\infty(M)$–module: for any $\mathfrak{X} \in \Gamma_H(TP_{|U})$ there exist dim(P) right–invariant functions $f_i \in \mathcal{C}^\infty_H(P_{|U})$ such that for $p \in P_{|U}$, $\mathfrak{X}_p = f_i(p)\mathfrak{X}^i$. This decomposition is unique.
- $\{\mathfrak{X}^i\}_i$ generates $\Gamma(TP_{|U})$ as a $\mathcal{C}^\infty(M)$–module and, for any $\tilde{\mathfrak{X}} \in \Gamma(TP_{|U})$, the decomposition $\tilde{\mathfrak{X}} = g_i\mathfrak{X}^i$ is unique.
- Two such families $\{\mathfrak{X}^i\}$ are related by linear combinations whose coefficients are in $\mathcal{C}^\infty_H(P_{|U}) \simeq \mathcal{C}^\infty(U)$.

This proposition will be used to pass from a description in terms of Atiyah algebroids to the one in terms of its underlying principal fibre bundle.

1.4.4 Erhesman Connections

The notion of connection (splitting) presented on transitive Lie algebroids naturally has an equivalent on principal fibre bundles, called Erhesman connections. The two notions coincide on Atiyah Lie algebroids. Let $A = TP/G$ be the Atiyah Lie algebroid of a certain principal fibre bundle, and let $\omega : \mathcal{A} \to \mathcal{L}$ be a connection 1–form. Recall that ω is normalized on the kernel, i.e. $\omega \circ \iota(\gamma) = -\gamma$, for any $\gamma \in \mathcal{L}$. Given such a connection, one constructs an object living on P by using

[e] I thank Iakovos Androulidakis for having reminded me this example.

the previous proposition 1.4.3. Let us define a map $\omega^{er} : TP \to \mathfrak{g}$, firstly locally over $U \subset M$, derived from ω. Since ω is defined only on right–invariant vector fields, one needs to decompose $\tilde{X} \in \Gamma(TP_{|U})$ as $\tilde{X} = g_i \mathfrak{X}^i$, see 1.4.3. Then, one defines the map $\omega^{er} \in \Omega^1(P_{|U}) \otimes \mathfrak{g}$ as

$$\omega^{er}(\tilde{X}) := -g_i \omega(\mathfrak{X}^i),$$

which is well defined on $P_{|U}$ and does not depend on the choice of a family of operators $\{\mathfrak{X}^i\}$. Indeed, if one takes $\{\mathfrak{Y}^i\}$, with functions h_i such that $\tilde{X} = h_i \mathfrak{Y}^i$, one knows from 1.4.3 that there exist a family of functions a_i^j such that $g_i = a_i^j h^i$ and $\mathfrak{Y}^i = a_j^i \mathfrak{X}^j$, and thus $g_i \omega(\mathfrak{X}^i) = a_i^j h_j \omega(\mathfrak{X}^i) = h_j \omega(a_i^j \mathfrak{X}^i) = h_i \omega(\mathfrak{Y}^i)$. Then, one takes a partition of unity associated to a covering $\{U_i\}_i$ of M, and defines this way ω^{er} on the whole TP.

Due to the normalization relation, ω^{er} realizes the isomorphism between vertical vector fields and elements of the Lie algebra $\mathfrak{g}$: $\omega_p^{er} : V_p P \xrightarrow{\sim} \mathfrak{g}$. ω^{er} satisfies also the condition: $R_g^* \omega^{er} = Ad_{g^{-1}} \omega^{er}$, due to the fact that ω takes values in equivariant maps $\Gamma_G(P, \mathfrak{g})$. ω^{er} is called an *Ehresman connection*.

Indeed, independently of Lie algebroids, an Ehresman connection defined on a principal bundle P is a 1–form $\omega^{er} : TP \to \mathfrak{g}$ such that:

- ω^{er} is the canonical isomorphism on vertical vector fields;
- $R_g^* \omega^{er} = Ad_{g^{-1}} \omega^{er}$.

The second relation is called the equivariance property of an Ehresman connection.

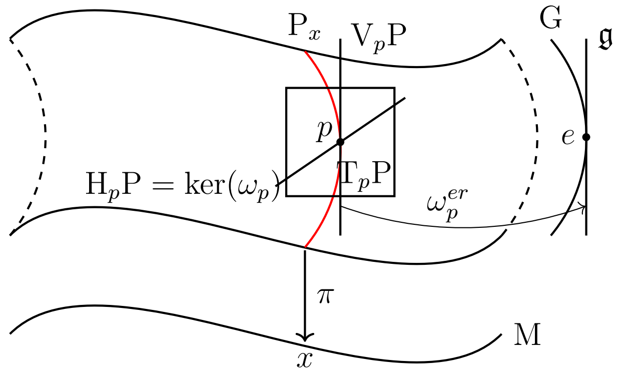


Figure 1.4: Ehresman connection on a principal fibre bundle

At each point $p \in P$, the vertical subspace V_pP of the tangent space T_pP is the space tangent to the fibre $P_{\pi(p)}$. However, there is an indetermination concerning the notion of horizontality, i.e. there are infinitely many possible choices of a complementary space for V_pP in T_pP. It is the same kind of indetermination about finding an antecedent of $X \in \Gamma(TM)$ by the anchor ρ in the framework of transitive Lie algebroid (a splitting eliminates such an indetermination). An Ehresman connection provides such a horizontal subspace by defining $H_pP :=$ ker(ω^{er}). The equivariance property of the connection 1–form passes to the horizontal subspaces, which means that if $\mathfrak{X}_{|p} \in H_pP$, then $R_{g*}\mathfrak{X}_{|p} \in H_{p\cdot g}P$, *idem* for V_pP.

1.5 Gauge Theories and Reduction of Gauge Symmetry

1.5.1 Mathematical Content of a Gauge Theory

The fundamental interactions are described in the framework of gauge theories, the geometric content of which is a principal fibre bundle P over a smooth manifold M, with structure Lie group H,[f] together with associated vector bundles $E = P \times_\alpha \mathbb{V}$ where α is a representation of H on the vector space $\mathbb{V}$. A (classical) matter field is then represented by a section ξ of E, while the interacting bosons fields are connections ω on P, and act on matter fields via the associated covariant derivative $D\xi = d\xi + \alpha_*(\omega)\xi$. The curvature associated to a connection ω will be denoted $\Omega := d\omega + \frac{1}{2}[\omega, \omega]$. These notations will be used throughout the paper.

As it has been already claimed, the central notion of a gauge theory is that of *local symmetry*. The latter is implemented by the local action of H, i.e. by the action of the (infinite dimensional) *gauge group*, see section 1.4.1.3.

Fields ξ, their covariant derivative $D\xi$, connections ω and their curvature Ω transform under the action of $\gamma \in \mathfrak{H}$ as:

$$\omega^\gamma = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma, \tag{1.63}$$

$$\Omega^\gamma = \gamma^{-1}\Omega\gamma + \gamma^{-1}d\gamma, \tag{1.64}$$

$$\xi^\gamma = \alpha(\gamma)^{-1}\xi, \tag{1.65}$$

$$(D\xi)^\gamma = \alpha(\gamma)^{-1}D\xi. \tag{1.66}$$

Knowing this, a gauge theory, in a physical sense, is the choice of an action functional $S[\omega, \xi]$ which has the property to be *invariant* under this action. The

[f] Throughout this section, the structure group of the gauge theories will be denoted H instead of G, just to harmonize the notations when passing to the application of the dressing method to the framework of Cartan geometry

theory is said to be *gauge invariant*. This feature translates the idea that two fields in the same gauge orbit are physically equivalent, i.e. are indistinguishable by any physical experiment. The gauge symmetry is the manifestation of an intrinsic mathematical redundancy in our formalism, as presented in the introduction.

The description of fundamental interactions on which modern physics is built starts then with the choice of symmetry Lie groups. Electroweak and strong interactions are ruled by the Lie group $U(1) \times SU(2) \times SU(3)$. Regarding the gravitational interaction, the fundamental symmetry group of General Relativity (GR) is the group of diffeomorphisms of the base manifold. Let us remark that one can also write GR under the form of an internal gauge theory[g], in which is added a local symmetry ruled by the local action of the Lorentz group $SO(1,3)$. This can be done in the framework of Cartan geometry, as it will be shown in chapter 2.

1.5.2 Dressing Field Method

The dressing field method has been developed over the past years. It was first used to show that the generation of masses in the electroweak sector can be performed without calling on a spontaneous symmetry breaking, see [37]. Then, it has been studied in the case of Cartan geometry and in particular in the case of conformal geometry, see [14], [19], [20] (the latter being the J. François' Phd Thesis), and a complete review of the method with all applications is available in [2]. One gives here a summary of the main results, all the proofs and details can be found in the cited papers.

1.5.2.1 Presentation

Although the symmetry group H is central and unavoidable in the construction of a gauge theory, one often needs to reduce its action, i.e. passing to a theory with less symmetry. There can be several reasons for that. For example, for a quantization purpose: the gauge symmetry group produces infinities in the path integral over all fields. Also, e.g. in the case of the electroweak sector of the Standard Model (SM) ($G = U(1) \times SU(2)$), the constraint imposed by the symmetry group is such that mass terms are not allowed, *a priori*, in the action. Thus, since massive gauge bosons are observed, one has to find a way to re-write the same theory but with a smaller symmetry group.

There exist many well-known ways of reducing a symmetry. The simplest is the *gauge fixing*: since all fields in a given gauge orbit are equivalent, one just can choose a particular one – which renders the computations easier, for example; the physical results should be, by definition, independent of the choice of gauge. Another one, which applies in the case of the electroweak sector is

[g] In the sense of using a connection on a fibre bundle.

the *spontaneous symmetry breaking*. In this case, the symmetry reduction is thought as a physical phenomenon, like a phase transition, induced by the fact that the ground state has less symmetry than the theory of which it is a solution.

The method of symmetry reduction presented here is called the *dressing field method*. It is a systematic way of finding, if they exist, new fields which are invariant under the action of the gauge group $\mathfrak{H}$ or of one of its subgroup. This method turns out to be a mere change of field variables. This change is performed with the help of a dressing field u which does not, in general, belong to the gauge group $\mathfrak{H}$. Thus, it is neither a gauge transformation nor a gauge fixing: the new field variables, called *composite fields*, belong in general to representation spaces – for the action of the remaining gauge (sub)group – different than the original variables.

The dressing field method is applicable to a lot of cases. In this framework, one can for example reinterpret the spontaneous symmetry breaking in the electroweak sector of the Standard Model as being a dressing field symmetry reduction. Also, the dressing field method finds applications in the framework of Cartan geometry. As one shall see, passing from the tetrad formalism in GR to the usual geometric formalism is a dressing. In chapter 3, one will apply the dressing field method to the case of conformal Cartan geometry, and recover tractors and twistors with a new and deeper comprehension of their geometric nature.

1.5.2.2 The formalism in a nutshell

As presented in section 1.4.1.3, the elements of the gauge group $\mathfrak{H}$ can also be seen as $H-$valued equivariant fields defined on P. Such an element γ is then transformed under the action of another element η as $\gamma^\eta = \eta^{-1}\gamma\eta$. Let K be a subgroup of H, possibly H itself, and $\mathfrak{K}$ its corresponding gauge group. A dressing field is a locally defined $H-$valued field u on P, which transforms under a gauge transformation $k \in \mathfrak{K}$ as $u^k = k^{-1}u$. Thus, $u \notin \mathfrak{H}$.

The existence of such a field ensures that the following *composite fields*:

- $\omega^u := u^{-1}\omega u + u^{-1}du$,
- $\Omega^u := u^{-1}\Omega u$,
- $\xi^u := \rho(u)^{-1}\xi$,
- $(D\xi)^u := \rho(u)^{-1}D\xi$,

are then $K-$gauge invariant as it can be checked by a straightforward computation. This fact is interpreted saying that actually, the subgroup K does not act anymore on the fields, or more exactly it acts trivially on them.

Thus, if one rewrites the theory (i.e. the gauge invariant action $S[\omega, \xi]$) in the new variables, one gets a theory for which the $K-$symmetry has been erased. It is a mere reconfiguration of the fields which redistributes the degrees of freedom of the theory. The latter are computed as follows: let #TOT, #Φ, #H and #$(\Theta = 0)$ be respectively the total number of degrees of freedom, the degrees of freedom related to the fields (ω and ξ) of the theory, the dimension of the symmetry group H, and the number of constraint equations, it there are some. Then:

$$\#TOT = \#\Phi - -\#H - -\#(\Theta = 0). \tag{1.67}$$

For example, if the operation of dressing leaves invariant the constraint equations, in the new variables the theory will have less symmetry and then necessarily less degrees of freedom coming from the fields. Let us stress that this is not a gauge transformation, even if it has formally the same form, for the dressing is not an element of the gauge group.

1.5.2.3 Residual Symmetry

One places oneself in the case where the subgroup K (the action of which is erased) is a normal subgroup of H: $K \trianglelefteq H$. In this case, the coset $J := H/K$ is a Lie group which plays the role of the residual structure group, and the quotient bundle $P' := P/K$ is a J–principal fibre bundle. Depending on how the dressing field u transforms under the action of the residual group J, the composite (dressed) fields can be either genuine gauge fields (in terms of the residual gauge action) or gauge fields of a new kind. Let see here two specific examples: the first one is the case where the composite fields are genuine gauge fields, and the second one is the case where the composite fields are *twisted–gauge fields*.

Genuine gauge composite fields Let us assume that the J–equivariance of the K–dressing field u is given by:

$$R_j^* u = Ad_{j^{-1}} u, \tag{1.68}$$

with $j \in \mathfrak{J}$. Then, the dressed connection ω^u is a J–principal connection on P', its curvature is given by Ω^u, and ϕ^u is a $(\rho, \mathbb{V})$–tensorial map and can be seen as a section of the associated vector bundle $P' \times_J \mathbb{V}$.

The transformations of the composite fields under the action of an element γ of the residual gauge group $\mathfrak{J}$ are the usual ones, that is:

$$(\omega^u)^\gamma = \gamma^{-1}\omega^u\gamma + \gamma^{-1}\mathrm{d}\gamma \tag{1.69}$$

$$(\Omega^u)^\gamma = \gamma^{-1}\Omega^u\gamma \tag{1.70}$$

$$(\phi^u)^\gamma = \rho(\gamma^{-1})\phi^u \tag{1.71}$$

$$(D^u\phi^u)^\gamma = \rho(\gamma^{-1})D^u\phi^u \tag{1.72}$$

Twisted–gauge composite fields A more exotic case, that one finds e.g. in the case of Tractors and Twistors, is when the K–dressing field u has the following *non usual* equivariance property under the action of $j \in J$:

$$(R_j^* u)(p) = j^{-1} u(p) C_p(j) \tag{1.73}$$

where C is a smooth map defined as follows. Let $H' \supset H$ for which representations $(\rho, \mathbb{V})$ of H are also representations of H'. Then, $C : P \times J \to H'$ is such that:

$$C_p(jj') = C_p(j) C_{pj}(j'). \tag{1.74}$$

One has, in particular, $C_p(e) = e$ (the identity element in both J and H'), and $C_p(j)^{-1} = C_{pj}(j^{-1})$. Then, given a dressing field u transforming under 1.73, the dressed connection ω^u is not a J–principal connection, but instead a kind of generalized connection for its equivariance property is now given by:

$$\omega_p^u(X_p^v) = c_p(X) := \frac{d}{dt} C_p(e^{tX})_{|t=0}, \text{ for } X \in \mathfrak{j} \text{ and } X_p^v \in V_p P; \tag{1.75}$$

$$R_j^* \omega^u = C(j)^{-1} \omega^u C(j) + C(j)^{-1} dC(j). \tag{1.76}$$

The equivariance of the other fields are:

$$R_j^* \Omega^u = C(j)^{-1} \Omega^u C(j) \tag{1.77}$$

$$R_j^* \phi^u = \rho(C(j)^{-1}) \phi^u \tag{1.78}$$

$$R_j^* D^u \phi^u = \rho(C(j)^{-1}) D^u \phi^u \tag{1.79}$$

For $\gamma \in \mathfrak{J}$, the residual *gauge transformations* of the fields are then:

$$(\omega^u)^\gamma = C(\gamma)^{-1} \omega^u C(\gamma) + C(\gamma)^{-1} dC(\gamma) \tag{1.80}$$

$$(\Omega^u)^\gamma = C(\gamma)^{-1} \Omega^u C(\gamma) \tag{1.81}$$

$$(\phi^u)^\gamma = \rho(C(\gamma)^{-1}) \phi^u \tag{1.82}$$

$$(D^u \phi^u)^\gamma = \rho(C(\gamma)^{-1}) D^u \phi^u \tag{1.83}$$

1.5.2.4 Conclusion

In both cases, the residual structure group is $J = H/K$. In the case of genuine gauge composite fields, the residual gauge group is given by the local action of J, i.e. after the dressing, the fibre bundle is still a principal fibre bundle, with now structure group J. In the case of twisted composite gauge fields, however, J acts via the map C, and thus the final structure is not a principal fibre bundle, but a kind of associated bundle, except that the association is not made thanks to a representation but with the map C. The case of Tractors and Twistors in chapter 3 will consist in a concrete instance of this case.

Chapter 2

Cartan Geometry and Gravitation Theories

One sums up in this section the basic features of Cartan geometry. In Chapter 5, in introduction, one will give more in depth material related to Cartan geometry. The material presented here is taken mainly from [50], and also from [53]. The latter gives a pedagogical introduction to Cartan geometry, directy oriented toward the formulation of gravitation theories, whereas the former gives a very general and more mathematical presentation. Equivalence of structures related to Cartan approach can be found in [30].

2.1 Klein Geometry

2.1.1 An intuitive introduction

The main idea of Klein's point of view about geometry is to substitute to the study of a homogeneous manifold M that of two symmetry groups G and H. Let us start with a very simple example, in an elementary geometric framework, just to give an intuitive idea of Klein's approach. Let $M_0 = S^2$ be the 2-sphere embedded in $\mathbb{R}^3$; $G = SO(3)$ consists in all rotations of vectors of $\mathbb{R}^3$, so to specify an element of this group one has to give two angles (ϕ, θ) denoting the direction of the rotation's axis, and an angle α being the angle of the rotation in the plane orthogonal to the direction (ϕ, θ). One can see this differently, saying that G acts transitively on M_0: one can go from any point of the sphere to any other by a certain rotation. Then, let us take $x_0 = (\phi, \theta) \in S^2$, and wonder: in $SO(3)$, which are the transformations that let x_0 invariant ? It is easy to see that they form a subgroup of $SO(3)$ which consists in all rotations of axis (ϕ, θ). One can thus associate to each point x, its 'stabilizer' $H_{x_0} \subset G$, which is nothing but $SO(2)$, the same for any x_0, denoted simply H. Now, one considers the coset

G/H, consisting of the set of rotations for which one identifies those of same axis: to specify an element of this set, one has just to give the angles (ϕ, θ) of the axis. So one can see that in a certain sense, the manifold S^2 and the coset G/H can be identified.[a]

Here is the Klein's idea: studying the pair $(SO(3), SO(2))$ instead of the sphere S^2 itself. The pair is called the Klein pair, or Klein geometry, and the sphere here is the homogeneous manifold of this Klein geometry.

2.1.2 General Case

Let now M_0 be a smooth manifold and G a Lie group acting (the action is denoted by a dot $\cdot$) *transitively* on M_0: given two points $x, y \in M_0$, there always exists an element $g \in G$ such that $y = g \cdot x$. M_0 is said to be a *homogeneous manifold*. Let $x_0 \in M_0$; then, one can define the map π_{x_0} which associates to each $g \in G$ the element obtained by its action on x_0. The transitivity of the action implies that this map is onto. Let H_{x_0} be the set of the elements of G which stabilizes x_0, called stabilizer: $H_{x_0} := \{g \in G, gx = x\}$. H_{x_0} is a closed subgroup of G, thus also a Lie group. In general, it is not reduced to the identity element and so the map π_x is not one-to-one.

Then, for each element of M_0, we can make the same reasonning and associate to it its stabilizer in G. We have the important property that, the manifold being homogeneous, all the stabilizers are isomorphic to each other, so we can call them simply H. This mathematically expresses the fact that by definition there is no prefered point on an homogeneous manifold. Then, let's consider the right coset G/H obtained from G by identifying all points in a same stabilizer. Thus, the map $\tilde{\pi} : G/H \to M_0$ induced by the different π_x for $x \in M_0$ is one-to-one. So it turns out that the manifolds G/H and M_0 can be identified. Thus, here comes the Klein's idea: studying the manifold M_0 is equivalent to studying the pair (G, H), called the Klein pair. The pair: $(\mathfrak{g}, \mathfrak{h})$ consisting in the corresponding Lie algebras is called the infinitesimal Klein pair.

Then, a Klein geometry is simply defined as a pair (G, H), where G is a Lie group and H a closed subgroup of G, such that the right coset G/H is an homogeneous connected manifold w.r.t. the action of G. G is called the principal symmetry group of the geometry, and H, as we have seen, is the stabilizer of a point.

[a] Let's just notice that this example is a bit imprecise: knowing that a stabilizer can be identified to two different points of the manifold, actually the right coset can be identified with only half of the sphere. But it is given only as a first approach of Klein's ideas.

2.1.3 Reductive Klein geometries

An important particular case of Klein geometries are the reductive ones. These are those the principal group's Lie algebra of which can be written as a direct sum of $\mathfrak{h}$ and of an H–invariant complement $\mathfrak{p}$: that is, if $(\mathfrak{g}, \mathfrak{h})$ is an infinitesimal Klein pair, one can write: $\mathfrak{g} = \mathfrak{h} \bigoplus \mathfrak{p}$ such that $\mathfrak{p}$ is invariant under the adjoint action of H, say $Ad_h x = h^{-1} x h$, for $h \in H$ and $x \in \mathfrak{p}$. Such a $\mathfrak{p}$ is also said to be a H–module. In this case, one has $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$.

2.1.4 Example: Minkowski Space-Time as a Reductive Klein Geometry

Let's give an example of a Klein Geometry. Let's consider $M_0 = \mathbb{R}^{1,3}$, i.e. the 4-dimensional Minkowski spacetime. To show out the Klein geometry behind it, let's call $H = SO(1,3)$, the Lorentz group, and $G = SO(1,3) \ltimes \mathbb{R}^{1,3}$, the Poincaré group. H is clearly a subgroup of G, and G/H is clearly isomorphic to M_0. Moreover, in terms of Lie algebras, one has: $\mathfrak{g} = \mathfrak{so}(1,3) \bigoplus \mathbb{R}^{1,3}$, i.e. $\mathfrak{g} = \mathfrak{h} \bigoplus \mathbb{R}^{1,3}$. Minkowski spacetime is thus a good (and the simplest) example of a homogeneous space of a reductive Klein geometry.

2.1.5 Principal fibre bundle's point of view

A Klein geometry (G, H) can also be seen as a principal H–bundle over the manifold M_0. To be more precise, one may define a Klein geometry to be a principal bundle $H \to G \to M_0$. That means that the principal symmetry group G is fibred in fibres diffeomorphic to H, on which H acts on the right (the action is still denoted $\cdot$), or that one puts over each point $x \in M_0$ its stabilizer in G, and reconstructs G over M_0.

2.1.6 Maurer-Cartan form of a Klein geometry

Let us recall first what is the Maurer-Cartan form ω_G of a Lie group G. Let $g \in G$, then one can define the left action of G on itself at g by $L_{g^{-1}}$, which sends g to e, the identity element. Let us consider now the differential map $\omega_G := L_{g^{-1}*} : T_g G \to \mathfrak{g}$. Since $L_{g^{-1}}$ is invertible, ω_G realises an isomorphism between the tangent plane of G at a point g and its Lie algebra. See figure 2.1.

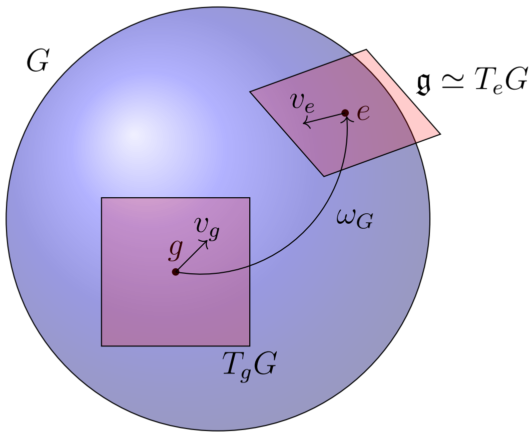


Figure 2.1: Maurer-Cartan Form of a Lie group G

Let us now consider a Klein pair (G, H) and see the principal symmetry group G as a principal H–bundle over $M_0 = G/H$. Restricted to the fibres, ω_G realizes an isomorphism between the vertical vectors (tangent to the fibres) and the Lie algebra of the structure group H. ω_G realizes in this very particular case the natural isomorphism which always exists between vertical vectors of a principal fibre bundle and the Lie algebra of the structure group.

The Maurer–Cartan form ω_G satisfies also:

$$\omega_{G_{|g \cdot h}} = Ad_{h^{-1}} \omega_{G_{|g}}. \tag{2.1}$$

Indeed: let X be a tangent vector to G, and $(g, t) \longmapsto \phi_g(t) \in G$ its flow, i.e.:

- $\phi_g(0) = g$;
- $\phi_{g \cdot h}(t) = \phi_g(t) \cdot h$;
- $\frac{d}{dt}_{|t=0} \phi_g(t) = X_g$;

for all $g \in G$, $h \in H$ and t a real parameter. Then, by definition, $L_{g^{-1}*}(X) := \frac{d}{dt}_{|t=0} L_{g^{-1}} \phi_g(t)$ so $L_{(g \cdot h)^{-1}*}(X) := \frac{d}{dt}_{|t=0} L_{(g \cdot h)^{-1}} \phi_{g \cdot h}(t) = \frac{d}{dt}_{|t=0} h^{-1} L_{g^{-1}} \phi_g(t) \cdot h = h^{-1} L_{g^{-1}*} h$.

Thus, the Maurer-Cartan form looks very like an Ehresman connection, but with the additional condition that it makes an isomorphism between each tangent plane of the fibre bundle (here, G) and the Lie algebra of the principal symmetry group $\mathfrak{g}$. One can also show that ω_G satisfies the structure equation:

$$\mathsf{d}\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0. \tag{2.2}$$

2.2 Cartan Geometry

2.2.1 Cartan Connection

Cartan geometries are natural generalizations of Klein geometries, working on general smooth manifolds M which locally look like a given homogeneous one. Riemannian geometry can be seen as a particular case of this, where the homogeneous manifold is simply $\mathbb{R}^n$. A Cartan geometry is said to be modeled on a certain Klein geometry (G, H), and characterized by a Cartan connection which is a generalization of the Maurer-Cartan form on a general H–principal fibre bundle. More precisely, a Cartan geometry over a smooth manifold M, denoted (P, ϖ), is a H–principal fibre bundle $H \to P \to M$ equipped with ϖ, a $\mathfrak{g}$–valued 1–form on P called Cartan connection, such that:

- For any $p \in P$, ϖ_p realises the natural isomorphism $V_pP \to \mathfrak{h}$, i.e. restricts to the Maurer–Cartan form of H on fibres: $\varpi(\xi^v) = \xi$ where $\xi \in \mathfrak{h}$ and ξ^v is its corresponding vertical vector field,
- $R_h^*\varpi = Ad_{h^{-1}}\varpi$,
- ϖ realizes an isomorphism from each tangent space of the bundle P into the Lie algebra $\mathfrak{g}$.

The main difference between this kind of connection and Ehresman connections is that these one does not take values in the Lie algebra of the structure group H, but in $\mathfrak{g} =$ Lie(G), and glue this Lie algebra to each tangent space of the bundle P. In this sense, Cartan geometries are more restrictive than ordinary (Ehresman) ones, for there is a stronger relation between the principal bundle and the base manifold.

2.2.2 Curvature and Geometric Interpretation

One can also define a curvature $\overline{\Omega}$ for such a connection:

$$\overline{\Omega} = \mathrm{d}\varpi + \tfrac{1}{2}[\varpi, \varpi].$$

which is a $\mathfrak{g}$–valued 2–form.

Naturally, in the case of a curvature–vanishing Cartan connection, i.e. a flat Cartan geometry, it turns out that the principal symmetry group G can be identified with the principal bundle P, the manifold M with the homogeneous space G/H, and the Cartan connection itself with the Maurer-Cartan form ω_G of G. Such a geometry can be seen as describing deformations of the homogeneous space G/H, in the same way (pseudo–)Riemannian geometry describes in a certain sense deformations of an Euclidian (or Minskowskian) space. The curvature is thus the measure of how much the given manifold M is far from being (locally) homogeneous.

2.2.3 Reductive Cartan Geometry

A reductive Cartan Geometry is based on a reductive Klein model (G, H). Since the Lie algebra $\mathfrak{g}$ can be written $\mathfrak{g} = \mathfrak{h} \bigoplus \mathfrak{p}$, with $\mathfrak{p}$ an H–invariant complement, and that a Cartan connection ϖ takes values in $\mathfrak{g}$, such a connection, in this reductive case, naturally splits into two parts: $\varpi = \omega + \theta$, with $\omega : TP \to \mathfrak{h}$ and $\theta : TP \to \mathfrak{p}$.

The three defining properties of a Cartan connection ϖ can be written in terms of the corresponding ω and θ:

- $R_h^* \varpi = Ad_{h^{-1}} \varpi$ implies that :
  - $R_h^* \omega = Ad_{h^{-1}} \omega \in \mathfrak{h}$
  - $R_h^* \theta = Ad_{h^{-1}} \theta \in \mathfrak{p}$
- $\varpi(\xi^v) = \xi$ implies that $(\omega + \theta)(\xi^v)$=$\omega(\xi^v) + \theta(\xi^v)$. But since the result is in $\mathfrak{h}$ and θ takes values in $\mathfrak{p}$ (the complement of $\mathfrak{h}$), then $\theta(\xi^v)$=0 and one gets: $\omega(\xi^v) = \xi$: the Maurer–Cartan–like ismorphism is realized by ω, and θ is said to be horizontal.
- The fact that $\varpi_p : T_pP \to \mathfrak{g}$ is an isomorphism implies that the following restrictions:
  - $\omega : V_pP \to \mathfrak{h}$
  - $\theta : H_pP \to \mathfrak{p}$

  are, each of them, isomorphisms as well.

It turns out that ω is then just an Ehresmann connection on the bundle P; θ is called a soldering form.

If one takes now a local section s over the bundle, it allows to pull the Cartan connection back on a open subset U of M: $s^*\varpi : TU \to \mathfrak{g}$, and thus, using the splitting of ϖ, we get two new objects over M: $s^*\varpi = s^*\omega + s^*\theta$, with $s^*\omega : TU \to \mathfrak{h}$ called Lorentz/Spin-Connection, and $s^*\theta : TU \to \mathfrak{p}$, called the vierbein.

2.3 Cartan Equivalence of Geometric Structures and Gravitation Theories

Klein geometry substitutes to the geometric study of a homogeneous manifold that of two related Lie groups of symmetry. Cartan geometry generalizes this approach and substitutes to the study of the geometry of a smooth manifold M (as it could be done in ordinary riemannian geometry) that of a principal bundle P, modeled on a homogeneous space, and the geometry of which is encoded in a certain Cartan connection ϖ. The equivalence process of Cartan consists in finding, given a geometric structure (like riemannian, conformal, ...) its equivalent (in terms of categories) in the language of Cartan geometry. Let us see in this section the simplest example: pseudo–Riemannian geometry and its equivalent Poincaré Cartan geometry. In the chapter 3, the same construction will be done for conformal geometry. For simplicity, in all this section, the dimension of the spacetime manifold will be taken to be $n = 4$. Moreover, all metrics will be Minkowskian, i.e. of signature (1,3).

2.3.1 Riemannian manifold, tetrad formulation, and Poincaré Cartan geometry

Recall that a Riemannian manifold is a smooth manifold M equipped with a non degenerate metric $g : TM \times TM \to \mathcal{C}^\infty(M)$. The Levi-Civita connection $\nabla : TM \times TM \to TM$ related to this metric is the only torsionless and metric affine connection. The Christoffel symbols $\Gamma^\rho_{\mu\nu}$ are defined as $\Gamma^\rho_{\mu\nu} b_\rho = \nabla_{b_\mu} b_\nu$, where $b_\mu := \partial_\mu$ are the holonomic vector basis related to a local coordinate system. Let $\{dx^\mu\}_\mu$ be its dual holonomic basis. Then, any $\{\tilde{\theta}^a\}_a$ defined by $\tilde{\theta}^a := e^a_\mu dx^\mu$, with $e \in GL_n(\mathbb{R})$, is another dual basis of the corresponding tangent space. This basis is called non–holonomic, for in general there is no coordinate system from which this basis can be derived as holonomic. Let us take e such that the resulting basis is orthogonal with respect to the metric g, which is always possible due to the Gramm–Schmidt procedure. Such a basis is called a local Lorentz frame, or a vierbein, and the forms $\tilde{\theta}^a$ are called tetrads. $\tilde{\theta}$ can be seen as a 1–form with values in $\mathbb{R}^{1,3}$. Written in this basis, the metric is flat at each point, that is to say:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{2.3}$$

Two orthogonal frames are related by a (local) SO transformation, and the orbit of all frames at all points of the base manifold define the SO–principal bundle of orthonormal 1–frames. This bundle will be presented in a slightly different way later. One can also define the so–called spin connection as:

$$\tilde{\omega}^a_{b\mu} = e^a_\nu \Gamma^\nu_{\sigma\mu} (e^{-1})^\sigma_b + e^a_\nu \partial_\mu (e^{-1})^\nu_b. \tag{2.4}$$

Let us set $\tilde{\omega}_{ab} := \eta_{ac}\tilde{\omega}^c_b$. The fact that the connection is metric implies, for the spin connection, that $\tilde{\omega}$ takes values in the Lie algebra of $\mathfrak{so}(1,3)$, i.e. that $\tilde{\omega}_{ab} + \tilde{\omega}_{ba} = 0$. $\tilde{\omega}$ can thus be seen as a 1–form on M with values in $\mathfrak{so}(1,3)$. The torsionless condition is translated as $d\tilde{\theta} + \tilde{\omega} \wedge \tilde{\theta} = 0$ in the tetrad formalism.

Conversely, the structure group $H = SO$ of the orthonormal frame bundle P is a subgroup of its affine extension (trivialy), the Poincaré group $G = \mathbb{R}^{1,3} \ltimes SO(1,3)$, the Lie algebra of which reads: $\mathfrak{g} = \mathfrak{so}(1,3) \oplus \mathbb{R}^{1,3}$. Let ϖ be a Cartan connection on P, i.e. $\varpi \in \Omega^1(P) \otimes \mathfrak{g}$. ϖ can thus be written

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & 0 \end{pmatrix} \tag{2.5}$$

and its curvature

$$\overline{\Omega} = \begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

with $R = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega$[b] the curvature of the connection, and $\Theta = d\theta + \omega \wedge \theta$ the torsion. These equations are deducible directly from the form of the Poincaré algebra, and from the definition of the Cartan curvature. θ is the *soldering form*. Then, one singles out the unique Cartan connection of this kind satisfying $\Theta = 0$. In this case, ω is directly determined by θ, the latter carrying all the degrees of freedom of the structure. It turns out that this particular Cartan geometry over $P \to M$ is equivalent to a riemannian geometry on M. Indeed, $\mathfrak{p} = \mathbb{R}^{1,3}$ is equipped with the Minkowski metric η. The local version of θ, $s^*\theta$ (which is nothing but the tetrad field $\tilde{\theta}$), with $s : U \to P$ a local section, makes an isomorphism between T_xU and $\mathbb{R}^{1,3}$. Then, define

$$g(X, Y) := \eta(\tilde{\theta}(X), \tilde{\theta}(Y)). \tag{2.7}$$

The choice of another θ is θ transformed by an element of $SO(1, 3)$, and since η is by definition invariant under such a transformation, it turns out that g is a well–defined (pseudo-riemannian) metric on the tangent bundle. This is the first example of equivalence in Cartan terms:

the data of a pseudo–riemannian geometry is equivalent to the data of a torsionless Cartan connection modeled on the Klein pair consisting in the Poincaré group and the Lorentz group.

2.3.2 General Relativity in Poincaré Geometry

2.3.2.1 Presentation

Usually (and historically), General Relativity is formulated in the framework of Riemannian geometry. It can also be rewritten in the so-called tetrad formulation, which is nothing but the Cartan geometry equivalent to Riemannian geometry. The field variable of the theory is the torsionless Cartan connection

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & 0 \end{pmatrix} \tag{2.8}$$

[b] For the Lie algebra here possesses a matrix representation

and the corresponding action functional reads:

$$S_{EH} = -\frac{1}{16\pi G}\int \text{tr}(R \wedge *(\theta \wedge \theta^t) + \frac{\Lambda}{6}\theta \wedge \theta^t \wedge *(\theta \wedge \theta^t)) \tag{2.9}$$

The notation θ^t stands for $\theta^t = \theta^T \eta$. The trace operator tr is just the Killing form of $\mathfrak{h} = \mathfrak{so}(1,3)$. Λ is the cosmological constant. Let us call $S_{matter} = \int L_{matter}$, with L_{matter} the lagrangian, a 4–form, describing matter and energy density (source of gravitation). Since the connection is torsionless, the degrees of freedom of the theory are all carried by θ. Thus, to derive the field equations from the action 2.9, one has to make a change $\theta \to \theta + \delta\theta$, defining $S = S_{EH} + S_{matter}$ and demanding that:

$$S(\theta) = S(\theta + \delta\theta) \tag{2.10}$$

in first order in $\delta\theta$. The energy-momentum 3-form τ is defined as :

$$\delta L_{matter} = -\delta\theta^a \wedge \tau_a \tag{2.11}$$

As it is shown for example in [22], 2.10 reduces to:

$$R^{ab}\theta^d \epsilon_{abcd} + \frac{\Lambda}{3}\theta^a\theta^b\theta^d\epsilon_{abcd} = -16\pi G\tau_c \tag{2.12}$$

which is shown to be exactly Einstein's Equations written in this more algebraic language. In [22], the Einstein-Cartan theory is also developed, where the torsion Θ is not set to zero. In this case, ω is an independent field variable, and the variations of the action with respect to ω has to be computed as well. The source of torsion is then seen as being a spin density, and where the spin density vanishes, so does the torsion, and thus gives in these regions the same predictions as General Relativity.

