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Noémie COMBE

On a new cell decomposition of a complement of the discriminant variety: application to the cohomology of braid groups

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Sur une nouvelle décomposition cellulaire de l'espace des polynômes a racines simples: application à la cohomologie des groupes de tresses.

Soutenue le 25/05/2018 devant le jury composé de:

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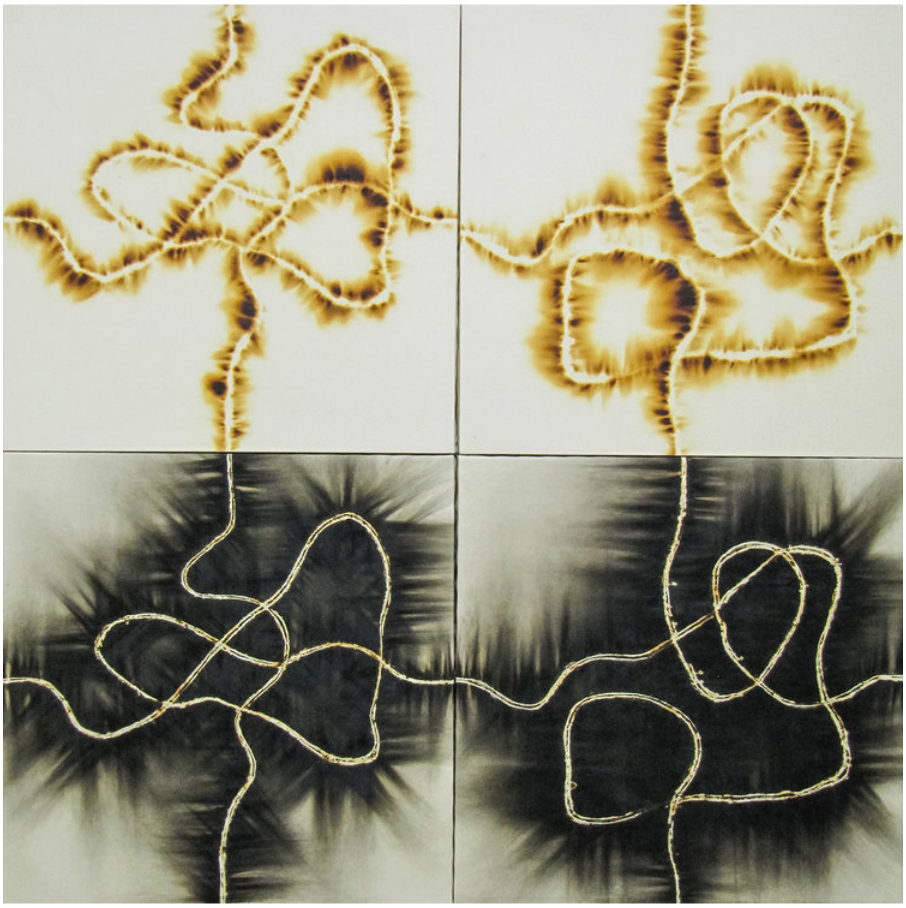

Entrelacs by Christian Jaccard

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Contents

Part 1

Introduction et sommaire

Cette thèse, est rédigée en anglais et comporte une introduction en français qui résume le travail effectué.

CHAPTER 1

Introduction de la thèse de doctorat

Noémie Cécile Combe

Sur une nouvelle décomposition de l'espace des polynômes à racines simples avec application au calcul de la cohomologie des groupes de tresses

Cette thèse concerne principalement deux objets classiques étroitement liés: d'une part la variété ${}^{\mathrm{D}}\mathrm{Pol}_d$ des polynômes complexes unitaires, de degré $d \geq 1$ à une variable, et à racines simples (donc de discriminant différent de zéro), et d'autre part, les groupes de tresses d'Artin $\mathcal{B}_d$ avec d brins. La richesse de leurs interactions a depuis longtemps été reconnue comme apportant des informations sur ces deux objets, l'information de base étant que l'espace ${}^{\mathrm{D}}\mathrm{Pol}_d$ est un espace $K(\pi, 1)$ de groupe fondamental $\mathcal{B}_d$. Ainsi l'espace ${}^{\mathrm{D}}\mathrm{Pol}_d$ peut être utilisé pour le calcul de la cohomologie des groupes de tresses non colorées.

Ce thème a inspiré de nombreux travaux, à commencer par ceux de V. Arnold dans les années 1970. Celui-ci utilise une suite spectrale pour calculer la cohomologie intégrale du groupe de tresses non colorées. Une autre version est apportée par D.B. Fuks, celle-ci permettant d'avoir la cohomologie du groupe de tresses à valeurs dans $\mathbb{Z}_2$. Enfin, F.V. Vainshtein propose un résultat plus général, reposant sur la decomposition de D.B. Fuks et sur les homomorphismes de Bockstein. Indépendamment, P. Deligne s'est également penché à cette époque là sur ce sujet, en introduisant sa fameuse notion d'immeubles de groupes de tresses. Plus récemment d'autres résultats ont suivis, notamment donnés par V. Lin, F. Cohen et A. Goryunov. Dans des thématiques proches on peut également noter des travaux de De Concini, C. Procesi, M. Salvetti.

Le travail présenté dans cette thèse propose une nouvelle approche permettant des calculs cohomologiques explicites à coefficients dans n'importe quel faisceau. On montre, entre autres choses, qu'à partir de la nouvelle stratification de l'espace des polynômes introduite dans cette thèse, la monodromie se lit de manière transparente.

En vue de calculs cohomologiques explicites, il est souhaitable d'avoir à sa disposition un bon recouvrement au sens de Čech. L'un des principaux objectifs de cette thèse est de construire un tel recouvrement basé sur des graphes qui sont associés à des polynômes complexes.classifiés par l'espace ${}^{\mathrm{D}}\mathrm{Pol}_d$.

1. Signatures

Si on note $P \in {}^{\mathrm{D}}\mathrm{Pol}_d$ un tel polynôme, le graphe correspondant est défini comme l'image inverse de la réunion $\mathbb{R} \cup i\mathbb{R}$ des axes réels et imaginaires. On définit ainsi, pour tout polynôme P de degré d à racines simples, un système de d courbes colorées en rouge et d courbes colorées en bleu, proprement plongées dans le plan. Les courbes rouges et bleues s'intersectent aux racines de P et ces intersections sont orthogonales du fait que P définit une application conforme.

On vérifie facilement que les graphes ainsi obtenus sont des forêts, autrement dit des réunions d'arbres. Les arêtes sont colorées en rouge ou bleu, et portent une orientation naturelle. Les 2-cellules, complémentaires du graphe dans le plan, se partitionnent naturellement en quatre couleurs respectivement notées A, B, C et D, correspondant aux images inverses par P des quatres quadrants du plan complexe. On appellera *signature* d'un tel graphe sa classe d'isotopie relativement aux $4d$ directions asymptotiques de $P^{-1}(\mathbb{R} \cup i\mathbb{R})$. Ces directions asymptotiques expriment que tout polynôme de degré d se comporte à l'infini comme une 'perturbation' du polynôme z^d (à constante multiplicative près).

2. Stratification de ${}^{\mathrm{D}}\mathrm{Pol}_d$

DEFINITION 1.1. *(Chapitre 2, Définition 2.3)* [**1**] *La partition $\mathcal{A}$ de $Dpol_d$ consiste en la réunion des sous-ensembles disjoints, notés A_σ, formés de tous les polynômes d'une signature donnée σ.*

Nous montrons que la décomposition $\mathcal{A}$ de ${}^{\mathrm{D}}\mathrm{Pol}_d$ fournit une stratification semi-algébrique 6.21, chapitre 6[1]

L'union des strates de dimension maximale, dites génériques, est formée de l'ensemble des polynômes complexes n'ayant aucune valeur critique sur les axes réel et imaginaire. Les composantes connexes de cette réunion sont en bijection avec les signatures comportant exactement d arbres. Les sommets internes de ces arbres sont tous de valence 4, avec localement une courbe bleue et une rouge se coupant transversalement (et même orthogonalement pour la métrique usuelle). L'union des strates de codimension 1 est formée de l'ensemble des polynômes ayant exactement un point critique z, dont la valeur critique associée $P(z)$ est sur l'union des deux axes et tel que $P''(z) \neq 0$.

Soit B_d le groupe des tresses d'Artin sur d brins, muni des générateurs usuels σ_i, $i = 1, ..., d-1$, avec les relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ si $|i-j| = 1$ et $\sigma_i\sigma_j = \sigma_j\sigma_i$ si $|i-j| > 1$. Dans leur article de 1997 [**8**], J. Birman, K. Ko and S. Lee ont introduit de nouveaux générateurs $a_{i,j}$ pour $1 \leq i < j \leq d$ connus sous le nom de générateurs de Birman-Ko-Lee (BKL), donnés par:

$$a_{i,j} = (\sigma_{i-1}\sigma_{i-2} \ldots \sigma_{j+1})\sigma_j(\sigma_{j+1}^{-1} \ldots \sigma_{i-2}^{-1}\sigma_{i-1}^{-1}).$$

[1]Notons en passant que cette stratification semi-algébrique fournit un bon exemple de décomposition semi-algébrique d'une variété algébrique dans laquelle les strates sont indexées par des objets topologico-combinatoires, à savoir ici les signatures des polynômes complexes à racines simples.

Un mot de Birman-Ko-Lee est un monôme en ces générateurs (à équivalence prés donnée par les relations entre les générateurs BKL, que nous rappelons explicitement au chapitre 3).

On établit une relation entre strates génériques et couples de mots de BKL.

PROPOSITION 1.1. *(Chapitre 3, Proposition 3.7) Il existe un morphisme injectif de l'ensemble des signatures de codimension 0 vers les couples de mots de Birman-Ko-Lee, un mot correspondant à la configuration des diagonales rouges, le deuxième à la configuration des diagonales bleues.*

Dans le chapitre 8 on se donne pour but de calculer le nombre de strates de degré et de codimension fixés. Nous donnons ce résulat sous la forme d'une série génératrice.

THEOREM 1.2. *(Chapitre 8, Théorème 8.5) Il existe deux séries en deux variables $\mathcal{N}_1(x,y)$ and $\mathcal{N}_2(x,y)$, vérifiant les propriétés suivantes:*

- *la série $\mathcal{N}_2$ est définie par l'équation*
$$(1-\mathcal{N}_2+y\mathcal{N}_2^4)(1+y\mathcal{N}_2^4-x^2y\mathcal{N}_2^5)+xy\mathcal{N}_2^6=0,$$
- *la série $\mathcal{N}_1$ est définie par:*
$$\mathcal{N}_1=1+y\mathcal{N}_2^4,$$
- *le coefficient de x^cy^d dans $\mathcal{N}_1$ est égal au nombre de signatures de degré d et de codimension c,*
- *la série $\mathcal{N}_1$ est algébrique et on peut calculer explicitement son polynôme minimal.*

On démontre comme corollaire de ce théorème le résultat suivant, qui apparaît aussi dans [**1**] (section 3).

COROLLARY 1.3. *(Chapitre 8, section 3) Le nombre de strates génériques de la strate ouverte (codimension nulle) est donné par le nombre dit de Fuss-Catalan.*

3. Bon recouvrement de ${}^{\mathrm{D}}\mathbf{Pol}_d$

La partition en strates ne fournit pas immédiatement un recouvrement adapté au calcul de la cohomologie de Čech (avec n'importe quels coefficients) pour deux raisons liées et évidentes: d'une part les sous-ensembles du recouvrement ne sont pas tous ouverts, et de plus ils sont disjoints puisqu'ils correspondent à différentes signatures. Pour remédier à ce problème, nous construisons (essentiellement au chapitre 6) un recouvrement à partir des strates A_σ en les épaississant. Nous disposons pour commencer du résultat suivant prouvé par N. A'Campo dans [**1**].

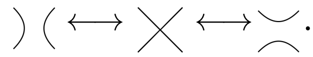
THEOREM 1.4. *(Chapitre 6, Théorème 6.1) Les composantes connexes A_σ de la partition $\mathcal{A}$ sont contractiles.*

Dans le but de construire un recouvrement convenable pour le calcul de la cohomologie de Čech, il faut toutefois beaucoup plus. Comme l'épaississement d'une strate A_σ contient son adhérence $\overline{A_\sigma}$, nous devons d'abord montrer que celle-ci reste contractile, puis étudier également la contractilité des intersections.

Pour étudier l'adhérence, nous commençons par en donner une description précise en utilisant les notions suivantes.

DEFINITION 1.2. *(Chapitre 2, Définition 2.20) Deux signatures de même codimension sont dites* adjacentes *s'il existe une déformation élémentaire de l'une à l'autre, représentée par un mouvement de Whitehead (définition 2.18) sur un ensemble de courbes de même couleur appartenant toutes au bord d'une 2-cellule du plan complexe privé du premier graphe.*

Pour effectuer un mouvement de Whitehead, on commence par déformer ces courbes jusqu'à ce qu'elles se rencontrent en un seul point critique; les deux signatures correspondantes sont alors dites incidentes. *On appelle cette opération un demi-mouvement de Whitehead contractant. On ouvre ensuite ce noeud dans l'autre sens; on appelle ce mouvement un demi-mouvement de Whitehead lissant.*

$\delta :$ 

Nous démontrons le résultat suivant qui décrit combinatoirement l'adhérence topologique d'une strate ouverte.

LEMMA 1.5. *(Chapitre 6, Lemme 6.12) Soit σ une signature. Alors l'adhérence topologique $\overline{A_\sigma}$ de la strate A_σ est la réunion disjointe de A_σ avec les strates A_τ pour tout τ incidente à σ.*

Nous utilisons ensuite ce résultat pour montrer que les adhérences sont aussi contractiles.

THEOREM 1.6. *(Chapitre 6, Lemme 6.14) Soient σ une signature générique. Alors, $\overline{A_\sigma}$ est contractile.*

Finalement, nous montrons que les intersections des adhérences restent contractiles.

THEOREM 1.7. *(Chapitre 6, Lemme 6.19) Soient $\sigma_1, \ldots, \sigma_r$ des signatures génériques distinctes. Alors, l'intersection multiple*

$$\cap_{i=1}^{r} \overline{A_{\sigma_i}}$$

est soit vide soit contractile.

Nous terminons la construction du bon recouvrement en épaississant les $\overline{A_\sigma}$ pour obtenir des ouverts. Les principaux outils sont le théorème de triangulation de Łojasiewicz et la double subdivision barycentrique.

THEOREM 1.8. *(Chapitre 6, Théorème 6.23) Il existe un recouvrement ouvert $\mathcal{U} = \{A_\sigma^+\}_{\sigma \in \Sigma_d}$ où les σ sont génériques et sont tels que:*

(1) *les éléments du recouvrement $A^+_{\sigma_i}$, où σ_i est générique, sont ouverts et contractiles;*
(2) *les intersections multiples $\cap_{i=1}^p A^+_{\sigma_i}$ sont soit vides ou contractiles.*

Dans le chapitre 7 nous montrons comment calculer explicitement des groupes de cohomologie de Čech à valeurs dans un faisceau quelconque par une méthode qui repose sur la construction du bon recouvrement et l'étude détaillée du complexe dual (voir plus bas). On détaille des exemples en basses dimensions des groupes de cohomologie à valeur dans $\mathbb{Z}$. Nous nous sommes restreints à ce cas car les calculs ont été faits à la main. L'utilisation d'un ordinateur permettrait certainement d'aller beaucoup plus loin.

4. Complexe dual à la stratification

L'étude de la combinatoire des signatures, essentielle dans le calcul de la cohomologie, nous a mèné à une analyse détaillée du complexe dual $\mathcal{W}$ de la stratification par les A_σ, contenue dans le chapitre 3 et 4. Les k-faces de $\mathcal{W}$ sont en bijection avec les signatures de codimension k, et les relations correspondent aux mouvements de Whitehead. Le chapitre 3 est consacré à étude de $\mathcal{W}$, qui définira le nerf associé au bon recouvrement.

Les cas de petites dimensions sont explicitement construits et illustrés. Le complexe $\mathcal{W}$ possède de belles propriétées; par exemple on montre le résultat suivant.

THEOREM 1.9. *(Chapitre 3, Théorème 3.15 et Corollaire 3.16) Le complexe dual $\mathcal{W}$ admet un groupe de symétries diédral $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$.*

On utilise également la notion de partitions d'ensembles non-croisées.

Une partition est dite non-croisée si pour chaque $i < j < k < l$ la partition de $\{i, j, k, l\}$ en blocs $\{i, k\} \cup \{j, l\}$ n'existe pas. L'ensemble de ces partitions non croisées forme un ensemble partiellement ordonné.

Comme il existe une bijection entre les partitions non-croisées d'un ensemble à d éléments et les mots de Birman-Ko-Lee pour d brins, on remarque que dans la structure du complexe dual pour les faces de dimension 0 et 1 il existe quatre sous-structures isomorphes au diagramme de Hasse de l'ensemble partiellement ordonné des partitions non-croisés d'un ensemble à d éléments, noté $NC(d)$. Cette structure est un invariant du complexe dual. On met aussi en évidence deux autres structures invariantes : la première, dite structure en pont, reliant deux $NC(d)$ opposés; la seconde est la structure dite en livre ouvert reliant un $NC(d)$ avec une structure en pont. En particulier on montre que:

PROPOSITION 1.10. *(Chapitre 3, Proposition 3.9) Pour tout $d > 1$, il existe quatre structures $NC(d)$ formant une structure en collier dans le complexe dual.*

Ici, le terme de *collier* indique qu'il existe un ensemble de m simplexes connectés entre eux de sorte à ce que le sommet terminal du i-ème simplexe se trouve relié au sommet initial du $(i+1)$-ème simplexe avec $i \in \mathbb{Z}_m$. Ces résultats sont exploités dans la suite, d'une part afin de simplifier les calculs cohomologiques, d'autre part pour illustrer la représentation de monodromie, expliquée au chapitre 5.

Part 2

The decomposition of the space of complex polynomials and the polyhedral structure

CHAPTER 2

The cell decomposition of the space of complex polynomials

We investigate a new decomposition of the d-th unordered configuration space of the complex plane space, introduced by N. A'Campo [**1**], using its natural relation with the space of complex monic degree $d > 0$ polynomials in one variable with simple roots and by introducing for each such polynomial, a topological object called its signature. A signature of a polynomial is an embedded decorated graph. Those signatures are the essence of this decomposition.

The aim of this chapter is to show the existence of signatures, its properties and prove the existing relations between different signatures.

• Let ${}^{\mathrm{D}}\mathrm{Pol}_d$ denote the space of complex monic degree $d > 0$ polynomials having one variable and simple roots with sum equal to zero (i.e. Tschirnhausen polynomials). The complex dimension of ${}^{\mathrm{D}}\mathrm{Pol}_d$ is $d-1$.

1. Polynomials and their signatures

Let us introduce in this part the notion of *picture* and *signature* of a polynomial (signature in short when no confusion is possible).

1.1. Pictures.

DEFINITION 2.1 (Picture). *Let P be a polynomial in ${}^{\mathrm{D}}Pol_d$. The inverse image under P of the union of $\mathbb{R}$ and $\imath\mathbb{R}$, is called the picture of the polynomial P. We denote a picture of a polynomial P by $\mathcal{C}_P$.*

REMARK 2.1. One could of course consider the pre-image of $\mathbb{R}$ and $\imath\mathbb{R}$ under any complex polynomial P. If P has multiple roots, multiple intersections of red and blue diagonals may appear. The reason for which we avoid this situation throughout the thesis is because our goal is to apply the cell decomposition to the study of the fundamental group of ${}^{\mathrm{D}}\mathrm{Pol}_d$ (the d-strand Artin braid group). A cell decomposition of $\mathbb{C}^d$, while combinatorially rich, would obviously not have any such application.

DEFINITION 2.2 (Codimension). *Let P be a polynomial in ${}^{\mathrm{D}}Pol_d$.*

- *A polynomial P with no critical values on $\mathbb{R} \cup \imath\mathbb{R}$ is called biregular or generic; such a polynomial is of codimension 0.*
- *The* special *critical points of P are the critical points z such that $P(z) \in \mathbb{R} \cup \imath\mathbb{R}$. The local index at a special critical point $z \in \mathbb{R}$ (resp. $\imath\mathbb{R}$) is equal to $2m-3$ where m is the number of red (or blue) diagonals crossing at the*

point z. The real codimension *of P is the sum of the local indices of all the special critical points.*

As a convention, we shall color the curves $P^{-1}(\mathbb{R})$ in red and the curves $P^{-1}(\imath\mathbb{R})$ in blue. Intersections of red and blue curves correspond to the roots of the polynomial P. Those curves intersect orthogonally, since P is a conformal map. At infinity, there exist $4d$ asymptotic directions at rays $\sqrt[4d]{r}, r > 0$, the colors of the rays alternating from red to blue.

EXAMPLE 2.1. An example for the picture of the generic polynomial

$$P(z) = z^6 - 0.71z^4 + 0.0063z^2 - \imath 0.38z^4 + \imath 0.0048z^2 - 0.0284 - \imath 0.0076.$$

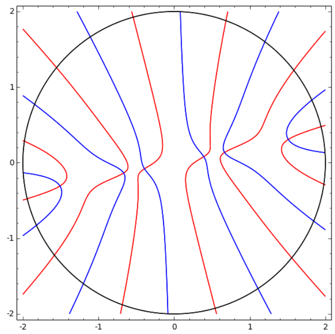


FIGURE 1. Example of a generic picture

We color the four regions of $\mathbb{P}^1 \setminus \mathbb{R} \cup \imath\mathbb{R}$ (resp. $\mathbb{C} \setminus \mathbb{R} \cup \imath\mathbb{R}$) in the colors A, B, C, D. The regions A and B lie in the upper half plane and the region of color A is bounded by both positive axis, as in the figure 2.

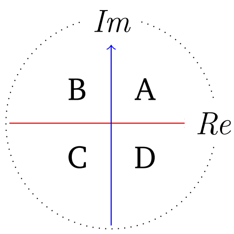


FIGURE 2. Partition of the complex plane

REMARK 2.2. Note that the pictures are represented as diagrams, without the point at infinity.

DEFINITION 2.3. *Two pictures $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent if and only if there exists an isotopy of $\mathbb{C}$ (a continuous family of homeomorphisms of $\mathbb{C}$) $u : \mathcal{C}_1 \to \mathcal{C}_2$ which preserves the labels of the $4d$ asymptotic directions and the colors. Moreover, the restriction of u on $\mathcal{C}_1$ is a graph isomorphism between $\mathcal{C}_1$ and $\mathcal{C}_2$.*

1.2. Signatures and properties of signatures. Let us also recall the notion of topological map from [**43**].

DEFINITION 2.4. [**43**] *A topological map σ is a graph Γ embedded into a surface X (considered as $\Gamma \subset X$) such that:*

- *vertices are distinct points on X;*
- *edges are curves which intersect only at vertices;*
- *if we cut long the graph we get a disjoint union of connected components (faces) each homeomorphic to an open disk.*

If the topological space X is the sphere then we qualify those topological graphs of "planar".

REMARK 2.3. Using definition 2.3 leads directly to the topological map associated to a picture. However, a supplementary structure needs to be added to those topological maps: the embedded graphs are decorated (since edges carry a red or blue coloring and an orientation. The 2-faces carry a coloring in four colors: A, B, C, D).

DEFINITION 2.5. *A decorated topological map is a topological map such that:*

(1) the edges of the embedded graph carry a coloring and an orientation;
(2) the 2-faces carry a coloring.

This leads to the notion of signature.

DEFINITION 2.6. *A signature is the equivalence class of pictures in the equivalence relation from definition 2.3. A signature is a decorated topological map. We shall denote a signature by σ and by $\mathcal{C}_P$ a representative of the class σ. We denote by Σ the set of signatures and by Σ_d the set of signatures corresponding to degree d polynomials.*

Recall that isotopy relations are also equivalence relations. Let $\mathcal{R}'$ be the equivalence relation on the space of pictures, given by the isotopy relation in definition 2.3.

LEMMA 2.1. *There exists an equivalence relation $\mathcal{R}$ on the space ${}^{\mathrm{D}}Pol_d$, induced from the equivalence relation $\mathcal{R}'$ on $\mathcal{C}$. Two polynomials $P_0, P_1 \in {}^{\mathrm{D}}Pol_d$ are equivalent for the equivalence relation $\mathcal{R}$ if their pictures $\mathcal{C}_0$ and $\mathcal{C}_1$ belong to the same isotopy class.*

PROOF. The proof follows from the definition of an equivalence class. □

PROPOSITION 2.2. *The number of signatures is finite.*

PROOF. Consider at first the generic case (i.e. graphs in the codimension 0 strata). There exist d inner vertices incident to edges of both red and blue colors and which are connected to the $4d$ terminal vertices. The number of different possible combinations follows from Hall's matching theorem. Thus it is a finite number of possibilities. In a signature, the distinct intersection points between curves of the same color correspond in ${}^{\mathrm{D}}\mathrm{Pol}_d$ to the critical points of the polynomials. There are at most $d-1$ distinct critical points for a polynomial P. So, there exist at most $d-1$ inner vertices in a signature with adjacent edges of the same color. Now, we show that the number of graphs of higher codimension than 0 is finite. Such a graph is obtained by adding k new vertices (where $1 \leq k \leq d-1$) in the complementary regions of the graph and by collapsing the edges of the same color lying in the boundary of this region to this point. The number of those new possible graphs is finite. □

The main result concerning the properties of a signature and the relations between signatures and pictures of polynomials are the following two theorems due to N. A'Campo. Theorem 2.3 appears as theorem 1.1 in [**1**] with a brief proof. We give complete details here using embedded graphs.

THEOREM 2.3. *If a decorated topological map is a signature, it satisfies the following combinatorial properties:*

(1) In $\mathbb{P}^1$ the topological map is a connected set.
(2) In $\mathbb{C}$ the decorated topological map is a bicolored forest (i.e. a union of bicolored trees)
(3) The 2-faces in $\mathbb{C} \setminus \sigma$ are colored A, B, C or D.
(4) The edges in the signature are oriented and carry a coloring red or blue. By convention the edge is colored red if it is adjacent to a couple of regions colored respectively D and A or B and C. The edge is colored blue if the it is adjacent to a couple of regions colored respectively A and B or C and D. The orientation of the red (resp. blue) edges goes towards infinity if one crosses the arc from the region D to A (resp. A to B). It comes from infinity if one crosses the region A to B (resp. C to D).
(5) At infinity the signature has $4d$ edges asymptotic to the rays $re^{k\pi i/4d}$, $r > 0$, $k = 0, 1, \cdots, 4d-1$, the red and blue colors alternate as well as their orientations. In the neighborhood of this point the incident regions are colored in the counterclockwise orientation by the 4-periodic sequence of symbols A, B, C, D.
(6) In the neighborhood of the valency four vertices incident to the blue and red edges there exist four incident 2-faces colored A, B, C, D. In the neighborhood of the vertices which are incident to an even number of edges of the same color the adjacent regions are colored $\{A, B\}$, $\{B, C\}$, $\{C, D\}$ or $\{D, A\}$.

PROOF. Using the conventions we shall color $P^{-1}(\mathbb{R})$ in red and $P^{-1}(\imath\mathbb{R})$ in blue and partition the complex plane into four regions A, B, C, D as in figure 2. We shall study the properties of the pictures and use systematically the definition 2.3 to obtain the signature.

(2) First observe that the function $Re(P) * Im(P) : \mathbb{C} \to \mathbb{R}$ is harmonic (this statement follows from the fact that $Re(P) * Im(P) = \frac{1}{2} Im(P^2)$ and the imaginary part of the holomorphic map P^2 is harmonic). Suppose by contradiction that there exists a cycle Z in the picture bounding an open region U of the complex plane. Since the function $Re(P) * Im(P)$ vanishes along Z, we would have by the maximum principle that the function $Re(P) * Im(P) = 0$ on U. It follows that the image $P(U)$ of P is inside the union of the real and imaginary axis in $\mathbb{C}$, contradicting the openness of the mapping P.

Let us prove the statements (3) and (4). The complement of the picture $\mathcal{C}_{P_0}$ (given by $\mathbb{C} \setminus \mathcal{C}_{P_0}$) decomposes the plane $\mathbb{C}$ into polygonal regions. The regions have piece-wise smooth curves as boundaries. The coloring of the regions and the orientation of the curves follows from the pull-back by the degree d map of $\mathbb{C}$ (we have d copies of the regions colored A, B, C, D and the orientation of the curves is induced from the one on the real and imaginary axis in the complex plane).

(5) Near infinity, $P^{-1}(\mathbb{R})$ and $P^{-1}(\imath\mathbb{R})$ have d branches each, and the branches are smooth curves, asymptotic to the rays $\sqrt[d]{\pm r}$, and $\sqrt[d]{\pm \imath r}, r \in \mathbb{R}^+$ (all complex roots of order d). The edges of the picture carry orientation induced by the natural orientation of $\mathbb{R}$ and $\imath\mathbb{R}$.

(6) Since P has distinct roots, the roots are not critical points. Near the root, P is holomorphic, conformal, invertible and $P^{-1}(\mathbb{R} \cup \imath\mathbb{R})$ is locally given by two small curves intersecting transversally, orthogonally. the incident regions are colored respectively A, B, C, D. Near a critical point z_0 where $P'(z_0) = 0, \dots, P^k(z_0) = 0, P^{k+1}(z_0) \neq 0$, after a smooth holomorphic change of the local coordinates, P is locally smoothly conjugated to the germ $z \mapsto z^{k+1}$. So, if $P(z_0) \in \mathbb{R}$ (resp. $P(z_0) \in \imath\mathbb{R}$) then, near the point z_0, $P^{-1}(\mathbb{R})$ is locally smoothly conjugated to $\sqrt[k]{r}$ (k distinct complex roots of r), $r \in \mathbb{R}$ (resp. $r \in \imath\mathbb{R}$). The point z_0 on $\mathcal{C}_P$ is said to be of multiplicity $k+1$, where $k \geq 1$. This gives a vertex of even valency which is incident to edges of the same color. The critical value belongs either to the real axis or to the imaginary axis. Therefore there are only two possible colors for the incident regions: $\{A, B\}, \{B, C\}, \{C, D\}$ or $\{D, A\}$.

Finally statement (1) follows from the previous ones and from the fact that we compactify $\mathbb{C}$ by adding the point at infinity. □

The next theorem, again due to A'Campo, gives the converse.

THEOREM 2.4. *(Theorem 1.3 in* [**1**]*) Let σ be a signature. Then there exists a polynomial P whose picture is isotopic to σ.*

1.3. Diagrams of signatures.

We replace signatures by diagrams consisting of a unit disk $\mathbb{D}$ with roots $\sqrt[4d]{1}$ replacing asymptotic rays. We contract pictures so that the roots of P and P' all lie inside the disk. A signature becomes an embedded graph in the disk.

DEFINITION 2.7. *Two signature diagrams are in the same equivalence class if there exists a homeomorphism of $\mathbb{D}$ fixed on the terminal vertices which takes one diagram to the other.*

DEFINITION 2.8. *The set of diagrams corresponding to the signatures Σ_d is a set of decorated forests embedded in the unit disk.*
Those diagrams verify the following extra structure:

1. *There exist $4d$ terminal vertices at the roots of order $4d$ of 1, ordered in counterclockwise order.*
2. *There exist d inner nodes, of valency 4, being adjacent to two edges of color red and two edges of color blue.*
3. *The inner vertices with edges of one color will be called meeting points.*
4. *The edges emerging from odd terminal vertices are colored blue, the orientation is from vertex number $3 \mod 4$ to vertex number $1 \mod 4$.*
5. *The edges emerging from even terminal vertices are colored red, the orientation is from vertex number $0 \mod 4$ to vertex number $2 \mod 4$.*
6. *The set of polygonal regions defined in $\mathbb{C} \setminus \mathcal{C}_P$ are labelled periodically A, B, C, D. The labels are inherited from the labels A, B, C, D on figure 2.*
   Each region has one arc lying on the boundary of the disc. If the edge on the boundary is an arc from vertex number 1 to vertex number 2 then it is colored A; if the edge is 2, 3 then the region is D; if the edge is $3, 0$ then the region is C, if the edge is $0, 1$ then the region is B. If the region has many edges on the boundary they are all of the same type.

In the following parts we shall keep confusing a signature with its diagram, diagrams being convenient tools for the description a given signature.

2. Combinatorics of signatures

2.1. Ordering signatures.

DEFINITION 2.9. *We denote by A_σ the set of polynomials with pictures in the isotopy class σ. The family $(A_\sigma)_{\sigma\in\Sigma}$ of subsets of ${}^{\mathrm{D}}Pol_d$ partitions the set ${}^{\mathrm{D}}Pol_d$. The partition of ${}^{\mathrm{D}}Pol_d$ into classes will be called stratification and the sets A_σ will be called strata, cells or classes.*

Here, on the basis of the notion of pictures and signatures we introduce incidence relations and adjacence relations between the connected components of the partition of ${}^{\mathrm{D}}\mathrm{Pol}_d$. This part is a stepping stone towards the Čech cohomology.

DEFINITION 2.10 (Arc). *Let σ_0 be a signature. By arc E, we mean an arc in a polygonal region $\mathbb{C} \setminus \sigma_0$, connecting two boundary points such that:*

1. *the graph $\sigma_0 \cup E$ is a forest;*
2. *the endpoints of E are smooth points of distinct edges, of the same color.*

The new graph $\sigma_0 \cup E$ obtained by adding E to σ_0 does not satisfy the properties of a picture. In particular two vertices are of degree 3. If we contract E to a point then we obtain a correct picture.

Incidence relations between two classes of polynomials are defined below.

DEFINITION 2.11 (Incidence relation between signatures). *The signature σ_1 is incident to the signature σ_2 if σ_2 is obtained from σ_1 by one of the following operations.*

(1) We add an arc E in σ_1 and contract it getting a new vertex. We get a new picture equivalent to σ_2.
(2) There is an edge in σ_1 connecting two vertices of the same color. We contract the edge and get a new vertex of higher multiplicity and a new signature equivalent to σ_2.

If the class A_{σ_1} is incident to A_{σ_2} then we denote this relation by $A_{\sigma_1} \prec A_{\sigma_2}$.

PROPOSITION 2.5. *There exists $P_2 \in A_{\sigma_2}$ in every neighborhood of every polynomial $P_1 \in A_{\sigma_1}$.*

PROOF. Let us consider the classes $A_{\sigma_1}, A_{\sigma_2}$ which are subsets of $\mathbb{C}^d$. In particular those subsets inherit the induced standard metric. If $A_{\sigma_1} \prec A_{\sigma_2}$ then for every $\epsilon > 0$ there exists a couple (B_ϵ, γ) where B_ϵ is a ball having non empty intersections with A_{σ_1}, A_{σ_2}, of radius ϵ and γ is a continuous path in B_ϵ, connecting $a \in A_{\sigma_1}$, to $a' \in A_{\sigma_2}$. The picture $\mathcal{C}_{a'}$ in the class σ_1 contains additional critical points, which do not exist on $\mathcal{C}_a$. □

We now introduce the notion of biregular (i.e. generic) polynomials. These polynomials play an important role in the decomposition of the space $^{\mathrm{D}}\mathrm{Pol}_d$ as will be seen in the next chapters.

THEOREM 2.6. *Let us consider the set of signatures for degree $d > 1$ polynomials. The incidence relation induces a partial order $\preceq$ on the set of signatures. We say that $\sigma \preceq \tau$ if there exists a sequence $\sigma = \sigma_1 \preceq \sigma_2 ... \preceq \sigma_n = \tau$.*

PROOF. Transitivity and reflexivity relations are obvious. We prove that the relation is antisymmetric. If one has $\sigma \preceq \tau$ then $codim(\tau) \geq codim(\sigma)$. So, if $\sigma \preceq \tau$ and $\tau \preceq \sigma$ then σ and τ have same codimension, and the equality of codimension is possible only if $n = 1$ in the sequence $\sigma_1 \preceq \sigma_2 ... \preceq \sigma_n$, so $\sigma = \sigma_1 = \sigma_n = \tau$. □

Notation: We shall extend the name incidence to the relation $\preceq$ and we shall denote it $\preceq$. Also a notation $A \prec B$ is equivalent to $B \succ A$.

2.2. Combinatorial presentation of diagrams. In order to describe rigorously the diagrams, we introduce some combinatorial definitions and tools. Let Σ_d be the set of all diagrams with $4d$ terminal vertices. A diagram in Σ_d is called of degree d.

2.2.1. *Families of trees in a diagram.*

Recall that the forests in the previous diagrams are composed of trees.

DEFINITION 2.12. *A graph G is defined by two finite sets: one non-empty set V of elements called vertices and one set E (E can be empty) of elements called edges, with two vertices x, y in V associated to each edge e and called the extremities of the edge. We denote the graph by $G = (V, E)$.*

DEFINITION 2.13. *A tree is a connected, undirected, acyclic graph.*

A tree has two types of vertices:

- the inner nodes (the inner vertices) which are of valency strictly greater than 1,
- the external vertices (also called leafs) which are of valency equal to 1.

DEFINITION 2.14. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if and only if there exists a bijection $\beta : V \to V'$ mapping each couple of end vertices x, y in V of a given edge $e \in E$ to an edge $e' \in E'$ of end vertices $\beta(x), \beta(y)$ in V'.*

Let $\mathcal{T} = \{T_i\}_{i\in J}$ be the set of planar trees of a diagram.

In $\mathcal{T}$, we distinguish two families of trees:

(1) simple trees consisting of a couple of red and blue diagonals crossing at a point,
(2) non-simple trees.

DEFINITION 2.15 (Short and long diagonals). *Let σ be a signature. By diagonal in a diagram we mean a coupling of two terminal vertices i, j which are of the same parity, denoted by (i, j). This coupling (i, j) is represented in the diagram as a 1-dimensional connected component of a given color (red or blue) joining two terminal vertices i and j.*

- *A diagonal is short if $|i - j| = 2$.*
- *A diagonal is long if it is not short.*

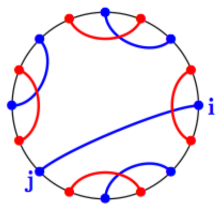


FIGURE 3. A long blue diagonal (i, j), with red short diagonals.

Let us present the properties of biregular diagrams. Biregular diagrams are composed of simple trees. There are three different types of simple trees.

- Trees consisting of two short (red and blue) diagonals are of type M.
- Trees consisting of one short and one long diagonal are of type F.
- Trees consisting of two long diagonals are of type S.

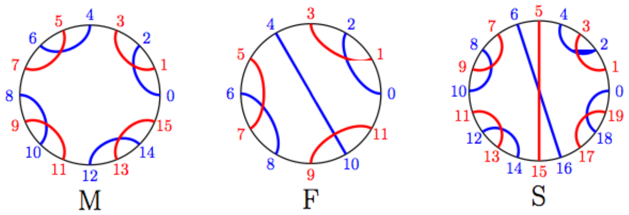
The M diagrams have only M trees. Some notation is introduced concerning these trees.

(1) An M tree is denoted by $\left|\begin{smallmatrix} i\\ i+2\end{smallmatrix}\right|$, where the number on the first line indicates that there exists a short diagonal $(i+1, i+3)$ crossing the diagonal $(i, i+2)$.

(2) An F tree is denoted by $\left|\begin{smallmatrix} j\\ i\end{smallmatrix}\right|$ where j and i are labels of the terminal vertices of the long diagonal; the number j, indicates that there exists a short diagonal of the opposite color joining the vertex $j-1$ to $j+1$. Two F diagrams are of opposite F trees if the indexes are switched, for instance $\left|\begin{smallmatrix} j\\ i\end{smallmatrix}\right|$ and $\left|\begin{smallmatrix} i\\ j\end{smallmatrix}\right|$ have opposite orientation.

(3) An S tree is denoted by $\left[\begin{smallmatrix} i,j\\ k,l\end{smallmatrix}\right]$, where the first line gives the coordinates of a long diagonal (i, j), the second line gives the coordinates of the second long diagonal (k, l).

EXAMPLE 2.2. An example of M, F, and S trees in biregular diagrams.



DEFINITION 2.16. *We call combinatorial presentation of a generic signature the collection of the d matrices* $\left[\begin{smallmatrix} i,j\\ k,l\end{smallmatrix}\right]$ *and* $\left|\begin{smallmatrix} j\\ i\end{smallmatrix}\right|$ *corresponding to the d trees of the signature.*

LEMMA 2.7. *The map from the set of generic signatures to the set of all combinatorial presentations is a bijection.*

PROOF. Let us show that this map is injective. Suppose that there exist two combinatorial presentations which are equal. These combinatorial presentations are equal if the elements in the matrices are equal: in other words the terminal vertices of the diagonals in the signature are equal. This defines a unique signature. The surjection follows from the construction of the diagrams. □

2.2.2. *Digression on trees in a diagram.*

Trees in $\mathcal{T}$ inherit properties from the diagrams above, since the trees in $\mathcal{T}$ satisfy a subset of the conditions in definition 2.8.

DEFINITION 2.17. *Let σ be a diagram.*

- *A tree bounds a 2-cell of $\mathbb{D} \setminus \sigma$ if a couple of edges of the tree lies in the boundary of this 2-cell.*
- *Two trees of a signature σ are called adjacent if they bound a common region in $\mathbb{D} \setminus \sigma$.*
- *A couple of diagonals $(i, j)(k, l)$ in a generic diagram σ are called successive if the terminal vertices satisfy one of the following conditions: $|i - k| = 2$ or $|i - l| = 2$ or $|j - k| = 2$ or $|j - l| = 2$.*

We shall define an operation on trees of different degrees.

Let $\mathcal{T}_n$ denote all signatures of degree n consisting of one tree. Then there exists an operation

$$\otimes : \Sigma_m \times \mathcal{T}_n \to \Sigma_{m+n},$$

not uniquely defined such that

$$\otimes : (\sigma_m, T_n) \mapsto \sigma_m \otimes T_n.$$

PROPOSITION 2.8. *The operation $\otimes : \Sigma_m \times \mathcal{T}_n \to \Sigma_{m+n}$ is non-commutative, except for finitely many cases.*

PROOF. Suppose that for any $m, n \in \mathbb{N}$ the union of trees $T_m \otimes T_n$ and $T_n \otimes T_m$ define the same signature σ in Σ_{m+n}. This statement is possible if and only if $T_n = T_m$. This contradicts that the statement holds for any $m, n \in \mathbb{N}$. ☐

COROLLARY 2.9. *Let $T_m, T_n \in \mathcal{T} \cup \{\emptyset\}$ and consider $T_m \otimes T_n \in \Sigma_{m+n}$. Then the trees T_m and T_n which are components of a given signature in Σ_{m+n} share a common region in $\mathbb{C} \setminus (T_m \cup T_n)$.*

2.3. Whitehead moves.

We introduce a topological operation called Whitehead move, on the signatures. This operation allows a modification of one signature σ to another one σ', in such a way that their codimensions remain equal. This operation is naturally related to the deformation of the coefficients of a given polynomial $P \in A_\sigma$ to another one belonging to the class $A_{\sigma'}$. If we deform the coefficients of a polynomial $P \in A_\sigma$, then this deformation is illustrated on the signature as a Whitehead move; reciprocally, if a Whitehead move is applied onto a signature σ giving in this way a new signature σ', then this corresponds to a deformation of the coefficients of a polynomial in the class A_σ into a second polynomial in the class $A_{\sigma'}$.

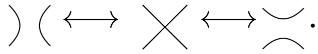
Using the Whitehead moves, we show that the space ${}^{\mathrm{D}}\mathrm{Pol}_d$ is simply-connected.

DEFINITION 2.18 (Half-Whitehead move of the first type).

(1) **Contraction step** *A half-Whitehead move of the first type on a signature σ_0 is a modification of the signature as follows: choose m diagonals of the same color which occur as part of the boundary of a given cell $\mathcal{R}$, add a polygon within the cell joining their centers, and contract this polygon to a point. This new vertex is called a meeting point. This gives a non-generic signature τ.*

(2) **Expansion step** *A smoothing half-Whitehead move of the first type on the signature τ is a modification applied to the meeting point, which is obtained by ungluing those m diagonals, giving m disjoint diagonals. The new signature may differ from σ_0.*

For two diagonals in $\mathcal{R}$, let us illustrate below a composition of a half-Whitehead move with a smoothing half-Whitehead move.

$\delta :$ 

Definition 2.19 (Half-Whitehead move of the second type).

(1) **Edge contraction step**- *Consider a non-simple tree in a signature σ_0 with at least two inner nodes, one arc joining them. All the edges (and arc) of the tree are of the same color. A half-Whitehead move of the second type on a signature σ_0 consists in contracting this arc to a point, glueing the two inner nodes together.*

(2) **Partial expansion step**- *Suppose that in a non-generic signature τ there exists a non-simple tree. Suppose without loss of generality that there exists one inner node of valency $2m$. A smoothing half-Whitehead move of the second type on the signature τ is obtained by un-glueing k (where $k > 0$) diagonals from the inner node along an edge of the tree, giving a non-simple tree with two inner nodes, one of valency $2(m-k)$ and the other one of valency $2(k+1)$.*

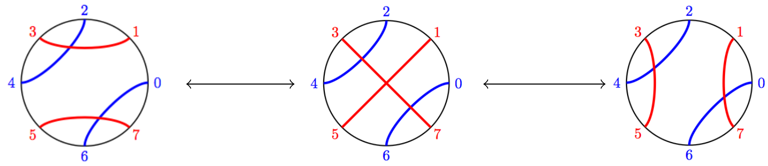
Definition 2.20. *Let σ_0 and σ_1 be signatures of the same codimension. A Whitehead move δ is a modification from σ_0 to σ_1 obtained from the composition of a half-Whitehead move and a smoothing half-Whitehead move on a set of m diagonals of the same color, occuring as part of the boundary of a given cell $\mathcal{R}$. Two classes A_{σ_0} and A_{σ_1} differing by a Whitehead move are called adjacent and are denoted by $A_{\sigma_0} \leftrightarrow A_{\sigma_1}$.*

Remark 2.4. Notice that in order to satisfy the theorem 2.3, the arcs with end-vertices on the centers of the couples of diagonal can not cross each-other.

Remark 2.5. Two classes A_{σ_0}, A_{σ_2} of bi-regular polynomials are adjacent if they are incident to one unique class of codimension 1. We have the relations $A_{\sigma_0} \prec A_{\sigma_1}$ and $A_{\sigma_2} \prec A_{\sigma_1}$.

Example 2.3.