2.3.2.2 Erasing Lorentz symmetry by dressing

Given the previous geometric framework, one can construct a dressing field which will erase the Lorentz symmetry, as it has already been claimed. The $\mathbb{R}^{1,3}$–valued 1–form θ locally reads: $\theta^a = e^a_\mu dx^\mu$. Due to the transformation of θ under a Lorentz transformation $S \in SO(1,3)$: $\theta^S = S^{-1}\theta$, it turns out that the $GL_4(\mathbb{R})$–valued function e transforms also as $e \to S^{-1}e$, and thus if one sets

$$u := \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix},$$

under a gauge transformation Gauge$(SO(1,3)) \ni \gamma = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$, one gets

$$u^{\gamma} = \gamma^{-1} u.$$

Thus, the locally defined field u is a dressing field for the Lorentz symmetry. The composite fields are then:

$$\varpi^u = \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix} \tag{2.13}$$

and

$$\overline{\Omega}^u = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \tag{2.14}$$

where $\Gamma = \Gamma^{\mu}_{\nu} = \Gamma^{\mu}_{\nu,\rho} dx^{\rho}$ are the Christoffel symbols and $R = R^{\mu}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\rho\sigma} \mathrm{dx}^{\rho} \wedge \mathrm{dx}^{\sigma}$ is the Riemannian tensor written in the coordinate system $\{x^{\mu}\}$.

As it has been said in the introduction, the Lorentz symmetry is erasable by a dressing. It is not a gauge fixing nor a gauge transformation, in the sense that the dressing field u generally does not belong to $SO(1,3)$ but to $GL_n(\mathbb{R})$. Thus, this Lorentz symmetry can be seen as artificial. However, following the philosophical discussion set in the introduction, this symmetry still manifests itself in the fact that certain fields possess what one calls *spin*, quantity which can only take half–integer or integer values. Indeed, this is directly tracked back as a manifestation of an underlying Lorentz symmetry.

2.3.3 Wise Approach to MacDowell Mansouri Gravity

In the late 1970s MacDowell and Mansouri ([35]) wrote down a formulation of General Relativity from a Yang-Mills-like lagrangian,[c] combining the Levi-Civita connection and the tetrad in a unique field. Even if this formulation turned to be the starting point of many theories, the geometric framework was not, in the original formulation, well understood. In 2009, Derek K.Wise pointed out in [53] the underlying Cartan geometry, giving a better interpretation to what initially appeared to be a "mathematical trick".

The framework here is the Cartan-de Sitter geometry, based on the Klein geometry (G, H) with $G = SO(1,4)$, the so-called "de Sitter group", and $H = SO(1,3)$, the Lorentz group as in the previous case. The homogeneous space $M_0 = G/H$ is isomorphic to the de Sitter space-time with cosmological constant

[c]Let us recall that if Ω is the curvature of an Ehresman connection, a Yang-Mills-type Lagrangian looks like $\mathrm{tr}(\Omega \wedge *\Omega)$, where $*$ is the Hodge star related to a metric on the base manifold. It is the typical purely gauge sector of the lagragian describing the other interactions, and one would like to recast Gravitation in a lagrangian of the same form.

Λ, which can be extrinsically described as:

$$M_0 = dS^{1,3}_{\Lambda} := \left\{ (\mathrm{t}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}) \in \mathrm{R}^{1,4} / -\mathrm{t}^2 + \mathrm{w}^2 + \mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 = \frac{3}{\Lambda} \right\} \tag{2.15}$$

with H stabilizing the point (0,0,0,0,$\sqrt{\frac{3}{\Lambda}}$). Then one considers the reductive Cartan geometry (P, ϖ) with P a principal H–bundle over a 4-dimensional space-time manifold M, and ϖ a Cartan connection. The Lie algebra is reductive and decomposes as:

$$\mathfrak{g} = \mathfrak{so}(1,4) = \mathfrak{h} \oplus \mathbb{R}^{1,3}. \tag{2.16}$$

One can write the connection as:

$$\varpi = \begin{pmatrix} \omega & \theta/l \\ \theta^t/l & 0 \end{pmatrix}$$

with $l = \sqrt{\frac{3}{\Lambda}}$, corresponding to the "radius" of the de Sitter space. Althought this connection carries the same amount of degrees of freedom as in the Poincaré case, encoded into θ, the structure of the Lie algebra $\mathfrak{g}$ makes it looking different, and this difference will be the key to write the Einstein-Hilbert action in a Yang-Mills form. Let us notice that in this geometry there is a natural lengthscale already included. The corresponding curvature 2-form reads:

$$\overline{\Omega} = \mathrm{d}\varpi + \varpi^2 = \begin{pmatrix} \Omega & \Theta \\ \Theta^t & 0 \end{pmatrix} \tag{2.17}$$

Θ is still called "torsion" and reads:

$$\Theta = \frac{1}{l}(d\theta + \frac{1}{2}[\omega, \theta]), \tag{2.18}$$

as the one defined in the Einstein-Cartan theory up to the factor $1/l$, while Ω is the analogeous of the Riemmanian curvature, and reads:

$$\Omega = R + \frac{\Lambda}{3}\theta \wedge \theta^t, \tag{2.19}$$

Remark here again the difference between the curvature Ω and the one obtained in the Poincaré case. Let us set $\Theta = 0$ from here. Then, one recovers again a Cartan geometry on P equivalent to a pseudo-Riemannian geometry on M.

The MacDowell-Mansouri action reads, in this framework:

$$S_{MM} = \frac{-3}{32\pi G\Lambda} \int \mathrm{tr}(\Omega \wedge *\Omega) \tag{2.20}$$

Let us remark several important things. First, one has only taken the Lorentz part of the curvature to write this action. This is more or less the translation of the "symmetry breaking" from $SO(1,4)$ to $SO(1,3)$ in the original idea of MacDowell and Mansouri. The trace is thus taken as previously, in the subspace $\mathfrak{h}$. Second, there is a trick used by D. Wise consisting in taking an *internal Hodge star* $*$. A usual hodge star, as in a Yang-Mills gauge theory, or even in the tetrad formulation of General Relativity, is supposed to act on spacetime indices, and to transform a q–form to a $n-q$ form. It turns out that by chance, in dimension 4 and for a 2–form, the genuine Hodge star (let us denote is $*_{ext}$ just in this session) makes an isomorphism between $\wedge^2\mathbb{R}^{1,3}$ and itself. Moreover, the space $\Omega^2(M)$ is isomorphic as a vector space to the space $\wedge^2\mathbb{R}^{1,3}$. Then, Wise defines the internal Hodge star $*$ as the map:

$$\Omega^2(M) \xrightarrow{\text{iso.}} \wedge^2\mathbb{R}^{1,3} \xrightarrow{*ext} \wedge^2\mathbb{R}^{1,3} \xrightarrow{iso.} \Omega^2(M)$$

Something important to notice is that acting on θ, $*$ and $*_{ext}$ gives exactly the same result! Thus, the Einstein Hilbert action in the previous section, for exemple, can be written with both of them. The difference is made on a 2–form like R: if $R_{ab} = \frac{1}{2}R_{abcd}\theta^c \wedge \theta^d$, then

$$(*R)_{ef} = \frac{1}{2}\epsilon^{ab}{}_{ef}R_{abcd}\theta^c \wedge \theta^d. \tag{2.21}$$

while for the "external" (usual) hodge star acting on such an object, one gets:

$$(*_{ext}R)_{ef} = \frac{1}{2}R_{efab}\epsilon^{ab}{}_{cd}\theta^c \wedge \theta^d. \tag{2.22}$$

These are quite similar, but actually it makes all the difference. Indeed, it turns out that a term like $\int$ tr$(R \wedge *R)$ is a *topological term*, which means that it has *no variation* under a change of the metric, i.e. of θ in the present case, whereas a term like $\int$ tr$(R \wedge *_{ext}R)$ is *not* a topological term and has no trivial variations under $\theta \to \theta + \delta\theta$. Finally, the MacDowell-Mansouri Lagrangian is of Yang-Mills type and knowing that:

$$\text{tr}(\Omega \wedge *\Omega) = \text{tr}(R \wedge *R) + 2\frac{\Lambda}{3}\text{tr}(R \wedge *(\theta \wedge \theta^t)) + \frac{\Lambda^2}{9}tr(\theta \wedge \theta^t \wedge *(\theta \wedge \theta^t)) \tag{2.23}$$

one gets, thus:

$$S_{MM} = \frac{-3}{32\pi G\Lambda}(\int \text{tr}(R \wedge *R) + L_{EH-\Lambda}) \tag{2.24}$$

One recovers exactly the same information as that encoded in the Einstein–Hilbert action with cosmological constant $S_{EH-\Lambda}$, since $\int$ tr$(R \wedge *R)$ gives no information. Even if the use of an internal star can be seen as an *ad hoc* procedure, it is nice that one gets the cosmological constant term "naturally", for Λ is already present at the level of the *geometry*.

2.4 Conclusion

Thus, gravitation theories are naturally written in the framework of Cartan geometry. The point to have the Lorentz group as a structure group is to be able to describe matter with spin. Indeed, as in internal gauge theories, matter fields in this framework are sections of associated vector bundle. Spin fields are thus just sections of the vector bundle associated to P with the representation of $SO(1,3)$ as $SL_2(\mathbb{C})$ acting on $\mathbb{C}^2$. In the following chapter, one will present the framework of conformal geometry in the language of Cartan, and at the end one will show that one can also write conformal gravity theories, like Weyl gravity, naturally from a Yang–Mills type lagrangian, and sections of associated bundle ("matter fields") will just be the well known Tractors and Twistors.

Chapter 3

Conformal Gauge Theories

After an introduction where one presents different hints which could suggest that conformal symmetry can be relevant for physics, one sets up all useful mathematical objects which could be used to write conformal gauge theories.

3.1 Introduction: Conformal Symmetry in Physics

Conformal symmetry is a potential candidate for a more fundamental framework in theoretical physics. This symmetry is actually already present in some of our current theories. However, this symmetry seems to be irrelevant as soon as it is about describing phenomena which contain any notion of scale, like mass terms for example. As Fulton pointed out in *Conformal Invariance in Physics* ([21]), conformal covariance requires that rest masses of particles transform under a conformal change of the metric.

3.1.1 Gauge Theories

Maxwell equations without source, i.e., in a geometric language:

$$dF = 0 \text{ and } d * F = 0,$$

are invariant under the action of the conformal group, that is the 15–dimensional Lie group of conformal diffeomorphisms of (conformally compactified) Minkowski spacetime (see below). It has been shown the first time by Cunningham and Bateman in 1900, as Fulton noticed. In general, gauge theories of fundamental interactions, if the interaction of the fields with the Higgs field is not taken into account, are scale invariant for there is no mass term in the action. See e.g. [49], in particular chapter 3.

3.1.2 Causal Structure

One of the most fundamental features of Science in general, and in particular of theoretical physics, is the notion of *causality*. One can wonder whether causality has an ontological status, however on an epistemological point of view, its status is clear: it seems to be impossible to built any predictive physical theory which would not respect causality (in the current state of our knowledge). In the geometric framework of special and general relativiy theories, causality is implemented by the lightcone structure, i.e. by the fact that the geometry is given by a Lorentzian metric. Two points of spacetime are said to be causally related if there is a timelike path joining them. Even if a point of spacetime has no intrinsic physical meaning, which is expressed by the Poincaré covariance in Special Relativity, and by diffeomorphisms covariance in General Relativity, the notion of being causally related is invariant under these latter transformations and thus has a physical meaning. In these contexts, the biggest geometric structure which respects the lightcone structure of spacetime is, in Special Relativity, the conformal group, and, in General Relativity, a conformal equivalence class of metrics $\{\phi^2 g, \phi \in \mathcal{C}^\infty(M)\}$. Thus, at the causality level, the relevant symmetry is the conformal symmetry.

3.1.3 Conformal Gravity

One calls generically *gravitational theory* a physical theory based on a smooth manifold M equipped with a certain geometric structure $\mathcal{G}$, covariant under the diffeomorphisms of M, and for which the state of $\mathcal{G}$ is not given a priori but follows from field equations. General Relativity is a particular case, for which $\mathcal{G}$ is a pseudo–Riemannian metric, and the field equations are Einstein equations. A conformal gravitational theory, or theory of conformal gravity, is then a gravitational theory for which the geometric structure $\mathcal{G}$ is a conformal structure over M, i.e. M is equipped with a conformal class of pseudo–Riemannian metrics **c** $:= \{\phi^2 g, \phi \in \mathcal{C}^\infty(M)\}$. For such a theory, the field equations are written with a certain metric g, but are invariant under a Weyl rescaling $g \to z^2 g$ of the metric, where z is a nowhere vanishing and positive smooth function on M.[a] The most known example of such a theory is Weyl gravity. It is based on a Lagrangian introduced by Bach in the early 1920's, which consists in the square of the Weyl tensor $W^\mu_{\nu\rho\sigma}$:

$$\int_M W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma} \sqrt{|g|} d^4x. \tag{3.1}$$

This action is written with a certain metric, but is constant on each gauge orbit (if one figures it into the space of all metrics). It is again the gauge principle

[a] Throughout the thesis, one will use either z^2 or f, a nowhere vanishing and positive smooth function, or $e^{2\lambda}$, with $\lambda \in \mathcal{C}^\infty(M)$, as a Weyl rescaling, depending on the context.

expressing itself here: one needs a particular metric g to write the action and compute, but the final result should not depend on the particular choice we made. Again in [21], it is recalled that Weyl introduced this new geometry few time after first confirmed prediction of General Relativity, in order to unify the only two forces known at this time, gravitation and electromagnetism. Einstein gave then a physical argument showing the irrelevance of such a framework. The work of Weyl and the argument of Einstein can be found in the book [52]. Later, one will present how to recast Weyl gravity in the geometric framework of Cartan geometry.

In some papers (like [23], [24]), one also finds a conformally invariant action constructed from Einstein–Hilbert action which reads:

$$S(\phi, g) = \int d^4x \sqrt{|g|}(6g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) + \phi^2 R(g)) \tag{3.2}$$

where ϕ is a scalar field, g a pseudo–Riemannian metric and $R(g)$ the Ricci scalar corresponding to g. It turns out that this action is invariant under the joint Weyl rescalings: $g \to z^2 g$ and $\phi \to z^{-1}\phi$. Is this action a good model for a conformal gravity theory? Actually, it is nothing but General Relativity. Indeed, this action can be rewritten as $S_{EH}(\phi^2 g)$, where S_{EH} is the Einstein–Hilbert action. This means that with the dressing field $u = \phi$, the composite field $\hat{g} = \phi^2 g$ is invariant under a Weyl rescaling in the dressing method sense, that is to say that Weyl rescalings do not act anymore on $\hat{g}$. It is possible to rewrite this theory in a conformally invariant way, which turns out to be General Relativity. From this viewpoint, trying to extract more physics than GR contains from this "conformal action" seems to be vain. Still from the dressing field method point of view, and to be a bit provocative, this action is not more conformally invariant than, e.g., $SO(42)$–invariant.

Let us remark that a dressing for Weyl rescaling is thus just a scalar field ϕ which transforms as $\phi \to \Omega^{-1}\phi$, i.e. a density. That means that any time such a field exists in a conformally covariant gravity theory, one knows that it is possible to erase the Weyl symmetry, ending up with a mere (pseudo–)Riemannian framework.

This chapter is devoted to conformal geometry. Firstly, one gives an overview of the conformal framework, reviewing the main ways of formulating conformal geometry, and giving the relation between them. Then, one sees how the dressing field method applies to conformal geometry, especially in its Cartan geometric formulation. It leads to a top-down construction of Twistors and Tractors, well known objects in the framework of conformal gravity. The section presents finally how Weyl gravity is possibly written as a Yang–Mills type gauge theory in this framework.

3.2 Overview of Conformal Symmetry, different equivalent approaches

3.2.1 First definitions

3.2.1.1 Conformal structure and Weyl structure

Let M be a n-dimensional smooth manifold.[b] A conformal structure on M is a conformal class **c** of metrics (with signature $(1, n-1)$ here) defined on M, that is to say a set of metrics such that for two $g, g' \in$ **c**, there exists a positive smooth function on M, f, such that $g' = fg$ (such a transformation is called a conformal change of the metric, or a Weyl rescaling as one will define it below). Historically, H.Weyl defined on such a conformal manifold $(M,$ **c**$)$ a connection ∇ such that for a given metric $g \in$ **c**:

$$\nabla_\alpha g_{\mu\nu} = W_\alpha g_{\mu\nu}, \tag{3.3}$$

with W being a 1-form on M, such that for another metric $\tilde{g} = e^{2\lambda} g \in$ **c**, with $\lambda : M \to \mathbb{R}$, W transforms as:

$$\tilde{W} = W - 2d\lambda. \tag{3.4}$$

This actually means that the fundamental relation 3.3 is well-defined on the given conformal manifold $(M,$ **c**$)$ (indeed, one can easily verify that this relation is unaffected by a conformal change of the metric provided that W transforms as required).

One can define more precisely a Weyl structure on M as a set $(M,$ **c**$, F)$, with:

- **c**: a conformal class of metrics;
- a map $F :$ **c** $\to \Omega^1(M)$ such that $F(e^{2\lambda} g) = F(g) - 2d\lambda$ for $g \in$ **c**, $\Omega^1(M)$ being the space of 1-forms on M.

3.2.1.2 Weyl rescalings and conformal group

Let us set, for a given n-dimensional smooth manifold M,

- $met(M) := \{$metrics of signature $(1, n-1)$ on $M\}$;
- $W_r := \mathcal{C}^\infty(M, \mathbb{R}^{+*})$

W_r, called the *group of Weyl rescalings*, acts on the set of metrics: $met(M) \ni g \to fg \in met(M)$, with $f \in W_r$. In other words, it rescales the metric by a

[b] In all the section the base manifold M will be of any dimension $n \geq 3$, except if something else is explicitly mentioned.

positive factor which depends on the point on M. A conformal class of metrics **c** $= [g]$ can then be seen as the orbit of a given metric g under the action of W_r.

Now, let us equip M with a given metric $g \in met(M)$. $C_0(M,g)$, called the *group of conformal transformations* of (M,g), or just its *conformal group*, is defined as the set of diffeomorphisms of M such that the metric is changed by a positive smooth function:

$$C_0(M,g) := \{\phi \in Diff(M), \exists f \in W_r, \phi^* g = fg\} \tag{3.5}$$

$C_0(M,g)$ and W_r, even if they are deeply related, are mathematically fundamentally different: the former is a set of diffeomorphisms of M, whereas W_r is an abelian group acting on the set of metrics. In order to emphase the difference between these two sets of transformations, let us consider Einstein's equations in the absence of matter, with $n \neq 2$:

$$R_{\mu\nu} = 0.$$

If one makes W_r acting on the metric by $g \to e^{2\lambda} g$, $R_{\mu\nu}$ becomes:

$$R_{\mu\nu} - (n-2)(\nabla_\mu \nabla_\nu \lambda - \nabla_\mu \lambda \nabla_\nu \lambda) + (\Delta\lambda - (n-2)(\nabla\lambda)^2))g_{\mu\nu}.$$

Thus, it turns out that Einstein's equations are not, unless λ is constant, Weyl-invariant. However, if one takes a conformal diffeomorphism, then one also has a conformal change of the metric, but in this case Einstein's equations remain invariant, because they are generally Diff(M)–invariant. This apparent contradiction is resolved by understanding that a diffeomorphism makes a change of coordinates, and *consequently* of the metric, but all-in-all since $R_{\mu\nu}$ is a tensor, it transforms regularly and Einstein's equations are preserved, while a Weyl rescaling acts only on the metric and not on the coordinates, and thus this conformal change of the metric cannot be "compensated" and $R_{\mu\nu}$ transforms in a completely different way. This remark shows that conformal diffeomorphisms and Weyl rescalings must not be confused, since they have nothing to do with each other.

Let us notice that, for a manifold endowed with a conformal class of metrics **c**, all pairs $(M, g \in$ **c**$)$ define the same conformal group. In other words, for $g \in met(M)$, the conformal group $C_0(M,g)$ can also be defined as the set of diffeomorphisms preserving the conformal class $[g]$.

3.2.1.3 Extended conformal group

Let us now set: $C(M) := Diff(M) \ltimes W_r$, called the *extended conformal group*[c], acting on $met(M)$ by:

$$(\phi, f) \cdot g := \phi^{*-1}(fg)$$

[c] We owe this definition to our colleague Christian Duval.

for $\phi \in$ Diff$(M), f \in \mathcal{C}^\infty(M, \mathbb{R}^{+*}) = W_r$ and $g \in met(M)$. Now, let us equip M with a given metric g. Then, the conformal group $C_0(M, g)$ can be seen as the stabilizer of g under the action of $C(M)$:

$$C_0(M, g) := \{(\phi, f) \in C(M), (\phi, f) \cdot g = g\}$$

One notices directly that a given element (ϕ, f) of $C_0(M)$ verifies $\phi^{*-1}(fg) = g$, i.e.:

$$\phi^* g = fg \tag{3.6}$$

and one recovers the definition of a conformal transformation of a manifold (M, g) given above.

Both Weyl group of rescalings and conformal group of a given manifold appear to be subgroups of the same (bigger) group, denoted $C(M)$. In general, for a given (M, g), it is not always straightforward to find its corresponding conformal group $C_0(M, g)$. It may be even trivial. There is one important theorem to keep in mind, however. (See e.g. [30]). Let set $n = dim(M)$. Then, $C_0(M, g)$ is a $m-$dimensional Lie group such that:

$$m \leq \frac{1}{2}(n+1)(n+2). \tag{3.7}$$

Thus, the group of conformal diffeomorphisms of any manifold is always a finite dimensional Lie group. It is of maximal dimension $m = \frac{1}{2}(n+1)(n+2)$ in the case of (compactified) Minkowski space-time M_0, as one shall see later.

3.2.2 Overview of conformal geometry

Let (M, g) be a n–dimensional smooth manifold which describes a space-time of a certain theory. One noticed that requiring invariance (or covariance) of gravitation field equations under conformal transformations of M (i.e. under $C_0(M, g)$) cannot constrain the theory at all, since the latter will naturally be invariant under Diff(M) and that conformal transformations are just particular diffeomorphisms. What is more interesting, however, is trying to construct a space-time theory invariant under Weyl rescalings, namely under W_r. In other words, a theory well-defined on a conformal manifold, i.e. on a manifold endowed with the orbit of a metric under W_r.

The purpose of this overview is to have a walk through different viewpoints of conformal geometry, with a guiding line: being able to do physics on a conformal manifold, i.e. building well-defined objects and operators on a manifold endowed with a conformal class of metrics.

For that, the way choosen here is the construction of the Cartan geometry corresponding to a given conformal manifold, that is to say finding the bijective correspondence between a certain set of *Cartan geometries* and conformal structures over a given manifold.

For this purpose, one will firstly present the objects usually defined on a conformal manifold, in a "tensor calculus style". Then, we will see how a conformal structure is uniquely defined by a CO–structure[d] over M, i.e. a CO–reduction of the 1-frame bundle over M. As it is well-known a Cartan connection on such a bundle is not uniquely determined by the conformal structure. However, one can build a Cartan connection corresponding to a Weyl structure $(M, \mathbf{c}, F)$, as one will firstly see. The corresponding Cartan geometry is modeled, in this case, on (G, H), where $H = CO$ and G is the affine extension of H. Then, the main work of this overview will be to present the Cartan equivalence problem for a conformal structure $(M, \mathbf{c})$. It will be necessary to prolongate the 1^{st}–order conformal bundle to a 2^{nd}–order bundle. The corresponding Klein pair of this second-order Cartan geometry is (G, H) where G is the full group of conformal transformations of compactified Minkowski space-time M_0, naturally endowed with a conformal class of metrics $\mathbf{c}_0$ (i.e. $G = C_0(M_0, \mathbf{c}_0)$), and H the stabilizer of a point of M_0. A Cartan geometry built on such a Klein pair is then called a (2^{nd}-order) conformal Cartan geometry. The latter can also be seen as a H–reduction of the 2-frames bundle. It turns out that on such a structure, one can build a unique Cartan connection, i.e. in a one-to-one correspondence with the conformal class of metrics one has started with. This part is inspired by [39].

There exist different representations of the groups G and H. Firstly, one will present the defining $\mathbb{R}^{n,2}$-representation, say the 1:2 homomorphism between G and $O(n, 2)$, and give in this representation the explicit form of the corresponding normal Cartan connection, its curvature, etc. In order to simplify the formalism and the use of it, one will then *dress* the connection and the associated objects. Then one will take an interest in sections of a certain associated vector bundle, which as we will see are just what some authors call *Tractors*.

Another representation exists, in dimension $n = 4$: in this case, there is an 1:4 homomorphism between G and $SU(2, 2)$, and an isomorphism between the respective Lie algebras: $\mathfrak{g} \cong \mathfrak{su}(2, 2)$. In this representation one will give the explicit form of the connection and associated objects. Then, sections of complex associated vector bundles (with the representation of $SU(2, 2)$ in $\mathbb{C}^4$) will appear to be deeply related to Penrose's *Twistors*.

3.2.3 Objects usually defined on a conformal manifold

Let $(M, \mathbf{c})$ be a n–dimensional smooth manifold endowed with a conformal class of (pseudo–)riemannian metrics $\mathbf{c}$. Let us present the usual tensors related to this particular framework w.r.t. their conformal transformation properties.

[d] As defined later, CO denotes the group $CO(1, n-1) = SO(1, 3) \times \mathbb{R}^{+*}$, which thus consists in Lorentz transformations and dilations.

3.2.3.1 The Schouten Tensor and the Weyl tensor

Given $g \in$ **c**, let Ric be the Ricci tensor related to g, then the Schouten tensor is also a 2–order symmetric tensor defined as:

$$P = \frac{-1}{n-2}(Ric - \frac{R}{2(n-1)}g) \tag{3.8}$$

where R is the Ricci scalar. The Schouten tensor has the following conformal transformation under a Weyl rescaling $g \to z^2 g$:

$$P_{\mu\nu} \to P_{\mu\nu} - \nabla_\mu \Upsilon_\nu + \Upsilon_\mu \Upsilon_\nu - \frac{1}{2}\Upsilon^2 \tag{3.9}$$

where ∇ is the Levi-Civita connection related to g, $\Upsilon_\mu = z^{-1}\partial_\mu z$ and $\Upsilon^2 = g^{\mu\nu}\Upsilon_\mu \Upsilon_\nu$.

The Weyl tensor (already mentioned in the introduction of this chapter) is defined as:

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu\rho}P_{\nu\sigma} + g_{\nu\rho}P_{\mu\sigma} + g_{\mu\sigma}P_{\nu\rho} - g_{\nu\sigma}P_{\mu\rho}. \tag{3.10}$$

It has the remarkable property to be strictly invariant under a rescaling of the metric.

3.2.3.2 Geometric Interpretation

If $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of a metric g, the equation $R_{\mu\nu\rho\sigma} = 0$ means that the manifold M is locally flat. Similarly, the equation $W_{\mu\nu\rho\sigma} = 0$ means that the manifold is locally *conformally flat*, i.e. that there exists locally a conformal factor Ω^2 such that the metric is locally proportional to $\Omega^2 \eta$.

A metric $g_{\mu\nu}$ is said to be an Einstein metric if and only if there exists $\Lambda > 0$ such that the Ricci tensor of $g_{\mu\nu}$ reads $R_{\mu\nu} = \Lambda g_{\mu\nu}$. It turns out that this property is also encoded in the Schouten tensor: $g_{\mu\nu}$ is Einstein if and only if the Schouten tensor $P_{\mu\nu}$ itself is proportional to the metric.

3.2.3.3 Conformal geodesics

Conformal geodesics are the equivalent of ordinary geodesics (i.e. in a riemannian context) in the conformal framework. They are defined by a 3^{rd}–order differential equations which are, naturally, conformal invariant. Let us present the main definitions found in the literature (e.g. see [51], [5]) and a nice result related to Einstein metrics.[e]

[e] I thank Paul Tod for his answer to my emails about conformal geodesics which helped me a lot.

General Parametrization Let γ be a curve on $(M, \mathbf{c})$ parametrized by a real parameter t, and U^μ its tangent vector. If ∇ is the Levi-Civita connection related to a metric $g \in \mathbf{c}$, one gets $U^\mu \nabla_\mu t = 1$. If A^μ is the acceleration vector: $A^\mu := U^\nu \nabla_\nu U^\mu$, one can easily check that A is not conformally invariant (neither covariant), but has the following transformation law under a conformal rescaling $g_{\mu\nu} \to z^2 g_{\mu\nu}$:

$$A^\mu \to A^\mu - U^2 \Upsilon^\mu + 2(U \cdot \Upsilon) U^\mu. \tag{3.11}$$

where Υ is defined as for 3.9. The contraction $U \cdot \Upsilon$ and U^2 are made by using the inverse metric. That means that contrary to riemannian geometry, "$A^\mu = f U^\mu$" cannot be a good geodesics equation, because it is not conformally invariant and thus not well-defined on such a manifold.[f]

γ is a conformal geodesics if, by definition, it verifies the following equation:

$$U^\mu \nabla_\mu A^\nu = 3 \frac{U \cdot A}{U^2} A^\nu - \frac{3}{2} \frac{A^2}{U^2} U^\nu + U^2 U^\mu P_\mu^\nu - P_{\rho\sigma} U^\rho U^\sigma U^\nu \tag{3.12}$$

where P is the Schouten tensor related to the given g which is used to write the equation. This equation, as announced, turns out to be conformally invariant. A solution of this equation always exists (at least locally) and is unique once given initial conditions, i.e. not only an initial position and an initial velocity but also an initial acceleration. These curves are invariant under projective reparametrizations, i.e. transformations:

$$t \to \frac{at + b}{ct + d} \tag{3.13}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}). \tag{3.14}$$

Proper Time Parametrization There exists another possible parametrization, more restrictive, which get the equations a bit simpler. It consists in taking, for a given $g \in \mathbf{c}$, the proper time defined by g as parameter. Recall that the proper time τ_g defined by a metric g is given by: $d\tau_g = \sqrt{-g_{\mu\nu} \dot{\gamma}^\mu(t) \dot{\gamma}^\nu(t)} dt$. Then, if U^μ is the tangent vector to γ parametrized by the corresponding proper time, one has $g_{\mu\nu} U^\mu U^\nu = -1$. In this case, a conformal change of metric $g_{\mu\nu} \longmapsto z^2 g_{\mu\nu}$ will change the definition of the proper time and thus $U^\mu \longmapsto z^{-1} U^\mu$. The acceleration vector, in this parametrization, has also another transformation:

$$A^\mu \to \Omega^{-2}(A^\mu - U^2 \Upsilon^\mu + (U \cdot \Upsilon) U^\mu). \tag{3.15}$$

[f] Recall that in this general parametrization, U^μ only depends on γ and thus is not affected by a conformal change of the metric, for the path γ is given witout any reference to the geometric structure.

As in riemannian geometry, one gets:

$$g_{\mu\nu}A^\mu U^\nu = 0 \tag{3.16}$$

and the conformal geodesics equation becomes:

$$U^\nu \nabla_\nu A^\mu = -A^2 U^\mu + U^\nu P^\mu_\nu - U^\rho U^\sigma P_{\rho\nu} U^\sigma \tag{3.17}$$

In particular, one easily gets the following equation satisfied by $A^2 = g_{\mu\nu}A^\mu A^\nu$ along the conformal geodesics:

$$\frac{d}{d\tau}(A^2) = 2P_{\mu\nu}U^\mu A^\nu \tag{3.18}$$

(Proof: recall that in this parametrization, $\frac{d}{d\tau_g} = U^\mu \nabla_\mu$. Apply it to $A^2 = g_{\mu\nu}A^\mu A^\nu$. Then, use $U \cdot A = 0$ and easily get the result.)

Thus, one has the following result: if $g_{\mu\nu}$ is Einstein, then $P_{\mu\nu} \propto g_{\mu\nu}$, and thus, as $U \cdot A = 0$, one gets that A^2 is constant along γ.

Relation with ordinary metric geodesics One calls metric geodesic a geodesic for a metric $g_{\mu\nu}$ in the Riemannian sense. Up to parametrizations, the space of metric geodesics is of dimension 4+3-1=6 (for a base manifold of dimension 4): 4 for the initial position, 3 for the initial velocity of square equal to 1, 1 for the fact that it is a curve (a submanifold of dimension 1). Concerning conformal geodesics, their space is of dimension (again, up to parametrizations) 4+3+3-1=9, the 3 dimensions added are for the initial acceleration of square equal to 1. Thus, both these spaces having different dimension, one may wonder at which condition a metric geodesic for a metric g is a conformal geodesic for the corresponding conformal structure $[g]$. It turns out that:

if g is an Einstein metric, then any g–geodesic
is a conformal geodesic for $[g]$.

3.2.3.4 Tractor and Twistor spaces and connections

Tractors and Twistors are typical objects defined on a conformal manifold, even if nowadays the term is used for cases wider than just conformal geometry. One presents here, in a nutshell, the usual bottom–up construction of these objects, deriving them from a defining differential equation. Later one will see that they appear naturally and with a clearer geometric nature from the application of dressing field method to conformal Cartan geometry.

Tractors The construction presented here is based on [12], [23], [24] and [10].[g] Tractors are built from a density σ, i.e. a scalar field which transforms as $\sigma \to z\sigma$ under $g_{\mu\nu} \to z^2 g_{\mu\nu}$. One imposes that σ satisfies the conformally invariant equation:

$$\nabla_\mu \nabla_\nu \sigma - P_{\mu\nu}\sigma - \frac{g_{\mu\nu}}{n}(\Delta\sigma - P\sigma) = 0 \tag{3.19}$$

where $\Delta := g^{\mu\nu}\nabla_\mu\nabla_\nu$, $P_{\mu\nu}$ is the Schouten tensor and $P := g^{\mu\nu}P_{\mu\nu}$. This equation is called the Almost Einstein Equation, because if σ and g are solutions of this equation, then the metric $\sigma^{-2}g$ is Einstein. One transforms this second order differential equation by introducing two other fields ℓ_μ, $\mathbb{R}^n$–valued, and ρ, another scalar field, such that:

$$\nabla_\mu \sigma - \ell_\mu = 0 \tag{3.20}$$

$$\nabla_\mu \ell_\nu - P_{\mu\nu}\sigma + g_{\mu\nu}\rho = 0 \tag{3.21}$$

$$\nabla_\mu \rho + g^{\alpha\beta}P_{\mu\alpha}\ell_\beta = 0 \tag{3.22}$$

This system of first order differential equations can be made more compact by defining the parallel tractor $t = \begin{pmatrix} \sigma \\ \ell_\mu \\ \rho \end{pmatrix}$ and the tractor connection $\nabla^T_\mu := \partial_\mu \mathbb{I}_{n+2} + \begin{pmatrix} 0 & -\delta^\alpha_\mu & 0 \\ -P_{\mu\nu} & -\Gamma^\alpha{}_{\mu\nu} & g_{\mu\nu} \\ 0 & g^{\alpha\beta}P_{\mu\beta} & 0 \end{pmatrix}$, and setting: $\nabla^T_\mu t = 0$. Knowing the transformation laws of each object, one can compute the transformation laws of t and $\nabla^T{}_\mu t$ under a Weyl rescaling and they turn out to be:

$$t \to \begin{pmatrix} z & 0 & 0 \\ z\Upsilon_\mu & z\mathbb{I}_n & 0 \\ -z^{-1}\frac{1}{2}\Upsilon^2 & -z^{-1}g^{\nu\alpha}\Upsilon_\alpha & z^{-1} \end{pmatrix} t. \tag{3.23}$$

and

$$\nabla^T{}_\mu t \to \begin{pmatrix} z & 0 & 0 \\ z\Upsilon_\mu & z\mathbb{I}_n & 0 \\ -z^{-1}\frac{1}{2}\Upsilon^2 & -z^{-1}g^{\nu\alpha}\Upsilon_\alpha & z^{-1} \end{pmatrix} \nabla^T{}_\mu t. \tag{3.24}$$

with usual notation. Now, a general tractor t is defined as the section of a vector bundle $\mathcal{T}$ with fibre $\mathbb{R}^{n+2}$ which transforms, *by decree*, with the latter transfor-

[g] I thank Rod Gover for having answered to my email which helped me to understand these constructions.

mation laws 3.23. However, an element like

$$\begin{pmatrix} z & 0 & 0 \\ z\Upsilon_\mu & z\mathbb{I}_n & 0 \\ -z^{-1}\frac{1}{2}\Upsilon^2 & -z^{-1}g^{\nu\alpha}\Upsilon_\alpha & z^{-1} \end{pmatrix}$$

cannot be seen merely as the localization of the action of a structure group as in the case of a principal bundle, because the elements $\Upsilon_\mu = \partial_\mu(\ln z)$ *cannot be defined pointwise*.[h] This particularity will be clarified later, when one shows that this transformation law turns out to be the *residual* transformation law of twisted composite gauge fields after the dressing of an underlying structure the geometry of which is very well understood.

By construction, parallel tractors are in bijective correspondence with solutions of the Almost Einstein Equation. The linear operator ∇^T is a covariant derivative on $\mathcal{T}$. The commutator of the tractor connection defines the tractor curvature:

$$[\nabla^T_\mu \nabla^T_\lambda - \nabla^T_\lambda \nabla^T_\mu]t = \begin{pmatrix} 0 & 0 & 0 \\ -C_{\mu\lambda,\nu} & W^\alpha_{\mu\lambda,\nu} & 0 \\ 0 & g^{\alpha\beta}C_{\mu\lambda,\beta} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \ell_\alpha \\ \rho \end{pmatrix} \tag{3.25}$$

where $C_{\mu\lambda,\beta} = \nabla_\lambda P_{\mu\nu}$ is the Cotton tensor, and $W^\alpha{}_{\mu\lambda,\nu}$ is the Weyl tensor. The tractor calculus provided by this construction can be thought as the analog for conformal manifolds of the Ricci tensorial calculus for Riemannian manifolds.

Twistors The construction presented here is mainly based on Penrose's work, it can be found in [45], [43], [44], [40], [41] and [42]. Like tractors, twistors are constructed from a defining differential equation transformed into a system of first order, but in the complex representation, based on the homomorphism between $\mathbb{R}^{1,3}$ and $H_2(\mathbb{C})$, the 2×2 hermitian matrices. This homomorphism is given by

$$x = x^a \to \overline{x} = \overline{x}^{AA'} := x^a \sigma_a^{AA'} = \frac{1}{2}\begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{3.26}$$

The A–indices are related to transformations by a matrix $\overline{S} \in SL_2(\mathbb{C})$, and A'–indices are related to transformations by its adjoint (transposed conjugate) $\overline{S}^*$. A description of this homomorphism can be found in [34].