- For $d = 2$, we illustrate a Whitehead move (c.f. definition 2.20) on a couple red diagonals. The two generic signatures, illustrated as diagrams on the right and on the left of the figure below, are incident to a codimension 1 class. This latter signature is illustrated as the diagram in the middle of the figure:



- We illustrate the natural relation between the Whitehead move (which is a topological operation) and the deformation on the set of coefficients of the a given polynomial. In this following example, we consider $Im(P) = 0$:

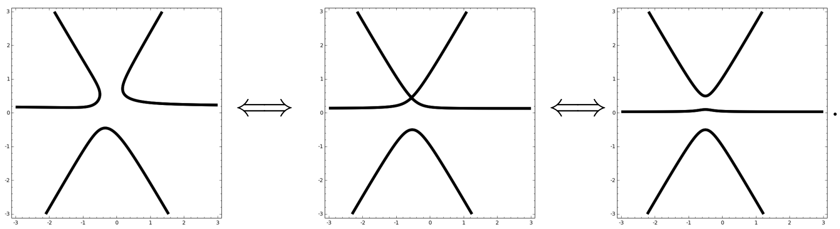


FIGURE 4. Whitehead move of $Im(P) = 0$ of monic polynomials, $P_1 = (z + 0.5 - 0.5\imath)(z + 0.5 + 0.5\imath)(z - 0.2 - 0.6\imath), P_2 = (z + 0.5 - 0.5\imath)(z + 0.5 + 0.5\imath)(z + 0.6 - 0.4\imath), P_3 = (z + 0.5 - 0.5\imath)(z + 0.5 + 0.5\imath)(z + 0.5 - 0.1\imath)$

2.4. Transitivity of Whitehead moves; connectedness of ${}^{\mathrm{D}}\mathbf{Pol}_d$.

LEMMA 2.10. *Let σ be a generic diagram such that all its red (resp. blue) diagonals are short. Let i, j, k be terminal vertices, i $(resp. j, k) \in \mathbb{Z}_{4d}$. Then, applying a Whitehead move onto the couple of adjacent blue (resp. red) diagonals (i, j) and $(j+2, k)$ increases by one the number of blue (resp. red) short diagonals, if $k \neq i-2$ and by two if $k = i - 2$.*

PROOF. Let us consider two cases:

(1) the case $k = i - 2$,
(2) the case $k \neq i - 2$.

Concerning the first case, by hypothesis the two adjacent long blue (resp. red) diagonals are given by (i, j) and $(j + 2, i - 1)$. Applying a Whitehead move onto this couple of diagonals induces the couple of diagonals $(i, i-2)(j, j+2)$, which are both short in the sense of the definition 2.15. So, the number of short blue (resp. red) diagonals is increased by two. Concerning the second case, the Whitehead move applied to the couple of diagonals $(i, j)(i-2, k)$ induces $(i, i-2)(j, k)$, where $(i, i - 2)$ is a short diagonal. So, the number of short blue (resp. red) diagonals is increased by one. □

REMARK 2.6. Suppose that the generic diagram σ has only short red (resp. blue) diagonals. Then, iterating successively lemma 2.10 onto couples of blue diagonals induces, in a finite number of steps, an M diagram.

LEMMA 2.11. *Let σ be a generic diagram such that all its red (resp. blue) diagonals are short. Let i, j, k, l be terminal vertices, i (resp.j, k, l) $\in \mathbb{Z}_{4d}$. Let (i, j) and (l, k) be a couple of non-successive blue diagonals in σ, which both occur as part of the boundary of a given 2-cell. Then, applying a Whitehead move onto the couple of diagonals (i, j) and (k, l):*

(1) increases by two the number of long blue (resp. red) diagonals, if both diagonals are short,
(2) increases by one the number of long blue (resp. red) diagonals, if one of the diagonals is short,
(3) gives a new couple of long diagonals $(i, l)(j, k)$, if both diagonals are long.

PROOF. Concerning the first statement, by hypothesis both diagonals are short and non-successive. Without loss of generality, suppose that $i < j < k < l$. The hypothesis implies that $|i - j| = |l - k| = 2$, where $k \neq j + 2 \mod 4d$, and $l \neq i - 2 \mod 4d$ (using definition 2.15 and definition 2.17). So, applying the Whitehead move onto the couple (i, j) and (k, l) induces the new couple of diagonals $(i, k)(j, l)$, where $|i - l| \neq 2$ and $|l - j| \neq 2$. Concerning the second statement, let us suppose, without loss of generality, that if (k, l) is short, then $l = k + 2$. Applying the Whitehead move onto (i, j) and $(k, k + 2)$ gives the couple of diagonals $(i, k + 2)(j, k)$. Since (i, j) and (k, l) are non-successive, then $l \neq i - 2$ and $k \neq j + 2$. Therefore, $|i - k - 2| \neq 2$ and $|j - k| \neq 2$. Since both diagonals are long, the number of long blue (resp. red) diagonals, is increased by one. Concerning the third statement, let us apply the definition of a Whitehead move (defintion 2.20) onto a couple of diagonals (i, j) and (k, l). In particular, this definition states that the codimension of the modified signatures must remain invariant, after the Whitehead move operation. Since (i, j) and (k, l) are disjoint and belong to a given signature σ, their terminal vertices verify $k \equiv i \equiv 1 \mod 4$ and $j \equiv l \equiv 3 \mod 4$ (see definition 2.8). This couple is first modified by a half-Whitehead move into a couple of diagonals meeting at one point: $(i, k)(j, l)$. This couple is then modified by a smoothing Whitehead move, which induces the unique possible couple of diagonals $(i, l)(j, k)$. □

THEOREM 2.12. *Let $\sigma \in \Sigma_d$ be a biregular diagram. Then, there exists a sequence of Whitehead move operations δ which lead from a generic diagram σ to an M-diagram.*

PROOF. The proof is by induction on the number of long blue diagonals. Let σ be biregular signature such that on the left side of a long blue diagonal there exist only short diagonals of red and blue color.

(1) *Base case.* Let σ have only one long blue diagonal. Then two cases are discussed:

(a) the red diagonals are all short.
(b) Not all red diagonals are short.

Consider the first case. Let us apply one of the lemma 2.10 onto the long blue diagonal and the blue short diagonals on its right side. Since each deformation step increases by one the number of short blue diagonals, so proceeding until there are d short diagonals in the diagram we obtain a diagram having only short blue diagonals. Hence, we have an M diagram. Consider the second case where we have k red long diagonals. Then this long blue diagonal and these k long red diagonals compartment the diagram into q disjoint adjacent polygonal regions. Each polygonal region can be interpreted as an M-diagram, locally. So, we have q adjacent locally M diagrams. We apply lemma 2.10 to the long blue diagonal in one compartment. Then, after a finite number of deformations using lemma 2.10 to the long blue diagonal, we have a local M-diagram in this compartment. The new long blue diagonal is now common to the adjacent compartment. We continue to apply lemma 2.10 to the new blue diagonal in this adjacent compartment until we obtain locally an M-diagram in this adjacent compartment. We continue in this way in all adjacent compartments. This gives in final all blue diagonals short. Now we apply the same procedure to the red long diagonals. We obtain after a finite number of deformations all red diagonals short. This is an M diagram.

(2) *Induction case.* Suppose that for a signature with m long diagonals there exists a path from the signature to an M diagram. Let us show that for $m+1$ long diagonals this statement is also true. Take a block of adjacent m long diagonals and apply the induction hypothesis to it. Then there exists a finite number of deformations such that these m long blue and red diagonals are all short, leaving only one long blue diagonal in the diagram. We can thus apply the case (1) from this discussion.

Remark 2.7. In the following chapters we will show that Whitehead moves correspond to isotopy classes of paths between strata. Thus the theorem 2.12 above can be interpreted as the path connectedness of ${}^{\mathrm{D}}\mathrm{Pol}_d$, from which we recover the fact that ${}^{\mathrm{D}}\mathrm{Pol}_d$ is connected since, as the complement of a hyperplane arrangement it is a open subset of $\mathbb{C}^d$.

□

CHAPTER 3

The geometric properties of the dual complex

The aim of this chapter is to represent geometrically the relations (incidence and adjacence) of the poset $(A_\sigma, \prec)$, these relations are illustrated by constructing the dual complex in such a way that each vertex of the dual complex corresponds to a biregular class, each edge of the dual complex corresponds to a codimension 1 class. In final, the construction of the dual complex corresponds to the nerve of the decomposition in the sense of Čech, which is essential for the calculation of Čech cohomology.

1. Construction of the dual complex

1.1. Vertices of the dual complex. We classify the generic diagrams into families. These generic diagrams are in bijection with the set of vertices in the dual complex.

(1) Type M diagram. The type M is a diagram which has only trees of type M. There are 4 such diagrams which we enumerate using the previous notations:

(a) $M_1 \leftrightarrow \left|\begin{smallmatrix}3\\1\end{smallmatrix}\right|\left|\begin{smallmatrix}7\\5\end{smallmatrix}\right|\dots\left|\begin{smallmatrix}4d-1\\4d-3\end{smallmatrix}\right|$.

(b) $M_2 \leftrightarrow \left|\begin{smallmatrix}1\\3\end{smallmatrix}\right|\left|\begin{smallmatrix}5\\7\end{smallmatrix}\right|\dots\left|\begin{smallmatrix}4d-3\\4d-1\end{smallmatrix}\right|$.

(c) $M_3 \leftrightarrow \left|\begin{smallmatrix}3\\5\end{smallmatrix}\right|\left|\begin{smallmatrix}7\\9\end{smallmatrix}\right|\dots\left|\begin{smallmatrix}1\\4d-1\end{smallmatrix}\right|$.

(d) $M_4 \leftrightarrow \left|\begin{smallmatrix}5\\3\end{smallmatrix}\right|\left|\begin{smallmatrix}9\\7\end{smallmatrix}\right|\dots\left|\begin{smallmatrix}4d-1\\1\end{smallmatrix}\right|$.

(2) Type F diagram.The F diagram, has only trees of type M and trees of type F.

A diagram of type $F^{\otimes m}$ has exactly m trees of type F (and other of type M).

(3) Type S diagram. The S diagram has only M trees and S trees. As for the previous family of diagrams, a diagram is of type $S^{\otimes m}$ if there exists m trees of type S, the other are M trees. We focus on S trees given by couples of diagonals (i, j) and $(i+1, j+1)$ or (i, j) and $(i-1, j-1)$ and

call the *narrow* S trees.

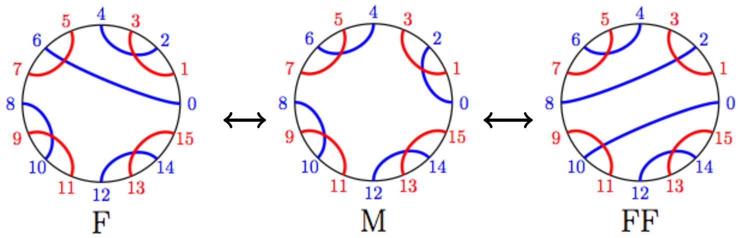
(4) Type FS diagram.The fourth type of diagrams is given by a combination of F and S simple trees.

In an S diagram (resp. F diagram) the maximal number of S trees (resp. F trees) is $d-2$. We call "Q" diagrams the $S^{\otimes d-2}$ (resp. $F^{\otimes d-2}$) diagram. All the S trees in a "Q" diagram are of type $(i,j)(i+1,j+1)$ or $(i,j)(i-1,j-1)$.

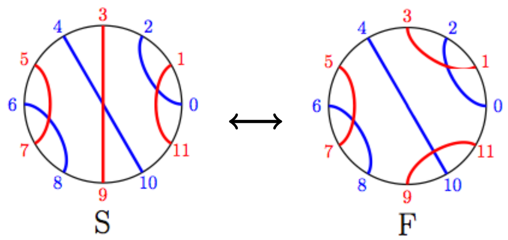
1.2. Edges of the dual complex. Recall from chapter 2 that two generic signatures which are both incident to a codimension 1 signature are adjacent. A codimension 1 signature, incident to two generic diagrams is represented by an edge with two vertices in its boundary. Those vertices correspond to the generic signatures.

A few examples of adjacent generic diagrams are proposed below.

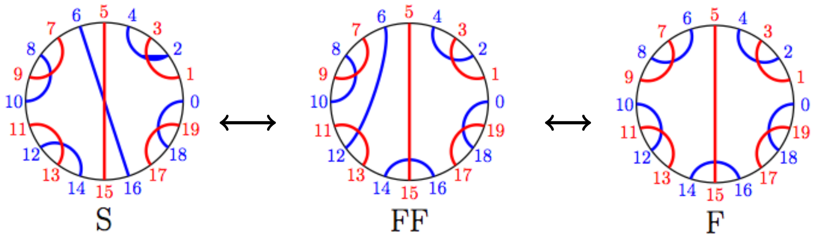
EXAMPLE 3.1 (Adjacence relations between different kinds of diagrams). • For degree 4, a sequence of deformations from the M diagram to an F diagram.



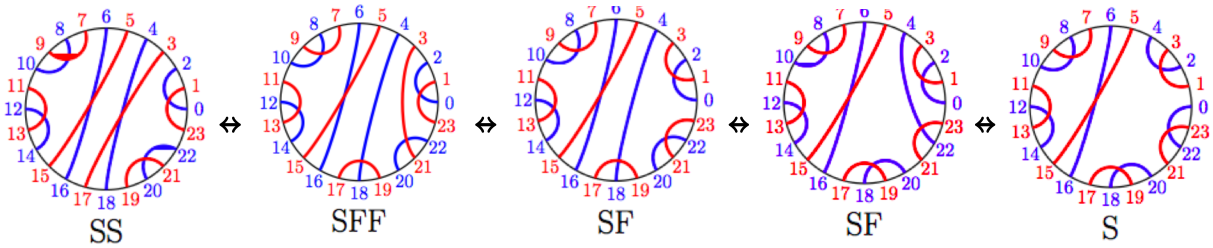
• For degree 3, a sequence of deformations from the S diagram to an F diagram :



• For degree 5 a sequence of deformations from the S diagram to an F diagram:



• For degree 6, a sequence of deformations from an SS diagram to an S diagram:



This can be written using the notation of chapter 2 as:

$$\left[\begin{smallmatrix}16 & 18\\ 6 & 4\end{smallmatrix}\right] \longleftrightarrow \left[\begin{smallmatrix}16\\ 6\end{smallmatrix}\right] \left| \begin{smallmatrix}18\\ 4\end{smallmatrix} \right| \left| \begin{smallmatrix}21\\ 3\end{smallmatrix} \right| \longleftrightarrow \left[\begin{smallmatrix}16\\ 6\end{smallmatrix}\right] \left| \begin{smallmatrix}18\\ 4\end{smallmatrix} \right| \longleftrightarrow \left[\begin{smallmatrix}16\\ 6\end{smallmatrix}\right] \left| \begin{smallmatrix}22\\ 4\end{smallmatrix} \right| \longleftrightarrow \left[\begin{smallmatrix}16\\ 6\end{smallmatrix}\right].$$

1.3. Geometric properties of dual complex.

DEFINITION 3.1 (Dual complex). *Let* $(A_\sigma)_{\sigma\in\Sigma_d}$ *be a cover of* ${}^{\mathrm{D}}Pol_d$. *Let* $\preceq$ *be the incidence relation between two classes. The dual complex* $(\mathcal{W}, \subset)$ *to* $(A_{\sigma\in\Sigma_d}, \prec)$ *is the complex such that:*

- *an* i*-face in* W_σ *is in one-to-one correspondence with a codimension* i*-class* A_σ*;*
- *a couple of faces* W_τ, W_σ *of the dual complex verifying* $W_\tau \subset W_\sigma$ *are incident in the dual complex if and only if the corresponding classes* A_τ, A_σ *of* $(A_\sigma)_{\sigma\in\Sigma_d}$ *verify the incidence relation:* $A_\tau \prec A_\sigma$.

LEMMA 3.1 (Edges and 2-faces of the dual complex). *Let* $(\mathcal{W}, \subset)$ *be the dual complex associated to* $(A_\sigma)_{\sigma\in\Sigma_d}$.

(1) Each 1-dimensional face in $\mathcal{W}$ *is bounded by 2 vertices.*
(2) Each 2-dimensional face in $\mathcal{W}$ *is a quadrangle.*

PROOF. Let us prove the first statement. Consider a codimension 1 signature β. Then, by definition 2.2 there exists one couple of diagonals of the same color, meeting at a point. Suppose that the set of terminal vertices of those diagonals is $\{i, j, k, l\}$ where $i < j < k < l$ and the numbers i, j, k, l are of the same parity. Moreover, from definition 2.8 we have the following relations between the terminal vertices: $i \equiv k \equiv 1 \mod 4$ and $j \equiv l \equiv 3 \mod 4$ (resp. $i \equiv k \equiv 2 \mod 4$

and $j \equiv l \equiv 0 \mod 4$). Again, from definition 2.8 we know that in a generic diagram each terminal vertex congruent to $1 \mod 4$ (resp. $2 \mod 4$) is attached by an edge to a terminal vertex which is congruent to $3 \mod 4$ (resp. $0 \mod 4$). So, using a smoothing half-Whitehead move the meeting point is smoothed and we obtain two different possible couples of diagonals: $(i, j)(k, l)$ or $(i, l)(j, k)$, with all the other diagonals of the signature remaining invariant. So, applying the definition 3.1 to construct the dual complex, we have that each 1-dimensional face (corresponding to a codimension 1 signature) in $\mathcal{W}$ is bounded by only 2 vertices (corresponding to the signatures obtained by smoothing the meeting point in β).

Let us prove the second statement. Consider a codimension 2 signature ω. By definition 2.2, there exist two critical points. Applying the half-Whitehead move onto one of the critical points gives two different possible codimension 1 signatures. This last statement follows from the first point above. So, applying the same to the second critical point, implies that there exist four codimension 1 signatures which are incident to ω. In other words: $\{\beta_0, \beta_1, \beta_2, \beta_3\} \prec \omega$ where $codim(\beta_i) = 1$ and $i \in \{0, ..., 3\}$. The deformation operations applied simultaneously to both critical points, defines four codimension 0 signatures which are all incident to ω: $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \prec \omega$. Applying (1) to every codimension 1 signature implies that there exist two codimension 0 signatures, which are incident to each codimension 1 signature. So, we obtain the following relations:

$$\{\sigma_0, \sigma_1\} \prec \beta_0,$$
$$\{\sigma_1, \sigma_2\} \prec \beta_1,$$
$$\{\sigma_2, \sigma_3\} \prec \beta_2,$$
$$\{\sigma_3, \sigma_0\} \prec \beta_3$$

The construction of the dual complex $\mathcal{W}$ from definition 3.1, implies that we have a quadrangle. □

Corollary 3.2.

- *Let σ_0 and σ_1 be two generic signatures. If σ_0 and σ_1 are both incident to a signature of codimension 1, then this codimension 1 signature is unique.*
- *Let $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ be four generic signatures. Then, if they are incident to a signature of codimension 2 , then this codimension 2 signature is unique.*

Example 3.2. See figure F:2-face for an example of a 2-face in $\mathcal{W}$ along with its incident faces of smaller codimenison.

1.4. Non-crossing subcomplex. Let us consider the set $[d] := \{1, 2, \ldots, d\}$. A partition of $[d] := \{1, 2, \ldots, d\}$ is a collection of (pairwise) disjoint subsets, called blocks, whose union is $[d]$. A non-crossing partition of $[d]$ is a partition of the vertices of a regular d-gon (labelled by the set $[d]$ in clockwise order) with the property that the convex hulls of its blocks are pairwise disjoint.

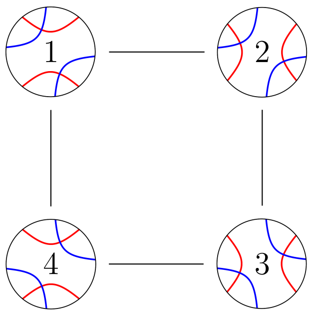


FIGURE 1. A 2-face of $\mathcal{W}$

DEFINITION 3.2. *Consider an ordered finite set* $\{1, ..., n\}$. *Consider partitions of this set into disjoint subsets called blocks. We say that two blocks* E, F *of a partition cross if there exists* $i < j < k < l$ *such that* i, k *belong to one of the blocks and* j, l *belong to the other block. A partition is non-crossing if no two blocks of the partition cross.*

REMARK 3.1. One can interpret the non-crossing condition pictorially. Suppose that we partition the set $\{1, 2, \ldots, n\}$. Let us place the numbers of $\{1, 2, \ldots, n\}$ on a circle, in order (such that n is adjacent to 1). By convex polygon, we mean either a point (if the number of vertices is 1), a segment (if the number of vertices is 2), or (if the number of vertices is 3 or more) a non degenerate convex polygon, i.e., such that three vertices never lie on the same line. The notion of non crossing extends as follows.

A finite nonempty subset is said to be convex if and only if it is the set of vertices of a convex polygon in $\mathbb{C}$. A partition π of $\{1, 2, \ldots, n\}$ is said to be non-crossing if and only if convex hulls of any two blocks do not intersect.

LEMMA 3.3. *Consider the set of all generic signatures having all their red diagonals short. Then, this set of generic signatures is in one-to-one correspondence with the non-crossing partitions of the set* $\{1, 2, ..., d\}$.

PROOF. Consider the set of terminal vertices with number verifying $0 \mod 4$. Those terminal vertices are colored red, according to the definition 2.8. To a number i of the set $\{1, 2, ..., d\}$ we assign a vertex number $4i$. Two such vertices belong to the same block if and only if they lie on the boundary of the same complementary region of the blue diagonals. Each such point belongs to exactly one region but there may be regions without such points on the boundary. It is easy to see that a partition uniquely determines the signature. □

THEOREM 3.4. *Let* σ *be a diagram. Let* $\mathcal{R}$ *be a polygonal region in* $\mathbb{D} \setminus \sigma$ *having in its boundary* m *sides of the boundary of the disk,* m *red arcs and* m *blue arcs. Consider the set of signatures which are obtained by applying a Whitehead move* *on pairs of blue (resp. red) arcs bounding $\mathcal{R}$ in σ. Then, the set of those signatures is in bijection with the set of non-crossing partitions $\{1, 2, ...m\}$.*

Proof. Let us show the bijection between the set of non-crossing partitions of $\{1, .., m\}$ and the set of signatures obtained by deforming couples of blue diagonals in $\mathcal{R}$. To show this bijection, we proceed similarly as in the proof of lemma 3.3 and consider $\mathcal{R}$ in σ as an M diagram of degree m (i.e. with only m couples of short blue and red diagonals). Being only concerned with the region $\mathcal{R}$, we use the cyclic notation from $\{1, ..., 4m\}$ on the $4m$ terminal vertices vertices lying in $\mathcal{R}$. The same argument is used as previously: consider all the terminal red vertices in $\mathcal{R}$ with number equal $0 \mod 4$. To a number i of $\{1, 2, ..., m\}$ we assign a vertex number $4i$. Two such vertices are in the same block if and only if they lie on the boundary of the same complementary region of the blue diagonals. Each such point belongs to exactly one region but there may be regions without such points on the boundary. We can see that a non-crossing partition of $\{1, ..., m\}$ determines uniquely the $\mathcal{R}$ polygonal region in the signature σ. □

Let $Cat(d) = \frac{1}{d+1}\binom{2d}{d}$ be the d-th Catalan number.

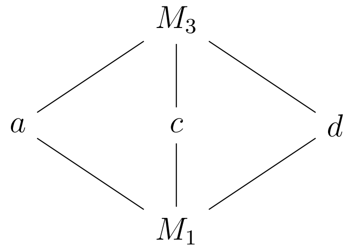
Lemma 3.5. *Let σ be a signature of codimension $2d-3$ in Σ_d having one unique meeting point of multiplicity d. Then, there exist $Cat(d)$ generic signatures incident to σ.*

Proof. To prove this statement, we use the bijection which maps a diagonal connecting two vertices i and j to a couple of left and right parenthesis "(i, j)".

Assume without loss of generality that the short diagonals are colored red. The blue vertices labeled $\{1, 3, 5, ..., 4d-1\}$ are drawn in order on a circle so that $4d-1$ is adjacent to 1. From definition 2.8 it follows that any blue diagonal connects one odd vertex i, congruent to $1 \mod 4$ with one odd vertex j, congruent to $3 \mod 4$. Since the diagonals in codimension 0 diagrams are non-crossing by lemma 3.3, we can apply the non-crossing condition to the vertex integers, from definition 3.2. In particular, if one has vertex integers $i < j < k < l$ belonging to the set $\{1, 3, 5, ..., 4d-1\}$, the couple of diagonals $(i, k)(j, l)$ can not exist in the same diagram since the non crossing condition is not satisfied, whereas $(i, j)(k, l)$ and $(i(j, k)l)$ can exist. Since there exist d diagonals, we count the number of possible combinations of those d couples of left and right parenthesis. These parenthesis are such that we insert d left parenthesis and d right parenthesis that define d binary operations. The d couples of left and right parenthesis verify (i, j) where $i \equiv 1 \mod 4$ and $j \equiv 3 \mod 4$ and $i, j \in \{1, 3, 5, ..., 4d-1\}$. Applying theorem 1.5.1 in *Catalan numbers* by Richard Stanley [**50**], the number of those combinations is the Catalan number $Cat(d)$. □

Remark 3.2. We showed that to each biregular signature having all short red fixed diagonals one can associate a non crossing partition of a set of d integers. The number of such non crossing partitions is the number of Catalan.

Let us give an example of the subcomplex of $\mathcal{W}$ connecting two M diagrams sharing a common set of short red diagonals for $d = 3$. There is one face of



codimension 3 which is incident to all the faces in this subcomplex, there are three codimension 2 sets, six codimension 1sets and $Cat(3) = 5$ codimension 0 signatures. This figure is called a" diamond" structure.

1.5. Digression on the composition of Whitehead moves applied to couples of diagonals.

LEMMA 3.6. *Consider three diagonals labeled* $\{1\}$,$\{2\}$ *and* $\{3\}$ *of the same color in an* M *diagram. Let* $\delta(k, m)$ *be the deformation operation acting on the diagonals* $\{k\}$ *and* $\{m\}$ *and giving a couple of diagonals. Then, the composition of two deformations operations verifies:*

$$\delta(\delta(1, 2), 3) = \delta(\delta(1, 3), 2) = \delta(\delta(2, 3), 1).$$

PROOF. Let the diagonal $\{1\}$ be defined by (i, j), the diagonal $\{2\}$ by (k, l) and the diagonal $\{3\}$ by (m, n). Applying the deformation operation $\delta(1, 2)$ gives a new couple of curves $(i, l)(j, k)$. Suppose with non loss of generality that $i < j < k < l < m < n$. Then $\delta(\delta(1, 2), 3)$ gives a deformation between (i, l) and (m, n). So $\delta(\delta(1, 2), 3) = (l, m)(i, n)(j, k)$. Apply the deformation operation $\delta(1, 3)$ to the curves $\{1\}$ and $\{3\}$. Then, $\delta(1, 3) = (i, n)(j, m)$. Applying $\delta(\delta(1, 3), 2)$ we deform $(i, n)(j, m)$ with (k, l). By the inequality above we deform (k, l) with (j, m). This gives $(j, k)(l, m)$, so $\delta(\delta(1, 3), 2) = (j, k)(l, m)(i, n)$. Hence $\delta(\delta(1, 3), 2) = \delta(\delta(1, 2), 3)$. Consider $\delta(2, 3)$. This deforms (k, l) and (m, n) to the couple of diagonals $(l, m)(k, n)$. Deform the curve $\{1\}$ with $(l, m)(k, n)$. This deforms (k, n) with (i, j). Therefore $\delta(\delta(2, 3), 1)$ is $(l, m)(n, i)(j, k)$. □

REMARK 3.3. Consider the family of biregular diagrams with d red short diagonals (resp. blue). Then, to each such biregular diagram one assigns a non-crossing partition of $[d]$.

To each short blue diagonal we associate a vertex on a circle. Recall that each blue diagonal (which is not short) is obtained by deformation operation applied to a couple of blue diagonals. Let us explain the relation to the non-crossing partition of $[d]$. Suppose that we apply a deformation operation to a couple of short blue diagonals, numbered by $\{k\}$ and $\{m\}$. Then according to definition

5, we draw an arc between the diagonals $\{k\}$ and $\{m\}$. We can interpret this deformation operation as a chord between the vertices k and m in the d-gon.

Then, this operation gives at most two long blue diagonals and at least one long blue diagonal. This operation corresponds to drawing a chord between the vertices k and m in the d-gon. Let us deform $\delta(m,k)$ with a short diagonal $\{i\}$, which is different from k and m. By the previous lemma we have $\delta(\delta(m,k),i) = \delta(\delta(i,m),k) = \delta(\delta(i,k),m)$. So this is drawn in the d-gon, as three chord diagrams connecting pairwise vertices $(i,m),(m,k),(k,i)$ and so defines a 3-gon.

Then, if we suppose that those diagonals are successive, then the deformation induces a new long diagonal. This composition of two deformation operations corresponds to drawing a 3-gon with vertices i,m,k.

Suppose, by induction, that an element of the partition of $[d]$, say $\{i_1,..,i_k\}$, is defined by a composition of $k-1$ deformations applied to the k short diagonals $\underbrace{\delta(...(\delta(i_1,i_2)...i_k))}_{k-1}$. This composition of $k-1$ deformations on the curves $\{i_1,..,i_k\}$ is interpreted as a convex hull of a k-gon of vertices $\{i_1,..,i_k\}$. Let us show that for $k+1$ short diagonals this statement is still true. Consider the set $\{i_1,..,i_k\} \cup \{i_{k+1}\}$. Let us apply the deformation operation onto the couple of diagonals $\{i_{k+1}\}$ and $\underbrace{\delta(...(\delta(i_1,i_2)...i_k))}_{k-1}$ sharing a common region. Then we obtain the diagonals given by $\underbrace{\delta(\delta(...(\delta(i_1,i_2)...i_k))i_{k+1})}_{k}$. We interpret this operation by drawing a $k+1$-gon in the d-gon of vertices $\{i_1,..,i_k\} \cup \{i_{k+1}\}$.

Remark 3.4. Consider a couple of M diagrams having fixed d short diagonals of the same color. Then the biregular diagrams which are adjacent to the M diagrams are F diagrams. In particular, M and S diagrams have no common codimension 1 diagrams, smaller than both in the partial order. Indeed, to obtain an M-diagram from an S-diagram, two deformation operations on a coulpe of red and blue diagonals are necessary.

1.6. Biregular signatures using Birman-Ko-Lee generators.

In this part we state results concerning a deep relation between Birman-Ko-Lee words and signatures of complex polynomials in ${}^{\mathrm{D}}\mathrm{Pol}_d$. These new results appear in our article [**16**].

Recall that the braid groups B_d with d strings, enjoy a presentation in terms of Artin generators $\sigma_1,...,\sigma_{d-1}$. J. Birman, K. Ko and S. Lee introduced new generators for the braid group in their paper [**8**]. These generators are now known as the Birman-Ko-Lee (BKL) generators. The BKL generators, $a_{i,j}$ for $1 \le i < j \le d$, and Artin generators verify:

$$a_{i,j} = (\sigma_{i-1}\sigma_{i-2}\dots\sigma_{j+1})\sigma_j(\sigma_{j+1}^{-1}\dots\sigma_{i-2}^{-1}\sigma_{i-1}^{-1}),$$

where σ_i are the Artin generators of B_d. In the next proposition, we recall the relations between the BKL generators.

PROPOSITION 3.7 (Birman-Ko-Lee [**8**]). *The braid group B_d has a presentation with generators $\{a_{ts}; d \geq t > s \geq 1\}$ and with defining relations:*

$$a_{ts}a_{rq} = a_{rq}a_{ts} \text{ if } (t-r)(t-q)(s-r)(s-q) > 0$$
$$a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} \text{ for all } t, s, r \text{ with } n \geq t > s > r \geq 1.$$

Let the fundamental word Δ_d^* be generated by Artin Braid generators $\sigma_{\mathbf{i}}$ as follows $\Delta_d^* = \sigma_{\mathbf{1}} * \cdots * \sigma_{\mathbf{d-1}}$ which is the new fundamental word in the sense of Birman-Ko-Lee, see [**8**]. We show the relation between degree d biregular signatures and a pairing of Birman-Ko-Lee words which are divisors of Δ_d^* and verifying a non-crossing condition.

We use the following proposition due to Bessis-Digne-Michel.

PROPOSITION 3.8 (Bessis-Digne-Michel [**6**]). *Let $\Delta_d^* = a_{d(d-1)}a_{(d-1)(d-2)}\cdots a_{21}$. Then the BKL words which are the divisors of Δ_d^* are in one-to-one correspondence with the non-crossing partitions of $\{1,..,d\}$.*

In the following construction, the main ingredient is to introduce a new notation on the terminal vertex numbers of the diagrams. For the even terminal vertices:

$$\sigma_1 : \{4,..,4d\} \to \{1,..,d\}$$
$$4i \mapsto i$$

Similarly for the odd terminal vertices:

$$\sigma_2 : \{3,..,4d-1\} \to \{1,..,d\}$$
$$4j-3 \mapsto j$$

PROPOSITION 3.9. *Each biregular signature is assigned a couple of words (a_s, b_t) where a_s and b_t are both Birman-Ko-Lee words, which are the left divisors of Δ_d^* and such that the indexes s and t satisfy a non-crossing condition.*

PROOF. Consider the set of red diagonals in the signature. Notice that the red diagonals compartment the diagram into disjoint polygonal regions. Each of the polygonal regions contain a given number of blue vertices lying in the boundary of the same complementary region of the red diagonals. Only the blue terminal vertices numbered by the σ_2 map are taken under account and blue vertices belonging to a common polygonal region define a block of the non-crossing partition of $\{1,...,d\}$. By the lemma 3.8, we assign a BKL word to each of these polygonal regions, indexed by the blue terminal vertices having the notation of σ_2. This gives one component of the couple of BKL words. Since these polygonal regions are disjoint, we have a product of BKL words given by $(*, b_{P_1}b_{P_2}...b_{P_k})$ where the P_i are disjoint and define the adjacent vertices of each polygon and k is the number of non degenerated polygons (i.e. polygons consisting of more than one vertex).

Consider now the set of blue diagonals in the signature. We proceed similarly as previously. The set of long blue diagonals partitions the diagram into disjoint polygonal regions P'_i. Each polygonal region contains red vertices. we consider the vertex numbers numbers obtained by σ_1. This defines a partition of the set of red vertices $\{1, .., d\}$ into disjoint non-crossing blocks. These disjoint blocks are each assigned to a BKL words by the lemma 3.8. Their product induces a component of the couple of BKL words: $(a_{P'_1}...a_{P'_{k'}}, b_{P_1}b_{P_2}...b_{P_k})$, where k, k' are the numbers of the non-degenerated polygons (i.e. polygons consisting of more than one vertex). □

Corollary 3.10. *Each biregular diagram with all red (resp. blue) diagonals being short is assigned to a Birman-Ko-Lee word, which is a left divisor of* Δ^*_d.

Proof. It was proven in lemma 3.3 that each biregular diagram with all red (or blue) diagonals being short is assigned to a non-crossing partition of a set $[d]$. We apply a result due to Bessis-Digne-Michel 3.8 [**6**] stating that non-crossing partitions of $[d]$ are in bijection with the divisors of Δ^*_d. Therefore to each biregular diagram with all red (or blue) diagonals being short one can assign a divisor of Δ^*_d. □

Example 3.3. A unique meeting point of multiplicity 3 corresponds to a face of dimension 3 in $\mathcal{W}$ which has 3 faces of dimension 2; 6 edges and 5 vertices. The faces are quadrangles.

Let us recall that a necklace structure is a collection of simplicial objects (called the beads) glued along there initial and end vertices such that the initial vertex of a i-th simplex is glued to the terminal vertex of the next $i+1$-th simplex where $i \in \mathbb{Z}_m$.

Proposition 3.11. *For all* $d > 1$*, there exist four structures* $NC(d)$ *forming a necklace structure in* $\mathcal{W}$.

Proof. There exist four $NC(d)$ structures in $\mathcal{W}$. The end vertices (initial and final vertex) of each $NC(d)$ is an M diagram. Since there exist four M diagrams, for any $d > 0$, therefore the union of those four $NC(d)$ structures forms a necklace of four beads. □

2. Symmetries of the dual complex

In the remaining part, we decompose the dual complex into different families of geometric structures and study how these parts are disposed one towards another. This investigation is interesting, because firstly one can construct from it the nerve (in the sense of the Čech cohomology), useful for the calculations of the Čech cohomology. Secondly, the number of faces in the dual complex is exponential (the counting is presented in chapter 8). This makes it impossible to draw, for higher degrees than 3. However, we solve this problem, by introducing a new method outlining a draft of this dual complex. Thirdly, the aim being the Čech cohomology groups, and because their calculation is very complicated, we are interested in a simplification. So, we look for symmetries and prove the existence of symmetries in this dual complex. These results are the object of our article [**17**].

DEFINITION 3.3 (Lego piece). *Let $Q = S^{\otimes d-2}$ be a diagram of type $S^{\otimes d-2}$. Let γ be one of its four shortest long diagonals. Consider all diagrams having only long diagonals parallel to the ones of Q and which are obtained from Q by a sequence of deformations such that γ is kept fixed. Do it for all four shortest long diagonals. We get the lego piece determined by Q which is a sub-graph in the dual complex.*

EXAMPLE 3.4. We present a detail of a lego piece.

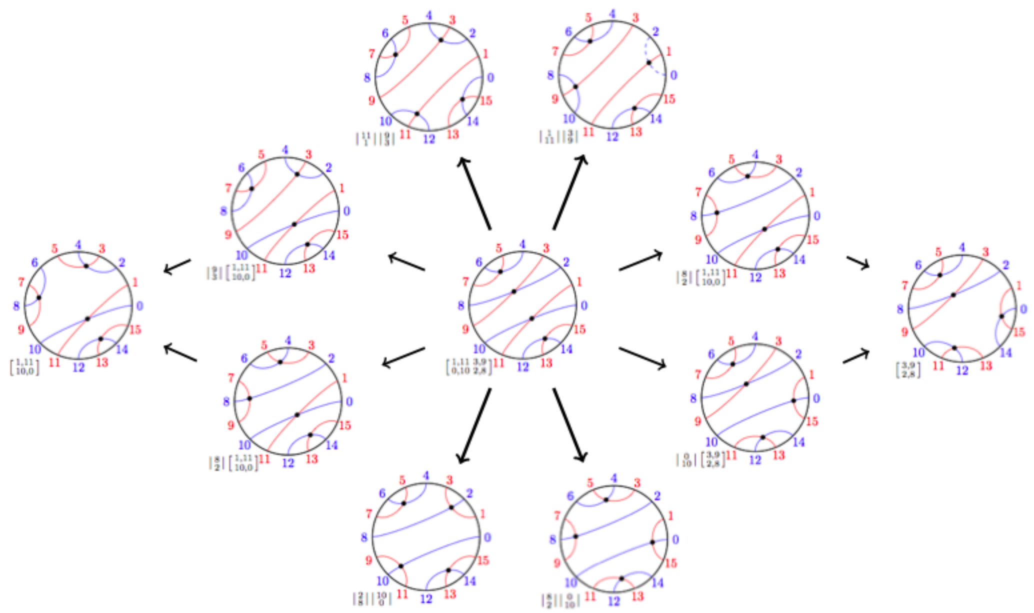


FIGURE 2. Lego detail for $d = 4$

REMARK 3.5. There exists $2d$ lego pieces, since there exist $2d$ rotations of Q.

DEFINITION 3.4 (Tower). *Consider two lego pieces and their respective Q diagrams. If these Q diagrams have a common set of red (resp. blue) diagonals, then these lego pieces are consecutive. The $2d$ consecutive lego pieces forms a subgraph of the dual complex $Y \subset \mathcal{W}$ called tower.*

DEFINITION 3.5 (Joints). *The sub-graph of the dual complex is called joint if it connects a couple of consecutive lego pieces.*

REMARK 3.6. The dual complex $\mathcal{W}$ contains two main parts: the disjoint union of the four $NC(d)$ lattices and the subpart of $\mathcal{W}$ containing the tower Y.The tower Y is constituted from three subparts:

- lego pieces,
- joints,
- sub-graphs of $\mathcal{W}$ connecting non consecutive lego pieces.

All types of diagrams do not appear in the lego pieces and joins. Those diagrams which do not belong to any lego pieces or join belong to the sub-graph of $\mathcal{W}$ establishing the connection between two non consecutive lego pieces.

Let us describe the diagrams of the tower using an algebraic method.

A "Q" diagram is associated to a matrix $\left[\begin{smallmatrix} L_0 \\ L_1 \end{smallmatrix}\right]$ where L_0 contains all the couples of terminal vertices connected by long diagonals of one color, L_1 contains all the couples of terminal vertices connected by long diagonals of the other color. The order in which the colors of L_0 or L_1 are taken does not matter. The couple of terminal vertices of two crossing diagonals lie in the same column of the matrix. The first and the last columns of the matrix contains the couple of terminal vertices of the shortest long diagonals. Two adjacent columns contain the terminal vertices of diagonals sharing a common region.

Using this approach we describe consecutive lego pieces.

LEMMA 3.12. *Let Q_1 and Q_2 be two consecutive "Q" diagrams. Then their associated matrices verify*

$$Q_1 = \left[\begin{smallmatrix} L_0 \\ L_1 \end{smallmatrix}\right], Q_2 = \left[\begin{smallmatrix} L_1 \\ L_2 := L_1 - 1 \end{smallmatrix}\right],$$

where $L_1 - 1$ means that all the indexes of the terminal vertices of the diagonals in L_2 are those of L_1 minus one, modulo $4d$.

PROOF. Suppose, with no loss of generality that in $Q_1 = \left[\begin{smallmatrix} L_0 \\ L_1 \end{smallmatrix}\right]$:

- L_0 consists of the collection of long diagonals
$$(2,8)(4d,10)...(4d-2(d-4), 8+2(d-3)),$$
- L_1 consists of the collection of long diagonals
$$(1,7)(4d-1,9)...(4d-1-2(d-4), 7+2(d-3)).$$

In particular the S trees in the diagram Q_1 are all narrow, i.e. defined by couples of long diagonals of type $(i+1, j+1)(i,j)$, where

- the couple (i,j) belongs to L_0
- the couple $(i-1, j-1)$ belongs to L_1 and where i, j are odd numbers modulo $4d$.

By definition, since Q_1 and Q_2 are consecutive, their matrices share a common line, say L_1. Consider the second line of Q_2. Since Q_2 is different from Q_1, their matrices are different. The Q_2 diagram is by definition of type $S^{\otimes d-2}$. The S trees are narrow and thus of type $(i,j)(i-1,j-1)$. In particular L_2 is $(4d,6)(4d-2,8)...(4d-2(d-3),8+2(d-4))$. So, we have $L_2 := L_1 - 1$. One generalizes by adding a number q to L_0: $(2+q,8+q)(4d+q,10+q)...(4d-2(d-4)+q,8+2(d-3)+q)$ mod $4d$ and to L_1. $\square$

EXAMPLE 3.5. For $d=4$ one has the following consecutive Q diagrams:

$$\begin{bmatrix} 1{,}11 & 3{,}9 \\ 0{,}10 & 2{,}8 \end{bmatrix}$$
$$\updownarrow$$
$$\begin{bmatrix} 0{,}10 & 2{,}8 \\ 15{,}9 & 1{,}7 \end{bmatrix}$$
$$\updownarrow$$
$$\begin{bmatrix} 15{,}9 & 1{,}7 \\ 14{,}8 & 0{,}6 \end{bmatrix}.$$

EXAMPLE 3.6. Example of lego tower $d=3$.

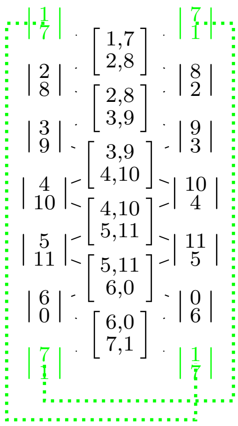


FIGURE 3. Lego tower for d=3

The example 4 of a entire lego tower is given at the end of this chapter for $d=4$. For $d=6$, the lego tower is given in the appendix B.

LEMMA 3.13. *A rotation by an angle of* $\frac{k\pi}{2d}$ *of a diagonal* (i,j) *in a diagram corresponds to adding* $+2k$ *to its terminal vertices, so we have* $(i+2k, j+2k)$.

PROOF. Le us proceed by induction on the angle $\frac{k\pi}{2d}$.

(1) *Base case* Let $k = 1$. Let us rotate by $\frac{k\pi}{2d}$ the diagonal (i,j). Since $i,j \in \{1,3,..,4d-1\}$ are the vertices of a regular $2d$-gon, then rotating by an angle $\frac{k\pi}{2d}$ a diagonal (i,j), sends it to the diagonal $(i+2, j+2)$.

(2) *induction case* Suppose that for k the statement, the diagonal (i,j) is sent to $(i+2k, j+2k)$. Let us show that for $k+1$ lemma is true. We rotate about an angle $\frac{k\pi}{2d}\frac{\pi}{2d}$ the diagonal (i,j). By induction hypothesis rotating by $\frac{k\pi}{2d}$ sends (i,j) to $(i+2k, j+2k)$. Rotating $(i+2k, j+2k)$ by an angle of $\frac{\pi}{2d}$ sends it to $(i+2k+2, j+2k+2)$ that is $(i+2(k+1), j+2(k+1))$.

□

Remark 3.7. In particular rotating all the diagonals in a diagram $[\,{}^{L_0}_{L_0-1}\,]$ by an angle of $\frac{k\pi}{2d}$ is equivalent to adding $+2k$ to all of the terminal vertices of the rotated diagonals, so we have $[\,{}^{L_0+2k}_{L_0-1+2k}\,]$. Rotating all the diagonals $[\,{}^{L_0}_{L_0-1}\,]$ by an angle of π is equivalent to adding $+4d$ to all of the terminal vertices of the rotated diagonals, so in the rotated diagram by an angle of π we have $[\,{}^{L_0+4d}_{L_0-1+4d}\,]$.

Lemma 3.14. *Let Q_1 and Q_2 be two consecutive "Q" diagrams. Then there exists a sequence of deformations of only one color from Q_1 to Q_2 containing a diagram $F^{\otimes d-2}$.*

Proof. We induct on d, where $d > 3$. Lower degrees are irrelevant since the Q diagrams do not exist.

- *Base case* $d = 4$. The Q diagram $[\,{}^{L_0}_{L_1}\,]$ has two long blue diagonals and two long red diagonals. Suppose that one of the S is given by $[\,{}^{i}_{j}\,]$. The couples of red and blue long diagonals share a common region, which implies that the deformation operation can be applied to one couple of diagonals of a given color, say red. The other couple of blue long diagonals remains fixed. Deforming the couple of long red diagonals induces a couple of short red diagonals. This gives a new diagram of type $F^{\otimes 2}$. We have thus showed the existence of a sequence connecting a Q diagram to a $F^{\otimes 2}$. Now, we show that starting from $F^{\otimes 2}$ there exists a sequence of deformations giving the consecutive Q diagram of matrix $[\,{}^{L_1}_{L_1-1}\,]$. Consider the couple of short red diagonals, sharing a region and different from the previous one. Deform it: one obtains a diagram SF, where the S is given by $[\,{}^{j}_{i}\,]$. Deform the remaining couple of red short diagonals. This gives a diagram SS with matrix $[\,{}^{L_1}_{L_1-1}\,]$.

- *Induction case*. Assume that the lemma works for two consecutive $S^{\otimes d-2}$ diagrams. We will prove that for two consecutive $S^{\otimes d-1}$ diagrams with $d-1$ couples of S trees there exists a sequence of deformations containing a diagram of type $F^{\otimes d-1}$. Consider the degree $d+1$ diagram $S^{\otimes d-1}$. Since the assumption of the lemma is true for $S^{\otimes d-2}$, then there exists a sequence of deformations from $S^{\otimes d-1}$ to a diagram $SF^{\otimes d-2}$, where S is a tree given by the crossing of the shortest long diagonals in the diagram. It remains to deform the couple of diagonals (one long, the other one short) sharing a region, so as to obtain $F^{\otimes d-1}$. Now starting from $F^{\otimes d-1}$, there exists by induction hypothesis a sequence of deformations giving the diagram $S^{\otimes d-2}F$. Finally, one more deformation induces the passage from F to S. This gives the $S^{\otimes d-1}$ diagram.

□

REMARK 3.8. Let X be a lego piece of the dual complex. Then, one deformation in X of the Q diagram gives an SF diagram. A second deformation step gives an S diagram with narrow S trees.

Indeed, let us enumerate all the three possible cases. Notice that a Q diagram has two S trees made of the shortest long diagonals in the diagram.

(1) $S^{\otimes d-2}$ is deformed in one deformation step to $S^{\otimes d-3}F$ (or to $FS^{\otimes d-3}$). This is obtained by deforming a couple diagonals (one in M and one in adjoining S, both being of the same color). From this diagram, by deforming the long diagonal in F with the diagonal in M of the corresponding color, we obtain $S^{\otimes d-3}M$, all the S being aligned.

(2) $S^{\otimes d-2}$ is deformed in one deformation step to $S^{\otimes d-4}FF$ (or to $FFS^{\otimes d-4}$). This is obtained by deforming a couple of long diagonals: one being the shortest long diagonal of the diagram, the other one being adjoining to it. Deforming once more the couple of long diagonals in FF reduces to the trees two MM. So the resulting diagram is $S^{\otimes d-4}MM$.

(3) $S^{\otimes d-2}$ is deformed in one deformation step to $S...SFFS...S$ where the number of S is $d-4$. This is obtained by deforming a couple of adjoining long diagonals (which are not the shortest long diagonals of the diagram). Deforming the remaining long diagonals in FF gives a couple of trees MM. This gives the diagram $S...SMMS...S$.

All the other diagrams are obtained iterating the procedure described above onto each of the remaining collection of narrow S trees in the diagram. Applying this method continuously, we obtain diagrams with only one narrow S tree and in particular we can obtain the F diagrams constructed from the shortest long diagonals (the γ diagrams).

The joins connect one lego to its consecutive one. The diagrams in a join have a set of parallel long diagonals of a given color. These long diagonals are common to the diagrams of the two consecutive lego pieces. Each diagram of S type in one lego is connected to its corresponding S diagram in the consecutive lego piece. These two diagrams have in common a fixed set of diagonals of a given color. In the join, the sequence of deformations between those two diagrams contains an F diagram. This F diagram has the common set of long diagonals in the S diagrams. For instance let us discuss the subgraph of a join, between two Q diagrams. Take $S^{\otimes d-2}$deform it to $S^{\otimes d-3}F$, deform it to $S^{\otimes d-5}FFF$. Continue deforming it to $F^{\otimes d-2}$. Now we deform the short diagonals differently than previously. A finite sequence of deformations gives the second Q diagram.

LEMMA 3.15. *Let X be a lego piece and G_1, G_2 be two subgraphs of X. Then there exists a decomposition of the graph X such that :*

- $X = G_1 \cup G_2$,

- *G_1, G_2 are isomorphic,*
- *the vertex corresponding to the Q diagram is contained in $G_1 \cap G_2$.*

PROOF. Let us describe the construction of X.

Take the Q diagram, numerate its long diagonals of a given color from 1 to $d-2$ (starting from one of its shortest long diagonals and ending on the second shortest long diagonal).

- Let G_1 be the subgraph of the lego connecting a Q diagram to all diagrams containing only one narrow S tree, consisting of long diagonals among the set $\{\frac{d-2}{2}+1, ..., d-2\}$.
- Let G_2 be the subgraph of the lego connecting a Q diagram to all diagrams containing only one narrow S tree consisting of long diagonals among the set $\{1, ..., \frac{d-2}{2}\}$.

We show using induction on the number of deformations which start from the Q diagram that G_1 and G_2 are isomorphic.

Let $A(n)$ be the hypothesis that after n deformations of the diagram Q respectively in G_1 and G_2 these subgraphs are isomorphic. Let us verify that for one deformation, $A(1)$ is true.

(1) Deforming Q induces two families of diagrams:
 (a) $S^{\otimes d-3}F$ and $FS^{\otimes d-3}$
 (b) $S^{\otimes d-4}FF$ or $S..SFFS...S$
 (Without loss of generality we assume that the deformation concerns only one fixed color of diagonals, the result being independent of this remark.)

(2) We partition this set of vertices into those belonging to the graph G_1 and those belonging to the graph G_2 according to the following criterion:
 - diagrams having a couple of trees FF (or only one F tree) among the set of long diagonals $\{1, ..., \frac{d-2}{2}\}$ belong to G_1;
 - diagrams having a couple of FF (or only one F tree) among the set of long diagonals $\{\frac{d-2}{2}+1, ..., d-2\}$ belong to G_2;
 - the diagram which verifies FF in $\{1, ..., \frac{d-2}{2}\}$ and$\{\frac{d-2}{2}+1, ..., d-2\}$ is a vertex belonging to $G_1 \cap G_2$.

 The number of couples of FF in the diagram is $d-3$ and one diagram is common to G_1 and G_2. So, for the first step there exist $\frac{d-4}{2}$ vertices in G_1 and G_2, each being connected to the Q diagram. Thus $A(1)$ is true.

 We deform the couple of FF or FM into a couple of MM trees and repeat this step each time that a new couple FF is obtained. This induces a new S diagram with a number of long diagonals strictly less than $d-2$.

(3) Let us show that $A(2)$ is true. Iterate the previous procedure on each of the S diagrams having $d-4$ S trees, obtained in G_1 and G_2 from the

first step. As previously, diagrams in G_1 are those having a deformation among the set of diagonals $\{1, ..., \frac{d-2}{2}\}$; diagrams in G_2 are those having a deformation among diagonals in $\{\frac{d-2}{2}+1, ..., d-2\}$. The diagrams having deformations in both sets $\{1, ..., \frac{d-2}{2}\}$ and $\{\frac{d-2}{2}+1, ..., d-2\}$ belong to the common set of vertices of G_1 and G_2. The number of diagrams obtained by a second deformation in G_1 and G_2 are the same (the number of long diagonals for diagrams in G_1 and G_2 after a second deformation are the same: $\frac{d-2}{2} - 6$). Considering only the diagonals of the color that is deformed, the diagrams in G_1 and G_2 are equivalent up to a rotation of π. Therefore, for each couple of diagrams connected by an edge in G_1, we have in G_2 this couple of connected diagrams rotated by an angle π. So, $A(2)$ is verified.

(4) Suppose that for a certain number n of deformations $A(n)$ is true. Let us show that $A(n+1)$ is true. For each diagram obtained after n deformations, we split diagrams from the $n+1$-th deformation step into those belonging to G_1 and those belonging to G_2 according to the previous criterion. Each diagram at the nth step in G_1 has an equivalent diagram in G_2, considering only the diagonals of the color we deform, rotated by an angle π. We apply the deformation procedure to the long diagonals of each diagram in G_1 and G_2, obtained by the n-th step of deformations. Two diagrams in G_1 of the nth and $n+1$th step are connected by an edge implies that the diagrams in G_2 equivalent by some rotation of π are also connected (considering only the color of deformed diagonals). Proceeding this way for all the diagrams in G_1 and G_2 we show that $A(n+1)$ is true.

□

Proposition 3.16. *Let Z be a couple of consecutive lego pieces and let Y be the tower. Then $\mathbf{p} : Y \to Z$ is a Galois covering, with Galois group of order d.*

Proof. Let us define $\mathbf{p} : Y \to Z$ where $Y = \cup_{\rho \in G} Z^{\rho}$ and G is a cyclic group of finite order d. From above it follows that Y is connected. Each inverse image $\mathbf{p}^{-1}(A_\sigma)$ of $A_\sigma \in Z$ is constituted from classes, having diagrams which are equivalent up to a rotation of σ. So, any action $\rho \in Aut_Z(Y)$ on σ gives a diagram σ' which belongs to $\mathbf{p}^{-1}(A_\sigma)$. We show that for $Y \times_Z Y = \{(z, z') \in Y \times Y, \mathbf{p}(z) = \mathbf{p}(z')\}$, the map

$$\phi : G \times Y \to Y \times_Z Y,$$
$$(g, z) \mapsto (z, gz)$$

is a homeomorphism.

- The map is a bijection. First let us show that the map is injective. Consider $\phi(g, z) = \phi(g', z')$ i.e. $(z, gz) = (z', g'z') \in Y \times_Z Y$. Then, $z = z'$ and $g' = g$ and in particular $(g, z) = (g', z')$ in $G \times Y$, so the map is injective.

The map is surjective since for every $(z, gz) \in Y \times_Z Y$, there exists at least one element in $(g, z) \in G \times Y$ such that $\phi((g, z)) = (z, gz)$.

- The map is bicontinuous because the group G continuously acts on Y.

So the map $\mathbf{p} : Y \to Z$ satisfies the definition of a Galois covering for an order d Galois group. $\square$

LEMMA 3.17. *Let* $S_1 = [\,{}^{L_0+2d}_{L_0+2d-1}\,]$ *and* $S_2 = [\,{}^{L_0-2d}_{L_0-2d-1}\,]$ *be two* S *diagrams. Then* $S_1 = S_2$.

PROOF. Consider any diagonal $(i, j) \in L_0$, where i, j are integers if the same parity modulo $4d$. We have $i \equiv i \mod 4d \iff i + 4d \equiv i \mod 4d \iff i + 2d \equiv i - 2d \mod 4d$. Proceeding similarly for j we have that $S_1 = S_2$. $\square$

THEOREM 3.18. *The tower is invariant under the group* $H = \mathbb{Z}_d \rtimes \mathbb{Z}_2$.

PROOF. We discuss the second term in H, the first term $\mathbb{Z}_2$ following from the proposition 3.16. Let us recall from lemma 3.15 that:

- the subgraphs G_1 and G_2 of a lego X are isomorphic;
- the diagrams in G_2 are those in G_1 rotated by an angle of π.

We attribute the matrices to each diagram of the lego piece. We follow the construction from lemma 3.12 and use the lemma 3.13. In particular, each couple of distinct S diagrams in $S_1 \in G_1$ in $S_2 \in G_2$ equivalent by a rotation of π are given by the matrices $S_1 = [\,{}^{L_0}_{L_0-1}\,], S_2 = [\,{}^{L_0+2d}_{L_0-1+2d}\,]$. Let us discuss the matrix relations between each couple of consecutive lego piece.

(1) Take any diagram in the subgraph G_1 of the first lego piece, associated to $[\,{}^{L_0}_{L_0-1}\,]$ and its rotated diagram by an angle π which is $[\,{}^{L_0+2d}_{L_0-1+2d}\,]$, in G_2. Then, applying lemma 3.12, the consecutive diagrams are respectively of type $[\,{}^{L_0-1}_{L_0-2}\,]$ and $[\,{}^{L_0+2d-1}_{L_0-1+2d-2}\,]$

(2) Take any diagram in the subgraph G_1 of the i-th lego piece, associated to $[\,{}^{L_0-i+1}_{L_0-i}\,]$ and its rotated diagram by an angle π which is $[\,{}^{L_0-i+1+2d}_{L_0-i+2d}\,]$, in G_2. Then, applying lemma 3.12, the consecutive diagrams are respectively of type $[\,{}^{L_0-i}_{L_0-(i+1)}\,]$ and $[\,{}^{L_0-i+2d}_{L_0-(i+1)+2d}\,]$.

(3) For the $2d-1$-th lego piece, we have $[\,{}^{L_0-(2d-1)}_{L_0-2d}\,]$ and $[\,{}^{L_0+2d-(2d-2)}_{L_0+2d-1-(2d-1)}\,]$. The consecutive diagrams are respectively $[\,{}^{L_0-(2d-1)}_{L_0-2d}\,]$ and $[\,{}^{L_0+2d-2d}_{L_0+2d-1-2d}\,]$. By Lemma 3.17 we have $L_0 - 2d = L_0 + 2d \mod 4d$. So, diagrams in the G_1 subgraph of the $2d-1$-th lego piece are connected by a join to the subgraph G_2, of the first ($2d$ th) lego piece. This switches the positions of the two subgraphs of G_1 and G_2 in the lego pieces numbered from to $2d$ to $4d-1$, compared to the positions of G_1 and G_2 in the lego pieces numbered from 1 to $2d-1$. So, the identity is obtained after $4d$ consecutive lego pieces.

Below we present a part of the tower, the joints are represented by thin vertical double arrow, deformations by a thick horizontal double arrow.

$$\xrightarrow{\hspace{6cm}}$$

$$\begin{array}{ccc}
\begin{bmatrix} L_0 \\ L_0-1 \end{bmatrix} \Leftrightarrow \dots & \Leftrightarrow \begin{bmatrix} Q_1 \end{bmatrix} \Leftrightarrow & \dots \Leftrightarrow \begin{bmatrix} L_0+2d \\ L_0-1+2d \end{bmatrix} \\
\updownarrow & \updownarrow & \updownarrow \\
\begin{bmatrix} L_0-1 \\ L_0-2 \end{bmatrix} \Leftrightarrow \dots & \Leftrightarrow \begin{bmatrix} Q_2 \end{bmatrix} \Leftrightarrow & \dots \Leftrightarrow \begin{bmatrix} L_0-1+2d \\ L_0-2+2d \end{bmatrix} \\
\updownarrow & \updownarrow & \updownarrow \\
\begin{bmatrix} L_0-2 \\ L_0-3 \end{bmatrix} \Leftrightarrow \dots & \Leftrightarrow \begin{bmatrix} Q_3 \end{bmatrix} \Leftrightarrow & \dots \Leftrightarrow \begin{bmatrix} L_0-2+2d \\ L_0-3+2d \end{bmatrix} \\
\dots & \dots & \dots \\
\updownarrow & \updownarrow & \updownarrow \\
\begin{bmatrix} L_0-(2d-2) \\ L_0-(2d-1) \end{bmatrix} \Leftrightarrow \dots & \Leftrightarrow \begin{bmatrix} Q_{2d-1} \end{bmatrix} \Leftrightarrow & \dots \Leftrightarrow \begin{bmatrix} L_0+2d-(2d-2) \\ L_0+2d-1-(2d-1) \end{bmatrix} \\
\updownarrow & \updownarrow & \updownarrow \\
\begin{bmatrix} L_0-(2d-1) \\ L_0-2d \end{bmatrix} \Leftrightarrow \dots & \Leftrightarrow \begin{bmatrix} Q_{2d} \end{bmatrix} \Leftrightarrow & \dots \Leftrightarrow \begin{bmatrix} L_0+2d-2d \\ L_0+2d-1-2d \end{bmatrix} = \begin{bmatrix} L_0 \\ L_0-1 \end{bmatrix}
\end{array}$$

$$\xleftarrow{\hspace{6cm}}$$

$\square$

Corollary 3.19. *Let $\mathcal{W}$ be the dual complex. Then $\mathcal{W}$ is invariant under $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$.*

Proof. From the argument in the proof of proposition 3.16, there exist d copies of Z. From theorem 3.18, the lego X is invariant under a cyclic group of order of 2. The diagrams of the decomposition are classified up to rotation and the smallest cardinality of diagrams belonging to the same rotation class is 4: it contains the four M diagrams. The other classes are of cardinalities of type $4p$ where p divides d. The only diagrams which are not considered in the tower are the M diagrams and each lego is connected to one M diagram. The procedure of consecutive lego pieces implies that the we have the following consecutive cyclic relations of M diagrams: M_1 is consecutive to M_2, M_2 is consecutive to M_4, M_4 is consecutive to M_3, M_3 is consecutive to M_1. So $\mathcal{W}$ is invariant under $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ since 2 is the smallest integer greater than one dividing the cardinality of the set of M diagrams. $\square$

2.1. The main classes of subcomplexes. Let $W(I)$ be the dual complex with I the set of its vertices. The vertices in $i_0 \in I$ correspond bijectively to the set of biregular classes $\sigma_{i_0} \in \Sigma_d$. Let us define the subcomplexes which generate the dual complex.

DEFINITION 3.6. *Consider the finite number of distinct points $i_0, ..., i_p$ in real affine space. We define the subcomplex $D(i_0, ..., i_p) \subset W$ as a compact and connected structure in real affine space of real dimension m, having vertices $I' = (i_0, ..., i_p) \subset I$ and such that:*

1. *there exists an edge between the vertices i_j and i_k in the sequence I' if $\{\sigma_{i_j}, \sigma_{i_k}\}$ are both incident to a common codimension 1 signature.*
2. *there exists a quadrangular 2-face between the vertices $(i_j, i_k, i_l, i_m) \subset I'$ if the signatures $\{\sigma_{i_j}, \sigma_{i_k}, \sigma_{i_l}, \sigma_{i_m}\}$ are incident to a common codimension 2 signature.*
3. *there exists an n-face of vertices $(i_{k_0}, ..., i_{k_q}) \subset I'$ if the codimension zero signatures $\{\sigma_{i_{k_0}}, ..., \sigma_{i_{k_q}}\}$ are all incident to one signature of higher codimension than 0.*
4. *there exists an m-face of vertices $(i_0, ..., i_p)$ if the codimension zero signatures $\{\sigma_{i_0}, ..., \sigma_{i_p}\}$ are all incident to one signature of high codimension.*

DEFINITION 3.7. *We introduce three main classes of structures in the dual complex called NC, B, O structures respectively.*

1. *$NC(d)$ structure is given the set of signatures which are obtained by smoothing a signature σ^N of codimension $2d-3$ having all blue (resp. red) short diagonals.*
2. *B structure is given the set of signatures which are obtained by smoothing*
   (a) *a signature σ^B of codimension $2d-4$ having one blue long diagonal and all the other diagonals short,*
   (b) *a signature having two long blue diagonals and all other short in σ^B*
   (c) *...*
   (d) *a signature having $d-2$ long diagonals in σ^B.*
3. *O structure is given the set of signatures which are obtained by smoothing a signature σ^O of codimension $2d-4$ with two critical points: one red and one blue. All the other diagonals are short. This class is subdivided into smaller parts such that:*
   (a) *there exists a blue (resp. red) long diagonal in σ^O,*
   (b) *there exists two long diagonals in σ^O*
   (c) *...*
   (d) *there exist d-2 long diagonals in σ^O.*

2.2. Geometry of the subcomplexes. We are interested in the geometry of the subcomplexes $D(I)$, contained in the dual complex $\mathcal{W}$. Let $D(i_0, ..., i_q)$ be the sub-complex of the dual complex $\mathcal{W}$ having $q+1$ vertices. The vertices correspond to the set $(A_{\sigma_{i_j}})_{j=0}^{q}$ and the intersection $\cap_{j=0}^{q} A^+_{\sigma_{i_j}}$ is non-empty.

In order to investigate the geometry of the subcomplexes, we use the geometric properties of the tower, introduced previously.

PROPOSITION 3.20. *Let $D(i_0, ..., i_{q+1})$ be a $NC(d)$ subcomplex of $(\mathcal{W}, \subset)$. Then, each couple of consecutive lego pieces Z in Y contains at least one vertex of this substructure.*

PROOF. By remark 3.4 each such subcomplex contains vertices corresponding to F diagrams and in the particular F diagrams with one shortest long diagonal. Since each lego piece contains such a diagram, then each copy of Z contains at least one vertex of $D(i_0, ..., i_{q+1})$. □

LEMMA 3.21. *For any $q \geq 0$, the set of vertices of a subcomplex $D(i_0, ..., i_q)$ is contained in the set of vertices of the subcomplex $D(i_0, ..., i_{q+1})$.*

PROOF. One has $D(i_0, ..., i_q) \subset D(i_0, ..., i_{q+1})$. So, the set of vertices of $\{i_0, ..., i_q\}$ is contained in the set of vertices of $\{i_0, ..., i_{q+1}\}$. □

COROLLARY 3.22. *For any $q \geq 0$, the set of vertices of $D(i_0, ..., i_q)$ is contained in r copies of Z, where r is a divisor of d.*

PROOF. From lemma 3.21, the set of vertices of $D(i_0, ..., i_q)$ is contained in the set of vertices of $D(i_0, ..., i_{q+1})$. Moreover, from proposition 3.16, the Galois group acts transitively on the set of vertices. Since the substructure formed by 0-faces and 1-faces in $\mathcal{W}$ is invariant under the group $\mathbb{Z}_d \rtimes \mathbb{Z}_2$ by theorem 3.19, therefore the set of vertices of $D(i_0, ..., i_q)$ is contained in r copies of X, where r is a divisor of d. □

COROLLARY 3.23. *Let $D(i_0, ..., i_q)$ be a subcomplex of $\mathcal{W}$ having vertices belonging to r_{q+1} copies of X, where r_{q+1} divides d. Then, the set of vertices of $D(i_0, ..., i_{q-1})$ is contained in r_q copies of X, where r_q is a divisor of r_{q+1}.*

PROOF. From lemma 3.21, the set of vertices of $D(i_0, ..., i_{q-1})$ is contained in the set of vertices of $D(i_0, ..., i_q)$. So, if the vertices belonging to $D(i_0, ..., i_q)$ are contained in r_{q+1} copies of X, then the set of vertices of $D(i_0, ..., i_{q-1})$ is contained in r_q copies of X, where $r_q < r_{q+1}$. Supposing that r_q is not a divisor of r_{q+1} would contradict the fact that Y is invariant under the group $\mathbb{Z}_d \rtimes \mathbb{Z}_2$. Indeed, if ones supposes that Y is invariant under a finite cyclic group G of order d then, Y is also invariant under the subgroups of G. By Lagrange's theorem the subgroups of G are cyclic groups and their order divides d. Therefore r_q is a divisor of r_{q+1}. □

LEMMA 3.24. *The sub-complex $D(i_0, ..., i_k)$ having only vertices in the lego pieces is invariant under a cyclic subgroup $\mathbb{Z}_b \rtimes \mathbb{Z}_2$ where b divides d.*

PROOF. By theorem 3.19, it follows that the structure in $\mathcal{W}$, formed from vertices and edges is invariant under $\mathbb{Z}_{2d} \rtimes \mathbb{Z}_2$. The subset of vertices belonging to $D(i_0, ..., i_k)$, does not belong to each copy of X. Indeed, from lemma 3.21, the vertices of $D(i_0, ..., i_k)$ belong to a smaller number of copies of X. Using theorem 3.19, Y is invariant under the group $\mathbb{Z}_{2d} \rtimes \mathbb{Z}_2$. From the same argument as previously, the number r of copies of X containing vertices of $D(i_0, ..., i_k)$, is a divisor of d. The theorem of Lagrange implies that $D(i_0, ..., i_k)$ is invariant under an order b cyclic subgroup of $\mathbb{Z}_d$, such that the equation $rb = d$ is verified. □

We now state the following conjecture:

Conjecture: The poset of signatures can be realized as the poset of cells of a CW-complex.

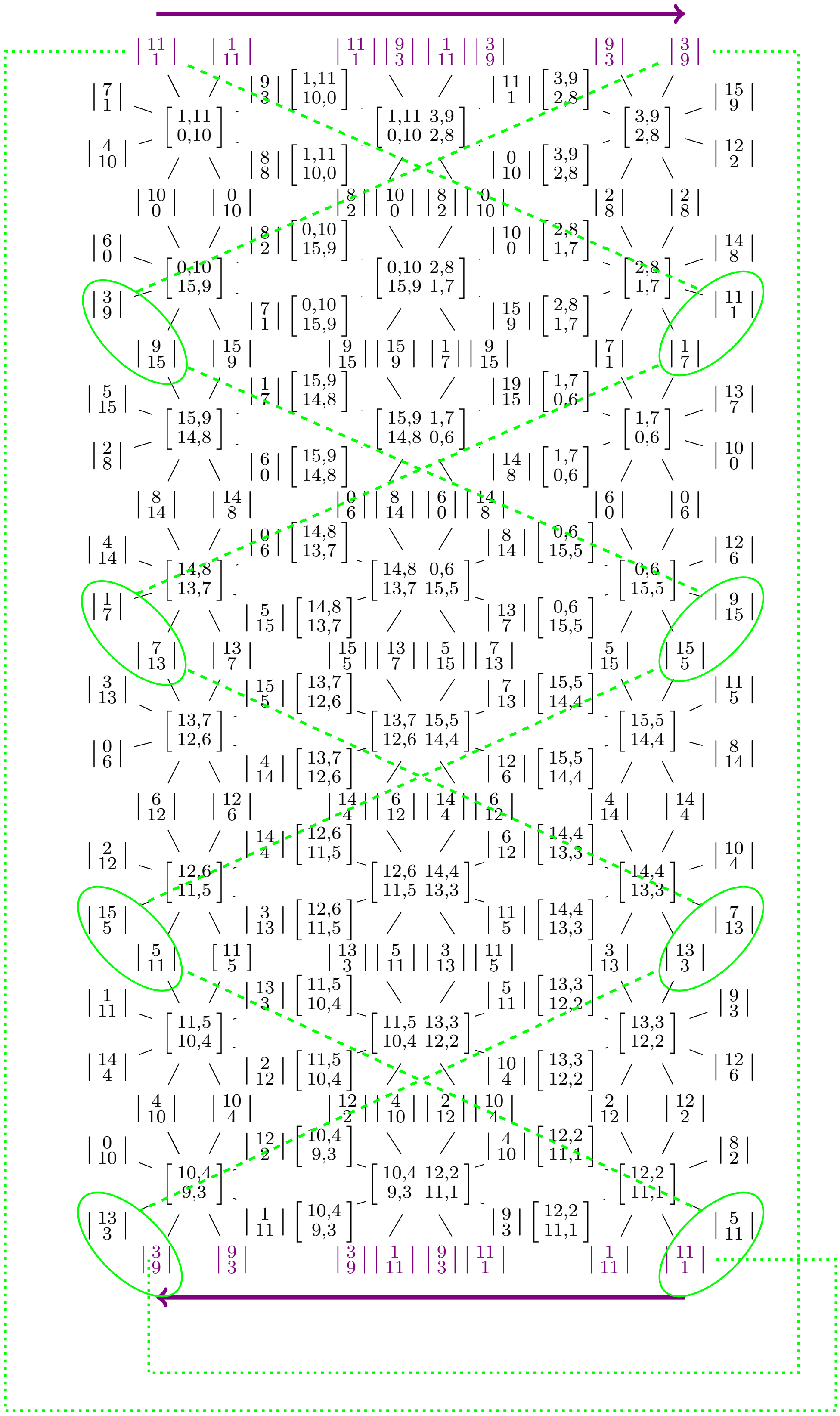


FIGURE 4. Lego tower for d=4

CHAPTER 4

Explicit construction of the dual complex for $d = 2, 3, 4$

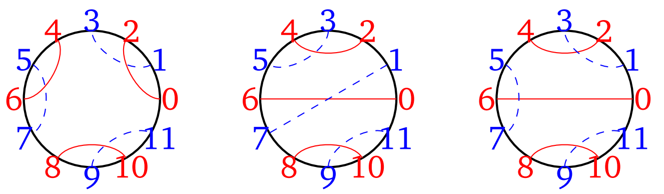
In the previous chapter we have introduced the dual complex, which is an essential tool for the calculation of the cohomology groups and useful to have the nerve in the sense of Čech. In this chapter, a detailed construction of it is proposed. In particular, we construct the isomorphic map between the i-faces of $(\mathcal{W}, \subset)$ in the dual complex and the codimension i classes of polynomials in $(A_\sigma, \prec)$.

The construction of the dual complex for $d = 3, 4$ is presented and in the appendix a detailed construction of the tower for $d = 6$ is given. In order to construct the dual complex we enumerate all the diagrams and give the incidence relations between them.

1. Dual complex for $d = 2, 3$

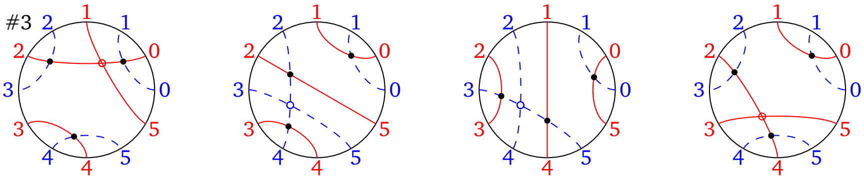
In the case where $d = 2$, the complex $\mathcal{W}$ is formed from 4 vertices and 4 edges. The dual complex associated to the decomposition into signatures of the space ${}^{\mathrm{D}}\mathrm{Pol}_2$ is represented in the figure 2 in chapter 2. See Figure 3 in chapter 7 for a detailed description of the signatures and elementary Reidemeister moves on the signatures. In the decomposition of ${}^{\mathrm{D}}\mathrm{Pol}_d$ when $d = 3$ there exist 22 biregular classes: four M diagrams, six S diagrams with one S tree, twelve F diagrams (six with one red long diagonal six with one long blue diagonal).

Representants of classes of biregular polynomials:



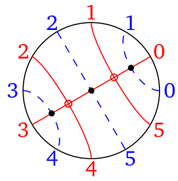
In addition to biregular classes there exist 48 codimension 1 classes (four families of twelve diagrams equivalent up to rotation), 30 codimension 2 classes and 4 codimension 3 classes. Here we did not use the standard notations on the terminal vertices introduced in the previous chapter, since these are the diagrams directly generated from the computer program.

Representants of codimension 1 classes:

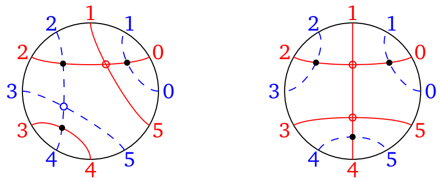


Representatives of codimension 2 classes:

– 1 family of size 6

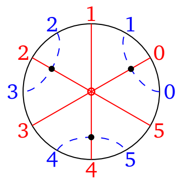


– 2 families of size 12

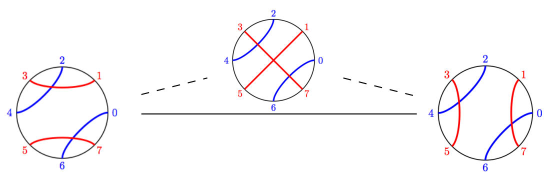


Representant of codimension 3 class:

– 1 family of size 4



Using the deformation operation (definition 2.11), we have the incidence relations between diagrams. For a couple of curves of the same color we have the following adjacence relations:



We use this operation onto all the diagrams. The figure 1 illustrates the dual complex in the $d = 3$ case. The dual complex contains two distinct parts.

(1) The four sub-complexes with black edges.
(2) The sub-complex with colored edges (corresponding to the tower)

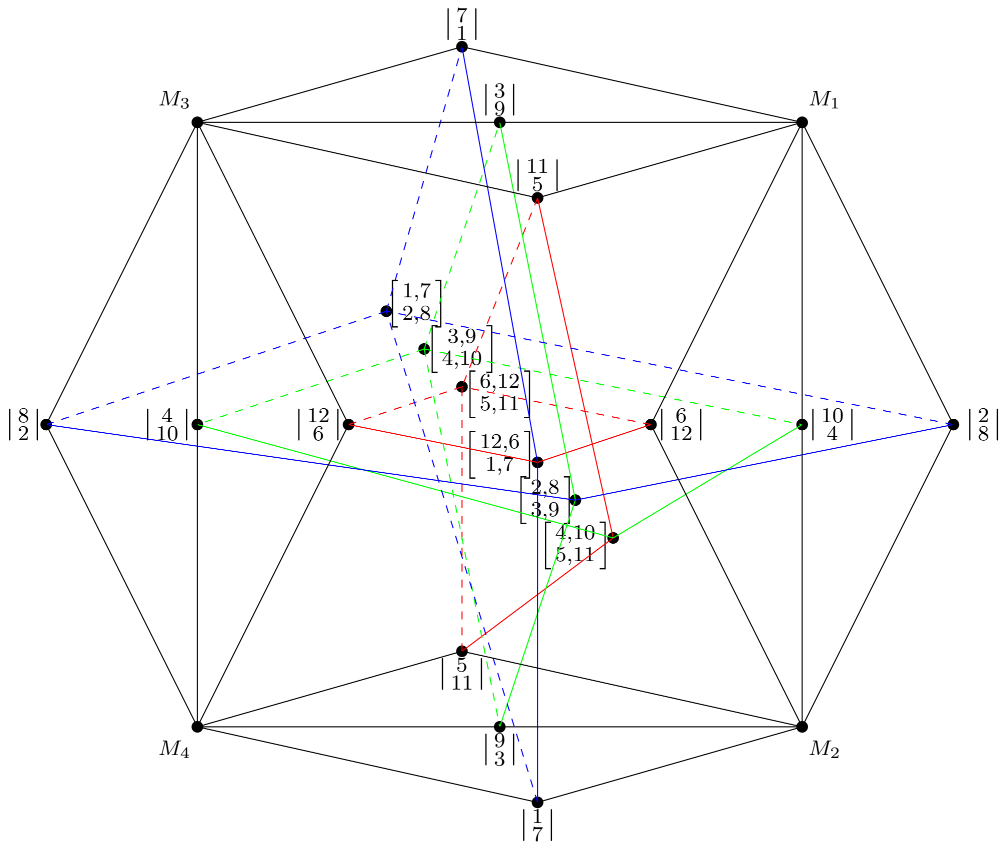


FIGURE 1. dual complex for d=3

- The four subcomplexes with black edges are the ones connecting a couple of M diagrams having a common set of short diagonals. These black subcomplexes contain three vertices corresponding to F diagrams. The 3-face of the black sub-complex corresponds to the codimension 3 diagram and it is incident to three 2-faces which correspond to the codimension 2 diagrams having two critical points z_0 and z_1 of codimension 1 verifying $Re(P)(z_i) = 0$ (resp. $Im(P)(z_i) = 0$), for $i = 0, 1$.
- The colored part of the dual complex corresponds to the parts in the tower, resumed in the lego construction 2. In particular the vertices in this colored part are the S diagrams. The 2-faces in the interior part correspond to codimension 2 classes with one codimension 1 point on $Re(P)(z_i) = 0$ and one on $Im(P)(z_i) = 0$.

This dual complex is resumed in the lego construction in Figure 2.

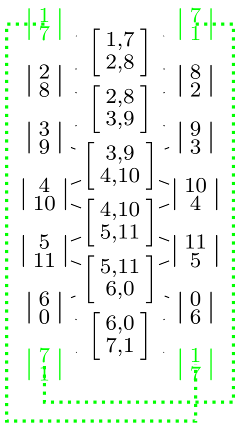


FIGURE 2. Lego tower for d=3

- The first quadrangle 2-face connecting the following two F diagrams and two S diagrams $\left|\begin{smallmatrix}2\\8\end{smallmatrix}\right|, \left|\begin{smallmatrix}8\\2\end{smallmatrix}\right|$ $\left[\begin{smallmatrix}1,7\\2,8\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2,8\\3,9\end{smallmatrix}\right]$ corresponds in figure 1 to the blue vertical cycle. The F diagrams have one long red diagonal.
- The second quadrangle 2-face connecting the following two F diagrams and two S diagrams diagrams $\left|\begin{smallmatrix}9\\3\end{smallmatrix}\right| \left|\begin{smallmatrix}3\\9\end{smallmatrix}\right|$, $\left[\begin{smallmatrix}2,8\\3,9\end{smallmatrix}\right]$ $\left[\begin{smallmatrix}3,9\\4,10\end{smallmatrix}\right]$ corresponds in figure 1 to the blue horizontal cycle. The F diagrams have two long blue diagonals.

REMARK 4.1. There does not exist any codimension 3 class of polynomials in the subcomplex corresponding to the tower.

The tower does not indicate the adjacence relations with the M diagrams. So, we illustrate these relations in the figure 3, where edges of the same color are glued together. The final structure is a union of cylinders.

2. Dual complex for $d = 4$

In this part the construction of the dual complex for $d = 4$ is presented. We give a detailed study of the non-empty intersections of the sets A_σ^+ in the good cover. Recall that the dual complex contains two subparts: the sub-complex connecting two M consecutive diagrams; the complementary sub-complex.

- The tower contains 8 lego pieces. Each lego piece follows the relations in figure 2. A subgraph of the lego piece is described by:

$$F \leftrightarrow S \leftrightarrow F \otimes S \leftrightarrow S \otimes S \leftrightarrow F \otimes S \leftrightarrow S \leftrightarrow F.$$

- Each lego piece is connected to a consecutive one using the join graph.
  - Two successive SS diagrams (SS^+ and SS^-) are related by the following commutative diagram: $\begin{smallmatrix} FF^+ & \leftrightarrow & SS^+ \\ \updownarrow & & \updownarrow \\ SS^- & \leftrightarrow & FF^- \end{smallmatrix}$, where the FF diagram is adjacent to the SS diagram in the following way:

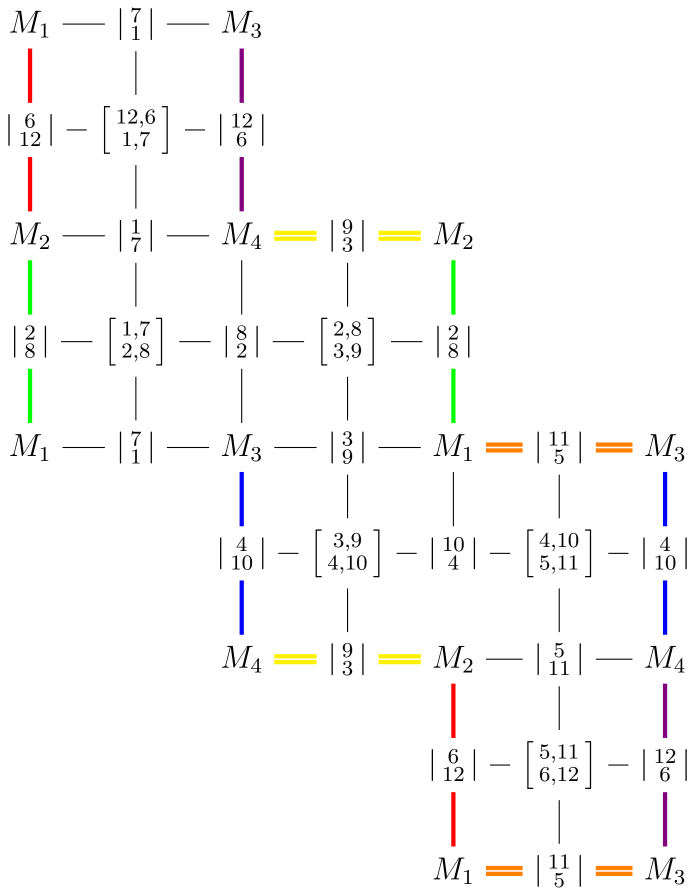
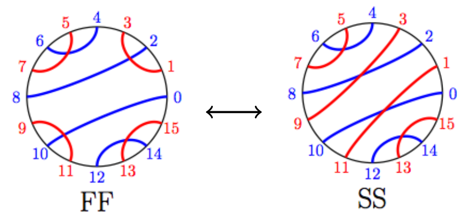


FIGURE 3. The 2-faces of the dual complex for $d = 3$



– Two consecutive S diagrams (S^+ and S^-) are related by the following commutative diagram: $\begin{matrix} F^+ & \leftrightarrow & S+ \\ \updownarrow & & \updownarrow \\ S^- & \leftrightarrow & F^- \end{matrix}$, where the F diagrams marked by F^+ and F^- have opposite orientation i.e. $F^+ = \left|\begin{smallmatrix} j \\ i \end{smallmatrix}\right|$ and $F^- = \left|\begin{smallmatrix} i \\ j \end{smallmatrix}\right|$.

- The last sub-complex is the connection between the four M diagrams. This subcomplex is represented by the double line and corresponds to

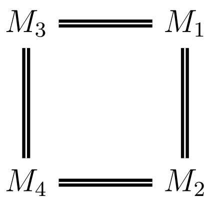


FIGURE 4. Four $NC(4)$ beads

the figure below.

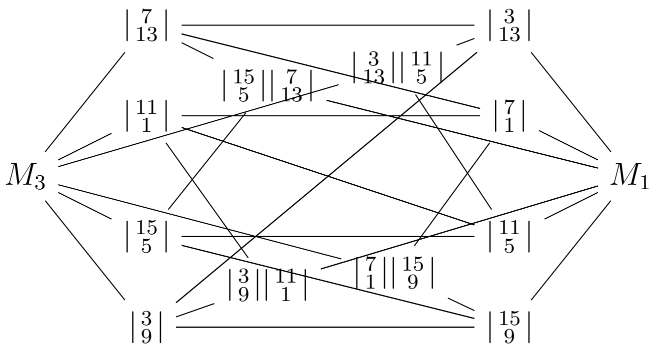
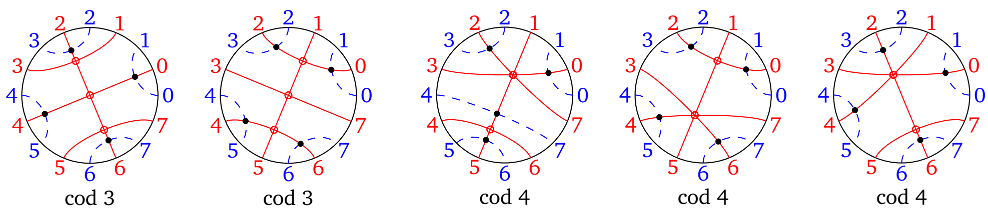


FIGURE 5. Detail of the connection between M_1 and M_3

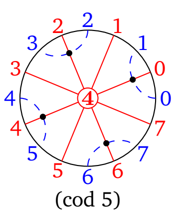
The subcomplex between M_1 and M_3 contains the following families of classes of codimension 3 and 4 :



,

There exist $Cat(4) = 14$ biregular classes in this subcomplex and one codimension 5 class. See the appendix A for the other diagrams.

Example of codimension 5 class in the subcomplex:



A detailed study of the tower is given. We present an algorithmic method to obtain the tower for the first two consecutive pieces (see figure 2 for an illustration). On the figure green circles indicate which vertices are glued together.

- Let us start with the first lego piece having a Q diagram $\left[\begin{smallmatrix}1,11 & 3,9\\0,10 & 2,8\end{smallmatrix}\right]$.
  (1) Deform in it the couple of red diagonals $(2,8),(4,6)$ in order to obtain $\left[\begin{smallmatrix}1,11\\0,10\end{smallmatrix}\right]\left|\begin{smallmatrix}9\\3\end{smallmatrix}\right|$. So, one obtains an FS diagram. Let us deform the long blue diagonal in the tree $\left|\begin{smallmatrix}9\\3\end{smallmatrix}\right|$ with the short blue diagonal in the M tree $\left|\begin{smallmatrix}5\\7\end{smallmatrix}\right|$. This turns it into an M tree and so, we have the S diagram $\left[\begin{smallmatrix}1,11\\0,10\end{smallmatrix}\right]$. The F diagrams which are obtained from this S diagram by a minimal number of deformation operations are $\left|\begin{smallmatrix}7\\1\end{smallmatrix}\right|, \left|\begin{smallmatrix}0\\10\end{smallmatrix}\right|, \left|\begin{smallmatrix}10\\0\end{smallmatrix}\right|, \left|\begin{smallmatrix}11\\1\end{smallmatrix}\right|, \left|\begin{smallmatrix}1\\11\end{smallmatrix}\right|, \left|\begin{smallmatrix}4\\10\end{smallmatrix}\right|$.
  (2) Deform in Q the couple of red diagonals $(0,10),(12,14)$ in order to obtain $\left|\begin{smallmatrix}11\\1\end{smallmatrix}\right|\left[\begin{smallmatrix}3,9\\2,8\end{smallmatrix}\right]$. So, one obtains an SF diagram. Let us deform the long blue diagonal in the tree $\left|\begin{smallmatrix}11\\1\end{smallmatrix}\right|$ with the short blue diagonal in the M tree $\left|\begin{smallmatrix}13\\15\end{smallmatrix}\right|$. This turns it into an M tree and so, we have the S diagram $\left[\begin{smallmatrix}3,9\\2,8\end{smallmatrix}\right]$.The F diagrams which are obtained from this S diagram by a minimal number of deformation operations are $\left|\begin{smallmatrix}15\\9\end{smallmatrix}\right|, \left|\begin{smallmatrix}2\\8\end{smallmatrix}\right|, \left|\begin{smallmatrix}8\\2\end{smallmatrix}\right|, \left|\begin{smallmatrix}12\\2\end{smallmatrix}\right|, \left|\begin{smallmatrix}3\\9\end{smallmatrix}\right|, \left|\begin{smallmatrix}9\\3\end{smallmatrix}\right|$.
- Let us consider the consecutive lego piece.
  (1) The consecutive lego piece contains the Q diagram $\left[\begin{smallmatrix}0,10 & 2,8\\15,9 & 1,7\end{smallmatrix}\right]$ and an adjacent S diagram $\left[\begin{smallmatrix}1,7\\2,8\end{smallmatrix}\right]$ obtained by deforming a couple of red diagonals giving the FS diagram $\left|\begin{smallmatrix}10\\0\end{smallmatrix}\right|\left[\begin{smallmatrix}2,8\\1,7\end{smallmatrix}\right]$, or a couple of blue diagonals giving an FS diagram $\left|\begin{smallmatrix}15\\9\end{smallmatrix}\right|\left[\begin{smallmatrix}2,8\\1,7\end{smallmatrix}\right]$. The S diagram $\left[\begin{smallmatrix}2,8\\1,7\end{smallmatrix}\right]$ is adjacent after one deformation operation to the following F diagrams $\left|\begin{smallmatrix}15\\9\end{smallmatrix}\right|, \left|\begin{smallmatrix}9\\15\end{smallmatrix}\right|, \left|\begin{smallmatrix}11\\1\end{smallmatrix}\right|, \left|\begin{smallmatrix}14\\8\end{smallmatrix}\right| \left|\begin{smallmatrix}2\\8\end{smallmatrix}\right| \left|\begin{smallmatrix}8\\2\end{smallmatrix}\right|$.
  (2) The lego piece contains the Q diagram $\left[\begin{smallmatrix}0,10 & 2,8\\15,9 & 1,7\end{smallmatrix}\right]$ and an adjacent S diagram $\left[\begin{smallmatrix}15,9\\0,10\end{smallmatrix}\right]$ obtained by deforming a couple of blue diagonals giving the FS diagram $\left|\begin{smallmatrix}10\\0\end{smallmatrix}\right|\left[\begin{smallmatrix}15,9\\0,10\end{smallmatrix}\right]$, or a couple of red diagonals giving an FS diagram $\left|\begin{smallmatrix}8\\2\end{smallmatrix}\right|\left[\begin{smallmatrix}2,8\\1,7\end{smallmatrix}\right]$. The S diagram $\left[\begin{smallmatrix}15,9\\0,10\end{smallmatrix}\right]$ is adjacent after one deformation operation to the following F diagrams $\left|\begin{smallmatrix}7\\1\end{smallmatrix}\right|, \left|\begin{smallmatrix}1\\7\end{smallmatrix}\right|, \left|\begin{smallmatrix}6\\0\end{smallmatrix}\right|, \left|\begin{smallmatrix}3\\9\end{smallmatrix}\right| \left|\begin{smallmatrix}0\\10\end{smallmatrix}\right| \left|\begin{smallmatrix}10\\0\end{smallmatrix}\right|$.

REMARK 4.2. The S diagram $\left[\begin{smallmatrix}1,7\\2,8\end{smallmatrix}\right]$ is not consecutive with the S diagram $\left[\begin{smallmatrix}1,11\\0,10\end{smallmatrix}\right]$ since these two diagrams do not share a common set of diagonals. However they belong to consecutive lego pieces and share a common set of F diagrams being adjacent to them in one deformation step. A detailed figure is illustrates this below.