Let ω^B be a $\mathbb{C}^2$–valued field over M, the twistor equation reads:

$$\nabla_{AA'}\omega^B - \frac{1}{2}\delta^B_A \nabla_{CA'}\omega^C = 0. \tag{3.27}$$

One introduces the new field $\pi_{A'} := \frac{i}{2}\nabla_{CA'}\omega^C$ and transforms the twistor equa-

[h] I thank Fréderic Hélein for having pointed this out during my visit at the IMJ–PRG in April 2018.

tion into a system:

$$\nabla_{AA'}\omega^B + i\frac{1}{2}\delta_A^B\pi_{A'} = 0 \tag{3.28}$$

$$\nabla_{AA'}\pi_{B'} - i\overline{P}_{AA'BB'}\omega^B = 0 \tag{3.29}$$

with $\overline{P}_{AA'BB'}$ the complex representation of the Schouten tensor. As in the tractor case, one calls $Z^\alpha := (\omega^B, \pi_{A'})$ a twistor, and rewrites the system as $\nabla^T_{AA'}Z^\alpha = 0$ by defining $\nabla^T_{AA'} := \nabla_{AA'}\mathbb{I}_4 + \begin{pmatrix} 0 & i\delta_A^B \\ -i\overline{P}_{AA'BB'} & 0 \end{pmatrix}$. Then, the transformations of such objects are:

$$\hat{Z}^\alpha = \begin{pmatrix} \mathbb{I}_2 & 0 \\ i\hat{\Upsilon}_{AA'} & \mathbb{I}_2 \end{pmatrix} Z^\alpha \tag{3.30}$$

and

$$\widehat{\nabla^T_{AA'}Z^\alpha} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ i\hat{\Upsilon}_{AA'} & \mathbb{I}_2 \end{pmatrix} \nabla^T_{AA'}Z^\alpha \tag{3.31}$$

where $\overline{\Upsilon}$ is the complex version (via the homomorphism) of Υ. Then, general twistors defined on a conformal manifold M are defined as being sections of a vector bundle with fibre $\mathbb{C}^4$ transforming as 3.30. The same remark applies here: the elements of the form

$$\begin{pmatrix} \mathbb{I}_2 & 0 \\ i\hat{\Upsilon}_{AA'} & \mathbb{I}_2 \end{pmatrix}$$

cannot be seen as the local version of the action a structure group, again for the same reasons. The gauge group, however, can be seen as a subgroup of the gauge group based on the group $SU(2,2)$. Indeed, the structure group turns out to be a subgroup of $SU(2,2)$. The commutator of the twistor connection $\nabla^T_{AA'}$ defines the local twistor curvature:

$$[\nabla^T, \nabla^T]Z = \begin{pmatrix} \overline{W}^* & 0 \\ -i\overline{C} & \overline{W} \end{pmatrix} \begin{pmatrix} \omega \\ \pi \end{pmatrix} \tag{3.32}$$

where $C = \nabla P$ is the Cotton tensor, and W the Weyl tensor.

Now, one presents the solution to the Cartan equivalence problem in conformal geometry. That is to say, the answer to the question: "Which Cartan geometry over a smooth manifold M is equivalent to the data of a conformal class of metrics on M ?" One will see that the solution is a certain ("normal") Cartan connection over a 2^{nd}–order frame bundle denoted $P(M,H)$. The first step is to construct this bundle, and in particular to present the corresponding structure group H. After having shown different ways to get to $P(M,H)$, one defines a normal Cartan connection (without showing why it is equivalent to a conformal structure, though). One can find a detailed proof of this last step in [30], or [39].

3.2.4 1^{st}-order conformal bundles: CO-structures

3.2.4.1 G-structures

Let M be a n-dimensional smooth manifold. Then one can always define a natural principal bundle over M which is the $GL(n)$-bundle $L(M)$ of linear frames. A linear frame over M, also called a 1–frame, is an isomorphism u_x given at each point $x \in M$:

$$u_x : \mathbb{R}^n \xrightarrow{\sim} T_x M \tag{3.33}$$

such that $GL(n)$ acts on it as: $u_x \to u_x \circ A$, with $A \in GL(n)$.

Then, given a Lie group $G \subset GL(n)$, a G-structure on M is a smooth subbundle $P \subset L(M)$ with G as structure group. That is to say, P is a subbundle of $L(M)$ composed of frames u related by elements $A \in G \subset GL(n)$. The existence of a G–structure, in general, is not something trivial. In the cases we are interested in, the existence of a G–structure is equivalent to the existence of a cross-section $M \to L(M)/G$. See for instance [30] for a more precise explanation.

As already presented, in the case of pseudo–riemannian geometry, the metric g given on the manifold M allows to reduce the $L(M)$–bundle over M to a $SO(1, n-1)$-subbundle. This reduction is used to write gravitation theories in the form of a $SO(1, n-1)$-gauge theory. One will explain in this section how a CO–structure uniquely defines a conformal class of metrics on M. Later on, one will construct the so-called first prolongation of such a structure.

3.2.4.2 A CO-structure $P \to M$ is equivalent to a conformal structure on M

The Weyl group (should not be confused with the Weyl group of rescaling W_r presented in introduction which has nothing to do with this one!) is defined as:

$$CO(1, n-1) := \{A \in \mathsf{GL}_n(\mathbb{R}),\ \exists \lambda > 0,\ A^T \eta A = \lambda \eta\} \tag{3.34}$$

One takes thus this matrix–defining–representation in the following. The Weyl group is thus composed of Lorentz transformations *plus* dilations, and any $A \in CO(1, n-1)$ can be written $A = \lambda S$, with $\lambda > 0$ and $S \in SO(1, n-1)$. It will also be shortly denoted: CO.

Let $P \to M$ be a CO-structure on M. One will see how such a structure defines a conformal class of metrics over M. Let $x \in M$, and u_x in the fibre over x. Then, as $\mathbb{R}^n$ is naturally endowed with a metric η, one can define a metric $g_x : T_x M \times T_x M \to \mathbb{R}$ by:

$$g_x(X, Y) := \eta(u_x^{-1}(X), u_x^{-1}(Y)), \text{ for } X, Y \in T_x M.$$

Now, one takes another $u'_x \in P$ over x, and defines as well a metric g'_x. Since u and u' are in the same fibre, there exists an $A \in CO$ such that $u' = u \circ A$. $A \in CO$ so there exist $S \in SO$ and $\lambda \in \mathbb{R}^*$ such that $A = zS$. Then, for $X, Y \in T_x M$, one gets:

$$\begin{aligned} g'_x(X,Y) &= \eta(u'^{-1}_x(X), u'^{-1}_x(Y)) \\ &= \eta(z^{-1}S^{-1}u(X), z^{-1}S^{-1}u(Y)) \\ &= z^{-2}\eta(S^{-1}u(X), S^{-1}u(Y)) \\ &= z^{-2}\eta(u(X), u(Y)) \\ &= z^{-2}g_x(X,Y), \end{aligned}$$

since $S^{-1} \in SO$. It turns out that a CO-structure over M defines not only a metric over M, like a SO-structure does, but a whole set of metrics all related by a positive function, i.e. a conformal class of metrics.

Let us now see the reverse property. Let $(M, \mathbf{c})$ be a conformal structure on M. Let $L(M)$ be the 1–frame principal fibre bundle over M. Let us define $P \subset L(M)$:

$$P := \{u \in L(M),\ \exists\, g \in \mathbf{c},\ u^T g u = \eta\}, \tag{3.35}$$

that is to say the subset of linear frames which are orthonormal for one metric in the conformal class. For $u \in L(M)$, u_x is an isomorphism between T_xM and $\mathbb{R}^n$, in these terms u_x^T is to be understood as the transposed matrix representing the isomorphism in a certain basis.

Now, let us show that P defined above is a CO-bundle over M, and thus that the given conformal class defines a CO-structure. Let us take $u, u' \in P$ over a same point $x \in M$. One knows that there exists $g, g' \in \mathbf{c}$ such that $\eta = u^T g u = u'^T g' u'$, and $A \in GL(n)$ such that $u' = u \circ A$ and finally $\lambda \in \mathbb{R}^*$ such that $g' = z^2 g$. It is now easy to show that $A \in CO$: $\eta = u^T g u = z^2 u'^T g u' = z^2 A^T u^T g u A = z^2 A^T \eta A$, thus $A^T \eta A = z^{-2}\eta$ i.e. $A \in CO$.

The question that naturally arises now is to know whether one can build on this CO-structure a Cartan connection which could also be equivalent to the given conformal structure. In other words a Cartan connection which would be uniquely defined by the data of a conformal structure. The answer is negative: as explained next, a Cartan geometry can only uniquely define a Weyl structure on M, which is less general than a conformal one.

3.2.4.3 A Cartan geometry on P is equivalent to a Weyl structure on M

See e.g. Chapt. 7 of [50]. Let us recall that, on the one hand, a Weyl structure is a set $(M, \mathbf{c}, F)$ with $F : \mathbf{c} \to \Omega^1(M)$ such that $F(e^{2\lambda}g) = F(g) - 2d\lambda$. On the other hand, a Cartan connection ϖ on a CO-structure $P \to M$ constructed on the affine extension of CO, namely $CO \ltimes \mathbb{R}^{1,n-1}$ takes values in the corresponding Lie algebra: $\mathfrak{co} \oplus \mathbb{R}^{1,n-1}$. One will use a convenient matrix notation and write:

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & a \end{pmatrix}$$

with $\omega \in \Omega^1(P) \otimes \mathfrak{so}$, $\theta \in \Omega^1(P) \otimes \mathbb{R}^{1,n-1}$ and $a \in \Omega^1(P)$. Let us take a Weyl structure and show how it can define a unique corresponding Cartan connection on P, and then see the reverse property.

Let $(M, \mathbf{c}, F)$ be a Weyl structure. Let us place in a certain gauge $g \in \mathbf{c}$ and write:

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & -\frac{1}{2}F(g) \end{pmatrix}$$

with ω being the unique solution to $\Theta = 0$, $\Theta = d\theta + \omega \wedge \theta + \theta \wedge a$ being the torsion, and θ the canonical form corresponding to the chosen metric g. Then, one computes a gauge transformation and sees that the form of this connection is unaltered: $\tilde{\varpi} = \begin{pmatrix} \tilde{\omega} & \tilde{\theta} \\ 0 & \tilde{a} \end{pmatrix}$ with:

- $\tilde{\omega}$;
- $\tilde{\theta} = zA^{-1}\theta$;
- $\tilde{F} = F - 2z^{-1}dz$;

with $\begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix} : M \to CO$ the gauge transformation element. One notices that if one sets $\tilde{g} = z^2 g$, i.e. $\lambda : M \to \mathbb{R}$ such that $e^\lambda = z$, then $\tilde{F}$ is just $F(\tilde{g})$, and thus $\tilde{\varpi} = \begin{pmatrix} \tilde{\omega} & \tilde{\theta} \\ 0 & -\frac{1}{2}F(\tilde{g}) \end{pmatrix}$ remains with the same form. This connection ϖ is then uniquely defined by the Weyl structure: θ is defined by the choice $g \in \mathbf{c}$, a is defined by $F(g)$ and ω is unique for a given θ and a.

Now, let us take a torsionless Cartan connection ϖ on a CO-structure $P \to M$ with values in the Lie algebra of the affine extension of CO. Thus,

$$\varpi = \begin{pmatrix} \omega & \theta \\ 0 & a \end{pmatrix},$$

and the curvature

$$\overline{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & f \end{pmatrix}.$$

One knows that a CO-structure defines a conformal class of metrics on M. The choice of a $g \in \mathbf{c}$ corresponds uniquely to a certain θ. ω is the unique solution to $\Theta = 0$, for a given a. Then, let us set $F(g) = -2a$, with g the unique metric associated to θ. One claims that this defines a Weyl structure on M. Under a gauge transformation, one gets:

$$\tilde{\varpi} = \begin{pmatrix} \tilde{\omega} & zA^{-1}\theta \\ 0 & a + z^{-1}dz \end{pmatrix}$$

with the same notations as above. Since g corresponds to θ, $\tilde{g}$ corresponding to $zA^{-1}\theta$ is $\tilde{g} = z^2 g$, and thus $a + z^{-1}dz$ defines $\tilde{F} = -2a - 2z^{-1}dz$, i.e. exactly the good transformation to define a Weyl structure.

Moreover, the covariant derivative defined from this Cartan connection has the good property in regard with the original definition of Weyl. Indeed, $\Theta = 0 = d\theta + \omega\theta - \frac{1}{2}\theta F$, where the wedge product is omitted. Thus, calling e the tetrad corresponding to θ, one gets: $de + \omega e = \frac{1}{2}eF$. One has also $g = e^T \eta e$. The covariant derivative applying on g is defined as:

$$\begin{aligned}
\nabla g :&= dg - \omega^T g - g\omega \\
&= d(e^T \eta e) - \omega^T e^T \eta e - e^T \eta e \omega \\
&= (de^T)\eta e + e^T \eta de - \omega^T e^T \eta e - e^T \eta e \omega \\
&= (de^T - \omega^T e^T)\eta e + e^T \eta (de - e\omega),
\end{aligned}$$

but since all these objects are 1-forms, $\omega^T e^T = -e^T \omega^T$ and $e\omega = -\omega e$ and so:

$$\begin{aligned}
\nabla g &= \frac{1}{2}F e^T \eta e + \frac{1}{2}e^T \eta e F = \frac{1}{2}F e^T \eta e - \frac{1}{2}e^T \eta F e \\
&= \frac{1}{2}F e^T \eta e + \frac{1}{2}F e^T \eta e = Fg.
\end{aligned}$$

Thus, one recovers the original definition of a Weyl connection.

3.2.4.4 From a Weyl structure to a (pseudo-)Riemannian one

In the Cartan geometrical framework, the part of the curvature related to the gauge field a is $f = da$. A particular Weyl structure where there exists a positive scalar field ϕ such that $a = \mathrm{dln}(\phi)$ (or equivalently, if the manifold is connected and simply connected, where $f = 0$) is called by certain authors, e.g. in [47], a "Weyl Integrable Spacetime" (WIST). However, by applying the dressing field method on such a case, one shows that a WIST is nothing but a mere Riemannian manifold, written in some exotic field variables. The dressing allows to do the inverse procedure and to make the good change of variables to exhibit the Riemannian structure.

Let us assume that the smooth manifold M is connected and simply connected, and that $\varpi = \begin{pmatrix} \omega & \theta \\ 0 & a \end{pmatrix}$ is a Cartan connection on a CO–fibre bundle, written in a given gauge, such that $f = 0$. Then, there exists ϕ, a positive scalar field over M, such that $a = \mathrm{dln}(\phi)$. Let $\begin{pmatrix} \mathbb{I} & 0 \\ 0 & z \end{pmatrix}$ be a Weyl gauge transformation. Under this gauge transformation, a transforms as : $a \longmapsto \tilde{a} = a + \mathrm{ln}(z)$. $\tilde{f} = 0$ thus there exists another $\tilde{\phi}$, positive scalar field such that $\tilde{a} = \mathrm{d}\ln(\tilde{\phi})$. ϕ and $\tilde{\phi}$ are thus related by $\ln(\tilde{\phi}) = \ln(\phi) + ln(z)$, i.e. without loss of generality, $\tilde{\phi} = z\phi$. Thus, $u := \phi^{-1}$ is a dressing field for the Weyl rescaling. A straightforward computation shows that $\varpi^u = \begin{pmatrix} \omega^u & \theta^u \\ 0 & 0 \end{pmatrix}$ with $\omega^u = u^{-1}\omega u + u^{-1}du$ and $\theta^u = u\theta$:

- invariant under the action of $z \in \mathbb{R}$;
- genuine SO–gauge fields w.r.t. Lorentz transformations;
- ω^u is the solution of $d\theta^u + \omega^u \wedge \theta^u = 0$.

The last property is just the torsionless condition written in the new variables. Thus, ϖ^u is a torsionless Cartan connection over a SO–principal bundle, i.e. defines uniquely a Riemannian structure over the smooth manifold M by the Cartan equivalence problem 2.3.1.

3.2.4.5 Conclusion

In conclusion of this section, let us count degrees of freedom of such structures: a conformal class of metrics has $n(n+1)/2 - 1$ degrees of freedom: $n(n+1)/2$ for a metric which is symmetric, and -1 for all of them are related by a one-dimensional parameter.

As for a Weyl structure, one has $n(n+1)/2 + n - 1$ degrees of freedom: $+n$ for the choice of n components of the 1-form F.

A torsionless Cartan connection built on a CO-structure has $(n(n-1)/2 + n + 1)n - n(n-1)/2 - 1 - n(n-1)n/2 = n(n+1)/2 + n - 1$ degrees of freedom: $(n(n-1)/2 + n + 1)n$ for the $\mathfrak{so} \oplus \mathbb{R} \oplus \mathbb{R}^n$-valued 1-form ϖ, $-n(n-1)/2 - 1$ for the symmetry group is CO, and $-n(n-1)n/2$ for the condition $\Theta = 0$, which is a $\mathbb{R}^n$-valued 2-form. One notices directly that as one could expect it, a torsionless Cartan geometry on a CO–bundle and a Weyl structure on M have the same total degrees of freedom. In the case of a "WIST", i.e. when $a = \ln(\phi)$, one passes from n degrees of freedom (for a is a scalar-valued 1-form) to only 1 (for ϕ is a scalar field), so there are $(n-1)$ less degrees of freedom. Thus, the total number of degrees of freedom for such a structure is

$$n(n+1)/2 + n - 1 - (n-1) = n(n+1)/2$$

i.e. exactly that of a (pseudo)–riemannian metric, as expected.

The aim of the next section is now to find a Cartan geometry corresponding uniquely to a conformal structure on M. One sees that this structure must have exactly $n(n+1)/2 - 1$ degrees of freedom. The idea is to prolongate the CO-structure. One will get a connection with values in a bigger Lie algebra, i.e. with more degrees of freedom. Yet, in the same time, the Lie symmetry group as well will be bigger. Thus, one can hope that under a certain condition of the type $\Theta = 0$ one could get a structure corresponding exactly (i.e., in particular, with the same degrees of freedom) to a conformal structure.

3.2.5 2^{nd}–order conformal bundles: prolongation of a CO-structure

See e.g.: chapter I (in part. examples 2.5 and 2.6) of [30]; first lecture of [12]; abundantly inspired by chapter I, section 1 and above all 2 of [39].

3.2.5.1 First prolongation of G-structures

Let $\mathfrak{g}$ be the Lie algebra of $G \subset GL(n)$. One defines the following objects:

- $\mathfrak{g}_0 := \mathfrak{g}$;
- $\mathfrak{g}_1 := \{t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ symmetric, such that $\forall v \in \mathbb{R}^n$, $u \to t(u, v) \in \mathfrak{g}\}$.

$\mathfrak{g}_1$ is called the first prolongation of the Lie algebra $\mathfrak{g}$. Notice that one can naturally extend this definition to define the k^{th}-prolongation $\mathfrak{g}_k$, but it will not be presented here since in our case one is interested in CO-structures which turn out to admit only a first prolongation, as one shall see.

These prolongations then can be glued together to form a *graded* Lie algebra:

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k \tag{3.36}$$

where one defined $\mathfrak{g}_{-1} := \mathbb{R}^n$, and one recalls that $\mathfrak{g}_0 = \mathfrak{g}$. *Graded* means that for $a \in \mathfrak{g}_p$ and $b \in \mathfrak{g}_q$, $[a, b] \in \mathfrak{g}_{p+q}$, with $[a, b] = 0$ if $p + q < -1$ or $p + q > k$, the Lie bracket being defined as usual (see [30]).

Given the first prolongation $\mathfrak{g}_1$ of $\mathfrak{g}$, one defines the first prolongation G_1 of the Lie group G as the group of linear transformations $\bar{t} \in GL(\mathbb{R}^{1,n-1} \oplus \mathfrak{g})$ induced by the elements $t \in \mathfrak{g}_1$ as follows:

$$\bar{t}(v) = v + t(., v) \text{ for } v \in \mathbb{R}^{1,n-1}$$
$$\bar{t}(x) = x \text{ for } x \in \mathfrak{g}$$

One symbolically represents $\bar{t}$ by the matrix:

$$\bar{t} = \begin{pmatrix} \mathbb{I}_n & 0 \\ t & \mathbb{I}_r \end{pmatrix}$$

where $r = dim(\mathfrak{g})$.

Now one sees how to construct the prolongation of the G-bundle $P \to M$ as a G_1-bundle over P.

3.2.5.2 Prolongation of a G-structure over M

The construction presented here can be found in a much precise form in [39] and [30]. It is still a bit detailed here for a better understanding of the final entities one gets. As for the application to our case of a CO-structure, just main results are given, the detailed construction being a bit too heavy for our purpose.

By construction, such a prolongation can be viewed as a reduction of the 2-frames bundle $P^2(M) \to M$. After having presented, in the following section, the $\mathbb{R}^{n,2}$-representation of the Klein pair of conformal geometry, one will present the construction of the conformal 2-frames bundle, and will see that the both description correspond.

3.2.5.3 Prolongation of $\mathfrak{so}(1, n-1)$

It is useful to start with the simpler case of $SO(1, n-1)$. The latter can be represented by $n \times n$-matrices $A \in GL(n)$ of determinant 1, such that $A^T \eta A = \eta$, η being the n-dimensional Minkowski metric. It is not difficult to show that the corresponding Lie algebra is made of $n \times n$-skew-symmetric matrices: $\mathfrak{so}(1, n-1) = \{a \in \mathfrak{gl}(n), a_{ij} + a_{ji} = 0\}$. Now, let us show that $SO(1, n-1)$ has a vanishing first prolongation (and so vanishing k^{th}-prolongation for any $k \neq 0$). These calculations can be found e.g. in [30], chapter I, or in [12], lecture 1.

Let $\mathfrak{g}_1$ be the first prolongation of $\mathfrak{so}(1, n-1)$, and $a \in \mathfrak{g}_1$. Then, a is a bilinear symmetric map with values in $\mathbb{R}^n$. Let us call ${a^i}_{jk}$ its components. By definition, one has: ${a^i}_{jk} = {a^i}_{kj}$, and for a given k, ${a^i}_{jk} \in \mathfrak{g}_0 = \mathfrak{so}(1, n-1)$, i.e. ${a^i}_{jk} = -{a^j}_{ik}$. Thus:

$$ {a^i}_{jk} = -{a^j}_{ik} = -{a^j}_{ki} = {a^k}_{ji} = {a^k}_{ij} = -{a^i}_{kj} = -{a^i}_{jk} $$

That is, ${a^i}_{jk} = 0$.

3.2.5.4 Prolongation of $\mathfrak{co}(1, n-1)$

The Lie algebra $\mathfrak{co}(1, n-1)$ of the Weyl group is given by the set of matrices $a \in \mathfrak{gl}(n)$ such that $a_{ij} + a_{ji} = \epsilon\eta$, with $\epsilon \in \mathbb{R}$. (This can be computed directy by demanding that the matrix $\mathbb{I}_n + a$ be in CO.) Let $\mathfrak{g}_1$ be the first prolongation of $\mathfrak{co}(1, n-1)$, and $a \in \mathfrak{g}_1$. One has ${a^i}_{jk} = {a^i}_{kj}$ by definition, and also that for a given k, ${a^i}_{jk} \in \mathfrak{co}(1, n-1)$. Then, let us define:

$$ \phi : \mathfrak{co}(1, n-1) \to \mathbb{R}, a \to tr(a) $$

$a \in Ker\phi \iff a^T + a = \epsilon\eta$ and $tr(a) = 0 \iff a^T + a = 0$ i.e. $Ker\phi = \mathfrak{so}(1, n-1)$. Then let us define the homomorphism:

$$ \psi : \mathfrak{g}_1 \to \mathbb{R}^{1,n-1*}, a = ({a^i}_{jk}) \to c_k = \frac{1}{n}{a^i}_{ik} $$

Let us show that it is one-to-one. If $a \in Ker\psi$, then for any k, $tr(a) = a^i{}_{ik} = 0$ i.e. $a^i{}_{jk} \in Ker\phi = \mathfrak{so}(1, n-1)$ for any k. In the same time, $a^i{}_{jk} = a^i{}_{kj}$. Thus, it turns out that $a^i{}_{jk}$ is in the first prolongation of $\mathfrak{so}(1, n-1)$, i.e. is equal to 0. That is, $Ker\psi = \{0\}$.

Now, let us consider $c = (c_k) \in \mathbb{R}^{1,n-1*}$,[i] and define:

$$a^i{}_{jk} := \delta^i{}_j c_k + \delta^i{}_k c_j - \eta_{jk}\eta^{im} c_m$$

and one gets $c = \psi(a)$. Thus, ψ is an isomorphism and one just found that the 1^{st}-prolongation of $\mathfrak{co}(1, n-1)$ is nothing but $\mathbb{R}^{1,n-1*}$. It can be shown in a similar manner that the 2^{nd} and more prolongations are all reduced to zero.

3.2.5.5 First prolongation of a CO-structure

A conformal structure over a manifold M defines uniquely a CO-structure $P \to M$. According to the construction above, it turns out that the first prolongation of P is unique, in the sense that in this particular case there is only one way to prolongate P as a $G \times G_1$–bundle $P_1 \to M$. Thus, one has just proven the following important result:

There is a one-to-one correspondence between conformal structures over M and $G \times G_1$–bundle over M, i.e. $G \times G_1 - 2^{nd}$–order structure over M.

Thus it is natural now, since it was not possible on the original CO–structure, to try to build a certain Cartan connection on the $G \times G_1$–bundle which would uniquely correspond to the conformal structure.

3.2.6 The Klein pair (G, H) of conformal geometry and its homogeneous space M_0

See e.g.: section 2 of [26]; Chapter 2, part 2.2 of [6]; [50].

One now presents the Klein pair (G, H) of conformal (or Moebius) geometry, and the corresponding homogeneous model on which they act, and one will then show what is the correspondence with the first prolongation presented above.

3.2.6.1 Defining-representation of G on $\mathbb{R}^{n,2}$

Let $\mathbb{R}^{n,2}$ be the $n+2$–dimensional Minkowski space, endowed with the metric:

$$\Sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[i] The star here denotes the dual space.

According to Σ's form, an element $X \in \mathbb{R}^{n,2}$ will be written: $X = \begin{pmatrix} a \\ x \\ b \end{pmatrix}$, with $x \in \mathbb{R}^{1,n-1}$ and $a, b \in \mathbb{R}$.

Let us set:

$$O(n,2) := \{A \in GL_{n+2}(\mathbb{R}), A^T \Sigma A = \Sigma\},$$

the orthogonal group of this $n+2-$Minkowski space. $O(n,2)$ preserves the metric Σ, thus it preserves the set of null (or isotropic) vectors of $\mathbb{R}^{n,2}$, the $n+2-$lightcone $N = \{X \in \mathbb{R}^{n,2}, X^T \Sigma X = 0\}$. The action of $O(n,2)$ on N is transitive.

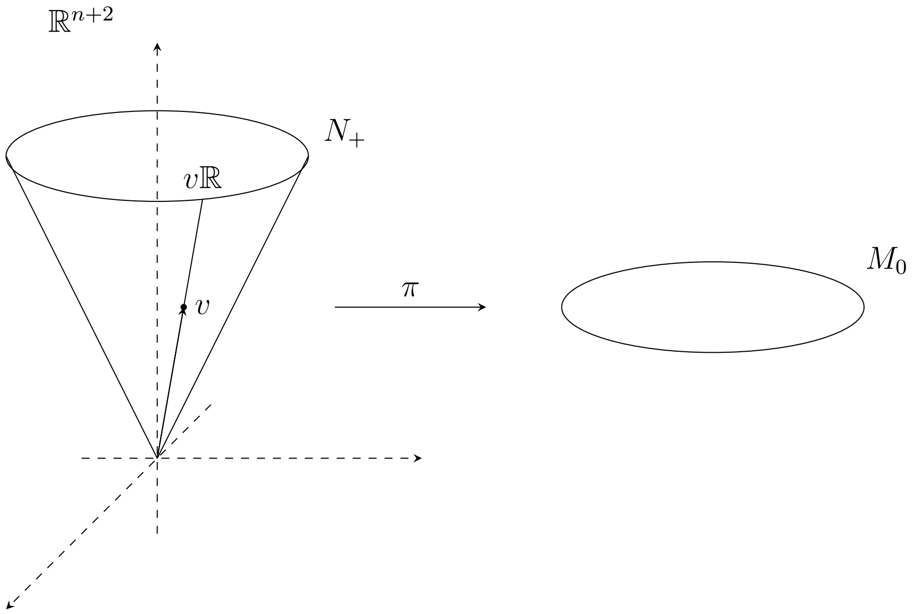


Figure 3.1: Projective Space of the $(\mathbb{R}^{n+2}, \Sigma)$–Lightcone

Now, let us define M_0 as the projective space of N, that is to say the space resulting of the identification, in N, of any two $n+2-$vectors $X, Y \in \mathbb{R}^{n,2}$ such that $Y = \lambda X$ for $\lambda \neq 0$. More precisely, defining the following equivalence relation between any two vectors of $\mathbb{R}^{n,2}$:

$$X \sim Y \text{ iif } Y = \lambda X, \lambda \neq 0, \tag{3.37}$$

then

$$M_0 := N/\sim.$$

One call N_+ the set of lines generated by the positive classes, i.e. that for which $\lambda > 0$. See figure ?? for a scheme of the process of projection. Since

$O(n,2)$ acts transitively on N, thus it does so on M_0 too. However, the kernel of its action ϕ is not reduced to $\mathbb{I}_{n+2}$, but equals $\{\pm\mathbb{I}_{n+2}\}$ (demonstration below), so the action is not effective. To see it more concretely, one can notice that the action of $O(n,2)$ on $X \in N$ is not the same as on $-X$, whereas once compactified, both these vectors are identified and so is the induced action. So, an element of the induced action corresponds to two elements of $O(n,2)$.

$Ker\phi = \{\pm\mathbb{I}_{n+2}\}$ See chapt. 7 of [50]. It is easy to see that $\pm\mathbb{I}_{n+2} \in Ker\phi$. Let us now take $A \in Ker\phi$: for all $x \in M_0, A.x = x$. That means that for all $x \in M_0$, there exists a vector $X_x \in N$ and a non-null real λ_x such that $AX_x = \lambda_x X_x$. But since the spectrum of A is necessary finite and discrete, λ_x cannot depend on $x \in M_0$. Thus, for all $v \in N$, $Av = \lambda v$. By testing this relation with vectors such that $e_0, e_{n+1}, e_i + \frac{1}{\sqrt{2}}(e_0 + e_{n+1})$, one finally finds that $A = \lambda\mathbb{I}_{n+2}$. Introducing now this in the defining relation of $O(n,2) : A^T\Sigma A = \Sigma$, one finds that $A = \pm\mathbb{I}_{n+2}$.

To fix this problem, one quotients $O(n,2)$ by the kernel, and then defines:

$$G := O(n,2)/\{\pm\mathbb{I}_{n+2}\}. \tag{3.38}$$

G, which is thus $1:2-$homomorphic to $O(n,2)$, is called the Möbius group, and it is the principal symmetry group (in Klein's terms) of conformal geometry. One has claimed two facts that remain to be shown:

- M_0 is the conformally compactified Minkowski space-time $\mathbb{R}^{1,n-1}$, naturally endowed with a conformal class of metrics $\mathbf{c}_0$;
- G is the group of transformations of M_0 which preserves its conformal structure: $G = C_0(M_0, \mathbf{c}_0)$.

Let us see that in detail. One will also show that $M_0 \simeq S^1 \times S^{n-1}/\mathbb{Z}_2$.

M_0 is the conformal compactification of $\mathbb{R}^{1,n-1}$ Let us set:

$$f : \mathbb{R}^{1,n-1} \ni x \to < X > = < \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \end{pmatrix} > \in M_0.$$

$< X >$ means: the direction in $\mathbb{R}^{n,2}$ spanned by $X \in \mathbb{R}^{n,2}$, or equivalently the class of X in M_0 with respect to the relation $\sim$ (3.37). f is a diffeomorphism and its inverse is well-defined:

$$f^{-1} : M_0 \ni \begin{pmatrix} X_0 \\ ... \\ X_{n+1} \end{pmatrix} \to \begin{pmatrix} \frac{X_1}{X_0} \\ ... \\ \frac{X_n}{X_0} \end{pmatrix} \in \mathbb{R}^{1,n-1}.$$

This map realises a conformal compactification of Minkowski space-time, see e.g. [32] or [29]. What does it mean? In this case, the compactification can be seen as the inverse of the stereographic projection to which is added a point at infinity. A good way to see that is to think of the example of the compactification of the plane $\mathbb{R}^2$ into the 2-sphere S^2: the northpole of the latter corresponds to the "point at infinity" of the plane.

$M_0 \simeq S^1 \times S^{n-1}/\mathbb{Z}_2$ Let us now show that $M_0 \simeq S^1 \times S^{n-1}/\mathbb{Z}_2$. It is easier to see that in $\mathbb{R}^{n,2}$ endowed with the metric:

$$\Sigma = \begin{pmatrix} I_n & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now one sets:

$$\phi : S^1 \times S^{n-1} \ni ((y_{n+1}, y_{n+2}), y) \to < (y_{n+1}, y_{n+2}, y) > \in M_0$$

where (y_{n+1}, y_{n+2}, y) is considered as a vector of the lightcone N_+ spanning a certain equivalence class.

Take an element $< (y_{n+1}, y_{n+2}, y) >$ of M_0, and then a representative (y_{n+1}, y_{n+2}, y). The latter verifies $y^2 = y_{n+1}^2 + y_{n+2}^2 \neq 0$. Let $h_y \in \mathbb{R}^*$ be such that $h_y^2 = y^2$ and one sets

$$X = (x_{n+1}, x_{n+2}, x) = (y_{n+1}/h_y, y_{n+2}/h_y, y/h_y).$$

$X \in S^1 \times S^{n-1}$ and it is the antecedent of $< (y_{n+1}, y_{n+2}, y) >$ by ϕ. This map is thus surjective. However, both the elements (y_{n+1}, y_{n+2}, y) and $(-y_{n+1}, -y_{n+2}, -y))$, distinct in $S^1 \times S^{n-1}$, are sent to the same image in M_0:

$$< (y_{n+1}, y_{n+2}, y) > = < (-y_{n+1}, -y_{n+2}, -y) > .$$

The map is thus not one-to-one yet. To render it so, one has, in $S^{n-1} \times S^1$, to indentify every pair $((y_{n+1}, y_{n+2}, y), (-y_{n+1}, -y_{n+2}, -y))$, i.e. to quotient by $\mathbb{Z}_2$. Once one has identified each double antecedent, ϕ is then one-to-one and the identification $M_0 \simeq S^1 \times S^{n-1}/\mathbb{Z}_2$ has a sense.

M_0 is naturally endowed with a conformal class of metrics One shows now how M_0, being the projective space of the lightcone of $\mathbb{R}^{n,2}$, is naturally endowed with a conformal class of metrics. By *naturally* one means *by construction*. Let q be the bilinear symmetric form in $\mathbb{R}^{n,2}$ corresponding to Σ. Then, the lightcone N_+ is made of $X \in \mathbb{R}^{n,2}$ such that $q(X, X) = 0$, by definition. Let $v \in N_+$, and $x \in T_v N_+$. Let us show that necessarily $q(x, v) = 0$. Indeed, let $\gamma(t)$ be a curve on N_+ such that $\gamma(0) = v$, $\dot{\gamma}(0) = x$. Then, for all t, $\gamma(t) \in N_+$ so $q(\gamma(t), \gamma(t)) = 0$.

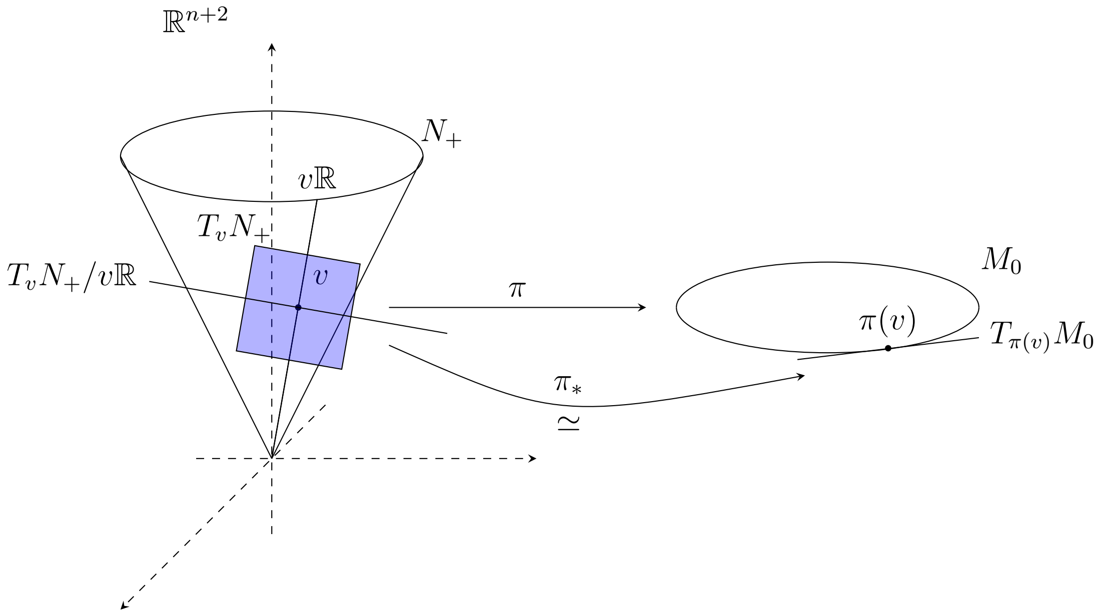


Figure 3.2: Conformal class of metrics on the projective space

Taking the derivative of this latter expression and evaluating at $t = 0$ gives the result.

Let us call π the projection $N_+ \to M_0$. In order to define a metric on $T_{\pi(v)}M_0$, one starts defining a bilinear form on the tangeant plane T_vN_+ and then use a certain isomorphism between one of its subspaces and $T_{\pi(v)}M_0$. One would like to define q_v on T_vN_+. Let set, for $x, y \in T_vN_+$: $q_v(x, y) := q(x + v, y + v)$. Indeed, $x \in T_vN_+$ so $x + v \in T_0N_+$, which can be seen as a subspace of $\mathbb{R}^{n,2}$,[j] on which q is thus well-defined. But thanks to the fact that $q(x, v) = q(v, y) = q(v, v) = 0$, at the end it does not matter and one can write $q_v(x, y) = q(x, y)$ without any ambiguity. The quotient space $T_vN_+/v\mathbb{R}$, in which any two vectors x and $x + \lambda v$ are identified, is important for two reasons. First, $ker(\pi) = v\mathbb{R}$ which allows to define correctly $\overline{q_v}$ on $T_vN_+/v\mathbb{R}$. Indeed, for $\overline{x}, \overline{y} \in T_vN_+/v\mathbb{R}$, the definition: $\overline{q_v}(\overline{x}, \overline{y}) := q_v(x, y)$ does not depend on the representant x and y in T_vN_+, thanks also to $q(x, v) = q(v, y) = 0$. Second, $ker(\pi_*) = v\mathbb{R}$, so $\pi_* : T_vN_+ \to T_{\pi(v)}M_0$ is not an isomorphism. One can render it one-to-one by composing it with the projection $T_vN_+ \to T_vN_+/v\mathbb{R}$ (one still call it π_* even if it is not really rigorous): $\pi_* : T_vN_+/v\mathbb{R} \to T_{\pi(v)}M_0$ thus defines an isomorphism.

Now one is ready to define a metric on $T_{\pi(v)}M_0$, and one will see that the definition depends, over a certain $x_0 \in M_0$, on the point v choosen in N_+ such that $x_0 = \pi(v)$, and thus on can only define a *class* of metric, all related by a positive factor. Let $x_0 \in M_0$, and $v \in N_+$ such that $x_0 = \pi(v)$. For $a, b \in T_{x_0}M_0$, one defines $\phi_v(a, b) := q_v(x, y)$ with $x, y \in T_vN_+$ such that $a = \pi_*(x), b = \pi_*(y)$,

[j] Let us precise here that we use implicitly the fact that the tangent bundle to $\mathbb{R}^{n,2}$ is trivial and thus that the tangent space to $\mathbb{R}^{n,2}$ at any point is isomorphic to $\mathbb{R}^{n,2}$ itself.

which is well-defined since π_* is an isomorphism between $T_vN_+/v\mathbb{R}$ and $T_{x_0}M_0$ and q_v does not depend on the representant choosen.

Take a and $x \in T_vN_+$ as above. For $\lambda > 0$, one will consider $\tilde{x} \in T_{\lambda v}N_+$ which is also sent to a via π_*, defined as follows: let γ be a curve on N_+ such that $\gamma(0) = v$ and $\dot{\gamma}\ (0) = x$. Then, let us define $\tilde{\gamma}$, another curve on N_+, such that for all t: $\tilde{\gamma}(t) = \lambda\gamma(t)$. Then, one naturally gets that $\tilde{\gamma}(0) = \lambda v$, and $T_{\lambda v}N_+ \ni \tilde{x} :=\dot{\tilde{\gamma}}\ (0) = \lambda x$. By construction, $\pi_*(\tilde{x}) = a$. Indeed,

$$\begin{aligned}\pi_*(\tilde{x}) &= \frac{d}{dt}|_{t=0}\pi(\tilde{\gamma}(t)) = \frac{d}{dt}|_{t=0}\pi(\lambda\gamma(t)) \\ &= \frac{d}{dt}|_{t=0}\pi(\gamma(t)) = \pi_*(x) = a\end{aligned}$$

$\tilde{x} = \lambda x$ defined this way is a particular element of $T_{\lambda v}N_+$ which is sent to a. But actually, since q_v does not make any difference between two elements of a class in $T_vN_+/v\mathbb{R}$, one can always choose this particular element to be sent to $a \in T_{x_0}M_0$.

However, this element $\tilde{x} \in T_{\lambda v}N_+$ then defines, via the isomorphism

$$T_{\lambda v}N_+/v\mathbb{R} \to T_{x_0}M_0,$$

another $\phi_{\lambda v}$ on $T_{x_0}M_0 \times T_{x_0}M_0$:

$$\begin{aligned}\phi_{\lambda v}(a,b) &= q_{\lambda v}(\tilde{x},\tilde{y}) \\ &= q_{\lambda v}(\lambda x, \lambda y) \\ &= \lambda^2 q_v(x,y) \\ &= \lambda^2\phi_v(a,b).\end{aligned}$$

One just has shown that indeed, the lorentzian ambient metric in $\mathbb{R}^{n,2}$ allows to define, on the projective space M_0, only a class of metrics, all related by a positive factor, i.e. a *conformal class of metrics*, denoted $\mathbf{c}_0$.

$G = C_0(M_0, \mathbf{c}_0)$ One has just seen that M_0 is naturally endowed with a conformal class of metrics: above each point $x \in M_0$, one can define a metric ϕ_v which depends on $v \in N$, where $\pi(v) = x$, and is such that $\phi_{\lambda v} = \lambda^2\phi_v$. Thus, one can consider that the lightcone N represents this conformal class of metric $\mathbf{c}_0$, in the sense that to each metric $g \in \mathbf{c}_0$ can be associated a unique map $M_0 \ni x \to v(x) \in N$, such that $g_x = \phi_{v(x)}$. By construction, the group G preserves N, thus, viewed as acting on M_0, it preserves as well the conformal class of metric $\mathbf{c}_0$.

3.2.6.2 The stabilizer H

From a Klein viewpoint, as it was already mentioned, describing an homogeneous space as M_0 is equivalent to studying two fundamental related groups: its principal Lie group of symmetry G, and a closed subgroup $H \subset G$ which stabilizes a point of M_0. In our case, since every point of M_0 is realised as a projection of a whole line over it, stabilizing a point is equivalent to stabilize the corresponding line. In this way one will get a representation of H on $\mathbb{R}^{n,2}$. Let us take the point (or line) $e_0 = \begin{pmatrix} 1 \\ \dots \\ 0 \\ 0 \end{pmatrix} \in N_+$, for it will be easier with it. One take $h \in H$ and write

$$h = \begin{pmatrix} a & D^T & b \\ B & A & C \\ c & E^T & d \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$, A $n \times n$–matrix and $B, C, D \in \mathbb{R}^n$ seen as n-vectors. he_0 has to be in the same class as e_0 itself, that is: $he_0 = \alpha e_0$ with $\alpha \neq 0$. That implies: $a \neq 0$, $B = 0$ and $c = 0$. Then, $h^T \Sigma h = \Sigma$, since H is a subgroup of G:

$$\begin{pmatrix} a & 0 & 0 \\ D^T & A^T & E \\ b & C^T & d \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & D^T & b \\ 0 & A & C \\ 0 & E^T & d \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

After some basic calculations, that leads to: $E = 0$, $A^T \eta A = \eta$, $b = \frac{C^T \eta C}{2d}$, $a = d^{-1}$ and $C = \frac{A^T \eta C}{d}$. As earlier, one denotes $C \cdot B := C^T \eta B = B^T \eta C$ the pseudo-scalar product between two n-vectors in Minkowski space-time, and thus $B^2 := B \cdot B = B^T \eta B$. Finally, any element h of the stabilizer H takes the form:

$$h = \begin{pmatrix} \lambda & \lambda c^T \eta \Lambda & \lambda \frac{c^2}{2} \\ 0 & \Lambda & c \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{R}^*, \Lambda \in SO(n-1,1), c \in \mathbb{R}^{n-1,1}$. Now on one shall use only these notations, so just forget about the former A,B,C,c, ... Notice that replacing c by Λc does not change anything, so one could use both representations.

3.2.6.3 The complex representation of G and H in dimension 4

In the framework of General Relativity, i.e. of Poincaré geometry, the accidental morphism in dimension 4 between $SO(1,3)$ and $SL_2(\mathbb{C})$ allows to get a spinorial description of GR. The same thing occurs here in conformal symmetry: there is an 1:2 homomorphism between $SO(2,4)$ and $SU(2,2)$, and thus a 1:4 homomorphism between the conformal group G and $SU(2,2)$, which allows to get the so–called twistorial representation.

Let us recall that $SU(2,2) = \{M \in GL_4(\mathbb{C}), M^*\Sigma M = \Sigma\}$, where $\Sigma = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$. This group plays the role of the principal symmetry group in the complex representation of the conformal Klein pair. To find the corresponding structure group $\overline{H}$, let us first recall the spinorial representation, i.e. the complex representation of the Lorentz group. The action of the Lorentz group $SO(1,3)$ on $M_0 = \mathbb{R}^{1,3}$ can be represented as the action of $SL_2(\mathbb{C})$ on the space of hermitian matrices of dimension 2 $H_2(\mathbb{C})$. Indeed, if $\{\sigma_\mu\}_{\mu=0..3}$ is a basis of $H_2(\mathbb{C})$, one has the isomorphism: $\mathbb{R}^{1,3} \ni x \longmapsto \overline{x} = \frac{1}{2}x^\mu\sigma_\mu$. If one chooses the Pauli matrices as basis, one gets

$$\overline{x} = \frac{1}{2}\begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix},$$

as presented before. For $S \in SO(1,3)$, one denotes $\overline{S} \in SL_2(\mathbb{C})$ such that $\overline{Sx} = \overline{S}\overline{x}$. The fact that S preserves the Minkowski metric is reflected in the fact that the corresponding $\overline{S}$ is in $SL_2(\mathbb{C})$, i.e. is of determinant 1, because for $x \in \mathbb{R}^{1,3}$, $x^2 = \det(\overline{x})$. Based on these notations, one can now define the homomorphism $H = K_0K_1 \to \overline{H} = \overline{K}_0\overline{K}_1$ respecting the (polar) decomposition of H by:

$$\overline{H} = \left\{ \begin{pmatrix} z^{1/2}\overline{S}^{-1*} & 0 \\ 0 & z^{-1/2}\overline{S} \end{pmatrix} \begin{pmatrix} \mathbb{I}_2 & -i\overline{r} \\ 0 & \mathbb{I}_2 \end{pmatrix} \middle| \; z \in \mathbb{R}^{+*}, \overline{S} \in SL_2(\mathbb{C}), \overline{r} \in H_2(\mathbb{C}) \right\}$$

One can easily check that $\overline{H}$ is indeed a subgroup of $SU(2,2)$.

3.2.6.4 Conformal transformations of Minkowski space-time $\mathbb{R}^{1,n-1}$

One uses again the map:

$$f : \mathbb{R}^{1,n-1} \ni x \to < X > = < \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \end{pmatrix} > \in M_0.$$

and more precisely its inverse:

$$f^{-1} : M_0 \ni \begin{pmatrix} X_0 \\ ... \\ X_{n+1} \end{pmatrix} \to \begin{pmatrix} \frac{X_1}{X_0} \\ ... \\ \frac{X_n}{X_0} \end{pmatrix} \in \mathbb{R}^{1,n-1}.$$

to see how H acts on the Minkowski space-time $\mathbb{R}^{1,n-1}$. Let us see, first, the explicit action of H on the homogeneous space M_0. For a $x \in \mathbb{R}^{1,n-1}$, one takes the corresponding $X = \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \end{pmatrix}$ and on this latter makes an element $h \in H$ acting:

$$hX = \begin{pmatrix} \lambda & \lambda c^T \eta \Lambda & \lambda \frac{c^2}{2} \\ 0 & \Lambda & c \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \end{pmatrix} = \begin{pmatrix} \lambda + \lambda c^T \eta \Lambda x + \frac{1}{4}\lambda c^2 x^2 \\ \Lambda x + \frac{1}{2} c x^2 \\ \lambda^{-1} \frac{x^2}{2} \end{pmatrix}$$

For $\lambda + \lambda c^T \eta \Lambda x + \frac{1}{4}\lambda c^2 x^2 \neq 0$ one can express the result in $\mathbb{R}^{1,n-1}$ with the inverse f^{-1} and one gets:

$$x \to \frac{\Lambda x + \frac{1}{2} c x^2}{\lambda + \lambda c^T \eta \Lambda x + \frac{1}{4}\lambda c^2 x^2}$$

Now, let us precise three particular cases, viewed as diffeomorphisms of n-dimensional Minkowski space-time:

- $\underline{\lambda = 1, c = 0}$. One has: $\mathrm{x}^\mu \to \Lambda^\mu{}_\nu \mathrm{x}^\nu$, Lorentz transformations;
- $\underline{\Lambda = \mathbb{I}_n, c = 0}$. One gets: $x^\mu \to \lambda x^\mu$, called dilations;
- $\underline{\Lambda = \mathbb{I}_n, \lambda = 1}$. Providing $c \to 2c$ (which does not change anything to the underlying group structure), one gets: $x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + c^2 x^2}$, called special conformal transformations (SCT).

Thus, acting on $\mathbb{R}^{1,n-1}$, H represents the group:

$$Lorentz \times Dilations \times SCT.$$

Adding translations of $\mathbb{R}^{1,n-1}$ to these three sets of transformations, one actually gets the group of conformal transformations of n–dimensional Minkowski space-time: $C_0(\mathbb{R}^{1,n-1}, \eta)$, with $\eta = diag(-1, 1, 1, ..., 1)$ (as it is presented e.g. in [26]). This is another way of deriving the conformal transformations of Minkowski spacetime, which one just sketches the proof: to characterize $C_0(\mathbb{R}^{1,n-1}, \eta)$, one is looking for diffeomorphisms $\phi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$ such that the target Minkowski space-time (still flat in the Riemannian sense) is equipped with a metric $f\eta$, with $f \in \mathcal{C}^\infty(M, \mathbb{R}^{+*})$. $(\mathbb{R}^{1,n-1}, f\eta)$ has to be both flat and conformally flat. Thus, its Riemann tensor together with its Weyl tensor vanish. These conditions lead to equations which then allow to characterize exactly the conformal transformations of n-dimensional Minkowski space-time as those presented above.

As one just has noticed, special conformal transformations are not well-defined on the whole $\mathbb{R}^{1,n-1}$, and for that the latter cannot be the homogeneous model of conformal geometry, and one cannot simply set "$G = \mathbb{R}^{1,n-1} \times Lorentz \times Dilations \times SCT$". The process of compactification has the property to render this map regular everywhere on the compact, that is to say on M_0.

3.2.6.5 Relation with the prolongation of CO–structures

See in part. [50]. H can also be written as a direct product $H = K_0 \times K_1$, with:

$$K_0 = \{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \}$$
$$K_1 = \{ \begin{pmatrix} 1 & c^T\eta & \frac{1}{2}c^2 \\ 0 & \mathbb{I}_n & c \\ 0 & 0 & 1 \end{pmatrix} \}$$

K_0 obviously represents CO. Let us show that K_1 actually represents the first prolongation of CO. At the Lie algebra's level, one has found $\mathfrak{g}_1 = \{a^i{}_{jk} := \delta^i{}_j c_k + \delta^i{}_k c_j - \eta_{jk}\eta^{im}c_m, c \in \mathbb{R}^{1,n-1*}\}$. These objects define an action which can be written, at first (non trivial) order:

$$x^i \to x^i + a^i{}_{jk}x^j x^k = x^i - 2(c.x)x^i + c^i x^2.$$

One notices that this is exactly the infinitesimal action of special conformal transformations as written above; indeed:

$$x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + c^2 x^2},$$

at first order in c, gives back the same transformation law. So both correspond at the Lie algebra's level. Now, let us construct a finite action generated by an $a^i{}_{jk}$. For that, and in order to compare the result with elements of K_1, let us first represent the action of $a^i{}_{jk}$ in $\mathbb{R}^{n,2}$. That is, one wants to find a $n+2 \times n+2$-matrix C such that $x \to f^{-1}((\mathbb{I}_{n+2} + C)X)$ be exactly the transformation law above. $C := \begin{pmatrix} 0 & c^T\eta & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ works. Indeed, for $X = \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \end{pmatrix}$, $(\mathbb{I}_{n+2} + C)X = \begin{pmatrix} 1 + c.x \\ x^i + c^i x^2/2 \\ \frac{x^2}{2} \end{pmatrix}$ which gives, in $\mathbb{R}^{1,n-1}$, $\frac{x^i + c^i x^2/2}{1 + c.x} \simeq_{1^{st} order} x^i + c^i x^2/2 - (c.x)x^i$. Thus, the Lie algebra $\mathfrak{g}_1$ can be represented, in $\mathbb{R}^{n,2}$, by the set of matrices $\begin{pmatrix} 0 & c^T\eta & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Now, to find the corresponding Lie group, let us exponentiate any element of the Lie algebra, taken with this form. For $C = \begin{pmatrix} 0 & c^T\eta & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, one notices that for $p > 2$, $C^p = 0$. Thus, $exp(C) = \mathbb{I}_{n+2} + C + C^2/2 = \begin{pmatrix} 1 & c^T\eta & c^2/2 \\ 0 & \mathbb{I}_n & c \\ 0 & 0 & 1 \end{pmatrix}$, i.e., one recovers elements of K_1. Thus, the Lie group corresponding to the first prolongation of the Lie algebra $\mathfrak{co}$ identifies with K_1.

Lie$G = \mathbb{R}^{1,n-1} \oplus \mathfrak{co}(1, n-1) \oplus \mathbb{R}^{1,n-1*}$ Let us show that G defined like that has got the expected graded Lie algebra $\mathfrak{g} = \mathbb{R}^{1,n-1} \oplus \mathfrak{co}(1, n-1) \oplus \mathbb{R}^{1,n-1*}$. For $M \in G$, $M^T \Sigma M = \Sigma$ so if one sets $M = \mathbb{I} + m$ with $m \in$ Lie$G$, m has to satisfy: $m^T \Sigma + \Sigma m = 0$. Writing explicitly m as a $(n+2) \times (n+2)$ matrix, the latter relation gives constraints on the components of m. This is a straightforward calculation and finally one gets that m is as follows:

$$m = \begin{pmatrix} a & \alpha & 0 \\ \theta & \omega & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}$$

with $\theta \in \mathbb{R}^{1,n-1}$, $a \in \mathbb{R}^*$, $\omega \in \mathfrak{so}(1, n-1)$, and $\alpha \in \mathbb{R}^{1,n-1*}$ (in the sense that it is written as a line matrix), and the so-called η-transposition ^{t} is defined by: $\theta^t := \theta^T \eta$ for a vector, $\alpha^t := \eta \alpha^T$ for a covector, and $\omega^t := \eta \omega^T \eta$ for a $n \times n$-matrix.

Such a matrix can then be seen as the following sum:

$$m = \begin{pmatrix} 0 & 0 & 0 \\ \theta & 0 & 0 \\ 0 & \theta^t & 0 \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \alpha^t \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, Lie G can be written as the direct sum $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the natural identification: $\mathfrak{g}_{-1} \cong \mathbb{R}^{1,n-1}, \mathfrak{g}_0 \cong \mathfrak{co}(1, n-1), \mathfrak{g}_1 \cong \mathbb{R}^{1,n-1*}$. One has obviously $\mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{h}$. One recognizes elements of $\mathfrak{g}_1$, in this representation, as the ones already found in the previous paragraph.

As for the graded aspect of $\mathfrak{g}$, it suffices to takes two matrices from different (or not) sectors and to do the commutator, and realise that the result belongs to the expected sector.

3.2.7 Conformal 2-frame bundle as a reduction of the 2–frame bundle

See e.g.: section 5 of chapter IV of [30]; section 3 of [26]. Let us present a last way of seeing the conformal 2–frame bundle, which consists of directly reducing the 2–frame bundle with a certain group, which turns out to be H itself, as we will show.

3.2.7.1 The fibre bundle of 1-frames and 2-frames

Let us define, one more time, the 1–frame bundle. Let M be a smooth manifold and $\{x^\mu\}$ a local coordinates system defined over an given open subset of M. Then, in any point $x \in M$, the set $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$ is a natural basis of the tangent plane $T_x M$, also called holonomic frame. Under a change of coordinates $x^\mu \to x'^\mu$, e_μ transforms as:

$$e'_\mu = e_\nu {a^\nu}_\mu$$

with ${a^\nu}_\mu \in GL(n)$. e'_μ is still a holonomic frame since it comes from the coordinates system x'^μ. Now, one can generalise this scheme in the following way: let us define ${P^1}_x = \{{a_\mu}^\nu e_\nu, a \in GL(n)\}$, i.e. the orbit of the given e_ν under the action of $GL(n)$, which is naturally isomorphic to $GL(n)$ itself, and then the union of all ${P^1}_x$, for $x \in M$:

$$P^1(M) := \cup_{x \in M} {P^1}_x.$$

$GL(n)$ acting on each ${P^1}_x$, one can naturally define an action of $GL(n)$ on $P^1(M)$. This define $P^1(M)$ as being a principal fibre bundle over M, of structure group $GL(n)$. $P^1(M)$ is usually denoted $L(M)$ and called the fibre bundle of linear frames over M, also called 1-frames.

Going into 2-frames implies to do the same construction, but considering now the 1-frames over $P^1(M)$ itself, seen as a manifold. Thus, the given coordinates system $\{x^\mu\}$ allows to define the holonomic 2-frames:

- $e_\mu = \frac{\partial}{\partial x^\mu}$
- $e_{\mu\nu} = \frac{\partial^2}{\partial x^\mu x^\nu}$

with the following laws under a change of coordinates:

- $e'_\mu = a^\nu_\mu e_\nu$
- $e'_{\mu\nu} = a^\sigma_{\mu\nu} e_\sigma + a^\sigma_\mu a^\tau_\nu e_{\sigma\tau}$

with $a^\nu_\mu \in GL(n)$ and $a^\sigma_{\mu\nu} = a^\sigma_{\nu\mu}$ with the following composition laws:

$$(a^\nu_\mu, a^\sigma_{\mu\nu})(b^\nu_\mu, b^\sigma_{\mu\nu}) = (a^\mu_\rho b^\rho_\nu, a^\sigma_\alpha b^\alpha_{\mu\nu} + a^\sigma_{\alpha\beta} b^\alpha_\mu b^\beta_\nu)$$

coming directly from the usual composition law of jacobian matrices. $(e'_\mu, e'_{\mu\nu})$ are still holonomic 2-frames for they come from the coordinates system x'^μ. One can again generalise this scheme by defining a new group:

$$G^2(n) = \{(a^i_j, a^i_{jk})\}$$

with:

- $a^i_j \in GL(n)$
- $a^i_{jk} = a^i_{kj}$
- $(a^i_j, a^i_{jk})(b^i_j, b^i_{jk}) = (a^i_l b^l_j, a^i_l b^l_{jk} + a^i_{lm} b^l_j b^m_k)$

Then, one considers the orbit ${P^2}_x$ of a given $(e_\mu, e_{\mu\nu})$ under the action of $G^2(n)$, and the union:

$$P^2(M) := \cup_{x\in M} {P^2}_x$$

called the 2-frames principal fibre bundle, of structure group $G^2(n)$. One can thus define a 2-frames (h_i, h_{ij}):

- $h_i = h_i^\mu e_\mu$
- $h_{ij} = h_{ji} = h_{ij}^\mu e_\mu + h_i^\mu h_j^\nu e_{\mu\nu}$

which is no more, in general, holonomic. Any two 2-frames over the same point of M are naturally related by an element of $G^2(n)$. h_i^μ and h_{ij}^μ are here some $G^2(n)$-valued functions, also called coordinate system on the bundle $P^2(M)$.

3.2.7.2 The conformal 2-frames bundle

See e.g. section 3 of [26];

As one shall see, it is possible to define the conformal bundle as a reduction of the 2-frames bundle. Indeed, let us consider the subgroup of $G^2_{conf} \subset G^2(n)$ composed of elements of the form: $({a^i}_j, {a^i}_{jk})$ such that:

- ${a^i}_j$ represents a $CO(1, n-1)$ matrix,
- ${a^i}_{jk} := {a^i}_j c_k + {a^i}_k c_j - \eta_{jk}\eta^{lm}{a^i}_l c_m$,
- $c_m \in \mathbb{R}^{1,n-1*}$.

It turns out that this subgroup identifies with H defined in the framework of conformal Klein geometry, composed of $CO(1, n-1)$ and $\mathbb{R}^{1,n-1}$, as one shall see.

Let us show it, first, at the Lie algebra level, by identifying the linearised form of ${a^i}_{jk}$ with the first prolongation of the Lie algebra of $CO(1, n-1)$. Let us set ${a^i}_j = \delta^i_j + \epsilon^i_j$.[k] Then one gets (at first order in ϵ):

$${a^i}_{jk} = {\delta^i}_j c_k + {\delta^i}_k c_j - \eta_{jk}\eta^{im} c_m$$

which is exactly the form of an element of the first prolongation $\mathfrak{g}_1$ presented in the lattest section.

One shall now make the correspondence at the level of the Lie group $G^2_{conf} \cong H$. For $h = \begin{pmatrix} \lambda & \lambda c^T\eta\Lambda & \lambda\frac{c^2}{2} \\ 0 & \Lambda & c \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \in H$, the corresponding element in G^2_{conf} is $({a^i}_j, {a^i}_{jk})$ with:

[k] c just becomes infinitesimal, because $\mathbb{R}^{1,n-1}$ is a vector space

- $a^i{}_j = \lambda^{-1}\Lambda^i{}_j$
- $a^i{}_{jk} = a^i{}_j c_k + a^i{}_k c_j - \eta_{jk}\eta^{lm} a^i{}_l c_m$

One just proves that both H and G^2_{conf} have the same composition law. Let us take $h = \begin{pmatrix} \lambda & \lambda c^T \eta \Lambda & \lambda \frac{c^2}{2} \\ 0 & \Lambda & c \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$ and $k = \begin{pmatrix} \gamma & \gamma d^T \eta \Gamma & \gamma \frac{d^2}{2} \\ 0 & \Gamma & d \\ 0 & 0 & \gamma^{-1} \end{pmatrix}$ and the corresponding $(a^i{}_j, a^i{}_{jk})$ and $(b^i{}_j, b^i{}_{jk})$. On the first hand, the product $h.k$ implies the following composition laws between their components:

- $\lambda.\gamma = \lambda\gamma$
- $\Lambda.\Gamma = \Lambda\Gamma$
- $c.d = d + c\gamma^{-1}\Gamma$

On the other hand, let us see what the composition law of $G^2(n)$ becomes in the case of G^2_{conf}. One has: $(a^\nu_\mu, a^\sigma_{\mu\nu})(b^\nu_\mu, b^\sigma_{\mu\nu}) = (a^\mu_\rho b^\rho_\nu, a^\sigma_\alpha b^\alpha_{\mu\nu} + a^\sigma_{\alpha\beta} b^\alpha_\mu b^\beta_\nu)$. Thus, $(a.b)^i{}_j = a^i{}_k b^k{}_j$, which corresponds exactly to the two first composition laws. One has also: $(a.b)^\sigma{}_{\mu\nu} = a^\sigma_\alpha b^\alpha_{\mu\nu} + a^\sigma_{\alpha\beta} b^\alpha_\mu b^\beta_\nu$. Yet, with the particular form:

$$a^\sigma{}_{\mu\nu} := a^\sigma{}_\mu c_\nu + a^\sigma{}_\nu c_\mu - \eta_{\mu\nu}\eta^{lm} a^\sigma{}_l c_m$$

and

$$b^\sigma{}_{\mu\nu} := b^\sigma{}_\mu d_\nu + b^\sigma{}_\nu d_\mu - \eta_{\mu\nu}\eta^{lm} b^\sigma{}_l d_m,$$

the transformation law becomes:

$$\begin{aligned}
(a.b)^\sigma{}_{\mu\nu} &= a^\sigma_\alpha (b^\alpha{}_\mu d_\nu + b^\alpha{}_\nu d_\mu - \eta_{\mu\nu}\eta^{lm} b^\alpha{}_l d_m) \\
&+ (a^\sigma{}_\alpha c_\beta + a^\sigma{}_\beta c_\alpha - \eta_{\alpha\beta}\eta^{lm} a^\sigma{}_l c_m) b^\alpha_\mu b^\beta_\nu \\
&= a^\sigma_\alpha b^\alpha{}_\mu (d_\nu + c_\beta b^\beta_\nu) + a^\beta_\alpha b^\beta{}_\nu (d_\mu \\
&+ c_\alpha b^\alpha{}_\mu) - a^\sigma{}_\alpha \eta_{\mu\nu}\eta^{lm} b^\alpha{}_l d_m - \eta_{\alpha\beta}\eta^{lm} a^\sigma{}_l c_m b^\alpha_\mu b^\beta_\nu.
\end{aligned}$$

Let us examine the last term

$$\eta_{\alpha\beta}\eta^{lm} a^\sigma{}_l c_m b^\alpha_\mu b^\beta_\nu,$$

to put it in the right form with the help of the structure equation of $b^\sigma{}_\alpha$. Let us write

$$b^\sigma{}_\alpha = \gamma^{-1}\Gamma^\sigma{}_\alpha,$$

with $\Gamma \in SO$.

$$\begin{aligned}\eta_{\alpha\beta}\eta^{lm}a^{\sigma}{}_{l}c_m b^{\alpha}_{\mu}b^{\beta}_{\nu} &= \eta_{\alpha\beta}\eta^{lm}(a^{\sigma}{}_{i}b^{i}{}_{j})(b^{-1})^{j}{}_{l}c_m b^{\alpha}_{\mu}b^{\beta}_{\nu}\\ &= \eta_{\mu\nu}\eta^{lm}(a^{\sigma}{}_{i}b^{i}{}_{j})\gamma^{-2}\gamma(\Gamma^{-1})^{j}{}_{l}c_m\\ &= \eta_{\mu\nu}(a^{\sigma}{}_{i}b^{i}{}_{j})\gamma c_m\eta^{jk}\Gamma^{m}{}_{k}\\ &= \eta_{\mu\nu}\eta^{lm}(a^{\sigma}{}_{\alpha}b^{\alpha}{}_{l})b^{\beta}{}_{m}c_{\beta}\end{aligned}$$

So finally, one can write:

$$(a.b)^{\sigma}{}_{\mu\nu} = a^{\sigma}_{\alpha}b^{\alpha}{}_{\mu}(d_{\nu} + c_{\beta}b^{\beta}_{\nu}) + a^{\beta}_{\alpha}b^{\beta}{}_{\nu}(d_{\mu} + c_{\alpha}b^{\alpha}{}_{\mu}) - \eta_{\mu\nu}\eta^{lm}(a^{\sigma}{}_{\alpha}b^{\alpha}{}_{l})(d_m + c_{\beta}b^{\beta}{}_{m}).$$

Thus, the composition law for c and d is $(c.d)_m = d_m + b^{\alpha}{}_{\mu}c_{\alpha}$, which corresponds exactly to that given by the $\mathbb{R}^{n,2}$-representation. Since both the bundles are principal, the corresponding fibres are isomorphic to the structure groups, that is why it is sufficient to make the calculations only on the latter. Finally, the conformal 2-frame bundle is a H–reduction of the 2-frame bundle.

3.2.8 Cartan geometries equivalent to conformal structures

In conclusion of this overview, let us recall the different possible approaches of conformal geometry, summarized in figure 3.3. First, at the first–order level, a CO–structure over a smooth manifold is equivalent to a conformal class of metrics. A certain Cartan connection over a CO–principal bundle can be made equivalent to a Weyl structure, but not to a conformal class of metrics. To do that, one has to go to a second order structure. There are two ways to do that. Either, one considers the prolongation of the first order CO–bundle, or a H–reduction of the second order frame bundle, H being the structure group corresponding to the Klein model of conformal geometry.

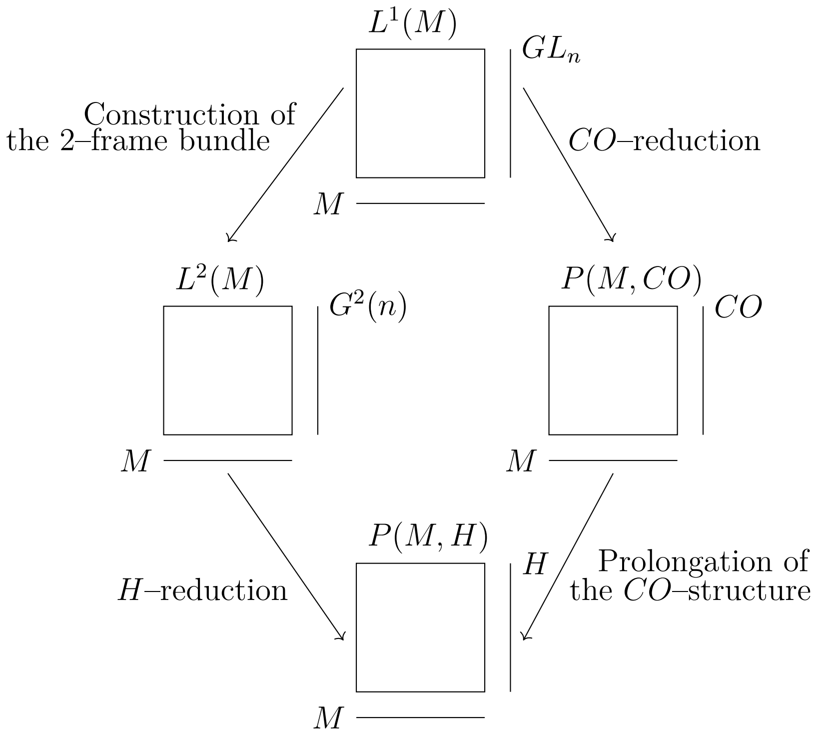


Figure 3.3: Different possible constructions of the 2^{nd}–order conformal structure.

Now, let us present the result of our questionning: what is the Cartan geometry equivalent to a given conformal structure over a smooth manifold M? A Cartan connection ϖ on a Cartan principal bundle modeled on the conformal Klein pair (G, H), taking values in the Lie algebra $\mathfrak{g}$, has thus the following form: $\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & \omega & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}$, with $\theta \in \mathbb{R}^{1,n-1}$, $a \in \mathbb{R}^*$, $\omega \in \mathfrak{so}(1, n-1)$, and $\alpha \in \mathbb{R}^{1,n-1*}$. Its curvature $\overline{\Omega}$ has a similar form:

$$\overline{\Omega} = \begin{pmatrix} f & \Pi & 0 \\ \Theta & \Omega & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix} \tag{3.39}$$

with:

$$f = da + \alpha \wedge \theta \tag{3.40}$$

$$\Omega = R + \theta \wedge \alpha + \alpha^t \wedge \theta^t \tag{3.41}$$

$$\Pi = d\alpha + a \wedge \alpha + \alpha \wedge \omega \tag{3.42}$$

$$\Theta = d\theta + \theta \wedge a + \omega \wedge \theta \tag{3.43}$$

On the first hand, a general Cartan connection ϖ over such a bundle thus posseses $\frac{n}{2}(n^3+3n+2)$ degrees of freedom (in terms of fields), minus $n+\frac{n(n-1)}{2}+1$ (which is the dimension of the structure group H), i.e. the structure as a whole posseses $\frac{n(n-1)^2}{2}+2n^2+1$ degrees of freedom. On the other hand, a conformal class of metrics over M represents $\frac{n(n-1)}{2}-1$ degrees of freedom. Thus, one sees that to get a Cartan connection which makes the structure equivalent to a conformal class of metrics, one has to impose restrictions on ϖ. Indeed, these so-called normality conditions are:

$$\Theta = 0 \tag{3.44}$$

$$\Omega^a{}_{bac} = 0 \tag{3.45}$$

This particular *normal* Cartan connection is unique, up to gauge transformations. One can show (see e.g. [30]) that added to Bianchi identity, these conditions 3.44 and 3.45 imply $f = 0$. In this particular case, let us show that the field α related to the special conformal transformations is completely determined and is equal to the so-called Schouten tensor: $\alpha_{ab} = -\frac{1}{2}(\mathcal{R}_{\text{ab}} - \frac{1}{6}\mathcal{R}\eta_{\text{ab}})$.

$$\begin{aligned}
\Omega^a{}_b &= (\frac{1}{2}\mathcal{R}^a{}_{bcd} + \delta^a{}_c\alpha_{bd} - \eta_{bc}\alpha^a{}_d)\theta^c \wedge \theta^d \\
&= \frac{1}{2}\Omega^a{}_{bcd}\theta^c \wedge \theta^d
\end{aligned}$$

with $\Omega^a{}_{bcd} = \frac{1}{2}R^a{}_{bcd} + \delta^a{}_{[c}\alpha_{|b|d]} - \eta_{b[c}\alpha^a{}_{d]}$. Then the condition $\Omega^a{}_{bac} = 0$ leads to: $\frac{1}{2}R_{bd} + \eta_{bd}\alpha^a{}_a + 2\alpha_{bd} = 0$ and contracting b and d then leads to: $\alpha^a{}_a = -\frac{1}{12}R$. Replacing $\alpha^a{}_a$ by its expression in the previous equations then gives what is expected.

One can also show that these conditions imply that Ω reduces to the Weyl 2–form. It turns out that such a normal Cartan connection defines uniquely a conformal class of metrics, and conversely the data of a conformal class of metrics defines a unique normal Cartan connection of this type.

If one wants to produce a theory of conformal gravity, then one can either work on a conformal manifold, or take such a normal Cartan connection over its corresponding bundle and take the different fields as dynamical. One will see later the possible use of such a viewpoint.

3.3 Dressing Field Method applied to Conformal Geometry, a Top-Down construction of Tractor and Twistor connections and bundles

It is possible and also very fruitful to reduce a part of the symmetry in the context of Cartan conformal geometry, i.e. to reduce the action of the structure group H to a smaller subgroup. Here one presents the possible reductions, in the real and complex representations. All this section is based on the two articles [18] and [17], which contain more details and proofs.

3.3.1 Real Representation and Tractors

Let $\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & \omega & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}$ be a conformal Cartan connection (not necessarily normal at this stage). The structure group (which implements the gauge transformations) can be factorized as follows:

$$H = K_0 K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & r & \frac{1}{2} r r^t \\ 0 & \mathbb{I}_n & r^t \\ 0 & 0 & 1 \end{pmatrix} \ \middle| \ z \in \mathbb{R},\ S \in SO(1, n-1),\ r \in \mathbb{R}^{n*} \right\}. \tag{3.46}$$

Its curvature has been given at 3.39. Let ψ be a section of the associated vector bundle $E = P \times_H \mathbb{V}$, where $\mathbb{V} = \mathbb{R}^{n+2}$ is the space of defining representation of H. Given the form of H, ψ has the form: $\psi = \begin{pmatrix} \rho \\ \ell \\ \sigma \end{pmatrix}$, and its covariant derivative reads $D\psi = d\psi + \varphi\psi$. Let us now show how the dressing method applies to this framework, and then how ψ become genuine tractor fields by this procedure, as they are usually defined. The form of the second order principal bundle suggests that the K_1–symmetry could be erased first. Then, one will show how to do an additional dressing operation, erasing the Lorentz–symmetry.