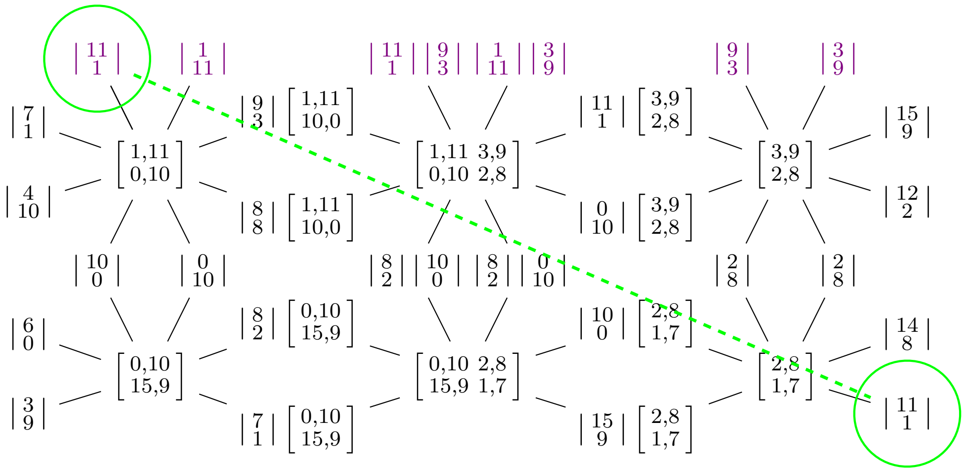
A more general approach is taken by considering the i-th couple of consecutive lego pieces. This is obtained by adding $+2i$ to the numbers $\{2, 4, ...4d\}$ (resp. $\{1, 3, ..., 4d-1\}$), which correspond to the terminal vertices colored red (resp. blue), modulo $4d$, in the first lego piece. This induces the following consecutive lego pieces. One may apply to these lego pieces the same remark as previously.

(1) Let the Q diagram be $\begin{bmatrix} 1+2i,11+2i & 3+2i,9+2i \\ 0+2i,10+2i & 2+2i,8+2i \end{bmatrix}$. Deform in it the couple of red diagonals $(2+2i, 8+2i), (4+2i, 6+2i)$ in order to obtain $\begin{bmatrix} 1+2i,11+2i \\ 0+2i,10+2i \end{bmatrix} \begin{vmatrix} 9+2i \\ 3+2i \end{vmatrix}$. So, one obtains an FS diagram. Let us deform the long blue diagonal in the tree $\begin{vmatrix} 9+2i \\ 3+2i \end{vmatrix}$ with the short blue diagonal in the M tree $\begin{vmatrix} 5+2i \\ 7+2i \end{vmatrix}$. This turns it into an M tree and so, we have the S diagram $\begin{bmatrix} 1+2i,11+2i \\ 0+2i,10+2i \end{bmatrix}$. The F diagrams which are obtained from this S diagram by a minimal number of deformation operations are $\begin{vmatrix} 7+2i \\ 1+2i \end{vmatrix}, \begin{vmatrix} 0+2i \\ 10+2i \end{vmatrix}, \begin{vmatrix} 10+2i \\ 0+2i \end{vmatrix}, \begin{vmatrix} 11+2i \\ 1+2i \end{vmatrix}, \begin{vmatrix} 1+2i \\ 11+2i \end{vmatrix}, \begin{vmatrix} 4+2i \\ 10+2i \end{vmatrix}$.

(2) Deform in Q the couple of red diagonals $(0+2i, 10+2i), (12+2i, 14+2i)$ in order to obtain $\begin{vmatrix} 11+2i \\ 1+2i \end{vmatrix} \begin{bmatrix} 3+2i,9+2i \\ 2+2i,8+2i \end{bmatrix}$. So, one obtains an SF diagram. Let us deform the long blue diagonal in the tree $\begin{vmatrix} 11+2i \\ 1+2i \end{vmatrix}$ with the short blue diagonal in the M tree $\begin{vmatrix} 13+2i \\ 15+2i \end{vmatrix}$. This turns it into an M tree and so, we have the S diagram $\begin{bmatrix} 3+2i,9+2i \\ 2+2i,8+2i \end{bmatrix}$. The F diagrams which are obtained from this S diagram by a minimal number of deformation operations are $\begin{vmatrix} 15+2i \\ 9+2i \end{vmatrix}, \begin{vmatrix} 2+2i \\ 8+2i \end{vmatrix}, \begin{vmatrix} 8+2i \\ 2+2i \end{vmatrix}, \begin{vmatrix} 12+2i \\ 2+2i \end{vmatrix}, \begin{vmatrix} 3+2i \\ 9+2i \end{vmatrix}, \begin{vmatrix} 9+2i \\ 3+2i \end{vmatrix}$.

(3) The consecutive lego piece contains the Q diagram $\begin{bmatrix} 0+2i,10+2i & 2+2i,8+2i \\ 15+2i,9+2i & 1+2i,7+2i \end{bmatrix}$ and an adjacent S diagram $\begin{bmatrix} 1+2i,7+2i \\ 2+2i,8+2i \end{bmatrix}$ obtained by deforming a couple of red diagonals giving the FS diagram $\begin{vmatrix} 10+2i \\ 0+2i \end{vmatrix} \begin{bmatrix} 2+2i,8+2i \\ 1+2i,7+2i \end{bmatrix}$, or a couple of blue diagonals giving an FS diagram $\begin{vmatrix} 15+2i \\ 9+2i \end{vmatrix} \begin{bmatrix} 2+2i,8+2i \\ 1+2i,7+2i \end{bmatrix}$. The S diagram $\begin{bmatrix} 2+2i,8+2i \\ 1+2i,7+2i \end{bmatrix}$ is adjacent after one deformation operation to the following F diagrams $\begin{vmatrix} 15+2i \\ 9+2i \end{vmatrix}, \begin{vmatrix} 9+2i \\ 15+2i \end{vmatrix}, \begin{vmatrix} 11+2i \\ 1+2i \end{vmatrix}, \begin{vmatrix} 14+2i \\ 8+2i \end{vmatrix} \begin{vmatrix} 2+2i \\ 8+2i \end{vmatrix} \begin{vmatrix} 8+2i \\ 2+2i \end{vmatrix}$.

(4) The lego piece contains the Q diagram $\begin{bmatrix} 0+2i,10+2i & 2+2i,8+2i \\ 15+2i,9+2i & 1+2i,7+2i \end{bmatrix}$ and an adjacent S diagram $\begin{bmatrix} 15+2i,9+2i \\ 0+2i,10+2i \end{bmatrix}$ obtained by deforming a couple of blue diagonals giving the FS diagram $\begin{vmatrix} 10+2i \\ 0+2i \end{vmatrix} \begin{bmatrix} 15+2i,9+2i \\ 0+2i,10+2i \end{bmatrix}$, or a couple of red diagonals giving an FS diagram $\begin{vmatrix} 8+2i \\ 2+2i \end{vmatrix} \begin{bmatrix} 2+2i,8+2i \\ 1+2i,7+2i \end{bmatrix}$. The S diagram $\begin{bmatrix} 15+2i,9+2i \\ 0+2i,10+2i \end{bmatrix}$ is adjacent after one deformation operation to the following F diagrams $\begin{vmatrix} 7+2i \\ 1+2i \end{vmatrix}, \begin{vmatrix} 1+2i \\ 7+2i \end{vmatrix}, \begin{vmatrix} 6+2i \\ 0+2i \end{vmatrix}, \begin{vmatrix} 3+2i \\ 9+2i \end{vmatrix} \begin{vmatrix} 0+2i \\ 10+2i \end{vmatrix} \begin{vmatrix} 10+2i \\ 0+2i \end{vmatrix}$.



3. Subcomplexes for $d = 4$ and tables of intersections of open sets

This part presents the different families of subcomplexes in the dual complex for $d = 4$. We study the common vertices of different subcomplexes, in the so-called intersections tables.

3.1. The subcomplexes. This part is devoted to showing explicitly the structures of subcomplexes of the dual complex for $d = 4$. The vertices of these subcomplexes correspond to the set of incident classes to a signature of codimension 4 (resp. codimension 5).

There is one family of diagrams of codimension 5 and two families of diagrams of codimension 4.

DEFINITION 4.1. *We call vertical (resp. horizontal) bridge the subcomplex connecting two opposite F diagrams, where the F diagram contains a single F tree having one long blue (resp. red) diagonal.*

The figure 6, presents a subcomplex of the dual complex W which consists of a quadrangle of vertices $M_1, ..., M_4$ where the edges represent the NC structure and the vertices $M_1, ..., M_4$ correspond to the four M diagrams. This quadrangle is divided into four regions by a green and red line. These lines represent respectively the vertical and horizontal bridges and connect two F diagrams oppositely oriented. The intersection of the red and green line is denoted by S, and corresponds to the common vertex to the vertical and horizontal bridges.

There exist 8 horizontal bridge structures and 8 vertical bridge structures. So, the vertical bridges are labeled from 1 to 8. A bridge subcomplex is drawn in the figure 7.

Notation We use the approach of proposition 3.9, chapter 2.

EXAMPLE 4.1. Take a diagram having only one long blue diagonal while all the others are short. If on one side of this long blue diagonal there exist two

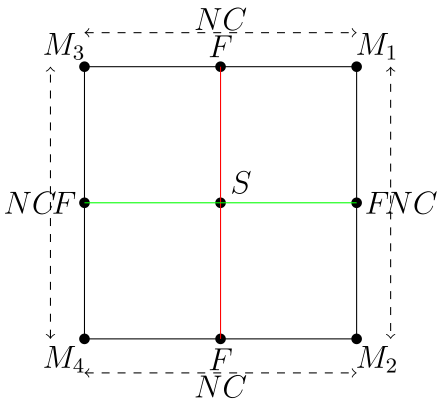


FIGURE 6. Horizontal, vertical bridge structures

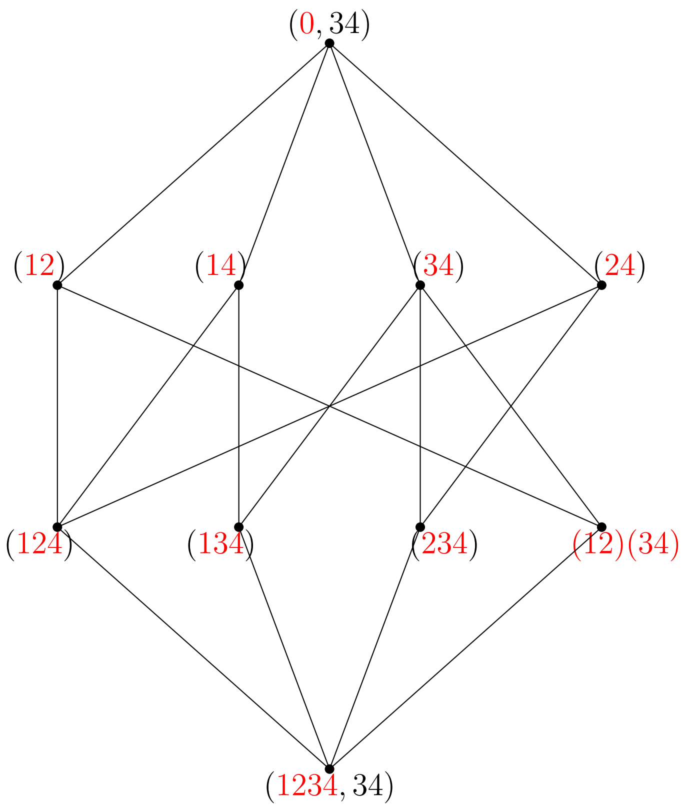


FIGURE 7. Bridge structures between two opposite $NC(4)$ subcomplexes

red vertices i, j and if each red diagonal delimits a region containing one blue vertex, then this diagram is assigned to the couple $(0, ij)$. The figure 8 presents the example of a long blue diagonal corresponding to $(*, 123)$

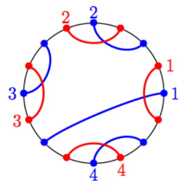


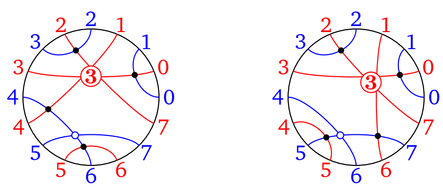
FIGURE 8. Example of $(*, 123)$ diagram

The blue diagonals are assigned to blocks colored in blue (noted “bl. bk"). Otherwise the block is colored red (noted “red bk").

There are three families of subcomplexes:

(1) The first family is composed by 4 subgraphs of W isomorphic to the lattice of non crossing partitions of $\{1, .., 4\}$, $NC(4)$ (the face of dimension 5 corresponds to the diagram of codimension 5).
(2) The second family is composed of 8 vertical bridge and 8 horizontal bridge structures, connecting a couple of opposite F diagrams in two lattices $NC(4)$. The face of dimension 4 corresponds to the diagram of codimension 4 with critical points on $ReP(z) = 0$ (resp. $ImP(z) = 0$).
(3) The third family is composed of 32 open book structures, connecting a couple of F diagrams (one with red long diagonal one with a long blue diagonal) in adjacent $NC(4)$ subcomplexes. The face of dimension 4 corresponds to a diagram of codimension 4 having critical points verifying $ReP(z) = 0$ and $ImP(z) = 0$.

The *open book* structures are labeled from i to viii and from a to h for the opposite color. The 4-face in an open book corresponds to one of the following diagrams of codimension 4:



The next table enumerates all the vertices of the vertical bridge subcomplex in the notation above. To each biregular diagram we assign a couple $(*, *)$ of red and blue blocks, partitioning $\{1, .., 4\}$. The diagrams of one horizontal bridge have a fixed red long diagonal equivalent to $(*, 123)$ up to rotation $\frac{k\pi}{8}$. The second table contains the vertices of the horizontal bridge structures and the vertices have a fixed blue long diagonal equivalent to $(*, 123)$ up to rotation $\frac{k\pi}{8}$.

BKL\ $N^{\circ}_{\text{polytope}}$	1	2	3	4	5	6	7	8
b. bk	34	41	12	23	123	124	134	234
red bk	0	0	0	0	1234	1234	1234	1234
red bk	12	23	34	41	134	124	123	234
red bk	23	34	41	12	124	134	234	123
red bk	24	13	24	13	(23)(41)	(12)(34)	(23)(41)	(12)(34)
red bk	34	14	12	23	123	234	134	124
red bk	124	123	234	134	23	34	41	12
red bk	134	124	123	234	12	23	34	41
red bk	234	134	124	123	41	12	23	34
red bk	(12)(34)	(23)(41)	(12)(34)	(23)(41)	13	24	13	24
red bk	1234	1234	1234	1234	0	0	0	0

	A	B	C	D	E	F	G	H
red block	34	41	12	23	123	124	134	234
bl. bk	0	0	0	0	1234	1234	1234	1234
bl. bk	12	23	34	41	134	124	123	234
bl. bk	23	34	41	12	124	134	234	123
bl. bk	13	24	13	24	(23)(41)	(12)(34)	(23)(41)	(12)(34)
bl. bk	34	14	12	23	123	234	134	124
bl. bk	124	123	234	134	23	34	41	12
bl. bk	134	124	123	234	12	23	34	41
bl. bk	234	134	124	123	41	12	23	34
bl. bk	(12)(34)	(23)(41)	(12)(34)	(23)(41)	13	24	13	24
bl. bk	1234	1234	1234	1234	0	0	0	0

We give the table corresponding to the open book subcomplexes. By $0, 34$ we mean that successively one must consider the first the case where the blue block is 0 and then the second case where the blue block is 34. We give the table for one of the two families of classes of diagrams above, due to their similarity.

	i	ii	iii	iv	v	vi	vii	viii
b. bk	0,34	0,41	0,12	0, 23	1,123	1,124	1,134	1,234
red bk	1234	1234	1234	1234	0	0	0	0
red bk	34	41	12	23	123	124	134	234
red bk	134	124	123	234	12	23	34	41
red bk	234	134	234	123	41	34	23	34
red bk	(12)(34)	(23)(41)	(12)(34)	(23)(41)	13	24	13	24

	a	b	c	d	e	f	g	h
b. bk	0,34	0,41	0,12	0, 23	1,123	1,124	1,134	1,234
b. bk	1234	1234	1234	1234	0	0	0	0
b. bk	34	41	12	23	123	124	134	234
b. bk	134	124	123	234	12	23	34	41
b. bk	234	134	234	123	41	34	23	34
b. bk	(12)(34)	(23)(41)	(12)(34)	(23)(41)	13	24	13	24

	9	10	11	12	13	14	15	16
b. bk	0,34	0,41	0,12	0, 23	1,123	1,124	1,134	1,234
red bk	0	0	0	0	1234	1234	1234	1234
red bk	124	123	234	341	12	14	34	23
red bk	12	23	34	41	134	234	123	124
red bk	14	12	23	34	234	123	124	134
red bk	24	13	24	13	(12)(34)	(23)(41)	(12)(34)	(23)(41)

	l	m	n	o	p	q	r	s
Black bk	0	0	0	0	1234	1234	1234	1234
Black bk	124	123	234	341	12	14	34	23
Black bk	12	23	34	41	134	234	123	124
Black bk	14	12	23	34	234	123	124	134
Black bk	24	13	24	13	(12)(34)	(23)(41)	(12)(34)	(23)(41)

The table corresponding to the four non-crossing partition subcomplex.

NC_1	NC_2	NC_3	NC_4
0	1234	0	1234
0	0	0	0
12	12	12	12
23	23	23	23
34	34	34	34
41	41	41	41
13	13	13	13
24	24	24	24
(12)(34)	(12)(34)	(12)(34)	(12)(34)
(23)(41)	(23)(41)	(23)(41)	(23)(41)
123	123	123	123
124	124	124	124
134	134	134	134
234	234	234	234

3.2. Intersections tables. This work aims at defining the non-empty intersections between the different families of substructures of W. This study serves to consider the cochains for the space ${}^{\mathrm{D}}\mathrm{Pol}_4$. Therefore, the intersections between the different families of subcomplexes are considered. The couple "$[\mathcal{I}], [\mathcal{K}]$" represents the intersection between one subcomplex of the family $\mathcal{I}$ and one subcomplex of the family $\mathcal{K}$. If their intersection is empty then it is not interesting for the Čech cohomology. So, we are interested only in the non-empty intersections between complexes. We construct the tables of intersections as follows. On the first vertical column we indicate the biregular diagram by "$*, *''$; the first line of the table contains the different families of subcomplexes. We have in the table:

$$\begin{cases} 0 \text{ if } \mathcal{I}, \mathcal{K} = \emptyset \\ \mathcal{I} \text{ if } \mathcal{I}, \mathcal{K} \neq \emptyset \text{ and } \mathcal{K} \text{ is on the first line or } \mathcal{I} \text{ is on the first line} \\ \mathcal{K} \text{ if } \mathcal{I}, \mathcal{K} \neq \emptyset \text{ and } \mathcal{I} \text{ is on the first line or } \mathcal{K} \text{ is on the first line} \end{cases}$$

There exist biregular signatures which do not belong to at least two structures enumerated above. Those points are called lonely points. Example of lonely points:

vertex	[A]	[a]	[i]	[1]
2334	A	0	0	0
1334	A	0	0	0

Other lonely points are enumerated in the following table.

vertex	[A]	[a]	[i]	[1]
1313	0	0	0	0
13(23)(41)	0	0	0	0
2424	0	0	0	0
24(12)(34)	0	0	0	0
(12)(34)(12)(34)	0	0	0	0
(12)(34)234	0	0	0	0
(12)(34)124	0	0	0	0
(34)(12)(34)	0	0	i	0
(23)(41)(23)(41)	0	0	0	0
341(23)(41)	0	0	0	0
123(23)(41)	0	0	0	0

REMARK 4.3. One may consider the vertices of W which are connected by one edge to 1434. This vertex is of valency 6. Its related vertices are not all lonely points. Let us list the non-lonely points: 23124$\in (F, f)$, 23123 $\in (iv, 4)$, 23412 $\in (C, viii)$, 12312$\in (C, i)$. The lonely points: 123412$\in C$,231234 $\in 4$.

We consider below the intersections between an NC structure and the other structures (bridge or open book structures). Below are given all the possible results concerning the intersection of subcomplexes.

REMARK 4.4. For the sake of simplicity, the block 1234 assigned to a biregular diagram corresponds to the notation "1".

Intersection between NC structures and other structures We give the intersections between NC and the other subcomplex structures i.e openbook structures and bridge structures. In the table, each column corresponds to a couple of intersecting structures. If they intersect at one vertex then we define the couple of structures and their intersections in the table. The last column contains the number of intersections of different couples of structures giving the same vertex.

vertex	NC, a	NC, i	NC, A	$NC, 1$	nb. inter
120	NC_1, c	0	0	$NC_1, 3$	2
230	NC_1, d	0	0	NC_1,4	2
340	NC_1, a	0	0	$NC_1, 1$	2
410	NC_1, b	0	0	$NC_1, 2$	2
(12)(34)0	$NC_1, \{a, c\}$	0	0	0	2
(23)(41)0	$NC_1, \{b, d\}$	0	0	0	2
1340	$NC_1, \{a, b\}$	0	NC_1, vii	$NC_1, 7$	4
2340	$NC_1, \{a, d\}$	0	$NC_1, viii$	$NC_1, 8$	4
1240	$NC_1, \{b, c\}$	0	NC_1, vi	$NC_1, 6$	4
1230	$NC_1, \{c, d\}$	0	NC_1, v	$NC_1, 5$	4
121	$NC_2, \{e, h\}$	0	NC_1, iii	$NC_1, 3$	4
231	$NC_2, \{f, g\}$	0	NC_2, iv	$NC_2, 4$	4
341	$NC_2, \{f, g\}$	0	NC_2, i	$NC_2, 1$	4
411	$NC_2, \{h, e\}$	0	NC_2, ii	$NC_2, 2$	4
131	$NC_2, \{e, g\}$	0	0	0	2
241	$NC_2, \{f, h\}$	0	0	0	2
1341	NC_2, g	0	0	$NC_2, 7$	2
2341	NC_2, h	0	0	$NC_2, 8$	2
1241	NC_2, f	0	0	$NC_2, 6$	2
1231	NC_2, e	0	0	$NC_2, 5$	2
034	NC_3, i	0	NC_3, A	0	nb. inter
041	NC_3, i	0	NC_3, B	0	2
012	NC_3, iii	0	NC_3, C	0	2
023	NC_3, iv	0	NC_3, D	0	2
0134	$NC_3, \{i, ii\}$	NC_3, g	NC_3, G	0	4
0234	$NC_3, \{i, iv\}$	NC_3, h	NC_3, H	0	4
0124	$NC_3, \{ii, iii\}$	NC_3, f	NC_3, F	0	4
0123	$NC_3, \{iii, iv\}$	NC_3, e	NC_3, E	0	4
0(12)(34)	$NC_3, \{i, iii\}$	0	0	0	2
0(23)(41)	$NC_3, \{ii, iv\}$	0	0	0	2
134	$NC_4, \{vi, vii\}$	NC_4, a	NC_4, A	0	4
141	$NC_4, \{v, vi\}$	NC_4, b	NC_4, B	0	4
112	$NC_4, \{v, viii\}$	NC_4, c	NC_4, C	0	4
123	$NC_4, \{vii, viii\}$	NC_4, d	NC_4, D	0	4
1134	NC_4, vii	0	NC_4, G	0	2
1234	NC_4, vii	0	NC_4, H	0	2
1124	$NC_4, \{vi\}$	0	NC_4, F	0	2
1123	$NC_4, \{v\}$	0	NC_4, E	0	2
113	$NC_4, \{v, vii\}$	0	0	0	2
124	$NC_4, \{vi, viii\}$	0	0	0	2

Below we are giving the intersection table corresponding to the couples of intersecting structures giving M diagrams (10 and 01 in the new notation).

$vertex \setminus inter$	NC, i	NC, a	NC, A	$NC, 1$	nb. inter
10	$NC_4, \{v, \ldots, viii\}$	$NC_1, \{a, \ldots, d\}$	0	0	8
01	$NC_3, \{i, \ldots, iv\}$	$NC_2, \{e, \ldots, h\}$	0	0	8

Intersections between bridge and open book structures. We present eight intersection tables (due to the eight possible rotations of the diagrams by an angle of $\frac{k\pi}{8}$) and some lonely points.

• Intersection of structures leaving the monochromatic diagram defined by 34 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
1234	A,3	0	0	0	0	0	1
(12)(34)34	0	A,a	0	0	0	0	1
034	0	0	A,i	0	0	0	1
123434	0	A,a	A,vii	0	0	a,vii	3
12434	A,6	0	A, vi	0	vi,6	0	3
23434	A,8	A,a	0	0	8,viii	a,viii	4
13434	A,7	A,a	A, vii	a,7	vii,7	a,i	6
3434	A,1	A,a	A,i	a,1	i,1	a,i	6

Lonely points:

vertex	[A]	[a]	[i]	[1]
1434	0	0	0	2
2334	A	0	0	0
1334	A	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 14 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
014	0	0	B,ii	0	0	0	1
(23)(41)14	0	B,b	0	0	0	0	1
2314	B,4	0	0	0	0	0	1
123414	0	B,b	0	0	ii,2	b,v	3
13414	B,7	B,b	0	b,7	0	0	3
12314	B,5	B,b	B,vii	0	v,5	0	4
4114	B,2	B,b	B,ii	b,2	ii,2	b,ii	6
12414	B,6	B,b	B, vi	b,6	vi,6	b,ii	6

Lonely points:

vertex	[A]	[a]	[i]	[1]
1214	B	0	0	0
3414	B	0	0	0
2414	B	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 12 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
012	0	0	C,iii	0	0	0	1
3412	C,1	0	0	0	0	0	1
(12)(34)12	0	C,c	0	0	0	0	1
123412	0	C,c	0	0	ii,2	c,v	3
12412	C,6	C,c	0	b,6	0	0	3
23412	C,8	0	0	b,7	8,Viii	h,viii	4
12312	C,5	C,c	C,i	c,5	v,5	c,iii	6
1212	C,3	C,c	C,iii	C,3	iii,3	c,iii	6

lonely points:

vertex	[A]	[a]	[i]	[1]
2312	C	0	0	0
4112	C	0	0	0
2412	C	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 23 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
023	0	0	D,iv	0	0	0	1
4123	C,2	0	0	0	0	0	1
(23)(41)23	0	D,d	0	0	0	0	1
123423	0	D,d	D,vi	0	0	d,vi	3
13423	D,7	0	D,vii	0	vii,7	0	3
12323	D,5	D,d	D,i	d,5	vi,6	0	4
23423	D,8	D,d	H,ii	b,7	8,Viii	h,viii	6
2323	D,4	D,d	D,i	c,3	iv,4	d,iv	6

Lonely points:

vertex	[A]	[a]	[i]	[1]
1223	D	0	0	0
3423	D	0	0	0
1323	D	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 123 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
13123	0	e,E	0	0	0	0	1
134123	E,7	0	0	0	0	0	1
1234123	0	0	E,v	0	0	0	1
41123	E,e	E,2	0	e,2	0	0	3
0123	e,E	0	E,iii	0	0	e,iii	3
23123	E,4	e,E	E,iii	0	iv,4	0	4
123123	E,5	E,e	E,v	e,5	v,5	e,vi	6
12123	E,3	e,E	E,iv	e,3	iii,3	e,v	6

Lonely points

vertex	[A]	[a]	[i]	[1]
123	E	0	0	0
234123	E	0	0	0
(23)(14)123	E	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 124 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
24124	0	f,F	0	0	0	0	1
234124	F,8	0	0	0	0	0	1
1234124	0	0	F,vi	0	0	0	1
0124	f,F	0	F,vi	0	0	f,ii	3
12124	F,3	0	F,iv	0	iii,3	0	3
34124	F,1	f,F	0	f,1	0	vii,4	4
41124	F,2	F,f	F,iii	f,2	ii,2	f,vii	6
124124	F,6	f,F	F,vi	f,6	vi,6	e,vi	6

Lonely points:

134124	F	0	0	0

• Intersection of structures leaving the red diagonals of the diagram defined by 134 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
1234134	0	0	0	0	0	0	1
123134	G,5	0	0	0	0	0	1
13134	0	g,G	0	0	0	0	1
0134	g,G	G,ii	0		0	g,ii	3
23134	G,4	0	0	g,4	iii,3	0	3
41134	G,2	G,g	G,ii	0	ii,2	0	4
34134	G,1	G,g	G,i	g,7	i,1	g,vii	6
134134	G,7	G,g	G,vii	g,7	vii,7	g,vii	6

Lonely points:

124134	G	0	0

• Intersection of structures leaving the monochromatic diagram defined by 234 fixed:

vertex	[A],[1]	[A],[a]	[A], [i]	[a], [1]	[i],[1]	[a],[i]	nb. inter
24234	0	H,h	0	0	0	0	1
123234	H,5	0	0	0	0	0	1
1234234	0	H,viii	0	0	0	0	1
0234	H,h	H,iv		h, iv	0	0	3
34234	H,1	0	H,i	0	i,1	0	3
12234	H,3	H,h	0	h,3	ii,2	0	4
23234	H,4	H,h	H, iv	h,4	iii,3	h,viii	6
234234	H,8	H,h	H,viii	h,8	viii,8	8,viii	6

Lonely points:

124234	H	0	0	0

CHAPTER 5

Signatures, invariants of polynomials and monodromy

In this chapter it is shown that the monodromy is explicitly obtained from the stratification $\{A_\sigma\}_{\sigma\in\Sigma}$ by signatures. Moreover, we show that the signatures highlight information concerning properties of the complex polynomial maps. Let us introduce a few notations.

- Let Tub be the tubular neighborhood of a picture embedded in the complex plane.
- Let Tub_0 be the tubular neighborhood of the union of the real and imaginary axis in the complex plane.
- Define $E = \mathbb{C} \setminus int(Tub)$ and $E_0 = \mathbb{C} \setminus Tub_0$.
- Let $\phi : \partial E \to \partial E_0$ be a smooth branched cover of degree $d > 0$. We denote by $\Phi : E \to E_0$ an extension of ϕ which is a holomorphic branched cover of degree d with branch points lying inside $E \setminus \partial E$.

LEMMA 5.1. *The number of 2-cells in $\mathbb{C} \setminus \sigma$ of a biregular signature of degree $d > 1$ is at least one and at most $d - 1$.*

PROOF.

(1) Counting the maximal number of common 2-cells.

Considering a tree as a simple curve with endpoints on the boundary of the disc (a chord), we have a system of d non intersecting chords on the plane. The maximal number of adjacent regions between chords that one can achieve is $d+1$. This case is possible only for a configuration of d parallel chords on the disc. Therefore counting only the regions shared between two chords, we find that there exists $d-1$ such shared regions.

(2) Counting the minimal number of common 2-cells.

Let us define $2d$ vertices on the boundary of the disc, numerated from 0 to $2d-1$. The configuration of chords with the least number of shared regions is given by d chords connecting vertex i with vertex $i+1 \mod 2d$. There exists one complementary region shared by the d chords.

□

LEMMA 5.2. *The image of a root of P' by a polynomial $P \in {}^{\mathrm{D}}Pol_d$ is different from zero.*

PROOF. Suppose by contradiction that the image by P of a root of P' is zero. The root of P' is then also a root of P. This implies that the discriminant of the polynomial is zero (the discriminant of a polynomial can be defined as the product of the values of the polynomial P' at the roots of P) and therefore that the polynomial P has multiple roots, which is a contradiction. □

Lemma 5.3. *For each smooth submersion $\phi : \partial E \to \partial E_0$ of degree $d > 0$ there exists a smooth extension $\Phi : E \to E_0$ with finitely many critical points. The germ of Φ at a critical point p is of the form $\Phi(p+h) = \Phi(p) + z(h)^k$. The real determinant of the differential $D\Phi$ is positive at each regular point.*

Proof. Consider the embedded graph σ in $\mathbb{C}\cup\{\infty\}$ satisfying the 7 properties of Theorem 2.3. Its tubular neighborhood being Tub, the boundary of the tubular neighborhood defines the boundary ∂E. The restriction of Φ to one edge of σ is an injective and regular map. Let us show that the smooth extension Φ has finitely many critical points lying in E. The E_0 is partitioned into four disjoint open 2-cells colored A, B, C, D, the inverse map $\Phi^{-1}(\partial E_0)$ copies d times ∂E_0 in E. We use a purely combinatorial argument to show that in the regions of $\mathbb{C} \setminus \sigma$ there exist the critical points of Φ: we count the number of regions in the complementary part of the signature and show that it is strictly smaller than $4d$. Let us consider the number of complementary regions of a generic signature. The counting is inspired from elementary combinatorial exercises: we place dots on a horizontal line which are compartmented by vertical lines. One tree of the signature plays the role of the vertical line (defining a compartment); the regions contained between two lines are marked by dots. One dot is attributed to each 2-cell which is a complementary region to a tree of the signature. Thus, for two trees sharing a common 2-cell, there exist 2 dots in the compartment. Since there exist d such trees in $\mathbb{C}$ the application of lemma 5.1 implies that there are at most $d-1$ compartments containing 2 dots and at least 1 compartment containing d dots . Now, assume that there exists a polygonal region $\mathcal{R}$ of color A containing m dots. This region $\mathcal{R}$ is connected and if $m \geq 2$, it contains the zeros of the derivative polynomial P' .Those points which are the zeros of P' (by lemma 5.2), lie in the regions containing $m > 1$ dots and are the branching points for Φ (since this polygonal region is connected this argument is true only if there exists at least one point glueing those regions together: the branching point of Φ). Therefore, Φ has critical points in the regions containing $m > 1$ dots and at these critical points p, the germ of Φ is a branching cover of type $\Phi(p+h) = \Phi(p) + z(h)^k$, where k is an integer greater than 1. In the case of co-dimension i signatures (i.e. when $i > 0$), the maps $\mathbf{Re}(\mathrm{P})$ and/or $\mathbf{Im}(\mathrm{P})$ have critical points. This follows from the Cauchy-Riemann equations: a zero of the gradient of $\mathbf{Re}(\mathrm{P})$ implies a zero of the gradient of $\mathbf{Im}(\mathrm{P})$, which signifies that the the zeros of Φ' are situated on the zero-loci of $\nabla\, \mathbf{Re}(\mathrm{P})$ and $\nabla\, \mathbf{Im}(\mathrm{P})$. These points are the critical points for Φ. At regular points, the smooth submersion is orientation preserving. Therefore, for all regular points de differential $D\Phi > 0$. □

1. Monodromy

First recall that the configuration space $Conf_d$ is defined as the set of d-tuples $(x_1, ..., x_d)$ such that

$$Conf_d = \{(x_1, ..., x_d) \in \mathbb{C}^d | x_i \neq x_j for\ i \neq j\}.$$

We are particularly interested in the subset of $Conf_d$ corresponding to the trajectories of a couple of roots x_i and x_j of polynomials with the following condition:

- the roots x_i and x_j are the inner nodes of a couple of trees belonging to the boundary of the same polygonal region;
- we consider the trajectories of those nodes obtained after the composition of Whitehead operations (acting on the couple of red and blue curves) and verifying $\rho \circ \beta \circ \rho \circ \beta = Id$.

We now consider the pure braid group (or colored braid group) with d strings which we identify with the fundamental group of ${}^{\mathrm{D}}\mathrm{Pol}_d$. We shall denote an elementary braid by x_{ij} if the i-th and j-th strings are involved. An elementary colored braid is drawn as follows:

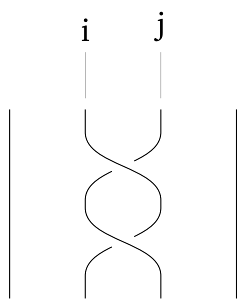


FIGURE 1. Elementary colored braid x_{ij}

LEMMA 5.4. *[Gauss-Lucas] Suppose that P is a complex polynomial such that its roots form a convex polygon. Then the roots of P' lie in the convex hull defined by the roots of P.*

PROPOSITION 5.5. *Consider two roots x_i and x_j of a polynomial being the crossing points of two couples of red and blue curves sharing a polygonal region. Let ρ (resp. β) be the Whitehead moves (i.e. deformation operations) acting on a couple of red curves (resp. blue curves). If we apply the Whitehead moves ρ (resp. β) successively onto the red couples of curves (resp. blue couples of curves) such that $\rho \circ \beta \circ \rho \circ \beta = Id$ then the roots x_i and x_j describe a trajectory which is in bijection with an elementary colored braid x_{ij}.*

PROOF. Let us denote by (#) the condition that $\rho \circ \beta \circ \rho \circ \beta = Id$. We first show that the condition (#) is true if the couple of deformed red and blue curves by ρ and β belongs to one couple of trees T_1 and T_2 sharing a polygonal region.

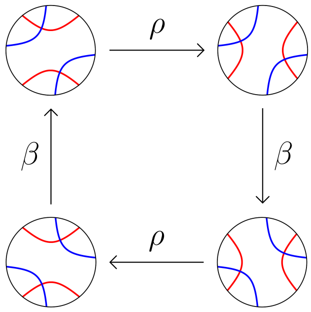


FIGURE 2. The composition of Whitehead deformations on a couple of trees

Suppose by contradiction that the minimal number of consecutive deformations ρ, β which alternate such that (#)

is true can be achieved considering more than two different trees sharing a common polygonal region, say three. Focusing on these three trees we get a partial signature of degree 3 (of type M for the degree 3). The adjacencies relations for the space of degree 3 polynomials are illustrated in the figure 1 in chapter 3. Starting from an M diagram we have only two types of quadrangles. First we there exist the quadrangles colored in black (and isomorphic to the Hasse diagram of an $NC(3)$ lattice) and secondly the quadrangles starting with an M diagram involving S and F diagrams. Let us investigate the first quadrangle. It is obtained using Whitehead moves which are all of one color β (resp. ρ). So the condition (#) is not satisfied. We now consider the second type of quadrangles which can be itself decomposed into two sub-cases.

We have one closed path starting (and ending) at the M diagram, such that the (#) condition is satisfied. One can easily verify that for this case among the three trees only a couple of trees are considered. Therefore, we are not interested in this case.

As for the second sub-case of quadrangles starting (and ending) at the M diagram, one can verify that it involves the three trees at the same time. However, the condition (#) is not satisfied as we have the following composition $\rho\circ\beta\circ\beta\circ\rho = Id$.

In conclusion for more than three trees sharing a polygonal region there do not exist any quadrangular relations verifying the condition (#). Since the dual complex corresponding to the decomposition of the space ${}^{\mathrm{D}}\mathrm{Pol}_{d-1}$ is contained in the one of the space ${}^{\mathrm{D}}\mathrm{Pol}_{d}$ this argument can be applied for any $d \geq 3$ and the number of deformation operations such that (#) is satisfied is achieved only when we consider them onto a couple of trees sharing a polygonal region.

Let us show that this condition implies the existence of homotopically non-nul paths which are defined by the couple of roots corresponding to the inner nodes of the two trees.

Suppose without loss of generality, that the polygonal region shared by the trees T_1 and T_2 is colored A and x_i , x_j are respectively the inner nodes of T_1 and T_2. We shall apply successively the two types of deformation operations β and ρ onto those trees T_1 and T_2. We shall only consider partial signatures, i.e. the piece of the signature containing the couple of trees T_1 and T_2. The codimesion 0 partial signatures are indexed by odd numbers and the partial signatures of codimesion 1 are indexed by even numbers. Using lemma 5.3 the region A of the partial signature σ_1 contains a root of P'. The first deformation operation β maps the partial signature σ_1 to the partial signature σ_3 by a Whitehead move on the couple of blue curves. By Cauchy-Riemann equations, this deformation implies that the root of P', defined at first in region A moves continuously onto the positive part of the imaginary axis (giving σ_2). Then it deforms again into a couple of smooth blue curves where the root P' belongs to the region colored B. This is signature σ_3. Now we apply ρ to the signature σ_3. Again the Whitehead move deforms the couple of red curves so that the root of P' continuously moves from the region B to the negative part of the real axis (signature σ_4) and then to the region of color C. This gives the signature σ_5. Applying again the deformation operation β on the couple of blue curves in σ_5 we obtain the signature σ_6. The root of P' which is in σ_5 and lies in the region colored C moves onto the negative part of the imaginary axis and into the region D. This defines the signature σ_7. Finally, applying again the deformation operation ρ onto σ_7, the root P' is moved onto the region A (giving back σ_1) and condition (#) is satisfied. Since the deformation operations are applied only onto one couple of trees sharing a polygonal region, the other trees remain invariant under the deformation operations β, ρ. Therefore the other $d-2$ roots of P' do not move into different regions. Now applying lemma 5.4, which states that the roots of the derivative lie in the convex hull of the roots of the polynomial, onto the degree 2 polynomial whose roots are the valency 4 inner nodes of the trees T_1 and T_2, we obtain that those two roots move along non-homotopically null closed paths, around the root of P' we considered. Therefore the trajectories of those two roots define a generator x_{ij} for the pure colored braid group, intertwining precisely these two roots x_i and x_j.

Note that we are computing the fundamental groupoid based at the generic pieces of the covering, using paths which cross codimension 1 strata only. ☐

2. The σ-sequences

We introduce an invariant of signatures.

Definition 5.1. *Let σ be a signature. To each signature is associated a σ-sequence which is an eight-tuple of positive integers (a, b, c, d, e, f, g, h) enumerating the number of critical values of the polynomials of A_σ respectively in the quadrants A, B, C, D and on the semi-axes $\mathbb{R}^+, \mathbb{R}^-, \imath\mathbb{R}^+, \imath\mathbb{R}^-$. The sum of those integers is $a + b + c + d + e + f + g + h = d$.*

EXAMPLE 5.1. For a degree 4 polynomial $P = (z - \imath 0.78251 - 0.2)(z - \imath 0.78405 + 0.21)(z + \imath 0.7299 - 0.36)(z + \imath 0.6903 + 0.349)$, its σ-sequence is $(0, 0, 0, 1, 2, 0, 0, 0)$. The signature is represented on the figure 3. The polygonal region A starts above the red horizontal curve, on the right of the diagram.

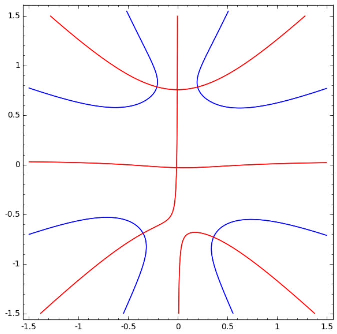


FIGURE 3. Signature of P

In a stratum corresponding to a generic signature there exist polygonal regions shared by k simple trees. A region common to k trees is called a region of multiplicity k.

EXAMPLE 5.2. Using the same example as above, the region D is of multiplicity 2.

PROPOSITION 5.6. *Let σ be a generic signature and P be a polynomial in A_σ. Then we have the following σ-sequences:*

(1) *If the signature is an M diagram, then the critical values of P are spread in one unique region of multiplicity d and the σ-sequences are of the following type: $(d-1, 0, 0, 0, 0, 0, 0, 0)$, $(0, d-1, 0, 0, 0, 0, 0, 0)$, $(0, 0, d-1, 0, 0, 0, 0, 0)$, $(0, 0, 0, d-1, 0, 0, 0, 0)$.*
(2) *If the signature is an F diagram (with one unique F tree), then the critical values of P are spread into at most two distinct quadrants, one of the them containing at least $\lceil \frac{d-1}{2} \rceil$ critical values of P. The σ-sequences associated to this diagram are of type $(m, d-1-m, 0, 0, 0, 0, 0, 0)$, $(0, m, d-1-m, 0, 0, 0, 0, 0)$, $(0, 0, m, d-1-m, 0, 0, 0, 0)$, $(d-1-m, 0, 0, m, 0, 0, 0, 0)$ where $0 \leq m \leq d-1$.*
(3) *If the signature is an S diagram (with one narrow S tree). Then, the critical values of P are spread among the four distinct quadrants: $(m_1, m_2, m_3, d-1-(m_1+m_2+m_3), 0, 0, 0, 0)$, where $1 \leq \{m_1, m_2, m_3\} \leq d-1$.*

PROOF. Recall that the map $P : \mathbb{C} \to \mathbb{C}$ is a holomorphic branched cover of degree d.

(1) If the signature of the polynomial is an M diagram then there exists one unique region of multiplicity d fo color A, B, C or D. It follows that the

roots of P' lie in this region of multiplicity d, giving the σ-sequences as in the statement.

(2) If the signature of the polynomial is an F diagram then F tree separates the diagram into two regions of multiplicity strictly greater than one. Using the lemmas above, these two regions contain the roots of P'. One region contains $1 \leq m \leq d-2$ simple trees (with short diagonals) and a second region constituted of $d-1-m$ other simple trees (with short diagonals). The roots of P' will be spread among the following couples of regions: $\{A,B\}$; $\{B,C\}$; $\{C,D\}$ and $\{D,A\}$. Indeed, if the long diagonal of the F tree is blue then the couple of adjacent regions to this long blue diagonal is colored $\{A,B\}$ or $\{C,D\}$. If the long diagonal of the F tree is red, the the couple of adjacent regions to this long red diagonal is colored $\{A,D\}$ or $\{B,C\}$. The number of roots of P' spread in each of the two compartments is respectively m and $d-1-m$.

(3) If the signature of the polynomial is an S diagram, where S is narrow then the roots of P' are spread into two regions from A,B,C,D which are $\{A,C\}$ and $\{B,D\}$. Indeed, if S is narrow then there exist two opposite regions which are of multiplicity 1. Therefore, the result follows.

□

Lemma 5.7. *Let P be a biregular polynomial of signature $F^{\otimes d-2}$ (resp. $S^{\otimes d-2}$) then there exist $d-1$ regions of multiplicity 2. The σ-sequences are one of the following:*

(1) $(\lfloor \frac{d-1}{2} \rfloor, \lceil \frac{d-1}{2} \rceil, 0,0,0,0,0,0)$,
(2) $(0, \lfloor \frac{d-1}{2} \rfloor, \lceil \frac{d-1}{2} \rceil, 0, 00, 0, 0)$,
(3) $(0,0, \lfloor \frac{d-1}{2} \rfloor, \lceil \frac{d-1}{2} \rceil, 0,0,0,0)$,
(4) $(\lceil \frac{d-1}{2} \rceil, 0,0, \lfloor \frac{d-1}{2} \rfloor, 0,0,0,0)$.

The σ-sequence for the signature $S^{\otimes d-2}$ is one of the following:

(1) $(\lfloor \frac{d-1}{2} \rfloor, 0, \lceil \frac{d-1}{2} \rceil, 0,0,0,0,0)$,
(2) $(0, \lfloor \frac{d-1}{2} \rfloor, 0,0, \lceil \frac{d-1}{2} \rceil, 0,0,0)$,
(3) $(\lceil \frac{d-1}{2} \rceil, 0, \lfloor \frac{d-1}{2} \rfloor, 0,0,0,0,0)$,
(4) $(0, \lceil \frac{d-1}{2} \rceil, 0, \lfloor \frac{d-1}{2} \rfloor, 0,0,0,0)$

Proof. The signature is constituted from $d-2$ simple F trees which divide the diagram into $d-1$ regions of multiplicity 2. Indeed, each such region is in the complement of two trees. The long diagonal of an F tree is adjacent to two adjacent regions colored $\{A,B\},\{C,D\}$ (if the long diagonal is blue) $\{B,C\},\{D,A\}$ (if the long diagonal is red). If d is odd then the number of roots of P' in both regions are equal. If d is even then the number of critical points in the regions are not equal. Reciprocally, for the signature $S^{\otimes d-2}$. There exist $d-1$ complementary regions which are colored respectively $\{A,C\}$ or $\{B,D\}$ and we apply the same argument as before. □

Part 3

Method for the computation of the Čech cohomology groups

CHAPTER 6

Čech cover indexed by signatures

Previously, the existence of a partition of ${}^{\mathrm{D}}\mathrm{Pol}_d$ indexed by signatures has been shown. In this part we prove that this partition is the basis, for a new *good* cover of ${}^{\mathrm{D}}\mathrm{Pol}_d$, in the sense of Čech; that is a covering by open subsets such that multiple intersections (including the pieces themselves) are contractible. On the one hand, we show that this new cover is a (semi-algebraic) stratification, using the double barycentric subdivision and the triangulation theorem of Łojasiewicz. On the other hand we give a sketch of proof the acyclicity condition. In this new cover the adjacence and incidence relations of the decomposition are preserved. As for the incidence relations they can now be read off the nerve of the covering.