3.3.1.1 Reduction of the special conformal transformations

Let

$$u_1 = \begin{pmatrix} 1 & q & \frac{1}{2} q^2 \\ 0 & \mathbb{I}_n & q^t \\ 0 & 0 & 1 \end{pmatrix}$$

with $q\theta = a$, i.e. $q_b = a_\mu (e^{-1})^\mu_b$. It is indeed a dressing field for the K_1–symmetry, as one can easily check. Then, the dressed Cartan connection reads:

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & \omega_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}.$$

The connection, together with the other fields, are well-behaved as K_0–gauge fields, so that the dressing amounts to a (local) reduction of the second–order conformal structure to the first–order conformal structure. Indeed, the dressing field u_1 transforms as

$$u_1^S = S^{-1} u_1 S$$

under a Lorentz transformation S. As one has noticed in 1.5.2.3, one can conclude that the dressed fields are genuine gauge fields w.r.t. the Lorentz symmetry, and also that the former can be erased.

3.3.1.2 Reduction of Lorentz symmetry

In order to erase the Lorentz symmetry one can use a dressing field which also works in the case of the tetrad formulation of general relativity, for one is formally in the same situation. That is to say, set

$$u_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with e the tetrad field such that $\theta^a = e^a_\mu dx^\mu$. Knowing that $\theta^S = S^{-1}\theta$, one has directly

$$u_0^S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} u_0.$$

Dressing the different fields with u_0, one ends up with:

$$\varpi_0 = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P \\ 0 & g \cdot dx & 0 \end{pmatrix},$$

the dressed connection which is now also invariant under SO.

3.3.1.3 Residual symmetry

Thus, one has two possible dressed descriptions of a conformal connection (and of the other related fields). One can work either with ϖ_1, which suffers both Weyl and Lorentz transformations, or with ϖ_0, on which only Weyl transformations apply. In both cases, however, one has to compute the residual Weyl symmetry. The first step is to compute the transformation of the dresssing field itself under a Weyl transformation $Z \in W$, i.e. $Z = \begin{pmatrix} z & & \\ & \mathbb{I}_n & \\ & & z^{-1} \end{pmatrix}$ $u_1^Z = Z^{-1}u_1C(z)$ where the map $C : W \to K_1W \subset H$ is defined by

$$C(z) = k_1(z)Z = \begin{pmatrix} 1 & \Upsilon \cdot e^{-1} & \frac{1}{2}\Upsilon^2 \\ & \mathbb{I}_n & (\Upsilon \cdot e^{-1})^t \\ & & 1 \end{pmatrix} \begin{pmatrix} z & & \\ & \mathbb{I}_n & \\ & & z^{-1} \end{pmatrix} \tag{3.47}$$

with $\Upsilon_\mu = z^{-1}\delta_\mu z$, $\Upsilon \cdot e^{-1} = \Upsilon_\mu e_a^\mu =: \Upsilon_a$, and $\Upsilon^2 = \Upsilon_a \eta^{ab} \Upsilon_b$. This is a particular instance of the second example of transformation see 1.5.2.3.

Thus, the dressed connection ϖ_1 transforms as

$$\varpi^Z = C(z)^{-1}\varpi_1 C(z) + C(z)^{-1}dC(z)$$

and so do the other fields. Thus, after the first dressing, the effective structure group is $C(W)$, i.e. composed of elements of the form $C(z)$. Let us remark that C does not realize a representation of the Weyl group, for $C(z)C(z') \neq C(zz')$. This group of transformations is exactly the one used in the usual bottom–up construction of tractor bundle.

Concerning the second dressing, one has to compute the transformation of $u := u_1u_0$ to get the residual symmetry. It turns out that $u^Z = Z^{-1}u\overline{C}(z)$ where the map $\overline{C} : W \to GL_{n+2}(\mathbb{R}) \supset H$ is defined as:

$$\overline{C}(z) = \overline{k}_1(z)Z = \begin{pmatrix} 1 & \Upsilon \cdot e^{-1} & \frac{1}{2}\Upsilon^2 \\ & \mathbb{I}_n & (\Upsilon \cdot e^{-1})^t \\ & & 1 \end{pmatrix} \begin{pmatrix} z & & \\ & z & \\ & & z^{-1} \end{pmatrix}. \tag{3.48}$$

As in the previous case, the effective gauge group is now composed of elements of the form $\overline{C}(z)$, for $z \in W$, which does not constitute a representation of W neither.

3.3.2 Complex Representation and Twistors

3.3.2.1 Reduction of the special conformal transformations

In the context of complex representation of conformal geometry, one can also reduce the Lorentz ($SL_2(\mathbb{C})$) symmetry by the same procedure. Moreover, the dressing field $\overline{u}_1$ used to erase it is the field corresponding to u_1 by the homomorphism. More precisely, one sets $\overline{u}_1 = \begin{pmatrix} \mathbb{I}_2 & -i\overline{q} \\ 0 & \mathbb{I}_2 \end{pmatrix}$, with $\overline{q}$ the image of $q = a \cdot e^{-1}$ by the homomorphism. It turns out that this defines a dressing field for the Special Conformal Transformations. One can form the $\overline{K}_1$–invariant composite fields:

$$\overline{\varpi}_1 := \overline{\varpi}^{u_1} = \begin{pmatrix} -\overline{A}_1^* & -i\overline{P}_1 \\ i\overline{\theta} & \overline{A}_1 \end{pmatrix}, \tag{3.49}$$

$$\overline{\Omega}_1 := \overline{\Omega}^{u_1} = \begin{pmatrix} -(\overline{W}_1^* - f_1/2\mathbb{I}_2) & -i\overline{C}_1 \\ i\overline{\Theta} & \overline{W}_1^* - f_1/2\mathbb{I}_2 \end{pmatrix} \tag{3.50}$$

In the normal conformal case (3.44 and 3.45) translated in the complex framework, these objets reduce to the usual objects found in Twistors theory:

$$\overline{\varpi}_1 := \overline{\varpi}^{u_1} = \begin{pmatrix} -\overline{A}_1^* & -i\overline{P}_1 \\ i\overline{\theta} & \overline{A}_1 \end{pmatrix}, \tag{3.51}$$

$$\overline{\Omega}_1 := \overline{\Omega}^{u_1} = \begin{pmatrix} -\overline{W}_1^* & -i\overline{C}_1 \\ 0 & \overline{W}_1^* \end{pmatrix} \tag{3.52}$$

3.3.2.2 Residual symmetry

Being $\overline{K_1}$–invariant after this dressing, the composite fields are expected to display a residual $\overline{K}_0$ (i.e. Weyl+Lorentz) transformation. The residual Lorentz ($SL_2(\mathbb{C})$) transformation is an ordinary one, for the $SL_2(\mathbb{C})$–equivariance of the dressing field $\overline{u}_1$ is $\overline{u}_1^S = S^{-1}\overline{u}_1 S$, and thus it is an instance of 1.68. As in the real case, however, the residual Weyl symmetry expresses in a non–standard way, i.e. without forming a representation of $\overline{K}_0$. Indeed, $\overline{u}_1$ transforms as $\overline{u}_1^Z = Z^{-1}\overline{u}_1 C(z)$ where the map $C : W \to \overline{K}_1 W \subset \overline{H}$ is defined by:

$$C(z) := \overline{k}_1(z)Z = \begin{pmatrix} \mathbb{I}_2 & -i\overline{\Upsilon} \\ 0 & \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} z^{1/2}\mathbb{I}_2 & 0 \\ 0 & z^{-1/2}\mathbb{I}_2 \end{pmatrix} \tag{3.53}$$

As in the real case, elements as $C(z)$ do not form a representation of the Weyl group W. The connection, the curvature and the twistors fields themselves transform with the help of the map C under a Weyl transformation as:

$$\overline{\varpi}_1^Z = C(z)^{-1}\overline{\varpi}_1 C(z) + C(z)^{-1}dC(z) \tag{3.54}$$

$$\overline{\Omega}_1^Z = C(z)^{-1}\overline{\Omega}_1 C(z) \tag{3.55}$$

$$\psi_1^Z = C(z)^{-1}\psi_1 = \begin{pmatrix} z^{-1/2}(\pi_1 + i\overline{\Upsilon}\omega_1) \\ z^{1/2}\omega_1 \end{pmatrix} \tag{3.56}$$

Notice that these transformations slightly differ from that one can find in the usual constructions: indeed, the factors $z^{-1/2}$ and $z^{1/2}$ do not appear in the latter. The difference is that in our construction, we are just following what geometry tells us, in the sense that we do not make any ad hoc assumption, at any stage of the construction. From this viewpoint, it is a more "natural" construction.

3.4 Weyl Gravity as a Yang–Mills Type Gauge Theory

A model of conformal gravity, called Weyl gravity, has been introduced by Bach in 1921 and is constructed with the Weyl tensor $W^{\mu}_{\nu\rho\sigma}$ or also with the 2–form Weyl curvature W:

$$L_{Weyl} = Tr(W \wedge *W) = \frac{1}{2} W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}. \tag{3.57}$$

The field equations coming from this lagrangian are called Bach equations, and are of 4^{th}–order. Here one shows that in the context of conformal Cartan geometry, this action is naturally of Yang–Mills type.

Indeed, let $\overline{\Omega}$ be the curvature of a normal Cartan connection after dressing:

$$\overline{\Omega} = \begin{pmatrix} 0 & P & O \\ 0 & W & P^t \\ 0 & 0 & 0 \end{pmatrix}.$$

The natural Yang–Mills lagrangian is thus

$$L = Tr_{\mathfrak{g}}(\overline{\Omega} \wedge *\overline{\Omega}) = Tr_{\mathfrak{so}}(W \wedge *W). \tag{3.58}$$

Thus, with the help of Cartan geometry, Weyl gravity is naturally formulated with a lagrangian of Yang–Mills type.

Chapter 4

Formulation of Gauge Theories on Transitive Lie Algebroids and Unified Lagrangians

In this chapter, we present two possible ways to formulate gauge theories on transitive Lie algebroids which naturally lead to *unified Lagrangians*. In a first section, one sets the useful additional material in order to be able to write a gauge theory on such a structure. Then, one presents a work inspired of that of Boroojerdian et al. [46]. The idea is to take a metric $\hat{g}$ on the transitive Lie algebroid as a dynamical field, and to study a notion of Levi–Civita connection $\hat{\nabla}$ on $\mathcal{A}$ associated to $\hat{g}$. This, with the tensorial calculus accompanying it, leads to a unified Lagrangian which contains in the same time General Relativity with cosmological constant and Yang–Mills theory. The notations used in this section will not be those of Boroojerdian et al., but those, more precise, defined in the first section of this chapter. Finally, one presents the approach of Fournel et al. [15] and Fournel's PhD Thesis [16], taking a background metric $\hat{g}$ on a transitive Lie algebroid $\mathcal{A}$ and studying the dynamics of a *generalized connection* ϖ on it, which leads to a unified Yang–Mills–Higgs–type Lagrangian.

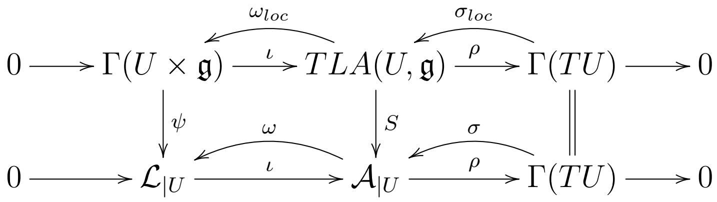
From now on, $\mathcal{A}$ is a transitive Lie algebroid over a base n–dimensional manifold M with trivializing Lie algebra $\mathfrak{g}$ of dimension m. $\mathfrak{g}$ should be thought as the infinitesimal generator of an internal gauge symmetry, even if we do not take a precise one. A will stand for the vector bundle such that $\mathcal{A} = \Gamma(A)$.

4.1 Transitive Lie algebroids: additional material

In this first section one presents additional material in order to write gauge theories on transitive Lie algebroid. First, one presents the mixed local basis, at the level of forms like in [15] and then its dual. One will see that this basis is well adapted to tensor calculus defined on a Lie algebroid $\mathcal{A}$. We present the definition of this calculus and the basis of Riemannian geometry on a Lie algebroid, which will be useful for the section 4.2. Then, one presents promptly material already presented in [15], about the theory of integration on a transitive Lie algebroid. We will see how to define a volume form, the notion of inner integration (integration along the kernel), and then integration itself and the Hodge star operator. This material will be useful for the section 4.3.

4.1.1 Mixed Local Basis: Definitions

Let us consider the trivialization of a transitive Lie algebroid over an open set $U \subset M$:



S is an isomorphism defining the trivialization, and ψ is its corresponding map on the kernel. σ is a splitting on $\mathcal{A}$, and ω is its corresponding connection 1–form. σ_{loc} and ω_{loc} are their respective local versions, defined by:

- $\sigma_{loc} := S^{-1} \circ \sigma$;
- $\omega_{loc} := \psi^{-1} \circ \omega \circ S$.

By the normalization condition $\omega \circ \iota = -\mathrm{id}_{\mathcal{L}}$, ω_{loc} reads:

$$\omega_{loc} = A - \mathrm{id}_{\Gamma(U \times \mathfrak{g})}$$

where $A : \Gamma(TU) \to \Gamma(U \times \mathfrak{g})$. σ_{loc} reads then:

$$\sigma_{loc}(X) = X \oplus A(X).$$

Recall that a splitting σ allows to realise an isomorphism:

$$\mathcal{A} \simeq \mathrm{Im}(\sigma) \oplus \mathcal{L}. \tag{4.1}$$

One would like to construct a local basis over the open set U adapted to the decomposition 4.1. First, let us define the *mixed local basis* at the level of forms. Let $\omega \in \Omega^q(A)$. Let $(dx^{\mu_1}, .., dx^{\mu_n})$ (or $\{dx^\mu\}_\mu$) be a basis of TU^* and $(\theta^{a_1}, .., \theta^{a_m})$ (or $\{\theta^a\}_a$) a basis of $\mathcal{C}^\infty(U) \otimes \mathfrak{g}^*$. The form ω decomposes, in the basis $(dx^{\mu_1}, .., dx^{\mu_n}, \theta^{a_1}, .., \theta^{a_m})$, as:

$$\omega = \sum_{r+s=q} \omega_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \dots \wedge \mathrm{d}x^{\mu_r} \wedge \theta^{a_1} \dots \wedge \theta^{a_2} \dots \wedge \theta^{a_s} \tag{4.2}$$

The problem of this decomposition is that the bi-gradation in not preserved under a change of trivialization of the Lie algebroid, for θ^a transforms to $\alpha^a_b\theta^b + \chi^a_\mu dx^\mu$, due to the change of trivialization 1.33. A nice way to avoid this problem is to write the forms in another basis, called mixed local basis, see e.g. [16] or [15], based on the components of the connection ω. The latter decomposed over an open subset U as $\omega = (A^a - \theta^a) \otimes E_a$, where $A^a \in \Omega^1(U)$, $\theta^a \in \mathfrak{g}^*$ and $\{E_a\}_a$ is its dual basis. Let us set $\omega^a_{loc} := A^a - \theta^a$. A family of such $\{\omega^a_{loc}\}_a$ is called a *mixed local basis*, and it is possible to rewrite any other form in the basis $(dx^{\mu_1}, .., dx^{\mu_n}, \omega^{a_1}_{loc}, .., \omega^{a_m}_{loc})$ by merely replacing θ^a in the decomposition 4.2 by $A^a - \omega^a_{loc}$. One obtains, for $\omega \in \Omega^q(A)$, the following (similar) decomposition:

$$\omega = \sum_{r+s=q} (\omega_{loc})_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \dots \wedge \mathrm{d}x^{\mu_r} \wedge \omega^{a_1}_{loc} \dots \wedge \omega^{a_2}_{loc} \dots \wedge \omega^{a_s}_{loc} \tag{4.3}$$

As we will see, the basis used in the decomposition is preserved by changes of trivialization. Moreover, decomposing forms in this basis is necessary in order to give a good definition of the notion of integration on a Lie algebroid. Given this mixed local basis for differential forms, its dual is defined as follows. Let $\{\partial_\mu\}_\mu$ be a basis of $\Gamma(TU)$ and $\{E_a\}_a$ be the basis of $\Gamma(U \times \mathfrak{g})$, dual to $\{\theta_a\}_a$. Then, let us define the following objects:

- $e_\mu := \sigma_{\partial_\mu} = \partial_\mu \oplus A_\mu$;
- $\tilde{E}_a := 0 \oplus (-E_a)$.

The basis $(\{e_\mu\}_\mu, \{\tilde{E}_a\}_a)$ is a basis of $TLA(U, \mathfrak{g})$, dual to the mixed local basis $(\{dx^\mu\}_\mu, \{\omega^a_{loc}\}_a)$. Indeed,

- $dx^\mu(e_\nu) = dx^\mu(\partial_\nu) = \delta^\mu_\nu$;
- $dx^\mu(\tilde{E}_a) = dx^\mu(0) = 0$;
- $\omega^a_{loc}(e_\mu) = \omega^a_{loc} \circ \sigma_{loc}(\partial_\mu) = 0$;
- $\omega^a_{loc}(\tilde{E}_b) = -\theta^a(-E_b) = \delta^a_b$.

Let us see now how such basis transform under a change of trivialization.

4.1.2 Change of Trivialisation

Let $S' : TLA(U, \mathfrak{g}) \to \mathcal{A}_{|U}$ be another trivialization over the same open set U. Any $\mathfrak{X} \in \mathcal{A}_{|U}$ can be written both ways: $\mathfrak{X} = S(X \oplus \gamma) = S'(X \oplus \gamma')$ (with the same X). One has:

$$X \oplus \gamma' = (S'^{-1} \circ S)(X \oplus \gamma) = X \oplus (\alpha(\gamma) + \chi(X)) \tag{4.4}$$

with the maps:

- $\alpha : \Gamma(U \times \mathfrak{g}) \to \Gamma(U \times \mathfrak{g})$ and
- $\chi : \Gamma(TU) \to \Gamma(U \times \mathfrak{g})$.

Also, one knows that S and S' can be written with the help of the corresponding flat splittings σ^0 and σ'^0 and of the maps ψ and ψ' as:

- $S(X \oplus \gamma) = \sigma^0_X + \iota \circ \psi(\gamma)$;
- $S(X \oplus \gamma') = \sigma'^0_X + \iota \circ \psi(\gamma') = \sigma'^0_X + \iota \circ \psi \circ \chi(X) + \iota \circ \psi' \circ \alpha(\gamma)$.

Thus one has the following relations:

- $\sigma'^0 = \sigma^0 + \iota \circ \psi \circ \chi$,
- $\psi' \circ \alpha = \psi$.

Thus, $\alpha = \psi'^{-1} \circ \psi$, which is coherent with the fact that α is a map from $\Gamma(U \times \mathfrak{g})$ to $\Gamma(U \times \mathfrak{g})$.

Let us compute how the connection and the splitting transform under a change of trivialisation over a given open set U: $\omega'_{loc} = \psi'^{-1} \circ \omega \circ S' = (\psi'^{-1} \circ \psi) \circ \omega_{loc} \circ (S^{-1} \circ S')$ thus

$$\omega^a_{loc} \to \alpha^a_b \omega^b_{loc} \circ (S^{-1} \circ S'),$$

where α^a_b are the coefficients of an invertible matrix (acting on Lie algebra elements). The same computation can be made for the splitting:

$$\sigma_{loc} \to \sigma'_{loc} = (S^{'-1} \circ S) \circ \sigma_{loc}.$$

We remark that since $\omega_{loc} \circ \sigma_{loc} = 0$, then $\omega'_{loc} \circ \sigma'_{loc} = 0$ as well. Thanks to these transformations, the mixed local basis and its dual transform as:

- $dx^\mu \to dx^\mu$,
- $\omega^a_{loc} \to \alpha^a_b \omega^b_{loc} \circ (S^{-1} \circ S')$,
- $e_\mu = \sigma_{loc}(\partial_\mu) \to e'_\mu = (S^{'-1} \circ S) \circ e_\mu$,
- $\tilde{E}_a \to \tilde{E}_a$,

under a change of trivialisation. Now, we want to write an element $\mathfrak{X}_{loc} = X \oplus \gamma$ in the dual mixed local basis, and see how do its components transform under a change of trivialisation. It will then allow to write any *tensor*

$$T \in \mathcal{A}^{\otimes^p} \otimes (\mathcal{A}^{\otimes^r})^*$$

in this local basis, and the homogeneous transformations of the components will ensure that this locally describes a globally well–defined object.

4.1.3 Transformation of $\mathfrak{X}_{loc} \in TLA(U, \mathfrak{g})$ under a change of trivialisation

Thanks to the trivialisation S, an element $\mathfrak{X}$ can locally be written: $\mathfrak{X} = S(\mathfrak{X}_{loc}) = S(X \oplus \gamma)$. Now, in the dual mixed local basis, $\mathfrak{X}_{loc}$ reads:

$$\mathfrak{X}_{loc} = X^\mu e_\mu + \gamma^a \tilde{E}_a. \tag{4.5}$$

Under a change of trivialisation, $\tilde{E}_a$ stays the same but as we have seen, e_μ changes. In the trivialisation S' such that $\mathfrak{X} = S'(\mathfrak{X}'_{loc})$, one has:

$$\mathfrak{X}'_{loc} = X'^\mu e'_\mu + \gamma'^a \tilde{E}_a. \tag{4.6}$$

However, one has also:

$$\begin{aligned}
\mathfrak{X}'_{loc} &= S'^{-1} \circ S(\mathfrak{X}_{loc}) \\
&= X^\mu (S'^{-1} \circ S)(e_\mu) + \gamma^a (S'^{-1} \circ S)(\tilde{E}_a) \\
&= X^\mu e'_\mu + \gamma^a (S'^{-1} \circ S)(\tilde{E}_a).
\end{aligned}$$

The element $\tilde{E}_a$ is not the localization of a global object, thus one cannot use here the formula 4.4. Yet, one knows that, by definition, $S(0 \oplus \gamma) = \iota \circ \psi(\gamma)$ and $S'(0 \oplus \eta) = \iota \circ \psi'(\eta)$, thus $S'^{-1} \circ \iota \circ \psi'(\eta) = 0 \oplus \eta$. If one takes a particular $\eta = \psi'^{-1}(\tilde{\eta})$ for $\tilde{\eta} \in \mathcal{L}_{|U}$, one thus has: $S'^{-1} \circ \iota(\tilde{\eta}) = 0 \oplus \psi'^{-1}(\tilde{\eta})$, and thus:

$$\begin{aligned}
S'^{-1} \circ S(0 \oplus \gamma) &= S'^{-1} \circ \iota\psi(\gamma) \\
&= 0 \oplus \psi^{-1} \circ \psi(\gamma) \\
&= 0 \oplus \alpha(\gamma).
\end{aligned}$$

Thus, one gets:

$$\begin{aligned}
S'^{-1} \circ S(\tilde{E}_a) &= S'^{-1} \circ S(0 \oplus (-E_a)) \\
&= 0 \oplus (-\alpha(E_a)) \\
&= 0 \oplus \alpha_b^a(-E_b) \\
&= \alpha_b^a \tilde{E}_a.
\end{aligned}$$

Finally, one has:

$$\mathfrak{X}'_{loc} = X^\mu e'_\mu + \gamma^a \alpha_a^b \tilde{E}_b. \tag{4.7}$$

Comparing this expression with 4.6, one concludes that:

- $X'^\mu = X^\mu$,
- $\gamma'^a = \alpha_b^a \gamma^b$.

Of course, one could also take into account changes of local charts, and in this case the greek indices would have to change too. Here, we use the fact that these changes of charts are taken in account in usual Riemannian geometry, and we rather focus on the purely algebraic part of these tensors.

4.1.4 Tensorial Calculus and Riemannian Geometry on Lie Algebroid

Let σ be a splitting and ω its corresponding connection. Then, one can write any tensor $T \in \mathcal{A}^{\otimes^p} \otimes (\mathcal{A}^{\otimes^r})^*$ in the mixed local basis and its dual, over a given open set U. Any tensor will have, in its local decomposition, two kinds of indices: either latin (internal–algebraic degrees of freedom) or greek (external–geometric degrees of freedom). The interest of this basis is that both kinds of indices appearing in this basis will be *respected by a change of trivialisation*. An upper or lower greek index will not transform under a change of trivialization (for here we stay over the same open set U), and a latin index will transform by:

- α_b^a for an upper index;
- ${\alpha^{-1}}_b^a$ for a lower one.

Now, let us take a non–degenerate metric $\hat{g}$ on the transitive Lie algebroid $\mathcal{A}$. One knows that this is equivalent to a triple (h, ω, g) with $h : \mathcal{L} \times \mathcal{L} \to \mathcal{C}^\infty(M)$ a non degenerate metric on $\mathcal{L}$, ω the connection 1–form of a splitting σ such that $\hat{g}(\iota(\gamma), \sigma_X) = 0$ for all $\gamma \in \mathcal{L}$ and $X \in \Gamma(TM)$, and $g : \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^\infty(M)$ is a metric on the tangent bundle. Globally, one has $\hat{g} = \sigma^* g + \omega^* h$. Locally, i.e. on $TLA(U, \mathfrak{g})$, $\hat{g}_{loc}$ then reads:

$$\hat{g}_{loc}(X \oplus \gamma, Y \oplus \eta) = g_{loc}(X, Y) + h_{loc}(\omega_{loc}(X \oplus \gamma), \omega_{loc}(Y \oplus \eta)). \tag{4.8}$$

In the mixed local basis and its dual, one defines:

- $g_{\mu\nu} := \hat{g}_{loc}(e_\mu, e_\nu) = \hat{g}_{loc}(\sigma_{\partial_\mu}, \sigma_{\partial_\nu}) = g_{loc}(\partial_\mu, \partial_\nu)$,
- $h_{ab} := \hat{g}_{loc}(\tilde{E}_a, \tilde{E}_b) = h_{loc}(E_a, E_b)$.

One remarks that thanks to the orthogonality condition, $\hat{g}_{loc}(e_\mu, \tilde{E}_a) = 0$, and thus $\hat{g}_{loc}$ is bloc diagonal in this basis. Indeed, it reads:

$$\hat{g}_{loc} = g_{\mu\nu} dx^\mu \otimes dx^\nu + h_{ab} \omega_{loc}^a \otimes \omega_{loc}^b \tag{4.9}$$

One denotes $g^{\mu\nu}$ and h^{ab} the respective inverse metrics, which will allow to raise indices of the corresponding kind. Let us compute how h_{ab} transforms under a change of trivialization over a given open set U, even if we already have some idea of its transformation. Remark first that $h_{ab} = h_{loc}(E_a, E_b) = h(\psi(E_a), \psi(E_b))$, and remind that $\psi' = \psi \circ \alpha^{-1}$. Thus:

$$\begin{aligned} h'_{ab} &= h(\psi'(E_a), \psi'(E_b)) \\ &= h(\psi \circ \alpha^{-1}(E_a), \psi \circ \alpha^{-1}(E_b)) \\ &= h(\psi({\alpha^{-1}}^c_a E_c), \psi({\alpha^{-1}}^d_b E_d)) \\ &= {\alpha^{-1}}^c_a {\alpha^{-1}}^d_b h(\psi(E_c), \psi(E_d)) \\ &= {\alpha^{-1}}^c_a {\alpha^{-1}}^d_b h_{cd}. \end{aligned}$$

As expected, the object h_{ab}, having two lower latin indices, transforms with two matrices α^{-1}, i.e. as a 2–covariant tensor. At the contrary, since $h_{ab}h^{bc} = \delta^c_a$, one can easily deduce that the inverse metric h^{ab} transforms as a 2–contravariant tensor, i.e. with two matrices like α.

Now, let us give a last example just to ensure that tensor calculus holds well in this basis. Let $B \in \Gamma(TM^*) \otimes \mathfrak{L}$, i.e. locally and in the good basis, B reads:

$$B_{loc} = B^a_\mu \tilde{E}_a \otimes dx^\mu.$$

Let us set $\overline{B}_{a\mu} := h_{ab}B^b_\mu$. We want to ensure that $\overline{B}_{a\mu}$ is the localisation of a globally well defined tensor $\overline{B} \in \Gamma(TM^*) \otimes \mathcal{L}^*$. One has the following transformations:

$$\begin{aligned} \overline{B}_{a\mu} \to ({\alpha^{-1}}^c_a {\alpha^{-1}}^d_b h_{cd})(\alpha^b_e B^e_\mu) &= \delta^d_e {\alpha^{-1}}^c_a h_{cd} B^e_\mu \\ &= {\alpha^{-1}}^c_a h_{cd} B^d_\mu \\ &= {\alpha^{-1}}^c_a \overline{B}_{c\mu}. \end{aligned}$$

Since $\overline{B}_{a\mu}$ transforms homogeneously, one is ensured that the object:

$$\overline{B}_{loc} = \overline{B}_{a\mu}\, \omega^a_{loc} \otimes dx^\mu$$

is the local formulation of a globally defined tensor $\overline{B} \in \Gamma(TM^*) \otimes \mathcal{L}^*$. This legitimates all the computations made in section 4.2.

4.1.5 Volume Form

A volume form on $\mathcal{A}$ is used to integrate forms on the Lie algebroid. One defines a volume form as a differential form of degree $m = \dim(\mathfrak{g})$, which allows to exhibit the maximal inner term associated to any form $\omega \in \Omega^q(A)$. Let us assume that A is inner–orientable, i.e. such that $\det(\alpha^{ij}) > 0$ where α^{ij} is the matrix used to make a change of local trivialization over the overlap of two trivializing open sets U_i and U_j, as defined above. Let us assume also that there exists a non–degenerate metric h on the kernel L. Let $\mathring{\omega}$ be a given background connection, and $\mathring{\omega}_{loc}$ its trivialization over an open subset U. Then, the volume form is defined as:

$$vol_{loc} := \sqrt{h_{loc}}\, \mathring{\omega}^1_{loc} \wedge \mathring{\omega}^2_{loc} \wedge ... \wedge \mathring{\omega}^m_{loc}. \tag{4.10}$$

Let us remark that the m–form $\mathring{\omega}^1_{loc} \wedge \mathring{\omega}^2_{loc} \wedge ... \wedge \mathring{\omega}^m_{loc}$ is defined on $TLA(U, \mathfrak{g})$ with values in $\mathcal{C}^\infty(U)$. It turns out that this volume form locally defined has a good transformation under a change of trivialization, which makes it into a well globally defined form $vol \in \Omega^m(A)$.

4.1.6 Maximal inner form

Let $\omega \in \Omega^q(A)$, written locally ω_{loc} over U in the mixed local basis as 4.3. One is interested in the components of ω_{loc} associated to the bi–graduation $(q-m, m)$ which can be written, in terms of the volume form vol_{loc} as:

$$\omega^{(q-m,m)}_{loc} = \frac{n!}{\sqrt{\mathrm{det}(h_{loc})}} (\omega_{loc})^{(q-m),m)}_{\mu_1\mu_2...\mu_{q-m}} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_{q-m}} \wedge vol_{loc}. \tag{4.11}$$

The maximal inner component associated to ω is defined as:

$$\omega^{m.i.}_{loc} := \frac{1}{\sqrt{\mathrm{det}(h_{loc})}} (\omega_{loc})^{(q-m,m)}_{\mu_1\mu_2...\mu_{q-m}} \tag{4.12}$$

"m.i." stands for "maximal inner". The maximal inner term does not transform under a change of trivialization, which turns it into the localization of a well globally defined $(q-m)$–form $\omega^{m.i.}$ on A. It can be proved that the existence of the maximal inner term is independent of the background connection $\mathring{\omega}$ used to define the mixed local basis, see e.g. section 2.3. of [15].

4.1.7 Integration

The integration over the Lie algebroid A is decomposed into two parts. The first one is the inner integration, which consists of getting rid of the inner (algebraic) part of the form one integrates. Then, one ends up with a form on M of maximal degree, that one can integrate on M as usually.

More precisely, the inner integration (along L) is defined as:

$$\int_{inner} : \Omega^\bullet(A) \to \Omega^{\bullet - m}(M); \omega \mapsto \omega^{m.i.}. \tag{4.13}$$

By construction, a lot of information (about the algebraic part of the integrant) is lost during the inner integration. For example, every component which does not contains vol_{loc} is killed by this integration. Then, one can define the integration of a form ω over A by:

$$\int_A \omega = \int_M \circ \int_{inner} \omega = \int_M \omega^{m.i.} \tag{4.14}$$

The integration thus gives non zero values only for forms of maximal degree (both geometric and algebraic) on A, i.e. of degree $m+n$, where $n = \text{dim}(M)$.

4.1.8 Hodge Star Operator

Given a metric $\hat{g}$ on the Lie algebroid $\mathcal{A}$, one can construct a Hodge star operator relating the space $\Omega^q(\mathcal{A}, E)$ and $\Omega^{n+m-q}(\mathcal{A}, E)$, where $n = \text{dim}(M)$, $m = \text{dim}(\mathfrak{g})$, and E is a vector bundle of representation of $\mathcal{A}$. If $\omega \in \Omega^q(\mathcal{A})$[a] can be decomposed as 4.3, i.e.:

$$\omega_{loc} = \sum_{r+s=q} (\omega_{loc})_{\mu_1 \dots \mu_r a_1 \dots a_s} \text{d}x^{\mu_1} \wedge \text{d}x^{\mu_2} \dots \wedge \text{d}x^{\mu_r} \wedge \mathring{\omega}^{a_1}_{loc} \dots \wedge \mathring{\omega}^{a_2}_{loc} \dots \wedge \mathring{\omega}^{a_s}_{loc} \tag{4.15}$$

then $*\omega_{loc} \in \Omega^{m+n-q}(A)$ is defined as:

$$\begin{aligned} *\omega_{loc} := \sum_{r+s=q} \frac{1}{r!s!} \sqrt{\text{det}(h_{loc})} \sqrt{\text{det}(g)} (\omega_{loc})_{\mu_1 \dots \mu_r a_1 \dots a_s} \epsilon_{\nu_1 \dots \nu_n} \epsilon_{b_1 \dots b_m} \\ \times\, g^{\mu_1 \nu_1} \dots g^{\mu_r \nu_r} h^{a_1 b_1} \dots h^{a_s b_s} \text{d}x^{\nu_{r+1}} \wedge \dots \wedge \text{d}x^{\mu_n} \wedge \mathring{\omega}^{b_{s+1}}_{loc} \wedge \dots \wedge \mathring{\omega}^{a_m}_{loc} \end{aligned} \tag{4.16}$$

This Hodge operator takes in account both kinds of degrees of form, the geometric ones and the algebraic ones. As in the usual (purely geometric) case, it will allow to construct forms of maximal degrees $(m+n)$ which can be integrated over A.

The method will thus be to construct maximal degrees form, then to write them in a way which makes the volume form appearing, and then to pick up the maximal inner term to end up with a maximal degree form on M which can be integrated as usual. One defines the following scalar product between two forms $\omega \in \Omega^p(\mathcal{A})$ and $\eta \in \Omega^q(\mathcal{A})$ with values in functions:

$$< \omega, \eta > := \int_A (\omega \wedge *\eta) \tag{4.17}$$

By construction, one gets $< \omega, \eta > = 0$ if $p \neq q$. If $p = q$, the degrees of forms of ω are completed by the degrees of forms of $*\eta$ and $< \omega, \eta >$ does not necessarily

[a] For simplicity one takes a $\mathcal{C}^\infty(M)$–valued q–form, but the formula stays the same in the general case.

vanish. Differential forms of distinct degrees are orthogonal with respect to this scalar product. This scalar product also leads to relations of orthogononality between the bi-graduations associated to the local decomposition of a differential form on the mixed local basis $(dx^\mu, \mathring{\omega}^a_{loc})$. The local (r,s)–form is orthogonal to any local form with bi-degree (r', s') except for $r = r'$ and $s = s'$. This scalar product can naturally be extended to forms with values in the kernel $\mathcal{L}$, if the latter is provided with a non–degenerate metric h, by:

$$< \omega, \eta >_h := \int_A h(\omega, *\eta) \tag{4.18}$$

This scalar product defined on $\Omega^\bullet(\mathcal{A}, \mathcal{L})$ is of primary importance in order to write gauge invariant action on transitive Lie algebroids.

4.2 Riemannian geometry on $\mathcal{A}$ and Einstein–Hilbert–Λ–Yang–Mills unified Lagrangian

In their work [13], [46], N. Boroojerdian et al. write a unified Lagrangian by using the generalized notion of Levi–Civita connection on a transitive Lie algebroid and its corresponding Riemann, Ricci and scalar curvatures. We rewrite here this work, recasted in our formalism, defining properly quantities we are dealing with.

4.2.1 The Mixed Local Basis

In this approach, one takes a non degenerate metric $\hat{g}$ on a transitive Lie algebroid $\mathcal{A}$ as field variable. One then knows that it is equivalent to a certain triple (h, ω, g). The key idea is to place in the *mixed local basis* (direct and dual) defined from ω and its corresponding splitting σ (see 4.1). All computations will be made on a trivial Lie algebroid $TLA(U, \mathfrak{g})$ over a certain open subset $U \subset M$, and in the basis: $(\{e_\mu\}_\mu, \tilde{E}_a\}_a)$ and its dual $(\{dx^\mu\}_\mu, \{\omega^a\}_a)$. The practical side of the use of this basis is that the decomposition of any element $\mathfrak{X}_{loc} = X^\mu e_\mu + \gamma^a \tilde{E}_a$ is a local decomposition respecting the orthogonality defined by $\hat{g}$. This makes all the difference, and render all computations easier.