1. Stratification of ${}^{\mathrm{D}}\mathbf{Pol}_d$ by signatures

1.1. The cell decomposition. The decomposition by signatures has the property that each stratum is contractible. This is proved in the first statement below.

THEOREM 6.1. [**1**] *Let σ be a signature in Σ_d. Then the set*

$$A_\sigma := \{P \in {}^{\mathrm{D}}Pol_d \mid \sigma(P) = \sigma\}$$

is contractible.

PROOF. If we equip the isotopy class Σ of graphs satisfying the 7 properties of theorem 2.3, with the topology induced by the topology of the arc-length parametrizations of the edges, then it is contractible by Theorems of Baer [**4**] [**5**] and Epstein [**31**]. For a given graph $\mathcal{C}_P$ in the isotopy class of these graphs Σ, the space $E_{\mathcal{C}_P}$ of functions $\Phi : \mathbb{C} \to \mathbb{C}$ is contractible, with Grothendieck's smooth topology. The space E_Σ of pairs $(\Phi, \mathcal{C}_P)$ with $\mathcal{C}_P \in \Sigma$ and $\Phi \in E_{\mathcal{C}_P}$ is by the fibration Theorem of J. Cerf the total space of a fiber bundle $\pi : E_\Sigma \to \Sigma\, (\Phi, \mathcal{C}_P) \mapsto \mathcal{C}_P$. It follows that the total space E_Σ is contractible by J.Cerf's theorem on the space of embeddings and jets [**11**]. The group $G_{\mathbb{C},\infty}$ of orientation preserving diffeomorphisms of $\mathbb{C}$ that extend to $\mathbb{C} \cup \{\infty\}$ as a diffeomorphism with the identity as differential at ∞ is contractible (Theorem of Eells [**30**]). The group $G_{\mathbb{C},\infty}$ acts without fix points on E_Σ, so the space $E_\Sigma / G_{\mathbb{C},\infty}$ is contractible [**10**]. By the Riemann mapping Theorem, there exists in every $G_{\mathbb{C},\infty}$-orbit a unique pair $(\Phi, \mathcal{C}_P)$ such that the pull back by Φ of the standard conformal structure on $\mathbb{C}$ is again the standard structure. In order to achieve unicity of the pair $(\Phi, \mathcal{C}_P)$ we require moreover that Φ, now by Rouché's Theorem a monic polynomial, is balanced. It follows that the space of monic and balanced polynomials with picture in Σ is a space of representatives for the quotient $E_\Sigma/G_{\mathbb{C},\infty}$. We conclude that the space of monic balanced polynomial mappings P with picture in the isotopy class Σ is contractible. The group of Tschirnhausen substitutions is contractible and acts fixed point free on the space of monic polynomial mappings P with picture in the isotopy class Σ. Hence, the space of monic polynomial mappings P with picture in the isotopy class Σ is contractible too. □

THEOREM 6.2. *The degree d signatures determine a stratification of* $^{\mathrm{D}}Pol_d$ *in which a signature of dimension k corresponds to a dimension k cell of* $^{\mathrm{D}}Pol_d$.

PROOF. A signature corresponds uniquely to a picture, a picture describes a region of polynomials (i.e. the set of polynomials having the same picture) and the statement from theorem 6.1 shows that these regions are contractible. □

1.2. Stars of strata.

• Recall that we write $\sigma \prec \tau$ when τ can be obtained from σ by a sequence of repeated half-Whitehead moves i.e. when τ is incident to σ. We call the union $\overline{\sigma} = \{\tau : \sigma \prec \tau\}$ the combinatorial closure of σ, and let $A_{\overline{\sigma}} = \cup_{\sigma \prec \tau} A_\tau$.

LEMMA 6.3. *Let τ be a signature with a given intersection of two red (or blue) diagonals. There are exactly two ways to smooth the intersection, which give two non-isotopic signatures σ_1 and σ_2 such that τ is incident to both σ_1 and σ_2.*

PROOF. Consider a couple of red (resp. blue) curves intersecting at one point with end-vertices $(i,k)(j,l)$ where $i < j < k < l$ on the circle (c.f. definition 2.15). The smoothing induced by the Whitehead move applied onto this couple of curves gives in a neighbourhood of this specific point either a couple of horizontal bands or a couple of vertical bands, which are attached to the boundary of the disc. This gives new diagonals $(i,j)(k,l)$ or $(i,l)(k,j)$ in a cell R. We recall that there does not exist an isotopy of the disc verifying definition 2.7 and mapping $(i,j)(k,l)$ to $(i,l)(k,j)$. Applying lemma 2.7 to the region R we obtain that σ_1 to σ_2 are two non-isotopic signatures. □

Similarly, we have the following for contracting half-Whitehead moves.

LEMMA 6.4. *Let τ_1 and τ_2 be obtained from a signature σ by two different half-Whitehead moves. Then τ_1 and τ_2 are different.*

PROOF. Let τ_1 and τ_2 be obtained from a signature σ by two different half-Whitehead moves and suppose by contradiction that τ_1 and τ_2 are the same signature. Consider two different couples of disjoint diagonals (i,j) and (k,l) (resp. (m,n) and (p,q)) sharing a polygonal region with the labels of the end vertices on the boundary of the polygonal region verifying $i < j < k < l$ and $m < n < p < q$.

We require that at least three of the four diagonals are distinct. Then one half-Whitehead move acting on σ deforms the couple of curves (i,j) and (k,l) into $(i,k)(j,l)$ and the second half-Whitehead move acting on σ deforms the couple of curves (m,n) and (p,q) into $(m,p)(n,q)$. There does not exist any isotopy verifying the definition 2.7. We obtain that these signatures define two non-isotopic signatures. □

Recall that the smoothings of an intersection were defined in definition 2.18, and can be of two types: the first separates the intersections of curves into more than one connected component and the second type does not.

LEMMA 6.5. *Let τ be a signature with a given red (or blue) intersection of $m > 2$ curves. The smoothings of τ obtained by smoothing a single half-Whitehead move, i.e. the smoothings which are of dimension 1 greater than τ, are all distinct.*

PROOF. Locally, around the intersection, the graph resembles a star shaped tree with one inner node and $2m$ leaves which we can number $1, .., 2m$. The possible smoothings are as follows. Of the first type, we can separate $i, i+1 \mod m$ from the tree as a separate diagonal; the resulting graphs are non-isotopic. Of the second type, we add an inner edge separating the leaves into two disjoint groups ; these are clearly non-isotopic, nor are they isotopic to the previous ones as they are locally connected.

□

2. $^{\mathrm{D}}\mathbf{Pol}_d$ as a covering of a non-compact stratified space

2.1. Critical values. Consider the space $V_d = (\mathbb{C}^{d-1} \setminus 0)/S_{d-1}$ where S_{d-1} is the group of permutations. If X denotes an equivalence class of points in $\mathbb{C}^{d-1}$, we can associate a unique σ-sequence (a,b,c,d,e,f,g,h) of positive integers to X enumerating the number of points in X in the quadrants A, B, C, D and on the semi-axes as in the definition 5.1 in chapter 5. The set of points X in V_d having a given σ-sequence (a,b,c,d,e,f,g,h) forms a polygonal cell in V_d isomorphic to

$$(1)\quad A^a/S_a \times B^b/S_b \times C^c/S_c \times D^d/S_d \times (\mathbb{R}^+)^e/S_e \times (\mathbb{R}^-)/S_f \times (\imath\mathbb{R}^+)^g/S_g \times (\imath\mathbb{R}^-)^h/S_h.$$

The real dimension of this cell is equal to $2(a+b+c+d)+(e+f+g+h)$. The cells are disjoint and thus form a stratification of V_d.

DEFINITION 6.1. *A subset V inside $\mathbb{C}^{d-1}/S_{d-1}$ for $d > 2$ is said to be a* $\mathbf{non-compact\ \ stratification}$ *if it is equipped with a stratification by a finite number of open cells of varying dimensions having the following properties:*

- *the relative closure of a k-dimensional cell of V is a union of cells in the stratification.*
- *the relative closure of a k-dimensional cell of V is a "semi-closed" polytope, i.e. the union of the interior of a closed polytope in $\mathbb{C}^{d-1}/S_{d-1}$ with a subset of its faces.*

LEMMA 6.6. *Let $d > 2$, and let $\mathcal{V}_d$ denote the space V_d equipped with the stratification by σ-sequences. Then $\mathcal{V}_d$ is a non-compact stratification.*

PROOF. The closure of the region of $\mathcal{V}_d$ described by 1 is given by

$$\overline{A}^a/S_a \times \overline{B}^b/S_b \times \overline{C}^c/S_c \times \overline{D}^d/S_d \times (\mathbb{R}^+)^e/S_e \times (\mathbb{R}^-)/S_f \times (\imath\mathbb{R}^+)^g/S_g \times (\imath\mathbb{R}^-)^h/S_h,$$

where $\overline{A}$ denotes the closure in $V_1 = \mathbb{C} \setminus 0$ of the quadrant A, namely the union of A with $\mathbb{R}^+$ and $\imath\mathbb{R}^+$, and similarly for $\overline{B}, \overline{C}, \overline{D}$. The direct product of semi-closed polytopes is again a semi-closed polytope, as is the quotient of a semi-closed polytope by a sub-group of its symmetry group. □

THEOREM 6.7. *The map ν that sends a polynomial in ${}^{\mathrm{D}}Pol_d$ to its critical values realizes ${}^{\mathrm{D}}Pol_d$ as a finite ramified cover of $\mathcal{V}_d$.*

PROOF. The image of ν contains only unordered tuples of $d-1$ complex numbers different from zero, since a polynomial can have 0 as a critical value if and only if it has multiple roots. Therefore the image of ν lies in $\mathcal{V}_d$. To show that ν is surjective, we use a theorem of R. Thom [**?**] (1963), stating that given $d-1$ complex critical values, there exists a complex polynomial P of degree d such that $P(r_i) = v_i$ for $1 \le i \le d-1$ where the r_i are the critical points of P, and $P(0) = 0$. To find a Tschirnhausen representative polynomial of ${}^{\mathrm{D}}\mathrm{Pol}_d$ having the same property it suffices to take $P(z - \frac{a_{d-1}}{d-1})$ where a_{d-1} is the coefficient of z^{d-1} in P. By a result of J. Mycielski [**47**], the map ν is a finite ramified cover, of degree $\frac{d^{d-1}}{d-1}$, see [**47**]. □

2.2. The cases $d = 2, 3, 4$. The exact nature of the ramified cover ${}^{\mathrm{D}}\mathrm{Pol}_d \to \mathcal{V}_d$ is complicated and interesting, especially in terms of describing the ramification using the signatures. In this section, we work out full details in the small dimensional cases, and for generic strata.

Let $d = 2$. The spaces ${}^{\mathrm{D}}\mathrm{Pol}_2$ and $\mathcal{V}_2$ are one-dimensional. The space $\mathcal{V}_2$ is $\mathbb{C} \setminus 0$ equipped with the stratification given by the four quadrants A, B, C, D and the four semi-axes. The only Tschirnhausen polynomial of degree 2 having given critical value v is $z^2 + v$, therefore the covering map ν is unramified of degree 1, an isomorphism. The four signatures corresponding to the strata of real dimensional 2 and the four corresponding to the one dimensional strata are illustrated in figure 3.

Let $d = 3$. In this case the covering map ν is of degree 3: explicitly, if $P(z) = z^3 + az + b$ has critical values v_1 and v_2 then so do the polynomials $P(\zeta z)$ and $P(\zeta^2 z)$ where $\zeta^3 = 1$. The 10 open cells of real dimension 4 in $\mathcal{V}_3$ are given by:

$$A \times A/S_2, \quad B \times B/S_2, \quad C \times C/S_2, D \times D/S_2, A \times B, A \times C, A \times D, B \times C, B \times D, C \times D.$$

Ramification occurs only above the first four; in fact exactly when the two critical values are equal, i.e. $P(z) = z^3 + b$. Thus above each of the first four cells, there is only one stratum, corresponding to the four rotations of the leftmost signature (see figure 1). In contrast, there are three distinct signatures above each of the remaining 6 cells. The six signatures in the middle of the figure 1 form two orbits under the $\frac{2\pi}{3}$ rotation which li above the cells, $A \times C$ and $B \times D$, whereas the twelve signatures on the right form four orbits lying ove the cells $A \times B; B \times C; C \times D; D \times A$.

Let $d = 4$. The degree of the covering map ν is 16. There are 20 generic cells in $\mathcal{V}_4$. Four generic cells correspond to taking three critical values in three different quadrants. There is no ramification above these cells; each of these have 16 distinct strata in the preimage of ν, corresponding to four rotations each of the fifth, sixth, eighth and eleventh signatures in the figure below.

Four more cells of $\mathcal{V}_4$ correspond to taking three critical values in the same quadrant. Only one stratum of $^{\mathrm{D}}\mathrm{Pol}_4$ lies above each of these cells, namely, the first signature in the figure below, with ramification of order sixteen. The remaining twelve cells correspond to two critical points in one quadrant and the third in a different quadrant. When the quadrants are adjacent, only six cells lie above the corresponding regions of $\mathcal{V}_4$. For example over the region A, A, B lie the two rotations of the second signature in the figure below with ramification of order 2, and the four rotations of the tenth signature each of ramification of order 3. The situation is analogous when the quadrants are opposed. For example over the region A, A, C, there are the two rotations of the third figure (below) each with ramification of order 2, and the four rotations of the ninth signature each of ramification order 3. the second signature in the figure 2.2 with ramification of order 2, and the third and seventh signatures

3. The good cover of $^{\mathrm{D}}\mathrm{Pol}_d$

In this key section we show how to use the stratification of $^{\mathrm{D}}\mathrm{Pol}_d$ to construct a good topological cover of $^{\mathrm{D}}\mathrm{Pol}_d$, suitable for calculating cohomology groups, which is the subject of the following chapter. We proceed in three steps. We will use theorem 6.1 above, stating that the strata are contractible. We will show that the topological closure of A_σ is given by the combinatorial closure of σ, and that it remains contractible. Next, we prove that finite intersections of generic strata are either contractible or empty. Finally, we will obtain the open sets of the good cover by a thickening the generic strata.

3.1. Closures of generic strata.

LEMMA 6.8. *Let τ be a non-generic signature and let A_τ be the corresponding strata of $^{\mathrm{D}}Pol_d$. Let $P_0 \in A_\tau$. Then for every generic signature σ such that τ is incident to σ, every $2d$-dimensional open ball B_ϵ containing P_0 intersects the generic stratum A_σ.*

PROOF. Suppose that there exists a generic signature σ which does not verify that B_ϵ containing P_0 intersects the generic stratum A_σ. Then there does not exist any path from a point $y \in A_\tau$ to $x \in A_\sigma$. Therefore A_σ and A_τ are disjoint. In

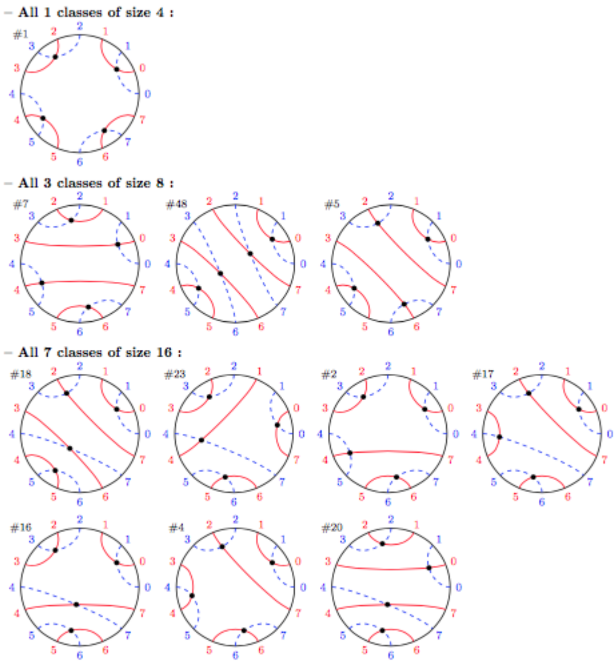


FIGURE 1. Generic signatures $d = 4$

particular this impies that A_τ is not in the closure of A_σ. So, τ is not incident to σ.

□

LEMMA 6.9. *Let σ be a generic signature and let $\tau \neq \sigma$. Then τ is incident to σ if and only if the following holds: for every pair of points $x, y \in {}^{\mathrm{D}}Pol_d$ with $x \in A_\sigma$ and $y \in A_\tau$, there exists a continuous path $\gamma : [0, 1] \to DPol_d$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(t) \in A_\sigma$ for all $t \in [0, 1)$. Any other such path ρ from a point $x' \in A_\sigma$ to a point $y' \in A_\tau$ is homotopic to γ.*

PROOF. The result follows from Lemma 6.8. Indeed, if τ is incident to σ then by Lemma 1, a ball around $y \in A_\tau$ necessarily intersects A_σ and therefore there is a path from y to a point $x' \in A_\sigma$. Composing this with a path from x' to x in A_σ we obtain γ. Conversely, if there is a path γ from $x \in A_\sigma$ to $y \in A_\tau$, then it is impossible to have an open ball containing y that does not intersect A_σ. □

We return to the original polynomials in order to define half-Whitehead moves in an analytic and then topological fashion.

Let $P \in DPol_{d+1}$; explicitly : $P(z) = z^{d+1} + a_1 z^{d-1} + \ldots + a_d$.

We denote the critical points by $\underline{z} = (z_1, \ldots, z_d)$ ($P'(z_i) = 0$) and the critical values by $\underline{w} = (w_1, \ldots, w_d)$, so that for any i there is a j such that $P(z_i) = w_j$.

There are no *a priori* constraints on the z_i's and w_j's other than $w_j \neq 0$ (for all j) because the roots of P are assumed to be simple.

Let $\mathbb{C}^d_w$ denote the (affine) space of the $\underline{w}$'s recall that there is a ramified cover

$$\pi_w : DPol_{d+1} \to \mathbb{C}^d_w$$

of degree $(d+1)^d$. We don't know much about the ramification locus (other than the fact that it is of pure codimension 1).

Let $\mathbb{C}^d_z$ denote the affine space of the critical points $\underline{z}$ and let $p : \mathbb{C}_z \to \mathbb{C}_w$ denote the natural map given by P ($P(z) = w$). Denote by $c(w_k)$ (or just $c(k)$) the number of distinct critical points above w_k and by $\underline{m} = (m_1, \ldots, m_{c(k)})$ their multiplicities ($m_i > 1$). Generically, above a point w there is just one critical point of multiplicity 2 and $d-1$ simple (non critical) points, i.e. $c(w) = 1$, $m_1 = 2$. Such a point we call a simple critical point (recall that Hurwitz investigated simple covers i.e. covers with only simple critical points and that these are still useful and much studied objects).

Now return to a given polynomial P and its critical values $\underline{w}$. It is "generic" if $w_j^2 \notin \mathbb{R}$ for all j. Assume it is *not* generic and let w_0 ($= w_j$ for some j) be such that $w_0^2 \in \mathbb{R}$; purely for notational simplicity assume that in fact w_0 is real (rather than pure imaginary). Let $c = c(w_0)$ be the number of distinct critical points above w_0, with multiplicities $\underline{m} = (m_1, \ldots, m_c)$ and denote these points $(z_1, \ldots, z_c)$ (here the z_i's are distinct).

We now wish to smooth out the critical value w_0 in a "universal" way. It is actually more natural to work "upstairs" since that means nothing but smoothing out every critical point above w_0. So choose one of the z_i ($1 \neq i \neq c$), and call it z_0, with multiplicity $m = m_0$. We now need a result which says that the universal expansion is obtained by "plugging" the generic polynomial of degree m in a small neighborhood of z_0. This means the following. First this is shown via holomorphic surgery and is thus *not* explicit ; that is we cannot explicitly write down a universal family in terms of the coefficients a_i of the polynomial P. But we do know that there exists such a local universal family and that it is biholomorphically equivalent to the one obtained by completing the polynomial $(z - z_0)^m$ into the generic polynomial in $(z - z_0)$ of degree m near z_0. In other words let $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$ be complex numbers with $|\varepsilon_i| < \varepsilon << 1$ for all i (with some $\varepsilon > 0$) and let

$$p_\varepsilon(z) = (z - z_0)^m + \varepsilon_1 (z - z_0)^{m-1} + \ldots + \varepsilon_m \, .$$

Then there is a biholomorphic map between the set of the $\underline{\varepsilon}$'s (i.e. $(0, \varepsilon)^m$), or equivalently the family p_ε, and a universal family $P_\varepsilon(z)$ with $P_0 = P$ (where $0 = (0, \ldots, 0)$). Here we ignored the normalization which says that the sum of the roots of $P_\varepsilon(z)$ vanishes and this has to be taken into account later.

Consider the generic case of a simple critical value $w_0 : c(w_0) = 1$, $m = 2$. Then $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, $p_\varepsilon(z) = (z - z_0)^2 + \varepsilon_1 (z - z_0) + \varepsilon_2$. However there is in fact only one deformation parameter, again because the sum of the roots of

$P_\varepsilon(z)$ vanishes. By varying $\underline{\varepsilon}$ let the critical point vary into $z_0(\varepsilon)$ (again ε is 1-dimensional), leading to a deformation $w_0(\varepsilon)$. The lemma says that $w_0(\varepsilon)$ has non zero derivative at $\varepsilon = 0$. By the implicit function theorem $w_0(\varepsilon)$ covers a neighborhood of w_0 in the w-plane and a half-whitehead move is nothing but the results of what happens when $w_0(\varepsilon)$ moves into the upper or lower half-plane.

When $m > 2$ it becomes richer and more complicated, one important point being that the space of deformation parameters, which looks like $(0, \varepsilon)^{m-1}$ (due to the normalization condition) is contractible, so that all the smoothings are so to speak equivalent.

LEMMA 6.10. *A polynomial* $P : \mathbb{C} \to \mathbb{C}; z \mapsto P = z^d + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0 \quad d \in \mathbb{N},\ a_i \in \mathbb{C}$ *is homotopic to* $f : z \mapsto z^d$, *in the space of unitary polynomial maps.*

PROOF. Explicitly, the homotopy is given by:

$$h(z,\ \tau) = z^d + (1-\tau)(a_{d-1}z^{d-1} + \cdots + a_1 z + a_0) \text{ for } z \text{ finite,}$$
$$h(\infty, \tau) = \infty.$$

for all finite τ and z, h is continuous and moreover $\lim_{z\to\infty} h(z,\tau) = \infty$ for all τ. Therefore, h is continuous on $\mathbb{P}^1 \times [0,1]$. □

LEMMA 6.11. *A contracting half-Whitehead move which takes a signature* σ *to an incident signature* τ *induces a retraction of the stratum* A_σ *onto the boundary stratum* A_τ.

PROOF. From the above discussion it follows that we can construct the retraction locally near A_τ; then we extend it to the whole of A_σ using the contractibility of this stratum. □

PROPOSITION 6.12. *If* σ *is a signature, the closure* $\overline{A}_\sigma$ *of the stratum* A_σ *in* ${}^{\mathrm{D}}Pol_d$ *is given by*

$$\overline{A}_\sigma = \cup_{\tau\in\overline{\sigma}} A_\tau,$$

where the boundary of σ *denoted* $\overline{\sigma}$ *consists of all incident signatures* τ.

PROOF. One direction is easy. Indeed, if $x \in A_\tau$ where τ is incident to σ then every $2d$-dimensional open set containing x must intersect A_σ, so $\cup_{\tau\in\overline{\sigma}} A_\tau \subset \overline{A}_\sigma$.

For the other direction, let $x \in \overline{A}_\sigma \setminus A_\sigma$ and let τ be the signature of x. We first note that the dimension of τ can not be equal to the dimension of σ because if they were equal, A_τ would be an open stratum disjoint from A_σ.

Therefore the dimension of τ is less than the dimension of σ. Let U be any small open neighborhood of x. Let $y \in U \cap A_\sigma$ and let γ be a path from y to x such that $\gamma \setminus x \subset A_\sigma$. Then every point $z \in \gamma \setminus x$ has the same signature σ. Using theorem 6.7, any path from the interior to any point not in A_σ must pass through the boundary of the polytope. Therefore any sequence of half-Whitehead moves and smoothings from σ to τ must begin with half-Whitehead moves bringing σ to a signature that is incident to σ. But x is the first point on γ where the signature changes and therefore τ must be incident to σ. □

LEMMA 6.13. *The set of signatures incident to a generic signature σ is equal to the set of signatures τ such that*

(i) $P_r^\sigma \subset P_r^\tau$ and $P_b^\sigma \subset P_b^\tau$ (with at least one of the containments being strict), and

(ii) the set of paths (i,j) in τ for the pairs in P_r^σ (resp. those for (k,l) in P_v^σ) intersect only at isolated points (no shared segments).

PROOF. Performing a half-Whitehead move on a signature can never eliminate a pair but can only add pairs, which shows that if τ is incident to σ then (i) holds. Furthermore, half-Whitehead moves cause the arcs of σ to cross only at isolated points.

Conversely, suppose that (i) and (ii) hold for τ. Consider the red forest of τ. We claim that the set P_r^σ provides a recipe for smoothing the red forest of τ to obtain the red forest σ (and subsequently the blue using P_b^σ), as follows.

Let us use the term "short arcs" for arcs joining two neighboring (even for red, odd for blue) points on the circle. We start with the short arcs, joining pairs of the form $(i,i+2)$ in P_r^σ. Because the paths cross only at points, smoothing the path $(i,i+2)$ by separating it off from the rest of the red forest of τ eliminates only paths from i or $i+2$ to other points; paths between all the other pairs of P_r^τ remain. We then erase the arcs $(i,i+2)$ from the signature and now treat the "new" short arcs (those from P_r^σ joining points j, $j+6$) separating them from the rest of the tree, then erase those and continue in the same way. The final result reduces the red forest of P_r^τ entirely to a set of d disjoint pairs, endpoints of red arcs, which is equal to P_r^σ since they all belong to P_r^σ. This proves that any signature τ satisfying (i) and (ii) can be smoothed to σ. □

THEOREM 6.14. *Let σ be a generic signature. Then, the topological closure $\overline{A}_\sigma$ of the stratum A_σ is contractible.*

PROOF. We know from the Theorem 6.2 that A_σ is a cell, so homeomorphic to the k dimensional open ball. Adding the disjoint incident strata of A_σ (of codimension 1) is thus equivalent to glueing a finite number of disjoint open subsets of the boundary of the open ball, which remains contractible. Adding the codimension 2 strata is homeomorphic to adding disjoint open subsets of the boundaries of the open subsets added in the previous step, which again remains contractible, and the same holds until all strata have been added. Altogether we see that the closure $\overline{A}_\sigma$ is homeomorphic to the open ball together with a subset of its boundary, which is contractible. □

This proof easily adapts to give the following result.

COROLLARY 6.15. *Let σ be any signature and let $\tilde{\sigma}$ be a subset of $\overline{\sigma}$. Let $A_{\tilde{\sigma}}$ be the union of A_ρ with ρ in $\tilde{\sigma}$. Then, $A_{\tilde{\sigma}}$ is contractible.*

3.2. The acyclicity condition. In what follows we show that multiple intersections of closures of strata are contractible or empty; this is in fact stronger than the acyclicity condition.

For this purpose as in chapter 2, we replace signatures by diagrams consisting of a unit disk $\mathbb{D}$ with 4d points on the boundary corresponding to the roots of unity $\sqrt{1}^{4d}$ replacing the asymptotic rays. By a slight abuse of notation, we continue to write σ_i for the diagram associated to the signature σ_i.

The idea is to superimpose diagrams $\sigma_0, ..., \sigma_p$ such that terminal vertices $\sqrt{1}^{4d}$ coincide and to apply contractions on the non identical diagonals, as follows.

Let $\sigma_0, \ldots, \sigma_p$ be generic signatures and let Θ denote their superimposition. This superimposition is not well-defined as the diagonals of the different signatures can be positioned differently with respect to each other, since the diagonals of each signature are given only up to isotopy, but we will consider only those having the following properties

(1) all intersections are crossings (but not tangents) of at most two diagonals,
(2) the superimposition Θ cuts the disk into polygonal regions; we require that no region is a bigon,
(3) the number of crossings of a given diagonal with other diagonals of Θ must be minimal, with respect to a possible isotopy. We take representatives of isotopy classes of arcs.

We call such superimpositions *admissible*.

LEMMA 6.16. *Let $\sigma_0, \ldots, \sigma_p$ be generic signatures. If there exists no superimposition Θ with the property that no diagonal has more than $p+1$ intersections with diagonals of the opposite color then there is no signature τ incident to all the σ_i.*

PROOF. The key point is the following. If $\sigma_0, ..., \sigma_p$ admit a signature τ incident to all of them, then τ has the following property: every segment of the tree τ (a segment is the part of an edge contained between two vertices, including terminal vertices) belongs to at most one diagonal (i, j) of each σ_i. Thus, in particular, each segment can be considered as belonging to at most $p+1$ diagonals, one from each σ_i. Thus if a red diagonal of τ crosses $p+2$ or more blue diagonals in the superimposition, there is no one segment of τ which can belong to all of them, so the red diagonal will necessarily cross more than one blue segment of τ, which is impossible. □

We will say that the set of signatures $\sigma_0, \ldots, \sigma_p$ is *compatible* if it admits an admissible superimposition Θ with the property that no diagonal crosses more than $p+1$ diagonals of the opposite color. (Note that a red diagonal (i, j) can never cross a blue diagonal (i', j') in more than one point.) Compatible sets of generic signatures may potentially have non-empty intersection. We will now show how to give a condition on Θ to see whether or not this is the case.

Graph associated to an intersection of generic signatures.

Let $\sigma_0, \ldots, \sigma_p$ be a set of compatible generic signatures and let Θ be an admissible superimposition. Then Θ cuts the disk into polygonal regions. Color a region red (resp. blue) if all its edges are red (resp. blue); the intersecting regions are purple. Construct a graph from Θ as follows : place a vertex in each red or blue region (but not purple) with number of sides greater than three. If two vertices lie in blue (resp. red) polygons that meet at a point, join them with a blue (resp. red) edge (even if this edge crosses purple regions). If two vertices lie in blue (resp. red) polygons that intersect along an edge of the opposite color, connect them with a blue (resp. red) edge. If two vertices lying in the same red (resp. blue) polygon can be connected by a segment inside the polygon which crosses only one purple region, add this segment. Connect each vertex to every terminal vertex lying in the same red (resp. blue) region, and also to any terminal vertex which can be reached by staying within the original red (resp. blue) polygon but crossing through a purple region formed by two blue (resp. red) diagonals emerging from that terminal vertex. Finally, if any vertex of the graph has valency 2, we ignore this vertex and consider the two emerging edges as forming a single edge. We call this graph the graph associated to the superimposition.

LEMMA 6.17. *The graph associated to Θ is independent of the actual choice of admissible superimposition Θ.*

PROOF. Let Θ and Θ' be admissible superimpositions, and consider a given diagonal (i, j). By the admissibility conditions the number of crossings of (i, j) with diagonals of the other color is equal in Θ and Θ', and in fact the set of diagonals of the other color crossed by (i, j) is identical in Θ and Θ' . Therefore, the only possible modifications of the Θ is to move (i, j) across an intersection of two diagonals of the other color. But this does not change the associated graph. $\square$

DEFINITION 6.2. *The canonical graph associated to a set of compatible signatures $\sigma_0, \ldots, \sigma_p$ is the graph associated to any admissible superimposition Θ.*

THEOREM 6.18. *Let $\sigma_0, \ldots, \sigma_p$ be generic signatures. Then there exists a signature incident to all the σ_i if and only if the set $\sigma_0, \ldots, \sigma_p$ is compatible and associated canonical graph is a signature.*

PROOF. Replacing a blue (or red) polygon by a graph having the shape of a star as in the construction above involves diagonals of the different σ_i which must be identified if we want to construct a common incident signature. The contracting moves may be stronger than strictly necessary (i.e. the signature τ may not be the signature of maximal dimension in the intersection), but any signature in the intersection must either have the same connected components as τ, i.e. be obtained from τ by applying only smoothing half-Whitehead moves which do not increase the number of connected components of τ (we call these smoothings *connected smoothings*), or lie in the closures of these. Thus, up to such smoothings, the moves constructing τ are necessary in order to identify the diagonals of the σ_i.

The contracting moves in the construction of the graph associated to Θ, restricted to just one of the signatures σ_i, has the effect of making a contracting half-Whitehead move on the blue (resp. red) curves of this signature. Thus, on each of the signatures, the graph construction reduces to a sequence of contracting half-Whitehead moves. Thus the σ_i possess a common incident signature if and only if τ is such a signature. □

THEOREM 6.19. *Let $\overline{A_{\sigma_0}}, ..., \overline{A_{\sigma_p}}$ be closures of strata in ${}^{\mathrm{D}}Pol_d$ corresponding to generic signatures $\sigma_0, \ldots, \sigma_p$, and assume that these admit a non empty intersection $\mathcal{I}$. Then the signatures σ_i are compatible, and admit a canonical signature τ, and $\mathcal{I}$ is given by the union of the closure $\overline{A_\tau}$ of the stratum A_τ and the closures $\overline{A_\rho}$ of strata A_ρ where ρ is obtained from τ by certain connected smoothings. The intersection $\mathcal{I}$ is contractible.*

PROOF. Since $\mathcal{I}$ is non-empty, by Theorem 6.18 the σ_i are compatible and their canonical graph is a signature τ. Therefore, the closure of A_τ lies in $\mathcal{I}$.As we saw in the first part of the proof of Theorem 6.18, any signature ρ in the intersection must either have the same connected components as τ, i.e. ρ must be obtained from τ by connected smoothings (but not all possible connected smoothings will give a stratum in the intersection), or be incident to such a signature.

Let B be the union of the strata A_ρ for those signatures ρ which are obtained from τ by connected smoothings and such that A_ρ lies in the intersection $\mathcal{I}$. We will show that B is contractible, and then that the closure $\overline{B}$ is contractible. By the above observation, $\overline{B}$ is equal to $\mathcal{I}$, so this will prove the theorem.

To show that B is contractible we first make the following observation. Let σ any non-generic signature, and C_σ be the union of σ with all signatures obtained from σ by smoothing. Then C_σ is an open neighborhood of A_σ of dimension $2d$, which retracts onto A_σ. By lemma 6.11, this retraction is induced by the contracting half-Whitehead moves on all the strata of C_σ, different from σ.

Let B_σ denote the subset of C_σ, consisting of only the strata obtained from σ by connected smoothings. Then B_σ is contractible because it is a subset of the contractible neighborhood C_σ such that the retraction of C_σ onto A_σ preserves B_σ; thus the restriction of the retraction from C_σ to B_σ remains a retraction.

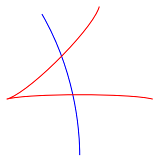
This shows that $B = B_\tau$ is contractible.

Let us show that the closure $\overline{B}$ remains contractible. By Proposition 6.12, the closure consists in the union of B with all strata incident any stratum in B. Observe that if μ is such a stratum, there is a unique stratum ρ in B such that μ is obtained from ρ only by non connected contracting half-Whitehead moves.

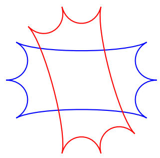
Indeed, connected and non connected contractions commute, so that if μ is obtained from any stratum ρ' by a sequence of contracting half-Whitehead moves, we can perform first all those that are connected, obtaining ρ, and then those that are non-connected. Let $\tilde{\rho}$ denote the union of ρ with all of its incident strata μ obtained by sequences of non-connected half-Whitehead moves. Then, as in corollary 6.15, $\tilde{\rho}$ retracts onto ρ. Thus, $\overline{B}$ retracts onto B, which is contractible by the previous argument. Therefore, $\overline{B}$ is contractible. □

3.3. Superimposition of signatures. Let $\sigma_0 \cup \cdots \cup \sigma_p$ be compatible generic signatures and Θ denote an admissible superimposition. In this subsection, we digress briefly in order to give a visual description of the conditions on the superimposition Θ for the associated graph to be a signature. In fact, it is quite rare for signatures to intersect. Almost always the canonical graph will not be a signature. Given a red polygon and a blue polygon of Θ, they must either be disjoint or intersect in one of exactly four possible ways:

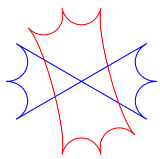
- the intersection is a three sided polygon with two red (resp. blue) edges and one blue edge (resp. red);



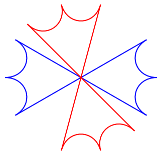
- the intersection is a four sided polygon with two red and two blue edges;



- the intersection is two triangles joined at a point, formed by two crossing blue (resp. red) diagonals, cut transversally on either side of the intersection by two red (resp. blue) diagonals; note that this means that two polygons of the same color meet at a point and both intersect the polygon of the other color;



- the intersection is a point at which two blue diagonals and two red diagonals all cross in the cyclic order red, red, blue, blue, red, red, blue, blue; note that this means that in fact four polygons meet at a point, each being the same color as the opposing one.



From such a disposition of diagonals, we obtain the graph described above.

• The graph must be a forest with even valency at every non-terminal vertex.

3.4. Semi-algebraic stratification. The next paragraph is devoted to show that the stratification is semi-algebraic. This result forms an important step towards the construction of the good cover.

DEFINITION 6.3. *A semi-algebraic subset of $\mathbb{R}^{2d}$ is a subset of the form*

$$\cup_{i=1}^{s} \cap_{j=1}^{r_i} \{x \in \mathbb{R}^{2d} | f_i = 0, g_j > 0\}$$

where $f_i, g_j \in \mathbb{R}[x_1, ..., x_n]$ for $i = 1, ..., s$ and $j = 1, ..., r_i$.

PROPOSITION 6.20. *The space ${}^{\mathrm{D}}Pol_d$ is a semi-algebraic set.*

PROOF. Let $(z_1, .., z_d) \in \mathbb{C}^d$ be the set of roots of one variable complex polynomials. The locus of non distinct roots defines an algebraic variety: $\{(z_1, .., z_d) \in \mathbb{C}^d | z_i = z_j$ for some $i \neq j\}$. So, this is a semi-algebraic set and since the complementary of a semi-algebraic set in the affine space is semi-algebraic then ${}^{\mathrm{D}}\mathrm{Pol}_d$ is a semi-algebraic set. □

THEOREM 6.21. *The decomposition ${}^{\mathrm{D}}Pol_d = \bigcup_{\sigma \in \Sigma_d} A_\sigma$ in $\mathbb{R}^{2d}$ is semi-algebraic.*

PROOF. Let us consider the set of classes of codimension c and of degree d. The codimension c classes verify a system of polynomial equations. Let z_i and z_j be some critical points of the polynomial P. Then, we have: $Re(P)(z_i) = 0$ and/or $Im(P)(z_j) = 0$, where $P'(z_i) = 0, ..., P^{(k_i)}(z_i) = 0, P^{((k+1)_i)}(z_i) \neq 0$ (resp. $P'(z_j) = 0, ..., P^{(n_j)}(z_j) = 0, P^{((n+1)_j)}(z_i) \neq 0$). So, it follows from the definition 6.3 that the set of codimension c classes is a semi-algebraic set. A union of semi-algebraic sets is semi-algebraic. Therefore, the decomposition ${}^{\mathrm{D}}\mathrm{Pol}_d = \bigcup_{\sigma \in \Sigma_d} A_\sigma$ is semi-algebraic. □

From S.Łojasiewicz's theorem 1 on triangulation [**45**], it follows that for a finite collection of semi-algebraic sets of an affine space $\{B_\nu\}_{\nu=1}^{k}$, there exists a finite simplicial complex K in the affine space $\mathbb{R}^{2d}$ and a semi-algebraic homeomorphism $\tau : |K| \to \cup_{\nu=1}^{k} \overline{B}_\nu$, where $|K| = \cup\{\mathrm{s} : \mathrm{s} \in K\}$ such that:

(1) $\forall s \in K, \tau(s)$ is an analytic submanifold of $\mathbb{R}^{2d}$ and $\tau : |K| \to \cup_{\nu=1}^{k} \overline{B}_\nu$ is an analytic isomorphism.
(2) $\forall s \in K$, and for all B_ν, $\tau(s) \subset B_\nu$ or $\tau(s) \subset \mathbb{R}^{2d} \setminus B_\nu$

The semi-algebraic sets $\{B_\nu\}_{\nu=1}^{k}$ in our case are the sets of a fixed codimension. These semi-algebraic sets are not connected: the number of connected components in a codimension c semi-algebraic set is the number of codimension c signatures.

The construction of the good cover is as follows. In the first place, we show theorem 6.21 which states that the decomposition is semi-algebraic. In the second place, we apply the classical result of S.Łojasiewicz on triangulation of semi-algebraic sets (Theorem 1 of [**45**]).

PROPOSITION 6.22. *Let σ be a generic signature, K the simplicial complex above and $\mathcal{K}''$ the simplicial complex obtained using the second barycentric subdivision of K. Let $A^+_\sigma = \overline{A_\sigma} \bigcup_{\mathsf{s}''\in\mathcal{K}''|\mathsf{s}''|\cap\overline{A_\sigma}\neq\emptyset} int(\mathsf{s}'')$ be an open set. Then $\overline{A_\sigma}$ is a deformation retract of A^+_σ.*

PROOF. Consider the dual complex $\mathcal{W}$ and a subcomplex $\mathcal{Z}$ of dimension n in $\mathcal{W}$. Let us consider the set of incident classes to $\mathcal{Z}$.

The open sets of the good cover are defined as follows. Take the second barycentric subdivision $\mathcal{K}''$ of K, and the union of all the simplicial interiors in $\mathcal{K}''$ that have non-empty intersections with the k-faces Z' of the subcomplex $\mathcal{Z}$ (where the dimension $0 \leq k \leq n$). The open sets are defined as follows:

$$U(Z') = \bigcup_{\mathsf{s}''\in\mathcal{K}''|\mathsf{s}''|\cap Z'\neq\emptyset} int(\mathsf{s}''),$$

such that the following conditions hold:

(1) the set $U(Z')$ is an open set of $|\mathcal{K}''|$
(2) the set $U(Z')$ retracts onto Z'.

Using the definition of the dual complex we come back to the stratification $\{A_\sigma\}_{\sigma\in\Sigma}$ of $^{\mathrm{D}}\mathrm{Pol}_d$ we have that $A^+_\sigma = \overline{A_\sigma} \bigcup_{\mathsf{s}''\in\mathcal{K}''|\mathsf{s}''|\cap\overline{A_\sigma}\neq\emptyset} int(\mathsf{s}'')$. □

We call this union of open sets the regular neighborhood in $\mathcal{W}$ of $\mathcal{Z}$. The open sets $A^+_\sigma \subset {}^{\mathrm{D}}\mathrm{Pol}_d$ are such that $\overline{A}_\sigma \subset A^+_\sigma$.

REMARK 6.1. A path exists in $^{\mathrm{D}}\mathrm{Pol}_d$ from A_{σ_1} to A_{σ_n} where $\sigma_1, .., \sigma_n$ are biregular, if there is a finite number of adjacent bi-regular sets A_{σ_i} such that $A^+_{\sigma_i} \cap A^+_{\sigma_{i+1}} \neq \emptyset$ for all $i \in \{1, ..., n\}$.

3.5. A good open cover of $^{\mathrm{D}}\mathbf{Pol}_d$ for the Čech cohomology.

THEOREM 6.23. *There exists an open cover $\mathcal{U} = \{A^+_\sigma\}_{\sigma\in\Sigma_d}$ where σ is generic of $^{\mathrm{D}}Pol_d$ with the desired properties:*

(1) The elements of the cover $A^+_{\sigma_i}$, where σ_i is generic, are open and contractible;
(2) the multiple intersections $\cap_{i=1}^p A^+_{\sigma_i}$ are either empty or contractible.

PROOF. (1) From the Theorem 6.14 it follows that $\overline{A}_\sigma$ is contractible. The thickening of $\overline{A}_\sigma$ presented in the proposition 6.22 is also contractible. Therefore, $A^+_{\sigma_i}$ is contractible.

(2) The multiple intersections

$$\cap_{i=1}^p A^+_{\sigma_i} = \cap_{i=1}^p (\overline{A}_{\sigma_i} \cup_{\mathsf{s}''\in\mathcal{K}''|\mathsf{s}''|\cap\overline{A_{\sigma_i}}\neq\emptyset} int(\mathsf{s}''))$$

and using the Theorem 6.19 the intersections in the equation above are either empty or equal to

$$\overline{A}_\tau \bigcup_{\mathsf{s}'' \in \mathcal{K}'' | \mathsf{s}''| \cap \overline{A_\tau} \neq \emptyset} int(\mathsf{s}'').$$

So, the multiple intersections $\cap_{i=1}^{p} A^+_{\sigma_i}$ are either empty or contractible. We have thus a good cover in the sense of Čech of the space ${}^{\mathrm{D}}\mathrm{Pol}_d$.

□

COROLLARY 6.24. *The thickening* $\{A^+_\sigma\}$ *is a Čech covering of* ${}^{\mathrm{D}}Pol_d$.

PROOF. By theorem 6.23 the cover by $\{A^+_\sigma\}$ satisfies the properties of a Čech cover: the multiple intersections contractible and the elements of the cover $A^+_{\sigma_i}$ are open and contractible. □

CHAPTER 7

General method for the Čech cohomology of braid groups

A general method to explicitly calculate the cohomology groups is givens, in this chapter.

The method for computing the i-th cohomology group relies on the structure of the stratification and it moreover requires two steps:

- the first step, consists in investigating the kernel of the coboundary map $d^i : C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$, where $\mathcal{F}$ is a sheaf. We show that the coboundary operator matrix of d^i is block diagonal and contains blocks which are circulant, for $i \geq 3$.

-The second step, consists in investigating the image of the coboundary map $d^{i-1} : C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$. To this end we show that the coboundary operator is a finite map.

1. Cohomology with values in a sheaf

This chapter is devoted to the presentation of a new general method which in principle should make it possible to compute the cohomology groups of the full braid groups with coefficients in a sheaf of abelian groups. However, here for simplicity we will confine ourselves to the case of a constant sheaf and indeed $\mathbb{Z}$. Starting first with a short overview on cohomology with values in a sheaf, we explain how to use it in our context. The short reminder below is essentially extracted from Jean-Pierre Serre's famous article *Faisceaux algébriques cohérents* (1956), see [**48**].

1.1. Cochains of the covering. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a good covering of the space ${}^{\mathrm{D}}\mathrm{Pol}_d$. The open sets U_i of the good cover correspond to the biregular sets A_{σ_i} along with their regular neighbourhood, that we have denoted previously by $A^+_{\sigma_i}$ (in chapter 6).

Let $p \geq 0$ be an integer. If $s = (i_0, i_1, ..., i_p)$, where s is a finite sequence of elements in I. We denote by $U_s = U_{i_0, i_1, ..., i_p}$ the intersection $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_p}$. We call a p-cochain of $\mathcal{U}$ with values in the sheaf $\mathcal{F}$ a function f assigning to every finite sequence $s = (i_0, i_1, ..., i_p)$ of $p+1$-elements of I a section $f_s = f_{i_0, i_1, ..., i_p}$ of $\mathcal{F}$ over $U_{i_0, i_1, ..., i_p}$. The p-cochains form an abelian group denoted by $C^p(\mathcal{U}, \mathcal{F})$. This group is given by the following product over all sequences s of $p+1$ elements of I: $\prod \Gamma(U_s, \mathcal{F})$, where $\Gamma(\mathcal{U}_s, \mathcal{F})$ is the set of sections of $\mathcal{F}$ over U_s.