4.2.2 Levi–Civita Connection

We recall that once a non–degenerate metric $\hat{g}$ is given on $\mathcal{A}$, the corresponding Levi–Civita connection $\hat{\nabla} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the object implicitly defined by the formula:

$$2\hat{g}(\hat{\nabla}_{\mathfrak{X}}\mathfrak{Y},\mathfrak{Z}) = \rho(\mathfrak{X})\cdot\hat{g}(\mathfrak{Y},\mathfrak{Z}) + \rho(\mathfrak{Y})\cdot\hat{g}(\mathfrak{Z},\mathfrak{X}) - \rho(\mathfrak{Z})\cdot\hat{g}(\mathfrak{X},\mathfrak{Y}) + \hat{g}([\mathfrak{X},\mathfrak{Y}],\mathfrak{Z}) - \hat{g}([\mathfrak{Y},\mathfrak{Z}],\mathfrak{X}) + \hat{g}([\mathfrak{Z},\mathfrak{X}],\mathfrak{Y}). \tag{4.19}$$

We are interested here in the local formulation of $\hat{\nabla}$. From now on, we will never put the label "loc" precising that we deal with the local writing of objects such that $\hat{\nabla}$ or $\hat{R}$ and so on, in order not to render the reading too heavy. It will be always clear with the context. We define the following generalized Christoffel symbols:

$$\hat{\nabla}_\mu e_\nu = \Gamma^\sigma_{\mu\nu} e_\sigma + \Gamma^a_{\mu\nu}\tilde{E}_a. \tag{4.20}$$

$$\hat{\nabla}_\mu \tilde{E}_b = \Gamma^\sigma_{\mu b} e_\sigma + \Gamma^a_{\mu b}\tilde{E}_a. \tag{4.21}$$

$$\hat{\nabla}_a e_\nu = \Gamma^\sigma_{a\nu} e_\sigma + \Gamma^b_{a\nu}\tilde{E}_b. \tag{4.22}$$

$$\hat{\nabla}_a \tilde{E}_b = \Gamma^\sigma_{ab} e_\sigma + \Gamma^c_{ab}\tilde{E}_c. \tag{4.23}$$

Here, we denoted $\hat{\nabla}_\mu := \hat{\nabla}_{\partial_\mu}$ and $\hat{\nabla}_a := \hat{\nabla}_{\tilde{E}_a}$ where the label "loc" is omitted. For example, since the basis is adapted to the orthogonality defined by $\hat{g}$, one has: $\hat{g}(\hat{\nabla}_\mu e_\nu, e_\rho) = \Gamma^\sigma_{\mu\nu}\hat{g}(e_\sigma, e_\rho) = \Gamma^\sigma_{\mu\nu} g_{\sigma\rho}$. Thus, one gets: $\Gamma^\sigma_{\mu\nu} = g^{\sigma\rho}\hat{g}(\hat{\nabla}_\mu e_\nu, e_\rho)$. The same computation for the other terms leads finally to the formulas:

$$\Gamma^\sigma_{\mu\nu} = g^{\sigma\rho}\hat{g}(\hat{\nabla}_\mu e_\nu, e_\rho) \tag{4.24}$$

$$\Gamma^c_{\mu\nu} = h^{ac}\hat{g}(\hat{\nabla}_\mu e_\nu, \tilde{E}_a) \tag{4.25}$$

$$\Gamma^\sigma_{\mu b} = g^{\sigma\rho}\hat{g}(\hat{\nabla}_\mu \tilde{E}_b, e_\rho) \tag{4.26}$$

$$\Gamma^c_{\mu b} = h^{ac}\hat{g}(\hat{\nabla}_\mu \tilde{E}_b, \tilde{E}_a) \tag{4.27}$$

$$\Gamma^\sigma_{a\nu} = g^{\sigma\rho}\hat{g}(\hat{\nabla}_a e_\nu, e_\rho) \tag{4.28}$$

$$\Gamma^c_{a\nu} = h^{bc}\hat{g}(\hat{\nabla}_a e_\nu, \tilde{E}_b) \tag{4.29}$$

$$\Gamma^\sigma_{ab} = g^{\sigma\rho}\hat{g}(\hat{\nabla}_a \tilde{E}_b, e_\rho) \tag{4.30}$$

$$\Gamma^c_{ab} = h^{cd}\hat{g}(\hat{\nabla}_a \tilde{E}_b, \tilde{E}_d) \tag{4.31}$$

Before performing the computations, one should notice some results. Let us compute $[e_\mu, e_\nu]$ using the definition of the Lie bracket on $TLA(U, \mathfrak{g})$:

$$\begin{aligned}
[e_\mu, e_\nu] &= [\partial_\mu \oplus A_\mu, \partial_\nu \oplus A_\nu] \\
&= [\partial_\mu, \partial_\nu] \oplus (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \\
&= 0 \oplus F_{\mu\nu}
\end{aligned}$$

where $F_{\mu\nu} = F^a{}_{\mu\nu} E_a$ is the local version of the curvature of the connection.

One will also need another object constructed from $F^a_{\mu\nu}$ defined by:

$$\tilde{F}_{b\mu}{}^\sigma := g^{\sigma\rho} h_{ab} F_{\mu\rho}{}^a. \tag{4.32}$$

After lenghty but quite straightforward computations, one gets finally:

$$\hat{\nabla}_\mu e_\nu = \Gamma^\sigma_{\mu\nu} e_\sigma + \frac{1}{2} F^a{}_{\mu\nu} \tilde{E}_a. \tag{4.33}$$

$$\hat{\nabla}_\mu \tilde{E}_b = -\frac{1}{2} \tilde{F}_{b\mu}{}^\sigma e_\sigma + C^a_{db} A^d_\mu \tilde{E}_a. \tag{4.34}$$

$$\hat{\nabla}_a e_\nu = -\frac{1}{2} \tilde{F}_{a\nu}{}^\sigma e_\sigma \tag{4.35}$$

$$\hat{\nabla}_a \tilde{E}_b = \frac{1}{2} C^c_{ab} \tilde{E}_c. \tag{4.36}$$

where $\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$ are the usual Christoffel symbols related to the metric g, and C^a_{bc} are the structure constants of the Lie algebra $\mathfrak{g}$.

4.2.3 Riemann Curvature

Given the Levi–Civita connection $\hat{\nabla}$, its corresponding Riemann curvature is defined as:

$$\hat{R}(\mathfrak{X}, \mathfrak{Y})\mathfrak{Z} := \hat{\nabla}_{\mathfrak{X}} \hat{\nabla}_{\mathfrak{Y}} \mathfrak{Z} - \hat{\nabla}_{\mathfrak{Y}} \hat{\nabla}_{\mathfrak{X}} \mathfrak{Z} - \hat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]} \mathfrak{Z} \tag{4.37}$$

As for the Levi–Civita connection, it is sufficient to make the computations on the elements of the local mixed basis, and decomposing the result in the same basis. One can define the following local components of the Riemann curvature:

$$\hat{R}(e_\nu, e_\rho) e_\sigma = \hat{R}^\mu{}_{\nu\rho\sigma} e_\mu + \hat{R}^a{}_{\nu\rho\sigma} \tilde{E}_a \tag{4.38}$$

$$\hat{R}(e_\nu, e_\rho)\tilde{E}_a = \hat{R}^\mu{}_{\nu\rho a} e_\mu + \hat{R}^b{}_{\nu\rho a}\tilde{E}_b \tag{4.39}$$

$$\hat{R}(e_\nu, \tilde{E}_a)e_\sigma = \hat{R}^\mu{}_{\nu a\sigma} e_\mu + \hat{R}^b{}_{\nu a\sigma}\tilde{E}_b \tag{4.40}$$

$$\hat{R}(e_\nu, \tilde{E}_a)\tilde{E}_b = \hat{R}^\mu{}_{\nu ab} e_\mu + \hat{R}^c{}_{\nu ab}\tilde{E}_c \tag{4.41}$$

$$\hat{R}(\tilde{E}_a, \tilde{E}_b)e_\sigma = \hat{R}^\mu{}_{ab\sigma} e_\mu + \hat{R}^c{}_{ab\sigma}\tilde{E}_c \tag{4.42}$$

$$\hat{R}(\tilde{E}_a, \tilde{E}_b)\tilde{E}_c = \hat{R}^\mu{}_{abc} e_\mu + \hat{R}^d{}_{abc}\tilde{E}_d \tag{4.43}$$

One defines a covariant derivative formed with the Christoffel symbols related to g and the structure constants of $\mathfrak{g}$ acting on $\tilde{F}_{a\rho}{}^\kappa$ by:

$$(\nabla_\nu F)_{a\rho}{}^\kappa := \frac{1}{2}(\partial_\nu \tilde{F}_{a\rho}{}^\kappa + \Gamma^\kappa_{\nu\sigma}\tilde{F}_{a\rho}{}^\sigma - \Gamma^\sigma_{\nu\rho}\tilde{F}_{a\sigma}{}^\kappa - C^c_{da}A^d_\nu \tilde{F}_{c\rho}{}^\kappa) \tag{4.44}$$

and another one acting on $F^c{}_{\rho\sigma}$:

$$(\nabla_\nu F)^c{}_{\rho\sigma} := \frac{1}{2}(\partial_\nu F^c{}_{\rho\sigma} - \partial_\rho F^c{}_{\nu\sigma} + \Gamma^\kappa_{\rho\sigma}F^c{}_{\nu\kappa} - \Gamma^\kappa_{\nu\sigma}F^c{}_{\rho\kappa} + C^e_{dc}F^c{}_{\rho\sigma}A^d_\nu - C^e_{dc}F^c{}_{\nu\sigma}A^d_\rho) \tag{4.45}$$

Let us insist that it makes sense to perform these computations with all these definitions, for $\hat{R}$ is a globally defined object on the Lie algebroid written in a basis which has homogeneous transformations under a change of trivialization. After even more lengthy but still straightforward computations, one finally gets:

$$\hat{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + \frac{1}{4}(-F^d{}_{\rho\sigma}\tilde{F}_{d\nu}{}^\mu + F^d{}_{\nu\sigma}\tilde{F}_{d\rho}{}^\mu + 2F^d{}_{\rho\nu}\tilde{F}_{d\sigma}{}^\mu) \tag{4.46}$$

$$\hat{R}^c{}_{\nu\rho\sigma} = (\nabla_\nu F)^c{}_{\rho\sigma} \tag{4.47}$$

$$\hat{R}^\mu{}_{\nu\rho a} = -(\nabla_\nu \tilde{F})_{a\rho}{}^\mu + (\nabla_\rho \tilde{F})_{a\nu}{}^\mu \tag{4.48}$$

$$\hat{R}^e{}_{\nu\rho a} = \frac{1}{4}(\tilde{F}_{a\nu}{}^\sigma F^e_{\rho\sigma} - \hat{F}_{a\rho}{}^\sigma F^e_{\nu\sigma}) + \frac{1}{2}F^d_{\rho\nu}C^e_{da} \tag{4.49}$$

$$\hat{R}^\mu{}_{\nu a\sigma} = -(\nabla_\nu \tilde{F})_{a\sigma}{}^\mu \tag{4.50}$$

$$\hat{R}^c{}_{\nu a\sigma} = -\frac{1}{4}(\tilde{F}_{a\sigma}{}^\rho F^c{}_{\nu\rho} + C^c_{ad}F^d{}_{\nu\sigma}) \tag{4.51}$$

$$\hat{R}^{\mu}{}_{\nu ab} = -\frac{1}{4}(\tilde{F}_{b\nu}{}^{\sigma}\tilde{F}_{a\sigma}{}^{\mu} + C^{c}_{ab}\tilde{F}_{c\nu}{}^{\mu}) \tag{4.52}$$

$$\hat{R}^{c}{}_{\nu ab=0} \tag{4.53}$$

$$\hat{R}^{\mu}{}_{ab\sigma} = \frac{1}{4}\tilde{F}_{b\sigma}{}^{\nu}\tilde{F}^{\mu}_{a\nu} - \frac{1}{4}\tilde{F}_{a\sigma}{}^{\nu}\tilde{F}^{\mu}_{b\nu} + \frac{1}{2}C^{c}_{ab}\tilde{F}_{c\sigma}{}^{\mu} \tag{4.54}$$

$$\hat{R}^{c}{}_{ab\sigma} = 0 \tag{4.55}$$

$$\hat{R}^{\mu}{}_{abc} = 0 \tag{4.56}$$

$$\hat{R}^{a}{}_{bcd} = \frac{1}{4}C^{a}_{de}C^{e}_{bc} \tag{4.57}$$

where $R^{\mu}{}_{\nu\rho\sigma}$ is the Riemann curvature related to the Levi–Civita connection of the metric g.

4.2.4 Ricci curvature

One can define the Ricci curvature as:

$$\widehat{Ric}(\mathfrak{X}_{loc}, \mathfrak{Y}_{loc}) :=< dx^{\rho}, \hat{R}(e_{\rho}, \mathfrak{Y}_{loc})\mathfrak{X}_{loc} > + < \omega^{a}_{loc}, \hat{R}(\tilde{E}_{a}, \mathfrak{Y}_{loc})\mathfrak{X}_{loc} > \tag{4.58}$$

where $< \, , \, >$ denotes the duality bracket. In particular, one has:

$$\widehat{Ric}_{\mu\nu} := \widehat{Ric}(e_{\mu}, e_{\nu}) =< dx^{\rho}, \hat{R}(e_{\rho}, e_{\nu})e_{\mu} > + < \omega^{a}_{loc}, \hat{R}(\tilde{E}_{a}, e_{\nu})e_{\mu} > \tag{4.59}$$

and

$$\widehat{Ric}_{ab} := \widehat{Ric}(\tilde{E}_{a}, \tilde{E}_{b}) =< dx^{\rho}, \hat{R}(e_{\rho}, \tilde{E}_{b})\tilde{E}_{a} > + < \omega^{a}_{loc}, \hat{R}(\tilde{E}_{a}, \tilde{E}_{b})\tilde{E}_{a} > \tag{4.60}$$

With the help of the results for the Riemann curvature, one straightforwardly gets:

$$\widehat{Ric}_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}g^{\rho\kappa}h_{ab}F^{b}_{\nu\kappa}F^{a}_{\mu\rho} \tag{4.61}$$

and

$$\widehat{Ric}_{ab} = \frac{1}{4}\tilde{F}_{[a|\nu|}{}^{\mu}\tilde{F}_{b]\mu}{}^{\nu} + \frac{1}{4}C^{c}_{ae}C^{e}_{bc} \tag{4.62}$$

where Ric is the Ricci curvature related to g.

4.2.5 Scalar Curvature

Finally, one can compute the scalar curvature related to the Levi–Civita connection $\hat{\nabla}$. One defines it as the trace of the Ricci, but since the metric $\hat{g}$ is bloc diagonal in the basis we use, one has:

$$\hat{R} := g^{\mu\nu}\widehat{Ric}_{\mu\nu} + h^{ab}\widehat{Ric}_{ab} \tag{4.63}$$

One finds eventually:

$$\hat{R} = R - \frac{1}{2}g^{\nu\sigma}g^{\rho\kappa}h_{ab}F^b_{\ \sigma\kappa}F^a_{\ \nu\rho} + \frac{1}{4}C^c_{ae}C^e_{bc}h^{ab} \tag{4.64}$$

$\widehat{R}$ is a scalar field defined on U, but by construction it is the localization of a globally defined scalar field. The same thing occurs for its different components: R, $\frac{1}{2}g^{\nu\sigma}g^{\rho\kappa}F^b_{\ \sigma\kappa}F^a_{\ \nu\rho}$ and $\frac{1}{4}C^c_{ae}C^e_{bc}h^{ab}$ are the local formulations of globally defined objects, that one write respectively R, $2 < F, F >$ and 2Λ. Indeed, one can verify that the three terms are invariant under a change of trivialisation.

Let us verify it for the last term $C^c_{ae}C^e_{bc}h^{ab}$, which is not so obvious. We are going to show that $C^c_{ae}C^e_{bc}$ is the local version of a trivialized but globally defined metric on $\mathcal{L}$, called the Killing metric. The first step is to define a trace–operator on the space of endomorphisms of $\mathcal{L}$, i.e. a well defined operator:

$$Tr : \Gamma(End(\mathcal{L})) \to \mathcal{C}^\infty(M).$$

Let us first define locally the trace–operator $tr : End(\mathfrak{g}) \to \mathbb{C}$. If $\phi \in End(\mathcal{L})$, $\phi_{loc} : U \to End(\mathfrak{g})$. The element $\phi(\gamma)$ will trivialize as $\phi_{loc}(\gamma_{loc})$ and $\phi'_{loc}(\gamma'_{loc})$ in another trivialization, such that $\phi'_{loc}(\gamma'_{loc}) = \phi'_{loc} \circ \alpha(\gamma_{loc})$ on the first hand and $\phi'_{loc}(\gamma'_{loc}) = \alpha \circ \phi_{loc}(\gamma_{loc})$ on the other hand. Thus,

$$\phi'_{loc} = \alpha \circ \phi_{loc} \circ \alpha^{-1}.$$

In a basis, $\phi_{loc} = {\phi_{loc}}^a_b$, and

$$tr(\phi_{loc}) := {\phi_{loc}}^a_a.$$

If one performs a change of trivialization, one gets: ${\phi'_{loc}}^a_b = \alpha^a_d {\phi_{loc}}^d_c {\alpha^{-1}}^c_{\ a}$, i.e. $tr(\phi'_{loc}) = {\phi_{loc}}'^a_a = {\phi_{loc}}^a_a = tr(\phi_{loc})$. This invariance under a change of trivialization allows to globally define an operator Tr which trivializes as tr.

Now, on can define on $\mathcal{L}$ the corresponding Killing metric:

$$K(\gamma_1, \gamma_2) := Tr(ad_{\gamma_1} \circ ad_{\gamma_2}), \tag{4.65}$$

for $\gamma_1, \gamma_1 \in \mathcal{L}$. This is a well defined metric on $\mathcal{L}$, since Tr and ad are well defined operators on $\mathcal{L}$. Let us place in a trivialization to see how does K look like. One writes the trivialization of elements of $\mathcal{L}$ without the label "loc" to render the reading easier. One has $(ad_{\gamma_1})^a_c = C^a{}_{bc}\gamma_1^b$ and thus $(ad_{\gamma_1} \circ ad_{\gamma_2})^a_b = C^a{}_{dc}C^c{}_{be}\gamma_1^d\gamma_2^e$. Thus,

$$K_{loc}(\gamma_1, \gamma_2) = C^c_{ae}C^e_{bc}\gamma_1^a\gamma_2^b.$$

One concludes that the quantity of interest: $C^c_{ae}C^e_{bc}$, being the trivialization of a well globally define metric K, transforms as a 2–covariant tensor under a change of trivialization. That is to say,

$$C^c_{ae}C^e_{bc} \to \alpha^{-1}{}^d_a\alpha^{-1}{}^f_b C^c_{de}C^e_{fc} \tag{4.66}$$

Since the metric h^{ab} transforms as a 2–contravariant tensor under a change of trivialization, it compensates the change of $C^c_{ae}C^e_{bc}$ and thus the quantity $C^c_{ae}C^e_{bc}h^{ab}$ is *invariant* under a change of trivialization. Actually, as we are going to show in the following, it can be a *constant* on the whole manifold (hence the notation Λ).

4.2.5.1 $C^c_{ae}C^e_{bc}$ is constant in any trivialization

The fact that one has the transformation 4.66 is actually quite strange. Indeed, the structure constants are constant by definition and defined without reference to any trivialization: thus the quantity $C^c_{ae}C^e_{bc}$ should not depend on a trivialization! Actually, as we are going to show now, this transformation reduces to the identity thanks to the compatibility of the isomorphism α and the Lie algebra structure. Indeed, α is an isomorphism of Lie algebras, thus it satisfies: $\alpha([E_a, E_b]) = [\alpha(E_a), \alpha(E_b)]$, i.e.:

$$\alpha^c_d C^d{}_{ab} = C^c{}_{de}\alpha^d_a\alpha^e_b.$$

One rewrites this identity under the form:

$$\alpha^d_a C^c{}_{de} = \alpha^c_d C^d{}_{ab}\alpha^{-1}{}^b_e \tag{4.67}$$

From this identity, we are going to show that $C^c_{ae}C^e_{bc} = \alpha^{-1}{}^d_a\alpha^{-1}{}^f_b C^c_{de}C^e_{fc}$ i.e. that:

$$C^c_{ae}C^e_{bc}\alpha^a_d\alpha^b_f = C^c_{de}C^e_{fc}.$$

Indeed,

$$\begin{aligned}
C^c_{ae}C^e_{bc}\alpha^a_d\alpha^b_f &= (\alpha^a_d C^c_{ae})(\alpha^b_f C^e_{bc}) \\
&= (\alpha^c_a C^a{}_{db}\alpha^{-1}{}^b_e)(\alpha^e_h C^h{}_{fg}\alpha^{-1}{}^g_c) \\
&= \delta^g_a\delta^b_h C^a{}_{db}C^h{}_{fg} \\
&= C^g{}_{dh}C^h{}_{fg}.
\end{aligned}$$

Thus, the quantity $C^c_{ae}C^e_{bc}$ is constant and equal to the same number in any trivialization.

4.2.6 Adjoint Connection and Parallel Metric on $\mathcal{L}$

Let us now show how one can render constant the quantity $C^c_{ae}C^e_{bc}h^{ab}$. Here, we see that one has to assume something more about the metric $\hat{g}$ and the Lie algebroid $\mathcal{A}$ in this aim.

First, as it is the case for gauge theories formulated in section 4.3, one has to assume that h is invariant under the adjoint action of $\mathfrak{g}$. This means that, for all $\gamma, \eta, \xi \in \mathcal{L}$, h satisfies:

$$h([\xi, \gamma], \eta) + h(\gamma, [\xi, \eta]) = 0. \tag{4.68}$$

Then, for any splitting $\sigma : \Gamma(TM) \to \mathcal{A}$, one can construct the *adjoint connection* $\nabla^\sigma : \Gamma(TM) \times \mathcal{L} \to \mathcal{L}$ which is an affine connection on $\mathcal{L}$. It is defined as follows (see e.g. [46], or [1]):

$$\nabla^\sigma_X \gamma := \iota^{-1}([\sigma_X, \iota(\gamma)]) \tag{4.69}$$

for $X \in \Gamma(TM)$ and $\gamma \in \mathcal{L}$. Since it is an affine connection on the vector bundle $\mathcal{L}$, one can define its action on geometric structures defined on $\mathcal{L}$. For e.g., if h is a non degenerate metric on $\mathcal{L}$, one says that h is parallel for σ if:

$$X \cdot h(\gamma, \eta) = h(\nabla^\sigma_X \gamma, \eta) + h(\gamma, \nabla^\sigma_X \eta). \tag{4.70}$$

It turns out that there are only two possibilities: either all splittings σ render h parallel w.r.t. ∇^σ, or none of them do. Indeed, if σ and σ' are two splittings, there exists a $\mathcal{L}$–valued form α such that $\sigma'_X - \sigma_X = \iota \circ \alpha(X)$. The corresponding adjoint connections are then related by:

$$\nabla^{\sigma'}_X \gamma - \nabla^\sigma_X \gamma = [\alpha(X), \gamma]$$

and thus we have the following identity:

$$\begin{aligned} h(\nabla^{\sigma'}_X \gamma, \eta) + h(\gamma, \nabla^{\sigma'}_X \eta) &= h(\nabla^\sigma_X \gamma, \eta) + h(\gamma, \nabla^\sigma_X \eta) + h([\alpha(X), \gamma], \eta) + h(\gamma, [\alpha(X), \eta]) \\ &= h(\nabla^\sigma_X \gamma, \eta) + h(\gamma, \nabla^\sigma_X \eta) \end{aligned}$$

thanks to the properties 4.68. Thus, one assumes that h (and thus $\hat{g}$) is *compatible* with the Lie algebroid structure, by taking it such that every splitting σ renders it parallel with respect to its corresponding adjoint connection ∇^σ.

Let us see what this means once written in a trivialisation and in the mixed local basis. For $X \in \Gamma(TM)$ and $\gamma \in \Gamma(U \times \mathfrak{g})$,

$$[\sigma_{locX}, \iota(\gamma)] = [X \oplus A(X), 0 \oplus \gamma] = 0 \oplus (X \cdot \gamma + [A(X), \gamma])$$

and thus

$$\nabla^\sigma_{locX} = X \cdot \gamma + [A(X), \gamma].$$

The dot " $\cdot$ " denotes merely the derivation by the vector field X.
Thus, one gets:

$$\begin{aligned} X \cdot h_{loc}(\gamma, \eta) &= h_{loc}(X \cdot \gamma + [A(X), \gamma], \eta) + h(\gamma, X \cdot \eta + [A(X), \eta]) \\ &= h_{loc}(X \cdot \gamma, \eta) + h([A(X), \gamma], \eta) + h(\gamma, X \cdot \eta) + h(\gamma, [A(X), \eta]) \\ &= h_{loc}(X \cdot \gamma, \eta) + h(\gamma, X \cdot \eta). \end{aligned}$$

Now, if one places in a local basis, by taking $\gamma = E_a$ and $\eta = E_b$ (basis elements) one has:

$$X \cdot h_{ab} = 0. \tag{4.71}$$

Thus, the compatibility condition of $\hat{g}$ w.r.t. the Lie algebroid structure, together with the adjoint–invariance 4.68, imply that h is locally constant, as defined in [15]. The quantity $\frac{1}{4} C^c_{ae} C^e_{bc} h^{ab}$ is thus locally constant, and since this does not depend on the trivialization, it is thus the localization of a constant on the whole manifold. We denoted this constant as 2Λ by anticipation.

4.2.7 Einstein–Hilbert–Λ–Yang–Mills Unified Lagrangian

From the metric g one can get a volume form vol_g defined on M and then write the action:

$$S = \int_M \hat{R} \, \text{vol}_g \tag{4.72}$$

Given the form of $\hat{R}$ written above, this action gives the following terms:

$$S = \int_M (R + 2\Lambda) \text{vol}_g + 2 \int_M < F, F > \text{vol}_g \tag{4.73}$$

i.e. nothing but the Einstein–Hilbert action with cosmological constant Λ together with the Yang–Mills action for the connection ω. It is quite interesting to notice that the cosmological constant Λ, in this unified Lagrangian, has an algebraic origin in this framework. That is to say, even if the cosmological constant describes a large scale gravitational physics, it comes from the symmetry generator of the other kinds of interactions, for it comes from the Lie algebra $\mathfrak{g}$.

4.2.8 Conclusion

Thus, by taking a non–degenerate metric $\hat{g}$ on a transitive Lie algebroid, one can encode both kinds of interaction and thus write a unified Lagrangian for physics. We present now another example of unified Lagrangian in the framework of transitive Lie algebroids, based on the notion of generalized connection developed in the work of C. Fournel et al., see e.g. [16].

4.3 Generalized connection and Yang–Mills–Higgs unified Lagrangian

4.3.1 Generalized connection

Any ordinary connection $\omega \in \Omega^1(\mathcal{A}, \mathcal{L})$, which thus fulfills the normalization condition $\omega \circ \iota(\gamma) = -\gamma$, comes from a certain splitting $\sigma : \Gamma(TM) \to \mathcal{A}$. One can define a generalized notion of connection, by just taking a map $\varpi \in \Omega^1(\mathcal{A}, \mathcal{L})$, without assuming the normalization condition. Thus, a generalized connection does not necessarily come from a splitting. Since $\varpi \circ \iota(\gamma) + \gamma$ does not necessarily vanish for all $\gamma \in \mathcal{L}$, one can define an endomorphism $\tau : \mathcal{L} \to \mathcal{L}$ such that:

$$\tau = \varpi \circ \iota + \mathrm{id}_{\mathcal{L}}. \tag{4.74}$$

Thus, a generalized connection carries algebraic degrees of freedom thanks to the part τ, unlike an ordinary one. If $\mathring{\omega}$ is an ordinary background connection on $\mathcal{A}$, then the generalized connection ϖ defines an ordinary connection by:

$$\omega := \varpi + \tau \circ \mathring{\omega}. \tag{4.75}$$

Indeed, one can easily verify that $\omega \circ \iota = -\mathrm{id}_{\mathcal{L}}$. For the ordinary connection $\mathring{\omega}$, one defines the covariant derivative:

$$\mathring{\Theta}(\mathfrak{X}) := \mathring{\sigma}_{\rho(\mathfrak{X})} = \mathfrak{X} + \iota \circ \mathring{\omega}(\mathfrak{X}) \tag{4.76}$$

where $\mathring{\sigma}$ is the splitting related to $\mathring{\omega}$. For the generalized connection ϖ, one can define a generalized covariant derivative $\overline{\Theta}$ by:

$$\overline{\Theta}(\mathfrak{X}) := \mathfrak{X} + \iota \circ \varpi(\mathfrak{X}), \tag{4.77}$$

even if there is no splitting related to it.

4.3.2 Curvature of a Generalized Connection

The curvature of a generalized connection ϖ is defined by the following usual formula:

$$\overline{\Omega} := \hat{\mathrm{d}}\varpi + \frac{1}{2}[\varpi, \varpi]. \tag{4.78}$$

It is a 2–form: $\overline{\Omega} \in \Omega^2(\mathcal{A}, \mathcal{L})$, and thus one can expect that its local version (i.e. $\overline{\Omega}_{loc} : \mathrm{TLA}(U, \mathfrak{g}) \times \mathrm{TLA}(U, \mathfrak{g}) \to \Gamma(U \times \mathfrak{g})$) decomposes as the following sum:

$$\overline{\Omega}_{loc} = \overline{\Omega}^{(2,0)}_{loc} + \overline{\Omega}^{(1,1)}_{loc} + \overline{\Omega}^{(0,2)}_{loc} \tag{4.79}$$

where:

- $\overline{\Omega}_{loc}^{(2,0)} : \Gamma(TU) \times \Gamma(TU) \to \Gamma(U \times \mathfrak{g})$
- $\overline{\Omega}_{loc}^{(1,1)} : \Gamma(TU) \times \Gamma(U \times \mathfrak{g}) \to \Gamma(U \times \mathfrak{g})$
- $\overline{\Omega}_{loc}^{(0,2)} : \Gamma(U \times \mathfrak{g}) \times \Gamma(U \times \mathfrak{g}) \to \Gamma(U \times \mathfrak{g})$

For the time being, on can compute $\overline{\Omega}$ as defined in 4.78 with the decomposition: $\varpi = \omega - \tau \circ \mathring{\omega}$, and arrange the different terms to reproduce the expected decomposition. The result is:

$$\overline{\Omega} = \mathcal{R} - (\mathcal{D}\tau) \circ \mathring{\omega} + \mathring{\omega}^* R_\tau \tag{4.80}$$

with:

$$\mathcal{R} = \Omega - \tau \circ \mathring{\Omega} \tag{4.81}$$

$$(\mathcal{D}\tau) \circ \mathring{\omega} = [\Theta, \tau \circ \mathring{\omega}] - \tau \circ [\mathring{\Theta}, \mathring{\omega}] \tag{4.82}$$

$$\mathring{\omega}^* R_\tau = \frac{1}{2}(\tau \circ [\mathring{\omega}, \mathring{\omega}] - [\tau \circ \mathring{\omega}, \tau \circ \mathring{\omega}]) \tag{4.83}$$

with:

- Ω: the curvature associated to the induced ordinary connection ω;
- $\mathring{\Omega}$: the curvature associated to the background ordinary connection $\mathring{\omega}$;
- Θ: the covariant derivative related to the induced ordinary connection ω;
- $\mathring{\Theta}$: the covariant derivative related to the background connection $\mathring{\omega}$.

4.3.3 Gauge Transformation

The Lie derivative along any element $\mathfrak{X} \in \mathcal{A}$ combines both an infinitesimal diffeomorphism and an infinitesimal gauge (internal) transformation. To see that, let us work on a trivialization, or on the trivial Lie algebroid TLA$(U, \mathfrak{g})$. Let ω be a connection 1–form on it, that is to say $\omega : \Gamma(TU) \bigoplus \Gamma(U \times \mathfrak{g}) \to \Gamma(U \times \mathfrak{g})$ such that $\omega(0 \oplus \gamma) = -\gamma$. Let call A the map from $\Gamma(TU)$ to $\Gamma(U \times \mathfrak{g})$ such that $\omega(X \oplus \gamma) = A(X) - \gamma$.

Then, let us take an element $X \oplus \gamma \in$ TLA$(U, \mathfrak{g})$: $L_{X \oplus \gamma}\omega$ represents the infinitesimal gauge transformations (encoding both internal one with γ and external one with X). Let us compute:

$$
\begin{aligned}
(\hat{\mathsf{d}} \circ i_{X\oplus\gamma}\omega)(Y \oplus \eta) &= (\hat{\mathsf{d}}\omega(X \oplus \gamma))(Y \oplus \eta) = (Y \oplus \eta) \cdot \omega(X \oplus \gamma) \\
(i_{X\oplus\gamma} \circ \hat{\mathsf{d}}\omega)(Y \oplus \eta) &= (\hat{\mathsf{d}}\omega)(X \oplus \gamma, Y \oplus \eta) \\
&= (X \oplus \gamma) \cdot \omega(Y \oplus \eta) - (Y \oplus \eta) \cdot \omega(X \oplus \gamma) - \omega([X \oplus \gamma, Y \oplus \eta])
\end{aligned}
$$

Recall that $[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])$, thus:

$$
\begin{aligned}
(L_{X\oplus\gamma}\omega)(Y \oplus \eta) &= (X \oplus \gamma) \cdot \omega(Y \oplus \eta) - \omega([X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])) \\
&= X \cdot (A(Y) - \eta) + [\gamma, A(Y) - \eta] - A([X, Y]) + X \cdot \eta - Y \cdot \gamma + [\gamma, \eta] \\
&= X \cdot A(Y) - d\gamma(Y) + [\gamma, A(Y)] - A([X, Y]).
\end{aligned}
$$

$X \cdot A(Y) - A([X, Y]) = (L_X A)(Y)$, thus this part represents the transformation of A under infinitesimal diffeomorphisms through the usual Lie derivative. The other part can be written:

$$A^\gamma = -(d\gamma + [A, \gamma]). \tag{4.84}$$

One can recognize the usual infinitesimal gauge transformation (up to a sign). This transformation is called *geometric* because it directly comes from the Lie derivative defined on the Lie algebroid. In intrinsic notation, one gets the following internal gauge transformation of a connection 1–form ω:

$$\omega^\gamma = \omega - \hat{\mathsf{d}}\gamma - [\gamma, \omega] \tag{4.85}$$

Let us remark that in the case of an Atiyah Lie algebroid coming from a principal fibre bundle P, this transformation is the infinitesimal version of a gauge transformation of the related Ehresman connection on P.

It is shown in [15] that if one tries to compute the gauge transformation of a generalized connection ϖ, one gets a non usual transformation, which does not give the good properties to the structure which would allow to construct generalized gauge theories. Thus, from considerations of covariant derivative type, see section 6.2.1 of [16] the gauge transformation of a generalized connection ϖ is *choosen* to be:

$$\varpi^\gamma = \varpi - \hat{\mathsf{d}}\gamma - [\gamma, \varpi] \tag{4.86}$$

This is called *algebraic* gauge transformation, for it does not come directly from geometric considerations. Given this transformation, the generalized curvature transforms homogeneously, like the curvature of an ordinary connection:

$$\overline{\Omega}^\gamma = -[\gamma, \overline{\Omega}] \tag{4.87}$$

4.3.4 Yang–Mills–Higgs Unified Lagrangian

In order to write a unified Lagrangian which contains both Yang–Mills theory and a Higgs potential, one takes a background inner non degenerate metric $\hat{g}$ such that the inner metric h be invariant under the adjoint action of the kernel, i.e. such that:

$$h([\xi, \omega], \eta) + h(\omega, [\xi, \eta]) = 0 \tag{4.88}$$

and a generalized connection $\varpi \in \Omega^1(\mathcal{A}, \mathcal{L})$ as field variable. ϖ will decompose with the help of the background connection $\mathring{\omega}$ coming from the triple related to the metric $\hat{g}$, to give an induced connection ω (locally A_μ) and an endomorphism τ. Knowing the gauge transformations of the generalized curvature, and the property 4.88, one can write the generalized *gauge invariant* Yang–Mills type Action:

$$S(\varpi) = < \overline{\Omega}, \overline{\Omega} >_h = \int_A h(\overline{\Omega}, *\overline{\Omega}). \tag{4.89}$$

The invariance of the action is mainly due to the invariance 4.88 of the metric h. Recall that the generalized curvature locally decomposes in three parts $\overline{\Omega}_{loc,(2,0)}$, $\overline{\Omega}_{loc,(1,1)}$ and $\overline{\Omega}_{loc,(0,2)}$ according to the bi–grading. Developing locally the action 4.89 will then, by the property of orthogonality of $*$, give only three terms, for only terms with the same bi-gradation will not annhilate each other. The strategy is first to compute explicitly $h(\overline{\Omega}, *\overline{\Omega})$ locally, by using the mixed local basis to make the volume form appear. Then, the integration along $\mathcal{L}$ is made by "reading" the maximal inner term in front of the volume form. Explicitly, directly taken from [15], the action reads finally:

$$\begin{aligned}
&S(A, \tau) = \int_M \lambda_1 g^{\mu_1\mu_2} g^{\nu_1\nu_2} h_{a_1a_2} \\
&(\partial_{\mu_1} A^{a_1}_{\nu_1} - \partial_{\nu_1} A^{a_1}_{\mu_1} + A^{b_1}_{\mu_1} A^{c_1}_{\nu_1} C^{a_1}_{b_1c_1} - \tau^{a_1}_{b_1}((\partial_{\mu_1} \mathring{A}^{b_1}_{\nu_1} - \partial_{\nu_1} \mathring{A}^{b_1}_{\mu_1} + \mathring{A}^{d_1}_{\mu_1} \mathring{A}^{e_1}_{\nu_1} C^{b_1}_{d_1e_1})). \\
&(\partial_{\mu_2} A^{a_2}_{\nu_2} - \partial_{\nu_2} A^{a_2}_{\mu_2} + A^{b}_{\mu_2} A^{c}_{\nu_2} C^{a_2}_{bc} - \tau^{a_2}_{b_2}((\partial_{\mu_2} \mathring{A}^{b_2}_{\nu_2} - \partial_{\nu_2} \mathring{A}^{b_2}_{\mu_2} + \mathring{A}^{d_1}_{\mu_2} \mathring{A}^{e_1}_{\nu_2} C^{b_2}_{d_1e_1})) \\
&+ \lambda_2 g^{\mu_2\mu_1} h^{a_2a_1} h_{b_1,b_2} \\
&(\partial_{\mu_1} \tau^{b_1}_{a_1} + A^{c_1}_{\mu_1} \tau^{d_1}_{a_1} C^{b_1}_{c_1d_1} - \mathring{A}^{c_1}_{\mu_1} \tau^{b_1}_{d_1} C^{d_1}_{c_1a_1}).(\partial_{\mu_2} \tau^{b_2}_{a_2} + A^{c_2}_{\mu_2} \tau^{d_2}_{a_2} C^{b_2}_{c_2d_2} - \mathring{A}^{c_2}_{\mu_2} \tau^{b_2}_{d_2} C^{d_2}_{c_2a_2}) \\
&+ \lambda_3 h_{c_1c_2} h^{a_1a_2} h^{b_1b_2} (\tau^{c_1}_{d_1} C^{d_1}_{a_1b_1} - \tau^{d_1}_{a_1} \tau^{e_1}_{b_1} C^{c_1}_{d_1e_1}).(\tau^{c_2}_{d_2} C^{d_2}_{a_2b_2} - \tau^{d_2}_{a_2} \tau^{e_2}_{b_2} C^{c_2}_{d_2e_2}) vol_g
\end{aligned} \tag{4.90}$$

where λ_1, λ_2 and λ_3 are constants coming from the definition of the Hodge operator, and vol_g is the volume form on M related to the metric g. The elements C^a_{bc} are structure constants of the Lie algebra $\mathfrak{g}$. This action describes massless vector bosons A_μ coupled to a multi-index scalar field τ^a_b embedded into a quartic potential given by the third term. It turns out that the first term gives the Yang–

Mills Lagrangian related to the induced connection ω, which plays the role of the Yang–Mills potential; the second term is the covariant derivative of the scalar field and gives a minimal coupling between the scalar fields τ_b^a and the gauge bosons A_μ.