1.2. Invariants of the polyhedral structure. Let $\mathcal{W}(I)$ be the polyhedral structure where I is the finite set of its vertices. The vertices $i_0 \in I$ are in bijection with the set of biregular classes $\sigma_{i_0} \in \Sigma_d$.

We first isolate two important building blocks in the recursive structure of the complex $\mathcal{W}(I)$. This study of incidence relations will bring some information concerning the non-empty intersection of open sets in the cover (see lemma ?? form chapter 6).

PROPOSITION 7.1. *The polyhedral complex $\mathcal{W}$ contains two invariant subcomplexes:*

(1) the 2-face, being a quadrangle denoted by $D(i_0, i_1, i_2, i_3)$,
(2) the 3-subcomplex, being a diamond denoted by $D(i_0, i_1, i_2, i_3, i_4)$,

PROOF. (1) The existence of the first invariant follows from the lemma 3.1.Let us study the relations of the chains associated to a 2-face in $\mathcal{W}$. From these incidence relations in the poset, it follows that if a 2-face exists then it is necessarily quadrangular.

Let $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ be the four generic signatures which correspond to the four vertices of a 2-face in $\mathcal{W}$. These incidence relations are stated two by two below.

- $\{\sigma_0, \sigma_1, \} \preceq \beta_{01}$, where $codim(\beta_{01}) = 1$.
- $\{\sigma_1, \sigma_2, \} \preceq \beta_{12}$, where $codim(\beta_{12}) = 1$.
- $\{\sigma_2, \sigma_3, \} \preceq \beta_{23}$, where $codim(\beta_{23}) = 1$.
- $\{\sigma_3, \sigma_0, \} \preceq \beta_{30}$, where $codim(\beta_{30}) = 1$.

$\{\sigma_0, \sigma_1, \sigma_2\} \preceq \{\beta_{01}, \beta_{12}, \} \preceq \alpha_{012}$, where $codim(\alpha_{012}) = 2$. Notice that the supremum of σ_0, σ_2 is a codimension 2 signature: $\{\sigma_0, \sigma_2\} \preceq \alpha_{012}$, instead of a codimension 1 signature.

The other chains of the poset up to the codimension 2 signatures are as follows:

$\{\sigma_1, \sigma_2, \sigma_3\} \preceq \{\beta_1, \beta_2, \} \preceq \alpha_{123}$, where $codim(\alpha_{123}) = 2$.
The supremum of σ_1, σ_3 is a codimension 2 signature: $\{\sigma_1, \sigma_3\} \preceq \alpha_{123}$, instead of a codimension 1 signature.

$\{\sigma_2, \sigma_3, \sigma_0\} \preceq \{\beta_{23}, \beta_{30}, \} \preceq \alpha_{230}$, where $codim(\alpha_{230}) = 2$.
The supremum of σ_0, σ_2 is a codimension 2 signature: $\{\sigma_0, \sigma_2\} \preceq \alpha_{012}$, instead of a codimension 1 signature.

$\{\sigma_3, \sigma_0, \sigma_1\} \preceq \{\beta_{30}, \beta_{01}, \} \preceq \alpha_{301}$, where $codim(\alpha_{301}) = 2$.
The supremum of σ_1, σ_3 is a codimension 2 signature: $\{\sigma_1, \sigma_3\} \preceq \alpha_{123}$, instead of a codimension 1 signature.

This implies that $\alpha_{123} = \alpha_{301}$ and $\alpha_{012} = \alpha_{230}$ which is possible if and only if $\alpha_{123} = \alpha_{301} = \alpha_{123} = \alpha_{0123}$ and $\alpha_{012} = \alpha_{230} = \alpha_{0123}$. These incidence relations are represented using a Hasse diagram as follows.

$$\begin{array}{ccc} \sigma_0 & \overset{\beta_{01}}{\longleftrightarrow} & \sigma_1 \\ \updownarrow\beta_{30} & \alpha_{0123} & \updownarrow\beta_{12} \\ \sigma_3 & \overset{\beta_{23}}{\longleftrightarrow} & \sigma_2 \end{array}$$

(2) The second type of invariant of the complex $\mathcal{W}$ is a so-called "diamond structure".

$$\begin{array}{ccccc} & & \sigma_0 & & \\ & \swarrow\beta_{01} & \downarrow\beta_{02} & \searrow\beta_{03} & \\ \sigma_1 & & \sigma_2 & & \sigma_3 \\ & \searrow\beta_{14} & \downarrow\beta_{24} & \swarrow\beta_{34} & \\ & & \sigma_4 & & \end{array}$$

In the complex it consists of one 3-face, three quadrangular 2-faces, six edges and five vertices. Let us show that for any degree $d > 2$ this structure exists. We first enumerate all the incidence relations in the poset given by this object. Let $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the generic signatures associated to the vertices in the complex.

Let the signatures $\beta_{ij}, \alpha_{ijk}, \gamma_{ijkl}$ verify $codim(\beta_{ij}) = 1$, $codim(\alpha_{ijk}) = codim(\alpha_{ijkl}) = 2$,$codim(\gamma_{ijklm}) = 3$ where $i \neq j \neq k \neq l \neq m$. We have the following chains up to the codimension 1 signatures:

- $\{\sigma_0, \sigma_1\} \preceq \beta_{01}$,
- $\{\sigma_0, \sigma_2\} \preceq \beta_{02}$,
- $\{\sigma_0, \sigma_3\} \preceq \alpha_{013}$,
- $\{\sigma_1, \sigma_4\} \preceq \beta_{14}$,
- $\{\sigma_2, \sigma_4\} \preceq \alpha_{024}$,
- $\{\sigma_3, \sigma_4\} \preceq \beta_{34}$.

The incidence relations for three generic (biregular) signatures:

(a) $\{\sigma_0, \sigma_1, \sigma_2\} \preceq \{\beta_{01}, \beta_{02}, \beta_{12} = \alpha_{012}\} \preceq \alpha_{012} \preceq \gamma_{0123}$,
(b) $\{\sigma_0, \sigma_1, \sigma_3\} \preceq \{\beta_{01}, \beta_{03}, \beta_{13} = \alpha_{013}\} \preceq \alpha_{013} \preceq \gamma_{0123}$,
(c) $\{\sigma_0, \sigma_2, \sigma_3\} \preceq \{\beta_{02}, \beta_{03}, \beta_{23} = \alpha_{023}\} \preceq \alpha_{023} \preceq \gamma_{0123}$,
(d) $\{\sigma_1, \sigma_2, \sigma_3\} \preceq \{\beta_{12} = \alpha_{012}, \beta_{23} = \alpha_{023}, \beta_{13} = \alpha_{013}\} \preceq \gamma_{0123}$.

Since from relation (b) $\beta_{13} = \alpha_{013}$ and from relation (d) $\alpha_{013} \preceq \alpha_{123}$. Now these signatures have same codimension 2. So, we have that $\alpha_{013} = \alpha_{123}$. Hence, $\alpha_{013} = \alpha_{123} = \gamma_{0123}$.

(e) $\{\sigma_4, \sigma_1, \sigma_2\} \preceq \{\beta_{41}, \beta_{42}, \beta_{12} = \alpha_{412}\} \preceq \alpha_{412} \preceq \gamma_{4123}$,
(f) $\{\sigma_4, \sigma_1, \sigma_3\} \preceq \{\beta_{41}, \beta_{43}, \beta_{13} = \alpha_{413}\} \preceq \alpha_{413} \preceq \gamma_{4123}$,
(g) $\{\sigma_4, \sigma_2, \sigma_3\} \preceq \{\beta_{42}, \beta_{43}, \beta_{23} = \alpha_{423}\} \preceq \alpha_{423} \preceq \gamma_{4123}$,

By relation (f) of the second block we have $\beta_{13} = \alpha_{413}$ and by relation (b) of the first block $\beta_{13} = \alpha_{013}$. So, $\alpha_{413} = \alpha_{013}$ and thus $\alpha_{413} = \alpha_{013} = \alpha_{0134}$. We have also that $\alpha_{013} \preceq \alpha_{123}$ implying that $\alpha_{0134} = \alpha_{123}$. So, $\alpha_{0134} = \alpha_{123} = \gamma_{01234}$. Similarly, we have: $\beta_{12} = \alpha_{412} = \alpha_{012}$, so $\beta_{12} =$

$\alpha_{412} = \alpha_{012} = \alpha_{0124}$ and $\beta_{23} = \alpha_{423} = \alpha_{023}$. So, $\beta_{23} = \alpha_{423} = \alpha_{023} = \alpha_{0234}$. Finally, we have $\beta_{0,4} = \{\alpha_{0134}, \alpha_{0123}, \alpha_{1234}\} \preceq \gamma_{01234}$.

□

We now exploit this information starting with the case of the quadrangular 2-face. These incidence relations have the following consequences on the open sets in $\mathcal{U}$.

Recall that the collection of open sets U_i in the good cover are in bijection with the set of vertices of the polyhedral complex and indexed by the set of codimension 0 signatures. The intersection of the three open sets associated to the signatures $\sigma_0, \sigma_1, \sigma_2$ is $U_{\sigma_0,\sigma_1,\sigma_2}$. So, we have: $U_{\sigma_0,\sigma_1,\sigma_2} = U_{\sigma_0,\sigma_1,\sigma_3} = U_{\sigma_0,\sigma_2,\sigma_3} = U_{\sigma_1,\sigma_2,\sigma_3} = U_{\sigma_0,\sigma_1,\sigma_2\sigma_3}$.

For the diamond structure, we have the following relations for the open sets: from the previous result we have the following non-empty intersections: $U_{01234} \subset \{U_{0234}, U_{0134}, U_{0124}\}$ and $U_{12} = U_{0124}, U_{23} = U_{0234}, U_{13} = U_{0134}$.

LEMMA 7.2. *Let $\mathcal{W}$ be the complex associated to the decomposition in signatures of ${}^{\mathrm{D}}Pol_d$. For any degree $d > 3$ there exists in $\mathcal{W}$ a subcomplex R which is connected, of topological genus 1 and obtained from the union of one subcomplex $NC(d)$, one bridge and two open book structures.*

PROOF. Going back to the definition of a bridge structure $D^{bridge}(u_0, ..., u_p)$ (cf definition 3.7), its vertices $\{u_0, ..., u_p\}$ are in bijection with the set of biregular signatures $\{\sigma_{u_0}, ..., \sigma_{u_p}\}$ which are incident to one signature σ_{bridge} of high codimension. The bridge structure is contained in the tower structure and it inherits invariance under the symmetry group $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$. So, we partition its set of vertices $\{u_0, ..., u_p\}$ into two disjoint sets which are of same cardinality and that we denote V_0 and V_1 (with $|V_0| = |V_1|$). Recall that the diagram of the signature σ_{bridge} contains two critical points: one of multiplicity 2, one of multiplicity $(d-1)$ and both critical points lie on curves of the same color, say red (resp. blue). There exist $d-1$ short diagonals colored blue (resp. red) and one long blue (resp. red) diagonal. Concerning the vertices of $D^{bridge}(u_0, ..., u_p)$, their BKL notation is as follows: $(*, a_{kl})$ (resp. $(a_{kl}, *)$), where * represents the word associated to the colored red diagonals. Notice that the signature $(*, a_{kl})$ is adjacent in one deformation step to either $(*, 0)$ or $(*, a_{d,d-1}a_{d-2,d-3}...a_{2,1})$ (reciprocally $(a_{kl}, *)$ is adjacent in one deformation step to $(0, *)$ or $(a_{d,d-1}a_{d-2,d-3}...a_{2,1}, *)$).

The two open book structures that we consider have vertices denoted by $(*, a_{kl})$ and $(*, 0)$ (if a_{kl} is adjacent in one deformation step to 0), and respectively $(*, a_{kl})$ $(*, a_{d,d-1}a_{d-2,d-3}...a_{2,1})$, if a_{kl} is adjacent in one deformation step to $a_{d,d-1}a_{d-2,d-3}...a_{2,1})$ (We proceed similarly for the opposite color, where diagrams are denoted by $(a_{kl}, *)$).

Now, any bridge subcomplex is glued to a couple of open book structures in the following manner: both open book structures share with the bridge a set of vertices. The first open book structure shares with the bridge structure the set of vertices V_0, the second open book structure shares with the bridge structure the set of vertices V_1. More precisely, the *openbook* subcomplexes denoted by $D^{openbook_1}$ (resp. $D^{openbook_2}$) verify that the following intersections are non-empty: $D^{openbook_1} \cap D^{bridge}(u_0, ..., u_p) = \{V_0\}$, resp. $D^{openbook_2} \cap D^{bridge}(u_0, ..., u_p) = \{V_1\}$. The remaining vertices which are contained *openbook*$_1$ (resp. in *openbook*$_2$) form the sets denoted by V_2 (resp. V_3). Now, since the signatures forming the sets V_2 and V_3 carry a BKL notation of the type $(*, 0)$ or $(*, a_{d,d-1}a_{d-2,d-3}...a_{2,1})$ (if a_{kl} is adjacent in one deformation step to $a_{d,d-1}a_{d-2,d-3}...a_{2,1})$) so those signatures belonging to the sets V_2 and V_3 are the ones from the $NC(d)$ subcomplex. So, the subcomplexes $NC(d)$ and $D^{openbook_1}$ have in common the set of vertices $\{V_2\}$; $NC(d)$ and $D^{openbook_2}$ have in common the set of vertices V_3. The union of these four subcomplexes forms one connected component in $\mathcal{W}$, denoted by R, which is of topological genus 1: the sets of vertices V_0, V_1, and V_2, V_3 are disjoint; reciprocally, the sets of vertices V_1, V_2, and V_3, V_0 are disjoint and $V_0 \cup \cdots \cup V_4$ forms one connected component. □

1.3. Coboundary operations.

1.3.1. *General definition.* Let us recall the definition of the coboundary operators for the Čech cohomology.

Let $D(J')$ be a subcomplex with the set J' of its vertices; an ordered subcomplex of $D(J')$ is a sequence $J' = (i_0, i_1, ..., i_p)$ of elements of I. The complex $\mathcal{W}$ is the sum of all the subcomplexes with set of vertices in I. We define $\mathcal{W}(I) = \oplus_{p=0}^{\infty} K_p(I)$ to be the complex defined by $D(I)$; $K_p(I)$ is the set of subcomplexes of p vertices of $D(I)$. If s is a subcomplex of $S(I)$ we denote by $|s|$ the set of vertices of s.

A mapping $h : K_p(I) \to K_q(I)$ is called a simplicial endomorphism if

(1) h is a homomorphism
(2) for any subcomplex s of p vertices of $D(I)$ we have

$$h(s) = \sum_{s'} c_s^{s'} \quad \text{with} \quad c_s^{s'} \in \mathbb{Z},$$

the sum being over all subcomplexes s' of q vertices such that $|s'| \subset |s|$.

This leads to the coboundary operator between cochains. Let h be the simplicial endomorphism and let $f \in C^q(\mathcal{U}, \mathcal{F})$ be a cochain of degree q. For any subcomplex s of p vertices put: $(h^t f)_s = \sum c_s^{s'} \rho_s^{s'}(f_{s'})$, where $\rho^{s'}$ denotes the restriction homomorphism: $\Gamma(\mathcal{U}_{s'}, \mathcal{F}) \to \Gamma(\mathcal{U}_s, \mathcal{F})$, which makes sense because $|s'| \subset |s|$. the mapping $s \mapsto h^t f$ is a homomorphism:

$$h^t : C^q(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F}).$$

Applying the above to the simplicial endomorphism

$$\partial : K_{p+1}(I) \to K_p(I)$$

defines by the formula: $\partial(i_0, i_1, ..., i_p) = \sum_{j=0}^{j=p+1} (-1)^j (i_0, i_1, ..\hat{i_j}., i_p)$.

The symbol " $\hat{i_j}$ " means that the subscript is omitted.

Thus we obtain a homomorphism $\partial^t : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ which we denote by d. From definition we have that:

$$(df)_{i_0,i_1,\ldots,i_{p+1}} = \sum_{j=0}^{j=p+1} (-1)^j \rho_j (f_{i_0,i_1,\ldots,\hat{i_j},\ldots,i_{p+1}}),$$

where ρ_j denotes the restriction homomorphism

$$\rho_j : \Gamma(\mathcal{U}_{i_0,i_1,\ldots,\hat{i_j},\ldots,i_{p+1}}, \mathcal{F}) \to \Gamma(\mathcal{U}_{i_0,i_1,\ldots,i_p}, \mathcal{F}).$$

Since $\partial \circ \partial = 0$, we have $d \circ d = 0$. Thus $C(\mathcal{U}, \mathcal{F})$ is equipped with a coboundary operator making it a complex. Note that we will omit ρ_j from our notation.

The q-cochain $C^q(\mathcal{U}, \mathcal{F})$ and the $q+1$-cochain $C^{q+1}(\mathcal{U}, \mathcal{F})$ being related by the coboundary operator:

$$d^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}),$$

where $(d^q f)_{i_0,\ldots,i_q} \in C^{q+1}(\mathcal{U}, \mathcal{F})$ is given by the equation (??):

$$(d^q f)_{i_0,\ldots,i_q} = \sum_{k=0}^{q+1} (-1)^k f_{i_0,\ldots,\hat{i}_k,\ldots,i_q}|_{A^+_{\sigma_{i_0,\ldots,i_{q+1}}}}, f \in C^q(\mathcal{U}, \mathcal{F}).$$

The matrix corresponding to the coboundary operator mapping $C^q(\mathcal{U}, \mathcal{F})$ to $C^{q+1}(\mathcal{U}, \mathcal{F})$ is represented by:

$$(2) \quad \underbrace{\begin{pmatrix} +1 & -1 & \ldots & (-1)^k & \ldots & (-1)^q & 0 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & \ldots & (-1)^k \ldots & (-1)^q \\ & & & \ldots & & & & & & \\ 0 & 0 & \ldots & \ldots & (-1)^m & \ldots & & & & \end{pmatrix}}_{A^q} \times \underbrace{\begin{pmatrix} f_{\hat{i}_0,\ldots,i_q} \\ \ldots \\ f_{i_0,\ldots,\hat{i}_k,\ldots,i_q} \\ \ldots \\ f_{i_0,\ldots,\hat{i}_q} \\ ----- \\ f_{\hat{j}_0,\ldots,j_q} \\ \ldots \\ ----- \\ f_{h_0,\ldots,\hat{h}_k,\ldots,h_q} \\ \ldots \end{pmatrix}}_{C^q(\mathcal{U},\mathcal{F})} = \underbrace{\begin{pmatrix} (d^q f)_{i_0,\ldots,i_q} \\ (d^q f)_{j_0,\ldots,j_q} \\ \ldots \\ (d^q f)_{h_0,\ldots,h_q} \end{pmatrix}}_{C^{q+1}(\mathcal{U},\mathcal{F})}$$

1.3.2. *Circulant matrices.* We now state and prove a proposition which essentially provides an explicit method for computing the coboundary operators.

Let us recall first the definition and properties of a circulant matrix.

A circulant matrix is specified by one vector column $v = (v_0, v_1, v_2, \ldots, v_{n-1})$. We define a shift operator $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T(v_0, v_1, v_2, \ldots, v_{n-1}) = (v_{n-1}, v_0, v_1, \ldots, v_{n-2}).$$

DEFINITION 7.1. *The circulant matrix associated to v is the $n \times n$ matrix whose rows are given by iterations of the shift operator acting on v.*

A circulant matrix thus looks like:

$$V = \begin{bmatrix} v_0 & v_1 & v_2 & \dots & v_{n-1} \\ v_{n-1} & v_0 & v_1 & \dots & v_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & v_0 \end{bmatrix}$$

By block circulant matrix we mean that the coefficient v_i is replaced by a block of the matrix.

Circulant matrices are tools that are elementary and well understood see in [**28**], a salient feature being the fact that the kernel is easily computable.

- Let ϵ be a primitive n-th root of unity, then the eigenvalue λ_l of V is such that: $\lambda_l = v_0 + \epsilon^l v_1 + \dots + \epsilon^{(n-1)l} v_{n-1}$ with normalized eigenvector $(1, \epsilon^l, \epsilon^{2l}, ..., \epsilon^{(n-1)l})$.
- The dimension of the kernel of the circulant matrix is the number of zero eigenvalues.
- the rank of a circulant matrix V is equal to $n-k$ where k is the degree of gcd $(X^n - 1, P(X))$ where $P(X)$ is the polynomial $v_0 + v_1 X + v_2 X^2 + ... + X^n$ associated to the circulant matrix V.

The generalization to block-circulant matrices is given in [**29**].

PROPOSITION 7.3. *The Čech coboundary operators d^q, $q \geq 3$ are associated to block diagonal matrices A^q (see picture (1) above) where the blocks are block circulant.*

PROOF. The matrix associated to the coboundary operator is block diagonal since each block corresponds to a substructure R as in lemma 7.2 and since these structures are disjoint. The blocks of the coboundary matrix are block circulant since each structure R is of topological genus 1. This latter argument can be proved by induction on the number of vertices in the R structure (and thus on the degree d of the ${}^{\mathrm{D}}\mathrm{Pol}_d$). $\square$

Since A^q is block diagonal the kernel is simply the direct sum of the kernels of the blocks. In conclusion this proposition makes it possible to describe the kernel of the coboundary operator.

2. Explicit calculation of the Čech cohomology groups

We now give the general method to calculate explicitly the cohomology groups using, naturally, the decomposition presented throughout this work. The following statement illustrates the application of this method for the constant sheaf $\mathbb{Z}$. However, the method presented in the proof remains independent from this choice.

2.1. Cohomology groups.

THEOREM 7.4. *Let* $d > 1$ *be the degree of the polynomials in* $^{\mathrm{D}}Pol_d$ *and let* q *be an integer verifying* $0 \leq q < d$. *Then the Čech cohomology groups* $H^q(^{\mathrm{D}}Pol_d, \mathcal{F})$ *are finite groups, for* $2 < q < d$ *and for* $\mathcal{F}$ *any coherent sheaf.*

PROOF. The statement in (1) follows from the theorem 2.12 in chapter 2.

To prove the statement (2), one must consider two steps. First, the kernel of $d^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ and show in particular that it is non-null. Secondly, the image of $d^{q-1} : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})$ and show that it is a finite map.

We are interested in the matrix associated to the coboundary operator

$$d^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}).$$

We show that this matrix is a block diagonal matrix with blocks being block circulant.

(a) Consider from lemma 7.2 an R structure in $\mathcal{W}$ and denote its set of vertices by the following collection of vertices $\{V_0, ..., V_3\}$, where the cardinalities of those sets are $|V_0| = ... = |V_3|$. For any sequence of vertices $(\sigma_{i_0}, ..., \sigma_{i_{q+1}})$ in the union of vertices $V_0 \cup V_1$ it follows from lemma 6.5 from chapter 6 that the open set $U_{\sigma^0_{i_0}, ..., \sigma^0_{i_{q+1}}}$ corresponding to those vertices is non-empty (the union of the vertices $V_0 \cup V_1$ belonging to one bridge structure). In particular this property implies that $f_{\sigma^0_{i_0}, ..., \sigma^0_{i_{q+1}}}$ is different from 0. We proceed similarly for the other couples of vertices, as depicted below.

The sequence of vertices $(\sigma^1_{i_0}, ..., \sigma^1_{i_{q+1}})$ in the union of vertices $V_1 \cup V_2$, corresponds by lemma 6.5 to a non-empty open set $U_{\sigma^1_{i_0}, ..., \sigma^1_{i_{q+1}}}$ since $V_1 \cup V_2$ is contained in an open book structure. Applying the sheaf properties we have that $f_{\sigma^1_{i_0}, ..., \sigma^1_{i_{q+1}}}$ is different from 0.

We notice that the coboundary relations defined for the $q + 2$-sequences of vertices in $V_0 \cup V_1$, are translated for $V_1 \cup V_2$, for $V_2 \cup V_3$ and $V_3 \cup V_0$. So, the matrix associated to the coboundary operator $d^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$, for the R structure is block circulant. Therefore from the properties of block circulant matrices it follows that the dimension of the kernel of this block is n where n is non-zero. Since there exist $4d$ such R-structures in the dual complex $\mathcal{W}$, the dimension of the kernel has the following lower bound:

$$dim(kerd^q) \geq 4dn.$$

(b) Now, we show that the image $Im(d^{q-1} : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})$ is a finite map. We point out an important property of our decomposition into signatures of the space $^{\mathrm{D}}\mathrm{Pol}_d$. Take a substructure of one R structure of the dual complex $\mathcal{W}$ (cf lemma 7.2) which is constituted from the union of a bridge structure and an open book structure. The sets of vertices are V_0, V_1 and V_2 where V_0, V_1 belong to the bridge structure and V_1, V_2 are the vertices in an open book structure. Let us consider a sequence of vertices $(i_0, ..., i_{q+1})$ in V_1 and its corresponding non-empty open set $U_{i_0,...,i_{q+1}}$.

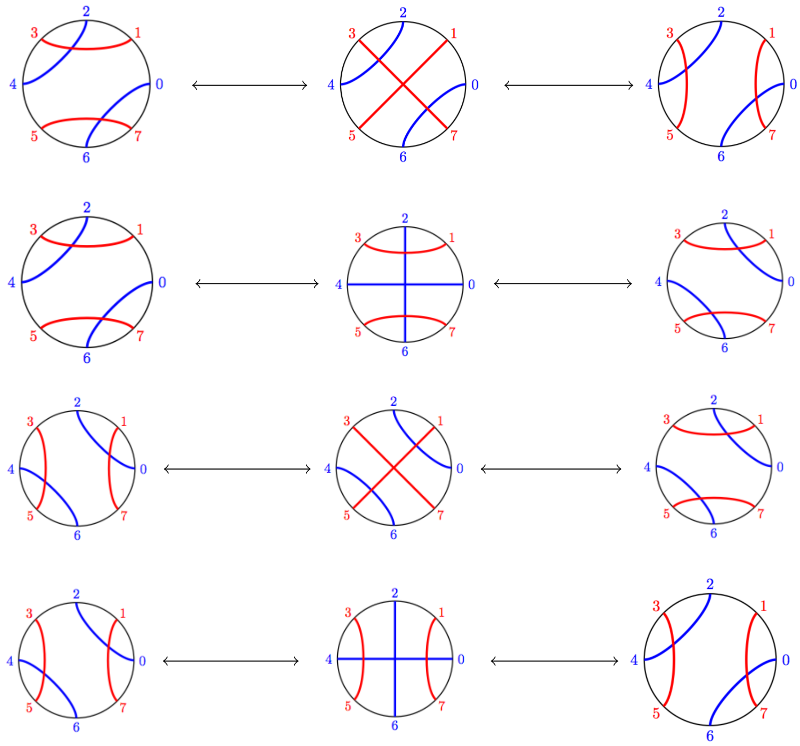
We have the following inclusion maps $U_J \hookrightarrow U_{i_0,...,i_q}$ and $U_{J'} \hookrightarrow U_{i_0,...,i_q}$, where J is composed from the $q + 1$ vertices in the bridge structure (contained in the union of V_0 with V_1) and similarly J' is composed from the $q + 1$ vertices in the open book structure (contained in the union of vertices in V_1 and V_2). Since J is associated to the bridge structure, the vertices in J are the generic signatures incident to σ_{bridge}. Similarly the set of vertices in J' are incident to $\sigma_{openbook}$. Applying lemma 6.5 the open sets U_J and $U_{J'}$ correspond respectively to the signatures σ_{bridge} and $\sigma_{openbook}$ and in particular we have that the intersection of U_J and $U_{J'}$ is empty. So, from the sheaf properties, we have the following maps: $\mathcal{F}(U_{i_0,...,i_q}) \to \mathcal{F}(U_J)$ and $\mathcal{F}(U_{i_0,...,i_q}) \to \mathcal{F}(U_{J'})$ where $\mathcal{F}(U_{J'}) \neq \mathcal{F}(U_J)$. In particular, taking $\mathcal{F}(U_{i_0,...,i_q}) \to \mathcal{F}(U_{i_0,...,i_{q+1}})$ where $i_0, ..., i_{q+1} \in V_1$, we have locally a finite map. Applying the same argument to the other couples of sets $\{V_2, V_3\}$ and $\{V_3, V_0\}$ and to all the other R structures, we define a finite map for coboundary operator. In order to explicitly compute the degree of the finite map we proceed by induction on the degree $d > 3$ using this property, in the following way. We have shown in lemma 7.1 that there exist building blocks which are invariant in the decomposition of ${}^{\mathrm{D}}\mathrm{Pol}_d$. So, firstly the structures $NC(d)$, *bridge*, and *openbook* in ${}^{\mathrm{D}}\mathrm{Pol}_d$, contain those invariants. Secondly, these three structures contain respectively also the subcomplexes $NC(d-1)$, *bridge*, and *openbook* structures which are in the signature decomposition of ${}^{\mathrm{D}}\mathrm{Pol}_{d-1}$. Then, we apply the property described above, by induction on the structures $NC(d-1)$, *bridge*, and *openbook* in ${}^{\mathrm{D}}\mathrm{Pol}_{d-1}$. On the one hand, since we have noticed that there exist $NC(d-1)$ structures in $NC(d)$ they are incident to a common signature $\sigma_{NC(d)}$, which is of the highest codimension. Similarly, the bridge structure of ${}^{\mathrm{D}}\mathrm{Pol}_d$ contains bridge structures of ${}^{\mathrm{D}}\mathrm{Pol}_{d-1}$ incident to a common signature σ_{bridge} of high codimension, the same argument is applied for the *openbook* structures. On the second hand, these structures remain invariant under the Klein group, as was shown in chapter 3. So, we can apply the property to the R structures in the decomposition of ${}^{\mathrm{D}}\mathrm{Pol}_d$. □

3. Čech cohomology for $d = 2, 3, 4$

Let us illustrate for $d = 2$ the Čech cohomology with values in the constant sheaf $\mathcal{F}(U) = \mathbb{Z}$ for all open sets A^+_σ. The dual complex (graph) $\mathcal{W}$ is a quadrangle (see Fig 2) constituted from

- four vertices, corresponding to the codimension 0 signatures,
- four edges, corresponding to the codimension 1 signatures

These signatures are explicitly drawn and enumerated below:



REMARK 7.1. For $d = 2$ the 2-face in the dual graph $\mathcal{W}$ does not exist, since it would be associated to a signature where both crossing points of the red and blue diagonals are not distinct and thus to a class of polynomials with non distinct roots.

Take the good cover $\mathcal{U} = \{A^+_{\sigma_1}, \ldots, A^+_{\sigma_4}\}$ where the open sets are the thickened co-dimension 0 sets. These sets have the following non-empty intersections: $A^+_{\sigma_{1,2}}$, $A^+_{\sigma_{2,3}}$, $A^+_{\sigma_{3,4}}$, $A^+_{\sigma_{1,4}}$.

(1) The 0-cochain is defined as

$$C^0(\mathcal{U}, \mathbb{Z}) := \{(e_1, \ldots, e_4) | e_i \text{ is a constant on } A^+_{\sigma_i}, i = 1, ..., 4\} = \mathbb{Z}^4.$$

(2) The 1-cochain is defined as

$$C^1(\mathcal{U}, \mathbb{Z}) = \{(w_{1,2}, \ldots, w_{4,1}) | w_{ij} = e_j - e_i \text{ is a constant on } A^+_{\sigma_{i,j}}\} \text{with } i, j \in \{1, \ldots, 4\} = \mathbb{Z}^4.$$

(3) Triple intersections are empty since there does not exist any 2-face in $\mathcal{W}$. $C^2(\mathcal{U}, \mathbb{Z}) = \{0\}$.

We have the following sequence of cochains:

$$0 \xrightarrow{d^{-1}} C^0(\mathcal{U}, \mathbb{Z}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathbb{Z}) \xrightarrow{d^1} C^2(\mathcal{U}, \mathbb{Z}) \xrightarrow{d^2} 0,$$

where the coboundary operator d^0 is:

$$d^0 : C^0(\mathcal{U}, \mathbb{Z}) \to C^1(\mathcal{U}, \mathbb{Z});$$

$(d^0e)_{jk} = w_k - w_j$ given by : $d^0(e_1, e_2, e_3, e_4) = (e_2 - e_1, e_3 - e_2, e_4 - e_3, e_4 - e_1)$.

The kernel of d^0 is $e_2 = e_1, e_3 = e_2, e_4 = e_3, e_4 = e_1$ So,

$$H^0(\mathcal{U}, \mathbb{Z}) = \frac{ker(d^0 : C^0 \to C^1)}{\{0\}} = \mathbb{Z}.$$

Let us consider $Im(d^0)$. We have $(d^0e_i)_{j,k} = \begin{cases} 0 \text{ if } & i \neq k \text{ and } i \neq j \\ 1 \text{ if } & i = k \\ -1 \text{ if } & i = j \end{cases}$ Suppose that $w = d^0(v)$ where $v = \sum_{i=1}^{4} c_i e_i$ and $c_i \in \mathbb{Z}$.

So, we have the following vectors:

$d^0(e_1) = [-1, 0, 0, -1]$
$d^0(e_2) = [-1, -1, 0, 0]$
$d^0(e_3) = [0, 1, -1, 0]$
$d^0(e_4) = [0, 0, 1, 1]$

$Im(d^0) = span\{(-1, 0, 0, -1), (1, -1, 0, 0), (0, -1, 1, 0), (0, 0, 1, 1)\}$. We have $kerd^1 = \mathbb{Z}^4$ and one vector in $span\{(-1, 0, 0, -1), (1, -1, 0, 0), (0, -1, 1, 0), (0, 0, 1, 1)\}$ is a linear combination of the three others. So, $H^1(\mathcal{U}, \mathbb{Z}) = \frac{ker(d^1:C^1 \to C^2)}{Im(d^0:C^0 \to C^1)} = \mathbb{Z}$. To conclude $H^2(\mathcal{U}, \mathbb{Z}) = \frac{ker(d^2:C^2 \to C^3)}{Im(d^1:C^1 \to C^2)} = 0$, since $ker(d^2 : C^2 \to C^3) = 0$.

REMARK 7.2. For the case $d = 3$, the results for the cohomology groups are the same as for $d = 2$.

3.1. The case $d = 4$. In order to explicitly compute the cohomology groups for $d = 4$ we study the structure of the complex. In particular, we count how many there exist bridge subcomplexes, $NC(4)$ subcomplexes and openbook complexes in the complex $\mathcal{W}$ (see chapter 3 for a detailed study of the subcomplexes).

Recall that there exist 4 $NC(4)$ structures, 16 bridge structures and 32 open book structures. The endvertex of one $NC(4)$, (i.e. one of the vertex corresponding to the M diagram) is the initial vertex of the second $NC(4)$. The end vertex of the second $NC(4)$ is glued to the initial vertex of the third $NC(4)$ and finally the fourth $NC(4)$ is glued to the endvertex of the third. The endvertex of the fourth $NC(4)$ is glued to the initial vertex of the first one.

There are 8 bridge subcomplexes connecting the first and the third $NC(4)$ structures and there exist 8 bridges connecting the second and fourth $NC(4)$ structure. There are 32 openbook structures establishing a connection between the 16 bridge structures and the $NC(4)$ structure. Each bridge has a couple of open books connecting respectively its upper part to the upper part of the $NC(4)$ and lower part to the lower part of the $NC(4)$ structure.

REMARK 7.3. The bridge and $NC(4)$ structure are invariant under the symmetry of $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$.

REMARK 7.4. The bridge structure contains two diamond structures.

Now, we show that $H^3({}^{\mathrm{D}}\mathrm{Pol}_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. To do this one must consider the image and kernel of the coboundary operators, $d^2 : C^2(\mathcal{U}, \mathbb{Z}) \to C^3(\mathcal{U}, \mathbb{Z})$ and $d^3 : C^3(\mathcal{U}, \mathbb{Z}) \to C^4(\mathcal{U}, \mathbb{Z})$.

Let us consider the coboundary operator $d^2 : C^2(\mathcal{U}, \mathbb{Z}) \to C^3(\mathcal{U}, \mathbb{Z})$. We show that:

PROPOSITION 7.5. *The coboundary operator d^2 is a twofold covering $C^2(\mathcal{U}, \mathbb{Z}) \xrightarrow{2} C^3(\mathcal{U}, \mathbb{Z})$.*

PROOF. We want to show that $C^2(\mathcal{U}, \mathbb{Z}) \xrightarrow{2} C^3(\mathcal{U}, \mathbb{Z})$. By definition, the homomorphism $d^2 : C^2(\mathcal{U}, \mathbb{Z}) \to C^3(\mathcal{U}, \mathbb{Z})$ follows from the simplicial endomorphism $\partial : K_{p+1} \to K_p$, where K_p is the set of subcomplexes with p-vertices contained in the dual complex $\mathcal{W}$, which was presented in the first section of this chapter. Therefore, we study the subcomplexes contained in K_{p+1} and K_p, where $p = 3$. Since there are four main families of subcomplexes with more than 3 vertices (diamond, bridge, $NC(4)$ and open book) we study these cases.

(1) Consider first the bridge structure $D^{bridge}(J)$ where $J = (i_0, ..., i_4, j_0, ..., j_4)$ are its set of 10 vertices. By the symmetry of the bridge structure, there is a set of 5 vertices on the upper half part and a set of 5 vertices on the lower part of the bridge.

Without loss of generality, take the bridge with the following set of vertices in the BKL notation:

$$bridge_1 = \{(0,34); (12,34); (14,34); (24;34); (34,34);$$
$$(124,34); (134,34); (234,34); ((12)(34),34); (1234,34)\}.$$

The vertices in its upper half part are the following:

$$\{\sigma_{i_0}; ...; \sigma_{i_4}\} = \{(0,34); (12,34); (14,34); (24;34); (34,34)\}.$$

The vertices in the lower half part are the remaining vertices.

This set of vertices correspond to an open non-empty intersection $U_{\sigma_{i_0},...,\sigma_{i_4}}$.
Indeed, the biregular signatures verify the incidence relation $\sigma_{i_0}, ..., \sigma_{i_4} \preceq \sigma_{bridge}$, where σ_{bridge} is the signature of maximal codimension contained the bridge structure.

From chapter 3, we find that there exists an open book structure containing those five vertices. This open book structure has the following vertices:

$$openbook_1 = \{(0,34); (12,34); (14,34); (24;34);$$
$$(34,34); (0,0); (12,0); (14,0); (24;0); (34,0)\}.$$

Since the 5 vertices belong to the open book structure, it implies that there exists the relation $\sigma_{i_0}, ..., \sigma_{i_4} \preceq \sigma_{openbook}$, where $\sigma_{openbook}$ is the signature of the highest codimension in the openbook structure. However, notice that

$$\sigma_{openbook} \neq \sigma_{bridge} \text{ with } codim(\sigma_{openbook}) = codim(\sigma_{bridge}).$$

So, this implies that for the open sets $U_{\sigma_{i_0},...,\sigma_{i_4}} \supset A_{\sigma_{openbook}}$ and $U_{\sigma_{i_0},...,\sigma_{i_4}} \supset A_{\sigma_{bridge}}$, where $A_{\sigma_{openbook}} \cap A_{\sigma_{bridge}} \neq \emptyset$.

Therefore, the sequence $(\sigma_{i_0}, ..., \sigma_{i_4})$ in I indexes two disjoint open sets. So, there exists a twofold covering, mapping two disjoint open sets to one open set

$$\pi_1 : (U_{\sigma_{i_0},...,\sigma_{i_4}})^{openbook} \sqcup (U_{\sigma_{i_0},...,\sigma_{i_4}})^{bridge} \to (U_{\sigma_{i_0},...,\sigma_{i_4}}),$$

where $(U_{\sigma_{i_0},...,\sigma_{i_4}})^{openbook}$ $(resp.(U_{\sigma_{i_0},...,\sigma_{i_4}})^{bridge}))$ is the open set corresponding to the open book structure (resp. is the open set corresponding to the bridge structure).

(2) Consider the lower part of the bridge. In our example, it consists of the following set of vertices:

$$\{\sigma_{j_0}; \ldots; \sigma_{j_4}\} = \{(1,34); (124,34); (134,34); (234;34); ((12)(34),34)\}.$$

Let us consider the second open book structure, which has non-empty intersection with this bridge subcomplex., Its vertices are as follows:

$$openbook_2 = \{(1234,34); (124,34); (134,34); (234;34); ((12)(34),34); ((12)(34),34); (1234,0); (124,0); (134,0); (234,0); ((12)(34),0)\}.$$

In particular, this contains the set of vertices $\{\sigma_{j_0}, \ldots, \sigma_{j_4}\}$. Therefore, as previously, to the sequence $(\sigma_{j_0}, \ldots, \sigma_{j_4})$, there exist two disjoint open sets, which correspond respectively to the open set belonging to the second open book structure and to the open set belonging to the lower part of the bridge

$$\pi_2 : (U_{\sigma_{j_0},...,\sigma_{j_4}})^{openbook_2} \sqcup (U_{\sigma_{j_0},...,\sigma_{j_4}})^{bridge} \to (U_{\sigma_{i_0},...,\sigma_{i_4}}).$$

(3) Let us consider now the $NC(4)$ structure which contains the following set of vertices:

$$NC(4)_1 = \{(1,0); (1,123); (1,124); (1,134); (1,234); (1,24); ((1,13); (1234,(12)(34)) (1234,(23)(41)); (1234,12); (1234,23); (1234,34); (1234,41)\}.$$

This set has a non-empty intersection with the set of vertices of the second open book structure (resp. first). These intersection sets are respectively:

$$\begin{aligned} \mathit{openbook}_2 \cap NC(4)_1 &= \{\sigma_{k_0}; \ldots; \sigma_{k_4}\} \\ &= \{(1,0); (124,0); (134,0); (234,0); ((12)(34),0)\}, \\ \mathit{openbook}_1 \cap NC(4)_1 &= \{\sigma_{m_0}; \ldots; \sigma_{m_4}\} \\ &= \{(0,0); (12,0); (14,0); (24;0); (34,0)\}. \end{aligned}$$

Those 5 signatures in $NC(4)$ are incident to two different signatures of high codimension which are distinct: $\sigma_{NC(4)_1}$ and $\sigma_{openbook_2}$ (and respectively $\sigma_{NC(4)_1}$ and $\sigma_{openbook_1}$) which implies, as previously, that there exists a twofold covering, mapping two disjoint open sets to one open set:

$$\pi_3 : (U_{\sigma_{k_0}}, \ldots, \sigma_{k_4})^{openbook_2} \sqcup (U_{\sigma_{k_0},\ldots,\sigma_{k_4}})^{NC(4)_1} \to (U_{\sigma_{k_0},\ldots,\sigma_{k_4}})$$

and

$$\pi_4 : (U_{\sigma_{m_0}}, \ldots, \sigma_{m_4})^{openbook_1} \sqcup (U_{\sigma_{m_0},\ldots,\sigma_{m_4}})^{NC(4)_1} \to (U_{\sigma_{m_0},\ldots,\sigma_{m_4}}).$$

The glueing of the bridge with $NC(4)$ by *openbook*$_1$ and *openbook*$_2$ forms a connected component of topological genus 1. We have defined local twofold coverings. In order to define a general twofold covering, we discuss the glueing properties of the open sets in the main structures. These glueing properties are considered by studying the 4-intersections of sets contained in the upper and lower part of the main structures. Consider the subcomplexes in the bridge structure having a combination of 4 vertices belonging to both sequences: $(i_0, \ldots, i_4), (j_0, \ldots, j_4)$. Since,

$$\{\sigma_{i_0}, \ldots, \sigma_{j_4}\} \prec \sigma_{bridge}, \text{ where } \sigma_{bridge} \text{ is the supremum,}$$

we have that $U_{\sigma_{i_0},\ldots,\sigma_{j_4}} \supset A_{\sigma_{bridge}}$.

In particular, any combination of four intersections contains the set $A_{\sigma_{bridge}}$.

There are two particular cases of substructures of the bridge which verify:

$$\sigma_{i_0}, \sigma_{i_1}, \sigma_{i_4}, \sigma_{j_0} \prec \sigma_{diamond_1} \prec \sigma_{bridge}.$$

$$\sigma_{j_1}, \sigma_{j_2}, \sigma_{j_3}, \sigma_{i_2} \prec \sigma_{diamond_2} \prec \sigma_{bridge}.$$

So, the open sets verify

$$U_{\sigma_{i_0},\sigma_{i_1},\sigma_{i_4},\sigma_{j_0}} \supset A_{\sigma_{diamond_1}} \supset A_{\sigma_{bridge}}$$

and

$$U_{\sigma_{j_1},\sigma_{j_2},\sigma_{j_3},\sigma_{i_2}} \supset A_{\sigma_{diamond_2}} \supset A_{\sigma_{bridge}}.$$

The other 4-combinations of vertices have as a supremum σ_{bridge}. So, this implies that for any other combinations of 4 vertexes different from $\sigma_{i_0}, \sigma_{i_1}, \sigma_{i_4}, \sigma_{j_0}$ and $\sigma_{j_1}, \sigma_{j_2}, \sigma_{j_3}, \sigma_{i_2}$ in the bridge structure we have:

$$\begin{aligned} U_{\sigma_{i_0},\ldots,\sigma_{i_4}} = U_{\sigma_{j_0},\ldots,\sigma_{j_4}} = U_{\sigma_{i_0},\ldots,\sigma_{i_3},\sigma_{j_1}} = \ldots = U_{\sigma_{i_0},\sigma_{i_1},\sigma_{i_2},\sigma_{j_0},\sigma_{j_1}} \\ = \ldots = U_{\sigma_{i_0},\sigma_{i_1},\sigma_{j_0},\sigma_{j_1},\sigma_{j_2}} = U_{\sigma_{i_0},\sigma_{j_0},\sigma_{j_1},\sigma_{j_2},\sigma_{j_3}}. \end{aligned}$$

We proceed similarly for the $NC(4)$ structure an open book structures.

There exist 16 bridges, 8 of which connect the structures NC_1 and NC_2 and 8 of which connect NC_3 and NC_4 (following the description in chapter 3 of the NC structures). We apply the procedure depicted above to the 16 bridges.

Finally, using the properties of the sheafs, see [**48**] and applying the procedure above to the 16 bridges, we have that the coboundary map d^2 defines a twofold covering. □

3.2. The kernel of d^3. We want to prove that the kernel of the coboundary map $d^3 : C^3(\mathcal{U}, \mathbb{Z}) \to C^4(\mathcal{U}, \mathbb{Z})$ is non-zero and calculate the kernel explicitly, in order to calculate the cohomology group $H^3(^{\mathrm{D}}\mathrm{Pol}_4, \mathbb{Z})$.

From the topological point of view, the structures $NC(4)$, bridge and openbook are balls. Each bridge structure is glued to two different openbook structures such that the upper half part of the bridge consisting of five vertices is glued to the five vertices of the first openbook structure, the five vertices on the lower half part are glued to the five vertices of the second openbook structure. The five remaining vertices of the second open book structure is glued the five vertices contained in the half lower part of the $NC(4)$ structure and the five remaining vertices of the first openbook structure are glued to five vertices contained in the upper half of the $NC(4)$ structure. This defines a structure closed and bounded of genus 1. Since there exist 16 bride structures, we have a genus 16 structure.

In a manner requiring calculations, the matrix associated to the coboundary operator d^3 is a block circulant one. This is illustrated as follows.

Let us decompose the set of vertices of the complex into four families. We enumerate the components of this decomposition. The first family of vertices consists of those vertices which belong to the upper half part of the $NC(4)$ structure ans to the first open book structure. This family is denoted by the sequence $(i_0, .., i_4)$. The order in which the vertices in the sub sequence of I are given does not matter. The second family consists of those vertices belonging to the upper half of vertices of the bridge structure and to the first open book structure. This family is denoted by the sequence $(j_0, ..., j_4)$. The third family consists of those vertices which belong to the lower half of the bridge structure an to the second openbook structure. It is denoted by $(k_0, .., k_4)$. Finally, the last family is a set of vertices belonging to the lower half part of the $NC(4)$ structure and to the second open book structure. These vertices are denoted by the sequence $(m_0, ..., m_4)$.

LEMMA 7.6. *The matrix associated to the coboundary operator d^3 is a block diagonal matrix where blocks are block circulant.*

PROOF. Let us introduce some notation. The sequence $(i_{p_0}, i_{p_1} i_{p_2} i_{p_3}) \in (i_0, .., i_4)$ means that we take four vertices among the five $(i_0, .., i_4)$. Similarly, $(j_{l_0}, j_{l_1}, ... j_{l_r}) \in (j_0, ..., j_4)$ means that we take $r + 1$ vertices among the ones in $(j_0, ..., j_4)$.

Let us consider the elements in the 4-cochain $C^4(\mathcal{U}, \mathbb{Z})$.

Let us illustrate a few of these elements, given by coboundary operator. One element $f \in C^4(\mathcal{U}, \mathbb{Z})$ is as follows :

$$(d^3 f)_{i_0,..,i_4} = \sum_{q=0}^{4}(-1)^q f(U_{i_0,..\hat{i_q},...,i_4}).$$

There exist 4 such relations for the sequences $(i_0, .., i_4)$, $(j_0, ..., j_4)$, $(k_0, .., k_4)$ and $(m_0, ..., m_4)$.

The other families of elements in the 4-cochain $C^4(\mathcal{U}, \mathbb{Z})$ including vertices belonging to the sequence of vertices $(i_0, .., i_4)$ and the sequence $(j_0, ..., j_4)$ of adjacent vertices in the first open book structure. These relations are as follows:

$$\begin{aligned}(d^3 f)_{i_{p_0},i_{p_1}i_{p_2}i_{p_3}j_r} =& f(U_{i_{p_1},...,i_{p_3},j_r}) - f(U_{i_{p_0},i_{p_2},i_{p_3},j_r}) \\ &+ f(U_{i_{p_0},i_{p_1},i_{p_3},j_r}) - f(U_{i_{p_0},i_{p_1},i_{p_2},j_r}) + f(U_{i_{p_0},i_{p_1},i_{p_2},i_{p_3}}),\end{aligned}$$

where $j_r \in (j_0, ..., j_4)$.

$$\begin{aligned}(d^3 f)_{i_{p_0},i_{p_1}i_{p_2}j_{l_0}j_{l_1}} =& f(U_{i_{p_1},i_{p_2},j_{l_0},j_{l_1}}) - f(U_{i_{p_0},i_{p_2},j_{l_0},j_{l_1}}) \\ &+ f(U_{i_{p_0},i_{p_1},j_{l_0},j_{l_1}}) - f(U_{i_{p_0},i_{p_1},i_{p_2},j_{l_1}}) + f(U_{i_{p_0},i_{p_1},i_{p_2},j_{l_0}}),\end{aligned}$$

where $j_{l_0}, j_{l_1} \in (j_0, ..., j_4)$.

$$\begin{aligned}(d^3 f)_{i_{p_0},i_{p_1}j_{l_0},j_{l_1},j_{l_2}} =& f(U_{i_{p_1},j_{l_0},j_{l_1},j_{l_2}}) - f(U_{i_{p_0},j_{l_0},j_{l_1},j_{l_2}}) \\ &+ f(U_{i_{p_0},i_{p_1},j_{l_1},j_{l_2}}) - f(U_{i_{p_0},i_{p_1},j_{l_0},j_{l_2}}) + f(U_{i_{p_0},i_{p_1},j_{l_0},j_{l_1}}),\end{aligned}$$

where $j_{l_0}, j_{l_1}, j_{l_2} \in (j_0, ..., j_4)$.

$$\begin{aligned}(d^3 f)_{i_{p_0},j_{l_0},j_{l_1},j_{l_2},j_{l_3}} =& f(U_{j_{l_0},j_{l_1},j_{l_2},j_{l_3}}) - f(U_{i_{p_0},j_{l_1},j_{l_2}j_{l_3}}) \\ &+ f(U_{i_{p_0},j_{l_0},j_{l_2},j_{l_3}}) - f(U_{i_{p_0},j_{l_0},j_{l_1},j_{l_3}}) + f(U_{i_{p_0},j_{l_0},j_{l_1}j_{l_2}}),\end{aligned}$$

where $j_{l_0}, j_{l_1}, j_{l_2} \in (j_0, ..., j_4)$.

For the other relations, they are copies of these relations, for couples $(j_0, ..., j_4)$, $(k_0, .., k_4)$; $(k_0, .., k_4)(m_0, ..., m_4)$ and $(m_0, ..., m_4)$,$(i_0, .., i_4)$.

This implies that we have a block circulant matrix. It follows from the properties of such matrices that the kernel is non-zero. □

Corollary 7.7. *The kernel of the coboundary map* $d^3 : C^3(\mathcal{U}, \mathbb{Z}) \to C^4(\mathcal{U}, \mathbb{Z})$ *is non zero.*

Proof. The matrix associated to the coboundary map d^3 is a block diagonal matrix where the block are block circulant. From the properties of such matrices, the kernel is non zero. So, the result follows. □

Let us illustrate, the first block, $Block_1$, in the matrix associated to the coboundary operator. This allows to have an explicit computation of the kernel of d^3.There are three other blocks, which are copies of $Block_1$ and obtained by translation of this block by 25 columns modulo the number of columns of the boundary matrix, which is 95 (there are $25 \times 3 + 20$ columns for the next blocks.

First block in d^3 =

$Block_1(a)$	$Block_1(b)$	0	0
0	$Block_1(a)$	$Block_1(b)$	0
0	0	$Block_1(a)$	$Block_1(b)$
$Block_1(b)$	0	0	$Block_1(a)$

$f(U_{i_{p_0}, i_{p_1} i_{p_2} i_{p_3}})$	$f(U_{i_{p_0}, i_{p_1}, i_{p_2} i_{p_3}, j_{l_0}})$	$f(U_{i_{p_0}, i_{p_1} i_{p_2}, j_{l_0}, j_{l_1}})$	$f(U_{i_{p_0}, i_{p_1}, j_{l_0}, j_{l_1}, j_{l_2}})$	$f(U_{i_{p_0}, j_{l_0}, j_{l_1}, j_{l_2}, j_{l_3}})$
1 −1 1 −1 1	0	0	0	0
1 −1 1 −1 0	1 0 0 0 0	0	0	
1 −1 1 −1 0	0 1 0 0 0	0	0	0
1 −1 1 −1 0	0 0 1 0 0	0	0	0
1 −1 1 −1 0	0 0 0 1 0	0	0	0
1 −1 1 −1 0	0 0 0 0 1	0	0	0
…			…	…
1 −1 1 0 0	0	1 −1 0 0 0	0	0
1 −1 1 0 0	0	0	0 1 −1 0 0	0
1 −1 1 0 0	0	0	0 0 1 −1 0	0
1 −1 1 0 0	0	0	0 0 0 1 −1	0
1 −1 1 0 0	0	0	1 0 −1 0 0	0
1 −1 1 0 0	0	0	1 0 0 −1 0	0
1 −1 1 0 0	0	0	1 0 0 0 −1	0
1 −1 1 0 0	0	0	0 1 0 −1 0	0
1 −1 1 0 0	0	0	0 1 0 0 −1	0
1 −1 1 0 0	0	0	0 0 1 0 −1	0
…			…	…
1 −1 0 0 0	0	0	1 −1 1 0 0	0
1 −1 0 0 0	0	0	1 −1 0 1 0	0
…				…
1 −1 0 0 0	0	0	0 0 1 −1 1	0
1 0 0 0 0	0	0	0	−1 1 −1 1 0
1 0 0 0 0	0	0	0	0 −1 1 −1 1
1 0 0 0 0	0	0	0	−1 1 −1 0 1
1 0 0 0 0	0	0	0	−1 1 0 −1 1
1 0 0 0 0	0	0	0	−1 0 1 −1 1
1 0 0 0 0	0	0	0	0 −1 1 −1 1
…				…
$\mathbf{Block_1(a)}$	$\mathbf{Block_1(b)}$			

$\mathbf{Block_1 = Block_1(a) \oplus Block_1(b)}$

The next blocks concern the other bridge structures.

EXAMPLE 7.1. We give an explicit example that the kernel of d^3 is non-zero. Indeed, take the vector lines :

$$v_1 = (1, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$v_2 = (1, 0, 0, 0, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$v_3 = (0, 0, 0, 0, 1, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$v_4 = (0, 0, 0, 0, 1, 0, 0, 0, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$
$$v_5 = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$
$$v_6 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 1, -1, 1, 0, 0, 0, 0, 0)$$
$$v_7 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, -1, 1, 0, 0, 0, 0, 0)$$
$$v_8 = (1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

These vectors follow the relation: $v_8 - v_1 + v_4 - v_5 + v_2 - v_3 + v_6 - v_7 = 0$, implying the result.

Part 4

Generating and counting signatures

CHAPTER 8

Counting diagrams

In this chapter, we count explicitly the number of diagrams for a fixed degree d and codimension c. Two different methods are employed. The first one, uses a computer program to draw those diagrams. The second one starts with a combinatorial calculation and leads to an efficient computer program, giving the number of diagrams of degree d and codimension c.

1. Generating diagrams

We count the number $N_{bc}(c,d)$ of degree d and co-dimension c diagrams.

1.1. Inductive methods. The number $N_{bc}(0,d)$ of bi-regular signatures of degree d is given by the recursive formula:

(3)

$$N_{bc}(0,d{+}1) = \sum_{0\le j,0\le k,0\le \ell,0\le m,j+k+\ell+m=d.} N_{bc}(0,j)N_{bc}(0,k)N_{bc}(\ell,0)N_{bc}(0,m), \quad N_{bc}(0,0)=1.$$

This recursion is obtained from the following splitting: let $\mathcal{L}_1$ be a red diagonal and $\mathcal{L}_2$ be a blue diagonal intersecting at a point. This pair of curves splits the complex plane in four regions. The summing indices j,k,ℓ,m are the number of roots of a bi-regular polynomial spread in those four regions.

This recursion gives the Catalan-Fuss sequence referred as $A002293$ in the Sloane OEIS Encyclopedia of Integral Sequences:

$$N_{bc}(0,d) = \frac{1}{3d+1}\binom{4d}{d}. \tag{4}$$

Similarly, the number $N_{bc}(1,d)$ of signatures with only one meeting point can be obtained by induction. Indeed, the two intersecting monochromatic lines create a codimension 1 point which splits the diagram into two smaller bi-regular diagrams of respective degrees a and $d-a$.

$$N_{bc}(1,d) = 2d\sum_{a=1}^{d-1} N_{bc}(0,a)N_{bc}(0,d-a), \quad d\ge 3, \tag{5}$$

verify

$$N_{bc}(1,d) = \mathbf{1}_{d\ge 2}4\binom{4d}{d-2}. \tag{6}$$

Until now this sequence was not classified in the Sloane's OEIS Encyclopedia of Integral Sequences. The number of this sequence is A283049.

Inductive methods present difficulties for higher codimension than 1. So, we investigate other methods.

1.2. Perfect matching method. A meeting point is characterized by an X-ing number which takes the values:

(1) 1 if the monochromatic intersections are of valency 4
(2) X if the valency of the meeting point is $2X, X > 2$.

The X-ing number of a diagrams is the sum of the X-ing numbers of each meeting point, $X = X_1 + \cdots + X_k$ for k meeting points.

EXAMPLE 8.1. For $X = 3$, let k be the number of I-monochromatic crossings and m the number of R-monochromatic crossings. We have the following cases:

- $k = 3, m = 0$
- $k = 2, m = 1$
- $k = 1, m = 2$
- $k = 0, m = 3$

For X-ing numbers 0,1,2,3, the perfect matching method is effective to solve the counting problem. However, for X-ing number greater than 3 the perfect matching method fails.

REMARK 8.1. The number of possible signatures $N_{bc}(X, d)$ where $X = \{0, 1, 2, 3\}$ can be solved by Hall's Marriage perfect matching theorem.

Let $d \geq 0$ be the degree of the diagram and let the $4d$ vertices on the circle be indexed $[0, ..., 4d - 1]$. One matches the even vertices together and the odd vertices together, under the constraint that the number of intersection points of curves of opposite color is d and the maximal number of monochromatic intersection points (meeting) is 3 for one color. A matching is drawn as a diagonal in the diagram. We can consider the numeration of the $4d$ vertices as d sets of four vertices on the disk. Each set is made of two starting vertices (one red and one blue) and two incoming vertices (one red, one blue). In counter-clockwise orientation we have:

- the set $4k + 0$ is the blue incoming vertex for the set k,
- the set $4k + 1$ is the red starting vertex for the set k,
- the set $4k + 2$ is the blue starting vertex for the set k,
- the set $4k + 3$ is the red incoming vertex for the set k.

2. Counting diagrams

The two next sections are devoted to counting of the diagrams using different methods.

$X\diagdown$d	0	1	2	3	4	5	6	7	8	9
0	1	1	4	22	140	969	7 084	53 820	420 732	3 362 260
1		0	4	48	480	4 560	42 504	393 120	3 624 768	33 390 720
2			0	30	608	8 740	109 296	1 269 450	14 096 320	151 927 776
3			0	4	344	8 760	157 504	2 388 204	32 737 984	

TABLE 1. Number of bi-chromatic diagrams of degree d and $X \leq 3$.

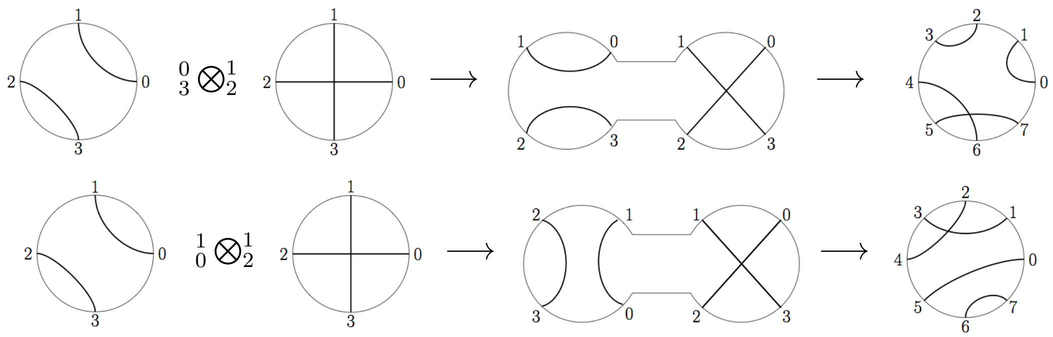
Firstly, we focus on computer algorithms which allow to draw diagrams. these are based on combinatorial matching, merging and superposition of diagrams. This approach is limited to lower degrees than 9. Indeed for $d = 7$ the number of signatures is equal to 8 780 632 and taking into account rotational symmetries and colors, we obtain 397 366 diagrams.

Secondly we investigate an analytic-combinatorial method which enables a general counting of the number of diagrams.

2.1. Drawing diagrams. We call a monochromatic diagram the diagram where only the blue (resp. the red) diagonals are considered. Let $\mathbb{W}$ be the space of monochromatic diagrams and $\mathbb{W}_d$ be the set of monochromatic diagrams of degree d (i.e. with $2d$ terminal vertices). Bi-chromatic diagrams are obtained from a superposition of two monochromatic red and blue diagrams.

Let us introduce two operations on the diagrams. The merging and the connected sum (called also union).

- *Connected sum*. The connected sum of two monochromatic diagrams is given by the binary operation $\otimes$ defined by $\mathbb{W}_n \times \mathbb{W}_m \to \mathbb{W}_n \otimes \mathbb{W}_m \subset \mathbb{W}_{n+m}$.The neutral element ϵ is the empty diagram. The connected sum is given by cutting open diagram 1 and diagram 2 at an arbitrary point and then connecting the two circles such that the orientations on the curves match.



- *Merging operation*. The *merging* concerns only monochromatic diagrams with crossings and is given by the binary operation $C_n \times C_m \to C_n \odot C_m \subset C_{n+m-1}$. The neutral element η is the diagram with no crossings. The *merging* operation of C_n with C_m consists in cutting open the two discs around C_n and C_m at two points on either sides of a chosen edge a for C_n and b for C_m.These two edges fuse together and the two points on either sides of the fused edges are glued.

REMARK 8.2. These operations are associative but not uniquely defined.

Let us now sketch algorithms to the draw diagrams.
2.1.1. *Merging of monochromatic trees.*

(1) We define a dictionary of planar trees with meeting points for a fixed color.
  - a diagram with one single curve corresponding to a degree 1 diagram of X-ing number 0.
  - a four branched tree corresponding to a degree 2 diagram of X-ing number 1
  - $2k$-branched tree for $k \geq 3$ corresponding to a degree k diagram of X-ing number k

  For the sake of brevity, k-star stands for $2k$-branched tree.
(2) We define the merging operation on the trees T_1, T_2 (defined in the algebra chapter) in order to obtain a new tree T.
  Let T_1 and T_2 be two trees in a dictionary. The merging of T_1 with T_2 consists in:
  - choosing one leaf f_1 of neighborhood v_1 in T_1
  - choosing a leaf f_2 of neighborhood v_2 in T_2
  - connecting T_1 to T_2 by suppressing f_1 and f_2 and replacing the edges (f_1, v_1) et (f_2, v_2) by the edge (v_1, v_2).
    ▷ If T_1 is a degree d_1 tree, of X-ing number X_1, and T_2 is a degree d_2 tree of X-ing number X_2 then the new tree T generated has a degree $d = (d_1 + d_2 - 1)$ and X-ing number $X = (X_1 + X_2)$.
(3) Now, we verify if the new tree is in the dictionary or not.
  ▷ If the representative of the class T is not in the dictionary then it is added to the dictionary.
(4) Iterating this procedure, generates all the trees of finite degree d and X-ing number.

2.1.2. *Connected sum: from trees to forests.* One may combine all the trees to generate forests. Therefore it is necessary to define a dictionary of forests. To obtain forests from trees the following steps must be fulfilled.

(1) Define a dictionary of forests.
(2) Obtain new forests from old, using the connected sum of a tree and a forest. More precisely, we combine a tree T_1 and a forest F_2, placing the leaves of a rotation of T_1 on the sector of the circle situated between two leaves of F_2.
(3) Check if the forest is in the dictionary:

▷ If the representative of the new forest is not in the forest dictionary, then it has to be added.

REMARK 8.3. Initially the forest dictionary is the final dictionary of trees.

2.2. Algorithm for the generation of monochromatic diagrams. The connected sum and merging operations have been defined above, we can now proceed to the generation of monochromatic diagrams, necessary for the bi-colored diagrams. As in the previous chapters, we consider on the boundary of the disc the $2d$ vertices of our graph. Take d be packages of 0 and 1 vertices, numerated from 0 to $d-1$ in counter-clockwise orientation. Each package is formed of a starting vertex 0 and of an incoming vertex 1 for the properly embedded oriented curves in the disk. For a given package, the incoming vertex follows immediately the starting vertex, in counter-clockwise orientation.

Let $M[d][X]$ be the set of monochromatic diagrams of degree d and X-ing number.

Two binary operations on the diagrams are allowed, as was describes previously: the union (connected sum) and the merging.

REMARK 8.4. The set $M[d][X]$ of chord diagrams of degree $d > 0$ and X-ing number $X > 0$ is obtained recursively by:

- all the unions (connected sums) of the diagrams $M[d-1][X]$ with a 1-star if $d > 0$.
- all the merges of diagrams of $M[d-1][X-1]$ with the 2-star, if $d > 0$ and $X > 0$.
- all the merges of diagrams if $M[d-k+1][X-k]$ with the k-stars ($k \geq 3$), if $d-k+1 \geq 0$ and $X-k \geq 0$.

2.3. Algorithm for bi-colored forests. In this section a guideline for the computer program is given along with the pseudo-codes, generating the signatures.

2.3.1. *Useful monochromatic diagrams.* In this subsection we introduce the Boolean matrix $useful[0 \dots d_{Max}][0 \dots X_{Max}]$ where $useful[d][X]$ indicates if it is necessary to construct $M[d][X]$ in order to construct $M[d_{max}][X_{max}]$. This tool avoids the generation of un useful diagrams when a particular class of monochromatic diagrams is required.

In order to minimize complexity in time (and space), we introduce a new matrix, instead of $M[d][X]$, which takes into account the set of equivalence classes of the signatures in $M[d][X]$ and the order of the rotation group leaving the representative of the diagram invariant. This new matrix is denoted by $K_1[0 \dots d_{Max}][0 \dots X_{Max}]$.

The elements of $K_1[d][X]$ are all the couples (r, s) where r is a canonical representative of an equivalence class in $M[d][X]$ and s is the order of the rotation group, leaving r invariant.

Initially only $K_1[d=0][X=0]$ which contains the 0-star (the empty diagram) is known.

One may remark that the sets $K_1[d][X]$ for which $useful[d][X]$ is not satisfied are empty.

For instance, for the generation of diagrams of degree 8 and X-ing number, $K_1[8][8]$, it is necessary to compute some $K_1[d][X]$ for smaller co-dimensions and degrees. Initially, the boolean matrix of d lines and X arrays is of the following type:

If one needs the $K_1[d][X]$, this implies that we will need also $K_1[d-1][X]$, $K_1[d-1][X-1]$ $K_1[d-2][X-3]$.

To generate the bi-colored graphs for fixed degree and X-ing number (X, d), one needs all the monochromatic classes $(0, d), (1, d), \ldots, (X, d)$. The next step is to combine all the superpositions of both monochromatic diagrams $\curlyvee_R, \curlyvee_I$ where $\curlyvee_R$ has degree and X-ing number (X_R, d) and $\curlyvee_I$ has degree and X-ing number (X_I, d). The superposition of the diagrams gives a degree d cell of X-ing number $X = X_R + X_I$.

If s_R is the size of the equivalence class $\curlyvee_R$ (order of the rotation group on the diagram) and s_I is the size of the class $\curlyvee_I$, one combines the $s = min(s_R, s_I)$ rotations when $\curlyvee_R$ is superposed with $\curlyvee_I$.

Below, we give two sub-programs to fill in the matrix $K_1[][]$. The first sub-program concerns the useful merges and the second sub-program concerns the useful unions (connected sums).

Generation of all the useful merges of (r, s) with a k-star x

Algorithm 1: Algorithm useful merge

To generate all the useful merges of (r, s) with a k-star x:
Let $X' = X - number(r) + X - number(x)$
Let $d' = degree(r) + degree(x) - 1$.
if *not* $useful[d'][X']$ **then**
  continue
for *all the points p from 0 to* $s-1$ **do**
  Merging of r (in p) with x (in 0) defining (r', s')
if (r', s') *is already in* $K_1[d'][X']$ **then**
  continue
else
  Add (r', s') in $K_1[d'][X']$.

To generate the useful unions of the couples (r, s) with the 1-star x:

Algorithm 2: Algorithm

```
Add (r = the 0-star, s = 1) in K_1[0][0].
for d from 0 to d_max − 1: do
    for X from 0 to i_max: do
        for all (r, s) in K[d][X]: do
            Generate all the useful merges of (r, s) with all the k-stars.
            Generate all the useful unions of (r, s) with the 1-star.
            REMARK 8.5. One may eventually suppress (r, s) from K[d][X] if
            d! = d_max or X! = X_max.
        End of the loop for (r, s)
    End of the loop for X
End of the loop d.
```

One can easily compute the Boolean matrix $useful[0, \dots, d_{Max}][0, \dots, X_{Max}]$, where $useful[d][X]$ indicates by the following procedure in the pseudo-codes.

Algorithm 3 Determines which classes are useful to compute K_1[dMax][cMax]

```
function COMPUTEUSEFUL(dMax, cMax)
    useful ← new boolean[0..dMax][0..cMax]
    INITUSEFUL(useful)
    POPULATEUSEFUL(useful)
    return useful

procedure INITUSEFUL(useful[0..dMax][0..cMax])
    useful[d][c] ← FALSE for all d in [0..dMax], c in [0..cMax]
    useful[dMax][cMax] ← TRUE

procedure POPULATEUSEFUL(useful[0..dMax][0..cMax])
    for d from dMax−1 downto 0 do
        for c in [0..cMax] do
            if ISREACHABLEFROM(useful,c,d) then
                useful[c][d]= True

function USEFULISREACHABLEFROM(useful[0..dMax][0..cMax], d, c)
    if useful[d+1][c] then return TRUE
    if c+1 > cMax then return FALSE
    if useful[d+1][c+1] then return TRUE
    if d+2 > dMax or c+3 > cMax then return FALSE
    if useful[d+2][c+3] then return TRUE
    return FALSE
```

Algorithm 4 Computes the monochromatic classes K_1

```
function COMPUTEMONOCHROMATICCLASSES(dMax, cMax)
    useful ← COMPUTEUSEFUL(dMax, cMax)
    K1 ← new set [0..dMax][0..cMax]
    INITCLASSES (K1)
    POPULATECLASSES (K1, useful)
    return K1

procedure INITCLASSES(K1[0..dMax][0..cMax])
    K1[d][c] ← ∅ for all d in [0 ..dMax], c in [0..cMax]
    Add class ⟨the 0-star, 1⟩ to the set K1[0][0]

procedure POPULATECLASSES(K1[0..dMax][0..cMax], useful[0..dMax][0..cMax])
    for d from 0 to dMax−1 do
        for c in [0..cMax] do
            for ⟨r, s⟩ in K1[d][c] do
                MERGECLASSWITHALLSTARS (⟨r, s⟩, K1, useful)
                UNITECLASSWITHTHE1STAR (⟨r, s⟩, K1, useful)

procedure MERGECLASSWITHALLSTARS(⟨r, s⟩, K1[0..dMax][0..cMax], useful[0..dMax][0..cMax])
    for k from 2 to dMax do
        x ← the k-star
        d'← DEGREE(r) + DEGREE(x) - 1
        c'← CODIM(r) + CODIM(x)
        if d' > dMax or c' > cMax then return
        if not useful[d'][c'] then continue
        for all p in [0..s-1] do
            ⟨r', s'⟩ ← MERGE (r, p, x, 0)
            if ⟨r', s'⟩ ∉ K1[d'][c'] then Add class ⟨r', s'⟩ to K1[d'][c']

procedure UNITECLASSWITHTHE1STAR(⟨r, s⟩, K1[0..dMax][0..cMax], useful[0..dMax][0..cMax])
    x ← the 1-star
    d'← DEGREE(r) + DEGREE(x)
    c'← CODIM(r) + CODIM(x)
    if d' > dMax or c' > cMax then return
    if not useful[d'][c'] then return
    for all p in [0..s-1] do
        ⟨r', s'⟩ ← UNITE (r, p, x)
        if ⟨r', s'⟩ ∉ K1[d'][c'] then Add class ⟨r', s'⟩ to K1[d'][c']
```

An example of the generated diagrams can be found in the appendix A using this computer program.

2.4. Numerical results. By the method developed above we generated the diagrams and numerical results are placed in the tabular below. Firstly is given the table for monochromatic diagrams, see Table 2 and secondly is given the number of signatures, see Table 3 and table 4. $N_{mc}(X, d)$ is the number of monochromatic diagrams of degree d and X-ing number. The results, up to $d = 11$, on the number of diagrams is given in table 2. Let us notice that the

$N_{mc}(0,d)$ Catalan numbers (sequence A002694 in the Sloane's OEIS Encyclopedia)

$$N_{mc}(0,d) = \frac{1}{d+1}\binom{2d}{d} = \frac{(2d)!}{d!(d+1)!},$$
$$= \sum_{0\le j\le d} N_{mc}(0,j)N_{mc}(0,d-j), d \ge 2, \quad N_{mc}(0,0) = N_{mc}(0,1) = 1.$$

The other sequences are now on the on line Integer Sequence Encyclopedia.

$X\diagdown d$	0	1	2	3	4	5	6	7	8	9	10	11
0	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786
1		0	1	6	28	120	495	2 002	8 008	31 824	125 970	497 420
2			0	3	28	180	990	5 005	24 024	111 384	503 880	2 238 390
3			0	1	20	195	1 430	9 009	51 688	278 460	1 434 120	7 141 530
4				0	9	155	1584	12 689	87 360	548 352	3 217 080	17 949 756
5					0	66	1 209	13 377	115 920	866 592	5 864 388	36 933 435
6						0	521	9 814	116 248	1 082 628	8 677 395	62 723 199
7							0	4 516	84 048	1 043 307	10 299 330	87 597 125
8								1 071	40 869	749 835	9 629 960	99 658 020
9								35	11 280	377 152	6 875 225	90 808 795
10								0	952	115 830	3 540 620	64 354 895
11									0	15 732	1 187 300	33 810 855
12										285	211 400	12 170 004
13										0	10 395	2 583 889
14											0	220 110
15												2 530
16												0

TABLE 2. $N_{mc}(X,d)$ number of monochromatic diagrams of degree d with X-ing number X.

2.4.1. *Bi-chromatic.* $N_{bc}(X,d)$ is the number of bi-chromatic diagrams of degree d and X-ing number X. The results on the number of quasi-cells is given up to $d = 9$ in table 3.

The numbers $N_{bc}(c,d)$ of bi-chromatic diagrams of degree d and of co-dimension c using the definition of codimension from chapter 2 is given by table 4. The last line gives the Euler-Poincaré characteristic number, which is the alternate sum of the cardinalities of the $N_{bc}(c,d)$ for fixed $d > 0$. Notice that the co-dimension number is $c \le 2d-3$.

3. Counting diagrams of any degree d

3.1. Canonical splitting. The previous approach gives a visual realization of signatures. However, this method to count diagrams is limited by the space and time of computation. To count diagrams of any degree d and co-dimension

$X \diagdown d$	0	1	2	3	4	5	6	7	8	9
0	1	1	4	22	140	969	7 084	53 820	420 732	3 362 260
1	0	0	4	48	480	4 560	42 504	393 120	3 624 768	33 390 720
2		0	0	30	608	8 740	109 296	1 269 450	14 096 320	151 927 776
3			0	4	344	8 760	157 504	2 388 204	32 737 984	419 969 088
4				0	84	4925	139 896	2 906 442	50 651 520	788 773 293
5					0	1404	77 376	2 379 048	54 885 376	1 063 824 300
6						50	23 216	1 279 026	42 051 904	1 054 350 990
7						0	1 980	399 452	21 996 336	764 136 072
8							0	51 870	7 140 924	391 047 426
9								840	1 168 480	131 199 712
10								0	49 200	24 819 480
11									0	1 759 500
12										16 215
13										0
$N_{bc}(d)$	0	1	8	104	1 656	29 408	568 856	8 780 632	226 748 766	4 433 855 265

TABLE 3. Number $N_{bc}(X, d)$ of bi-chromatic diagrams of degree d and X-ing number X

$c \diagdown d$	0	1	2	3	4	5	6	7	8	9
0	0	1	4	22	140	969	7 084	53 820	420 732	3 362 260
1		0	4	48	480	4 560	42 504	393 120	3 624 768	33 390 720
2			0	30	608	8 740	109 296	1 269 450	14 096 320	151 927 776
3				4	344	8 760	157 504	2 388 204	32 737 984	419 969 088
4				0	80	4 845	138 792	2 893 442	50 507 680	787 265 325
5					4	1 380	75 600	2 340 744	54 275 296	1 055 436 228
6					0	150	24 016	1 258 362	41 238 464	1 036 993 650
7						4	3 816	430 052	21 941 488	750 708 252
8						0	240	85 316	7 906 016	396255 366
9							4	8 512	1 815 704	148 591 440
10							0	350	242 080	37 896 876
11								4	16 528	6 163 560
12								0	480	5 865 533
13									4	29 124
14									0	630
15										4
16										0
$N(d)$	0	1	8	104	1 656	29 408	568 856	8 780 632	226 748 766	4 433 855 265
Euler	0	1	0	0	0	0	0	0	0	0

TABLE 4. Number $N_{bc}(c, d)$ of cells of degree d and co-dimension c.

c, as defined in chapter 2, we use a method based upon combinatorial generating functions, developed in [**19**].

From the previous sections, it appears that an important combinatorial notion is the splitting of signatures. Every bi-regular signature has one unique splitting, somehow non bi-regular signatures σ may have several splittings, as illustrated in Figure 1. A natural way to associate a non bi-regular signature to a unique

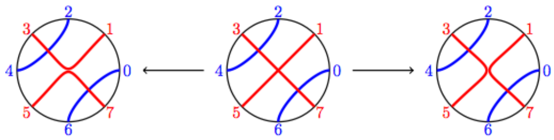


FIGURE 1. Two splittings of a degree 2 signature, with one meeting point

bi-regular one is to define a canonical splitting.

DEFINITION 8.1. *We call canonical splitting a splitting for which the connected components in the disk* $\mathbb{D}$ *are oriented by theorem 2.3*

LEMMA 8.1. *Let* σ *be a signature;* $\mathbb{D}$ *be the unit disk, and let* $\mathcal{C}$ *be a connected component of* $\mathbb{D} \setminus \sigma$. *Then there exists a (unique) integer* $k \in \mathbb{Z}/4\mathbb{Z}$ *such that the border* $\partial\mathcal{C} \cap \partial\mathbb{D}$ *is a non-empty union of arcs of circles of the form* $[P_a, P_{a+1}]$ *with* $a \equiv k \pmod 4$. *The integer* k *is called the* index *of* $\mathcal{C}$.

PROOF. First, since σ is cycle-free, it follows that $\partial\mathcal{C}\cap\partial\mathbb{D}$ is non-empty. Hence, we write it as a disjoint union $[P_{a_1}, P_{a_1+1}] \cup \ldots \cup [P_{a_i}, P_{a_i+1}]$ of arcs of circles, with $a_1 < a_2 < \ldots < a_i$ (and $P_{4d} = P_0$). If $i \geq 2$, let L be the fragment of $\partial\mathcal{C}$ that joins P_{a_1+1} to P_{a_2}. The line L splits the disk $\mathbb{D}$ into two parts: one part $\mathcal{D}_1$ (which excludes the line L) that contains $\mathcal{C}$ and one part $\mathcal{D}_2$ (which contains L) that does not. Let $\mathcal{S}$ be a splitting of σ, and let $\mathcal{S}'$ be the collection of those lines of $\mathcal{S}$ that intersect the area $\mathcal{D}_2$. By construction, no line of $\mathcal{S}$ may go inside the interior of $\mathcal{C}'$, and therefore all lines in $\mathcal{S}'$ belong entirely to the area $\mathcal{D}_2$. If the collection $\mathcal{S}'$ contains ℓ crossing points, then it contains exactly ℓ odd and ℓ even lines, which join 4ℓ points on $\partial\mathbb{D}$ overall. This means that the set of endpoints $\{P_{a_1+1}, \ldots, P_{a_2}\}$ has cardinality 4ℓ, and therefore that $a_2 \equiv a_1 \pmod 4$. This completes the proof. □

3.2. Counting signatures. Canonical splittings of signatures pave the way to recursive decompositions of signatures, which will then allow enumerating them. However, taking into account meeting points forces us to introduce auxiliary combinatorial objects: the partial signatures.

DEFINITION 8.2. *Let* d *be a non-negative integer. Let* $P_0, P_1, \ldots, P_{4d}$ *be points lying in the counter-clockwise order on the unit circle* $\partial\mathbb{D}$. *In addition, let us consider* $2d+1$ *piecewise-affine lines lying inside the unit disk* $\mathbb{D}$, *which we divide between* d odd *(R),* d even *(I) lines and one* base *line, so that:*

- *every point* P_a *belongs to one unique line, and every line touches* $\partial\mathbb{D}$ *at its endpoints only;*

- *the base line joins the points* P_{-1} *and* P_{4d}*;*
- *every odd (respectively, even) line joins points* P_a *and* P_b *such that* $\{a,b\} \equiv \{0,2\} \pmod 4$ *(respectively,* $\{a,b\} \equiv \{1,3\} \pmod 4$*);*
- *every line touches one unique line of the opposite parity, at a point that we call* crossing point*, and it must cross that line at that point;*
- *two lines of the same parity may touch each other at some point, which we call* meeting point*, and they may not cross each other at that point;*
- *the base line may touch even lines only, and may not cross them;*
- *the union of these* $2d+1$ *lines is cycle-free.*

The union of these $2d+1$ *lines is called a* partial signature *of degree d, and the collection of these* $2d+1$ *lines is called a* splitting *of the partial signature. If, furthermore, the points* P_{-1} *and* P_{4d-1} *belong to the same connected component of the partial signature, then the partial signature is said to be* widespread*.*

Like in the case of signature, the co-dimension of a partial signature is the sum of the co-dimension of its meeting points, and two partial signature σ and σ' are *equivalent* if some homeomorphism of the unit disk maps σ to σ' and maps each point P_a to itself.

We denote by $\mathfrak{N}_1(c,d)$ (respectively, $\mathfrak{N}_2(c,d)$ and $\mathfrak{N}_3(c,d)$) the set of signatures (respectively, partial signature and widespread partial signature) with co-dimension c and degree d.

By extension, for $i \in \{1,2,3\}$, we also denote by $\mathfrak{N}_i$ the set $\bigcup_{c,d\geq 0} \mathfrak{N}_i(c,d)$ and by $\mathcal{N}_i$ the associated bivariate generating function, defined by:

$$\mathcal{N}_i(x,y) = \sum_{c,d\geq 0} \#\mathfrak{N}_i(c,d) x^c y^d.$$

Let us notice that $\#\mathfrak{N}_1(c,d) = N_{bc}(c,d)$, introduced above.

We investigate now recursive decompositions of (standard, partial) signatures, which will give rise to equations involving the generating functions $\mathcal{N}_i$.

LEMMA 8.2. *We have* $\mathcal{N}_1 = 1 + y\mathcal{N}_2^4$*. Furthermore, we can associate unambiguously every partial signature with a splitting, which we call* canonical *splitting of this signature.*

PROOF. We define a bijection $\varphi : \mathfrak{N}_1 \mapsto \{\emptyset\} \cup \mathfrak{N}_2^4$, such that $\varphi(\sigma) = \emptyset$ if σ has degree 0, and such that $\varphi(\sigma) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ where $\deg \sigma = \sum_{i=0}^3 \deg \sigma_i + 1$ and $\operatorname{codim}(\sigma) = \sum_{i=0}^3 \operatorname{codim}(\sigma_i)$ if σ has degree at least 1.

First, there exists one unique signature σ of degree 0, hence we safely associate it with $\emptyset$. Then, if σ has degree at least 1, let $\mathcal{S}$ be the canonical splitting of σ. Let L_1 be the line of $\mathcal{S}$ with endpoint P_0, let L_2 be the line of $\mathcal{S}$ that crosses L_1, and let χ be the crossing point at which L_1 and L_2 cross each other.

Observe that L_1 has two endpoints P_{a_0} and P_{a_2}, and L_2 has two endpoints P_{a_1} and P_{a_3}, with and $a_k \equiv k \pmod 4$ for all k. It is straightforward that $0 = a_0 < a_1 < a_2 < a_3 < a_4 = 4d$. Furthermore, the set $\mathbb{D} \setminus (L_1 \cup L_2)$ is formed of

4 connected components $\mathcal{C}_0, \ldots, \mathcal{C}_3$, where $\mathcal{C}_k$ contains the arc of circle $[P_{a_k}, P_{a_{k+1}}]$ on its border.

For $0 \leq k \leq 3$, we do the following: we delete all the points P_x that do not belong to the arc of circle $[P_{a_k}, P_{a_{k+1}}]$ then we renumber each of the points P_x (with $a_k \leq x \leq a_{k+1}$) to P_{x-a_k-1}. Doing so, we observe that $\sigma \cap \overline{\mathcal{C}_k}$ is a partial signature, which we denote by σ_k. Hence, we define $\varphi(\sigma)$ as the tuple $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$.

By construction, every crossing point (beside the point χ) and every meeting point of σ belongs to one unique partial signature σ_k. Furthermore, the mapping φ is clearly bijective. This proves the equality $\mathcal{N}_1 = 1 + y\mathcal{N}_2^4$.

Furthermore, let $\mathcal{B}$ be a partial decomposition and let $\mathcal{B}_\emptyset$ be the unique partial decomposition of degree 0. The canonical splitting of $\varphi^{-1}(\mathcal{B}, \mathcal{B}_\emptyset, \mathcal{B}_\emptyset, \mathcal{B}_\emptyset)$ induces a splitting of $\mathcal{B}$, which is the above-mentioned canonical splitting of $\mathcal{B}$. ☐

LEMMA 8.3. *We have* $\mathcal{N}_2 = \mathcal{N}_1\mathcal{N}_3$.

PROOF. We define a bijection $\varphi : \mathfrak{N}_2 \mapsto \mathfrak{N}_3 \times \mathfrak{N}_1$, such that $\varphi(\sigma) = (\sigma_0, \sigma_1)$ where $\deg \sigma = \deg \sigma_0 + \deg \sigma_1$ and $\mathrm{codim}(\sigma) = \mathrm{codim}(\sigma_0) + \mathrm{codim}(\sigma_1)$ for all partial signature σ.

Let σ be a partial signature of degree d and let $\mathcal{C}$ be the connected component of σ that contains the point P_{-1}. Let a be the greatest element of $\{-1, \ldots, 4d-1\}$ such that $P_a \in \mathcal{C}$. Furthermore, let $\mathcal{D}$ be the unique connected component of $\mathbb{D} \setminus \mathcal{C}$ whose border contains the arc of circle $[P_a, P_{4d}]$.

First, deleting the points P_x that do not belong to the arc of circle $[a+1, 4d-1]$ and renaming every point P_x (with $a < x < 4d$ to P_{x-a-1}), we observe that $\sigma \cap \mathcal{D}$ is a signature (whence $a \equiv 3 \pmod 4$). Second, deleting the points P_x with $a < x < 4d$ and renaming the point P_{4d} to P_{a+1}, we also observe that $\sigma \setminus \mathcal{D}$ is a widespread partial signature. Hence, we define $\varphi(\sigma)$ as the pair $(\sigma \setminus \mathcal{D}, \sigma \cap \mathcal{D})$.

By construction, every crossing point or meeting point of σ belongs to either $\sigma \setminus \mathcal{D}$ or $\sigma \cap \mathcal{D}$, and φ is clearly bijective. Lemma 8.3 follows. ☐

LEMMA 8.4. *We have* $\mathcal{N}_2 = \mathcal{N}_1 + \sum_{k \geq 1} x^{2k-1} y^k \mathcal{N}_2^{4k+1} \mathcal{N}_3^k$.

PROOF. Before defining suitable bijections, we first partition the set $\mathfrak{N}_2$ as follows. Let σ be a partial signature and let L be the base line of σ. If L contains no meeting point, then we say that σ has type 0. Otherwise, let χ be the first meeting point of L (while going from P_{-1} to P_{4d}): if $k \geq 1$ even lines of σ meet L at that point, then we say that σ has type k. Now, for all $k \geq 0$, we denote by $\mathfrak{N}_{2,k}$ the set of partial signatures of type k.

We first define a bijection $\varphi_0 : \mathfrak{N}_{2,0} \mapsto \mathfrak{N}_1$ as follows: if σ has type 0 and degree d, then $\varphi(\sigma)$ is the signature obtained by deleting the points P_{-1}, P_{4d} and the base line of σ. The mapping φ_1 is clearly bijective, and leaves both the degree and the co-dimension unchanged.

Then, for all $k \geq 1$, we define a bijection $\varphi_k : \mathfrak{N}_{2,k} \mapsto \mathfrak{N}_2 \times (\mathfrak{N}_2^4 \times \mathfrak{N}_3)^k$ such that $\varphi(\sigma) = (\sigma_i)_{0 \leq i \leq 5k}$, where

$$\deg \sigma = \sum_{i=0}^{5k} \deg \sigma_i + k \text{ and } \operatorname{codim}(\sigma) = \sum_{i=0}^{5k} \operatorname{codim}(\sigma_i) + k + \mathbf{1}_{k \geq 2}. \tag{7}$$

Let σ be a partial signature of type k and degree d, let $\mathcal{S}$ be the canonical splitting of σ; hereafter we consider exclusively lines in $\mathcal{S}$. Let $L_1^e, \ldots, L_k^e$ be the even lines that touch L at the meeting point χ; let $L_1^o, \ldots, L_k^o$ be the odd lines that cross respectively $L_1^e, \ldots, L_k^e$. For $i \in \{1, \ldots, k\}$, let $P_{a_{0,i}} < P_{a_{2,i}}$ be the endpoints of L_i^e and let $P_{a_{1,i}} < P_{a_{3,i}}$ be the endpoints of L_i^o. In addition, let x be the greatest integer such that $P_{a_{3,i}}$ and $P_{a_{4,i}}$ belong to the same connected component of $\sigma \setminus \{\chi\}$: we set $a_{4,i} = x + 1/2$ and we add a new point P_x on the open arc of circle $(P_{a_{4,i}}, P_{a_{4,i}+1})$.

Assuming that $a_{0,1} < a_{0,2} < \ldots < a_{0,k}$, one checks easily that

$$0 \leq a_{0,1} < a_{1,1} < a_{2,1} < a_{3,1} < a_{4,1} < a_{0,2} < \ldots < a_{4,k} < 4d.$$

Then, let L_i^z be a (new) piecewise-affine line with endpoints X and $P_{a_{4,i}}$ and that lies in $\mathbb{D} \setminus \sigma$. The set $L \cup \bigcup_{i=1}^k (L_i^e \cup L_i^o \cup L_i^z)$ splits the unit disk $\mathbb{D}$ into $5k+2$ connected components:

- for all $0 \leq u \leq 4$ and $1 \leq i \leq k$, one component $\mathcal{C}_{u,i}$ whose border contains the arc of circle $[P_{a_{u,i}}, P_{a_{u+1,i}}]$ (with the convention that $a_{5,i} = a_{0,i+1}$ when $i < k$, and $a_{5,k} = 4d$);
- one component $\mathcal{C}_0$ whose border contains the arc of circle $[P_{-1}, P_{a_{0,1}}]$;
- one component $\mathcal{C}_{-1}$ whose border contains the arc of circle $[P_{4d}, P_{-1}]$.

Let $\mathcal{C} \neq \mathcal{C}_{-1}$ be one such component. Up to deleting the points P_x (where x is an integer or a half-integer of the form $a_{4,i}$) that do not belong to $\partial\mathcal{C}$ and to renumbering the other points P_x from P_{-1} to P_ℓ (where there are $\ell + 2$ points P_x on $\partial\mathcal{C}$), we observe that $(\sigma \cup \bigcup_{i=1}^k L_i^z) \cap \overline{\mathcal{C}}$ is a partial signature; and is even a widespread partial signature if $\mathcal{C} = \mathcal{C}_{3,i}$ for some i. Hence, we denote this partial signature by $\sigma_{u,i}$ if $\mathcal{C} = \mathcal{C}_{u,i}$, or by σ_0 if $\mathcal{C} = \mathcal{C}_0$.