4.3.5 Conclusion

Thus, with the help of a generalized connection ϖ on the transitive Lie algebroid $\mathcal{A}$, one can encode both a Yang–Mills term and a Higgs potential term in a unified generalized Yang–Mills Lagrangian. Unlike in the previous example, the metric $\hat{g}$ and in particular the connection $\mathring{\omega}$ is taken as background data, i.e. is not dynamical.

Chapter 5

Cartan Geometry in the Framework of Transitive Lie Algebroids

5.1 Introduction – Cartan geometry: additional material

One presents here additional material concerning Cartan geometry, which can still be found in the already cited references [50], and [53]. Recall that a reductive Cartan geometry is given by:

- Two Lie groups (G, H), with $H \subset G$, such that $M_0 := G/H$ is an homogeneous manifold w.r.t. the action of G,
- $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{h}$ is a H–module, i.e. $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$,
- A H–principal bundle P over the base smooth manifold M, such that dim$(P) =$ dim$(\mathfrak{g})$.

A Cartan connection ϖ on P is a 1–form $\varpi : TP \to \mathfrak{g}$ such that:

- $\varpi(\xi^v) = \xi$, with ξ^v and $\xi \in \mathfrak{h}$ the vertical vector and its canonically associated Lie algebra element,
- $R_h^* \varpi = Ad_{h^{-1}} \varpi$,
- $\varpi_p : T_pP \to \mathfrak{g}$ is an isomorphism of vector spaces for all $p \in P$.

One shall see in the introduction of this chapter the additional material related to a Cartan connection, in particular the underlying G–principal bundle and its Ehresman connection.

5.1.1 The G–principal bundle Q related to a Cartan H–principal bundle P

Let us define the following fibre bundle: $Q := P \times_H G$, associated to P via the left action of H on G. That means that one has quotiented $P \times G$ by the equivalence relation:

$$(p, g) \sim (p \cdot h, h^{-1} g) \tag{5.1}$$

with $h \in H$. One calls π_Q the map : $P \times G \to Q$ which realizes the quotient. One gets the corresponding tangent map: $T\pi_Q : T(P \times G) = TP \oplus TG \to TQ$. One denotes $\mathfrak{X}$ an element in TQ and X and ξ its corresponding elements in $TP \oplus TG$, i.e. such that

$$T_{(p,g)}\pi_Q(X_{|p} \oplus \xi_{|g}) = \mathfrak{X}_q \tag{5.2}$$

A vector field $X \oplus \xi \in \Gamma(TP \oplus TG)$ induces a well defined vector on Q if and only if it is well defined on each $T_{|[p,g]}Q$, so if and only if its definition does not depend on the representative $(p, g) \in [p, g]$, i.e. under the condition:

$$T_{(p,g)}\pi_Q(X_{|p} \oplus \xi_{|g}) = T_{(ph,h^{-1}g)}\pi_Q(X_{|ph} \oplus \xi_{|h^{-1}g}) \tag{5.3}$$

for any $p \in P$, $g \in G$ and $h \in H$.

Q is a G–principal bundle. One has the natural inclusion homomorphism $\zeta : P \hookrightarrow Q$ given by $\zeta(p) := [p, e]$, with e the identity element of G.

5.1.2 The Ehresman connection related to a Cartan connection

Given ϖ a Cartan connection on P, one can construct a corresponding Ehresman connection ω on Q. Let $\pi_P : P \times G \to P$ and $\pi_G : P \times G \to G$ the projections on respectively P and G. One gets the corresponding tangent maps $T\pi_P$ and $T\pi_G$:

- $T_p\pi_P : TP \oplus TG \to TP$,
- $T_p\pi_G : TP \oplus TG \to TG$.

One has to define a 1–form on G, i.e. a map $\omega : TQ \to \mathfrak{g}$. There is a possibility to define it on $T(P \times G) = TP \oplus TG$ such that it naturally passes to the quotient. Let us define:

$$\hat{\omega}_{|(p,g)}(X_{|p} \oplus \xi_{|g}) := Ad_{g^{-1}}(\varpi_p(X_{|p})) + \omega_{G|g}(\xi_{|g}) \tag{5.4}$$

where ω_G is the Maurer–Cartan form on G. Then, one pointwisely defines:

$$\omega_{|q}(\mathfrak{X}_{|q}) := \hat{\omega}_{|(p,g)}(X_{|p} \oplus \xi_{|g}) \tag{5.5}$$

with $T_{(p,g)}\pi_Q(X_{|p} \oplus \xi_{|g}) = \mathfrak{X}_q$. It turns out that ω is an Ehresman connection well defined on Q. One remarks that from a field theory perspective, both ω and ϖ contain the same amount of degrees of freedom, since what is added to ϖ is the Maurer–Cartan form which is given with the group G, and unique. Thus, whereas Q can always be defined, an Ehresman connection on Q is something more general than a Cartan connection on P. Thus, a natural question is to wonder at which condition(s) an Ehresman connection on Q is equivalent to a Cartan connection on P. Let us remark that this construction works also to pass from any *Ehresman* connection on P to an Ehresman connection on Q.

5.1.3 Ehresman connections on Q which are Cartan connections on P

The main difference between an Ehresman connection ω and a Cartan connection ϖ is that the latter has no kernel, for $\varpi_p : T_pP \to \mathfrak{g}$ is an isomorphism, whereas ω has. $\zeta : P \hookrightarrow Q$ gives the tangent map $\zeta_* = T\zeta : TP \hookrightarrow TQ$. If ω is such that it can be constructed from a certain Cartan connection ϖ on P, one expects that the restriction of ω to $\zeta_*(TP) \subset TQ$ gives all information encoded in ϖ. In particular, ω is expected to realize an isomorphism from $T_{\zeta(p)}\zeta(TP)$ to $\mathfrak{g}$ for each $p \in P$, that is to say:

$$\ker(\omega_{|\zeta(p)}) \cap T_{\zeta(p)}\zeta(TP) = \{0\} \tag{5.6}$$

It turns out that this condition is necessary and sufficient for ω to be derived from a Cartan connection ϖ on P, i.e. to be of the form 5.5. One has indeed the following result:

$$\left\{ \begin{array}{l} \text{Ehresman connections } \omega \text{ on } Q \text{ s.t.} \\ \ker(\omega_{|\zeta(p)}) \cap T_{\zeta(p)}\zeta(TP) = \{0\},\ \forall p \in P \end{array} \right\} \overset{1:1}{\longleftrightarrow} \{\text{Cartan connections } \varpi \text{ on P.}\} \tag{5.7}$$

the symbol $\overset{1:1}{\longleftrightarrow}$ means that there is a one–to–one correspondence between these two sets.

5.2 The commutative diagram of transitive Lie algebroids related to a Cartan geometry

Let us present now the basics of our framework to recast Cartan geometry in the language of Lie algebroids which takes the form of a commutative diagram of transitive Lie algebroids. All this section is based on the forthcoming paper [3].

5.2.1 Two short exact sequences of Transitive Lie algebroid for a Cartan geometry

Given a Cartan geometry with H–principal fibre bundle P modeled on (G, H), one can construct the corresponding G–principal bundle $Q = P \times_H G$. Thus, one has the two following corresponding Atiyah sequences of Lie algebroids:

$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{\iota_P} \Gamma_H(TP) \xrightarrow{\rho_P} \Gamma(TM) \longrightarrow 0 \tag{5.8}$$

$$0 \longrightarrow \Gamma_G(Q, \mathfrak{g}) \xrightarrow{\iota_Q} \Gamma_G(TQ) \xrightarrow{\rho_Q} \Gamma(TM) \longrightarrow 0 \tag{5.9}$$

One will now show that one can define natural maps between these objects which are homomorphisms of $\mathcal{C}^\infty(M)$–modules and turn the two latter short exact sequences into a commutative diagram in which the construction of a Cartan connection naturally takes place.

5.2.2 The diagram

The diagram is made in subsequent steps.

5.2.2.1 $\Gamma_G(Q, \mathfrak{g}) \simeq \Gamma_H(P, \mathfrak{g})$

There is an isomorphism of Lie algebras and $\mathcal{C}^\infty(M)$–modules:

$$\hat{\imath} : \Gamma_H(P, \mathfrak{g}) \xrightarrow{\simeq} \Gamma_G(Q, \mathfrak{g}) \tag{5.10}$$

given by

$$\hat{\imath}(v)([p, g]) := Ad_{g^{-1}} \circ v(p) \tag{5.11}$$

with $v \in \Gamma_H(P, \mathfrak{g})$, and $[p, q] \in Q$. This isomorphism means that the degrees of freedom given by the fact one works on Q are compensated by the G–equivariance and then equivalent to working on P with a H–equivariance. $\hat{\imath}$ allows to define the injective map $\iota : \Gamma_H(P, \mathfrak{h}) \to \Gamma_G(TQ)$:

$$\iota := \iota_Q \circ \hat{\imath}.$$

5.2.2.2 Inclusions i and $j : \Gamma_H(P, \mathfrak{h}) \to \Gamma_G(Q, \mathfrak{g})$

Let us call i the natural inclusion $i : \Gamma_H(P, \mathfrak{h}) \hookrightarrow \Gamma_H(P, \mathfrak{g})$, directly coming from the inclusion $\mathfrak{h} \subset \mathfrak{g}$. There is then a natural map $j : \Gamma_H(P, \mathfrak{h}) \to \Gamma_G(Q, \mathfrak{g})$ defined by:

$$j(v)([p, q]) := Ad_{g^{-1}} \circ i \circ v(p) \tag{5.12}$$

which is an injection and a morphism of Lie algebras and $\mathcal{C}^\infty(M)$–modules. j reads also merely: $j = \hat{\imath} \circ i$.

5.2.2.3 Surjection $r : \Gamma_H(P, \mathfrak{g}) \to \Gamma_H(P, \mathfrak{g}/\mathfrak{h})$

There is a surjection $r : \Gamma_H(P, \mathfrak{g}) \to \Gamma_H(P, \mathfrak{g}/\mathfrak{h})$ such that the short sequence of $\mathcal{C}^\infty(M)$–modules:

$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{i} \Gamma_H(P, \mathfrak{g}) \xrightarrow{r} \Gamma_H(P, \mathfrak{g}/\mathfrak{h}) \longrightarrow 0 \tag{5.13}$$

is exact. Let us call $[\xi]$ the element of $\mathfrak{g}/\mathfrak{h}$ corresponding to $\xi \in \mathfrak{g}$. Let us remark that this is defined in any case, not only in the reductive case. Then, r is merely defined by:

$$r(v)(p) := [v(p)]. \tag{5.14}$$

5.2.2.4 The map $J : \Gamma_H(TP) \to \Gamma_G(TQ)$

One now defines a very important map, the inclusion map $J : \Gamma_H(TP) \to \Gamma_G(TQ)$ which realizes $\Gamma_H(TP)$ as a *subalgebroid* of $\Gamma_G(TQ)$ and which extends the map j.

Let $\mathfrak{X} \in \Gamma_H(TP)$, the vector field $\mathfrak{X} \oplus 0 \in \Gamma(TP \oplus TG)$ satifies 5.3 and is invariant under the right action of G, thus defines an element $J(\mathfrak{X})$. The map J defined like this is a morphism of Lie algebras and $\mathcal{C}^\infty(M)$–modules such that:

$$J \circ \iota_P = \iota_Q \circ j, \tag{5.15}$$

$$\rho_Q \circ J = \rho_P. \tag{5.16}$$

$J(\Gamma_H(TP)) \subset \Gamma_G(TQ)$ is itself an algebroid, for J respects the Lie algebra structure, and its kernel is $j(\Gamma_H(P, \mathfrak{h}))$. The map J thus realizes $\Gamma_H(TP)$ as a Lie subalgebroid of $\Gamma_G(TQ)$.

Recall that one has defined the canonical inclusion $\zeta : P \to Q$ with $\zeta(p) = [p, e]$. For $\mathfrak{X} \in \Gamma_H(TP)$, let $t \to \phi_{\mathfrak{X}}(p, t)$ be the flow of $\mathfrak{X}$ through $p \in P$. Then, the flow of $J(\mathfrak{X})$ through $[p, g] \in Q$ will be defined as $t \to [\phi_{\mathfrak{X}}(p, t), g]$. In particular, the flow of $J(\mathfrak{X})$ through $\zeta(p) = [p, e]$ is $t \to [\phi_{\mathfrak{X}}(t, p), e]$, i.e.

$$J(\mathfrak{X})_{|\zeta(p)} = T_p\zeta(\mathfrak{X}) \tag{5.17}$$

$T\zeta$ is defined on any tangent vector field in TP, and its restriction to right-invariant vector fields gives the map J in $\zeta(TP)$.

Moreover, for any $v \in \Gamma_H(P, \mathfrak{h})$, $\hat{v} \in \Gamma_H(P, \mathfrak{g})$, and $\mathfrak{X} \in \Gamma_H(TP)$, one has:

$$j(\mathfrak{X} \cdot v) = J(\mathfrak{X}) \cdot j(v), \tag{5.18}$$

$$\hat{\imath}(\mathfrak{X} \cdot \hat{v}) = J(\mathfrak{X}) \cdot \hat{\imath}(\hat{v}). \tag{5.19}$$

That means that these different maps map also the actions of the Lie algebroid $\Gamma_H(TP)$ and $\Gamma_G(TQ)$ on their respective kernel to each other.

Let σ^P be a splitting of the short exact sequence 5.8 involving $\Gamma_H(TP)$, and ω^P its corresponding 1–form with values in the kernel, in a one–to–one correspondence with Ehresman connections on the underlying fibre bundle P (see chapter 1). As seen in the introduction, such a connection can define another Ehresman connection on Q, and thus a splitting σ^Q on the second short exact sequence 5.9. It turns out that these two connections are related via J, in the sense that

$$J \circ \sigma^P = \sigma^Q.$$

Moreover, if $\mathfrak{X} = \sigma^P_X + \iota_P(v)$ with $v \in \Gamma_H(P, \mathfrak{h})$, then $J(\mathfrak{X})$ reads:

$$J(\mathfrak{X}) = \sigma^Q_X + \iota_Q \circ j(v) \tag{5.20}$$

and the right hand side of this equation is independent of the choosen splitting σ^P.

5.2.2.5 The map R

For any $\hat{\mathfrak{X}}$ and $\hat{\mathfrak{Y}} \in \Gamma_G(TQ)$, one defines the following equivalence relation:

$$\hat{\mathfrak{X}} \sim \hat{\mathfrak{Y}} \Leftrightarrow \hat{\mathfrak{X}} = \hat{\mathfrak{Y}} + J(\mathfrak{Z}) \text{ for some } \mathfrak{Z} \in \Gamma_H(TP). \tag{5.21}$$

One defines then the map R which realizes the quotient, i.e. $R : \Gamma_G(TQ) \to \Gamma_G(TQ)/\sim$. Let us choose a splitting σ^P and its corresponding splitting σ^Q. Let $\hat{\mathfrak{X}} = \sigma^Q_X + \iota_Q(v)$ and $\hat{\mathfrak{Y}} = \sigma^Q_Y + \iota_Q(w)$ in the same equivalence class, and $\mathfrak{Z} = \sigma^P_X + \iota_P(z)$. Then one has: $\sigma^Q_X - \sigma^Q_Y + \iota_Q(v-w) = \sigma^Q_Z + \iota_Q \circ j(z)$. Applying ρ to this relation leads to $X - Y = Z$, and the relation reduces to $v - w = j(z)$. To v and w correspond uniquely $\hat{v}$ and $\hat{w} \in \Gamma_H(P, \mathfrak{g})$ by the isomorphism $\hat{i}$, and thus the equivalence relation $\mathfrak{X} \sim \mathfrak{Y}$ reduces to the equivalence relation $\hat{v} \sim \hat{w}$ which defined r. Thus,

$$R(\hat{\mathfrak{X}}) = R(\sigma^Q_X + \iota(\hat{v})) = r(\hat{v}), \tag{5.22}$$

which shows that:

$$\Gamma_G(TQ)/\sim \simeq \Gamma_H(P, \mathfrak{g}/\mathfrak{h}). \tag{5.23}$$

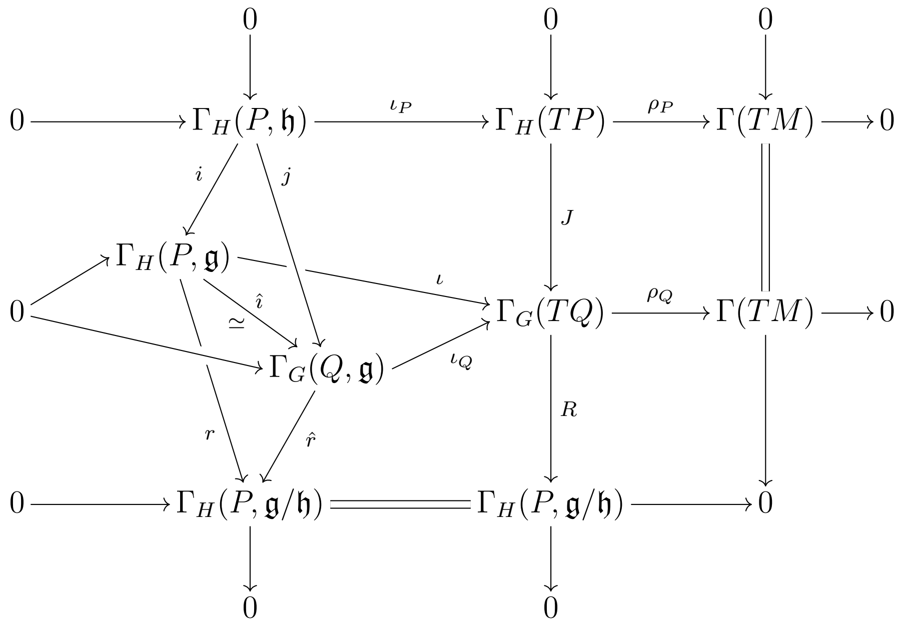
It is straightforward to show that this definition does not depend on the choice of the splitting σ^P.

One obtains, thus, the following short exact sequence of $\mathcal{C}^\infty(M)$–modules:

$$0 \longrightarrow \Gamma_H(TP) \xrightarrow{J} \Gamma_G(TQ) \xrightarrow{R} \Gamma_H(P, \mathfrak{g}/\mathfrak{h}) \longrightarrow 0 \tag{5.24}$$

5.2.2.6 The whole diagram

Collecting all the maps so far defined, one gets the following diagram:



(5.25)

From now on, one works with the diagram written in the following form:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{h}) & \xrightarrow{\iota_P} & \Gamma_H(TP) & \xrightarrow{\rho_P} & \Gamma(TM) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle J} & & \| & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{g}) & \xrightarrow{\iota_Q} & \Gamma_G(TQ) & \xrightarrow{\rho_Q} & \Gamma(TM) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle R} & & \downarrow & & \\
0 & \longrightarrow & \Gamma_H(P,\mathfrak{g}/\mathfrak{h}) & = & \Gamma_H(P,\mathfrak{g}/\mathfrak{h}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5.26}$$

5.3 Cartan connection as an isomorphism of $\mathcal{C}^\infty(M)$–modules

5.3.1 Definition of a Cartan connection in our approach

5.3.1.1 Definition

A Cartan connection on the Lie algebroid $\Gamma_H(TP)$ is an isomorphism $\varpi_{Lie} : \Gamma_H(TP) \overset{\simeq}{\to} \Gamma_H(P, \mathfrak{g})$, such that the following diagram, related to the top left square of 5.26:

$$\begin{array}{ccc} \Gamma_H(P,\mathfrak{h}) & \xrightarrow{\iota_P} & \Gamma_H(TP) \\ \downarrow i & \swarrow \varpi_{Lie}\ \simeq & \\ \Gamma_H(P,\mathfrak{g}) & & \end{array} \tag{5.27}$$

commutes, i.e. such that:

$$\varpi_{Lie} \circ \iota_P = i \tag{5.28}$$

ϖ_{Lie} is a one–form on $\Gamma_H(TP)$ with values in the kernel $\Gamma_H(P, \mathfrak{g})$, not in $\Gamma_H(P, \mathfrak{h})$ as a mere ordinary (Ehresman) connection on P.

5.3.1.2 The isomorphism $\tilde{\varpi}_{Lie}$

Given a Cartan connection ϖ_{Lie} one can define an isomorphism at the level of $\Gamma(TM)$: $\tilde{\varpi}_{Lie} : \Gamma(TM) \xrightarrow{\simeq} \Gamma_H(P, \mathfrak{g}/\mathfrak{h})$, related to the bottom right square of 5.26 by:

$$\tilde{\varpi}_{Lie}(X) := r \circ \varpi_{Lie}(\mathfrak{X}) \tag{5.29}$$

where $\mathfrak{X}$ is *any* element of $\Gamma_H(TP)$ such that $\rho_P(\mathfrak{X}) = X$. It is indeed well–defined because if we take another $\mathfrak{X}'$ such that $\rho(\mathfrak{X}') = X$, then $\mathfrak{X} - \mathfrak{X}' = \iota_P(\gamma)$ for a certain $\gamma \in \Gamma_H(P, \mathfrak{h})$. Thus, $r \circ \varpi_{Lie}(\mathfrak{X}') = r \circ \varpi_{Lie}(\mathfrak{X}) + r \circ \varpi_{Lie} \circ \iota_P(\gamma) = r \circ \varpi_{Lie}(\mathfrak{X}) + r \circ i(\gamma) = r \circ \varpi_{Lie}(\mathfrak{X})$, for $r \circ i = 0$.

One can show that $\tilde{\varpi}_{Lie}$ is an isomorphism by applying the five lemma to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_H(P,\mathfrak{h}) & \xrightarrow{\iota_P} & \Gamma_H(TP) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\ \| & & \| & & \varpi_{Lie}\downarrow\simeq & & \tilde{\varpi}_{Lie}\downarrow\simeq & & \| \\ 0 & \longrightarrow & \Gamma_H(P,\mathfrak{h}) & \longrightarrow & \Gamma_H(P,\mathfrak{g}) & \longrightarrow & \Gamma_H(P,\mathfrak{g}/\mathfrak{h}) & \longrightarrow & 0 \end{array} \tag{5.30}$$

Thus, a Cartan connection ϖ_{Lie} allows to glue the "geometric" short exact sequence (first horizontal line in 5.26):

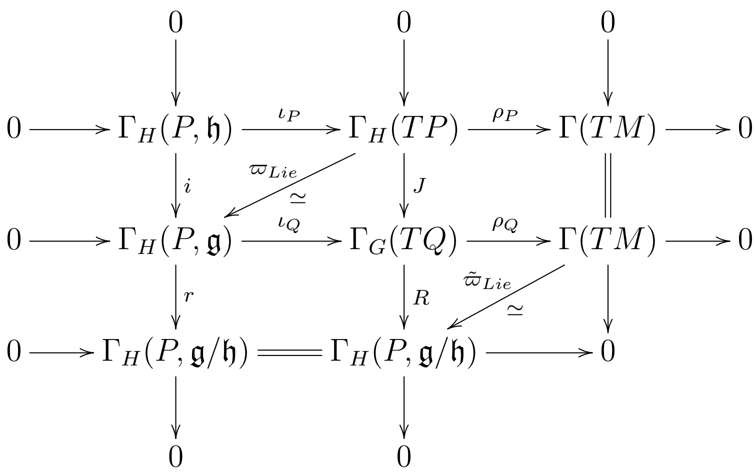
$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{\iota_P} \Gamma_H(TP) \xrightarrow{\rho_P} \Gamma(TM) \longrightarrow 0 \tag{5.31}$$

and the "algebraic" short exact sequence (first vertical line in 5.26):

$$0 \longrightarrow \Gamma_H(P, \mathfrak{h}) \xrightarrow{i} \Gamma_H(P, \mathfrak{g}) \xrightarrow{r} \Gamma_H(P, \mathfrak{g}/\mathfrak{h}) \longrightarrow 0 \tag{5.32}$$

One will see that such a Cartan connection allows then to transpose certain structures (like metrics) from the algebraic sequence to the geometric sequence. The isomorphism $\tilde{\varpi}_{Lie}$ encodes the soldering form found in usual Cartan geometries, for it glues the tangent bundle of the Klein geometry to the tangent bundle of the base manifold.

It is worth replacing the both maps in the whole diagram:



(5.33)

5.3.1.3 Curvature

The curvature 2–form related to ϖ_{Lie} is defined as:[a]

$$\overline{\Omega}_{Lie} := \hat{d}\varpi_{Lie} - \frac{1}{2}[\varpi_{Lie}, \varpi_{Lie}]. \tag{5.34}$$

$\overline{\Omega}_{Lie}$ vanishes on the kernel, in the sense that $\overline{\Omega}_{Lie}(\mathfrak{X}, \iota_P(\gamma)) = 0$, for all $\mathfrak{X} \in \Gamma_H(TP)$ and $\gamma \in \Gamma_H(P, \mathfrak{h})$. Indeed, recall that for $\mathfrak{Y} \in \Gamma_H(TP)$ and $\eta \in \Gamma_H(P, \mathfrak{h})$, $\mathfrak{Y} \cdot \eta$ is defined as the unique element of $\Gamma_H(P, \mathfrak{h})$ such that $\iota_P(\mathfrak{Y} \cdot \eta) = [\mathfrak{Y}, \iota_P(\eta)]$.

[a] The unconventional sign will be made clear later.

The map i being a mere canonical injection, one has $\mathfrak{Y}\cdot i(\eta) = i(\mathfrak{Y}\cdot\eta)$. Thus:

$$\begin{aligned}
\overline{\Omega}_{Lie}(\mathfrak{X}, \iota_P(\gamma)) &= \hat{d}\varpi_{Lie}(\mathfrak{X}, \iota_P(\gamma)) - [\varpi_{Lie}(\mathfrak{X}), \iota_P(\gamma)] \\
&= \mathfrak{X}\cdot\varpi_{Lie}\circ\iota_P(\gamma) - \iota_P(\gamma)\cdot\varpi_{Lie}(\mathfrak{X}) \\
&\qquad - \varpi_{Lie}([\mathfrak{X}, \iota_P(\gamma)]) - [\varpi_{Lie}(\mathfrak{X}), i(\gamma)] \\
&= \mathfrak{X}\cdot i(\gamma) - [i(\gamma), \varpi_{Lie}(\mathfrak{X})] \\
&\qquad - \varpi_{Lie}\circ\iota_P(\mathfrak{X}\cdot\gamma) - [\varpi_{Lie}(\mathfrak{X}), i(\gamma)] \\
&= \mathfrak{X}\cdot i(\gamma) - [i(\gamma), \varpi_{Lie}(\mathfrak{X})] \\
&\qquad - i(\mathfrak{X}\cdot\gamma) - [\varpi_{Lie}(\mathfrak{X}), i(\gamma)] = 0
\end{aligned} \tag{5.35}$$

5.3.1.4 The reductive case

The Lie algebra $\mathfrak{g}$ is reductive if it reads $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, with $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Let us recall that one always has $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. However, one does not know anything about $[\mathfrak{p}, \mathfrak{p}]$. As we will see, it can be zero as in Lorentz geometry, or belonging to $\mathfrak{h}$ as in the de Sitter case. A Cartan geometry based on such a reductive Lie algebra has some specific properties worth noting. Let us call $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $\pi_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$ the corresponding projections. One identifies

$$\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{h},$$

and $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{p}}$ induce projections:

$$\pi_{\mathfrak{h}} : \Gamma_H(P, \mathfrak{g}) \to \Gamma_H(P, \mathfrak{h})$$

and

$$\pi_{\mathfrak{p}} : \Gamma_H(P, \mathfrak{g}) \to \Gamma_H(P, \mathfrak{p}).$$

There is also a natural inclusion $i_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{g}$ which induces an inclusion $\bar{\imath} : \Gamma_H(P, \mathfrak{p}) \to \Gamma_H(P, \mathfrak{g})$.

One has the following relations, with respect to the maps already defined on the diagram:

$$\pi_{\mathfrak{h}}\circ i = \mathrm{Id}, \pi_{\mathfrak{p}} = r, \pi_{\mathfrak{p}}\circ\bar{\imath} = \mathrm{Id} \tag{5.36}$$

Since ϖ_{Lie} takes values in $\Gamma_H(P, \mathfrak{g})$, it accordingly splits into two parts:

$$\varpi_{Lie} = i\circ\omega_{Lie} + \bar{\imath}\circ\beta_{Lie}, \tag{5.37}$$

with:

$$\omega_{Lie} := r_{\mathfrak{h}}\circ\varpi_{Lie} : \Gamma_H(TP) \to \Gamma_H(P, \mathfrak{h}), \tag{5.38}$$

and

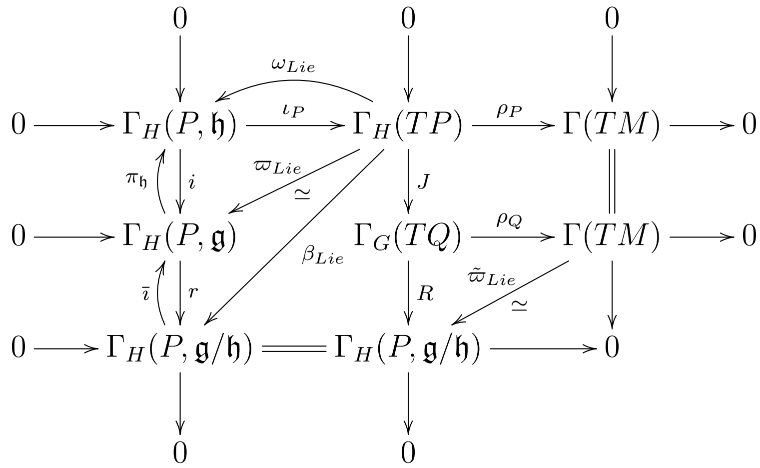
$$\beta_{Lie} := \pi_{\mathfrak{p}}\circ\varpi_{Lie} : \Gamma_H(TP) \to \Gamma_H(P, \mathfrak{p}). \tag{5.39}$$

It turns out that ω_{Lie} is an ordinary (Ehresman) connection on the Lie algebroid $\Gamma_H(TP)$. Let σ_{Lie} be its corresponding splitting; the part β_{Lie} plays the role of the soldering form as it reads:

$$\tilde{\varpi}_{Lie}(X) = \beta_{Lie} \circ \sigma_{Lie}(X). \tag{5.40}$$

One gets the following maps of $\mathcal{C}^\infty(M)$–modules taking place in the diagram:

(5.41)



5.3.2 Comparison with the bundle definition

Let us show now that there is an equivalence of structures between isomorphisms ϖ_{Lie} such that $\varpi_{Lie} \circ \iota_P = i$ and Cartan connections ϖ on the underlying principal bundle P.

$\varpi \Rightarrow \varpi_{Lie}$ Let ϖ be a Cartan connection on P. We associate to ϖ the map $\varpi_{Lie} : \Gamma_H(TP) \to \Gamma_H(P, \mathfrak{g})$ defined by:

$$\varpi_{Lie}(\mathfrak{X})(p) := -\varpi_p(\mathfrak{X}_p), \tag{5.42}$$

for $\mathfrak{X} \in \Gamma_H(TP)$ and $p \in P$.

The map $p \mapsto \varpi_{Lie}(\mathfrak{X})(p)$ is indeed H–equivariant:

$$\begin{aligned}
\varpi_{Lie}(\mathfrak{X})(ph) &= -\varpi_{ph}(\mathfrak{X}_{ph}) \\
&= -\varpi_{ph}(T_p R_h(\mathfrak{X}_p)) \\
&= -(R_h^* \varpi)_p(\mathfrak{X}_p) \\
&= -Ad_{h^{-1}} \varpi_p(\mathfrak{X}_p) \\
&= Ad_{h^{-1}} \circ \varpi_{Lie}(\mathfrak{X})(p).
\end{aligned}$$

ϖ_{Lie} is an isomorphism of $\mathcal{C}^\infty(M)$–modules and $\varpi_{Lie} \circ \iota_P = i$. Indeed, using the isomorphism ϖ_p, to any $v \in \Gamma_H(P, \mathfrak{g})$ we associate $\mathfrak{X} \in \Gamma(TP)$ by

$$\mathfrak{X}_p := -\varpi_p^{-1}(v(p)).$$

Let us check that $\mathfrak{X} \in \Gamma_H(TP)$: one has

$$\begin{aligned}\varpi_{ph}(T_p R_h^P(\mathfrak{X}_p)) &= (R_h^{P*}\varpi_p)(\mathfrak{X}_p) \\ &= \mathsf{Ad}_{h^{-1}}\varpi_p(\mathfrak{X}_p) \\ &= -\mathsf{Ad}_{h^{-1}}v(p) \\ &= -v(ph) = \varpi_{ph}(\mathfrak{X}_{ph})\end{aligned}$$

so that $\mathfrak{X}_{ph} = T_p R_h^P(\mathfrak{X}_p)$. By construction $\varpi_{Lie}(\mathfrak{X}) = v$, which proves the surjectivity. Suppose that $\mathfrak{X} \in \Gamma_H(TP)$ is such that $\varpi_{Lie}(\mathfrak{X}) = 0$. Then, for any $p \in P$, $\varpi_p(\mathfrak{X}_p) = 0$, so that $\mathfrak{X}_p = 0$ since ϖ_p is injective. This proves the injectivity. For any $v \in \Gamma_H(P, \mathfrak{h})$, one has $\iota_P(v)_p = -[v(p)]_p^v$ so that $\varpi_{Lie}(\iota_P(v))(p) = -\varpi_p(\iota_P(v)_p) = \varpi_p\left([v(p)]_p^v\right) = v(p)$. Since i is just the inclusion map, we get $\varpi_{Lie} \circ \iota_P = i$.

$\varpi_{Lie} \Rightarrow \varpi$ The proof that a Cartan connection ϖ_{Lie} on this Atiyah Lie algebroid defines a Cartan connection ϖ on P is exactly the same as the proof in 1.4.4 (chapter 1) that an ordinary connection ω on an Atiyah Lie Algebroid gives an Ehresman connection ω^{er} on P. The straightforward adaptation is thus left to the reader.

It is straightforward to check that the curvature $\overline{\Omega}_{Lie}$ is related to the curvature $\overline{\Omega}$ of ϖ by

$$\overline{\Omega}_{Lie}(\mathfrak{X}, \mathfrak{Y})(p) = -\overline{\Omega}_p(\mathfrak{X}_p, \mathfrak{Y}_p).$$

5.3.3 Condition for an ordinary connection on $\Gamma_G(TQ)$ to reduce to a Cartan connection ϖ_{Lie} on $\Gamma_H(TP)$

One has seen in introduction of this chapter that at the level of fibre bundles, one has a 1–to–1 correspondence between Cartan connections on the H–principal bundle P and certain Ehresman connections on the G–bundle Q. The condition for an Ehresman connection ω^{er} on Q to reduce to a Cartan connection on P is:

$$\forall p \in P,\ \mathsf{ker}(\omega^{er}_{|\zeta(p)}) \cap T_p\zeta(T_pP) = \{0\} \tag{5.43}$$

One would like to recast this condition in our section–based framework, rather that this geometric pointwise one. As one is going to show now, an ordinary connection $\omega : \Gamma_G(TQ) \to \Gamma_G(Q, \mathfrak{g})$ reduces to a Cartan connection ϖ_{Lie} on $\Gamma_H(TP)$ (which defines a Cartan connection ϖ on P as one has shown) if and only if:

$$\text{ker}(\omega) \cap J(\Gamma_H(TP)) = \{0\}. \tag{5.44}$$

Both the spaces ker(ω) and $J(\Gamma_H(TP))$ are subspaces of $\Gamma_G(TQ)$ such that their intersection is well defined, and does not need to place at a certain point: it is a global condition, which the sections–based formalism naturally allows.

ω and ω^{er} being uniquely related, let us show that both conditions 5.43 and 5.44 are equivalent. Let us first recall the following important relations:

- $\omega^{er}_q(\hat{\mathfrak{X}}_q) := \omega(\mathfrak{X})(q)$,
- $J(\mathfrak{X})_{\zeta(p)} = T_p\zeta(\mathfrak{X}_p)$,
- $\hat{\mathfrak{X}}_{[p,g]} = T_pR_g\hat{\mathfrak{X}}_{\zeta(p)}$,

where ω^{er} is an Ehresman connection at the level of the bundle Q, ω is its corresponding connection at the level of the Lie algebroid $\Gamma_G(TQ)$, $\hat{\mathfrak{X}} \in \Gamma_G(TQ)$, $\mathfrak{X} \in \Gamma_H(TP)$, $p \in P$, $[p, g] \in Q$.

5.43 $\Rightarrow$ 5.44 Let us assume that ω^{er} satisfies 5.43, and let us show that then, ω satifies 5.44. Let us take $\hat{\mathfrak{X}} \in$ ker$(\omega) \cap J(\Gamma_H(TP))$, and show that $\hat{\mathfrak{X}} = 0$. Let us place first in any $q = \zeta(p)$, $p \in P$. We have:$\omega(\hat{\mathfrak{X}})(\zeta(p)) = 0$ by assumption, i.e. $\omega^{er}_{\zeta(p)}(\hat{\mathfrak{X}}_{\zeta(p)}) = 0$, that is $\mathfrak{X}_{\zeta(p)} \in$ ker$(\omega^{er}_{\zeta(p)})$.

$\hat{\mathfrak{X}} \in J(\Gamma_H(TP))$ thus there exists $\mathfrak{X} \in \Gamma_H(TP)$ such that $\hat{\mathfrak{X}} = J(\mathfrak{X})$, in particular $\hat{\mathfrak{X}}_{\zeta(p)} = J(\mathfrak{X})_{\zeta(p)} = T_p\zeta(\mathfrak{X}_p)$, i.e. $\hat{\mathfrak{X}}_{\zeta(p)} \in T_p\zeta(T_pP)$. Thus, since 5.43 holds, $\hat{\mathfrak{X}}_{\zeta(p)} = 0$. By equivariance, for any $q = [p, g]$, $\hat{\mathfrak{X}}_q = T_pR_g\hat{\mathfrak{X}}_{\zeta(p)} = 0$, and thus 5.44 holds.