Hence, we set $\varphi_k(\sigma) = (\sigma_0, \sigma_{0,1}, \sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}, \sigma_{4,1}, \sigma_{0,2}, \ldots, \sigma_{4,k})$. It is easy to check that φ_k is bijective. Furthermore, every crossing point of σ besides those between lines L_i^e and L_i^o belongs to some $\sigma_{u,i}$ (or to σ_0) and every meeting point of σ besides χ belongs to some $\sigma_{u,i}$ (or to $\mathcal{C}_0$) as well, with the same co-dimension number. This proves that φ_k satisfies (7) and lemma 8.4 follows. □

The following result follows immediately.

THEOREM 8.5. *The bivariate generating series $\mathcal{N}_1(x, y)$ and $\mathcal{N}_2(x, y)$ are solutions of:*

$$\mathcal{N}_1 = 1 + y\mathcal{N}_2^4 \textit{ and } (\mathcal{N}_2 - \mathcal{N}_1)(\mathcal{N}_1 - x^2 y \mathcal{N}_2^5) = xy\mathcal{N}_2^6.$$

In particular, $\mathcal{N}_1$ is algebraic, and therefore holonomic.

PROOF. Lemmas 8.4 state explicitly the equality

$$\mathcal{N}_2 = \mathcal{N}_1 + \sum_{k\geq 1} x^{2k-1} y^k \mathcal{N}_2^{4k+1} \mathcal{N}_3^k = \mathcal{N}_1 + \frac{xy\mathcal{N}_2^5 \mathcal{N}_3}{1 - x^2 y \mathcal{N}_2^4 \mathcal{N}_3}.$$

Lemmas 8.3 state $\mathcal{N}_2 = \mathcal{N}_1 v_3$ then

$$(\mathcal{N}_2 - \mathcal{N}_1)(\mathcal{N}_1 - x^2 y \mathcal{N}_2^5) = xy\mathcal{N}_2^6$$

which completes the proof. □

From lemma 8.2 we have $\mathcal{N}_1 = 1 + y\mathcal{N}_2^4$, which allows to rewrite the second equality of theorem 8.5

$$(\mathcal{N}_2 - 1 - y\mathcal{N}_2^4)(1 + y\mathcal{N}_2^4 - x^2 y \mathcal{N}_2^5) = xy\mathcal{N}_2^6.$$

REMARK 8.6. The sum of the signatures of co-dimension k weighted by $(-1)^k$ is equivalent to look at the previous equation for the value $x = -1$, that is

$$(\mathcal{N}_2 - 1 - y\mathcal{N}_2^4)(1 + y\mathcal{N}_2^4 - y\mathcal{N}_2^5) + y\mathcal{N}_2^6 = 0.$$

which factorize as

$$(\mathcal{N}_2 - 1)(1 + y\mathcal{N}_2^4)^2 = 0.$$

because $1 + y\mathcal{N}_2^4$ is a sequence with strictly positive coefficients, we have $\mathcal{N}_2 = 1$, then

$$\mathcal{N}_1 = 1 + y\mathcal{N}_2^4 = 1 + y.$$

It follows that the alternate sum, with respect to the co-dimensions, of the number of signatures of degree d takes the value 0 for any $d > 2$.

From the formulas for generating functions of theorem 8.5 we can use computer programs to compute the number of cells for any d and co-dimension c. This program is polynomial in time, see appendix B for the first values of d.

4. Counting simple signatures

Theorem 8.5 provides us with polynomial equations, defining implicitly the series $\mathcal{N}_1$ and $\mathcal{N}_2$. However, the size of these equations (including their degree) do not allow extracting simple general formulas for the coefficients $\#\mathfrak{N}_1(X, d)$.

Therefore, we investigate more closely a restricted class of signatures, which we will subsequently be able to count efficiently.

DEFINITION 8.3. *A signature σ is said to be* simple *if every connected component of σ contains at most one meeting point, and if exactly two lines touch each other at that point.*

We denote by $\#\mathfrak{N}_4(c, d)$ the number of equivalence classes of simple signatures of co-dimension c and degree d, and let $\mathcal{N}_4(x, y) = \sum_{c,d\geq 0} \#\mathfrak{N}_4(c, d) x^c y^d$ be the associated bivariate generating function.

LEMMA 8.6. *We have* $\mathcal{N}_4 = 1 + y\mathcal{N}_4^4 + 4xy^2\mathcal{N}_4^8$.

PROOF. We prove the statement using a bijective proof. Let σ be a signature of degree $d \geq 1$, with canonical splitting $\mathcal{S}$. Let L^e be the even line (in $\mathcal{S}$) with endpoint P_0, and let L^o be the odd line that crosses L^e. We say that σ has type 1 if the union $L^e \cup L^o$ contains no meeting point of σ, and that σ has type 2 otherwise. For $k = 1, 2$, let $\mathfrak{N}_{4,k}$ be the set of signatures of type k.

We first define a bijection $\varphi_1 : \mathfrak{N}_{4,1} \mapsto \mathfrak{N}_4^4$ such that $\varphi_1(\sigma) = (\sigma_i)_{0 \leq i \leq 3}$, where $\deg \sigma = \sum_{i=0}^3 \deg \sigma_i + 1$ and $\operatorname{codim}(\sigma) = \sum_{i=0}^3 \operatorname{codim}(\sigma_i)$.

This bijection is very similar to that of Lemma 8.2: the set $\mathbb{D} \setminus (L^e \cup L^o)$ is formed of 4 connected components $\mathcal{C}_0, \ldots, \mathcal{C}_3$ and, for $i = 0, 1, 2, 3$, up to deleting the points P_x that do not belong to $\partial\mathcal{C}_i$ and to renaming those that belong to $\partial\mathcal{C}_i$, we observe that $\sigma \cap \mathcal{C}_i$ is a simple signature. Denoting this signature by σ_i, we set $\varphi_1(\sigma) = (\sigma_i)_{0 \leq i \leq 3}$. Observing that φ_1 is bijective and satisfies the above equalities (involving co-dimensions and degrees) is then straightforward.

In the same vein, we define another bijection $\varphi_2 : \mathfrak{N}_{4,2} \mapsto \{0, 1, 2, 3\} \times \mathfrak{N}_4^8$ such that $\varphi_2(\sigma) = (k, (\sigma_i)_{0 \leq i \leq 7})$, where $k \in \{0, 1, 2, 3\}$, $\deg \sigma = \sum_{i=0}^7 \deg \sigma_i + 2$ and $\operatorname{codim}(\sigma) = \sum_{i=0}^7 \operatorname{codim}(\sigma_i) + 1$.

Let X be the point at which L^e crosses L^o. This point splits both L^e and L^o into four half-lines $L_0, \ldots, L_3$, with respective endpoints P_{a_0}, P_{a_1}, P_{a_2} and P_{a_3} (with $0 = a_0 < a_1 < a_2 < a_3 < 4d$ and $a_i \equiv i \pmod 4$). One of these half-lines (which we denote L_u in the sequel) contains a meeting point Y, at which it touches another line L'_0 of σ. The line L'_0 itself is crossed by a line L'_1, and none of L'_0 or L'_1 contains a meeting point different from Y. Hence, the set $\mathbb{D} \setminus (L^e \cup L^o \cup L'_0 \cup L'_1)$ is formed of 8 connected components $\mathcal{C}_0, \ldots, \mathcal{C}_7$. Up to reordering these components, we assume that, for all i, the border $\partial\mathcal{C}_i \cap \partial\mathbb{D}$ is an arc of circle $[P_{a_i}, P_{a_{i+1}}]$, with $0 = a_0 < a_1 < \ldots < a_9 = 4d$.

Furthermore, we can again check that, up to deleting and renaming points P_x, $\sigma \cap \mathcal{C}_i$ is a simple signature for all i, and we denote it by σ_i. Hence, we set $\varphi_2(\sigma) = (u, (\sigma_i)_{0 \leq i \leq 7})$, where we recall that L_u was the line to which belongs the point Y. Again, φ_2 is bijective and satisfies the above requirement about degrees and co-dimensions.

Observing that the simple signature of degree 0 is the unique simple signature that does not belong to $\mathfrak{N}_{4,1} \cup \mathfrak{N}_{4,2}$ completes the proof. ☐

THEOREM 8.7. *For all integers* $c, d \geq 0$, *we have*

$$\#\mathfrak{N}_4(c, d) == \mathbf{1}_{d \geq 2c} \frac{4^c}{c + 3d + 1} \binom{4d}{c, d - 2c, c + 3d}.$$

In particular, for a fixed value of c *and when* $d \to +\infty$, *we have*

$$\#\mathfrak{N}_4(c, d) \sqrt{\frac{2}{27\pi}} 4e3c4^4 3^3 d \frac{d^{c-3/2}}{c!}.$$

PROOF. Consider the function $G : (u, z) \mapsto \mathsf{N}_4((2z)^{-2}u, z)$. Setting $u = (2z)^2 t$, we have $1 - G + zG^4 - uG^8 = 0$. Consider also the bivariate functions $h : (u, z) \mapsto 1 + u + z$, $g_u : (u, z) \mapsto h(u, z)^8$ and $g_z : (u, z) \mapsto h(u, z)^4$, and let G_u and G_z be the solutions of the equations

$$G_u(u, z) = u g_u(G_u(u, z), G_z(u, z)) \text{ and } G_z(u, z) = z g_z(G_u(u, z), G_z(u, z)).$$

Note that the function $\mathbf{G} : (u, z) \mapsto h(G_u(u, z), G_z(u, z))$ is solution of the equation

$$\mathbf{G}(u, z) = 1 + G_u(u, z) + G_z(u, z) = 1 + u\mathbf{G}(u, z)^8 + z\mathbf{G}(u, z)^4,$$

i.e. that $G = \mathbf{G}$.

The bivariate Lagrange inversion formula states, for all integers $k, \ell \geq 0$, that

$$\begin{aligned}
[u^k, z^\ell]h(G_u, G_z) &= \frac{1}{k\ell}[u^{k-1}, z^{\ell-1}] \left((\partial_u h)(\partial_z g_u^k)g_z^\ell + (\partial_z h)(\partial_u g_z^\ell)g_u^k + (\partial_u \partial_z h)g_u^k g_z^\ell\right) \\
&= \frac{1}{k\ell}[u^{k-1}, z^{\ell-1}](8k + 4\ell)(1 + u + z)^{8k+4\ell-1} \\
&= \frac{8k + 4\ell}{k\ell}\binom{8k + 4\ell - 1}{k - 1, \ell - 1, 7k + 3\ell + 1} \\
&= \frac{1}{7k + 3\ell + 1}\binom{8k + 4\ell}{k, \ell, 7k + 3\ell}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{N}_4(t, z) &= \mathbf{G}(4z^2 t, z) \\
&= \sum_{k,\ell \geq 0} \frac{1}{7k + 3\ell + 1}\binom{8k + 4\ell}{k, \ell, 7k + 3\ell}(4z^2 t)^k z^\ell \\
&= \sum_{k,m \geq 0} \mathbf{1}_{m \geq 2k} \frac{4^k}{k + 3m + 1}\binom{4m}{k, m - 2k, k + 3m} t^k z^m
\end{aligned}$$

This proves the equality

$$\#\mathfrak{N}_4(c, d) = \mathbf{1}_{d \geq 2c} \frac{4^c}{c + 3d + 1}\binom{4d}{c, d - 2c, c + 3d}.$$

Using the Stirling formula when $d \to +\infty$ then provides the asymptotic estimation of Theorem 8.8, where $\binom{4d}{c, d-2c, c+3d} = \frac{(4d)!}{c!(d-2c)!(c+3d)!}$ stands for the multinomial coefficient. □

Since all signatures of co-dimension $c \leq 1$ are simple, Theorem 8.8 follows immediately.

THEOREM 8.8. *For all integers $d \geq 0$, and co-dimension $c = 0, 1$ we have*

$$\#\mathfrak{N}_1(0, d) = N_{bc}(0, d) = \frac{1}{4d + 1}\binom{4d + 1}{d} \text{ and } \#\mathfrak{N}_1(1, d) = N_{bc}(1, d) = \mathbf{1}_{d \geq 2} 4\binom{4d}{d - 2}.$$

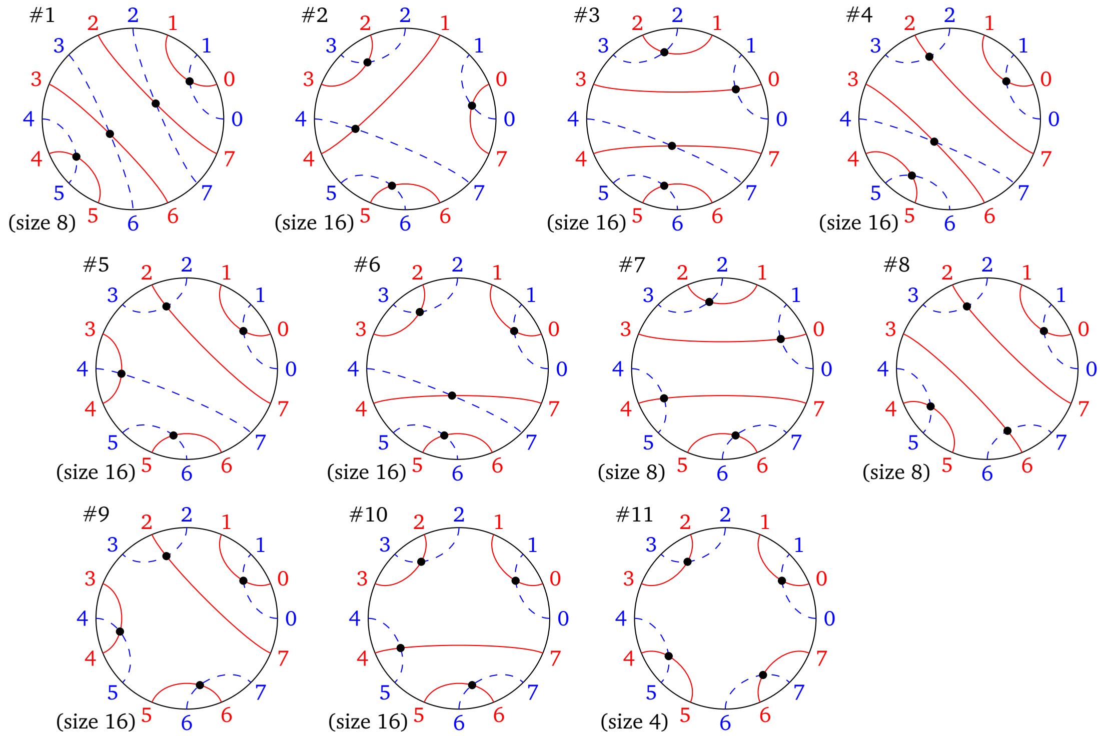
The first part of the statement (the numbers $\#\mathfrak{N}_1(0,d)$) were already computed in section 3 of [?]. We list the number of the new integer sequences emerging from this work which are now on the online Sloane Data base, (OEIS):

- A002293 (Catalan Fuss)
- A283049 codimension 1
- A277877 codimension 2
- A283101 codimension 3
- A283102 codimension 4
- A283103 codimension 5.

Bichromatic diagrams of degree 4

Bichromatic diagrams of degree 4 and codim 0

- 140 diagrams, 11 classes, 3 sizes
- 1 class of size 4
- 3 classes of size 8
- 7 classes of size 16



Bichromatic diagrams of degree 4 and codim 1

- 480 diagrams, 32 classes, 2 sizes
- 4 classes of size 8
- 28 classes of size 16

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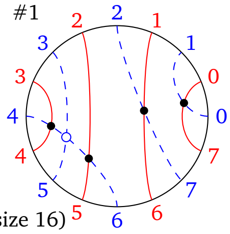
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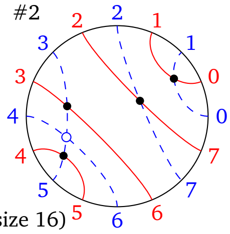
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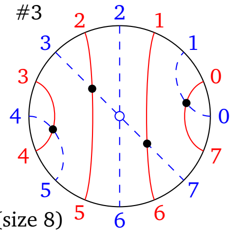
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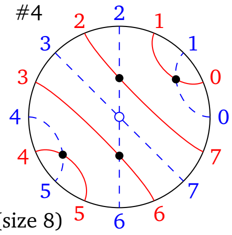
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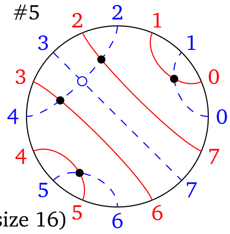
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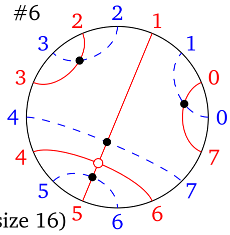
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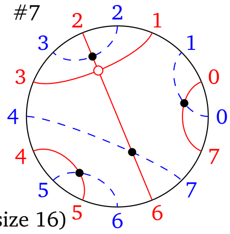
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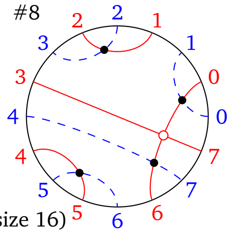
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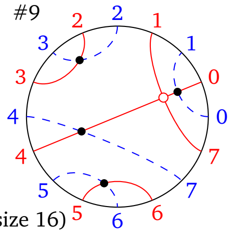
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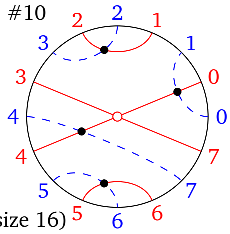
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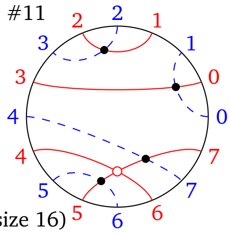
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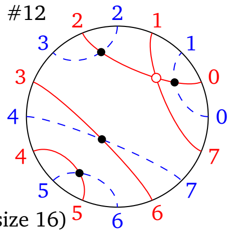
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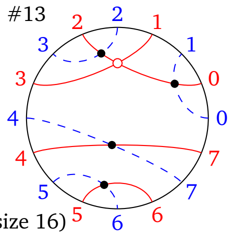
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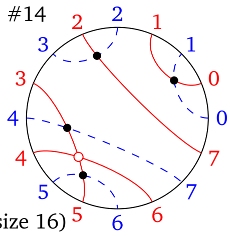
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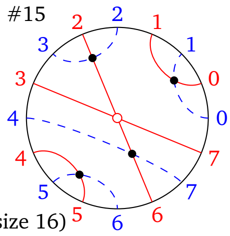
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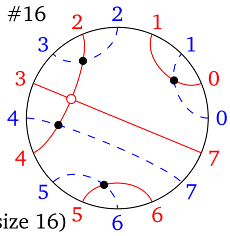
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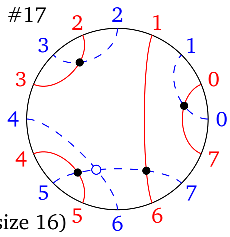
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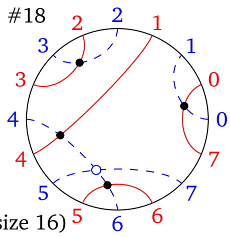
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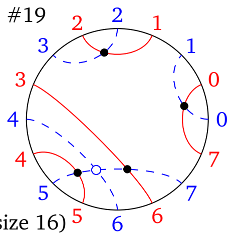
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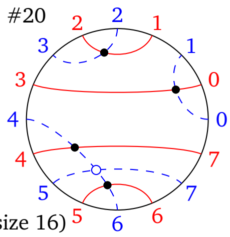
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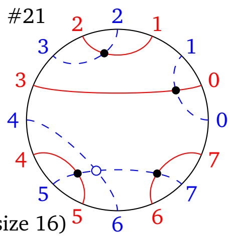
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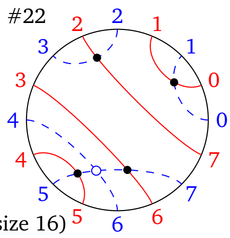
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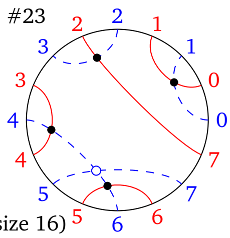
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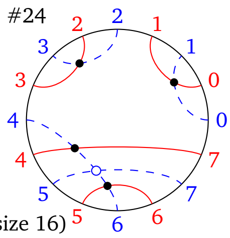
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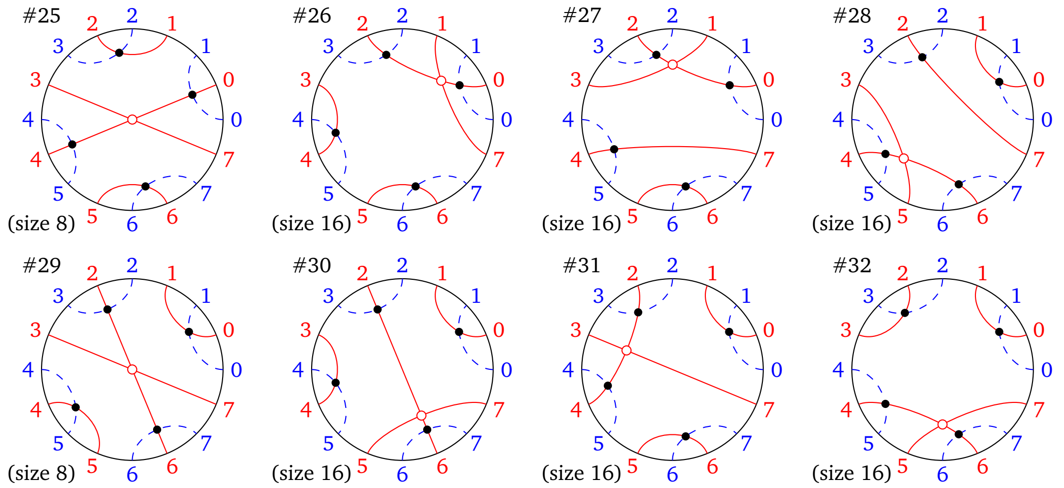
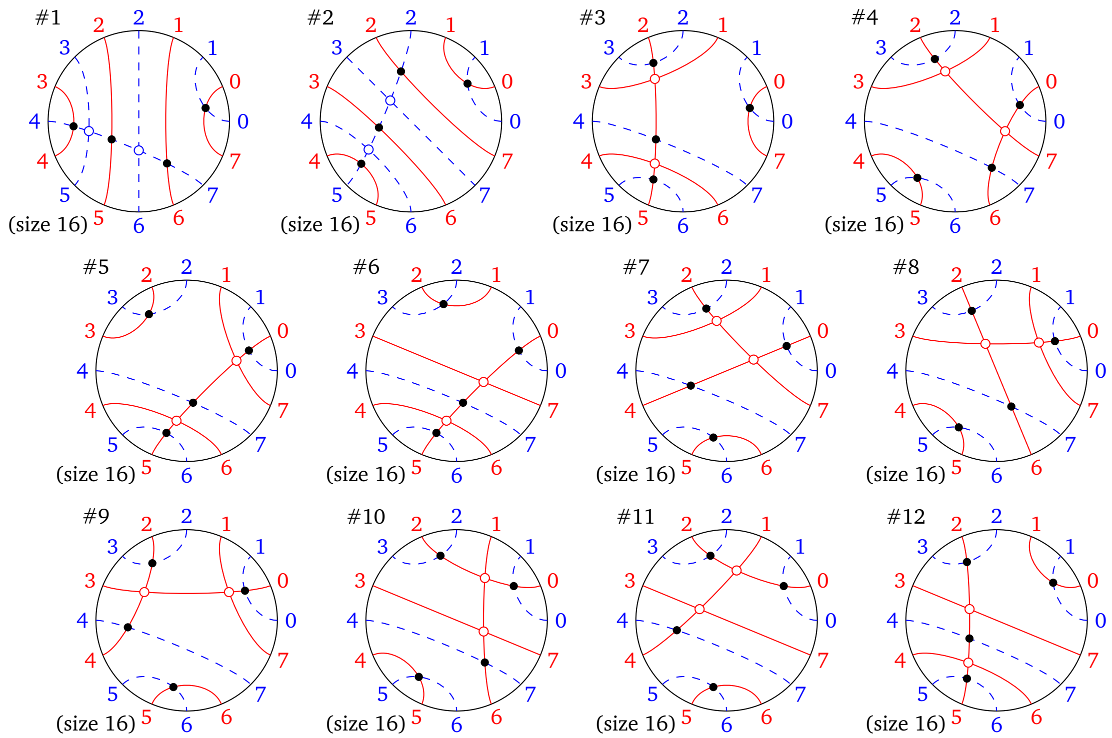


Bichromatic diagrams of degree 4 and codim 2

- 608 diagrams, 40 classes, 2 sizes
- 4 classes of size 8
- 36 classes of size 16



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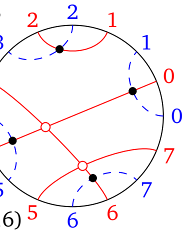
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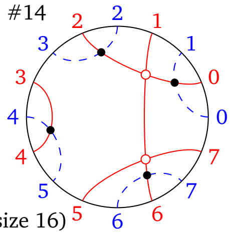
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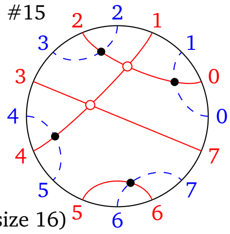
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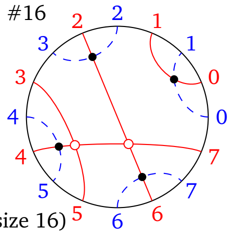
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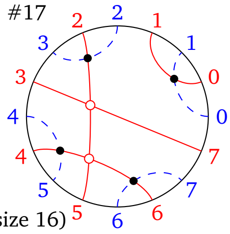
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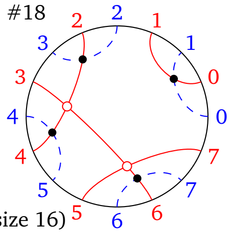
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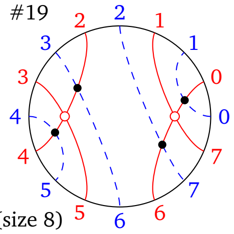
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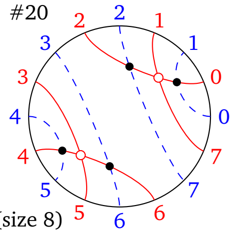
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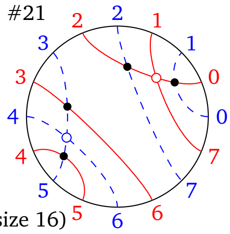
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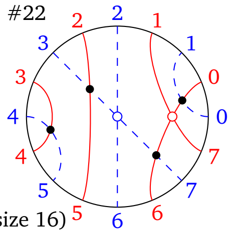
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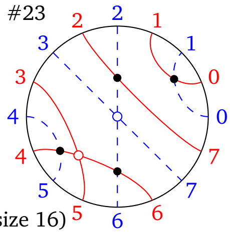
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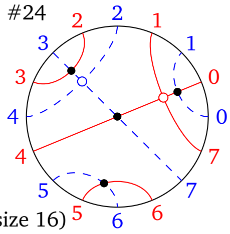
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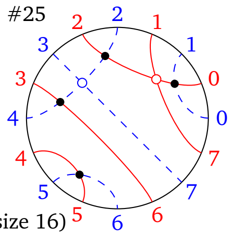
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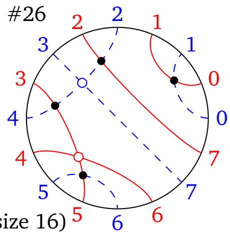
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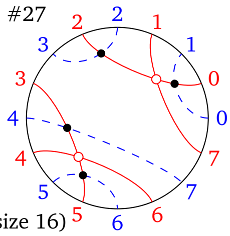
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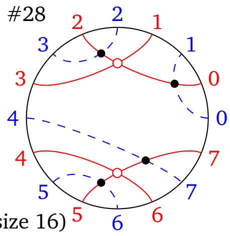
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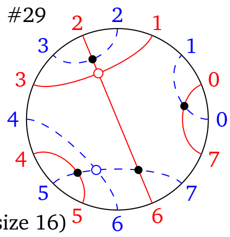
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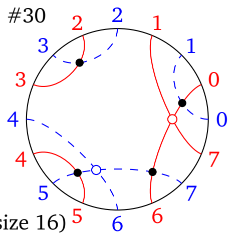
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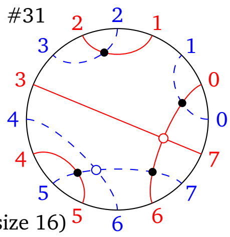
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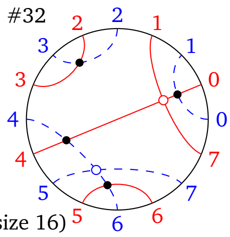
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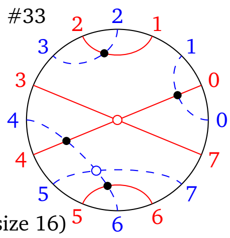
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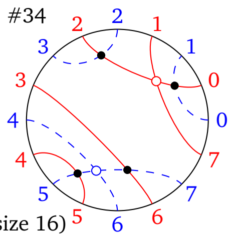
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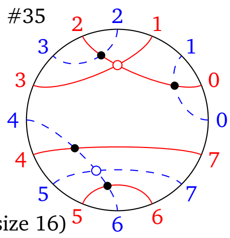
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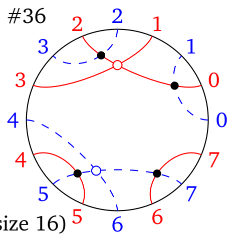
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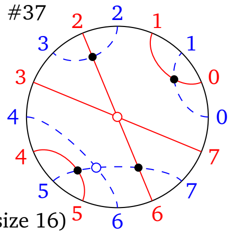
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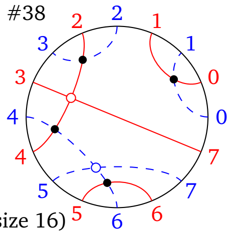
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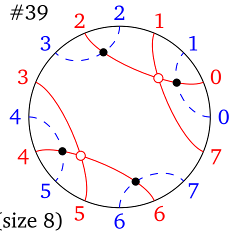
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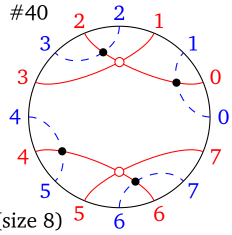
(size 8)

Bichromatic diagrams of degree 4 and codim 3

- 344 diagrams, 24 classes, 2 sizes
- 5 classes of size 8
- 19 classes of size 16

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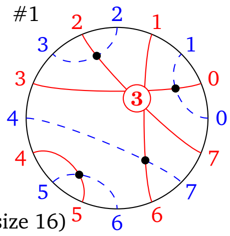
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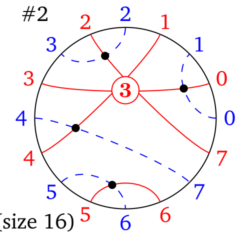
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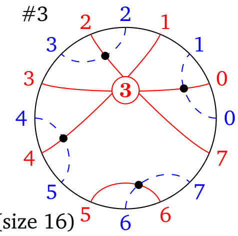
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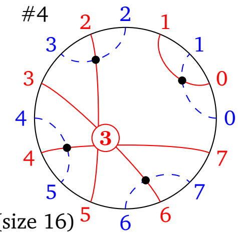
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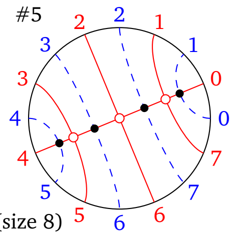
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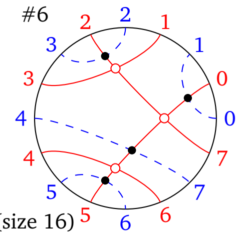
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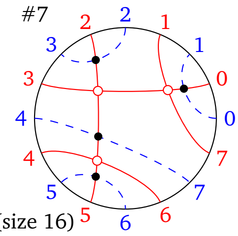
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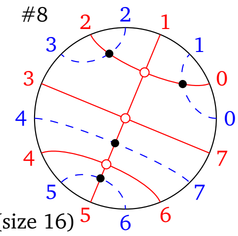
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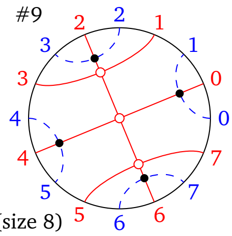
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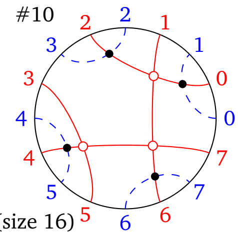
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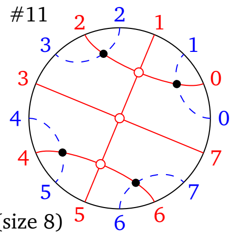
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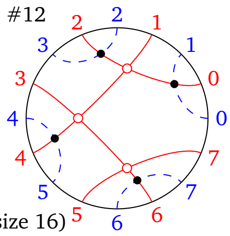
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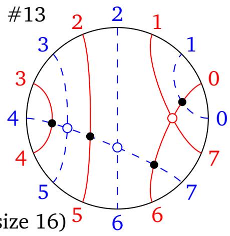
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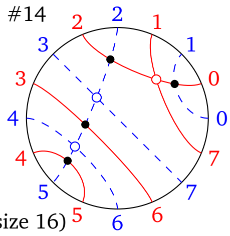
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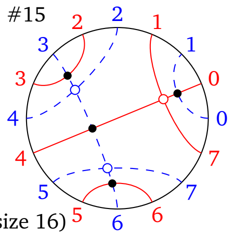
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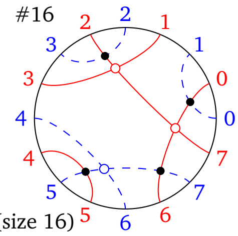
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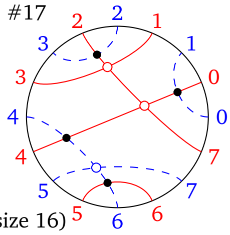

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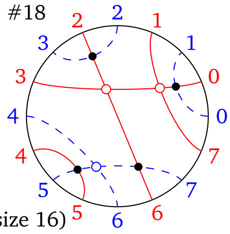

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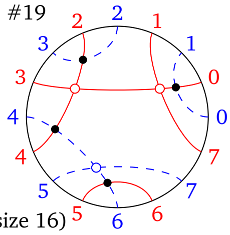

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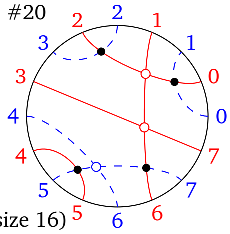

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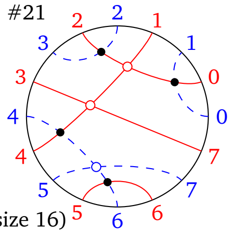

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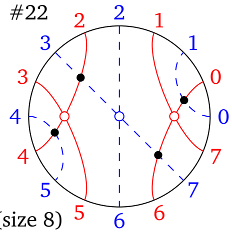

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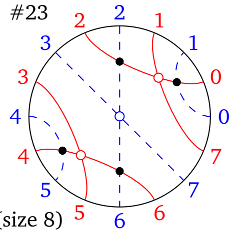

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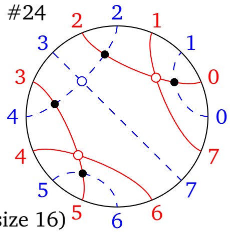
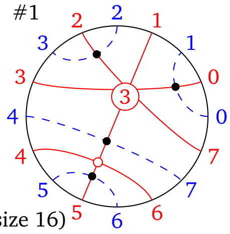


(size 16)

Bichromatic diagrams of degree 4 and codim 4

- 80 diagrams, 6 classes,
- 5 classes of size 16

#1


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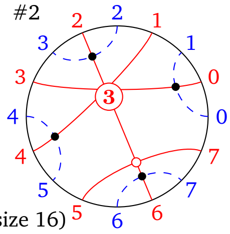

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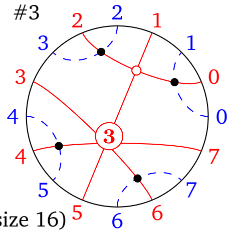

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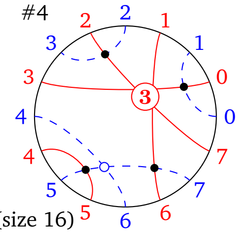

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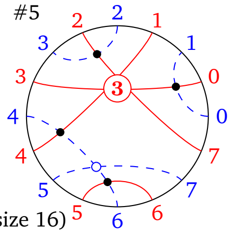
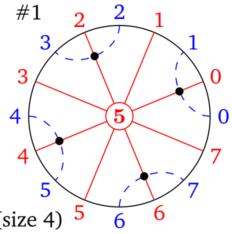


(size 16)

Bichromatic diagrams of degree 4 and codim 5

- 4 diagrams,
- 1 class of size 4

#1


(size 4)

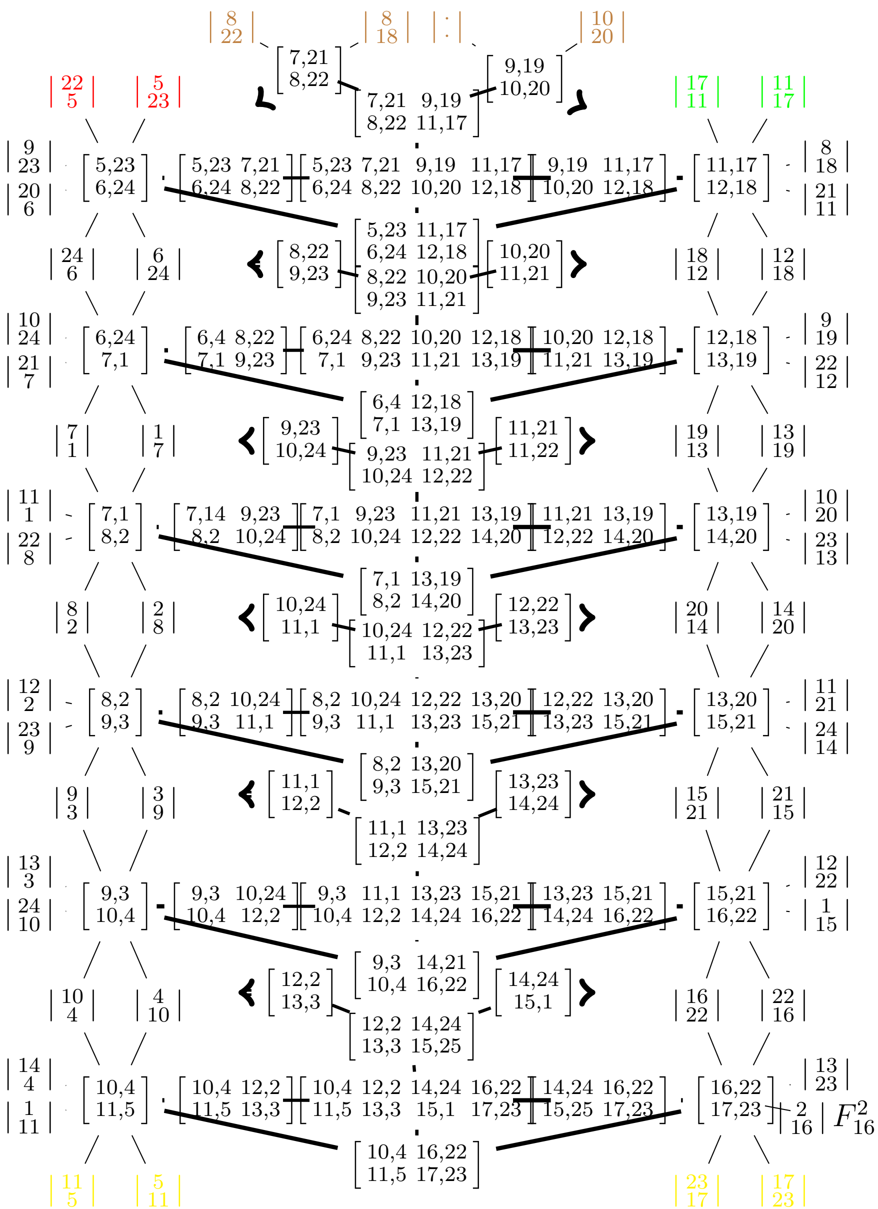
Computer program values - number of cells

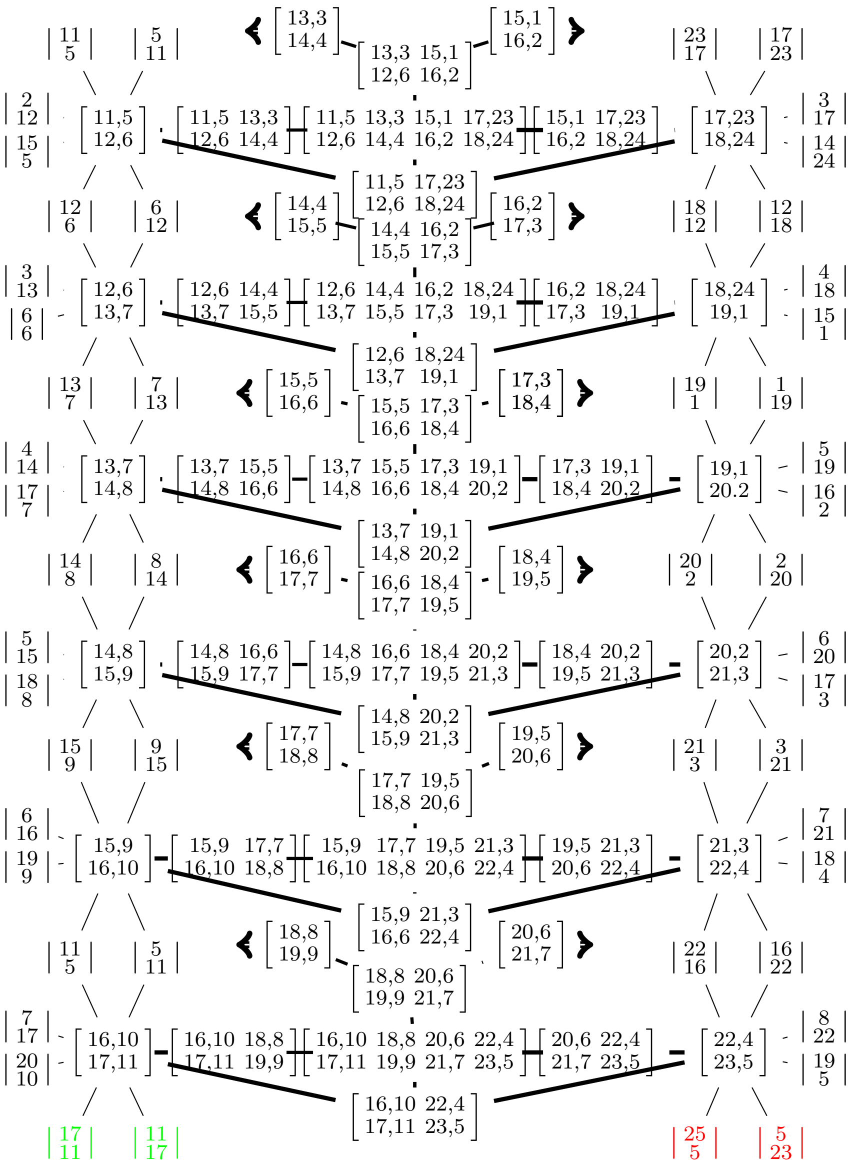
d	c	$N_{bc}(c,d)$
1	0	1
2	0	4
2	1	4
3	0	22
3	1	48
3	2	30
3	3	4
4	0	969
4	1	480
4	2	608
4	3	344
4	4	80
4	5	4
5	0	969
5	1	4 560
5	2	8 740
5	3	8 760
5	4	4 845
5	5	1 380
5	6	150
5	7	4
6	0	7 084
6	1	42 504
6	2	109 296
6	3	157 504
6	4	138 792
6	5	75 600
6	6	24 016
6	7	3 816
6	8	240
6	9	4
7	0	53 820
7	1	393 120
7	2	1 269 450
7	3	2 388 204
7	4	2 893 442
7	5	2 340 744
7	6	1 258 362
7	7	430 052
7	8	85 316
7	9	8 512
7	10	350
7	11	4
8	0	420 732
8	1	3 624 768
8	2	14 096 320
8	3	32 737 984
8	4	50 507 680
8	5	54 275 296
8	6	41 238 464
8	7	21 941 488
8	8	7 906 016
8	9	1 815 704
8	10	242 080
8	11	16 528
8	12	480
8	13	4
9	0	3 362 260
9	1	33 390 720
9	2	151 927 776
9	3	419 969 088
9	4	787 265 325
9	5	1 055 436 228
9	6	1 036 993 650
9	7	750 708 252
9	8	396 255 366
9	9	148 591 440
9	10	37 896 876
9	11	6 163 560
9	12	5 865 533
9	13	29 124
9	14	630
9	15	4
10	0	27 343 888
10	1	307 618 740
10	2	1 603 346 160
10	3	5 141 235 840
10	4	11 345 154 600
10	5	18 230 184 752
10	6	21 792 863 120
10	7	20 126 905 552

d	c	$N_{bc}(c,d)$
10	8	14 021 232 420
10	9	7 354 776 260
10	10	2 845 096 800
10	11	785 431 920
10	12	147 319 320
10	13	17 773 320
10	14	12 650 040
10	15	47 760
10	16	800
10	17	4
11	0	225 568 798
11	1	2 835 722 032
11	2	16 659 866 938
11	3	60 795 581 132
11	4	154 362 306 956
11	5	289 150 871 152
11	6	412 908 658 612
11	7	457 460 566 948
11	8	395 957 692 834
11	9	267 354 108 384
11	10	139 532 842 240
11	11	55 339 181 040
11	12	16 256 668 362
11	13	3 414 327 224
11	14	489 540 436
11	15	45 209 560
11	16	2 495 856
11	17	74 096
11	18	990
11	19	4
12	0	1 882 933 364
12	1	26 162 863 584
12	2	171 064 877 280
12	3	700 024 311 536
12	4	2 010 147 294 672
12	5	4 300 858 168 200
12	6	7 099 049 144 352
12	7	9 225 783 741 888

d	c	$N_{bc}(c,d)$
12	8	9 541 810 226 232
12	9	7 881 383 144 384
12	10	5 185 446 594 624
12	11	2 695 068 395 136
12	12	1 090 946 146 544
12	13	336 954 018 096
12	14	77 243 088 672
12	15	12 683 065 664
12	16	1 427 076 480
12	17	104 154 312
12	18	4 589 376
12	19	109 992
12	20	1 200
12	21	4
13	0	15 875 338 990
13	1	241 614 915 360
13	2	1 740 202 664 200
13	3	7 892 352 548 080
13	4	25 288 375 607 950
13	5	60 843 411 796 440
13	6	113 967 746 771 060
13	7	169 931 096 233 080
13	8	1 043 370 697 693 320
13	9	1 995 773 262 454 440
13	10	1 584 838 661 637 792
13	11	102 019 613 173 088
13	12	52 839 076 931 524
13	13	21 757 054 387 016
13	14	7 003 483 629 012
13	15	1 237 077 740 456
13	16	315 384 324 853
13	17	41 438 992 088
13	18	3 748 508 088
13	19	221 492 544
13	20	7 970 885
13	21	157 508
13	22	1 430
13	23	4

Lego tower for $d = 6$





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13,3 14,4
13,3 15,1 12,6 16,2
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