5.44 $\Rightarrow$ 5.43 The converse property is a bit more technical to prove, because one assumes a property holding for sections and has to prove something at a particular point. To do that, one is going to use again the proposition given in chapter 1 about the local decomposition of a vector field in a basis of right–invariant vector field, see section 1.4.3. Assuming that 5.44 holds, one takes $\hat{\mathfrak{X}}_{\zeta(p)} \in T_{\zeta(p)}Q$ such that

- $\omega^{er}_{\zeta(p)}(\hat{\mathfrak{X}}_{\zeta(p)}) = 0$,
- There exists $\mathfrak{X}_p \in T_pP, \hat{\mathfrak{X}}_{\zeta(p)} = T_p\zeta(\mathfrak{X}_p)$.

The aim is of course to show that $\hat{\mathfrak{X}}_{\zeta(p)} = 0$, proving that 5.43 holds. The strategy is to fix $p_0 \in P$, and from the vector (not vector field!) $\hat{\mathfrak{X}}_{\zeta(p_0)}$, to construct a right–invariant vector field $\hat{\mathfrak{X}}$ on $\zeta(TP)$ (it is sufficient) which coincide with $\hat{\mathfrak{X}}_{\zeta(p_0)}$

at p_0, and which is in ker$(\omega) \cap J(\Gamma_H(TP))$, i.e. which is zero. Thus, in particular, one will conclude that $\hat{\mathfrak{X}}_{\zeta(p_0)} = 0$ too.

Let us take $U \subset M$ which contains $\pi(p_0)$, and $\{\mathfrak{X}^i\}_i$ the family of $\Gamma_H(TP_{|U})$ defined by the proposition 1.4.3. Applied to $\mathfrak{X}_{p_0}$, there exist numbers $f_i \circ \pi(p_0)$ such that $\mathfrak{X}_{p_0} = f_i \circ \pi(p_0)\mathfrak{X}^i_{p_0}$, and thus $\hat{\mathfrak{X}}_{\zeta(p_0)} = f_i \circ \pi(p_0) T_{p_0}\zeta(\mathfrak{X}^i_{p_0})$. Since $\mathfrak{X}^i$ is a right–invariant vector field, one can write $T_{p_0}\zeta(\mathfrak{X}^i_{p_0}) = J(\mathfrak{X}^i)_{\zeta(p_0)}$. Thus the numbers $f_i \circ \pi(p_0)$ are such that

$$f_i \circ \pi(p_0)\omega^{er}_{\zeta(p_0)}(J(\mathfrak{X}^i)_{\zeta(p_0)}) = 0. \tag{5.45}$$

The aim is to prove that it is possible to find functions $f_i \circ \pi$ defined on $P_{|U}$ (not only at p_0) which equal $f_i \circ \pi(p_0)$ at p_0, satisfying 5.45, and such that:

$$f_i \circ \pi(p)\omega^{er}_{\zeta(p)}(J(\mathfrak{X}^i)_{\zeta(p)}) = 0, \text{ for all } p \in P_{|U}. \tag{5.46}$$

If these functions exist, then one takes $h \in \mathcal{C}^\infty(U)$ such that there exists $V \subset U$, $p_0 \in V$, with $h_{|V} = 1$, and $h = 0$ on $M \setminus \overline{U}$, where $\overline{U}$ denotes the closure of U. It is always possible to find such a function for U small enough. Then, define the vector field on P:

$$\hat{\mathfrak{X}}_{\zeta(p)} := \begin{cases} h(\pi(p)) f_i \circ \pi(p) J(\mathfrak{X}^i_{\zeta(p)}) & \text{if } p \in P_U \\ 0, & \text{otherwise.} \end{cases} \tag{5.47}$$

The last thing to prove is that it is possible to find such functions. Denoting $\alpha_i = f_i \circ \pi$, with $\alpha_i : U \to \mathbb{R}, i = 1..\text{dim}(P)$, and dim$(P) =$ dim$(\mathfrak{g})$. For each i and $p \in P_{|U}$, $\omega^{er}_{\zeta(p)}(J(\mathfrak{X}^i)_{\zeta(p)}) \in \mathfrak{g}$, let us call this element $M^{ij}(p)E_j$, with $\{E_j\}_j$ a basis of $\mathfrak{g}$. Then, α_i have to satisfy

$$\alpha_i(p)M^{ij}(p) = 0, \tag{5.48}$$

with $\alpha_i(p_0)$ given as initial conditions. The argument is to say that the kernel of the matrix $M^{ij}(p)$, for each p, is never reduced to zero and thus this equation always has a solution in a neighbourhood of p_0.

We have just proved that ω verifying the condition 5.44 is equivalent to the data of a Cartan connection on P, and thus to a Cartan connection ϖ_{Lie}. Thus, a Cartan connection could also be defined as an ordinary connection ω on $\Gamma_G(TQ)$ which verifies 5.44, or, closer to the Crampin & Saunders approach presented later, as a splitting σ of $\rho_Q : \Gamma_G(TQ) \to \Gamma(TM)$ such that:

$$\text{Im}(\sigma) \cap J(\Gamma_H(TP)) = \{0\} \tag{5.49}$$

Let us show now how such a connection ω concretely generates a Cartan connection ϖ_{Lie}. Let ω an ordinary connection on $\Gamma_G(TQ)$ such that it satisfies 5.44. Then, set $\varpi_{Lie} := \omega \circ J$. We claim that this is a Cartan connection on $\Gamma_H(TP)$, i.e. that this is an isomorphism between $\Gamma_H(TP)$ and $\Gamma_H(P, \mathfrak{g})$ and

$\varpi_{Lie} \circ \iota_P = i$. The condition 5.44 is only used to prove the injectivity, as we shall see. Indeed, for any ω, $\omega \circ J$ is surjective and $\omega \circ J \circ \iota_P = \omega \circ \iota_Q \circ i = i$, for $J \circ \iota_P = \iota_Q \circ i$. This property always holds, due to the underlying diagram itself. To prove the surjectivity, let us take $v \in \Gamma_H(P, \mathfrak{g})$, and take $\mathfrak{X} \in \Gamma_H(TP)$ an antecedant of v by any isomorphism α (one knows that $\Gamma_H(P, \mathfrak{g})$ and $\mathfrak{X} \in \Gamma_H(TP)$ are isomorphic, thus such an α always exist.) Let us construct, from $\mathfrak{X}$, an antecedant of v by $\omega \circ J$, of the form $\iota_P(w)$ with $w \in \Gamma_H(P, \mathfrak{h})$. One wants w to satisfy: $\omega \circ J(\mathfrak{X} + \iota_P(w)) = v$. This algebraic equation is easy to solve, it suffices to take $w = v - \omega \circ J(\mathfrak{X})$. Thus, for any $v \in \Gamma_H(P, \mathfrak{g})$, the element $\mathfrak{X} + \iota_P(w)$ is an antecedant by $\omega \circ J$, the latter is thus surjective. Let us show the injectivity: take $\mathfrak{X} \in \Gamma_H(TP)$ such that $\omega \circ J(\mathfrak{X}) = 0$. That means that $J(\mathfrak{X}) \in \ker(\omega)$, but $J(\mathfrak{X})$ obviously belongs to $J(\Gamma_H(TP))$, and thus $J(\mathfrak{X}) = 0$, and by injectivity of J, $\mathfrak{X} = 0$.

5.4 Crampin & Saunders' approach of Cartan geometry via Lie algebroids

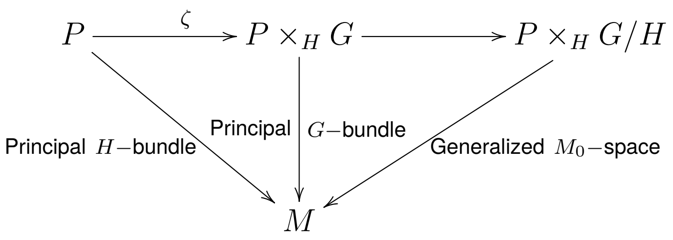
Crampin and Saunders' work [9] recasts Cartan geometries in the framework of Lie groupoids and Lie algebroids. One presents here a short summary of a little part of their work, in order to compare it with our approach, what they call *Infinitesimal Cartan Geometry*, which means the Lie algebroid part.

5.4.1 The generalized space *à la Cartan*

The approach of Crampin and Saunders wants to be closer to the original idea of Cartan of an *espace généralisé*, which consists of gluing an homogeneous manifold $M_0 = G/H$ at each point of a base manifold M. Let P be the principal bundle of a Cartan geometry modeled on $M_0 = G/H$. They call generalized M_0–space, what Wise calls the bundle of tangent Klein geometries, the bundle

$$E := P \times_H G/H. \tag{5.50}$$

This is a bundle with typical fibre M_0. Wise ([53]) places it in the following sequence of bundles, which summarizes the different bundles associated to a Cartan geometry:



(5.51)

Let us remark that in their textbook [9], they use the letter P for the G–principal bundle, and the letter Q for the H–principal bundle (that is, exactly the contrary of us and of Wise and Sharpe), and the letter G_0 for our H.

The G–bundle Q being, by construction, reducible to the H–bundle P, the generalized space $E = P \times_H G/H$ automatically admits a global section ξ. Then $\xi(M)$ is identified, as a submanifold of the generalized space, with M itself, and in this sense is seen as gluing a fibre diffeomorphic to M_0 at each point of M, closer, according to Crampin and Saunders, to the idea of Cartan.

Starting from the G–bundle $Q = P \times_H G$, one can also consider the generalized space $Q \times_G (G/H)$. It turns out that:

$$E = P \times_H (G/H) \simeq Q \times_G (G/H) \tag{5.52}$$

Indeed, let $\psi : P \times_H (G/H) \to Q \times_G (G/H)$ defined as:

$$\psi([p,[g]]) := [\zeta(p),[g]] = [[p,e],[g]]. \tag{5.53}$$

$[g] \in G/H$ is defined as $[g] = [gh]$ for $h \in H$. Then, let $[[p,g'],[g]] \in Q \times_G (G/H)$.[b] One has:

$$[[p,g'],[g]] = [[p,g']g'^{-1}, g'[g]] = [[p,e],[g'g]]. \tag{5.54}$$

Thus, the map $Q \times_G (G/H) \ni [[p,g'],[g]] \mapsto [p,[g'g]] \in P \times_H (G/H)$ is the inverse of ψ.

5.4.2 The Lie groupoids of Fibre Morphisms

The Lie groupoids considered in [9] are not directly the respective gauge groupoids of P and Q, but the so–called groupoids of fibre morphisms of E, respecting the action of H and G respectively. The Lie groupoid of fibre morphisms of E respecting the action of G is defined as the set of diffeomorphisms $\phi : E_x \to E'_x$, for any $x, x' \in M$, such that in a local trivialization, the action of ϕ (which can be mapped to an action on the fibre M_0) corresponds to an action of the Lie group G, see section 3.1. of their book [9]. This Lie groupoid is denoted $\mathcal{G}$. Then, the Lie groupoid $\mathcal{H}$ of fibre morphisms of E respecting the action of the Lie subgroup $H \subset G$ is also a Lie groupoid, and it is a Lie subgroupoid of $\mathcal{G}$ by the proposition 3.1.3. of [9].

An important remark to make now is to notice the relation between these Lie groupoids and the gauge groupoids that one can construct from the principal bundles P and Q, let us denote them respectively $Gauge(P)$ and $Gauge(Q)$. In section 3.5. of their textbook, Crampin and Saunders show that they are actually respectively isomorphic. Indeed, for $p \in P_x$, define $\hat{p} : M_0 \to E_x$ by $\hat{p}(y) := [p, y]$.

[b] Let us recall that $q = [p,g'] = [ph, h^{-1}g]$, and $[q,[g]] = [qg, g^{-1}[g]] = [qg,[g^{-1}g]]$

One has, in particular,

$$\widehat{p \cdot g} = \hat{p} \circ L_g. \tag{5.55}$$

It turns out that the following map:

$$Gauge(P) \to \mathcal{H}, [p, g] \mapsto \phi := \hat{q} \circ \hat{p}^{-1} \tag{5.56}$$

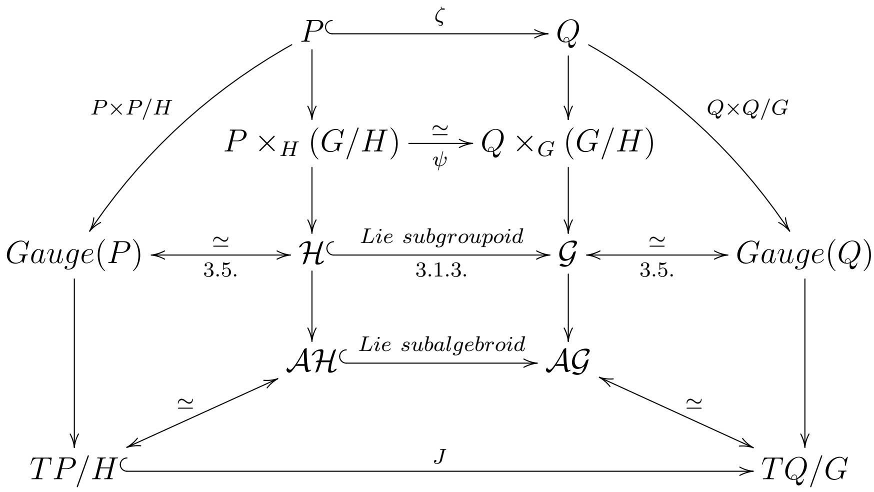
is well defined thanks to 5.55, and is an isomorphism of Lie groupoids. Of course, the same construction, *mutatis mutandis*, shows that $Gauge(Q) \simeq \mathcal{G}$.

5.4.3 The Lie algebroids

Crampin and Saunders consider the transitive Lie algebroids corresponding to the Lie groupoids $\mathcal{G}$ and $\mathcal{H}$, denoted respectively $\mathcal{AG}$ and $\mathcal{AH}$. If α_G is the source of $\mathcal{G}$, then let us recall that $\mathcal{AG} := \cup_{x \in M} T_x \alpha_G^{-1}(x)$. $\mathcal{G}$ and $Gauge(Q)$ being isomorphic as Lie groupoids, the corresponding fibre–sources will also be isomorphic as submanifolds, and so will be their respective tangent space. Thus, the Atiyah Lie algebroid of Q, that we denote $\Gamma_G(TQ)$, is isomorphic to $\mathcal{AG}$. The same reasonning, *mutatis mutandis* holds also to show that $\Gamma_H(TP)$ is isomorphic to $\mathcal{AH}$. $\mathcal{AH}$ is a Lie subalgebroid of $\mathcal{AG}$, and the inclusion is nothing but our map J defining $J(\Gamma_H(TP))$ as a Lie subalgebroid of $\Gamma_G(TQ)$.

5.4.4 Summary of Approaches

Let us summarize now the comparison between their approach and ours in the following diagram:



We make an abuse of notations by using the same letter J for the map between vector bundles while it is originally defined for spaces of sections. The name of corresponding sections in the book [9] is recalled under the arrows when necessary.

5.4.5 Infinitesimal Cartan Connection

From the viewpoint of Crampin and Saunders, a Cartan connection is defined as a splitting $\gamma : TM \to \mathcal{AG}$ such that:

$$\gamma_x(T_xM) \cap \mathcal{AH}_x = \{0\} \tag{5.57}$$

They show in chapter 7 the equivalence of this definition with several usual approaches of Cartan geometry. From our discussion 5.3.3, one directly recognizes the condition 5.49:

$$\mathsf{Im}(\sigma) \cap J(\Gamma_H(TP)) = \{0\} \tag{5.58}$$

for a splitting σ (and thus its corresponding ordinary connection ω) to generate a Cartan connection ϖ_{Lie}, written in a geometric pointwise language. The definition of Crampin and Saunders is then just the condition found in the appendix of Sharpe [50] for an Ehresman connection on Q to generate a Cartan connection on P, but rewritten in the language of transitive Lie algebroids.

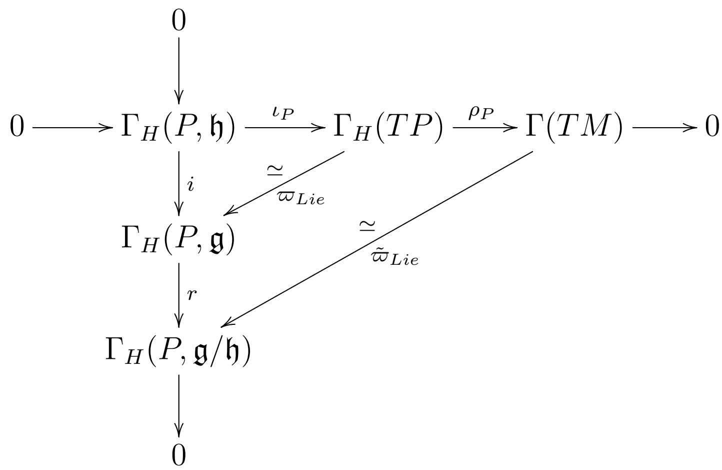
5.5 Metric

Since the Lie algebra $\mathfrak{g}$ is given from the begining and can also be given with a metric $\hat{h}$, then one gets a metric on the space $\Gamma_H(P, \mathfrak{g})$,

$$\hat{h} : \Gamma_H(P, \mathfrak{g}) \times \Gamma_H(P, \mathfrak{g}) \to \mathcal{C}^\infty(M).$$

This metric reduces to $i^*\hat{h}$, which is a metric on $\Gamma_H(P, \mathfrak{h})$. Let us consider the following (simplified) diagram, equipped with a Cartan connection:

(5.59)



5.5.1 Metric $\hat{g}$ on $\Gamma_H(TP)$

One can then define a metric $\hat{g}$ on the Lie algebroid $\Gamma_H(TP)$ by:

For $\mathfrak{X}, \mathfrak{Y} \in \Gamma_H(TP)$, $\hat{g}(\mathfrak{X}, \mathfrak{Y}) := \hat{h}(\varpi_{Lie}(\mathfrak{X}), \varpi_{Lie}(\mathfrak{Y}))$.

$\hat{g}$ and $\hat{h}$ being related by an isomorphism, the degeneracy of one of them implies the degeneracy of the other. From now on, one will take $\hat{h}$ to be non–degenerate.

5.5.2 Equivalent triple on the Lie algebroid exact sequence

$\hat{g}$ being a metric on the Lie algebroid, one knows that then one can define the equivalent triple $(h, \mathring{\omega}, g)$ with:

- h a metric on the kernel $\Gamma_H(P, \mathfrak{h})$: $h(u, v) := \hat{g}(\iota_P(u), \iota_P(v))$.

  $h(u, v) = \hat{g}(\iota_P(u), \iota_P(v)) = \hat{h}(\varpi_{Lie} \circ \iota_P(u), \varpi_{Lie} \circ \iota_P(v)) = \hat{h}(i(u), i(v))$, i.e.

  $$h = i^* \hat{h}$$

  $\hat{g}$ is supposed to be inner non-degenerate, i.e. h is supposed to be non-degenerate (which is always the case if (stronger hypothesis) so does $\hat{h}$).

- $\mathring{\omega}$ the unique ordinary connection such that

  $$\hat{g}(\iota_P(u), \mathring{\sigma}_X) = 0, \ \forall u \in \Gamma_H(P, \mathfrak{h}) \text{ and } X \in \Gamma(TM),$$

  where $\mathring{\sigma}$ is the splitting associated to $\mathring{\omega}$ by:

  $$\mathring{\sigma}_X := \mathfrak{X} + \iota_P \circ \mathring{\omega}(\mathfrak{X})$$

  where $\mathfrak{X}$ is any representative of $X \in \Gamma_H(TP)$, i.e. such that $\rho_P(\mathfrak{X}) = X$. One has defined $\mathring{\omega}$ in the first chapter, but let us give here another definition, as in [33] and [15], adapting it to the present case to prove an important property: let us set, for $\mathfrak{X} \in \Gamma_H(TP)$, $f_{\mathfrak{X}}(u) := -\hat{g}(\mathfrak{X}, \iota_P(u))$, and call $\mathring{\omega}(\mathfrak{X})$ the unique element of $\Gamma_H(P, \mathfrak{h})$ such that $f_{\mathfrak{X}}(u) = h(\mathring{\omega}(\mathfrak{X}), u)$ (Cf Riesz thm) and thus $-\hat{g}(\mathfrak{X}, \iota_P(u)) = h(\mathring{\omega}(\mathfrak{X}), u)$. It turns out then, that $-\hat{h}(\varpi_{Lie}(\mathfrak{X}), i(u)) = \hat{h}(i \circ \mathring{\omega}(\mathfrak{X}), i(u))$, i.e.

  $$\hat{h}(i \circ \mathring{\omega}(\mathfrak{X}) + \varpi_{Lie}(\mathfrak{X}), i(u)) = 0, \forall u \in \Gamma_H(P, \mathfrak{h}) \tag{5.60}$$

  thus, as $\hat{h}$ is supposed to be non-degenerate, one gets:

  $$i \circ \mathring{\omega}(\mathfrak{X}) + \varpi_{Lie}(\mathfrak{X}) \in \mathfrak{h}^{\perp} \tag{5.61}$$

- g a metric on $\Gamma(TM)$: $g(X, Y) := \hat{g}(\mathring{\sigma}_X, \mathring{\sigma}_Y)$.

5.6 Gravitational Theories

5.6.1 Einstein–Hilbert Action

In the same spirit as in chapter 4, in which formulations of (internal) gauge theories are given in the language of Lie algebroids, it is possible here to recast in this language the Einstein–Hilbert action given in chapter 2. One takes a Cartan connection ϖ_{Lie} as in 5.3, with a reductive underlying Lie algebra as in 2.3.2: $\mathfrak{g} = \mathfrak{so}(1,3) \oplus \mathbb{R}^{1,3}$. Thus, the Cartan connection looks like:

$$\varpi_{Lie} = i \circ \omega_{Lie} + \bar{\imath} \circ \beta_{Lie}.$$

The same computation as in 4.3.3 can be pursued, and one gets that ϖ_{Lie} transforms as an ordinary connection under an infinitesimal gauge transformation $\xi \in \Gamma_H(P;\mathfrak{h})$, i.e.:

$$\varpi^{\xi}_{Lie} = \varpi_{Lie} + \hat{\mathsf{d}}\xi + [\xi, \varpi_{Lie}]. \tag{5.62}$$

The curvature, due to this fact, transforms homogeneously under a gauge transformation, and reads at the infinitesimal level:

$$\overline{\Omega}^{\xi}_{Lie} = \overline{\Omega}_{Lie} + [\xi, \Omega_{Lie}]. \tag{5.63}$$

$\mathfrak{g}$ being reductive, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. The both parts of ϖ_{Lie} transform then as:

$$\omega^{\xi}_{Lie} = \omega_{Lie} + \hat{\mathsf{d}}\xi + [\xi, \omega_{Lie}] \tag{5.64}$$

and

$$\beta^{\xi}_{Lie} = \beta_{Lie} + [\xi, \beta_{Lie}]. \tag{5.65}$$

Recall that ξ takes values in $\mathfrak{h}$, and β in $\mathfrak{p} = \mathbb{R}^{1,3}$, thus in the matrix representation of this Lie algebra, the commutator $[\xi, \beta_{Lie}]$ reads:

$$[\xi, \beta_{Lie}] = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta_{Lie} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi\beta_{Lie} \\ 0 & 0 \end{pmatrix} \tag{5.66}$$

so it will be often written just "$\xi\beta_{Lie}$". For the same reasons, one has

$$[\xi, \beta_{Lie}]^t = -\beta^t_{Lie}\xi = [\xi, \beta_{Lie}].$$

The $\mathfrak{h}$–valued 2–form $\beta_{Lie} \wedge \beta^t_{Lie}$ transforms as:

$$(\beta_{Lie} \wedge \beta^t_{Lie})^{\xi} = \beta_{Lie} \wedge \beta^t_{Lie} + [\xi, \beta_{Lie} \wedge \beta^t_{Lie}]. \tag{5.67}$$

Let Ω_{Lie} being the $\mathfrak{h}$–part of the curvature $\overline{\Omega}$, and one sets the $\mathfrak{p}$–part to zero (torsionless condition). Then, the action:

$$S(\varpi_{Lie}) = S(\beta_{Lie}) = \int_A h(\Omega_{Lie}, *(\beta_{Lie} \wedge \beta_{Lie}^t)) \tag{5.68}$$

is the Einstein–Hilbert action written in the language of Lie algebroids. It is invariant due to the ad–invariance of the metric h together with the gauge transformations of the different parts.

This formulation is a bit artificial, in the sense that since $\beta_{Lie} \wedge \beta_{Lie}^t$ have only geometric degrees of freedom, the hodge star will make appear directly the maximal inner term, once combined with Ω, which will disappear straight after by integrating along the kernel.

5.7 Conclusion: Generalized Cartan Connection and Gravity Theory

Let us recall the link between a Cartan connection ϖ_{Lie} on $\Gamma_H(TP)$ and its corresponding ordinary connection ω on $\Gamma_G(TQ)$. Let us set $\varpi_{Lie} := \omega \circ J$ with $\omega \in \Omega^1(\Gamma_G(TQ), \Gamma_H(P, \mathfrak{g}))$. If ω is an ordinary connection such that ker$(\omega) \cap J(\Gamma_H(TP)) = \{0\}$, then ϖ_{Lie} is a Cartan connection. From this viewpoint, two ways of generalizing the notion of a Cartan connection may be to demand either:

- ω to be a generalized connection in the sense of [16], while the other condition holds, or
- ω to be still an ordinary connection but ker$(\omega) \cap J(\Gamma_H(TP)) \neq \{0\}$. In this case ϖ_{Lie} would be still surjective and normalized, but no more injective.

If ω is a generalized connection (first possibility), then $\omega \circ \iota \neq \mathrm{Id}_{\Gamma_H(P,\mathfrak{g})}$ and then $\varpi_{Lie} \circ \iota_P \neq i$. Thus, the first way of generalizing is equivalent to demand ϖ_{Lie} to be an isomorphism but without the normalization condition. We tried to work with a generalized notion of Cartan connection in this way, but we did not succeed to obtain a nice formulation of a generalized Einstein–Hilbert action which could have been easily interpreted. The first problem is to get a nice gauge transformation. Indeed, as in the case of a generalized connection in the sense of [16] (as in chapter 4, see 4.3.3), the geometric gauge transformation (i.e. the one coming directly from the computation with the Lie derivative) is not usual and give something very complicated once combined with the other terms, and leads to something hardly interpretable. Thus, one takes usual gauge transformations by decree, even if we could not justify it from covariant derivative considerations, as in ??. Then, one explicitely computes the generalized Einstein–Hilbert action by replacing Ω_{Lie} by the $\mathfrak{h}$–part of the curvature and β_{Lie} by the projection of ϖ_{Lie} on $\mathfrak{p}$. The lagrangian gives three terms, for in this case algebraic degrees of freedom enter also into the game. Unfortunately, as far as we have worked out the lagrangian explicitely, it gives terms which are really complicated and hardly interpretable. The way of generalizing the Cartan connection seems very natural, however, so maybe we could just take another lagrangian from which we will recover known physical terms. This work is to be done, though.

The second possibility has not been tested yet. We could expect that the non–injectivity of ϖ_{Lie} generate a degenerate metric on the geometric sequence. In the reductive case, maybe it could be interesting to make only one part of the Cartan connection (for exemple, only β_{Lie}) carrying the degeneracy. This work could also be done in future research.

Conclusion

Let us summarize our exploration throught new geometrico-framework for theoretical physics, and then present some possible further developments and generalization.

Summary

In this thesis, we have presented several non–usual frameworks for the formulation of physical theories.

Cartan Geometry

Cartan geometry allows to write gravitation theories in the form of "internal" gauge theories, i.e. formulated in terms of a connection on a certain principal fibre bundle. In these cases, however, the structure group is still a "spacetime–type" structure group, i.e. is a Lie subgroup of a bigger Lie group acting on an homogeneous space which plays the role of the "flat" model for spacetime. For example, in the case of General Relativity, the structure group is the Lorentz group, i.e. a group of "rotations" in a space which one has a physical intuition about, unlike internal gauge theories for which the structure group, of type $SU(n)$, acts on an abstract space to which one cannot give a direct interpretation.

Conformal Geometry

We have presented conformal geometry from the point of view of Cartan geometry, giving the equivalent description of a conformal structure in terms of Cartan connection on the 2–frame conformal bundle. Applying the dressing field method of symmetry reduction to this framework allows to recover the usual objects playing the role of matter fields for this case, say Tractors and Twistors. The fact to recover them from this "top-down" construction allows to give a better understanding of their geometric nature. In particular, one shows that their transformations under the action of the Weyl group of rescalings W, which are not direct representations of W, are actually *residual symmetries*, after dressing, of a genuine representation of the conformal group.

Transitive Lie algebroids

After having presented a framework for gravitation theories where one generalizes the structure group, one has exposed a generalization of the framework itself in which are usually formulated gauge theories, say a generalization of principal fibre bundles: transitive Lie algebroids. These objects are a particular case of vector bundles, and can be seen as generalizations either of Lie algebras, of tangent bundles or of the infinitesimal data encoded in principal fibre bundles. One works with sections of Lie algebroids more than with the vector bundles themselves, and thus one deals with short exact sequences of Lie algebras and modules on the space of functions. Among transitive Lie algebroids, there are those which are the infinitesimal version of an underlying structure, like Lie algebras are the infinitesimal versions of Lie groups. For these particular cases, there exists a principal fibre bundle from which one can reconstruct the transitive Lie algebroids. The constructions presented in this thesis are valid for any transitive Lie algebroids, thus these are actual generalizations of usual gauge theories. Two examples of unified lagrangians are presented. The first uses a metric on the Lie algebroid as a background, and formulates a gauge theory in terms of a *generalized* connection; it allows to write a lagrangian which encodes both Yang–Mills theory and the Higgs sector of the Standard Model, i.e. a scalar field embbeded in a quartic potential. The second one takes the metric as a field variable, and uses its Levi-Civita connection (on the Lie algebroid, the direct generalization of what one can define on the tangent space) to encode both the degree of freedom related to gravitation and those related to a Yang–Mills theory; it allows to write a unified lagrangian containing both General Relativity (i.e. Einstein–Hilbert term) with cosmological constant, and Yang–Mills lagrangian. Transitive Lie algebroids are thus a rich framework where unified lagrangians naturally appear.

Cartan Geometry and Lie algebroids

A last work presented in this thesis is our formulation of Cartan geometry in the framework of (Atiyah) Lie algebroids, that we compare to another approach in the literature. One construct a commutative diagram associated to Cartan principal bundles, and then define a Cartan connection as an isomorphism of vector space and modules on functions. As usual, one can define the notion of metric on such a transitive Lie algebroid, but in this case it can be generated by the "algebraic" short exact sequence associated to a Cartan geometry. Hodge star operator and integration can also be defined as usual, and gravitation theories in this framework can be formulated aswell.

Further in Algebra

Atiyah Lie algebroids provide, with regard to classical aspects, a compact geometrico-algebraic framework for encoding gauge theories (encompassing also gravitation theories via Cartan connections). A natural idea is then to generalize these constructions to more general structures, which mimic in some way the former. There are at least two possible ways to do that.

- *Working on a non-integrable transitive Lie algebroid.* Any transitive Lie algebroid which is integrable to a Lie groupoid is the Atiyah Lie algebroid of some principal fibre bundles. Thus, an actual generalization of gauge theories would be using the same constructions on a non-integrable Lie algebroid, and seeing which physical consequences are derivable from the non-integrability conditions. These integrability conditions have been promptly presented in chapter 1 and, for our case of interest, the non–integrability would imply that the topology of the base manifold, in particular the second homotopy group $\pi_2(M)$, be highly non-trivial. A deeper investigation is still to be done. In this framework, the notion of smooth base manifolds remains a basic ingredient of the theory, and thus that of space-time as a primary notion. A symmetry principle is still present as well, because any transitive Lie algebroid is given with a certain Lie algebra, which encodes it infinitesimally.

- *Working on a Lie-Rinehart pair.* A more radical framework is that of a Lie-Rinehart pair $(A, \mathfrak{B})$, also called Lie-Rinehart *algebra*, where A is an associative algebra and $\mathfrak{B}$ is a Lie algebra and a Der(A)–module. See e.g. [28]. Lie-Rinehart pairs for which there exists a manifold M such that $A = \mathcal{C}^\infty(M)$ are in 1–to–1 correspondance with Lie algebroids, i.e. in this case $\mathfrak{B}$ corresponds to the space of sections of a certain Lie algebroid. Lie-Rinehart pairs are, from this viewpoint, natural generalizations of Lie algebroids. The fact that one works with sections of Lie algebroids instead of the vector bundles themselves will render the generalization even more straightforward. The usual notions that exist on a Lie algebroid, as that of connection or curvature, have been developed in the case of Lie-Rinehart pairs (see e.g. [27]). A natural idea with respect to my researchline is applying our way of building gauge theories on transitive Lie algebroids to this more abstract and purely algebraic framework. Here, the very notion of base manifolds, and thus of spacetime as a primary notion, disappears, clearly in the spirit of (derivation based) noncommutative geometry. The notion of spacetime would hopefully emerge from a purely algebraic construction mimicking *ad minima* that of principal bundle and usual (classical) gauge theories.

These two generalizations, which would lead, for the former, into more topology, and for the latter, into more abstract algebra and possibly noncommutative geometrical notions, are appealing directions we would like to explore at this point of our work. It seems clear that one should find a theory in which spacetime is not given as a primary notion, even if it is intellectually very hard to imagine such a framework. Yet this is the main teaching of modern physics over the past century: spacetime, in the framework of general relativity as in that of quantum physics, is an entity which does not have any intrinsic physical meaning. Given this idea: one should be able to think of physics without giving to spacetime an ontological status, in particular, without thinking that spacetime exists independently of physical phenomena, the contribution of mathematical physics would be providing new mathematical frameworks to encompass it. Exaclty like language is what enables us to think, maybe this is in part a lack of words (mathematical structures) which prevents us from figuring out and to concretely *embody* new and promising physical ideas.

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Résumé

Notre connaissance actuelle de l'Univers repose sur l'existence de quatre interactions fondamentales, qui sont la gravitation, l'électromagnétisme, l'interaction forte et l'interaction faible. Elles forment la base conceptuelle de la physique moderne depuis un demi-siècle. Je m'intéresse dans ma thèse à l'aspect classique des théories physiques sous-jacentes, appelées “ théories de jauge ”. Ma démarche est celle d'un physicien mathématicien. Dans un premier temps, elle consiste à étudier les théories de jauge dans leur formulation mathématique, afin de mettre en lumière certaines structures géométriques et algébriques sous-jacentes. Dans un second temps, on propose de nouveaux cadres mathématiques possibles pour formuler des théories de jauge. On a exploré pour cela la géométrie conforme et les théories de jauge de la gravitation conforme associées, pour lesquelles le groupe de symétrie est élargi, passant du groupe de Lorentz au groupe conforme. Le tout est formulé dans le langage de la géométrie de Cartan. En appliquant la méthode de l'habillage, qui consiste à réduire la symétrie de jauge d'une théorie par un simple changement de variable, on retrouve les objets habituellement définis dans une telle géométrie, comme les Tractors et les Twistors, avec en prime une meilleure compréhension de leur nature géométrique. On présente également le cadre des algébroïdes de Lie transitifs, et différentes façons de formuler des théories de jauge en son sein. Premièrement, on développe une notion de tenseur sur les algébroïdes de Lie, le choix d'une base locale adaptée étant fondamentale dans la poursuite des calculs. On parvient, reprenant dans une formulation plus claire un travail de N. Boroojerdian, à décrire dans un unique lagrangien la relativité générale avec constante cosmologique ainsi que les théories de Yang-Mills pour les autres interactions. Le travail de C. Fournel est également présenté, dans lequel la notion de connexion généralisée sur des algébroïdes de Lie permet d'écrire un lagrangien contenant à la fois la théorie de Yang-Mills et un terme de Higgs plongé dans un potentiel quartique. Finalement, nous présentons un travail récent consistant à combiner géométrie de Cartan et algébroïdes de Lie transitifs. Pour cela, on écrit les suites d'Atiyah correspondant aux deux fibrés principaux sous-jacents à une géométrie de Cartan, puis nous donnons la définition d'une connexion de Cartan dans ce langage. Nous démontrons l'équivalence de cette définition avec la définition usuelle sur les fibrés principaux. Nous comparons également notre approche avec celle, récente également, de M. Crampin et D. Saunders.

Abstract

Our current knowledge about Universe rests on the existence of four fundamental interactions. These are : gravitation, electromagnetism, weak interaction and strong interaction. They have formed the conceptual basis of modern physics since half a century. I am interested in the classical aspect of the underlying physical theories : “ gauge theories ”. My approach is that of a mathematical physicist. First, this consists in studying gauge theories in their mathematical formulation, in order to enlighten some underlying geometric and algebraic structures. Second, new mathematical frameworks are proposed to formulate gauge theories, generalizing the previous ones. In this aim, we explored conformal geometry and its associated conformal gauge theories. These are gravitational gauge theories for which one passes from the Lorentz group to the conformal as structure group. The whole work is formulated in the language of Cartan geometry. Applying the dressing field method, which consists in reducing the gauge symmetry of a theory by a mere change of variables, we recover some objects usually defined in this geometry, as Tractors and Twistors. The bonus is that we get a deeper understanding of their geometric nature. We also present the theory of transitive Lie algebroids, and different ways of formulating gauge theories in this framework. First, we develop a notion of tensors on Lie algebroids, with an adapted basis which is fundamental in order to facilitate computations. It is possible, as N. Boroojerdian already did, to describe in a unique lagrangian General Relativity with cosmological constant together with Yang-Mills theories for other interactions. We recast this work in our clearer notations. The work of C. Fournel is also presented, in which the notion of generalized connection on Lie algebroids allows to write a lagrangian which contains both Yang-Mills theory and a Higgs term embedded in a quartic potential. Finally, we present a recent work which consists in combining Cartan geometry and transitive Lie algebroids. For this, we write Atiyah Lie sequences corresponding to both principal bundle related to a Cartan geometry, and then we give the definition of a Cartan connection in this language. We show the equivalence of this definition with the usual one on principal fibre bundles. We also compare our approach with that of M. Crampin and D. Saunders, also quite recent.